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Algebraic differential equations and nonlinear control systems

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ALGEBRAIC DIFFERENTIAL EQUATIONS AND NONLINEAR CONTROL SYSTEMS

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ABSTRACT OF THE DISSERTATION

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Dissertation Director: Professor Eduardo D. Sontag

This dissertation establishes a precise correspondence between realizability of operators defined by convergent generating series and the existence of high order differential equations (*"i/o equations"*) relating derivatives of inputs and outputs.

State space models are central to modern nonlinear control theory, since they permit the application of techniques from various mathematics branches such as differential equations, dynamical systems and optimization theory. A natural question is to decide when a given i/o operator admits a representation by an initialized state space system (the operator is *realizable*).

To investigate the relation between i/o equations and realizability, we introduce and study the structures of observation spaces, observation algebras and observation fields. In realization theory and many other areas of nonlinear control, the concept of observation space plays a central role. One may define observation spaces in two very different ways. Roughly, one possibility is to take the functions corresponding to derivatives with respect to switching times in piecewise constant controls, and the other is to take high-order derivatives at the final time , if smooth controls are used. It turns out that the existence of algebraic i/o equations is closely related to the finiteness properties of the observation algebra and field associated with the first type of observation space,

while realizability is closely related to the finiteness properties of the algebraic objects associated with the other type of observation space. One of the central technical results, given in Chapter 3, shows that the two types of spaces coincide.

Based on the results mentioned above, we get our main results: Realizability by singular polynomial systems is equivalent to existence of algebraic i/o equations. We also provide other results relating various special kinds of i/o equations to some specific classes of realizations, for instance, what are called recursive i/o equations are related to realizability by polynomial systems.

In Chapter 7, our results relating algebraic i/o equations to realizability by "rational" systems are extended to analytic i/o equations and local realization by analytic systems. By studying properties of meromorphically finitely generated field of functions, together with the application of some known facts in the literature of nonlinear realization, we conclude that the existence of analytic i/o equations implies local realizability by analytic systems.

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Chapter 1

Introduction

In this dissertation we establish a precise correspondence between realizability of input/output operators and the existence of high order differential equations relating derivatives of inputs and outputs.

In many experimental situations involving systems, it is often the case that one can model system behavior through differential equations, which are referred to as *input/output ("i/o") equations* in this work, of the type

$$E\left(u(t), u'(t), u''(t), \ldots, u^{(r)}(t), y(t), y'(t), y''(t), \ldots, y^{(r)}(t)\right) = 0 \qquad (1.1)$$

where $u(\cdot)$ and $y(\cdot)$ are the input and output signals respectively. An i/o operator $F: u(\cdot) \mapsto y(\cdot)$ is said to *satisfy* the equation (1.1) if the equation holds for each input u and the corresponding output $y = F[u]$ of F. (Precise definitions will be given later.)

The functional relation E is usually estimated, for instance through least squares techniques, if a parametric general form (e.g. polynomials of fixed degree) is chosen. For example, in linear systems theory one often deals with degree-one polynomials E:

$$y^{(k)}(t) = a_1 y(t) + \ldots + a_k y^{(k-1)}(t) + b_1 u(t) + \ldots + b_k u^{(k-1)}(t) \qquad (1.2)$$

(or their frequency-domain equivalent, transfer functions; the difference equation analogue is sometimes called an "autoregressive moving average" representation). In the linear case, such representations form the basis of much of modern systems analysis and identification theory.

On the other hand, in most theoretical developments in nonlinear control, one uses a state-space formalism, where inputs and outputs are related by a system of *first* order differential equations

$$x'(t) = f(x(t)) + G(x(t))u(t)\,, \quad y(t) = h(x(t)) \qquad (1.3)$$

where the state $x(t)$ is multidimensional, and no derivatives of controls are allowed. These descriptions are central to the modern nonlinear control theory, as they permit the application of techniques from differential equations, dynamical systems, and optimization theory. Thus a basic question is that of deciding when a given i/o operator admits a representation of this form. This is the area of *realization theory*, which is closely related, especially when stochastic effects are included, to *systems identification*. Roughly speaking, if such a state space description does exist for a given i/o operator, then we say that the i/o operator is *realizable*. The precise definitions for various notions of realizability will be given in Chapters 5 and 7.

It is a classical (and easy) fact that an equation such as (1.2) can be reduced, by adding state variables for enough derivatives of the output y, to a system (1.3) of first order equations, with $f(x)$ linear and $G(x)$ constant, i.e., a linear finite-dimensional system. In frequency-domain terms, rationality of the transfer function is equivalent to realizability. (For references on the linear theory, see eg [19], [33] and [43].) One of the methods for obtaining a linear realization from a given linear i/o equation relies on Lord Kelvin's principle for solving differential equations by means of mechanical analog computers (cf. [19]). The principle, which was suggested a hundred years ago, provided a way for simulating a system without using differentiators.

For nonlinear systems this reduction presents a far more difficult problem, one that is to a great extent unsolved. The problem is basically that of in some sense replacing a nontrivial equation (1.1) by a system of first-order equations (1.3) which does not involve the derivatives of the inputs.

1.1 Previous Nonlinear Work

The work [36] used a differential geometry approach to develop a theory of realization of i/o operators, including results on existence and uniqueness of "minimal" realizations. The papers [18] and [17] characterized existence conditions for realizations in terms of smoothness (or analyticity) of the i/o operator plus a rank condition. In [4] and [3], tools from algebraic geometry were employed in order to study the structure of observation algebras and observation fields. The results there related finiteness properties of the various algebraic objects to realizability, in strict analogy to the relations that hold in discrete time ([34]).

In fact, the discrete-time work [34] provided one approach to relating these two types of representations —with difference equations appearing instead,— and this was used as a basis of identification algorithms by other authors; see for instance [22] and [8]. (The former reference shows also how to include stochastic effects in the resulting approach.) These results have recently been extended to continuous-time for the very special case of *bilinear* systems: A theorem showed that realizability by such systems is equivalent to the existence of an E of a special form, namely affine on y (see [32]). However, the techniques in [32] were linear-algebraic, and hence not powerful enough to handle the extension of [34] to the general nonlinear case.

The present work completes the development of this extension of the result in [34] to continuous-time. A number of partial results were already available about the relation between (1.1) and (1.3); see for instance [39], [7] or [14]. It is easy to show, by elementary arguments involving finite transcendence degree, that any i/o operator realizable by a rational state space system satisfies some i/o equation of type (1.1), with E a polynomial. In [9] it was remarked –as a consequence of theorems from differential algebra, that in order to characterize the i/o behavior of a state space system *uniquely*, one needs to add inequality constraints to (1.1). In [26] and [40] it was shown that, under some constant rank conditions, the outputs of an observable smooth state space system can be described by an equation of type (1.1) for which E is a smooth function. Local i/o equations were shown to exist, for generic initial states of (1.3), in [6]; however, in contrast to the algebraic case, it is generally not true that every state space system gives rise to a global i/o equation, even under analyticity assumptions. (This is discussed later through an example.)

1.2 Our Approach

The view proposed in [30], [34] for discrete-time, and followed here in the differential equation case, is that one should attack the problem as follows. One should separate the issue of existence of a realization from the question of "well-posedness" of the equation.

For example, the equation

$$u(t)y'(t) = 1$$

can never be satisfied by all the input/output pairs corresponding to a state space system, as remarked in [32], nor is this true for

$$y''(t) = u'(t)^2 \ .$$

In both of these cases, not only cannot the equation be reduced to state-space form but —as one can easily prove— even more basically, it cannot be satisfied by any "input/output map" of the type that we shall consider. Indeed, our main contribution is to show that *if* the equation would have been well-posed, in the sense that it is an equation satisfied by all input/output pairs corresponding to what we will call a *Fliess operator* —i.e. one described by a convergent generating series— and if E is a polynomial, then it is always realizable by a singular polynomial system, or a rational system with possible poles. (Singular systems appear naturally in control theory, for instance in robotics; see [24] for many examples.)

In the special case when equation (1.1) is recursive —i.e. the coefficient of the highest derivative of y in (1.1) does not depends on the lower derivatives of y,— our construction will provide a polynomial realization (no poles). In the general case, we shall prove that about every singular point of the realization there is another system, locally defined in terms of analytic functions, that realizes (locally) the desired behavior. The picture that emerges then is that, at least, one can cover the possibly singular part with local analytic realizations. In a computer simulation, this would be achieved by passing to a subroutine to deal with trajectories near this set. In fact we have proved that any equation (1.1) for which E is just analytic gives rise to a local analytic realization. (We emphasize, this is always subject to the hypothesis that the i/o pairs arise from some Fliess operator.)

Our formalism is based on the *generating series* suggested by Fliess in the late 70's, who was in turn motivated by Chen's work on power series solutions of differential equations. The i/o operators induced by convergent generating series form a very general class of causal operators, capable of representing a variety of nonlinear systems.

We shall call them "Fliess operators". For instance, any i/o operator induced by an initialized analytic state space system affine in controls can be described in this manner. In the development of linear control theory, transfer functions and transfer matrices were used first in the analysis of linear systems and state-space approaches were only later introduced. State space approaches have always played a role central in nonlinear control theory. However, other descriptions of i/o behaviors, such as Volterra series and generating series, are also appropriate and often useful. The better understanding of the relations between these different descriptions of i/o behaviors is essential. It has been known that any Fliess operator is realizable (locally, by an analytic system,) if and only if the "Lie rank" of the series is finite (see [11] and [35]). What we do in this work is to provide a link between existence of i/o equations and realizability for such operators.

Our results also provide a link with the differential-algebraic work of Fliess, who in [12] *defined* realizability by the requirement that outputs be differentiably dependent on inputs, in other words, that an equation such as (1.1) hold. We show then that this is basically the same as realizability in the more classical sense. Yet another link is with the recent work of Willems and his school. Consider the *behavior* $w(\cdot) = (u(\cdot), y(\cdot))$ associated to an input/output description. If we write the equation as

$$E\left(w(t), w'(t), w''(t), \ldots, w^{(r)}(t)\right) = 0$$

as preferred in some of the recent system-theoretic literature (see [42]), then what we do is to relate the fact that the behavior satisfies an algebraic differential equation to realizability.

In addition to single operators, it is also natural to study *families of i/o maps*, defined by a *family of convergent generating series*. To study a single i/o map is natural as a formal description of a initialized *black box*, but in general, a system may induce more than one i/o map. For example, a system described by an ordinary differential equation on a manifold may induce infinitely many i/o maps, each of them corresponding to some initial state. One should study all the i/o maps induced by the system simultaneously rather than individually, unless a fixed initial state is of particular interest. This leads to the concept of families of i/o maps. One question arises naturally: when can a family of i/o maps be realized by *one* state space system? i.e., when can all the members of the family be realized by some singular polynomial system in such a way that each member of the family is associated to some initial state of the system? We will prove that a family of i/o maps is realizable in this sense if and only if all the members of the family satisfy a common i/o equation.

The proofs are based on a careful analysis of the concept of *observation space*, introduced in [23] (and [34] for discrete-time), developed further in [13], and later rediscovered by many authors. One of the central technical results, given in Chapter 3, relates two different definitions of this space, one in terms of smooth controls and another in terms of piecewise constant ones; these two definitions are seen to coincide. One of them immediately relates to i/o equations, while the other is related to realizability through the notion of *observation algebras* and *observation fields*. The latter are the analogues of the corresponding discrete-time concepts studied in [34]. For differential equations they were first employed in [4] and [3].

1.3 Outline of This Work

The organization of this work is as follows:

In Chapter 2 we first introduce the basic terminology regarding series, convergence, and so forth, and introduce an algebraic structure on series, the shuffle product. Then for operators defined by evaluation of these series, we study smoothness properties of their output functions. Although several of the results presented there have been known and used often by previous authors, it is difficult to find complete proofs in the literature.

In Chapter 3, we consider observation spaces. Since their introduction in the mid 70's (see [23] and [13], as well as [34] for the discrete time analogue), observation spaces for nonlinear control systems

$$x' = f(x) + \sum u_i g_i(x)\ , \quad y = h(x) \tag{1.4}$$

have played a central role in the understanding of realization theory. For the system

(1.4), one defines the observation space $\mathcal{F}$ as the linear span of the Lie derivatives

$$L_{X_1} \cdots L_{X_k} h,$$

where each X_i is either f or one of the g_i's. (Here we are taking states $x(t)$ in a manifold, $f, g_1, \ldots, g_m$ smooth vector fields, and h a function from the manifold to $\mathbb{R}$, the output map. The linear span is understood in the space of smooth functions into $\mathbb{R}$.)

It is known that many important properties of systems, such as the possibility of simulating such a system by one described by linear vector fields (the "bilinear immersion" problem, [13]), are characterized by properties of this space.

It was shown in [32] that a different type of "observation space" is much more important when one studies i/o equations satisfied by (1.4), i.e. equations of the type (1.1) that hold for all those pairs of functions $(u(\cdot), y(\cdot))$ that arise as solutions of (1.4). This alternative observation space is obtained by taking the derivatives $y(t), y'(t), \ldots$ as functions of initial states, over all $u(t), u'(t), \ldots$. This space is obtained by considering differentiable controls and time-derivatives, while the space previously considered is based on derivatives with respect to switching times in piecewise constant controls.

The central fact used in [32] in order to relate i/o equations to realizability is the equality of the two observation spaces defined in the above manners. This equality is fundamental not only for the results in that paper, which hold under the assumption that the spaces are finite-dimensional, but also for more general results in this work. However, the techniques used in [32] are based on a linear-algebraic and a topological argument, involving closure in the weak topology, which does not in any way extend to the more general case of infinite dimensional observation spaces. Since the latter are the norm rather than the exception (unless the system can be simulated by a bilinear system to start with), one needs to establish the equality of these two types of spaces using totally different combinatorial techniques. That is achieved in Chapter 3. Also in Chapter 3, we extend the result to families of i/o operators, which, in turn, is applied to state space systems.

In Chapter 4, we study i/o equations satisfied by i/o operators. For this purpose, we find it useful to introduce the algebraic concepts of observation algebra and observation field corresponding to a given series. Motivated by [34], we show that the existence of an i/o equation implies that the observation field is a finitely generated field extension of $\mathbb{R}$. The real meaning of this result is that if the transcendence degree of the observation field over $\mathbb{R}$ is finite, then the field is a finitely generated extension of $\mathbb{R}$, which is not true in general for arbitrary fields. It does hold here basically due to the fact that the observation field is a "differential field" —a field with derivation operators. However, it is not a usual differential field in the standard sense of the differential algebra literature (cf. [20]) since the derivation operators used here are not commutative in general. Nonetheless, one still finds some special properties of the field that help in proving the results.

In Chapter 5, realizability by polynomial systems and singular polynomial systems are considered. One may consider a singular polynomial system as a "rational" system in the sense of [3]. However, to be cautious about the possible poles in the right-hand side of the equation, we prefer to study singular systems.

As existence of i/o equations is closely related to the structure of observation space, algebra and field, realizability forces the study of the structures of the algebraic objects related to the second type of observation space that we introduced in Chapter 3. Though it turns out that the two types of the algebraic objects are the same, due to the results in Chapter 3, the results of this Chapter are independent of the fact and they are more readily understood in terms the second type of the objects. The main result in Chapter 5 is that realizability by singular polynomial systems is guaranteed by the condition that the observation field is a finitely generated extension of $\mathbb{R}$. The approach pursued there is to use the generators of the field as state variables and use the equalities which hold among the generators to construct the needed vector fields.

Realizability for families of operators is also studied. To achieve our goal, some technical conditions are imposed on the parameter dependence and the parameter sets. These conditions are usually satisfied by families of i/o operators described by a state space system with different initial states.

In Chapter 6, based on the results obtained in the previous Chapters, we give the main results of our dissertation, establishing the equivalence between realizability by a singular polynomial system and the existence of an algebraic i/o equation for both operators and families of operators.

We also show there that recursive i/o equations lead to realization by polynomial systems. However, as opposed to the previous case, the converse of this fact is not true in general. A counterexample is provided to illustrate the fact that realizability by a polynomial system may not lead to a recursive i/o equation. For this purpose we show that for an operator that arise from an initialized accessible state space system, the two observation algebras, one defined in terms of Lie derivatives of the observation map in the state space, the other one defined in terms of the series which defines the same operator, are isomorphic to each other.

In Chapter 7, our previous results relating algebraic i/o equations to internal realizability are extended to analytic i/o equations and local internal realizability. One of the motivations of this Chapter is to get rid of the singularities in singular polynomial realizations, but we do not know yet whether one can always get a polynomial realization without singularities. (This is perhaps one of the most interesting problems for further research.) The result of this Chapter serves to show that if one can get a singular polynomial realization, then around each singular point there is another analytic system realizing the i/o operator locally.

The approach to proving the results in this Chapter is to first construct a "meromorphic" realization by studying the properties of meromorphically finitely generated field extensions. Then by a perturbation approach, together with the Lie rank condition for realizability (cf. [11] and [35]), we show that around each point there is local analytic realization. This can also be done by using the rank condition studied in [17]. In contrast to the algebraic case, it is not true that every operator realizable by an analytic system satisfies an analytic i/o equation. A modification of an example due to [28] is presented to illustrate the fact.

A final Chapter summarizes conclusions and gives suggestions for further research.

Chapter 2

Generating Series and I/O Operators

In this chapter we introduce the basic terminology regarding series and i/o operators as well as most of the elementary facts to be used later. Although several of the results presented here have been known and used often by previous authors, it is difficult to find complete proofs of many of them in the literature. The Chapter introduces an algebraic structure on series, the shuffle product, and studies convergence properties. For operators defined by evaluating these series, we study smoothness properties of the corresponding output functions.

2.1 Generating Series

The "input/output maps" that we use are defined by a certain type of power series. The most convenient way to introduce them is by first defining and studying the abstract algebra of such power series.

Let m be a fixed integer and $I = \{0, 1, \ldots, m\}$. For any integer $k \geq 1$, we define

$$I^k = \{(i_1\, i_2\, \ldots\, i_k) : i_s \in I,\ 1 \leq s \leq k\}.$$

For $k = 0$, we use I^0 to denote the set whose only element is the empty sequence ϕ. Let

$$I^* = \bigcup_{k \geq 0} I^k. \tag{2.1}$$

Then I^* is a free monoid with the composition rule:

$$(i_1 i_2 \ldots i_k)(j_1 j_2 \ldots j_l) = (i_1 i_2 \ldots i_k j_1 j_2 \ldots j_l).$$

If $\iota \in I^l$, then we say that the *length* of ι, denoted by $|\iota|$, is l.

Consider now the "alphabet" set

$$P = \{\eta_0, \eta_1, \ldots, \eta_m\}$$

and P^*, the free monoid generated by P, where the neutral element of P^* is the empty word, denoted by 1, and the product is concatenation. Let

$$P^k = \{\eta_{i_1}\eta_{i_2}\ldots\eta_{i_k} : 0 \le i_s \le m,\ 1 \le s \le k\}$$

for each $k \ge 0$. We define $\mathcal{P}$ to be the $\mathbb{R}$-algebra generated by P^*, i.e, the set of all polynomials in the variables η_i's. A *power series in the noncommutative variables* $\eta_0, \eta_1, \ldots, \eta_n$ is a formal power series

$$c = \sum_{\iota \in I^*} \langle c,\ \eta_\iota \rangle \eta_\iota, \tag{2.2}$$

where

$$\eta_\iota = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_l} \quad \text{if } \iota = i_1 i_2 \cdots i_l,$$

and $\langle c,\ \eta_\iota\rangle \in \mathbb{R}$ for each multiindex ι. Note that c is a polynomial if and only if there are only finitely many $\langle c,\ \eta_\iota\rangle$'s which are non-zero. A power series is nothing more than a mapping from I^* to $\mathbb{R}$; as we shall see later, however, the algebraic structures suggested by the series formalism are very important. We use $\mathcal{S}$ to denote the set of all power series (over a fixed but arbitrary alphabet P).

For $c,\ d \in \mathcal{S}$ and $\gamma \in \mathbb{R}$, $\gamma c + d$ is the series defined as follows:

$$\langle \gamma c + d,\ \eta_\iota \rangle = \gamma \langle c,\ \eta_\iota \rangle + \langle d,\ \eta_\iota \rangle.$$

With these operations, $\mathcal{S}$ forms a vector space over $\mathbb{R}$. In addition, we can introduce an algebra structure on $\mathcal{S}$ by defining the *shuffle product* on $\mathcal{S}$. First of all, we define the shuffle product on words,

$$\text{ш} : P^* \times P^* \longrightarrow \mathcal{P}$$

inductively on length in the following way:

$$1 \,\text{ш}\, \eta = \eta \,\text{ш}\, 1 = \eta \quad \text{for any} \quad \eta \in P\ ,$$

$$\eta_i\eta_\iota \sqcup\!\sqcup \eta_j\eta_\kappa = \eta_i(\eta_\iota \sqcup\!\sqcup \eta_j\eta_\kappa) + \eta_j(\eta_i\eta_\iota \sqcup\!\sqcup \eta_\kappa) \quad \text{for any} \quad \eta_\iota, \eta_\kappa \in P^*, \eta_i, \eta_j \in P. \tag{2.3}$$

It can be proved by induction that an equivalent way to define the shuffle product is to replace (2.3) by the following:

$$\eta_\iota\eta_i \sqcup\!\sqcup \eta_\kappa\eta_j = (\eta_\iota \sqcup\!\sqcup \eta_\kappa\eta_j)\eta_i + (\eta_\iota\eta_i \sqcup\!\sqcup \eta_\kappa)\eta_j \quad \text{for any} \quad \eta_\iota, \eta_\kappa \in P^*, \eta_i, \eta_j \in P. \tag{2.4}$$

Then we extend the shuffle product to power series in the following way: For

$$c = \sum \langle c,\ \eta_\iota\rangle \eta_\iota \quad \text{and} \quad d = \sum \langle d,\ \eta_\kappa\rangle \eta_\kappa,$$

we define

$$c \sqcup\!\sqcup d = \sum \langle c,\ \eta_\iota\rangle \langle d,\ \eta_\kappa\rangle \eta_\iota \sqcup\!\sqcup \eta_\kappa. \tag{2.5}$$

Note that (2.5) can be written in an alternative manner which is often very useful, as follows:

$$c \sqcup\!\sqcup d = \sum_{\iota,\kappa,\rho \in I^*} \langle c,\ \eta_\iota\rangle \langle d,\ \eta_\kappa\rangle \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ \eta_\rho\rangle \eta_\rho,$$

that is,

$$\langle c \sqcup\!\sqcup d,\ \eta_\rho\rangle = \sum_{\iota,\kappa \in I^*} \langle c,\ \eta_\iota\rangle \langle d,\ \eta_\kappa\rangle \langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ \eta_\rho\rangle.$$

With the operations "+" and "$\sqcup\!\sqcup$" defined as above, $\mathcal{S}$ forms a commutative $\mathbb{R}$-algebra. Moreover, we have the following fact:

Lemma 2.1.1 The algebra $\mathcal{S}$ is an integral domain.

Proof. First we order the basis elements $(\eta_{i_1}, \ldots, \eta_{i_k})$ of P^* lexicographically with respect to

$$k, i_1, i_2, \ldots, i_k.$$

Then take two nonzero series c and d and let

$$z_1 = \eta_{i_1} \cdots \eta_{i_m}$$

and

$$z_2 = \eta_{j_1} \cdots \eta_{j_n}$$

be the smallest basis element of P^* appearing in c and d, respectively, with nonzero coefficients. Let

$$w := \eta_{l_1} \cdots \eta_{l_{m+n}}$$

be the smallest basis elements of P^* appearing in $z_1 \sqcup\!\sqcup z_2$. Then the coefficient of w in $c \sqcup\!\sqcup d$ is:

$$\langle c \sqcup\!\sqcup d,\ w\rangle = \sum_{\iota,\kappa} \langle c,\ \eta_\iota\rangle\langle d,\ \eta_\kappa\rangle\langle \eta_\iota \sqcup\!\sqcup \eta_\kappa,\ w\rangle.$$

Using the minimality property of w, z_1, z_2, we get

$$\langle c \sqcup\!\sqcup d,\ w\rangle = \langle c,\ z_1\rangle\langle d,\ z_2\rangle\langle z_1 \sqcup\!\sqcup z_2,\ w\rangle,$$

which is nonzero since $\langle c,\ z_1\rangle$, $\langle d,\ z_2\rangle$, $\langle z_1 \sqcup\!\sqcup z_2,\ w\rangle$ are all nonzero. ∎

The method used in the above proof is similar to the method used in [27], where the author proved that the ring of polynomials in $\eta_0, \eta_1, \ldots, \eta_m$ is an integral domain. In [27], the author used the greatest basis elements (the "degree") for polynomials while here we used the smallest basis elements (the "order") for power series.

We shall say that the power series c is *convergent* if there exist K, $M \geq 0$ such that

$$|\langle c,\ \eta_\iota\rangle| \leq KM^k k! \quad \text{for each } \iota \in I^k, \text{ and each } k \geq 0. \tag{2.6}$$

This growth condition is the natural generalization of the one used for Taylor series in one variable, and it has been suggested before in the present context by Fliess [10].

To each monomial $z = \eta_\kappa$, we associate a "shift" operator $c \mapsto z^{-1}c$ defined by

$$\langle z^{-1}c,\ \eta_\iota\rangle = \langle c,\ z\eta_\iota\rangle \quad \text{for } \eta_\iota \in P^*.$$

That is to say, η_κ is "erased" from all terms that start with η_κ, and other terms are deleted. For instance,

$$\begin{aligned}(\eta_0\eta_1)^{-1}(1 + \eta_1 - 2\eta_0\eta_1 + \eta_1\eta_1 - \eta_0\eta_1\eta_0 + 3\eta_0\eta_1\eta_1 + \cdots)\\ = -2 - \eta_0 + 3\eta_1 + \cdots\ .\end{aligned}$$

Note that $z_2^{-1}z_1^{-1}c = (z_1z_2)^{-1}c$. By definition, for any $z, w \in P^*$,

$$z^{-1}w = \begin{cases} w_1 & \text{if } w = zw_1 \text{ for some } w_1 \in P^*, \\ 0 & \text{otherwise,} \end{cases}$$

and for any $c \in \mathcal{S}$,

$$z^{-1}c = \sum_{\iota \in I^*} \langle c,\ z\eta_\iota\rangle \eta_\iota \ = \ \sum_{\iota \in I^*} \langle c,\ \eta_\iota\rangle z^{-1}\eta_\iota.$$

Lemma 2.1.2 For any c, $d \in \mathcal{S}$ and $z \in P$,

$$z^{-1}(c \sqcup\!\sqcup d) \ = \ (z^{-1}c) \sqcup\!\sqcup d \ + \ c \sqcup\!\sqcup (z^{-1}d). \tag{2.7}$$

Proof. Take any w_1, $w_2 \in P^*$ and any $z \in P$, and write $w_1 = z_1 w_1'$ and $w_2 = z_2 w_2'$ for some $w_1', w_2' \in P^*$, $z_1, z_2 \in P$. This can always be done unless w_1 or w_2 is the empty sequence, in which case the formula to be proved is trivial. Then,

$$z^{-1}(w_1 \sqcup\!\sqcup w_2) \ = \ (z^{-1}z_1)(w_1' \sqcup\!\sqcup w_2) \ + \ (z^{-1}z_2)(w_1 \sqcup\!\sqcup w_2').$$

Notice that

$$z^{-1}z_i \ = \ \begin{cases} 1 & \text{if } z = z_i, \\ 0 & \text{if } z \neq z_i \end{cases}$$

for all $z, z_i \in P^*$, $i = 1, 2$. It follows that

$$z^{-1}(w_1 \sqcup\!\sqcup w_2) = (z^{-1}w_1) \sqcup\!\sqcup w_2 \ + \ w_1 \sqcup\!\sqcup (z^{-1}w_2).$$

Therefore, for any c and $d \in \mathcal{S}$,

$$\begin{aligned} z^{-1}(c \sqcup\!\sqcup d) \ &= \ \sum \langle c,\ \eta_\iota\rangle \langle d,\ \eta_\kappa\rangle z^{-1}(\eta_\iota \sqcup\!\sqcup \eta_\kappa) \\ &= \ \sum \langle c,\ \eta_\iota\rangle \langle d,\ \eta_\kappa\rangle \left((z^{-1}\eta_\iota) \sqcup\!\sqcup \eta_\kappa \ + \ \eta_\iota \sqcup\!\sqcup (z^{-1}\eta_\kappa)\right) \\ &= \ \sum \langle c,\ z\eta_\iota\rangle \langle d,\ \eta_\kappa\rangle \eta_\iota \sqcup\!\sqcup \eta_\kappa \ + \ \sum \langle c,\ \eta_\iota\rangle \langle d,\ z\eta_\kappa\rangle \eta_\iota \sqcup\!\sqcup \eta_\kappa, \end{aligned}$$

which implies (2.7). ∎

Remark 2.1.3 Lemma 2.1.2 implies that z^{-1} is a derivation operator over the ring $\mathcal{S}$. Later we will see that z^{-1} *is* indeed closely related to derivatives of certain functions associated to c. □

2.2 I/O Operators

We are now ready to define input/output maps, using the formalism introduced in the last section.