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GENERATING SERIES FOR INTERCONNECTED ANALYTIC NONLINEAR SYSTEMS*

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Abstract. Given two analytic nonlinear input-output systems represented as Fliess operators, four system interconnections are considered in a unified setting: the parallel connection, product connection, cascade connection, and feedback connection. In each case, the corresponding generating series is produced and conditions for the convergence of the corresponding Fliess operator are given. In the process, an existing notion of a *composition product* for formal power series has its set of known properties significantly expanded. In addition, the notion of a *feedback product* for formal power series is shown to be well defined in a broad context, and its basic properties are characterized.

Key words. Chen–Fliess series, formal power series, nonlinear operators, nonlinear systems

AMS subject classifications. 47H30, 93C10

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1. Introduction.

Let $X = \{x_0, x_1, \ldots, x_m\}$ denote an alphabet and X^* the set of all words over X (including the empty word $\emptyset$). A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate an m-input, ℓ-output operator F_c in the following manner. Let $p \geq 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_p$-norm and $B_p^m(R)[a, b] := \{u \in L_p^m[a, b] : \|u\|_p \leq R\}$. With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset = 1$, and

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0(t) \equiv 1$. The input-output operator corresponding to c is then

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator*. All Volterra operators with analytic kernels, for example, are Fliess operators. In the classical literature, where these operators first appeared [7, 9, 10, 26], it is normally assumed that there exist real numbers $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|}|\eta|!$ for all $\eta \in X^*$, where $|z| = \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, and $|\eta|$ denotes the number of letters in η. This growth condition on the coefficients of c ensures that there exist positive real numbers R and T_0 such that, for all piecewise continuous u with $\|u\|_\infty \leq R$ and $T \leq T_0$, the series defining

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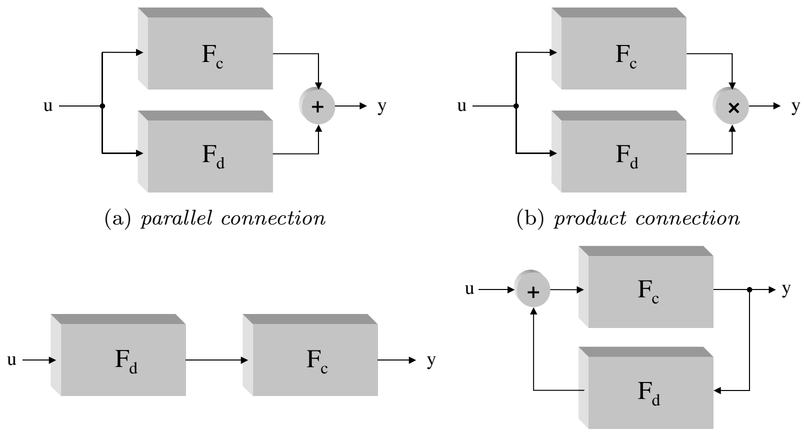

(a) *parallel connection* (b) *product connection*

(c) *cascade connection* (d) *feedback connection*

FIG. 1.1. *Elementary system interconnections.*

F_c converges uniformly and absolutely on $[t_0, t_0+T]$. Therefore, a power series c is said to be *locally convergent* when its coefficients satisfy such a growth condition. The set of all locally convergent series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ will be denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. More recently, it was shown in [13] that local convergence also implies that F_c constitutes a well-defined operator from $B_p^m(R)[t_0,\, t_0+T]$ into $B_q^\ell(S)[t_0,\, t_0+T]$ for sufficiently small $R, S, T > 0$, where the numbers $p, q \in [1,\infty]$ are conjugate exponents, i.e., $1/p + 1/q = 1$ with $(1,\infty)$ being a conjugate pair by convention.

In many applications, input-output systems are interconnected in a variety of ways. Given two Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, Figure 1.1 shows four elementary interconnections. The product connection is defined componentwise, and in the case of the feedback connection it is assumed that $\ell = m > 0$. The general goal of this paper is to describe in a unified manner the generating series for each elementary interconnection and conditions under which it is locally convergent. The clear antecedent to this work is that of Ferfera, who first described the generating series for such connections (implicitly in the case of feedback) and, in particular, introduced the composition product $c \circ d$ of two formal power series c and d [5, 6]. In each case, however, the local convergence of the new generating series or, equivalently, the convergence of the corresponding Fliess operator, was not explicitly addressed. It is trivial to show that the parallel connection of F_c and F_d always produces a locally convergent generating series when c and d are locally convergent. The same conclusion was later provided in [28] for the product connection via an analysis involving the shuffle product. In this paper, an analogous result is developed for the composition product by producing an explicit expression for one pair of growth constants, $K_{c\circ d}$ and $M_{c\circ d}$. In the process, the set of known properties of the composition product is significantly expanded. (An interesting parallel development has appeared in [3, 11] regarding a composition product for formal power series motivated by the composition of two analytic functions (see, e.g., [18]) rather than two Fliess (integral) operators. Its definition is quite distinct and not clearly related to the composition product described in this paper.)

The feedback connection is a fundamentally more difficult case to analyze. For example, when F_c is a linear operator, the formal solution to the feedback equation

$$y = F_c[u + F_d[y]] \tag{1}$$

is

$$y = F_c[u] + F_c \circ F_d \circ F_c[u] + \cdots.$$

It is not immediately clear that this series converges in any manner and, in particular, converges to another Fliess operator, say, $F_{c@d}$, for some $c@d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. When F_c is nonlinear, the problem is further complicated by the fact that operators of the form $I + F_d$, where I denotes the identity map, *never* have a Fliess operator representation. In this paper, the problem is circumvented by introducing a simple variation of the composition product so that an appropriate *feedback product*, $c@d$, is well defined, and $y = F_{c@d}[u]$ satisfies the feedback equation (1) in the sense that every analytic input u produces an analytic output y with (u, y) satisfying (1). In this case, $c@d$ is referred to as being *input-output locally convergent*, and explicit expressions are derived for one set of growth constants, K_{c_y} and M_{c_y}, for the series representation of the output function, c_y.

It should be stated that Ferfera's primary interest in [5, 6] was rational series and their corresponding bilinear realizations. In a state space setting, the issue of local convergence is rather straightforward. If c and d each have finite Lie rank, in addition to being locally convergent, then the mappings F_c and F_d each have a finite-dimensional analytic state space realization, and therefore so does each interconnected system (see [16, 21] for a basic treatment of nonlinear realization theory). The literature then provides that the corresponding generating series can be computed by successive Lie derivatives and, in particular, it must be locally convergent [26, Lemma 4.2]. (Additional analysis of interconnected state space systems using a chronological product together with Hall–Viennot bases appears in [17].) While the state space formalism is clearly dominant in modern control theory, other system descriptions like Volterra series [10, 16, 21] or input-output differential equations [28, 29, 30] are sometimes useful. In such settings, the convergence analysis of interconnected systems is a natural application for the main results of this paper. But even in a pure state space setting, as illustrated by Examples 3.2 and 4.11, knowledge of the growth constants for the generating series of a given interconnection permits one to compute a lower bound on any finite escape time. This is particularly useful in physical problems, like the one described in [12], as it provides computable limitations on the applicability of the underlying mathematical models.

The paper is organized as follows. In section 2 the composition product is introduced and developed independently of the system interconnection problem. First, its various fundamental properties are presented. Then, in preparation for the feedback analysis, it is shown that the composition product produces a contractive mapping on the set of all formal power series using a familiar ultrametric. In section 3, the three *nonrecursive* connections, parallel, product, and cascade, are analyzed primarily by applying results from section 2. In section 4 the feedback connection is considered. The main focus is on showing when the feedback product of two formal power series is well defined and in precisely what sense it is locally convergent.

2. The composition product. The composition product of two formal power series over an alphabet $X = \{x_0, x_1, \ldots, x_m\}$ is defined recursively in terms of the *shuffle product*. The shuffle product of two words $\eta, \xi \in X^*$ is defined recursively by

$$\eta \sqcup\!\sqcup \xi = (x_j\eta') \sqcup\!\sqcup (x_k\xi') := x_j[\eta' \sqcup\!\sqcup \xi] + x_k[\eta \sqcup\!\sqcup \xi']$$

with $\emptyset \sqcup\!\sqcup \emptyset = \emptyset$ and $\xi \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \xi = \xi$. It is easily verified that $\eta \sqcup\!\sqcup \xi$ is always a polynomial consisting of words each having length $|\eta|+|\xi|$. The definition is extended to any two series $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ by

$$c \sqcup\!\sqcup d = \sum_{\eta,\xi\in X^*} [(c,\eta)(d,\xi)]\, \eta \sqcup\!\sqcup \xi. \tag{2}$$

For a fixed $\nu \in X^*$, the coefficient $(\eta \sqcup\!\sqcup \xi, \nu) = 0$ if $|\eta| + |\xi| \neq |\nu|$. Hence, the infinite sum in (2) is well defined since the family of polynomials $\{\eta \sqcup\!\sqcup \xi : \eta, \xi \in X^*\}$ is locally finite [2]. In general, the shuffle product is commutative. It is also associative and distributes over addition. Thus, the vector space $\mathbb{R}\langle\langle X\rangle\rangle$ with the shuffle product forms a commutative $\mathbb{R}$-algebra, the so-called *shuffle algebra*, with multiplicative identity element $\emptyset$. The shuffle product on $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is defined componentwise, i.e., $(c \sqcup\!\sqcup d, \nu)_i = (c_i \sqcup\!\sqcup d_i, \nu)$ for $i = 1, 2, \ldots, \ell$.

For any $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the *composition product* is defined recursively as

$$\eta \circ d = \begin{cases} \eta & : \ |\eta|_{x_i} = 0 \ \ \forall i \neq 0, \\ x_0^{n+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \ \eta = x_0^n x_i \eta', \ \ n \geq 0, \ \ i \neq 0, \end{cases}$$

where $|\eta|_{x_i}$ denotes the number of letters in η equivalent to x_i and $d_i : \xi \mapsto (d, \xi)_i$, the ith component of the coefficient (d, ξ). Consequently, if

$$\eta = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \cdots x_0^{n_1} x_{i_1} x_0^{n_0}, \tag{3}$$

where $i_j \neq 0$ for $j = 1, \ldots, k$, then it follows that

$$\eta \circ d = x_0^{n_k+1}[d_{i_k} \sqcup\!\sqcup x_0^{n_{k-1}+1}[d_{i_{k-1}} \sqcup\!\sqcup \cdots x_0^{n_1+1}[d_{i_1} \sqcup\!\sqcup x_0^{n_0}] \cdots]].$$

Alternatively, for any $\eta \in X^*$, one can uniquely associate a set of right factors $\{\eta_0, \eta_1, \ldots, \eta_k\}$ by the iteration

$$\eta_{j+1} = x_0^{n_{j+1}} x_{i_{j+1}} \eta_j, \ \ \eta_0 = x_0^{n_0}, \ \ i_{j+1} \neq 0, \tag{4}$$

so that $\eta = \eta_k$ with $k = |\eta| - |\eta|_{x_0}$. In which case, $\eta \circ d = \eta_k \circ d$, where

$$\eta_{j+1} \circ d = x_0^{n_{j+1}+1}[d_{i_{j+1}} \sqcup\!\sqcup (\eta_j \circ d)]$$

and $\eta_0 \circ d = x_0^{n_0}$. The theorem below ensures that the composition product of two series described subsequently is well defined.

THEOREM 2.1. *Given a fixed $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the family of series $\{\eta \circ d : \eta \in X^*\}$ is locally finite, and therefore summable.*

Proof. Given an arbitrary $\eta \in X^*$ expressed in the form (3), it follows directly that

$$\operatorname{ord}(\eta \circ d) = n_0 + k + \sum_{j=1}^{k} n_j + \operatorname{ord}(d_{i_j}) = |\eta| + \sum_{j=1}^{|\eta|-|\eta|_{x_0}} \operatorname{ord}(d_{i_j}), \tag{5}$$

where the *order* of c is defined as

$$\mathrm{ord}(c) = \begin{cases} \inf\{|\eta| : \eta \in \mathrm{supp}(c)\} & : \quad c \neq 0, \\ \infty & : \quad c = 0, \end{cases}$$

and $\mathrm{supp}(c) := \{\eta \in X^* : (c, \eta) \neq 0\}$ denotes the *support* of c. Hence, for any $\xi \in X^*$,

$$\begin{aligned} I_d(\xi) &:= \{\eta \in X^* : (\eta \circ d, \xi) \neq 0\} \\ &\subset \{\eta \in X^* : \mathrm{ord}(\eta \circ d) \leq |\xi|\} \\ &= \left\{ \eta \in X^* : |\eta| + \sum_{j=1}^{|\eta| - |\eta|_{x_0}} \mathrm{ord}(d_{i_j}) \leq |\xi| \right\}. \end{aligned}$$

Clearly this last set is finite, and thus $I_d(\xi)$ is finite for all $\xi \in X^*$. This fact implies summability. □

For any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$, the composition product is defined as

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\ \eta \circ d.$$

The summation can also be written using the set of all right factors as described by (4). Let X^i be the set of all words in X^* of length i. For each word $\eta \in X^i$, the jth right factor, η_j, has exactly j letters not equal to x_0. Therefore, given any $\nu \in X^*$,

$$(c \circ d, \nu) = \sum_{i=0}^{|\nu|} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} (c, \eta_j)(\eta_j \circ d, \nu). \tag{6}$$

The third summation is understood to be the sum over the set of all possible jth right factors of words of length i. This set has a familiar combinatoric interpretation. A *composition* of a positive integer N is an ordered set of positive integers $\{a_1, a_2, \ldots, a_K\}$ such that $N = a_1 + a_2 + \cdots + a_K$. (For example, the integer 3 has the compositions $1+1+1$, $1+2$, $2+1$, and 3). For a given N and K, it is well known that there are $\mathcal{C}_K(N) = \binom{N-1}{K-1}$ possible compositions. Now each factor $\eta_j \in X^i$, when written in the form

$$\eta_j = x_0^{n_j} x_{i_j} x_0^{n_{j-1}} x_{i_{j-1}} \cdots\ x_0^{n_1} x_{i_1} x_0^{n_0},$$

maps to a unique composition of $i+1$ with $j+1$ elements:

$$i + 1 = (n_0 + 1) + (n_1 + 1) + \cdots + (n_j + 1).$$

Thus, there are exactly $\mathcal{C}_{j+1}(i+1)m^j = \binom{i}{j} m^j$ possible factors η_j in X^i, and the total number of terms in the summations of (6) is $((m+1)^{|\nu|+1} - 1)/m \approx (m+1)^{|\nu|}$. As will be seen shortly, this provides a conservative lower bound on the growth rate of the coefficients of $c \circ d$.

It is easily verified that the composition product is linear in its first argument, but not its second. A special exception are *linear series*. A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is called linear if

$$\mathrm{supp}(c) \subseteq \{\eta \in X^* : \eta = x_0^{n_1} x_i x_0^{n_0},\ \ i \in \{1, 2, \ldots, m\},\ \ n_1, n_0 \geq 0\}.$$

It was shown in [5] that the composition product is associative and distributive from the right over the shuffle product. But in general it is neither commutative nor has an identity element. This lack of an identity element is precisely the reason the identity map I is not realizable as a Fliess operator. Other elementary properties concerning the composition product are summarized below.

LEMMA 2.2. *The following identities hold* ($\mathbb{1}$ *is a column vector with* m *ones*):
1. $0 \circ d = 0$ for all $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$.
2. $c \circ 0 = c_0 := \sum_{n\geq 0}(c, x_0^n)\, x_0^n$. (*Therefore,* $c \circ 0 = 0$ *if and only if* $c_0 = 0$.)
3. $c_0 \circ d = c_0$ for all $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$. (*In particular,* $1 \circ d = 1$.)
4. $c \circ \mathbb{1} = c_{\mathbb{1}} := \sum_{\eta\in X^*}(c, \eta)\, x_0^{|\eta|}$. (*Therefore,* $c \circ \mathbb{1} = c$ *if and only if* $c_0 = c$.)

The set $\mathbb{R}^m\langle\langle X\rangle\rangle$ forms a metric space under the ultrametric

$$\begin{aligned} \mathrm{dist} \quad &: \quad \mathbb{R}^m\langle\langle X\rangle\rangle \times \mathbb{R}^m\langle\langle X\rangle\rangle \to \mathbb{R}^+ \cup \{0\}, \\ &: \quad (c, d) \mapsto \sigma^{\mathrm{ord}(c-d)}, \end{aligned}$$

where $\sigma \in (0, 1)$ is arbitrary [2]. The following theorem states that the composition product on $\mathbb{R}^m\langle\langle X\rangle\rangle \times \mathbb{R}^m\langle\langle X\rangle\rangle$ is continuous in its left argument. (Right argument continuity will be addressed later.)

THEOREM 2.3. *Let* $\{c_i\}_{i\geq 1}$ *be a sequence in* $\mathbb{R}^m\langle\langle X\rangle\rangle$ *with* $\lim_{i\to\infty} c_i = c$. *Then* $\lim_{i\to\infty}(c_i \circ d) = c \circ d$ *for any* $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$.

Proof. Define the sequence of nonnegative integers $k_i = \mathrm{ord}(c_i - c)$ for $i \geq 1$. Since c is the limit of the sequence $\{c_i\}_{i\geq 1}$, the sequence $\{k_i\}_{i\geq 1}$ must have an increasing subsequence $\{k_{i_j}\}$. Now observe that

$$\mathrm{dist}(c_i \circ d, c \circ d) = \sigma^{\mathrm{ord}((c_i - c)\, \circ d)}$$

and

$$\begin{aligned} \mathrm{ord}((c_{i_j} - c) \circ d) &= \mathrm{ord}\left(\sum_{\eta\in\mathrm{supp}(c_{i_j}-c)} (c_{i_j} - c, \eta)\, \eta \circ d\right) \\ &\geq \inf_{\eta\in\mathrm{supp}(c_{i_j}-c)} \mathrm{ord}(\eta \circ d) \\ &\geq \inf_{\eta\in\mathrm{supp}(c_{i_j}-c)} |\eta| + (|\eta| - |\eta|_{x_0})\, \mathrm{ord}(d) \\ &\geq k_{i_j}. \end{aligned}$$

Thus, $\mathrm{dist}(c_{i_j} \circ d, c \circ d) \leq \sigma^{k_{i_j}}$ for all $j \geq 1$, and $\lim_{i\to\infty} c_i \circ d = c \circ d$. $\square$

The ultrametric space $(\mathbb{R}^m\langle\langle X\rangle\rangle, \mathrm{dist})$ is known to be complete [2]. Given a fixed $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, consider the mapping $\mathbb{R}^m\langle\langle X\rangle\rangle \to \mathbb{R}^m\langle\langle X\rangle\rangle : d \mapsto c \circ d$. The goal is to show that this mapping is always a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$, i.e., that

$$\mathrm{dist}(c \circ d, c \circ e) \leq \sigma\, \mathrm{dist}(d, e) \;\; \forall d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle,$$

so that fixed point theorems can be applied in later analysis [14, 22, 23, 24]. Any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$ can be written unambiguously in the form

$$c = c_0 + c_1 + \cdots, \tag{7}$$

where $c_k \in \mathbb{R}^m\langle\langle X\rangle\rangle$ has the defining property that $\eta \in \mathrm{supp}(c_k)$ only if $|\eta| - |\eta|_{x_0} = k$. Some of the series c_k may be the zero series. When $c_0 = 0$, c is referred to as being *homogeneous*. When $c_k = 0$ for $k = 0, 1, \ldots, l-1$ and $c_l \neq 0$, then c is called *homogeneous of order* l. In this setting consider the following lemma.

LEMMA 2.4. *For any* c_k *in* (7),

$$\mathrm{dist}(c_k \circ d, c_k \circ e) \leq \sigma^k\, \mathrm{dist}(d, e) \;\; \forall d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle.$$

Proof. The proof is by induction for the nontrivial case, where $c_k \neq 0$. First suppose $k=0$. From the definition of the composition product it follows directly that $\eta \circ d = \eta$ for all $\eta \in \mathrm{supp}(c_0)$. Therefore,

$$c_0 \circ d = \sum_{\eta \in \mathrm{supp}(c_0)} (c_0, \eta)\, \eta \circ d = \sum_{\eta \in \mathrm{supp}(c_0)} (c_0, \eta)\, \eta = c_0,$$

and

$$\mathrm{dist}(c_0 \circ d, c_0 \circ e) = \mathrm{dist}(c_0, c_0) = 0 \leq \sigma^0 \, \mathrm{dist}(d, e).$$

Now fix any $k \geq 0$ and assume the claim is true for all $c_0, c_1, \ldots, c_k$. In particular, this implies that

$$\mathrm{ord}(c_k \circ d - c_k \circ e) \geq k + \mathrm{ord}(d - e). \tag{8}$$

For any $j \geq 0$, words in $\mathrm{supp}(c_j)$ have the form η_j as defined in (4). Observe then that

$$\begin{aligned}
c_{k+1} \circ d - c_{k+1} \circ e &= \sum_{\eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1})\, \eta_{k+1} \circ d - (c_{k+1}, \eta_{k+1})\, \eta_{k+1} \circ e \\
&= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1}) \;\Big[{x_0}^{n_{k+1}+1} [d_{i_{k+1}} \sqcup\!\sqcup [\eta_k \circ d]] \\
&\qquad\qquad - {x_0}^{n_{k+1}+1} [e_{i_{k+1}} \sqcup\!\sqcup [\eta_k \circ e]] \Big] \\
&= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1}) \;\Big[{x_0}^{n_{k+1}+1} [d_{i_{k+1}} \sqcup\!\sqcup [\eta_k \circ d]] \\
&\qquad\qquad - {x_0}^{n_{k+1}+1} [d_{i_{k+1}} \sqcup\!\sqcup [\eta_k \circ e]] \\
&\quad + {x_0}^{n_{k+1}+1} [d_{i_{k+1}} \sqcup\!\sqcup [\eta_k \circ e]] - {x_0}^{n_{k+1}+1} [e_{i_{k+1}} \sqcup\!\sqcup [\eta_k \circ e]] \Big] \\
&= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1}) \;\Big[{x_0}^{n_{k+1}+1} [d_{i_{k+1}} \sqcup\!\sqcup [\eta_k \circ d - \eta_k \circ e]] \\
&\qquad\qquad + {x_0}^{n_{k+1}+1} [(d_{i_{k+1}} - e_{i_{k+1}}) \sqcup\!\sqcup [\eta_k \circ e]] \Big],
\end{aligned}$$

using the fact that the shuffle product distributes over addition. Next, applying the identity (5) and the inequality (8) with $c_k = \eta_k$, it follows that

$$\begin{aligned}
\mathrm{ord}(c_{k+1} \circ d - c_{k+1} \circ e) &\geq \min \Bigg\{ \inf_{\eta_{k+1} \in \mathrm{supp}(c_{k+1})} n_{k+1} + 1 + \mathrm{ord}(d) + k + \mathrm{ord}(d - e), \\
&\qquad \inf_{\eta_{k+1} \in \mathrm{supp}(c_{k+1})} n_{k+1} + 1 + \mathrm{ord}(d - e) + |\eta_k| + k \; \mathrm{ord}(e) \Bigg\} \\
&\geq k + 1 + \mathrm{ord}(d - e),
\end{aligned}$$

and thus,

$$\mathrm{dist}(c_{k+1} \circ d, c_{k+1} \circ e) \leq \sigma^{k+1} \, \mathrm{dist}(d, e).$$

Hence, $\mathrm{dist}(c_k \circ d, c_k \circ e) \leq \sigma^k \, \mathrm{dist}(d, e)$ holds for any $k \geq 0$. ∎

Applying the above lemma leads to the following result.

LEMMA 2.5. *If $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, then for any series $c_0' \in \mathbb{R}^m\langle\langle X_0\rangle\rangle$,*

$$\mathrm{dist}((c_0'+c)\circ d, (c_0'+c)\circ e) = \mathrm{dist}(c\circ d, c\circ e) \ \ \forall d,e \in \mathbb{R}^m\langle\langle X\rangle\rangle. \tag{9}$$

(Here X_0 denotes the single letter alphabet $\{x_0\}$.) If c is homogeneous of order $l \geq 1$ then

$$\mathrm{dist}(c\circ d, c\circ e) \leq \sigma^l\,\mathrm{dist}(d,e) \ \ \forall d,e \in \mathbb{R}^m\langle\langle X\rangle\rangle. \tag{10}$$

Proof. The equality is proved first. Since the ultrametric dist is shift-invariant, observe that

$$\begin{aligned}\mathrm{dist}((c_0'+c)\circ d, (c_0'+c)\circ e) &= \mathrm{dist}\left(c_0'\circ d + c\circ d, c_0'\circ e + c\circ e\right)\\ &= \mathrm{dist}\left(c_0' + c\circ d, c_0' + c\circ e\right)\\ &= \mathrm{dist}\left(c\circ d, c\circ e\right).\end{aligned}$$

The inequality is proved next by first selecting any fixed $l \geq 1$ and showing inductively that it holds for any partial sum $\sum_{i=l}^{l+k} c_i$, where $k \geq 0$. When $k = 0$, Lemma 2.4 implies that

$$\mathrm{dist}(c_l\circ d, c_l\circ e) \leq \sigma^l\,\mathrm{dist}(d,e).$$

If the result is true for partial sums up to any fixed $k \geq 0$, then using the ultrametric property

$$\mathrm{dist}(d,e) \leq \max\{\mathrm{dist}(d,f), \mathrm{dist}(f,e)\} \ \ \forall d,e,f \in \mathbb{R}^m\langle\langle X\rangle\rangle,$$

it follows that

$$\begin{aligned}&\mathrm{dist}\left(\left(\sum_{i=l}^{l+k+1} c_i\right)\circ d, \left(\sum_{i=l}^{l+k+1} c_i\right)\circ e\right)\\ &\quad= \mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right)\circ d + c_{l+k+1}\circ d, \left(\sum_{i=l}^{l+k} c_i\right)\circ e + c_{l+k+1}\circ e\right)\\ &\quad\leq \max\left\{\mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right)\circ d + c_{l+k+1}\circ d, \left(\sum_{i=l}^{l+k} c_i\right)\circ d + c_{l+k+1}\circ e\right),\right.\\ &\qquad\qquad\left.\mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right)\circ d + c_{l+k+1}\circ e, \left(\sum_{i=l}^{l+k} c_i\right)\circ e + c_{l+k+1}\circ e\right)\right\}\\ &\quad= \max\left\{\mathrm{dist}(c_{l+k+1}\circ d, c_{l+k+1}\circ e), \mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right)\circ d, \left(\sum_{i=l}^{l+k} c_i\right)\circ e\right)\right\}\\ &\quad\leq \max\left\{\sigma^{l+k+1}\,\mathrm{dist}(d,e), \sigma^l\,\mathrm{dist}(d,e)\right\}\\ &\quad\leq \sigma^l\,\mathrm{dist}(d,e).\end{aligned}$$

Hence, the result holds for all $k \geq 0$. Inequality (10) is proved by noting that $c = \lim_{k\to\infty}\sum_{i=l}^{l+k} c_i$ and using the left argument continuity of the composition product, proved in Theorem 2.3, and the continuity of the ultrametric. ∎

The main result regarding contractive mappings is below.

THEOREM 2.6. *For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $d \mapsto c\circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$.*

Proof. Choose any series $d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle$. If c is homogeneous of order $l \geq 1$, then the result follows directly from (10). Otherwise, observe that, via (9),

$$\operatorname{dist}(c \circ d, c \circ e) = \operatorname{dist}\left(\left(\sum_{l=1}^{\infty} c_i\right) \circ d, \left(\sum_{l=1}^{\infty} c_i\right) \circ e\right) \leq \sigma \operatorname{dist}(d, e). \qquad \square$$

An immediate result of this theorem is the right argument continuity of the composition product.

THEOREM 2.7. *Let $\{d_i\}_{i\geq 1}$ be a sequence in $\mathbb{R}^m\langle\langle X\rangle\rangle$ with $\lim_{i\to\infty} d_i = d$. Then $\lim_{i\to\infty}(c \circ d_i) = c \circ d$ for all $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$.*

Proof. Trivially,

$$\lim_{i\to\infty} \operatorname{dist}(c \circ d_i, c \circ d) \leq \sigma \lim_{i\to\infty} \operatorname{dist}(d_i, d) = 0. \qquad \square$$

The final property considered in this section is local convergence. If all the summands in the defining expression (6) are unity, i.e., c and d have no coefficient growth whatsoever, then earlier combinatoric analysis shows that $(c \circ d, \nu)$ grows at least at the rate $(m+1)^{|\nu|}$. Of course, in general, much faster growth rates are possible when c and d are simply locally convergent. The analysis begins by considering the local convergence of the shuffle product. It provides a point of reference and some important tools. The following theorem was proved in [28].

THEOREM 2.8. *Suppose $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ with growth constants K_c, M_c and K_d, M_d, respectively. Then $c \sqcup\!\sqcup d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ with*

$$|(c \sqcup\!\sqcup d, \nu)| \leq K_c K_d M^{|\nu|}(|\nu|+1)! \;\; \forall \nu \in X^*, \tag{11}$$

where $M = \max\{M_c, M_d\}$.

Noting that $n + 1 \leq 2^n$ for all $n \geq 0$, (11) can be written more conventionally as

$$|(c \sqcup\!\sqcup d, \nu)| \leq K_c K_d (2M)^{|\nu|}|\nu|! \;\; \forall \nu \in X^*.$$

The specific goal here is to show that $c \circ d$ is also locally convergent, when the series c and d are locally convergent, and to produce an inequality analogous to (11). The following properties of the shuffle product are essential.

LEMMA 2.9 (see [28]). *For $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ and any $\nu \in X^*$,*

1. $(c \sqcup\!\sqcup d, \nu) = \displaystyle\sum_{\xi,\bar\xi\in X^*} (c,\xi)(d,\bar\xi)(\xi \sqcup\!\sqcup \bar\xi, \nu) = \sum_{i=0}^{|\nu|} \sum_{\substack{\xi\in X^i \\ \bar\xi \in X^{|\nu|-i}}} (c,\xi)(d,\bar\xi)(\xi \sqcup\!\sqcup \bar\xi, \nu)$;

2. $\displaystyle\sum_{\substack{\xi\in X^i \\ \bar\xi \in X^{|\nu|-i}}} (\xi \sqcup\!\sqcup \bar\xi, \nu) = \binom{|\nu|}{i}, \;\; 0 \leq i \leq |\nu|.$

Now given any $\eta \in X^*$, the set of right factors $\{\eta_0, \eta_1, \dots, \eta_k\}$ defined by (4) produces a corresponding family of real-valued functions:

$$S_{\eta_0}(n) = \frac{1}{|\eta_0|!}, \;\; n \geq 0,$$
$$S_{\eta_1}(n) = \frac{1}{(n)_{n_1+1}} S_{\eta_0}(n), \;\; 1 \leq |\eta_1| \leq n,$$
$$S_{\eta_j}(n) = \frac{1}{(n)_{n_j+1}} \sum_{i=0}^{n-|\eta_j|} S_{\eta_{j-1}}(n - (n_j+1) - i), \;\; j \leq |\eta_j| \leq n, \;\; 2 \leq j \leq k,$$

where $(n)_i = n!/(n-i)!$ denotes the falling factorial. The next two lemmas form the core of the local convergence proof for the composition product.

LEMMA 2.10. *Suppose $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ with growth constants K_c, M_c and K_d, M_d, respectively. Then*

$$|(c \circ d, \nu)| \leq K_c\, \psi_{|\nu|}(K_d)\, M^{|\nu|}|\nu|!\ \ \forall \nu \in X^*, \tag{12}$$

where $M = \max\{M_c, M_d\}$, and $\{\psi_n(K_d)\}_{n\geq 0}$ is the set of degree n polynomials in K_d,

$$\psi_n(K_d) = \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{\eta_j \in X^i} K_d^j\, S_{\eta_j}(n)\, |\eta_j|!,\ \ n \geq 0.$$

Proof. The proof has two main steps. It is first shown that for any integer $l > 0$ and any $\eta \in X^*$ with $|\eta| \leq l$ and right factors $\{\eta_0, \eta_1, \ldots, \eta_k\}$ as defined in (4),

$$|(\eta_j \circ d, \nu)| \leq K_d^j M_d^{-|\eta_j|} M_d^{|\nu|}\, |\nu|!\, S_{\eta_j}(|\nu|) \tag{13}$$

for all $0 \leq j \leq k$ and $|\eta_j| \leq |\nu| \leq l$. (Note that when $|\nu| < |\eta_j|$, the coefficients $(\eta_j \circ d, \nu) = 0$, and $S_{\eta_j}(|\nu|)$ is simply not defined.) This is shown by induction on j. The case $j = 0 < l$ is trivial. When $j = 1 \leq l$, the left-shift operator $x_0^{-(n_1+1)} := (x_0^{n_1+1})^{-1}$ is employed, where, in general, for any $\xi, \nu \in X^*$,

$$\xi^{-1}(\nu) = \begin{cases} \nu' & : \quad \nu = \xi\nu', \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Observe the following for any ν with $|\eta_1| \leq |\nu| \leq l$ and containing the left factor $x_0^{n_1+1}$ (otherwise the claim is trivial):

$$\begin{aligned}
|(\eta_1 \circ d, \nu)| &= \left|(x_0^{n_1+1}(d_{i_1} \sqcup\!\sqcup\, x_0^{n_0}), \nu)\right| \\
&= \left|\Big(d_{i_1} \sqcup\!\sqcup\, x_0^{n_0}, \underbrace{x_0^{-(n_1+1)}(\nu)}_{\nu'}\Big)\right| \\
&= \left|\sum_{\xi \in X^{|\nu'|-n_0}} (d_{i_1}, \xi)(\xi \sqcup\!\sqcup\, x_0^{n_0}, \nu')\right| \\
&\leq \sum_{\xi \in X^{|\nu'|-n_0}} (K_d M_d^{|\xi|}|\xi|!)\, (\xi \sqcup\!\sqcup\, x_0^{n_0}, \nu') \quad (\text{since } 0 \leq |\xi| < l) \\
&\leq K_d M_d^{|\nu'|-n_0}(|\nu'| - n_0)! \binom{|\nu'|}{n_0} \\
&= K_d M_d^{-|\eta_1|} M_d^{|\nu|}|\nu|!\, S_{\eta_1}(|\nu|).
\end{aligned}$$

Now assume that the result holds up to some fixed j, where $1 \leq j \leq k-1$. Then in a similar fashion for $|\eta_{j+1}| \leq |\nu| \leq l$,

$$\begin{aligned}
|(\eta_{j+1} \circ d, \nu)| &= \left|\Big(d_{i_{j+1}} \sqcup\!\sqcup\, (\eta_j \circ d), \underbrace{x_0^{-(n_{j+1}+1)}(\nu)}_{\nu'}\Big)\right| \\
&= \left|\sum_{i=0}^{|\nu'|} \sum_{\substack{\xi \in X^i \\ \bar{\xi} \in X^{|\nu'|-i}}} (d_{i_{j+1}}, \xi)(\eta_j \circ d, \bar{\xi})(\xi \sqcup\!\sqcup\, \bar{\xi}, \nu')\right|.
\end{aligned}$$

Since $(\eta_j \circ d, \bar{\xi}) = 0$ for $|\bar{\xi}| < |\eta_j|$, it follows that, by using the coefficient bounds for d (because $0 \leq |\xi| \leq l - (j+1)$) and Lemma 2.9 (since $|\eta_j| \leq |\bar{\xi}| < l - (n_{j+1}+1)$),

$$\begin{aligned}
|(\eta_{j+1} \circ d, \nu)| &\leq \sum_{i=0}^{|\nu'|-|\eta_j|} \sum_{\substack{\xi \in X^i \\ \bar{\xi} \in X^{|\nu'|-i}}} (K_d M_d^{|\xi|} |\xi|!) \cdot \left(K_d^j M_d^{-|\eta_j|} M_d^{|\bar{\xi}|} |\bar{\xi}|!\, S_{\eta_j}(|\bar{\xi}|) \right) (\xi \sqcup\!\sqcup \bar{\xi}, \nu') \\
&= K_d^{j+1} M_d^{-|\eta_{j+1}|} M_d^{|\nu|} \sum_{i=0}^{|\nu'|-|\eta_j|} i!\, (|\nu'|-i)!\, S_{\eta_j}(|\nu'|-i) \binom{|\nu'|}{i} \\
&= K_d^{j+1} M_d^{-|\eta_{j+1}|} M_d^{|\nu|} |\nu|!\, \frac{1}{(|\nu|)_{n_{j+1}+1}} \sum_{i=0}^{|\nu'|-|\eta_j|} S_{\eta_j}(|\nu| - (n_{j+1}+1) - i) \\
&= K_d^{j+1} M_d^{-|\eta_{j+1}|} M_d^{|\nu|} |\nu|!\, S_{\eta_{j+1}}(|\nu|).
\end{aligned}$$

Hence, the claim is true for all $0 \leq j \leq k$.

In the second step of the proof, the claimed upper bound on $(c \circ d, \nu)$ is produced in terms of the polynomials $\psi_n(K_d)$. Since $\eta \in I_d(\nu)$ only if $|\eta| \leq |\nu|$, using the inequality (13), it follows that

$$\begin{aligned}
|(c \circ d, \nu)| &= \left| \sum_{i=0}^{|\nu|} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} (c, \eta_j)(\eta_j \circ d, \nu) \right| \\
&\leq \sum_{i=0}^{|\nu|} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} (K_c M^{|\eta_j|} |\eta_j|!) \cdot (K_d^j M^{-|\eta_j|} M^{|\nu|} |\nu|!\, S_{\eta_j}(|\nu|)) \\
&= K_c\, \psi_{|\nu|}(K_d)\, M^{|\nu|} |\nu|!. \qquad \square
\end{aligned}$$

LEMMA 2.11. *For each right factor η_j as defined in* (4) *of a given word $\eta \in X^*$, the following bounds apply:*

$$0 < S_{\eta_j}(n) \leq \frac{(\alpha+1)^{n-|\eta_j|+j}}{\alpha^j |\eta_j|!}$$

for any $\alpha > 0$ and all $n \geq |\eta_j|$.

Proof. The proof is again by induction. The $j = 0$ case is trivial. When $j = 1$, observe that

$$\begin{aligned}
S_{\eta_1}(n) &= \frac{1}{(n)_{n_1+1} |\eta_0|!} \\
&\leq \frac{1}{(|\eta_1|)_{n_1+1} |\eta_0|!}, \quad n \geq |\eta_1|, \\
&= \frac{1}{|\eta_1|!} \\
&\leq \left(\frac{\alpha+1}{\alpha} \right) \frac{(\alpha+1)^{n-|\eta_1|}}{|\eta_1|!}, \quad n \geq |\eta_1|.
\end{aligned}$$

Now suppose the lemma is true up to some fixed $j \geq 1$. Then

$$S_{\eta_{j+1}}(n) = \frac{1}{(n)_{n_{j+1}+1}} \sum_{i=0}^{n-|\eta_{j+1}|} S_{\eta_j}(n - (n_{j+1}+1) - i)$$

$$
\begin{aligned}
&\le \frac{1}{(n)_{n_{j+1}+1}} \sum_{i=0}^{n-|\eta_{j+1}|} \frac{(\alpha+1)^{(n-(n_{j+1}+1)-i)-|\eta_j|+j}}{\alpha^j |\eta_j|!} \\
&\le \frac{(\alpha+1)^j}{\alpha^j |\eta_{j+1}|!} \sum_{i=0}^{n-|\eta_{j+1}|} (\alpha+1)^{n-|\eta_{j+1}|-i}, \quad n \ge |\eta_{j+1}|, \\
&\le \frac{(\alpha+1)^{n-|\eta_{j+1}|+j+1}}{\alpha^{j+1} |\eta_{j+1}|!}.
\end{aligned}
$$

So the result holds for all $j \ge 0$. ∎

The main local convergence theorem for the composition product follows.

THEOREM 2.12. *Suppose $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ with growth constants K_c, M_c and K_d, M_d, respectively. Then $c \circ d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ with*

$$|(c \circ d, \nu)| \le K_c((\phi(mK_d)+1)M)^{|\nu|}(|\nu|+1)! \ \ \forall \nu \in X^*,$$

where $\phi(x) := x/2 + \sqrt{x^2/4 + x}$ and $M = \max\{M_c, M_d\}$.

Proof. In light of Lemma 2.10, the goal is to show that $\psi_n(K_d) \le (\phi(mK_d) + 1)^n(n+1)$ for all $n \ge 0$. Observe that applying Lemma 2.11 gives, for any $\alpha > 0$,

$$
\begin{aligned}
\psi_n(K_d) &\le \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{\substack{\eta_j \in X^i \\ i \ge j}} K_d^j \, \frac{(\alpha+1)^{n-|\eta_j|+j}}{\alpha^j} \\
&= (\alpha+1)^n \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{i}{j} \left(\frac{mK_d}{\alpha}\right)^j \left(\frac{1}{\alpha+1}\right)^{i-j} \\
&= (\alpha+1)^n \sum_{i=0}^{n} \beta^i,
\end{aligned}
$$

where $\beta := mK_d/\alpha + 1/(\alpha+1)$. Setting $\beta = 1$ corresponds to letting $\alpha = \phi(mK_d)$, and the theorem is proved. (Note that $\phi(1) = \phi_g := (1+\sqrt{5})/2$, the golden ratio, and $\phi(mK_d) \approx mK_d$ when $mK_d \gg 1$.) ∎

Example 2.13. In some cases, the coefficient boundaries given in Theorem 2.12 are conservative; i.e., smaller growth constants might be produced by exploiting particular features of the series under consideration. For example, given linear series $c = \sum_{n\ge 0}(c, x_0^n x_1)x_0^n x_1$ and $d = \sum_{n\ge 0}(d, x_0^n x_1)x_0^n x_1$ in $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ with $X = \{x_0, x_1\}$, it can be shown directly that, by writing the composition product as a convolution sum and using the fact that $\sum_{k=0}^{n} \binom{n}{k}^{-1} < 3$ for any $n \ge 0$,

$$|(c \circ d, \nu)| < K_c K_d M^{|\nu|} |\nu|! \ \ \forall \nu \in X^*.$$

3. The nonrecursive connections. In this section the generating series are produced for the three nonrecursive interconnections, and their local convergence is characterized.

THEOREM 3.1. *If $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, then each nonrecursive interconnected input-output system shown in* Figure 1.1(a)–(c) *has a Fliess operator representation generated by a locally convergent series as indicated:*

1. $F_c + F_d = F_{c+d}$;
2. $F_c \cdot F_d = F_{c \,\sqcup\!\sqcup\, d}$;
3. $F_c \circ F_d = F_{c\circ d}$, *where $\ell = m$.*

Proof.
1. Observe that

$$F_c[u](t) + F_d[u](t) = \sum_{\eta\in X^*} [(c,\eta)+(d,\eta)]\, E_\eta[u](t,t_0) = F_{c+d}[u](t).$$

Since c and d are locally convergent, define $M = \max\{M_c, M_d\}$. Then it follows that

$$|(c+d,\eta)| = |(c,\eta)+(d,\eta)| \le (K_c + K_d)M^{|\eta|}|\eta|! \ \ \forall \eta \in X^*,$$

or $c + d$ is locally convergent.

2. In light of the componentwise definition of the product interconnection and the shuffle product, it can be assumed without loss of generality that $\ell = 1$. Therefore,

$$\begin{aligned} F_c[u](t)F_d[u](t) &= \sum_{\eta\in X^*} (c,\eta)E_\eta[u](t,t_0) \sum_{\xi\in X^*} (d,\xi)E_\xi[u](t,t_0) \\ &= \sum_{\eta,\xi\in X^*} (c,\eta)(d,\xi)\, E_{\eta \,\sqcup\!\sqcup\, \xi}[u](t,t_0) \\ &= F_{c \,\sqcup\!\sqcup\, d}[u](t). \end{aligned}$$

Local convergence of $c \sqcup\!\sqcup\, d$ is provided by Theorem 2.8.

3. It is first shown by induction that $F_\eta \circ F_d = F_{\eta\circ d}$ for any $\eta \in X^*$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. Choose any $\eta \in X^*$, and let $\{\eta_0, \eta_1, \ldots, \eta_k\}$ be the corresponding set of right factors defined in (4). Clearly,

$$(F_{\eta_0} \circ F_d[u])(t) = E_{\eta_0}[u](t,t_0) = F_{\eta_0}[u](t) = F_{\eta_0\circ d}[u](t).$$

Now assume that

$$(F_{\eta_j} \circ F_d[u])(t) = F_{\eta_j\circ d}[u](t)$$

holds up to some fixed factor η_j. Then

$$\begin{aligned} (F_{\eta_{j+1}} \circ F_d[u])(t) &= E_{x_0^{n_{j+1}} x_{i_{j+1}} \eta_j}[F_d[u]](t,t_0) \\ &= \underbrace{\int_{t_0}^{t} \cdots \int_{t_0}^{\tau_2}}_{n_{j+1}+1 \text{ times}} F_{d_{i_{j+1}}}[u](\tau_1) E_{\eta_j}[F_d[u]](\tau_1,t_0)\, d\tau_1 \cdots d\tau_{n_{j+1}+1} \\ &= \underbrace{\int_{t_0}^{t} \cdots \int_{t_0}^{\tau_2}}_{n_{j+1}+1 \text{ times}} F_{d_{i_{j+1}} \,\sqcup\!\sqcup\, (\eta_j\circ d)}[u](\tau_1)\, d\tau_1 \cdots d\tau_{n_{j+1}+1} \\ &= F_{x_0^{n_{j+1}+1}[d_{i_{j+1}} \,\sqcup\!\sqcup\, (\eta_j\circ d)]}[u](t) \\ &= F_{\eta_{j+1}\circ d}[u](t). \end{aligned}$$

Thus, the claim holds for $\eta = \eta_{j+1}$ and, by induction, for $\eta = \eta_k$. Finally,

$$\begin{aligned} (F_c \circ F_d[u])(t) &= \sum_{\eta\in X^*} (c,\eta)E_\eta[F_d[u]](t,t_0) = \sum_{\eta\in X^*} (c,\eta)F_{\eta\circ d}[u](t) \\ &= \sum_{\eta\in X^*} (c,\eta) \left[\sum_{\nu\in X^*} (\eta\circ d,\nu)E_\nu[u](t,t_0) \right] \end{aligned}$$

$$= \sum_{\nu \in X^*} \left[\sum_{\eta \in X^*} (c, \eta)(\eta \circ d, \nu) \right] E_\nu[u](t, t_0)$$
$$= \sum_{\nu \in X^*} (c \circ d, \nu)\, E_\nu[u](t, t_0)$$
$$= F_{c \circ d}[u](t).$$

Local convergence of $c \circ d$ was proved in Theorem 2.12. ∎

Example 3.2. Let $X = \{x_0, x_1\}$, $c = \sum_{k\geq 0} K_c M_c^k k!\, x_1^k$, and $d = \sum_{k \geq 0} K_d M_d^k k!\, x_1^k$, where $K_c, M_c > 0$ and $K_d, M_d > 0$ are arbitrary growth constants. It is easily verified that the state space systems,

$$\dot{z}_c = M_c z_c^2 u_c, \quad z_c(0) = 1, \qquad \dot{z}_d = M_d z_d^2 u_d, \quad z_d(0) = 1,$$
$$y_c = K_c z_c, \qquad y_d = K_d z_d,$$

realize the operators $F_c : u_c \mapsto y_c$ and $F_d : u_d \mapsto y_d$, respectively, for sufficiently small inputs and intervals of time. Letting $z = [z_c^T \;\; z_d^T]^T$, it follows directly that $F_{c \circ d}$ is realized by

$$\dot{z} = f(z) + g(z)u, \quad z(0) = [1 \;\; 1]^T, \tag{14}$$
$$y = h(z), \tag{15}$$

where

$$f(z) = \begin{pmatrix} K_d M_c z_c^2 z_d \\ 0 \end{pmatrix}, \quad g(z) = \begin{pmatrix} 0 \\ M_d z_d^2 \end{pmatrix}, \quad h(z) = K_c z_c.$$

The first few coefficients of c, d, and $c \circ d$ are given in Table 3.1 along with the upper bounds on the coefficients of $c \circ d$ predicted by Theorem 2.12. Since these upper bounds hold for *any* series c and d with the given growth constants, they can be

TABLE 3.1
Some coefficients (c, ν), (d, ν), $(c \circ d, \nu)$ and upper bounds for $(c \circ d, \nu)$ in Example 3.2.

ν	(c, ν)	(d, ν)	$(c \circ d, \nu)$	Upper bounds for $(c \circ d, \nu)$
$\emptyset$	K_c	K_d	K_c	K_c
x_0	0	0	$K_c(K_d M_c)$	$K_c((\phi(K_d)+1)M)\ 2!$
x_1	$K_c M_c$	$K_d M_d$	0	$K_c((\phi(K_d)+1)M)\ 2!$
x_0^2	0	0	$K_c(K_d M_c)^2\ 2!$	$K_c((\phi(K_d)+1)M)^2\ 3!$
$x_0 x_1$	0	0	$K_c(K_d M_c) M_d$	$K_c((\phi(K_d)+1)M)^2\ 3!$
$x_1 x_0$	0	0	0	$K_c((\phi(K_d)+1)M)^2\ 3!$
x_1^2	$K_c M_c^2 2!$	$K_d M_d^2 2!$	0	$K_c((\phi(K_d)+1)M)^2\ 3!$
x_0^3	0	0	$K_c(K_d M_c)^3\ 3!$	$K_c((\phi(K_d)+1)M)^3\ 4!$
$x_0^2 x_1$	0	0	$K_c(K_d M_c)^2 M_d\ 2^2$	$K_c((\phi(K_d)+1)M)^3\ 4!$
$x_0 x_1 x_0$	0	0	$K_c(K_d M_c)^2 M_d\ 2$	$K_c((\phi(K_d)+1)M)^3\ 4!$
$x_0 x_1^2$	0	0	$K_c(K_d M_c) M_d^2\ 2$	$K_c((\phi(K_d)+1)M)^3\ 4!$
$x_1 x_0^2$	0	0	0	$K_c((\phi(K_d)+1)M)^3\ 4!$
$x_1 x_0 x_1$	0	0	0	$K_c((\phi(K_d)+1)M)^3\ 4!$
$x_1^2 x_0$	0	0	0	$K_c((\phi(K_d)+1)M)^3\ 4!$
x_1^3	$K_c M_c^3 3!$	$K_d M_d^3 3!$	0	$K_c((\phi(K_d)+1)M)^3\ 4!$

TABLE 3.2
$T_{\max}$ and $t_{\rm esc}$ for specific examples of $c \circ d$ with $\bar{u} = 1$.

Case	K_c	M_c	K_d	M_d	$M_{c\circ d}$	$T_{\max}$	$t_{\rm esc}$	$t_{\rm esc}/T_{\max}$
1	4	2	2	2	7.46	0.03349	0.1967	5.873
2	2	4	2	2	14.93	0.01675	0.1105	6.598
3	2	2	4	2	11.66	0.02145	0.1105	5.152
4	2	2	2	4	14.93	0.01675	0.1580	9.435

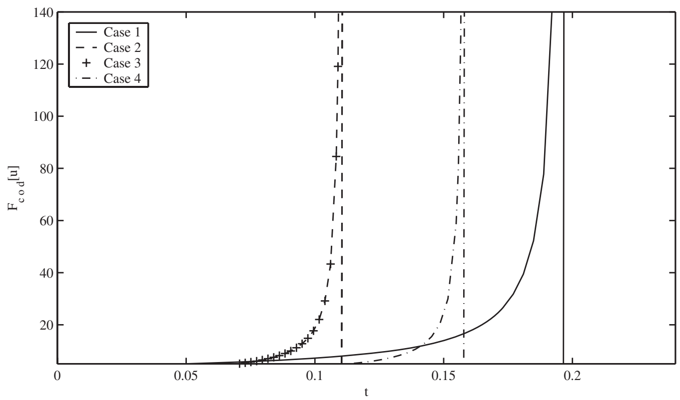

FIG. 3.1. *The output of $F_{c\circ d}[u]$ when $u(t) = \bar{u} = 1$ for Cases* 1–4 *of Table* 3.2.

conservative in specific cases. In [13] it is shown that given any series $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, where $X = \{x_0, x_1, \ldots, x_m\}$ and $|(c,\nu)| \leq K_c M_c^{|\nu|}\, |\nu|!$ for all $\nu \in X^*$, if

$$\max\{||u||_1, T\} \leq \frac{1}{(m+1)^2 M_c},$$

then $F_c[u]$ converges absolutely and uniformly on $[0,T]$. The result still holds if one has the slightly more generous growth condition $|(c,\nu)| \leq K_c M_c^{|\nu|}\,(|\nu|+1)!$. For a constant input $u(t) = \bar{u}$, where $|\bar{u}| \geq 1$, define

$$T_{\max} = \frac{1}{(m+1)^2 M_c |\bar{u}|}. \tag{16}$$

Then it follows from Theorem 2.12 that when $m = 1$, $F_{c\circ d}[\bar{u}]$ will always be well defined on at least the interval $[0, T_{\max})$, where

$$T_{\max} = \frac{1}{4 M_{c\circ d} |\bar{u}|}$$

and $M_{c\circ d} = (\phi(K_d)+1)\max\{M_c, M_d\}$. Four specific cases are described in Table 3.2. Here each $T_{\max}$ is compared against the finite escape time, $t_{\rm esc}$, of the state space system (14)–(15) with $u(t) = \bar{u} = 1$, which is determined numerically (see Figure 3.1). In each case, the value of $T_{\max} < t_{\rm esc}$, but, as expected, $T_{\max}$ is conservative since the coefficient upper bounds for $c \circ d$ are conservative.

Example 3.3. The composition product provides an alternative interpretation of the symbolic calculus of Fliess [8, 10, 19]. Specifically, consider an input-output system represented by F_c with $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$. Any input u, which is analytic at $t = t_0$, can be represented near t_0 by a series $c_u \in \mathbb{R}^{m}_{LC}\langle\langle X_0\rangle\rangle$, i.e., $u = F_{c_u}[v]$ for some locally convergent series $c_u = \sum_{k\geq 0}(c_u, x_0^k)\, x_0^k$ and all $\nu \in B_p^m(R)[t_0, t_0+T]$. In effect, c_u is the formal Laplace–Borel transform of the input u. (See [20] for more analysis of this example using the formal Laplace–Borel transform.) The analyticity of $y = F_c[u]$ follows from [28, Lemma 2.3.8], and therefore the formal Laplace–Borel transform of y, namely, c_y, can be related to c and c_u via

$$F_{c_y}[v] = y = F_c[F_{c_u}[v]] = F_{c\circ c_u}[v].$$

From [28, Corollary 2.2.4], it follows directly that $c_y = c \circ c_u$.

This last example motivates the following definition.

DEFINITION 3.4. *A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is* input-output locally convergent *if for every $c_u \in \mathbb{R}^m_{LC}\langle\langle X_0\rangle\rangle$ it follows that $c \circ c_u \in \mathbb{R}^\ell_{LC}\langle\langle X_0\rangle\rangle$.*

It is immediate that every locally convergent series is input-output locally convergent, but the converse claim is only known to hold at present in certain special cases.

LEMMA 3.5. *Let $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ be an input-output locally convergent series with nonnegative coefficients. Then c is locally convergent.*

Proof. Set $c_u = \mathbb{1}$ and let K, M be the growth constants for the series $c \circ \mathbb{1}$. Then from Lemma 2.2, property 4,

$$|(c\circ \mathbb{1}, x_0^n)| = \max_i \sum_{\eta\in X^n} (c_i, \eta) \leq KM^n\, n! \;\; \forall n \geq 0.$$

Thus, $|(c,\eta)| = \max_i (c_i, \eta) \leq KM^n\, n!$ for all $n \geq 0$. $\square$

LEMMA 3.6. *Let $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ be an input-output locally convergent linear series of the form $c = \sum_{j\geq 0}(c, x_0^j x_{i_j})\, x_0^j x_{i_j}$, where $i_j \in \{1, 2, \ldots, m\}$ for all $j \geq 0$. Then c is locally convergent.*

Proof. Again set $c_u = \mathbb{1}$ and let K, M be the growth constants for the series $c \circ \mathbb{1}$. Then

$$|(c\circ \mathbb{1}, x_0^n)| = \max_i |(c_i, x_0^{n-1} x_{i_n})| \leq KM^n\, n! \;\; \forall n \geq 0,$$

and the conclusion follows. $\square$

4. The feedback connection. Given any $c, d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$, the general goal of this section is to determine when there exists a y which satisfies the feedback equation (1) and, in particular, when there exists a generating series e so that $y = F_e[u]$ for all admissible inputs u. In the latter case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d\circ e}[u]], \tag{17}$$

and the *feedback product* of c and d is defined by $c@d = e$. It is assumed throughout that $m > 0$; otherwise the feedback connection is degenerate. An initial obstacle in this analysis is that F_e is required to be the composition of two operators, F_c and $I + F_{d\circ e}$, where the second operator is *never* realizable by a Fliess operator due to the direct feed term I. This does not prevent the composition from being a Fliess operator, but to compensate for the presence of this term a *modified* composition product is needed. Specifically, for any $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, define the modified composition product as

$$\eta \,\tilde{\circ}\, d = \begin{cases} \eta & : \;\; |\eta|_{x_i} = 0 \;\; \forall i \neq 0, \\ x_0^n x_i(\eta' \,\tilde{\circ}\, d) + x_0^{n+1}[d_i \sqcup\!\sqcup (\eta' \,\tilde{\circ}\, d)] & : \;\; \eta = x_0^n x_i \eta', \;\; n \geq 0, \;\; i \neq 0. \end{cases}$$

For $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the definition is extended as

$$c \,\tilde{\circ}\, d = \sum_{\eta \in X^*} (c, \eta)\, \eta \,\tilde{\circ}\, d.$$

It can be verified in a manner completely analogous to the original composition product that the modified composition product is always well defined (summable), continuous in both arguments, and locally convergent when both c and d are. In particular, the following theorems are central to the analysis in this section.

THEOREM 4.1. *For any $c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$, it follows that*

$$F_{c\,\tilde{\circ}\,d}[u] = F_c[u + F_d[u]]$$

for all admissible u.

Proof. The result is verified simply by inserting the direct feed term into the proof of Theorem 3.1, part 3. □

THEOREM 4.2. *For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $d \mapsto c \,\tilde{\circ}\, d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$.*

Proof. This is also a minor variation of previous results concerning the composition product, in particular, Lemma 2.4, Lemma 2.5, and Theorem 2.6. The contraction coefficient, σ, is unaffected by the required modifications. □

The first main result of this section is given next.

THEOREM 4.3. *Let c, d be fixed series in $\mathbb{R}^m\langle\langle X\rangle\rangle$. Then the following propositions hold:*

1. *The mapping*

$$\begin{aligned} S : \mathbb{R}^m\langle\langle X\rangle\rangle &\to \mathbb{R}^m\langle\langle X\rangle\rangle \\ : e_i &\mapsto e_{i+1} = c \,\tilde{\circ}\, (d \circ e_i) \end{aligned} \tag{18}$$

has a unique fixed point in $\mathbb{R}^m\langle\langle X\rangle\rangle$, $c@d = \lim_{i\to\infty} e_i$, which is independent of e_0.

2. *If c, d, and $c@d$ are locally convergent, then $F_{c@d}$ satisfies the feedback equation* (17).

Proof.

1. The mapping S is a contraction since, by Theorems 2.6 and 4.2,

$$\mathrm{dist}(S(e_i), S(e_j)) \leq \sigma\, \mathrm{dist}(d \circ e_i, d \circ e_j) \leq \sigma^2\, \mathrm{dist}(e_i, e_j).$$

Therefore, the mapping S has a unique fixed point, $c@d$, that is independent of e_0, i.e.,

$$c@d = c \,\tilde{\circ}\, (d \circ (c@d)). \tag{19}$$

2. From the stated assumptions concerning c, d, and $c@d$, it follows that

$$F_{c@d}[u] = F_{c\,\tilde{\circ}\,(d\circ(c@d))}[u] = F_c[u + F_d[F_{c@d}[u]]]$$

for any admissible u. □

The obvious question is whether $c@d$ is always locally convergent, or at least input-output locally convergent, when both c and d are locally convergent. The local convergence of c and d guarantees that the feedback system in Figure 1.1(d) is at least *well-posed* in the sense described in [1, 27] since F_c and F_d are well-defined causal analytic operators. That is, there exist sufficiently small $R, S, T > 0$ such that for any $u \in B_p^m(R)[t_0, t_0+T]$, there exists a $y \in B_q^m(S)[t_0, t_0+T]$ which satisfies the feedback equation (1). But whether $y = F_{c@d}[u]$ on some ball of input functions of nonzero radius over a nonzero interval of time is not immediate. The following example shows that $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ is not a closed subset of $\mathbb{R}^m\langle\langle X\rangle\rangle$ in the ultrametric topology.

Example 4.4. Let $X = \{x_0, x_1\}$ and consider the following sequence of polynomials in $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$:

$$e_i = x_1 + 2^2 2!\, x_1^2 + 3^3 3!\, x_1^3 + \cdots + i^i i!\, x_1^i, \quad i \geq 1.$$

Clearly, $e = \lim_{i\to\infty} e_i$ is not locally convergent.

A central issue is whether such an example can be produced by repeated compositions of a locally convergent series. It will be first shown that the answer to this question is *no*. Then the more general case described by (18) is examined. This leads to the main conclusion that the feedback product of two locally convergent series is always input-output locally convergent.

Observe first that if $e = c \circ e$, then it follows that e must have the form $e = \sum_{n\geq 0}(e, x_0^n)\, x_0^n$. Furthermore, since e appears on both sides of the expression $e = c\circ e$, it is possible by repeated substitution to express each coefficient (e, x_0^n) in terms of the coefficients $\{(c,\nu) : |\nu| \leq n\}$. For example, if $X = \{x_0, x_1\}$, the first few coefficients of e are

$$\begin{aligned}
(e,\emptyset) &= (c,\emptyset),\\
(e,x_0) &= (c,x_0) + (c,\emptyset)(c,x_1),\\
(e,x_0^2) &= (c,x_0^2) + (c,x_0)(c,x_1) + (c,\emptyset)(c,x_1)^2 + (c,\emptyset)(c,x_0x_1) + (c,\emptyset)(c,x_1x_0)\\
&\quad +(c,\emptyset)^2(c,x_1^2),\\
(e,x_0^3) &= (c,x_0^2)(c,x_1) + (c,x_0)(c,x_1)^2 + (c,\emptyset)(c,x_1)^3 + (c,\emptyset)(c,x_1)(c,x_0x_1)\\
&\quad +(c,\emptyset)(c,x_1)(c,x_1x_0) + (c,\emptyset)^2(c,x_1)(c,x_1^2) + (c,x_0)(c,x_0x_1)\\
&\quad +(c,\emptyset)(c,x_1)(c,x_0x_1) + 2(c,x_0)(c,x_1x_0) + 2(c,\emptyset)(c,x_1)(c,x_1x_0)\\
&\quad +3(c,\emptyset)(c,x_0)(c,x_1^2) + 3(c,\emptyset)^2(c,x_1)(c,x_1^2) + (c,x_0^3) + (c,\emptyset)(c,x_0^2x_1)\\
&\quad +(c,\emptyset)(c,x_0x_1x_0) + (c,\emptyset)^2(c,x_0x_1^2) + (c,\emptyset)(c,x_1x_0^2) + (c,\emptyset)^2(c,x_1x_0x_1)\\
&\quad +(c,\emptyset)^2(c,x_1^2x_0) + (c,\emptyset)^3(c,x_1^3)\\
&\;\;\vdots
\end{aligned}$$

If c is locally convergent with growth constants K_c, M_c, then

$$\begin{aligned}
|(e,\emptyset)| &\leq K_c,\\
|(e,x_0)| &\leq K_c(K_c+1)M_c,\\
|(e,x_0^2)| &\leq K_c\left(\frac{3}{2}K_c^2 + \frac{5}{2}K_c + 1\right)M_c^2\, 2!,\\
|(e,x_0^3)| &\leq K_c\left(\frac{5}{2}K_c^3 + \frac{35}{6}K_c^2 + \frac{13}{3}K_c + 1\right)M_c^3\, 3!\\
&\;\;\vdots
\end{aligned}$$

This suggests that a variation of inequality (12) is possible, namely, that

$$|(e,x_0^n)| \leq K_c\, \tilde{\psi}_n(K_c)\, M_c^n\, n! \;\; \forall n \geq 0,$$

Table 4.1
The first few polynomials $\tilde{S}_{\eta_j}(K_c,n)$ and $\tilde{\psi}_n(K_c)$ when $m=1$.

n	η_j	$\tilde{S}_{\eta_0}(K_c,n),\dots,\tilde{S}_{\eta_j}(K_c,n)$	$\tilde{\psi}_n(K_c)$
0	$\emptyset$	$\tilde{S}_{\emptyset}(K_c,0)=1$	1
1	x_0 x_1	$\tilde{S}_{x_0}(K_c,1)=1$ $\tilde{S}_{\emptyset}(K_c,1)=1,\ \tilde{S}_{x_1}(K_c,1)=1$	K_c+2
2	x_0^2 x_0x_1 x_1x_0 x_1^2	$\tilde{S}_{x_0^2}(K_c,2)=\frac{1}{2}$ $\tilde{S}_{\emptyset}(K_c,2)=1,\ \tilde{S}_{x_0x_1}(K_c,2)=\frac{1}{2}$ $\tilde{S}_{x_0}(K_c,2)=1,\ \tilde{S}_{x_1x_0}(K_c,2)=\frac{1}{2}$ $\tilde{S}_{\emptyset}(K_c,2)=1,\ \tilde{S}_{x_1}(K_c,2)=\frac{1}{2}K_c+1,$ $\tilde{S}_{x_1^2}(K_c,2)=\frac{1}{2}$	$\frac{3}{2}K_c^2+3K_c+3$
3	x_0^3 $x_0^2x_1$ $x_0x_1x_0$ $x_0x_1^2$ $x_1x_0^2$ $x_1x_0x_1$ $x_1^2x_0$ x_1^3	$\tilde{S}_{x_0^3}(K_c,3)=\frac{1}{6}$ $\tilde{S}_{\emptyset}(K_c,3)=1,\ \tilde{S}_{x_0^2x_1}(K_c,3)=\frac{1}{6}$ $\tilde{S}_{x_0}(K_c,3)=1,\ \tilde{S}_{x_0x_1x_0}(K_c,3)=\frac{1}{6}$ $\tilde{S}_{\emptyset}(K_c,3)=1,\ \tilde{S}_{x_1}(K_c,3)=\frac{1}{2}K_c^2+K_c+1,$ $\tilde{S}_{x_0x_1^2}(K_c,3)=\frac{1}{6}$ $\tilde{S}_{x_0^2}(K_c,3)=\frac{1}{2},\ \tilde{S}_{x_1x_0^2}(K_c,3)=\frac{1}{6}$ $\tilde{S}_{\emptyset}(K_c,3)=1,\ \tilde{S}_{x_0x_1}(K_c,3)=\frac{1}{6}K_c+\frac{1}{3},$ $\tilde{S}_{x_1x_0x_1}(K_c,3)=\frac{1}{6}$ $\tilde{S}_{x_0}(K_c,3)=1,\ \tilde{S}_{x_1x_0}(K_c,3)=\frac{1}{3}K_c+\frac{2}{3},$ $\tilde{S}_{x_1^2x_0}(K_c,3)=\frac{1}{6}$ $\tilde{S}_{\emptyset}(K_c,3)=1,\ \tilde{S}_{x_1}(K_c,3)=\frac{1}{2}K_c^2+K_c+1,$ $\tilde{S}_{x_1^2}(K_c,3)=\frac{1}{2}K_c+1,\ \tilde{S}_{x_1^3}(K_c,3)=\frac{1}{6}$	$\frac{5}{2}K_c^3+7K_c^2+6K_c+4$

where each $\tilde{\psi}_n(K_c)$ is a polynomial in K_c of degree n. The next lemma establishes the claim using a family of polynomials of the form

$$\tilde{\psi}_n(K_c)=\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{\eta_j\in X^i}K_c^j\tilde{S}_{\eta_j}(K_c,n)|\eta_j|!,\quad n\geq 0.$$

Given a fixed n, every word η_j in the innermost summation satisfies $j\leq|\eta_j|\leq n$ and has a corresponding set of right factors $\{\eta_0,\eta_1,\dots,\eta_j\}$. When $j>0$, each polynomial $\tilde{S}_{\eta_j}(K_c,n)$ is computed iteratively using its right factors and the previously computed polynomials $\{\tilde{\psi}_0(K_c),\tilde{\psi}_1(K_c),\dots,\tilde{\psi}_{n-1}(K_c)\}$:

$$\begin{aligned}
\tilde{S}_{\eta_0}(K_c,n)&=\frac{1}{|\eta_0|!},\quad 0\leq|\eta_0|\leq n,\\
\tilde{S}_{\eta_1}(K_c,n)&=\frac{1}{(n)_{n_1+1}}\,\tilde{\psi}_{n-|\eta_1|}(K_c)\,\tilde{S}_{\eta_0}(K_c,n),\quad 1\leq|\eta_1|\leq n,\\
\tilde{S}_{\eta_2}(K_c,n)&=\frac{1}{(n)_{n_2+1}}\sum_{i=0}^{n-|\eta_2|}\tilde{\psi}_i(K_c)\,\tilde{S}_{\eta_1}(K_c,n-(n_2+1)-i),\quad 2\leq|\eta_2|\leq n,\\
&\ \vdots\\
\tilde{S}_{\eta_j}(K_c,n)&=\frac{1}{(n)_{n_j+1}}\sum_{i=0}^{n-|\eta_j|}\tilde{\psi}_i(K_c)\,\tilde{S}_{\eta_{j-1}}(K_c,n-(n_j+1)-i),\quad 2\leq j\leq|\eta_j|\leq n.
\end{aligned}$$

See Table 4.1 for the case where $m=1$.

LEMMA 4.5. *Let $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants K_c, M_c, and $e \in \mathbb{R}^m\langle\langle X\rangle\rangle$ such that $e = c \circ e$. Then*

$$|(e, x_0^n)| \leq K_c\, \tilde{\psi}_n(K_c)\, M_c^n\, n!\ \ \forall n \geq 0. \tag{20}$$

Proof. The proof has some elements in common with that of Lemma 2.10, except here it is not assumed a priori that e is locally convergent. The basic approach employs *nested inductions*. The outer induction is on n. It is clear from the discussion above that the claim holds when $n = 0$ and $n = 1$ for $m = 1$. A similar calculation can be done for arbitrary $m \geq 1$. Now suppose (20) holds up to some fixed $n - 1 \geq 1$. Given any η_j, where $j \leq |\eta_j| \leq n$, it will first be shown by induction on j (the inner induction) that

$$|(\eta_j \circ e, x_0^n)| \leq K_c^j M_c^{-|\eta_j|} M_c^n\, n!\, \tilde{S}_{\eta_j}(K_c, n),\ \ 0 \leq j \leq n. \tag{21}$$

The $j = 0$ case is trivial. Suppose $j = 1$. Then $0 \leq n - |\eta_1| \leq n - 1$ and

$$\begin{aligned}
|(\eta_1 \circ e, x_0^n)| &= \left|\left(x_0^{n_1+1}(e_{i_1} \sqcup\!\sqcup\, x_0^{n_0}), x_0^n\right)\right| \\
&= \left|\left(e_{i_1} \sqcup\!\sqcup\, x_0^{n_0}, x_0^{n-(n_1+1)}\right)\right| \\
&= \left|\left(e_{i_1}, x_0^{n-|\eta_1|}\right)\left(x_0^{n-|\eta_1|} \sqcup\!\sqcup\, x_0^{n_0}, x_0^{n-(n_1+1)}\right)\right| \\
&\leq \left(K_c\, \tilde{\psi}_{n-|\eta_1|}(K_c)\, M_c^{n-|\eta_1|}(n-|\eta_1|)!\right)\binom{n-(n_1+1)}{n-|\eta_1|} \\
&= K_c M_c^{-|\eta_1|} M_c^n\, n!\, \tilde{S}_{\eta_1}(K_c, n).
\end{aligned}$$

Now assume that inequality (21) holds up to some fixed j, where $1 \leq j \leq n-1$. Then $0 \leq n - |\eta_{j+1}| \leq n - (j+1)$ and

$$\begin{aligned}
|(\eta_{j+1} \circ e, x_0^n)| &= \left|\left(e_{i_{j+1}} \sqcup\!\sqcup\, (\eta_j \circ e), x_0^{n-(n_{j+1}+1)}\right)\right| \\
&= \left|\sum_{i=0}^{n-(n_{j+1}+1)} \left(e_{i_{j+1}}, x_0^i\right)\left(\eta_j \circ e, x_0^{n-(n_{j+1}+1)-i}\right)\binom{n-(n_{j+1}+1)}{n-(n_{j+1}+1)-i}\right|.
\end{aligned}$$

Since $(\eta_j \circ e, x_0^{n-(n_{j+1}+1)-i}) = 0$ when $n - (n_{j+1}+1) - i < |\eta_j|$ or, equivalently, $i > n - |\eta_{j+1}|$, it follows that, using the coefficient bound (20) for e (because $0 \leq i \leq n-1$) and the bound (21) for $\eta_j \circ e$,

$$\begin{aligned}
|(\eta_{j+1} \circ e, x_0^n)| \leq \sum_{i=0}^{n-|\eta_{j+1}|} &\left(K_c \tilde{\psi}_i(K_c) M_c^i\, i!\right)\Big(K_c^j M_c^{-|\eta_j|} M_c^{n-(n_{j+1}+1)-i} \\
&\cdot (n-(n_{j+1}+1)-i)!\, \tilde{S}_{\eta_j}(K_c, n-(n_{j+1}+1)-i)\Big) \\
&\cdot \binom{n-(n_{j+1}+1)}{n-(n_{j+1}+1)-i} \\
= K_c^{j+1} M_c^{-|\eta_{j+1}|} &M_c^n\, n!\, \frac{1}{(n)_{n_{j+1}+1}} \\
&\cdot \sum_{i=0}^{n-|\eta_{j+1}|} \tilde{\psi}_i(K_c)\, \tilde{S}_{\eta_j}(K_c, n-(n_{j+1}+1)-i) \\
= K_c^{j+1} M_c^{-|\eta_{j+1}|} &M_c^n\, n!\, \tilde{S}_{\eta_{j+1}}(K_c, n).
\end{aligned}$$

Hence, the claim is true for all $0 \leq j \leq n$.

To complete the outer induction with respect to n, observe that

$$\begin{aligned}
|(e, x_0^n)| = |(c \circ e, x_0^n)| &= \left| \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} (c, \eta_j)(\eta_j \circ e, x_0^n) \right| \\
&\leq \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} \left(K_c M_c^{|\eta_j|} |\eta_j|! \right) \left(K_c^j M_c^{-|\eta_j|} M_c^n \, n! \, \tilde{S}_{\eta_j}(K_c, n) \right) \\
&= K_c \, \tilde{\psi}_n(K_c) \, M_c^n \, n!.
\end{aligned}$$

Therefore, inequality (20) holds for all $n \geq 0$. ∎

The next lemma provides an upper bound on the growth of the sequence $\tilde{\psi}_n(K_c)$, $n \geq 0$, when K_c is fixed.

Lemma 4.6. *For any $K_c \geq 1$, it follows that*

$$\tilde{\psi}_n(K_c) \leq \phi_g (m K_c (2 + \phi_g) + 1)^n s_n \quad \forall n \geq 0, \tag{22}$$

where $s_0 = 1/\phi_g$, and s_n, $n \geq 1$, is an integer sequence equivalent to the binomial transform of the sequence of Catalan numbers, C_n, $n \geq 1$ (specifically, sequence A007317 *in* [25]).

Proof. The proof has two main parts. First, it is shown by a nested induction that, for any $\epsilon > 0$, there exists a sequence of positive real numbers, $\xi_n(\epsilon)$, such that

$$\tilde{\psi}_n(K_c) \leq (m K_c (2 + \epsilon) + 1)^n \xi_n(\epsilon), \quad n \geq 0, \quad K_c \geq 1. \tag{23}$$

Then inequality (22) is produced for $n \geq 1$ by setting $\epsilon = \phi_g$ and showing that $\xi_n(\phi_g) = \phi_g s_n$ when $n \geq 1$. ($n = 0$ is a trivial special case.)

Let $\epsilon > 0$ and define two sequences of positive real numbers, $\xi_n(\epsilon)$ and $\Gamma_n(\epsilon)$, via the recurrence equations

$$\xi_{n+1}(\epsilon) = \xi_n(\epsilon) + \Gamma_{n+1}(\epsilon), \quad n \geq 0, \quad \xi_0 = 1, \quad \Gamma_1 = 1/\epsilon, \tag{24}$$

$$\Gamma_{n+1}(\epsilon) = \frac{1}{\epsilon} \left[\xi_n(\epsilon) + \sum_{i=1}^{n} \xi_i(\epsilon) \Gamma_{n-i+1}(\epsilon) \right], \quad n \geq 1. \tag{25}$$

By definition, $\Gamma_0 = 1$. In light of Table 4.1, inequality (23) clearly holds when $n = 0$ and $n = 1$ for $m = 1$ and $K_c \geq 1$. (It is easily verified to also hold when $m \geq 1$.) Now suppose the inequality holds up to some fixed $n - 1 \geq 1$. Given any word η_j, where $j \leq |\eta_j| \leq n$, an inner induction with respect to j will now show that

$$\tilde{S}_{\eta_j}(K_c, n) \leq \frac{(m K_c (2 + \epsilon) + 1)^{n - |\eta_j|} \, (2 + \epsilon)^j \, \Gamma_{n - |\eta_j|}(\epsilon)}{|\eta_j|!}, \quad 0 \leq j \leq |\eta_j| \tag{26}$$

(cf. the proof of Lemma 2.11, where some of the computational details are similar). The $j = 0$ case is trivial. Suppose $j = 1$. Since $n - |\eta_1| < n$, it follows that

$$\begin{aligned}
\tilde{S}_{\eta_1}(K_c, n) &= \frac{1}{(n)_{n_1+1}} \frac{\tilde{\psi}_{n - |\eta_1|}(K_c)}{|\eta_0|!} \\
&\leq \frac{(m K_c (2 + \epsilon) + 1)^{n - |\eta_1|} \, \xi_{n - |\eta_1|}(\epsilon)}{|\eta_1|!} \\
&\leq \frac{(m K_c (2 + \epsilon) + 1)^{n - |\eta_1|} \, (2 + \epsilon) \, \Gamma_{n - |\eta_1|}(\epsilon)}{|\eta_1|!}, \quad n \geq |\eta_1|.
\end{aligned}$$

This last inequality employs the general properties for any $j \geq 0$ that $\xi_{n-|\eta_j|}(\epsilon) = \Gamma_{n-|\eta_j|}(\epsilon)$ when $n = |\eta_j|$ and

$$\sum_{i=0}^{n-|\eta_j|} \xi_i(\epsilon)\Gamma_{n-|\eta_j|-i}(\epsilon) = (2+\epsilon)\,\Gamma_{n-|\eta_j|}(\epsilon) \tag{27}$$

when $n > |\eta_j|$. Now suppose that inequality (26) holds up to some fixed $j \geq 1$. Then

$$\begin{aligned}
\tilde{S}_{\eta_{j+1}}(K_c, n) &= \frac{1}{(n)_{n_{j+1}+1}} \sum_{i=0}^{n-|\eta_{j+1}|} \tilde{\psi}_i(K_c)\,\tilde{S}_{\eta_j}(K_c, n-(n_{j+1}+1)-i) \\
&\leq \frac{1}{|\eta_{j+1}|!} \sum_{i=0}^{n-|\eta_{j+1}|} (mK_c(2+\epsilon)+1)^i \xi_i(\epsilon) \\
&\qquad \cdot \left[(mK_c(2+\epsilon)+1)^{n-|\eta_{j+1}|-i}\,(2+\epsilon)^j\,\Gamma_{n-|\eta_{j+1}|-i}(\epsilon)\right] \\
&= \frac{(mK_c(2+\epsilon)+1)^{n-|\eta_{j+1}|}(2+\epsilon)^j}{|\eta_{j+1}|!} \sum_{i=0}^{n-|\eta_{j+1}|} \xi_i(\epsilon)\Gamma_{n-|\eta_{j+1}|-i}(\epsilon) \\
&= \frac{(mK_c(2+\epsilon)+1)^{n-|\eta_{j+1}|}\,(2+\epsilon)^{j+1}\,\Gamma_{n-|\eta_{j+1}|}(\epsilon)}{|\eta_{j+1}|!}, \quad |\eta_j| < |\eta_{j+1}| \leq n,
\end{aligned}$$

where again identity (27) was used to derive the final equality above. Hence, inequality (26) holds for all $0 \leq j \leq |\eta_j|$. To complete the outer induction with respect to n, observe that

$$\begin{aligned}
\tilde{\psi}_{n+1}(K_c) &= \sum_{i=0}^{n+1} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} K_c^j \tilde{S}_{\eta_j}(K_c, n+1)|\eta_j|! \\
&\leq \sum_{i=0}^{n+1} \sum_{j=0}^{i} \binom{i}{j} \left[\frac{(mK_c(2+\epsilon)+1)^{n+1-i}\,(mK_c(2+\epsilon))^j\,\Gamma_{n+1-i}(\epsilon)}{i!}\right] i! \\
&= (mK_c(2+\epsilon)+1)^{n+1} \sum_{i=0}^{n+1} \Gamma_{n+1-i}(\epsilon) \\
&= (mK_c(2+\epsilon)+1)^{n+1} \xi_{n+1}(\epsilon).
\end{aligned}$$

Thus, inequality (23) must hold for all $n \geq 0$.

Now consider setting $\epsilon = \phi_g$ in the system of equations (24)–(25). Eliminating by substitution the sequence $\Gamma_n(\phi_g)$ gives the recurrence relation

$$\xi_{n+1}(\phi_g) = \phi_g + \frac{1}{\phi_g} \sum_{i=1}^{n} \xi_i(\phi_g)\xi_{n-i+1}(\phi_g), \quad n \geq 1, \quad \xi_1(\phi_g) = \phi_g,$$

or, equivalently,

$$\left(\frac{\xi_{n+1}(\phi_g)}{\phi_g}\right) = 1 + \sum_{i=1}^{n} \left(\frac{\xi_i(\phi_g)}{\phi_g}\right)\left(\frac{\xi_{n-i+1}(\phi_g)}{\phi_g}\right), \quad n \geq 1, \quad \frac{\xi_1(\phi_g)}{\phi_g} = 1.$$

It is known that s_n satisfies the recurrence equation

$$s_{n+1} = 1 + \sum_{i=1}^{n} s_i s_{n-i+1}, \quad n \geq 1, \quad s_1 = 1 \tag{28}$$

(see [25] and the references therein). Hence, the conclusion that $\xi_n(\phi_g) = \phi_g s_n$, $n \geq 1$, is immediate. □

The recurrence equation (28) can be derived from the well-known recurrence relation for the Catalan numbers: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ with $C_0 = 1$ [4], which in turn is equivalent to Segner's recurrence formula given in the year 1758 as a solution to Euler's polygon division problem [31]. It is also worth noting that the sequence $t_n := \Gamma_n(\phi_g)/\phi_g$, $n \geq 1$, the increments of s_n, is sequence A002212 in [25]. The positive integer sequences C_n, s_n, and t_n each have a variety of combinatoric interpretations in graph theory and the theory of formal languages. Of particular interest to system theorists, for example, is the fact that C_n is equivalent to the number of ways to binary bracket the letters in a word of length $n+1$ [31, 32]. The asymptotic behavior of s_n,

$$s_n \sim \frac{1}{8}\sqrt{\frac{5}{\pi}}\,\frac{5^n}{n^{3/2}}$$

(see [15, sequence 124]), motivates the following central result concerning local convergence.

THEOREM 4.7. *If $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants K_c, M_c, and $e = c \circ e$, then $e \in \mathbb{R}^m_{LC}\langle\langle X_0\rangle\rangle$. Specifically, for any $K_c \geq 1$,*

$$|(e, x_0^n)| \leq K_c((mK_c(2+\phi_g)+1)5M_c)^n\; n! \;\; \forall n \geq 0. \tag{29}$$

Proof. The result is trivial when $n = 0$. When $n \geq 1$, it is first necessary to show by induction that $s_{n+1} < 5s_n$. The claim is clearly true when $n = 1$ or $n = 2$. Suppose it is known to hold up to some fixed integer $n+1 \geq 2$. Sequence s_n is known to satisfy another recurrence equation [15, 25]:

$$(n+2)s_{n+2} = (6n+4)s_{n+1} - 5ns_n.$$

Therefore,

$$s_{n+2} < [(6n+4)s_{n+1} - ns_{n+1}]/(n+2) < 5s_{n+1},$$

which proves the claim for all $n \geq 1$. Next, substituting the upper bound $\phi_g s_n \leq 5^n, n \geq 0$, into (22) gives

$$\tilde{\psi}_n(K_c) \leq ((mK_c(2+\phi_g)+1)5)^n \;\; \forall n \geq 0. \tag{30}$$

The theorem is finally proved by simply applying Lemma 4.5. □

In most cases the upper bound in (29) is quite conservative because the upper bound (30) is conservative. Figure 4.1 shows $\tilde{\psi}_n(K_c)$ (generated symbolically via MAPLE) and upper bound (30) versus n for various values of K_c.

The final step of the analysis is to use Theorem 4.7 to prove the input-output local convergence of the feedback product.

THEOREM 4.8. *If $c, d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$, then $c@d$ is input-output locally convergent. Specifically, when $K_c \geq 1$, then*

$$((c@d)\circ b, x_0^n) \leq K_c([mK_c(2+\phi_g)+1][\phi(m(K_b+K_d))+1]10M)^n\; n!$$

for any $b \in \mathbb{R}^m_{LC}\langle\langle X_0\rangle\rangle$ and where $M = \max\{M_b, M_c, M_d\}$.

Proof. Select any series $b \in \mathbb{R}^m_{LC}\langle\langle X_0\rangle\rangle$. It follows from (19) that

$$(c@d)\circ b = (c\,\tilde{\circ}\,(d\circ(c@d)))\circ b = c\circ(b+d)\circ((c@d)\circ b).$$

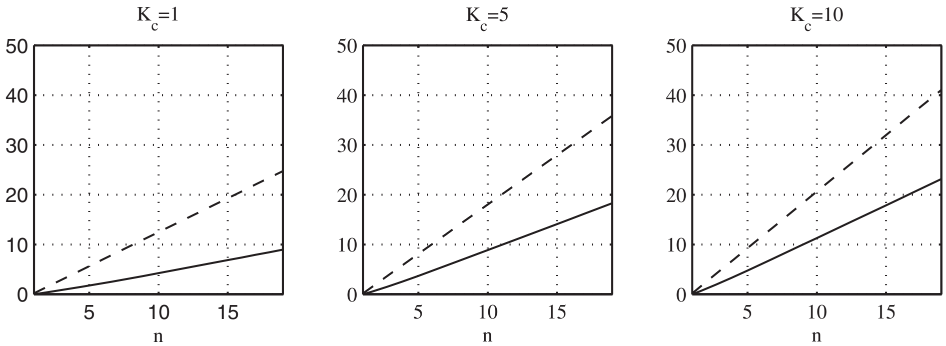

FIG. 4.1. *A plot of* $\log_{10}(\tilde{\psi}_n(K_c))$ (*solid lines*) *and the logarithm* (*base* 10) *of the upper bound in* (30) (*dashed lines*) *versus* n *for various values of* K_c.

Since b, c, and d are all locally convergent, so is the series $c \circ (b + d)$. Now apply Theorem 4.7, replacing c with $c \circ (b + d)$ and e with $(c@d) \circ b$. This implies that $(c@d) \circ b$ is always locally convergent, and therefore $c@d$ must be input-output locally convergent. To produce the given growth condition for the output series, note that

$$K_{c\circ(b+d)} = K_c \qquad M_{c\circ(b+d)} = 2(\phi(m(K_b + K_d)) + 1)M,$$

using Theorem 2.12 and the fact that $n + 1 \leq 2^n$ for all $n \geq 0$. Substituting these growth constants for K_c and M_c, respectively, in Theorem 4.7 produces the desired result. $\square$

Example 4.9. Suppose c and d are linear series in $\mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$. Then $c@d = \lim_{i\to\infty} e_i$, where

$$e_{i+1} = c \,\tilde{\circ}\, (d \circ e_i) = c + (c \circ d) \circ e_i.$$

Setting $e_0 = c$ gives

$$c@d = c + \sum_{k=1}^{\infty} (c \circ d)^{\circ k} \circ c,$$

where $c^{\circ k}$ denotes k copies of c composed $k - 1$ times. It is easily verified since $(c, \emptyset) = 0$ that $((c \circ d)^{\circ k}, \nu) = 0$ for all $k > |\nu|$. Hence,

$$(c@d, \nu) = (c, \nu) + \sum_{k=1}^{|\nu|-1} ((c \circ d)^{\circ k} \circ c, \nu).$$

Example 4.10. For any $c, d \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$, a self-excited feedback loop can be described by $F_{c@d}[0] = F_{(c@d)\circ 0}[u] = F_{(c@d)_0}[u]$ (cf. Lemma 2.2, property 2). In this case $(c@d)_0 = \lim_{i\to\infty} e_i$, where $e_{i+1} = (c \circ d) \circ e_i$. Using the $m = 0$ version of (16)

Table 4.2
Some coefficients (c,ν), (d,ν), *and* $(c@d,\nu)$ *in Example* 4.11.

ν	(c,ν)	(d,ν)	$(c@d,\nu)$
$\emptyset$	K_c	K_d	K_c
x_0	0	0	$K_cK_dM_c$
x_1	K_cM_c	K_dM_d	K_cM_c
x_0^2	0	0	$K_c((K_dM_c)^2\,2! + K_cK_dM_cM_d)$
x_0x_1	0	0	$K_cK_dM_c^2\,2!$
x_1x_0	0	0	$K_cK_dM_c^2\,2!$
x_1^2	$K_cM_c^2 2!$	$K_dM_d^2 2!$	$K_cM_c^2\,2!$
x_0^3	0	0	$K_c((K_dM_c)^3\,3! + K_c(K_dM_c)^2M_d\,7 + K_c^2K_dM_cM_d^2\,2!)$
$x_0^2x_1$	0	0	$K_c((K_dM_c)^2M_c\,3! + K_cK_dM_c^2M_d\,3)$
$x_0x_1x_0$	0	0	$K_c((K_dM_c)^2M_c\,3! + K_cK_dM_c^2M_d\,2!)$
$x_0x_1^2$	0	0	$K_cK_dM_c^3\,3!$
$x_1x_0^2$	0	0	$K_c((K_dM_c)^2M_c\,3! + K_cK_dM_c^2M_d\,2!)$
$x_1x_0x_1$	0	0	$K_cK_dM_c^3\,3!$
$x_1^2x_0$	0	0	$K_cK_dM_c^3\,3!$
x_1^3	$K_cM_c^3 3!$	$K_dM_d^3 3!$	$K_cM_c^3\,3!$

(since the closed-loop system has, in effect, no external input) and Theorem 4.7, $F_{(c@d)_0}[u]$ will converge at least on the interval $[0,T_{\max})$, where

$$T_{\max} = \frac{1}{M_{(c@d)_0}} = \frac{1}{(K_{c\circ d}(2+\phi_g)+1)5M_{c\circ d}}.$$

Of course, if the series $(c@d)_0$ can be computed explicitly, a potentially better estimate $T'_{\max} = 1/M'_{(c@d)_0}$ is possible. For example, when $c\circ d = 1+x_1$, it is easily verified that $(c@d)_0 = \sum_{k\geq 0} x_0^k$ so that $F_{c@d}[0](t) = e^t$ for $t\geq 0$. In this case, both $T_{\max} = 0.04331$ and $T'_{\max} = 1$ are very conservative. On the other hand, when $c\circ d = 1+2x_1+2x_1^2$, it follows that $(c@d)_0 = \sum_{k\geq 0}(k+1)!\,x_0^k$ and $F_{c@d}[0](t) = 1/(1-t)^2$ for $0\leq t<1$. Here $T_{\max} = 0.02428$ is less conservative and $T'_{\max} = 1$ is exact.

Example 4.11. Reconsider the state space systems in Example 3.2. The operator $F_{c@d}[u]$ then has the analytic state space realization

$$f(z) = \begin{pmatrix} K_dM_cz_c^2z_d \\ K_cM_dz_cz_d^2 \end{pmatrix},\quad g(z) = \begin{pmatrix} M_cz_c^2 \\ 0 \end{pmatrix},\quad h(z) = K_cz_c$$

near $z(0) = [1\ \ 1]^T$. The first few coefficients of $c@d$ are given in Table 4.2. Since $c@d$ is a nonnegative series in this case, local convergence and input-output local convergence are equivalent as a consequence of Lemma 3.5. Setting $u(t) = \bar{u} = 1$ is equivalent to letting $b = 1$ in Theorem 4.8. Therefore, using again the $m = 0$ version of (16) and the growth condition from Theorem 4.8, a lower bound on the finite escape time for this system is

$$T_{\max} = \frac{1}{M_{(c@d)\circ 1}} = \frac{1}{[K_c(2+\phi_g)+1][\phi(1+K_d)+1]10M}.$$

Four specific cases of $T_{\max}$ are given in Table 4.3 and compared against the numerically determined escape times. The conservativeness in these estimates is a consequence of *accumulated* conservativeness in various intermediate upper bounds, for example inequality (30), as compared to the cascade connection in Example 3.2.

TABLE 4.3
$T_{\max}$ and t_{esc} for specific examples of $(c@d) \circ 1$.

Case	K_c	M_c	K_d	M_d	$M_{(c@d)\circ 1}$	$T_{\max}$	t_{esc}	$t_{\text{esc}}/T_{\max}$
1	4	2	2	2	1483	0.6745e−03	0.07556	112.0
2	2	4	2	2	1579	0.6335e−03	0.06606	104.3
3	2	2	4	2	1129	0.8857e−03	0.07387	83.4
4	2	2	2	4	1579	0.6335e−03	0.07556	119.3

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