# EXAMPLE COS1501 MCQ EXAMINATION PAPER WITH DISCUSSIONS

**NOTE: Please read this page before you continue with the paper. This paper contains discussions of the answers. I suggest you first try the paper, *Example MCQ exam.pdf* under Additional Resources, before looking at this discussion.**

Dear COS1501 students,

As from 2019, the COS1501 exam paper is an MCQ paper. You can still work through old exam papers, as you would still have to, for example, do a Venn diagram on rough, do a proof on rough, etc before you will be able to choose the correct alternative for a question.

I suggest you work through the Practice fill-in paper under *Additional Resources* on myUnisa first.

There is also a practice MCQ paper on myUnisa under *Additional Resources*. This paper will also be discussed on your e-tutor site.

The example paper in this document is discussed in detail. Please work through every question. The examination paper that you will be writing in October/November will be similar in format. This does not mean that questions will be repeated. We cannot test all theory in one exam paper, so note that all content has not been included. In a next paper we may test content that was not tested in this paper. We have removed all the rough work spaces from this document, but the exam paper that you will be writing will contain enough spaces for rough work, similar to the practice MCQ paper on myUnisa under Additional Resources.

The exam consists of 50 MCQ questions. If you do not understand why an alternative is correct/incorrect, please contact the lecturer or e-tutor for an explanation. You need to understand all definitions, properties etc., as explained in the study guide in order to answer the questions. If you do not prepare properly, you will find the paper too long. It is not too long. This is a difficult module where you need to apply theory, definitions and properties in order to choose the correct alternative to a question. Knowing the content, will allow you to make decisions about an alternative. Working through this discussion will give you an idea of what you need to look out for when choosing an alternative. You will clearly see that guessing an answer is not going to help you at all. You need to apply the knowledge obtained through the semester. The discussions of the assignment solutions will assist you as well. Spend the time to read through all the explanations – it will help you considerably in the exam.

You will notice that the explanations given in the discussions are not as formal as in the assignment discussions or the study guide. We do however, give references to the study guide, where you can read the formal definitions.

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**SECTION 1**
**SETS AND RELATIONS**
**(Questions 1 to 12) (12 marks)**

**Questions 1 to 8 relate to the following sets:**

Suppose **U = {1, {2, 3}, 3, d, {d, e}, e}** is a universal set with the following subsets:

**A = {{2, 3}, 3, {d, e}}**, **B = {1, {2, 3}, d, e}** and **C = {1, 3, d, e}.**

Before attempting the questions, let us write down the sets U, A, B and C, by adding spaces between elements, so that common elements are vertically grouped:

**U = {1, {2, 3}, 3, d, {d, e}, e}**
**A = {    {2, 3}, 3,     {d, e}}**
**B = {1, {2, 3},     d,         e}**
**C = {1,           3, d,         e}**

We can clearly see, for example, that element {d, e} in U appears in subset A only. Or that elements 1, d and e in U, also appear in subsets B and C. Or that the intersection of A and C contains element 3 only. If you find it difficult to see which elements are in which set, it may help you to write it in this way on rough in the exam.

**Question 1**

Which one of the following sets represents A ∪ B?

1. {1, 2, 3, d, e}
2. {1, {2, 3}, 3, d, e}
3. {1, {2, 3}, 3, {d, e}}
4. {1, {2, 3}, 3, {d, e}, d, e}

**Discussion**:

**A = {    {2, 3}, 3,     {d, e}}**
**B = {1, {2, 3},     d,         e}**

A ∪ B represents the union of the sets A and B. This means, it contains elements that are in A or in B or in both A and B. (Study guide p. 41).

Thus A ∪ B = {1, {2, 3}, 3, d, {d, e}, e}. This corresponds to **alternative 4**. Remember that the order of the elements in the set does not matter, as long as all elements are in the set. We also do not repeat the same element in the set – although the element {2, 3} is in both A and B, it only appears once in the set A ∪ B.

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**Question 2**

Which one of the following sets represents B ∩ C?

1. {1, 3, d, e}
2. {1, d, e}
3. {d, e}
4. {3, {2, 3}}

**Discussion**:

**B = {1, {2, 3}, d, e}**
**C = {1, 3, d, e}**

The intersection of B and C contains all the elements that are in **both** subsets B and C, ie elements that are common in B and C (Study guide p. 42).

**Thus** B ∩ C = {1, d, e}, corresponding to **alternative 2**.

**Question 3**

Which one of the following sets represents C – A?

1. {1, {2, 3}, d, e, {d, e}}
2. {3, d, e}
3. { }
4. {1, d, e}

**Discussion**:

**A = { {2, 3}, 3, {d, e}}**
**C = {1, 3, d, e}**

C – A (Set difference / C without A) is the set of all elements that are in C, but **not** in A (Study guide p. 42). This means that if an element appears in both A and C, it will be removed from C to get C – A. It is clear that element 3 is in both A and C and should be removed from C, thus

C – A = {1, d, e}, corresponding to **alternative 4**.

**Question 4**
Which one of the following sets represents U + B?

1. U
2. {3, {d, e}}
3. {1, {2, 3}, d, e}
4. (U – A) – C

**Discussion**:

**U = {1, {2, 3}, 3, d, {d, e}, e}**
**B = {1, {2, 3}, d, e}**

U + B (symmetric set difference) is the set of elements that are in U or in B, but not in both (Study guide p. 43). This means we have to remove the elements that are in both U and B, which are elements 1, {2, 3}, d and e. We then remain with elements 3 and {d, e}, thus U + B = {3, {d, e}}, which corresponds to **alternative 2**.

**Question 5**
Which one of the following sets represents C ∩ B′ ?

1. {3}
2. {1, d, e}
3. {1, 3, d, e}
4. {1, 3, {d, e}}

**Discussion**:
**U = {1, {2, 3}, 3, d, {d, e}, e}**
**B = {1, {2, 3}, d, e}**
**C = {1, 3, d, e}**

We first determine B′(the complement of B – study guide p. 42). The complement of B is the set of all elements that is in U but not in B. From the above it is clear that if we remove the elements 1, {2, 3}, d and e in B from U, we are left with elements 3 and {d, e}, therefore, B′ = {3 , {d, e}}. Now we can determine which elements should be in C ∩ B′.
C ∩ B′ = {1, 3, d, e} ∩ {3, {d, e}} = {3}, corresponding to **alternative 1**. (See definition of intersection in Study guide p.42).

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**Question 6**
Which one of the following statements regarding $\mathcal{P}$(A) is true?

1. $\mathcal{P}$(A) = { { }, {{2, 3}}, {3}, {{d, e}}, {{2, 3}, 3}, {{2, 3}, {d, e}}, {3, {d, e}} }
2. The cardinality of $\mathcal{P}$(A) is 16.
3. 3 $\in$ $\mathcal{P}$(A)
4. {{ }} $\subset$ $\mathcal{P}$(A)

**Discussion**:

**A = {{2, 3}, 3, {d, e}}**

To answer the question, we must first determine $\mathcal{P}$(A), the power set of A (Study guide p. 45). The power set of a set consists of all the possible **subsets**, including the empty set, that can be formed from the set A. The assignment solutions and the study guide describe in detail how to do this.

Set A has three elements: {2, 3}, 3 and {d, e}. We first form subsets that contain 1 element:
As A have three elements, there are three subsets of A with one element:
{{2, 3}} contains the element {2, 3} of A;
{3} contains the element 3 of A, and
{{d, e}} contains the element {d, e} of A.
Next we determine the subsets containing 2 elements of A:
{{2, 3}, 3} contains elements {2, 3} and 3 of A;
{{2, 3}, {d, e}} contains elements {2, 3} and {d, e} of A, and
{3, {d, e}} contains the elements 3 and {d, e} of A.
Next we determine the subsets containing 3 elements of A:
{{2, 3}, 3, {d, e}} contains all three elements {2, 3}, 3 and {d, e} of A.
In order to complete $\mathcal{P}$(A), we need to add the empty set and all the subsets that we determined above, thus
$\mathcal{P}$(A) = { { }, {{2, 3}}, {3}, {{d, e}}, {{2, 3}, 3}, {{2, 3}, {d, e}}, {3, {d, e}}, {{2, 3}, 3, {d, e}} }.
We can also use the symbol Ø for the empty set, thus
$\mathcal{P}$(A) = { Ø, {{2, 3}}, {3}, {{d, e}}, {{2, 3}, 3}, {{2, 3}, {d, e}}, {3, {d, e}}, {{2, 3}, 3, {d, e}} }.
The powerset of a set has $2^n$ elements, where n is the number of the elements in the original set. This means that $\mathcal{P}$(A) has $2^3 = 8$ elements, because A has three elements. We also say that the cardinality of $\mathcal{P}$(A) is 8.

Looking at the alternatives, you should eliminate alternative 1, because it only has 7 elements. Alternative 2 is clearly also incorrect as we have determined that the cardinality is 8.

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What about alternative 3? If $3 \in \mathcal{P}(A)$, it means that 3 must be in the set $\mathcal{P}(A)$. Do you see an element 3 in $\mathcal{P}(A)$? No, the elements of $\mathcal{P}(A)$ are **subsets** of A, and 3 is not a subset. It is true that $\{3\} \in \mathcal{P}(A)$, but $3 \in \mathcal{P}(A)$ is false.

How about alternative 4? $\{\{\ \}\} \subset \mathcal{P}(A)$. This means that the set containing the empty set is a subset of $\mathcal{P}(A)$. We know $\{\ \} \in \mathcal{P}(A)$. If we want to form a subset of A containing the empty set as an element, what do we do? We include the element in a set, thus
$\{\{\ \}\}$ is a subset containing one element, namely the element $\{\ \}$ of $\mathcal{P}(A)$. It is therefore true that
$\{\{\ \}\} \subset \mathcal{P}(A)$, and **alternative 4** should be selected.

**Question 7**
Let T = {(1, 1), (3, d), (e, d), (1, d), (e, 1), (e, 3)} be a relation on the set C. Which one of the following statements is **true**?

1. T satisfies trichotomy.
2. T is reflexive.
3. T is transitive.
4. T is symmetric.

**Discussion**:
**C = {1, 3, d, e}, and** T = {(1, 1), (3, d), (e, d), (1, d), (e, 1), (e, 3)} is a relation on C.

We look at each alternative to determine whether it is true or false.
Alternative 1 states that T satisfies trichotomy. A relation on a set A satisfies trichotomy if, for every x and y, with $x \neq y$, x and y is comparable. This means that the relation must contain ordered pairs for pairing each element of A to every other element of A. For the formal definition of trichotomy, see Study guide p. 78.
T = {(1, 1), (3, d), (e, d), (1, d), (e, 1), (e, 3)}. The elements of C are 1, 3, c and d. This means 1 must appear in ordered pairs with 3, c and d; 3 must appear in ordered pairs with 1, c and d, etc. The order of the coordinates does not matter.
Does 1 pair up with 3, d and e? Well there is no ordered pair (1, 3) or (3, 1) in T. So we can stop testing straight away, because T does not satisfy trichotomy.
The relation S = {(1, 3), (3, 1), (3, d), (e, d), (1, d), (e, 1), (e, 3)} on C satisfies trichotomy. Do you agree? The fact that both (1, 3) and (3, 1) are in the relation makes no difference – at least one of the pairs must be in the relation.

Alternative 2 states that T is reflexive. A relation on a set A is reflexive iff for each element x in A, the pair (x, x) is in the relation (Study guide p. 75). (1, 1) is in the relation, but (3, 3), (d, d) and (e, e) is not. So T cannot be reflexive.

Alternative 3 states that T is transitive (Study guide p.77). A relation T on a set A is transitive, when, for all x, y, z in A, if (x, y) ∈ T and (y, z) ∈ T then (x, z) ∈ T. So how do we test this? We start with the first ordered pair (x, y) in T, find pairs, if any, of the form (y, z), and see if (x, z) is in the relation. In other words, we find pairs which have y as the first coordinate:
T = {(1, 1), (3, d), (e, d), (1, d), (e, 1), (e, 3)}.
(1, 1) and (1, d) are in T, thus (1, d) must be in T. It is in fact in T.
(e, 1) and (1, 1) are in T, thus (e, 1) must be in T. It is in fact in T.
(e, 1) and (1, d) are in T, thus (e, d) must be in T. It is in fact in T.
(e, 3) and (3, d) are in T, thus (e, d) must be in T. It is in fact in T.
There are no more pairs that satisfy the criteria. Therefore, T is transitive and **alternative 3** is the correct alternative.
Alternative 4 states that T is symmetric. A relation on a set is symmetric, if for every (x, y) in the relation, (y, x) is also in the relation (Study guide p. 76). We give one counterexample but you can find many for this relation. (3, d) ∈ T, but (d, 3) ∉ T, so T cannot be symmetric.

**Question 8**
Let S = {(1, 3), (1, d), (1, e), (d, e), (3, d)} be a relation on set C. Which one of the following statements regarding S is true?

1. S is a strict total order.
2. S is a weak partial order.
3. If (e, 3) is added to S, S would satisfy trichotomy.
4. If (e, 3) is added to S, it would make S transitive.

**Discussion**:
**C = {1, 3, d, e}, and** S = {(1, 3), (1, d), (1, e), (d, e), (3, d)}.
A strict total order is a relation that is irreflexive, transitive and antisymmetric, and satisfies trichotomy. A weak partial order is reflexive, transitive and antisymmetric (Study guide p.87-88).

So let us see which properties S has:
- S is irreflexive (Study guide p. 75), because there is no ordered pair (x, x) in S for each x in C.
- S is antisymmetric, because for each pair (x, y) in S, (y, x) is not in S.
- S is not transitive, because pairs (3, d) and (d, e) are in S but (3, e) is not in S.
- S does not satisfy trichotomy. There is no pairing between 3 and e.

Looking at alternative 1, we can see that it is not true, because to be a strict partial order, S has to be irreflexive, antisymmetric and transitive, and satisfy trichotomy. We have just shown that it is not transitive, and does not satisfy trichotomy.

Alternative 2 is not true. To be a weak partial order, S has to be reflexive, transitive and antisymmetric. We've shown that it is irreflexive, it is not transitive and it is antisymmetric. All the criteria are not met, so S cannot be a weak partial order.

Alternative 3: We have shown above that S does not satisfy trichotomy because neither of the pairs ((3, e) and (e, 3) are in S. This alternative, however, adds (e, 3), which means S will now satisfy trichotomy. So **alternative 3** is the correct answer.

Let us also check alternative 4. We have shown above that to be transitive, ordered pair (3, e) must be added. Adding (e, 3) as the alternative suggests, will therefore not make S transitive.

**Questions 9 to 12 are based on set A = {1, 2, {2}, {{1}, 3}}.**

**Question 9**
Which one of the following statements provides some of the **elements** in A?

1. 1, 3
2. {1, {2}}
3. {1}, 3
4. 1, {{1}, 3}

**Discussion:**
**A = {1, 2, {2}, {{1}, 3}}.** To list all the elements in A, we remove the outer set of brackets, and we are left with elements 1, 2, {2} and {{1}, 3} of A. Alternative 1 is clearly incorrect, because 3 is not an element of A. Similarly {1, {2}} in alternative 2 is not an element of A. In alternative 3, {1} and 3 are elements of {{1}, 3}, but they are not elements of A. **Alternative 4** gives two elements of A namely 1 and {{1}, 3}, so this is the correct alternative. Make sure that you understand the difference between an element and a set/subset. Two of the elements in A, namely {2}, and {{1}, 3} happen to be sets, but they are still elements of A. Read the questions carefully to determine when we ask for elements and when we ask for subsets.

**Question 10**
Which one of the following is NOT a valid relation on A?

1. {(2, {2})}
2. {({1}, 3), (1, 2), ({2}, {2})}
3. {(1, 1), (1, 2), ({{1}, 3}, {2})}
4. {({{1}, 3}, {{1}, 3})}

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**Discussion:**

**A = {1, 2, {2}, {{1}, 3}}.**

A relation on a set A consists of ordered pairs (x, y) such that all x and y are in A. Let us list the ordered pairs in each of the alternatives to see if the coordinates in each pair, are in A. Alternative 1 is a relation containing ordered pair (2, {2}). Are 2 and {2} elements of A? Yes.

Alternative 2 is a relation containing ordered pairs ({1}, 3), (1, 2) and ({2}, {2}). Is {1} and 3 of ordered pair ({1}, 3) elements of A? No. This means that **alternative 2** is not a relation on A.

Alternative 3 is a relation containing ordered pairs (1, 1), (1, 2) and ({{1}, 3}, {2}). In the first two ordered pairs, 1 and 2 are both in A. In the last ordered pair both {{1}, 3} and {2} are elements in A. Alternative 3 is therefore a valid relation on A.

Alternative 4 is a relation containing ordered pair ({{1}, 3}, {{1}, 3}) with x = y. {{1}, 3} is an element in A, which makes this alternative a valid relation on A.

**Question 11**

Which one of the following is NOT a **subset** of the set A?

1. {{{1}, 3}}
2. {2, {2}, {1}, 3}
3. {1, 2, {2}, {{1}, 3}}
4. {2, {2}}

**Discussion**:

**A = {1, 2, {2}, {{1}, 3}}.**

You can find the definition of a subset in the Study guide on p. 40. A set B is a subset of a set A, only if every element in B is also an element in A.

Alternative 1: Is {{{1}, 3}} ⊆ {1, 2, {2}, {{1}, 3}}? Yes, element {{1}, 3} in {{{1}, 3}} is also in A.

Alternative 2: Is {2, {2}, {1}, 3} ⊆ {1, 2, {2}, {{1}, 3}}? Are elements 2, {2}, {1} and 3 in the set in this alternative in A? While 2 and {2} are elements in A, {1} and 3 are not elements in A. This alternative is therefore not a subset of A. **Alternative 2** should therefore be selected.

Alternative 3: Is {1, 2, {2}, {{1}, 3}} ⊆ {1, 2, {2}, {{1}, 3}}? Yes, in fact, the subset in this alternative contains all the elements in A.

Alternative 4: Is {2, {2}} ⊆ {1, 2, {2}, {{1}, 3}}? Yes, elements 2 and {2} in this set, are also elements in set A.

**Question 12**
Which one of the following sets is a partition of the set A?

1. {{{1} ,3}, {1, 2, {2}}}
2. {{2}, {1, 2}, {{{1}, 3}}}
3. {1, 2, {2}, {{1}, 3}}
4. {{{{1}, 3}}, {1, {2}}, {2}}

**Discussion:**
A partition P of a non-empty set A, has as its elements (also called members), a number of subsets of A such that: (i) there are no empty subsets, (ii) for any two non-equal subsets in P, the intersection of the two subsets is empty, and (iii) the union of all the subsets in P is equal to set A (see study guide p. 94 for the formal definition).

**A = {1, 2, {2}, {{1}, 3}}**

We will check each alternative against the criteria given above.

Alternative 1 : {{{1} ,3}, {1, 2, {2}}}. The set consists of two subsets. Are both these subsets indeed subsets of A?
The set {1, 2, {2}} contains the elements 1, 2 and {2} from A.
The set {{1} ,3} is not a subset of A, but an element of A. So we do not have to check any further. The set in this alternative is not a partition of A.

Alternative 2 : {{2}, {1, 2}, {{{1}, 3}}}. The set consists of three subsets. Are all of these subsets indeed subsets of A?
The set {2} is a subset of A containing the element 2 of A.
The set {1, 2} is a subset of A containing the elements 1 and 2 of A.
The set {{{1}, 3}} is a subset of A containing the element {{1}, 3} of A.

Does the set contain an empty set? No
Is {2} ∩ {1, 2} ∩ {{{1}, 3}} = { }? Yes
Is {2} ∪ {1, 2} ∪ {{{1}, 3}} = A? No. {2} ∪ {1, 2} ∪ {{{1}, 3}} = {1, 2, {{1}, 3}} ≠ A. Therefore alternative 2 is not a partition of A. The element {2} in A has not been included in any of the subsets.

Alternative 3 : {1, 2, {2}, {{1}, 3}}. The set consists of all the elements of A which are not subsets. We do not have to look any further. This set cannot be a partition of A.

Alternative 4 : {{{{1}, 3}}, {1, {2}}, {2}}. The set consists of three subsets. Are all of these

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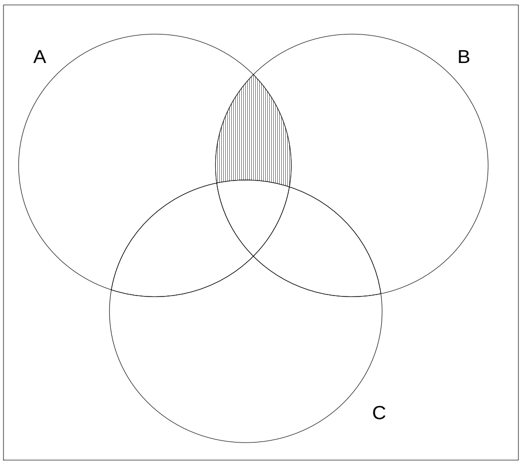
subsets indeed subsets of A?
The set {{{1}, 3}} is a subset of A containing the element {{1}, 3} of A.
The set {1, {2}} is a subset of A containing the elements 1 and {2} of A.
The set {2} is a subset of A containing the element 2 of A.

Does the set contain an empty set? No.
Is {{{1}, 3}} ∩ {1, {2}} ∩ {2} = { }? Yes.
Is {{{1}, 3}} ∪ {1, {2}} ∪ {2} = A? Yes. Therefore **alternative 4** is a partition of A.

**SECTION 2**
**SET THEORY**
**(Questions 13 to 17) (5 marks)**

**Question 13**
Consider the following Venn diagram with A, B and C sets from the universal set U:




Which one of the following alternatives describes the set represented by the Venn diagram correctly? (***Hint***: Draw the Venn-diagrams in the alternatives on rough to find a match.)

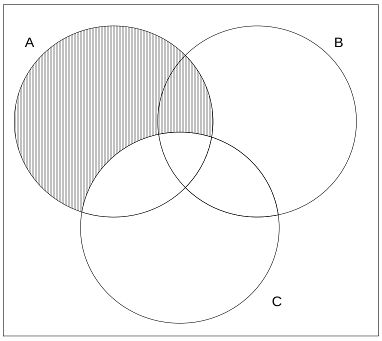
1. (A – C) ∩ (B ∪ C)
2. (A ∩ C) ∩ (B ∪ C)
3. (A – C) ∪ (B ∪ C)
4. (C – A) ∪ (B ∪ C)

**Discussion**:

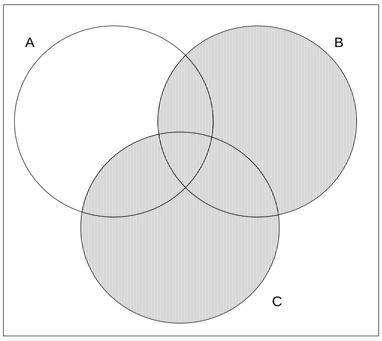
We draw Venn diagrams of the sets in the different alternative step by step:

Alternative 1: (A – C) ∩ (B ∪ C)

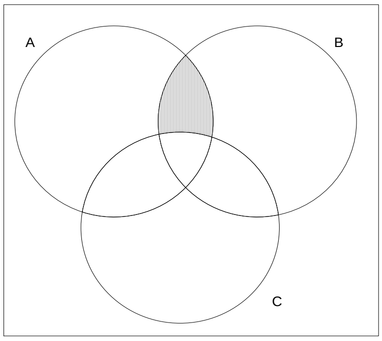
(A – C)




(B ∪ C)



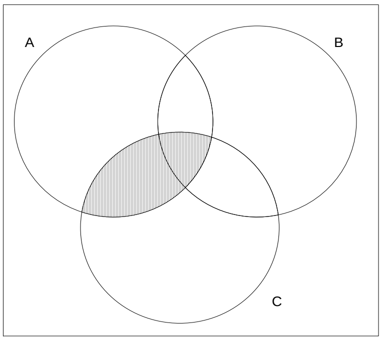

(A – C) ∩ (B ∪ C)



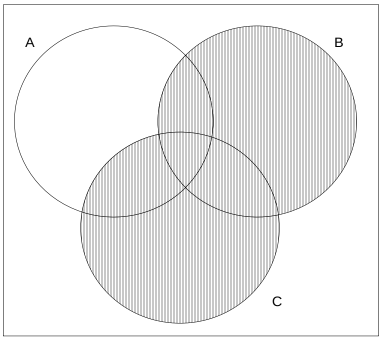

It is clear that the result is the same as the given Venn diagram. **Alternative 1** should therefore be selected.

Alternative 2: (A ∩ C) ∩ (B ∪ C)

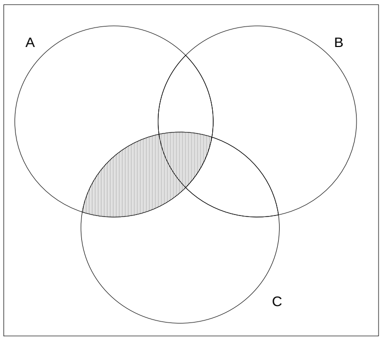
(A ∩ C)




(B ∪ C)



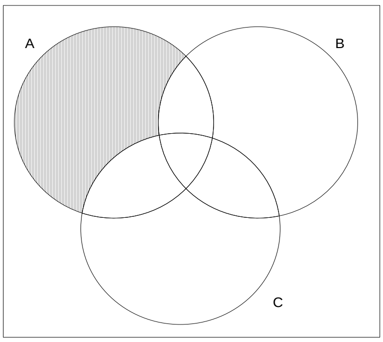

(A ∩ C) ∩ (B ∪ C)



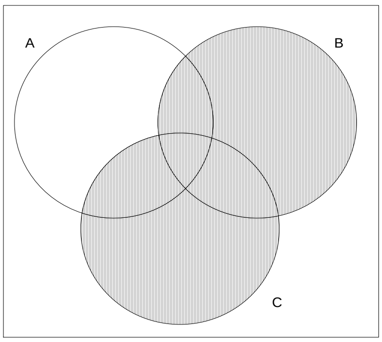

It is clear that the result is not the same as the given Venn diagram.

Alternative 3: (A – C) ∪ (B ∪ C)

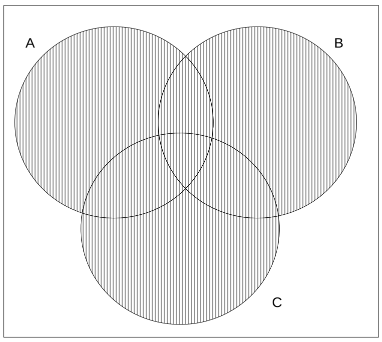
(A – C)




(B ∪ C)




(A – C) ∩ (B ∪ C)

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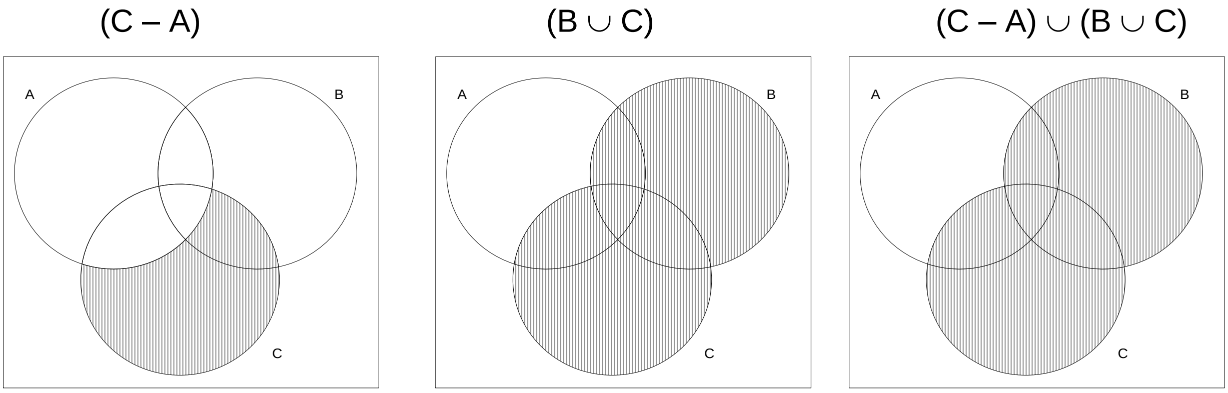
It is clear that the result is not the same as the given Venn diagram.

Alternative 4: (C – A) ∪ (B ∪ C)



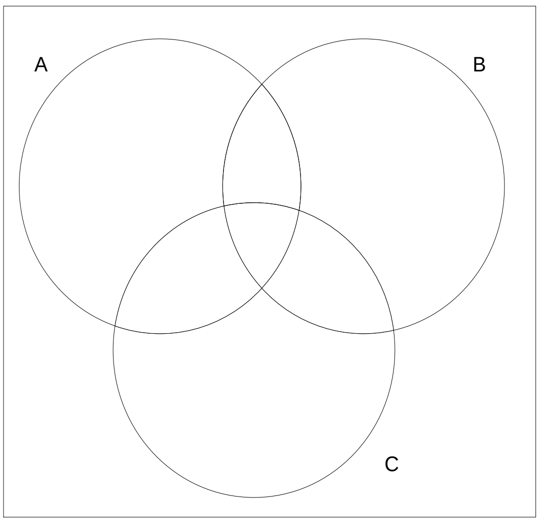

It is clear that the result is not the same as the given Venn diagram.

**Question 14**

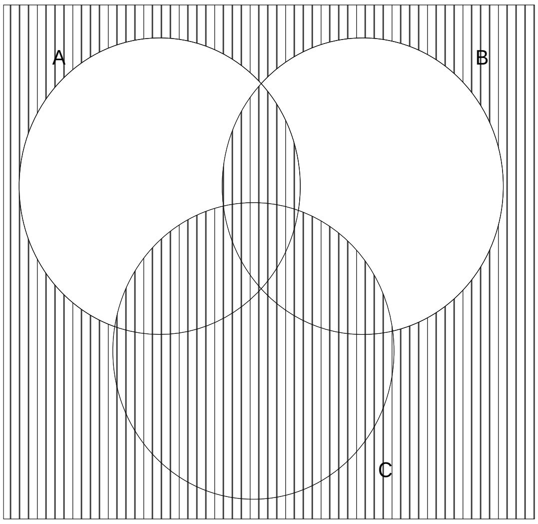
Which one of the Venn diagrams in the following alternatives represents the set

**(A – B′) ∪ (C + (B ∩ A))**

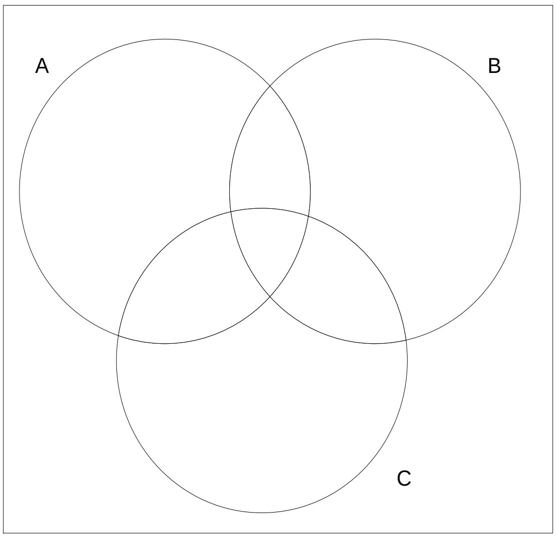
1.



2.

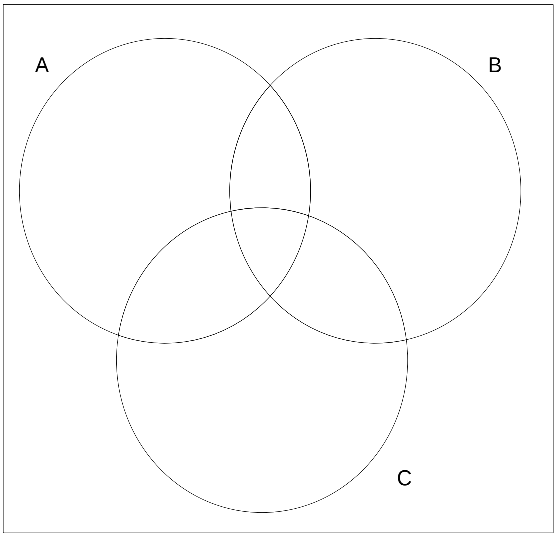



3.



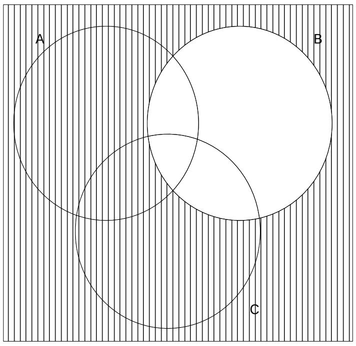

**[TURN OVER]**

4.

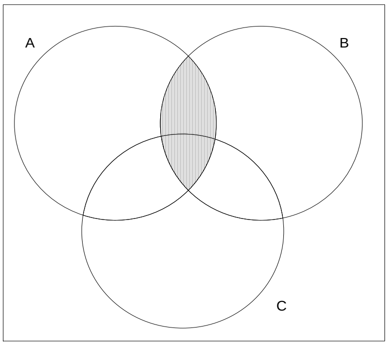



**Discussion:** We draw the Venn diagram for **(A – B′) ∪ (C + (B ∩ A))** step by step:

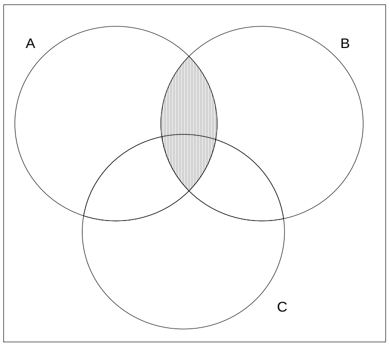
B′



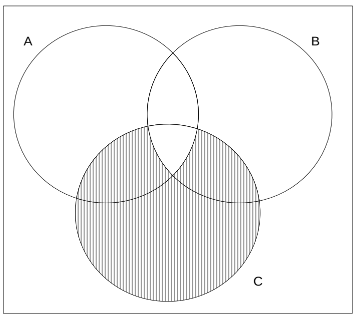

(A – B′)



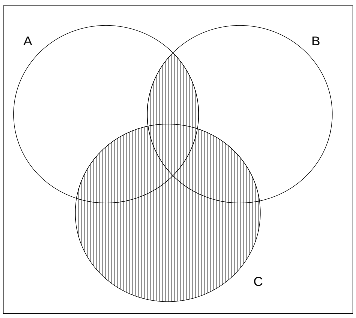

(B ∩ A)




(C + (B ∩ A))




(A – B′) ∪ (C + (B ∩ A))




From the diagrams above it is clear that **alternative 3** is the correct alternative.

**[TURN OVER]**

**Question 15**

We want to prove that for all A, B, C $\subseteq$ U,

$(A \cup C) - (C \cap B) = (A - C) \cup [(A - B) \cup (C - B)]$ is an identity.

Consider the following incomplete proof:

$z \in (A \cup C) - (C \cap B)$

iff $(z \in A \text{ or } z \in C) \text{ and } (z \notin (C \cap B))$

iff $(z \in A \text{ or } z \in C) \text{ and } (z \notin C \text{ or } z \notin B)$

Step 4

iff $[(z \in A \text{ or } z \in C) \text{ and } (z \in C')] \text{ or } [(z \in A \text{ or } z \in C) \text{ and } (z \in B')]$

Step 6

iff $[(z \in A \text{ and } z \in C')] \text{ or } [(z \in A \text{ and } z \in B') \text{ or } (z \in C \text{ and } z \in B')]$

iff $[(z \in A - C)] \text{ or } [(z \in (A - B) \text{ or } (z \in C - B)]$

iff $z \in (A - C) \cup [(A - B) \cup (C - B)]$

Which one of the following alternatives contain the correct Step 4 and Step 6 to complete the proof correctly?

1. **Step 4**: iff $(z \in A \text{ or } z \in C) \text{ and } (z \in C' \text{ or } z \in B')$
   **Step 6**: iff $[(z \in A \text{ and } z \in C') \text{ or } (z \in C \text{ and } z \in C')] \text{ or }$
   $[(z \in A \text{ and } z \in B') \text{ or } (z \in C \text{ and } z \in B')]$

**Step 4**: iff $(z \in A \text{ or } z \in C) \text{ and } (z \in C' \text{ and } z \in B')$
   **Step 6**: iff $[(z \in A \text{ or } z \in C') \text{ and } (z \in C \text{ or } z \in C')] \text{ and }$
   $[(z \in A \text{ or } z \in B') \text{ and } (z \in C \text{ or } z \in B')]$

2. **Step 4**: iff $(z \in A \text{ or } z \in C) \text{ and } z \in (z \in C' \text{ and } z \in B')$
   **Step 6**: iff $[(z \in A \text{ and } z \in C') \text{ or } (z \in C \text{ and } z \in C')] \text{ or }$
   $[(z \in A \text{ and } z \in B') \text{ or } (z \in C \text{ and } z \in B')]$

4. **Step 4**: iff $(z \in A \text{ or } z \in C) \text{ and } (z \in C' \text{ or } z \in B')$
   **Step 6**: iff $[(z \in A \text{ or } z \in C') \text{ and } (z \in C \text{ or } z \in C')] \text{ and }$
   $[(z \in A \text{ or } z \in B') \text{ and } (z \in C \text{ or } z \in B')]$

**Discussion**:

For these types of proofs you need to know the content in section 4.3 and on pages 44-62 in the study guide. You also need to know the definitions of union, intersection, set difference, etc., and understand distributive, associative and commutative properties. Also note that there are activities that you can work through, as well as similar questions in your assignments. We look at the left hand side of the equation and see how we can get to the right hand side.

Step 3: iff$(z \in A \text{ or } z \in C) \text{ and } (z \notin C \text{ or } z \notin B)$

If we look at step 3, ($z \in A$ or $z \in C$) on the left is in its simplest form, but ($z \notin C$ or $z \notin B$) on the right hand side of step 3 can be simplified more. If z is not element of C, it must be an element of the complement of C. Similarly, if z is not an element of B, it must be an element of the complement of B, thus step 4 becomes

Step 4: iff ($z \in A$ or $z \in C$) and ($z \in C'$ or $z \in B'$).
Note that we have not changed any logic, so the rest of the statement stays intact.
Step 5: iff[($z \in A$ or $z \in C$) and ($z \in C'$)] or [($z \in A$ or $z \in C$) and ($z \in B'$)].
If you look closely, you will see that we have two statements of similar form here: $[(X \cup Y) \cap Z]$. (Remember $\cup$ is equal to union which is OR and $\cap$ is equal to intersection which is AND)
From the distributive rule we know that $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$ (See study guide activities 4.5 to 4.12, pp.54 – 68 where this principle is continuously used). So we apply this principle to both parts of the statement in step 5 which leads us to step 6.

Step 6: iff [($z \in A$ and $z \in C'$) or ($z \in C$ and $z \in C'$)] or
[($z \in A$ and $z \in B'$) or ($z \in C$ and $z \in B'$)]
If you look at the alternatives given, you will see that this corresponds to **alternative 1.**

**Question 16**
Let U = {1, 2, 3} and A, B and C be subsets of U. The set **$(B - C) = (A \cap (B \cap C'))$** is NOT an identity. Which of the following alternatives contains sets A, B and C that can be used as counterexample to prove that the set **$(B - C) = (A \cap (B \cap C'))$** is not an identity.

1. A = {1}, B = {1}, C = {2, 3}
2. A = {1, 2}, B = { }, C = {1, 2}
3. A = {2, 3}, B = { }, C = {2}
4. A = {3}, B = {1, 3}, C = {2}

**Discussion**:
If you have to prove that a statement is not an identity, and you are given sets for A, B and C as in this question, you need to substitute the given sets in the left hand and right hand side of the statement separately, and then see for which alternative LHS ≠ RHS. Only when LHS ≠ RHS, have you found a counterexample. We will test alternatives 1 and 4 below, and leave alternatives 2 and 3 as an exercise.
Alternative 1:
Let A = {1}, B = {1} and C = {2, 3}
**LHS**: (B – C) = ({1} – {2, 3})
= {1} (See definition of set difference in study guide p. 42).

**[TURN OVER]**

**RHS**: $(A \cap (B \cap C')) = (\{1\} \cap (\{1\} \cap \{1\}))$ (U = {1, 2, 3}. If C = {2, 3}, then C′ = {1})
$= (\{1\} \cap \{1\}$
$= \{1\}$

Therefore LHS = RHS. This does not show that the statement is not an identity.

Alternative 4:

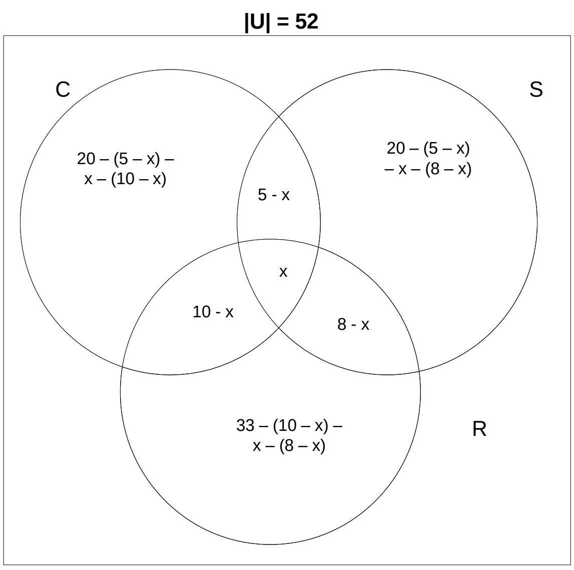
**LHS**: $(B – C) = (\{1, 3\} – \{2\})$
$= \{1, 3\}$

**RHS**: $(A \cap (B \cap C')) = (\{3\} \cap (\{1, 3\} \cap \{1, 3\}))$
$= (\{3\} \cap \{1, 3\}$
$= \{3\}$

Clearly, LHS ≠ RHS. That means that we have found a counterexample to show that the statement is not an identity. Therefore **alternative 4** should be selected. If you do the same for alternatives 2 and 3, you will see that in both cases LHS = RHS.

**Question 17**

The Venn diagram below represents the sports (C = Cricket, R = Rugby, S = Soccer) that a group of 52 boys partake in.




Which one of the alternatives is true?

1. 6 kids play cricket, soccer and rugby, and
   4 kids play both cricket and rugby but not soccer.

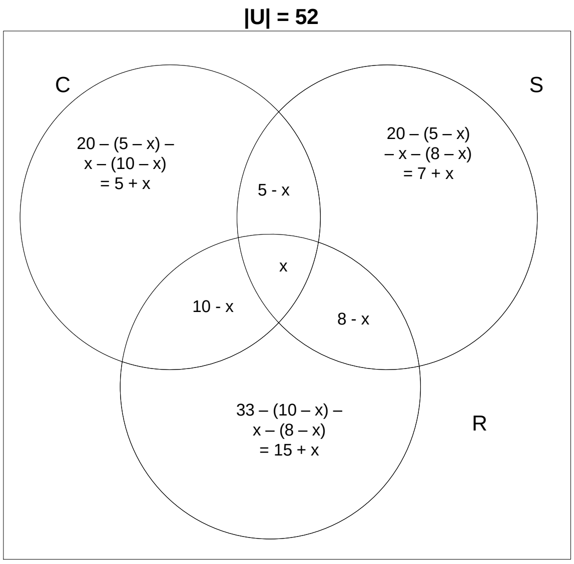
2. 6 kids play cricket, soccer and rugby, and
   34 kids play both cricket and rugby but not soccer.

3. 2 kids play cricket, soccer and rugby, and
   32 kids play both cricket and rugby but not soccer.

4. 2 kids play cricket, soccer and rugby, and
   8 kids play both cricket and rugby but not soccer.

**Discussion**:

We first simplify the entries in the Venn diagram. This is not a necessity, but it does make the calculation of the value of x less complicated.




First we need to calculate the value of x, ie we have to find out how many kids play all three sports. We know that |U| = 52. That means all areas in the Venn diagram must add up to 52.

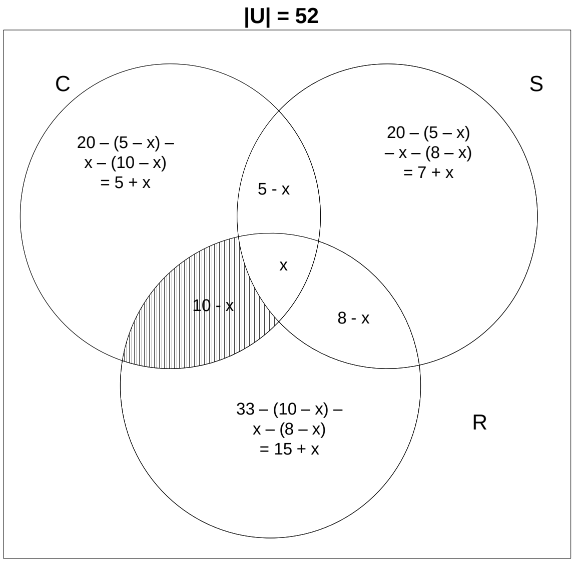
5 + x + 7 + x + 15 + x + 5 – x + 10 – x +8 – x + x = 52

ie 50 + x = 52

ie x = 2.

There are therefore 2 kids that play cricket, soccer and rugby. This answers the first part of each alternative. Can you see that alternatives 1 and 2 are already ruled out?

For the second part of each alternative, we have to calculate how many kids play both cricket and rugby, but not soccer. Do you agree that it is the shaded area below?




The number of kids playing both cricket and rugby, but not soccer is 10 – x. We have calculated the value of x as 2. Therefore 10 – 2 = 8, ie kids play both cricket and rugby, but not soccer. **Alternative 4** is therefore the alternative that should be selected.

**SECTION 3**
**RELATIONS AND FUNCTIONS**
**(Questions 18 to 32) (15 marks)**

**Question 18**
Let **C = {1, 3, d, e}** and let **R = {(1, 1), (3, d), (3, 3), (d, 1), (d, 3), (e, e)}** be a relation on C. Which one of the following alternatives is needed to make R reflexive and symmetric?

1. Add the ordered pairs (d, d) and (1, d) to R.
2. Add the ordered pair (d, d) to R.
3. Add the ordered pair (1, d) to R.
4. Nothing needs to be added – R is already reflexive and symmetric.

**Discussion**:
A relation on a set A is reflexive if for each x in A, the ordered pair (x, x) is part of the relation. See the formal definition in the study guide p. 75.
A relation is symmetric if for each ordered pair (x, y) , with x not equal to y in the relation, (y, x) is also in the relation. See the formal definition in the study guide p. 76.
**C = {1, 3, d, e}** , **R = {(1, 1), (3, d), (3, 3), (d, 1), (d, 3), (e, e)}**
For R to be reflexive, ordered pairs (1, 1), (3, 3), (d, d) and (e, e) must be part of R, therefore we need to add (d, d) to the relation. To be symmetric we need to add (1, d) to the relation.

**Alternative 1** should therefore be selected.

**Question 19**
Let **A = {1, 2, c, 3}.** Which one of the following relations is a functional relation on A?

1. {(1, 2), (c, 3), (3, c), (1, 3)}
2. {(3, 2), (c, 4), (2, 1), (1, 3)}
3. {(1, c), (3, 3), (2, 1)}
4. {(3, 1), (3, 3), (3, c)}

**Discussion**:
A relation on a set A is functional iff all first coordinates are elements in A, and each first coordinate are only used once in the relation (study guide p. 98).
In alternative 1 {(1, 2), (c, 3), (3, c), (1, 3)} element 1 occurs more than once as a first coordinate, so this relation is not functional.
In alternative 2 {(3, 2), (c, 4), (2, 1), (1, 3)} the ordered pair (c, 4) is invalid, as the element 4 is not an element of the set A. Therefore this relation on A is invalid.
**Alternative 3** {(1, c), (3, 3), (2, 1)} is a functional relation because the first coordinates 1, 3 and 2 are elements in A, and they appear in only one ordered pair.
In alternative 4 {(3, 1), (3, 3), (3, c)} the element 3 appears more than once as a first coordinate, therefore the relation is not functional.

Let **U = {1, {2}, 3, {1, 2}, 4}.** Let **A = {1, {2}, 3}**, **B = {{2}, 3, {1, 2}, 4}** and **C = {{2}, 1, 4}**.
**Questions 20 to 23 are based on U, A, B and C.**

**Question 20**
Which one of the following relations is **not** a function from C to U?

1. {(4, {2}), (1, {1, 2}), ({2}, 3)}
2. {({2}, {2}), (1, 1), (4, 4)}
3. {({1, 2}, {2}), (1, 4), (3, 1)}
4. {({2}, {2}), (1, {2}), (4, {2})}

**Discussion**:
A relation R from X to Y is a function iff each first coordinate in the relation is used once only and **every** element from X (the domain) is used in the relation as a first coordinate. For the formal definition, see study guide p. 99.
The relation in question is from **C = {{2}, 1, 4}** to **U = {1, {2}, 3, {1, 2}, 4}.**
In alternative 1 the relation {(4, {2}), (1, {1, 2}), ({2}, 3)} is a function because the set of first coordinates is exactly the set C, and each first coordinate is used in an ordered pair only once.
Similar to alternative 1, the relation in alternative 2 {({2}, {2}), (1, 1), (4, 4)} is a function because the set of first coordinates is exactly the set C, and each first coordinate is used in an ordered pair only once.

The relation in **alternative 3** {({1, 2}, {2}), (1, 4), (3, 1)} is a relation from U to C, not from C to U, therefore this is the alternative to choose.
Similar to alternatives 1 and 2, the relation in alternative 4 {({2}, {2}), (1, {2}), (4, {2})} is a function because the set of first coordinates is exactly the set C, and each first coordinate is used in an ordered pair only once.

**Question 21**
Which one of the following relations is functional from B to A?
1. {(4, {2}), (4, 1), (4, 3)}
2. {({2}, 1)}
3. {(3, 3), ({1, 2}, {2}), (3, 1)}
4. {({2}, {2}), (3, 3), (4, {2}), ({1, 2}, {2}), (3, {2})}

**Discussion:**
A relation R from X to Y is functional iff all first coordinates in R, are elements in X, and each first coordinate are only used once in the relation as a first coordinate (study guide p. 98).
The relation in question should be a relation from **B = {{2}, 3, {1, 2}, 4}** to **A = {1, {2}, 3}**.
Alternative 1: {(4, {2}), (4, 1), (4, 3)}
- Are all first coordinates elements in B? Yes 4 ∈ B.
- Are all first coordinates used only once? No, 4 is used in every ordered pair, so this relation is not functional.

Alternative 2: {({2}, 1)}
- Are all first coordinates elements in B? Yes {2} ∈ B.
- Are all first coordinates used only once? Yes, {2} is used only once, therefore we can conclude that this relation is functional. **Alternative 2** should therefore be selected.

Alternative 3: {(3, 3), ({1, 2}, {2}), (3, 1)}
- Are all first coordinates elements in B? No. 1 ∉ B. We do not have to check any further because this is not a relation from B to A.

Alternative 4: {({2}, {2}), (3, 3), (4, {2}), ({1, 2}, {2}), (3, {2})}
- Are all first coordinates elements in B? Yes {2}, 3, 4, {1, 2} ∈ B.
- Are all first coordinates used only once? No, 3 is used more than once as a first coordinate, so this relation is not functional.

**Question 22**
Which one of the following relations on C satisfies trichotomy?
1. {(4, 1), ({2}, 4), ({2}, 1)}
2. {(4, 1), ({2}, {2}), (4, {2}), (1, 1), (1, 4)}
3. {(4, 1), (1, 4), ({2}, 4), (4, {2}), (1, 1), (4, 4)}
4. {({2}, {2}), (1, 1), (4, 4)}

**[TURN OVER]**

**Discussion:**

A relation on a set A satisfies trichotomy if, for every x and y, with x ≠ y, x and y is comparable. This means that the relation must contain ordered pairs for pairing each element of A to every other element of A. For the formal definition of trichotomy, see Study guide p. 78.

**C = {{2}, 1, 4}**

Alternative 1: {(4, 1), ({2}, 4), ({2}, 1)}

- Is {2} paired up with 1? Yes, in ordered pair ({2}, 1).
- Is {2} paired up with 4? Yes in ordered pair ({2}, 4).
- Is 1 paired up with 4? Yes in ordered pair (4, 1).

There are no more element to pair, so we can conclude that this relation on C satisfies trichotomy. **Alternative 1** should therefore be selected.

Alternative 2: {(4, 1), ({2}, {2}), (4, {2}), (1, 1), (1, 4)}

- Is {2} paired up with 1? No, neither ordered pair ({2}, 1) nor (1, {2}) is part of the relation. We need not look any further. The relation does not satisfy trichotomy.

Alternative 3: {(4, 1), (1, 4), ({2}, 4), (4, {2}), (1, 1), (4, 4)}

- Is {2} paired up with 1? No, neither ordered pair ({2}, 1) nor (1, {2}) is part of the relation. We need not look any further. The relation does not satisfy trichotomy.

Alternative 4: {({2}, {2}), (1, 1), (4, 4)}

- Is {2} paired up with 1? No, neither ordered pair ({2}, 1) nor (1, {2}) is part of the relation. We need not look any further. The relation does not satisfy trichotomy.

**Question 23**

Which one of the following relations on A is symmetric and neither reflexive nor irreflexive?

1. {1, 1), ({2}, {2}), (3, 1)}
2. {(3, 3), (1, {2}), ({2}, 1), (1, 1), ({2}, 3)}
3. {(3, 3), (1, 1), ({2}, {2})}
4. {(1, 3), (3, 1), (3, 3)}

**Discussion:**

**A = {1, {2}, 3}**

A relation on a set A is symmetric, if for every (x, y) in the relation, (y, x) is also in the relation (Study guide p. 76). A relation on a set A is reflexive iff for each element x in A, the pair (x, x) is in the relation (Study guide p. 75). A relation on a set A is irreflexive iff for each element x in A, the ordered pair (x, x) is NOT in the relation (Study guide p. 75).

Alternative 1: {1, 1), ({2}, {2}), (3, 1)}

- Is the relation symmetric? No, the ordered pair (3, 1) is in the relation, but (1, 3) is not. To be symmetric, both ordered pairs must be in the relation. This alternative is therefore not applicable to the question.

Alternative 2: {(3, 3), (1, {2}), ({2}, 1), (1, 1), ({2}, 3)}

- Is the relation symmetric? No. The ordered pair ({2}, 3) is in the relation but the pair (3, {2}) is not. To be symmetric, both ordered pairs must be in the relation. This alternative is therefore not applicable to the question.

Alternative 3: {(3, 3), (1, 1), ({2}, {2})}

- Is the relation symmetric? Yes. There are no pairs (x, y) with x not equal to y, for which the pair (y, x) is not present in the relation.
- Is the relation reflexive? Yes for each first coordinate x in A, the ordered pair (x, x) is in the relation. This alternative therefore does not satisfy the criteria for the question.

Alternative 4: {(1, 3), (3, 1), (3, 3)}

- Is the relation symmetric? Yes. There are no pairs (x, y) with x not equal to y, for which the pair (y, x) is not present in the relation.
- Is the relation reflexive? No. It does not contain all of the pairs (1, 1), (3, 3) and ({2}, {2}), as is required for the relation to be symmetric.
- Is the relation irreflexive? No. To be irreflexive, no ordered pair (x, x) for each x from A must be present in the relation. (3, 3) is present in this relation, so it cannot be irreflexive. In this alternative we have established that the relation is symmetric, but neither reflexive nor irreflexive. **Alternative 4** therefore satisfies the criteria (study guide pp. 74-77).

Let **A = {a, b, c}, B = {b, c, d}** and **C = {a, b, c, d}.**

**Answer Questions 24 and 25, based on these given sets:**

**Question 24**

Which one of the following alternatives represents an injective function from B to C?

1. {(c, c), (d, a), (b, b), (c, d)}
2. {(a, b), (b, c), (c, d), (d, b)}
3. {(d, d), (b, c), (c, b)}
4. {(b, a), (c, d), (a, c)}

**Discussion:**

A function f: A → B is *injective* iff f has the property that

whenever a1 ≠ a2 then f(a1) ≠ f(a2). Also see study guide page 104 for an alternative definition.

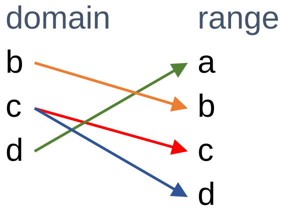
What does this mean? It means that there is a one-to-one relation between elements in the domain and elements in the range.

**B = {b, c, d}** and **C = {a, b, c, d}.**

Alternative 1: {(c, c), (d, a), (b, b), (c, d)}

**[TURN OVER]**

Let us write this differently:



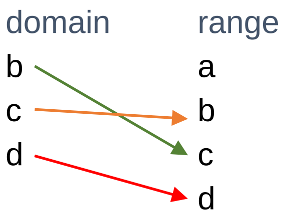

Is this an injective function? Well we first have to check that the relation is a function. If it is, then we can determine whether it is an injective function. A relation R from X to Y is a function iff each first coordinate in the relation is used once only and every element from X (the domain) is used in the relation as a first coordinate. For the formal definition, see study guide p. 99. Can you see that this relation is not a function? The first coordinate c is used twice in the relation. If the relation is not a function, it also cannot be an injective function.

Alternative 2: {(a, b), (b, c), (c, d), (d, b)}

The question states that the relation is from B to C. This relation is from C to B, so it is not a valid relation from B to C.

Alternative 3: {(d, d), (b, c), (c, b)}

Let us write this differently:




Is this relation a function? Yes. All the first coordinates are only used once, and all elements in B are used as first coordinates.

Is it an injective function? Yes. Each first coordinate points to only one second coordinate, and each second coordinate is pointed to by only one first coordinate. See Study guide p.108 for further explanation. Note that the definition of an injective function does not require all the elements of the range to be used in ordered pairs, but if they are used, they should each be pointed to by only one unique element from the domain. **Alternative 3** is therefore the alternative to be selected. (See definition of domain and rage of a function in Study guide pp. 104 and 105).

Alternative 4: {(b, a), (c, d), (a, c)}

This is not a relation from B to C, because the first coordinate a in the ordered pair(a, c) is not in the domain (study guide pp.105-112).

**Question 25**

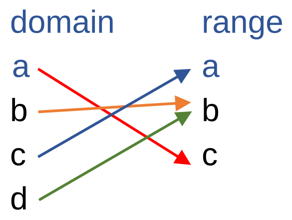
Let F = {(b, b), (a, c), (d, b), (c, a)} be a relation from C to A.
Which one of the following alternatives regarding F is TRUE?

1. F is an injective function from C to A.
2. F is a surjective function from C to A.
3. F is a bijective function from C to A.
4. F is neither an injective nor a surjective function from C to A.

**Discussion**:

A function f: A → B is *surjective* iff f has the property that the range of function f (see definition of the range of a function in Study guide p. 104) is equal to the codomain of f (Study guide p. 105). In this question A = {a, b, c} is the codomain of f.

**C = {a, b, c, d}** and **A = {a, b, c}**
**f = {(b, b), (a, c), (d, b), (c, a)}**




Is this a surjective function? What is the range of f? Ran(f) = {a, b, c}, ie all the elements that appear as second coordinates in the relation. It is clear that ran(f) = A = {1, 2, 3}, ie the range of f and the codomain are equal. f is therefore a surjective function.
Is f an injective function? From the description in the previous question it should be clear that f is not injective, because both the first coordinates b and d map to second coordinate b, which means the function is not one-to-one. This renders alternatives 1 and 4 incorrect. Alternative 3 can also not be correct, because a bijective function is both injective and surjective, which is not the case for f. **Alternative 2** should therefore be selected (study guide pp.105-112).

Let **A = {1, 2, 3}** and **B = {2, 3, 4, 5}**. Consider the following two relations from **A** to **B**:

**L = {(1, 4), (2, 2), (2, 3), (3, 2), (3, 5)}** and
**M = {(3, 3), (3, 2), (1, 3), (2, 4), (1, 5)}**.

**Question 26**

Which one of the following alternatives represents M ○ L (ie L; M)?

1. {(3, 2), (3, 5), (3, 2), (3, 3), (1, 2), (1, 5)}
2. {(2, 4), (2, 3), (2, 2), (3, 4)}
3. {(2, 3), (2, 2), (2, 5), (3, 2), (3, 3)}
4. {(3, 2), (3, 4), (1, 3), (1, 2)}

**[TURN OVER]**

**Discussion**:

Given two relations R from A to B and S from B to C, the composition of the two relations (ie R followed by S or R;S) is defined as S ○ R = {(a, c) | there is some b ∈ B such that (a, b) ∈ R and (b, c) ∈ S}.(Study guide p. 79).

This means that if (a, b) is an ordered pair in R, and (b, c) is an ordered pair in S, then (a, c) is an ordered pair in the composition R;S (or S ○ R). That is, we look at each ordered pair (a, b) in R, and find all ordered pairs in S that has b as a first coordinate:

**L = {(1, 4), (2, 2), (2, 3), (3, 2), (3, 5)}** and

**M = {(3, 3), (3, 2), (1, 3), (2, 4), (1, 5)}.**

For this question we want to determine L;M, so we start with relation L:

(1, 4): Is there an ordered pair in M that has 4 as first coordinate? No, continue;

(2, 2): Is there an ordered pair in M that has 2 as first coordinate? Yes, (2, 4). That means if (2, 2) is in L and (2, 4) is in M, then (2, 4) must be in L;M. Continue.

(2, 3): Is there an ordered pair in M that has 3 as first coordinate? Yes, both the pairs (3, 3) and (3, 2) in M has 3 as a first coordinate. That means if (2, 3) is in L and (3, 3) is in M, then (2, 3) must be in L;M. Similarly, if (2, 3) is in L and (3, 2) is in M, then (2, 2) must be in L;M. Continuing in this manner we determine that L;M = {(2, 4), (2, 3), (2, 2), (3, 4)}, which make **alternative 2** the correct alternative (study guide pp.79, 108).

**Questions 27 to 32 are based on the following functions:**

Let f and g be functions on Z defined by:

**(x, y) ∈ g iff $y = 2x^2 - 5$** and **(x, y) ∈ f iff y = -3x + 7**.

**Question 27**

Which one of the following statements regarding functions f and g is TRUE?

1. Function f is injective, but function g is not injective.
2. Function f is surjective, but function g is not surjective.
3. Neither function f nor function g is injective.
4. Function f is bijective, but function g is not bijective.

**Discussion**:

Refer to the discussion for injective and surjective functions in questions 24 and 25 again.

It is very important to notice that f and g are functions on Z. This should immediately tell you that function g cannot be injective. Why? Well, all the positive and negative numbers are part of the domain of g. If we substitute for example -2 and 2 in g, we get:

$y = 2x^2 - 5$

$= 2(-2)^2 – 5$

$= 2(4) – 5$

**[TURN OVER]**

$= 3$
ie (-2, 3) is in g.

$y = 2x^2 - 5$
$= 2(2)^2 – 5$
$= 2(4) – 5$
$= 3$
ie (2, 3) is in g.
This means that the x values -2 and 2 both map to the y value 3, so g cannot be injective.
On the other hand, f is injective. If we substitute a few values for x, we will quickly see that for each x value, there is one unique y value:

**y = -3x + 7**

| x | y | |
|---|---|---|
| -3 | -3(-3) + 7 = 9 + 7 = 16 | (-3, 16) is in f |
| -2 | -3(-2) + 7 = 6 + 7 = 13 | (-2, 13) is in f |
| -1 | -3(-1) + 7 = 3 + 7 = 10 | (-1, 10) is in f |
| 0 | -3(0) + 7 = 0 + 7 = 7 | (0, 7) is in f |
| 1 | -3(1) + 7 = -3 + 7 = 4 | (1, 4) is in f |
| 2 | -3(2) + 7 = -6 + 7 = 1 | (2, 1) is in f etc |

If you draw a graph for this function, you will see that it is a straight line graph and indeed injective.
What about surjectivity? Remember that for surjectivity the range of a function must be equal to the codomain, which is $\mathbb{Z}$ in this case. From the table above you should immediately see that, for example, there is no x value in $\mathbb{Z}$ for y = 14 and y = 15 (see highlighted y values above)..
We can test that:
Let y = 14,
Then 14 = -3x + 7,
ie 3x = 7 – 14
ie 3x = -7
ie x = -7/3 which is not in $\mathbb{Z}$. The range of f is therefore not equal to the codomain $\mathbb{Z}$

In a similar way it is easy to see that function g is also not surjective. This leaves **alternative 1** as the only correct alternative (study guide units 6 and 7).

**Question 28**
Which one of the following alternatives represents g ∘ f(x) (ie g(f(x))?

1. $18x^2 + 42x + 49$
2. $18x^2 - 84x + 93$
3. $-6x^2 + 22$
4. $18x^2 + 2$

**Discussion**:

$g \circ f(x) = g(f(x))$
$= g(-3x + 7)$
$= 2(-3x + 7)^2 – 5$
$= 2(9x^2 + 2(-3x)(7) + 49) – 5$
$= 2(9x^2 – 42x + 49) – 5$
$= 18x^2 - 84x + 98 – 5$
$= 18x^2 - 84x + 93$, which is equavalent to **alternative 2** (study guide p. 108, assesment exercises, assignments 2 and 3).

**Question 29**
Which one of the following alternatives represents f ∘ f(x) (ie f(f(x))?

1. $9x^2 - 42x + 49$
2. $-9x^2 + 42x + 49$
3. $9x - 21$
4. $9x - 14$

**Discussion**:

$f \circ f(x) = f(f(x))$
$= f(-3x + 7)$
$= -3(-3x + 7) + 7$
$= 9x -21 + 7$
$= 9x – 14$, which means **alternative 4** is the correct alternative (study guide p.108).

**Question 30**
Which one of the following alternatives represents an ordered pair that does NOT belong to f?

1. (-2, 1)
2. (9, -20)
3. (3, -2)
4. (-4, 19)

**Discussion**:

If you want to know if an ordered pair is an ordered pair in a function, all you need to do is replace the x value given in the function, and see if that yields the y value given.
We look at each alternative:
Alternative 1:
Let x = -2, then
y = -3x + 7
= -3(-2) + 7
= 6 + 7 = 13
Therefore (-2, 13) is in f, but (-2, 1) is not. **Alternative 1** is therefore the correct alternative to choose.

Alternative 2:
Let x = 9, then
y = -3x + 7
= -3(9) + 7
= -27 + 7 = -20.
Therefore (9, -20) is in f. The question, however, wants an ordered pair that is not in f.

Alternative 3:
Let x = 3, then
y = -3x + 7
= -3(3) + 7
= -9 + 7 = -2.
Therefore (3, -2) is in f. The question, however, wants an ordered pair that is not in f.

Alternative 4:
Let x = 9, then
y = -3x + 7
= -3(-4) +19
= 12 + 7 = 19
Therefore (-4, 19) is in f. The question, however, wants an ordered pair that is not in f.

**Question 31**
Which one of the following statements regarding functions f and g is TRUE?

1. The ordered pair (-2, 3) is an ordered pair in both functions f and g.
2. Both the ordered pairs (-2, 3) and (2, 3) are ordered pairs in function g.
3. Both the ordered pairs (1, 4) and (-4, 1) are ordered pairs in function f.
4. The ordered pair (-1, -3) is in function g, but (1, -3) is not in function g.

**Discussion**:

In a similar way to question 30, we substitute the given pairs in the alternatives in f and/or g to determine which alternative is true.

We look at each alternative:

Alternative 1:

Let x = -2, then

y = -3(-2) + 7 = 13.

Therefore (-2, 13) is in f, but (-2, 3) is not. We do not have to test the second half of the statement, because the statement is already false.

Alternative 2:

Let x = -2, then

$= 2x^2 – 5$

$= 2(-2)^2 – 5$

= 8 – 5 = 3. Therefore(ordered pair (-2, 3) is in g.

Let x = 2, then

$= 2x^2 – 5$

$= 2(2)^2 – 5$

= 8 – 5 = 3. Therefore(ordered pair (2, 3) is in g.

**Alternative 2** is therefore true because both ordered pairs (-2, 3) and (2, 3) is in g.

Alternative 3:

Let x = 1, then

= -3(1) + 7 = -3 + 7

= 4

Therefore ordered pair (1, 4) is in f.

Let x = -4, then

= -3(-4) + 7 = 12 + 7

= 19

Therefore the ordered pair (-4, 19) is in f, but ordered pair (-4, 1) is not.

Alternative 4:

Let x = -1, then

$= 2x^2 – 5$

$= 2(-1)^2 – 5$

= 2 – 5 = -3

Therefore ordered pair (-1, -3) is in function g.

Let x = 1, then

$= 2x^2 – 5$

$= 2(1)^2 – 5$

= 2 – 5 = -3
Therefore ordered pair (1, -3) is in function g.
Both ordered pairs (-1, -3) and (1, -3) are in g, contrary to the alternative (study units 6 and 7, self-assesment exercises, assignments 2 and 3).

**Question 32**
Which one of the following alternatives represents the range of g (ie ran(g))?

1. {y | for some y $\in$ Z, y = $2x^2$ - 5 $\in$ Z}
2. {y | $2x^2$ - 5 $\in$ Z }
3. {y | $\sqrt{\frac{y+5}{2}}$ $\in$ Z }
4. Z

**Discussion**:

When we want to determine the range of a function, we want determine y in terms of x.
ran(f) = {y | for some **x** $\in$ **Z**, y = $2x^2$ - 5 $\in$ $\mathbb{Z}$}

ie {y | for some **x** $\in$ **Z**, x = $\sqrt{\frac{y+5}{2}}$ $\in$ $\mathbb{Z}$}

ie {y | $\sqrt{\frac{y+5}{2}}$ $\in$ $\mathbb{Z}$}

Please work through self-assessment exercises, assignments etc where you will find more examples (study guide p. 104).

**SECTION 4**
**OPERATIONS AND MATRICES**
**Questions 33 – 38 (6 marks)**

**Question 33**

Consider the following matrices:

$$Let\ A = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix} \quad and\ C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Which one of the following alternatives regarding operations on the given matrices is FALSE?

1. Performing the operation A • C will result in a 1 x 3 matrix.
2. The result of C • B is the matrix $\begin{bmatrix} 13 & 11 \\ 11 & 13 \end{bmatrix}$.
3. Calculating C + B will result in a 2 x 2 matrix.
4. The result of B • C is the matrix $\begin{bmatrix} 7 & 6 & 5 \\ 12 & 12 & 12 \\ 5 & 6 & 7 \end{bmatrix}$.

**Discussion**:

Alternative 1:

A is a 1 x 2 matrix and C is a 2 x 3 matrix. The result of multiplying A with C is a 1 x 3 matrix.

This alternative is therefore true. Note that the same is not true for multiplying C with A. The number of columns in the first matrix must be equal to the number of rows in the second matrix before the multiplication operation is valid.

Alternative 2:

Is C • B a valid multiplication?

C is a 2 x 3 matrix and B is a 3 x 2 matrix. The result of multiplying C with B is a 2 x 2 matrix, because the number of columns in the first matrix is equal to the number of rows in the second matrix. Let the resulting matrix be D, then

$D_{11}$ = (1 * 1) + (2 * 3) + (3 * 2) = 13
$D_{12}$ = (1 * 2) + (2 * 3) + (3 * 1) = 11
$D_{21}$ = (3 * 1) + (2 * 3) + (1 * 2) = 11
$D_{22}$ = (3 * 2) + (2 * 3) + (1 * 1) = 13

Therefore C • B = $\begin{bmatrix} 13 & 11 \\ 11 & 13 \end{bmatrix}$

Alternative 2 is therefore also true.

Alternative 3: You can only add to matrices if they have exactly the same rows and columns.

**[TURN OVER]**

**Alternative 3** is therefore false because C is a 2 x 3 matrix and B is a 3 x 2 matrix, so they cannot be added.

Alternative 4:

Is B • C a valid multiplication? B is a 3 x 2 matrix and C is a 2 x 3 matrix. The result of will therefore be a 3 x 3 matrix, because B has the same number of columns as the number of rows of C. Let the resulting matrix be D, then

$D_{11}$ = (1 * 1) + (2 * 3) = 7
$D_{12}$ = (1 * 2) + (2 * 2) = 6
$D_{13}$ = (1 * 3) + (2 * 1) = 5
$D_{21}$ = (3 * 1) + (3 * 3) = 12
$D_{22}$ = (3 * 2) + (3 * 2) = 12
$D_{23}$ = (3 * 3) + (3 * 1) = 12
$D_{31}$ = (2 * 1) + (1 * 3) = 5
$D_{32}$ = (2 * 2) + (1 * 2) = 6
$D_{33}$ = (2 * 3) + (1 * 1) = 7

This gives us exactly the matrix $\begin{bmatrix} 7 & 6 & 5 \\ 12 & 12 & 12 \\ 5 & 6 & 7 \end{bmatrix}$ as given in the alternative (study guide pp. 127-128).

**Question 34**

Consider the following matrices:

A = $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ -2 & 1 & 3 \end{bmatrix}$ B = $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 6 & 2 \\ 0 & 0 & 6 \end{bmatrix}$

Which one of the following alternatives provides a matrix D such that A + D = B.

1. $\begin{bmatrix} 3 & -2 & -1 \\ 0 & 6 & 0 \\ -4 & -1 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 0 \\ 2 & -1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 1 & -1 \\ -1 & 3 & 0 \\ 4 & -1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 0 \\ 2 & -1 & 3 \end{bmatrix}$

**Discussion**:

If we need a matrix D such that A + D = B,, then D = B – A, that is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -1 \\ 0 & 6 & 2 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 3 & 0 \\ 2 & -1 & 3 \end{bmatrix}$$

**Alternative 2** the therefore the correct alternative to choose (study guide p. 127).

**Question 35**

What is the result of the operation $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 & 5 \end{bmatrix}$ ?

1. It is not possible to do the multiplication on these two matrices.

2. $\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}$

4. $\begin{bmatrix} 21 & 30 \end{bmatrix}$

**Discussion**:

Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 & 5 \end{bmatrix}$ a valid multiplication operation?

Multiplying a 3 x 1 matrix and a 1 x 2 matrix, results in a 3 x 2 matrix, because the first matrix has the same number of columns than the number of rows of the second matrix. If we call D the resulting matrix, then we calculate D as follows (study guide pp. 127-128):

$D_{11} = (1 * 4) = 4$

$D_{12} = (1 * 5) = 5$

**[TURN OVER]**

$D_{21}$ = (2 * 4) = 8
$D_{22}$ = (2 * 5) = 10
$D_{31}$ = (3 * 4) = 12
$D_{32}$ = (3 * 5) = 15

resulting in matrix $\begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}$

**Question 36**

Consider the following binary operation * on the set {a, b, c}:

| * | a | b | c |
|---|---|---|---|
| a | b | a | a |
| b | c | b | c |
| c | a | c | b |

Which one of the following statements regarding the binary operation * is TRUE?

1. (a * b) * c  = c * ((c * c ) * b ).
2. The identity element of the operation * is **b**.
3. (c * a) = (a * c) = a, which proves that * is commutative.
4. (b * c) * a ≠ b * (c * a) can be used as a counterexample to prove that * is not associative.

**Discussion**:

Alternative 1:
We apply the operation to the left- and right hand side of the equation to see if it is true:
LHS: (a * b) * c  = a * c = **a**
RHS: c * ((c * c ) * b ) = c * (b * b) = c * b = **c**
It is clear that this statement is not true.

Alternative 2:

| * | a | b | c |
|---|---|---|---|
| a | b | a | a |
| b | c | b | c |
| c | a | c | b |

If **b** is the identitiy element, then the vertical yellow line under **b** in column 2 should be equal to the vertical grey line, AND the horizontal yellow line next  to **b** in row 2, must be equal to the horizontal grey line. Clearly this is not the case.

Alternative 3:

| * | a | b | c |
|---|---|---|---|
| a | b | a | a |
| b | c | b | c |
| c | a | c | b |

Firstly, you cannot take one example as in the alternative that satisfies commutativity and assume the binary operation as a whole is commutative. You would have to test each and every possible combination first. There is however, a much easier way to determine if an operation is commutative. If you look at the area above and the area below the diagonal line (indicated in grey in the figure above), they should be mirror images of each other. Clearly the values highlighted in yellow are not equal. Therefore, the binary operation cannot be commutative.

Alternative 4:

(b * c) * a ≠ b * (c * a).

We apply the binary operation to the left and right hand side to see if it is true:
LHS: (b * c) * a = c * a = a.
RHS: b * (c * a) = b* a = c.
Indeed the statement is true. It means that we found a counterexample that proves that the binary operation cannot be associative. **Alternative 4** is therefore the correct alternative to choose (studyguide pp. 116-120).

**Question 37**

Consider the incomplete binary operation ◊ on the set {a, b, c} below:

| ◊ | a | b | c |
|---|---|---|---|
| a | | | a |
| b | b | | |
| c | | b | c |

Which one of the following tables represents the binary operation ◊ with the following properties:

(i) The operation ◊ is commutative.

(ii) The operation ◊ has an identity element.

1.

| ◊ | a | b | c |
|---|---|---|---|
| a | b | b | a |
| b | b | a | b |
| c | a | b | a |

2.

| ◊ | a | b | c |
|---|---|---|---|
| a | b | b | a |
| b | b | a | b |
| c | c | b | c |

3.

| ◊ | a | b | c |
|---|---|---|---|
| a | c | b | a |
| b | b | a | b |
| c | a | b | c |

4.

| ◊ | a | b | c |
|---|---|---|---|
| a | b | c | a |
| b | b | a | b |
| c | a | b | c |

**Discussion**:

Alternative 1:

| ◊ | a | b | c |
|---|---|---|---|
| a | b | b | a |
| b | b | a | b |
| c | a | b | a |

This binary operation is commutative because the area above and below the diagonal line are mirror images of each other. You could also have tested all the possible combinations, but it is really not needed. However, the binary operation has no identity element, because for no element in the set {a, b, c} is it true that the column underneath the element is equal to the leftmost column, and the row next to the element is equal to the topmost row, as we have explained in a previous question as well (studyguide pp. 116-120).

Alternative 2:

| ◊ | a | b | c |
|---|---|---|---|
| a | b | b | a |
| b | b | a | b |
| c | c | b | c |

From the discussion in alternative 1, it should be clear that the binary operation is neither commutative, nor does t have an identity element.

Alternative 3:

| ◊ | a | b | c |
|---|---|---|---|
| a | c | b | a |
| b | b | a | b |
| c | a | b | c |

From the discussion in alternative 1 it is clear that this binary operation is commutative, and it has an identity element. **Alternative 3** is therefore the alternative to select.

Alternative 4:

| ◊ | a | b | c |
|---|---|---|---|
| a | b | c | a |
| b | b | a | b |
| c | a | b | c |

Although this binary operation has the identity element c, it is not commutative, as the highlighted values indicate.

**Question 38**

Consider the following representation of binary operation ☼ on the set {a, b}:

| ☼ | a | b |
|---|---|---|
| a | a | a |
| b | a | b |

**[TURN OVER]**

Which one of the following alternatives gives the list notation for the binary operation ☼ ?

1. {{a, a}, a, {a, b}, a, {b, a}, a, {b, b}, b}
2. {({a, a}, a), ({a, b}, a), ({b, a}, a), ({b, b}, b)}
3. {{(a, a), a}, {(a, b), a}, {(b, a), a}, {(b, b), b}}
4. {((a, a), a), ((a, b), a), ((b, a), a), ((b, b), b)}

**Discussion**:

☼ is a binary operation. That means that it operates on two elements of a set. The list notation can be seen as a type of a relation on the set {a, b}. What does a relation consists of? It consists of ordered pairs. The first ordered pair element in each ordered pair of the list notation contains the ordered pair on which the operation is performed. Four operations are possible in this case: a ☼ a, a ☼ b, b ☼ a and b ☼ b. Thus the first elements in the 4 possible ordered pairs of this operation is (a, a), (a, b), (b, a), and (b, b). So far our list notation looks as follows: {((a, a), ?), ((a, b), ?), ((b, a), ?), ((b, b), ?)}. Remember, just as a relation is a set, the list notation is also a set, so we enclose the ordered pairs in curly bracket { }. The second element in each ordered pair consists of the result after applying the binary operation. If we now apply the operations a ☼ a, a ☼ b, b ☼ a and b ☼ b, we get the respective results a, a, a and b. We can now replace the ? with the results, giving us the list notation
{((a, a), a), ((a, b), a), ((b, a), a), ((b, b), b)}. This is equivalent to **alternative 4** (study guide pp.116-122).

## SECTION 5
## TRUTH TABLES AND SYMBOLIC LOGIC
## Questions 39 – 45 (7 marks)

### Question 39

Consider the incomplete truth table below.

| p | q | ¬p | ¬q | ¬q → ¬p |
|---|---|---|---|---|
| T | T | F | F | |
| T | F | F | T | |
| F | T | T | F | |
| F | F | T | T | |

Which one of the following alternatives provides the correct completed truth table?

1.

| p | q | ¬p | ¬q | ¬q → ¬p |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | T | T |

2.

| p | q | ¬p | ¬q | ¬q → ¬p |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

3.

| p | q | ¬p | ¬q | ¬q → ¬p |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

4.

| p | q | ¬p | ¬q | ¬q → ¬p |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | T | T | T |

[TURN OVER]

**Discussion**:

The definition and truth table of the conditional operator $\rightarrow$ are discussed in the study guide pp.139-140. If you apply that you will see that **alternative 1** is the correct alternative. The $\rightarrow$ operator is false only when the first operand is true and the second operand is false. In all other cases it is true.

**Question 40**

Which one of the statements in the following alternatives is NOT equivalent to $p \wedge q$?
(***Hint***: simplify the statement in each alternative using de Morgan's rules or a truth table to find the statement that is NOT equivalent to $p \wedge q$.) **(**study guide pp.146-148, self-assessment exercises, assignment 3).

1. $p \wedge (\neg(p \rightarrow \neg q))$
2. $p \wedge (\neg(\neg p \rightarrow \neg q))$
3. $\neg(\neg p \vee (p \rightarrow \neg q))$
4. $p \wedge (\neg(\neg(p \wedge q)))$

**Discussion**:

We look at alternative 1 and 2 and simplify the statement as much as we can. We leave alternative 3 and 4 as an exercise.

Alternative 1:

$p \wedge (\neg(p \rightarrow \neg q))$

| | |
|---|---|
| $\equiv p \wedge (\neg(\neg p \vee \neg q))$ | See example on p. 149 of the study guide |
| $\equiv p \wedge (\neg\neg p \wedge \neg\neg q))$ | See item (f) under 'Important logical equivalences in study guide p. 147 |
| $\equiv p \wedge (p \wedge q)$ | Law of double negation, item (e) in study guide p. 147 |
| $\equiv (p \wedge p) \wedge q)$ | Associative law item (b) in study guide p. 147 |
| $\equiv (p \wedge q)$ | Idempotent law item (d) in study guide p. 149 |

This is clearly equivalent to the given statement.

Alternative 2:

$p \wedge (\neg(\neg p \rightarrow \neg q))$

| | |
|---|---|
| $\equiv p \wedge (\neg(\neg\neg p \vee \neg q))$ | See example on p. 149 of the study guide |
| $\equiv p \wedge (\neg(p \vee \neg q))$ | Law of double negation, item (e) in study guide p. 147 |
| $\equiv p \wedge (\neg p \wedge \neg\neg q))$ | See item (f) under 'Important logical equivalences in study guide p. 147 |
| $\equiv p \wedge (\neg p \wedge q))$ | Law of double negation, item (e) in study guide p. 147 |
| $\equiv (p \wedge \neg p) \wedge q$ | Associative law item (b) in study guide p. 147 |

| | |
|---|---|
| $\equiv$ FALSE $\wedge$ q | See item (g) in study guide p. 147 |
| $\equiv$ FALSE | |

This is clearly not the same as the given statement, thus **alternative 2** is the correct alternative to choose.

You can apply the rules in a similar way to alternatives 3 and 4.

**Question 41**

Which one of the statements in the following alternatives is equivalent to $\neg(\neg(p \rightarrow q)) \vee q$?
(***Hint***: simplify the given statement using de Morgan's rules.)

1. $p \vee q$
2. $\neg p \vee q$
3. $\neg p \wedge q$
4. $p \wedge q$

**Discussion**:

We simplify the statement as much as we can using the laws on p.147 of the study guide:
Alternative 1:

$\neg(\neg(p \rightarrow q)) \vee q$

| | |
|---|---|
| $\equiv \neg(\neg(\neg p \vee q)) \vee q$ | See example on p. 149 of the study guide |
| $\equiv \neg(\neg\neg p \wedge \neg q) \vee q$ | See item (f) under 'Important logical equivalences in study guide p. 147 |
| $\equiv \neg(p \wedge \neg q) \vee q$ | Law of double negation, item (e) in study guide p. 147 |
| $\equiv \neg p \vee \neg\neg q \vee q$ | See item (f) under 'Important logical equivalences in study guide p. 147 |
| $\equiv \neg p \vee q \vee q$ | Law of double negation, item (e) in study guide p. 147 |
| $\equiv \neg p \vee q$ | Idempotent law item (d) in study guide p. 149 |

Clearly, **alternative 2** is the correct alternative to select.

**Question 42**

Consider the following statement

$$[p \wedge (r \rightarrow q)] \leftrightarrow [(r \vee q) \wedge (p \rightarrow q)]$$

and the incomplete truth table for the given statement below:

| p | q | r | r → q | p ∧ (r → q) | ↔ | (r ∨ q) | ∧ | (p → q) |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | T |
| T | F | T | | | | T | | |
| T | F | F | | | F | F | | |
| F | T | T | T | F | F | T | T | T |
| F | T | F | T | F | F | T | T | T |
| F | F | T | | | F | T | | |
| F | F | F | | | | F | | |

Which one of the following alternatives gives the correct completed truth table? The values that were given in the table above are highlighted in each alternative.

1.

| p | q | r | r → q | p ∧ (r → q) | ↔ | (r ∨ q) | ∧ | (p → q) |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | T |
| T | F | T | F | F | F | T | F | F |
| T | F | F | T | T | F | F | F | F |
| F | T | T | T | F | F | T | T | T |
| F | T | F | T | F | F | T | T | T |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | F | F | F | F | T |

2.

| p | q | r | r → q | p ∧ (r → q) | ↔ | (r ∨ q) | ∧ | (p → q) |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | T |
| T | F | T | F | F | T | T | F | F |
| T | F | F | T | T | F | F | F | F |
| F | T | T | T | F | F | T | T | T |
| F | T | F | T | F | F | T | T | T |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | F | T | F | F | T |

**[TURN OVER]**

3.

| p | q | r | $r \to q$ | $p \wedge (r \to q)$ | $\leftrightarrow$ | $(r \vee q)$ | $\wedge$ | $(p \to q)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | T |
| T | F | T | F | F | T | T | F | F |
| T | F | F | F | F | F | F | F | F |
| F | T | T | T | F | F | T | T | T |
| F | T | F | T | F | F | T | T | T |
| F | F | T | F | F | F | T | F | F |
| F | F | F | T | F | T | F | F | F |

4.

| p | q | r | $r \to q$ | $p \wedge (r \to q)$ | $\leftrightarrow$ | $(r \vee q)$ | $\wedge$ | $(p \to q)$ |
|---|---|---|---|---|---|---|---|---|
| .<br>T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | T |
| T | F | T | T | T | T | T | T | T |
| T | F | F | T | T | F | F | F | T |
| F | T | T | T | F | F | T | T | T |
| F | T | F | T | F | F | T | T | T |
| F | F | T | T | F | F | T | T | T |
| F | F | F | T | F | T | F | F | T |

**Discussion**:

We leave this as an exercise for you to complete. The correct alternative is alternative 2. Apply the theory and definitions in your study guide pages 139 to 147. You need to practice this so that you do not spend a lot of time on this type of question in the exam.

**Question 43**

Consider the two statements below:

Statement 1: $\forall x \in \mathbb{Z}^+$, $[(2x + 1 > 3) \vee (2x^2 - 1 \geq 1)]$

Statement 2: $\exists x \in \mathbb{Z}$, $[(x^2 - 1 < 0) \wedge (2x - 2 \geq 0)]$

Which one of the following alternatives is true regarding statements 1 and 2?

1. Statement 1 is true and statement 2 is false.
2. Statement 1 is false and statement 2 is true.
3. Both statements 1 and 2 are false.
4. Both statements 1 and 2 are true.

**Discussion**:

Make sure that you understand the definitions of the universal and the existential quantifiers on pp. 152 and 153 of the study guide, because you will use these definitions in all these type of questions.

In statement 1, let A = ($2x + 1 > 3$), and B = ($2x^2 – 1 \geq 1$), then the statement reads as follows: For ALL values of x from the positive integers, it is true that A OR B is true. Remember the definition of $\vee$ states that both A and B may also be true, but at least one should be true for the statement to be true. We do the same for statement 2.
In statement 2, let C = ($x^2 – 1 < 0$) and D = ($2x - 2 \geq 0$), then the statement reads as follows: There exists a value of x from the integer set, such that C AND D are true. So in this case BOTH parts of the statement must be true.

In statement 1, where only A OR B needs to be true, we don't have to test B if A is true, because, only one of them needs to be true. We can test this in various ways. If we substitute a few values from the positive integers, in A (or B), we will quickly see a pattern or find a counterexample:

| **Value chosen for x** | **Result : statement A =** $2x + 1 > 3$ |
|---|---|
| $x = 1$ | $2(1) + 1 = 3$, which is not greater than 3 |
| $x = 2$ | |
| $x = 3$ | |
| $x = 4$ | |

We don't need to test any further. x = 1 is counterexample to show that A is false. So we test B next.

| **Value chosen for x** | **Result : statement B =** $2x^2 – 1 \geq 1$ |
|---|---|
| $x = 1$ | $2(1)^2 – 1 = 1 \geq 1$ |
| $x = 2$ | $2(2)^2 – 1 = 7 \geq 1$ |
| $x = 3$ | $2(3)^2 – 1 = 17 \geq 1$ |
| $x = 4$ | ……… |

Can you see that B will always be true, because the higher the value of x, the higher the result. Remember for statement 1, only values from the positive integers are allowed. Statement 1 is therefore true, because B is true.

For statement 2, we only need to find one value from the set of integers for which C and D both are true. In this case we always want to test some negative and positive values for x.

| **Value chosen for x** | **Result : statement C =** $(x^2 - 1 < 0)$ |
|---|---|
| $x = -2$ | $(-2)^2 - 1 = 3$ which is not less than 0 |
| $x = -1$ | $(-1)^2 - 1 = 0$ which is not less than 0 |
| $x = 0$ | $(0)^2 - 1 = -1$, which is less than 0 |
| $x = 1$ | $(1)^2 - 1 = 0$ which is not less than 0 |
| $x = 2$ | $(2)^2 - 1 = 3$ which is not less than 0 |

You should be able to see that statement C is true iff $x = 0$. At this point we check if $x = 0$ will also make D true:

| **Value chosen for x** | **Result : statement D =** $2x - 2 \geq 0$ |
|---|---|
| $x = 0$ | $2(0) - 2 = -2$ which is not greater than or equal to 0 |

Therefore statement 2 is false. **Alternative 2** is therefore the correct alternative to choose.

**Question 44**

Consider the following statement:

$$\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]$$

Which one of the following alternatives provides the correct simplification of the negation of the given statement such that the *not*-symbol ($\neg$) does not occur to the left of any quantifier?

1. $\neg[\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]]$

   $\equiv \forall x \in \mathbb{Z}, \neg[(2x + 4 > 0) \vee (4 - x^2 \leq 0)]$

   $\equiv \forall x \in \mathbb{Z}, [\neg(2x + 4 > 0) \wedge \neg(4 - x^2 \leq 0)]$

   $\equiv \forall x \in \mathbb{Z}, [(2x + 4 \leq 0) \wedge (4 - x^2 > 0)]$

2. $\neg[\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]]$

   $\equiv \exists x \in \mathbb{Z}, \neg[(2x + 4 > 0) \vee (4 - x^2 \leq 0)]$

   $\equiv \exists x \in \mathbb{Z}, [\neg(2x + 4 > 0) \wedge \neg(4 - x^2 \leq 0)]$

   $\equiv \exists x \in \mathbb{Z}, [(2x + 4 < 0) \wedge (4 - x^2 \geq 0)]$

3. $\neg[\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]]$

   $\equiv \exists x \in \mathbb{Z}, \neg[(2x + 4 > 0) \vee (4 - x^2 \leq 0)]$

   $\equiv \exists x \in \mathbb{Z}, [\neg(2x + 4 > 0) \wedge \neg(4 - x^2 \leq 0)]$

   $\equiv \exists x \in \mathbb{Z}, [(2x + 4 \leq 0) \wedge (4 - x^2 > 0)]$

4. $\neg[\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]]$

   $\equiv \exists x \in \mathbb{Z}, \neg[(2x + 4 > 0) \wedge (4 - x^2 \leq 0)]$

   $\equiv \exists x \in \mathbb{Z}, [\neg(2x + 4 > 0) \vee \neg(4 - x^2 \leq 0)]$

   $\equiv \exists x \in \mathbb{Z}, [(2x + 4 \leq 0) \vee (4 - x^2 > 0)]$

**Discussion**:

If p(x) is a statement, then $\neg[\forall x \in \mathbb{Z}, p(x)] = \exists x \in \mathbb{Z}, \neg[p(x)]]$.

Similarly, $\neg[\exists x \in \mathbb{Z}, p(x)] = \forall x \in \mathbb{Z}, \neg[p(x)]]$.

Then for this question you also need to know de Morgan's rules:

If p(x) and q(x) are two statements, then $\neg[p(x) \vee q(x)] = \neg p(x) \wedge \neg q(x)$.

Similarly, $\neg[p(x) \wedge q(x)] = \neg p(x) \vee \neg q(x)$ (study guide 147, 152-153, additional exercises, assignment 3).

We use these rules, and determine the negative of the given statement:

$\neg[\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]]$

$\equiv \exists x \in \mathbb{Z}, \neg[(2x + 4 > 0) \vee (4 - x^2 \leq 0)]$

$\equiv \exists x \in \mathbb{Z}, [\neg(2x + 4 > 0) \wedge \neg(4 - x^2 \leq 0)]$

$\equiv \exists x \in \mathbb{Z}, [(2x + 4 \leq 0) \wedge (4 - x^2 > 0)]$

**Question 45**

Which one of the following alternatives provides a statement that is TRUE?

1. $\exists x \in \mathbb{Z}^+, [(2x - 3 < 0) \wedge (x^2 + 1 \geq 10)]$
2. $\forall x \in \mathbb{Z}^+, [(2x - 3 < 0) \vee (x^2 + 1 \geq 10)]$
3. $\exists x \in \mathbb{Z}^+, [(2x - 3 > 0) \wedge (x^2 + 1 \leq 10)]$
4. $\forall x \in \mathbb{Z}^+, [(2x - 3 > 0) \vee (x^2 + 1 \geq 10)]$

**[TURN OVER]**

**Discussion**:

Similar to Question 43, you can go through the exercise of substituting values for x to see if there are any values of x for which both the statements in each alternative given. Note that $x \in \mathbb{Z}^+$ for each alternative. It could be different. Always check what the set is that x is defined for. Also note whether the alternative uses the universal or the existential quantifiers. (See definition for these quantifiers in Study guide pp152-153). For the universal quatifier, the given statement must be true for ALL values of x. For the existential quantifier, we only need to find ONE value of x for which the statement is true. Also note which logical operator is used in the statement. For $\wedge$, both sub-statements must be true for the whole statement to be true. For , at least one of the sub-statements must be true for the whole statement to be true.

Alternative 1:

It is clear that $(2x - 3 < 0)$ is only true for $x = 1$, but $(x^2 + 1 \geq 10)$ is not true for $x = 1$. Therefore no value of x can make this statement true, because BOTH $(2x - 3 < 0)$ and $(x^2 + 1 \geq 10)$ must be true for the whole statement to be true.

Alternative 2:

We give a counterexample: for $x = 2$ neither $(2x - 3 < 0)$ nor $(x^2 + 1 \geq 10)$ is true, so the statement as a whole cannot be true.

Alternative 3:

For any value of x greater than 1 $(2x - 3 > 0)$ is true. For $x = 1, 2$ and $3$, $(x^2 + 1 \leq 10)$ is true. We therefore have find at least one value of x for which both $(2x - 3 > 0)$ and $(x^2 + 1 \leq 10)$ are true. **Alternative 3** is therefore the correct alternative to choose.

Alternative 4:

We give a counterexample: for $x = 1$ neither $(2x - 3 > 0)$ **nor** $(x^2 + 1 \geq 10)$ is true.

**SECTION 6**
**MATHEMATICAL PROOFS**
**QUESTIONS 46 – 50 (5 marks)**

**Question 46**
Consider the statement

**If n is even, then $4n^2 + 2n – 7$ is odd.**

Which one of the following statements provides the **converse** of the given statement?

1. If n is odd, then $4n^2 + 2n – 7$ is even.
2. If $4n^2 + 2n – 7$ is even, then n is odd.
3. If $4n^2 + 2n – 7$ is odd, then n is even.
4. If n is odd, then $4n^2 + 2n – 7$ is odd.

**Discussion**:

The converse of $p \to q$, is $q \to p$. If p represents 'n is even' and q represents $4n^2 + 2n – 7$ is odd, then **alternative 3** represents $q \to p$ (study guide p. 161).

**Question 47**
Consider the statement

**If n is a multiple of 3, then $3n^2 + 6n + 9$ is even.**

Which one of the following statements provides the **contrapositive** of the given statement?

1. If n is a multiple of 3, then $3n^2 + 6n + 9$ is even.
2. If n is not a multiple of 3, then $3n^2 + 6n + 9$ is odd.
3. If $3n^2 + 6n + 9$ is even, then n is a multiple of 3.
4. If $3n^2 + 6n + 9$ is odd, then n is not a multiple of 3.

**Discussion**:

The contrapositive of $p \to q$, is $\neg q \to \neg p$. If p represents 'n is a multiple of 3' and q represents **$3n^2 + 6n + 9$ is even**, then **alternative 4** represents $\neg q \to \neg p$ (study guide p 160).

**[TURN OVER]**

**Question 48**

Which of the following alternatives provides a **direct** proof to show that for all n $\in \mathbb{Z}$,

**if n + 1 is even, then $n^2 + 3n + 4$ is even.**

1. Let n be even, then n = 2k, for some k $\in \mathbb{Z}$,
   then $n^2 + 3n + 4$
   ie $(2k)^2 + 3(2k) + 4$,
   ie $4k^2 + 6k + 4$,
   ie $2(2k^2 + 3k + 2)$, which is even.

2. Let n + 1 = 2, which is even, then n = 2 – 1 = 1,
   then $n^2 + 3n + 4$
   ie $(1)^2 + 3(1) + 4$,
   ie 8 which is even.
3. Let n + 1 be even,
   then $n^2 + 3n + 4$
   ie $(n + 1)^2 + 3(n + 1) + 4$,
   ie $n^2 + 2n + 1 + 3n + 3 + 4$,
   ie $n^2 + 5n + 1 + 3 + 4$,
   ie $n^2 + 5n + 2(4)$, which is even because odd + odd + even is even.

4. Let n + 1 be even, then n + 1 = 2k, for some k $\in \mathbb{Z}$, then n = 2k – 1,
   then $n^2 + 3n + 4$
   ie $(2k - 1)^2 + 3(2k - 1) + 4$,
   ie $4k^2 - 4k + 1 + 6k - 3 + 4$,
   ie $4k^2 + 2k + 2$
   ie $2(2k^2 + k + 1)$, which is even.

**Discussion**:

if n + 1 is even, then $n^2 + 3n + 4$ is even.

In a direct proof, we assume the left hand side is true, and then try and prove that the right hand side is also true (study guide p. 159).

Alternative 1:

Let n be even, then n = 2k, for some k $\in \mathbb{Z}$,
   then $n^2 + 3n + 4$
   ie $(2k)^2 + 3(2k) + 4$,
   ie $4k^2 + 6k + 4$,
   ie $2(2k^2 + 3k + 2)$, which is even.

The left hand side of the statement says that ‘if n + 1 is even”. If n + 1 is even, it means that n is odd. This alternative wrongly assumes that n is even.

Alternative 2:

When we want to prove something, we cannot just substitute a value for n. It must be true for all values of n. This alternative is therefore not a proof.

Alternative 3:

. Let n + 1 be even,
then $n^2 + 3n + 4$
ie $(n + 1)^2 + 3(n + 1) + 4$,
ie $n^2 + 2n + 1 + 3n + 3 + 4$,
ie $n^2 + 5n + 1 + 3 + 4$,
ie $n^2 + 5n + 2(4)$, which is even because odd + odd + even is even.

This alternative starts off correctly by assuming that n + 1 is even, but it does not equate n + 1 to 2k, for some value k. This means that there is no way that we can determine whether n is even or not, as you can see from the final line of the proof.

Alternative 4:

If you work through the proof in this alternative, you will see that it gives the correct direct proof. **Alternative 4** should therefore be selected.

**Question 49**

Consider the following statement, for all $x \in \mathbb{Z}$:

If $x^3 - 2x$ is odd, then x is odd.

Which one of the following alternatives contains the correct way to start a proof by **contradiction** to proof the statement?

1. Assume x is odd, then $x = 2k + 1$ for some $k \in \mathbb{Z}$,
   ie $x^3 - 2x = (2k + 1)^3 – 2(2k +1)$,
   ie ………..

2. Assume x is even, then $x = 2k$ for some $k \in \mathbb{Z}$,
   ie $x^3 - 2x = (2k)^3 – 2(2k)$,
   ie ………..

3. Assume $x^3 - 2x$ is odd, then x can be odd or even. We will assume that x is even.
   Let x be even, then $x = 2k$ for some $k \in \mathbb{Z}$,
   ie ………..

4. Assume $x^3 - 2x$ is odd,

**[TURN OVER]**

We know that an odd number minus an even number is odd,
ie $x^3$ must be odd, because odd * odd * odd is odd,
ie let $x = 2k + 1$ for some $k \in \mathbb{Z}$,
ie ………..

**Discussion**:

A proof by contradiction assumes that the first half of the statement is true, and then tries to prove that the opposite of the second half of the statement is true, leading to a contradiction.
Alternative 1:
This alternative tries to prove that the converse of the original statement is true.
Alternative 2:
This alternative tries to do a contrapositive proof (study guide p. 160-161).
Alternative 3:
This alternative assumes that the first half of the statement is true, and then tries to prove the opposite of the second half of the statement. **Alternative 3** is therefore the alternative that tries to do a proof by contradiction (Study guide p.160).
Alternative 4:
This alternative does not have a valid proof.

**Question 50**

Which one of the following values for x can be used in a counter-example to prove that the statement $\exists x \in \mathbb{Z}^+$, $-x^3 - 5x - 7 > 0$ is FALSE.

1. -2
2. -1
3. 0
4. 1

**Discussion**:

$x \in \mathbb{Z}^+$, therefore $x \in \{1, 2, 3, 4.....\}$. It should be clear that alternatives 1, 2 and 3 cannot be correct.
We test if alternative 4 will indeed prove that the given statement is false:
Let x = 1, then
$-x^3 - 5x - 7 = -(1)^3 - 5(1) - 7 = -13$, which is not greater than 0. **Alternative 4** therefore proves that the given statement is false. It is very important to check the domain of x in these types of questions. (study guide 152-156 plus assignment 3).

- - - - - - -oooooooOOOOoooooooo- - - - - -