Peut-on factoriser suffisamment rapidement les nombres en facteurs premiers?

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Plan

Les nombres *RSA*

- Factoriser $N = pq$ où p et q sont premiers et très grands.
- Dernier nombre non factorisé: RSA-260 (260 chiffres)

$N =$ 2211282552952966643528108525502623092761208950247001539441374831912882294140200198651272972656974659908590033003140005117074220456085927635795375718595429883895870922923849100670303412462054578456641366454068421436129301769402084639106587591479425143514445819

Plan

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Congruence de carrés

$N = pq$, p premier. Supp. $x^2 \equiv y^2 \pmod{N}$ et $x \neq \pm y$.

- On a $x^2 - y^2 \equiv 0 \pmod{N}$ i.e. $N \mid (x-y)(x+y)$
- Donc $p \mid (x-y)(x+y)$
- Lemme d'Euclide: par exemple $p \mid x - y$
- Alors p divise N et $x-y$: $p \mid N \wedge (x-y)$, ce qui donne $\mathbf{N \wedge (x-y) \neq 1}$

Conclusion

$N \wedge (x-y)$ et $N \wedge (x+y)$ sont des facteurs <u>non-triviaux</u> de N

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Plan

$\boxed{b \in \mathbb{N}}$

2

3

5

$\cdot$

$\cdot$

$\cdot$

p_b

$\boxed{b \in \mathbb{N}}$

$$\boxed{(x_1, \quad x_2, \quad x_3, \quad \ldots, \quad x_{b+1})}$$

2

3

5

$\cdot$

$\cdot$

$\cdot$

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$$\begin{tikzcd} & \boxed{(x_1, \quad x_2, \quad x_3, \quad \dots, \quad x_{b+1})} \arrow[dl] \\ \boxed{x_1^2 \pmod{N}} & \end{tikzcd}$$

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•

•

•

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$\boxed{b \in \mathbb{N}}$

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2	$v_1^{(1)}$
3	$v_1^{(2)}$
5	$v_1^{(3)}$
$\cdot$	$\cdot$
$\cdot$	$\cdot$
$\cdot$	$\cdot$
p_b	$v_1^{(b)}$

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$$\left.\begin{array}{cc}
2 & v_1^{(1)} \\
3 & v_1^{(2)} \\
5 & v_1^{(3)} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
p_b & v_1^{(b)}
\end{array}\right\} x_1^2 \pmod{N} = \prod_{i=1}^{b} p_i^{v_1^{(i)}} = 2^{v_1^{(1)}} 3^{v_1^{(2)}} \ldots p_b^{v_1^{(b)}}$$

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$$\begin{tikzcd} \boxed{(x_1, \quad x_2, \quad x_3, \quad \ldots, \quad x_{b+1})} \arrow[d] \\ \boxed{x_1^2 \pmod N} \arrow[d] \\ \begin{array}{cc} 2 & v_1^{(1)} \\ 3 & v_1^{(2)} \\ 5 & v_1^{(3)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ p_b & v_1^{(b)} \end{array} \arrow[d] \\ v_1 \end{tikzcd}$$

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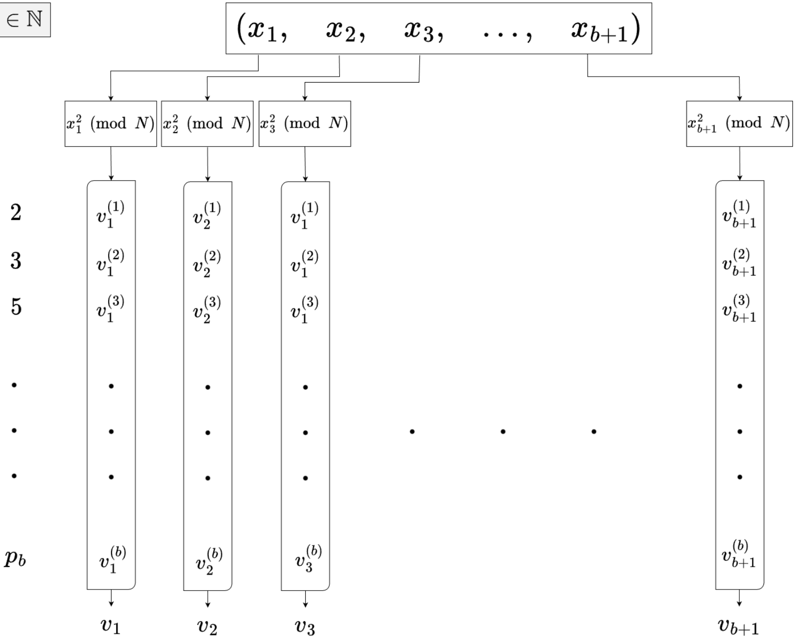

Construction de y - Pivot de Gauss

- $b+1$ vecteurs de $\mathbb{F}_2^b$, système lié:

$$\exists(\lambda_i)_{i\in[\![1,b+1]\!]} \in \{0,1\}^{b+1} \mid \sum_{i=1}^{b+1} \lambda_i v_i = 0_{\mathbb{F}_2^b} = (2\alpha_1, \ldots, 2\alpha_b)$$

- On pose $y = \prod_{j=1}^{b} p_j^{\alpha_j}$ et $x = \prod_{j=1}^{b+1} x_j^{\lambda_j}$

Résultat admis (calcul en Annexe)

$x^2 \equiv y^2 \pmod{N}$

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On peut trouver les λ_i avec un système que l'on résout avec un **pivot de Gauss**

- $N = 20382493 = 3467 \times 5879$ et $b = 4$.
- $x_j^2 \bmod N = 2^{v_j^{(1)}} \cdots 7^{v_j^{(4)}}$ pour $j \in [\![1, 5]\!]$

x_j	v_j
16853	$(6,5,2,2)$
32877	$(3,0,7,0)$
35261	$(5,3,0,1)$
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- On résout dans $\mathbb{F}_2^5$

$$\begin{cases} 6\lambda_1 + 3\lambda_2 + 5\lambda_3 + 3\lambda_4 + 0\lambda_5 = 0_{\mathbb{F}_2} \\ 5\lambda_1 + 0\lambda_2 + 3\lambda_3 + 2\lambda_4 + 2\lambda_5 = 0_{\mathbb{F}_2} \\ 2\lambda_1 + 7\lambda_2 + 0\lambda_3 + 1\lambda_4 + 3\lambda_5 = 0_{\mathbb{F}_2} \\ 2\lambda_1 + 0\lambda_2 + 1\lambda_3 + 0\lambda_4 + 1\lambda_5 = 0_{\mathbb{F}_2} \end{cases}$$

$\lambda = (1,1,1,0,1)$ solution.

- $x = \prod_{j=1}^{b+1} x_j^{\lambda_j} = 7248176$
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Ce qu'il faut retenir

L'enjeu principal

Étant donné $b \in \mathbb{N}$, trouver $b + 1$ nombres tels que
$\forall j \in [\![1, b+1]\!], x_j^2$ mod N a ses facteurs premiers inférieurs à p_b

Plan

L'algorithme final

Algorithme 1 Recherche de nombres

Entrée: $N \in \mathbb{N}$ composé, $b \in \mathbb{N}$
Sortie: $(v_i)_{i\in[\![1,b+1]\!]}, (x_i)_{i\in[\![1,b+1]\!]}$

```
1: pour $i \leftarrow 1 \dots b+1$ faire
2:     $en\_cours \leftarrow V$
3:     tant que $en\_cours$ faire
4:         $x_i \leftarrow \mathbb{U}(1, N-1)$
5:         si $x_i^2$ mod $N$ est factorisable alors          ▷ par algorithme naïf
6:             $en\_cours \leftarrow F$
7:             $v_i \leftarrow (v_i^{(1)}, \dots, v_i^{(b)})$

   renvoyer $(v_i)_{i\in[\![1,b+1]\!]}, (x_i)_{i\in[\![1,b+1]\!]}$
```

L'algorithme final

Algorithme 2 Factorisation par la méthode de Dixon

Entrée: $N \in \mathbb{N}$ composé, $b \in \mathbb{N}$

Sortie: p et q, tels que $p \mid N$ et $q \mid N$

```
1: $(v_i)_{i \in [\![1, b+1]\!]}, (x_i)_{i \in [\![1, b+1]\!]} \leftarrow RechercheNombres(N, b)$
2: $(\lambda_i)_{i \in [\![1, b+1]\!]} \leftarrow PivotdeGauss((v_i)_{i \in [\![1, b+1]\!]})$
3: $x \leftarrow \prod_{j=1}^{b+1} x_j^{\lambda_j}$
4: $y \leftarrow \prod_{j=1}^{b} p_j^{\alpha_j}$

   renvoyer $N \wedge (x - y), N \wedge (x + y)$
```

Etude théorique (Louise Nguyen)

Une minoration de la densité des B-friables

Soit $B : \mathbb{N}^* \to \mathbb{N}^*$ une fonction telle que $\ln n = o(B(n))$ et $\ln B(n) = o(\ln n)$. Alors on a, pour $n \to +\infty$,

$$\Psi(B(n), n) \geq n \exp\left(\left(\frac{\ln n}{\ln B(n)} \ln \ln n\right)(-1 + o(1))\right)$$

Une complexité sous-exponentielle

$$\exp\left((1 + o(1))2\sqrt{2}(\ln n \ln \ln n)^{1/2}\right)$$

lorsque $B = \exp\left(\frac{1}{\sqrt{2}}(\ln n \ln \ln n)^{1/2}\right)$

Plan

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Principe

- Utilisation d'un polynôme $Q = (\lfloor\sqrt{N}\rfloor + X)^2 - N$ pour générer les x_i
- Résolution de $Q(x) \equiv 0 \pmod{p}$ grâce à Tonelli-Shanks, 2 solutions x_1 et x_2 dans $[\![1, p]\!]$.
- $p|Q(x) \implies \forall k \in \mathbb{N}, p|Q(x+kp)$ (Démonstration en Annexe)
- Cribler sur un intervalle $[\![1, S]\!]$, puis sur $[\![S+1, 2S]\!]$ etc...

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$\boxed{S = 10}$ $\boxed{N = 20382493}$

$$T = [Q(1), Q(2), Q(3), Q(4), Q(5), Q(6), Q(7), Q(8), Q(9), Q(10)]$$

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{[2732,} \arrow[rr, bend left, "+2"] \arrow[d, "\div 2^2"', "\boxed{v_1^{(1)} = 2}"] \arrow[d, bend right=60] & {11736,} & {20796,} \arrow[rr, bend left, "+2"] \arrow[d, "\div 2^2"', "\boxed{v_3^{(1)} = 2}"] & {29831,} & {38868,} \arrow[d, "\div 2^2"', "\boxed{v_5^{(1)} = 2}"] & {\ldots,} & {Q(10)]} \\
{[683,} & {11736,} & {5199,} & {29831,} & {9717,} & {\ldots,} & {Q(10)]}
\end{tikzcd}$$

$\boxed{S = 10}$ $\boxed{N = 20382493}$

$$T = [Q(1), Q(2), Q(3), Q(4), Q(5), Q(6), Q(7), Q(8), Q(9), Q(10)]$$

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$\boxed{p = 3}$ $\boxed{Q(2) \equiv 0 \pmod{3}}$ $\boxed{Q(3) \equiv 0 \pmod{3}}$

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$$[683, \quad 11736, \quad 5199, \quad 29831, \quad 9717, \quad \ldots, \quad Q(10)]$$

$$\vdots$$

$\boxed{p = p_b}$ $\boxed{Q(x_1) \equiv 0 \pmod{p_b}}$ $\boxed{Q(x_2) \equiv 0 \pmod{p_b}}$

Algorithme 3 Algorithme du crible quadratique

```
Entrée: $N \in \mathbb{N}^*$, $b \in \mathbb{N}^*$, $S \geq 1$
Sortie: $(v_i)_{i \in [\![1,k]\!]}, (x_i)_{i \in [\![1,k]\!]}$, $k \in [\![0, S]\!]$
 1: $T \leftarrow$ tableau tel que $T[i] \leftarrow (i + \lfloor\sqrt{N}\rfloor)^2 - N$ pour $i \in [\![1, S]\!]$
 2: $V \leftarrow$ tableau tel que $V[i] \leftarrow (0, \dots, 0) \in \mathbb{N}^b$ pour $i \in [\![1, S]\!]$
 3: pour $p \in \{p_1, \dots, p_b\}$ tel que $N$ est un carré modulo $p$ faire
 4:     $x_1, x_2 \leftarrow$ les racines de $(X + \lfloor\sqrt{N}\rfloor)^2 - N$ modulo $p$
 5:     pour $i \in \{1, 2\}$ faire
 6:         $q \leftarrow x_i$
 7:         tant que $q \leq S$ faire
 8:             tant que $T[q]$ mod $p = 0$ faire
 9:                 $T[q] \leftarrow T[q]/p$
10:                 $V[q] \leftarrow V[q] + (0, \dots, 1, \dots, 0)$ (en position $p$)
11:             $q \leftarrow q + p$
    renvoyer L'ensemble des $(i + \lfloor\sqrt{N}\rfloor, V[i])$ tels que $T[i] = 1$ pour
    $i \in [\![1, S]\!]$
```

Plan

Justification

- $O(n)$ au lieu de $O(n^2)$, voire $O(n \log n)$
- $Q(x) = \prod_{i=1}^{k} p_i^{\alpha_i}$, soit $\ln(Q(x)) = \sum_{i=1}^{k} \alpha_i \ln(p_i)$.
  Idée: soustraire par $\alpha_i \ln(p_i)$ au lieu de diviser par $p_i^{\alpha_i}$
- $\log_2(Q(x)) \approx \text{nb\_bits}(Q(x))$
- Problème: on ne connaît pas α_i.
  Solution: on soustrait par $\log_2(p_i)$ seulement. Des approximations nécessitent déjà un **seuil**

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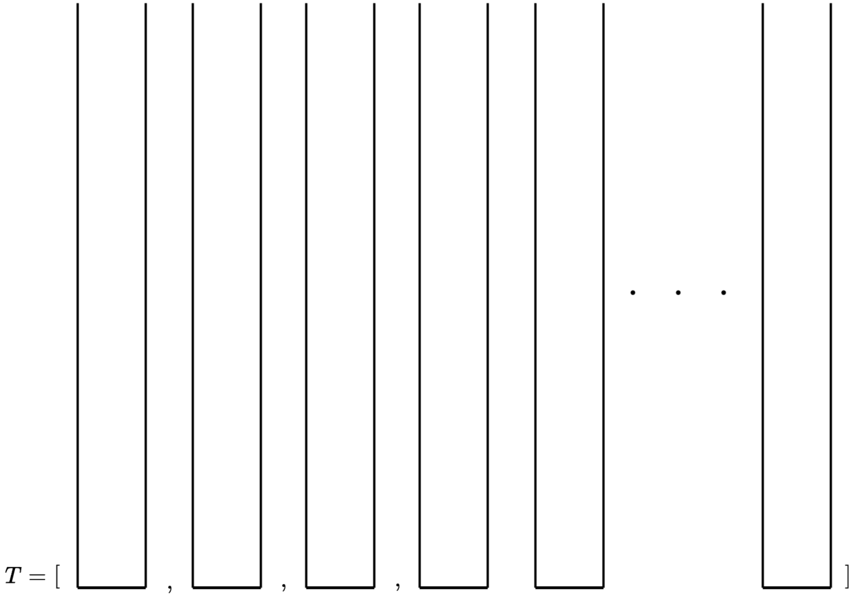
$\boxed{S = 10}$ $\boxed{N = 20382493}$



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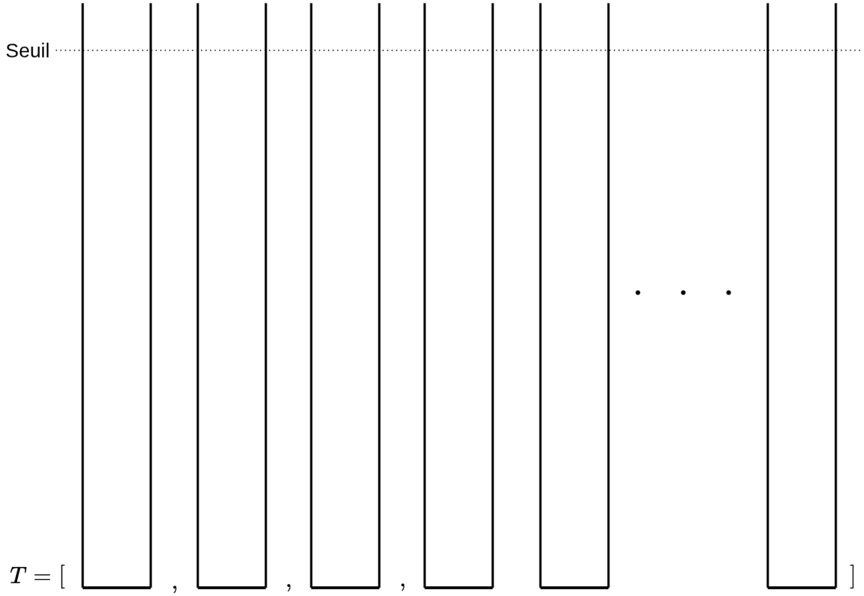
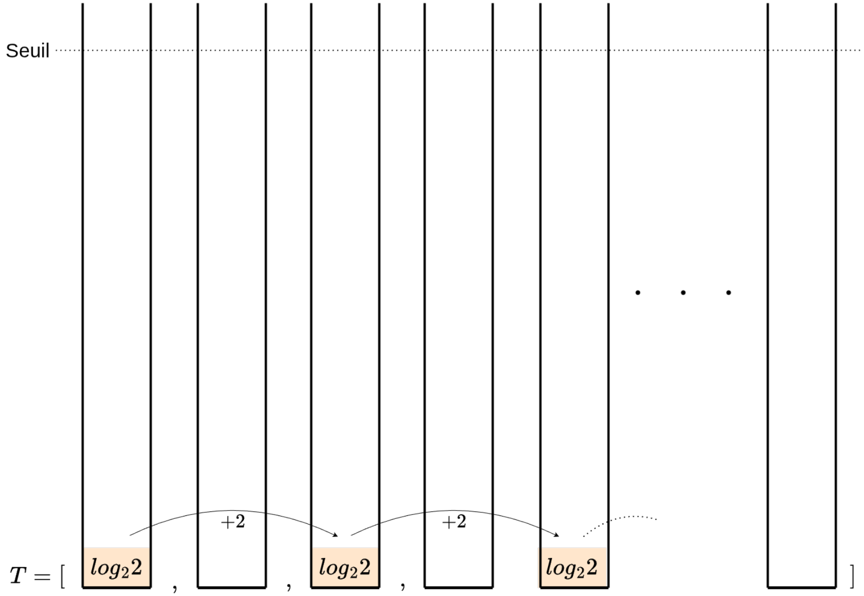
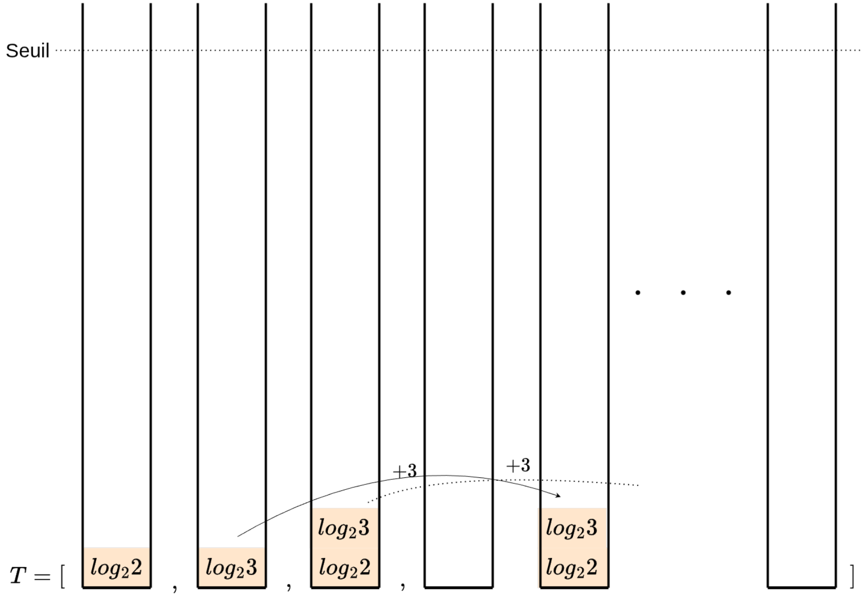


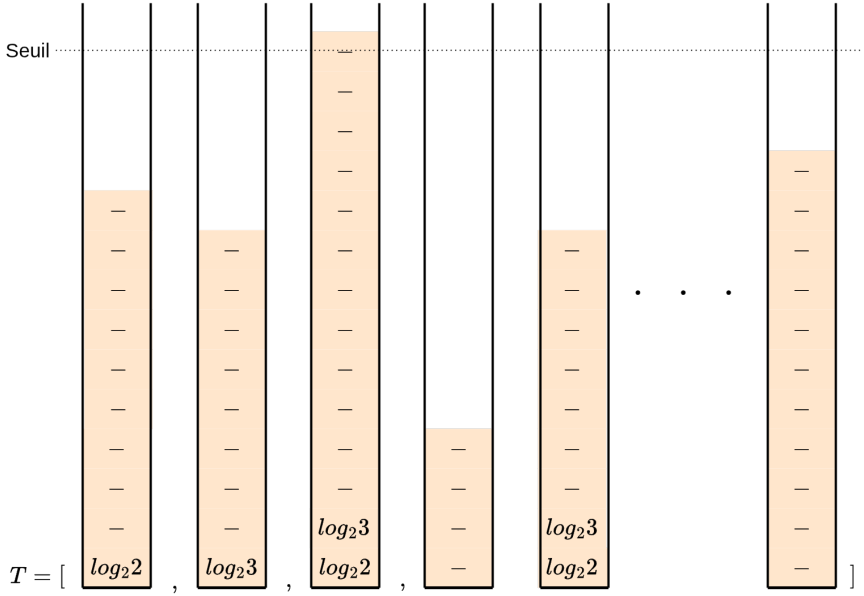
$\boxed{S = 10}$ $\boxed{N = 20382493}$ $\boxed{Q(1) \equiv 0 \pmod{2}}$



$\boxed{S = 10}$ $\boxed{N = 20382493}$ $\boxed{Q(2) \equiv 0 \pmod 3}$ $\boxed{Q(3) \equiv 0 \pmod 3}$



$\boxed{S = 10}$ $\boxed{N = 20382493}$



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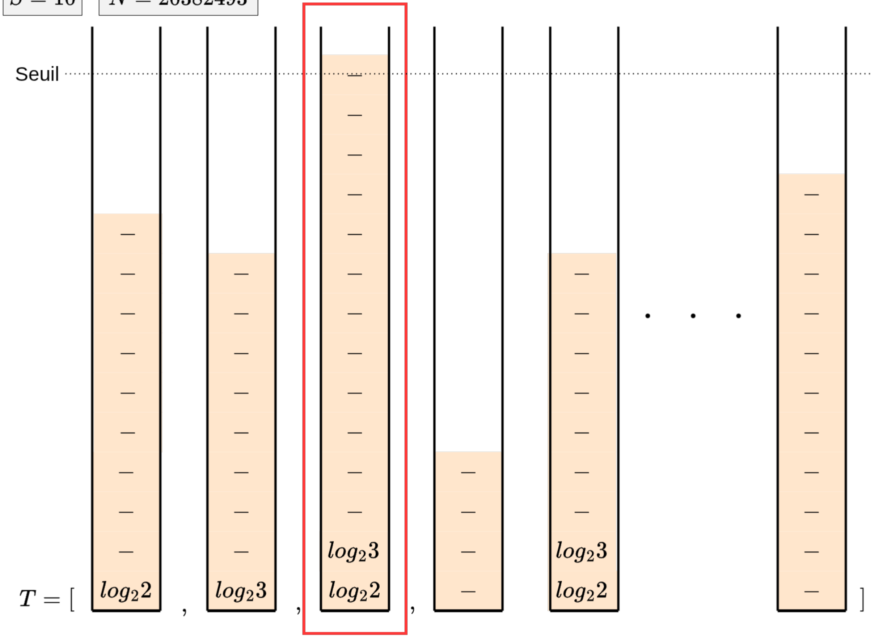


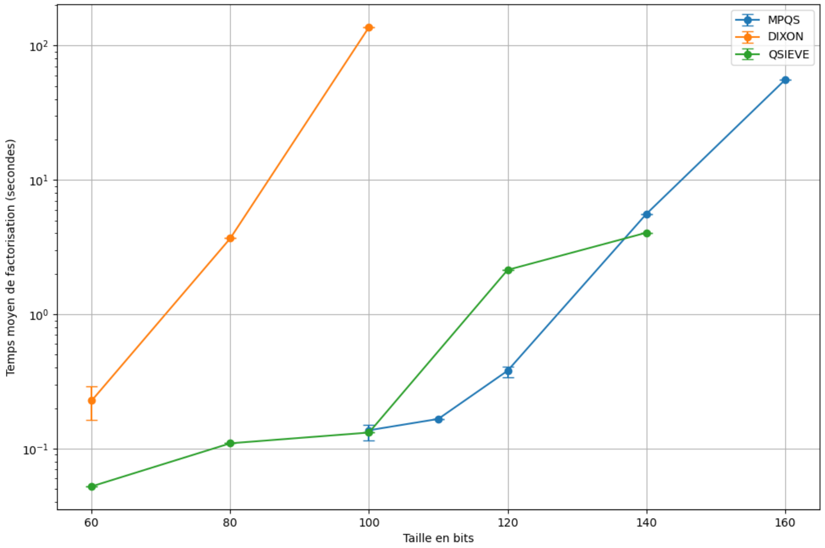
Plan

Résultats

Après plusieurs centaines de tests, on a les résultats suivants:

Bits	Dixon	QSIEVE	MPQS
60	0.5s	0.05s	-
80	5s	0.1s	-
100	100s	0.1s	0.1s
120	-	2s	0.6s
140	-	5s	5s
160	-	-	80s

Graphique final



Plan

Proposition

Soient $b \in \mathbb{N}, (x_i)_{i \in [\![1,b+1]\!]} \in \mathbb{N}^{b+1}$ et $(v_i)_{i \in [\![1,b+1]\!]} \in \mathbb{F}_2^b$ les vecteurs valuations de $x_i^2 \pmod N$ pour $i \in [\![1, b+1]\!]$ et finalement $(\lambda_i)_{i \in [\![1,b+1]\!]} \in \{0,1\}^{b+1}$ tels que,

$$\sum_{i=1}^{b+1} \lambda_i v_i = 0_{\mathbb{F}_2^b} = (2\alpha_1, \dots, 2\alpha_b)$$

On pose $y = \prod_{j=1}^{b} p_j^{\alpha_j}$ et $x = \prod_{j=1}^{b+1} x_j^{\lambda_j}$, alors $x^2 \equiv y^2 \pmod N$

Démonstration

$$\begin{aligned}
x^2 = (\prod_{i=1}^{b+1} x_i^2)^{\lambda_i} &\equiv \prod_{i=1}^{b+1} \prod_{j=1}^{b} p_j^{\lambda_i v_i^{(j)}} \pmod N \\
&\equiv \prod_{j=1}^{b} \prod_{i=1}^{b+1} p_j^{\lambda_i v_i^{(j)}} \pmod N \\
&\equiv \prod_{j=1}^{b} p_j^{\sum_{i=1}^{b+1} \lambda_i v_i^{(j)}} \pmod N \\
&\equiv (\prod_{j=1}^{b} p_j^{\alpha_j})^2 \pmod N \qquad \text{(déf de } \alpha_j\text{)} \\
&\equiv y^2 \pmod N
\end{aligned}$$

Proposition

Si $Q = (\lfloor\sqrt{N}\rfloor + X)^2 - N$, alors
$p \mid Q(x) \implies \forall k \in \mathbb{N}, p \mid Q(x + kp)$

Démonstration

En effet, supposons $p \mid Q(x)$, on a:

$$\begin{aligned} Q(x+kp) &= (\lfloor\sqrt{N}\rfloor + x + kp)^2 - N \\ &= Q(x) + 2kp(\lfloor\sqrt{N}\rfloor + x) + k^2p^2 \\ &= Q(x) + p \times (2k(\lfloor\sqrt{N}\rfloor + x) + k^2p) \end{aligned}$$

d'où $p \mid Q(x + kp)$

../c/vector.h

```
#pragma once
#include <gmp.h>

void mod_vect(int* v, int mod, int n1);
void add_vect(int* sum, int* op, int n1);
void div_vect(int* v, int d, int n1);
void sub_vect(int** v, int i, int j, int n1);
void prod_vect(mpz_t prod, mpz_t* z, int n1,
        system_t s);
```

../c/vector.c

```
#include <gmp.h>
#include <assert.h>
#include <stdlib.h>
#include "system.h"

void mod_vect(int* v, int mod, int n1){
    for(int i = 0; i<n1; i++){
        v[i] = abs(v[i]) % mod;
    }
}

void add_vect(int* sum, int* op, int n1){
    for(int i = 0; i<n1; i++){
        sum[i] += op[i];
    }
}


void div_vect(int* v, int d, int n1){
    for(int i = 0; i<n1; i++){
        assert(v[i]%d == 0);
        v[i] /= d;
    }
}

void sub_vect(int** v, int i, int j, int n1){
    for(int k = 0; k<n1; k++){
        v[i][k] = v[i][k] - v[j][k];
    }
}

void prod_vect(mpz_t prod, mpz_t* z, int n1,
      system_t s){
    mpz_set_ui(prod, 1);
    for(int i = 0; i<n1; i++){
        if(s->sol[i]){
            mpz_mul(prod, prod, z[s->perm[i]]);
        }
    }
}
```

../c/tonellishanks.h

```
#pragma once

#include <gmp.h>

void tonelli_shanks_ui(mpz_t n, int p, int* x1, int* x2
        );
void tonelli_shanks_mpz(mpz_t a, mpz_t p, mpz_t
        x1, mpz_t x2);
```

../c/tonellishanks.c

```c
#include <stdint.h>
#include <gmp.h>
#include <stdio.h>
#include <assert.h>
#include <stdlib.h>

uint64_t modpow(uint64_t a, uint64_t b, uint64_t n)
        {
    uint64_t x = 1, y = a;
    while (b > 0) {
        if (b % 2 == 1) {
            x = (x * y) % n; // multiplying with base
        }
        y = (y * y) % n; // squaring the base
        b /= 2;
    }
    return x % n;
}

void tonelli_shanks_ui(mpz_t n, unsigned long int p,
        int* x1, int* x2) {
    uint64_t q = p - 1;
    uint64_t ss = 0;
    uint64_t z = 2;
    uint64_t c, r, t, m;


    while ((q & 1) == 0) {
        ss += 1;
        q >>= 1;
    }

    mpz_t temp, pj;
    mpz_init(temp);
    mpz_init_set_ui(pj, p);

    if (ss == 1) {
        //uint64_t r1 = modpow(n, (p + 1) / 4, p);
        mpz_powm_ui(temp, n, (p+1)/4, pj);
        uint64_t r1 = mpz_get_ui(temp);

        *x1 = r1;
        *x2 = p - r1;
        mpz_clears(temp, pj, NULL);
        return;
    }

    while (modpow(z, (p - 1) / 2, p) != (unsigned
            long int) p - 1) { // uint_64 only there
            for the compiler to stop complaining
        z++;
    }

    c = modpow(z, q, p);

    //r = modpow(n, (q + 1) / 2, p);
    mpz_powm_ui(temp, n, (q+1)/2, pj);
    r = mpz_get_ui(temp);

    //t = modpow(n, q, p);
    mpz_powm_ui(temp, n, q, pj);
    t = mpz_get_ui(temp);

    m = ss;

    while(1){
        uint64_t i = 0, zz = t;
        uint64_t b = c, e;
        if (t == 1) {
            *x1 = r;
            *x2 = p - r;
            mpz_clears(temp, pj, NULL);
            return;
        }
        while (zz != 1 && i < (m - 1)) {
            zz = zz * zz % p;
            i++;
        }
        e = m - i - 1;
        while (e > 0) {
            b = b * b % p;
            e--;
        }
        r = r * b % p;
        c = b * b % p;
        t = t * c % p;
        m = i;
    }
}

void tonelli_shanks_mpz(mpz_t n, mpz_t p, mpz_t
        x1, mpz_t x2){
    assert(mpz_legendre(n, p) == 1);

    mpz_t q, z;
    mpz_init_set(q, p);
    mpz_sub_ui(q, q, 1);
    int ss = 0;
    mpz_init_set_ui(z, 2);

    while(mpz_divisible_ui_p(q, 2) != 0){
        ss += 1;
        mpz_divexact_ui(q, q, 2);
    }

    mpz_t op1;
    mpz_init(op1);

    if (ss == 1) {
        //uint64_t r1 = modpow(n, (p + 1) / 4, p);
        mpz_add_ui(op1, p, 1);
        mpz_divexact_ui(op1, op1, 4);
        mpz_powm(op1, n, op1, p);

        mpz_set(x1, op1);
        mpz_sub(x2, p, x1);

        mpz_clears(q, z, op1, NULL);
        return;
    }

    mpz_t op2, op3;
    mpz_inits(op2, op3, NULL);

    mpz_sub_ui(op1, p, 1);
    mpz_divexact_ui(op1, op1, 2);
    mpz_powm(op2, z, op1, p);

    mpz_sub_ui(op3, p, 1);
    while(mpz_cmp(op2, op3) != 0){
        mpz_add_ui(z, z, 1);
        mpz_powm(op2, z, op1, p);
    }

    mpz_t c, r, t, m, i, zz, b, e;
    mpz_inits(c, r, t, m, i, zz, b, e, NULL);
    mpz_powm(c, z, q, p);

    mpz_add_ui(op1, q, 1);
    mpz_divexact_ui(op1, op1, 2);
    mpz_powm(r, n, op1, p);

    mpz_powm(t, n, q, p);

    mpz_set_ui(m, ss);

    while(1){
        mpz_set_ui(i, 0);
        mpz_set(zz, t);
        mpz_set(b, c);

        if(mpz_cmp_ui(t, 1) == 0){
            mpz_set(x1, r);
            mpz_sub(x2, p, x1);

            mpz_clears(c, r, t, m, i, zz, b, e, op1, op2
                    , op3, q, z, NULL);
            return;
        }

        mpz_sub_ui(op1, m, 1);
        while(mpz_cmp_ui(zz, 1) != 0 && mpz_cmp
                (i, op1)<0){
            mpz_mul(zz, zz, zz);
            mpz_mod(zz, zz, p);
            mpz_add_ui(i, i, 1);
        }

        mpz_sub(e, m, i);
        mpz_sub_ui(e, e, 1);
        while(mpz_sgn(e)>0){
            mpz_mul(b, b, b);
            mpz_mod(b, b, p);
            mpz_sub_ui(e, e, 1);
        }

        mpz_mul(r, r, b);
        mpz_mod(r, r, p);

        mpz_mul(c, b, b);
        mpz_mod(c, c, p);

        mpz_mul(t, t, c);
        mpz_mod(t, t, p);

        mpz_set(m, i);
    }
}
```

../c/system.h

```
#pragma once
#include <stdbool.h>

typedef struct system {
    int** m;
    int* perm;
    int* sol;
    bool done;
    int n1, n2, arb;
} system_s;

typedef system_s* system_t;

system_t init_gauss(int** v, int n1, int n2);
void gaussian_step(system_t s);
void free_system(system_t s);
```

../c/system.c

```
#include "system.h"
#include "vector.h"
#include "list_matrix_utils.h"
#include <stdlib.h>
#include <stdio.h>
#include <stdbool.h>

void swap_lines_horz(system_t s, int i, int j){
    int* temp = s->m[i];
    s->m[i] = s->m[j];
    s->m[j] = temp;
}

void swap_lines_vert(system_t s, int i, int j){
    int temp = s->perm[i];
    s->perm[i] = s->perm[j];
    s->perm[j] = temp;

    for(int k = 0; k<s->n1; k++){
        int temp = s->m[k][i];
        s->m[k][i] = s->m[k][j];
        s->m[k][j] = temp;
    }
}

int find_index(system_t s, int from, int look){
    for(int i = from; i < s->n1; i++){
        if(s->m[i][look]){
            return i;
        }
    }
    return -1;
}

system_t transpose(int** v, int n1, int n2){
    system_t s = malloc(sizeof(system_s));

    s->m = malloc(n2*sizeof(int*));
    for(int i = 0; i<n2; i++){
        s->m[i] = malloc(n1*sizeof(int));
        for(int j = 0; j<n1; j++){
            s->m[i][j] = v[j][i];
        }
    }

    s->n1 = n2;
    s->n2 = n1;
    return s;
}

void triangulate(system_t s){
    s->perm = malloc(s->n2*sizeof(int));
    for(int i = 0; i<s->n2; i++){
        s->perm[i] = i;
    }

    int i = 0;
    int j = 0;
    while(i<s->n1 && j<s->n2){
        int k = find_index(s, i, j);
        if(k != −1){
            if(i != j){
                swap_lines_vert(s, i, j);
            }

            swap_lines_horz(s, i, k);

            for(int l = i + 1; l < s−>n1; l++){
                if(s−>m[l][i] == 1){
                    sub_vect(s−>m, l, i, s−>n2);
                    mod_vect(s−>m[l], 2, s−>n2);
                }
            }
            i++;
            j = i;
        }
        else{
            j++;
        }
    }
}

void get_arbitary(system_t triangulated){
    for(int i = triangulated−>n1−1; i>=0; i−−){
        int j = 0;
        while(j < triangulated−>n2 && !triangulated
               −>m[i][j]){
            j++;
        }
        if(j<triangulated−>n2){
            triangulated−>arb = j+1;
            return;
        }
    }

    fprintf(stderr, "ERROR:␣All␣vectors␣are␣zero␣in␣
           system\n");
    exit(1);
}

void init_sol(system_t s){
    s−>sol = malloc(s−>n2*sizeof(int));
    for(int i = s−>arb; i<s−>n2; i++){
        s−>sol[i] = 0;
    }
}

void iter_sol(system_t s){
    int i = s−>arb;
    while(i<s−>n2 && (s−>sol[i] == 1)){
        s−>sol[i] = 0;
        i++;
    }
    if(i >= s−>n2){
        s−>done = true;
        return;
    }
    s−>sol[i] = 1;
}

system_t init_gauss(int** v, int n1, int n2){
    //printf("Initial vectors\n");
    //print_ll(v, n1, n2);

    system_t s = transpose(v, n1, n2);
    s−>done = false;
```

```c
    //printf("Transposed\n");
    //print_ll(s->m, s->n1, s->n2);

    for(int i = 0; i<s->n1; i++){
        mod_vect(s->m[i], 2, s->n2);
    }

    //printf("Modded\n");
    //print_ll(s->m, s->n1, s->n2);

    triangulate(s);

    //printf("Triangulated\n");
    //print_ll(s->m, s->n1, s->n2);

    get_arbitary(s);
    init_sol(s);

    return s;
}

void gaussian_step(system_t s){
    iter_sol(s);

    for(int i = s->n1-1; i>=0; i--){
        int j = 0;
        while(j < s->n2 && !s->m[i][j]){
            j++;
        }

        if(j<s->n2){
            s->sol[j] = 0;

            for(int k = s->n2-1; k>j; k--){
                s->sol[j] -= s->m[i][k] * s->sol[k
                        ];
            }
            s->sol[j] = abs(s->sol[j]) % 2;
        }
    }
}

void free_system(system_t s){
    for(int i = 0; i<s->n1; i++){
        free(s->m[i]);
    }
    free(s->m);
    free(s->sol);
    free(s->perm);
    free(s);
}
```

../c/parse_input.h

```
#pragma once
#include <gmp.h>
#include <stdbool.h>

typedef enum {DIXON, QSIEVE, MPQS, PMPQS}
        TYPE;

typedef struct input_s {
    char* output_file;
    int bound, sieving_interval;
    mpz_t N;
    bool quiet;
    TYPE algorithm;
    int extra;
    int delta;
} input_t;

input_t* parse_input(int argc, char** argv);
void free_input(input_t* input);
```

../c/parse_input.c

```c
#include "parse_input.h"
#include <stdlib.h>
#include <string.h>
#include <gmp.h>
#include <stdbool.h>

input_t* init_input(void){
    input_t* input = malloc(sizeof(input_t));
    input->bound = -1;
    input->output_file = NULL;
    input->sieving_interval = -1;
    input->extra = -1;
    input->quiet = false;
    input->algorithm = QSIEVE;
    input->delta = 0;
    mpz_init_set_ui(input->N, 0);
    return input;
}

bool valid_int(char* str){
    int i = 0;
    char c = str[i];
    while(c != '\0'){
        if(c<48 || c>57) return false;
        c = str[++i];
    }

    return true;
}
```

```c
void free_input(input_t* input){
    if(input->output_file) free(input->output_file);
    mpz_clear(input->N);
    free(input);
}

input_t* parse_input(int argc, char** argv){
    input_t* input = init_input();

    int i = 1;
    while(i<argc){
        if(strcmp(argv[i], "-b") == 0 || strcmp(argv[i
            ], "--bound") == 0){
            i++;
            if(i<argc){
                if(valid_int(argv[i])) input->bound
                    = atoi(argv[i]);
                else return NULL;}
            else return NULL;
        }

        else if(strcmp(argv[i], "-s") == 0 || strcmp(
            argv[i], "--sieving_interval") == 0){
            i++;
            if(i<argc){
                if(valid_int(argv[i])) input->
                    sieving_interval = atoi(argv[i])
                    ;
                else return NULL;}
            else return NULL;
    }

    else if(strcmp(argv[i], "-e") == 0 || strcmp(
            argv[i], "--extra") == 0){
        i++;
        if(i<argc){
            if(valid_int(argv[i])) input->extra =
                    atoi(argv[i]);
            else return NULL;}
        else return NULL;
    }

    else if(strcmp(argv[i], "-n") == 0 || strcmp(
            argv[i], "--number") == 0){
        i++;
        if(i<argc){
            if(valid_int(argv[i])) mpz_set_str(
                    input->N, argv[i], 10);
            else return NULL;}
        else return NULL;
    }

    else if(strcmp(argv[i], "-d") == 0 || strcmp(
            argv[i], "--delta") == 0){
        i++;
        if(i<argc){
            if(valid_int(argv[i])) input->delta =
                    atoi(argv[i]);
            else return NULL;}
        else return NULL;
    }

    else if(strcmp(argv[i], "-o") == 0){
        i++;
        if(i<argc) input->output_file = argv[i];
        else return NULL;
    }

    else if(strcmp(argv[i], "-t") == 0 || strcmp(
            argv[i], "--type") == 0){
        i++;
        if(i<argc) {
            if(strcmp(argv[i], "dixon") == 0)
                    input->algorithm = DIXON;
            else if(strcmp(argv[i], "qsieve") ==
                    0) input->algorithm =
                    QSIEVE;
            else if(strcmp(argv[i], "mpqs") == 0)
                     input->algorithm = MPQS;
            else if(strcmp(argv[i], "pmpqs") ==
                    0) input->algorithm =
                    PMPQS;
            else return NULL;}
        else return NULL;
    }

    else if(strcmp(argv[i], "-q") == 0 ||
            strcmp(argv[i], "-stfu") == 0 /*
                    easter egg*/ ||
            strcmp(argv[i], "--quiet") == 0){
        input->quiet = true;
    }

    else return NULL;

    i++;
```

```
}                                   }

return input;
```

../c/list_matrix_utils.h

```c
#pragma once

void print_list(int* l, int n);
void print_ll(int** ll, int n1, int n2);
void free_ll(int** m, int n1);
```

../c/list_matrix_utils.c

```c
#include <stdio.h>
#include <stdlib.h>

void print_list(int* l, int n){
    for(int i = 0; i<n; i++){
        printf("%d␣", l[i]);
    }
    printf("\n");
}

void print_ll(int** ll, int n1, int n2){
    for(int i = 0; i<n1; i++){
        print_list(ll[i], n2);
    }
    printf("\n");
}

void free_ll(int** m, int n1){
    for(int i = 0; i<n1; i++){
        free(m[i]);
    }
    free(m);
}
```

../c/factorbase.h

```c
#pragma once
#include <gmp.h>

// bruh
bool is_prime(int n);

// calculates pi(n), the number of prime numbers <=
        n
int pi(int n);

// returns a list of piB first primes
int* primes(int piB, int B);

/** Reduces the factor base of the algorithm, refer to:
 * Quadratic sieve factorisation algorithm
 * Bc. OndĚĞrej Vladyka
 * Section 2.3.1 (p.16)
*/
int* prime_base(mpz_t n, int* pb_len, int* primes,
        int piB);
```

../c/factorbase.c

```c
#include <stdbool.h>
#include <gmp.h>
#include <stdlib.h>

bool is_prime(int n) {
    // Corner cases
    if (n <= 1)
        return false;
    if (n <= 3)
        return true;

    // This is checked so that we can skip
    // middle five numbers in below loop
    if (n % 2 == 0 || n % 3 == 0)
        return false;

    for (int i = 5; i * i <= n; i = i + 6)
        if (n % i == 0 || n % (i + 2) == 0)
            return false;

    return true;
}

int pi(int n) {
    int k = 0;
    for (int i = 2; i <= n; i++) {
        if (is_prime(i)) k++;
    }
    return k;
}

int* primes(int piB, int B){
    int* p = malloc(piB*sizeof(int));
    int k = 0;
    for (int i = 2; i <= B; i++) {
        if (is_prime(i)){
            p[k] = i;
            k++;
        }
    }
    return p;

}

/* Used for legendre symbol, exists in gmp already
bool euler_criterion(mpz_t n, int p){
    int e = (p-1)/2;
    mpz_t r, p1;
    mpz_init(r);
    mpz_init_set_ui(p1, p);
    mpz_powm_ui(r, n, e, p1);
    return(mpz_cmp_ui(r, 1) == 0);
}
*/

int* prime_base(mpz_t n, int* pb_len, int* primes,
       int piB){

    int* pb = malloc(piB*sizeof(int));
    pb[0] = 2;
    int j = 1;
    mpz_t p1;
    mpz_init(p1);
    for(int i = 1; i<piB; i++){
        mpz_set_ui(p1, primes[i]);
        if(mpz_legendre(n, p1) == 1){
            //printf("%d\n", primes[i]);
            pb[j] = primes[i];
            j++;
        }
    }
    *pb_len = j;
    pb = realloc(pb, (j+1)*sizeof(int)); // +1 used
            for mpqs

    mpz_clear(p1);
    return pb;
}
```

```
#include <stdbool.h>
#include <gmp.h>
#include <sys/time.h>
#include <stdio.h>
#include <stdlib.h>
#include <assert.h>
#include "system.h"
#include "vector.h"
#include "parse_input.h"
#include "factorbase.h"
#include "list_matrix_utils.h"

// Include algorithms
// Dixon's method
#include "./dixon/dixon.h"

// The Quadratic Sieve
#include "./qsieve/qsieve.h"

// Multipolynomial Quadratic Sieve
#include "./mpqs/polynomial.h"
#include "./mpqs/mpqs.h"
#include "./mpqs/parallel_mpqs.h"


/**
 *
 *
 * START OF ALGORITHM
 *
 */


void rebuild_mpqs(mpz_t prod, mpz_t* d, int* v, int*
        primes, int n1, system_t s){
    mpz_set_ui(prod, 1);
    mpz_t temp;
    mpz_init(temp);
    for(int i = 0; i<n1; i++){
        if(s->sol[i]){
            mpz_mul(prod, prod, d[s->perm[i]]);
        }
        mpz_ui_pow_ui(temp, primes[i], v[i]);
        mpz_mul(prod, prod, temp);
    }
    mpz_clear(temp);
}

void rebuild(mpz_t prod, int* v, int* primes, int n1){
    /** Rebuilds the product of primes to the power of
            half
     * the solution found by the gaussian solve

     * EX:
     * v = (1, 2, 3, 1)
     * primes = [2, 3, 5, 7]
     * prod = 2**1 * 3** 2 * 5**3 * 7**1
     * returns prod
     *
    */
    mpz_set_ui(prod, 1);
    mpz_t temp;
    mpz_init(temp);
    for(int i = 0; i<n1; i++){
        mpz_ui_pow_ui(temp, primes[i], v[i]);
        mpz_mul(prod, prod, temp);
    }
    mpz_clear(temp);
}

void sum_lignes(int* sum, int** v, system_t s){
    /** Sums the lines of vectors into 'sum' according
            the solution of the
     * output of the system 's', such that each power
            is even
     */
    for(int i = 0; i<s->n1; i++){
        sum[i] = 0;
    }

    for(int i = 0; i<s->n2; i++){
        if(s->sol[i]){
            add_vect(sum, v[s->perm[i]], s->n1);
        }
    }
}

void factor(input_t* input){
    int piB = pi(input->bound);
    if(!input->quiet) printf("pi(B)␣=␣%d\n", piB);
    int* p = primes(piB, input->bound);
    int pb_len;
    int* pb;
    switch(input->algorithm){
        case DIXON:
            pb = p;
            pb_len = piB;
            break;
        case QSIEVE:
            pb = prime_base(input->N, &pb_len, p,
                    piB);
            if(!input->quiet) printf("base␣reduction␣
                    %f%%\n", (float)pb_len/piB
                    *100);
            free(p);
            break;
        case MPQS:
            pb = prime_base(input->N, &pb_len, p,
                    piB);
            pb[pb_len] = -1;
            if(!input->quiet) printf("base␣reduction␣
                    %f%%\n", (float)pb_len/piB
                    *100);
            free(p);
            break;
        case PMPQS:
            pb = prime_base(input->N, &pb_len, p,
                    piB);
            pb[pb_len] = -1;
            if(!input->quiet) printf("base␣reduction␣
                    %f%%\n", (float)pb_len/piB
                    *100);
            free(p);
            break;
```

```
}
int target_nb = pb_len + input->extra;

mpz_t* z = malloc((target_nb)*sizeof(mpz_t));
for(int i = 0; i < target_nb; i++){
    mpz_init(z[i]);
}

//Getting zis
int** v;
mpz_t* d;
struct timeval t1, t2;
gettimeofday(&t1, 0);
switch(input->algorithm){
    case DIXON:
        v = dixon(z, input->N, pb_len, pb, input
                ->extra, input->quiet);
        break;
    case QSIEVE:
        v = qsieve(z, input->N, pb_len, pb,
                input->extra, input->
                sieving_interval, input->quiet);
        break;
    case MPQS:
        d = malloc(target_nb*sizeof(mpz_t));
        for(int i = 0; i < target_nb; i++){
            mpz_init(d[i]);
        }
        v = mpqs(z, d, input->N, pb_len, pb,
                input->extra, input->
                sieving_interval, input->delta,
                input->quiet);
        break;
    case PMPQS:
        d = malloc(target_nb*sizeof(mpz_t));
        for(int i = 0; i < target_nb; i++){
            mpz_init(d[i]);
        }
        v = parallel_mpqs(z, d, input->N,
                pb_len, pb, input->extra, input
                ->sieving_interval, input->delta,
                 input->quiet);
        break;
}

gettimeofday(&t2, 0);
long seconds = t2.tv_sec - t1.tv_sec;
long microseconds = t2.tv_usec - t1.tv_usec;
double time_spent = seconds + microseconds*1e
        -6;
if(!input->quiet) printf("Time to get zi: %fs\n",
        time_spent);

mpz_t f, Z1, Z2, test1, test2;
mpz_inits(f, Z1, Z2, test1, test2, NULL);

//gaussian init
system_t s;
int* sum;
switch(input->algorithm){
    case DIXON:
        s = init_gauss(v, target_nb, pb_len);
        sum = malloc(pb_len*sizeof(int));
        break;
    case QSIEVE:
        s = init_gauss(v, target_nb, pb_len);
        sum = malloc(pb_len*sizeof(int));
        break;
    case MPQS:
        // for -1
        s = init_gauss(v, target_nb, pb_len+1);
        sum = malloc((pb_len+1)*sizeof(int));
        break;
    case PMPQS:
        // for -1
        s = init_gauss(v, target_nb, pb_len+1);
        sum = malloc((pb_len+1)*sizeof(int));
        break;
}
if(!input->quiet) printf("2^%d␣solutions␣to␣
       iterate\n", s->n2 - s->arb);

bool done = false;
while(!done){
    gaussian_step(s);

    prod_vect(Z1, z, target_nb, s);
    sum_lignes(sum, v, s);
    div_vect(sum, 2, pb_len);

    switch(input->algorithm){
        case DIXON:
            rebuild(Z2, sum, pb, pb_len);
            break;
        case QSIEVE:
            rebuild(Z2, sum, pb, pb_len);
            break;
        case MPQS:
            rebuild_mpqs(Z2, d, sum, pb, pb_len,
                    s);
            break;
        case PMPQS:
            rebuild_mpqs(Z2, d, sum, pb, pb_len,
                    s);
            break;
    }

    // TEST
    mpz_set(test1, Z1);
    mpz_mul(test1, test1, test1);
    mpz_set(test2, Z2);
    mpz_mul(test2, test2, test2);
    assert(mpz_congruent_p(test1, test2, input
           ->N) != 0);
    // END TEST

    mpz_sub(f, Z1, Z2);
    mpz_gcd(f, f, input->N);

    if(mpz_cmp_ui(f, 1) != 0 && mpz_cmp(f,
           input->N) != 0){
        assert(mpz_divisible_p(input->N, f));
        if(!input->quiet) gmp_printf("%Zd␣=␣0
               ␣[%Zd]\n", input->N, f);
        done = true;
    }

    mpz_add(f, Z1, Z2);
    mpz_gcd(f, f, input->N);

    if(mpz_cmp_ui(f, 1) != 0 && mpz_cmp(f,
           input->N) != 0){
        assert(mpz_divisible_p(input->N, f));
        if(!input->quiet) gmp_printf("%Zd␣=␣0
                ␣[%Zd]\n", input->N, f);
        done = true;
    }

    if(s->done){
        if(!input->quiet) fprintf(stderr, "ERROR:
                ␣no␣solution␣for␣this␣set␣of␣zi\n"
                );
        exit(1);
    }
}

free(sum);
free(pb);
free_system(s);
free_ll(v, target_nb);
for(int i = 0; i < target_nb; i++){
    mpz_clear(z[i]);
}
free(z);
switch(input->algorithm){
    case DIXON:
        break;
    case QSIEVE:
        break;
    case MPQS:
        for(int i = 0; i < target_nb; i++)
                mpz_clear(d[i]);
        free(d);
        break;
    case PMPQS:
```

```
            for(int i = 0; i < target_nb; i++)
                    mpz_clear(d[i]);
            free(d);
            break;
    }


    mpz_clears(f, Z1, Z2, test1, test2, NULL);
}

int main(int argc, char** argv){
    input_t* input = parse_input(argc, argv);
    if(input==NULL){
        fprintf(stderr, "ERROR:␣Invalid␣input\n");
        return 1;
    }

    if(mpz_cmp_ui(input->N, 0) == 0){
        fprintf(stderr, "ERROR:␣No␣input␣number,␣
                use␣-n␣%%number%%\n");
        return 1;
    }

    if(input->bound == -1) input->bound =
            10000;
    if(input->sieving_interval == -1) input->
            sieving_interval = 100000;
    if(input->extra == -1) input->extra = 1;

    struct timeval t1, t2;
    gettimeofday(&t1, 0);
    factor(input);
    gettimeofday(&t2, 0);
```

```
long seconds = t2.tv_sec - t1.tv_sec;
long microseconds = t2.tv_usec - t1.tv_usec;
double time_spent = seconds + microseconds*1e
        -6;
if(!input->quiet) printf("Total␣time:␣%fs\n",
        time_spent);

    free_input(input);

    return 0;
}
```

../c/dixon/dixon.h

```c
#pragma once

int** dixon(mpz_t* z, mpz_t N, int pb_len, int* pb,
            int extra, bool tests);
```

../c/dixon/dixon.c

```
#include <gmp.h>
#include <stdbool.h>
#include <stdio.h>
#include <stdlib.h>

bool vectorize_dixon(mpz_t n, int∗ v, int pb_len, int∗
        pb){
    /∗∗ Attemps naive factorisation to 'n' with the
            primes in
     ∗ the prime base 'pb' and putting the result into '
            v', vector of powers of
     ∗ the primes in the prime base
     ∗ If it succeeds, returns true, otherwise, returns
            false
    ∗/
    for(int i = 0; i<pb_len; i++){
        v[i] = 0;
    }

    for(int i = 0; i<pb_len && (mpz_cmp_ui(n, 1)
            != 0); i++){
        while (mpz_divisible_ui_p(n, pb[i])){
            v[i]++;
            mpz_divexact_ui(n, n, pb[i]);
        }
    }

    if(mpz_cmp_ui(n, 1) == 0)
        return true;
    return false;
}

int∗∗ dixon(mpz_t∗ z, mpz_t N, int pb_len, int∗ pb,
        int extra, bool tests){
    /∗∗ Gets pb_len+extra b−smooth realtions
            definied at:
     ∗ Quadratic sieve factorisation algorithm
     ∗ Bc. OndĚĞrej Vladyka
     ∗ Definition 1.11 (p.5)
     ∗/

    //ceil(sqrt(n))
    mpz_t sqrt_N;
    mpz_init(sqrt_N);
    mpz_sqrt(sqrt_N, N);
    mpz_add_ui(sqrt_N, sqrt_N, 1);

    mpz_t zi;
    mpz_t zi_cpy;
    mpz_init_set(zi, sqrt_N);
    mpz_init(zi_cpy);

    int∗∗ v = malloc((pb_len+extra)∗sizeof(int∗));

    for(int i = 0; i < pb_len+extra; i++){
        bool found = false;
        int∗ vi = malloc(pb_len∗sizeof(int));

        while(!found){
```

```
        mpz_add_ui(zi, zi, 1);
        mpz_mul(zi_cpy, zi, zi);
        mpz_mod(zi_cpy, zi_cpy, N);

        found = vectorize_dixon(zi_cpy, vi,
                pb_len, pb);
    }
    if(!tests){
        printf("\r");
        printf("%.1f%%", (float)i/(pb_len+extra
                -1)*100);
        fflush(stdout);
    }
```

```
            v[i] = vi;
            mpz_set(z[i], zi);
        }
        if(!tests) printf("\n");

        mpz_clears(sqrt_N, zi, zi_cpy, NULL);


        return v;
    }
```

../c/qsieve/qsieve.h

```c
#pragma once
#include <gmp.h>
#include <stdbool.h>

bool vectorize_qsieve(mpz_t n, int* v, int pb_len, int*
        pb);
int** qsieve(mpz_t* z, mpz_t N, int pb_len, int* pb,
        int extra, int s, bool tests);
```

```c
#include <gmp.h>
#include <stdbool.h>
#include <stdio.h>
#include <stdlib.h>
#include <assert.h>
#include <math.h>

#include "../system.h"
#include "../tonellishanks.h"

bool vectorize_qsieve(mpz_t n, int* v, int pb_len, int*
        pb){
    /** Attemps naive factorisation to 'n' with the
            primes in
     * the prime base 'pb' and putting the result into '
            v', vector of powers of
     * the primes in the prime base
     * If it succeeds, returns true, otherwise, returns
            false
    */
    for(int i = 0; i<pb_len; i++){
        v[i] = 0;
    }

    for(int i = 0; i<pb_len && (mpz_cmp_ui(n, 1)
            != 0); i++){
        while (mpz_divisible_ui_p(n, pb[i])){
            v[i]++;
            mpz_divexact_ui(n, n, pb[i]);
        }
    }

    if(mpz_cmp_ui(n, 1) == 0)
        return true;
    return false;
}

float* prime_logs(int* pb, int pb_len){
    float* plogs = malloc(pb_len*sizeof(float));

    for(int i = 0; i<pb_len; i++){
        plogs[i] = log2(pb[i]);
    }

    return plogs;
}

int calculate_threshhold(mpz_t N, mpz_t sqrt_N, int
        s, int loop_number, int* pb, int pb_len){

    mpz_t qstart;
    mpz_init_set_ui(qstart, s);
    mpz_mul_ui(qstart, qstart, loop_number);
    mpz_add(qstart, qstart, sqrt_N);
    mpz_mul(qstart, qstart, qstart);
    mpz_sub(qstart, qstart, N);

    int t = mpz_sizeinbase(qstart, 2) - (int) log2(pb[
            pb_len-1]);
    mpz_clear(qstart);
    return t;
}

int** qsieve(mpz_t* z, mpz_t N, int pb_len, int* pb,
        int extra, int s, bool quiet){
    /** Gets pb_len+extra zis that are b-smooth,
            definied at:
     * Quadratic sieve factorisation algorithm
     * Bc. OndËĞrej Vladyka
     * Definition 1.11 (p.5)
     */

    //ceil(sqrt(n))
    mpz_t sqrt_N;
    mpz_init(sqrt_N);
    mpz_sqrt(sqrt_N, N);
    mpz_add_ui(sqrt_N, sqrt_N, 1);

    mpz_t zi;
    mpz_init_set(zi, sqrt_N);
    mpz_t qx;
    mpz_init(qx);

    int** v = malloc((pb_len+extra)*sizeof(int*));
    for(int i = 0; i<pb_len+extra; i++){
        v[i] = malloc(pb_len*sizeof(int*));
    }
    float* sinterval = malloc(s*sizeof(float));
    float* plogs = prime_logs(pb, pb_len);


    // TESTS
    mpz_t temp;
    mpz_init(temp);
    // END TESTS


    int* x1 = malloc(pb_len*sizeof(int));
    int* x2 = malloc(pb_len*sizeof(int));

    // find solution for 2
    mpz_set(temp, sqrt_N);
    mpz_mul(temp, temp, temp);
    mpz_sub(temp, temp, N);
    x1[0] = 0;
    if(mpz_divisible_ui_p(temp, 2) == 0) x1[0] = 1;

    int sol1, sol2;
    for(int i = 1; i < pb_len; i++){

            tonelli_shanks_ui(N, pb[i], &sol1, &sol2);
            x1[i] = sol1;
            x2[i] = sol2;

            // change solution from xÂš = n [p] to (
                    sqrt(N) + x)Âš = n [p]
            mpz_set_ui(temp, x1[i]);
            mpz_sub(temp, temp, sqrt_N);
            mpz_mod_ui(temp, temp, pb[i]);

            x1[i] = mpz_get_ui(temp);

            mpz_set_ui(temp, x2[i]);
            mpz_sub(temp, temp, sqrt_N);
            mpz_mod_ui(temp, temp, pb[i]);
            x2[i] = mpz_get_ui(temp);
    }
    mpz_clear(temp);

    int loop_number = 0;
    int relations_found = 0;
    int tries = 0;
    while(relations_found < pb_len + extra){

        for(int i = 0; i<s; i++){
            sinterval[i] = 0;
        }

        // sieve for 2
        while(x1[0]<s){
            sinterval[x1[0]] += plogs[0];
            x1[0] += pb[0];
        }
        x1[0] = x1[0] - s;

        // sieve other primes
        for(int i = 1; i < pb_len; i++){

            while(x1[i]<s){
                sinterval[x1[i]] += plogs[i];
                x1[i] += pb[i];
            }

            while(x2[i]<s){

                sinterval[x2[i]] += plogs[i];
                x2[i] += pb[i];
            }

            //next interval
            x1[i] = x1[i] - s;
            x2[i] = x2[i] - s;
        }

        int t = calculate_threshhold(N, sqrt_N, s,
               loop_number, pb, pb_len);
        //printf("t = %d\n", t);

        bool found;
        for(int i = 0; i<s && relations_found <
               pb_len + extra; i++){
            if(sinterval[i] > t){
                tries++;

                // zi = sqrt(n) + x where x = s*
                       loopnumber + i
                mpz_set_ui(zi, s);
                mpz_mul_ui(zi, zi, loop_number);
                mpz_add_ui(zi, zi, i);
                mpz_add(zi, zi, sqrt_N);

                // qx = zi**2 - N
                mpz_mul(qx, zi, zi);
                mpz_sub(qx, qx, N);

                found = vectorize_qsieve(qx, v[
                       relations_found], pb_len, pb);

                if(found){
                    mpz_set(z[relations_found], zi);
                    relations_found++;
                found = false;
                if(!quiet){
                    printf("\r");
                    printf("%.1f%%␣|␣%.1f%%",
                             (float)
                            relations_found/(
                            pb_len+extra)*100, (
                            float)relations_found/
                            tries*100);
                    fflush(stdout);
                }
            }
        }
    }
```

```
        loop_number++;
    }

    if(!quiet) printf("\n");

    mpz_clears(sqrt_N, zi, qx, NULL);
    free(x1);
    free(x2);
    free(sinterval);
    free(plogs);

    return v;
}
```

../c/mpqs/common_mpqs.h

```c
#pragma once
#include <gmp.h>
#include <stdbool.h>

int calculate_threshhold_mpqs(mpz_t sqrt_N, int s,
        int* pb, int pb_len, int delta);
float* prime_logs_mpqs(int* pb, int pb_len);
bool vectorize_mpqs(mpz_t n, int* v, int pb_len, int*
         pb);
bool already_added(mpz_t zi, mpz_t* z, int
        relations_found);
```

../c/mpqs/common_mpqs.c

```
#include <gmp.h>
#include <stdbool.h>
#include <math.h>
#include <stdlib.h>
#include <stdio.h>

int calculate_threshhold_mpqs(mpz_t sqrt_N, int s,
        int* pb, int pb_len, int delta){

    mpz_t qstart;
    mpz_init_set_ui(qstart, s);
    mpz_mul(qstart, qstart, sqrt_N);

    int t = mpz_sizeinbase(qstart, 2) - (int) log2(pb[
            pb_len-1]) - delta;
    mpz_clear(qstart);
    return t;
}

float* prime_logs_mpqs(int* pb, int pb_len){
    float* plogs = malloc(pb_len*sizeof(float));

    for(int i = 0; i<pb_len; i++){
        plogs[i] = log2(pb[i]);
    }

    return plogs;
}

bool vectorize_mpqs(mpz_t n, int* v, int pb_len, int*
    pb){
/** Attemps naive factorisation to 'n' with the
        primes in
 * the prime base 'pb' and putting the result into '
        v', vector of powers of
 * the primes in the prime base
 * If it succeeds, returns true, otherwise, returns
        false
*/
for(int i = 0; i<pb_len; i++){
    v[i] = 0;
}
if(mpz_sgn(n)<0){
    v[pb_len] = 1;
    mpz_neg(n, n);
}
else{
    v[pb_len] = 0;
}

for(int i = 0; i<pb_len && (mpz_cmp_ui(n, 1)
        != 0); i++){
    while (mpz_divisible_ui_p(n, pb[i])){
        v[i]++;
        mpz_divexact_ui(n, n, pb[i]);
    }
}

if(mpz_cmp_ui(n, 1) == 0)
    return true;
    return false;
}

bool already_added(mpz_t zi, mpz_t* z, int
        relations_found){
    for(int i = 0; i<relations_found; i++){
        if(mpz_cmp(zi, z[i]) == 0){
            return true;
        }
    }
    return false;
}
```

../c/mpqs/polynomial.h

```
#pragma once
#include <gmp.h>
#include <stdbool.h>

struct poly_s {
    mpz_t d;
    mpz_t N;

    mpz_t a;
    mpz_t b;
    mpz_t c;

    mpz_t zi;
    mpz_t qx;

    // used to make operations without declaring and
        freeing everytime
    mpz_t op1, op2, op3;
};

typedef struct poly_s* poly_t;

void get_next_poly(poly_t p);
poly_t init_poly(mpz_t N, int M);
void calc_poly(poly_t p, mpz_t x);
poly_t copy_poly(poly_t p);
void free_poly(poly_t p);
```

../c/mpqs/polynomial.c

```
#include "polynomial.h"
#include <gmp.h>
#include <stdlib.h>
#include <assert.h>
#include <stdio.h>

#include "../tonellishanks.h"

void calc_coefficients(poly_t p){
    mpz_mul(p->a, p->d, p->d);

    mpz_t x1, x2;
    mpz_inits(x1, x2, NULL);
    tonelli_shanks_mpz(p->N, p->d, x1, x2);

    // getting ready for congruence solve for raising
        solution
    mpz_mul_ui(p->op1, x1, 2);

    mpz_mul(p->op2, x1, x1);
    mpz_sub(p->op2, p->op2, p->N);
    mpz_divexact(p->op2, p->op2, p->d);
    mpz_neg(p->op2, p->op2);
    mpz_mod(p->op2, p->op2, p->d);

    mpz_t g, n, m;
    mpz_inits(g, n, m, NULL);
    mpz_gcdext(g, n, m, p->d, p->op1);
    assert(mpz_cmp_ui(g, 1) == 0);
    mpz_mul(p->op1, p->op2, m); // t
    mpz_clears(g, n, m, NULL);

    mpz_set(p->b, p->d);
    mpz_mul(p->b, p->b, p->op1);
    mpz_add(p->b, p->b, x1);

    mpz_mul(p->op1, p->b, p->b);
    assert(mpz_congruent_p(p->op1, p->N, p->a)
        != 0);

    mpz_sub(p->c, p->op1, p->N);
    mpz_divexact(p->c, p->c, p->a);

    mpz_clears(x1, x2, NULL);
}

void get_next_poly(poly_t p){
    mpz_nextprime(p->d, p->d);
    while(mpz_legendre(p->N, p->d) != 1){
        mpz_nextprime(p->d, p->d);
    }
    calc_coefficients(p);
}

poly_t init_poly(mpz_t N, int M){
    poly_t p = malloc(sizeof(struct poly_s));

    mpz_inits(p->d, p->N, p->a, p->b, p->c, p
        ->op1, p->op2, p->op3, p->zi, p->
        qx, NULL);
    mpz_set(p->N, N);

    // choose value of d according to 2.4.2
    // sqrt( (sqrt(2N))/M )
    mpz_mul_ui(p->op1, N, 2);
    mpz_sqrt(p->op1, p->op1);
    mpz_div_ui(p->op1, p->op1, M);
    mpz_sqrt(p->op1, p->op1);
    mpz_prevprime(p->d, p->op1);

    // get next prime such that (n/p) = 1
    while(mpz_legendre(N, p->d) != 1){
        mpz_nextprime(p->d, p->d);
    }

    calc_coefficients(p);
    return p;
}

void calc_poly(poly_t p, mpz_t x){
    mpz_mul(p->zi, p->a, x);
    mpz_add(p->zi, p->zi, p->b);

    mpz_mul(p->qx, x, x);
    mpz_mul(p->qx, p->qx, p->a);

    mpz_mul(p->op1, p->b, x);
    mpz_mul_ui(p->op1, p->op1, 2);
    mpz_add(p->qx, p->qx, p->op1);

    mpz_add(p->qx, p->qx, p->c);

}

void free_poly(poly_t p){
    mpz_clears(p->d, p->N, p->a, p->b, p->c,
            p->op1, p->op2, p->op3, p->zi, p->
            qx, NULL);
    free(p);
}

poly_t copy_poly(poly_t p){
    poly_t cpy = malloc(sizeof(struct poly_s));

    mpz_inits(cpy->d, cpy->N, cpy->a, cpy->b,
            cpy->c, cpy->op1, cpy->op2, cpy->
            op3, cpy->zi, cpy->qx, NULL);

    mpz_set(cpy->d, p->d);
    mpz_set(cpy->N, p->N);

    mpz_set(cpy->a, p->a);
    mpz_set(cpy->b, p->b);
    mpz_set(cpy->c, p->c);

    return cpy;
}
```

../c/mpqs/mpqs.h

```c
#pragma once

#include <gmp.h>
#include <stdbool.h>

int** mpqs(mpz_t* z, mpz_t* d, mpz_t N, int pb_len
        , int* pb, int extra, int s, int delta, bool quiet);
```

../c/mpqs/mpqs.c

```c
#include <gmp.h>
#include <stdbool.h>
#include <stdio.h>
#include <stdlib.h>
#include <assert.h>
#include <math.h>
#include <time.h>

#include "polynomial.h"
#include "common_mpqs.h"
#include "../system.h"
#include "../tonellishanks.h"

int** mpqs(mpz_t* z, mpz_t* d, mpz_t N, int pb_len
        , int* pb, int extra, int s, int delta, bool quiet){
    /** Gets pb_len+extra zis that are b-smooth,
            definied at:
     * Quadratic sieve factorisation algorithm
     * Bc. OndĚĞrej Vladyka
     * Definition 1.11 (p.5)
     */

    //ceil(sqrt(n))
    mpz_t sqrt_N;
    mpz_init(sqrt_N);
    mpz_sqrt(sqrt_N, N);
    mpz_add_ui(sqrt_N, sqrt_N, 1);

    mpz_t x;
    mpz_init(x);
    poly_t Q = init_poly(N, s);

    int** v = malloc((pb_len+extra)*sizeof(int*));
    for(int i = 0; i<pb_len+extra; i++){
        v[i] = malloc((pb_len+1)*sizeof(int*)); //
                +1 for -1
    }
    float* sinterval = malloc(2*s*sizeof(float));
    float* plogs = prime_logs_mpqs(pb, pb_len);
    int t = calculate_threshhold_mpqs(sqrt_N, s, pb,
            pb_len, delta);


    // TESTS
    mpz_t temp;
    mpz_init(temp);
    // END TESTS


    int* r = malloc(pb_len*sizeof(int));
    int* x1 = malloc(pb_len*sizeof(int));
    int* x2 = malloc(pb_len*sizeof(int));

    int sol1, sol2;
    for(int i = 1; i < pb_len; i++){
        tonelli_shanks_ui(N, pb[i], &sol1, &sol2);
        r[i] = sol1;
    }
```

```
mpz_t g, m, n, pi;
mpz_inits(g, m, n, pi, NULL);

int relations_found = 0;
clock_t start;
start = clock();
int tries = 0;
while(relations_found < pb_len + extra){

    // for 2
    mpz_set_ui(temp, 0);
    calc_poly(Q, temp);
    x1[0] = 0;
    if(mpz_divisible_ui_p(Q->qx, 2) == 0) x1
            [0] = 1;

    //others
    for(int i = 1; i<pb_len; i++){
        mpz_set_ui(pi, pb[i]);
        mpz_gcdext(g, m, n, Q->a, pi);
        if(mpz_cmp_ui(g, 1) != 0){
            fprintf(stderr, "ERROR: Number is 
                    too small for the current 
                    implementation of MPQS\n")
                    ;
            exit(1);
        }

        mpz_set_ui(temp, r[i]);
        mpz_sub(temp, temp, Q->b);
        mpz_mul(temp, temp, m);
        mpz_mod(temp, temp, pi);

        x1[i] = mpz_get_ui(temp);

        //calc_poly(Q, temp);
        //assert(mpz_divisible_ui_p(Q->qx, pb[i
                ]) != 0);

        mpz_set_ui(temp, pb[i]);
        mpz_sub_ui(temp, temp, r[i]);
        mpz_sub(temp, temp, Q->b);
        mpz_mul(temp, temp, m);
        mpz_mod(temp, temp, pi);

        x2[i] = mpz_get_ui(temp);

        //calc_poly(Q, temp);
        //assert(mpz_divisible_ui_p(Q->qx, pb[i
                ]) != 0);


        //realign sieving interval to [-s, s]
        int k = (x1[i] + s)/pb[i];
        x1[i] -= k * pb[i];
        x1[i] += s;

        k = (x2[i] + s)/pb[i];
        x2[i] -= k * pb[i];
        x2[i] += s;

        //mpz_set_si(temp, -s);
        //mpz_add_ui(temp, temp, x1[i]);
        //calc_poly(Q, temp);
        //assert(mpz_divisible_ui_p(Q->qx, pb[i
                ]) != 0);
```

```
}

for(int i = 0; i<2*s; i++){
     sinterval[i] = 0;
}

/*
// sieve for 2
while(x1[0]<2*s){
     sinterval[x1[0]] += plogs[0];
     x1[0] += pb[0];
}
*/

// sieve other primes
for(int i = 30; i < pb_len; i++){

     while(x1[i]<2*s){
          sinterval[x1[i]] += plogs[i];
          x1[i] += pb[i];
     }

     while(x2[i]<2*s){
          sinterval[x2[i]] += plogs[i];
          x2[i] += pb[i];
     }
}


bool found;
bool update_time = false;
for(int i = 0; i<2*s && relations_found <
        pb_len + extra; i++){
```

```
if(sinterval[i] > t){
     tries++;
     mpz_set_si(x, -s);
     mpz_add_ui(x, x, i);
     calc_poly(Q, x);

     if(!already_added(Q->zi, z,
               relations_found)){
          found = vectorize_mpqs(Q->qx,
                    v[relations_found],
                   pb_len, pb);
          if(found){
               mpz_set(z[relations_found],
                        Q->zi);
               mpz_set(d[relations_found],
                        Q->d);
               relations_found++;
               update_time = true;
               found = false;
               if(!quiet){
                    printf("\r");
                    printf("%.1f%%␣|␣%.1f
                            %%", (float)
                            relations_found/(
                            pb_len+extra)
                            *100, (float)
                            relations_found/
                            tries*100);
                    fflush(stdout);
               }
          }
     }
}
```

```
        }

        if(update_time && !quiet) printf("␣(~%.0fs␣
                left)␣␣␣␣␣␣␣␣␣" , (double)(clock() −
                start)/CLOCKS_PER_SEC/
                relations_found∗((pb_len+extra −
                relations_found)));
        get_next_poly(Q);
    }

    if(!quiet) printf("\n");

        mpz_clears(sqrt_N, temp, g, m, n, pi, x, NULL);
        free(x1);
        free(x2);
        free(r);
        free(sinterval);
        free(plogs);
        free_poly(Q);

        return v;
    }
```

../c/mpqs/parallel_mpqs.h

```c
#pragma once
#include <gmp.h>
#include "polynomial.h"
#include <sys/time.h>
#include <stdint.h>

struct sieve_arg_s {
    // used for sieveing
    int* pb;
    int pb_len;
    int extra;
    int* r;
    float* plogs;
    int s;
    int t;
    int* relations_found;
    int** v;
    bool quiet;
    mpz_t* z;
    mpz_t* d;
    poly_t Qinit;

    // used to print progress and predicted time left
    struct timeval begin;
    uint_fast64_t* tries;

    // used to constantly have a certain number of
        threads running
    int thread_id;
    bool* threads_running;
};
typedef struct sieve_arg_s sieve_arg_t;

bool already_added(mpz_t zi, mpz_t* z, int
        relations_found);
void* sieve_100_polys (void* args);
int** parallel_mpqs(mpz_t* z, mpz_t* d, mpz_t N,
        int pb_len, int* pb, int extra, int s, int delta,
        bool quiet);
```

../c/mpqs/parallel_mpqs.c

```
#include <gmp.h>
#include <stdbool.h>
#include <stdio.h>
#include <stdlib.h>
#include <assert.h>
#include <math.h>
#include <time.h>
#include <pthread.h>
#include <sys/time.h>

#include "polynomial.h"
#include "common_mpqs.h"
#include "parallel_mpqs.h"
#include "../system.h"
#include "../tonellishanks.h"

pthread_mutex_t mutex;


void* sieve_100_polys (void* args){
    sieve_arg_t* arg = (sieve_arg_t*) args;

    poly_t Q = copy_poly(arg->Qinit);

    mpz_t temp, g, m, n, pi, x;
    mpz_inits(temp, g, m, n, pi, x, NULL);
    float* sinterval = malloc(2*arg->s*sizeof(float));
    int* x1 = malloc(arg->pb_len*sizeof(int));
    int* x2 = malloc(arg->pb_len*sizeof(int));
```

```
for(int i = 0; i<100 && *(arg->relations_found)
        < arg->pb_len + arg->extra; i++){
    get_next_poly(Q);

    //get sol for 2
    mpz_set_ui(temp, 0);
    calc_poly(Q, temp);
    x1[0] = 0;
    if(mpz_divisible_ui_p(Q->qx, 2) == 0) x1
            [0] = 1;

    //get sol for others
    for(int i = 1; i<arg->pb_len; i++){
        mpz_set_ui(pi, arg->pb[i]);
        mpz_gcdext(g, m, n, Q->a, pi);
        if(mpz_cmp_ui(g, 1) != 0){
            fprintf(stderr, "ERROR:␣Number␣is␣
                    too␣small␣for␣the␣current␣
                    implementation␣of␣MPQS\n")
                    ;
            exit(1);
        }

        mpz_set_ui(temp, arg->r[i]);
        mpz_sub(temp, temp, Q->b);
        mpz_mul(temp, temp, m);
        mpz_mod(temp, temp, pi);

        x1[i] = mpz_get_ui(temp);
        //calc_poly(Q, temp);
        //assert(mpz_divisible_ui_p(Q−>qx, arg
                −>pb[i]) != 0);

        mpz_set_ui(temp, arg−>pb[i]);
        mpz_sub_ui(temp, temp, arg−>r[i]);
        mpz_sub(temp, temp, Q−>b);
        mpz_mul(temp, temp, m);
        mpz_mod(temp, temp, pi);

        x2[i] = mpz_get_ui(temp);

        //calc_poly(Q, temp);
        //assert(mpz_divisible_ui_p(Q−>qx, arg
                −>pb[i]) != 0);

        //realign sieving interval to [−s, s]
        int k = (x1[i] + arg−>s)/arg−>pb[i];
        x1[i] −= k ∗ arg−>pb[i];
        x1[i] += arg−>s;

        k = (x2[i] + arg−>s)/arg−>pb[i];
        x2[i] −= k ∗ arg−>pb[i];
        x2[i] += arg−>s;

        //mpz_set_si(temp, −arg−>s);
        //mpz_add_ui(temp, temp, x1[i]);
        //calc_poly(Q, temp);
        //assert(mpz_divisible_ui_p(Q−>qx, arg
                −>pb[i]) != 0);
    }

    //reset sieveing_interval
    for(int i = 0; i<2∗arg−>s; i++){
        sinterval[i] = 0;
    }

    /∗
    // sieve for 2
    while(x1[0]<2∗arg−>s){
        sinterval[x1[0]] += arg−>plogs[0];
        x1[0] += arg−>pb[0];
    }
    ∗/

    // sieve other primes
    for(int i = 30; i < arg−>pb_len; i++){
        while(x1[i]<2∗arg−>s){
            sinterval[x1[i]] += arg−>plogs[i];
            x1[i] += arg−>pb[i];
        }
        while(x2[i]<2∗arg−>s){
            sinterval[x2[i]] += arg−>plogs[i];
            x2[i] += arg−>pb[i];
        }
    }

    bool found;
    bool update_time = false;
    pthread_mutex_lock(&mutex);
    for(int i = 0; i<2∗arg−>s && ∗(arg−>
            relations_found) < arg−>pb_len +
            arg−>extra; i++){
        if(sinterval[i] > arg−>t){
            ∗(arg−>tries) += 1;
            mpz_set_si(x, −arg−>s);
```

```
mpz_add_ui(x, x, i);
calc_poly(Q, x);

if(!already_added(Q->zi, arg->z,
        *(arg->relations_found))){
    found = vectorize_mpqs(Q->qx,
            arg->v[*(arg->
            relations_found)], arg->
            pb_len, arg->pb);
    if(found){
        mpz_set(arg->z[*(arg->
                relations_found)], Q
                ->zi);
        mpz_set(arg->d[*(arg->
                relations_found)], Q
                ->d);
        *(arg->relations_found)
                += 1;
        found = false;
        update_time = true;
        if(!arg->quiet){
            printf("\r");
            printf("%.1f%%␣|␣%.1f
                    %%", (float)(*(
                    arg->
                    relations_found))
                    /(arg->pb_len+
                    arg->extra)
                    *100, (float)(*(
                    arg->
                    relations_found))
                    /(*(arg->tries))
                    *100);
                                fflush(stdout);
                            }
                        }
                    }
                }
            }

            struct timeval current;
            gettimeofday(&current, 0);
            long seconds = current.tv_sec - arg->begin.
                    tv_sec;
            long microseconds = current.tv_usec - arg
                    ->begin.tv_usec;
            double elapsed = seconds + microseconds*1e
                    -6;
            if(update_time && !arg->quiet) printf("␣
                    (~%.0fs␣left)␣␣␣␣␣␣␣␣␣" , elapsed/(*arg
                    ->relations_found)*(arg->pb_len+
                    arg->extra - (*arg->
                    relations_found)));
            pthread_mutex_unlock(&mutex);
        }

        mpz_clears(temp, g, m, n, pi, x, NULL);
        free(x1);
        free(x2);
        free(sinterval);
        free_poly(Q);

        arg->threads_running[arg->thread_id] = false;
        return NULL;
    }
```

```
int** parallel_mpqs(mpz_t* z, mpz_t* d, mpz_t N,
        int pb_len, int* pb, int extra, int s, int delta,
        bool quiet){
    /** Gets pb_len+extra zis that are b-smooth,
            definied at:
     * Quadratic sieve factorisation algorithm
     * Bc. OndĔĞrej Vladyka
     * Definition 1.11 (p.5)
     */

    //ceil(sqrt(n))
    mpz_t sqrt_N;
    mpz_init(sqrt_N);
    mpz_sqrt(sqrt_N, N);
    mpz_add_ui(sqrt_N, sqrt_N, 1);

    poly_t Q = init_poly(N, s);

    int** v = malloc((pb_len+extra)*sizeof(int*));
    for(int i = 0; i<pb_len+extra; i++){
        v[i] = malloc((pb_len+1)*sizeof(int*)); //
                +1 for -1
    }
    float* plogs = prime_logs_mpqs(pb, pb_len);


    int* r = malloc(pb_len*sizeof(int));
    int sol1, sol2;
    for(int i = 1; i < pb_len; i++){
        tonelli_shanks_ui(N, pb[i], &sol1, &sol2);
        r[i] = sol1;
    }
    int t = calculate_threshhold_mpqs(sqrt_N, s, pb,
            pb_len, delta);

    sieve_arg_t* args = malloc(8*sizeof(sieve_arg_t)
            );
    pthread_t* threads = malloc(8*sizeof(pthread_t))
            ;
    bool* threads_running = malloc(8*sizeof(bool));
    for(int i = 0; i<8; i++){
        threads_running[i] = false;
    }

    int relations_found = 0;
    uint_fast64_t tries = 0;
    struct timeval begin;
    gettimeofday(&begin, 0);
    while(relations_found < pb_len + extra){
        for(int i = 0; i<8; i++){
            if(!threads_running[i]){
                args[i] = (sieve_arg_t) {
                    pb,
                    pb_len,
                    extra,
                    r,
                    plogs,
                    s,
                    t,
                    &relations_found,
                    v,
                    quiet,
                    z,
                    d,
                    Q,
                    begin,
                    &tries,
                    i,
                    threads_running
                };
                threads_running[i] = true;
                pthread_create(threads+i, NULL,
                        sieve_100_polys, args+i);
            }
            for(int i = 0; i<100; i++){
                get_next_poly(Q);
            }
        }
    }
    if(!quiet) printf("\n");

    for(int i = 0; i<8; i++){
        pthread_join(threads[i], NULL);
    }

    free(threads);
    free(args);
    free(r);
    free(plogs);
    free(threads_running);
    free_poly(Q);
    mpz_clear(sqrt_N);

    return v;
}
```