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Algebraic differential equations and nonlinear control systems

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ALGEBRAIC DIFFERENTIAL EQUATIONS AND NONLINEAR CONTROL SYSTEMS

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ABSTRACT OF THE DISSERTATION

Algebraic Differential Equations and Nonlinear Control Systems

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Dissertation Director: Professor Eduardo D. Sontag

This dissertation establishes a precise correspondence between realizability of operators defined by convergent generating series and the existence of high order differential equations (*"i/o equations"*) relating derivatives of inputs and outputs.

State space models are central to modern nonlinear control theory, since they permit the application of techniques from various mathematics branches such as differential equations, dynamical systems and optimization theory. A natural question is to decide when a given i/o operator admits a representation by an initialized state space system (the operator is *realizable*).

To investigate the relation between i/o equations and realizability, we introduce and study the structures of observation spaces, observation algebras and observation fields. In realization theory and many other areas of nonlinear control, the concept of observation space plays a central role. One may define observation spaces in two very different ways. Roughly, one possibility is to take the functions corresponding to derivatives with respect to switching times in piecewise constant controls, and the other is to take high-order derivatives at the final time , if smooth controls are used. It turns out that the existence of algebraic i/o equations is closely related to the finiteness properties of the observation algebra and field associated with the first type of observation space,

while realizability is closely related to the finiteness properties of the algebraic objects associated with the other type of observation space. One of the central technical results, given in Chapter 3, shows that the two types of spaces coincide.

Based on the results mentioned above, we get our main results: Realizability by singular polynomial systems is equivalent to existence of algebraic i/o equations. We also provide other results relating various special kinds of i/o equations to some specific classes of realizations, for instance, what are called recursive i/o equations are related to realizability by polynomial systems.

In Chapter 7, our results relating algebraic i/o equations to realizability by "rational" systems are extended to analytic i/o equations and local realization by analytic systems. By studying properties of meromorphically finitely generated field of functions, together with the application of some known facts in the literature of nonlinear realization, we conclude that the existence of analytic i/o equations implies local realizability by analytic systems.

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Chapter 1

Introduction

In this dissertation we establish a precise correspondence between realizability of input/output operators and the existence of high order differential equations relating derivatives of inputs and outputs.

In many experimental situations involving systems, it is often the case that one can model system behavior through differential equations, which are referred to as *input/output ("i/o") equations* in this work, of the type

$$E\left(u(t), u'(t), u''(t), \ldots, u^{(r)}(t), y(t), y'(t), y''(t), \ldots, y^{(r)}(t)\right) = 0 \tag{1.1}$$

where $u(\cdot)$ and $y(\cdot)$ are the input and output signals respectively. An i/o operator $F: u(\cdot) \mapsto y(\cdot)$ is said to *satisfy* the equation (1.1) if the equation holds for each input u and the corresponding output $y = F[u]$ of F. (Precise definitions will be given later.)

The functional relation E is usually estimated, for instance through least squares techniques, if a parametric general form (e.g. polynomials of fixed degree) is chosen. For example, in linear systems theory one often deals with degree-one polynomials E:

$$y^{(k)}(t) = a_1 y(t) + \ldots + a_k y^{(k-1)}(t) + b_1 u(t) + \ldots + b_k u^{(k-1)}(t) \tag{1.2}$$

(or their frequency-domain equivalent, transfer functions; the difference equation analogue is sometimes called an "autoregressive moving average" representation). In the linear case, such representations form the basis of much of modern systems analysis and identification theory.

On the other hand, in most theoretical developments in nonlinear control, one uses a state-space formalism, where inputs and outputs are related by a system of *first* order differential equations

$$x'(t) = f(x(t)) + G(x(t))u(t)\,, \quad y(t) = h(x(t)) \tag{1.3}$$

where the state $x(t)$ is multidimensional, and no derivatives of controls are allowed. These descriptions are central to the modern nonlinear control theory, as they permit the application of techniques from differential equations, dynamical systems, and optimization theory. Thus a basic question is that of deciding when a given i/o operator admits a representation of this form. This is the area of *realization theory*, which is closely related, especially when stochastic effects are included, to *systems identification*. Roughly speaking, if such a state space description does exist for a given i/o operator, then we say that the i/o operator is *realizable*. The precise definitions for various notions of realizability will be given in Chapters 5 and 7.

It is a classical (and easy) fact that an equation such as (1.2) can be reduced, by adding state variables for enough derivatives of the output y, to a system (1.3) of first order equations, with $f(x)$ linear and $G(x)$ constant, i.e., a linear finite-dimensional system. In frequency-domain terms, rationality of the transfer function is equivalent to realizability. (For references on the linear theory, see eg [19], [33] and [43].) One of the methods for obtaining a linear realization from a given linear i/o equation relies on Lord Kelvin's principle for solving differential equations by means of mechanical analog computers (cf. [19]). The principle, which was suggested a hundred years ago, provided a way for simulating a system without using differentiators.

For nonlinear systems this reduction presents a far more difficult problem, one that is to a great extent unsolved. The problem is basically that of in some sense replacing a nontrivial equation (1.1) by a system of first-order equations (1.3) which does not involve the derivatives of the inputs.

1.1 Previous Nonlinear Work

The work [36] used a differential geometry approach to develop a theory of realization of i/o operators, including results on existence and uniqueness of "minimal" realizations. The papers [18] and [17] characterized existence conditions for realizations in terms of smoothness (or analyticity) of the i/o operator plus a rank condition. In [4] and [3], tools from algebraic geometry were employed in order to study the structure of observation algebras and observation fields. The results there related finiteness properties of the various algebraic objects to realizability, in strict analogy to the relations that hold in discrete time ([34]).

In fact, the discrete-time work [34] provided one approach to relating these two types of representations —with difference equations appearing instead,— and this was used as a basis of identification algorithms by other authors; see for instance [22] and [8]. (The former reference shows also how to include stochastic effects in the resulting approach.) These results have recently been extended to continuous-time for the very special case of *bilinear* systems: A theorem showed that realizability by such systems is equivalent to the existence of an E of a special form, namely affine on y (see [32]). However, the techniques in [32] were linear-algebraic, and hence not powerful enough to handle the extension of [34] to the general nonlinear case.

The present work completes the development of this extension of the result in [34] to continuous-time. A number of partial results were already available about the relation between (1.1) and (1.3); see for instance [39], [7] or [14]. It is easy to show, by elementary arguments involving finite transcendence degree, that any i/o operator realizable by a rational state space system satisfies some i/o equation of type (1.1), with E a polynomial. In [9] it was remarked –as a consequence of theorems from differential algebra, that in order to characterize the i/o behavior of a state space system *uniquely*, one needs to add inequality constraints to (1.1). In [26] and [40] it was shown that, under some constant rank conditions, the outputs of an observable smooth state space system can be described by an equation of type (1.1) for which E is a smooth function. Local i/o equations were shown to exist, for generic initial states of (1.3), in [6]; however, in contrast to the algebraic case, it is generally not true that every state space system gives rise to a global i/o equation, even under analyticity assumptions. (This is discussed later through an example.)

1.2 Our Approach

The view proposed in [30], [34] for discrete-time, and followed here in the differential equation case, is that one should attack the problem as follows. One should separate the issue of existence of a realization from the question of "well-posedness" of the equation.

For example, the equation

$$u(t)y'(t) = 1$$

can never be satisfied by all the input/output pairs corresponding to a state space system, as remarked in [32], nor is this true for

$$y''(t) = u'(t)^2 \ .$$

In both of these cases, not only cannot the equation be reduced to state-space form but —as one can easily prove— even more basically, it cannot be satisfied by any "input/output map" of the type that we shall consider. Indeed, our main contribution is to show that *if* the equation would have been well-posed, in the sense that it is an equation satisfied by all input/output pairs corresponding to what we will call a *Fliess operator* —i.e. one described by a convergent generating series— and if E is a polynomial, then it is always realizable by a singular polynomial system, or a rational system with possible poles. (Singular systems appear naturally in control theory, for instance in robotics; see [24] for many examples.)

In the special case when equation (1.1) is recursive —i.e. the coefficient of the highest derivative of y in (1.1) does not depends on the lower derivatives of y,— our construction will provide a polynomial realization (no poles). In the general case, we shall prove that about every singular point of the realization there is another system, locally defined in terms of analytic functions, that realizes (locally) the desired behavior. The picture that emerges then is that, at least, one can cover the possibly singular part with local analytic realizations. In a computer simulation, this would be achieved by passing to a subroutine to deal with trajectories near this set. In fact we have proved that any equation (1.1) for which E is just analytic gives rise to a local analytic realization. (We emphasize, this is always subject to the hypothesis that the i/o pairs arise from some Fliess operator.)

Our formalism is based on the *generating series* suggested by Fliess in the late 70's, who was in turn motivated by Chen's work on power series solutions of differential equations. The i/o operators induced by convergent generating series form a very general class of causal operators, capable of representing a variety of nonlinear systems.

We shall call them "Fliess operators". For instance, any i/o operator induced by an initialized analytic state space system affine in controls can be described in this manner. In the development of linear control theory, transfer functions and transfer matrices were used first in the analysis of linear systems and state-space approaches were only later introduced. State space approaches have always played a role central in nonlinear control theory. However, other descriptions of i/o behaviors, such as Volterra series and generating series, are also appropriate and often useful. The better understanding of the relations between these different descriptions of i/o behaviors is essential. It has been known that any Fliess operator is realizable (locally, by an analytic system,) if and only if the "Lie rank" of the series is finite (see [11] and [35]). What we do in this work is to provide a link between existence of i/o equations and realizability for such operators.

Our results also provide a link with the differential-algebraic work of Fliess, who in [12] *defined* realizability by the requirement that outputs be differentiably dependent on inputs, in other words, that an equation such as (1.1) hold. We show then that this is basically the same as realizability in the more classical sense. Yet another link is with the recent work of Willems and his school. Consider the *behavior* $w(\cdot) = (u(\cdot), y(\cdot))$ associated to an input/output description. If we write the equation as

$$E\left(w(t), w'(t), w''(t), \ldots, w^{(r)}(t)\right) = 0$$

as preferred in some of the recent system-theoretic literature (see [42]), then what we do is to relate the fact that the behavior satisfies an algebraic differential equation to realizability.

In addition to single operators, it is also natural to study *families of i/o maps*, defined by a *family of convergent generating series*. To study a single i/o map is natural as a formal description of a initialized *black box*, but in general, a system may induce more than one i/o map. For example, a system described by an ordinary differential equation on a manifold may induce infinitely many i/o maps, each of them corresponding to some initial state. One should study all the i/o maps induced by the system simultaneously rather than individually, unless a fixed initial state is of particular interest. This leads to the concept of families of i/o maps. One question arises naturally: when can a family of i/o maps be realized by *one* state space system? i.e., when can all the members of the family be realized by some singular polynomial system in such a way that each member of the family is associated to some initial state of the system? We will prove that a family of i/o maps is realizable in this sense if and only if all the members of the family satisfy a common i/o equation.

The proofs are based on a careful analysis of the concept of *observation space*, introduced in [23] (and [34] for discrete-time), developed further in [13], and later rediscovered by many authors. One of the central technical results, given in Chapter 3, relates two different definitions of this space, one in terms of smooth controls and another in terms of piecewise constant ones; these two definitions are seen to coincide. One of them immediately relates to i/o equations, while the other is related to realizability through the notion of *observation algebras* and *observation fields*. The latter are the analogues of the corresponding discrete-time concepts studied in [34]. For differential equations they were first employed in [4] and [3].

1.3 Outline of This Work

The organization of this work is as follows:

In Chapter 2 we first introduce the basic terminology regarding series, convergence, and so forth, and introduce an algebraic structure on series, the shuffle product. Then for operators defined by evaluation of these series, we study smoothness properties of their output functions. Although several of the results presented there have been known and used often by previous authors, it is difficult to find complete proofs in the literature.

In Chapter 3, we consider observation spaces. Since their introduction in the mid 70's (see [23] and [13], as well as [34] for the discrete time analogue), observation spaces for nonlinear control systems

$$x' = f(x) + \sum u_i g_i(x)\,, \quad y = h(x) \tag{1.4}$$

have played a central role in the understanding of realization theory. For the system

(1.4), one defines the observation space $\mathcal{F}$ as the linear span of the Lie derivatives

$$L_{X_1} \cdots L_{X_k} h,$$

where each X_i is either f or one of the g_i's. (Here we are taking states $x(t)$ in a manifold, $f, g_1, \ldots, g_m$ smooth vector fields, and h a function from the manifold to $\mathbb{R}$, the output map. The linear span is understood in the space of smooth functions into $\mathbb{R}$.)

It is known that many important properties of systems, such as the possibility of simulating such a system by one described by linear vector fields (the "bilinear immersion" problem, [13]), are characterized by properties of this space.

It was shown in [32] that a different type of "observation space" is much more important when one studies i/o equations satisfied by (1.4), i.e. equations of the type (1.1) that hold for all those pairs of functions $(u(\cdot), y(\cdot))$ that arise as solutions of (1.4). This alternative observation space is obtained by taking the derivatives $y(t), y'(t), \ldots$ as functions of initial states, over all $u(t), u'(t), \ldots$. This space is obtained by considering differentiable controls and time-derivatives, while the space previously considered is based on derivatives with respect to switching times in piecewise constant controls.

The central fact used in [32] in order to relate i/o equations to realizability is the equality of the two observation spaces defined in the above manners. This equality is fundamental not only for the results in that paper, which hold under the assumption that the spaces are finite-dimensional, but also for more general results in this work. However, the techniques used in [32] are based on a linear-algebraic and a topological argument, involving closure in the weak topology, which does not in any way extend to the more general case of infinite dimensional observation spaces. Since the latter are the norm rather than the exception (unless the system can be simulated by a bilinear system to start with), one needs to establish the equality of these two types of spaces using totally different combinatorial techniques. That is achieved in Chapter 3. Also in Chapter 3, we extend the result to families of i/o operators, which, in turn, is applied to state space systems.

In Chapter 4, we study i/o equations satisfied by i/o operators. For this purpose, we find it useful to introduce the algebraic concepts of observation algebra and observation field corresponding to a given series. Motivated by [34], we show that the existence of an i/o equation implies that the observation field is a finitely generated field extension of $\mathbb{R}$. The real meaning of this result is that if the transcendence degree of the observation field over $\mathbb{R}$ is finite, then the field is a finitely generated extension of $\mathbb{R}$, which is not true in general for arbitrary fields. It does hold here basically due to the fact that the observation field is a "differential field" —a field with derivation operators. However, it is not a usual differential field in the standard sense of the differential algebra literature (cf. [20]) since the derivation operators used here are not commutative in general. Nonetheless, one still finds some special properties of the field that help in proving the results.

In Chapter 5, realizability by polynomial systems and singular polynomial systems are considered. One may consider a singular polynomial system as a "rational" system in the sense of [3]. However, to be cautious about the possible poles in the right-hand side of the equation, we prefer to study singular systems.

As existence of i/o equations is closely related to the structure of observation space, algebra and field, realizability forces the study of the structures of the algebraic objects related to the second type of observation space that we introduced in Chapter 3. Though it turns out that the two types of the algebraic objects are the same, due to the results in Chapter 3, the results of this Chapter are independent of the fact and they are more readily understood in terms the second type of the objects. The main result in Chapter 5 is that realizability by singular polynomial systems is guaranteed by the condition that the observation field is a finitely generated extension of $\mathbb{R}$. The approach pursued there is to use the generators of the field as state variables and use the equalities which hold among the generators to construct the needed vector fields.

Realizability for families of operators is also studied. To achieve our goal, some technical conditions are imposed on the parameter dependence and the parameter sets. These conditions are usually satisfied by families of i/o operators described by a state space system with different initial states.

In Chapter 6, based on the results obtained in the previous Chapters, we give the main results of our dissertation, establishing the equivalence between realizability by a singular polynomial system and the existence of an algebraic i/o equation for both operators and families of operators.

We also show there that recursive i/o equations lead to realization by polynomial systems. However, as opposed to the previous case, the converse of this fact is not true in general. A counterexample is provided to illustrate the fact that realizability by a polynomial system may not lead to a recursive i/o equation. For this purpose we show that for an operator that arise from an initialized accessible state space system, the two observation algebras, one defined in terms of Lie derivatives of the observation map in the state space, the other one defined in terms of the series which defines the same operator, are isomorphic to each other.

In Chapter 7, our previous results relating algebraic i/o equations to internal realizability are extended to analytic i/o equations and local internal realizability. One of the motivations of this Chapter is to get rid of the singularities in singular polynomial realizations, but we do not know yet whether one can always get a polynomial realization without singularities. (This is perhaps one of the most interesting problems for further research.) The result of this Chapter serves to show that if one can get a singular polynomial realization, then around each singular point there is another analytic system realizing the i/o operator locally.

The approach to proving the results in this Chapter is to first construct a "meromorphic" realization by studying the properties of meromorphically finitely generated field extensions. Then by a perturbation approach, together with the Lie rank condition for realizability (cf. [11] and [35]), we show that around each point there is local analytic realization. This can also be done by using the rank condition studied in [17]. In contrast to the algebraic case, it is not true that every operator realizable by an analytic system satisfies an analytic i/o equation. A modification of an example due to [28] is presented to illustrate the fact.

A final Chapter summarizes conclusions and gives suggestions for further research.

Chapter 2

Generating Series and I/O Operators

In this chapter we introduce the basic terminology regarding series and i/o operators as well as most of the elementary facts to be used later. Although several of the results presented here have been known and used often by previous authors, it is difficult to find complete proofs of many of them in the literature. The Chapter introduces an algebraic structure on series, the shuffle product, and studies convergence properties. For operators defined by evaluating these series, we study smoothness properties of the corresponding output functions.

2.1 Generating Series

The "input/output maps" that we use are defined by a certain type of power series. The most convenient way to introduce them is by first defining and studying the abstract algebra of such power series.

Let m be a fixed integer and $I = \{0, 1, \ldots, m\}$. For any integer $k \geq 1$, we define

$$I^k = \{(i_1 \, i_2 \, \ldots \, i_k) : i_s \in I, \; 1 \leq s \leq k\}.$$

For $k = 0$, we use I^0 to denote the set whose only element is the empty sequence ϕ. Let

$$I^* = \bigcup_{k \geq 0} I^k. \tag{2.1}$$

Then I^* is a free monoid with the composition rule:

$$(i_1 i_2 \ldots i_k)(j_1 j_2 \ldots j_l) = (i_1 i_2 \ldots i_k j_1 j_2 \ldots j_l).$$

If $\iota \in I^l$, then we say that the *length* of ι, denoted by $|\iota|$, is l.

Consider now the "alphabet" set

$$P = \{\eta_0, \eta_1, \ldots, \eta_m\}$$

and P^*, the free monoid generated by P, where the neutral element of P^* is the empty word, denoted by 1, and the product is concatenation. Let

$$P^k = \{\eta_{i_1}\eta_{i_2}\ldots\eta_{i_k} : 0 \le i_s \le m, \ 1 \le s \le k\}$$

for each $k \ge 0$. We define $\mathcal{P}$ to be the $\mathbb{R}$-algebra generated by P^*, i.e, the set of all polynomials in the variables η_i's. A *power series in the noncommutative variables* $\eta_0, \eta_1, \ldots, \eta_n$ is a formal power series

$$c = \sum_{\iota \in I^*} \langle c, \eta_\iota \rangle \eta_\iota, \tag{2.2}$$

where

$$\eta_\iota = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_l} \quad \text{if } \iota = i_1 i_2 \cdots i_l,$$

and $\langle c, \eta_\iota \rangle \in \mathbb{R}$ for each multiindex ι. Note that c is a polynomial if and only if there are only finitely many $\langle c, \eta_\iota \rangle$'s which are non-zero. A power series is nothing more than a mapping from I^* to $\mathbb{R}$; as we shall see later, however, the algebraic structures suggested by the series formalism are very important. We use $\mathcal{S}$ to denote the set of all power series (over a fixed but arbitrary alphabet P).

For c, $d \in \mathcal{S}$ and $\gamma \in \mathbb{R}$, $\gamma c + d$ is the series defined as follows:

$$\langle \gamma c + d, \eta_\iota \rangle = \gamma \langle c, \eta_\iota \rangle + \langle d, \eta_\iota \rangle.$$

With these operations, $\mathcal{S}$ forms a vector space over $\mathbb{R}$. In addition, we can introduce an algebra structure on $\mathcal{S}$ by defining the *shuffle product* on $\mathcal{S}$. First of all, we define the shuffle product on words,

$$ш : P^* \times P^* \longrightarrow \mathcal{P}$$

inductively on length in the following way:

$$1 ш \eta = \eta ш 1 = \eta \quad \text{for any} \quad \eta \in P\ ,$$

$$\eta_i\eta_\iota ш\eta_j\eta_\kappa = \eta_i(\eta_\iota ш\eta_j\eta_\kappa) + \eta_j(\eta_i\eta_\iota ш\eta_\kappa) \quad \text{for any} \quad \eta_\iota, \eta_\kappa \in P^*, \eta_i, \eta_j \in P. \tag{2.3}$$

It can be proved by induction that an equivalent way to define the shuffle product is to replace (2.3) by the following:

$$\eta_\iota\eta_i ш\eta_\kappa\eta_j = (\eta_\iota ш\eta_\kappa\eta_j)\eta_i + (\eta_\iota\eta_i ш\eta_\kappa)\eta_j \quad \text{for any} \quad \eta_\iota, \eta_\kappa \in P^*, \eta_i, \eta_j \in P. \tag{2.4}$$

Then we extend the shuffle product to power series in the following way: For

$$c = \sum\langle c,\ \eta_\iota\rangle\eta_\iota \quad \text{and} \quad d = \sum\langle d,\ \eta_\kappa\rangle\eta_\kappa,$$

we define

$$c ш d = \sum\langle c,\ \eta_\iota\rangle\langle d,\ \eta_\kappa\rangle\eta_\iota ш\eta_\kappa. \tag{2.5}$$

Note that (2.5) can be written in an alternative manner which is often very useful, as follows:

$$c ш d = \sum_{\iota,\kappa,\rho\in I^*} \langle c,\ \eta_\iota\rangle\langle d,\ \eta_\kappa\rangle\langle\eta_\iota ш\eta_\kappa,\ \eta_\rho\rangle\eta_\rho,$$

that is,

$$\langle c ш d,\ \eta_\rho\rangle = \sum_{\iota,\kappa\in I^*} \langle c,\ \eta_\iota\rangle\langle d,\ \eta_\kappa\rangle\langle\eta_\iota ш\eta_\kappa,\ \eta_\rho\rangle.$$

With the operations "+" and "ш" defined as above, $\mathcal{S}$ forms a commutative $\mathbb{R}$-algebra. Moreover, we have the following fact:

Lemma 2.1.1 The algebra $\mathcal{S}$ is an integral domain.

Proof. First we order the basis elements $(\eta_{i_1}, \ldots, \eta_{i_k})$ of P^* lexicographically with respect to

$$k, i_1, i_2, \ldots, i_k.$$

Then take two nonzero series c and d and let

$$z_1 = \eta_{i_1}\cdots\eta_{i_m}$$

and

$$z_2 = \eta_{j_1}\cdots\eta_{j_n}$$

be the smallest basis element of P^* appearing in c and d, respectively, with nonzero coefficients. Let

$$w := \eta_{l_1} \cdots \eta_{l_{m+n}}$$

be the smallest basis elements of P^* appearing in $z_1 \sqcup\!\sqcup z_2$. Then the coefficient of w in $c \sqcup\!\sqcup d$ is:

$$\langle c \sqcup\!\sqcup d,\ w\rangle = \sum_{\iota,\kappa} \langle c,\ \eta_\iota\rangle \langle d,\ \eta_\kappa\rangle \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ w\rangle.$$

Using the minimality property of w, z_1, z_2, we get

$$\langle c \sqcup\!\sqcup d,\ w\rangle = \langle c,\ z_1\rangle \langle d,\ z_2\rangle \langle z_1 \sqcup\!\sqcup z_2,\ w\rangle,$$

which is nonzero since $\langle c,\ z_1\rangle$, $\langle d,\ z_2\rangle$, $\langle z_1 \sqcup\!\sqcup z_2,\ w\rangle$ are all nonzero. ∎

The method used in the above proof is similar to the method used in [27], where the author proved that the ring of polynomials in $\eta_0, \eta_1, \ldots, \eta_m$ is an integral domain. In [27], the author used the greatest basis elements (the "degree") for polynomials while here we used the smallest basis elements (the "order") for power series.

We shall say that the power series c is *convergent* if there exist $K, M \geq 0$ such that

$$|\langle c,\ \eta_\iota\rangle| \leq K M^k k! \quad \text{for each } \iota \in I^k \text{, and each } k \geq 0. \tag{2.6}$$

This growth condition is the natural generalization of the one used for Taylor series in one variable, and it has been suggested before in the present context by Fliess [10].

To each monomial $z = \eta_\kappa$, we associate a "shift" operator $c \mapsto z^{-1}c$ defined by

$$\langle z^{-1}c,\ \eta_\iota\rangle = \langle c,\ z\eta_\iota\rangle \quad \text{for } \eta_\iota \in P^*.$$

That is to say, η_κ is "erased" from all terms that start with η_κ, and other terms are deleted. For instance,

$$\begin{aligned}&(\eta_0\eta_1)^{-1}(1 + \eta_1 - 2\eta_0\eta_1 + \eta_1\eta_1 - \eta_0\eta_1\eta_0 + 3\eta_0\eta_1\eta_1 + \cdots)\\ &\quad = -2 - \eta_0 + 3\eta_1 + \cdots\ .\end{aligned}$$

Note that $z_2^{-1}z_1^{-1}c = (z_1z_2)^{-1}c$. By definition, for any $z, w \in P^*$,

$$z^{-1}w = \begin{cases} w_1 & \text{if } w = zw_1 \text{ for some } w_1 \in P^*, \\ 0 & \text{otherwise,} \end{cases}$$

and for any $c \in \mathcal{S}$,

$$z^{-1}c = \sum_{\iota \in I^*} \langle c,\ z\eta_\iota \rangle \eta_\iota\ =\ \sum_{\iota \in I^*} \langle c,\ \eta_\iota \rangle z^{-1}\eta_\iota.$$

Lemma 2.1.2 For any c, $d \in \mathcal{S}$ and $z \in P$,

$$z^{-1}(c \sqcup\!\sqcup d)\ =\ (z^{-1}c) \sqcup\!\sqcup d\ +\ c \sqcup\!\sqcup (z^{-1}d). \tag{2.7}$$

Proof. Take any w_1, $w_2 \in P^*$ and any $z \in P$, and write $w_1 = z_1 w_1'$ and $w_2 = z_2 w_2'$ for some $w_1', w_2' \in P^*$, $z_1, z_2 \in P$. This can always be done unless w_1 or w_2 is the empty sequence, in which case the formula to be proved is trivial. Then,

$$z^{-1}(w_1 \sqcup\!\sqcup w_2)\ =\ (z^{-1}z_1)(w_1' \sqcup\!\sqcup w_2)\ +\ (z^{-1}z_2)(w_1 \sqcup\!\sqcup w_2').$$

Notice that

$$z^{-1}z_i\ =\ \begin{cases} 1 & \text{if } z = z_i, \\ 0 & \text{if } z \neq z_i \end{cases}$$

for all $z, z_i \in P^*$, $i = 1, 2$. It follows that

$$z^{-1}(w_1 \sqcup\!\sqcup w_2) = (z^{-1}w_1) \sqcup\!\sqcup w_2\ +\ w_1 \sqcup\!\sqcup (z^{-1}w_2).$$

Therefore, for any c and $d \in \mathcal{S}$,

$$\begin{aligned} z^{-1}(c \sqcup\!\sqcup d)\ &=\ \sum \langle c,\ \eta_\iota \rangle \langle d,\ \eta_\kappa \rangle z^{-1}(\eta_\iota \sqcup\!\sqcup \eta_\kappa) \\ &=\ \sum \langle c,\ \eta_\iota \rangle \langle d,\ \eta_\kappa \rangle \left((z^{-1}\eta_\iota) \sqcup\!\sqcup \eta_\kappa\ +\ \eta_\iota \sqcup\!\sqcup (z^{-1}\eta_\kappa) \right) \\ &=\ \sum \langle c,\ z\eta_\iota \rangle \langle d,\ \eta_\kappa \rangle \eta_\iota \sqcup\!\sqcup \eta_\kappa + \sum \langle c,\ \eta_\iota \rangle \langle d,\ z\eta_\kappa \rangle \eta_\iota \sqcup\!\sqcup \eta_\kappa, \end{aligned}$$

which implies (2.7). ∎

Remark 2.1.3 Lemma 2.1.2 implies that z^{-1} is a derivation operator over the ring $\mathcal{S}$. Later we will see that z^{-1} *is* indeed closely related to derivatives of certain functions associated to c. □

2.2 I/O Operators

We are now ready to define input/output maps, using the formalism introduced in the last section.

Let $T > 0$ be a fixed real number, to be thought of as the duration of the inputs to be applied. For any such T, let $\mathcal{U}_T$ be the set of all essentially bounded measurable functions

$$u : [0, T] \to \mathbb{R}^m$$

endowed with the L^1 norm. We write $\|u\|_1$ for

$$\max\{\|u_i\|_1 :, 1 \le i \le m\}$$

and $\|u\|_\infty$ for

$$\max\{\|u_i\|_\infty :, 1 \le i \le m\}$$

where u_i is the i-th component of u, and $\|u_i\|_1$ is the L^1 norm of u_i, $\|u_i\|_\infty$ is the L^∞ norm of u_i. For each $u \in \mathcal{U}_T$ and each $\iota \in I^l$, we define inductively the functions

$$V_\iota = V_\iota[u] \in \mathcal{C}[0, T]$$

by

$$V_{i_1 \cdots i_{l+1}}[u](t) = \int_0^t u_{i_1}(s) V_{i_2 \cdots i_{l+1}}(s)\, ds, \tag{2.8}$$

where $V_\phi = 1$ and u_i is the i-th coordinate of $u(t)$ for $i = 1, 2, \ldots, m$ and $u_0(t) \equiv 1$.

For each formal power series c in $\eta_0, \eta_1, \ldots, \eta_m$, we define a formal operator on $\mathcal{U}_T$ in the following way:

$$F_c[u](t) = \sum \langle c,\ \eta_\iota \rangle V_\iota[u](t). \tag{2.9}$$

For convergent power series c, the series of functions defined in (2.9) converges in the sense of the following lemma:

Lemma 2.2.1 Suppose that c is convergent and let K and M be as in (2.6). Then for any fixed

$$T < (Mm + M)^{-1} \tag{2.10}$$

the series of functions (2.9) converges uniformly and absolutely for all $t \in [0, T]$ and all those $u \in \mathcal{U}_T$ such that $\|u\|_\infty \le 1$.

Proof. First notice that for $\iota = i_1 i_2 \cdots i_k$,

$$|V_\iota[u](t)| = \left| \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} u_{i_1}(s_1) u_{i_2}(s_2) \cdots u_{i_k}(s_k)\, ds_k \cdots ds_2\, ds_1 \right| \le \frac{t^k}{k!}$$

for any u such that $\|u\|_\infty \leq 1$. Thus for $t \in [0, T]$,

$$\sum_{\iota \in I^k} |\langle c, \eta_\iota \rangle V_\iota[u](t)| \leq \sum_{k=0}^{\infty} (m+1)^k K M^k k! \frac{|t|^k}{k!} \leq \sum_{k=0}^{\infty} K M^k (m+1)^k T^k,$$

since the number of the elements in I^k is at most $(m+1)^k$. Therefore, the series (2.9) converges uniformly and absolutely on $[0, T]$ if $\|u\|_\infty \leq 1$. ∎

Remark 2.2.2 The same proof in fact shows that the series (2.9) converges uniformly and absolutely on an interval $[0, T]$ provided that

$$T \max\{1, \|u\|_\infty\} < \frac{1}{M(m+1)}.$$

Thus, for a convergent power series c and any $u \in \mathcal{U}_T$, not necessarily of norm less than one, there is always some small $\tau > 0$ such that the series (2.9) converges uniformly for $t \in [0, \tau]$. (An analogue can also be proved for L_1 controls, but we well not need it.) □

For each $T > 0$, we define

$$\mathcal{V}_T = \{u \in \mathcal{U}_T : \|u\|_\infty < 1\}, \tag{2.11}$$

and we shall say that T is *admissible for* c if there exist some $M > 0$, $K > 0$ such that (2.6) and (2.10) hold.

So far we have seen that F_c is always well defined on $\mathcal{V}_T$ if T is admissible for c. We shall call F_c an *input/output operator* defined on $\mathcal{V}_T$ if T is admissible for c. Hence, every convergent power series defines an i/o map, or more precisely, one such map on each $\mathcal{V}_T$ for which T is admissible. (We often identify any two such operators, when there is no danger of confusion, dealing in effect with "germs" of such operators.)

Consider piecewise constant controls in L_1, and use the notation

$$\beta(\mu, t, k) = (\mu_1, t_1)(\mu_2, t_2)\cdots(\mu_k, t_k)$$

to denote the piecewise constant control whose value is μ_i on the time interval

$$\left(\sum_{j=0}^{i-1} t_j \,,\ \sum_{j=0}^{i} t_j\right)$$

where

$$\mu_j = (\mu_{1j}, \mu_{2j}, \ldots, \mu_{mj}) \in \mathbb{R}^m, \ |\mu_{ij}| < 1, \ 1 \le j \le k, \ 1 \le i \le m$$

and $t_0 = 0$. Assume c is a convergent power series and T is admissible to c. Let

$$\psi_k(\mu, t) = F_c[\beta(\mu, t, k)](t_1 + \cdots + t_k).$$

Then ψ_k is defined on the set

$$D_{k,T} = \Big\{(\mu, t) \in \mathbb{R}^{m\times k} \times \mathbb{R}^k : \ |\mu_{ij}| < 1, \ 1 \le i \le m, \ 1 \le j \le k, \\ t_j \ge 0, \ 1 \le j \le k, \ t_1 + \cdots + t_k \le T\Big\}.$$

The conclusion of the next result will be used repeatedly later.

Lemma 2.2.3 Suppose c is a convergent power series and T is admissible for c. Then for each fixed k, $\psi_k(\mu, t)$ can be extended to a neighborhood of $D_{k,T}$ as an analytic function and, for any given sequence $i_1, \ldots, i_s, j_1, \ldots, j_s$ for which $j_r \ne j_q$ if $r \ne q$,

$$\left.\frac{\partial^s}{\partial \mu_{i_1 j_1} \cdots \partial \mu_{i_s j_s}}\right|_{\mu=0} \left.\frac{\partial^k}{\partial t_1 \cdots \partial t_k}\right|_{t=0} \psi_k(\mu, t) = \langle c, \eta_{l_1} \cdots \eta_{l_k}\rangle, \tag{2.12}$$

where

$$l_p = \begin{cases} i_r & \text{if } k-(p-1) = j_r, \\ 0 & \text{if } k-(p-1) \notin \{j_1, \ldots, j_s\}. \end{cases}$$

Proof. First we fix any $\delta > 0$ so that

$$(1+\delta)(T+\delta)M(m+1) < 1,$$

where M is defined as in (2.6) for the series c. For each multiindex ι,

$$V_\iota[\beta(\mu, t, k)](t_1 + \cdots + t_k)$$

is a polynomial in the variables μ_{ij} and t_i defined on the set $D_{k,T}$. Now let

$$\tilde{D} = \Big\{(\mu, t) \in \mathbb{C}^{m\times k} \times \mathbb{C}^k : \ |t_1| + \cdots + |t_k| < 1 + \delta \\ |\mu_{ij}| < 1 + \delta, \ 1 \le i \le m, \ 1 \le j \le k\Big\}.$$

Let

$$\tilde{V}_\iota(\mu_{11}, \mu_{12}, \ldots, \mu_{1k}, \mu_{21}, \ldots, \mu_{mk}, t_1, \ldots, t_k)$$

be the polynomial function defined on $\tilde{D}$, whose restriction to the set $D_{k,T}$ is

$$V_\iota[\beta(\mu,t,k)](t_1+\cdots+t_k).$$

It can be seen then

$$\begin{aligned}&\left|\tilde{V}_\iota(\mu_{11},\mu_{12},\ldots,\mu_{1k},\mu_{21},\ldots,\mu_{mk},t_1,\ldots,t_k)\right|\\&\leq\left|\tilde{V}_\iota(|\mu_{11}|,\,|\mu_{12}|,\,\ldots,|\mu_{1k}|,\,|\mu_{21}|,\,\ldots,|\mu_{mk}|,\,|t_1|,\,\ldots,|t_k|)\right|\\&\leq(1+\delta)^{|\iota|}\frac{(T+\delta)^{|\iota|}}{|\iota|!}\end{aligned}$$

for $(\mu,\, t)\in\tilde{D}$. So

$$\sum_{|\iota|\geq N}^{\infty}|\langle c,\,\eta_\iota\rangle\tilde{V}_\iota(\mu,t)|\leq\sum_{l\geq N}^{\infty}KM^l(1+m)^l(1+\delta)^l(T+\delta)^l$$

for $(\mu,t)\in\tilde{D}$, which implies that the series

$$\sum\langle c,\,\eta_\iota\rangle\tilde{V}_\iota(\mu,t)\tag{2.13}$$

converges uniformly on $\tilde{D}$. Let Ψ denote the function defined by (2.13). Applying Theorem 5.1 in [1], one knows that Ψ is analytic in each variable μ_{ij} or t_j when the values of other variables are fixed. By Hartogs' Theorem (cf. [15]) Ψ is analytic in $\tilde{D}$, which implies that $\psi_k(\mu,t)$ can be extended in a neighborhood of $D_{k,T}$ as an analytic function.

Now we are ready to prove formula (2.12). It follows directly from (2.8) that

$$\frac{\partial^k}{\partial t_1\cdots\partial t_k}\Big|_{t=0}\psi_k(\mu,t)=\sum\langle c,\,\eta_{l_1}\cdots\eta_{l_k}\rangle\mu_{l_1k}\cdots\mu_{l_k1}.\tag{2.14}$$

One can see that if

$$\{(i_1,j_1),\cdots,(i_s,j_s)\}\subseteq\{(l_1,k),\cdots,(l_k,1)\}$$

and

$$l_p=0\quad\text{for }p\notin\{j_1,\ldots,j_s\},$$

then

$$\frac{\partial^s}{\partial\mu_{i_1j_1}\cdots\partial\mu_{i_sj_s}}\Big|_{\mu=0}\mu_{l_1k}\cdots\mu_{l_k1}=1,$$

and,

$$\frac{\partial^s}{\partial\mu_{i_1j_1}\cdots\partial\mu_{i_sj_s}}\Big|_{\mu=0}\mu_{l_kk}\cdots\mu_{l_11}=0$$

otherwise.

Combining this fact and (2.14), we get (2.12). ∎

One of the important corollaries of Lemma 2.2.3 is that each convergent power series is uniquely determined by the induced i/o operators, in the following sense:

Corollary 2.2.4 Suppose that c and d are two convergent power series. If $F_c = F_d$ on $\mathcal{V}_T$ for some $T > 0$, then $c = d$.

Proof. Because $F_c - F_d = F_{c-d}$, it is enough to show that if c is convergent and if $F_c = 0$ on $\mathcal{V}_T$ for some small T, then $c = 0$. By assumption, for any μ_i, t_i, such that $\sum t_i < T$, $|\mu_{ij}| \le 1$,

$$F_c[(\mu_1, t_1)(\mu_2, t_2)\cdots(\mu_k, t_k)](t) = 0,$$

where $t = \sum t_i$. Take $y = F_c[u]$ as a function of $\mu_1,\ldots,\mu_k$ and $t_1,\ldots,t_k$. Then

$$\frac{\partial^s}{\partial\mu_{i_1j_1}\cdots\partial\mu_{i_sj_s}}\bigg|_{\mu=0}\frac{\partial^k}{\partial t_1\cdots\partial t_k}\bigg|_{t=0^+} y = 0, \tag{2.15}$$

for all $i_1,\ldots,i_s,j_1,\ldots j_s$. Combining (2.15) and (2.12), we know that

$$\langle c,\ \eta_\iota\rangle = 0,\ \text{ for any } \eta_\iota \in P^*$$

which implies that $c = 0$. ∎

For $u \in \mathcal{V}_\tau$ ($\tau < T$) and $v \in \mathcal{V}_{T-\tau}$, we use $u\#_\tau v$ to denote the concatenated control:

$$(u\#_\tau v)(t) = \begin{cases} u(t) & \text{if } 0 \le t \le \tau, \\ v(t-\tau) & \text{if } \tau < t \le T. \end{cases}$$

Lemma 2.2.5 Suppose c is a convergent series which satisfies (2.6) and T is admissible for c. For any $0 \le \tau < T$, $u \in \mathcal{V}_\tau$, let d be the series defined by

$$\langle d,\ \eta_\iota\rangle = F_{\eta_l^{-1}c}[u](\tau). \tag{2.16}$$

Then d is also a convergent power series and $T-\tau$ is admissible for d. Furthermore, for each $v \in \mathcal{V}_{T-\tau}$,

$$F_c[u\#_\tau v](t+\tau) = F_d[v](t). \tag{2.17}$$

Proof. It follows from (2.6) that

$$|\langle \eta_\iota^{-1} c,\ \eta_\kappa\rangle| \le K M^{l+k}(l+k)! \quad \text{for} \quad \iota \in I^l,\ \kappa \in I^k.$$

Thus for any $0 \le \tau \le T$ and any $u \in \mathcal{V}_\tau$,

$$\begin{aligned} |F_{\eta_\iota^{-1}c}[u](\tau)| &\le \left| \sum_{\kappa \in I^*} \langle c,\ \eta_\iota \eta_\kappa \rangle V_\iota[u](\tau) \right| \\ &\le \sum_{k=0}^{\infty} K M^{l+k}(l+k)!(m+1)^k \frac{\tau^k}{k!} \\ &= K M^l \sum_{k=0}^{\infty} \frac{s^k}{k!}(l+k)!\,, \end{aligned} \tag{2.18}$$

where $s = M(m+1)\tau$. For power series (2.18), we have

$$\sum_{k=0}^{\infty} \frac{s^k}{k!}(l+k)! = \frac{d^l}{ds^l} \sum_{k=0}^{\infty} \frac{s^k}{k!} = \frac{d^l}{ds^l} \frac{s^l}{1-s}\,,$$

for all $s \in (-1,\ 1)$. Now write

$$\frac{s^l}{1-s} = a_l s^{l-1} + \cdots + a_2 s + a_1 + \frac{a_0}{1-s}\,. \tag{2.19}$$

Multiplying by $(1-s)$ on both sides of (2.19) and then letting $s = 1$, we get $a_0 = 1$. Hence,

$$\begin{aligned} \frac{d^l}{ds^l} \frac{s^l}{1-s} &= \frac{d^l}{ds^l}(a_l s^{l-1} + \cdots + a_2 s + a_1 + \frac{1}{1-s}) \\ &= \frac{l!}{(1-s)^{l+1}}\,, \end{aligned}$$

for $|s| < 1$. Therefore,

$$|F_{\eta_\iota^{-1}c}[u](\tau)| \le \frac{K M^l l!}{(1 - M(m+1)\tau)^{l+1}}\,,$$

i.e.,

$$|\langle d,\ \eta_\iota \rangle| \le K_\tau M_\tau^l l! \quad \text{for} \quad \iota \in I^l,\ l \ge 0\,, \tag{2.20}$$

where

$$K_\tau = \frac{K}{1 - M(m+1)\tau}\,,$$

$$M_\tau = \frac{M}{1 - M(m+1)\tau}\,,$$

and the constants M and K are as in (2.6). Since

$$\frac{1}{M_\tau(m+1)} = \frac{1}{M(m+1)} - \tau > T - \tau,$$

it follows that $T-\tau$ is admissible for d.

Formula (2.17) will follow from the following formula:

$$V_\rho[u\#_\tau v](t+\tau) = \sum_{\iota\kappa=\rho} V_\iota[v](t)V_\kappa[u](\tau), \tag{2.21}$$

for any $\rho \in I^*$, since if we assume (2.21) holds, then

$$\begin{aligned} F_c[u\#_\tau v](t+\tau) &= \sum_\rho \langle c,\ \eta_\rho\rangle \sum_{\iota\kappa=\rho} V_\iota[v](t)V_\kappa[u](\tau) &(2.22)\\ &= \sum_\iota\sum_\kappa \langle c,\ \eta_\iota\eta_\kappa\rangle V_\iota[v](\tau)V_\kappa[u](t) &(2.23)\\ &= \sum_\iota\sum_\kappa \langle \eta_\iota^{-1}c,\ \eta_\kappa\rangle V_\kappa[u](\tau)V_\iota[v](t) &\\ &= \sum_\iota F_{\eta_\iota^{-1}c}[u](\tau)V_\iota[v](t) &\\ &= F_d[v](t). &(2.24)\end{aligned}$$

Note here that we can rearrange the terms in (2.22) to get (2.23) because the series of functions in (2.22) is absolutely convergent for $0 \le \tau + t \le T$.

We now return to prove (2.21) by induction on the length of ρ. Equation (2.21) is true when $\rho = j \in I^1$ because

$$\begin{aligned} V_\rho[u\#_\tau v](t+\tau) &= \int_0^{t+\tau} (u_j\#_\tau v_j)(s)\,ds \\ = \int_0^\tau u_j(s)\,ds + \int_0^t v_j(s)\,ds &= V_\rho[u](\tau) + V_\rho[v](t). \end{aligned}$$

Now assume that (2.21) holds for $\rho \in I^n$. For any

$$\rho = i_1 i_2 \cdots i_{n+1} \in I^{n+1},$$

we have

$$\begin{aligned} &V_\rho[u\#_\tau](t+\tau) = \int_0^{t+\tau} (u_{i_1}\#_\tau v_{i_1})(s)V_{i_2i_3\cdots i_{n+1}}[u\#_\tau v](s)\,ds \\ &= \int_0^\tau u_{i_1}(s)V_{i_2i_3\cdots i_{n+1}}[u\#_\tau v](s) + \int_0^t v_{i_1}(s)V_{i_2i_3\cdots i_{n+1}}[u\#_\tau v](\tau+s)\,ds \\ &= V_\rho[u](\tau) + \sum_{i_1\iota\kappa=\rho} \int_0^t v_{i_1}(s)V_\iota[v](s)V_\kappa(\tau)\,ds \end{aligned}$$

$$= V_\rho[u](\tau) + \sum_{|\iota|\geq 1, \iota\kappa=\rho} V_\iota[v](t)V_\kappa[u](\tau)$$
$$= \sum_{\iota\kappa=\rho\cdot} V_\iota[v](t)V_\kappa[u](\tau).$$

We completed the proof of (2.21) by induction. ∎

Remark 2.2.6 Formula (2.21) can also be proved by using Lemma 3.1 in [37]. Part of the conclusions of the Lemma says the following:

$$\sum_\rho \sum_{\iota\kappa=\rho} V_{\iota*}[u](\tau)V_{\kappa*}[v](t)\eta_\rho = \sum_\rho V_{\rho*}[u\#_\tau v](t+\tau)\,\eta_\rho, \tag{2.25}$$

for any u, v which are integrable over some finite interval, where

$$\rho* = (i_k i_{k-1}\cdots i_1) \text{ if } \rho = (i_1 i_2 \cdots i_k)$$

for any $\rho \in I^*$.

In fact, one can show that (2.17) and (2.25) are equivalent as follows:

Assume (2.25) is true; then

$$\sum_{\iota\kappa=\rho} V_{\iota*}[u](\tau)V_{\kappa*}[v](t) = V_{\rho*}[u\#_\tau v](\tau + t)$$

for any $\rho \in I^*$, which implies that

$$\sum_{\iota\kappa=\rho} V_\kappa[u](\tau)V_\iota[v](t) = \sum_{\kappa*\iota*=\rho*} V_{(\kappa*)*}[u](\tau)V_{(\iota*)*}[v](t)$$
$$= V_{(\rho*)*}[u\#_\tau v](t+\tau) = V_\rho[u\#_\tau v](t+\tau), \quad \text{for any } \rho \in I^*.$$

Conversely, if (2.17) is true, then

$$\begin{aligned}\sum_{\iota\kappa=\rho} V_{\iota*}[u](\tau)V_{\kappa*}[v](t) &= \sum_{\kappa*\iota*=\rho*} V_{\kappa*}[v](t)V_{\iota*}[u](\tau) \\ &= V_{\rho*}[u\#_\tau v](t+\tau) \qquad \text{for any } \rho \in I^*,\end{aligned}$$

which implies (2.25). □

If c, d are two convergent power series, then it can be easily seen that $c + d$ is also a convergent power series. The next lemma shows that the shuffle product of two convergent power series is still convergent.

Lemma 2.2.7 Suppose that c and d are two convergent power series and T is admissible for both of them. Then $c \sqcup\!\sqcup d$ is again convergent and T is also admissible for $c \sqcup\!\sqcup d$.

Proof. Assume T is admissible for both c and d. Then there exist some K and M so that (2.6) and (2.10) hold for c and d. Let $e = c \sqcup\!\sqcup d$. We will prove our conclusion by showing the following formula:

$$|\langle e,\ \eta_\iota\rangle| \leq K^2 M^n (n+1)!, \quad \text{for any } \iota \in I^n. \tag{2.26}$$

Supposing that (2.26) holds, we pick any number M_1 such that $M < M_1$ but still

$$\frac{1}{M_1(m+1)} < T.$$

From

$$\frac{M^n(n+1)}{M_1^n} \longrightarrow 0, \text{ as } n \to \infty$$

we know that there exists some L such that

$$M^n(n+1) \leq L M_1^n$$

for all $n \geq 0$. Take $K_1 = K^2 L$. It follows from (2.26) that

$$|\langle e,\ \eta_\iota\rangle| \leq K_1 M_1^n n!, \text{ for any } \iota \in I^n.$$

By the choice of M_1, T is admissible for e.

Now we turn back to prove (2.26). We will first prove the following inequality by induction:

$$\sum_{|\iota|=l,|\kappa|=k} \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ w\rangle \leq \binom{n}{l}, \tag{2.27}$$

for any $w \in P^n$, where $n = k + l$. Inequality (2.27) is true if $l = n$, since in this case, (2.27) becomes

$$\sum_{|\iota|=n,|\kappa|=0} \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ w\rangle = \langle \omega \sqcup\!\sqcup 1,\ w\rangle = 1.$$

Similarly, (2.27) holds if $l = 0, k = n$. The case of $n = 1$ is trivial. For $n = 2, l = 1$, write $w = z_1 z_2$. Then

$$\sum_{|\iota|=1,|\kappa|=1} \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ z_1 z_2\rangle = \begin{cases} \langle z_1 \sqcup\!\sqcup z_2,\ z_1 z_2\rangle + \langle z_2 \sqcup\!\sqcup z_1,\ z_1 z_2\rangle = 1 + 1 = 2 & \text{if } z_1 \neq z_2 \\ \langle z_1 \sqcup\!\sqcup z_1,\ z_1 z_1\rangle = 2 & \text{if } z_1 = z_2 \end{cases}.$$

Thus (2.27) holds for $n = 2$.

Now suppose (2.27) holds for $n-1$ and any $l \le n-1$. Take $w \in P^n$ and write it as zw_1 for some $z \in P, w_1 \in P^{n-1}$. We have seen that (2.27) for any n if $l = n$ or $l = 0$. So we may assume that $1 \le l \le n-1$. Then

$$\begin{aligned}
\sum_{|\iota|=l,|\kappa|=n-l} \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\, w\rangle &= \sum_{|\iota|=l,|\kappa|=n-l} \langle z^{-1}(\eta_\iota \sqcup\!\sqcup \eta_\kappa),\, w_1\rangle \\
&= \sum_{|\iota|=l,|\kappa|=n-l} \langle z^{-1}\eta_\iota \sqcup\!\sqcup \eta_\kappa,\, w_1\rangle + \sum_{|\iota|=l,|\kappa|=n-l} \langle \eta_\iota \sqcup\!\sqcup z^{-1}\eta_\kappa,\, w_1\rangle \\
&\le \sum_{|\iota|=l-1,|\kappa|=n-l} \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\, w_1\rangle + \sum_{|\iota|=l,|\kappa|=n-l-1} \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\, w_1\rangle \\
&\le \binom{n-1}{l-1} + \binom{n-1}{l} = \binom{n}{l}.
\end{aligned}$$

Thus (2.27) was proved by induction.

We are now ready to prove (2.26). For any nonnegative integer n and $w \in P^n$,

$$\begin{aligned}
|\langle e,\, w\rangle| &= \left|\sum_{l=0}^{n} \sum_{w_1 \in P^l} \langle c,\, w_1\rangle\langle d,\, w_2\rangle\langle w_1 \sqcup\!\sqcup w_2,\, w\rangle\right| \\
&\le \sum_{l=0}^{n} \sum_{w_1 \in P^l} KM^l l! KM^{n-l}(n-l)!\langle w_1 \sqcup\!\sqcup w_2,\, w\rangle \\
&\le \sum_{l=0}^{n} K^2M^n l!(n-l)! \binom{n}{l} \\
&= \sum_{l=0}^{n} K^2M^n n! = K^2M^n(n+1)!.
\end{aligned}$$

Thus (2.26) holds for all n and all $w \in P^*$, concluding the proof of admissibility. ∎

As in the standard context of power series, the convergence radius of a product of two series may be greater than either of the convergence radii of the two factor series.

Example 2.2.8 Consider the series

$$c = 1 + \frac{1}{2}\eta_1 - \frac{1}{2}\eta_1^{(2)} + \frac{3}{2^2}\eta_1^{(3)} + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{2^{n-1}}\eta_1^{(n)} + \cdots$$

where $\eta_\iota^{(n)}$ denotes $\underbrace{\eta_\iota \cdots \eta_\iota}_{n}$ for any $\iota \in I^*$, $n \ge 1$, and

$$k!! = \begin{cases} k(k-2)\cdots 3 \cdot 1 & \text{if } k \text{ is odd,} \\ k(k-2)\cdots 4 \cdot 2 & \text{if } k \text{ is even} \end{cases}$$

for any integer k. Let c_k be the coefficient of $\eta_1^{(k)}$ in the series. Then

$$c_k = (-1)^{k-1}\frac{(2k-3)!!}{2^{k-1}}$$

for $k \geq 2$, and

$$|c_k| = \frac{(2k-3)!!}{2^{k-1}} \geq \frac{(2(k-2))!!}{2^{k-1}} = \frac{(k-2)!}{2}.$$

Notice that for $u \equiv 1$,

$$\sum_{\iota} |\langle c,\, \eta_\iota\rangle V_\iota[u](t)| \geq \sum_{l=2}^{\infty} \frac{(k-2)!}{2}\frac{t^k}{k!} = \sum_{k=2}^{\infty} \frac{t^k}{2k(k-1)}.$$

So we know that T cannot be admissible for c if $T \geq 1$. However, the following computation shows that no matter how large T is, it is always admissible for $c ⧢ c$. By definition,

$$\langle c ⧢ c,\, \eta_1^{(k)}\rangle = \sum_{i+j=k} c_i c_j \langle \eta_1^{(i)} ⧢ \eta_1^{(j)},\, \eta_1^{(k)}\rangle$$
$$= \sum_{i=0}^{k} \binom{k}{i} c_i c_j = k! \sum_{i=0}^{k} \frac{c_i}{i!}\frac{c_j}{j!}.$$

Note here that

$$(1+x)^{1/2} = \sum_{i=1}^{\infty} \frac{c_i}{i!} x^i$$

and

$$(1+x)^{1/2}(1+x)^{1/2} = 1 + x.$$

It follows that

$$\sum_{i+j=k} \frac{c_i}{i!}\frac{c_j}{j!} = 0 \text{ if } k \geq 2$$

and

$$c_0\, c_0 = 1, \quad c_0\, c_1 + c_1\, c_0 = 1.$$

Therefore,

$$\langle c ⧢ c,\, \eta_1^{(k)}\rangle = \begin{cases} 0 & \text{if } k \geq 2, \\ 1 & \text{if } k = 0, \\ 2 & \text{if } k = 1, \end{cases}$$

that is, $c ⧢ c = 1 + 2\,\eta_1$ Hence, T is always admissible for $c ⧢ c$ though it may not be admissible for c. □

Now for any positive integer n, denote

$$c^n = \underbrace{c \sqcup\!\sqcup c \sqcup\!\sqcup \cdots \sqcup\!\sqcup c}_{n},$$

and $c^0 = 1$.

The next result shows that the assignment $c \mapsto F_c$ is a homomorphism from the set of all convergent series, seen as an algebra under the shuffle product, into the set of input/output operators. (More precisely, identifying operators with their restrictions to smaller time intervals.) By Lemma 2.2.3, this homomorphism in one-to-one.

Lemma 2.2.9 For any polynomial $p \in \mathbb{R}[X_1, X_2, \ldots, X_s]$ and any s convergent power series $c_1, \ldots, c_s$,

$$p(F_{c_1}, F_{c_2}, \ldots, F_{c_s}) = F_{p(c_1, c_2, \ldots, c_s)}. \tag{2.28}$$

Proof. We shall prove the conclusion first for letters, then for monomials, and finally for power series. Take η_i, $\eta_j \in P$. Integrating by parts, we get

$$\begin{aligned}
F_{\eta_i}[u](t)\, F_{\eta_j}[u](t) &= \int_0^t u_i(s)\, ds \int_0^t u_j(s)\, ds \\
&= \int_0^t u_i(\tau) \int_0^\tau u_j(s)\, ds\, d\tau + \int_0^t u_j(\tau) \int_0^\tau u_i(s)\, ds\, d\tau \\
&= F_{\eta_i \eta_j}[u](t) + F_{\eta_j \eta_i}[u](t) \\
&= F_{\eta_i \sqcup\!\sqcup \eta_j}[u](t).
\end{aligned}$$

Now suppose that for any η_ι $\eta_\kappa \in P^*$ such that $|\iota| \le n$ and $|\kappa| \le n+1$,

$$F_{\eta_\iota}[u](t)\, F_{\eta_\kappa}[u](t) = F_{\eta_\iota \sqcup\!\sqcup \eta_\kappa}[u](t).$$

Take η_i, $\eta_j \in P$ and consider $F_{\eta_i \eta_\iota} \cdot F_{\eta_j \eta_\kappa}$. Integrating again by parts, we get

$$\begin{aligned}
&F_{\eta_i \eta_\iota}[u](t)\, F_{\eta_j \eta_\kappa}[u](t) \int_0^t u_i(s)\, F_{\eta_\iota}[u](s)\, ds \int_0^t u_j(s)\, F_{\eta_\kappa}[u](s)\, ds \\
&= \int_0^t \left(\int_0^s u_j(\tau) F_{\eta_\kappa}[u](\tau)\, d\tau \right) u_i(s) F_{\eta_\iota}[u](s)\, ds \\
&\qquad + \int_0^t \left(\int_0^s u_i(\tau) F_{\eta_\iota}[u](\tau)\, d\tau \right) u_j(s) F_{\eta_\kappa}[u](s)\, ds \\
&= \int_0^t u_i(s) F_{\eta_\iota}[u](s)\, F_{\eta_j \eta_\kappa}[u](s)\, ds \\
&\qquad + \int_0^t u_j(s) F_{\eta_\kappa}[u](s)\, F_{\eta_i \eta_\iota}[u](s)\, ds
\end{aligned}$$

$$= \int_0^t u_i(s) F_{\eta_\iota \sqcup\!\sqcup \eta_j \eta_\kappa}[u](s)\,ds + \int_0^t u_j(s) F_{\eta_\kappa \sqcup\!\sqcup \eta_i \eta_\iota}[u](s)\,ds$$
$$= F_{\eta_i \eta_\iota \sqcup\!\sqcup \eta_j \eta_\kappa}[u](t).$$

Thus we proved by induction that

$$F_{\eta_\iota}[u](t)\, F_{\eta_\kappa}[u](t) = F_{\eta_\iota \sqcup\!\sqcup \eta_\kappa}[u](t), \tag{2.29}$$

for any $\eta_\iota,\ \eta_\kappa \in P^*$ and $u \in \mathcal{V}_T$.

Finally suppose c, d are two convergent power series and T is admissible for both c and d. Then for $u \in \mathcal{V}_T$ and $t \in [0, T]$,

$$\begin{aligned} F_c[u](t)\, F_d[u](t) &= \sum_\iota \langle c,\ \eta_\iota\rangle F_{\eta_\iota}[u](t) \sum_\iota \langle d,\ \eta_\kappa\rangle F_{\eta_\kappa}[u](t) \\ &= \sum_{\iota,\kappa} \langle c,\ \eta_\iota\rangle\langle d,\ \eta_\kappa\rangle\, F_{\eta_\iota}[u](t)\, F_{\eta_\kappa}[u](t) \\ &= \sum_{\iota,\kappa} \langle c,\ \eta_\iota\rangle\langle d,\ \eta_\kappa\rangle\, F_{\eta_\iota \sqcup\!\sqcup \eta_\kappa}[u](t) \\ &= F_{c \sqcup\!\sqcup d}[u](t). \end{aligned}$$

It follows then that for any polynomial $p \in \mathbb{R}[X_1, \ldots, X_s]$ and any convergent power series $c_1, \ldots, c_s$, (2.28) holds. ∎

Remark 2.2.10 Note that

$$\eta_\iota \sqcup\!\sqcup \eta_\kappa = \sum_\rho \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ \eta_\rho\rangle\, \eta_\rho\,.$$

It follows that (2.29) can be written alternatively as follows:

$$V_\iota[u](t)\, V_\kappa[u](t) = \sum_\rho \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ \eta_\rho\rangle\, V_\rho[u](t)\,. \tag{2.30}$$

□

2.3 Properties of I/O operators

So far we have seen that every convergent power series c determines an i/o operator F_c. In this section we will study properties of such i/o operators. We shall first show that $F_c : \mathcal{V}_T \to \mathcal{C}[0, T]$ is a continuous operator with respect to the L^1 norm in $\mathcal{V}_T$ and the $\mathcal{C}^0$ norm in $\mathcal{C}[0, T]$. For this purpose, we need to establish the following lemma:

Lemma 2.3.1 For every multiindex $\iota \in I^*$, the map

$$V_\iota : \mathcal{V}_T \to \mathcal{C}[0,T], \ \ u \mapsto V_\iota[u]$$

is continuous with respect to the L^1 norm in $\mathcal{V}_T$ and the $\mathcal{C}^0$ norm in $\mathcal{C}[0,T]$.

Proof. We use induction on the length of ι. For $\iota = i \in I^1$, we have

$$V_\iota[u](t) = \int_0^t u_i(s)\,ds.$$

It follows that for any $u, v \in \mathcal{V}_T$,

$$\|V_\iota[u] - V_\iota[v]\|_\infty \le \|u - v\|_1,$$

where "$\|\cdot\|_\infty$" denotes the $\mathcal{C}^0$ norm in $[0,T]$. Thus V_ι is continuous for any $\iota \in I^1$.

Suppose the conclusion is true for all ι with $|\iota| \le n$. Then for $\kappa = i\iota \in I^{n+1}$ and $u, v \in \mathcal{U}_T$, we have

$$\begin{aligned}
|V_\kappa[v](t) - V_\kappa[u](t)| &= \left|\int_0^t v_i(s)V_\iota[v](s)\,ds - \int_0^t u_i(s)V_\iota[u](s)\,ds\right| \\
&\le \left|\int_0^t (v_i(s) - u_i(s))V_\iota[v](s)\,ds\right| + \left|\int_0^t u_i(s)(V_\iota[v](s) - V_\iota[u](s))\,ds\right| \\
&\le \|u - v\|_1 \|V_\iota[v]\|_\infty + \|u\|_1 \|V_\iota[v] - V_\iota[u]\|_\infty.
\end{aligned}$$

Notice V_ι is continuous, thus for any $\varepsilon > 0$ given, there exists some $\tau > 0$ such that

$$\|V_\iota[v]\|_\infty \le \|V_\iota[u]\|_\infty + 1 \quad \text{and} \quad \|u\|_1 (\|V_\iota[v] - V_\iota[u]\|_\infty) < \varepsilon/2,$$

for all $v \in \mathcal{B}_\tau(u)$, where $\mathcal{B}_\tau(u)$ is the ball of radius τ centered at u in $\mathcal{U}_T$. Now let

$$\delta = \min\left\{\tau,\ \frac{\varepsilon}{2(1 + \|V_\iota[u]\|)}\right\}.$$

Then for any $v \in \mathcal{B}_\delta(u)$,

$$|V_\kappa[v](t) - V_\kappa[u](t)| < \varepsilon$$

for all $t \in [0,T]$, which implies that $\|V_\kappa[v] - V_\kappa[u]\|_\infty < \varepsilon$. This shows that V_κ is continuous, completing the induction step. ∎

Now let c be a convergent series, and pick any T admissible for c. Then for $u,\ v \in \mathcal{V}_T$ and $0 \le t \le T$,

$$|F_c[u](t) - F_c[v](t)| = \left|\sum_{\iota} \langle c,\ \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right|$$

$$\le \left|\sum_{|\iota|\le s} \langle c,\ \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right| + \left|\sum_{|\iota|> s} \langle c,\ \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right|$$

$$\le \left|\sum_{|\iota|\le s} \langle c,\ \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right| + 2\sum_{i\ge s} (M(m+1)T)^i$$

for any $s \ge 0$. Since $V_\iota : \mathcal{V}_T \to \mathcal{C}[0, T]$ is continuous and

$$\sum_{i\ge s} (M(m+1)T)^i \longrightarrow 0 \text{ as } s \to \infty,$$

it follows that for any $\varepsilon > 0$ given, there exists some $\delta > 0$ such that

$$|F_c[u](t) - F_c[v](t)| < \varepsilon$$

for any $v \in \mathcal{V}_T$ satisfying $\|u - v\|_1 < \delta$. Thus, we get the following conclusion:

Lemma 2.3.2 Assume that c is a convergent power series and T is admissible for c. Then the operator

$$F_c : \mathcal{V}_T \to \mathcal{C}[0, T]$$

is continuous with respect to the L^1 norm in $\mathcal{V}_T$ and the $\mathcal{C}^0$ norm in $\mathcal{C}[0, T]$. □

We now turn to considering the smoothness properties of $F_c[u](t)$ as a function of time t. Notice that, for every multiindex ι, $V_\iota[u](t)$ is absolutely continuous as a function of t. It follows immediately from the fact that $F_c[u]$ defined by (2.9) converges uniformly that $F_c[u](t)$ is continuous on $[0, T]$. If fact, we can prove the following stronger fact:

Lemma 2.3.3 Assume that T is admissible for a convergent power series. Then, for any $u \in \mathcal{V}_T$, $F_c[u](\cdot)$ is absolutely continuous.

Proof. First we note that for any given sequence

$$0 \le a_1 \le b_1 \le a_2 \le \cdots \le b_k \le T$$

and any $u \in \mathcal{V}_T$, $\iota = j_1 \cdots j_n \in I^n$,

$$\sum_i |V_\iota[u](b_i) - V_\iota[u](a_i)| \le \sum_i \int_{a_i}^{b_i} |u_j(s) V_{j_2\cdots j_n}(s)|\, ds$$
$$\le \int_0^T |u_j(s) V_{j_2\cdots j_n}(s)|\, ds \le \frac{T^n}{n!}$$

For any $\varepsilon > 0$ given, one can always choose an $N > 0$ so that

$$\sum_{n \ge N} K M^n (m+1)^n T^n \; < \; \frac{\varepsilon}{2},$$

where K and M are defined as in (2.6) for the series c. Since $V_\iota[u]$ is absolutely continuous for every $\iota \in I^*$, one can always choose an $\delta > 0$ so that

$$\sum_i \sum_{|\iota| \le N} |\langle c,\, \eta_\iota \rangle|\; |V_\iota[u](b_i) - V_\iota[u](a_i)| \; < \; \frac{\varepsilon}{2}$$

whenever

$$\sum_i |b_i - a_i| \; < \; \delta. \tag{2.31}$$

For such a choice of δ, we have

$$\sum_i |F_c[u](b_i) - F_c[u](a_i)| \le \sum_i \sum_\iota |\langle c,\, \eta_\iota \rangle|\; |V_\iota[u](b_i) - V_\iota[u](a_i)|$$
$$\le \sum_i \sum_{|\iota| \le N} |\langle c,\, \eta_\iota \rangle|\; |V_\iota[u](b_i) - V_\iota[u](a_i)|$$
$$+ \sum_i \sum_{|\iota| \ge N} |\langle c,\, \eta_\iota \rangle|\; |V_\iota[u](b_i) - V_\iota[u](a_i)|$$
$$\le \frac{\varepsilon}{2} + \sum_{|\iota| \ge N} \langle c,\, \eta_\iota \rangle \frac{T^{|\iota|}}{|\iota|!}$$
$$\le \frac{\varepsilon}{2} + \sum_{n \ge N} K M^n (m+1)^n T^n$$
$$< \varepsilon,$$

provided (2.31) holds. Hence we proved that $F_c[u](t)$ is absolutely continuous on $[0,\, T]$.

∎

To study the differentiability of the function $F_c[u](t)$, we need the following lemma:

Lemma 2.3.4 Let c be a convergent power series and pick any T admissible for c. Then T is admissible for $z^{-1}c$, for any $z \in P^*$.

Proof. Suppose T is admissible for c, and let K and M be so that (2.6) and (2.10) hold. Thus there exists some $\varepsilon > 0$ such that

$$T(m+1)M(1+\varepsilon) < 1. \tag{2.32}$$

Fix any such ε. For any

$$\alpha = \eta_{i_1} \cdots \eta_{i_s} \in P^*$$

and any convergent power series c satisfying (2.6), we have

$$|\langle \alpha^{-1}c,\ \eta_\kappa\rangle| = |\langle c,\ \alpha\eta_\kappa\rangle| \leq KM^{k+s}(k+s)! \leq KM^{k+s}k!(k+s)^s, \tag{2.33}$$

for each $\kappa \in I^k$ and each $k \geq 0$. Let

$$M_1 = (1+\frac{\varepsilon}{2})M.$$

Then we have

$$\frac{M^{k+s}(k+s)^s}{M_1^k} = \frac{M^s(k+s)^s}{(1+\frac{\varepsilon}{2})^k} \longrightarrow 0 \text{ as } k \to \infty.$$

Thus there exists some $K_1 > 0$ such that

$$KM^{k+s}(k+s)^s \leq K_1 M_1^k \text{ for all } k > 0. \tag{2.34}$$

It follows from (2.32), (2.33) and (2.34) that T is admissible for $\alpha^{-1}c$. ∎

Now suppose that $u \in \mathcal{V}_T$ is continuous. Then for $\iota = i_1 i_2 \cdots i_k \in I^*$,

$$\frac{d}{dt}V_\iota[u](t) = \frac{d}{dt}\int_0^t u_{i_1}(s)V_{\iota'}[u](s)\,ds = u_{i_1}(t)V_{\iota'}[u](t), \tag{2.35}$$

where $\iota' = i_2 \cdots i_k$. By Lemma 2.3.4,

$$\sum_{j=0}^m \sum_\iota u_j(t)\langle \eta_j^{-1}c,\ \eta_\iota\rangle V_\iota[u](t) = \sum_\iota \frac{d}{dt}\langle c,\ \eta_\iota\rangle V_\iota[u](t)$$

converges absolutely and uniformly on $[0, T]$. By Theorem 7.17 in [29], we get the following conclusion:

Lemma 2.3.5 Suppose c is convergent and T is admissible for c. Then $F_c[u]$ is continuously differentiable if $u \in \mathcal{V}_T$ is continuous, and

$$\frac{d}{dt}F_c[u](t) = F_{\eta_0^{-1}c}[u](t) + \sum_{j=1}^m u_j(t)\,F_{\eta_j^{-1}c}[u](t) \tag{2.36}$$

for all $t \in [0, T]$, and each continuous $u \in \mathcal{V}_T$. □

Applying Lemma 2.3.4 and Lemma 2.3.5, one knows that $F_{z^{-1}c}[u]$ is continuously differentiable for any continuous u and any $z \in P^*$ if c is convergent, and moreover,

$$\frac{d}{dt}F_{z^{-1}c}[u](t) = \sum_{j=0}^{m} u_j(t)\, F_{(z\eta_j)^{-1}c}[u](t).$$

Therefore, one can prove by induction the following fact:

Corollary 2.3.6 Suppose T is admissible for c. Then $F_c[u]$ is of class $\mathcal{C}^k$ if $u \in \mathcal{V}_T$ is of class $\mathcal{C}^{k-1}$ for $k \geq 1$. □

We shall call a pair

$$(u,\, y) = (u,\, F_c[u])$$

a $\mathcal{C}^k$ i/o pair of F_c if u is of class $\mathcal{C}^k$.

Remark 2.3.7 If u is merely in L^1, the above argument does not work. But we do know $F_c[u]$ is absolutely continuous, therefore, differentiable almost everywhere. To find its derivative, write

$$F_c[u](t) = \sum_{i=0}^{m} \sum_{\iota} \langle \eta_i^{-1}c,\ \eta_\iota \rangle \int_0^t u_i(s) V_\iota(s)\, ds.$$

It follows that

$$\frac{F_c[u](t+\Delta t) - F_c[u](t)}{\Delta t} = \sum_{i=0}^{m} \sum_{\iota} \langle \eta_i^{-1}c,\ \eta_\iota \rangle \frac{1}{\Delta t} \int_t^{t+\Delta t} u_i(s) V_\iota(s)\, ds. \tag{2.37}$$

Notice that $|V_\iota(s)| \leq \dfrac{s^{|\iota|}}{|\iota|!}$, thus

$$\begin{aligned} &\left| \sum_{|\iota| \geq n} \langle \eta_i^{-1}c,\ \eta_\iota \rangle \frac{1}{\Delta t} \int_t^{t+\Delta t} u_i(s) V_\iota(s)\, ds \right| \\ &\leq \sum_{|\iota| \geq n} \left| \langle \eta_i^{-1}c,\ \eta_\iota \rangle \right| \left| \frac{1}{\Delta t} \int_t^{t+\Delta t} \frac{s^{|\iota|}}{|\iota|!}\, ds \right| \\ &\leq \sum_{l \geq n} K M^{l+1} (m+1)^l \frac{t_1^l}{l!}, \end{aligned}$$

where $t_1 = \max\{t, t+\Delta t\}$. Hence the series in (2.37) converges uniformly with respect to Δt for small enough Δt. Taking limits on both sides of (2.37) and noting that (2.35) holds almost everywhere for an L^1 control u, we conclude that

$$\frac{d}{dt}F_c[u](t) = F_{\eta_0^{-1}c}[u](t) + \sum_{i=1}^{m} u_i\, F_{\eta_i^{-1}c}[u](t)$$

almost everywhere. □

The next lemma says that for a (real-)analytic input u, $F_c[u]$ is also analytic. The proof, however, is less trivial.

Lemma 2.3.8 Suppose c is a convergent series and T is admissible to c. Then $F_c[u]$ is analytic if $u \in \mathcal{V}_T$ is analytic.

Proof. Take an analytic control u in $\mathcal{V}_T$. First notice that if T is admissible for c, then there exists some $\varepsilon > 0$ such that $T + \varepsilon$ is also admissible for c. Assume ε is so small that u is analytic in $(-\varepsilon, T + \varepsilon)$. Let $T_1 = T + \varepsilon$ and

$$\tilde{u}(z) = (\tilde{u}_1(z), \ldots, \tilde{u}_m(z))$$

be the complex analytic function whose restriction to the real interval $(-\varepsilon, T_1)$ is u. One would like to say that the output is the restriction to real t of the complex output corresponding to $\tilde{u}$, from which analyticity would follow by the above differentiability (extended to the complexes). However, it is not necessary that $\tilde{u}$ be bounded by 1 for $|z| \le T_1$, so a local analysis is needed.

For any $0 \le t_0 \le T$, there exist some $\delta > 0$ and $\sigma > 0$ such that

$$|\tilde{u}(z)| M(m+1)T_1 \le 1 - \sigma \quad \text{if} \quad z \in B_\delta(t_0)$$

since

$$|\tilde{u}(t_0)| M(m+1)T_1 < 1$$

where $B_\delta(t_0)$ is the ball of radius δ centered at t_0 and M is as in (2.6). Assume here that $\delta < \varepsilon$. By Lemma 2.2.5,

$$F_c[u](t) = F_d[v](t - t_0)$$

where d is defined as in (2.16) and $v(t) = u(t + t_0)$. Let $\tilde{v}(z) = \tilde{u}(z + t_0)$. For any complex vector function of dimension m,

$$w(z) = (w_1(z), \ldots, w_m(z))$$

which is defined and analytic for all z in a ball around the origin, we define

$$V_{i_1 \cdots i_{l+1}}[w](z) = \int_0^z w_{i_1}(s) V_{i_2 \cdots i_{l+1}}(s)\, ds,$$

inductively, where $V_\phi = 1$ and $w_0(z) :\equiv 1$. By induction, the integrand is analytic, so the integral is independent of the path and the result in analytic too. Then

$$\left|\sum_l \langle d,\ \eta_\iota\rangle V_l[\bar{v}](z-t_0)\right| \leq \sum_{l=0}^{\infty} K_{t_0} M_{t_0}^l (m+1)^l N^l |z-t_0|^l \tag{2.38}$$

where M_{t_0}, K_{t_0} are defined as in Lemma 2.2.5 and

$$N = \max_{z\in B_\delta(t_0)} \bar{u}(z).$$

Notice that the series

$$\sum_{l=0}^{\infty} M_{t_0}^l (m+1)^l N^l |z-t_0|^l$$

converges uniformly for

$$|z-t_0| \leq \frac{1-\sigma}{NM_{t_0}(m+1)}.$$

Now let

$$\bar{\delta} = \min\left\{\delta,\ \frac{1-\sigma}{NM_{t_0}(m+1)}\right\}.$$

Then the series of complex functions

$$\sum \langle d,\ \eta_\iota\rangle V_l[\bar{v}](z-t_0)$$

defines an analytic function in $B_{\bar{\delta}}(t_0)$ since it converges uniformly, (cf: Theorem 5.1 in [1]). For t real,

$$F_d[v](t-t_0) = \sum \langle d,\ \eta_\iota\rangle V_\iota[v](t-t_0) = \sum \langle d,\ \eta_\iota\rangle V_\iota[\bar{v}](t-t_0).$$

Thus, $F_d[v](t-t_0)$, i.e. $F_c[u](t)$, is analytic for $|t-t_0| < \bar{\delta}$. Since t_0 can be chosen arbitrarily in $[0, T]$, we get the desired conclusion. ∎

Observe here that we have not claimed the following stronger statement: if u has a single convergent power series representation on $[0, T]$ then $F_c[u]$ also does. We only proved that $F_c[u]$ is analytic, that is, it has a local power series expansion around each point. The following example shows that the above stronger statement is not true in general.

Example 2.3.9 Consider the series

$$c = 1 + \eta_1 + 2\eta_1^{(2)} + 3!\eta_1^{(3)} + \cdots + k!\,\eta_1^{(k)} + \cdots.$$

It is not hard to see that any $T < 1$ is admissible for c, and

$$\begin{aligned} F_c[u](t) &= \sum_{k=0}^{\infty} k! \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} u(s_1)\cdots u(s_k)\, ds_k \cdots ds_1 \\ &= \sum_{k=0}^{\infty} \left(\int_0^t u(s)\, ds \right)^k = \frac{1}{1 - \int_0^t u(s)\, ds}. \end{aligned}$$

Let $u = -\sin \pi t$. Then $\int_0^t u(s)\, ds = \frac{2}{\pi} \sin^2 \frac{\pi}{2} t$. Hence

$$F_c[u](t) = \frac{\frac{\pi}{2}}{\frac{\pi}{2} + \sin^2 \frac{\pi}{2} t} .$$

Consider the equation

$$\frac{\pi}{2} + \sin^2 \frac{\pi}{2} \theta = 0 \tag{2.39}$$

on the complex plane. When $\theta = bj$ where $j = \sqrt{-1}$, equation (2.39) becomes

$$\frac{e^{b\pi/2} - e^{-b\pi/2}}{2} = \pm \sqrt{\frac{\pi}{2}} .$$

Let $f(b) = \frac{e^{b\pi/2} - e^{-b\pi/2}}{2}$. Then $f(0) = 0$ and

$$f(1) = \sum_{k=0}^{\infty} \frac{(\frac{\pi}{2})^{2k+1}}{(2k+1)!} \geq \frac{\pi}{2} \geq \sqrt{\frac{\pi}{2}} .$$

Therefore there exists some $b \in (0, 1)$ such that $f(b) = \sqrt{\frac{\pi}{2}}$ which implies that there exists some $\theta \in \mathbb{C}$ with $\|\theta\| < 1$ such that (2.39) holds. Therefore, the complex function

$$g(z) = \frac{\frac{\pi}{2}}{\frac{\pi}{2} + \sin^2 \frac{\pi}{2} z}$$

has at least one singularity inside the unit disc. It then follows that $F_c[u](t)$ cannot have a global convergent power series representation on $[0, T]$ if $0 < 1 - T < \delta$ for δ small enough, even though u has a global convergent power series representation. □

2.4 Families of Generating Series and I/O Operators

In this section we consider families of power series. Let Λ be a index set. We say that $\underline{c}$ is a *family of power series* (parameterized by $\lambda \in \Lambda$) if

$$\underline{c} = \{c^\lambda : \lambda \in \Lambda\},$$

where c^λ is a power series for each fixed λ. A family $\underline{c}$ can also be viewed as a power series with coefficients belonging to a ring of functions from Λ to $\mathbb{R}$, i.e,

$$\underline{c} = \sum \langle \underline{c}, \eta_\iota \rangle \eta_\iota,$$

where

$$\langle \underline{c}, \eta_\iota \rangle : \Lambda \to \mathbb{R}, \; \lambda \mapsto \langle c^\lambda, \eta_\iota \rangle.$$

Thus one may treat families of power series as power series over some ring R. In our context, R may be any ring of functions from Λ to $\mathbb{R}$. We use $\mathcal{S}_R$ to denote the set of all power series over R. Then $\mathcal{S}_R$ is a ring with "+" and "ш" defined as the following:

$$\gamma \underline{c} + \underline{d} = \{\gamma c^\lambda + d^\lambda : \lambda \in \Lambda\},$$

$$\underline{c} ш \lambda = \{\underline{c}^\lambda ш \underline{d}^\lambda : \lambda \in \Lambda\},$$

for all $\underline{c}$, $\underline{d} \in \mathcal{S}_R$, $\gamma \in \mathbb{R}$.

Unlike the set $\mathcal{S}$ of power series over $\mathbb{R}$, $\mathcal{S}_R$ may not be an integral domain. This is due to the fact that ring R may not be an integral domain. However, by following the same steps in the proof of Lemma 2.1.1, one can get the following conclusion:

Lemma 2.4.1 The ring $\mathcal{S}_R$ is an integral domain if R is an integral domain. □

It follows from the principle of analytic continuation that any ring of analytic functions from a connected analytic manifold to $\mathbb{R}$ is an integral domain. So we have the following fact:

Corollary 2.4.2 If Λ is a connected analytic manifold and R is a ring of analytic functions from Λ to $\mathbb{R}$, then $\mathcal{S}_R$ is an integral domain. □

Definition 2.4.3 We say a family $\underline{c}$ is a *convergent family* if:

(a) Each member of the family is convergent;

(b) Λ is a topological space, $\langle c^\lambda, \eta_\iota \rangle$ depends on λ continuously, for each $\eta_\iota \in P^*$, and the constants K_λ, M_λ as in (2.6) can be chosen continuously depending on λ.

Since each convergent series induces an i/o operator, each convergent family c of power series induces a family of i/o operators

$$\{F_{c^\lambda} : \lambda \in \Lambda\}$$

which we denote by $\mathbf{F}_{\underline{c}}$. It follows from part (b) of Definition (2.4.3) that if T is admissible for c^{λ_0}, then

$$TM_\lambda(m+1) < 1$$

for all λ near λ_0, therefore, T is admissible for c^λ for all λ in a small neighborhood of λ_0. Now take $\lambda_0 \in \Lambda$ and suppose T is admissible for c^λ if $\lambda \in U_{\lambda_0}$, where U_{λ_0} is some neighborhood of λ_0. Then for any $u \in \mathcal{V}_T, t_0 \in [0, T]$, we have

$$\begin{aligned} &|F_{c^{\lambda_0}}[u](t_0) - F_{c^\lambda}[u](t)| \\ &\qquad \leq |F_{c^{\lambda_0}}[u](t_0) - F_{c^{\lambda_0}}[u](t)| + |F_{c^{\lambda_0}}[u](t) - F_{c^\lambda}[u](t)| . \end{aligned} \tag{2.40}$$

Since $F_{c_0^\lambda}[u](t)$ is continuous as a function of t, thus for any $\varepsilon > 0$ given, there exists some $\delta > 0$ such that

$$|F_{c^{\lambda_0}}[u](t_0) - F_{c^{\lambda_0}}[u](t)| < \frac{\varepsilon}{2},$$

if $|t - t_0| < \delta$. Shrinking U_{λ_0} if necessary, we may assume that

$$M_\lambda(m+1)T \leq 1 - \sigma$$

for some fixed $\sigma > 0$, for all $\lambda \in U_{\lambda_0}$. Then we can estimate the second term in (2.40) as follows:

$$\begin{aligned} |F_{c^{\lambda_0}}[u](t) - F_{c^\lambda}[u](t)| &\leq \sum_{\iota} \left|\langle c^{\lambda_0}, \eta_\iota\rangle - \langle c^\lambda, \eta_\iota\rangle\right| |V_\iota[u](t)| \\ \leq \sum_{|\iota|\leq s} \left|\langle c^{\lambda_0}, \eta_\iota\rangle - \langle c^\lambda, \eta_\iota\rangle\right| \frac{T^{|\iota|}}{|\iota|!} &+ \sum_{|\iota|>s} K_{\lambda_0} M_{\lambda_0}^{|\iota|} T^{|\iota|} + \sum_{l>s} K_\lambda M_\lambda^{|\iota|} T^{|\iota|} \\ \leq \sum_{|\iota|\leq s} \left|\langle c^{\lambda_0}, \eta_\iota\rangle - \langle c^\lambda, \eta_\iota\rangle\right| \frac{T^{|\iota|}}{|\iota|!} &+ \sum_{l>s} K_{\lambda_0}(1-\sigma)^l + \sum_{l>s} K_\lambda(1-\sigma)^l . \end{aligned}$$

From here one can see that one can always choose some neighborhood $\tilde{U}_{\lambda_0}$ of λ_0, which may be smaller than U_{λ_0}, so that

$$|F_{c^{\lambda_0}}[u](t) - F_{c^\lambda}[u](t)| < \frac{\varepsilon}{2}.$$

Therefore, if $|t - t_0| < \delta$ and $\lambda \in \bar{U}_{\lambda_0}$, then

$$|F_{c^{\lambda_0}}[u](t_0) - F_{c^{\lambda}}[u](t)| < \varepsilon.$$

Thus we get the following conclusion:

Lemma 2.4.4 Assume that $\underline{c}$ is a convergent family. If T is admissible for c^{λ_0}, then T is admissible for c^{λ} for all λ in a small neighborhood of λ_0, and, $F_c^{\lambda}[u](t)$ depends (jointly) continuously on t and λ. $\square$

Chapter 3

Observation Spaces

In realization theory and many other areas of nonlinear control, the concept of observation space plays a central role. One may define observation spaces in two very different ways, as discussed in this Chapter. Roughly, one possibility is to take the functions corresponding to derivatives with respect to switching times in piecewise constant controls, and the other is to take high-order derivatives at the final time , if smooth controls are used. We will show however that both definitions lead to the same concept, and this equivalence will provide one of the main technical tools that we used in order to establish the main result.

3.1 Definitions of Observation Spaces

We now introduce the first approach.

3.1.1 First Type of Observation Space

For each power series c, we define the observation space $\mathcal{F}_1$ to be the linear subspace of the set of all power series spanned by all the elements of the form $z^{-1}c$, i.e.,

$$\mathcal{F}_1(c) = \mathrm{span}_{\mathbb{R}}\{z^{-1}c : z \in P^*\}. \tag{3.1}$$

According to Lemma 2.3.4, $F_1(c)$ consists of convergent series if c is a convergent series.

For a convergent power series c, the elements of $\mathcal{F}_1(c)$ are closely related to the derivatives of $F_c[u]$ with respect to switching times in piecewise constant controls, in the sense to be made precise next.

For any $\mu \in \mathbb{R}^m$, we define $P^\mu : \mathbf{F} \to \mathbf{F}$, where $\mathbf{F}$ is the set of all germs of i/o operators induced by convergent generating series, in the following way:

$$(P^\mu \circ F_c)[u](t) = \frac{d}{dt}\Big|_{\tau=0^+} F_c[u\#_t\omega_\mu](t+\tau),$$

where $\omega_\mu(\tau) \equiv \mu$, a constant control. Note that $(P^\mu \circ F_c)[u]$ is defined if u is in the domain of F_c. In fact, by Lemma 2.3.5, one has the following easy relation:

$$P^\mu \circ F_c = F_{\eta_0^{-1}c} + \sum_{j=1}^{m} \mu_j F_{\eta_j^{-1}c},$$

for any $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$.

For a convergent power series c, let $\mathcal{G}_1(c)$ be the smallest subspace of operators which contains F_c and which is invariant under P^μ for any $\mu \in \mathbb{R}^m$. By Corollary 2.2.4, $\mathcal{G}_1(c)$ is isomorphic to $\mathcal{F}_1(c)$.

3.1.2 Second Type of Observation Space

The second approach to defining observation space is a bit more complicated. To introduce it, let's consider, for each $q \geq 1$, the following set of $2 \times q$ matrices:

$$S_q = \left\{ \begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix} : \right.$$
$$\left. i_s, j_s \in \mathbb{Z},\ 1 \leq i_s \leq m,\ j \geq 0,\ (1,0) \leq (i_1, j_1) \leq \cdots \leq (i_q, j_q) \right\}, \tag{3.2}$$

where "$\leq$" is the lexicographic order on the set $\{(i,j) : i, j \in \mathbb{Z}\}$. For each element

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix}$$

in S_q and each $n \geq q + \sum j_r$, we define

$$\Gamma_{i_1 \cdots i_q}^{j_1 \cdots j_q}(n) = \eta_0^{(k)} ш \eta_{i_1} X^{(j_1)} ш \eta_{i_2} X^{(j_2)} ш \cdots ш \eta_{i_q} X^{(j_q)}\Big|_{X=1}, \tag{3.3}$$

where $k = n - q - \sum j_s$. The evaluation is interpreted as follows: first introduce a new variable X, then perform all shuffles, and finally delete X from the result. Note that (3.3) is different from $\eta_{i_1} ш \eta_{i_2} ш \cdots ш \eta_{i_q}$, for example,

$$\eta_0 ш \eta_1 X|_{X=1} = \eta_0\eta_1 + 2\eta_1\eta_0,$$

while

$$\eta_0 ш \eta_1 = \eta_0\eta_1 + \eta_1\eta_0.$$

For any word $w \in P^*$ and and each series $c \in \mathcal{S}$, we define $\psi_c(w) = w^{-1}c$, and more generally, for any polynomial $d = \sum \langle d,\ \eta_\kappa \rangle \eta_\kappa$, we let

$$\psi_c(d) = \sum \langle d,\ \eta_\kappa \rangle {\eta_\kappa}^{-1} c.$$

Now let $X_j = (X_{1j}, \ldots, X_{mj})$ be m indeterminates over $\mathbb{R}$, for $j \geq 0$. For any $n > 0$, let

$$c_n(X_0, \ldots, X_{n-1}) = \psi_c(\eta_0^{(n)}) + \sum_{q=1}^{n} \sum \frac{1}{s_1! \cdots s_p!} \psi_c \left(\Gamma_{i_1 \cdots i_q}^{j_1 \cdots j_q}(n) \right) X_{i_1 j_1} \cdots X_{i_q j_q}, \quad (3.4)$$

where the second sum is taken over the set of all those

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix} \in S_q$$

such that $\sum j_s + q \leq n$, and where $s_1, \ldots, s_p$ are integers so that

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix} = \begin{pmatrix} \beta_1 \cdots \beta_1 & \beta_2 \cdots \beta_2 & \cdots & \beta_p \cdots \beta_p \\ \underbrace{\alpha_1 \cdots \alpha_1}_{s_1} & \underbrace{\alpha_2 \cdots \alpha_2}_{s_2} & \cdots & \underbrace{\alpha_p \cdots \alpha_p}_{s_p} \end{pmatrix}$$

and $(\alpha_1, \beta_1) < (\alpha_2, \beta_2) < \cdots < (\alpha_p, \beta_p)$. For $n = 0$, we simply define

$$c_0 := c.$$

We are now ready to introduce the second type of observation space associated to c, $\mathcal{F}_2(c)$. This is defined as follows:

$$\mathcal{F}_2(c) = \operatorname{span}_{\mathbb{R}} \{ c_n(\mu_0, \ldots, \mu_{n-1}) : \ \mu_i \in \mathbb{R}^m, 0 \leq i \leq n-1,\ n \geq 0 \}. \quad (3.5)$$

We will see below that the elements of $\mathcal{F}_2(c)$ are closely related to the derivatives of $F[u](t)$ with respect to time t. The formula given below is an analogue, proved by using different techniques, of a similar formula given for state space systems in the paper [21].

Theorem 1 *If $u \in \mathcal{V}_T$ is of class C^{n-1} and T is admissible for c, then we have*

$$\frac{d^n}{dt^n} F_c[u](t) = F_{c_n(u(t), \ldots, u^{n-1}(t))}[u](t). \quad (3.6)$$

Before proving this formula, we look at an example to illustrate its meaning.

Example 3.1.1 For $n = 2$, we have

$$\begin{aligned} c_2(X_0, X_1) &= \psi_c(\eta_0\eta_0) + \sum_{i=1}^{m} \psi_c(\Gamma_i^0(2))\, X_{i0} \\ &\quad + \sum_{i<j} \psi_c(\Gamma_{ij}^{00}(2)) X_{i0}X_{j0} + \sum_{i=1}^{m} \frac{1}{2}\psi_c(\Gamma_{ii}^{00}(2)) X_{i0}^2 + \sum_{i=1}^{m} \psi_c(\Gamma_i^1(2)) X_{i1} \\ &= (\eta_0\eta_0)^{-1}c + \sum \left((\eta_0\eta_i)^{-1}c + (\eta_i\eta_0)^{-1}c\right) X_{i0} \\ &\quad + \sum_{i<j} \left((\eta_i\eta_j)^{-1}c + (\eta_j\eta_i)^{-1}c\right) X_{i0}X_{j0} \\ &\quad + \sum (\eta_i\eta_i)^{-1}c\, X_{i0}^2 + \sum \eta_i^{-1}c\, X_{i1}. \end{aligned}$$

Thus, for $n = 2$, formula (3.6) becomes:

$$\begin{aligned} y''(t) &= F_{c_2(u(t),u'(t))}[u](t) \\ &= F_{(\eta_0\eta_0)^{-1}c}[u](t) + \sum \left(F_{(\eta_0\eta_i)^{-1}c}[u](t) + F_{(\eta_i\eta_0)^{-1}c}[u](t)\right) u_i(t) \\ &+ \sum_{i<j} \left(F_{(\eta_i\eta_j)^{-1}c}[u](t) + F_{(\eta_j\eta_i)^{-1}c}[u](t)\right) u_i(t)u_j(t) + \sum F_{(\eta_i\eta_i)^{-1}c}[u](t)\, u_i^2 \\ &+ \sum F_{\eta_i^{-1}c}[u](t)\, u_i'(t). \end{aligned}$$ ∎

To prove Theorem 1, we need first to establish the following:

Lemma 3.1.2 Suppose $w_1, \ldots, w_n \in P^*$ and $w_i = w_i' z_i$ with $w_i' \in P^*$, $z_i \in P$. Then

$$\sum_{s=1}^{n} (w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_s' \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_n) z_s = w_1 \sqcup\!\sqcup w_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_n. \tag{3.7}$$

Proof. : Use induction on n. When $n = 2$, equation (3.7) becomes

$$(w_1' \sqcup\!\sqcup w_2) z_1 + (w_1 \sqcup\!\sqcup w_2') z_2 = w_1 \sqcup\!\sqcup w_2,$$

which is the same as (2.4). Thus the conclusion is true for $n = 2$. Now suppose the conclusion is true for $n \le l$. For $n = l + 1$, we have:

$$\begin{aligned} &\sum_{s=1}^{l+1} (w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_s' \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_{l+1}) z_s \\ = &\sum_{s=1}^{l} (w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_s' \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_{l+1}) z_s + (w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_l \sqcup\!\sqcup w_{l+1}') z_{l+1} \\ = &\sum_{s=1}^{l} \left((w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_s' \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_l) z_s \sqcup\!\sqcup w_{l+1}\right) + (w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_l \sqcup\!\sqcup w_{l+1}') z_{l+1} \\ &\qquad - \sum_{s=1}^{l} \left((w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_s' \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_l) z_s \sqcup\!\sqcup w_{l+1}'\right) z_{l+1}. \end{aligned} \tag{3.8}$$

By the induction assumption,

$$\sum_{s=1}^{l}((w_1 ш \cdots ш w'_s ш \cdots ш w_l)z_s ш w_{l+1}) = w_1 ш \cdots ш w_l ш w_{l+1}, \tag{3.10}$$

and

$$\sum_{s=1}^{l}((w_1 ш \cdots ш w'_s ш \cdots ш w_l)z_s ш w'_{l+1})\, z_{l+1} = (w_1 ш \cdots ш w_l ш w'_{l+1})z_{l+1}. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we get

$$\sum_{s=1}^{l+1}(w_1 ш \cdots ш w'_s ш \cdots ш w_{l+1})z_s = w_1 ш w_2 ш \cdots ш w_{l+1}.$$

Thus our conclusion is proved by induction. ∎

We now return to prove Theorem 1. For each $\eta_\iota \in P^*$, define $\theta_c(\eta_\iota) = F_{\eta_\iota^{-1}c}$, and for any polynomial

$$d = \sum \langle d,\ \eta_\kappa\rangle \eta_\kappa,$$

define

$$\theta_c(d) = \sum \langle d,\ \eta_\kappa\rangle \theta_c(\eta_\iota) = \sum \langle d,\ \eta_\kappa\rangle F_{\eta_\kappa^{-1}c}.$$

Then (3.6) is equivalent to

$$y^{(n)}(t) = \frac{d^n}{dt^n}F_c[u](t) = \sum_{q=0}^{n}\sum_{S_q}\frac{1}{s_1!\cdots s_p!}\theta_c\left(\Gamma_{i_1\cdots i_q}^{j_1\cdots j_q}(n)\right)[u](t)\,u_{i_1}^{(j_1)}(t)\cdots u_{i_q}^{(j_q)}(t), \tag{3.12}$$

in the other words, $y^{(n)}(t)$ is a polynomial in

$$u(t),\ u'(t),\ \ldots,\ u^{(n)}(t)$$

whose coefficients are the $\theta_c(\eta_\iota)(t)$'s, and the coefficient of $u_{i_1}^{(j_1)}(t)\cdots u_{i_q}^{(j_q)}(t)$ in $y^{(n)}(t)$ is

$$\frac{1}{s_1!\cdots s_p!}\theta_c\left(\Gamma_{i_1\cdots i_q}^{j_1\cdots j_q}(n)\right)(t). \tag{3.13}$$

Note that (3.13) can also be written as

$$\frac{1}{s_1!\cdots s_p!}\theta_c\left((\eta_0^{(k)} ш^{s_1} \eta_{\alpha_1}X^{(\beta_1)} ш^{s_2} \eta_{\alpha_2}X^{(\beta_2)} ш \cdots ш^{s_p} \eta_{\alpha_p}X^{(\beta_p)})|_{X=1}\right)(t)$$

if $u_{i_1}^{(j_1)}\cdots u_{i_q}^{(j_q)} = \left(u_{\alpha_1}^{(\beta_1)}\right)^{s_1}\cdots\left(u_{\alpha_p}^{(\beta_p)}\right)^{s_p}$, where

$$w_1 ш^s w_2 = w_1 ш \underbrace{w_2 ш w_2 ш \cdots ш w_2}_{s},$$

for $w_1, w_2 \in P^*$.

We now use induction to prove the lemma. From (2.36) we see that the conclusion is true for $n = 1$.

Suppose the conclusion is true for $n-1$. Consider the coefficient of

$$u_{i_1}^{(j_1)} \cdots u_{i_q}^{(j_q)}$$

in $y^{(n)}$. By induction from formula (2.36) it can be seen that $\sum j_s + q \leq n$. First we assume that $\sum j_s + q < n$. Let $k = n - \sum j_s - q$. Suppose

$$u_{i_1}^{(j_1)} \cdots u_{i_q}^{(j_q)} = (u_{\alpha_1}^{(\beta_1)})^{s_1} \cdots (u_{\alpha_p}^{(\beta_p)})^{s_p},$$

where $(\alpha_1, \beta_1) < \cdots < (\alpha_p, \beta_p)$. Further, we assume that $\beta_r = 0$ for $r \leq l$. Let

$$\hat{y}_1(t) = \sum_{r=1}^{p} \frac{1}{\tau_r} \theta(w_r)(t) \left(u_{\alpha_1}^{(\beta_1)}\right)^{s_1} \cdots \left(v_{\alpha_r}^{(\beta_r)}\right)^{s_r} \cdots \left(u_{\alpha_p}^{(\beta_p)}\right)^{s_p},$$

where

$$\left(v_{\alpha_r}^{(\beta_r)}\right)^{s_r} = \begin{cases} u_{\alpha_r}^{s_r - 1} & \text{if } \beta_r = 0, \\ \left(u_{\alpha_r}^{(\beta_r)}\right)^{s_r - 1} \cdot u_{\alpha_r}^{(\beta_r - 1)} & \text{if } \beta_r \geq 1, \end{cases}$$

and $\tau_r = s_1'! \cdots s_{p'}'!$ if

$$\left(u_{\alpha_1}^{(\beta_1)}\right)^{s_1} \cdots \left(v_{\alpha_r}^{(\beta_r)}\right)^{s_r} \cdots \left(u_{\alpha_p}^{(\beta_p)}\right)^{s_p} = \left(u_{\alpha_1'}^{(\beta_1')}\right)^{s_1'} \cdots \left(u_{\alpha_{p'}'}^{(\beta_{p'}')}\right)^{s_{p'}'},$$

and,

$$w_r = \begin{cases} \eta_0^{(k)} ш^{s_1} \eta_{\alpha_1} ш \cdots ш^{(s_r - 1)} \eta_{\alpha_r} ш \cdots ш^{s_p} \eta_{\alpha_p} X^{(\beta_p)} & \text{if } \beta_r = 0, \\ \eta_0^{(k)} ш^{s_1} \eta_{\alpha_1} ш \cdots ш^{s_r - 1} \eta_{\alpha_r} X^{(\beta_r)} ш \eta_{\alpha_r} X^{(\beta_r - 1)} ш \cdots ш^{s_p} \eta_{\alpha_p} X^{(\beta_p)} & \text{if } \beta_r \neq 0. \end{cases}$$

Note that the coefficient of $u_{i_1}^{(j_1)} \cdots u_{i_q}^{(j_q)}$ in

$$\frac{d}{dt}\left\{\frac{1}{\tau_r}\theta(w_r)(t) \left(u_{\alpha_1}^{(\beta_1)}\right)^{s_1} \cdots \left(v_{\alpha_r}^{(\beta_r)}\right)^{s_r} \cdots \left(u_{\alpha_p}^{(\beta_p)}\right)^{s_p}\right\}$$

is

$$\frac{1}{s_1 \cdots (s_r - 1)! \cdots s_p!} \theta(w_r \eta_{\alpha_r})(t) \qquad \text{if } r \leq l,$$

and

$$\frac{1}{s_1! \cdots (s_r - 1)! \cdots s_p!} \theta(w_r)(t) \qquad \text{if } r > l.$$

Let

$$y_1(t) = \hat{y}_1(t) + \frac{1}{s_1! \cdots s_p!} \theta_c \left(\Gamma^{j_1 \cdots j_q}_{i_1 \cdots i_q}(n-1) \right) u^{(j_1)}_{i_1}(t) \cdots u^{(j_q)}_{i_q}(t).$$

By induction assumption, the coefficient of $u^{(j_1)}_{i_1} \cdots u^{(j_q)}_{i_q}$ in $y^{(n)}(t)$ is the same as in $y_1'(t)$. Thus, this coefficient is $\theta_c(w)(t)$, where

$$\begin{aligned} w = & \left\{ \sum_{r=1}^{l} b_r \left((\eta_0^{(k)} ш^{s_1} \eta_{\alpha_1} ш \cdots ш^{s_r-1} \eta_{\alpha_r} ш \cdots ш^{s_p} \eta_{\alpha_p} X^{(\beta_p)} \right) \eta_{\alpha_r} \right. \\ & + \sum_{r=l+1}^{p} b_r \left(\eta_0^{(k)} ш^{s_1} \eta_{\alpha_1} ш \cdots ш^{s_r-1} \eta_{\alpha_r} X^{(\beta_r)} ш \eta_{\alpha_r} X^{(\beta_r-1)} ш \cdots ш^{s_p} \eta_{\alpha_p} X^{(\beta_p)} \right) \\ & \left. \left. + \frac{1}{s_1! s_2! \cdots s_p!} (\eta_0^{(k-1)} ш^{s_1} \eta_{\alpha_2} ш \cdots ш^{s_p} \eta_{\alpha_p} X^{(\beta_p)}) \eta_0 \right\} \right|_{X=1}, \end{aligned} \tag{3.14}$$

and $\quad b_r = \dfrac{1}{s_1! \cdots (s_r-1)! \cdots s_p!}$.

Notice that

$$w_1 \, ш^{r-1} \, w_2 = \frac{1}{r} \sum_{t=0}^{r-1} w_1 \, ш^t w_2 \, ш \, 1 \, ш^{r-1-t} w_2$$

and

$$\begin{aligned} \left\{ w_1 ш^{r-1} w_2 X^{(\beta)} ш \, w_2 X^{(\beta-1)} \right\}\Big|_{X=1} &= \left\{ (w_1 ш^{r-1} w_2 X^{(\beta)} ш \, w_2 X^{(\beta-1)}) X \right\}\Big|_{X=1} \\ &= \frac{1}{r} \left\{ \left(\sum_{t=0}^{r-1} w_1 ш^t w_2 X^{(\beta)} ш \, w_2 X^{(\beta-1)} ш^{r-1-t} w_2 X^{(\beta)} \right) X \right\}\Bigg|_{X=1}. \end{aligned}$$

Applying Lemma (3.1.2) to (3.14), we get

$$\begin{aligned} w &= \frac{1}{s_1! \cdots s_p!} \left\{ \eta_0^{(k)} ш^{s_1} \eta_{\alpha_1} X^{(\beta_1)} ш \cdots ш^{s_p} \eta_{\alpha_p} X^{(\beta_p)} \right\}\Big|_{X=1} \\ &= \frac{1}{s_1! \cdots s_p!} \Gamma^{j_1 \cdots j_q}_{i_1 \cdots i_q}(n). \end{aligned}$$

In the case $q + \sum j_s = n$, the proof is virtually the same except that $k = 0$, which leads to the fact that the coefficient of $u^{(j_1)}_{i_1} \cdots u^{(j_q)}_{i_q}$ in $y^{(n-1)}$ is 0, so the last term in (3.14) disappears. ∎

3.2 Relation between $\mathcal{F}_1$ and $\mathcal{F}_2$

In last section we defined $c_n(X_0, \ldots, X_{n-1})$. One can see that $c_n(X_0, \ldots, X_{n-1})$ is a polynomial on the X_i's with coefficients belonging to $\mathcal{F}_1(c)$. Thus, $c_n(\mu_0, \ldots, \mu_{n-1})$ is a linear combination of elements of $\mathcal{F}_1(c)$ for each fixed value of $(\mu_0, \ldots, \mu_{n-1})$. Therefore,

$$\mathcal{F}_2(c) \subseteq \mathcal{F}_1(c).$$

But in fact, these two spaces are the same as we can see in the following Theorem.

Theorem 2 *For any power series c, $\mathcal{F}_1(c) = \mathcal{F}_2(c)$.*

We've shown that $\mathcal{F}_2(c) \subseteq \mathcal{F}_1(c)$. The other direction is, however, less trivial. To prove our conclusion, we need to establish the following fact:

For any fixed positive integers k and $i_1, i_2, \ldots, i_q$ such that

$$1 \le i_1 \le i_2 \le \cdots \le i_q \le m,$$

let

$$S^k(i_1, i_2, \ldots, i_q) = \left\{ \sigma(\underbrace{0, \ldots, 0}_{k}, i_1, i_2, \ldots, i_q) : \sigma \in S_n \right\},$$

where $n = k + q$ and S_n is the permutation group on a set of n elements. Let

$$T_k(i_1, i_2, \ldots, i_q) = \left\{ w = \eta_{l_1} \eta_{l_2} \cdots \eta_{l_n} : (l_1, \ldots, l_n) \in S^k(i_1, i_2, \ldots, i_q) \right\} ,$$

and order the elements of $T_k(i_1, i_2, \ldots, i_q)$ as

$$W_1, W_2, \ldots, W_r .$$

Then for any $j_1, \ldots, j_q$ given,

$$\begin{aligned} \Upsilon^{j_1 \cdots j_q}_{i_1 \cdots i_q}(k) &:= \Gamma^{j_1 \cdots j_q}_{i_1 \cdots i_q}(j_1 + \cdots + j_q + k + q) \\ &= \eta_0^{(k)} \sqcup\!\sqcup\, \eta_{i_1} X^{(j_1)} \sqcup\!\sqcup\, \eta_{i_2} X^{(j_2)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, \eta_{i_q} X^{(j_q)} \Big|_{X=1} \end{aligned}$$

is a linear combination of the elements in $T_k(i_1, i_2, \ldots, i_q)$. We now define

$$\Delta_k(i_1, \ldots, i_q) = \left\{ \Upsilon^{j_1 \cdots j_q}_{i_1 \cdots i_q}(k) : j_s \ge 0,\ 1 \le s \le q \right\} .$$

Lemma 3.2.1 Every element of $T_k(i_1, i_2, \ldots, i_q)$ is a linear combination of elements in $\Delta_k(i_1, i_2, \ldots, i_q)$ for any $i_1, \ldots, i_q$ and k given.

Proof. First of all for each fix k and fixed $i_1, i_2, \ldots, i_q$, we put the lexicographic order on

$$\Delta_k(i_1, i_2, \ldots, i_q)$$

according to the order of

$$\left(\sum j_s,\ j_1, \ldots, j_q\right)$$

and we use $Q_1, Q_2, \ldots$ to denote the elements of $\Delta_k(i_1, i_2, \ldots, i_q)$. Then for each Q_i, there exist $a_{ij}, j = 1, \ldots, r$ such that

$$Q_i = \sum_{j=1}^{r} a_{ij} W_j .$$

Let A be the matrix of r columns and infinitely many rows whose (i,j)-th entry is a_{ij}, i.e, $A = (a_{ij})$.

We claim that A is of full column rank in the sense that there is no nonzero vector $v \in \mathbb{R}^r$ such that $Av = 0$. Suppose there is some $v \neq 0$ such that $Av = 0$. Let a be the polynomial defined by

$$a = v_1 W_1 + v_2 W_2 + \cdots, + v_r W_r$$

where v_i is the i-th component of v. Then for any $w \in P^*$,

$$\langle w^{-1} a,\ \phi\rangle \neq 0$$

if and only if $w = W_i$ for some i. Hence

$$\langle \psi_a \left(\Upsilon_{s_1 \cdots s_p}^{j_1 \cdots j_p}(l)\right),\ \phi\rangle = 0 \tag{3.15}$$

if $l \neq k$, or $p \neq q$ or $s_t \neq i_t$ for some t. In the other words, (3.15) holds if

$$\Upsilon_{s_1 \cdots s_p} j_1 \cdots j_p(k) \notin \Delta_k(i_1, i_2, \ldots, i_q) .$$

For $Q_i \in \Delta_k(i_1, i_2, \ldots, i_q)$, we have

$$\langle \psi_a(Q_i),\ \phi\rangle = \sum_{j=1}^{r} a_{ij} \langle W_j^{-1} a,\ \phi\rangle = \sum_{j=1}^{r} a_{ij} \langle a,\ W_j\rangle = \sum_{j=1}^{r} a_{ij} v_j .$$

But by assumption, $\sum a_{ij} v_j = 0$ for any i. Therefore, (3.15) holds for any choice of $s_1, \ldots, s_p$, $j_1, \ldots, j_p$ and any l. It then follows directly from the definition of $a_n(X_0, \ldots, X_{n-1})$ that

$$\langle a_n(\mu_0, \mu_1, \ldots, \mu_{n-1}),\ \phi\rangle = 0 \tag{3.16}$$

for any n and any value of $\mu_0, \ldots, \mu_{n-1}$. Note that $F_a[u]$ is always defined for any u since a is a polynomial. It then follows from (3.6) that for analytic input u,

$$\frac{d^l}{dt^l} F_a[u](0) = F_{a_n(\mu_0,\ldots,\mu_{n-1})}[u](0) = \langle a_n(\mu_0, \mu_1, \ldots, \mu_{n-1}), \phi \rangle.$$

Thus (3.16) implies that

$$\frac{d^l}{dt^l} F_a[u](0) = 0$$

for any l, which in turn implies that $F_a[u](t) \equiv 0$ for any analytic u since $F_a[u]$ is analytic. Since analytic controls are dense in $\mathcal{V}_T$ (under the L^1 topology) and, by Lemma 2.3.2, F_a is continuous, it follows that $F_a \equiv 0$. By Corollary (2.2.4), $a = 0$. Thus, $v = 0$, a contradiction to the assumption. Hence, A is of full column rank.

Now let $\mathcal{A}_s$ be the subspace of $\mathbb{R}^r$ spanned by the first s row vectors of A. Then

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_s \subseteq \cdots.$$

Since $\mathcal{A}_r \subseteq \mathbb{R}^r$ for any r, there exists some $s_0 > 0$ such that $\mathcal{A}_s = \mathcal{A}_{s_0}$ for every $s \geq s_0$. Let A_1 be the $s_0 \times r$ submatrix of A consisting the first s_0 rows of A. Then

$$A = TA_1$$

for some matrix T. Therefore

$$\operatorname{rank} A_1 = r.$$

By the construction of A_1, we know that

$$A_1 \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_r \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{s_0} \end{pmatrix}. \tag{3.17}$$

Then the fact that A_1 is full rank implies that every W_i is a linear combination of $Q_1, \ldots, Q_{s_0}$. ∎

We now return to prove Theorem 2.

Proof. For any integer k and $i_1, i_2, \ldots, i_q$, we let

$$\Omega_k(i_1, i_2, \ldots, i_q) = \left\{ w^{-1}c : \ w \in T_k(i_1, i_2, \ldots, i_q) \right\}.$$

Then

$$\mathcal{F}_1(c) = \text{span}_{\mathbf{R}}\{d \in \Omega_k(i_1, i_2, \ldots, i_q) : k \geq 0, q \geq 0\}.$$

Thus the theorem can be proved by showing that

$$\Omega_k(i_1, i_2, \ldots, i_q) \subseteq \mathcal{F}_2(c) \tag{3.18}$$

for any k, q, and $(i_1, i_2, \ldots, i_q)$. Let

$$\Pi_k(i_1, i_2, \ldots, i_q) = \{\psi_c(Q) : Q \in \Delta_k(i_1, i_2, \ldots, i_q)\} .$$

Then we have

$$\Pi_k(i_1, i_2, \ldots, i_q) \subseteq \mathcal{F}_2(c)$$

since for every choice $i_1, \ldots, i_q$, $j_1, \ldots, j_1$ and k, $\psi_c(\Upsilon_{i_1 \cdots i_q}^{j_1 \cdots j_q}(k))$ is the coefficient of $\mu_{i_1 j_1} \cdots \mu_{i_1 j_q}$ in $c_l(\mu_0, \ldots, \mu_{l-1})$ where $l = j_1 + \cdots + j_q + q + k$. Thus to prove our result, it is enough to show that each element in $\Omega_k(i_1, i_2, \ldots, i_q)$ is a linear combination of the elements in $\Pi_k(i_1, i_2, \ldots, i_q)$.

By definition, for any $d \in \Omega_k(i_1, i_2, \ldots, i_q)$, there exists some $w \in T_k(i_1, i_2, \ldots, i_q)$ such that $d = w^{-1}c = \psi_c(w)$. By Lemma 3.2.1, there exist some r and some constants b_1, b_2, b_r, and $Q_1, \ldots, Q_r \in \Delta_k(i_1, i_2, \ldots, i_q)$ such that

$$w = b_1 Q_1 + b_2 Q_2 + \cdots + b_r Q_r .$$

It follows from the linearity of ψ_c that

$$w^{-1}c = b_1 y_1 + b_2 y_2 + \cdots + b_r y_r$$

where $y_i = \psi_c(Q_i) \in \Pi_k(i_1, i_2, \ldots, i_q)$. Since k, q and $(i_1, i_2, \ldots, i_q)$ were arbitrary, we get the desired conclusion $\mathcal{F}_1(c) = \mathcal{F}_2(c)$. ∎

3.3 Families of Power Series

It will be useful later to have analogues of the above results for families of power series.

For a family $\underline{c}$ of power series, we define $z^{-1}c$ to be the family $\{z^{-1}c : \lambda \in \Lambda\}$, for any $z \in P^*$. For any $n \geq 0$, $\underline{c}_n(X_0, \ldots, X_{n-1})$ is defined to be the family

$$\left\{c_n^\lambda(X_0, \ldots, X_{n-1}) : \lambda \in \Lambda\right\},$$

where $X_i = (X_{i1}, \ldots, X_{im})$ are m indeterminates over $\mathbb{R}$, $i \geq 0$. Applying Theorem 1, we have that

$$\frac{d^n}{dt^n} F_c^\lambda[u](t) = F_{c_n^\lambda(u(t),\ldots,u^{n-1}(t))}[u](t), \tag{3.19}$$

for each λ.

As in the case of single power series, we associate to $\underline{c}$ two types of observation spaces in the following way:

$$\tilde{\mathcal{F}}_1(\underline{c}) := \operatorname{span}_{\mathbb{R}} \left\{ \alpha^{-1}\underline{c} : \alpha \in P^* \right\}.$$

$$\tilde{\mathcal{F}}_2(\underline{c}) := \operatorname{span}_{\mathbb{R}} \left\{ \underline{c}_n(\mu_0, \ldots, \mu_{n-1}) : \mu_i \in \mathbb{R}^m, 0 \leq i \leq n-1, n \geq 0 \right\}.$$

Note here the elements of $\tilde{\mathcal{F}}_1(\underline{c})$ and $\tilde{\mathcal{F}}_2(\underline{c})$ are families of series. For instance, if $\underline{c}$ is given by

$$c^\lambda = \lambda^2 + 2\lambda\, \eta_0 - \lambda^3 \eta_1, \quad \lambda \in \mathbb{R}.$$

Then $\tilde{\mathcal{F}}_1(\underline{c})$ is spanned by three elements: $\underline{c}$, 2λ and λ^3, thus, $\tilde{\mathcal{F}}_1(\underline{c})$ is a three dimensional $\mathbb{R}$-space.

Treating families of series as single series over a ring and following the same steps in the proof of Theorem 2, one can obtain an analogue of Theorem 2 for families. The precise proof goes as follows:

First we use

$$\left\{ w^{-1}\underline{c} : w \in T_k(i_1, i_2, \ldots, i_q) \right\}$$

to replace the Ω_k, $\Psi_{\underline{c}}$ to replace ψ_c, and

$$\left\{ \Psi_{\underline{c}}(Q) : Q \in \Delta_k(i_1, i_2, \ldots, i_q) \right\}$$

to replace $\Pi_k(i_1, i_2, \ldots, i_q)$ in the proof of Theorem 2, where

$$\Psi_{\underline{c}}(d) = \sum \langle d, \eta_\kappa \rangle \eta_\kappa^{-1}\underline{c}$$

for any single polynomial

$$d = \sum \langle d, e_\kappa \rangle \eta_\kappa.$$

Then as in the single series case, $\tilde{\mathcal{F}}_1(\underline{c})$ is spanned by all the elements in $\Omega_k(i_1, i_2, \ldots, i_q)$, and $\tilde{\mathcal{F}}_2(\underline{c})$ is spanned by all the elements in $\Pi_k(i_1, i_2, \ldots, i_q)$ for all k, q and $i_1, \ldots, i_q$.

Since each $w \in T_k(i_1, i_2, \ldots, i_q)$ is a linear combination of elements in $\Delta_k(i_1, i_2, \ldots, i_q)$, it follows that each element in $T_k(i_1, i_2, \ldots, i_q)$ is a linear combination of elements in $\Pi_k(i_1, i_2, \ldots, i_q)$. Therefore, each element in $\Omega_k(i_1, i_2, \ldots, i_q)$ is a linear combination of elements in $\Delta_k(i_1, i_2, \ldots, i_q)$ for all k q and $i_1, \ldots, i_q$. Thus we proved the following conclusion:

Theorem 3 *For any family* $\underline{c}$ *of power series,* $\tilde{\mathcal{F}}_1(\underline{c}) = \tilde{\mathcal{F}}_2(\underline{c})$. ∎

3.4 State – Space Systems

Now consider a state space system

$$\begin{aligned} x' &= g_0(x) + \sum u_i g_i(x) \\ y &= h(x) \end{aligned} \tag{3.20}$$

where $x(t) \in \mathcal{M}$, a C^ω manifold, $g_0, g_1, \ldots, g_m$ are C^ω vector fields, and h is a C^ω function from $\mathcal{M}$ to $\mathbb{R}$. For each $\iota = i_1 i_2 \cdots i_l \in I^*$, $L_\iota h$ denotes

$$L_{g_{i_l}} L_{g_{i_{l-1}}} \cdots L_{g_{i_1}} h.$$

By Lemma (4.2) in [37], the generating series

$$c = \sum_\iota L_\iota h(x^0)\, \eta_\iota \tag{3.21}$$

is a convergent power series for any $x^0 \in \mathcal{M}$.

The next Theorem shows that the output function of system (3.20) is given by the i/o operator induced by the series (3.21). The fact is due to Fliess (see [10] and [16]). For completeness, we shall prove the Theorem in detail.

Theorem 4 *For the power series c defined as in (3.21), there exists some $T > 0$ admissible for c such that for any $u \in \mathcal{V}_T$,*

$$y(t) = h(\varphi(t, x^0, u)) = F_c[u](t)$$

for t small enough, where $\phi(t, x^0, u)$ denotes the solution of (3.20) with the initial state $x(0) = x^0$.

To prove this theorem, we first establish the following Lemma:

Lemma 3.4.1 Assume $\varphi_1, \varphi_2, \ldots, \varphi_k$ are n analytic functions defined on $\mathcal{M}$ and for $i = 1, \ldots, n$,

$$c_i = \sum_{\iota} L_\iota \varphi_i(x^0)\, \eta_\iota$$

for some $x_0 \in \mathcal{M}$. Suppose $y_1(t), y_2(t), \ldots, y_k(t)$ are given by

$$y_i(t) = F_{c_i}[u](t) \tag{3.22}$$

for all $u \in V_T$, $i = 1, \ldots, n$. Then for any analytic function Φ defined in a neighborhood of $y^0 = (\varphi_1(x^0), \ldots, \varphi_k(x^0))$ in $\mathbb{R}^n$, and any $u \in V_T$,

$$\Phi(y_1(t), y_2(t), \ldots, y_k(t)) = F_c[u](t) \tag{3.23}$$

for t small enough, where

$$c = \sum_{\iota} L_\iota\, \Phi(\varphi_1, \ldots, \varphi_2)(x^0)\, \eta_\iota.$$

Proof. We shall prove the conclusion by first showing that it holds for the function $\Phi(y) = y_1 y_2$. For this purpose we need to establish the following formula:

$$L_\rho \psi_1 \psi_2 = \sum_{\kappa_1, \kappa_2} L_{\kappa_1} \psi_1\, L_{\kappa_2} \psi_2 \langle \eta\kappa_1 \sqcup\!\sqcup\, \kappa_2,\ \eta_\rho \rangle \tag{3.24}$$

for any analytic functions ψ_1, ψ_2 and $\kappa \in I^*$.

We use induction on the length of ρ to prove (3.24). It is true for $\rho \in I^1$ since

$$L_\rho (\psi_1 \psi_2) = L_\rho \psi_1\, \psi_2 + \psi_1\, L_\rho \psi_2.$$

if $|\rho| = 1$. Now Assume (3.24) holds for $\rho \in I^n$. Then for $\rho = i\kappa$ with $\kappa \in I^n$ we have

$$\begin{aligned}
& L_{i\kappa}(\psi_1 \psi_2) = L_\kappa (L_i \psi_1\, \psi_2 + \psi_1\, L_i \psi_2) \\
= & \sum_{\kappa_1, \kappa_2} (L_{\kappa_1} L_i \psi_1\, L_{\kappa_2} \psi_2 + L_{\kappa - 1} \psi_1\, L_{\kappa_2} L_i \psi_2) \langle \eta_{\kappa_1} \sqcup\!\sqcup\, \eta_{\kappa_2},\ \rho \rangle \\
= & \sum_{\kappa_1, \kappa_2} (L_{i\kappa_1} \psi_1\, L_{\kappa_2} \psi_2 + L_{\kappa_1} \psi_1\, L_{i\kappa_2}) \langle \eta_{\kappa_1} \sqcup\!\sqcup\, \eta_{\kappa_2},\ \rho \rangle \\
= & \sum_{\kappa_1, \kappa_2} L_{\kappa_1} \psi_1 L_{\kappa_2} \psi_2 \langle (\eta_i^{-1} \eta_{\kappa_1}) \sqcup\!\sqcup\, \eta_{\kappa_2},\ \rho \rangle + \sum_{\kappa_1, \kappa_2} L_{\kappa_1} \psi_1 L_{\kappa_2} \psi_2 \langle \eta_{\kappa_1} \sqcup\!\sqcup\, (\eta_i^{-1} \eta_{\kappa_2}),\ \rho \rangle \\
= & \sum_{\kappa_1, \kappa_2} L_{\kappa_1} \psi_1\, L_{\kappa_2} \psi_2 \langle \eta_i^{-1} (\eta_{\kappa_1} \sqcup\!\sqcup\, \eta_{\kappa_2}),\ \rho \rangle \\
= & \sum_{\kappa_1, \kappa_2} L_{\kappa_1} \psi_1\, L_{\kappa_2} \psi_2 \langle \eta_{\kappa_1} \sqcup\!\sqcup\, \eta_{\kappa_2},\ \eta_i \rho \rangle
\end{aligned}$$

We completed the proof of (3.24) by induction.

Now assume that $y_1(t), y_2(t)$ are given by (3.22), then

$$y_1(t)y_2(t) = F_{c_1 ш c_2}[u](t).$$

But

$$\langle c_1 ш c_2, \eta_\rho\rangle = \sum_{\kappa_1,\kappa_2} L_{\kappa_1}\varphi_1(x^0)\, L_{\kappa_2}\varphi_2(x^0)\langle \eta_{\kappa_1} ш \eta_{\kappa_2}, \eta_\rho\rangle = L_\rho(\varphi_1\varphi_2)(x^0)$$

Therefore,

$$y_1(t)y_2(t) = \sum_\rho L_\rho(\varphi_1\varphi_2)(x^0)\, V_\rho[u](t).$$

By induction, one can prove that for any integers $i_1, i_2, \ldots, i_k$, $1 \le i_s \le k$,

$$y_{i_1}(t)y_{i_2}(t)\cdots y_{i_k}(t) = \sum_\rho L_\rho(\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_k})(x^0)\, V_\rho[u](t).$$

Now we are ready to prove (3.23). Without loss of generality, we may assume that

$$y^0 = (y_1(0), y_2(0), \ldots, y_k(0)) = (\varphi_1(x^0), \varphi_2(x^0), \ldots, \varphi_k(x^0)) = 0.$$

Assume

$$\Phi(x_1, x_2, \ldots, x_k) = \sum_I a_I\, x^I$$

for $|x_i| \le \alpha$, $i = 1, 2, \ldots, n$, where

$$a_I = a_{i_1\ldots i_k} \in \mathbb{R}, \quad x^I = x_1^{i_1}\cdots x_k^{i_k}$$

for $I = i_1 i_2 \cdots i_k$. It follows from the analyticity of the functions involved that for any $\iota \in I^*$,

$$L_\iota\Phi(\varphi_1, \ldots, \varphi_k)(x^0) = L_\iota \sum_I a_I\, \varphi^I = \sum_I a_I\, L_\iota\varphi^I(x^0).$$

Hence

$$\begin{aligned}
\Phi(y_1(t), y_2(t), \ldots, y_k(t)) &= \sum_I a_I\, y(t)^I \\
&= \sum_I a_I \sum_\rho L_\rho\varphi^I(x^0)\, V_\rho[u](t) && (3.25)\\
&= \sum_\iota\sum_I a_I\, L_\rho\varphi^I(x^0)\, V_\rho[u](t) && (3.26)\\
&= \sum_\iota L_\rho\Phi(\varphi_1, \varphi_2, \ldots\varphi_k)(x^0)\, V_\rho[u](t)
\end{aligned}$$

for t small enough. Note that we can switch the order of the sums in (3.25) to get (3.26) because the series (3.25) converges absolutely for t small enough. To prove this, we need first to show that for any $I = i_1 i_2 \cdots i_n$,

$$\sum_{\iota} \left| L_\rho \varphi^I(x_0)\, V_\rho[u](t) \right| \leq \sum_{\iota_1, \iota_2, \ldots, \iota_s} |L_{\iota_1} \psi_{j_1}(x^0)| \cdots |L_{\iota_s} \psi_{j_s}(x^0)|\, |V_{\iota_1}[u](t)| \cdots |V_{\iota_s}[u](t)| \quad (3.27)$$

where $s = i_1 + \cdots + i_k$ and ψ_{i_l} is one of the $\varphi_1, \cdots, \varphi_k$. Clearly (3.27) holds when $s = 1$. Now assume (3.27) holds for $s-1$. Take some $I = i_1 \cdots i_k$ such that $i_1 + \cdots + i_k = s$. Without loss of generality, we may assume $i_k = j_k + 1$ for some $j_k \geq 0$ and let I' denote $i_1 \cdots j_k$. By Remark (2.2.10),

$$\begin{aligned} &\sum_\rho \left| L_\psi \varphi^I(x_0)\, V_\rho[u](t) \right| \\ &\leq \sum_\rho \sum_{\rho_1, \iota_s} \langle \eta_{\rho_1} \sqcup\!\sqcup\, \eta_{\iota_s}, \eta_\rho \rangle\, |L_{\rho_1} \varphi^{I'}(x_0)|\, |L_{\iota_s} \varphi_k(x_0)|\, |V_\rho[u](t)| \\ &\leq \sum_{\rho_1, \iota_s} |L_{\rho_1} \varphi^{I'}(x_0)|\, |L_{\iota_s} \varphi_k(x_0)|\, |V_{\rho_1}[u](t)|\, |V_{\iota_s}[u](t)|. \end{aligned} \quad (3.28)$$

Applying the induction assumption to (3.28), one gets (3.27). Inequality (3.27) is proved by induction.

Finally, By Lemma 4.2 in [37], we know that for any $I = i_1 i_2 \cdots i_k$ and any $u \in V_T$, there exists a constant M so that

$$|L_\iota \varphi_i(x^0)| \leq M^{|\iota|} |\iota|!$$

for $i = 1, \ldots, k$ and any $\iota \in I^*$. So from (3.27) we know that

$$\begin{aligned} &\sum_\rho \left| L_\rho \varphi^I(x_0)\, V_\rho[u](t) \right| \\ &\leq \sum_{k_1 \geq 1, \ldots, k_s \geq 1} (M(m+1))^{k_1} t^{k_1} (M(m+1))^{k_2} \cdots (M(m+1))^{k_s} t^{k_s} \\ &= \left(\frac{M(m+1)t}{1 - M(m+1)t} \right)^s. \end{aligned}$$

Pick up any number $\beta < \alpha$ and choose an $t_1 > 0$ so that

$$\frac{M(m+1)t_1}{1 - M(m+1)t_1} \leq \beta.$$

Then

$$\sum_{I,\iota} |a_I|\,|L_\rho \varphi^I(x^0)|\,|V_\rho[u](t)| \leq \sum_I |a_I| \beta^{i_1+i_2+\cdots+i_k}$$

which is convergent. Therefore, series (3.25) converges absolutely for t small enough. ∎

Remark 3.4.2 The more general form of (3.24) is as follows: For any k analytic functions $\psi_1, \ldots, \psi_k$ defined on $\mathcal{M}$ and any $\rho \in I^*$,

$$L_\rho(\psi_1 \psi_2 \cdots \psi_k) = \sum_{\iota_1, \iota_2, \ldots, \iota_k} L_{\iota_1}\psi_1 L_{\iota_2}\psi_2 \cdots L_{\iota_k}\psi_k \langle \eta_{\iota_1} ш \eta_{\iota_2} ш \cdots ш \eta_{\iota_k},\, \eta_\rho \rangle. \tag{3.29}$$

This formula can be proved by using induction on k. We have shown it holds for $k = 2$. Assume it holds for $k \leq l$. Then

$$\begin{aligned} L_\rho(\psi_1 \psi_2 \cdots \psi_{(l+1)}) &= \sum_{\rho_1, \iota_{l+1}} \langle \eta_{\rho_1} ш \eta_{\iota_{l+1}},\, \eta_\rho \rangle L_{\rho_1}(\psi_1 \psi_2 \cdots \psi_l)\, L_{\iota_{l+1}} \psi_{l+1} \\ &= \sum_{\rho_1} \sum_{\iota_1, \ldots, \iota_{l+1}} \langle \eta_{\iota_1} ш \cdots ш \eta_{\iota_l},\, \eta_{\rho_1} \rangle \langle \eta_{\rho_1} ш \eta_{\iota_{l+1}},\, \eta_\rho \rangle\, L_{\iota_1}\psi_1 \ldots L_{\iota_{l+1}} \psi_{l+1}. \end{aligned} \tag{3.30}$$

Notice that for any polynomials w_1 and w_2, and any word $z \in P^*$,

$$\langle w_1 ш w_2,\, z \rangle = \sum_\iota \langle w_1,\, \eta_\iota \rangle \langle \eta_\iota ш w_2,\, z \rangle \tag{3.31}$$

since $w_1 = \sum_\iota \langle w_1,\, \eta_\iota \rangle$. Combining (3.30) and (3.31) we get (3.29). □

We now return to prove Theorem 4.

Proof. Without loss of generality, we may assume $\mathcal{M}$ is a neighborhood of x^0 in $\mathbb{R}^n$. The conclusion will follow from the following fact: for any solution of (3.20) corresponding to u with initial state $x(0) = x^0$,

$$x_i(t) = \sum_\iota (L_\iota x_i)(x_0)\, V_\iota[u](t), \tag{3.32}$$

for $i = 1, 2, \ldots, n$, because if (3.32) holds, then by Lemma 3.4.1,

$$y(t) = \sum_\iota L_\iota h(x^0)\, V_\iota[u](t)$$

as desired.

To prove (3.32), we let

$$\varphi_i(t) = \sum_\iota (L_\iota x_i)(x^0)\, V_\iota[u](t)$$

for $i = 1, 2, , \ldots, n$. Then

$$\varphi_i(0) = x_i(0), \quad \text{for } i = 1, 2, \ldots, n$$

By Lemma 2.3.5,

$$\begin{aligned} \frac{d}{dt}\varphi_i(t) &= \sum_{j=0}^{m} \sum_{\iota} u_j(t)\,(L_\iota L_j x_i)(x^0)\, V_\iota[u](t) \\ &= \sum_{j=0}^{m} u_j(t)\, L_\iota g_{ij}(x^0)\, V_\iota[u](t), \end{aligned} \tag{3.33}$$

where g_{ij} is the i-th component of g_j for $i = 1, \ldots, n,\ j = 0, 1, \ldots, m$. Applying Lemma 3.4.1 to (3.33) one gets

$$\frac{d}{dt}\varphi_i(t) = \sum_{j=0}^{m} u_j(t)\, g_{ij}(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)).$$

It then follows from the uniqueness of solution to the differential equation that

$$x_i(t) = \varphi_i(t)$$

for $i = 1, 2, \ldots, n$. ∎

For the system (3.20), we can also define two types of observation space. The first type of observation space associated with (3.20) is

$$F_1 := \text{span}_{\mathbf{R}}\{L_{g_{i_1}} \cdots L_{g_{i_k}} h : \ k \geq 0\}.$$

To define the second type, we introduce the following notations: For $\mu_0, \ldots, \mu_{k-1}$ given, we let

$$y^{\mu_0 \cdots \mu_{k-1}} := \left.\frac{d^k}{dt^k}\right|_{t=0} y(t),$$

where $y(t)$ is the output corresponding to any C^∞ input u such that $u^{(j)}(0) = \mu_j$ for $0 \leq j \leq k-1$.

We associate to (3.20) a second type of observation space, as follows:

$$F_2 := \text{span}_{\mathbf{R}}\{y^{\mu_1 \cdots \mu_{k-1}} : \mu_i \in \mathbf{R}^m, k \geq 0\}.$$

By Theorem 4, the input/output map of (3.20) can be written as

$$y(t) = \mathbf{F}_{\underline{c}}[u](t),$$

where $\underline{c}$ is defined by

$$\langle \underline{c},\ \eta_{i_1} \cdots \eta_{i_k} \rangle = L_{g_{i_k}} \cdots L_{g_{i_1}} h,$$

or, equivalently, for the output corresponding to the initial state x,

$$y_x(t) = F_{c^x}[u](t),$$

where

$$\langle c^x,\ \eta_{i_1} \cdots \eta_{i_k} \rangle = L_{g_{i_k}} \cdots L_{g_{i_1}} h(x).$$

Thus,

$$L_{g_{i_k}} \cdots L_{g_{i_1}} h = \langle \underline{c},\ \eta_{i_1} \cdots \eta_{i_k} \rangle = \langle (\eta_{i_1} \cdots \eta_{i_k})^{-1} \underline{c},\ \phi \rangle,$$

i.e.,

$$L_{g_{i_k}} \cdots L_{g_{i_1}} h(x) = \langle (\eta_{i_1} \cdots \eta_{i_k})^{-1} \underline{c}^x,\ \phi \rangle,$$

and, therefore,

$$F_1 = \{\langle \mathrm{d},\ \phi \rangle :\ \mathrm{d} \in \tilde{\mathcal{F}}_1(\underline{c})\}.$$

By (3.19), we know that

$$y_x^{\mu_0 \cdots \mu_{k-1}} = F_{c_k^x(\mu_0,\ldots,\mu_{k-1})}[u](0) = \langle c_k^x(\mu_0, \ldots, \mu_{k-1}),\ \phi \rangle.$$

Hence,

$$F_2 = \{\langle \mathrm{d},\ \phi \rangle :\ \mathrm{d} \in \tilde{\mathcal{F}}_2(\underline{c})\}.$$

So the following conclusion follows immediately from Theorem 3:

Corollary 3.4.3 For the state space system (3.20), $F_1 = F_2$. □

3.5 An example

To illustrate the ideas used in the proof of Lemma 3.2.1 and in the proof of Theorem 2 which give rise to Corollary 3.4.3, we show how to prove that $L_f L_g h$ and $L_g L_f h$ are linear combination of elements of F_2 for system (3.20) in the single input case. First of all, we have the following formulae:

$$y = h(x)$$

$$
\begin{aligned}
y^{\mu_0} &= L_f h(x) + L_g h(x)\mu_0 \\
y^{\mu_0\mu_1} &= L_f^2 h(x) + (L_g L_f h(x) + L_f L_g h(x))\mu_0 + L_g^2 h(x)\mu_0^2 + L_g h(x)\mu_1 \\
y^{\mu_0\mu_1\mu_2} &= L_f^3 h(x) + (L_f^2 h(x) + L_f L_g L_f h(x) + L_g L_f^2 h(x))\mu_0 \\
&\quad (L_f L_g^2 h(x) + L_g L_f L_g h(x) + L_g^2 L_f h(x))\mu_0^2 + L_g^3 h(x)\mu_0^3 \\
&\quad 2L_g^2 h(x)\mu_0\mu_1 + (L_g L_f h(x) + 2L_f L_g h(x))\mu_1 + L_g h(x)\mu_2 \\
&\cdots\cdots
\end{aligned}
$$

Let

$$Q_k = \eta_0 ш \eta_1 X^k\Big|_{X=1} .$$

For any polynomial $P = w_1 + \cdots + w_r$, where $w_i \in P^*$, define

$$L_Q h(x) = L_{w_1} h(x) + L_{w_2} h(x) + \cdots + L_{w_r} h(x)$$

where $L_w h(x) = L_{g_{i_s}} \cdots L_{g_{i_1}} h(x)$ if $w = \eta_{i_1} \cdots \eta_{i_s}$.

Applying Theorem 1 to the series

$$c^x = \sum_{\iota} L_\iota h(x)\, \eta_\iota ,$$

we know that $L_{Q_k} h(x)$ is the coefficient of μ_k in $y^{\mu_0 \cdots \mu_{k+1}}$, in the other words, $L_{Q_k} h(x)$ is the coefficient of $u^{(k)}(t)$ in $y^{(k+1)}$ for each $k \geq 0$. It then follows that

$$L_{Q_k} h(x) = y_{k1} - y_{k0} \tag{3.34}$$

where $y_{k1} = y^{\mu_0 \cdots \mu_{k+2}}$ with $\mu_i = 0$ if $i \neq k$ and $\mu_k = 1$, and $y_{k0} = y^{\mu_0 \cdots \mu_{k+2}}$ with $\mu_i = 0$ for all i. Hence $L_{Q_k} h(x) \in F_2$ for each k.

From the definition of Q_k, it is not hard to see that each Q_k is a linear combination of $\eta_1\eta_0$ and $\eta_0\eta_1$. In fact

$$Q_k = \eta_0\eta_1 + \binom{k+1}{1}\eta_1\eta_0 = \eta_0\eta_1 + (k+1)\eta_1\eta_0 .$$

Therefore,

$$L_{Q_k} h(x) = \varphi_1(x) + (k+1)\varphi_2(x)$$

for each k where $\varphi_1(x) = L_{g_1} L_{g_0} h(x)$ and $\varphi_2(x) = L_{g_0} L_{g_1} h(x)$. Now Let A be the matrix of 2 columns and infinitely many rows defined by $a_{k1} = 1$, $a_{k2} = k + 1$. It is very clear that A is full column rank, but to sketch the ideas in the proof of Lemma 3.2.1, we assume that there exists some nonzero 2-vector v such that $Av = 0$. Now let

$$a = v_1\eta_0\eta_1 + v_2\eta_1\eta_0$$

and consider the derivatives of $F_a[u]$ at $t = 0$ for any analytic control u. It is easy to calculate the derivatives of the first three orders:

$$
\begin{aligned}
&F_a[u](0) = 0\,,\\
&\left.\frac{d}{dt}\right|_{t=0} F_a[u](t) = v_1F_{\eta_1}[u](0) + u(0)v_2F_{\eta_0}[u](0) = 0\,,\\
&\left.\frac{d^2}{dt^2}\right|_{t=0} F_a[u](t) = (v_1+v_2)u(0) + u'(0)v_2F_{\eta_0}[u](0) \;=\; (v_1+v_2)u(0)\,,\\
&\left.\frac{d^3}{dt^3}\right|_{t=0} F_a[u](t) = (v_1+2v_2)u'(0) + u''(0)v_2F_{\eta_0}[u](0) \;=\; (v_1+2v_2)u'(0)\,.
\end{aligned}
$$

(We are using the fact that $F_c[u](0) = 0$ for every series c with no constant term.) In fact, for any n,

$$\left.\frac{d^n}{dt^n}\right|_{t=0} F_a[u](t) \;=\; (v_1 + (n-1)v_2)u^{(n-2)}(0)\,.$$

But v_1 and v_2 were chosen so that $Av = 0$, i.e., $v_1 + (k+1)v_2 = 0$ for any $k \geq 0$. Therefore,

$$\left.\frac{d^n}{dt^n}\right|_{t=0} F_a[u](t) \;=\; 0$$

for any n which implies, by analyticity, that $F_a[u](t) \equiv 0$. Since analytic controls are dense in L_1 with respect to the L_1 norm, it follows that $F_a[u] = 0$ for any L_1 control u. By Corollary 2.2.4, $a = 0$, which contradicts the assumption that $v \neq 0$. Therefore, A is full column rank. In fact, since the first two rows of A already form a full rank matrix, let A_1 be this matrix, i.e.,

$$A_1 \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$A_1^{-1} \;=\; \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

It then follows that

$$\eta_0\eta_1 \;=\; 2Q_0 - Q_1\,,\;\; \eta_1\eta_0 \;=\; -Q_0 + Q_1\,.$$

Therefore

$$L_gL_fh(x) = 2L_{Q_0}h(x) - L_{Q_1}h(x), \quad L_fL_gh(x) = -L_{Q_0}h(x) + L_{Q_0}h(x)\,.$$

Applying (3.34), we get

$$L_gL_fh(x) = 2y^{01} - 2y^{00} - y^{010} + y^{000}\,,$$

$$L_fL_fh(x) = -y^{01} + y^{00} + y^{010} + y^{000}\,.$$

Thus we proved that $L_fL_gh(x)$ and $L_gL_fh(x)$ belong to F_2.

Chapter 4

Input/Output Equations

In this Chapter, we study high–order differential equations satisfied by inputs and outputs arising from i/o operators. To carrry out this study, we find it useful to introduce the algebraic concepts of observation algebra and observation field corresponding to any given series c.

The *observation algebra* $\mathcal{A}_2(c)$ is defined as the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_2(c)$. By Lemma 2.1.1, $\mathcal{A}_2(c)$ is an integral domain, so its quotient field is well defined; we define the *observation field* of c as this quotient field. We shall see later that elementary properties of these algebraic objects serve to characterize the existence of i/o equations.

4.1 Definitions

By an *algebraic input/output equation of order k* we shall mean an equation of the type

$$P\left(u(t),\ldots,u^{(k)}(t),y(t),\ldots,y^{(k)}(t)\right)=0, \tag{4.1}$$

where

$$P\in\mathbb{R}[S_0,\ldots,S_k,L_0,\ldots,L_k]$$

is a polynomial nontrivial in L_k, and S_i denotes the set of m variables

$$(S_{1i},\ldots,S_{mi}).$$

Definition 4.1.1 We say that a polynomial P as above is

(a) *rational* when $P(S_0,\ldots,S_k,L_0,\ldots,L_k)=$

$$P_0(S_0,\ldots,S_{k-1},L_0,\ldots,L_{k-1})\,L_k+P_1(S_0,\ldots,S_k,L_0,\ldots,L_{k-1}); \tag{4.2}$$

(b) *recursive* when $P(S_0, \ldots, S_k, L_0, \ldots, L_k) =$

$$= P_0(S_0, \ldots, S_{k-1})\, L_k + P_1(S_0, \ldots, S_k, L_0, \ldots, L_{k-1}); \tag{4.3}$$

(c) *affine* when $P(S_0, \ldots, S_k, L_0, \ldots, L_k) =$

$$P_0(S_0, \ldots, S_{k-1})L_k + \sum_{i=1}^{k} P_i(S_0, \ldots, S_k)\, L_{k-i} + P_{k+1}(S_0, \ldots, S_k); \tag{4.4}$$

(d) *linear* when P is affine as in (4.4), and $P_{k+1} = 0$, i.e.,

$$\begin{aligned} &P(S_0, \ldots, S_k, L_0, \ldots, L_k) \\ &\qquad = P_0(S_0, \ldots, S_{k-1})L_k + \sum_{i=1}^{k} P_i(S_0, \ldots, S_k)\, L_{k-i}. \end{aligned} \tag{4.5}$$

□

Definition 4.1.2 Assume that c is a convergent power series. We say that the i/o operator F_c *satisfies an algebraic i/o equation (4.1)* if (4.1) holds for every possible $\mathcal{C}^k$ i/o pair

$$(u(t),\, y(t)) := (u(t), F_c[u](t))$$

of F_c for all $t \in [0, T]$ and for any T admissible for c. In such a case, (4.1) is called an *i/o equation of F_c*.

An i/o operator F_c satisfies a *recursive* (respectively *affine, linear*) i/o equation if there is some such equation for which P is recursive (respectively affine, linear). An i/o operator F_c satisfies a *rational* i/o equation if P can be chosen rational so that $P_0 = 0$ is *not* an i/o equation of F_c, in another word, there exists some i/o pair (u, y) of F_c such that

$$P_0(u(t),\, u'(t), \ldots, u^{(k)}(t),\, y(t),\, y'(t), \ldots, y^{(k-1)}(t)) \neq 0, \tag{4.6}$$

for some t. □

Assume c is a convergent power series. For any integer n and $\mu_0, \mu_1, \ldots, \mu_{n-1} \in \mathbb{R}^m$ given, we let

$$w_\mu(t) = \sum_{i=0}^{n-1} \mu_i \frac{t^i}{i!}\,.$$

Notice that $\omega_\mu^{(i)}(0) = \mu_i$ for $0 \le i \le n-1$. Now take any $u \in \mathcal{V}_T$, (where T is admissible for c,) and consider $u\#_t\omega_\mu$. One may notice that $u\#_t\omega_\mu$ may not be in the set $\mathcal{V}_T$ since it may happen that $|\mu_0| > 1$. But the proof of Lemma 2.2.5 and Remark 2.2.2 indicate that $F_c[u\#_t\omega_\mu](t+\tau)$ is always well defined for $0 \le t \le T$ and small enough τ. By Theorem 1,

$$\frac{d^n}{d\tau^n}\Big|_{\tau=0^+} F_c[u\#_t\omega_\mu](t+\tau) = F_{c_n(\mu_0,\mu_1,\ldots,\mu_{n-1})}[u](t), \tag{4.7}$$

for any $0 \le t \le T$.

Remark 4.1.3 In Theorem 1 We required that u be of class C^{n-1}, while here $u\#_\tau\omega$ may not be C^{n-1}. But if one checks the steps in the proof of Theorem 1, one will find that equation (3.6) holds at any point t at which u is $n-1$ times continuously differentiable even if u is not in C^{n-1} in the whole interval. □

Lemma 4.1.4 F_c satisfies i/o equation (4.1) if and only if

$$P\left(\mu_0,\ldots,\mu_k, F_c, F_{c_1(\mu_0)},\ldots,F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) = 0, \tag{4.8}$$

for any $\mu_0,\mu_1,\ldots,\mu_k \in \mathbb{R}^m$.

Remark 4.1.5 Equation (4.8) means that

$$P\left(\mu_0,\ldots,\mu_k, F_c[u](t), F_{c_1(\mu_0)}[u](t),\ldots,F_{c_k(\mu_0,\ldots,\mu_{k-1})}[u](t)\right) = 0, \tag{4.9}$$

for any $u \in \mathcal{V}_T$ and any $t \in [0,T]$. By definition, (4.9) holds for those u such that $u\#_t\omega_\mu \in C^k$. But in general, $u\#_t\omega_\mu$ is not of class C^k. The following proof of Lemma 4.1.4 in fact shows that equation (4.1) holds for all C^k i/o pairs of F_c if and only if it holds at any point at which $u(t)$ is C^k for every $u \in \mathcal{V}_T$. □

To prove Lemma 4.1.4, we need the following Lemma:

Lemma 4.1.6 Assume $f \in C[0,1]$ and $f(0) = 0$. Then for each given integer $n \ge 0$, there exists a sequence of polynomial functions $\{f_k\}$ such that

$$f_k \longrightarrow f, \quad \text{as } k \to \infty$$

uniformly and $f_k^{(i)}(0) = 0$ for all k and $0 \le i \le n-1$.

Proof. Suppose $f \in C[0,1]$ and $f(0) = 0$. Let $\hat{f}(x) = f(x^{1/n})$. Then, by Weierstrass' Theorem, there exists a sequence of polynomials approaching to $\hat{f}$; in particular, the sequence of polynomials can be chosen as the Bernstein polynomials

$$\hat{f}_k(x) = \sum_{j=0}^{k} \hat{f}\left(\frac{j}{k}\right)\binom{k}{j} x^j (1-x)^{k-j}.$$

Notice that

$$\hat{f}_k(0) = 0$$

for all k. Now let $f_k(x) = \hat{f}_k(x^n)$. Then

$$f_k(x) \longrightarrow f(x) \quad \text{as } k \to \infty$$

uniformly and f_k's are polynomials of x^n. Since $f_k(0) = 0$, it follows that $f_k^{(i)}(0) = 0$ for $0 \le i \le n-1$. ∎

We are now ready to prove Lemma 4.1.4.

Proof. Assume equation (4.9) holds for any $u \in \mathcal{V}_T$ and $t \in [0, T]$. Take an input

$$u \in \mathcal{V}_T \bigcap C^{k-1}$$

and pick any $t \in [0, T]$. Assume

$$u^{(s)}(t) = \mu_s$$

for $0 \le s \le k$. Then

$$y^{(s)}(t) = \frac{\partial^s}{\partial t^s} F_c[u](t) = F_{c_s(\mu_0,\ldots,\mu_{s-1})}[u](t)$$

for $0 \le s \le k$. Therefore,

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0.$$

Since u and t can be picked arbitrarily, it follows that $P = 0$ is an i/o equation for F_c.

Now assume $P = 0$ is an i/o equation for F_c. Take any fixed $u \in \mathcal{V}_T$ and consider

$$\hat{u} := u \#_t \omega_\mu,$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_k)$. We shall prove the Lemma by first showing that (4.9) holds for all $u \in \mathcal{V}_T$ and t and all those μ's such that $|\mu_0| < 1$. For this purpose, we will first find a sequence $\{v_j\} \in \mathcal{C}^k$ such that

$$\|v_j - \hat{u}\|_1 \to 0, \text{ as } j \to \infty, \tag{4.10}$$

$$v_j(s) = \hat{u}(s) = \omega(s - t) \quad \text{for } s \geq t,$$

and there exists some fixed $\delta > 0$ such that

$$|v_j(s)| < 1 \quad \text{for} \quad s \in [0, t + \delta]$$

for all large enough j.

Assuming for now that there exists such a sequence $\{v_j\}$, we show how to complete the proof. The output $F_c[v_j]$ is defined and differentiable in $[0, t + \delta]$ for large enough j. Applying (4.1) to the $\mathcal{C}^k$ pair (v_j, y_j) at time t, we get

$$P\left(v_j(t), \ldots, v_j^{(k)}(t), y_j(t), \ldots, y_j^{(k)}(t)\right) = 0, \tag{4.11}$$

where $y_j = F_c[v_j]$.

By Lemma 2.3.2, we know that for $t \leq \tau \leq t + \delta$ for some $\delta > 0$

$$y_j^{(s)}(\tau) \to \frac{d^s}{dt^s} F_c[\hat{u}](\tau) \quad \text{as} \quad j \to \infty.$$

Letting

$$j \to \infty \quad \text{and} \quad \tau \to t^+$$

and taking the limits on both sides of (4.11), we get

$$P\left(\mu_0, \ldots, \mu_{k-1}, F_c[\hat{u}](t), \left.\frac{d}{d\tau}\right|_{\tau=0+} F_c[u](t+\tau), \ldots, \left.\frac{d^k}{d\tau^k}\right|_{\tau=0+} F_c[\hat{u}](t+\tau)\right) = 0.$$

Since u and t can be chosen arbitrarily, we get (4.8) under the assumption that $|\mu_0| < 1$. To remove this assumption, notice that for any fixed $u \in \mathcal{V}_T$ and t, the function

$$P\left(\mu_0, \ldots, \mu_{k-1}, F_c[u](t), F_{c_1(\mu_0)}[u](t), \ldots, F_{c_k(\mu_0, \ldots, \mu_{k-1})}[u](t)\right) \tag{4.12}$$

is analytic in μ, in fact, a polynomial function in μ. Hence (4.9) holds for *all* μ, not only for those with $|\mu_0| < 1$. Again, as u and t can be chosen arbitrarily, we get (4.8).

Now we return to prove the existence of $\{v_j\}$. Take $u \in \mathcal{V}_T$ and $\mu_0, \ldots, \mu_k \in \mathbb{R}^m$, where $|\mu_0| < 1$. Notice that the set

$$\hat{\mathcal{V}}_T := \left\{u \in \mathcal{V}_T \cap \mathcal{C}[0,T] : u(t) = \mu_0,\ \|u\|_\infty < 1\right\}$$

is dense in $\mathcal{V}_T$ (still using the L^1 norm), so we may assume that $u \in \hat{\mathcal{V}}_T$. Now let

$$\tilde{u}(s) = \begin{cases} u(s) - \omega_\mu(s-t) & \text{if } s \leq t, \\ 0 & \text{if } t < s \leq T. \end{cases}$$

By Lemma 4.1.6, there exists a sequence $\tilde{v}_j$ in $\mathcal{C}^k[0,T]$ such that

$$\tilde{v}_j(s) \rightarrow \tilde{u}(s)$$

uniformly and $\tilde{v}_j^{(i)}(t) = 0$ for $0 \leq i \leq k$. Let

$$v_j(s) = \tilde{v}_j(s) + \omega_\mu(s-t).$$

Then $\{v_j\} \rightarrow u\#_t\omega_\mu$ uniformly. Since

$$|u\#_t\omega_\mu(s)| < 1 \quad \text{for} \quad s \in [0, t+\delta]$$

for some small δ, it follows that $|v_j(s)| < 1$ for $s \in [0, t+\delta]$ if j is large enough. ∎

Lemma 4.1.7 If F_c satisfies an algebraic i/o equation, then it satisfies a rational i/o equation.

Proof. Assume that F_c satisfies an algebraic i/o equation

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0.$$

Without loss of generality, we may assume that $\deg_{L_k} P$ is smallest possible among all such nontrivial equation. Assume that P is not rational. Taking the derivative with respect to time t on both side of the equation, we can see that F_c satisfies the following equation:

$$\begin{aligned} &P_0\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) y^{(k+1)}(t) \\ &\quad + P_1\left(u(t), \ldots, u^{(k+1)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0, \end{aligned} \tag{4.13}$$

where $P_0 = \dfrac{\partial P}{\partial L_k}$ and

$$P_1 = \sum_{i=0}^{k} \frac{\partial P}{\partial S_i} S_{i+1} + \sum_{i=0}^{k-1} \frac{\partial P}{\partial L_i} L_{i+1}.$$

Notice that

$$\deg_{L_k} P_0 < \deg_{L_k} P.$$

It follows from the minimality of $\deg_{L_k} P$ that

$$P_0 = 0$$

is not an i/o equation of F_c (recall that this is a requirement in the definition of rationality). Thus F_c satisfies an order $(k+1)$ rational i/o equation, namely, equation (4.13). ∎

Note that without the assumption of minimality of $\deg_{L_k} P$, (4.13) would not necessarily give an i/o equation. As an illustration for this, take F_c as an example, where c is defined as in Example 2.2.8:

$$c = 1 + \frac{1}{2}\eta_1 - \frac{1}{2}\eta_1^{(2)} + \frac{3}{2^2}\eta_1^{(3)} + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{2^{n-1}}\eta_1^{(n)} + \cdots.$$

We have shown in Example 2.2.8 that

$$c ⧢ c = 1 + \eta_1.$$

So

$$\eta_1^{-1} c ⧢ c = \frac{1}{2}\eta_1^{-1}(c ⧢ c) = 1.$$

Since

$$\frac{d}{dt} F_c[u](t) = F_{\eta_0^{-1}c}[u](t) + F_{\eta_1^{-1}c}[u](t)\, u(t) = F_{\eta_1^{-1}c}[u](t)\, u(t),$$

it follows that

$$\begin{aligned} y'(t)\, y(t) &= u(t) F_{\eta_1^{-1}c}[u](t)\, F_c[u](t) \\ &= u(t) F_{\eta_1^{-1}c ⧢ c}[u](t) = \frac{1}{2} u(t). \end{aligned}$$

Now let

$$P(S_0, L_0, L_1) = L_0^2 L_1^2 - S_0 L_0 L_1 + \frac{1}{4} S_0^2.$$

Then

$$P(u,y,y') = y^2y'^2 - uyy' + \frac{1}{4}u^2 = yy'(yy' - \frac{1}{2}u) + \frac{1}{2}(yy' + \frac{1}{2}u) = 0 \tag{4.14}$$

which implies that $P = 0$ is an algebraic i/o equation of F_c. Taking the derivatives with respect to t on both sides of (4.14), we get

$$2y(yy' - \frac{1}{2}u)y'' + 2yy'^3 - uy'^2 - u'yy' + \frac{1}{2}uu' = 0. \tag{4.15}$$

But

$$P(S_0, S_1, L_0, L_1, L_2) = 2L_0(L_0L_1 - \frac{1}{2}S_0)L_2 + 2L_0L_1^3 - S_0L_1^2 - S_1L_0L_1 + \frac{1}{2}S_0S_1 = 0$$

is not a rational i/o equation of F_c, though it is linear in L_2, since the coefficient of L_2 vanishes for every i/o pair of F_c, which is caused by the fact that $\deg_{L_1} P$ is not minimal.

4.2 Properties of I/O Equations

We start this section by introducing the field

$$K = \mathbb{R}(\{S_{ij},\ i = 1, \ldots, m, j \geq 1\})$$

obtained by adjoining the indeterminates S_{ij} to $\mathbb{R}$. Let $\mathcal{F}^K$, $\mathcal{A}^K$ be the *K-space* and *K-algebra* generated by $c_n(S_0, \ldots, S_{n-1})$ for all n. Let $\mathcal{Q}^K$ be the quotient field of $\mathcal{A}^K$.

Remark 4.2.1 The field $\mathcal{Q}^K$ is defined since $\mathcal{A}^K$ is an integral domain. The reason for this is essentially because $\mathcal{A}^K$ can be naturally identified to the tensor product $\mathcal{A}_2 \otimes K$. We prove it explicitely next. Assume that there exist some nonzero D, $E \in \mathcal{A}^K$ such that $D \cdot E = 0$. Suppose

$$D, E \in \mathcal{S}[S_{i_1j_1}, S_{i_2j_2}, \ldots, S_{i_pj_p}],$$

the ring of polynomials in

$$\Sigma = \{S_{i_1j_1}, S_{i_2j_2}, \ldots, S_{i_pj_p}\}$$

over $\mathcal{S}$. Order the elements of Σ as $X_1, X_2, \ldots, X_q$ and define

$$\Gamma = \{X_{i_1} X_{i_2} \cdots X_{i_n} : 1 \leq i_1 \leq i_2 \leq i_n \leq q, n \geq 0\}.$$

Use X_ι to denote $X_{i_1} \cdots X_{i_l}$ if $\iota = i_1 \cdots i_l$. Order the elements $X_{i_1} X_{i_2} \cdots X_{i_n}$ of Γ lexicographically with respect to

$$n, i_1, i_2, \ldots, i_n.$$

Assume X_ι and X_κ are smallest elements of Γ appearing in D and E, and let their coefficients be c_1 and c_2 respectively. Let X_τ be the element of Γ corresponding to the product of X_ι and X_κ. Then the coefficient of X_τ in $D \cdot E$ is $c_1 \cdot c_2$. Since $\mathcal{S}$ is an integral domain, it follows that $c_1 \cdot c_2 \neq 0$, so $D \cdot E$ cannot be 0. □

Lemma 4.2.2 Assume c is a convergent power series. Then:

(a) $\mathcal{F}^K$ is a finite dimensional K-space if F_c satisfies an affine i/o equation.

(b) $\mathcal{A}^K$ is a finitely generated K-algebra if F_c satisfies a recursive i/o equation.

(c) $\mathcal{Q}^K$ is a finitely generated field extension of K if F_c satisfies an algebraic i/o equation.

Proof. We begin to prove the Lemma by making some general remarks. Observe first that if F_c satisfies an i/o equation

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0 \tag{4.16}$$

of order k, then by taking derivatives r times with respect to t on both sides of equation (4.16), one can see that F_c satisfies an i/o equation of order $k+r$ for any $r \geq 0$. Moreover, the i/o of order $k+r$ obtained by taking derivatives is the same type as the type of (4.16), i.e., if F_c satisfies an affine (recursive, rational, respectively) equation of order k, then, for any $r \geq 0$, F_c satisfies an affine (recursive, rational, respectively) i/o equation of order $k+r$, with

$$P_0 = \frac{\partial P}{\partial L_k}$$

as coefficient of $y^{(k+r)}$. For rational equations, the nondegeneracy property (4.6) still holds. Indeed, assume that F_c satisfies the ration i/o equation

$$P_0(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t))\, y^{(k)}(t)$$
$$= P_1(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t)). \tag{4.17}$$

Taking derivative with respect to time t on both sides of the equation, one gets

$$P_0(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t))\, y^{(k+1)}(t)$$
$$= P_2(u(t), \ldots, u^{(k+1)}(t), y(t), \ldots, y^{(k)}(t)),$$

where

$$P_2 = -\sum_{i=0}^{k} S_{i+1}\frac{\partial P_0}{\partial S_i} - \sum_{i=0}^{k-1} L_{i+1}\frac{\partial P_0}{\partial S_i} + \sum_{i=0}^{k} S_{i+1}\frac{\partial P_1}{\partial S_i} + \sum_{i=0}^{k-1} L_{i+1}\frac{\partial P_1}{\partial S_i}$$

By induction, one can show that for any $r > 0$, there exists some polynomial

$$P_{r+1}(S_0, \ldots, S_{k+r}, L_0, \ldots, L_{k+r-1})$$

so that

$$P_0(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t))\, y^{(k+r)}(t)$$
$$= P_{r+1}(u(t), \ldots, u^{(k+r)}(t), y(t), \ldots, y^{(k+r-1)}(t)). \tag{4.18}$$

It follows from the fact that (4.17) is a rational i/o equation of F_c that $P_0 = 0$ is not an i/o equation of F_c. Therefore, (4.18) is a rational i/o equation of F_c of order $k + r$, and (4.6) holds for this equation as well.

Secondly, consider $\hat{\mathcal{F}}^K$ and $\hat{\mathcal{A}}^K$, the K-space and K-algebra generated by

$$F_{c_n(S_0,\ldots,S_{n-1})}$$

for all n. By Lemma 2.2.4, the assignment $\psi : c_n(\mu_0, \cdots \mu_{n-1}) \mapsto F_{c_n(\mu_0,\ldots,\mu_{n-1})}$ is an isomorphism from $\mathcal{F}_2(c)$ onto $\hat{\mathcal{F}}_2(c)$; the $\mathbb{R}$-space generated by $F_{c_n(\mu_0,\ldots,\mu_{n-1})}$. Thus ψ induces an isomorphism from $\mathcal{F}^K$ onto $\hat{\mathcal{F}}^K$. By Lemma 2.2.9 we know that ψ also induces a ring isomorphism from $\mathcal{A}^K$ onto $\hat{\mathcal{A}}^K$. Consequently, $\hat{\mathcal{Q}}^K$, the quotient field of $\hat{\mathcal{A}}^K$, is isomorphic to $\mathcal{Q}^K$.

We now prove (a). Let F_c satisfy equation (4.16) with P as in (4.4). It follows from Lemma 4.1.4 that

$$\begin{aligned} & P_0(S_0,\ldots,S_k)F_{c_k(S_0,\ldots,S_{k-1})} \\ = & -\sum_{i=1}^{k} P_i(S_0,\ldots,S_k)F_{c_i(S_0,\ldots,S_{i-1})} - P_{k+1}(S_0,\ldots,S_k). \end{aligned}$$

Notice that

$$P_0(S_0,\ldots,S_k) \neq 0,$$

so

$$F_{c_k(S_0,\ldots,S_{k-1})} \in \hat{\mathcal{F}}_{k-1}^K + \operatorname{span}_K\{1\},$$

where $\hat{\mathcal{F}}_p^K$ denotes the K-space generated by

$$F_c, F_{c_1(S_0)}, \ldots, F_{c_{p-1}(S_0,\ldots,S_{p-2})}$$

for any $p \geq 0$. Therefore,

$$\hat{\mathcal{F}}_k^K \subseteq \hat{\mathcal{F}}_{k-1}^K + \operatorname{span}_K\{1\}.$$

By the remark made at the beginning of the proof, F_c satisfies an affine i/o equation of order $k+1$. By the same argument, one can show that

$$F_{c_{k+1}(S_0,\ldots,S_k)} \in \hat{\mathcal{F}}_k^K + \operatorname{span}_K\{1\} \subseteq \hat{\mathcal{F}}_{k-1}^K + \operatorname{span}_K\{1\},$$

and, therefore,

$$\hat{\mathcal{F}}_{k+1}^K \subseteq \hat{\mathcal{F}}_{k-1}^K + \operatorname{span}_K\{1\}.$$

By induction, it can be proved that

$$\hat{\mathcal{F}}_{k+r}^K \subseteq \hat{\mathcal{F}}_{k-1}^K + \operatorname{span}_K\{1\},$$

for any $r \geq 0$. Thus,

$$\hat{\mathcal{F}}^K \subseteq \hat{\mathcal{F}}_{k-1}^K + \operatorname{span}_K\{1\}.$$

Since $\hat{\mathcal{F}}_{k-1}^K$ is a finite dimensional K-space, it follows that $\hat{\mathcal{F}}^K$ is a finite dimensional K-space, and hence, $\mathcal{F}^K$ is a finite dimensional K-space.

Part (b) can be proved via the same arguments, by showing that

$$\hat{\mathcal{A}}^K = \hat{\mathcal{A}}_{k-1}^K$$

if F_c satisfies a recursive i/o equation of order k, where $\hat{\mathcal{A}}_k^K$ is the K-algebra generated by $\hat{\mathcal{F}}_{k-1}^K$.

Part (c): Suppose F_c satisfies an algebraic i/o equation. Then, it satisfies a rational equation as in (4.2). Again, by Lemma 4.1.4, we know that

$$\begin{aligned} &P_0\left(S_0,\ldots,S_{k-1},F_c,\ldots,F_{c_{k-1}(S_0,\ldots,S_{k-2})}\right)F_{c_k(S_0,\ldots,S_{k-1})} \\ &= -P_1\left(S_0,\ldots,S_k,F_c,\ldots,F_{c_{k-1}(S_0,\ldots,S_{k-2})}\right). \end{aligned}$$

Notice that since $P_0 = 0$ is not an i/o equation of F_c, there must exist some vector $(\mu_0,\ldots,\mu_{k-1})$ such that

$$P_0\left(\mu_0,\ldots,\mu_{k-1},F_c,\ldots,F_{c_{k-1}(\mu_0,\ldots,\mu_{k-2})}\right) \neq 0$$

which in turn implies that

$$P_0\left(S_0,\ldots,S_{k-1},F_c,\ldots,F_{c_{k-1}(S_0,\ldots,S_{k-2})}\right) \neq 0$$

as a polynomial in $S_0,\ldots,S_{k-1}$. It follows from this discussion that

$$F_{c_k(S_0,\ldots,S_{k-1})} \in \hat{\mathcal{Q}}_{k-1}^K,$$

where $\hat{\mathcal{Q}}_r^K$ denotes the quotient field of the K-algebra generated by $\hat{\mathcal{F}}_r^K$ for any $r \geq 0$. By using the same method used in part (a), one can prove that

$$\hat{\mathcal{Q}}^K = \hat{\mathcal{Q}}_{k-1}^K.$$

Since $\hat{\mathcal{Q}}_{k-1}^K$ is a finitely generated field extension of K, — the generators are the coefficients of S_{ij} $i = 1,\ldots,m$; $j = 0,1,\ldots,k-2$, in $F_c, F_{c_1},\ldots,F_{c_{k-1}}$,— we get our conclusion that $\mathcal{Q}^K$ is also a finitely generated field extension of K. ∎

Lemma 4.2.3 Assume that c is a convergent power series. Then

(a) $\mathcal{F}_2(c)$ is a finite dimensional $\mathbb{R}$-space if $\mathcal{F}^K$ is a finite dimensional K-space.

(b) $\mathcal{A}_2(c)$ is a finitely generated $\mathbb{R}$-algebra if $\mathcal{A}^K$ is a finitely generated K-algebra.

(c) $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$ if $\mathcal{Q}^K$ is a finitely generated field extension of K.

To prove the Lemma, we need to establish first two technical Lemmas.

Lemma 4.2.4 Let $\mathcal{A}$, $\mathcal{L}$ be vector space over $\mathbb{R}$ with $\mathcal{L}$ a space of functions $Z \to \mathbb{R}$ for some set Z. Assume that

$$c_1, c_2, \ldots, c_n$$

are linearly independent elements of $\mathcal{L}$ and let

$$a_1, a_2, \ldots, a_n$$

be n elements of $\mathcal{A}$. Then the linear subspace G_1 of $\mathcal{A}$ generated by

$$\left\{\sum c_i(z)\, a_i : z \in Z\right\}$$

coincides with the subspace G_2 generated by $a_1, \ldots, a_n$.

Proof. Clearly we have

$$G_1 \subseteq G_2$$

since $\sum c_i(z)\, a_i \in G_2$ for each $z \in Z$. It suffices to show that for each $a_i \in \mathcal{A}$, say a_1, there exist some constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$a_1 = \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} c_i(z_j)\, a_i \tag{4.19}$$

for some $z_1, \ldots, z_n \in Z$. Rewriting (4.19) as

$$a_1 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_j\, c_i(z_j) \right) a_i,$$

one sees that (4.19) is solvable if $\operatorname{rank} T = n$, where

$$T = \begin{pmatrix} c_1(z_1) & c_1(z_2) & \ldots & c_1(z_n) \\ c_2(z_1) & c_2(z_2) & \ldots & c_2(z_n) \\ \vdots & \vdots & \ddots & \vdots \\ c_n(z_1) & c_n(z_2) & \ldots & c_n(z_n) \end{pmatrix}. \tag{4.20}$$

In that case, one can choose λ so that

$$T \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which implies (4.19). Hence, our conclusion will follow from the following fact: For the n linearly independent functions $c_1(z), \ldots, c_n(z)$, one can always find n points $z_1, \ldots, z_n \in Z$ so that the matrix defined in (4.20) is nonsingular. We prove this fact by using induction on n.

The fact is trivial for the case $n = 1$, since $\dim_{\mathbf{R}} \operatorname{span}\{c_1(z)\} = 1$ implies that there is at least one point z so that $c_1(z) \neq 0$.

Now assume that the conclusion is true for $n-1$, but it is false for n. By the induction assumption, there exist $z_1, \ldots, z_{n-1} \in Z$ so that the matrix

$$T_1 = \begin{pmatrix} c_1(z_1) & c_1(z_2) & \cdots & c_1(z_{n-1}) \\ c_2(z_1) & c_2(z_2) & \cdots & c_2(z_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1}(z_1) & c_{n-1}(z_2) & \cdots & c_{n-1}(z_{n-1}) \end{pmatrix}$$

is full rank. Since for any $z \in Z$ the matrix

$$T = \begin{pmatrix} c_1(z_1) & c_1(z_2) & \cdots & c_1(z) \\ c_2(z_1) & c_2(z_2) & \cdots & c_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ c_n(z_1) & c_n(z_2) & \cdots & c_n(z) \end{pmatrix}$$

is singular, there exist $n-1$ functions $b_1(z), \ldots, b_{n-1}(z)$ so that

$$\begin{pmatrix} c_1(z) \\ c_2(z) \\ \vdots \\ c_n(z) \end{pmatrix} = \sum_{i=1}^{n-1} b_i(z) \begin{pmatrix} c_1(z_i) \\ c_2(z_i) \\ \vdots \\ c_n(z_i) \end{pmatrix}$$

which implies that

$$\operatorname{span}_{\mathbf{R}} \{c_1(\cdot), \ldots, c_n(\cdot)\} = \operatorname{span}_{\mathbf{R}} \{b_1(\cdot), \ldots, b_{n-1}(\cdot)\} ,$$

a contradiction to the fact that $c_1, \ldots, c_n$ are independent. Therefore, one can always choose $z_1, \ldots, z_n \in Z$ so that the matrix T defined in (4.20) is full rank, as desired. ∎

We now give the second technical Lemma needed for proving Lemma 4.2.3.

Lemma 4.2.5 Let $\mathcal{A}$ be a $\mathbb{R}$-algebra and take a polynomial Q in $\mathcal{A}[T_1, \ldots, T_r]$, for some $r \geq 0$. Let D be a dense subset of $\mathbb{R}^r$. Then the linear subspace of $\mathcal{A}$ spanned by

$$\{Q(v) : v \in D\}$$

is equal to the linear span of

$$\{Q(v) : v \in R^r\}.$$

Proof. We may assume that $Q \neq 0$ and write

$$Q(T_1, \ldots, T_r) = \sum_{i=1}^{s} a_i c_i(T_1, \ldots, T_n)$$

with all $c_i \in \mathbb{R}[T_1, \ldots, T_r]$ and all $a_i \in \mathcal{A}$, and with s smallest possible. Minimality of s implies that both

$$\{a_1, \ldots, a_s\} \text{ and } \{c_1, \ldots, c_s\}$$

are linearly independent over $\mathbb{R}$. By the continuity of the functions c_i we know that $c_i|_D$ are independent functions defined on D, where $c_i|_D$ denotes the restriction of c_i to D. Applying Lemma 4.2.4, we know that

$$\operatorname{span}_{\mathbb{R}} \left\{ \sum_{i=1}^{s} a_i c_i(v) : v \in D \right\} = \operatorname{span}_{\mathbb{R}} \{a_1, a_2, \ldots, a_s\}.$$

Applying Lemma 4.2.4 again, we know that

$$\operatorname{span}_{\mathbb{R}} \left\{ \sum_{i=1}^{s} a_i c_i(v) : v \in \mathbb{R}^r \right\} = \operatorname{span}_{\mathbb{R}} \{a_1, a_2, \ldots, a_s\}$$

which implies that

$$\operatorname{span}_{\mathbb{R}} \left\{ \sum_{i=1}^{s} a_i c_i(v) : v \in D \right\} = \operatorname{span}_{\mathbb{R}} \left\{ \sum_{i=1}^{s} a_i c_i(v) : v \in \mathbb{R}^r \right\},$$

as we wanted. ∎

We are now ready to prove Lemma 4.2.3.

Proof. (a): The fact that $\mathcal{F}^K$ is a finite dimensional K-space implies that, for some fixed integer $n > 0$, it holds for any $r \geq 0$, there exist $Q_1, \ldots, Q_n \in K$ such that

$$c_{n+r}(S_0, S_1, \ldots, S_{n+r-1}) = \sum_{i=1}^{n} Q_i c_{n-i}(S_0, S_1, \ldots, S_{n-i-1}). \tag{4.21}$$

Without loss of generality, we may assume that each $Q_i \in \mathbb{R}(S_0, \ldots, S_k)$ and

$$Q_i = \frac{P_i}{P_0},$$

where P_i's are polynomials and $P_0 \neq 0$. Multiplying by P_0 on both sides of (4.21), we get

$$P_0(S_0, \ldots, S_k) c_{n+r}(S_0, \ldots, S_{n+r-1}) = \sum_{i=1}^{n} P_i(S_0, \ldots, S_k) c_{n-i}(S_0, \ldots, S_{n-i-1}).$$

From the fact that $c_i(S_0, \ldots, S_{i-1})$ does not depend on S_i, we may assume that $k = n + r - 1$. Indeed, if this is not the case, we can see that

$$\begin{aligned} &P_0(S_0, \ldots, S_{n+r-1}, \mu_{n+r}, \ldots, \mu_k) c_{n+r}(S_0, \ldots, S_{n+r-1}) \\ &= \sum_{i=1}^{n} P_i(S_0, \ldots, S_{n+r-1}, \mu_{n+r}, \ldots, \mu_k) c_{n-i}(S_0, \ldots, S_{n-i-1}) \end{aligned}$$

holds for all $\mu_{n+r}, \ldots, \mu_k$. Choose $\bar{\mu}_{n-r}, \ldots, \bar{\mu}_k$ such that

$$P_0(S_0, \ldots, S_{n+r-1}, \bar{\mu}_{n+r}, \ldots, \bar{\mu}_k) \neq 0.$$

Replacing

$$P_i(S_0, \ldots, S_{n+r-1}, S_{n+r}, \ldots, S_k)$$

by

$$P_i(S_0, \ldots, S_{n+r-1}, \bar{\mu}_{n+r}, \ldots, \bar{\mu}_k),$$

one gets

$$\begin{aligned} &P_0(S_0, \ldots, S_{n+r-1}) c_{n+r}(S_0, \ldots, S_{n+r-1}) \\ &\qquad = \sum_{i=0}^{n} P_i(S_0, \ldots, S_{n+r-1}) c_{n-i}(S_0, \ldots, S_{n-i-1}). \end{aligned}$$

This proves that we can assume $k = n + r - 1$, as claimed. Now let

$$\Omega = \left\{ \mu_0, \ldots, \mu_{n+r-1} \, : \, P_0(\mu_0, \ldots, \mu_{n+r-1}) \neq 0 \right\}. \tag{4.22}$$

It follows that Ω is an open dense subset of $\mathbb{R}^{m \times (n+r)}$ since P_0 is a polynomial in the μ'_{ij}s. For $(\mu_0, \ldots, \mu_{n+r-1}) \in \Omega$,

$$c_{n+r}(\mu_0, \ldots, \mu_{n+r-1}) \in \mathcal{K}_{n-1},$$

where

$$\mathcal{K}_{n-1} = \operatorname{span}_{\mathbb{R}} \left\{ c_i(\mu_0, \ldots, \mu_{i-1}) : \mu_0, \ldots, \mu_{i-1} \in \mathbb{R}^m, \ 0 \le i \le n-1 \right\}.$$

Notice that $c_{n+r}(\mu_0, \ldots, \mu_{n+r-1})$ is a polynomial in μ^{n+r-1} over the ring $\mathcal{S}$, where

$$\mu^{n+r-1} = (\mu_0, \ldots, \mu_{n+r-1}).$$

By Lemma 4.2.5 we have

$$\begin{aligned} &\operatorname{span}_{\mathbb{R}} \left\{ c_{n+r}(\mu^{n+r-1}) : \mu^{n+r-1} \in \mathbb{R}^{m \times (n+r)} \right\} \\ &\qquad = \operatorname{span}_{\mathbb{R}} \left\{ c_{n+r-1}(\mu^{n+r-1}) : \mu^{n+r} \in \Omega \right\}. \end{aligned} \tag{4.23}$$

Thus

$$c_{n+r}(\mu^{n+r-1}) \in \mathcal{K}_{n-1}, \quad \text{for all } \mu^{n+r-1} \in \mathbb{R}^{m \times (n+r)}.$$

Since r can be chosen arbitrarily, we reach the conclusion that

$$\mathcal{F}_2(c) = \mathcal{K}_{n-1}.$$

Now let G_i be the subset of $\mathcal{S}$ consists of all the coefficients of μ_{i_j}'s in c_i and let $\mathcal{G}_i$ be the $\mathbb{R}$-space spanned by elements of G_i. Since $c_i(\mu^{i-1})$ is a polynomial in μ^{i-1}, it follows from Lemma 4.2.4 that

$$\mathcal{G}_i = \operatorname{span}_{\mathbb{R}} \left\{ c_i(\mu^{i-1}) : \mu^{i-1} \in \mathbb{R}^{m \times i} \right\}.$$

So we get

$$\mathcal{K}_{n-1} = \bigcup_{i=0}^{n-1} \mathcal{G}_i.$$

Since each G_i is a set of finitely many elements, $\mathcal{K}_{n-1}$ is a finite dimensional $\mathbb{R}$-space, which implies that $\mathcal{F}_2(c)$ is a finite dimensional $\mathbb{R}$-space.

(b): Assume $\mathcal{A}^K$ is a finitely generated $\mathbb{R}$-algebra. Similar to part (a), one can show that there exists some integer $n > 0$ such that for each integer $r \ge 0$, there exist polynomials P_0 and P_1 so that

$$\begin{aligned} &P_0(S_0, \ldots, S_{n+r-1})\, c_{n+r-1}(\mu^{n+r}) \\ &\qquad = P_1\left(S_0, \ldots, S_{n+r-1}, c_0, c_1(\mu^0), \ldots, c_{n+r-1}\left(\mu^{n+r-2}\right)\right). \end{aligned}$$

Thus

$$c_{n+r}(\mu^{n+r-1}) \in \mathcal{B}_{n-1}$$

for $\mu^{n+r} \in \Omega$, where $\mathcal{B}_{n-1}$ is the $\mathbb{R}$-algebra generated by $\mathcal{K}_{n-1}$ and Ω is the same as defined in (4.22). Using (4.23) again, we get

$$c_{n+r}(\mu^{n+r-1}) \in \mathcal{B}_{n-1}$$

for all μ^{n+r}. Since $\mathcal{A}_2(c)$ is generated by all the $c_k(\mu^k)$, we get the conclusion that

$$\mathcal{A}_2(c) = \mathcal{B}_{n-1}.$$

Notice that $\mathcal{B}_{n-1}$ is the $\mathbb{R}$-algebra generated by $\mathcal{K}_{n-1}$ and, in turn, $\mathcal{K}_{n-1}$ is spanned by the elements of G where

$$G = \bigcup_{i=0}^{n-1} G_i.$$

Thus $\mathcal{B}_{n-1}$ is an $\mathbb{R}$-algebra generated by the elements of G; therefore, $\mathcal{A}_2(c)$ is a finitely generated $\mathbb{R}$-algebra.

(c): Assume that $\mathcal{Q}^K$ is a finitely generated field extension of K. Then there exists some $n > 0$ so that for any $r \geq 0$, there exist two polynomials Q_0, Q_1 over K with

$$Q_0\,(c_0, c_1(S_1), \ldots, c_{n-1}(S_0, \ldots, S_{n-2})) \neq 0$$

such that

$$\begin{aligned} Q_0\,(c_0, c_1(S_0), \ldots, c_{n-1}(S_0, \ldots, S_{n-1})\; c_{n+r}(S_0, \ldots, S_{n+r-1}) \\ = Q_1\,(c_0, c_1(S_0), \ldots, c_{n-1}(S_0, S_1, \ldots, S_{n-2})) \end{aligned}$$

After clearing denominators and getting rid of extra μ_j's as we did in part (a), there results an equation

$$\begin{aligned} & P_0\,(S_0, \ldots, S_{n+r-1}, c_0, c_1(S_0), \ldots, c_{n-1}(S_0, \ldots, S_{n-2}))\; c_{n+r}(S_0, \ldots, S_{n+r-1}) \\ = {} & P_1\,(S_0, \ldots, S_{n+r-1}, c_0, c_1(S_0), \ldots, c_{n-1}(S_0, \ldots, S_{n-2})) \end{aligned}$$

with

$$P_0\,(S_0, \ldots, S_{n+r-1}, c_0, c_1(S_0), \ldots, c_{n-1}(S_0, \ldots, S_{n-2})) \neq 0,$$

which implies that there exists some $(\mu_0, \ldots, \mu_{n+r-1})$ so that

$$P_0\left(\mu_0, \ldots, \mu_{n+r-1}, c_0, c_1(\mu_0), \ldots, c_{n-1}(\mu_0, \ldots, \mu_{n-2})\right) \neq 0,$$

or equivalently,

$$P_0\left(\mu_0, \ldots, \mu_{n+r-1}, F_c, F_{c_1(\mu_0)}, \ldots, F_{c_{n-1}(\mu_0, \ldots, \mu_{n-2})}\right) \neq 0.$$

This is an equation involving operators. Its meaning is that there exists some $u \in \mathcal{V}_T$, where T is admissible to c, and t such that

$$P_0\left(\mu_0, \ldots, \mu_{n+r-1}, F_c[u](t), \ldots, F_{c_{n-1}(\mu_0, \ldots, \mu_{n-2})}[u](t)\right) \neq 0.$$

Since for each fixed u and t,

$$P_0\left(\mu_0, \ldots, \mu_{n+r-1}, F_c[u](t), \ldots, F_{c_{n-1}(\mu_0, \ldots, \mu_{n-2})}[u](t)\right)$$

is a polynomial in μ^{n+r-1}, the set

$$\Omega_1 := \left\{\mu^{n+r-1} : P_0\left(\mu^{(n+r-1)}, F_c[u](t), \ldots, F_{c_{n-1}(\mu^{n-2})}[u](t)\right) \neq 0.\right\}$$

is dense in $\mathbb{R}^{m\times(n+r)}$. Define

$$\Omega = \left\{\mu^{n+r-1} : P_0\left(\mu^{n+r-1}, c_0, \ldots, c_{n-1}(\mu^{n-2})\right) \neq 0\right\}.$$

Then

$$\Omega_1 \subseteq \Omega.$$

It follows immediately that Ω is dense in $\mathbb{R}^{n+r}$.

By using the same techniques used in part (a) and part (b), one proves that

$$\mathcal{Q}_2(c) = \mathcal{T}_{n-1},$$

where $\mathcal{T}_{n-1}$ is the quotient field of $\mathcal{B}_{n-1}$. It follows that $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. ∎

Combining Lemma 4.2.2 and 4.2.3, we get our main result in this section:

Theorem 5 *Assume that c is a convergent power series. Then* *(a) $\mathcal{F}_2(c)$ is a finite dimensional $\mathbb{R}$-space if F_c satisfies an affine i/o equation.*

(b) $\mathcal{A}_2(c)$ is a finitely generated $\mathbb{R}$-algebra if F_c satisfies a recursive i/o equation.

(c) $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$ if F_c satisfies an algebraic i/o equation. ∎

Noticing that a linear i/o equation is in particular an affine one and a rational one is in particular an algebraic one, we get the following:

Corollary 4.2.6 Assume c is a convergent power series. Then

(a) $\mathcal{F}_2(c)$ is a finite dimensional $\mathbb{R}$-space if F_c satisfies a linear i/o equation.

(b) $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$ if F_c satisfies a rational i/o equation. ∎

Remark 4.2.7 Generally speaking, a field extension over $\mathbb{R}$ with finite transcendence degree is not necessarily a finitely generated field extension of $\mathbb{R}$. But by using Theorem 5, one can show that if the transcendence degree of $\mathcal{Q}_2(c)$ is finite, then it follows that $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. The reasoning is as follows:

Assume that

$$\operatorname{trdeg}_{\mathbb{R}} \mathcal{Q}_2(c) < \infty,$$

where $\operatorname{trdeg}_{\mathcal{K}} \mathcal{Q}$ denotes the transcendence degree of $\mathcal{Q}$ over $\mathcal{K}$ for any fields $\mathcal{Q}$ and $\mathcal{K}$.

Now let $\mathcal{L}_n$ be the set of all the coefficients of $c_n(S_0, \ldots, S_{n-1})$, seen as a polynomial in $S_0, \ldots, S_{n-1}$ over $\mathcal{S}$, the ring of all series. Let

$$\mathcal{L} = \bigcup_n \mathcal{L}_n .$$

Then, by Lemma 4.2.4, we know that $\mathcal{Q}_2(c) = \mathbb{R}(\mathcal{L})$. On the other hand, $\mathcal{Q}^K = K(\mathcal{L})$. Therefore,

$$\operatorname{trdeg}_{\mathbb{R}} \mathcal{Q}_2(c) < \infty$$

implies that

$$\operatorname{trdeg}_K \mathcal{Q}^K < \infty. \tag{4.24}$$

If (4.24) holds, then there exists some n such that

$$c,\ c_1(S_0),\ \ldots,\ c_n(S_0,\ldots,S_{n-1})$$

are algebraically dependent over K, i.e., there exists some polynomial P over K such that

$$P\,(c,\ c_1(S_0),\ldots,\ c_n(S_0,\ldots,S_{n-1})) = 0.$$

After clearing denominators and getting rid of the extra S_{ij}, one gets the following equation:

$$Q\,(S_0,\ldots,S_k,\ c,\ c_1(S_0),\ \ldots,\ c_n(S_0,\ldots,S_{n-1})) = 0\,. \tag{4.25}$$

Notice that if a convergent series c satisfies (4.25), then (4.25) is an algebraic i/o equation of F_c. Then, by Theorem 5, $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. Hence, the above discussion shows that that if the transcendence degree of $\mathcal{Q}_2(c)$ over $\mathbb{R}$ is finite, then $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. □

We conclude this section by the following example.

Example 4.2.8 Let c be the series defined in Example 2.2.8:

$$c = 1 + \frac{1}{2}\eta_1 - \frac{1}{2}\eta_1^{(2)} + \frac{3}{2^2}\eta_1^{(3)} + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{2^{n-1}}\eta_1^{(n)} + \cdots.$$

We have shown before that F_c satisfies the i/o equation

$$y'(t)\,y(t) \;=\; \frac{1}{2}\,u(t).$$

By Theorem 5, we know that $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. Below we shall find out the field $\mathcal{Q}_2(c)$ explicitely.

First of all, we knew that

$$c \sqcup\!\sqcup (\eta_1^{-1}c) \;=\; \frac{1}{2},$$

in other words,

$$\eta_1^{-1}c \;=\; \frac{1}{2\,c}. \tag{4.26}$$

Claim: If a series d can be written as $\dfrac{c_1}{c_2}$ for some series c_1, c_2, then

$$z^{-1}\,d = \frac{c_2 \sqcup\!\sqcup z^{-1}c_1 - c_1 \sqcup\!\sqcup z^{-1}c_2}{c_2^2} \tag{4.27}$$

for any $z \in P$. The proof goes as follows: to say that

$$d = \frac{c_1}{c_2}$$

is equivalent to say that

$$d \sqcup\!\sqcup c_2 = c_1.$$

So, by Lemma 2.1.2, we have

$$z^{-1} d \sqcup\!\sqcup c_2 + d \sqcup\!\sqcup z^{-1} c_2 = z^{-1} c_1, \quad \text{for any } z \in P,$$

which implies (4.27).

Combining (4.26) and (4.27), we get

$$(\eta_1\eta_1)^{-1} c = -\frac{\eta_1^{-1} c}{2\,c^2} = -\frac{1}{2^2\,c^3}.$$

It is not hard to see

$$\left(\eta_1^{(k)}\right)^{-1} c = \frac{(-1)^{k-1}}{2^k\,c^{2k-1}}$$

for any integer k. Since $\eta_0^{-1}\left(\eta_1^{(k)}\right)^{-1} c = 0$ for any integer k, it follows that

$$c_n(\mu_0, \ldots, \mu_{n-1}) = \left(\eta_1^{(n)}\right)^{-1} c\, \mu_{n-1}.$$

Therefore,

$$\mathcal{Q}_2(c) = \mathbb{R}\left(\left\{\left(\eta_1^{(k)}\right)^{-1} c\right\}_{k \geq 0}\right) = \mathbb{R}\,(c)$$

which is a field extension of $\mathbb{R}$ with one generator. □

4.3 I/O Equations for Families of I/O Operators

In this section, we study i/o equation for families of i/o operators. Suppose that

$$\underline{c} = \{c^\lambda : \lambda \in \Lambda\}$$

is a convergent family of generating series. We say that the family of i/o operator

$$\mathbf{F}_{\underline{c}} = \{F_{c^\lambda} : d \in \Lambda\}$$

satisfies an algebraic i/o equation if there exists some polynomial

$$P \in \mathbb{R}\,[S_0, \ldots, S_k, L_0, \ldots, L_k],$$

nontrivial in L_k so that the equation

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0 \tag{4.28}$$

is an i/o equation for F_{c^λ} for each $\lambda \in \Lambda$.

Similarly to the case of a single i/o operator, we say that $\mathbf{F}_{\underline{c}}$ satisfies a *recursive* (*affine, linear* respectively) equation if (4.28) is *recursive* (*affine, linear* respectively). A family $\mathbf{F}_{\underline{c}}$ of i/o operators satisfies an *rational* i/o equation if

$$\begin{aligned} & P(S_0, \ldots, S_k, L_0, \ldots, L_k) \\ &= P_0(S_0, \ldots, S_k, L_0, \ldots, L_{k-1})\, L_k + P_1(S_0, \ldots, S_k, L_0, \ldots, L_{k-1}) \end{aligned}$$

for some polynomials P_0 and P_1, and P_0 is not an i/o equation for $\mathbf{F}_{\underline{c}}$, i.e., there exists some $\lambda \in \Lambda$ and some i/o pair (u, y) of F_{c^λ} which does not satisfy equation (4.28).

For a family of generating series $\underline{c}$, we associate with it an observation algebra $\tilde{\mathcal{A}}_2(\underline{c})$ defined as the $\mathbb{R}$-algebra generated by the elements of $\tilde{\mathcal{F}}_2(\underline{c})$. Recall that $\tilde{\mathcal{F}}_2(\underline{c})$ is the $\mathbb{R}$-space generated by

$$\underline{c}_n(\mu_0, \ldots, \mu_{n-1})$$

for all n and all μ.

To be able to define the observation field, we need the assumption that $\tilde{\mathcal{A}}_2(\underline{c})$ is an integral domain.

Definition 4.3.1 We say that a convergent family

$$\underline{c} = \{c^\lambda : \lambda \in \Lambda\}$$

is an ***analytic*** family if Λ is a connected analytic manifold and $\langle c^\lambda, \eta_\iota \rangle$ is an analytic function defined on Λ for all $\iota \in P^*$. □

By Corollary 2.4.3, $\tilde{\mathcal{A}}_2(\underline{c})$ is an integral domain, therefore, its quotient field is well defined.

For an analytic family $\underline{c}$, we define the observation field $\tilde{\mathcal{Q}}_2(\underline{c})$ of $\underline{c}$ as the quotient field of $\tilde{\mathcal{A}}_2(\underline{c})$.

The following is our main result in this section.

Theorem 6 *Assume that $\underline{c}$ is an analytic family of power series. Then*

(a) $\tilde{\mathcal{F}}_2(\underline{c})$ *is a finite dimensional* $\mathbb{R}$*-space if* $\tilde{\mathcal{F}}_{\underline{c}}$ *satisfies an affine i/o equation.*

(b) $\tilde{\mathcal{A}}_2(\underline{c})$ *is a finitely generated* $\mathbb{R}$*-algebra if* $\tilde{\mathcal{F}}_{\underline{c}}$ *satisfies a recursive i/o equation.*

(c) $\tilde{\mathcal{Q}}_2(\underline{c})$ *is finitely generated field extension of* $\mathbb{R}$ *if* $\tilde{\mathcal{F}}_{\underline{c}}$ *satisfies an algebraic i/o equation.*

As in the proof of single i/o operator, we need to establish some lemmas to prove the Theorem.

We still use K to denote the field $\mathbb{R}(\{S_{ij}\})$, obtained by adjoining the indeterminates S_{ij} to $\mathbb{R}$, and in this section we again use $\mathcal{F}^K$, $\mathcal{A}^K$ to denote the K-space and K-algebra generated by families of series

$$\underline{c}_n(S_0, S_1, \ldots, S_{n-1})$$

for all n. By the same argument used in Remark 4.2.1, one shows that $\mathcal{A}^K$ is an integral domain if $\underline{c}$ is an analytic family. In this section we use $\mathcal{Q}^K$ to denote the quotient field of $\mathcal{A}^K$.

By following exactly the same steps in the proof of Lemma 4.2.2, one gets the following lemma.

Lemma 4.3.2 Assume that $\underline{c}$ is an analytic family of power series. Then:

(a) $\mathcal{F}^K$ is a finite dimensional K-space if $\mathbf{F}_{\underline{c}}$ satisfies an affine i/o equation.

(b) $\mathcal{A}^K$ is a finitely generated K-algebra if $\mathbf{F}_{\underline{c}}$ satisfies a recursive i/o equation.

(c) $\mathcal{Q}^K$ is a finitely generated field extension of K if $\mathbf{F}_{\underline{c}}$ satisfies an algebraic i/o equation.

The following lemma is an analoge of Lemma 4.2.3 for families of power series.

Lemma 4.3.3 Assume that $\underline{c}$ is an analytic series. Then

(a) $\tilde{\mathcal{F}}_2(\underline{c})$ is a finite dimensional $\mathbb{R}$-space if $\mathcal{F}^K$ is a finite dimensional K-space.

(b) $\tilde{\mathcal{A}}_2(\underline{c})$ is a finitely generated $\mathbb{R}$-algebra if $\mathcal{A}^K$ is a finitely generated K-algebra.

(c) $\tilde{\mathcal{Q}}_2(\underline{c})$ is a finitely generated field extension of $\mathbb{R}$ if $\mathcal{Q}^K$ is a finitely generated field extension of K.

Proof. The ideas of the proof are the same as those in the proof of Lemma 4.2.3. The only significant difference is to show that

$$\begin{aligned} &P_0\left(S_0,\ldots,S_{n+r-1},\underline{c}_0,\underline{c}_1(S_0),\ldots,\underline{c}_{n-1}(S_0,\ldots,S_{n-2})\right)\underline{c}_{n+r}(S_0,\ldots,S_{n+r-1}) \\ &= P_1\left(S_0,\ldots,S_{n+r-1},\underline{c}_0,\underline{c}_1(S_0),\ldots,\underline{c}_{n-1}(S_0,\ldots,S_{n-2})\right) \end{aligned} \tag{4.29}$$

implies that

$$\underline{c}_{n+r}(\mu_0,\ldots,\mu_{n+r-1}) \in \mathcal{T}_{n-1} \quad \text{for all } \mu_0,\ldots,\mu_{n+r-1} \in \mathbb{R}^m, \tag{4.30}$$

where $\mathcal{T}_{n-1}$ is the quotient field of $\mathcal{B}_{n-1}$, the K-algebra generated by elements

$$\underline{c},\ \underline{c}_1(\mu^0),\ \ldots,\ \underline{c}_{n-1}(\mu_0,\ \ldots,\ \mu_{n-1})$$

for all choices of $\mu_0, \ldots, \mu_{n-1}$.

Below we show how to prove this statement.

Assume that (4.29) holds. Then

$$\underline{c}_{n+r} \in \mathcal{T}_{n-1} \ \text{ if } \ (\mu_0,\ \ldots,\ \mu_{n+r-1}) \in \Omega, \tag{4.31}$$

where

$$\Omega = \left\{\mu^{n+r-1} :\ P_0\left(\mu^{n+r-1}, \underline{c},\ \underline{c}_1(\mu^0), \ldots, \underline{c}_{n+r-1}(\mu^{n+r-2})\right) \neq 0\right\}.$$

Since $P_0 = 0$ is not an i/o equation for $\mathbf{F}_{\underline{c}}$, there exists some $\lambda \in \Lambda$ so that

$$P_0\left(\mu_0,\ \ldots,\ \mu^{n+r-1}, \underline{c},\ F_{c^\lambda}, \ldots, F_{c^\lambda_{n1}(\mu_0,\ldots,\mu_{n-2})}\right) \neq 0.$$

We have shown in the proof of Lemma 4.2.3 that for such a choice λ, the set

$$\Omega_1 := \left\{\mu^{n+r-1} :\ P_0\left(\mu^{(n+r-1)}, c^\lambda,\ c_1^\lambda(\mu_0) \ldots, c_{n-1}^\lambda(\mu^{n-2})\right) \neq 0\right\}.$$

is dense in $\mathbb{R}^{m\times(n+r)}$. It follows from the fact that

$$\Omega_1 \subseteq \Omega$$

that Ω is dense in $\mathbb{R}^{m\times(n+r)}$. Applying Lemma 4.2.5 with $\mathcal{A}$ being the algebra of analytic families of series, we get the following:

$$\operatorname{span}_{\mathbb{R}}\left\{\underline{c}_{n+r}(\mu^{n+r-1}) : \mu^{n+r-1} \in \mathbb{R}^{m\times(n+r)}\right\}$$
$$= \operatorname{span}_{\mathbb{R}}\left\{\underline{c}_{n+r-1}(\mu^{n+r-1}) : \mu^{n+r} \in \Omega\right\}. \quad (4.32)$$

It then can be seen that (4.30) follows immediately from (4.32) and (4.31). ∎

Combining Lemma 4.3.2 and Lemma 4.3.3 concludes the proof of Theorem 6.

Corollary 4.3.4 Assume that $\underline{c}$ is an analytic family of power series. Then

(a) $\tilde{\mathcal{F}}_2(\underline{c})$ is a finite dimensional $\mathbb{R}$-space if $\mathbf{F}_{\underline{c}}$ satisfies a linear i/o equation.

(b) $\tilde{\mathcal{Q}}_2(\underline{c})$ is a finitely generated field extension of $\mathbb{R}$ if $\mathbf{F}_{\underline{c}}$ satisfies a rational i/o equation. ∎

Chapter 5

Realizability

We wish to study realization by "rational" systems, such as those studied in Bartosiewicz [3]. However, the question of possible poles in the right-hand side of the equation is very delicate, and it seems better to study instead a "singular" polynomial model, as we do next.

Just as i/o equations turn out to be related to the structure of $\mathcal{F}_2(c)$ $\mathcal{A}_2(c)$ and $\mathcal{Q}_2(c)$, realizability forces the study of the observation algebra and observation field corresponding to the other type of observation space, $\mathcal{F}_1(c)$. For a given power series c, we associate with it an *observation algebra* $\mathcal{A}_1(c)$ defined as the $\mathbb{R}$-algebra generated by the elements of $F_1(c)$, and associate with it an *observation field* $\mathcal{Q}_1(c)$ defined as the quotient field of $\mathcal{A}_1(c)$. Again, we know that $\mathcal{Q}_1(c)$ is defined since $\mathcal{A}_1(c)$ is an integral domain. It turns out, because of previous results that $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{Q}_1 = \mathcal{Q}_2$ for every c, but the facts in this Chapter do not depend on the equality, and they are more readily understood in terms of $\mathcal{A}_1$ and $\mathcal{Q}_1$.

5.1 Realizability of a Single I/O Operator

Definition 5.1.1 Suppose that c is a convergent series and T is admissible for c. The i/o operator F_c is *realizable* by a *singular polynomial state-space system*

$$\Sigma = ((g_0, \ldots, g_m),\, x_0,\, q,\, h)$$

if there exists an integer n, some $x_0 \in \mathbb{R}^n$, polynomial vector fields

$$g_0, g_1, \ldots, g_m$$

on $\mathbb{R}^n$, and two polynomial functions

$$q,\, h : \mathbb{R}^n \to \mathbb{R}$$

such that the following properties hold:

(a) For each $u \in \mathcal{V}_T$ and $y = F_c[u]$, there is some absolutely continuous function $x(\cdot)$ defined on $[0, T]$ and satisfying $x(0) = x_0$, such that

$$q(x(t))x'(t) = g_0(x(t)) + \sum_{i=1}^{m} u_i(t)g_i(x(t))$$

for almost all $t \in [0, T]$, and,

$$y(t) = h(x(t)),$$

for all $t \in [0, T]$.

(b) The solution $x(\cdot)$ in part (a) is of class C^ω if u is of class C^ω, and $x(\cdot)$ is of class C^{k+1} if u is of class C^k.

(c) There holds the following *regularity* condition: there exists some set Ω of analytic inputs which is dense in $C^\infty[0, T]$ (with respect to the Whitney topology) such that for any $u \in \mathcal{V}_T \cap \Omega^m$, there exists some C^ω solution $x(\cdot)$ as in (a) so that

$$q(x(\cdot)) \not\equiv 0.$$

If F_c can be realized by a singular polynomial system with

$$q(x) \equiv 1,$$

we say that F_c is realizable by a *polynomial* state-space system. If in addition, the vector fields $g_0, g_1, \ldots g_m$ are linear in x, then we say that F_c is realizable by a *bilinear* state-space system. □

It can be seen from the definition (5.1.1) that if

$$q(x) \neq 0$$

for any $x \in \mathbb{R}^n$, then F_c is realizable (globally) by an analytic system in the usual sense. If

$$q(x_0) \neq 0,$$

then F_c is realizable locally by an analytic system.

The nondegeneracy condition turns out to be equivalent (as shown in the proof below) to the fact that for "almost every" i/o pair it holds that

$$q(x(t)) \neq 0$$

for almost every t. It could happen, however, that q vanishes along some trajectories.

The following Theorem is our main result in this section. It constitutes a converse to Theorem 5, but in terms of different algebraic objects.

Theorem 7 *The following properties hold for a convergent power series c:*

(a) F_c is realizable by a bilinear system if $\mathcal{F}_1(c)$ is a finite dimensional $\mathbb{R}$-space.

(b) F_c is realizable by a polynomial system if $\mathcal{A}_1(c)$ is a finitely generated $\mathbb{R}$-algebra.

(c) F_c is realizable by a singular polynomial system if $\mathcal{Q}_1(c)$ is a finitely generated field extension of $\mathbb{R}$.

To prove this theorem, we need first to prove the following lemma. We consider the set of all polynomial inputs of degree less than or equal to $(n-1)$ with $\mathbb{R}^{mn}$ in the following way: for each $(n-1)$-degree polynomial input

$$u_a := a_1 + a_2 t + \cdots + a_n t^{n-1} \ ,$$

we associate with it the point

$$a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^{mn}.$$

Now let

$$\mathcal{W}_{n,T} = \left\{ (a_1, \ldots, a_n) : \left| \sum_{j=1}^{n} a_{ij} t^{j-1} \right| < 1, \quad \text{for} \quad 0 \leq t \leq T,\ 1 \leq i \leq m \right\} \subset \mathbb{R}^{mn}.$$

Then $\mathcal{W}_{n,T}$ is an open set of $\mathbb{R}^{mn}$. If $u_{a_1} \in \mathcal{W}_{n,T}$ and $u_{a_2} \in \mathcal{W}_{n,T}$, then $u_{a^s} \in \mathcal{W}_{n,T}$ for all $s \in [0, 1]$ where

$$a^s = s\, a_1 + (1-s)\, a_2.$$

Therefore $\mathcal{W}_{n,T}$ is a connected analytic manifold.

Lemma 5.1.2 Assume c is a convergent series and T is admissible for c. Then for polynomial inputs of degree less than or equal to $n-1$, $F_c[u_a](t_0)$ is an analytic function defined on $\mathcal{W}_{n,T}$ for any fixed $t_0 \in [0,\ T]$.

Proof. Suppose T is admissible for c as defined in (2.10). Take $a = (a_1, \dots, a_n) \in W_{n,T}$, viewed as a point in $\mathbb{C}^{mn}$. Then there exists a neighborhood $B_\delta(a)$ of a in $\mathbb{C}^{mn}$ such that

$$\left|\sum \beta_i t^{i-1}\right| < 1$$

for $(b_1, b_2, \dots, b_n) \in B_\delta(a)$. Use u_b to denote the polynomial control

$$b_1 + b_2 t + \dots + b_n t^{n-1}.$$

Then for any fixed $t_0 \in [0, T]$ we have the estimate

$$\left|\sum_{|\iota| \geq n} \langle c,\ \eta_\iota\rangle V_\iota[u_b](t_0)\right| \leq \sum_{k \geq n}^{\infty} KM^k (m+1)^k (1+\sigma)^k T^k, \tag{5.1}$$

for some K as defined in (2.6). Thus, the series of analytic complex functions

$$\sum_\iota \langle c,\ \eta_\iota\rangle V_\iota[u_b](t_0)$$

converges uniformly for $b \in B_\delta(a)$. Therefore,

$$\alpha(b) := \sum_\iota \langle c,\ \eta_\iota\rangle V_\iota[u_b](t_0) \tag{5.2}$$

is an analytic function in $B_\delta(a)$, which implies that $F_c[u_a](t_0)$ is analytic in a neighborhood of a in $\mathcal{W}_{n,T}$. The conclusion of the lemma follows since a can be chosen arbitrarily in $W_{n,T}$. ∎

Remark 5.1.3 Assume that T is admissible for the convergent series $c_1, \dots, c_k$ and q is an analytic function. Then for any $t_0 \in [0,\ T]$, the function

$$\gamma(a) := q(F_{c_1}[u_a](t_0), \dots, F_{c_k}[u_a](t_0))$$

depends analytically on $a \in \mathcal{W}_{n,T}$. Thus if $\gamma(a_0) \neq 0$, then

$$\gamma(a) \neq 0$$

in an open dense subset of $\mathcal{W}_{n,T}$. Noticing that the set of polynomial controls is dense in $\mathcal{V}_T$ (in the L^1 topology) and F_c is continuous in $\mathcal{V}_T$, it follows that if

$$q(F_{c_1}[u], \ldots, F_{c_k}[u]) \neq 0, \tag{5.3}$$

holds for some control in $\mathcal{V}_T$, then it holds for at least one polynomial control, which implies that (5.3) holds in a dense set of $\mathcal{V}_T$. Again, by the continuity of F_c, we know that the set on which F_c does not vanish is an open set of $\mathcal{V}_T$. So we get our conclusion that if (5.3) holds for some $u \in \mathcal{V}_T$, then it holds for every control in an open dense set of $\mathcal{V}_T$. $\square$

We now return to prove Theorem 7.

Proof. (a): Suppose

$$\mathcal{F}_1(c) = \operatorname{span}_{\mathbb{R}}\{c_1, c_2, \ldots, c_n\}.$$

Then for each $i \in \{0, 1, \ldots, m\}$, there exist some linear functions $g_{ij} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\eta_j^{-1} c_i = g_{ij}(c_1, c_2, \ldots, c_n),$$

and there exists some linear function $h : \mathbb{R}^n \to \mathbb{R}$ so that

$$c = h(c_1, c_2, \ldots, c_n).$$

Let

$$x_0 = (\langle c_1, \phi\rangle, \langle c_2, \phi\rangle, \ldots, \langle c_n, \phi\rangle)',$$

where "$'$" denotes the transpose, and let

$$g_j = (g_{1j}, g_{2j}, \ldots, g_{nj})'$$

for $j = 0, 1, \ldots, m$. For $u \in \mathcal{V}_T$, let

$$x(t) = (F_{c_1}[u](t), F_{c_2}[u](t), \ldots, F_{c_n}[u](t))'.$$

It follows from Lemma (2.3.3) that $x(\cdot)$ is absolutely continuous. Property (b) of Definition 5.1.1 follows from Corollary 2.3.6 and Lemma 2.3.8. Notice that for any convergent power series d,

$$F_d[u](0) = \langle d, \phi\rangle$$

for all L^∞ inputs u, so we have $x(0) = x_0$. Moreover,

$$\begin{aligned}\frac{d}{dt}F_{c_i}[u](t) &= F_{\eta_0^{-1}c_i}[u](t) + \sum_{j=1}^{m} u_j\, F_{\eta_j^{-1}c_i}[u](t) \\ &= g_{i0}(x(t)) + \sum_{j=1}^{m} u_j\, g_{ij},\end{aligned}$$

for almost all $t \in [0, T]$ and $i = 1, 2, \ldots, n$. Hence,

$$x'(t) = g_0(x(t)) + \sum_{j=1}^{m} u_j(t) g_j(x(t))$$

for almost all t, and

$$y(t) = F_c[u](t) = x_1(t).$$

Therefore, F_c is realizable by a bilinear system.

By the same method used above, one can prove part (b).

(c): Suppose that $Q_1(c)$ is a finitely generated field extension of $\mathbb{R}$, i.e, there exist some $c_1, c_2, \ldots, c_n$ such that

$$Q_1(c) = \mathbb{R}(c_1, c_2, \ldots, c_n).$$

Without loss of generality, we may assume that $c_i \in \mathcal{A}_1(c)$ for $i = 1, 2, \ldots, n$ and $c_1 = c$. For each c_i and η_j, there exist some $q_{ij}, g_{ij} \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ such that

$$q_{ij}(c_1, c_2, \ldots, c_n)(\eta_j^{-1}c) = g_{ij}(c_1, c_2, \ldots, c_n),$$

for $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, m$, and

$$q_{ij}(c_1, c_2, \ldots, c_n) \neq 0.$$

Without loss of generality, we may assume that $q_{ij} = q$ for all i, j. Otherwise, we may let

$$q(c_1, c_2, \ldots, c_n) = \prod_{ij} q_{ij}(c_1, c_2, \ldots, c_n)$$

and change the g_{ij} accordingly. It follows from the fact that $\mathcal{S}$ is an integral domain that

$$q(c_1, c_2, \ldots, c_n) \neq 0. \tag{5.4}$$

Now let

$$g_j = (g_{1j}, g_{2j}, \ldots, g_{nj})'$$

for $j = 0, 1, \ldots, m$,

$$x_0 = (\langle c_1, \phi\rangle, \langle c_2, \phi\rangle, \ldots, \langle c_n, \phi\rangle)',$$

and

$$h(x) = x_1.$$

For $u \in \mathcal{V}_T$, let

$$x(t) = (F_{c_1}[u](t), F_{c_2}[u](t), \ldots, F_{c_n}[u](t))'. \tag{5.5}$$

As we have shown before,

$$\begin{aligned} x(0) &= x_0 \\ q(x(t))x'(t) &= g_0(x(t)) + \sum_{j=1}^{m} u_j(t) g_j(x(t)) \end{aligned}$$

for almost all $t \in [0, T]$ and

$$y(t) = h(x(t)).$$

To verify the regularity condition for this realization, let

$$d = q(c_1, c_2, \ldots, c_n).$$

As we have shown before, $F_d \neq 0$. Since polynomial controls are dense in the $C[0, T]$ with respect to the C^0 topology and $C[0, T]$ is dense in $\mathcal{V}_T$ with respect to the L^1 topology, polynomial controls are dense in $\mathcal{V}_T$ with respect to the L^1 topology. By Lemma 2.3.2, we know there is at least one polynomial control $p \in \mathbb{R}[t]$ such that

$$F_d[p] \neq 0.$$

It follows from the analyticity of $F_d[p]$ that

$$F_d[p](t) \neq 0$$

for almost all $t \in [0, T]$. Fix such an t and assume that $\deg p = k_0$. It follows from Lemma 5.1.2 that, for any $k \geq k_0$, there is some open dense set Ω_k in $\mathbb{R}^{mk}$ such that for any $u \in \Omega_k$

$$F_d[u](t_0) \neq 0. \tag{5.6}$$

Let

$$\Omega = \bigcup_{k=k_0}^{\infty} \Omega_k. \tag{5.7}$$

For any C^∞ input u and $k \geq 0$ given, there exists some sequence $\{p_j\}$ of polynomial inputs such that

$$p_j(t) \to u^{(k)}(t)$$

uniformly as $j \to \infty$. Define $p_{j,s}$ for $0 \leq s \leq k$ inductively in the following way:

$$p_{j,0}(t) = p_j(t),$$
$$p_{j,s}(t) = u^{(k-s)}(0) + \int_0^t p_{j,s-1}(\tau)\, d\tau.$$

Let $u_j = p_{j,k}$. From the construction of the $p_{j,s}$'s, it can be seen that

$$u_j^{(s)} = p_{j,k-s} \quad \text{for } s = 0, 1, \ldots, k$$

and by induction, one can show that for $0 \leq s \leq k$,

$$|u_j^{(s)}(t) - u^{(s)}(t)| \leq T_1 \|p_j - u^{(k)}\| \leq \|p_j - u^{(k)}\| T^k,$$

where

$$T_1 = \begin{cases} T^k & \text{if } T \geq 1\,, \\ T & \text{if } T < 1 \end{cases} .$$

Therefore we have

$$\max_{1 \leq s \leq k} \|u_j^{(s)} - u^{(s)}\| \to 0 \text{ as } j \to \infty.$$

(This is a proof of the well known fact that the set of polynomial inputs is dense in $C^\infty[0, T]$ with respect to the Whitney topology.)

For any polynomial input u and a sequence $\{u_k\}$, if

$$u_k \to u \text{ as } k \to \infty$$

with respect to the Euclidean norm, it follows that

$$u_k \to u \text{ as } k \to \infty$$

with respect to the Whitney topology. Thus the set Ω defined by (5.7) is dense in the set of polynomial inputs. Therefore, Ω is dense in $C^\infty[0, T]$ with respect to Whitney topology. From (5.6) one can see that for any $u \in \Omega$,

$$q(x(\cdot)) \neq 0$$

where $x(\cdot)$ is defined by (5.5). This is the desired regularity property. ∎

To conclude this section, we consider the following example:

Example 5.1.4 Consider the power series

$$c = 1 + \frac{1}{2}\eta_1^{(2)} - \frac{1}{2}\eta_1^{(4)} + \frac{3}{2^2}\eta_1^{(6)} + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{2^{n-1}}\eta_1^{(2n)} + \cdots .$$

It is easy to see that F_c is defined on any $\mathcal{V}_T$ for which $T < 1$. Using the same method as used in Example 2.2.8, one concludes that

$$c \sqcup\!\sqcup c = 1 + \eta_1\eta_1 .$$

By Lemma 2.1.2,

$$\eta_1^{-1} c \sqcup\!\sqcup c = \frac{1}{2}\eta_1^{-1}(c \sqcup\!\sqcup c) = \frac{1}{2}\eta_1 . \tag{5.8}$$

Applying η_1^{-1} again, we get

$$(\eta_1\eta_1)^{-1} c \sqcup\!\sqcup c + \eta_1^{-1} c \sqcup\!\sqcup \eta_1^{-1} c = \frac{1}{2} . \tag{5.9}$$

Noticing formula (4.27), we reach the know that

$$\mathcal{Q}_1(c) = \mathbb{R}(c,\ \eta_1^{-1}c) .$$

Thus F_c is realizable. To find a system realizing F_c, we let

$$x_1(t) = F_c[u](t), \quad x_2(t) = F_{\eta_1^{-1}c}[u](t)$$

for $u \in \mathcal{V}_T$, where $T < 1$. Note that

$$x_2'(t) = u(t)\, F_{(\eta_1\eta_1)^{-1}c}[u](t) .$$

It then follows from (5.9) that

$$x_1(t)\, x_2'(t) + u(t)\, x_1^2(t) = \frac{1}{2}u(t) .$$

Since $c \neq 0$, $q(x) = x_1$ satisfies the regularity property. Therefore, the following singular polynomial system

$$\begin{aligned} x_1' &= x_2\, u\,, \\ x_1\, x_2' &= (\frac{1}{2} - x_1^2)\, u(t)\,, \\ y &= x_1 \end{aligned}$$

realizes F_c. □

5.2 Realizability of Families of I/O Operators

In this section we study the realizability problem for families of, rather than single, i/o operators.

Definition 5.2.1 We say that a family $\mathbf{F}_{\underline{c}}$ of i/o operators is *realizable by a singular polynomial state space system*

$$\Sigma = ((g_0, g_1, \ldots, g_m), X, q, h)$$

where

$$g_0, g_1, \ldots, g_m$$

are polynomial vector fields of $\mathbb{R}^n$, X is a subset of $\mathbb{R}^n$, q and h are polynomial functions defined on $\mathbb{R}^n$, if the following properties hold:

(a) For each $\lambda \in \Lambda$ and each $u \in \mathcal{V}_{T_\lambda}$, where T_λ is admissible for c^λ, there exists some absolutely continuous function $x^\lambda(\cdot)$ defined on $[0, T]$ satisfying $x^\lambda(0) = x_0^\lambda$ for some $x_0^\lambda \in X$, such that

$$q(x^\lambda(t))\,(x^\lambda(t))' = g_0(x^\lambda(t)) + \sum_{j=1}^{m} g_j(x^\lambda(t))u_j(t)$$

for almost all $t \in [0, T]$, and,

$$F_{c^\lambda}[u](t) = h(x^\lambda(t))$$

for all $t \in [0, T]$ and all $\lambda \in \Lambda$.

(b) The solution $x^\lambda(\cdot)$ in part (a) is of class C^ω if u is of class C^ω, and $x^\lambda(\cdot)$ is of class C^{k+1} if u is of class C^k.

(c) There holds the following *regularity* condition: There exists some open dense set Λ_1 of Λ such that for $\lambda \in \Lambda_1$, there exists some set Ω_λ of analytic functions which is dense in $C^\infty[0, T_\lambda]$ (with respect to Whitney topology) such that for any $u \in \mathcal{V}_{T_\lambda} \cap \Omega_\lambda^m$, there exists some C^ω solution $x^\lambda(\cdot)$ as in (a) so that

$$q(x^\lambda(\cdot)) \neq 0$$

If $\mathbf{F}_{\underline{c}}$ can be realized by a singular polynomial system with

$$q(x) = 1 \quad \text{for all } x \in \mathbb{R}^n,$$

we say that $\mathbf{F}_{\underline{c}}$ is realizable by a polynomial system, and if in addition, the vector fields $g_0, \ldots, g_m$ are linear in x, then we say that $\mathbf{F}_{\underline{c}}$ is realizable by a bilinear system. □

For an analytic family of power series $\underline{c}$, we associated with it an observation algebra $\tilde{\mathcal{A}}_1(\underline{c})$ defined as the $\mathbb{R}$-algebra generated by the elements of $\tilde{\mathcal{F}}_1(\underline{c})$ and an observation field $\tilde{\mathcal{Q}}_1(\underline{c})$ defined as the quotient field of $\tilde{\mathcal{A}}_1(\underline{c})$. Note here that the analyticity of the family implies that the quotient field of $\tilde{\mathcal{A}}_1(\underline{c})$ is well defined.

The following is our main result in this section:

Theorem 8 *Let $\underline{c}$ be an analytic family of power series. Then*

(a) The family of i/o operators $\mathbf{F}_{\underline{c}}$ is realizable by a bilinear system if $\tilde{\mathcal{F}}_1(\underline{c})$ is a finite dimensional $\mathbb{R}$-space.

(b) The family of i/o operators $\mathbf{F}_{\underline{c}}$ is realizable by a polynomial system if $\tilde{\mathcal{A}}_1(\underline{c})$ is a finitely generated $\mathbb{R}$-algebra.

(c) The family of i/o operators $\mathbf{F}_{\underline{c}}$ is realizable by a singular polynomial system if $\tilde{\mathcal{Q}}_1(\underline{c})$ is a finitely generated field extension of $\mathbb{R}$.

Proof. Assume that

$$\{\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_n\}$$

is a basis of $\tilde{\mathcal{F}}_1(\underline{c})$ and then choose $g_0, g_1, \ldots, g_n$ and h in the same way as in the proof of Theorem 7, and let

$$X = \left\{\left(\langle c_1^\lambda, \phi\rangle, \ldots, \langle c_n^\lambda, \phi\rangle\right) : \lambda \in \Lambda\right\} \subset \mathbb{R}^n.$$

Then one can show that the bilinear system

$$\Sigma\ ((g_0, g_1, \ldots, g_m), X, h)$$

realizes the family of $\mathbf{F}_{\underline{c}}$.

We omit here the proof for part (b) since one can prove it by the same idea used for proving part (a). We now prove part (c).

Assume that $\tilde{Q}_1(\underline{c})$ is a finitely field extension of $\mathbb{R}$ with generators

$$\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_n.$$

Without loss of generality, we may assume that $\underline{c} = \underline{c}_1$ and $\underline{c}_i \in \tilde{A}_1(\underline{c})$ for $1 \leq i \leq n$. Then for each $\underline{c}_i$ and each η_j, there exist some polynomials p_{ij} and q_{ij} such that

$$q_{ij}(\underline{c}_1, \ldots, \underline{c}_n) \neq 0$$

so that

$$\eta_j^{-1}\underline{c} = \frac{p_{ij}(\underline{c}_1, \ldots, \underline{c}_n)}{q_{ij}(\underline{c}_1, \ldots, \underline{c}_n)}.$$

As in the proof of Theorem 7, we may assume that $q_{ij} = q$ for all i and j since $\tilde{A}_1(\underline{c})$ is an integral domain. We define the vector fields $g_0, \ldots, g_n$ of $\mathbb{R}^n$ as follows;

$$g_j(x) = \sum_{i=1}^{n} g_{ij}(x)\frac{\partial}{\partial x_i} \quad \text{for } j = 0, \ldots, m$$

and let

$$X = \left\{(\langle c_1^\lambda, \phi\rangle, \ldots, \langle c_n^\lambda, \phi\rangle) \in \mathbb{R}^n : \lambda \in \Lambda\right\}.$$

Let $h(x) = x_1$, then it can be seen that

$$\Sigma\ (g_0, \ldots, g_n, X, q, h)$$

is a realization of $\mathbf{F}_{\underline{c}}$ if we can prove the regularity of the realization. We now prove regularity. Let

$$d^\lambda = q\left(c_1^\lambda, \ldots, c_n^\lambda\right)$$

and let $\underline{d}$ denote the family $\{d^\lambda\}_\lambda$. Since

$$\underline{d} \neq 0,$$

it follows that there exists at least one λ so that

$$c^\lambda \neq 0 \tag{5.10}$$

which implies that there exists at least one $\eta^\iota \in P^*$ such that

$$\langle d^\lambda, \eta_\iota \rangle \neq 0. \tag{5.11}$$

Then it follows from analyticity that there exists some dense set Λ_1 so that (5.11) holds for every $\lambda \in \Lambda_1$ which implies that (5.10) holds for every $\lambda \in \Lambda_1$. Following the same steps in the proof of regularity in Theorem 7, one completes the proof of the regularity.

∎

Remark 5.2.2 Note that in the proof of Theorem 8, analyticity was not used in parts (a) and (b). Thus the conclusions of parts (a) and (b) of the Theorem also hold for continuous families, that is, if the observation space $\tilde{\mathcal{F}}_1(\underline{c})$ (respectively, the observation algebra $\tilde{\mathcal{A}}_1(\underline{c})$) of a continuous family $\underline{c}$ is finite dimensional (respectively, finite generated), then the family of i/o operators $\mathbf{F}_{\underline{c}}$ is realizable by a bilinear (respectively, polynomial) system. □

Chapter 6

Main Results

In this Chapter, we establish the equivalence between realizability and the existence of input/output equations.

6.1 I/O Equations and Realizability for Operators

Recall that any convergent series c induces an i/o operator F_c on $\mathcal{V}_T$ for which T is admissible for c. The following is our main result in this work.

Theorem 9 *Assume that c is a convergent power series, let $T > 0$ be admissible for c, and let F_c be the i/o operator induced by c on $\mathcal{V}_T$. Then:*

(a) The following statements are equivalent:

(i) F_c satisfies an algebraic i/o equation;

(ii) F_c satisfies a rational i/o equation;

(iii) F_c is realizable by a singular polynomial system.

(b) The following statements are equivalent:

(i) F_c satisfies an affine i/o equation;

(ii) F_c satisfies a linear i/o equation;

(iii) F_c is realizable by a bilinear system.

(c) F_c is realizable by a polynomial system if F_c satisfies a recursive i/o equation.

The realizability implications will follow from the material developed in the previous chapters. The converses, i.e. the existence of equations assuming realizability, are quite straightforward exercises in elimination theory, and the details are given next.

Lemma 6.1.1 Assume that c is a convergent power series. Then

(a) F_c satisfies an algebraic i/o equation if F_c is realizable by a singular polynomial system;

(b) F_c satisfies a linear i/o equation if F_c is realizable by a bilinear system.

Proof. (a): Assume c is a convergent power series. We need to prove that F_c satisfies some i/o equation

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0 \tag{6.1}$$

valid for C^k i/o pairs (u, y) with $u \in \mathcal{V}_T$, and any T admissible for c. Form now on we shall fix such a T and we assume that F_c is realized by the singular polynomial system

$$q(x)x' = g_0(x) + \sum_{j=0}^{m} u_j\, g_j(x), \qquad x \in \mathbb{R}^n \tag{6.2}$$

$$x(0) = x_0, \qquad x_0 \in \mathbb{R}^n \tag{6.3}$$

$$y = h(x), \qquad y \in \mathbb{R}\ . \tag{6.4}$$

Assume for now that $q(x_0) \neq 0$. Then there exists some neighborhood $\mathcal{N}$ of x_0 in $\mathbb{R}^n$ such that

$$q(x) \neq 0 \qquad \text{for all } x \in \mathcal{N}.$$

Note that on $\mathcal{N}$, equation (6.2) can be written as

$$x' = p_0(x) + \sum_{j=0}^{m} u_j\, p_j(x), \tag{6.5}$$

where

$$p_j = \frac{g_j}{q}, \qquad j = 0, 1, \ldots, m.$$

We shall prove that the i/o operator F_c satisfies an i/o equation

$$P\left(u(t), \ldots, u^{(k)}(t), F_c[u](t), \ldots, \frac{d^k}{dt^k} F_c[u](t)\right) = 0 \tag{6.6}$$

by first showing that for each *analytic* input u in $\mathcal{V}_T$, there exists some $t_1 > 0$ (which may depend on u) so that $(u, F_c[u])$ satisfies (6.6) for $t < t_1$. Assume for now that this has been shown, for some P. It then follows from the fact that

$$P\left(u(t), \ldots, u^{(k)}(t), F_c[u](t), \ldots, \frac{d^k}{dt^k} F_c[u](t)\right)$$

is an analytic function of t (cf Lemma 2.3.8) that (6.6) holds for *all* $t \in [0, T]$. By the same method used in the proof of Theorem 7, one can prove that for any integer k given and any C^k input u, there exists a sequence of analytic (if fact, polynomial) inputs $\{\omega_l\}_{l\geq 1}$ in $\mathcal{V}_T$ so that

$$\max_{0\leq i\leq n} \|\omega_l^{(i)} - u^{(i)}\|_\infty \longrightarrow 0 \qquad \text{as } l \to \infty.$$

By Lemma 2.3.2 and Theorem 1, one proves that

$$\frac{d^i}{dt^i} F_c[\omega_l](t) \longrightarrow F_c[u](t). \qquad \text{as } l \to \infty, \tag{6.7}$$

for all $t \in [0, T]$. Since $(\omega_l,\ F_c[\omega_l])$ satisfies (6.6), it follows that $(u,\ F_c[u])$ also satisfies (6.6). Since u can be chosen arbitrarily, it follows that F_c satisfies the equation (6.6). Thus we now show the result for analytic inputs and small times.

For any analytic input u in $\mathcal{V}_T$ and any $\lambda \in [0, T]$, we define an i/o operator $F_c^{\lambda,u}$ by

$$F_c^{\lambda,u}[v](t) \;=\; F_c[u\#_\lambda v](t+\lambda)$$

for any $v \in \mathcal{V}_{T-\lambda}$. Then $F_c^{\lambda,u}$ is realized by the system given by (6.2) and (6.4), with the initial state

$$x(0) = \varphi(\lambda, x_0, u)\,,$$

where we are using $\varphi(t, x, u)$ to denote the solution of (6.5) with the initial state $\varphi(0, x, u) = x$. Notice that if λ is small enough, $\varphi(\lambda, x_0, u)$ still remains in $\mathcal{N}$. Therefore, for any analytic $u \in \mathcal{V}_T$ and any t small enough, there exists some $x \in \mathcal{N}$ such that

$$F_c[u](t) = h(\varphi(0, x, u))\,.$$

Hence, to show that $(u(t),\ F_c[u](t))$ satisfies the algebraic i/o equation (6.6) for t small enough, it suffices to show that, for all vectors $\mu_0, \ldots, \mu_k$,

$$P\left(\mu_0, \ldots, \mu_k, y_x(0), \ldots, y_x^{(k)}(0)\right) \;=\; 0$$

for each $x \in \mathcal{N}$ and any analytic input u with

$$u^{(i)}(0) = \mu_i, \qquad i = 0, 1, \ldots k,$$

where

$$y_x(t) = h(\varphi(t, x, u))\,. \tag{6.8}$$

Differentiating with respect to t on both sides of (6.8), one gets

$$y_x'(t) = L_{p_0}h(\varphi(t)) + \sum_{j=1}^{m} u_j\, L_{g_j}h(\varphi(t)),$$

where $\varphi(t)$ denotes $\varphi(t, x, u)$. In particular,

$$y_x'(0) = L_{p_0}h(x) + \sum_{j=1}^{m} u_j(0)\, L_{p_j}h(x).$$

Differentiating $y_x(t)$ again with respect to time t, one sees that there exists some rational function $F_2 \in \mathbb{R}(X, S_0, S_1)$ where $S_i = (S_{i1}, \ldots, S_{im})$ are m indeterminates over $\mathbb{R}$, and $X = (X_1, \ldots, X_n)$ are n indeterminates over $\mathbb{R}$, so that

$$y_x''(t) = F_2(\varphi(t), u(t), \dot{u}(t))$$

and, in particular,

$$y_x''(0) = F_2(x, u(0), u'(0)).$$

It not hard to see that for any integer k, there exists some rational function

$$F_k \in \mathbb{R}(X, S_0, \ldots, S_{k-1})$$

so that

$$y_x^{(k)}(t) = F_k(\varphi(t), u(t), \ldots, u^{(k-1)}(t))$$

and

$$y_x^{(k)}(0) = F_k(x, u(0), \ldots, u^{(k-1)}(0)).$$

Now we view $F_k(X, \mu_0, \mu_{k-1})$, $k = 0, \ldots, n$ as rational functions in the variables X with coefficients over the field K, obtained by adjoining

$$S_0, S_1, \ldots, S_{n-1}$$

to $\mathbb{R}$. Since the transcendence degree of $K(X)$ over K is n, the $n+1$ rational functions $F_0, F_1, \ldots, F_n$ must be algebraically dependent over K, i.e., there exists some nontrivial polynomial Q over K such that

$$Q\,(F_0(X), F_1(X), \ldots, F_n(X)) = 0.$$

Clearing the denominators in the coefficients (rational functions in the variables $S_0, \ldots,$ S_{n-1}), one gets

$$P(S_0, \ldots, S_{n-1}, F_0(X), \ldots, F_n(X)) = 0$$

where $P \in \mathbb{R}[X, S_0, \ldots, S_{n-1}]$ is some polynomial over $\mathbb{R}$. Note here that P is non-trivial in X since Q is nontrivial. Evaluating $X, S_0, \ldots, S_{n-1}$ at $x, \mu_0, \ldots, \mu_{n-1}$, one gets

$$P\left(\mu_0, \ldots, \mu_{n-1}, y_x(0), \ldots, y_x^{(n)}(0)\right) = 0, \tag{6.9}$$

Since P was chosen independent of the initial state x, equation (6.9) holds for all $x \in \mathcal{N}$. The above discussion shows that

$$P\left(u(t), \ldots, u^{(n-1)}(t), y(t), \ldots, y^{(n)}(t)\right) = 0 \tag{6.10}$$

is an i/o equation for F_c.

Finally, we show how to overcome the restriction $q(x_0) \neq 0$. Assume now $q(x_0) = 0$. Then by definition, there exists a set Ω of analytic inputs in C^∞, open dense with respect to the Whitney topology, so that for each $u \in \Omega \cap \mathcal{V}_T$, there exists some analytic function $\varphi(t)$ satisfying (6.2) and (6.3) such that

$$q(\varphi(\cdot)) \neq 0$$

and

$$F_c[u](t) = h(\varphi(t)).$$

It follows from analyticity that

$$q(\varphi(t)) \neq 0 \qquad \text{for } t \in (0, \delta),$$

for some $\delta > 0$. Since $F_c^{\lambda,u}$ is also realized by (6.5) and (6.4) with the initial state

$$x(0) = \varphi(\lambda)$$

for each fixed λ, it follows that $F_c^{\lambda,u}$ satisfies the i/o equation (6.10) for any $\lambda \in (0, \delta)$. Therefore,

$$P\left(u(\lambda), \ldots, u^{(n-1)}(\lambda), y(\lambda), \ldots, y^{(n)}(\lambda)\right) = 0$$

for all $\lambda \in (0, \delta)$. Using analyticity again, one knows that $(u(t), F_c[u](t))$ satisfies (6.10) for all $t \in [0, T]$.

Now assume that $u \in \mathcal{V}_T$ is an arbitrary C^n input. By the following exactly the same steps in the proof of Theorem 7, one shows that there exists a sequence of analytic inputs $\{\omega_l\}_{l\geq 1}$ such that

$$\max_{0\leq i\leq n} \|\omega_l^{(i)} - u^{(i)}\|_\infty \longrightarrow 0 \qquad \text{as } l \to \infty. \tag{6.11}$$

Since for each analytic input ω, there exists a sequence $\{\nu_j\}_{j\geq 0}$ in Ω such that

$$\max_{0\leq i\leq n} \|\nu_j^{(i)} - w^{(i)}\|_\infty \longrightarrow 0 \qquad \text{as } j \to \infty,$$

it follows, in turn, that the sequence $\{\omega_l\}_{l\geq 1}$ satisfying (6.11) can be chosen in Ω. Since $(\omega_l,\ F_c[\omega_l])$ satisfies the i/o equation (6.10) for each l, a continuity argument implies that $(u,\ F_c[u])$ also satisfies the i/o equation (6.10), as desired.

Part (b) can be proved by an analogous argument. The proof, however, is much simpler, since there is no need to deal with singular points in this case. The basic idea of the proof is as follows:

Since the functions

$$y_x(0), y_x'(0), \ldots, y_x^{(n)}(0)$$

are linear combinations of n variables $x_1, x_2, \ldots, x_n$ over the field K, it then follows that they are linear dependent over the field K, i.e., there exist $a_0, a_2, \ldots, a_n \in K$, not all zero, such that

$$a_0 y_x(0) + a_1 y_x'(0) + \cdots + a_n y_x^{(n)}(0) = 0.$$

Clearing the denominators in a_i if necessary, we obtain a linear equation

$$b_0(\mu_0, \ldots, \mu_{n-1}) y_x(0) + \cdots + b_1(\mu_0, \ldots, \mu_{n-1}) y_x^{(n)}(0) = 0$$

for any x, any $\mu_0, \ldots, \mu_{n-1}$ and any input u such that $u^{(i)} = \mu_i$, $i = 0, 1, \ldots, n-1$. By the same argument used in the proof of part (a), one shows that the linear equation

$$b_0(u, \ldots, u^{(n-1)}) y + \cdots + b_n(u, \ldots, u^{(n-1)}) y^{(n)} = 0$$

is an i/o equation for F_c. ∎

We are now ready to prove our main result, Theorem 9.

Proof. (a) (i) $\Longrightarrow$ (ii): by Lemma 4.1.7.

(ii) $\Longrightarrow$ (iii): by Corollary 4.2.6 (b) we know that $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. By Theorem 2, we get the conclusion that $\mathcal{Q}_1(c)$ is a finitely generated field extension of $\mathbb{R}$. Applying Theorem 7 (c), we know that F_c is realizable by a singular polynomial system.

(iii) $\Longrightarrow$ (i): by Lemma 6.1.1 (a).

(b): (i) $\Longrightarrow$ (iii): by Theorem 5 (a), Theorem 2, and Theorem 7 (a).

(iii) $\Longrightarrow$ (ii): by Lemma 6.1.1 (b).

(ii) $\Longrightarrow$ (i): trivial.

(c): by Theorem 5 (b), $\mathcal{A}_2(c)$ is a finitely generated $\mathbb{R}$-algebra, which implies, by Theorem 2, that $\mathcal{A}_1(c)$ is also finitely generated as an $\mathbb{R}$-algebra. By Theorem 7 (b), F_c is realizable by a polynomial system. ∎

Note that, in contrast to the cases of rational i/o equation and linear i/o equations, the converse of part (c) does not hold in general, i.e, realizability by polynomials system does not necessarily imply the existence of a recursive i/o equation. This can be illustrated by the following example:

Example 6.1.2 Consider the following system

$$\begin{aligned} x_1' &= x_1x_2, \; x_1(0) = x_{10} = 1; \\ x_2' &= u, \qquad x_2(0) = x_{20} = 0; \\ y &= x_1. \end{aligned} \tag{6.12}$$

By Theorem 4, we know that there exists some $T > 0$ such that for all $u \in \mathcal{V}_T$, $y(t) = F_c[u](t)$, where c is given by

$$\langle c, \eta_{i_1}\eta_{i_2}\cdots\eta_{i_l}\rangle = L_{g_{i_l}}\cdots L_{g_{i_2}}L_{g_{i_1}}h(x_0)$$

and

$$g_0 = x_1x_2\frac{\partial}{\partial x_1}, \;\; g_1 = \frac{\partial}{\partial x_2}$$

and $h(x) = x_1$. (In fact, every $T > 0$ is admissible for this example). In the other words, F_c is realizable by the polynomial system (6.12).

To show that the operator F_c does not satisfy any recursive i/o equation, we need first establish the following fact.

Recall that we have associated in Chapter 3 to each analytic state space system

$$\begin{aligned} x' &= g_0(x) + \sum_{i=1}^{m} g_i(x), \qquad x \in \mathcal{M} \\ y &= h(x), \end{aligned} \tag{6.13}$$

the observation space F_1 defined as $\mathbb{R}$-space spanned by all the functions

$$L_{g_{i_1}} L_{g_{i_2}} \cdots L_{g_{i_k}} h(x)\,, \qquad k \geq 0,\;\; 0 \leq i_1, i_2, \ldots, i_k \leq m$$

We define the *observation algebra* $\mathcal{A}$ of (6.13) as the $\mathbb{R}$-algebra generated by the elements of F_1.

For each $x_0 \in \mathcal{M}$, let c_h be the generating series defined by

$$\langle c_h,\, \eta_\iota \rangle \;=\; L_\iota h(x)\,. \tag{6.14}$$

According to the discussion in §3.4, c_h is a convergent power series and for any $u \in \mathcal{V}_T$,

$$y(t) = h(x(t, x_0, u)) = F_{c_h}[u](t)$$

for t small enough, where $x(t, x, u))$ is the solution of (6.13) corresponding to the control u with the initial condition $x(0, x_0, u) = x_0$.

We say that the system (6.13) is *accessible* at x_0 if for any neighborhood $\mathcal{B}$ of x_0, there exists an open subset of $\mathcal{U}$ of $\mathcal{B}$ such that for any $p \in \mathcal{U}$, there exist some $\tau \geq 0$ and some $u \in L^m_\infty[0, \tau]$ such that $x(\tau, x_0, u) = p$.

Lemma 6.1.3 Assume that the analytic system (6.13) is accessible at x_0. Assume that $\mathcal{M}$ is connected. Let c_h be the series defined by (6.14). Then the observation algebra $\mathcal{A}_1(c_h)$ associated with c_h is isomorphic to the observation algebra $\mathcal{A}$ associated with (6.13). □

Proof. We will prove our conclusion by first showing that for any function $\varphi \in \mathcal{A}$, the series c_φ defined by (6.14) for φ and x_0 belongs $\mathcal{A}_1(c_h)$.

Clearly, for any function $\varphi(x) = L_\iota h(x)$, $c_\varphi \in \mathcal{A}_1(c_h)$ because $c_\varphi = \eta_\iota^{-1}c$. Now assume that

$$\varphi(x) = P(\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))$$

for some polynomial P, where $\varphi_i(x) = L_{\iota_i} h(x)$ for some multiindex ι_i, $i = 1, \ldots, n$. Let

$$d = P(c_1, c_2, \ldots, c_n)\,,$$

where $c_i = \eta_{\iota_i}^{-1}c$. Note that $\langle \eta_\kappa^{-1}c_i,\, \phi\rangle = L_\kappa \varphi_i(x_0)$ for any $\kappa \in I^*$. The fact that

$$\langle d_1 \sqcup\!\sqcup\, d_2,\, \phi\rangle = \langle d_1,\, \phi\rangle\langle d_2,\, \phi\rangle \tag{6.15}$$

for any series d_1, d_2 implies that

$$\langle d,\, \phi\rangle = P(\varphi_1(x_0), \varphi_2(x), \ldots, \varphi_n(x_0)) = \varphi(x_0)\,.$$

For any $i \in \{0, 1, \ldots, m\}$, the shift operator η_i^{-1} is a derivation operator (cf. Lemma 2.1.2) for series, therefore,

$$\eta_i^{-1}\, d = \sum_{j=1}^{n} \frac{\partial}{\partial X_j} P(c_1, c_2, \ldots, c_n) \sqcup\!\sqcup\, \eta_i^{-1} c_j\,.$$

Applying (6.15) again, one knows that

$$\langle \eta_i^{-1} d,\, \phi\rangle = \sum_{j=1}^{n} \langle \frac{\partial}{\partial X_j} P(c_1, c_2, \ldots, c_n),\, \phi\rangle\, \langle \eta_i^{-1} c_j,\, \phi\rangle = L_{g_i}\varphi(x_0)\,.$$

Applying the above procedure repeatedly, one sees that for any $w = \eta_{j_1} \cdots \eta_{j_k}$

$$\langle w^{-1} P(c_1, c_2, \ldots, c_n),\, \phi\rangle = L_{g_{j_k}} \cdots L_{g_{j_1}} h(x_0)\,. \tag{6.16}$$

Therefore, $d = c_\varphi$ which implies that $c_\varphi \in \mathcal{A}_1(c_h)$ as desired.

Let $\psi : \mathcal{A} \to \mathcal{A}_1(c)$ be defined by $\varphi \mapsto c_\varphi$ for any $\varphi \in \mathcal{A}$. The map ψ is well defined since if $\varphi = 0$, then $L_\iota f(x_0) = 0$ for any ι which implies that $c_\varphi = 0$.

For any $\varphi_1, \varphi_2 \in \mathcal{A}$, we have, by formula (3.24),

$$L_\iota(\varphi_1 \varphi_2) = \sum_{\kappa_1, \kappa_2} L_{\kappa_1}\varphi_1(x_0)\, L_{\kappa_2}\varphi_2(x_0) \langle \eta_{\kappa_1} \sqcup\!\sqcup\, \eta_{\kappa_2},\, \eta_\iota\rangle$$

from which it follows that

$$\langle c_{\varphi_1\varphi_2},\, \eta_\iota\rangle = \sum_{\kappa_1, \kappa_2} \langle c_{\varphi_1},\, \eta_{\kappa_1}\rangle\langle c_{\varphi_2},\, \eta_{\kappa_2}\rangle\langle \eta_{\kappa_1} \sqcup\!\sqcup\, \eta_{\kappa_2},\, \eta_\iota\rangle\,.$$

Therefore, $\psi(\varphi_1\varphi_2) = \psi(\varphi_1) ш \psi(\varphi_2)$ for any $\varphi_1, \varphi_2 \in \mathcal{A}$. Hence ψ is a homomorphism.

Finally, we show that ψ is an isomorphism. Assume that $\psi(\varphi) = 0$ for some $\varphi \in \mathcal{A}$. It then follows that

$$L_{g_{i_k}} \cdots L_{g_{i_1}} \varphi(x_0) = 0$$

for any $i_1, \ldots, i_k$. Applying Theorem 4, we know that for any $u \in L_\infty^m$,

$$\varphi(x(t)) = F_{c_\varphi}[u](t) = 0 \tag{6.17}$$

for t small enough. It then follows that (6.17) holds for all t if u is analytic. Since the system 6.13 is accessible at x_0, there exists some open subset $\mathcal{U}$ of $\mathcal{M}$ such that for any $p \in \mathcal{U}$, there exists some $\tau \geq 0$ and some u such that $p = x(\tau, x_0, u)$. Since any L_∞ control u defined on any finite interval $[0, T]$ can be approximated by analytic controls, it follows that there exists a dense subset $\mathcal{U}_1$ of $\mathcal{U}$ such that for any $p \in \mathcal{U}_1$, there exists some analytic control u and some $\tau \geq 0$ so that $p = x(\tau, x_0, u)$ (cf. Chapter 2 in [33]). Hence, $\varphi(p) = 0$ for any $p \in \mathcal{U}_1$. The continuity of φ then implies that $\varphi(p) = 0$ for any $p \in \mathcal{U}$, which in turn implies, by analyticity, that φ is identically zero on $\mathcal{M}$. Therefore ψ is an isomorphism. Obviously ψ is onto, thus, $\mathcal{A}$ is isomorphic to $\mathcal{A}_1(c)$. ∎

We now return to complete our example. First of all one can see that the system (6.2) is accessible at $x_0 = (1, 0)$ since the accessibility rank condition (see for instance [38]) holds:

$$\text{rank}\,\big(\, g_0(x_0) \quad [g_0, g_1](x_0) \,\big) = 2\,.$$

If F_c would satisfy some recursive i/o equation, then the observation algebra $\mathcal{A}_2(c)$ would be finitely generated, which by Lemma 6.1.3, would imply that $\mathcal{A}$ is also finitely generated as an $\mathbb{R}$-algebra. But this is false as $\mathcal{A}$ is the algebra generated by

$$x_1,\ x_1x_2,\ x_1x_2^2,\ \ldots,\ x_1x_2^k, \ldots \quad k \geq 0\ .$$

Thus F_c cannot satisfy any recursive i/o equation, even though it is realized by the polynomial system (6.12). □

6.2 Equivalence between I/O Equation and Realizability for Families of I/O Operators

In this section we wish to give an analogue to Theorem 9, for families of i/o operators. For this purpose, we first establish an analogue of Lemma 6.1.1.

Lemma 6.2.1 Assume that $\underline{c}$ is a convergent family of power series. Then

(a) $\mathbf{F}_{\underline{c}}$ satisfies an algebraic i/o equation if $\mathbf{F}_{\underline{c}}$ is realizable by a singular polynomial system;

(b) $\mathbf{F}_{\underline{c}}$ satisfies a linear i/o equation if $\mathbf{F}_{\underline{c}}$ is realizable by a bilinear system.

Proof. (a): Assume that a convergent family of i/o operators

$$\mathbf{F}_{\underline{c}} = \{F_{c^\lambda} : \lambda \in \Lambda\}$$

is realizable by a singular polynomial system

$$(g_0, g_1, \ldots, g_m, X, q, h).$$

Then by definition, there exists some dense set Λ_1 of Λ such that for any $\lambda \in \Lambda$, F_{c^λ} is realizable by the system

$$\left(g_0, g_1, \ldots, g_m, x^\lambda, q, h\right)$$

in the sense of Definition 5.1.1. It then follows that for each $\lambda \in \Lambda_1$, F_{c^λ} satisfies an algebraic equation

$$P\left(u(t), \ldots, u^{(k)}(t), F_c[u](t), \ldots, \frac{d^k}{dt^k}F_{c^\lambda}[u](t)\right) = 0. \tag{6.18}$$

Checking the proof of Lemma 6.1.1 carefully, one sees that the polynomial P in the i/o equation can be chosen independent of the choice of λ. This is basically because (6.11) holds for any value of

$$S_0, S_1, \ldots, S_{n-1}$$

and any value of X. Therefore, F_{c^λ} satisfies the i/o equation for all $\lambda \in \Lambda_1$.

Below we show that (6.2) holds for all λ, not just for λ in Λ_1.

Take $\lambda \in \Lambda$. Then there exists a sequence of points $\{\lambda_j\}_{j\geq 1}$ in Λ_1 so that

$$\lambda_j \longrightarrow \lambda, \text{ as } j \to \infty.$$

By Lemma 2.4.4, we know that

$$F_{c^{\lambda_j}}[u](t) \longrightarrow F_{c^\lambda}[u](t) \quad \text{as } j \to \infty$$

for all $u \in \mathcal{V}_T$ and each T admissible for λ. By the same method used in the proof of Lemma 2.4.4, one can also prove that

$$F_{z^{-1}c^{\lambda_j}}[u](t) \longrightarrow F_{z^{-1}c^\lambda}[u](t) \quad \text{as } j \to \infty$$

for all $z \in P^*$. It then follows that

$$F_{c_n^{\lambda_j}(\mu_0,\ldots,\mu_{n-1})}[u](t) \longrightarrow F_{c_n^\lambda(\mu_0,\ldots,\mu_{n-1})}[u](t) \quad \text{as } j \to \infty$$

for any $\mu_0, \mu_1, \ldots, \mu_{n-1}$ which implies that

$$\frac{d^k}{dt^k} F_{c^{\lambda_j}}[u](t) \longrightarrow \frac{d^k}{dt^k} F_{c^\lambda}[u](t) \quad \text{as } j \to \infty.$$

Since $F_{c^\lambda_j}$ satisfies the i/o equation (6.18), it follows from the continuity of the operators that F_{c^λ} also satisfies equation (6.18). Since λ in the above discussion can be picked up arbitrarily, we proved our conclusion that the family of i/o operators $\mathbf{F}_{\underline{c}}$ satisfies an algebraic i/o equation.

Part (b) of the Lemma follows immediately from the fact that every member F_{c^λ} of the family satisfies a linear i/o equation and the equation can be chosen independent of the parameter λ if the family is realizable by a bilinear system. We omit here the detailed and by now routine proof. ∎

We now give our main results for families of i/o operators.

Theorem 10 *Assume that $\underline{c}$ is an analytic families of series. Then*

(a) The following statements are equivalent:

(i) $\mathbf{F}_{\underline{c}}$ satisfies an algebraic i/o equation;

(ii) $\mathbf{F}_{\underline{c}}$ satisfies a rational i/o equation;

(iii) $\mathbf{F}_{\underline{c}}$ *is realizable by a singular polynomial system.*

(b) The following statements are equivalent:

(i) $\mathbf{F}_{\underline{c}}$ *satisfies an affine i/o equation;*

(ii) $\mathbf{F}_{\underline{c}}$ *satisfies a linear i/o equation;*

(iii) $\mathbf{F}_{\underline{c}}$ *is realizable by a bilinear system.*

(c) $\mathbf{F}_{\underline{c}}$ *is realizable by a polynomial system if* $\mathbf{F}_{\underline{c}}$ *satisfies a recursive i/o equation.*

Proof. (a): (i) $\Longrightarrow$ (ii): by the same proof of Lemma 4.1.7.

(ii) $\Longrightarrow$ (iii): by Theorem 6 (c), Theorem 3 and Theorem 8 (c).

(iii) $\Longrightarrow$ (i): by Lemma 6.2.1 (a).

(b): (i) $\Longrightarrow$ (iii): by Theorem 6 (a), Theorem 3 and Theorem 8 (a).

(iii) $\Longrightarrow$ (ii): by Lemma 6.2.1.

(ii) $\Longrightarrow$ (i): trivial.

(c): by Theorem 6 (b), $\tilde{\mathcal{A}}_2(\underline{c})$ is a finitely generated $\mathbb{R}$-algbra, which implies, by Theorem 3, that $\tilde{\mathcal{A}}_1(\underline{c})$ is a finitely generated $\mathbb{R}$-algebra. By Theorem 8 (b), $\mathbf{F}_{\underline{c}}$ is realizable by a polynomial system. ∎

Chapter 7

Realization and Input/Output Relations: The Analytic Case

In this Chapter, our previous results relating algebraic i/o equations to (global and rational) internal realizability are extended to analytic i/o equations and (local and analytic) internal realizability.

7.1 Analytic I/O Equations

By an *analytic input/output equation of order k* we shall mean an equation of type

$$A\left(u(t),\ldots,u^{(k)}(t),y(t),\ldots,y^{(k)}(t)\right)=0, \tag{7.1}$$

where A is an analytic function defined on $\mathbb{R}^{m(k+1)}\times\mathbb{R}^{(k+1)}$ and nontrivial in the last variable. The latter means that there exists some point

$$\bar{\mu}_0,\bar{\mu}_1,\ldots,\bar{\mu}_k,\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}$$

in $\mathbb{R}^{m(k+1)}\times\mathbb{R}^k$ such that

$$A\left(\bar{\mu}_0,\bar{\mu}_1,\ldots,\bar{\mu}_k,\bar{\nu}_0,\ldots,\bar{\nu}_{k-1},\cdot\right)$$

is not a constant function.

We say that A is

(a) *meromorphic* when

$$\begin{aligned} & A(\mu_0,\ldots,\mu_k,\nu_0,\ldots,\nu_k) \\ = \;& A_0(\mu_0,\ldots,\mu_k,\nu_0,\ldots,\nu_{k-1})\nu_k + A_1(\mu_0,\ldots,\mu_k,\nu,\ldots,\nu_{k-1}). \end{aligned} \tag{7.2}$$

(b) *analytically recursive* when

$$\begin{aligned} & A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k) \\ = \; & A_0(\mu_0, \ldots, \mu_k)\nu_k + A_1(\mu_0, \ldots, \mu_k, \nu, \ldots, \nu_{k-1}). \end{aligned} \tag{7.3}$$

Definition 7.1.1 Assume that c is a convergent series. We say that the i/o operator F_c *satisfies the analytic i/o equation* (7.1) if (7.1) holds for every possible C^k i/o pair (u, y) of F_c. In such a case, (7.1) is called an *analytic i/o equation for* F_c.

An i/o operator F_c satisfies an *analytically recursive i/o equation* if there is some such equation for which A is analytically recursive. An i/o operator F_c satisfies a *meromorphic i/o equation* if A can be chosen meromorphic so that $A_0 = 0$ is *not* an i/o equation of F_c, in the other words, there exists some i/o pair (u, y) of F_c such that

$$A_0(u(t), u'(t), \ldots, u^{(k)}(t), y(t), y'(t), \ldots, y^{(k-1)}(t)) \neq 0$$

for some t. □

Using the same arguments as in the proof of Lemma 4.1.4, we get the following:

Lemma 7.1.2 F_c satisfies the i/o equation (7.1) if and only if

$$A\left(\mu_0, \ldots, \mu_k, F_c, F_{c_1(\mu_0)}, \ldots, F_{c_k(\mu_0, \ldots, \mu_{k-1})}\right) = 0, \tag{7.4}$$

for any $\mu_0, \mu_1, \ldots, \mu_k \in \mathbb{R}^m$. □

Similar to the algebraic case, we have the following result:

Lemma 7.1.3 If F_c satisfies an analytic i/o equation, then it satisfies a meromorphic i/o equation.

Proof. Assume that F_c satisfies an analytic i/o equation of order k;

$$A\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0.$$

Without loss of generality, we may assume that k is smallest possible among all analytic i/o equations for F_c. Now let, for each $i \geq 0$,

$$Q_i(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k) = \frac{\partial^i}{\partial \nu_k^i} A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k).$$

Claim: There exists some i such that $Q_i = 0$ is not an i/o equation for F_c.

Suppose by way of contradiction that $Q_i = 0$ is an i/o equation of F_c for all i, i.e.,

$$Q_i(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)) = 0 \tag{7.5}$$

for all C^k i/o pairs (u, y) of F_c and for all $i \geq 0$. For each fixed u and each fixed t, we let

$$(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_k) = (u(t), \ldots, u^{(k)}(t), y(t), \ldots, \ldots, y^{(k)}(t)).$$

Then (7.5) means that

$$\frac{\partial^i}{\partial \nu_k} A(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_{k-1}, \bar{\nu}_k) = 0$$

for all i. It then follows from the analyticity of A that

$$A(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_{k-1}, \alpha) = 0$$

for all $\alpha \in \mathbb{R}$. Since u and t can be chosen arbitrarily, one concludes that

$$A(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t), \alpha) = 0$$

for any C^k i/o pair (u, y) of F_c and any constant α. Choose an α such that the function

$$A_1(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}) := A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}, \alpha)$$

is not identically zero. (Such an α exists because A is not identically zero, by definition of i/o equations.) It follows immediately that

$$A_1(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t)) = 0 \tag{7.6}$$

for all i/o pairs of F_c, and this is a nontrivial equation, by the choice of α. We may assume that

$$\frac{\partial}{\partial \nu_j} A_1(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}) \neq 0$$

for some $j = 0, 1, \ldots, k-1$. Otherwise, if

$$\frac{\partial}{\partial \nu_i} A_1(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}) = 0$$

for any $i = 0, 1, \ldots, k-1$, then there exist some $\nu_0, \ldots, \rho_{k-1}$ so that

$$A_2(\mu_0, \mu_1, \ldots, \mu_k) = A_1(\mu_0, \mu_1, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1})$$

is not identically zero and

$$A_2(u(t), u'(t), \ldots, u^{(k)}(t)) = 0$$

holds for all input functions and all t, which is impossible. Let j be as large as possible. Applying to this j the same arguments in the proof of Lemma 4.2.3 to equation (7.6), one knows that there exists some analytic function

$$A_3(\mu_0, \ldots, \mu_j, \nu_0, \ldots, \nu_j)$$

such that $A_3 = 0$ is an i/o equation for F_c of order $j < k$, which contradicts the assumed minimality of k.

Thus we proved by induction that there exists some i such that $Q_i = 0$ is not an i/o equation of F_c. Now let $r \geq 1$ be the smallest number for which $Q_r = 0$ is not an i/o equation of F_c. Then

$$Q_{r-1}(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}) = 0 \tag{7.7}$$

is an i/o equation of F_c. Taking derivative with respect to time t on both sides of (7.7), one sees that F_c satisfies the following meromorphic i/o equation:

$$\begin{aligned} &Q_r\left(u(t), \ldots, u_k(t), y(t), \ldots, y^{(k)}\right) y^{(k+1)} \\ &\qquad = P_3\left(u(t), \ldots, u_k(t), y(t), \ldots, y^{(k)}\right), \end{aligned}$$

where P_3 is some analytic function defined in $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{(k+1)}$. ∎

Remark 7.1.4 Without the assumption that k is smallest possible, one may fail to find a meromorphic i/o equation by the approach used in the above proof. The simplest example can be taken as the following: Consider the i/o operator

$$y(t) = F_c[u](t) = \int_0^t u(s)\, ds.$$

This satisfies

$$A(u, u', y, y', y'') = (y' - u)e^{y''} = 0 \qquad (7.8)$$

which is an analytic i/o equation. In the above proof, for each i,

$$Q_i = (\nu_1 - \mu_0)e^{\nu_2},$$

and $Q_i = 0$ is an i/o equation for F_c for every i, so the construction in the proof would fail. This is caused by the fact that the order of the equation is not smallest possible, since

$$y' - u = 0$$

is a lower order i/o equation for F_c. □

Recall that for any given series c, the observation space $\mathcal{F}_2(c)$ is defined to be the $\mathbb{R}$-space spanned by $c_n(\mu_0, \ldots, \mu_{n-1})$ for all n and all values of $\mu_0, \ldots, \mu_{n-1}$; the observation algebra $\mathcal{A}_2(c)$ is defined to be the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_2(c)$, and the observation field $\mathcal{Q}_2(c)$ is defined to be the quotient field of $\mathcal{A}_2(c)$.

For any given convergent power series c, we say that $\mathcal{A}_2(c)$ is an *analytically finitely generated* $\mathbb{R}$-algebra if there exist an integer n and n elements

$$c_1, c_2, \ldots, c_n$$

of $\mathcal{A}_2(c)$ such that for every element d in $\mathcal{A}_2(c)$, there exists some analytic function φ defined on $\mathbb{R}^n$ such that

$$F_d[u](t) = \varphi(F_{c_1}[u](t), \ldots, F_{c_n}[u](t))$$

for all $u \in \mathcal{V}_T$, $t \in [0, T]$ and for any T admissible for c.

We say that the observation field $\mathcal{Q}_2(c)$ is a *meromorphically finitely generated field extension of* $\mathbb{R}$ if there exists an integer n and

$$c_1, c_2, \ldots, c_n \in \mathcal{A}_2(c)$$

such that for each element d in $\mathcal{Q}_2(c)$, there exist some analytic functions φ_0 and φ_1 defined on $\mathbb{R}^n$ such that

$$\varphi_0(F_{c_1}[u](t), \ldots, F_{c_n}[u](t))\, F_d[u](t) = \varphi_1(F_{c_1}[u](t), \ldots, F_{c_n}[u](t))$$

for all $u \in \mathcal{V}_T$, $t \in [0, T]$ and for any T admissible for c, and,

$$\varphi_0(F_{c_1}[u], \ldots, F_{c_n}[u]) \neq 0$$

for some $u \in \mathcal{V}_T$, and some T admissible for c. If this is the case, we call $c_1, \ldots, c_n$ the generators of the field, or, we say that the field is generated by $c_1, \ldots, c_n$. Informally speaking, then, a meromorphically finitely generated field extension of $\mathbb{R}$ is one for which every element can be expressed as a meromorphic function of a finite set of generators.

The following Theorem relates the existence of analytic i/o equations to finiteness properties of the observation algebra and field.

Theorem 11 *Assume c is a convergent power series. Then*

(a) $\mathcal{Q}_2(c)$ is meromorphically finitely generated if F_c satisfies an analytic i/o equation;

(b) $\mathcal{A}_2(c)$ is analytically finitely generated if F_c satisfies an analytically recursive i/o equation.

Proof. We shall only provide the proof of part (a). Part (b) can be proved by the same approach.

Assume F_c satisfies an analytic i/o equation. By Lemma 7.1.3, F_c satisfies a meromorphic i/o equation

$$\begin{aligned}A_0&\left(u(t), \ldots, u^{(k+1)}(t), y(t), \ldots, y^{(k)}(t)\right) y^{(k+1)}(t)\\ &= A_1\left(u(t), \ldots, u^{(k+1)}(t), y(t), \ldots, y^{(k)}(t)\right).\end{aligned} \tag{7.9}$$

Taking derivative with respect to time t on both sides of the equation, one gets

$$\begin{aligned}&A_0\left(u(t), \ldots, u^{(k+1)}(t), y(t), \ldots, y^{(k)}(t)\right) y^{(k+2)}(t)\\ &= \tilde{A}_2\left(u(t), \ldots, u^{(k+2)}(t), y(t), \ldots, y^{(k)}(t)\right)\\ &+ \hat{A}_2\left(u(t), \ldots, u^{(k+2)}(t), y(t), \ldots, y^{(k)}(t)\right) y^{(k+1)}(t),\end{aligned} \tag{7.10}$$

where

$$\begin{aligned}\tilde{A}_2 &= \sum_{i=0}^{k+1} \frac{\partial}{\partial \mu_i}(A_1 - A_0)\,\mu_{i+1} + \sum_{i=0}^{k-1} \frac{\partial}{\partial \nu_i}(A_1 - A_0)\,\nu_{i+1}\,,\\ \hat{A}_2 &= \frac{\partial}{\partial \nu_k}(A_1 - A_0).\end{aligned}$$

Multiplying by A_0 on both sides of (7.10) and replacing $y^{(k+1)}$ by (7.9), one knows that there exists some analytic function A_2 such that

$$A_0^2\left(u(t), \ldots, u^{(k+1)}(t), y(t), \ldots, y^{(k)}(t)\right) y^{(k+2)}(t) \\ = A_2\left(u(t), \ldots, u^{(k+2)}(t), y(t), \ldots, y^{(k)}(t)\right).$$

using the above arguments repeatedly, one proves that for each $r > 0$ there exists some analytic function A_r so that

$$A_0^r\left(u(t), \ldots, u^{(k+1)}(t), y(t), \ldots, y^{(k)}(t)\right) y^{(k+r)}(t) \\ = A_r\left(u(t), \ldots, u^{(k+r)}(t), y(t), \ldots, y^{(k)}(t)\right).$$

According to Lemma 7.1.2, we have

$$\begin{aligned} &A_0^r\left(\mu_0, \ldots, \mu_{k+1}, F_c, \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) F_{c_{k+r}(\mu_0,\ldots,\mu_{k+r-1})} \\ &= A_r\left(\mu_0, \ldots, \mu_{k+r}, F_c, \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) \end{aligned} \tag{7.11}$$

for any $r > 0$ and any $\mu_0, \ldots, \mu_{k+r} \in \mathbb{R}^m$. Let

$$\Omega = \Big\{(\mu_0, \ldots, \mu_{k+1}) : \\ A_0\left(\mu_0, \ldots, \mu_{k+1}, F_c, \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) \neq 0\Big\}.$$

It follows from the fact that $A_0 = 0$ is not an i/o equation of F_c that there exists some

$$(\mu_0, \ldots, \mu_{k+1}) \in \mathbb{R}^{m(k+2)},$$

some $u \in \mathcal{V}_T$, and some $\tau \in [0, T]$ so that

$$A_0\left(\mu_0, \ldots, \mu_{k+1}, F_c[u](\tau), \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}[u](\tau)\right) \neq 0.$$

Since the function

$$\psi(\mu_0, \mu_2 \ldots, \mu_{k+1}) \\ := A_0\left(\mu_0, \ldots, \mu_{k+1}, F_c[u](\tau), \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}[u](\tau)\right)$$

is an analytic function,

$$\Omega_1 := \Big\{(\mu_0, \ldots, \mu_{k+1}) : \psi(\mu_0, \ldots, \mu_{k+1}) \neq 0\Big\}$$

is an open dense subset of $\mathbb{R}^{m(k+2)}$. As $\Omega_1 \subseteq \Omega$, Ω is itself an open dense set of $\mathbb{R}^{m(k+2)}$.

Now we let Φ be the set of all the coefficients of μ_{ij} that appear in $c_n(\mu_0, \ldots, \mu_{n-1})$, seen as a polynomial in μ_{ij} over the ring of power series in variables $\eta_0, \ldots, \eta_m$ for all $n \leq k+1$. Note that Φ is a finite set of power series.

Pick up an arbitrary $r \geq 2$. Equation (7.11) implies that $c_{k+r}(\mu_0, \ldots, \mu_{k+r-1})$ is meromorphically generated by the elements of Φ if

$$(\mu_0, \mu_1, \ldots, \mu_{k+1}, \mu_{k+2}, \ldots, \mu_{k+r-1}) \in \Omega \times \mathbb{R}^{r-2} .$$

Since Ω is dense in $\mathbb{R}^{k+2}$, it follows that $\Omega \times \mathbb{R}^{r-2}$ is dense in $\mathbb{R}^{k+r}$. By Lemma 4.2.5, we know that for any choice of $\mu_0, \ldots, \mu_{k+r-1}$,

$$c_{k+r}(\mu_0, \mu_1, \ldots, \mu_{k+r}) \tag{7.12}$$

is a linear combination of the elements in the set

$$\mathcal{B} := \left\{ c_{k+r}(\mu_0, \ldots, \mu_{k+r-1}) : (\mu_0, \ldots, \mu_{k+r-1}) \in \Omega \times \mathbb{R}^{r-2} \right\} .$$

It follows immediately that (7.12) is meromorphically generated by the elements of Φ for any $\mu_0, \ldots, \mu_{k+r-1}$. Since r can be chosen arbitrarily, we get our conclusion that all of $Q_2(c)$ is meromorphically generated by the finite set Φ. ∎

7.2 Realizability by Analytic Systems

For any given convergent series c, we say that F_c is *realizable by an analytic system*

$$\Sigma = (\mathcal{M}, (g_0, \ldots, g_m), x_0, h) \tag{7.13}$$

if there exist some analytic manifold $\mathcal{M}$, some $x_0 \in \mathcal{M}$, $(m+1)$ analytic vector fields

$$g_0, g_1, \ldots, g_m$$

on $\mathcal{M}$ and an analytic function

$$h : \mathcal{M} \to \mathbb{R}$$

such that for each $u \in \mathcal{V}_T$ with T admissible for c, there exists a solution $x(\cdot)$ of the equation

$$
\begin{aligned}
x' &= g_0(x) + \sum_{j=1}^{m} g_j(x) u_j \ , \\
x(0) &= x_0
\end{aligned}
$$

defined on all of $[0, T]$, and

$$F_c[u](t)(t) = h(x(t)) \,, \ \ t \in [0, T] \,. \tag{7.14}$$

We shall say that F_c is *locally* realizable by an analytic system (7.13) if the solution $x(\cdot)$ of (7.13) is only defined for, and (7.14) only holds for, t small enough.

Recall that the observation space $\mathcal{F}_1(c)$ associated with a series c is defined to be the $\mathbb{R}$-space spanned by all the series $\alpha^{-1}c$, the observation algebra $\mathcal{A}_1(c)$ is the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_(c)$, and the observation field is the quotient field of $\mathcal{A}_1(c)$. The following Theorem shows that certain finiteness properties imply realizability.

Theorem 12 *Assume that c is a convergent series. Then*

(a) F_c is realizable by an analytic system if $\mathcal{A}_1(c)$ is analytically finitely generated.

(b) F_c is locally realizable by an analytic system if $\mathcal{Q}_1(c)$ is a meromorphically finitely generated field extension of $\mathbb{R}$.

Proof. (a): Assume that $\mathcal{A}_1(c)$ is generated by

$$c_1, c_2, \ldots, c_n,$$

for some integer n. It follows that there exist an analytic function g_{ij} such that

$$F_{\eta_j^{-1}c_i} = g_{ij}(F_{c_1}, \ldots, F_{c_n}) \tag{7.15}$$

for any $i = 1, \ldots, n$, each $j = 0, 1, \ldots, m$, and an analytic function h such that

$$F_c = h(F_{c_1}, F_{c_2}, \ldots, F_{c_n}) \ .$$

Take $\mathcal{M}$ to be the Euclidean space $\mathbb{R}^n$ and let

$$x_0 = (\langle c_1, \phi\rangle, \langle c_2, \phi\rangle \ldots, \langle c_n, \phi\rangle)' . \tag{7.16}$$

It follows from (7.15) and formula (2.36) that the function

$$x(t) = (F_{c_1}[u](t), F_{c_2}[u](t), \ldots, F_{c_n}[u](t))'$$

satisfies equations

$$\begin{aligned} x' &= g_0(x) + \sum_{j=1}^{m} g_j(x)\, u_j \ , \\ x(0) &= x_0 \ , \end{aligned}$$

for any $u \in \mathcal{V}_T$, and (7.16) implies that

$$y(t) = F_c[u](t) = h(x(t))$$

for all $t \in [0, T]$. We proved that F_c is realizable by an analytic system.

(b): Assume that $\mathcal{Q}_1(c)$ is meromorphically generated by

$$c_1, c_2, \ldots, c_n .$$

Without loss of generality, we may assume that $c = c_1$ and $c_i \in \mathcal{A}_1(c)$ for all i. Similarly to part (a), one knows that for each $i = 1, 2, \ldots, n,\ j = 0, 1, \ldots, m$, there exist analytic functions g_{ij} and q_{ij} such that

$$q_{ij}(F_{c_1}, \ldots, F_{c_n})\, F_{\eta_j^{-1}c} = g_{ij}(F_{c_1}, \ldots, F_{c_n}) \tag{7.17}$$

and

$$q_{ij}(F_{c_1}, \ldots, F_{c_n}) \neq 0.$$

(These are all equations among operators.) By Remark 5.1.3, for each fixed i and j, the functions

$$q_{ij}(F_{c_1}[u], \ldots, F_{c_n}[u])$$

are not identically zero for infinitely many inputs u. Hence we may assume that $q_{ij} = q$ for any i and j and

$$q(F_{c_1}[u], \ldots, F_{c_n}[u]) \neq 0 \tag{7.18}$$

by letting

$$q = \prod_{i,j} q_{ij}$$

and changing g_{ij} accordingly. As we did in the algebraic case, one shows that for any $u \in \mathcal{V}_T$, the function

$$x(t) = (F_{c_1}[u](t), F_{c_2}[u](t), \ldots, F_{c_n}[u](t))' \tag{7.19}$$

satisfies the equation

$$q(x(t))\, x'(t) = g_0(x(t)) + \sum_{j=0}^{m} g_j(x(t))\, u_j(t) \tag{7.20}$$

$$x(0) = x_0 = (\langle c_1, \phi\rangle, \langle c_2, \phi\rangle \ldots, \langle c_n, \phi\rangle)', \tag{7.21}$$

and,

$$y(t) = F_c[u](t) = x_1(t).$$

It is clear that *if*

$$q(x_0) \neq 0\,, \tag{7.22}$$

then $q(x) \neq 0$ in a neighborhood $\mathcal{N}$ of x_0, thus, F_c is locally realized by the analytic system

$$(\mathcal{N}, \tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_m, x_0, h)\,,$$

where $\tilde{g}_i = \dfrac{g_i}{q}$ and $h(x) = x_1$.

We now assume $q(x_0) = 0$. Note that the function $x(\cdot)$ defined by (7.19) is of class C^k if u is of class C^{k-1} for any $k \geq 1$, and $x(\cdot)$ is analytic if u is analytic (because of Lemma 2.3.8 and Corollary 2.3.6 in Chapter). Since analytic controls is dense in $\mathcal{V}_T$ (with respect to the L^1 topology), (7.19) implies that there is at least one analytic input u_0 for which the function

$$q\,(F_{c_1}[u_0], \ldots, F_{c_n}[u_0])$$

is not identically zero. Fix such an u_0. Then the analyticity of $F_{c_i}[u](t)$ and the analyticity of q imply that there exists some $\delta > 0$ such that

$$q\,(F_{c_1}[u_0](t), \ldots, F_{c_n}[u_0](t)) \neq 0 \tag{7.23}$$

for all $t \in (0, \delta)$. For each $\lambda \in (0, \delta)$, we define a series c^λ by

$$\langle c^\lambda, \eta_i \rangle = F_{\eta_i^{-1}c}[u_0](\lambda).$$

According to Lemma 2.2.5, c^λ is a convergent series and

$$F_{c^\lambda}[u](t) = F_c[u_0 \#_\lambda u](\lambda + t) \tag{7.24}$$

for any $u \in \mathcal{V}_{T-\lambda}$ and any $\lambda \in (0, \delta)$, which implies that, for each $u \in \mathcal{V}_{T-\lambda}$, the function

$$x^\lambda(t) = \left(F_{c_1^\lambda}[u](t),\ F_{c_2^\lambda}[u](t),\ \ldots,\ F_{c_n^\lambda}[u](t)\right)'$$

also satisfies equation (7.20) with the initial state

$$x^\lambda(0) = (F_{c_1}[u_0](\lambda),\ F_{c_2}[u_0](0),\ \ldots,\ F_{c_n}[u_0](0))'$$

and

$$F_c^\lambda[u](t) = x_1^\lambda(t)\,.$$

Notice that (7.24) means that

$$q(x_0^\lambda) \neq 0$$

for each $\lambda \in (0, \delta)$. thus for each fixed λ, there exists a neighborhood $\mathcal{N}_\lambda$ of x^λ such that

$$q(x) \neq 0, \text{ for all } x \in \mathcal{N}_\lambda.$$

Hence, each of these "perturbed" operators F_{c^λ} is locally realized by the analytic system

$$\left(\mathcal{N}_\lambda,\ (\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_m),\ x^\lambda,\ h\right).$$

To show that F_{c^λ} is still realizable when $\lambda = 0$, we need to introduce more notations. Recall that $\mathcal{P}$ is the set of polynomials in $\eta_0, \eta_1, \ldots, \eta_m$. We now define the *Lie bracket* $[\cdot\,,\,\cdot]$ on $\mathcal{P}$ as follows:

$$[P_1,\ P_2] = P_1 \cdot P_2 - P_2 \cdot P_1$$

where "$\cdot$" denotes the standard product defined for polynomials. With $[\cdot\,,\,\cdot]$ defined as above, $\mathcal{P}$ forms a Lie algebra. Let $\mathcal{L}$ be the subalgebra of $\mathcal{P}$ generated by $\eta_0, \eta_2, \ldots, \eta_m$. The elements of $\mathcal{L}$ will be called *Lie polynomials*.

We defined in §3.1.2 the notation

$$\psi_c(w) = \sum \langle w,\ \eta_\kappa \rangle \eta_\kappa^{-1} c$$

for any polynomial $w = \sum \langle w,\ \eta_\kappa \rangle$. Now for a given series c, we define the *Lie rank* $\rho(c)$ of c to be the dimension of the $\mathbb{R}$-space spanned by $\psi_c(w)$, over all Lie polynomials $w \in \mathcal{L}$, i.e.,

$$\rho(c) = \dim \left(\operatorname{span}_{\mathbb{R}} \{ \psi_c(w) : \ w \in \mathcal{L} \} \right) .$$

It is well-known that the i/o operator F_c is locally realizable by an analytic system if and only if the Lie rank $\rho(c)$ is finite, and, if F_c is realizable by a system of dimension n, then the Lie rank $\rho(c)$ is less than or equal to n (cf [11], [16], [35]).

It follows from the second part of the above statement that the Lie rank $\rho(c^\lambda)$ of c^λ is bounded by n for any $\lambda \in (0,\ \delta)$.

Claim: $\rho(c) \leq n$.

Suppose $\rho(c) > n$. Then there exist $w_1, \ldots, w_{n+1}$ such that the $n+1$ series $\psi_c(w_1), \ldots, \psi_c(w_{n+1})$ are linearly independent.

We now enumerate the elements of P^*, the set of all monomials in η_i's, as $z_1,\ z_2, \ldots$ and we let A_λ be the matrix of $n+1$ columns and infinitely many rows whose (i,j)-th entry is $\langle \psi_{c^\lambda}(w_j),\ z_i \rangle$. Then A_0 is full column rank in the sense that there is no nonzero $(n+1)$-vector v such that $A_0\, v = 0$. Following exactly the same steps in the proof of Theorem 2, one can prove that there exists some submatrix B of A_0 consisting of the first k rows of A_0 such that $\operatorname{rank} B = n+1$. Now let the B_λ be the submatrix of A_λ consisting of the first k rows of A_λ. Note that for the (ij)-th entry a_{ij}^λ of B_λ, we have

$$\begin{aligned} a_{ij}^\lambda &= \langle \psi_{c^\lambda}(w_j),\ z_i \rangle \\ &= \langle c^\lambda,\ w_j z_i \rangle = F_{(w_j z_i)^{-1} c}[u_0](\lambda). \end{aligned}$$

It follows that the entries of matrix B_λ are continuous functions of λ, which implies that the rank of B_λ is a semicontinuous function of λ. Therefore,

$$\operatorname{rank} B_\lambda = n+1 \tag{7.25}$$

for $\lambda \in (0,\ \varepsilon)$ for some $\varepsilon > 0$. It follows immediately from (7.25) that the series

$$\psi_{c^\lambda}(w_1),\ \psi_{c^\lambda}(w_2) \ldots, \psi_{c^\lambda}(w_{n+1})$$

are linearly independent. This contradicts the fact that $\rho(c^\lambda)$ is bounded by n. Therefore, the Lie rank $\rho(c)$ is bounded by n. Applying the results in [11], [16], [35] cited above, we conclude that F_c is locally realizable by an analytic system. Furthermore, since $\rho(c) \leq n$, we know that also F_c is locally realizable by a system of dimension less than or equal to n. ∎

Combining Theorem 2, Theorem 11 and Theorem 12 we get our main results in this Chapter.

Theorem 13 *Assume that c is a convergent series. Then*

(a) F_c *is realizable by an analytic system if* F_c *satisfies an analytically recursive i/o equation.*

(b) F_c *is locally realizable by an analytic system if* F_c *satisfies an analytic i/o equation.*

∎

7.3 An Example

We have shown in last section that the existence of an analytic i/o equation implies realizability by analytic systems. However, in contrast to the algebraic analogues of these results, the converse of Theorem 13 does not hold in general. In this section, we give some examples to illustrate this fact.

Consider any fixed sequence of analytic functions $f_0, f_1, \ldots$ of one variable, defined on a neighborhood of $|x| \leq 1$, that satisfying the following conditions:

1. The functions $f_0, f_1, \ldots$ are algebraically independent over $\mathbb{R}$;
2. for each k, the algebra $\mathbb{R}(f_0, f_1, \ldots, f_k)$ is differentiably closed, i.e., for each k, there is a polynomial P_k such that

$$f_k'(x) = P_k(f_0, f_1(x), \ldots, f_k(x))\,;$$

3. there exists some fixed complex neighborhood $\mathcal{U}$ of the unit interval of the real line such that f_k has a complex analytic extension $\varphi_k(z)$ on $\mathcal{U}$.

Let $\mathbf{C}$ be a compact set so that

$$\mathcal{U}_1 \subset \mathbf{C} \subset \mathcal{U}\,,$$

where $\mathcal{U}_1$ is a complex neighborhood of the unit interval of the real line. Let, for each k, a_k be such that

$$a_k \max_{z\in\mathbf{C}} \varphi_k(z) \le 1$$

and let

$$h_1(x_1, x_2) = \sum_{k=0}^{\infty} a_k f_k(x_1) \frac{x_2^k}{k!}\,.$$

Then h_1 is an analytic function defined on $[0,1]\times\mathbb{R}$. This is because that the complex analytic extension of h_1 to $\mathcal{U}_1 \times \mathbb{C}$ is given by the series $\sum a_k \varphi_k(z_1)\frac{z_2^k}{k!}$, which converges uniformly on any compact set of $\mathcal{U}_1 \times \mathbb{C}$.

Consider the system with $\mathcal{M} = \mathbb{R}^3$ and equations

$$\begin{aligned} x_1' &= u_1, \\ x_2' &= u_2, \\ x_3' &= u_3, \end{aligned} \tag{7.26}$$

with initial state $x(0) = 0$. Take the output function

$$h(x) = e^{x_1} h_1(x_2, x_3) = e^{x_1} \sum_{k=0}^{\infty} a_k f_k(x_2) \frac{x_3^k}{k!}\,. \tag{7.27}$$

Lemma 7.3.1 Let F be the i/o operator defined by system (7.26). Then, the following conclusions holds:

(a) $F[u]$ is defined for $0 \le t \le 1$ and all u for which $\|u_2\|_\infty \le 1$.

(b) F does not satisfy any analytic i/o equation.

(c) There is no singular polynomial system realizing F.

Proof. (a): Clearly $|x_2(t)|$ is bounded by 1 on $[0, 1]$ if $|u_2|$ is bounded by 1. Then the fact that

$$h(x(t)) = e^{x_1(t)} \sum a_k f_k(x_2(t)) \frac{x_3(t)^k}{k!}$$

is a composition of analytic functions implies that $h(x(t))$ is analytic on $[0, 1]$.

(b): Assume that F satisfies an analytic i/o equation

$$A\left(u(t), u'(t), \ldots, u^{(n-1)}(t),\, y(t), \ldots, y^{(n)}(t)\right) = 0. \tag{7.28}$$

where $u = (u_1, u_2, u_3)$. Write A as

$$\begin{aligned} &A\left(u(t), u'(t), \ldots, u^{(n-1)}(t), y(t), \ldots, y^{(n)}(t)\right) \\ &\qquad = \sum_{i=0}^{\infty} A_i\left(u(t), u'(t), \ldots, u^{(n-1)}(t), y(t), \ldots, y^{(n)}(t)\right) \end{aligned}$$

where A_i is a homogeneous polynomial of degree i in the variables $y, y', \ldots, y^{(n)}$ and is analytic in the first $3n$ variables. Note that since $y, y', \ldots y^{(n)}$ all depend on e^{x_1} linearly, (7.28) can be written as

$$\sum_{i=0}^{\infty} (e^{x_1(t)})^i A_i\left(u(t), u'(t), \ldots, u^{(n-1)}(t), y_0(t), \ldots, y_n(t)\right) = 0, \tag{7.29}$$

where

$$y_i(t) = y^{(i)}(t)e^{-x_1(t)}.$$

Taking $t = 1/2$ and using μ_{ij} to denote $u_i^{(j)}(1/2)$ and μ_j for $\mu_{1j}, \mu_{2j}, \mu_{3j}$, we get

$$\sum_{i=0}^{\infty} (e^{x_1(\frac{1}{2})})^i A_i\left(\mu_0, \ldots, \mu_{n-1}, \ldots, y_0(\frac{1}{2}), \ldots, y_n(\frac{1}{2})\right) = 0, \tag{7.30}$$

for any μ_j's. But for any $\mu_{10}, \ldots, \mu_{1n-1}$ and $b \in \mathbb{R}$ given, there exists some analytic input function u such that

$$u^{(j)}(\frac{1}{2}) = \mu_{1j} \quad \text{and} \quad \int_0^{\frac{1}{2}} u(s)\, ds = b.$$

For instance, one such function can be taken as

$$\omega(t) = \sum_{j=1}^{n-1} \frac{(t-\frac{1}{2})^j}{j!}\mu_{1j} + \frac{(t-\frac{1}{2})^n}{n!}\mu_{1n}\,,$$

where

$$\mu_{1n} = -b + (n+1)! \sum_{j=1}^{n-1} \frac{(-\frac{1}{2})^{(j-n)}}{(j+1)!}\mu_{1j}\,.$$

Thus (7.30) implies that

$$\sum_{i=0}^{\infty} a^i A_i\left(\mu_0, \ldots, \mu_{n-1}, y_0(\frac{1}{2}), \ldots, y_n(\frac{1}{2})\right) = 0, \tag{7.31}$$

for any μ_i and $a > 0$. It then follows that

$$A_i\left(\mu_0, \ldots, \mu_{n-1}, y_0(\frac{1}{2}), \ldots, y_n(\frac{1}{2})\right) = 0, \tag{7.32}$$

for any i. Notice that

$$x_3(\frac{1}{2}) = \int_0^{\frac{1}{2}} u_3(s)ds.$$

Thus for any $\mu_{30}, \ldots, \mu_{3(n-1)}$, there exists some u_3 such that

$$u_3^{(j)}(\frac{1}{2}) = \mu_{3j} \quad \text{and} \quad x_3(\frac{1}{2}) = 0.$$

With such a choice of u_3,

$$y_i(\frac{1}{2}) = \mu_{30} f_i(\beta) + Y_i$$

where $\beta = \int_0^{\frac{1}{2}} u_2(s)\, ds$ and Y_i is some function which is polynomial on $f(\beta)$, $f_1(\beta)$, ..., $f_{i-1}(\beta)$ and their derivatives and analytic in the μ_j's. Again for any μ_{2j}, $(j = 0, 1, \ldots, n-1)$ and $\beta \in \mathbb{R}$ given, there exists some analytic input function u_2 satisfying $u_2^{(j)}(\frac{1}{2}) = \mu_{2j}$ and $\int_0^{\frac{1}{2}} u_2(s)\, ds = \beta$. It follows that

$$A_i\left(\mu_0, \ldots, \mu_{n-1}, f_0(b), \mu_{30} f_1(b) + Y_1, \ldots, \mu_{30}^n f_n(b) + Y_n\right) = 0, \tag{7.33}$$

for all i and and any choice of μ_i's b. To complete this example, we need the following simple algebraic lemma:

Lemma 7.3.2 Assume that $P : \mathbb{R}^{n+2} \to \mathbb{R}$ an analytic function which depends on the last $n+1$ variables polynomially, i.e., $P(x_0, x_1, \ldots, x_{n+1})$ is a polynomial in $x_1, \ldots, x_{n+1}$ for each fixed value of x_0. Let f_k be defined as before and assume that Y_k is some function which is polynomial in $f_1, f_2, \ldots, f_{k-1}$ and analytic in some parameter μ. Then the following equation

$$P(\mu, f_0(x), \mu f_1(x) + Y_1, \ldots, \mu^n f_n(x) + Y_n) = 0, \quad \text{for any} \quad \mu, x \in \mathbb{R}$$

implies that P is zero identically. □

It will follow from this Lemma immediately that A_i vanishes identically, which in turn implies that A vanishes identically. Thus we obtain the desired conclusion that F cannot satisfy any nontrivial analytic i/o equation.

(c): By Theorem 9 (a), if F were realizable by a singular polynomial system, F would satisfy a polynomial i/o equation, which contradicts conclusion (b). ∎

We now return to prove Lemma 7.3.2.

Proof. We use induction in n. The lemma is true for $n = 0$ since $P(\mu, f_0)$ is a polynomial in x for each fixed value of μ. Now assume the conclusion is true for $n \leq N$. Suppose we have

$$P(\mu,\ f_0(x),\ \mu f_1(x) + Y_1,\ \ldots,\ \mu^{N+1} f_{N+1}(x) + Y_N) = 0.$$

Assume P is not a zero function and

$$\begin{aligned} P(x_0,\ x_1,\ \ldots,\ x_{N+1}) &= P_k(x_0,\ x_1, \ldots,\ x_N)\, x_{N+1}^k \\ &\quad + P_{k-1}(x_0,\ x_1,\ \ldots,\ x_N)\, x_{N+1}^{k-1} + \ldots + P_0(x_0,\ x_1,\ \ldots,\ x_N) \end{aligned}$$

where $P_k(x_0, \ldots, x_N)$ is not a zero function. Write

$$\begin{aligned} &P(\mu,\ f_0(x),\ \mu f_1(x) + Y_1,\ \ldots,\ \mu^{N+1} f_{N+1}(x) + Y_N) \\ &\quad = Q_k(\mu,\ f_0(x),\ f_1(x),\ \ldots,\ f_N(x))\, f_{N+1}(x)^k \\ &\quad + Q_{k-1}(\mu,\ f_0(x),\ f_1(x),\ \ldots,\ f_N(x))\, f_{N+1}(x)^{k-1} \\ &\quad + \ldots + Q_0(\mu,\ f_0(x),\ f_1(x),\ \ldots,\ f_N(x)). \end{aligned}$$

Then we have

$$\begin{aligned} &Q_k(\mu,\ f_0(x),\ f_1(x),\ \ldots,\ f_N(x)) \\ &\qquad = \mu^k P_k(\mu,\ f_0(x),\ \mu f_1(x) + Y_1,\ \ldots,\ \mu^N f_N(x)). \end{aligned}$$

Since $f_0, f_1, \ldots, f_{N+1}$ are algebraically independent over the field $\mathbb{R}$, it follows that for any fixed μ,

$$Q_k(\mu, f_0(x), f_1(x), \ldots, f_N(x)) \equiv 0,$$

which implies that for any $\mu \neq 0$ and any x

$$P_k(\mu, f_0(x), \mu f_1(x) + Y_1, \ldots, \mu^N f_N(x) + Y_N) = 0\,. \tag{7.34}$$

By continuity, one knows that (7.34) holds identically. This fact contradicts the induction assumption. Therefore P must be identically zero. ∎

To construct a concrete example, we let

$$f_k(x) = a_k \underbrace{\exp(\exp(\cdots(\exp(x))\cdots))}_{k},$$

for $k \geq 1$, and $f_0(x) = 1$, and take $a_k = (f_k(1))^{-1}$, $k = 0, 1, \ldots$. It is a standard fact that f_0, f_1, ... are algebraically independent. This fact can be shown by induction as follows:

Clearly, f_0 and f_1 are algebraically independent. Assume that $f_0, f_1, \ldots, f_n$ are independent but $f_0, \ldots, f_n, f_{n+1}$ are algebraically dependent. Then there would exist some polynomial P which is nontrivial in the last variable such that

$$P(f_1(x), f_2(x), \ldots, f_{n+1}(x)) = 0 \quad \text{for all } x. \tag{7.35}$$

Write P as

$$P_s(X_1, \ldots, X_n)X_{n+1}^s + P_{s-1}(X_1, \ldots, X_n)X_{n+1}^{s-1} + \cdots + P_0(X_1, \ldots, X_n).$$

Without loss of generality, we may assume that P_0 is a nontrivial polynomial. Dividing both sides of equation (7.35) by f_{n+1}, letting $x \to \infty$ and taking limits on both sides of (7.35), we get

$$\lim_{x\to\infty} P_0(f_1(x), f_2(x) \ldots, f_n(x)) = 0\,. \tag{7.36}$$

Claim: There is no nonzero polynomial P_0 such that (7.36) holds.

The induction proof will be completed if we show the above claim is true. We now proof the claim. It is not hard to see that the claim is true for $k = 1$. We now assume that the conclusion is true for n but fails for $n+1$. Then there would exist some nonzero polynomial Q, nontrivial in the last variable, so that

$$\lim_{x\to\infty} Q(f_1(x), \ldots, f_{n+1}(x)) = 0. \tag{7.37}$$

Write Q as $\sum_{i=0}^{l} Q_i(X_1, \ldots, X_n)X_{n+1}^i$. Then

$$\begin{aligned} &Q(f_1(x), \ldots, f_{n+1}(x)) \\ &= f_{n+1}^l(x)\,(Q_l(f_1(x), \ldots, f_n(x)) + \frac{Q_{l-1}(f_1(x), \ldots, f_n(x))}{f_{n+1}(x)} \\ &\qquad + \cdots + \frac{Q_0(f_1(x), \ldots, f_n(x))}{f_{n+1}^l(x)})\,. \end{aligned} \tag{7.38}$$

Since $\lim_{x\to\infty} \frac{T(f_1(x),\ldots,f_n(x))}{f_{n+1}(x)} = 0$ for any polynomial T, it follows that

$$\lim_{x\to\infty} Q(f_1(x),\ldots,f_{n+1}(x)) = \lim_{x\to\infty} f_{n+1}^l(x)\, Q_l(f_1(x),\ldots,f_n(x)). \tag{7.39}$$

The fact $\lim_{x\to\infty} f_{n+1}(x) = \infty$ and the induction assumption imply that the limit in (7.39) cannot be zero, which implies that (7.37) is impossible.

For such a choice of $\{f_k\}$, $f_k' \in \mathbb{R}(f_0, f_1, \ldots, f_k)$ for any k. By Lemma 7.3.1, the i/o operator given by (7.26) does not satisfy any analytic i/o equation.

Simpler systems are enough in order to provide counterexamples to stronger statements. For instance, if we consider only the 2-dimensional system

$$\begin{aligned} x_1' &= u_1, \\ x_2' &= u_2, \\ y &= \sum_{i=0}^{\infty} a_k f_k(x_1) \frac{x_2^k}{k!} \end{aligned} \tag{7.40}$$

with the initial state $x(0) = 0$, where f_i and a_i are still as in Lemma 7.3.1, then by the same approach used before one can show that there is no *algebraic* i/o equation for the i/o operator F defined by (7.40). This is enough to imply that there is no singular polynomial system realizing F.

The same example without the x_1 coordinate is

$$\begin{aligned} x' &= u, \\ y &= e^x \end{aligned}$$

with $x(0) = 0$, whose i/o operator F satisfies a recursive i/o equation

$$y' = yu$$

and is realizable by a polynomial system. Indeed, if we let $z = e^x$, then $z' = zu$ and therefore, F is realized by the polynomial system

$$\begin{aligned} z' &= zu, \\ y &= z \end{aligned}$$

with $z(0) = 1$.

Chapter 8

Final Remarks

In this Chapter we summarize our main results, discuss their possible applications and indicate some ideas for further research in this area.

8.1 Main Conclusions

This work has provided an equivalence between realizability and existence of i/o equations.

There are two different definitions of observation spaces, one in terms of smooth controls and the other in terms of piecewise constant ones. In Chapter 4 we showed that existence of i/o equations is closely related to the finiteness properties of the observation algebra and field associated with the first type of observation space; and in Chapter 5 we showed that realizability is closely related to the finiteness properties of the algebraic objects associated with the other observation space. A result provided in Chapter 3 showed that the two observation spaces coincide.

Based on the results mentioned above, we obtained our main conclusion: Realizability by singular polynomial systems is equivalent to existence of algebraic i/o equations. We also provided results relating various special kinds of i/o equations to some specific classes of realizations, for instance, what are called recursive i/o equations were related to realizability by polynomial systems. These results were also extended to families of operators. In Chapter 7, our results were extended also to analytic i/o equations and local realization by analytic systems; we concluded that the existence of analytic i/o equations implies local realizability by analytic systems.

8.2 Possible Applications

We envision our results being used as follows. The idea is very similar to that employed in the discrete case, and explored in some detail in [8]. If there are reasons to believe that the system producing the observed data is well-posed, then an equation E may be fit to the data. We are *assured* that there is then a realization of the type to be considered, and we then try to find this realization. Efficient techniques for obtaining the realization are a major topic for further research involving symbolic computation, but the following example illustrates the basically constructive character of the proofs.

Consider the input/output equation

$$uy'' = y^2u^2 + y'u' \tag{8.1}$$

and assume that it is "well-posed" in the sense mentioned above, that is, that there is a Fliess operator $y = F_c[u]$ for which every pair $(u, F_c[u])$ satisfies the equation. Then we know that F_c can be realized by some polynomial state space system

$$x' = f(x) + g(x)u, \tag{8.2}$$

$$y = h(x)\,. \tag{8.3}$$

with some fixed initial state. We try to deduce now what f, g, h should be. We have

$$\begin{aligned} y' &= L_fh(x) + L_gh(x)u, \\ y'' &= L_f^2h(x) + (L_fL_gh(x) + L_gL_fh(x))\,u + L_g^2h(x)u^2 + L_gh(x)u'. \end{aligned}$$

Substituting y, y', y'' into equation (8.1) we get the following formulas:

$$L_fh = 0, \tag{8.4}$$

$$L_fL_gh + L_gL_fh = h^2, \tag{8.5}$$

$$L_g^2h = 0. \tag{8.6}$$

Formulas (8.4) and (8.5) suggest that $L_f^2h = 0$ and $L_fL_gh = h^2$. Now let

$$z_1 = h(x), \quad z_2 = L_gh(x).$$

Then along any trajectory $x(t)$ of (8.2),

$$\begin{aligned} z_1'(t) &= L_f h(x(t)) + L_g h(x(t))u(t) = z_2(t)u(t), \\ z_2'(t) &= L_f L_g h(x(t)) + L_g^2 h(x(t))u(t) = z_1(t)^2. \end{aligned}$$

Hence, F_c can be realized by the following polynomial system

$$\begin{aligned} z_1' &= z_2 u, \\ z_2' &= z_1^2, \\ y &= z_1, \end{aligned}$$

where the choice of initial state will depend on additional data (such as the knowledge of $y(0)$ and $y'(0)$ for some nonzero control).

Of course, for practical applications, it is not clear when one would be justified in assuming well-posedness. But we take the position that postulating well-posedness is a far weaker assumption than assuming that the data was produced by a linear system. The assumption of linearity underlies most applications of control theory, on the other hand.

8.3 Ideas for Further Research

Among future directions in this area, perhaps one of the most interesting problems is to study the possibility of realizability by nonsingular polynomial systems (those for which q is identically 1) as well as by "rational systems" (those for which q never vanishes). It is very possible that under extremely weak conditions every i/o operator might be realizable by some such systems, or at worst a system which can be naturally decomposed into a finite number of components of this type. Essentially this is the case for discrete-time in [34], where it is shown that realizability is equivalent to the existence of realizations whose state spaces are (Grothendieck) schemes which can be stratified in this fashion. In the present context, the treatment is complicated by the problem of understanding the meaning of differential equations over such spaces, as well the difficulties that arise when trying to prove existence and uniqueness results for differential equations on such spaces, i.e. integrability results for vector fields defined by formal derivation operators in the corresponding algebra of functions. The work [2] (and other papers by the same author) has already given preliminary results in that direction, however, and we plan to exploit these.

Another major topic is to find efficient techniques for obtaining the realizations. In the discrete time case, the work in [34] and [31] has been used as a basis of identification algorithms by other authors; see for instance [22] and [8] (The former also shows how to include stochastic effects in the resulting approach). Such algorithms for continuous time systems could have wide applications in system identification. See, for instance, [25].

In [34], what are called *integral* i/o equations are related to realizability by polynomial systems. For continuous time, an integral i/o equation should be of the following type:

$$P_0(u(t), \ldots, u^{(k)}(t))(y^{(k)}(t))^r = P_1(u(t), \ldots, u^{(k)}(t), y(t), \ldots . y^{(k-1)}(t)),$$

where P_0, P_1 are some polynomials with $P_0 \neq 0$ and k, r are some integers. We are still not clear whether the analogue for continuous time systems holds, —i.e., does existence of integral an i/o equation imply realizability by a polynomial system? One difference between discrete time systems and continuous time systems is that in the discrete time case, a nontrivial (i.e., $P_0 \neq 0$) k-th order integral i/o equation automatically gives rise to a nontrivial $(k+1)$-th order integral i/o equation, while in continuous time case this may not be true.

There is another topic which was treated in [34] but we didn't cover here. In [34], a Jacobian condition was presented for checking if a given observation field has finite transcendence degree over $\mathbb{R}$, for the discrete time case. It is very possible that, for the continuous time case, such a Jacobian condition should also exist. As we have shown in Chapter 4, if the transcendence degree of the observation field associated with a given i/o operator is finite, then the operator satisfies an i/o equation. But generally speaking, it is very hard to check whether a field has a finite transcendence degree or not. Thus such a Jacobian condition should provide a useful tool for checking whether an i/o operator admits an i/o equation. But to find out a Jacobian condition, one should first choose some suitable indeterminates. In discrete time case discussed in [34], the indeterminates were chosen to be the control variables since the output functions are polynomials in the control variables. The question for continuous time case is how to choose indeterminates properly?

For singular polynomial systems, instead of systems of the type considered before, one may also consider systems of the following type:

$$E(x)x' = p_0(x) + \sum_{i=1}^{m} p_i(x)u_i, \qquad x \in R^n \tag{8.7}$$
$$y = h(x), \qquad y \in \mathbb{R}$$

where $E(x)$ is a matrix whose entries are polynomial functions of x. This generalizes the linear "descriptor systems" model, (see eg. [5] and [41].) The question is when an i/o operator induced by system (8.7) with some initial state $x(0) = 0$ admits an algebraic i/o equation? If we let $q(x) = \det E(x)$, then it can be seen that the operator admits an algebraic i/o equation under the following conditions:

1. the initialized singular system $((p_0, \ldots, p_m),\ x_0,\ q,\ h)$ satisfies the regularity condition in Definition 5.1.1;

2. the operator is a Fliess operator.

The question is whether these conditions can be relaxed, and, if they can, to what extent?

The work presented here are *"local"* in some sense: First of all, we asked that each i/o operator has a *global* power series representation. However, in the same manner as an analytic function may not have a global Taylor series representation, an *"analytic"* i/o operator may not have a global power series representation. Of course, even how to define an analytic i/o operator is still not very clear. Secondly, for each i/o operator F_c, we only deal with controls in $\mathcal{V}_T$ and the corresponding outputs for which T is admissible for the series c. For a given series c, this T may be too conservative in the sense that the output functions corresponding to inputs in a generic set are well defined in a greater interval. For instance, an output function of an operator that arises from a state space system may blow up in a small time interval for certain input, but its response to the other inputs may still be well defined in a greater interval. One interesting question is to investigate the "global" problem. If one could define *analytic* i/o operator properly, then, based on results established in [18], it should not be hard to obtain a global version of this work.

As a final remark, we would like to point out that one may also consider formal realizations such as those discussed in [18]. For rigorous definition of formal realization, we refer the readers to [18]. Roughly speaking, for a formal power series c, —which can be thought of as a nonconvergent Taylor expansion,— if there exists some formal vector fields $\{g_i\}_{i=0}^m$ and some formal analytic function h and some point x_0 such that

$$L_{g_{i_k}} \cdots L_{g_{i_1}} h(x_0) = \langle c, \eta_{i_1} \cdots \eta_{i_k} \rangle \tag{8.8}$$

for any k and any $0 \le i_1, \ldots, i_k \le m$, then we say that c is formally realizable. In [18], it was shown that c is formally realizable if the Lie rank of c is finite. Can we use observation algebras and observation fields to characterize the realizability? The following conclusion can be easily proved:

Assume that c is a formal power series and the observation algebra $\mathcal{A}_1(c)$ is finitely generated as an $\mathbb{R}$ algebra. Then there exists an integer n, some point $x_0 \in \mathbb{R}^n$, some $m+1$ polynomial vector fields on $\mathbb{R}^n$

$$p_0, p_1, \ldots, p_m$$

and some polynomial function $h : \mathbb{R}^n \to \mathbb{R}$ such that (8.8) holds.

Notice here that $p_0, p_1, \ldots, p_m$ and h are not formal vector fields and function, indeed, they are well defined polynomial vector fields and function. We now prove this fact.

Assume that $\mathcal{A}_1(c) = \mathbb{R}[c_1, c_2, \ldots, c_n]$. Then for each i and j there exists some polynomial function p_{ij} such that

$$\eta_j^{-1} c_i = p_{ij}(c_1, \ldots, c_n)\,.$$

We define, for $j = 0, 1, \ldots, m$, polynomial vector fields on $\mathbb{R}^n$ as follows:

$$p_j(x) = (p_{1j}(x), p_{2j}(x), \ldots, p_{nj}(x))'$$

and we let

$$z_0 = (\langle c_1, \phi \rangle, \langle c_2, \phi \rangle, \ldots, \langle c_n, \phi \rangle)' .$$

Now assume that $c = h(c_1, c_2, \ldots, c_n)$ for some polynomial. Then we know that

$$\langle c, \phi \rangle = h(\langle c_1, \phi \rangle, \langle c_2, \phi \rangle, \ldots, \langle c_n, \phi \rangle) = h(z_0) .$$

Below we show by induction that for any $w \in P^*$,

$$\omega^{-1} c = h_w(c) \tag{8.9}$$

where $h_w(x) = L_{g_{i_k}} \cdots L_{g_{i_1}} h(x)$ if $w = \eta_{i_1} \cdots \eta_{i_k}$, which implies (8.8). Clearly the conclusion holds for $w = \phi$. Now assume that the conclusion holds for $w \in P^k$. Take some word $w \in P^{k+1}$ and assume that $w = w'\eta_j$ for some $w' \in P^k$ and some j. Then

$$\begin{aligned}
w^{-1}c = \eta_j^{-1}(w'^{-1}c) &= \eta_j^{-1} h_{w'}(c_1, c_2 \ldots, c_n) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} h_{w'}(c_1, c_2 \ldots, c_n)\, \eta_j^{-1} c_i \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} h_{w'}(c_1, c_2 \ldots, c_n)\, p_{ij}(c_1, c_2, \ldots, c_n) \\
&= h_w(c_1, c_2, \ldots, c_n) .
\end{aligned}$$

Thus (8.9) holds, as desired.

Applying Lemma (4.2) in [37] together with the above discussion, one sees that if the observation algebra $\mathcal{A}_1(c)$ of a series c is finitely generated as an $\mathbb{R}$ algebra, then necessarily c is convergent. Now the question is whether the analogue for observation field holds? That is, will the fact that $\mathcal{Q}_1(c)$ is a finitely generated field extension imply realizability or formal realizability? We are still not clear about the answer yet. Better understanding of this problem should help to have a deeper insight into the realizability problem.

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