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Evaluation transform

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Abstract

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Given a nonlinear control system, one can view its output function as a signal, parametrized by the primitives of the input functions. This signal can be formally described by Fliess' power series, that is a formal power series on noncommuting variables. The temporal behaviour of the system can be derived from this symbolic description by a transform, called *Evaluation transform*, which generalizes the inverse Laplace transform to the nonlinear area. We develop here the basic tools of that symbolic calculus. We prove a correspondence theorem between certain convolutions of signals and Cauchy products of generating power series.

1. Introduction

Let $Z=\{z_0, z_1, \ldots, z_m\}$ be a finite alphabet. Let w be a word of Z^*:

- if $w=\varepsilon$, then A_ε is the identity,
- if $w=w_1z$, then A_w is the differential operator $A_{w_1}A_z$.

Let (S) be a nonlinear control system described in the following form

$$(S)\qquad \begin{cases}\dot q(t)=\sum_{z\in Z} a^z(t)A_z(q),\\ y(t)=h(q(t)),\end{cases}$$

where

- q is an element of the real analytic manifold Q of dimension N,
- $\forall z\in Z$, A_z is an analytic vector field over Q. We note A the vector ${}^{\mathrm{T}}(A_{z_0}\ \ A_{z_1}\ \ \ldots\ \ A_{z_m})$,
- $\forall z\in Z$, a^z is a continuous piecewise mapping from $\mathbb{R}_+$ to $\mathbb{R}$. In particular a^{z_0} is a constant mapping and equal to 1. We note a the vector $(a^{z_0}\ \ a^{z_1}\ \ \ldots\ \ a^{z_m})$,
- the observation $h={}^{\mathrm{T}}(h_1\ \ h_2\ \ \ldots\ \ h_p)$ is an analytic mapping from the real analytic manifold Q to $\mathbb{R}^p$.

We can associate to the observation h its generating series

$$\sigma h=\sum_{w\in Z^*}\langle\sigma h|w\rangle w,$$

that is a formal power series in noncommuting variables belonging to the finite alphabet Z. By the fundamental formula of Fliess (also called the Peano-Baker formula, [3]), the output $y(t)$ defined by the observation h is obtained by the replacement in σh of each word w by the associated *iterated integral* $\int_0^t \delta_a w$ relative to the input $a=(a^{z_0} \quad a^{z_1} \quad \dots \quad a^{z_m})$ defined over $[0, t]$, $t \geqslant 0$ (the input $a^{z_0}(t) \equiv 1$ encodes the autonomous part of the system). Here we will call *Evaluation* of the word w, this associated iterated integral, $\mathscr{E}_a(w)=\int_0^t \delta_a w$, and we will call Evaluation of the power series S (submitted to some convergence conditions) *the output* $\mathscr{E}_a(S)$, obtained by replacing each word w in S by its Evaluation $\mathscr{E}_a(w)$. Then the Evaluation of S can be viewed as a signal, depending on the time t, and on the m independent parameters $\xi_z(t)=\int_0^t a^z(\tau)\,\mathrm{d}\tau$, $z \in Z$.

This point of view leads naturally to develop a noncommuting *symbolic calculus* in the nonlinear area, that generalizes the Heaviside calculus [4]. So, the notions of *transfer function* (generating series on one variable) and *impulsive response*, coding signals produced by linear or multilinear systems, can be generalized to generating series on $m+1$ variables and Volterra series, coding signals produced by nonlinear control systems. The Evaluation function $\mathscr{E}_a$ corresponds to the *inverse Laplace transform*.

Our goal is to develop here the basic tools of this symbolic calculus on noncommuting variables. We prove a correspondence theorem between certain convolutions of signals and Cauchy products of generating power series. We give also some applications of this theorem. This Evaluation transform allows us to obtain a simple calculation of the Taylor expansion of Volterra kernels [6]. This systematic treatment has been used in [7] (via some kernel function) to give a concise implementation in the computer algebraic system MACSYMA, allowing a particularly quick computation.

Recall that $Z=\{z_0, z_1, \dots, z_m\}$ is a finite alphabet. An element of Z is called a *letter*. A *word* is a finite sequence w of letters $w=z_{j_1}z_{j_2}\dots z_{j_k}$. The *length* of w, noted $|w|$, is its length as a sequence of letters. The *empty word* ε is the empty sequence of letters ($|\varepsilon|=0$). We note Z^* the set of all words over Z. The *concatenation product* of $u=z_{j_1}z_{j_2}\dots z_{j_k}$ and $v=z_{i_1}z_{i_2}\dots z_{i_l}$ is the juxtaposition of u and v. Thus we have $uv=z_{j_1}z_{j_2}\dots z_{j_k}z_{i_1}z_{i_2}\dots z_{i_l}$. This product is associative, and admits ε as the identity element. It is easy to verify that Z^* is the *free monoid generated by the alphabet Z*. Any subset of Z^* is called a *language*.

A *formal power series on the associative variables* $z \in Z$ (*noncommuting if* card $Z \geqslant 2$) with coefficients in A [1], is any mapping

$$S: Z^* \to A$$
$$w \mapsto \langle S|w\rangle,$$

and the set of all formal power series over Z is denoted by $A\langle\langle Z\rangle\rangle$.

A formal power series S will be written as a formal sum:

$$S=\sum_{w \in Z^*} \langle S|w\rangle w,$$

where $\langle S|w\rangle$ is the *coefficient* of the word w in S. A formal power series $S \in A\langle\langle Z\rangle\rangle$ will be said to be *quasiregular* if and only if its constant term vanishes. For any quasiregular formal power series on noncommutative variables S in $A\langle\langle Z\rangle\rangle$, S^* represents classically the formal power series $\sum_{n \geq 0} S^n$. In commutative variables, it coincides with the rational fraction $1/(1-S)$, and in this case we have $S^{*n} = (1/(1-S))^n$.

Let S be a formal power series in $A\langle\langle Z\rangle\rangle$. The *support* of S is the language over Z defined by

$$\mathrm{supp}(S) = \{w \in Z^* \mid \langle S|w\rangle \neq 0\}.$$

A formal power series P with finite support is called a *polynomial*.

The *sum* of two formal power series S,T in $A\langle\langle Z\rangle\rangle$ is the formal power series $S+T$ defined by

$$\forall w \in Z^*, \quad \langle S+T|w\rangle = \langle S|w\rangle + \langle T|w\rangle.$$

The *Cauchy product*, noted by ".", of two formal power series S,T in $A\langle\langle Z\rangle\rangle$ is the formal power series $S.T$ defined by

$$\forall w \in Z^*, \quad \langle S.T|w\rangle = \sum_{u,\, v \in Z^*,\, uv = w} \langle S|u\rangle\langle T|v\rangle.$$

The symbol "." will be omitted when there is no ambiguity.

The *shuffle product*, noted by "$\sqcup\!\sqcup$", of two formal power series S,T in $A\langle\langle Z\rangle\rangle$ is the formal power series $S \sqcup\!\sqcup T$ defined by

$$S \sqcup\!\sqcup T = \sum_{u,\, v \in Z^*} \langle S|u\rangle\langle T|v\rangle u \sqcup\!\sqcup v,$$

where $u \sqcup\!\sqcup v$ is the polynomial defined as follows:

for any word u: $\quad u \sqcup\!\sqcup \varepsilon = \varepsilon \sqcup\!\sqcup u = u$

for any words u, v, for any letters x, y:
$$ux \sqcup\!\sqcup vy = [(ux) \sqcup\!\sqcup v]y + [u \sqcup\!\sqcup (vy)]x.$$

The coefficients of the polynomial $u \sqcup\!\sqcup v$ are positive integers.

We note $\mathrm{som}(P)$ the sum of the coefficients of the polynomial P:

$$\mathrm{som}(P) = \sum_{w \in \mathrm{supp}(P)} \langle P|w\rangle.$$

Let u,v be two words in Z^*. The sum of the coefficients of the polynomial $u \sqcup\!\sqcup v$ is

$$\mathrm{som}(u \sqcup\!\sqcup v) = \binom{|u|+|v|}{|u|} = \frac{(|u|+|v|)!}{|u|!|v|!}.$$

In particular, given a letter z in Z, one has:

$$\forall \lambda, \mu \geqslant 0, \quad z^{\lambda} ш z^{\mu} = \binom{\lambda+\mu}{\mu} z^{\lambda+\mu}.$$

2. Calculus of the formal power series Evaluation

2.1. Evaluation of formal power series

Let $Z = \{z_0, z_1, \ldots, z_m\}$ be a finite alphabet.

Definition 2.1. We call *input related to* Z the given of a vector $a = (a^{z_0} \quad a^{z_1} \quad \ldots \quad a^{z_m})$ of piecewise continuous real valued functions defined over $[0, t]$, $(t \geqslant 0)$. Conventionally the 0-component of any input is $a^{z_0} \equiv 1$.

Following [2], we call *path associated to the input* $a = (a^{z_0} \quad a^{z_1} \quad \ldots \quad a^{z_m})$, the time dependent vector $\xi = {}^{\mathrm{T}}(\xi_{z_0} \quad \xi_{z_1} \quad \ldots \quad \xi_{z_m})$ defined by

$$\forall z \in Z, \quad \xi_z(\tau) = \int_0^{\tau} \mathrm{d}\xi_z(\rho) = \int_0^{\tau} a^z(\rho)\,\mathrm{d}\rho.$$

Thus we have $\xi_{z_0}(\tau) = \int_0^{\tau} \mathrm{d}\rho = \tau$, and for any letter $z \in Z$, $\xi_z(0) = 0$.

Any formal (resp. analytical) control system can be viewed as a functional defined on the inputs, whose value is generally called the *output function* of the system. More specifically, following [3], we will call *causal analytical* any functional of the entry that can be expressed as a convergent Peano-Baker series $\sum_{w \in Z^*} \langle S|w\rangle \int_0^t \delta_a w$, where the iterated integrals of the path ξ occur, and can be defined as follows:

$$\int_0^t \delta_a \varepsilon = 1 \quad \text{and} \quad \int_0^t \delta_a(vz) = \int_0^t \left(\int_0^{\tau} \delta_a v \right) \mathrm{d}\xi_z(\tau)$$

(we use the symmetric order of Fliess notations for some practical programming opportunity [7]).

In other words, any analytical functional can be encoded by some non-commutative power series, and its output can be obtained by the Evaluation procedure described below.

Here we shall call Evaluation of S for the input a at time t, the value of the functional encoded by S for the input a and the time t, also called the *output function* of S. Hence the Evaluation of any formal power series in noncommuting variables can be interpreted as a signal depending on the independent parameters ξ_z, $z \in Z^*$. In fact, Evaluation functions can be viewed as a generalization of the inverse Laplace transform, as already pointed out by Fliess et al. [3, 4]. The Evaluation function is defined as follows:

Definition 2.2. We will call *Evaluation of the word* w *in* Z^*, for the input $a = (a^{z_0} \quad a^{z_1} \quad \ldots \quad a^{z_m})$ related to the finite alphabet Z, the iterated integral, noted

$\int_0^t \delta_a w$, where this notation is defined by induction on the length of w:

$$\mathscr{E}_a(w)(t)=\int_0^t \delta_a w=\begin{cases}1 & \text{if } w=\varepsilon,\\ \displaystyle\int_0^t \mathscr{E}_a(v)(\tau)\,\mathrm{d}\xi_z(\tau) & \text{if } w=vz.\end{cases}$$

This definition is extended to $A\langle\langle Z\rangle\rangle$ in the following way.

Definition 2.3. We will call *Evaluation of the formal power series S in A* $\langle\langle Z\rangle\rangle$, for the input $a=(a^{z_0}\quad a^{z_1}\quad\ldots\quad a^{z_m})$ related to the finite alphabet Z, when it is defined, the function

$$\mathscr{E}_a(S)(t)=\sum_{w\in Z^*}\langle S|w\rangle\mathscr{E}_a(w)(t).$$

Theorem 2.4 (uniqueness of Evaluation, Fliess [3]). *Given two formal power series S and T in* $A\langle\langle Z\rangle\rangle$. *For any input* $a=(a^{z_0}\quad a^{z_1}\quad\ldots\quad a^{z_m})$ *related to the finite alphabet Z such that the series* $\sum_{w\in Z^*}\langle S|w\rangle\mathscr{E}_a(w)$ *and* $\sum_{w\in Z^*}\langle T|w\rangle\mathscr{E}_a(w)$ *are normally convergent, then we have*

$$\mathscr{E}_a(S)=\mathscr{E}_a(T)\iff S=T.$$

2.2. Shuffle product and product of Evaluations

Let a be an input related to the finite alphabet $Z=\{z_0,z_1,\ldots,z_m\}$.

Lemma 2.5. *Let u and v be two words in* Z^*. *Then the Evaluation of the polynomial* $u\sqcup\!\sqcup v$, *for the input* $a=(a^{z_0}\quad a^{z_1}\quad\ldots\quad a^{z_m})$ *related to the finite alphabet Z, is given by* $\mathscr{E}_a(u\sqcup\!\sqcup v)=\mathscr{E}_a(u)\mathscr{E}_a(v)$.

Proof. (a) The result is immediate for $|u|=0$ or $|v|=0$ because $u\sqcup\!\sqcup\varepsilon=u$, $\varepsilon\sqcup\!\sqcup v=v$ and $\mathscr{E}_a(\varepsilon)=1$.

(b) Now, suppose the result is true for any words u,v in Z^* that satisfy $|uv|\leqslant n$.

(c) If $|uv|=n+1$, then we can write $u=u_1x$ and $v=v_1y$ with $z,y\in Z$, $u_1,v_1\in Z^*$ and $|u_1v|=|uv_1|=n$. Using the fact that $u\sqcup\!\sqcup v=(u\sqcup\!\sqcup v_1)y+(u_1\sqcup\!\sqcup v)z$, we have

$$\begin{aligned}\mathscr{E}_a(u\sqcup\!\sqcup v)(t)&=\mathscr{E}_a((u\sqcup\!\sqcup v_1)y)(t)+\mathscr{E}_a((u_1\sqcup\!\sqcup v)z)(t)\\
&=\int_0^t \mathscr{E}_a(u\sqcup\!\sqcup v_1)(\tau)\,\mathrm{d}\xi_y(\tau)+\int_0^t \mathscr{E}_a(u_1\sqcup\!\sqcup v)(\tau)\,\mathrm{d}\xi_x(\tau)\\
&=\int_0^t [\mathscr{E}_a(u)(\tau)\mathscr{E}_a(v_1)(\tau)]\,\mathrm{d}\xi_y(\tau)+\int_0^t [\mathscr{E}_a(u_1)(\tau)\mathscr{E}_a(v)(\tau)]\,\mathrm{d}\xi_x(\tau)\\
&\qquad\text{(by induction hypothesis)}\\
&=\int_0^t \mathscr{E}_a(u)(\tau)\,\mathrm{d}[\mathscr{E}_a(v_1y)(\tau)]+\int_0^t \mathscr{E}_a(v)(\tau)\,\mathrm{d}[\mathscr{E}_a(u_1x)(\tau)]\\
&=[\mathscr{E}_a(u_1x)(\tau)\mathscr{E}_a(v_1y)(\tau)]_0^t\quad\text{(integration by parts)}\\
&=\mathscr{E}_a(u)(t)\mathscr{E}_a(v)(t).\qquad\square\end{aligned}$$

Corollary 2.6. *Let z be a letter in Z. For any positive integer n, we have $\mathscr{E}_a(z^n) = \xi_z^n / n!$. In particular case, $\mathscr{E}_a(z_0^n) = t^n / n!$.*

Proof. The result is immediate for $n = 0$. We suppose that the result is true for all ν, $0 \leq \nu \leq n$. For $\nu = n+1$, since $z^n ш z = (n+1) z^{n+1}$ (see Section 1), and by Lemma 2.5, we have

$$\begin{aligned}\mathscr{E}_a(z^{n+1}) &= \frac{1}{n+1} \mathscr{E}_a(z^n ш z) = \frac{1}{n+1} \mathscr{E}_a(z^n) \mathscr{E}_a(z) \\ &= \frac{1}{n+1} \frac{\xi_z^n}{n!} \xi_z = \frac{\xi_z^{n+1}}{(n+1)!}.\end{aligned}$$

The expression corresponding to the particular case of $z = z_0$ is obtained using the fact that $\xi_{z_0}(t) = t$. □

Theorem 2.7 (Fliess [3]). *Let S and T be two formal power series in $A\langle\langle Z \rangle\rangle$. Then the Evaluation of the shuffle product $S ш T$, for the input $a = (a^{z_0} \quad a^{z_1} \quad \ldots \quad a^{z_m})$ related to the finite alphabet Z, is the product of the Evaluations of S and T:*

$$\mathscr{E}_a(S ш T) = \mathscr{E}_a(S) \mathscr{E}_a(T).$$

Proof. By the definition of $S ш T$ and Lemma 2.5, we have

$$\begin{aligned}\mathscr{E}_a(S ш T) &= \sum_{u, v \in Z^*} \langle S|u \rangle \langle T|v \rangle \mathscr{E}_a(u ш v) \\ &= \sum_{u \in Z^*} \sum_{v \in Z^*} \langle S|u \rangle \langle T|v \rangle \mathscr{E}_a(u) \mathscr{E}_a(v) \\ &= \left(\sum_{u \in Z^*} \langle S|u \rangle \mathscr{E}_a(u) \right) \left(\sum_{v \in Z^*} \langle T|v \rangle \mathscr{E}_a(v) \right) \\ &= \mathscr{E}_a(S) \mathscr{E}_a(T). \qquad \square\end{aligned}$$

2.3. Cauchy product and convolution

Let a be an input related to the finite alphabet $Z = \{z_0, z_1, \ldots, z_m\}$.

Lemma 2.8. *Let $\mathscr{E}_a(S)$ be the Evaluation of the formal power series S, for the input $a = (a^{z_0} \quad a^{z_1} \quad \ldots \quad a^{z_m})$ related to the finite alphabet Z, and let z be a letter. For any integer n greater than or equal to 1, the Evaluation of the formal power series Sz^n is*

$$\mathscr{E}_a(Sz^n)(t) = \int_0^t \mathscr{E}_a(S)(\tau) \frac{(\xi_z(t) - \xi_z(\tau))^{n-1}}{(n-1)!} \, \mathrm{d}\xi_z(\tau).$$

In particular, for $z = z_0$, we have

$$\mathscr{E}_a(Sz_0^n)(t) = \int_0^t \mathscr{E}_a(S)(\tau) \frac{(t-\tau)^{n-1}}{(n-1)!} \, \mathrm{d}\tau.$$

Proof. For $n=1$, we have

$$\begin{aligned}
\mathscr{E}_a(Sz)(t) &= \sum_{w\in Z^*} \langle S|w\rangle \mathscr{E}_a(wz)(t) \\
&= \sum_{w\in Z^*} \langle S|w\rangle \int_0^t \mathscr{E}_a(w)(\tau)\,\mathrm{d}\xi_z(\tau) \\
&= \int_0^t \Big(\sum_{w\in Z^*} \langle S|w\rangle \mathscr{E}_a(w)(\tau)\Big)\,\mathrm{d}\xi_z(\tau) \\
&= \int_0^t \mathscr{E}_a(S)(\tau)\,\mathrm{d}\xi_z(\tau).
\end{aligned}$$

We suppose that the result is true for any ν, $1\leq \nu\leq n$. For $\nu=n+1$, by induction hypothesis, we have

$$\begin{aligned}
\mathscr{E}_a((Sz)z^n)(t) &= \int_0^t \mathscr{E}_a(Sz)(\tau)\frac{(\xi_z(t)-\xi_z(\tau))^{n-1}}{(n-1)!}\,\mathrm{d}\xi_z(\tau) \\
&= \int_0^t \mathscr{E}_a(Sz)(\tau)\,\mathrm{d}\Big[-\frac{(\xi_z(t)-\xi_z(\tau))^n}{n!}\Big] \\
&= -\Big[\mathscr{E}_a(Sz)(\tau)\frac{(\xi_z(t)-\xi_z(\tau))^n}{n!}\Big]_0^t \\
&\quad + \int_0^t \frac{(\xi_z(t)-\xi_z(\tau))^n}{n!}\,\mathrm{d}[\mathscr{E}_a(Sz)(\tau)].
\end{aligned}$$

The first term is vanishing (after one integration by parts); then we have

$$\mathscr{E}_a(Sz^{n+1})(t) = \int_0^t \mathscr{E}_a(S)(\tau)\frac{(\xi_z(t)-\xi_z(\tau))^n}{n!}\,\mathrm{d}\xi_z(\tau).$$

The particular form obtained for $z=z_0$ follows, since we have $\xi_0(t)=t$. □

Theorem 2.9. *Given $G\in A\langle\langle Z\rangle\rangle$ a* quasiregular *formal power series, $H\in A\langle\langle z\rangle\rangle$ a formal power series, we set*

$$\psi(\xi(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_a(G)(t), \qquad h(\xi_z(t)) = \mathscr{E}_a(H)(t).$$

With these notations, the Evaluation of the formal power series GH, for the input $a=(a^{z_0}\ \ a^{z_1}\ \ \ldots\ \ a^{z_m})$ related to the finite alphabet Z, is

$$\mathscr{E}_a(GH)(t) = \int_0^t \psi(\xi(\tau))h(\xi_z(t)-\xi_z(\tau))\,\mathrm{d}\tau.$$

In particular, if H is a formal power series in $A\langle\langle z_0\rangle\rangle$, then

$$\mathscr{E}_a(GH)(t) = \int_0^t \psi(\xi(\tau))h(t-\tau)\,\mathrm{d}\tau.$$

Proof. (i) First case: $H = z^n$, $n \geq 0$. For $n = 0$, the result is immediate. If $n > 0$ then we have

$$\mathscr{E}_a(Gz^n)(t) = \int_0^t \mathscr{E}_a(G)(\tau) \frac{(\xi_z(t) - \xi_z(\tau))^{n-1}}{(n-1)!} \mathrm{d}\xi_z(\tau) \quad \text{(by Lemma 2.9)}$$

$$= \int_0^t \mathscr{E}_a(G)(\tau)\, \mathrm{d}\left[-\frac{(\xi_z(t) - \xi_z(\tau))^n}{n!} \right]$$

$$= -\left[\frac{(\xi_z(t) - \xi_z(\tau))^n}{n!} \mathscr{E}_a(G)(\tau) \right]_0^t$$

$$+ \int_0^t \frac{(\xi_z(t) - \xi_z(\tau))^n}{n!} \mathrm{d}[\mathscr{E}_a(G)(\tau)] \quad \text{(integration by parts)}$$

$$= \int_0^t \psi(\xi(\tau)) \frac{(\xi_z(t) - \xi_z(\tau))^n}{n!} \mathrm{d}\tau \quad \text{(the first term vanishes)}$$

$$= \int_0^t \psi(\xi(\tau)) h(\xi_z(t) - \xi_z(\tau))\, \mathrm{d}\tau.$$

(ii) General case: $H = \sum_{n \geq 0} H_n z^n$ and $h(\xi_z(t)) = \sum_{n \geq 0} H_n \xi_z^n(t)/n!$:

$$\mathscr{E}_a(GH)(t) = \sum_{n \geq 0} H_n \mathscr{E}_a(Gz^n)(t)$$

$$= \sum_{n \geq 0} H_n \int_0^t \psi(\xi(\tau)) \frac{(\xi_z(t) - \xi_z(\tau))^n}{n!} \mathrm{d}\tau \quad \text{(see (i))}$$

$$= \int_0^t \psi(\xi(\tau)) \sum_{n \geq 0} H_n \frac{(\xi_z(t) - \xi_z(\tau))^n}{n!} \mathrm{d}\tau$$

$$= \int_0^t \psi(\xi(\tau)) h(\xi_z(t) - \xi_z(\tau))\, \mathrm{d}\tau.$$

In particular, if $z = z_0$ then $\xi_0(t) = t$, hence we have the expected result. □

2.4. Evaluations of some usual formal power series

Let z be a letter in Z. If the support of the formal power series S is a subset of z^*, then we have, for the input $a = (a^{z_0} \ \ a^{z_1} \ \ \ldots \ \ a^{z_m})$ related to the finite alphabet Z, the Evaluations shown in Table 1.

So the Evaluation of the generating series $S = \sum_{n \geq 0} c_n z^n$ is the associated *exponential generating series* $\mathscr{E}_a(S) = \sum_{n \geq 0} c_n(\xi_z^n/n!)$, and thus the Evaluation function is reduced to the usual exponential transform as studied in [5]. In particular, if $z = z_0$ then we have $\xi_{z_0}(t) = t$; in this case we obtain the inverse Laplace-Borel transform.

Table 1

S	$\mathscr{E}_a(S)$
ε	1
z^n	$\frac{\xi_z^n(t)}{n!}$
$z^{*n}, n \geqslant 1$	$\exp(\xi_z(t)) \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(\xi_z(t))^j}{j!}$
$\sum_{n\geqslant 0} c_n z^n$	$\sum_{n\geqslant 0} c_n \frac{\xi_z^n(t)}{n!}$
Then, in particular	
$z^* = \sum_{n\geqslant 0} z^n$	$\exp(\xi_z(t))$
$\sum_{n\geqslant 0} nz^n = z^*zz^*$	$\xi_z(t)\exp(\xi_z(t))$
$\sum_{n\geqslant 0} (-1)^n z^{2n}$	$\cos(\xi_z(t))$
$\sum_{n\geqslant 0} (-1)^n z^{2n+1}$	$\sin(\xi_z(t))$

Let G be a quasiregular formal power series. Its Evaluation, for the input $a = (a^{z_0} \quad a^{z_1} \quad \ldots \quad a^{z_m})$ related to the finite alphabet Z, can be described by

$$\mathscr{E}_a(G)(t) = \int_0^t \psi(\xi(\tau))\,\mathrm{d}\tau.$$

Let z be a letter in Z. We have, for the input $a = (a^{z_0} \quad a^{z_1} \quad \ldots \quad a^{z_m})$ related to the finite alphabet Z, the following Evaluation (see the convolution theorem):

$$\mathscr{E}_a\left(G \sum_{n\geqslant 0} c_n z^n\right)(t) = \int_0^t \psi(\xi(\tau)) \sum_{n\geqslant 0} c_n \frac{[\xi_z(t) - \xi_z(\tau)]^n}{n!}\,\mathrm{d}\tau,$$

and consequently, we have the Evaluations shown in Table 2.

Table 2

S	$\mathscr{E}_a(S)$
Gz^n	$\int_0^t \psi(\xi(\tau)) \frac{(\xi_z(t) - \xi_z(\tau))^n}{n!}\,\mathrm{d}\tau$
$Gz^* = G \sum_{n\geqslant 0} z^n$	$\int_0^t \psi(\xi(\tau)) \exp(\xi_z(t) - \xi_z(\tau))\,\mathrm{d}\tau$
$G \sum_{n\geqslant 0} nz^n$	$\int_0^t \psi(\xi(\tau))(\xi_z(t) - \xi_z(\tau)) \exp(\xi_z(t) - \xi_z(\tau))\,\mathrm{d}\tau$
$G \sum_{n\geqslant 0} (-1)^n z^{2n}$	$\int_0^t \psi(\xi(\tau)) \cos(\xi_z(t) - \xi_z(\tau))\,\mathrm{d}\tau$
$G \sum_{n\geqslant 0} (-1)^n z^{2n+1}$	$\int_0^t \psi(\xi(\tau)) \sin(\xi_z(t) - \xi_z(\tau))\,\mathrm{d}\tau$

3. Computing examples

In the following examples $a = (a^{z_0} \quad a^{z_1} \quad \dots \quad a^{z_m})$ is an input related to the given finite alphabet Z.

3.1. Example

This example gives the computation of the Evaluation of the formal power series z^{*n}, for any letter and for any z positive integer n.

Lemma 3.1. *For any integer $n \geqslant 1$, we have*

$$\mathscr{E}_a(z^{*n})(t) = \exp(\xi_z(t))g_n(\xi_z(t)),$$

where the g_n are polynomials in $\xi_z(t)$, and verifies the following inductive equations:

$$g_n(\xi_z(t)) = \begin{cases} 1 & \text{if } n = 1, \\ g_{n-1}(\xi_z(t)) + \int_0^t g_{n-1}(\xi_z(\tau))\, \mathrm{d}\xi_z(\tau) & \text{if } n > 1. \end{cases}$$

Proof. Since $\mathscr{E}_a(z^*)(t) = \exp(\xi_z(t))$, we can write $g_1(\xi_z(t)) = 1$. We suppose that the result is true for any ν, $1 \leqslant \nu < n$. For $\nu = n$, recall the following identities:

$$\forall n \geqslant 1, \quad z^{*n} = z^{*n-1}z^* = z^{*n-1}(1 + zz^*) = z^{*n-1} + z^{*n-1}zz^*.$$

Recall also that for any formal power series $S = \sum_{w \in Z^*} \langle S|w\rangle\, w$ in $\mathbb{R}\langle\langle Z\rangle\rangle$, we have

$$\begin{aligned} \mathscr{E}_a(Sz)(t) &= \sum_{w \in Z^*} \langle S|w\rangle \int_0^t \mathscr{E}_a(w)(\tau)\, \mathrm{d}\xi_z(\tau) \\ &= \int_0^t \sum_{w \in Z^*} \langle S|w\rangle \mathscr{E}_a(w)(\tau)\, \mathrm{d}\xi_z(\tau) \\ &= \int_0^t \mathscr{E}_a(S)(\tau)\, \mathrm{d}\xi_z(\tau) \\ &= \int_0^t \mathscr{E}_a(S)(\tau) a^z(\tau)\, \mathrm{d}\tau. \end{aligned}$$

Thus for $S = z^{*n-1}$, by induction hypothesis, we have

$$\mathscr{E}_a(z^{*n-1}z)(t) = \int_0^t \exp(\xi_z(\tau))g_{n-1}(\xi_z(\tau))a^z(\tau)\, \mathrm{d}\tau.$$

Using the convolution theorem, we obtain

$$\begin{aligned} \mathscr{E}_a(z^{*n-1}zz^*)(t) &= \int_0^t \exp(\xi_z(\tau))g_{n-1}(\xi_z(\tau))a^z(\tau)\exp(\xi_z(t) - \xi_z(\tau))\, \mathrm{d}\tau \\ &= \exp(\xi_z(t)) \int_0^t g_{n-1}(\xi_z(\tau))\, \mathrm{d}\xi_z(\tau). \end{aligned}$$

Now we have, for any integer $n \geqslant 1$, the following equalities:

$$\mathscr{E}_a(z^{*n})(t) = \mathscr{E}_a(z^{*n-1})(t) + \mathscr{E}_a(z^{*n-1}zz^*)(t)$$

$$= \exp(\xi_z(t))\left[g_{n-1}(\xi_z(t)) + \int_0^t g_{n-1}(\xi_z(\tau))\,\mathrm{d}\xi_z(\tau)\right].$$

Hence the family $(g_n)_{n \geqslant 1}$ is the unique solution of the inductive equations

$$g_n(\xi_z(t)) = \begin{cases} 1 & \text{if } n = 1, \\ g_{n-1}(\xi_z(t)) + \int_0^t g_{n-1}(\xi_z(\tau))\,\mathrm{d}\xi_z(\tau) & n > 1. \end{cases} \quad \square$$

Lemma 3.2. *The family*

$$g_n(\xi_z(t)) = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\xi_z^j(t)}{j!},$$

for any integer $n \geqslant 1$, is the unique solution of the inductive equations

$$g_n(\xi_z(t)) = \begin{cases} 1 & \textit{if } n = 1, \\ g_{n-1}(\xi_z(t)) + \int_0^t g_{n-1}(\xi_z(\tau))\,\mathrm{d}\xi_z(\tau) & \textit{if } n > 1. \end{cases}$$

Proof. Let $G_1 = 1 \in \mathbb{R}\langle\langle Z \rangle\rangle$. We have clearly,

$$g_1(\xi_z(t)) = \mathscr{E}_a(G_1)(t).$$

Suppose that for any integer $n \geqslant 1$, $g_n(\xi_z(t)) = \mathscr{E}_a(G_n)(t)$ where G_n is a formal power series in $\mathbb{R}\langle\langle Z \rangle\rangle$. Thus we have

$$\mathscr{E}_a(G_n)(t) = \mathscr{E}_a(G_{n-1})(t) + \mathscr{E}_a(G_{n-1}z)(t).$$

This equation is true if G_n satisfies $G_n = G_{n-1}(1+z)$. In other terms,

$$G_n = (1+z)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} z^j.$$

Since $\mathscr{E}_a(z^j)(t) = \xi_z^j(t)/j!$, $j \geqslant 0$ (Corollary 2.6), we have

$$\forall n \geqslant 1, \quad g_n(\xi_z(t)) = \mathscr{E}_a(G_n)(t) = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\xi_z^j(t)}{j!}. \qquad \square$$

Finally we obtain, as corollaries, the two following propositions.

Proposition 3.3. *For any positive integer n, we have*

$$\mathscr{E}_a(z^{*n})(t)=\begin{cases}1 & \text{if } n=0\\ \exp(\xi_z(t))\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{\xi_z^j(t)}{j!} & \text{if } n>0.\end{cases}$$

Proposition 3.4. *For any positive integer n, for any complex number α, we have*

$$\mathscr{E}_a((\alpha z)^{*n})(t)=\begin{cases}1 & \text{if } n=0,\\ \exp(\alpha\xi_z(t))\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{(\alpha\xi_z(t))^j}{j!} & \text{if } n>0.\end{cases}$$

3.2. Example

Let G be a quasiregular formal power series in $\mathbb{R}\langle\langle Z\rangle\rangle$ such that its evaluation is

$$\mathscr{E}_a(G)(t)=\int_0^t \psi(\xi(\tau))\,\mathrm{d}\tau.$$

Let z be a letter in Z. Let S be a formal power series in $\mathbb{R}\langle\langle Z\rangle\rangle$ that verifies the following polynomial equation

$$S+\beta_1 Sz+\beta_2 Sz^2+\cdots+\beta_n Sz^n=G;$$

that is $SK=G$, where K is the formal power series in $\mathbb{R}\langle\langle Z\rangle\rangle$ defined by

$$K=\sum_{k=0}^{n}\beta_k z^k \quad \text{with } \beta_0=1.$$

Since the constant term $\langle K|\varepsilon\rangle=\beta_0=1$ does not vanish, then the formal power series K^{-1} exists and it is a formal power series in the single commutative variable z. Suppose that K admits r complex distinguished roots $\mu_1,\ldots,\mu_r$ of respective multiplicity order $m_1,\ldots,m_r$ $(\sum_{l=1}^r m_l=n)$. One can express unically

$$K^{-1}=\frac{1}{\prod_{l=1}^r (z-\mu_l)^{m_l}}$$

under partial fraction decomposition form

$$K^{-1}=\sum_{l=1}^{r}\sum_{k=1}^{m_l}\frac{\lambda_{l,k}}{(-\mu_l)^k}H_{l,k},$$

where for any $l\in[1\,..\,r]$ and for any $k\in[1\,..\,m_l]$, $\lambda_{l,k}\in\mathbb{C}$ and each $H_{l,k}$ is of the following form:

$$H_{l,k}=\left(\frac{z}{\mu_l}\right)^{*k}.$$

So $S = GK^{-1}$ can be expressed as

$$S = \sum_{l=1}^{r} \sum_{k=1}^{m_l} \frac{\lambda_{l,k}}{(-\mu_l)^k} GH_{l,k},$$

and by the convolution theorem, we obtain the Evaluation of S:

$$\mathscr{E}_a(S)(t) = \sum_{l=1}^{r} \sum_{k=1}^{m_l} \frac{\lambda_{l,k}}{(-\mu_l)^k} \int_0^t \psi(\xi(\tau)) h_{l,k}(\xi_z(t) - \xi_z(\tau))\, \mathrm{d}\tau,$$

where $h_{l,k}(\xi_z(t))$ is the Evaluation of $H_{l,k}$:

$$h_{l,k}(\xi_z(t)) = \exp\left(\frac{\xi_z(t)}{\mu_l}\right) \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{1}{j!} \left(\frac{\xi_z(t)}{\mu_l}\right)^j \quad \text{(Proposition 3.4).}$$

Let us indicate that if $z = z_0$ then the above calculation corresponds mutatis mutandis to the *inverse Laplace transform* in the study of the systems discribed by one linear differential equation in the inputs and outputs.

3.3. Example

Theorem 3.5. *For any positive integer k, G_k is supposed a formal power series on only one letter z_{j_k} in Z and we note $g_k(\xi_{z_{j_k}}(t))$ its Evaluation. We set for any positive integer k, $S_k = G_0 z_{i_1} G_1 \ldots z_{i_k} G_k$, where $z_{i_1}, z_{i_2}, \ldots, z_{i_k}$ are k letters in Z. With these notations, for any integer $k > 0$, we have*

$$\mathscr{E}_a(S_k)(t) = \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} g_0(\xi_{z_{j_0}}(\tau_1)) g_1(\xi_{z_{j_1}}(\tau_2) - \xi_{z_{j_1}}(\tau_1)) \ldots$$
$$g_k(\xi_{z_{j_k}}(t) - \xi_{z_{j_k}}(\tau_k))\, \mathrm{d}\xi_{z_{i_1}}(\tau_1) \ldots \mathrm{d}\xi_{z_{i_k}}(\tau_k).$$

Proof. For $k = 1$, by Lemma 2.3, we have

$$\mathscr{E}_a(G_0 z_{i_1})(t) = \int_0^t g_0(\xi_{z_{j_0}}(t)) a^{z_{i_1}}(\tau)\, \mathrm{d}\tau,$$

and by the convolution theorem, we have

$$\mathscr{E}_a(G_0 z_{i_1} G_1)(t) = \int_0^t g_0(\xi_{z_{j_0}}(\tau)) a^{z_{i_1}}(\tau) g_1(\xi_{z_{j_1}}(t) - \xi_{z_{j_1}}(\tau))\, \mathrm{d}\tau$$
$$= \int_0^t g_0(\xi_{z_{j_0}}(\tau)) g_1(\xi_{z_{j_1}}(t) - \xi_{z_{j_1}}(\tau))\, \mathrm{d}\xi_{z_{i_1}}(\tau).$$

The result is supposed true for any ν, $1 \leq \nu \leq k-1$. For $\nu = k$, by Lemma 2.8, we have

$$\mathscr{E}_a(S_{k-1}z_{i_k})(t) = \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} g_0(\xi_{z_{i_0}}(\tau_1))g_1(\xi_{z_{i_1}}(\tau_2) - \xi_{z_{i_1}}(\tau_1)) \cdots$$
$$\cdots g_{k-1}(\xi_{z_{i_{k-1}}}(\tau_k) - \xi_{z_{i_{k-1}}}(\tau_{k-1}))\, \mathrm{d}\xi_{z_{i_1}}(\tau_1) \ldots \mathrm{d}\xi_{z_{i_k}}(\tau_k).$$

Finally, by the convolution theorem, we have

$$\mathscr{E}_a(S_k)(t) = \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} g_0(\xi_{z_{i_0}}(\tau_1))g_1(\xi_{z_{i_1}}(\tau_2) - \xi_{z_{i_1}}(\tau_1)) \cdots$$
$$\cdots g_k(\xi_{z_{i_k}}(t) - \xi_{z_{i_k}}(\tau_k))\, \mathrm{d}\xi_{z_{i_1}}(\tau_1) \ldots \mathrm{d}\xi_{z_{i_k}}(\tau_k). \qquad \square$$

We deduce the two following results.

Corollary 3.6. *Let $p_0, p_1, \ldots, p_k$ be $k+1$ positive integers. Let $z_{i_1}, z_{i_2}, \ldots, z_{i_k}$, be letters in $Z_0 = Z\backslash\{z_0\}$. Then the Evaluation of the word $z_0^{p_0} z_{i_1} z_0^{p_1} \ldots z_{i_k} z_0^{p_k}$ is* (*see* Corollary 2.6)

$$\int_0^t \int_0^{\tau_k} \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} \frac{\tau_1^{p_0}(\tau_2 - \tau_1)^{p_1} \ldots (t-\tau_k)^{p_k}}{p_0!\, p_1! \ldots p_k!} \mathrm{d}\xi_{z_{i_1}}(\tau_1) \ldots \mathrm{d}\xi_{z_{i_k}}(\tau_k).$$

Corollary 3.7. *Let $c_0, c_1, \ldots, c_k$ be $k+1$ complex numbers. Let $p_0, p_1, \ldots, p_k$ be $k+1$ positive integers. Let $z_{i_1}, z_{i_2}, \ldots, z_{i_k}$, be letters in $Z_0 = Z\backslash\{z_0\}$. Then the Evaluation of the rational fraction $(c_0 z_0)^{*p_0} z_{i_1}(c_1 z_0)^{*p_1} \ldots z_{i_k}(c_k z_0)^{*p_k}$ is*

$$\int_0^t \int_0^{\tau_k} \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} f_0(\tau_1) f_1(\tau_2 - \tau_1) \ldots f_k(t-\tau_k)\, \mathrm{d}\xi_{z_{i_1}}(\tau_1) \ldots \mathrm{d}\xi_{z_{i_k}}(\tau_k),$$

with (*see Proposition* 3.4)

$$\forall n \in [0\,..\,k], \quad f_n(t) = \begin{cases} 1 & \text{if } p_n = 0, \\ \exp(c_n t) \displaystyle\sum_{j=0}^{p_n - 1} \binom{p_n - 1}{j} \frac{(c_n t)^j}{j!} & \text{if } p_n > 0. \end{cases}$$

The first result is used in [6] to give the Taylor expansion of Volterra kernels. The second result is also presented in [4]; it allows us to compute iteratively the Volterra kernels of the solution of certain nonlinear differential equations with forcing terms. In [7], we give a concise MACSYMA program, allowing a particularly quick computation of the Evaluation of these series S_k.

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