2009 EXAM PAPER

SECTION 1 : TRUTH TABLES AND SYMBOLIC LOGIC
Write your answer to each question out in full in the answer book. [16 marks]

Question 1.1 **(7)**

a) Give the truth table for the following symbolic compound statement:

$$((p \vee q) \wedge \neg (p \rightarrow r) \wedge (q \rightarrow r)) \leftrightarrow r$$

b) Is the expression a tautology, a contradiction or neither?

Question 1.2 **(5)**

Write down the negation of the following expression. Simplify the expression so that the *not*-simbol (¬) does not occur to the left of any quantifier. The *not*-symbol may also not occur outside of parentheses.

$$\forall x \in \mathbb{Z}, (x > 0) \vee (x \leq 12)$$

N.B.: Show all steps. Which is true, the original statement, the negation of it or neither?

Question 1.3 **(4)**

Given the implication $P(x) \rightarrow Q(x)$, write down

a) the converse, and
b) the contrapositive

of the implication. After you have written down the converse and the contrapositive, simplify the expressions where possible, e.g. where negations are involved.

(**Hint:** Use the identity $A(x) \rightarrow B(x) = \neg A(x) \vee B(x)$ in the simplification process.)

SECTION 2 : MATHEMATICAL PROOFS
Write your answer to each question out in full in the answer book. [12 marks]

Question 2.1 **(3)**

Provide a counterexample to prove that the following statement is not true for all integers $x \geq 0$.

If $x \geq 0$ then $x^2 - 2x + 3$ is a multiple of 3.

Question 2.2 **(4)**

Provide a direct proof to show that, for all $n \in \mathbb{Z}$,

if n is even, then $n^3 + 2n$ is divisible by 4.

Question 2.3 **(5)**

Provide a *proof by contrapositive* to prove that, for all $x \in \mathbb{Z}$, if $x^2 + x + 1$ is even, then x is even.

SECTION 3 : SET THEORY

Write your answer to each question out in full in the answer book. [28 marks]

Question 3.1 **(10)**

Use Venn diagrams to determine whether, for all sets X, Y and W

$(X \cap Y) \cup (W - X) = (X \cup W) \cap (Y + X)$.

If it seems to be an identity, give a proof. Otherwise give a counterexample.

Question 3.2 **(8)**

Determine whether or not the following holds for all sets X, Y and W:

$(W \times X') \cap (W \times Y') = W \times (X \cup Y)'$

(**Hint:** Start your proof with $(a,b) \in W \times (X \cup Y)'$)

Question **(10)**

A car manufacturer uses 32 robots. Of these robots

24 are used for welding,
10 are used for painting, and
10 are used for sanding.

Furthermore, some of these robots can be programmed as follows:

5 to weld and to paint,
6 to weld and to sand, and
4 to sand and to paint.

Answer the following two questions:

a) How many robots can perform all three functions?

b) How many can only sand?

SECTION 4 : SETS AND RELATIONS (Multiple-Choice Questions)
Each question comprises 2 marks.
Choose only one alternative per question and then write the question number and the alternative that you regard as the correct answer in the answer book. [14 marks]

Suppose U = {1,2,3,4,5,6}, A = {1,3,5} and B ={1,2,4}. Answer questions 4.1 to 4.4 using these three sets.

Question 4.1
Which one of the following sets represents A + B?

1. {1,2,3,4,5}
2. {3,5}
3. {2,5,9}
4. {2,3,4,5}

Question 4.2
Which one of the following sets represents A - B?

1. {0,1,1}
2. {3,5}
3. {1}
4. {1,2,3,4,5}

Question 4.3
Which one of the following sets represents U - A' ?

1. {2,4}
2. {1,2,3,4,5}
3 {1,3,5}.
4. ∅

Question 4.4
Which one of the following sets represents A' ∩ B' ?

1. {6}
2. {1,2,3,4,5}
3. {1}
4. {2,3,4,5}

Question 4.5
Let A = {∅, {1},{3}}. Which one of the following sets represents the power set of A, i.e. $\mathcal{P}$(A)?

1. { ∅, {∅}, {{1}}, {{3}} }
2. { ∅, {1}, {3} }
3. { ∅, {∅}, {{1}}, {{3}}, {∅,{1}}, {∅,{3}}, {{1},{3}}, {∅,{1},{3}} }
4. { {∅}, {{1}}, {{3}}, {{1},{3}} }

Question 4.6
Consider the following relation on T = {1, 2, 3}: R = {(1, 1), (2, 1), (3, 2), (3, 1)}.
Which one of the following statements regarding the relation R is TRUE?

1. R is reflexive on T and antisymmetric.
2. R is symmetric and transitive.
3. R is irreflexive and antisymmetric
4. R is antisymmetric and transitive.

Question 4.7
Consider the following relation on T = {1, 2, 3}: S = {(1, 2), (2, 3), (1, 3)}.
Which one of the following statements regarding the relation S is TRUE?

1. S is a partial order on T.
2. S is a strict total order on T.
3. S is an equivalence relation on T.
4. S is a strict equivalence relation on T.

SECTION 5 :FUNCTIONS AND RELATIONS	
Write your answer to each question out in full in the answer book.	**[30 marks]**

Question 5.1 **(7)**
Let A = {1, 2, 3} and B = {1, 2, {a}, {b}}.
a) Provide an example of a relation on B that satisfies trichotomy.
b) Provide an example of a partition of A.
c) Consider the following relation S on the set B: S = { (1, 2), ({b}, {a}), (2, {b}) }
   (i) Determine the inverse relation S^{-1}.
   (ii) Determine the composition $S;S^{-1}$ (i.e. $S^{-1} \circ S$).

Question 5.2:

This question in the paper is included as question E in self-assessment 2, therefore we provide another question: Let X be the set of letters of the English alphabet and let R be a relation on X such that (x, y) ∈ R iff x has as its value a letter coming before the value of y in the usual alphabet rhyme. By doing the appropriate tests, prove that R is a total strict order relation. (Note: Do not use examples in your proof.)

Question 5.3 **(7)**

Suppose $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = x - 2$, and
$g: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $g(x) = x^2 - 1$.

a) Is *f* injective (one-to-one)? Justify your answer fully.
b) Determine ran(*f*) (i.e. the range of *f*).
c) Determine the rule $f \circ g(x)$ for the composition $f \circ g$.

Question 5 **(6)**

Consider the binary operation $\cup$ (union) in the following table:

$\cup$	$\emptyset$	$\{\emptyset\}$
$\emptyset$	$\emptyset$	$\{\emptyset\}$
$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$

a) Does the operation have an identity element? Justify your answer fully.
b) Write the operation in list notation.

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