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AN ALGEBRAIC CRITERION FOR THE ONSET OF CHAOS IN NONLINEAR DYNAMICAL SYSTEMS

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1. INTRODUCTION

CHAOTIC regimes of nonlinear dynamical systems are becoming more and more important as we find new applications. When we are interested in designing the operation of a dynamical system so as to avoid its chaotic regimes, we must know in advance the ranges of parameters for which the system's response will be chaotic. This explains the need for a criterion.

A dynamical system will respond according to the values of the physical parameters in play. We can always specify a space of physical parameters over which the system operates and imagine traversing paths within such a space. On any segment of a path, the system's dynamical response will be characterized by the long-term behaviour of its trajectories in a phase space, and this characterization may change at critical points on the path. For example, over a certain segment of parameter space, the system's response may become steady at long time. Corresponding trajectories in a phase space will aproach a fixed point. We say that trajectories are *attracted* to such a point; accordingly, we call it a fixed-point attractor. On the other hand, over another segment of parameter space, the system's long-term behaviour may be periodic in time; corresponding trajectories in a phase space will approach a closed curve or limit cycle. Since this curve is defined by one dimension, its length, we call it a one-dimensional attractor. Over yet another segment of parameter space, the system's long-term response in time may contain two incommensurate periods. Corresponding trajectories in phase space will be attracted to a two-dimensional figure, a torus; we call it a two-torus. Finally, one or more segments of parameter space may exist where the system's long-term response appears to be random or *chaotic*. Corresponding trajectories in phase space will be attracted to a limiting set, the dimension of which is noninteger. We call this attracting set a *strange attractor*. To avoid chaotic regimes of the system, we must know the ranges of parameters over which the system's response is characterizable by a strange attractor.

In this paper, we shall seek necessary and sufficient conditions for the existence of a point in parameter space defining the onset of a chaotic regime in a given dynamical system. At the moment, we have two leads suggesting how we might arrive at such a criterion. First, if we knew how to tell when the sign of the largest Lyapunov coefficient became positive, we would have a criterion in our hands. At present, however, we do not know how to do this without computing the coefficients. The trouble here is that we must repeat the computation for each possible set of physical parameters, and we may never exhaust the possibilities. Numerical computations of this nature, therefore, are not suitable for developing a criterion which must be capable of picking out the relevant set of parameter values from the infinity of possible combinations. The second lead suggests itself from the observation that, in phase space, the trajectories will have zero autocorrelation at the onset of chaos. Here, rather than carry out numerical computations it is possible to derive an analytical expression for the autocorrelation function at the onset of chaos, and seek a criterion for the onset from the expression itself. To this end, we shall find it convenient to transform from the time domain to the Laplace-Borel transform domain.

Laplace-Borel transforms lend themselves to the treatment of iterated integrals. The connection between iterated integrals and nonlinear dynamical systems can be brought to light by giving the example of (let us say) the Duffing equation. We can integrate this equation twice and express the same amount of dynamics in the form of (twice) repeated integrals. Repeated integrals have been studied extensively by Chen in [1-3]. There exists a formal correspondence between the repeated integrals and a noncommutative algebra, as explained by Fliess and his colleagues in [4] and [5]. We shall utilize this correspondence to recast the dynamical system from its repeated-integral form to an algebraic form in the Laplace-Borel transform domain. The resulting set of algebraic equations can be solved, in principle, precisely for the physical parameters existent at the onset of a chaotic regime.

This paper contains a derivation of the criterion along with a brief summary of the necessary mathematical background. Subsequent work should explore applications of the criterion to widely known dynamical systems as well as the novel flight dynamical system that we have discussed in [6-9].

2. BACKGROUND

The following sections contain a brief summary on the use of iterated integrals, a noncommutative algebra, and Laplace-Borel transforms. More complete information can be found in [1-5].

2.1. *Iterated integrals*

Adopting the notation introduction by Chen [1-3], we define an iterated integral as follows:

$$\int_0^t \mathrm{d}\xi_{j\mu} \cdots \mathrm{d}\xi_{j0} = \int_0^t \mathrm{d}\xi_{j\mu}(\tau) \int_0^\tau \mathrm{d}\xi_{j\mu-1} \cdots \mathrm{d}\xi_{j0} \tag{1}$$

and

$$\int_0^t \mathrm{d}\xi_j = \xi_j(t) - \xi_j(0) \tag{2}$$

where $j = 0, 1, \ldots, n$ and $0 \le t \le T$ and $\xi_0, \xi_1, \ldots, \xi_n$: $[0, T] \to \mathbf{R}$ are $(n + 1)$ continuous functions with bounded variations. We can associate the forcing terms of the dynamical system with iterated integrals as follows. Let u_i; $u_1, u_2, \ldots, u_n$: $[0, T] \to \mathbf{R}$; then a forcing term u_i can be represented by the integral

$$\int_0^t \mathrm{d}\xi_{i\mu} \cdots \mathrm{d}\xi_{i0} \tag{3}$$

where

$$\xi_0(\tau) = \tau \tag{4}$$

$$\xi_i(\tau) = \int_0^\tau u_i(\sigma)\,d\sigma \tag{5}$$

and $i = 1, \ldots, n$.

2.2. *A noncommutative algebra*

We shall define a noncommutative algebra and its correspondence with the iterated integrals. Let $X = \{x_0, \ldots, x_n\}$ be a finite set called an *alphabet* and let $\tilde{X} = \{w_0, \ldots, w_n\}$ be a free *monoid*, another set, generated by X. If w_i is an element of $\tilde{X}$ ($w_i \in \tilde{X}$), then w_i is a *word*. The word w_i consists of a finite sequence $x_{i_\mu} \cdots x_{i0}$ of letters of an alphabet.

The product of two words will be defined as a *concatenation* or *juxtaposition* of these two words. Concatenation is a noncommutative operation.

The *neutral element*, which can be interpreted as an empty word, will be denoted by 1.

The length of a word, say w_i, will be denoted by $|w_i|$, $w_i \in \tilde{X}$, and it will be defined by the number of its letters. For example the word $x_0x_1x_1x_1x_0$ will have a length of 5, i.e. $|x_0x_1^3x_0| = 5$. The length of the neutral element is $|1| = 0$.

The product in this algebra is called the *shuffle product* (le mélange). We denote it by II and define it by induction on the length of words as:

$$1 \text{ II } 1 = 1 \tag{6}$$

$$x \text{ II } 1 = 1 \text{ II } x = x \qquad \forall\, x \in X \tag{7}$$

$$(xw) \text{ II } (x'w') = x[w \text{ II } (x'w')] + x'[(xw) \text{ II } w'] \tag{8}$$

$$\forall\, x, x' \in X \qquad \text{and} \qquad \forall\, w, w' \in \tilde{X}. \tag{9}$$

We shall give an example of the shuffle product to familiarize the reader with this form. Let $w_1 = x_0x_1$ be the first word and $w_2 = x_1x_0$ the second word. Let us determine w_1 II w_2 (read w_1 shuffled into w_2):

$$x_0x_1 \text{ II } x_1x_0 = x_0[x_1 \text{ II } x_1x_0] + x_1[x_0x_1 \text{ II } x_0]. \tag{10}$$

The first term of the right-hand side generates the following terms:

$$x_1[1 \text{ II } x_1x_0] + x_1[x_1 \text{ II } x_0]. \tag{11}$$

The second term of (11) generates the following terms:

$$x_1[1 \text{ II } x_0] + x_0[x_1 \text{ II } 1]. \tag{12}$$

Similarly, the second term of the right-hand side in (10) will give the following expression:

$$x_0[x_1 \text{ II } x_0] + x_0[x_0x_1 \text{ II } 1], \tag{13}$$

which finally yields

$$x_0[x_1(1 \text{ II } x_0)] + x_0[x_1 \text{ II } 1] + x_0x_0x_1. \tag{14}$$

Next we add like terms on the right-hand side to obtain:

$$x_0[x_1x_1x_0 + x_1x_1x_0 + x_1x_0x_1] + x_1[x_0x_1x_0 + x_0x_0x_1 + x_0x_0x_1]. \tag{15}$$

Further regrouping yields:

$$2x_0x_1x_1x_0 + x_0x_1x_0x_1 + x_1x_0x_1x_0 + 2x_1x_0x_0x_1. \tag{16}$$

Finally we obtain for the right-hand side of (10):

$$2x_0x_1^2x_0 + x_0x_1x_0x_1 + x_1x_0x_1x_0 + 2x_1x_0^2x_1. \tag{17}$$

The reader should not be alarmed by the tediousness of this operation since a computer programmed for a symbolic language can carry out the shuffle product for us.

Next we shall define the real algebra of formal polynomials and the real algebra of power series with real coefficients and noncommutative variables, $x_j \in X$. Let $\mathfrak{R}\langle x\rangle$ be the $\mathfrak{R}$-algebra of a formal polynomial and $\mathfrak{R}\langle\langle x\rangle\rangle$ be the $\mathfrak{R}$-algebra of a power series with real coefficients and noncommutative variables. Let $\mathcal{G} \in \mathfrak{R}\langle\langle x\rangle\rangle$ be an element of $\mathfrak{R}\langle\langle x\rangle\rangle$, which can be written as

$$\mathcal{G} = \sum\{[(\mathcal{G}, w)]w \mid w \in \tilde{X}\} \tag{18}$$

where $(\mathcal{G}, w) \in \mathfrak{R}$. We define the following three operations (addition, Cauchy multiplication and shuffle product) on two elements of $\mathfrak{R}$, namely, $\mathcal{G}_1, \mathcal{G}_2$:

$$\mathcal{G}_1 + \mathcal{G}_2 = \sum\{[(s_1, w) + (s_2, w)]w \mid w \in \tilde{X}\} \tag{19}$$

$$\mathcal{G}_1\mathcal{G}_2 = \sum\left\{\left[\sum_{w_1w_2 = w}(s_1, w_1)(s_2 w_2)\right]w \mid w \in \tilde{X}\right\} \tag{20}$$

$$\mathcal{G}_1 \amalg \mathcal{G}_2 = \sum\{(s_1, w_1)(s_2, w_2)w_1 \amalg w_2 \mid w_1, w_2 \in \tilde{X}\}. \tag{21}$$

Let us give an interpretation of the meaning of a noncommutative power series $\mathcal{G}$ which is an element of $\mathfrak{R}\langle\langle x\rangle\rangle$. This noncommutative power series actually defines a *causal* (nonanticipative) functional of the forcing functions u_i of the dynamical system if we replace the word $x_{j\mu}\cdots x_{j0}$ by the corresponding iterated integral $\int_0^t \mathrm{d}\xi_{j\mu}\cdots \mathrm{d}\xi_{j0}$. Let $y(t; u_1, \ldots, u_n)$ be a causal functional. We can write this causal functional as

$$y(t; u_1, \ldots, u_n) = (\mathcal{G}, 1) + \sum_{\mu \geq 0}\sum_{j0,\ldots,j\mu = 0}^{n}(\mathcal{G}, x_{j\mu}\cdots x_{j0})\int_0^t \mathrm{d}\xi_{j\mu}\cdots \mathrm{d}\xi_{j0}. \tag{22}$$

Such a causal functional is said to be analytic with generating power series $\mathcal{G}$. Equation (22) should be considered as a representation of a causal functional in terms of a noncommutative generating power series and iterated integrals.

Let us consider an example to better understand what all this means. Let u_1 represent the forcing function of a dynamical system. Let u_1 be given as

$$u_1 = t^\alpha. \tag{23}$$

Then we shall have (cf. equations 3–5)

$$\int_0^t \mathrm{d}\xi_0\,\mathrm{d}\xi_1 = \int_0^t \mathrm{d}\tau\int_0^\tau \sigma^\alpha\,\mathrm{d}\sigma \tag{24}$$

$$= \int_0^t \frac{\tau^{\alpha+1}}{\alpha+1}\,\mathrm{d}\tau \tag{25}$$

$$= \frac{t^{\alpha+2}}{(\alpha+1)(\alpha+2)}. \tag{26}$$

Now let us consider the following integral instead

$$\int_0^t \mathrm{d}\xi_1\,\mathrm{d}\xi_0 = \int_0^t \sigma^\alpha\,\mathrm{d}\sigma \int_0^\sigma \mathrm{d}\tau = \int_0^t \sigma^\alpha \cdot \sigma\,\mathrm{d}\sigma \tag{27}$$

$$= \int_0^t \sigma^{\alpha+1}\,\mathrm{d}\sigma \tag{28}$$

$$= \frac{t^{\alpha+2}}{\alpha+2}. \tag{29}$$

These two repeated integrals are different iff $\alpha \neq 0$ in the same way that the corresponding series $x_0 x_1 \neq x_1 x_0$. This is what we mean by noncommutativity.

We shall give some fundamental theorems and remarks from [4, 5] and urge the curious reader to consult the references for the details of the proofs.

1. A causal functional is said to be analytic iff it is defined by a noncommutative formal power series called the generating power series.
2. The notion of an analytic causal functional generalizes in some sense the notion of an analytic function. We can represent an analytic function by its Taylor-series expansion and in a similar fashion we can *represent an analytic functional by its generating power series.*
3. The product of two analytic causal functionals is a functional of the same kind, the generating power series of which is the shuffle product of the two generating power series.

2.3. *Laplace–Borel transforms*

In this section we shall define the Laplace-Borel transformation $\mathcal{LB}$. Let us consider an analytic function $h(t)$:

$$h(t) = \sum_{n\geq 0} h_n \frac{t^n}{n!}. \tag{30}$$

We also have

$$\frac{t^n}{n!} = \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \cdots \int_0^{\tau_2} \mathrm{d}\tau_1. \tag{31}$$

Hence, we can rewrite $h(t)$ as

$$h(t) = \sum_{n\geq 0} h_n \int_0^t \mathrm{d}\tau_n \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} \mathrm{d}\tau_1. \tag{32}$$

Let us define the generating power series of noncommutative variable x_0 as

$$\mathcal{G} = \sum_{n\geq 0} h_n x_0^n. \tag{33}$$

We say that the Laplace-Borel transformation of $h(t)$ is $\mathcal{G}$ and we write

$$\mathcal{LB}[h(t)] \equiv \mathcal{G} = \sum_{n\geq 0} h_n x_0^n. \tag{34}$$

We have introduced a number of new concepts (new even to some mathematicians). An example may help to fix and identify them. Let us consider the Duffing equation:

$$\ddot{y}(t) + \alpha\dot{y}(t) + y(t) + \beta y^3(t) = u_1(t). \tag{35}$$

The nonlinearity is a cubic polynomial, so that the equation is sometimes referred to as a cubic harmonic oscillator. To obtain the repeated integral form, we integrate twice.

1. Integrate from 0 to t

$$\dot{y}|_0^t + \alpha y|_0^t + \int_0^t y(\tau)\,\mathrm{d}\tau + \beta\int_0^t y(\tau)^3\,\mathrm{d}\tau = \int_0^t u_1(\tau)\,\mathrm{d}\tau \tag{36}$$

and rearrange the terms:

$$\dot{y}(t) + \alpha y(t) + \int_0^t y(\tau)\,\mathrm{d}\tau + \beta\int_0^t y(\tau)^3\,\mathrm{d}\tau = \dot{y}(0) + \alpha y(0) + \int_0^t u_1(\tau)\,\mathrm{d}\tau. \tag{37}$$

2. Integrate again:

$$y(t) + \alpha\int_0^t y(\tau)\,\mathrm{d}\tau + \int_0^t \mathrm{d}\tau_2\int_0^{\tau_2} y(\tau_1)\,\mathrm{d}\tau_1 + \beta\int_0^t \mathrm{d}\tau_2\int_0^{\tau_2} y(\tau_1)^3\,\mathrm{d}\tau_1 \tag{38}$$

$$= \int_0^t \dot{y}(0)\,\mathrm{d}\tau + \int_0^t \alpha(y(0))\,\mathrm{d}\tau + y(0) + \int_0^t \mathrm{d}\tau_2\int_0^{\tau_2} u_1(\tau_1)\,\mathrm{d}\tau_1. \tag{39}$$

3. Use the correspondence between the iterated integrals and noncommutative variables to obtain:

$$\mathcal{G} + \alpha x_0\mathcal{G} + x_0^2\mathcal{G} + \beta x_0^2\mathcal{G}\amalg\mathcal{G}\amalg\mathcal{G} = b + (\alpha a + b)x_0 + x_0x_1 \tag{40}$$

where

$$a \equiv y(0) \tag{41}$$

$$b \equiv \dot{y}(0) \tag{42}$$

are the initial conditions. We can rewrite this as

$$(1 + \alpha x_0 + x_0^2)\mathcal{G} = b + (\alpha a + b)x_0 + x_0x_1 - \beta x_0^2\mathcal{G}\amalg\mathcal{G}\amalg\mathcal{G}. \tag{43}$$

The important point is that in the $\mathcal{LB}$ (Laplace–Borel) transform domain, the given nonlinear dynamical system is now represented as an algebraic equation. This algebraic equation can be solved, for example, by iteration. We consider it very important to acknowledge that, indeed, we are able to reduce a given nonlinear differential equation to an algebraic equation in much the same way that we use ordinary integral transforms (Laplace or Fourier) to reduce a given *linear* differential equation to an algebraic equation. An iterative scheme to solve (43) can be formulated as follows. Let us try a solution of the form

$$\mathcal{G} = \mathcal{G}_0 + \beta\mathcal{G}_1 + \beta^2\mathcal{G}_2 + \cdots. \tag{44}$$

Substitution into the orginal equation yields

$$\mathcal{G}_0 = \frac{b + (\alpha a + b)x_0 + x_0x_1}{1 + \alpha x_0 + x_0^2} \tag{45}$$

$$\mathcal{G}_1 = -\frac{x_0^2}{1 + \alpha x_0 + x_0^2}[\mathcal{G}_0 \amalg \mathcal{G}_0 \amalg \mathcal{G}_0] \tag{46}$$

$$\mathcal{G}_2 = -\frac{x_0^2}{1 + \alpha x_0 + x_0^2}[\mathcal{G}_0 \amalg \mathcal{G}_1 \amalg \mathcal{G}_0 + \mathcal{G}_1 \amalg \mathcal{G}_0 \amalg \mathcal{G}_0 + \mathcal{G}_0 \amalg \mathcal{G}_0 \amalg \mathcal{G}_1]. \tag{47}$$

We can rewrite $\mathcal{G}_2$ as

$$\mathcal{G}_2 = -\frac{x_0^2}{1 + \alpha x_0 + x_0^2}[3\mathcal{G}_0 \amalg \mathcal{G}_0 \amalg \mathcal{G}_1] \tag{48}$$

$$\mathcal{G}_3 = -\frac{x_0^2}{1 + \alpha x_0 + x_0^2} \times [\mathcal{G}_0 \amalg \mathcal{G}_2 \amalg \mathcal{G}_0 + \mathcal{G}_1 \amalg \mathcal{G}_0 \amalg \mathcal{G}_0 + \mathcal{G}_2 \amalg \mathcal{G}_0 \amalg \mathcal{G}_0 + \mathcal{G}_0 \amalg \mathcal{G}_1 \amalg \mathcal{G}_1 + \mathcal{G}_1 \amalg \mathcal{G}_0 \amalg \mathcal{G}_1]. \tag{49}$$

Let us introduce

$$\mathcal{S}(x_0) \equiv -\frac{x_0^2}{1 + \alpha x_0 + x_0^2}. \tag{50}$$

Then we have

$$\mathcal{G}_0 = -\mathcal{S}(x_0)\left(\frac{x_1}{x_0} + \frac{\alpha a + b}{x_0} + \frac{b}{x_0^2}\right) \tag{51}$$

$$\mathcal{G}_1 = \mathcal{S}(x_0)\mathcal{G}_0 \amalg \mathcal{G}_0 \amalg \mathcal{G}_0 \tag{52}$$

$$\mathcal{G}_2 = 3\mathcal{S}(x_0)\mathcal{G}_0 \amalg \mathcal{G}_0 \amalg \mathcal{G}_1 \tag{53}$$

$$\mathcal{G}_3 = \mathcal{S}(x_0)[\mathcal{G}_0 \amalg \mathcal{G}_2 \amalg \mathcal{G}_0 + \mathcal{G}_1 \amalg \mathcal{G}_0 \amalg \mathcal{G}_0 + \mathcal{G}_2 \amalg \mathcal{G}_0 \amalg \mathcal{G}_0 + \mathcal{G}_0 \amalg \mathcal{G}_1 \amalg \mathcal{G}_1 + \mathcal{G}_1 \amalg \mathcal{G}_0 \amalg \mathcal{G}_1]. \tag{54}$$

We cannot ignore the connection between these iterations and *Feynman* diagrams which enable one to figure out the next iteration if one does not want to use a computer. We repeat that a computer can be programmed to carry out such iterations for us up to whatever order we request.

To return to the time domain we must: (a) decompose $\mathcal{G}_i$ into partial fractions; (b) use the following lemma obtained in [1].

LEMMA. The rational fraction $(1 - ax_0)^{-p}$ corresponds to the exponential polynomial

$$\left[\sum_{j=0}^{p-1}\binom{j}{p-1}\frac{a^j t^j}{j!}\right]e^{at}. \tag{55}$$

2.4. *A list of $\mathcal{LB}$ transforms*

We give a *partial* list of $\mathcal{LB}$ transforms:

$$\mathcal{LB}[\text{unit step}] = 1$$

$$\mathcal{LB}[e^{at}] = \frac{1}{1 - ax_0}$$

$$\mathcal{L}\mathcal{B}[e^{at}(1 + at)] = \frac{1}{(1 - ax_0)^2}$$

$$\mathcal{L}\mathcal{B}\left[e^{at}\left(1 + at = \frac{a^2t^2}{2}\right)\right] = (1 - ax_0)^3$$

$$\mathcal{L}\mathcal{B}\left[\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{a^it^i}{i!}\right)e^{at}\right] = (1 - a_0x_0)^{-n}$$

$$\mathcal{L}\mathcal{B}[\sin wt] = \frac{iwx_0}{1 + w^2x_0^2}$$

$$\mathcal{L}\mathcal{B}[\cos wt] = \frac{1}{1 + w^2x_0^2}$$

$$\mathcal{L}\mathcal{B}[\sinh wt] = \frac{wx_0}{1 - w^2x_0^2}$$

$$\mathcal{L}\mathcal{B}[\cosh wt] = \frac{1}{1 - w^2x_0^2}.$$

3. CHAOTIC REGIMES OF NONLINEAR DYNAMICAL SYSTEMS

Chaotic regimes correspond to strange attractors. Strange attractors are limiting sets with noninteger dimension in the phase space of the trajectories of the dynamical system. Numerically speaking, given the physical parameters we can compute: (a) the power spectrum; (b) the Lyapunov coefficients; or (c) the dimension of the attractor to check whether the regime is chaotic or not. For (a) a broad band, for (b) a positive Lyapunov coefficient, and for (c) a noninteger dimension will indicate a random-like response which is the characteristic of the so-called chaotic regime. A fourth check would be (d) the autocorrelation function $AC(\tau)$. The autocorrelation function $AC(\tau)$ vanishes in the limit as $\tau \to \infty$ for chaotic regimes. Initially infinitesimally close trajectories in the phase space diverge exponentially and hence fail to correlate at large times. More precisely, the autocorrelation is a measure of the similarity of a trajectory at a given time t with its value at a later time $t + \tau$. We define $AC(\tau)$ as the arithmetic mean of a large number of products such as $x(t) \cdot x(t + \tau)$, i.e.

$$AC(\tau) = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x(t) \cdot x(t + \tau)\, dt. \tag{56}$$

In a more compact way we can write:

$$AC(\tau) = \langle x(t) \cdot x(t + \tau)\rangle. \tag{57}$$

We call $AC(\tau)$ a *temporal autocorrelation function*. By varying the interval τ, we can construct $AC(\tau)$. It defines the degree of similarity of the trajectory $x(t)$ with itself as time evolves.

The Wiener–Kintchine theorem states that $AC(\tau)$ is the Fourier transform of the power spectrum. As a result of this we see that for the regimes represented by an attractor which is: (a) a fixed point; (b) a limit cycle; (c) a n-torus we have

$$\lim_{\tau\to\infty} AC(\tau) \neq 0, \tag{58}$$

since in these three cases the power spectrum is formed of distinct rays. In other words, periodic or quasiperiodic trajectories keep their internal similarity with the evolution of time. This means that the behaviour of the system is predictable. In contrast, for a chaotic regime, where the power spectrum has a broad band,

$$\lim_{\tau\to\infty} AC(\tau) = 0, \tag{59}$$

(cf. Fig. 1).

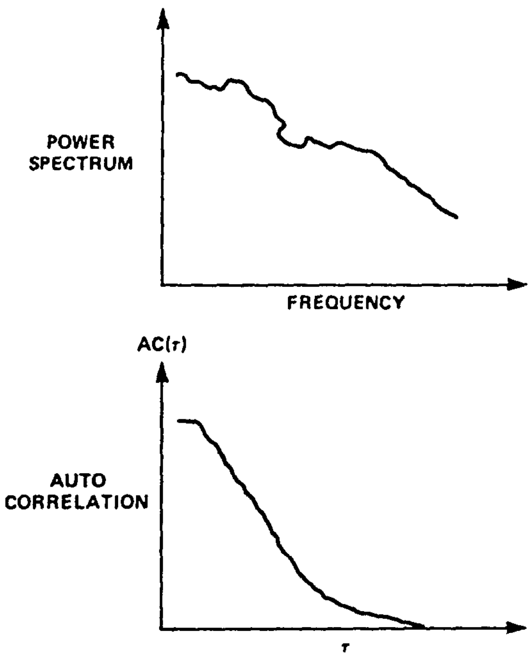

Fig. 1. Power spectrum and temporal autocorrelation.

As Fig. 1 illustrates, the temporal similarity of the trajectory with itself tends towards zero at sufficiently large times. It follows that knowing $x(t)$ for a long lapse of time (as long as one wishes!) does not allow the prediction of the future of $x(t)$. We can say that the chaotic regime is unpredictable because of the progressive loss of internal similarity of trajectories. We observe a loss of memory in terms of initial conditions (IC) with attractors of the fixed point, limit cycle and n-tori types since for each of these types, trajectories originating from diverse initial conditions outside the attractor converge on the attractor. For example, any initial condition will result in the same constant steady motion if the attractor is a fixed point. In chaotic regimes, we have just the opposite: almost the same initial conditions result in different final states. Figure 2

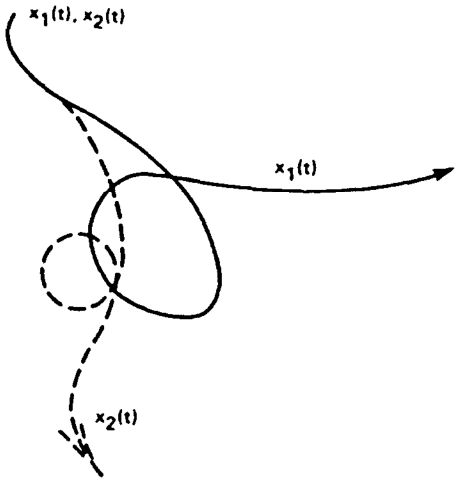

Fig. 2. Two initially close trajectories.

depicts the phase space in which two trajectories initially very close diverge from each other and hence lose their similarity in a finite time.

We shall give the following definition of chaos and its connection with the property called sensitivity to initial conditions (SIC).

Definition. If a regime is represented by an attractor such that infinitesimally close neighboring trajectories diverge exponentially, such a regime is characterized as chaotic. The property of exponential amplification of errors or uncertainties in the initial conditions (IC) which is called the *sensitivity to initial conditions* (SIC), characterizes the regime as chaotic.

To summarize, the existence of a chaotic regime of a dissipative nonlinear dynamical system is characterized by any one of the following conditions: (a) an attractor with a noninteger dimension; (b) a sensitivity to initial conditions; (c) a broad-band spectrum; (d) a vanishing autocorrelation function.

We shall make use of the last condition for our criterion. We have observed that the $\mathcal{LB}$ transformation will enable us to have a semi-analytic criterion. The following basic properties of $\mathcal{LB}$ transforms are needed in the formulation of the criterion.

SHIFTING THEOREM. The Laplace–Borel transformation of a function whose argument is shifted by an amount τ is related to its Laplace–Borel transform as follows:

$$\mathcal{LB}[f(t+\tau)] = \left(\sum_{k\geq 0} \frac{\tau^k}{k!x_0^k}\right)\mathcal{LB}[f(t)]. \tag{60}$$

Proof. Let us write the Taylor-series expansion for this function:

$$f(t+\tau) = f(t) + \tau f'(t) + \frac{\tau^2}{2!}f''(t) + \cdots. \tag{61}$$

Since we started with a function for which there exists an $\mathcal{LB}$ transformation, we can write

$$f(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!} \tag{62}$$

or, there is a corresponding generating power series $\mathcal{G}$ of the noncommutative variable x_0:

$$\mathcal{G} = \sum_{n \geq 0} a_n x_0^n. \tag{63}$$

This $\mathcal{G}$ by definition is the Laplace-Borel transformation of the original function, $f(t)$. The Laplace-Borel transformation of the derivative of $f(t)$ follows immediately as:

$$f'(t) = \sum_{n \geq 0} n \frac{a_n}{n!} t^{n-1} \tag{64}$$

$$f'(t) = \sum_{n \geq 0} \frac{a_n}{(n-1)!} t^{n-1} \tag{65}$$

$$f'(t) = \sum_{n \geq 0} a_n \frac{t^{n-1}}{(n-1)!}. \tag{66}$$

Hence its generating power series will be

$$\sum_{n \geq 0} a_n x_0^{n-1} \tag{67}$$

or

$$\frac{1}{x_0} \sum_{n \geq 0} a_n x_0^n \tag{68}$$

or we write

$$\mathcal{LB}[f'(t)] = \frac{1}{x_0} \mathcal{LB}[f(t)]. \tag{69}$$

Now we can write down the Laplace-Borel transformation of

$$f(t + \tau) = f(t) + \tau f'(t) + \frac{\tau^2}{2!} f''(t) \tag{70}$$

$$\mathcal{LB}[f(t + \tau)] = \sum_{n \geq 0} a_n x_0^n + \frac{\tau}{x_0} \sum_{n \geq 0} a_n x_0^n + \frac{\tau^2}{2! x_0^2} \sum_{n \geq 0} a_n x_0^n + \cdots \tag{71}$$

$$\mathcal{LB}[f(t + \tau)] = \left(1 + \frac{\tau}{x_0} + \frac{\tau^2}{2! x_0^2} + \cdots\right) \sum_{n \geq 0} a_n x_0^n \tag{72}$$

$$\mathcal{LB}[f(t + \tau)] = \left(\sum_{k \geq 0} \frac{\tau^k}{k! x_0^k}\right) \sum_{n \geq 0} a_n x_0^n \tag{73}$$

or we write

$$\mathcal{LB}[f(t + \tau)] = \left(\sum_{k \geq 0} \frac{\tau^k}{k! x_0^k}\right) \mathcal{LB}[f(t)] \tag{74}$$

which completes the proof of the shifting theorem.

Next we show a limiting property under the name of the *limiting theorem.*

LIMITING THEOREM. The temporal limit as $t \to \infty$ corresponds to the limit of the noncommutative variable the Laplace–Borel transformation as $x_0 \to \infty$.

Proof. The proof follows from the definition of the $\mathcal{LB}$ transform. If we have an analytic function

$$f(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!} \tag{75}$$

then its $\mathcal{LB}$ transform is given by

$$\mathcal{LB}[f(t)] = \sum_{n \geq 0} a_n x_0^n. \tag{76}$$

Therefore if $t \to \infty$ then in the transform domain $x_0 \to \infty$. With the help of these theorems we are ready to present the main theorem of this paper, which represents the criterion for the onset of chaos.

MAIN THEOREM. The following algebraic criterion has to be satisfied for the onset of chaos:

$$\lim_{\substack{\tau \to \infty \\ x_0 \to \infty}} \left(\sum_{k \geq 0} \frac{\tau^k}{k! x_0^k} \right) \mathcal{G} \sqcup\!\sqcup \mathcal{G} = 0 \tag{77}$$

where $\mathcal{G}$ is the generating power series for the trajectories of the nonlinear dynamical system. In other words, $\mathcal{G}$ is the Laplace–Borel transformation of the trajectories of the given nonlinear dynamical system.

Proof. We adopt as definition of the autocorrelation function

$$AC(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t + \tau) \, dt. \tag{78}$$

As we discussed previously, we must have

$$\lim_{\tau \to \infty} AC(\tau) = 0$$

for the onset of chaos

Let us take the Laplace–Borel transformation of both sides;

$$\mathcal{LB}[AC(\tau)] = \lim_{x_0 \to \infty} x_0 \mathcal{LB}[x(t)] \sqcup\!\sqcup \mathcal{LB}[x(t + \tau)]. \tag{79}$$

Notice that we have utilized a result in [1] to take the Laplace–Borel transformation of a product of two functions. The Laplace–Borel transformation of the product is equal to the shuffle product of the individual transformations. Let us denote $\mathcal{LB}[x(t)]$ by $\mathcal{G}$. Then by the shifting theorem we have:

$$\mathcal{LB}[x(t + \tau)] = \left(\sum_{k \geq 0} \frac{\tau^k}{k! x_0^k} \right) \mathcal{G}. \tag{80}$$

Hence the criterion for the onset of chaos becomes

$$\lim_{\substack{\tau \to \infty \\ x_0 \to \infty}} \left(\sum_{k \geq 0} \frac{\tau^k}{k! x_0^k} \right) \mathcal{G} \amalg \mathcal{G} = 0 \tag{81}$$

which completes the proof of the main theorem.

4. CONCLUDING REMARKS

The criterion obtained as a main theorem is a computer-algebraic one. It is noteworthy that the algebra required in obtaining the two generating power series and their shuffle product, together with the two limits, all can be done on a computer with a symbolic language. At the moment we have the option of using the following symbolic languages: PLI, REDUCE, MACSYMA, LISP.

Although the criterion is simple in concept, its application by no means will be a trivial affair. A considerable effort doubtless will be necessary to handle the difficulty of bifurcation points. These are points in parameter space where analyticity will be lost, and hence where the generalized series expansions will not be valid. Such points will reveal themselves beforehand by slowing down the rate of convergence of the iterations. We anticipate the necessity of considering slow convergence as a signal to change the set of parameters so as to skip over a bifurcation point. Since we are not interested in the identification of such a point, it should not be necessary to come too close to it. We assume that there will be a finite and a small number of bifurcation points which it should be possible to skip successively if necessary, bringing us ultimately to a parameter regime that borders on the object of our search. This is one of the difficulties we can foresee; there may well be others.

Our criterion, which is computer-algebraic, can be utilized to characterize the ranges of the physical parameters for which chaotic regimes will take place. Hence, it can be used in designing the operation of a nonlinear dynamical system to avoid such regimes. In other words, we can control chaos in our nonlinear dynamical system by staying outside of the ranges defined by our criterion.

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