COS1501

Unit 1 & 2 : Numbers

Integers

$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2 \ldots\}$

* Positive : 2k
* Negative : 2k+1

Positive Integers

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

Non-negative Integers

$\mathbb{Z}^{\geq} = \{0, 1, 2, \ldots\}$ ($\mathbb{N}$)

Rational numbers

$\mathbb{Q} = a; \quad a = \frac{p}{q}; \quad p \,\&\, q \in \mathbb{Z}; \quad q \neq 0$

Real numbers

$\mathbb{R} = a; \quad a \,\exists$

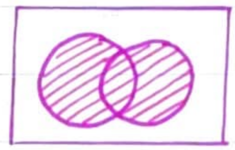
Unit 3: Sets

Subset

↳ Improper: $A \subseteq B$ → every element of A, also elemet of B
* All B is A

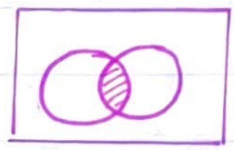
↳ Proper: $A \subset B$ → every element of A is an element of B, but not all elements of B is an element of A

Union



↳ $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

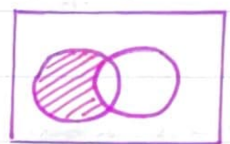
Intersection



↳ $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

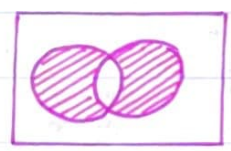
↳ Disjointness: $A \cap B = \emptyset$

Set Difference



↳ $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Symmetric set Difference



↳ $A + B = \{x \mid x \in A \text{ or } x \in B, \text{ but not both}\}$

Power Set

↳ The set that has as its members all subsets of A

↳ $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ * $A = \{1,2,3\}$

Cardinality

↳ Number of elements in set

↳ $n(A)$ or $|A|$

↳ $|P(A)| = 2^{n}$ → elements in A

Unit 4 : Proofs involving sets

Venn-diagram proof

If and only if

- $x \in (A \cup B)$
  iff $x \in A$ or $x \in A$

- $x \in (A \cup B)'$
  iff $x \notin (A \cup B)$
  iff $x \notin A$ and $x \notin B$
  iff $x \in A'$ and $x \in B'$
  iff $x \in (A' \cap B')$

- $x \in (A - B)$
  iff $x \in A$ and $x \notin B$
  iff $x \in A$ and $x \in B'$
  iff $x \in (A \cap B')$

- $|A \cup B| = |A| + |B| - |A \cap B|$

- $(u, v) \in X \times (Y \cup W)'$
  iff $u \in X$ and $v \in (Y \cup W)'$

- $x \in (A \cap B) \cup (C - B)$
  iff $(x \in A$ and $x \in B)$ or $(x \in C$ and $x \notin B)$
  iff $(x \in A$ or $x \in C)$ and $(x \in A$ or $x \notin B)$ and $(x \in B$ or $x \in C)$ and $(x \in B$ or $x \notin B)$

Unit 5: Relations

Relation $A = \{2, 3, 5\}$

$R = \{(2,3), (2,5), (3,5)\}$

$(2,3) \in R \Rightarrow 2R3$

Cartesian product : $A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$

Relation A to B : $R = \{(x_a, y_b), \dots\}$ * B is codomain of R

Domain : $dom(A) = \{x \mid \forall x \in Y,\ (x,y) \in A\}$

Range : $ran(A) = \{y \mid \forall x \in X,\ (x,y) \in A\}$

Properties of relations

Reflexive : $(x,x) \in R$

Irriflexive : $(x,x) \notin R$

Symmetric : if $(x,y) \in R$, then $(y,x) \in R$

can be neither

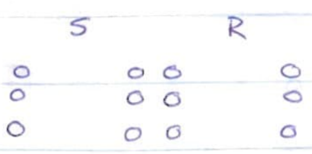
Antisymmetric : if $(x,y) \in R$, then $(y,x) \notin R$

Transitive : if $(x,y) \in R$ & $(y,z) \in R$, then $(x,z) \in R$

Trichotomy : $(x,y) \in R$ OR $(y,x) \in R$

Inverse (f^{-1}): $(x,y) \in R$ if $(y,x) \in R'$ $\quad$ *$\{(y,x) \mid (x,y) \in R\}$

Composition : $R \circ S$ $\Rightarrow$



Unit 6: Special kinds of relation

Order

	Weak	Strict
Partial	→ Refexive → Antisymmetric → Transitive	→ Irreflexive → Antisymmetric → Transitive
Total (linear)	→ Reflexive → Antisymmetric → Transitive → Trichotomy	→ Irreflexive → Antisymmetric → Transitive → Trichotomy

Equivalance

Relation
- ↳ Reflexive
- ↳ Symmetric
- ↳ Transitive

Class
- ↳ $[x] = \{y \mid \text{——}\}$

Theorem

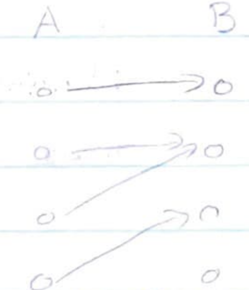
i) If R is equivalance relation on A, then $x \in [x]$ $\forall x \in A$

ii) If xRy, then $[x]=[y]$ and visa versa

iii) Either • $[x]=[y]$ or • $[x] \cap [y] = \emptyset$

Partitions: A set comprising of subsets of A
- → No empty members
- → Different parts have no members in common (No element repeated)
- → Every element in A used (contains all elements of A)

Functional: Binary relation R from A to B
- → 1st co-ordinate appears only once
- → dom(R) = A



~~→ B can have many A~~

→ A can't have many B

$\therefore R: A \to B$

Unit 7: More about functions

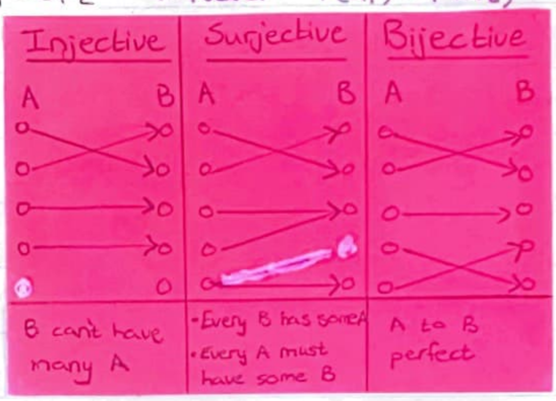
Surjective function

↳ the range of the function is equal to the codomain of the function $*f[A]=B$

↳ from (B) to X

Injective function

↳ whenever $f(a_1)=f(a_2)$ then $a_1=a_2$ ∴ Never $f(a_1)=f(a_2)$



Bijective function

↳ Both surjective and injective

Composition function

↳ $S \circ R$

↳ Theorems: - Composition of 2 functions is also a function

- Composition of 2 surjectives is also surjective
- composition of 2 injectives is also injective

↳ $f: A \to B$ & $g: B \to C \Rightarrow g \circ f: A \to C \Rightarrow g \circ f(x) = g(f(x))$

Invertible function

↳ the inverse relation of $f: A \to B$ is $f^{-1} = A \to B$

↳ Theorem: f is invertible iff f is bijective

Identity function

For any set A, define the function $i_A : A \to A$ by requiring that $i_A(x) = x$ for all $x \in A$. This function is called the identity function on A, since what it spits out is identical to what it eats

$y = x$

Unit 8: Operations

Binary operations

$\hookrightarrow f : P(u) \times P(u) \rightarrow P(u)$

Notation : Prexix $\Rightarrow f(x, y)$

Infix $\Rightarrow x f y$

Identity element: $e \diamond x = x \diamond e = x$

$\circledast \; \diamond : X \times X \rightarrow X$

$u = \{a\} \therefore P(u) = \{\emptyset, \{a\}\}$

$P(u) \times P(u) = \{(\emptyset, \emptyset), (\emptyset, \{a\}), (\{a\}, \emptyset), (\{a\}, \{a\})\}$

$\underline{g : P(u) \times P(u) \rightarrow P(u)}$

$(\emptyset, \emptyset) \rightarrow \emptyset \quad \emptyset \cup \emptyset$

$(\emptyset, \{a\}) \rightarrow \{a\} \quad \emptyset \cup \{a\}$

$(\{a\}, \emptyset) \rightarrow \{a\} \quad \{a\} \cup \emptyset$

$(\{a\}, \{a\}) \rightarrow \{a\} \quad \{a\} \cup \{a\}$

$\therefore g : \{((\emptyset, \emptyset), \emptyset), ((\emptyset, \{a\}), \{a\}), ((\{a\}, \emptyset), \{a\}), ((\{a\}, \{a\}), \{a\})\}$

Vectors

Sum : $u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$

Scalar : $r \times u = (r u_1, r u_2, \dots, r u_n)$

Dot product: $u \cdot v = (u_1 v_1, u_2 v_2, \dots, u_n v_n)$

Matrices

Addition : $A + B = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1m} + b_{1m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{nm} + b_{nm} \end{bmatrix}$

Multiplication: $r A = \begin{bmatrix} r a_{11} & \cdots & r a_{1m} \\ \vdots & & \vdots \\ r a_{n1} & \cdots & r a_{nm} \end{bmatrix}$

$A B = \begin{bmatrix} a_{11}, a_{12}, a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31} \end{bmatrix}$

$\overbrace{n \times \underbrace{m \; X \; n}_{\text{MUST}} \times m}^{\text{answ}}$

Identity : $I A = A I = A$

Unit 9: Logic

logical connectives

- $\wedge$: Conjunction (and)
- $\vee$: Disjunction (or)
- $\rightarrow$: Conditional/Implication (if... then...)
- $\leftrightarrow$: Biconditional (iff)
- $\neg$: Negation (not)

Truth Table

- Both T
- One T / Both T
- Both T / Both F / aF & bT
- Both T / Both F

Truth tables

Compound statements: Truth table columns

Tautology: Compound statements always true

Contradiction: Compound statements always false

Logical equivalence: $a \equiv b$ iff $a \leftrightarrow b$ is a Tautology

Identities

Cummutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Associative
- $p \vee (q \vee r) \equiv (p \vee q) \vee r$
- $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

De Morgan
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Implication
- $p \rightarrow q \equiv \neg p \vee q$

Negation
- $p \vee \neg p \equiv T_0$
- $p \wedge \neg p \equiv F_0$

Identity
- $p \vee F_0 \equiv p$
- $p \wedge T_0 \equiv p$

Universal Bound
- $p \vee T_0 \equiv T_0$
- $p \wedge F_0 \equiv F_0$

Conditionals $(p \rightarrow q)$
- ↳ Inverse $\Rightarrow \neg p \rightarrow \neg q$
- ↳ Converse $\Rightarrow q \rightarrow p$
- ↳ Contrapositive $\Rightarrow \neg q \rightarrow \neg p$

Unit 10: Predicates, Quantifiers and Proof Strategies

Predicate

↳ $\forall x \in \mathbb{Z},\ x > 2 \Rightarrow \forall x \in \mathbb{Z},\ P(x)$

Quantifiers

↳ Universal : for all ($\forall$)

↳ Existential: There exists ($\exists$)

Fundamental rule: $\forall x \in \mathbb{R},\ (x > 2) \Rightarrow (x^2 > 4) \equiv \forall y \in \mathbb{R},\ (y > 2) \Rightarrow (y^2 > 4)$

Negation

- $\neg(\forall x \in A,\ P(x)) \Rightarrow \exists x \in A,\ \neg P(x)$
- $\neg(\exists x \in A,\ P(x)) \Rightarrow \forall x \in A,\ \neg P(x)$

Proof strategies

Direct
- Assume "if"
- Reason until "then"

Contradiction
- Assume "if"
- Assume ¬ "then"
- Reason until "if" contradicted

Contrapositive
- Assume ¬"then"
- Reason until ¬ "if"

Quantifiers
- If [qualifier], then ___
- Reason

Vacuous

Counter example

Implication

⊛ If we cannot prove statement as true, provide 1 counterexample