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A SIMPLE CRITERION FOR ESTABLISHING AN UPPER LIMIT TO THE HARMONIC EXCITATION LEVEL OF THE DUFFING OSCILLATOR USING THE VOLTERRA SERIES

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Higher order frequency response functions, based on the Volterra series, are employed to represent the input-output characteristics of the Duffing oscillator subject to sinusoidal excitation. From these a series representation of a first order frequency response function of the non-linear system is obtained and used to derive a simple criterion which provides an estimate of the magnitude of the input sine-wave above which the predicted response begins to diverge. The criterion results in a small overestimate of the magnitude of the input force excitation at the undamped natural frequency condition. Although the derived criterion strictly applies only to the Duffing oscillator, the procedure can be applied generally and is extended to a two-degree-of-freedom system with a non-linear cubic stiffness producing a similar criterion to that of the single degree-of-freedom system.

1. INTRODUCTION

Stiffness non-linearities are widely encountered in the dynamic testing of structures, particularly when the input excitation levels are high, producing large response levels. Such situations can occur in the dynamic testing of structures incorporating panels, shells, membranes and beams, commonly found in aircraft, marine and automotive structures. In cases where the resonance frequencies are well separated, such systems can be represented as a single-degree-of-freedom. An equation frequently employed to model a single-degree-of-freedom system with non-linear stiffness is the well-known Duffing oscillator. Subject to sinusoidal constant magnitude force excitation, this model, for a given set of constant coefficients, can display considerable distortion in the response characteristics. The cause of the distortion has been explained to some extent [1, 2] by the use of higher order frequency response functions (FRFs) which have been used to describe the frequency domain characteristics of systems with polynomial type non-linearities, expressed in terms of an infinite Volterra series [3–7]. In practice a truncated series is used and the number of terms to be included to provide a given accuracy

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of prediction of the response as a function of frequency is dependent on the series converging [8–10]. However, even if convergence occurs, the Volterra series cannot predict the bifurcation characteristics of the Duffing oscillator. There are situations where it may be necessary to use the higher order FRFs to calculate the system response [11] and in such cases it is important to know the limitations of these procedures. In this paper the response of the non-linear oscillator is expressed as a sum of the FRF of the corresponding linear system and higher order FRFs (which are input level dependent) and a methodology is described for estimating the magnitude of the sinusoidal forcing function at which the predicted response begins to diverge. By comparing the analytical results for the response of the Duffing oscillator with those predicted from the Volterra series approximation with an increasing number of higher order FRF terms, a mean square error is obtained for a range of input excitation levels. This offers a means of establishing the requisite number of higher order FRF terms for a defined error and excitation level and indicates the conditions under which the approach fails due to the series approximation predicting a divergent response.

2. VOLTERRA SERIES AND HIGHER ORDER FRFs

In order to provide an introduction to higher order FRFs the basic equations defining these are introduced and expressed in a general form of a first order FRF for a non-linear system subject to a sine wave input. Detailed derivations of the calculation of the response of the Volterra series to a sine wave input can be found in reference [5].

The higher order FRFs are defined as the multi-dimensional Fourier transforms of the higher order impulse response functions:

$$H_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} \mathrm{e}^{-\mathrm{j}\omega_i \tau_i} \, \mathrm{d}\tau_1, \ldots, \mathrm{d}\tau_i . \tag{1}$$

If the input is expressed as a simple harmonic of magnitude X,

$$x(t) = X \cos(\omega t) = (X/2)\,\mathrm{e}^{\mathrm{j}\omega t} + (X/2)\,\mathrm{e}^{-\mathrm{j}\omega t}$$

the respective components of the response become

$$\begin{aligned}
y_1(t) &= \int_{-\infty}^{\infty} h_1(\tau_1) x(t-\tau_1)\,\mathrm{d}\tau_1 = \int_{-\infty}^{\infty} h_1(\tau_1) \left\{ \frac{X}{2} \mathrm{e}^{\mathrm{j}\omega(t-\tau_1)} + \frac{X}{2} \mathrm{e}^{-\mathrm{j}\omega(t-\tau_1)} \right\} \mathrm{d}\tau_1 \\
&= H_1(\omega)(X/2)\,\mathrm{e}^{\mathrm{j}\omega t} + H_1(-\omega)(X/2)\,\mathrm{e}^{-\mathrm{j}\omega t}, \\
y_2(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) \left\{ \frac{X}{2} \mathrm{e}^{\mathrm{j}\omega(t-\tau_1)} + \frac{X}{2} \mathrm{e}^{-\mathrm{j}\omega(t-\tau_1)} \right\} \left\{ \frac{X}{2} \mathrm{e}^{\mathrm{j}\omega(t-\tau_2)} + \frac{X}{2} \mathrm{e}^{-\mathrm{j}\omega(t-\tau_2)} \right\} \mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\
&= H_2(\omega, \omega)(X/2)^2\,\mathrm{e}^{\mathrm{j}2\omega t} + 2H_2(\omega, -\omega)(X/2)^2 + H_2(-\omega, \omega)(X/2)^2\,\mathrm{e}^{-\mathrm{j}2\omega t}, \\
y_3(t) &= H_3(\omega, \omega, \omega)(X/2)^3\,\mathrm{e}^{\mathrm{j}3\omega t} + 3H_3(\omega, \omega, -\omega)(X/2)^3\,\mathrm{e}^{\mathrm{j}\omega t} \\
&\quad + 3H_3(\omega, -\omega, -\omega)(X/2)^3\,\mathrm{e}^{-\mathrm{j}\omega t} + H_3(-\omega, -\omega, -\omega)(X/2)^3\,\mathrm{e}^{-\mathrm{j}3\omega t}, \\
&\ldots
\end{aligned} \tag{2}$$

and

$$y(t) = \sum_{n=1}^{\infty} y_n(t). \tag{3}$$

If the following notation is used to denote the higher order FRFs,

$$H_{n,i} = H_n(\underbrace{\omega, \ldots, \omega}_{n-i \text{ times}}, \underbrace{-\omega, \ldots, -\omega}_{i \text{ times}}),$$

$y_n(t)$ can be expressed as

$$y_n(t) = {}_nC_0 H_{n,0}(X/2)^n \,\mathrm{e}^{\mathrm{j}n\omega t} + {}_nC_1 H_{n,1}(X/2)^n \,\mathrm{e}^{\mathrm{j}(n-2)\omega t} + {}_nC_2 H_{n,2}(X/2)^n \,\mathrm{e}^{\mathrm{j}(n-4)\omega t} + \cdots$$

$$= \sum_{i=0}^{n} {}_nC_i H_{n,i}(X/2)^n \,\mathrm{e}^{\mathrm{j}(n-2i)\omega t}, \tag{4}$$

where ${}_nC_i$ is a binomial coefficient defined as

$${}_nC_i = n!/i!(n-i)!. \tag{5}$$

Hence,

$$y(t) = \sum_{n=1}^{\infty} y_n(t) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} {}_nC_i H_{n,i} \left(\frac{X}{2}\right)^n \mathrm{e}^{\mathrm{j}(n-2)\omega t}. \tag{6}$$

2.1. FIRST ORDER FREQUENCY RESPONSE FUNCTION

The first order frequency response function is the complex ratio of the response at the excitation frequency ω to the excitation at the same frequency ω, which represents the classical FRF if the system is linear. By considering only the terms associated with the input $(X/2)\,\mathrm{e}^{\mathrm{j}\omega t}$ in equation (6) one obtains an estimate of the first order FRF for a non-linear system:

$$\hat{H}_1(\omega) = Y(\omega)/X(\omega)$$

$$= H_1(\omega) + {}_3C_1 H_3(\omega, \omega, -\omega)(X/2)^2 + {}_5C_2 H_5(\omega, \omega, \omega, -\omega, -\omega)(X/2)^4 + \cdots$$

$$= \sum_{m=0}^{\infty} {}_{2m+1}C_m H_{2m+1,m}(X/2)^{2m}. \tag{7}$$

It can be seen from equation (7) that the estimate of the first order FRF is dependent on the input amplitude. If the amplitude is small, higher order terms are negligible and the FRF approaches that of the linear system, $H_1(\omega)$. However, as the excitation amplitude increases the higher order terms add significant contributions to the estimate of the FRF which begins to deviate from that of the linear case.

2.2. APPLICATION TO THE DUFFING OSCILLATOR

The Duffing oscillator model subject to sinusoidal excitation is represented by the equation

$$m\ddot{y}(t) + c\dot{y}(t) + k_1 y(t) + k_3 y(t)^3 = (X/2)\,\mathrm{e}^{\mathrm{j}\omega t} + (X/2)\,\mathrm{e}^{-\mathrm{j}\omega t} \tag{8}$$

and can be employed to investigate the characteristics of the infinite series representation of the first order FRF of equation (7). Substituting equation (2) in equation (8) gives the expression

$$\begin{aligned}
&H_1(\omega)(X/2)\,\mathrm{e}^{\mathrm{j}\omega t}[m(\mathrm{j}\omega)^2+c(\mathrm{j}\omega)+k_1]+H_1(-\omega)(X/2)\,\mathrm{e}^{-\mathrm{j}\omega t}[m(-\mathrm{j}\omega)^2+c(-\mathrm{j}\omega)+k_1]\\
&\quad+H_2(\omega,\omega)(H/2)^2\,\mathrm{e}^{\mathrm{j}2\omega t}[m(\mathrm{j}2\omega)^2+c(\mathrm{j}2\omega)+k_1]+\cdots\\
&\quad\;+H_3(\omega,\omega,\omega)(X^2)^3\,\mathrm{e}^{\mathrm{j}3\omega t}[m(\mathrm{j}3\omega)^2+c(\mathrm{j}3\omega)+k_1]+3H_3(\omega,\omega-\omega)(X/2)^3\,\mathrm{e}^{\mathrm{j}\omega}[m(\mathrm{j}\omega)^2\\
&\quad+c(\mathrm{j}\omega)+k_1]+\cdots+k_3[H_1(\omega)^3(X/2)^3\,\mathrm{e}^{\mathrm{j}3\omega t}+H_1(-\omega)^3(X/2)^3\,\mathrm{e}^{-\mathrm{j}3\omega t}\\
&\quad+3H_1(\omega)^2H_1(-\omega)(X/2)^3\,\mathrm{e}^{\mathrm{j}\omega t}+3H_1(\Omega)H_1(-\omega)^2(X/2)^3\,\mathrm{e}^{-\mathrm{j}\omega t}+\cdots]\\
&=(X/2)\,\mathrm{e}^{\mathrm{j}\omega t}+(X/2)\,\mathrm{e}^{-\mathrm{j}\omega t}.
\end{aligned}\tag{9}$$

Equating coefficients of $(X/2)\,\mathrm{e}^{\mathrm{j}\omega t}$ gives

$$H_1(\omega)=(k_1-m\omega^2+\mathrm{j}c\omega)^{-1}.\tag{10}$$

Equating coefficients of $(X/2)^3\,\mathrm{e}^{\mathrm{j}\omega t}$ gives

$$3H_3(\omega,\omega-\omega)[k_1-m\omega^2+\mathrm{j}c\omega]+k_3[3H_1(\omega)^2H_1(-\omega)]=0;\tag{11}$$

i.e.,

$$H_3(\omega,\omega-\omega)=k_3H_1(\omega)^3H_1(-\omega).\tag{12}$$

Extending this method of harmonic probing [3] gives the following general expression:

$${}_nC_1H_{n,i}H_1[(n-2i)\omega]^{-1}+k_3\sum_{n_1}C_{i_1\;n_2}C_{i_2\;n_3}C_{i_3}H_{n_1,i_1}H_{n_2,i_2}H_{n_3,i_3}=0.\tag{13}$$

In equation (13) the summation must be performed over all sets of ns and is such that $n_1+n_2+n_3=n$ and $i_1+i_2+i_3=i$. Equation (13) shows that a higher order FRF is expressed in terms of FRFs of lower orders. Therefore, by starting with the linear first order FRF $H_1(\omega)$, a higher order FRF of any order can be calculated. By substituting these higher order FRFs into equation (7), an estimate of the first order FRF, $H_1(\omega)$, can be obtained.

The estimated first order FRF, of the Duffing oscillator given by equation (8) was calculated by using equations (5), (7) and (13) with the parameters $m=1$ kg, $c=10$ Ns/m, $k_1=10^4$ N/m, and $k_3=10^{10}$ N/m^3. In order to determine an initial value for the input excitation level the analytical solution of the Duffing oscillator was used to generate the displacement response. The response amplitude of equation (8) due to a sinusoidal input force $X\cos\omega t$ can be obtained from [12]

$$[(k_1-m\omega^2)Y+\tfrac{3}{4}k_3Y^3]^2+c^2\omega^2Y^2=X^2,$$

where Y is the magnitude of the response at the excitation frequency ω. Figure 1 shows the modulus of the response divided by the excitation level, defined as the frequency response function, over the frequency range which encompasses the undamped natural frequency for excitation levels ranging from $X=0{\cdot}01$ N to $X=0{\cdot}75$ N. From these results a value of $X=0{\cdot}22$ N peak was chosen as this was the lowest value that produced a noticeable change in the frequency response characteristics.

With the value $X=0{\cdot}22$ N, the first order FRF given by equation (7) was calculated by considering the first n terms only. Figure 2 shows the modulus and phase results for $n=2,5,15$ ($n=1$ being the linear case) compared with the analytical result. It can be seen that for $n=2$ and 5 the analytical and the estimated first order FRF differ only in the

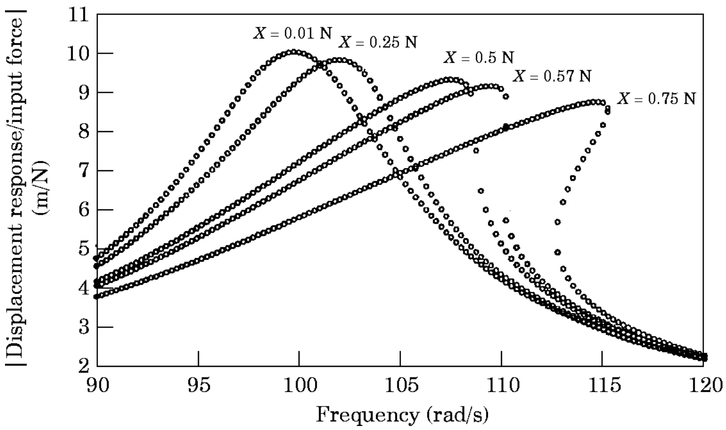

Figure 1. Analytical frequency response functions of the Duffing oscillator with the parameters $m = 1$ kg, $c = 10$ Ns/m, $k_1 = 10^4$ N/m, $k_3 = 10^{10}$ N/m^3 for the input amplitudes $X = 0{\cdot}01$, $0{\cdot}25$, $0{\cdot}50$, $0{\cdot}57$, $0{\cdot}75$ N.

region of the undamped natural frequency where the response levels reach a maximum. However, as the terms are increased to $n = 15$, which corresponds to a higher order FRF of H_{29}, the estimated first order FRF converges to the analytical one. Repeating the procedure for an input level of $X = 0{\cdot}25$ N produces the results in Figure 3. In this case the estimated first order FRF and the analytical FRF do not converge as the number of terms is increased.

In order to investigate the conditions under which the estimated first order FRF fails to predict the correct results, a mean square error criterion was employed, defined as

$$\text{mean square error} = \sqrt{(1/N)\,\Sigma|f_1 - f_2|^2}/\sqrt{(1/N)\,\Sigma|f_1|^2}, \tag{14}$$

where f_1 and f_2 represent the magnitudes of the analytical and estimated FRF, respectively, and N is the number of frequency points. The mean square error was calculated for different values of the viscous damping coefficient and the non-linear stiffness coefficient with the mass and linear stiffness coefficients the same as before. Figures 4 and 5 show the mean square error as a function of the input excitation level and the number of terms in the series approximation of equation (7). In both figures it can be seen that for a given number of terms and excitation level the mean square error begins to increase.

By using the condition

$$|\sigma_{n+1}|/|\sigma_n| < 1{\cdot}0, \tag{15}$$

where $|\sigma_n|$ represents the magnitude of the nth term, with the series of equation (7), the magnitude of the input excitation level can be estimated which results in failure to predict the true value of the first order FRF in the region of the undamped natural frequency.

3. ESTIMATING THE INPUT EXCITATION LEVEL

For the condition given in equation (15), the series of equation (7) results in

$$|\sigma_{n+1}|/|\sigma_n| = ({}_{2m+3}C_{m+1}/{}_{2m+1}C_m)(X/2)^2|H_{2m+3,m+1}|/|H_{2m+1,m}|. \tag{16}$$

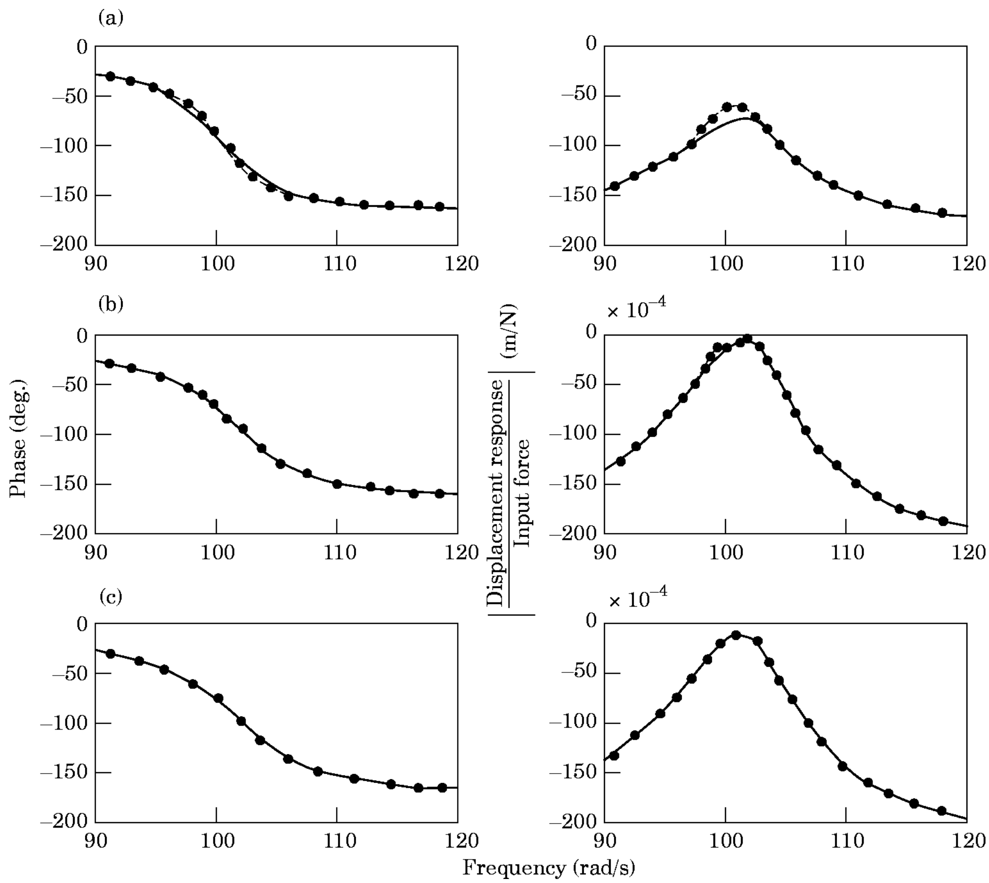

Figure 2. Comparison of the analytical (——) and n-term approximated (---) frequency response functions of the Duffing oscillator with the parameters $m=1$ kg, $c_2=10$ Ns/m, $k_1=10^4$ N/m, $k_3=10\times10^{10}$ N/m^3, $X=0{\cdot}22$ N. (a) $n=2$; (b) $n=5$; (c) $n=15$.

For $n=2m+3$ and $i=m+1$, equation (13) can be written as

$$\begin{aligned}{}_{2m+3}C_{m+1}\,H_{2m+3,m+1}\,H_1(\omega)^{-1} = -k_3[6\;{}_{2m+1}C_m\,H_{2m+1,m}\,H_1(\omega)H_1(-\omega)]\\ +3\;{}_{2m+1}C_{m+1}\,H_{2m+1,m}\,H_1(\omega)^2\\ +3\;{}_{2m+1}C_{m-1}\,H_{2m+1,m-1}\,H_1(-\omega)^2\\ +6_{2m-1}C_{m+1}\,H_{2m+1,m+1}\,H_1(\omega)H_3(\omega,\omega,\omega)+\cdots.\end{aligned} \tag{17}$$

It is clear from equation (17) that if the terms on the right-hand side after the square brackets could be ignored it would be a simple matter to obtain an expression in the form of equation (16).

Equation (17) can be simplified by noting that it is only in the region of the undamped natural frequency that the predicted response deviates from the true response. As a consequence, terms in the series expansion of equation (17) which contain any $H_1(n\omega)$ where $|n|\neq1$ may be disregarded and the predominant terms are those composed entirely of $H_1(\omega)$ and $H_1(-\omega)$ components.

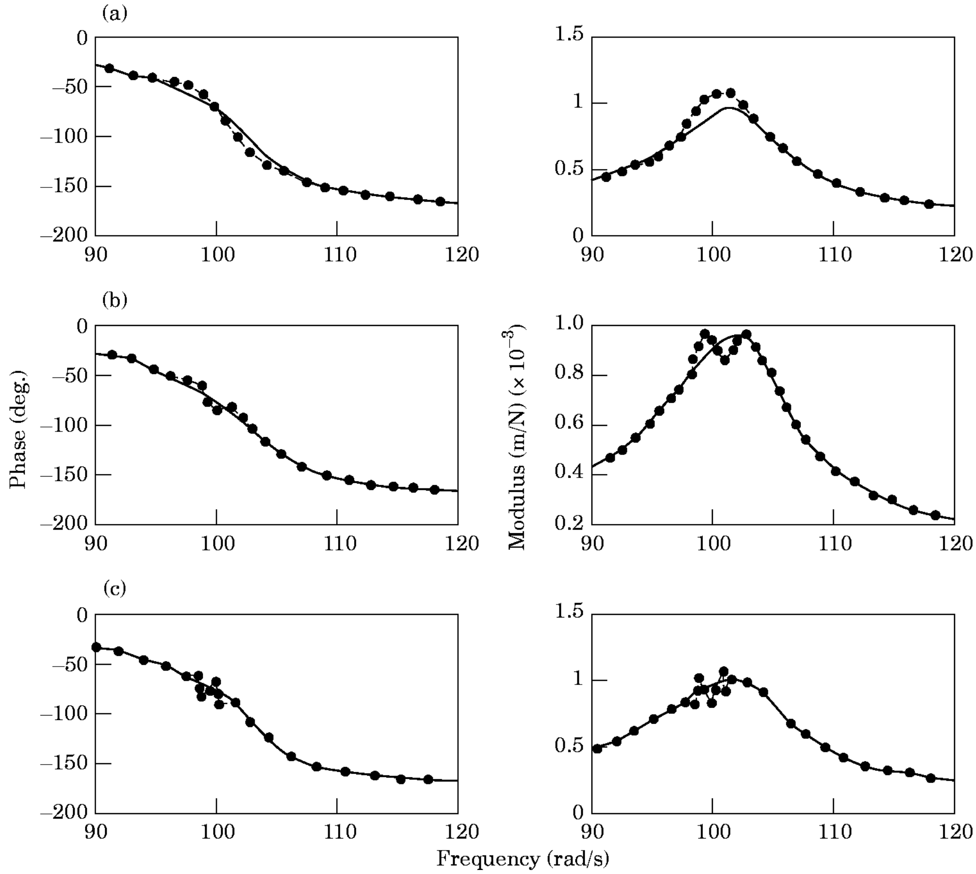

Figure 3. As Figure 2 but with an input excitation level increased to $X=0{\cdot}25$ N.

Consider the full expressions for $H_{5,2}$ ($m=1$) and $H_{7,3}$ ($m=2$) from equation (17):

$$H_{5,2}=-\tfrac{3}{10}k_3 H_1(\omega)(6H_{3,1}H_1(\omega)H_1(-\omega)+3H_{3,2}H_1(\omega)^2+H_{3,0}H_1(-\omega)^2), \tag{18}$$

$$H_{7,3}=-\tfrac{3}{35}k_3 H_1(\omega)(20H_{5,2}H_1(\omega)H_1(-\omega)+10H_{5,3}H_1(\omega)^2+5H_{5,1}H_1(-\omega)^2$$
$$+2H_{3,3}H_{3,0}H_1(\omega)+6H_{3,2}H_{3,0}H_1(-\omega)+9H_{3,1}^2H_1(-\omega)+H_{3,2}H_{3,1}H_1(\omega)). \tag{19}$$

Reducing equations (18) and (19) to $H_1(\omega)$ and $H_1(-\omega)$ terms only gives, for $H_{5,2}$,

$$H_{5,2}=\tfrac{9}{10}k_3^2(2H_1(\omega)^5H_1(-\omega)^2)+H_1(\omega)^4H_1(-\omega)^3), \tag{20}$$

and, for $H_{7,3}$,

$$H_{7,3}=-\tfrac{27}{35}k_3^3(4H_1(\omega)^7H_1(-\omega)^3+2H_1(\omega)^6H_1(-\omega)^4)+2H_1(\omega)^5H_1(-\omega)^5$$
$$+3H_2(\omega)^6H_1(-\omega)^4+H_1(\omega)^7H_1(-\omega)^3). \tag{21}$$

At the undamped natural frequency,

$$H_1(\omega_n)=-(i/\omega_n c), \quad H_1(-\omega_n)=i/\omega_n c. \tag{22}$$

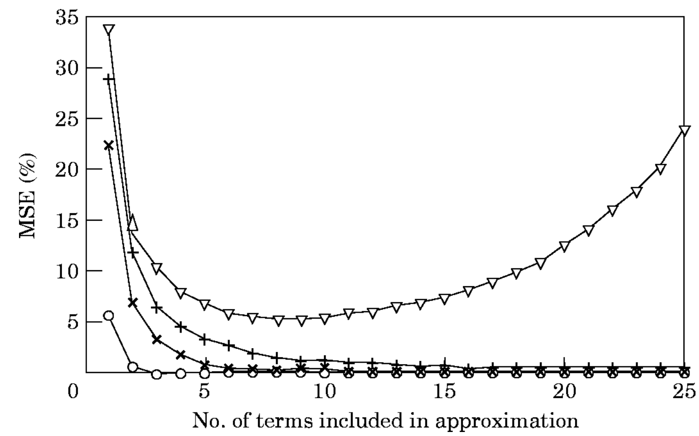

Figure 4. Mean square errors between the analytical and approximated frequency response functions of the Duffing oscillator with the parameters $m=1$ kg, $c=10$ Ns/m, $k_1=10^4$ N/m, $k_3=10^{10}$ N/m^3, ○, $X=0{\cdot}10$; ×, $X=0{\cdot}20$ N; +, $X=0{\cdot}23$ N, ▽, $X=0{\cdot}25$ N.

Substituting into equations (20) and (21) gives

$$H_{5,2}=\tfrac{9}{10}\{k_3^2/(\omega_n c)^7\}(2(-i)^5+(-i)^4 i^3) \tag{23}$$

$$H_{7,3}=-\tfrac{27}{35}\{k_3^3/(\omega_n c)^{10}\}(4(-i)^7 i^3+2(-i)^6 i^4+2(-i)^5 i^5+3(-i)^6 i^4+(-i)^7 i^3). \tag{24}$$

If only the square bracketed term of equation (17), referred to as the truncated term, had been considered, equations (23) and (24) would have become

$$\hat{H}_{5,2}=\tfrac{9}{10}\{k_3^2/(\omega_n c)^7\}(2(-i)^5 i^2), \tag{25}$$

$$\hat{H}_{7,3}=-\tfrac{27}{35}\{k_3^3/(\omega_n c)^{10}\}(4(-i)^7 i^3+2(-i)^6 i^4). \tag{26}$$

It can be seen from equation (25) that the truncated term for $H_{5,2}$ gives a value of $\frac{9}{5}k_3^2/(\omega_n c)^7$ at the undamped natural frequency. This is double the value from the full expression of equation (23). However, for the $H_{7,3}$ term equations (24) and (26) return the same value of $-54k_3^3/35(\omega_n c)^{10}$.

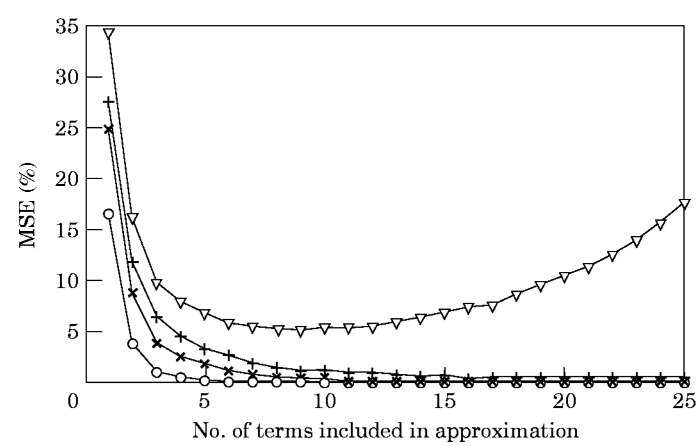

Figure 5. Mean square errors between the analytical and approximated frequency response functions of the Duffing oscillator with the parameters $m=1$ kg, $c=40$ Ns/m, $k_1=10^4$ N/m, $k_3=10^7$ N/m^3. ○, $X=40$ N, ×, $X=50$ N; +, $X=55$ N; ▽, $X=60$ N.

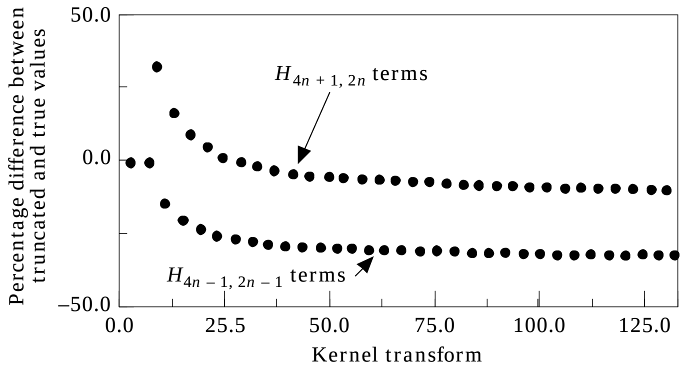

Figure 6. Upper and lower limits of the truncated term of equation (17) as a function of the kernel transformation order.

By continuing this process and including more higher order FRF terms it is found that the truncated term tends to an underestimate of approximately 10% of the true value for the $H_{4n+1,2n}$ terms and an underestimate of approximately 32% for the $H_{4n-1,2n-1}$ terms. Figure 6 shows these upper and lower bounds where the percentage difference between the truncated and true values is plotted as a function of the kernel transformation order. Figure 7 shows a comparison between the truncated values for selected $H_{2m+3,m+1}$ terms and the true values up to $H_{25,12}$ within the region of the undamped natural frequency. In Figure 7 it can be seen how, at the undamped natural frequency condition denoted by the vertical dotted line, the truncated value oscillates about the true value, as shown in Figure 6. Figure 6 also indicates that using only the truncated term results in a mean difference of about -20% on the value obtained from the full expression as the number of terms in the series exceeds 100. No attempt has been made in this paper to explain the oscillating results of Figure 6 since the objective was to establish the significance of the truncation term with respect to the full expression. The analysis does indicate

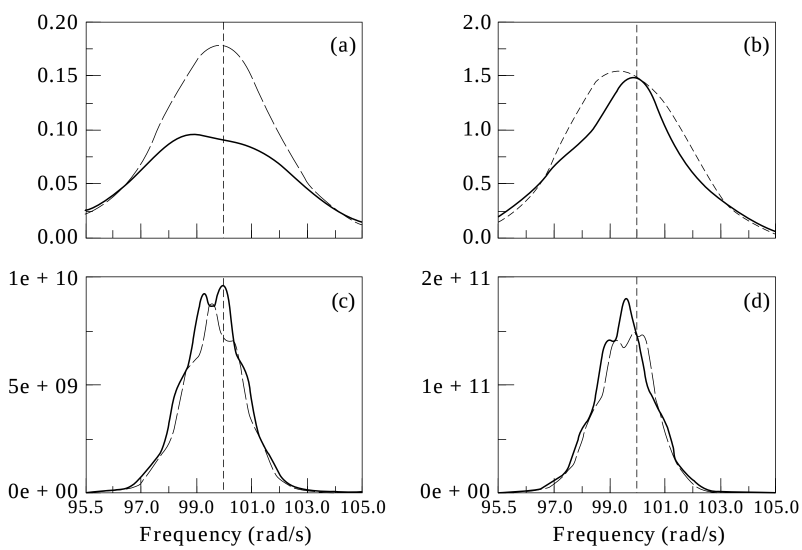

Figure 7. Comparison between the truncated values (- - -) of selected higher order FRF terms and the true values (——) based on the full expansion of equation (17). The vertical dashed lines indicate the natural frequency. (a) H (5,2); (b) H (7,3); (c) H (23, 11); (d) H (25,12).

that the square bracketed term on the right-hand side of equation (17) is predominant under the conditions used: i.e., it holds for equation (8) at the undamped natural frequency condition.

With these restrictions, equations (15)–(16) give

$$|H_{2m+3,m+1}|/|H_{2m+1,m}| = (6k_3\, {}_{2m+1}C_m |H_1(\omega)|^3)/({}_{2m+3}C_{m-1}), \tag{27}$$

$$|\sigma_{n+1}|/|\sigma_n| < 1 < 6k_3 (X/2)^2 |H_1(\omega)|^3: \tag{28}$$

i.e.,

$$X < \tfrac{2}{3}(k_3 |H_1(\omega)|^3)^{-1}. \tag{29}$$

The criterion defined by expression (29) provides a simple estimate of the magnitude of the input excitation level in order for the condition of equation (15) to hold. Applying this result with the parameters used to produce Figures 4 and 5 gives the following: inserting $k_3 = 10^{10}$ N/m^3 and $|H_1(\omega_n)| = 10^{-3}$ m/N into equation (29) gives $X < 0{\cdot}258$ N in order that the predicted response does not diverge. From Figure 4, $X = 0{\cdot}23$ N; thus the simple criterion overestimates the true maximum value of the input before the prediction diverges by 12%.

Repeating this for the parameters of Figure 5 where the non-linear stiffness coefficient was reduced by a factor of 10^3 and the damping coefficient was increased by a factor of four gives, for equation (29), with $|H_1(\omega_n)| = 0{\cdot}25 \times 10^{-3}$ m/N, the maximum escitation level $X = 65$ N. From Figure 5 the true value was found to be $X = 56$ N; thus an overestimate of 16% occurred in the predicted maximum value of the excitation.

4. TWO-DEGREE-OF-FREEDOM SYSTEM

The above procedure was applied to a two-degree-of-freedom, harmonically forced, non-linear system defined by the equations

$$m_1 \ddot{y}_1 + 2c\dot{y}_1 - c\dot{y}_2 + 2k_1 y_1 - k_1 y_2 - k_3 (y_2 - y_1)^3 = (X/2)\, \mathrm{e}^{\mathrm{j}\omega t} + (X/2)\, \mathrm{e}^{-\mathrm{j}\omega t},$$

$$m_2 \ddot{y}_2 + 2c\dot{y}_2 - c\dot{y}_1 + 2k_1 y_2 - k_1 y_1 + k_3 (y_2 - y_1)^3 = 0 \tag{30}$$

The system model is composed of linear elements except for a cubic stiffness spring between the two masses, with the mass m_1 being excited by a sine wave.

By following the same procedure as before the response of each mass can be expressed as

$$y_1(t) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} {}_nC_i\, {}_{11}H_{n,i} \left(\frac{X}{2}\right)^n \mathrm{e}^{\mathrm{j}(n-2i)\omega t}, \qquad y_2(t) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} {}_nC_i\, {}_{12}H_{n,i} \left(\frac{X}{2}\right)^n \mathrm{e}^{\mathrm{j}(n-2i)\omega t}, \tag{31}$$

where the subscripts of ${}_{rs}H_{n,i}$ represent the excitation point and the response point, respectively. Substituting equations (31) into equations (30) and using the harmonic probing method, one obtains the following equations for the higher order FRFs, ${}_{11}H_{n,i}$ and ${}_{12}H_{n,i}$:

$$[-m_1((n-2_i)\omega)^2 + \mathrm{j}2c\omega(n-2i) + 2k_1]_n C_i\, {}_{11}H_{n,i} - [-\mathrm{j}c\omega(n-2i) + k_1]_n C_i\, {}_{12}H_{n,i}$$

$$= k_3 \Sigma_{n_1} C_{i_1} C_{i_2 n_3} C_{i_3} [{}_{12}H_{n_1,i_1} - {}_{11}H_{n_1,i_1}][{}_{12}H_{n_3,i_3}], \tag{32}$$

$$-[\mathrm{j}c\omega(n-2i) + k_1]_n C_i\, {}_{11}H_{n,i} + [-m_2((n-2i)\omega)^2 + \mathrm{j}2c\omega(n-2i) + 2k_1] C_i\, {}_{12}H_{n,i}$$

$$= -k_2 \Sigma_{n_1} C_{\;i_1 n_2} C_{i_2 n_3} C_{i_3} [{}_{12}H_{n_2,i_2} - {}_{11}H_{n_2,i_2}][{}_{12}H_{n_3,i_3} - {}_{12}H_{n_3,i_3}]. \tag{33}$$

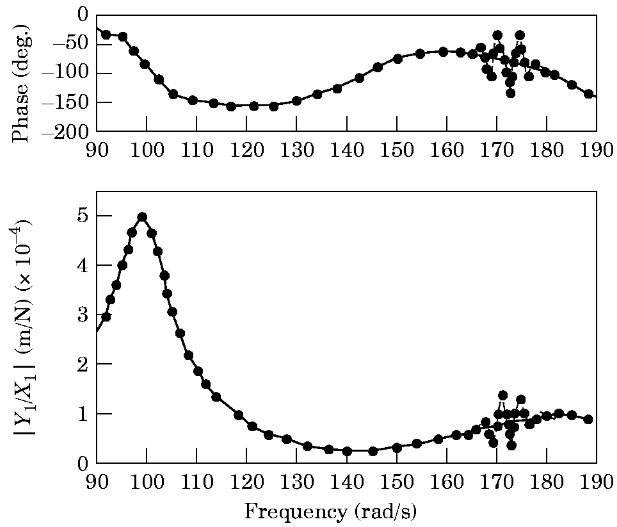

Figure 8. Comparison of the simulated (——) and predicted (- - -) frequency response characteristics of the two-degree-of-freedom non-linear system with an input excitation at mass m_1 of $X=1{\cdot}0$ N.

By using equation (7) the first order FRFs of the two-degree-of-freedom non-linear system were estimated with the parameters $m_1=m_2=1$ kg, $c=10$ Ns/m, $k_1=10^4$ N/m, and $k_3=10^{10}$ N/m^3. The FRF at the driving point was obtained by considering the first 15 terms in equation (7) and the result was compared with the numerically simulated FRF. With the input amplitude set to $X=1{\cdot}0$ N, the two FRFs coincide with each other. However, as the input amplitude is increased the predicted FRF deviates from the simulated result within the frequency zone around the second natural frequency, as shown in Figure 8. In Figure 8 the system appears to be linear at the first natural frequency. This is due to the symmetry of the system whereby the first mode results in equal in-phase magnitudes of the two masses and the non-linear spring is not activated at the first natural frequency. The same procedure was followed as for the single DOF case for obtaining a simple criterion for the level of the input excitation. When $m_1=m_2$, one can obtain simplified relations from the equations (32) and (33):

$$ {}_{12}H_{n,2} = -{}_{11}H_{n,i}, \tag{34}$$

$$[-m_1((n-2i)\omega^2+\mathrm{j}3c\omega(n-2i)+3ki_1]_n C_{i\,11} H_{n,i}$$
$$=(-2)^3 k_3 \Sigma_{n1} C_{i_1} n_2 C_{i_2} n_3 C_{i_3\ 11} H_{n_1,i_1\ 11} H_{n_3,i_3}. \tag{35}$$

For $n=2m+3$ and $i=m+1$, equation (35) can be written as

$$[3k_1-m_1\omega^2+\mathrm{j}3c\omega]_{2m+3}C_{m+1}H_{2m+3,m+1}$$
$$=(-2)^3 k_3[6_{2m+1}C_m H_{2m+1,m} H_1(\omega)H_1(-\omega)]. \tag{36}$$

Omitting the subscripts $_{11}$ of the Hs for brevity, one obtains,

$$|H_{2m+3,m+1}|/|H_{2m+1,m}| = 48k_{3\,2m+1}C_m |H_1(\omega)|^2/_{2m+3}C_{m+1}. \tag{37}$$

Combining equations (35), (36) and (37) gives

$$X<\tfrac{1}{12}|3k_1-m_1\omega^2+\mathrm{j}3c\omega|(k_3|H_1(\omega)|^2)^{-1}. \tag{38}$$

For the two-degree-of-freedom non-linear system considered, at the second undamped natural frequency of $\omega=173{\cdot}2$ rads. $|\mathrm{H}_1(\omega_n)|=1{\cdot}015\times10^{-4}$ m/N. Substituting these values into equation (38) gives the value of the input amplitude $X = 2{\cdot}08$ N. From numerical simulation the value of X above which the response, in the region of the second mode undamped natural frequency began to diverge, was found to be $X = 2{\cdot}0$ N. Thus the simple criterion, which applies only to the specific conditions studied, overestimates the true value by 4%.

5. CONCLUSIONS

A procedure for establishing a simple criterion which can be used to estimate the value of the magnitude of the sinusoidal input excitation to the Duffing oscillator above which the predicted response diverges, based on the Volterra series expansion, has been proposed. The procedure was applied to the specific case of the single-degree-of-freedom Duffing oscillator and to a two-degree-of-freedom non-linear system with a cubic stiffness term. In each case the criterion resulted in an overestimate of the true maximum value of the excitation level, obtained from numerical simulations. However, the results are surprisingly accurate considering the very simple formulae which are derived.

The methodology employed could be extended to any type of polynomial non-linearity and to any number of degrees-of-freedom, although the algebra would become exceedingly tedious for systems with more than two- or three-degrees-of-freedom.

The procedures described in the paper also provide a useful means of identifying how many terms are required in the Volterra series to compute a given accuracy of response for the Duffing oscillator, after having established the limit of the sinusoidal input excitation level by using the simple criterion presented.

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