Tutorial letter 203/0/2021

Theoretical Computer Science 1

COS1501

Year module

School of Computing

Discussion of assignment 03

Dear Student,

By this time you should have received the tutorial matter listed below. These can be downloaded from *my*Unisa.

- COSALLP/301/4/2021 General information regarding the School of Computing including lecturers' information;
- COS1501/101/0/2021 General information about the module and the assignments;
- COS1501/201/0/2021 Solutions to the first assignment, and **examination information**;
- COS1501/202/0/2021 Solutions to the second assignment;
- COS1501/203/0/2021 This tutorial letter;
- COS1501/102/0/2021 Solutions to self-assessment questions in assignments 02 and 03, example assignments & solutions, example examination paper & solutions, and extra questions & solutions.

Everything of the best with the exam!
Regards
COS1501 team

Discussion of assignment 03

Suppose U = {1, 2, 3, 4, 5, a, b, c} is a universal set with the subset A = {a, b, c, 1, 2, 3, 4}. Answer questions 1 and 2 by using the given sets U and A.

Question 1
Which one of the following relations on A is NOT functional?

1. {(1, 3), (b, 3), (1, 4), (b, 2), (c, 2)}
2. {(a, c), (b, c), (c, b), (1, 3), (2, 3), (3, a)}
3. {(a, a), (c, c), (2, 2), (3, 3), (4, 4)}
4. {(a, c), (b, c), (1, 3), (3, 3)}

Answer: Alternative 1

Discussion
First we look at the definition of functionality:
Suppose R ⊆ B × C is a binary relation from a set B to a set C. We may call R functional if the elements of B that appear as first co-ordinates of ordered pairs in R do not appear in more than one ordered pair of R.

We consider the relations provided in the different alternatives:

1. Let L = {(**1**, 3), (**b**, 3), (**1**, 4), (**b**, 2), (c, 2)} (say). L is not functional since the elements 1 and b appear more than once as first co-ordinates in ordered pairs of L.

2. Let M = {(a, c), (b, c), (c, b), (1, 3), (2, 3), (3, a)} (say).
   M is a relation on A = {a, b, c, 1, 2, 3, 4}: {a, b, c, 1, 2, 3} = dom(M) $\subseteq$ A and ran(M) = {a, b, c, 3} $\subseteq$ A. Each first co-ordinate appears only once as first co-ordinate thus M is functional.

3. Let N = {(a, a), (c, c), (2, 2), (3, 3), (4, 4)} (say).
   N is a relation on A = {a, b, c, 1, 2, 3, 4}: {a, c, 2, 3, 4} = dom(N) $\subseteq$ A and ran(N) = {a, c, 2, 3, 4} $\subseteq$ A. Each first co-ordinate appears only once as first co-ordinate thus N is functional.

4. Let S = {(a, c), (b, c), (1, 3), (3, 3)} (say).
   S is a relation on A = {a, b, c, 1, 2, 3, 4}: {a, b, 1, 3} = dom(S) $\subseteq$ A and ran(S) = {c, 3} $\subseteq$ A. Each first co-ordinate appears only once as first co-ordinate thus S is functional.

From the arguments provided we can deduce that alternative 1 should be selected.

Refer to study guide, p 98.

Question 2
Which one of the following alternatives represents a **surjective function** from **U** to **A**?

1. {(1, 4), (2, b), (3, 3), (4, 3), (5, a), (a, c), (b, 1), (c, b)}
2. {(a, 1), (b, 2), (c, a), (1, 4), (2, b), (3, 3), (4, c)}
3. {(1, a), (2, c), (3, b), (4, 1), (a, c), (b, 2), (c, 3)}
4. {(1, a), (2, b), (3, 4), (4, 3), (5, c), (a, a), (b, 1), (c, 2)}

Answer: Alternative 4

Discussion
We look at the definitions of a function and of surjectivity:
Suppose R $\subseteq$ B × C is a binary relation from a set B to a set C. We may call R a **function** from B to C if every element of B appears exactly once as the first co-ordinate of an ordered pair in R (i.e. f is functional), and the domain of R is exactly the set B, ie dom(R) = B.
A function R from B to C is **surjective** iff ran(R) = C.

We have U = {1, 2, 3, 4, 5, a, b, c} and A = {a, b, c, 1, 2, 3, 4}.

We consider the relations provided in the different alternatives:

1. Let L = {(1, 4), (2, b), (3, 3), (4, 3), (5, a), (a, c), (b, 1), (c, b)} (say). It is the case that 2 $\in$ A but 2 is not a second co-ordinate in any ordered pair of L thus ran(L) ≠ A. Thus L is not surjective.

2. Let M = {(a, 1), (b, 2), (c, a), (1, 4), (2, b), (3, 3), (4, c)} (say). It is the case that 5 $\in$ U but 5 is not a first co-ordinate in any element of M. We have dom(M) = {1, 2, 3, 4, a, b, c} ≠ U. Thus M is not a function from U to A and thus cannot be a surjective function.

3. Let N = {(1, a), (2, c), (3, b), (4, 1), (a, c), (b, 2), (c, 3)} (say). For the same reason as provided in alternative 2, N is not a function. Thus N is not a surjective function.

4. Let S = {(1, a), (2, b), (3, 4), (4, 3), (5, c), (a, a), (b, 1), (c, 2)} (say). S is a function since it is functional and dom(S) = U, and furthermore, ran(S) = A thus S is a surjective function.

From the arguments provided we can deduce that alternative 4 should be selected.

Refer to study guide, p 105.

Question 3

Let G and L be relations on A = {1, 2, 3, 4} with

G = {(1, 2), (2, 3), (4, 3)} and L = {(2, 2), (1, 3), (3, 4)}.

Which one of the following alternatives represents the relation L ○ G = G; L?

1. {(2, 3), (3, 3)}
2. {(1, 2), (2, 4), (4, 4)}
3. {(1, 2), (2, 1), (3, 3), (4, 4)}
4. {(2, 4), (4, 4)}

Answer: Alternative 2

Discussion
We first look at a definition of a composition relation:
Given relation P from A to B and R from B to C, the composition of P followed by R
(R ○ P or P; R) is the relation from A to C defined by

R ○ P = P; R = {(a, c) | there is some b $\in$ B such that (a, b) $\in$ P and (b, c) $\in$ R}.

G = {(1, 2), (2, 3), (4, 3)} and L = {(2, 2), (1, 3), (3, 4)} are defined on A = {1, 2, 3, 4}.

To determine G; L we start with the pair (1, **2**) of G, and then we look for a pair in L that has as first co-ordinate an **2**, and then see where it takes us.

Link (1, **2**) of G with (**2**, 2) of L, then (1, 2) $\in$ G; L,
link (2, **3**) of G with (**3**, 4) of L, then (2, 4) $\in$ G; L, and
link (4, **3**) of G with (**3**, 4) of L, then (4, 4) $\in$ G; L.

No other pairs can be linked, so G; L = {(1, 2), (2, 4), (4, 4)}. Thus alternative 2 should be selected.

Refer to study guide, pp 79, 108, 109.

Let g be a function from $\mathbb{Z}^+$ (the set of positive integers) to $\mathbb{Q}$ (the set of rational numbers) defined by

$$(x, y) \in g \quad \text{iff} \quad y = 4x - \tfrac{3}{7} \quad (g \subseteq \mathbb{Z}^+ \times \mathbb{Q}) \text{ and}$$

let f be a function on $\mathbb{Z}^+$ defined by

$$(x, y) \in f \quad \text{iff} \quad y = 5x^2 + 2x - 3 \quad (f \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+).$$

Answer questions 4 to 7 by using the given functions g and f.
Hint: Draw graphs of the f and g before answering the questions. Keep in mind that $g \subseteq \mathbb{Z}^+ \times \mathbb{Q}$ and $f \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$.

Question 4
Consider the function f on $\mathbb{Z}^+$. For which values of x is it the case that $5x^2 + 2x - 3 > 0$?
Hint: Solve $5x^2 + 2x - 3 > 0$ and keep in mind that $x \in \mathbb{Z}^+$.

1. $x < 5, x \in \mathbb{Z}^+$
2. $\frac{3}{5} < x < 1, x \in \mathbb{Z}^+$
3. $x \geq 1, x \in \mathbb{Z}^+$
4. $x < 1, x \in \mathbb{Z}^+$

Answer: Alternative 3

We solve for x:
$5x^2 + 2x - 3 > 0$
ie $(5x - 3)(x + 1) > 0$
ie $[(5x - 3) > 0 \text{ and } (x + 1) > 0]$ OR $[(5x - 3) < 0 \text{ and } (x + 1) < 0]$
ie $[x > \frac{3}{5} \text{ and } x > -1]$ OR $[x < \frac{3}{5} \text{ and } x < -1]$
ie $x > \frac{3}{5}$ OR $x < -1$ (which is impossible)

It is not possible for x to be less than –1 since f is defined on $\mathbb{Z}^+$. It is also stated that $x > \frac{3}{5}$, but x cannot be a rational number less than 1 since f is defined on $\mathbb{Z}^+$ so we have that $x \geq 1, x \in \mathbb{Z}^+$ (If $x > \frac{3}{5}$, then it is also true that $x \geq 1$). This is the correct statement provided in alternative 3.

Question 5

Which one of the following is an ordered pair belonging to f?

1. (–1, 0)
2. (2, 21)
3. (1, 5)
4. (3, 44)

Answer: Alternative 2

Discussion

The first and second co-ordinates of elements (x, y) of f are elements of $\mathbb{Z}^+$.

We consider the ordered pairs provided in the different alternatives:

1. $(x, y) \in f$ iff $y = 5x^2 + 2x - 3$. Is $(-1, 0) \in f$?

   No, f is defined on $\mathbb{Z}^+$ but neither –1 nor 0 are elements of $\mathbb{Z}^+$.

   Thus $(-1, 0) \notin f$.

2. Is $(2, 21) \in f$?

   Let x = 2 then

   $y = 5(2)^2 + 2(2) - 3$
   $\;\;= 20 + 4 - 3$
   $\;\;= 21$

   Thus $(2, 21) \in f$.

3. Is $(1, 5) \in f$?

   Let x = 1 then

   $y = 5(1)^2 + 2(1) - 3$
   $\;\;= 4$

   $(1, 4) \in f$ but $(1, 5) \notin f$.

4. Is $(3, 44) \in f$?

   Let x = 3 then

   $y = 5(3)^2 + 2(3) - 3$
   $= 48$

   $(3, 48) \in f$ but $(3, 44) \notin f$.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, pp 71, 72.

Question 6

Which one of the following alternatives represents the **image of x under g ○ f** (ie g ○ f(x)))?

1. $20x^2 + 8x - 12\frac{3}{7}$
2. $80x^2 + 4\frac{4}{7}x - \frac{180}{49}$
3. $20x^2 + 8x + 3\frac{3}{7}$
4. $80x^2 + 4\frac{4}{7}x - 3$

Answer: Alternative 1

Discussion

Given the functions f: A → B and g: B → C the composite function

g ○ f: A → C is defined by g ○ f(x) = g(f(x)).

g ○ f(x)

= g(f(x))

$= 4(f(x)) - \frac{3}{7}$

$= 4(5x^2 + 2x - 3) - \frac{3}{7}$

$= 20x^2 + 8x - 12 - \frac{3}{7}$

$= 20x^2 + 8x - 12\frac{3}{7}$

Note that we define the **function g ○ f** by g ○ f: $\mathbb{Z}^+ \to \mathbb{Q}$ defined by g ○ f(x) = $20x^2 + 8x - 12\frac{3}{7}$.

From the above we can deduce that alternative 1 should be selected.

Refer to study guide, p 110.

Question 7

Which one of the following statements regarding the function g is TRUE?

(Remember, g $\subseteq \mathbb{Z}^+ \times \mathbb{Q}$.)

1. g can be presented as a **straight line** graph.
2. g is injective.
3. g is surjective.
4. g is bijective.

Answer: Alternative 2

We consider the statements provided in the different alternatives:

1. g is not defined on the set of real numbers thus g **cannot** be depicted as a straight **line** graph. Only positive integers can be present in the domain of g, ie dom(g) = $\mathbb{Z}^+$. It is the case that

ordered pairs such as $(1, 3\frac{4}{7})$, $(2, 7\frac{4}{7})$, $(3, 11\frac{4}{7})$,… belong to g and these pairs can be presented as **dots** in a graph.

2. We prove that g is indeed injective:

   Assume g(u) = g(v) then
   $4u - \frac{3}{7} = 4v - \frac{3}{7}$
   ie $4u = 4v$
   ie $u = v$

3. The function g is NOT surjective since ran(g) ≠ $\mathbb{Q}$. A counterexample provides a value $y \in \mathbb{Q}$ for which there is **no** element x such that $x = (y + \frac{3}{7})/4$ and $x \in \mathbb{Z}^+$.
   (If $y = 4x - \frac{3}{7}$ then $4x = y + \frac{3}{7}$, ie $x = (y + \frac{3}{7})/4$.)

   Let $y = \frac{4}{7}$ then we determine whether a positive integer x can live together with this y in an ordered pair: $(x, \frac{4}{7})$.

   $x = (y + \frac{3}{7})/4$
   $= (\frac{4}{7} + \frac{3}{7})/4$
   $= (\frac{7}{7})/4$
   = ¼ which is not a positive integer.

   We see that for $y = \frac{4}{7}$ there is **no** positive integer x such that $x = (y + \frac{3}{7})/4$.

   Thus $y = \frac{4}{7} \notin$ ran(g), therefore ran(g) ≠ $\mathbb{Q}$.
   Since ran(g) ≠ $\mathbb{Q}$ we may conclude that g is not surjective.

4. In alternative 3 we proved that g is not surjective thus g is not bijective since it is not injective **and** surjective.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, pp 98, 105, 106, 112.

Let A = {a, b, c, d} and let * be a binary operation from A × A to A presented by the following table:

*	a	b	c	d
a	**a**	**d**	**c**	**b**
b	**b**	**a**	**d**	**a**
c	**c**	**b**	**a**	**c**
d	**d**	**b**	**c**	**a**

Answer questions 8 and 9 by referring to the table for the binary operation *.

Question 8

Which one of the following alternatives gives the format for the list notation of

1. (((a, a), a), ((a, b), d), ((a, c), c), ((a, d), b), ((b, a), b), ((b, b), a), ((b, c), d), ((b, d), a), ((c, a), c), ((c, b), b), ((c, c), a), ((c, d), c), ((d, a), d), ((d, b), b), ((d, c), c), ((d, d), a))
2. {((a, a), a), ((a, b), d), ((a, c), c), ((a, d), b), ((b, a), b), ((b, b), a), ((b, c), d), ((b, d), a), ((c, a), c), ((c, b), b), ((c, c), a), ((c, d), c), ((d, a), d), ((d, b), b), ((d, c), c), ((d, d), a)}
3. {{(a, a), a}, {(a, b), d}, {(a, c), c}, {(a, d), b}, {(b, a), b}, {(b, b), a}, {(b, c), d}, {(b, d), a}, {(c, a), c}, {(c, b), b}, {(c, c), a}, {(c, d), c}, {(d, a), d}, {(d, b), b}, {(d, c), c}, {(d, d), a}}
4. {({a, a}, a), ({a, b}, d), ({a, c}, c), ({a, d}, b), ({b, a}, b), ({b, b}, a), ({b, c}, d), ({b, d}, a), ({c, a}, c), ({c, b}, b), ({c, c}, a), ({c, d}, c), ({d, a}, d), ({d, b}, b), ({d, c}, c), ({d, d}, a)}

Answer: Alternative 2

Discussion:

The binary operation * is a function from A x A to A. In other words A x A is formed by an operation on two elements in A. The result is then again an element in A. So we have a kind of a relation, consisting of ordered pairs of which the first element is also an ordered pair. That is, each ordered pair in the relation has the format ((x, y), y). In other words, we should use round brackets here. Each of these ordered pairs is part of a set, which in this case is called the list notation for the operation. As you know, a set is enclosed in curly bracket { }. It should be clear the the use of brackets in alternatives 1, 3 and 4 is incorrect.

Question 9

Which one of the following options regarding the binary operation * is FALSE?

1. (a * b) * (c * d) = (a * (b * d)) * (d * c)
2. (a * b) ≠ (b * a) can be used as a counterexample to prove that the binary operation * is not commutative.
3. (a * b) * d = a * (b * d) proves that the binary operation * is associative.
4. The binary operation * does not have an identity element.

Answer: Alternative 3

*Discussion***:**

We look at each alternative.

Alternative 1: Here we look at the LHS and RHS of the statement separately to see if the two sides are in fact equal.

LHS: (a * b) * (c * d)	**RHS**: (a * (b * d)) * (d * c)
= d * c	= (a * a) * c
= c	= a * c
	= c

So LHS = RHS and this statement is true.

Alternative 2: To be commutative, it must be true that for every x, y in A, x * y = y * x. It is indeed true that (a * b) = d is not equal to (b * a) = b. We therefore have found a counterexample to prove that the binary operation * is not commutative, so alternative 2 is true.

Alternative 3: (a * b) * d = a * (b * d) proves that the binary operation * is associative.
This statement is not true. By finding one example that is associative, does not mean that the operation is associative. We would have to test every possible combination, to make sure that there is no counterexample that would render it not associative. The alternative is therefore false, and should have been chosen.

Alternative 4: This operation indeed does not have an identity element. A detailed way of finding out whether an operation has an identity element, is given in the next question. Follow the same route to see if you can prove that this operation * does not have an identity element.

Let A = {□, ◊, ☼, ⌂} and let # be a binary operation from A × A to A presented by the following table:

#	□	◊	☼	⌂
□	□	◊	☼	⌂
◊	◊	□	◊	□
☼	☼	◊	☼	⌂
⌂	⌂	□	⌂	⌂

Answer questions 10 and 11 by referring to the table for #.

Question 10

Which one of the following statements pertaining to the binary operation # is TRUE?

1. ☼ is the identity element for **#.**
2. From the table it can be observed that # is commutative.
3. \# is associative.
4. (⌂ # ◊) # ☼ = ⌂ # (◊ # ☼)

Answer: Alternative 2

We consider the statements provided in the different alternatives:

1. Definition of an identity element of a binary operation:
An element e of X is an identity element in respect of the binary operation
** : X × X → X iff e * **x** = **x** * e = **x** for all x ∈ X. (Note that the output is **x**.)*

Is it possible to identify an element *e* in A such that *e* # x = x # *e* = x for all x ∈ A?

Yes, e = □ is such an element of A:

	e # x	= x # e	= x
x = □:	□ # □	= □ # □	= □,
x = ◊:	□ # ◊	= ◊ # □	= ◊,
x = ☼:	□ # ☼	= ☼ # □	= ☼, and
x = ⌂:	□ # ⌂	= ⌂ # □	= ⌂.

So □ acts as an identity element for #.

The highlighted row and column in the table confirm that □ is the identity element for #. For an explanatory example, refer to study guide, p 121.

To **prove** that ☼ is **not** an identity element for #, we provide a counterexample:

Let e = ☼ and **x** = □. Is it the case that e # **x** = **x** # e = **x**?
e # **x** = **x** # e becomes
☼ # □ = □ # ☼ = ☼, but if ☼ was the identity element, the output should have been
x = □ which is not the case. Thus ☼ is **not** the identity element.

2. From the table it can be observed that # is commutative since there is symmetry about the diagonal from the top left to the bottom right corners of the table. In a formal proof it can be shown that # is indeed commutative by proving that x # y = y # x for all x, y $\in$ A.

3. Definition of an associative binary operation:
The binary operation * : $X \times X \to X$ is associative iff (x * y) * z = x * (y * z) for all x, y, z $\in$ X.

is **not** an associative binary operation. A counterexample is provided in alternative 4.

4. We determine (⌂ # ◊) # ☼ and ⌂ # (◊ # ☼) then compare the results:
(⌂ # ◊) # ☼
= □ **#** ☼
= ☼

⌂ # (◊ # ☼)
= ⌂ # ◊
= □

We see that [(⌂ # ◊) # ☼] ≠ [⌂ # (◊ # ☼)] since ☼ ≠ □.
This is a counterexample which proves that # is not associative.

From the above we can deduce that alternative 2 should be selected.

Refer to study guide, pp 116-122.

Question 11
can be written in list notation. Which one of the following ordered pairs is an element of the list notation set representing #?
1. ((□, ◊), ⌂)
2. ((⌂, ☼), ◊)
3. ((☼, ◊), ◊)
4. ((⌂, ◊), ◊)

Answer: Alternative 3

We consider the ordered pairs provided in the different alternatives:

1. From the table □ # ◊ = ◊
   thus ((□, ◊), ◊) $\in$ # but ((□, ◊), ⌂) $\notin$ #.

2. ⌂ # ☼ = ⌂
   thus ((⌂, ☼), ⌂) $\in$ # but ((⌂, ☼), ◊) $\notin$ #.

3. From the table ☼ # ◊ = ◊
   thus ((☼, ◊), ◊) $\in$ #.

4. From the table ⌂ # ◊ = □
   thus ((⌂, ◊), □) $\in$ # but ((⌂, ◊), ◊) $\notin$ #.

From the arguments provided we can deduce that alternative 3 should be selected.

Refer to study guide, pp 118, 119.

Question 12

Perform the following matrix multiplication operation:

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \\ -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer: Alternative 3

Discussion
A 3 $\times$ 3 matrix multiplied by a 3 $\times$ 3 matrix gives a 3 $\times$ 3 matrix.

We determine a_{ij}:
$a_{11} = 2(-3) + 1(-1) + 0(1) = -7$
$a_{12} = 2(0) + 1(0) + 0(1) = 0$
$a_{13} = 2(0) + 1(-1) + 0(1) = -1$
$a_{21} = 3(-3) + 4(-1) + 1(1) = -12$
$a_{22} = 3(0) + 4(0) + 1(1) = 1$
$a_{23} = 3(0) + 4(-1) + 1(1) = -3$
$a_{31} = -1(-3) + 0(-1) + -1(1) = 2$
$a_{32} = -1(0) + 0(0) + -1(1) = -1$
$a_{33} = -1(0) + 0(-1) + -1(1) = -1$

Thus the answer to the multiplication of the given matrices is

$$\begin{bmatrix} -7 & 0 & -1 \\ -12 & 1 & -3 \\ 2 & -1 & -1 \end{bmatrix}$$

From the above we can deduce that alternative 3 should be selected.

Refer to study guide, pp 131, 132.

Question 13

Consider the following matrices:

$$\text{Let } A = \begin{bmatrix} 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -4 & 3 \\ -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 & -3 \\ 4 & 0 & -1 \end{bmatrix}$$

Which one of the following statements is FALSE?

Answer: Alternative 3
We look at each alternative separately and do the multiplication where necessary.

Alternative 1: We calculate B · C:

A 3 $\times$ 2 matrix multiplied by a 2 $\times$ 3 matrix gives a 3 $\times$ 3 matrix.

We determine a_{ij}:
a_{11} = (-1)(2) + 0(4) = −2
a_{12} = (-1)(2) + 0(0) = -2
a_{13} = (-1)(-3) + 0(−1) = 3
a_{21} = (-4)(2) + 3(4) = 4
a_{22} = (-4)(2) + 3(0) = -8
a_{23} = (-4)(-3) + 3(−1) = 9
a_{31} = −2(2) + 0(4) = -4
a_{32} = −2(2) + 0(0) = −4
a_{33} = −2(-3) + 0(−1) = 6
This produces exactly the matrix given for this alternative.

Alternative 2: We calculate A · C:

A 1 $\times$ 2 matrix multiplied by a 2 $\times$ 3 matrix gives a 1 $\times$ 3 matrix.

We determine a_{ij}:
$a_{11} = (4)(2) + -1(4) = 4$
$a_{21} = (4)(2) + -1(0) = 8$
$a_{31} = (4)(-3) + -1(-1) = 11$

Clearly, this results in the matrix given in alternative 2.

Alternative 3: Is it true that B · C = C · B? We have already calculated B · C in alternative 1. We will now calculate C · B:

A 2 $\times$ 3 matrix multiplied by a 3 $\times$ 2 matrix gives a 2 $\times$ 2 matrix.

$a_{11} = (2)(-1) + (2)(-4) + (-3)(-2) = -4$
$a_{12} = (2)(0) + (2)(3) + (-3)(0) = 6$
$a_{21} = (4)(-1) + 0(-4) + (-1)(-2) = -2$
$a_{22} = (4)(0) + 0(3) + (-1)(0) = 0$

Clearly, it is false that B · C = C · B. This alternative should therefore be selected.

Alternative 4: The operation (C · B) · A is not possible. Is this true?
(C · B) is a 2 x 2 matrix as we have seen in alternative 3. A is a 1 x 2 matrix. These two matrices cannot be multiplied. A 2 x 2 matrix can only be multiplied by a 2 x 1, 2 x 2, 2 x 3 etc matrix. This statement is therefore true.

Question 14
Consider the truth table for the connective '$\leftrightarrow$' with two simple declarative statements p and q.

p	**q**	**p $\leftrightarrow$ q**
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

Which one of the given alternatives represents '$\leftrightarrow$' as a binary operation on the set of truth values {T, F}? (b) Does the operation '$\leftrightarrow$' have an identity element?

Answer: Alternative 4

(a) Discussion

We see that in the table for the connective '$\leftrightarrow$' with two simple declarative statements p and q, when p and q are true or when p and q are false that p $\leftrightarrow$ q is true. In the table below we see that T $\leftrightarrow$ T = T and F $\leftrightarrow$ F = T. When p or q is false, then p $\leftrightarrow$ q is false. In the table below we see that T $\leftrightarrow$ F = F and F $\leftrightarrow$ T = F. (We can write $\leftrightarrow$ in list notation: { ((T, T), T), ((T, F), F), ((F, T), F), ((F, F), T) }.)

So we can present the binary operation $\leftrightarrow$ in the following table:

$\leftrightarrow$	**T**	**F**
T	T	F
F	F	T

(b) The binary operation '$\leftrightarrow$' has an identity element, namely **T**:

We have T $\leftrightarrow$ **T** = T and **T** $\leftrightarrow$ T = T; **T** $\leftrightarrow$ F = F and F $\leftrightarrow$ **T** = F.

From the above we can deduce that alternative 4 should be selected.

Refer to study guide, pp 136 – 146.

Question 15

Let p, q and r be simple declarative statements. Which alternative provides the truth values for the biconditional '$\leftrightarrow$' of the compound statement provided in the given table?

Hint: Determine the truth values of $p \rightarrow r$, $q \vee r$, $(p \rightarrow r) \wedge (q \vee r)$, $q \rightarrow p$, $\neg(q \rightarrow p)$ and $\neg(q \rightarrow p) \wedge r$ in separate columns before determining the truth values of

$[(p \rightarrow r) \wedge (q \vee r)] \leftrightarrow [\neg(q \rightarrow p) \wedge r]$. **p**	**q**	**r**	**$[(p \rightarrow r) \wedge (q \vee r)] \leftrightarrow [\neg(q \rightarrow p) \wedge r]$**
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

1.

$\leftrightarrow$
F
T
F
T
T
F
F
T

2.

$\leftrightarrow$
F
T
F
T
F
T
F
T

3.

$\leftrightarrow$
T
F
F
F
T
F
F
F

4.

$\leftrightarrow$
F
F
T
F
T
F
F
T

Answer: Alternative 1

Discussion
We apply the **definitions** of the logical connectives which are discussed and also provided in **truth tables** in the study guide. Refer to these to determine the truth values of the statement provided in the question.

We look at the truth values highlighted in one row of the table below:
With reference to the statement $[(p \to r) \wedge (q \vee r)] \leftrightarrow [\neg(q \to p) \wedge r]$ we first want to determine the truth value of $p \to r$. If **p** is T and **r** is F it means that the hypothesis **p** is true and the conclusion is **r** false, thus the conditional statement 'if p then r' ie $p \to r$ is false. Next determine the truth value of $q \vee r$. If **q** is F and **r** is F it means that the disjunction of q or r is false. Now $(p \to r)$ and $(q \vee r)$ are false, so the conjunction $(p \to r) \wedge (q \vee r)$ is false. And so we can go on to determine the truth values of all the expressions. (Refer to the study guide for the tables of all the logical connectives.)

We compile a truth table for the given expression:

p	**q**	**r**	$\mathbf{p \to r}$	$\mathbf{q \vee r}$	$\mathbf{(p \to r) \wedge (q \vee r)}$	$\leftrightarrow$	$\mathbf{q \to p}$	$\mathbf{\neg(q \to p)}$	$\mathbf{\neg (q \to p) \wedge r}$
T	T	T	T	T	T	**F**	T	F	F
T	T	F	F	T	F	**T**	T	F	F
T	F	T	T	T	T	**F**	T	F	F
T	F	F	F	F	F	**T**	T	F	F
F	T	T	T	T	T	**T**	F	T	T
F	T	F	T	T	T	**F**	F	T	F
F	F	T	T	T	T	**F**	T	F	F
F	F	F	T	F	F	**T**	T	F	F

(Considering the final column containing Ts and Fs, it is clear that the expression is neither a tautology nor a contradiction.)

From the above table we can deduce that alternative 1 should be selected.

Refer to study guide, pp 136 – 146.

Question 16

Consider the following quantified statement:

$$\forall x \in \mathbb{Z}, [(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]$$

Which one of the alternatives provides a true statement regarding the given statement or its negation?

1. The negation $\exists x \in \mathbb{Z}, [(x^2 < 0) \vee (x^2 + 2x - 8 \leq 0)]$ is not true.

2. $x = -3$ would be a counterexample to prove that the negation is not true.

3. $x = -6$ would be a counterexample to prove that the statement is not true.

4. The negation $\exists x \in \mathbb{Z}, [(x^2 < 0) \wedge (x^2 + 2x - 8 \leq 0)]$ is true.

Answer: Alternative 2

Discussion

Firstly we will derive the negation of the given statement step by step:

$\neg[\forall x \in \mathbb{Z}, [(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]]$ (always write down this step)

$\equiv \exists x \in \mathbb{Z}, \neg[(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]$

$\equiv \exists x \in \mathbb{Z}, \neg(x^2 \geq 0) \wedge \neg(x^2 + 2x - 8 > 0)$ (de Morgan's law)

$\equiv \exists x \in \mathbb{Z}, (x^2 \ngeq 0) \wedge (x^2 + 2x - 8 \ngtr 0)$

$\equiv \exists x \in \mathbb{Z}, (x^2 < 0) \wedge (x^2 + 2x - 8 \leq 0)$

Let us look at the statement first:

$$\forall x \in \mathbb{Z}, [(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]$$

This statement states that for all integers x, it is true that $(x^2 \geq 0)$ **or** $(x^2 + 2x - 8 > 0)$. This means that at least $(x^2 \geq 0)$ **or** $(x^2 + 2x - 8 > 0)$ must be true for the statement to be true. Both may also be true. (Refer to the truth table for the disjunction '∨'.)

We substitute a few integers for x in both ($x^2 \geq 0$) and ($x^2 + 2x – 8 > 0$) and see where that takes us:

x	$x^2 \geq 0$	$x^2 + 2x – 8 > 0$
-1	$(-1)^2 \geq 0$ which is true	$(-1)^2 + 2(-1) – 8 > 0$ which is not true
-2	$(-2)^2 \geq 0$ which is true	$(-2)^2 + 2(-2) – 8 > 0$ which is not true
3	$3^2 \geq 0$ which is true	$(3)^2 + 2(3) – 8 > 0$ which is true
4	$4^2 \geq 0$ which is true	$(4)^2 + 2(4) – 8 > 0$ which is true
	etc…	etc…

We know that $x^2 \geq 0$ is true for all integers x. Thus $(x^2 \geq 0) \vee (x^2 + 2x – 8 > 0)$ is true for all integers x.

What about the negation statement?

$$\exists x \in \mathbb{Z}, (x^2 < 0) \wedge (x^2 + 2x – 8 \leq 0)$$

The negation statement states that there exists an integer x such that **both** ($x^2 < 0$) **and** ($x^2 + 2x – 8 \leq 0$) are true. But we know that there is no integer x such that $x^2 < 0$, thus the negation statement is not true. Any integer could play a role in a counterexample to prove that the negation statement is not true, thus $x = – 3$ can play this role. Can you determine why alternatives 3 and 4 are false?

From the discussion above, alternative 2 should be selected.

Refer to study guide, pp 152-158.

Question 17

Consider the following proposition:

For any predicates P(x) and Q(x) over a domain D, the negation of the statement

$$\exists x \in D, P(x) \wedge Q(x)$$

is the statement

$$\forall x \in D, P(x) \rightarrow \neg Q(x).$$

We can use this truth to write the negation of the following statement:

“There exist integers a and d such that a and d are negative and $a/d = 1 + d/a$.”

Which one of the alternatives provides the negation of this statement?

1. There exist integers a and d such that a and d are positive and a/d = 1 + d/a.
2. For all integers a and d, if a and d are positive then a/d ≠ 1 + d/a.
3. For all integers a and d, if a and d are negative then a/d ≠ 1 + d/a.
4. For all integers a and d, a and d are positive and a/d ≠ 1 + d/a.

Answer: Alternative 3

Discussion

We provide the following substitutions:

Let D be a set such that a, d ∈ D (a and d are the only elements of D); and

let P(x) be the predicate 'a and d are negative'; and

let Q(x) be the predicate 'a/d = 1 + d/a'.

The statement 'There exist integers a and d such that a and d are negative and a/d = 1 + d/a.' can now be written as

$\exists x \in D, P(x) \wedge Q(x)$. We are required to write down the negation of this statement:

In the question statement it is given that

$\neg [\exists x \in D, P(x) \wedge Q(x)]$

$\equiv \forall x \in D, P(x) \rightarrow \neg Q(x)$. By using our initial substitutions we can write this statement as:

For all integers a and d belonging to D, if a and d are negative then it is not the case that a/d = 1 + d/a.

i.e. For all integers a and d, if a and d are negative then a/d ≠ 1 + d/a.

From the discussion provided, alternative 3 should be selected.
This was a difficult question. You will not get a question like this in the exam.

Question 18

Which one of the alternatives is a proof by contrapositive of the statement

"If $x^3 - x + 4$ is not divisible by 4 then x is even."

1. Required to prove: If $x^3 - x + 4$ is not divisible by 4 then x is even.
   Proof:
   Suppose x is odd. Let x = 2k + 1, then we have to prove that $x^3 - x + 4$ is divisible by 4.
   $x^3 - x + 4 = (2k + 1)^3 - (2k + 1) + 4$
   $= (2k + 1)(4k^2 + 4k + 1) - 2k - 1 + 4$
   $= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 - 2k - 1 + 4$
   $= 8k^3 + 12k^2 + 4k + 4$
   $= 4(2k^3 + 3k^2 + k + 1)$, which is divisible by 4. (4 multiplied by any integer is divisible by 4)

2. Required to prove: If $x^3 - x + 4$ is not divisible by 4 then x is even.

Proof:
Assume that $x^3 – x + 4$ is not divisible by 4. Then x can be even or odd. We assume that x is odd.
Let $x = 2k + 1$, then
$x^3 – x + 4 = (2k+1)^3 – (2k + 1) + 4$
$= (2k + 1)(4k^2 + 4k +1) – 2k – 1 + 4$
$= 8k^3 + 8k^2 + 2k + 4k^2 + 4k +1 – 2k – 1 + 4$
$= 8k^3 + 12k^2 + 4k + 4$
$= 4(2k^3 + 3k^2 + k + 1)$, which is divisible by 4. (4 multiplied by any integer is divisible by 4)

But this is a contradiction to our original assumption. Therefore x must be even if $x^3 – x + 4$ is not divisible by 4.

3. Required to prove: If $x^3 – x + 4$ is not divisible by 4 then x is even.
Proof:
Let x = 4 be an even element of $\mathbb{Z}$. We can replace x with 4 in the expression $x^3 – x + 4$.
$x^3 – x + 4$
$= (4)^3 – (4) + 4$
$= 64 – 4 + 4$
$= 64$ which is divisible by 4.

4. Required to prove: If $x^3 – x + 4$ is not divisible by 4 then x is even.
Proof:
Assume that x is even, ie $x = 4k$,
then $x^3 – x + 4$
$= (4k)^3 – (4k) + 4$
$= 64k^3 – 4k + 4$
$= 4(16k^3 – k + 1)$, which is divisible by 4.

Answer: Alternative 1

Discussion: It is very important that you know how to apply each of the proof methods discussed in the study guide.

We look at each of the alternatives:

1. The proof provided in this alternative is a proof by contrapositive. Another way to look at this proof method is the following:

The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. This means that $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$.

The contrapositive of the statement "If $x^3 – x + 4$ is not divisible by 4 then x even."
is "If x odd then $x^3 – x + 4$ is divisible by 4."
This is exactly what is proven in alternative 1. Thus this is the correct alternative.

2. This alternative provides a proof by contradiction. We assume the ‘if’ part of the given statement is true, ie “$x^3 – x + 4$ is not divisible by 4” is true, then we assume the opposite of the ‘then’ part. The ‘then’ part states that “x is even”, so we assume the opposite, ie “x is odd”, and then try to get to a contradiction.
This alternative does not provide the required contrapositive proof.

3. This alternative is not a proof. One cannot substitute values for x in a proof. One example (ie choosing a value for x and substituting it in the expression) does not provide a general proof to show that
“If $x^3 – x + 4$ is not divisible by 4 then x is even”.

4. This proof is not valid. The proof shows that “If x is divisible by 4 then $x^3 – x + 4$ is divisible by 4.”

Refer to study guide, pp 152 - 163.

Question 19

By using logical equivalences and de Morgan’s rules, we can show that the statements **¬p ∨ q** and **(p ∧ ¬q) → (¬p ∨ q)** are equivalent.

Correct alternative: Alternative 1

Discussion: Refer to the logical equivalences on p. 147 of the study guide.

We start with **(p ∧ ¬q) → (¬p ∨ q)** and see if simplification using the logical equivalences results in **(p ∧ ¬q) → (¬p ∨ q)**:

(p ∧ ¬q) → (¬p ∨ q)	
ie **¬**(p ∧ ¬q) ∨ (¬p ∨ q)	We know that p → q is logically equivalent to ¬p ∨ q
ie **¬p ∨ ¬¬q)** ∨ (¬p ∨ q)	de Morgan’s laws (f)
ie ¬p ∨ **q** ∨ (¬p ∨ q)	law of double negation
ie ¬p ∨ q ∨ ¬p ∨ q	brackets can be removed because only one logical symbol
ie ¬p ∨ **¬p ∨ q** ∨ q	commutative law
ie ¬p ∨ q	idempotent law

This statement is therefore true and alternative 1 should be selected.

Question 20

The statement **[p $\wedge$ (r $\rightarrow$ q)] $\leftrightarrow$ [(r $\vee$ q) $\wedge$ (p $\rightarrow$ q)]** is a contradiction.

Correct alternative: Alternative 2

*Discussion***:**

The easiest way to determine whether the statement is true, is to draw a truth table:

p	q	r	r $\rightarrow$ q	p $\wedge$ (r $\rightarrow$ q)	$\leftrightarrow$	(r $\vee$ q)	$\wedge$	(p $\rightarrow$ q)
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	F	T	T	F	F
T	F	F	T	T	F	F	F	F
F	T	T	T	F	F	T	T	T
F	T	F	T	F	F	T	T	T
F	F	T	F	F	F	T	T	T
F	F	F	T	F	T	F	F	T

This statement is not a contradiction, because the highlighted column contains a combination of **T**'s and **F**'s. When a statement is a tautology, the final result contains **T**'s only. When a statement is a contradiction, the final result contains **F**'s only. This original statement is therefore false.

Question 21

Consider the two statements below:

Statement 1: $\forall x \in \mathbb{Z}^+$, $[(2x + 1 > 3) \vee (2x^2 - 1 \geq 1)]$

Statement 2: $\exists x \in \mathbb{Z}$, $[(x^2 - 1 < 0) \wedge (2x - 2 \geq 0)]$

It is true that both statements 1 and 2 are false. Is this statement true?

Correct alternative: alternative 2

Discussion:

Make sure that you understand the definitions of the universal and the existential quantifiers on pp. 152 and 153 of the study guide, because you will use these definitions in all these type of questions.

In statement 1, let A = $(2x + 1 > 3)$, and B = $(2x^2 - 1 \geq 1)$, then the statement reads as follows:

For ALL values of x from the positive integers, it is true that A OR B is true. Remember the definition of $\vee$ states that both A and B may also be true, but at least one should be true for the statement to be true. We do the same for statement 2.

In statement 2, let C = $(x^2 - 1 < 0)$ and D = $(2x - 2 \geq 0)$, then the statement reads as follows:

There exists a value of x from the integer set, such that C AND D are true. So in this case BOTH parts of the statement must be true.

In statement 1, where only A OR B needs to be true, we don't have to test B if A is true, because, only one of them needs to be true. We can test this in various ways. If we substitute a few values from the positive integers, in A (or B), we will quickly see a pattern or find a counterexample:

Value chosen for x	**Result : statement A =** $2x + 1 > 3$
$x = 1$	$2(1) + 1 = 3$, which is not greater than 3
$x = 2$	
$x = 3$	
$x = 4$	

We don't need to test any further. x = 1 is counterexample to show that A is false. So we test B next.

Value chosen for x	**Result : statement B =** $2x^2 – 1 \geq 1$
$x = 1$	$2(1)^2 – 1 = 1 \geq 1$
$x = 2$	$2(2)^2 – 1 = 7 \geq 1$
$x = 3$	$2(3)^2 – 1 = 17 \geq 1$
$x = 4$	………

Can you see that B will always be true, because the higher the value of x, the higher the result. Remember for statement 1, only values from the positive integers are allowed. Statement 1 is therefore true, because B is true. At this point we can already determine that the given statement is false and that alternative 2 should be selected. We do however continue and also test the second statement as an exercise.

For statement 2 to be true, we only need to find one value from the set of integers for which C and D both are true. In this case we always want to test some negative and positive values for x.

Value chosen for x	**Result : statement C =** $(x^2 – 1 < 0)$
$x = -2$	$(-2)^2 – 1 = 3$ which is not less than 0
$x = -1$	$(-1)^2 – 1 = 0$ which is not less than 0
$x = 0$	$(0)^2 – 1 = -1$, which is less than 0
$x = 1$	$(1)^2 – 1 = 0$ which is not less than 0
$x = 2$	$(2)^2 – 1 = 3$ which is not less than 0

You should be able to see that statement C is true iff x = 0. At this point we check if x = 0 will also make D true:

Value chosen for x	**Result : statement D =** $2x - 2 \geq 0$
x = 0	2(0) – 2 = -2 which is not greater than or equal to 0

Therefore statement 2 is false.

Question 22

Consider the following statement:

$$\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]$$

Correct alternative: Alternative 1

Discussion:

The negation of the above statement is:

$\neg[\forall x \in \mathbb{Z}, [(2x + 4 > 0) \vee (4 - x^2 \leq 0)]]$

$\equiv \exists x \in \mathbb{Z}, \neg[(2x + 4 > 0) \vee (4 - x^2 \leq 0)]$

$\equiv \exists x \in \mathbb{Z}, [\neg(2x + 4 > 0) \wedge \neg(4 - x^2 \leq 0)]$

$\equiv \exists x \in \mathbb{Z}, [(2x + 4 \leq 0) \wedge (4 - x^2 > 0)]$

If you cannot see why this is true, please study pp.153 – 158 in the study guide again, and try some of the self-assessment exercises on negation on p.158.

Question 23

Consider the statement

If n is even, then $4n^2$ - 3 is odd.

The **contrapositive** of the given statement is:

If $4n^2$ - 3 is odd, then n is even.

Correct alternative: alternative 2

Discussion:

If p and q are two variables, then the contrapositive of the statement $\mathbf{p \rightarrow q}$ is $\mathbf{\neg q \rightarrow \neg p}$.

Let **p** represent **"n is even"** and let **q** represent **"$4n^2$ - 3 is odd"**. That is, $\mathbf{p \rightarrow q}$ represents

If n is even, then $4n^2$ - 3 is odd.

Thus the contrapositive $\mathbf{\neg q \rightarrow \neg p}$ is therefore:

If ¬($4n^2$ - 3 is odd), then ¬(n is even)

ie if $4n^2$ - 3 is even, then n is odd.

The given contrapositive statement is therefore false, and alternative 2 should be selected.

Question 24

Consider the statement

If n is a multiple of 3, then 2n + 2 is not a multiple of 3.

The converse of the given statement is:

If n is not a multiple of 3, then 2n + 2 is a multiple of 3.

Correct alternative: alternative 2

Discussion:

If p and q are two variables, then the converse of the statement $\mathbf{p \rightarrow q}$ is $\mathbf{q \rightarrow p}$.

Let **p** represent **"n is a multiple of 3"** and let **q** represent **"2n + 2 is not a multiple of 3"**. That is, $\mathbf{p \rightarrow q}$ represents

If n is a multiple of 3, then 2n + 2 is not a multiple of 3.

Thus the converse $\mathbf{q \rightarrow p}$ is therefore:

If 2n + 2 is not a multiple of 3, then n is a multiple of 3.

The given converse statement is therefore false, and alternative 2 should be selected.

Question 25

Consider the following statement, for all $x \in \mathbb{Z}$:

If x + 1 is even, then $3x^2$ - 4 is odd.

The correct way to start a **direct** proof to determine if the statement is true is as follows:

Assume x is even, then x = 2k for some $k \in \mathbb{Z}$,

then $3x^2 – 4$

ie $3(2k)^2 - 4$

ie ………..

Correct alternative: alternative 2

Discussion:

In a direct proof, one assumes the **if** part, and tries to prove the **then** part of the statement.

If we assume that x + 1 is even, then **x + 1 = 2k** for some k $\in \mathbb{Z}$. The given start to the proof is therefore wrong. I suggest you try and complete the proof. Alternative 2 should therefore be selected.