# Tutorial letter 202/0/2021

## Theoretical Computer Science 1

# COS1501

## Year module

## School of Computing

**Discussion of Assignment 02**

Dear Student,

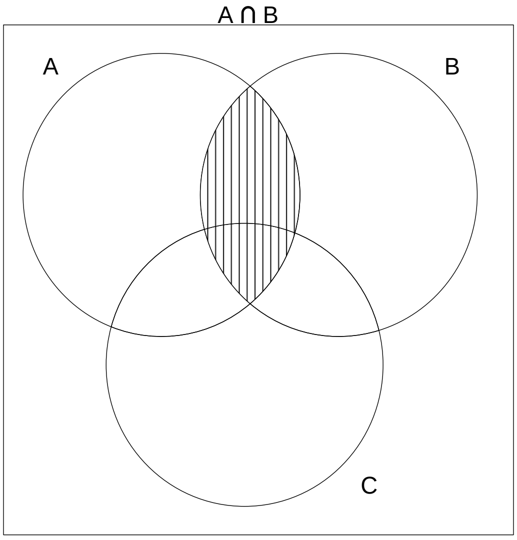
The solutions to the second assignment MCQ questions are discussed in this tutorial letter. A discussion of the self-assessment questions is provided in tutorial letter 102. It is very important that you also do all the self-assessment questions. Take note of the hints provided in tutorial letter 101 since these hints will help you to avoid making common errors in the exam.

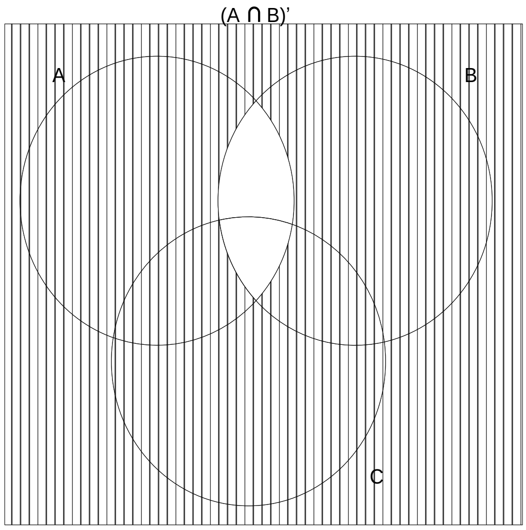
Regards,
COS1501 Team

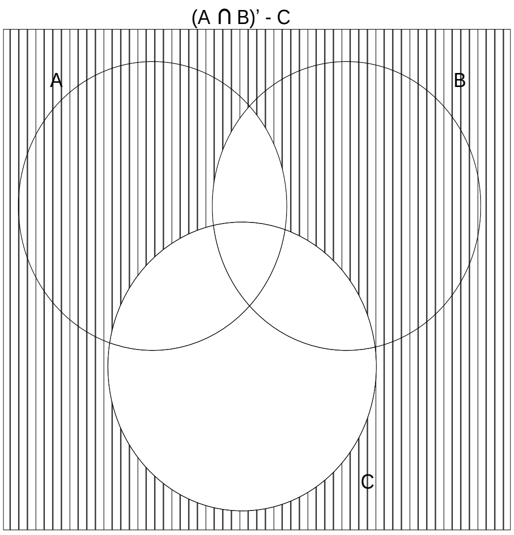
## Discussion of assignment 02

**Question 1** **Alternative 4**

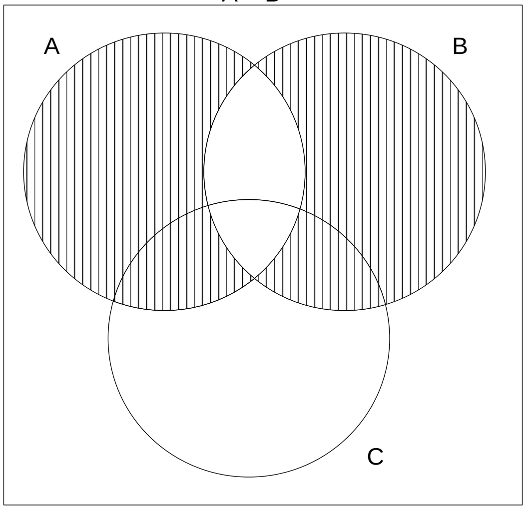
We determine the Venn diagram for the set $[(A \cap B)' - C] \cap [(A + B) - C)]$ step by step:



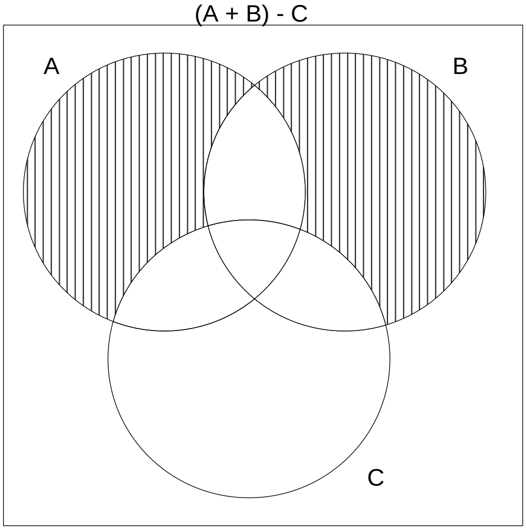



A + B

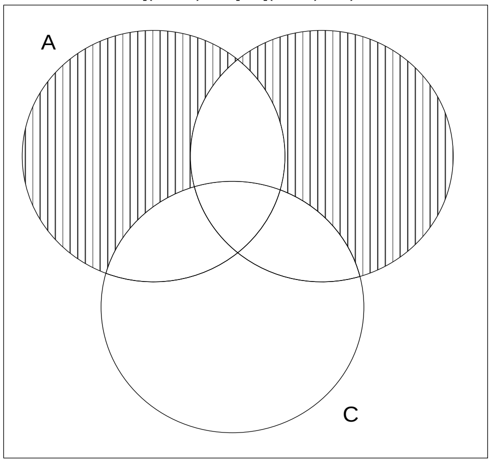



(A + B) - C




Answer:

[(A ∩ B)' – C] ∩ [(A + B) – C)




Did you notice that [(A $\cap$ B)′ – C] $\cap$ [(A + B) – C) = (A + B) – C]?

*Refer to study guide, pp 50, 51.*

**Question 2** **Alternative 1**

Let A, B and C be subsets of a universal set U = {1, 2, 3, 4}.

The statement (A – B) $\cup$ C′ = (C′ – B) + A is NOT an identity. Which of the following sets A, B and C can be used in a *counterexample* to prove that the given statement is not an identity?

1. A = {1}, B = {2} & C = {3}
2. A = {1}, B = {1} & C = {2}
3. A = {1, 2}, B = {1, 2} & C = {3}
4. A = {3}, B = {3, 4} & C = {4}

*Discussion*

*Given: A, B, C $\subseteq$ U with U = {1, 2, 3, 4}, and (A – B) $\cup$ C′ = (C′ – B) + A is not an identity.*

*We do **not** start our counterexample solution with (A – B) $\cup$ C′ ≠ (C′ – B) + A.*

***First*** *we* ***determine (A – B) ∪ C′****,* ***then*** *we* ***determine (C′ – B) + A*** *by using the sets provided in the different alternatives, then we* ***compare the answers*** *and come to a conclusion.*

We consider the different alternatives:

1. We use the sets $A = \{1\}$, $B = \{2\}$ & $C = \{3\}$ to determine $(A - B) \cup C'$ and $(C' - B) + A$ then we compare the answers. Note that we use **curly brackets** for **sets**.

$$\begin{aligned}(A - B) \cup C' &= [\{1\} - \{2\}] \cup \{3\}' \\ &= \{1\} \cup \{1, 2, 4\} \\ &= \{1, 2, 4\} \\ (C' - B) + A &= [\{3\}' - \{2\}] + \{1\} \\ &= [\{1, 2, 4\} - \{2\}] + \{1\} \\ &= \{1, 4\} + \{1\} \\ &= \{4\}\end{aligned}$$

Clearly $\{1, 2, 4\} \neq \{4\}$ thus $(A - B) \cup C' \neq (C' - B) + A$.

2. We use the sets $A = \{1\}$, $B = \{1\}$ & $C = \{2\}$ to determine $(A - B) \cup C'$ and $(C' - B) + A$ then we compare the answers.

$$\begin{aligned}(A - B) \cup C' &= [\{1\} - \{1\}] \cup \{2\}' \\ &= \{\ \} \cup \{1, 3, 4\} \\ &= \{1, 3, 4\} \\ (C' - B) + A &= [\{2\}' - \{1\}] + \{1\} \\ &= [\{1, 3, 4\} - \{1\}] + \{1\} \\ &= \{3, 4\} + \{1\} \\ &= \{1, 3, 4\}\end{aligned}$$

Thus $(A \cap B) - C' = (C' - B) + A$.

3. We use the sets $A = \{1, 2\}$, $B = \{1, 2\}$ & $C = \{3\}$ to determine $(A - B) \cup C'$ and $(C' - B) + A$ then we compare the answers.

$$\begin{aligned}(A - B) \cup C' &= [\{1, 2\} - \{1, 2\}] \cup \{3\}' \\ &= \{\ \} \cup \{1, 2, 4\} \\ &= \{1, 2, 4\} \\ (C' - B) + A &= [\{3\}' - \{1, 2\}] + \{1, 2\} \\ &= [\{1, 2, 4\} - \{1, 2\}] + \{1, 2\} \\ &= \{4\} + \{1, 2\} \\ &= \{1, 2, 4\}\end{aligned}$$

Thus $(A - B) \cup C' = (C' - B) + A$.

4. We use the sets $A = \{3\}$, $B = \{3, 4\}$ & $C = \{4\}$ to determine $(A - B) \cup C'$ and $(C' - B) + A$ then we compare the answers.

$$
\begin{aligned}
(A - B) \cup C' &= [\{3\} - \{3, 4\}] \cup \{4\}' \\
&= \{\ \} \cup \{1, 2, 3\} \\
&= \{1, 2, 3\} \\
(C' - B) + A &= [\{4\}' - \{3, 4\}] + \{3\} \\
&= [\{1, 2, 3\} - \{3, 4\}] + \{3\} \\
&= \{1, 2\} + \{3\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

Thus $(A - B) \cup C' = (C' - B) + A$.

Alternatives 2, 3 and 4 do not provide counterexamples, but a counterexample is provided in alternative 1, thus this alternative should be selected.

*Refer to study guide, pp 60, 61.*

**Question 3** **Alternative 2**

We want to prove that for all $A, B, C \subseteq U$,

$(A \cap B) \cup (C - B) = (A \cup C) \cap (A \cup B') \cap (B \cup C)$ is an identity.

*Discussion:*
*In the proof we apply the definitions for union, intersection, difference and complement of sets. The notation should be correct and all the necessary steps should appear in the proof.*

We complete the proof by including the correct steps 4 & 6 from alternative 2:

$z \in (A \cap B) \cup (C - B)$

iff $(z \in A$ and $z \in B)$ or $(z \in C$ and $z \notin B)$

iff $(z \in A$ or $z \in C)$ and $(z \in A$ or $z \notin B)$ and $(z \in B$ or $z \in C)$ and $(z \in B$ or $z \notin B)$

iff $(z \in A$ or $z \in C)$ and $(z \in A$ or $z \in B')$ and $(z \in B$ or $z \in C)$ and $(z \in B$ or $z \in B')$

iff $z \in (A \cup C)$ and $z \in (A \cup B')$ and $z \in (B \cup C)$ and $z \in (B \cup B')$

iff $z \in (A \cup C)$ and $z \in (A \cup B')$ and $z \in (B \cup C)$ and $z \in U$

iff $z \in (A \cup C) \cap (A \cup B') \cap (B \cup C) \cap U$

iff $z \in (A \cup C) \cap (A \cup B') \cap (B \cup C)$ [For any sets U and G, $(G \cap U) = G$.]

*Refer to study guide, pp 41-43, 55-57.*

**Question 4** **Alternative 1**

Fourty (40) students go to a party wearing red, white and blue.
Of these students,

17 wear red,
22 wear white,
25 wear blue.
(Students do not necessarily wear only one colour.)
Furthermore,
7 wear red and white,
12 wear blue and white, and
9 wear red and blue.

Which one of the following alternatives is true? (Hint: first calculate the value of x, the unknown)

1. 5 students wear red only.
   8 students wear white and blue, but not red.
   3 students wear red and white, but not blue.
2. 2 students wear red only.
   11 students wear white and blue, but not red.
   6 students wear red and white, but not blue.
3. 2 students wear red only.
   8 students wear white and blue, but not red.
   3 students wear red and white, but not blue.
4. 5 students wear red only.
   11 students wear white and blue, but not red.
   6 students wear red and white, but not blue.

*Solution:*

$|U| = 40$, $|R| = 17$, $|B| = 25$, $|W| = 22$, (U = universal set; R = red; B = blue;)

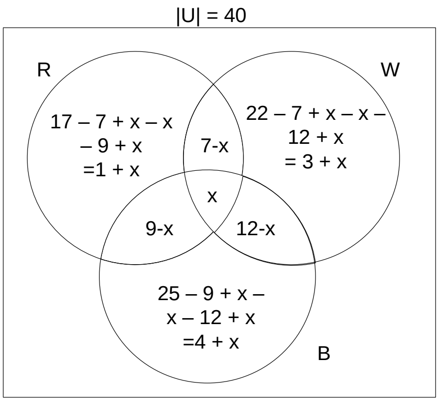
$|R \cap B| = 9$,

$|R \cap W| = 7$,

$|B \cap W| = 12$,

Let **x** students wear red, white and blue, ie $|R \cap W \cap B| = \mathbf{x}$.

*Now we can fill in the various regions in the following Venn diagram.*
*We initially fill in **x** for* $|R \cap W \cap B|$.



Now we can calculate the value of x:

$|U| = 40 = 1 + x + 7 – x + x + 9 – x + 3 + x + 12 – x + 4 + x$
ie $40 = 36 + x$, ie 4 students wear white, red and blue.

How many students wear red only? From the Venn diagram $1 + x = 1 + 4 = 5$ students wear read only.
How many students wear white and blue, but not red? From the Venn diagram $12 – x = 12 – 4 = 8$ students wear white and blue, but not red.

How many students wear read and white, but not blue? From the Venn diagram $7 – x = 7 – 4 = 3$ students wear red and white, but not blue.

We can therefore conclude that alternative 1 is the correct alternative to choose.*Refer to study guide, pp 63 – 66.*

**Let T be a relation from A = {0, 1, 2, 3} to B = {0, 1, 2, 3, 4} such that**
**(a, b) $\in$T iff $b^2 – a^2$ is an odd number. (A, B $\subseteq$ U = $\mathbb{Z}$.)**
**(Hint: Write down all the elements of T. Eg, if 4 $\in$ B and 1 $\in$ A then $4^2 – 1^2 = 16 – 1 = 15$ which is an odd number, thus (1, 4) $\in$ T.)**
**Answer questions 5 and 6 by using the defined relation T.**

**Question 5** **Alternative 3**

Which one of the following alternatives provides only elements belonging to T?
1. (3, 1), (4, 1), (3, 2)
2. (0, 1), (2, 4), (2, 3)
3. (3, 0), (1, 2), (3, 4)
4. (1, 0), (1, 2), (1, 3)

We consider ordered pairs provided in the different alternatives:

1. Is (3, 1) $\in$ T? No, $1^2 – 3^2 = –8$ which is an even number (not odd) thus (3, 1) $\notin$ T.
Since (3, 1) $\notin$ T, this alternative does not provide only elements belonging to T.

2. Is (2, 4) $\in$ T? No, $4^2 – 2^2 = 12$ which is an even number thus (2, 4) $\notin$ T.
Since (2, 4) $\notin$ T, this alternative does not provide only elements belonging to T.

3. Is (3, 0) $\in$T? Yes, $0^2 – 3^2 = –9$ which is an odd number thus (3, 0) $\in$ T.
Is (1, 2) $\in$T? Yes, $2^2 – 1^2 = 3$ which is an odd number thus (1, 2) $\in$ T.
Is (3, 4) $\in$T? Yes, $4^2 – 3^2 = 7$ which is an odd number thus (3, 4) $\in$ T.
Thus all the given ordered pairs in this alternative are elements of T.

4. Is (1, 3) $\in$ T? No, $3^2 – 1^2 = 8$ which is an even number thus (1, 3) $\notin$ T.
Since (1, 3) $\notin$ T, this alternative does not provide only elements belonging to T.

From the arguments provided we can deduce that alternative 3 should be selected.

*Refer to study guide, p 73.*

**Question 6** **Alternative 4**

Which one of the following statements regarding the relation T is true?

1. T is transitive.
2. T is symmetric.
3. T is antisymmetric.
4. T is irreflexive.

*Discussion*

*We first provide definitions using some relation R on A:*

*Irreflexive:*

*We ask the question: Is it true that for all $x \in A$ that we have $(x, x) \notin R$?*

*(For **no** element $x \in A$ we have that $(x, x) \in R$.)*

*Symmetric: We ask the question: Is it true that for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$?*

*Antisymmetric: We ask the question: Is it true that for all $x, y \in A$,*
*if $x \neq y$ and $(x, y) \in R$ then $(y, x) \notin R$?*

*Transitive: R is transitive iff it has the property that for all $x, y, z \in A$,*
*whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.*
*We ask the question: Is it true that for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$?*

*T is a relation from $A = \{0, 1, 2, 3\}$ to $B = \{0, 1, 2, 3, 4\}$ such that*
*$(a, b) \in T$ iff $b^2 - a^2$ is an odd number. We provide the set T:*
*$T = \{(1, 0), (3, 0), (0, 1), (2, 1), (1, 2), (3, 2), (0, 3), (2, 3), (1, 4), (3, 4)\}$*

We consider the different alternatives:

1. We provide a counterexample to prove that T is *not transitive*:
   $(3, 0) \in T$ and $(0, 1) \in T$ but $(3, 1) \notin T$.

2. We provide a counterexample to prove that T is *not symmetric*:
   $(1, 4) \in T$ but $(4, 1) \notin T$.

3. We provide a counterexample to prove that T is *not antisymmetric*:
   $(1, 0) \in T$ and $(0, 1) \in T$.

4. There is no element $x$ in A and B such that $(x, x) \in T$ thus *T is irreflexive*.
   (**No** elements such as (1, 1), (2, 2), … belong to T.)

From the arguments provided it is clear that alternative 4 should be selected.

*Refer to study guide, pp 75-78.*

**Consider the following relation on the set B = {a, b, {a}, {b}, {a, b}}:**

**P = {(a, b), (b, {a, b}), ({a, b}, a), ({b}, a), (a, {a})}.**

**Answer questions 7 to 10 by using the given relation P and the set B.**

**Question 7** **Alternative 1**

Which one of the following alternatives represents the range of P (ran(P))?

1. {a, b, {a}, {a, b}}
2. {a, b, {a}, {b}, {a, b}}
3. {a, b, {b}, {a, b}}
4. {a, b, {a, b}}

*Let's first look at the definition for the range of a function:*
*Given a function T from X to Y, the range of T is defined by:*
*ran(T) = {y* | *for some x ∈ X, (x, y) ∈ T}, ie the set of second coordinates.*

We see that only the elements a, b, {a}, {a, b} appear as second co-ordinates, thus alternative 1 provides the range of P, ie ran(P) = {a, b, {a}, {a, b}}.

**Question 8** **Alternative 3**

Which one of the following relations represents the composition relation P ○ P (ie P; P)?

1. {(a, {a, b}), (b, a), ({a, b}, a), ({b}, {a}), (a, {a})}
2. {(a, {a, b}), (b, a), ({a, b}, a), ({b}, {a})}
3. {(a, {a, b}), (b, a), ({a, b}, {a}), ({a, b}, b), ({b}, b), ({b}, {a})}
4. {(a, {a, b}), (b, a), ({a, b}, {a}), ({a, b}, b), ({b}, {a})}

*Discussion*
*We first look at the definition of a composition relation:*
*Given relation* **P** *from B to B and P from B to B, the composition of* **P** *followed by P*

*(P ○* **P** *or* **P***; P) is the relation from B to B defined by*

*P ○* **P** = **P**; *P = {(m, o)* | *there is some n ∈ B such that (m, n) ∈* **P** *and (n, o) ∈ P}.*

*(P and* **P** *is exactly the same relation, but for the purpose of our explanations we make the subtle distinction.)*

P = {(a, b), (b, {a, b}), ({a, b}, a), ({b}, a), (a, {a})} is defined on B = {a, b, {a}, {b}, {a, b}}.

To determine **P**; *P* we start with the pair (a, **b**) of **P**, and then we look for a pair in *P* that has as first co-ordinate an **b**, and then see where it takes us.
Link (a, **b**) of **P** with (**b**, {a, b}) of *P*, then (a, {a, b}) ∈ **P**; *P*.

Link (b, {**a**, **b**}) of **P** with ({**a**, **b**}, a) of *P*, then (b, a) $\in$ **P**; *P*.
Link ({a, b}, **a**) of **P** with (**a**, {a}) of *P*, then ({a, b}, {a}) $\in$ **P**; *P*.
Link ({a, b}, **a**) of **P** with (**a**, b) of *P*, then ({a, b}, b) $\in$ **P**; *P*.
Link ({b}, **a**) of **P** with (**a**, b) of *P*, then ({b}, b) $\in$ **P**; *P*.
Link ({b}, **a**) of **P** with (**a**, {a}) of *P*, then ({b}, {a}) $\in$ **P**; *P*.
No other pairs can be linked, so
P $\circ$ P = {(a, {a, b}), (b, a), ({a, b}, {a}), ({a, b}, b), ({b}, b), ({b}, {a})}

*Refer to study guide, pp 79, 108, 109.*

**Question 9** **Alternative 3**

The relation P does not satisfy trichotomy. Which ordered pairs can be included in P so that an extended relation $P_1$ (say) would satisfy trichotomy?

1. (b, {a}), (b, {b}), (b, a), ({a}, {a, b}) & ({a, b}, {b})
2. ({a}, b), (b, {b}), ({b}, {a, b}) & ({a, b}, {a})
3. (b, {a}), ({b}, b), ({a}, {b}), ({b}, {a, b}) & ({a, b}, {a})
4. (b, {a}), (b, {b}), ({a}, {b}) & ({a, b}, {a, b})

*We ask the question: Which ordered pairs should be included in $P_1$ so that it will be true that for **all** x, y $\in$ B with **x ≠ y**, we have **(x, y) $\in$ $P_1$ or (y, x) $\in$ $P_1$**?*

P = {(a, b), (b, {a, b}), ({a, b}, a), ({b}, a), (a, {a})} is defined on B = {a, b, {a}, {b}, {a, b}}.

We compare the elements of B: b ≠ {a}, {b} ≠ b, {a} ≠ {b}, {b} ≠ {a, b} & {a, b} ≠ {a} but these elements are not grouped in ordered pairs of P.
We only have (a, b), (b, {a, b}), ({a, b}, a), ({b}, a), (a, {a}) $\in$ P. We can include the ordered pairs (b, {a}), ({b}, b), ({a}, {b}), ({b}, {a, b}) & ({a, b}, {a}) in $P_1$ which will then satisfy trichotomy.

$P_1$ = {(a, b), (b, {a, b}), ({a, b}, a), ({b}, a), (a, {a}), (b, {a}), ({b}, b), ({a}, {b}), ({b}, {a, b}),
({a, b}, {a})} satisfies trichotomy.

From the arguments provided we can deduce that alternative 3 should be selected.

*Refer to study guide, p 78.*

**Question 10** **Alternative 1**

Which one of the following sets is a partition of B = {a, b, {a}, {b}, {a, b}}?

1. {**{**a, b, {a}, {b}**}**, **{**{a, b}**}**}
2. {{a}, {b}, {a, b}}
3. {**{**a, b, {a}**}**, **{**{a}, {b}, {a, b}**}**}
4. {a, b, {a}, {b}, {a, b}}

*Discussion*
*Refer to the definition of a partition provided in the question. We test the sets provided in each alternative against this definition:*

1. Let P = $\{\mathbf{\{}a, b, \{a\}, \{b\}\mathbf{\}}, \mathbf{\{}\{a, b\}\mathbf{\}}\}$ (say).

   We test whether P is a partition of B = $\{a, b, \{a\}, \{b\}, \{a, b\}\}$:

   a. $\mathbf{\{}a, b, \{a\}, \{b\}\mathbf{\}}$ and $\mathbf{\{}\{a, b\}\mathbf{\}}$ are two **non-empty** subsets of B,
   b. $\mathbf{\{}a, b, \{a\}, \{b\}\mathbf{\}} \cap \mathbf{\{}\{a, b\}\mathbf{\}} = \emptyset$, and
   c. $\mathbf{\{}a, b, \{a\}, \{b\}\mathbf{\}} \cup \mathbf{\{}\{a, b\}\mathbf{\}} = B$.

   Since P has all the necessary properties, it is a partition of B.

2. Let T = $\{\{a\}, \{b\}, \{a, b\}\}$ (say).
   We test whether T is a partition of B = $\{a, b, \{a\}, \{b\}, \{a, b\}\}$:

   a. $\{a\}$, $\{b\}$ and $\{a, b\}$ are **non-empty** subsets of B,
   b. $\{a\} \cap \{b\} \cap \{a, b\} = \emptyset$, but
   c. $\{a\} \cup \{b\} \cup \{a, b\} = \{a, b\} \neq B$.

   Since T does not have all the necessary properties, it is not partition of B.

3. Let M = $\{\mathbf{\{}a, b, \{a\}\mathbf{\}}, \mathbf{\{}\{a\}, \{b\}, \{a, b\}\mathbf{\}}\}$ (say).
   We test whether M is a partition of B = $\{a, b, \{a\}, \{b\}, \{a, b\}\}$:

   a. $\mathbf{\{}a, b, \{a\}\mathbf{\}}$ and $\mathbf{\{}\{a\}, \{b\}, \{a, b\}\mathbf{\}}$ are two **non-empty** subsets of U, but
   b. $\mathbf{\{}a, b, \{a\}\mathbf{\}} \cap \mathbf{\{}\{a\}, \{b\}, \{a, b\}\mathbf{\}} = \mathbf{\{}\{a\}\mathbf{\}} \neq \emptyset$.
   c. We do have that $\mathbf{\{}a, b, \{a\}\mathbf{\}} \cup \mathbf{\{}\{a\}, \{b\}, \{a, b\}\mathbf{\}} = B = \{a, b, \{a\}, \{b\}, \{a, b\}\}$ but
   since M does not have all the necessary properties, it is not partition of B.

4. Let M = $\{a, b, \{a\}, \{b\}, \{a, b\}\}$ (say).
   We test whether M is a partition of B = $\{a, b, \{a\}, \{b\}, \{a, b\}\}$:

   a. The elements a, b, $\{a\}$, $\{b\}$ and $\{a, b\}$ of M are not **subsets** of B, but rather **elements** of B.
   Since M does not have all the necessary properties, it is not partition of B.

From the arguments provided we can deduce that alternative 1 should be selected.

*Refer to study guide, pp 94, 95.*

**Suppose U = {1, 2, 3, a, b, c} is a universal set with the subset A = {a, c, 2, 3}.**
**Let R = { (a, a), (a, c), (3, c), (3, a), (2, 3), (2, a), (2, c), (c, 2) } be a relation on A.**
**Answer questions 11 & 12 by using the given sets A, U and the relation R.**

**Question 11** **Alternative 4**

Which one of the following statements regarding relation **R** is true?

1. R is a weak total order.
2. R is a strict total order.
3. R is a weak partial order.
4. R is **not** an equivalence relation.

The relations mentioned in alternatives 1-3 should have different properties which includes *antisymmetry*:
A weak total order is reflexive, *antisymmetric* and transitive, and satisfies trichotomy;
A strict total order is irreflexive, *antisymmetric* and transitive, and satisfies trichotomy; and
A weak partial order is reflexive *antisymmetric* and transitive.

However, R is **not** antisymmetric:
We provide a counterexample: (2, c) is an element of R but the mirror image (c, 2) is also an element of R.

From the argument provided it follows that R is not a weak or strict total order, nor is it a weak partial order.

Furthermore, R is **not** an equivalence relation since it is not reflexive nor is it symmetric. We provide counterexamples:
Not reflexive: eg (c, c) $\notin$ R; and
not symmetric, eg (3, c) $\in$ R but (c, 3) $\notin$ R.

From the above discussions it is clear that alternative 4 should be selected.

*Refer to study guide, pp 75 – 78, 88.*

**Question 12** **Alternative 3**

Which ordered pairs should be removed from relation R in order for the changed relation $R_1$ (say) to be a strict partial order?

1. only (2, c)
2. only (a, a)
3. (a, a) & (c, 2)
4. (a, a) & (2, c)

A strict partial order is irreflexive, antisymmetric and transitive.

*Discussion*
*Irreflexive:*
*We ask the question: Is it true that for all $x \in A$ that we have $(x, x) \notin R$?*
No, R is not irrreflexive since we have that (a, a) $\in$ R. If we remove (a, a) from R then we can start forming the new irreflexive relation $R_1$:
$R_1$ = {~~(a, a),~~ (a, c), (3, c), (3, a), (2, 3), (2, a), *(2, c)?*, *(c, 2)?*}

*Antisymmetric:*
*We ask the question: Is it true that for all $x, y \in A$, if $x \neq y$ and $(x, y) \in R$ then $(y, x) \notin R$?*
No, since (2, c) and its mirror image (c, 2) are both elements of $R_1$, we have to remove one of the pairs (2, c) or (c, 2) to form an antisymmetric relation. We cannot remove (2, c) because we will then loose the transitivity property:

*Transitive:*
*We ask the question: Is it true that for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$?*
We see that (2, 3), (3, c) $\in$ $R_1$ thus (2, c) should also be an element of $R_1$. We keep (2, c) as an element of $R_1$ and then we can safely remove (c, 2) from $R_1$ = {~~(a, a),~~ (a, c), (3, c), (3, a), (2, 3), (2, a), (2, c), ~~(c, 2)~~ } so that $R_1$ satisfies antisymmetry.

We can now form the new relation: $R_1$ = {(a, c), (3, c), (3, a), (2, 3), (2, a), (2, c)}, but is this relation transitive?

We investigate:
(2, a), (a, c) $\in$ R then also (2, c) $\in$ R,
(3, a), (a, c) $\in$ R then also (3, c) $\in$ R,
(2, 3), (3, c) $\in$ R then also (2, c) $\in$ R, and
(2, 3), (3, a) $\in$ R then also (2, a) $\in$ R
thus R is transitive.

From the arguments provided we can deduce that alternative 3 should be selected.

*Refer to study guide, pp 75 – 78, 88.*

**Let R be the relation on $\mathbb{Z}^{\geq}$ (the set of integers) defined by**

**$(x, y) \in R$ iff $x^2 + y^2 = 2k$ for some integers $k \geq 0$.**

**Answer questions 13 to 15 by using the given relation R.**

**Question 13** **Alternative 4**

Which one of the following is an ordered pair in R?
1. (1, 0)
2. (2, 9)
3. (3, 8)

4. (5, 7)

Relation R on $\mathbb{Z}^{\geq}$ is defined by $(x, y) \in R$ iff $x^2 + y^2 = 2k$ for some integers $k \geq 0$.
We consider the ordered pairs provided in the different alternatives:

1. Let $x = 1$ and $y = 0$ then
   $1^2 + 0^2 = 1$ which is not a multiple of 2.
   thus $(1, 0) \notin R$.

2. Let $x = 2$ and $y = 9$ then
   $2^2 + 9^2 = 85$ which is not a multiple of 2.
   thus $(2, 9) \notin R$.

3. Let $x = 3$ and $y = 8$ then
   $3^2 + 8^2 = 73$ which is not a multiple of 2.
   thus $(3, 8) \notin R$.

4. Let $x = 5$ and $y = 7$ then
   $5^2 + 7^2 = 74$ which is a multiple of 2.
   thus $(5, 7) \in R$.

From the arguments provided we can deduce that alternative 4 should be selected.

*Refer to study guide, pp 71-73.*

**Question 14** **Alternative 2**

R is symmetric. Which one of the following is a valid proof showing that R is symmetric?

1. Let $x, y \in \mathbb{Z}^{\geq}$ be given.
   Suppose $(x, y) \in R$
   then $x^2 + y^2 = 2k$ for some $k \geq 0$.
   ie $y^2 + x^2 = 2k$ for some $k \geq 0$.
   thus $(x, y) \in R$.

2. Let $x, y \in \mathbb{Z}^{\geq}$ be given.
   Suppose $(x, y) \in R$
   then $x^2 + y^2 = 2k$ for some $k \geq 0$.
   ie $y^2 + x^2 = 2k$ for some $k \geq 0$.
   thus $(y, x) \in R$.

3. Let $x, y \in \mathbb{Z}^{\geq}$ be given.
   Suppose $(x, y) \in R$
   then $x^2 + y^2 = 2k$ for some $k \geq 0$.

thus $(y, x) \in R$.

4. Let $x, y \in \mathbb{Z}^{\geq}$ be given.
   Suppose $(x, x) \in R$
   then $x^2 + y^2 = 2k$ for some $k \geq 0$.
   ie $y^2 + x^2 = 2k$ for some $k \geq 0$.
   thus $(y, y) \in R$.

*Discussion*
*Refer to the definition of symmetry provided previously.*

Alternative 1: The conclusion "$(x, y) \in R$" states the same as the supposition: $(x, y) \in R$". This is not a valid proof.

Alternative 2 provides a valid proof:
Let $x, y \in \mathbb{Z}^{\geq}$ be given.
Suppose $(x, y) \in R$
then $x^2 + y^2 = 2k$ for some $k \geq 0$.
ie $y^2 + x^2 = 2k$ for some $k \geq 0$.
thus $(y, x) \in R$.

Alternative 3: In a proof each step must follow logically from the previous step. In this proof the step "$y^2 + x^2 = 2k$" is omitted.

Alternative 4: The first step "$(x, x) \in R$" does not relate to the definition of symmetry.

*Refer to study guide, p 76.*

**Question 15** **Alternative 4**

R is not antisymmetric. Which of the following ordered pairs can be used together in a counterexample to prove that R is **not** antisymmetric? (Remember that R is defined on $\mathbb{Z}^{\geq}$.)

1. (–1, 1) & (1, –1)
2. (5, 9) & (13, 15)
3. (8, 7) & (7, 8)
4. (3, 1) & (1, 3)

*Discussion*
*The definition for antisymmetry requires that the mirror images of ordered pairs $(x, y)$ with $x \neq y$ may not appear in a relation. In a counterexample we will show that $x \neq y$ and $(x, y) \in R$ but then it is also the case that $(y, x) \in R$, ie R is* ***not*** *antisymmetric.*

We discuss the different alternatives:

1. Firstly we must ask the question: Are (–1, 1) and (1, –1) elements of R? No, since R is defined on $\mathbb{Z}^2$. Thus these ordered pairs cannot be used in a counterexample to prove that R is not antisymmetric.

2. The ordered pairs (5, 9) & (13, 15) are both elements of R (we have that the sums $5^2 + 9^2$ and $13^2 + 15^2$ are multiples of 2), but these ordered pairs cannot be used in a counterexample to prove that R is not antisymmetric since these pairs are not mirror images of each other.

3. The ordered pairs (8, 7) & (7, 8) are not elements of R (we have $8^2 + 7^2 = 7^2 + 8^2 = 113$ which is not a multiple of 2), thus these ordered pairs cannot be used in a counterexample to prove that R is not antisymmetric.

4. In ordered pairs (3, 1) & (1, 3) are elements of R (we have $3^2 + 1^2 = 1^2 + 3^2 = 4$ which is a multiple of 2). It is the case that $1 \neq 3$ and $(3, 1) \in R$ but the mirror image (1, 3) is also an element of R, thus R is not antisymmetric.

From the arguments provided alternative 4 should be selected as the correct one.

*Refer to study guide, pp 76, 77.*

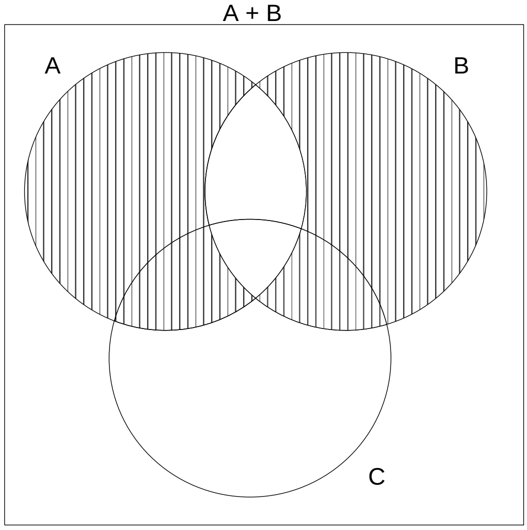
**Question 16** **Alternative 2**

The statement (A + B) – (C′ ∩ B) = (A + C) is an identity. (*Hint*: draw a Venn diagram of the LHS and RHS).

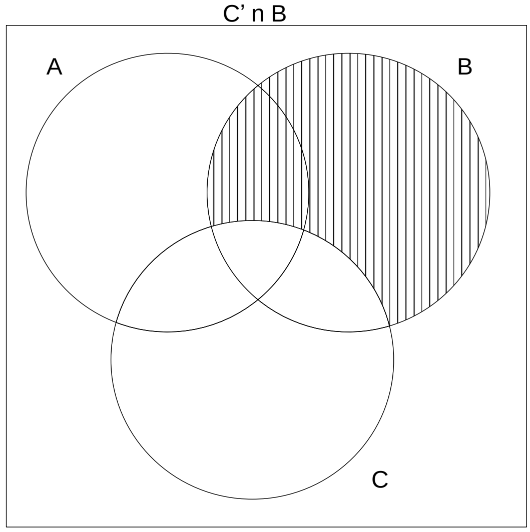
*Discussion:*

We draw Venn diagrams for the LHS and RHS separately:
**LHS**: (A + B) – (C′ ∩ B). We do this in 3 steps:

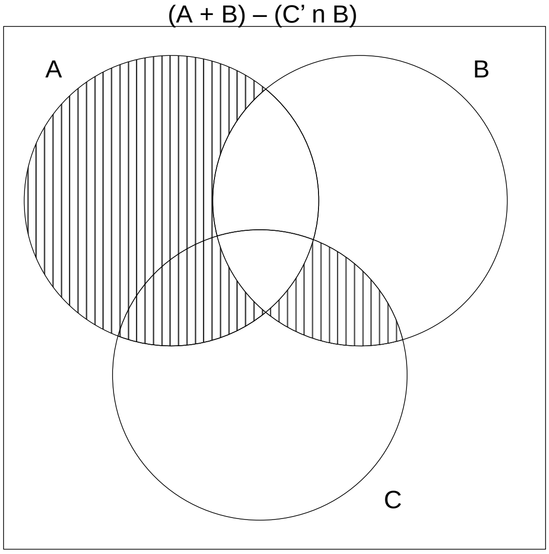
A + B




C’ n B

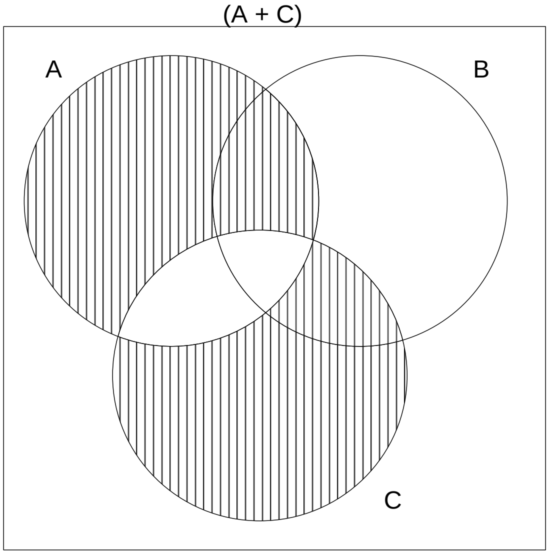


(A + B) – (C’ n B)




**RHS**: A + C

(A + C)




Clearly, LHS ≠ RHS. The given statement is therefore FALSE because it not an identity. Alternative 2 should be selected.

**Questions 17 to 20 are based on the following:**

**Let A = {1, {1}, 3, 4, {5, 6}} be a set, and let B and C be the following relations on A:**

**B = {(1, 3), (4, 1), (3, {5, 6}), ({1}, 4), ({5, 6}, {1})}**
**C = {(1, 1), (3, 4), (4, 3), ({5, 6}, {1})**

**Question 17** **Alternative 1**

The relation B is antisymmetric.

*Discussion*:
We look at the definition of antisymmetry:

*A relation R $\subseteq$ A $\times$ A is* antisymmetric *iff R has the property that, for all x, y $\in$ A, if x $\neq$ y and (x, y) $\in$ R then (y, x) $\notin$ R.*

*We also look at an alternative definition:*
A relation R $\subseteq$ A $\times$ A is *antisymmetric* iff R has the property that, for all x, y $\in$ A, if (x, y) $\in$ R and (y, x) $\in$ R, then x = y. See study guide p. 76.

According to the definition above there are no ordered pairs (x, y) and (y, x) in B. Therefore, B is antisymmetric, and alternative 1 should be selected.

**Question 18** **Alternative 2**

The relation C is irreflexive. Is this true?

*Discussion*:

**B = {(1, 3), (4, 1), (3, {5, 6}), ({1}, 4), ({5, 6}, {1})}**
**C = {(1, 1), (3, 4), (4, 3), ({5, 6}, {1})**

We look at the definition of irreflexitivity:

A relation R $\subseteq$ A × A is called *irreflexive* iff there is *no* x $\in$ A such that (x, x) $\in$ R. In other words, for any x $\in$ A, (x, x) $\notin$ R.

According to the definition above, relation C cannot be irreflexive, because it contains the ordered pair (1, 1). Alternative 2 should therefore be selected.

**Question 19** **Alternative 1**
The composition relation **B;C** is {(1, 4), (4, 1), (3, {1}), ({1}, 3)}. Is this true?

*Discussion*:

- **B = {(1, 3), (4, 1), (3, {5, 6}), ({1}, 4), ({5, 6}, {1})}**
- **C = {(1, 1), (3, 4), (4, 3), ({5, 6}, {1})**

We start with an ordered pair in B, and then see if there are ordered pairs in C with a first coordinate that is the same as the second coordinate of the ordered pair in B. We do this for all ordered pairs in B.
Ordered pair (1, 3) is in B and ordered pair (3, 4) is in C. The resulting ordered pair that is part of B;C is (1, 4);

Ordered pair (4, 1) is in B and ordered pair (1, 1) is in C. The resulting ordered pair that is part of B;C is (4, 1);
Ordered pair (3, {5, 6}) is in B and ordered pair ({5, 6}, {1}) is in C. The resulting ordered pair that is part of B;C is (3, {1});
Ordered pair ({1}, 4) is in B and ordered pair (4, 3) is in C. The resulting ordered pair that is part of B;C is ({1}, 3).

Therefore the resulting B;C is the relation {(1, 4), (4, 1), (3, {1}), ({1}, 3)} which is identical to the given relation. Alternative 1 should therefore be selected.

**Question 20** **Alternative 2**

The composition relation **C;B** is {(1, 3), (3, 1), (3, {5, 6}), ({5, 6}, 4)}. Is this true?
*Discussion*:

- **B = {(1, 3), (4, 1), (3, {5, 6}), ({1}, 4), ({5, 6}, {1})}**
- **C = {(1, 1), (3, 4), (4, 3), ({5, 6}, {1})**

In a similar way as in Question 19, we start with ordered pairs in C. We will leave this as an exercise for you. **C;B** = {(1, 3), (3, 1), (4, {5, 6}), ({5, 6}, 4)}. The given statement is therefore false and alternative 2 should be selected.