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The Formal Laplace-Borel Transform, Fliess Operators and the Composition Product

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***Abstract*— In this paper, the formal Laplace-Borel transform of an analytic nonlinear input-output system is defined, specifically, an input-output system that can be represented as a Fliess operator. Using this concept and the composition product, an explicit relationship is then derived between the formal Laplace-Borel transforms of the input and output signals. This provides an alternative interpretation of the symbolic calculus introduced by Fliess to compute the output response of such systems. Finally, it is shown that the formal Laplace-Borel transform provides an isomorphism between the semigroup of all well defined Fliess operators under composition and the semigroup of all locally convergent formal power series under the composition product.**

I. Introduction

The one-sided integral Laplace transform pair

$$x^k \overset{\mathscr{L}}{\Longleftrightarrow} k!\,(s^{-1})^{k+1}$$

naturally suggests a definition for the *formal Laplace-Borel transform* of a formal power series in one variable:

$$\begin{aligned} \mathscr{L}_f &: \mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle X\rangle\rangle \\ &: c \mapsto \tilde{c} \\ \mathscr{B}_f &: \mathbb{R}_{LC}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle X\rangle\rangle \\ &: \tilde{c} \mapsto c, \end{aligned}$$

where $\mathbb{R}\langle\langle X\rangle\rangle$ is the set of formal power series over the alphabet $X=\{x_0\}$, $\tilde{c}$ is a series with coefficients $(\tilde{c},\emptyset)=0$ and

$$(\tilde{c},x_0^{k+1}) = k!\,(c,x_0^k),\quad \forall k\geq 0,$$

and $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ denotes the collection of all *locally convergent* formal power series over X [15]. That is, $\tilde{c}\in\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ if its coefficients satisfy the growth condition $|(\tilde{c},x_0^k)|\leq KM^k k!$, $k\geq 0$, for some finite real numbers $K,M>0$. The formal Laplace-Borel transform was first used in [4]–[6], [12] and later in [14] to produce a type of symbolic calculus for computing the output response of a nonlinear system given various inputs. What is absent in this framework, however, is the explicit notion of computing the formal transform of the input-output *operator*, an idea very familiar in the linear setting. Thus, the general purpose of this paper is to define this type of transform and then show how it can provide an alternative interpretation of the symbolic calculus of Fliess when combined with the so call *composition product* of two formal power series [2], [3], [7]–[10].

The specific class of nonlinear input-output systems considered are known as Fliess operators. Let $X=\{x_0,x_1,\ldots,x_m\}$ denote an arbitrary alphabet and X^* the set of all words over X. For each power series $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate a corresponding m-input, ℓ-output operator F_c in the following manner. Let $p\geq 1$ and $a<b$ be given. For a measurable function $u:[a,b]\rightarrow\mathbb{R}^m$, define $\|u\|_p=\max\{\|u_i\|_p:\ 1\leq i\leq m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a,b]$. Let $L_p^m[a,b]$ denote the set of all measurable functions defined on $[a,b]$ having a finite $\|\cdot\|_p$ norm and $B_p^m(R)[a,b]:=\{u\in L_p^m[a,b]:\|u\|_p\leq R\}$. With $t_0,T\in\mathbb{R}$ fixed and $T>0$, define inductively for each $\eta=x_i\eta'\in X^*$ the mapping $E_\eta: L_1^m[t_0,t_0+T]\rightarrow \mathcal{C}[t_0,t_0+T]$ by

$$E_\eta[u](t,t_0)=\int_{t_0}^t u_i(\tau)E_{\eta'}[u](\tau,t_0)\,d\tau,$$

where $E_\emptyset\equiv 1$ and $u_0\equiv 1$. The input-output operator corresponding to c is the Fliess operator

$$F_c[u](t)=\sum_{\eta\in X^*}(c,\eta)\,E_\eta[u](t,t_0).$$

All Volterra operators with analytic kernels, for example, are Fliess operators. When c is locally convergent, it is known that F_c constitutes a well defined operator from $B_p^m(R)[t_0,\,t_0+T]$ into $B_q^\ell(S)[t_0,\,t_0+T]$ for sufficiently small $R,S,T>0$, where the numbers $p,q\in[1,\infty]$ are conjugate exponents, i.e., $1/p+1/q=1$ with $(1,\infty)$ being a conjugate pair by convention [11]. Therefore, the specific operator class of interest is the set of Fliess operators $\mathscr{F}:=\{F_c: c\in\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle\}$. This set forms a semigroup under composition, as does the set $\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$ under the composition product. It will be shown that the formal Laplace-Borel transform provides an isomorphism between these two semigroups.

The paper is organized as follows. In Section 2, the notion of a formal Laplace-Borel transform of a Fliess operator is defined. Then it basic properties are explored and a set of examples is given. In Section 3, the composition product is introduced and its relationship to the formal Laplace-Borel transform is developed. Another set of examples is provided.

II. The Formal Laplace-Borel Transform of a Fliess Operator

In linear time-invariant system analysis, a causal homogeneous input-output mapping is expressed in terms of a convolution of the system impulse response with the input signal

$$y(t) = \int_{t_0}^{t} h(t-\tau)u(\tau)\, d\tau.$$

This mapping is also uniquely characterized by its system function $H(s) = \mathscr{L}\{h(t)\} = \sum_{k>0} h_k s^{-k}$. In an analogous fashion, given a well defined Fliess operator, its generating series uniquely defines the corresponding input-output mapping. So the following definition is given.

Definition 2.1: The **formal Laplace transform on** $\mathscr{F}$ is defined as

$$\begin{aligned} \mathscr{L}_f \quad &: \quad \mathscr{F} \to \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle \\ &: \quad F_c \mapsto c. \end{aligned}$$

The corresponding **formal Borel transform** is

$$\begin{aligned} \mathscr{B}_f \quad &: \quad \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle \to \mathscr{F} \\ &: \quad c \mapsto F_c. \end{aligned}$$

It is next shown that many of the properties of the integral Laplace transform have counterparts in this context. To facilitate the analysis two concepts are needed.

Definition 2.2: For any $x_i \in X$, the **left-shift operator**, $x_i^{-1}(\cdot)$, of a formal power series is defined as

$$x_i^{-1}(c) = \sum_{\eta \in X^*} (c,\eta)\, x_i^{-1}(\eta),$$

where

$$x_i^{-1}(\eta) = \begin{cases} \eta' & : \quad \text{if } \eta = x_i\eta' \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Definition 2.3: A **Dirac series**, δ_i, is a generalized series with the defining property that $F_{\delta_i}[u] = u_i(t)$ for any $1 \leq i \leq m$.

Now the first result can be stated.

Proposition 2.1: Given any $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and scalars $\alpha, \beta \in \mathbb{R}$, the following identities hold:

1) Linearity

$$\begin{aligned} \mathscr{L}_f\left[\alpha F_c + \beta F_d\right] &= \alpha \mathscr{L}_f\left[F_c\right] + \beta \mathscr{L}_f\left[F_d\right] \\ \mathscr{B}_f\left[\alpha c + \beta d\right] &= \alpha \mathscr{B}_f\left[c\right] + \beta \mathscr{B}_f\left[d\right] \end{aligned}$$

2) Integration

$$\begin{aligned} \mathscr{L}_f\left[\int_0^t \cdots \int_0^{\tau_{n-1}} F_c[u](\tau_n)\, d\tau_n \cdots d\tau_1\right] &= x_0^n c \\ \mathscr{B}_f\left[x_0^n c\right] &= \int_0^t \cdots \int_0^{\tau_{n-1}} F_c[u](\tau_n)\, d\tau_n \cdots d\tau_1 \end{aligned}$$

3) Differentiation
In general, for the first derivative

$$\begin{aligned} \mathscr{L}_f\left[\frac{d}{dt}F_c[u](t)\right] &= x_0^{-1}(c) + \sum_{i=1}^{m} \delta_i ⧢ (x_i^{-1}(c)) \\ \mathscr{B}_f\left[x_0^{-1}(c) + \sum_{i=1}^{m} \delta_i ⧢ (x_i^{-1}(c))\right] &= \frac{d}{dt}F_c[u](t), \end{aligned}$$

where ⧢ denotes the shuffle product. If x_0^n is a left factor of c then

$$\begin{aligned} \mathscr{L}_f\left[\left(\frac{d}{dt}\right)^n F_c[u](t)\right] &= x_0^{-n}(c) = x_0^{-n}\left(\mathscr{L}_f\left[F_c[u]\right]\right) \\ \mathscr{B}_f\left[x_0^{-n}(c)\right] &= \left(\frac{d}{dt}\right)^n \mathscr{B}_f\left[c\right]. \end{aligned}$$

4) Multiplication

$$\begin{aligned} \mathscr{L}_f\left[F_c \cdot F_d\right] &= \mathscr{L}_f\left[F_c\right] ⧢ \mathscr{L}_f\left[F_d\right] \\ \mathscr{B}_f\left[c ⧢ d\right] &= \mathscr{B}_f\left[c\right] \cdot \mathscr{B}_f\left[d\right] \end{aligned}$$

Proof: The properties of linearity and integration are trivial. The multiplication property follows from results in the literature [14], [16]. Only the differentiation property remains to be justified. It is shown in [16] that the derivative of a Fliess operator is

$$\frac{d}{dt}F_c[u](t) = F_{x_0^{-1}(c)}[u](t) + \sum_{i=1}^{m} u_i F_{x_i^{-1}(c)}[u](t).$$

Applying the formal Laplace transform to this equality gives the first pair of equations. Now if x_0 is a left factor of c then $F_{x_i^{-1}(c)}[u](t) = 0$ for $i = 1, 2, \cdots, m$. In this case $\frac{d}{dt}F_c[u](t) = F_{x_0^{-1}(c)}[u](t)$. Proceeding inductively, the second pair of equations follows. ■

The following definition is utilized in the examples which follow.

Definition 2.4: [1] The **star operator** applied to a formal power series is defined as

$$c^* := \sum_{n\geq 0} c^n := (1-c)^{-1},$$

where c^n denotes the catenation power.

If a formal power series c is proper, that is, if $(c,\phi) \neq 0$, it is always possible to write $c = (c,\phi)(1-c')$. Then it follows that there exists a $c^{-1} \in \mathbb{R}\langle\langle X\rangle\rangle$ such that the catenation product gives $cc^{-1} = 1$ and $c^{-1}c = 1$. Specifically,

$$c^{-1} = \frac{1}{(c,\phi)}(1-c')^{-1} = \frac{1}{(c,\phi)}(c')^*.$$

Example 2.1: Let $X = \{x_0, x_1, x_2\}$ and $F_c[u](t) = \exp\left[\int_0^t u_1(t) + u_2(t)\, dt\right]$. Observe that F_c can be expanded

TABLE I: SOME FORMAL LAPLACE TRANSFORM PAIRS

$F_c[u](t)$	$\mathscr{L}_f\left[F_c[u](t)\right]$
1	1
t^n	$n!\ x_0^n$
$\left(\sum_{i=0}^{n-1}\frac{\binom{i}{n-1}}{i!}a^i t^i\right)e^{at}$	$(1-ax_0)^{-n}$
$\frac{1}{n!}\left(\int_{t_0}^t\sum_{j=1}^k u_{i_j}(\tau)d\tau\right)^n$	$(x_{i_1}+\cdots+x_{i_k})^n$
$e^{\int_{t_0}^t\sum_{j=1}^k u_{i_j}(\tau)d\tau}$	$(x_{i_1}+\cdots+x_{i_k})^*$
$\int_{t_0}^t\sum_{j=1}^k u_{i_j}(\tau)d\tau\, e^{\int_{t_0}^t\sum_{j=1}^k u_{i_j}(\tau)d\tau}$	$\frac{x_{i_1}+\cdots+x_{i_k}}{1-(x_{i_1}+\cdots+x_{i_k})^2}$
$\cos\left(\int_{t_0}^t\sum_{j=1}^k u_{i_j}(\tau)d\tau\right)$	$\frac{1}{1+(x_{i_1}+\cdots+x_{i_k})^2}$
$\sin\left(\int_{t_0}^t\sum_{j=1}^k u_{i_j}(\tau)d\tau\right)$	$\frac{x_{i_1}+\cdots+x_{i_k}}{1+(x_{i_1}+\cdots+x_{i_k})^2}$

as

$$F_c[u](t)=\sum_{n\geq 0}\frac{1}{n!}\left(\int_0^t u_1(t)+u_2(t)\,dt\right)^n$$
$$=\sum_{n\geq 0}\int_0^t[(u_1(\tau_1)+u_2(\tau_1)]\int_0^{\tau_1}[u_1(\tau_2)+u_2(\tau_2)]$$
$$\cdots\int_0^{\tau_{n-1}}[u_1(\tau_n)+u_2(\tau_n)]\ d\tau_n\cdots d\tau_2\ d\tau_1.$$

Therefore,

$$\mathscr{L}_f[F_c]=\sum_{n\geq 0}(x_1+x_2)^n=(x_1+x_2)^*.$$

Other formal Laplace transform pairs are given in Table I. (See also [16, Example 2.3.9] for discussion related to this example.) □

Example 2.2: Let $X=\{x_0,x_1,\ldots,x_m\}$. Suppose F_c has the generating series $c=\sum_{\eta\in X^*}\eta$, and F_ξ is given for some fixed word $\xi\in X^*$. Then

$$\begin{aligned}\mathscr{L}_f[F_c\cdot F_\xi] &= \mathscr{L}_f[F_c]\sqcup\!\sqcup\,\mathscr{L}_f[F_\xi]\\ &= c\sqcup\!\sqcup\,\xi\\ &= \sum_{\nu\in X^*}\binom{\nu}{\xi}\nu,\end{aligned}$$

where $\binom{\nu}{\xi}$ denotes the binomial coefficients over words in X^* (see [13, p. 127]). □

III. FORMAL LAPLACE-BOREL TRANSFORMS AND THE COMPOSITION PRODUCT

The composition product of two series $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$ is defined recursively in terms of the shuffle product. The definition of the composition product for the single-input, single-output case first appeared in [2], [3]. It was generalized for the multivariable setting in [7]–[10].

Definition 3.1: For any $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$, the **composition product** is defined as

$$c\circ d=\sum_{\eta\in X^*}(c,\eta)\,\eta\circ d,$$

where

$$\eta\circ d=\begin{cases}\eta & : |\eta|_{x_i}=0,\ \forall i\neq 0\\ x_0^{n+1}[d_i\sqcup\!\sqcup(\eta'\circ d)] & : \eta=x_0^n x_i\eta',\ n\geq 0,\ i\neq 0.\end{cases}$$

(Here $|\eta|_{x_i}$ denotes the number of symbols in η equivalent to x_i, and $d_i:\xi\mapsto(d,\xi)_i$ is the i-th component of (d,ξ).) Consequently, if

$$\eta=x_0^{n_k}x_{i_k}x_0^{n_{k-1}}x_{i_{k-1}}\cdots x_0^{n_1}x_{i_1}x_0^{n_0},$$

where $i_j\neq 0$ for $j=1,\ldots,k$, it follows that

$$\eta\circ d=x_0^{n_k+1}[d_{i_k}\sqcup\!\sqcup x_0^{n_{k-1}+1}[d_{i_{k-1}}\sqcup\!\sqcup\cdots x_0^{n_1+1}[d_{i_1}\sqcup\!\sqcup x_0^{n_0}]\cdots]].$$

Alternatively, for any $\eta\in X^*$ one can uniquely define a set of right factors $\{\eta_0,\eta_1,\ldots,\eta_k\}$ of η by the iteration

$$\eta_{j+1}=x_0^{n_{j+1}}x_{i_{j+1}}\eta_j,\ \ \eta_0=x_0^{n_0},\ \ i_{j+1}\neq 0,$$

so that $\eta=\eta_k$ with $k=|\eta|-|\eta|_{x_0}$. In which case $\eta\circ d=\eta_k\circ d$, where $\eta_{j+1}\circ d=x_0^{n_{j+1}+1}[d_{i_{j+1}}\sqcup\!\sqcup(\eta_j\circ d)]$ and $\eta_0=x_0^{n_0}$. It is shown in [8] that the composition product of two series is always well defined since the family of series $\{\eta\circ d:\eta\in X^*\}$ is summable for any fixed $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$.

It is easily verified for any series $c,d,e\in\mathbb{R}^m\langle\langle X\rangle\rangle$ that

$$(c+d)\circ e=c\circ e+d\circ e,$$

but in general $c\circ(d+e)\neq c\circ d+c\circ e$. An exception is *linear series*. A series c is called linear if

$$\begin{aligned}\mathrm{supp}(c)\subseteq\{\eta\in X^*:\eta=x_0^{n_1}x_ix_0^{n_0},\\ i\in\{1,2,\ldots,m\},\ \ n_1,n_0\geq 0\}.\end{aligned}$$

The composition product is associative, i.e., $(c\circ d)\circ e=c\circ(d\circ e)$, hence $(\mathbb{R}\langle\langle X\rangle\rangle,\circ)$ forms a semigroup. In [10], it is shown that the composition of two locally convergent formal power series is always locally convergent, therefore the set $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ is closed under composition, and $(\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle,\circ)$ also forms a semigroup. Perhaps the most important property of the composition product is its relationship to the cascade interconnection of two Fliess operators. Namely, for any $c\in\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$

$$F_c\circ F_d=F_{c\circ d}.\tag{1}$$

It is this property that provides the link to the symbolic calculus presented in [4]–[6], [12] and [14].

Theorem 3.1: Let F_c be a Fliess operator with $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, and $y = F_c[u]$ with u analytic. If c_u denotes the formal Laplace transform of the input u then y is analytic with formal Laplace transform $c_y = c \circ c_u$.

Proof: The analyticity of y follows from [16, Lemma 2.3.8]. The identity follows from equation (1). Specifically, for any admissible input v:

$$\begin{aligned} F_{c_y}[v] &= y \\ &= F_c[F_{c_u}[v]] \\ &= F_{c\circ c_u}[v]. \end{aligned}$$

Then by [16, Corollary 2.2.4] it follows that $c_y = c \circ c_u$. ■

In the next theorem it is shown that the formal Laplace-Borel transform provides an isomorphism between the two semigroups $(\mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle, \circ)$ and $(\mathscr{F}, \circ)$.

Theorem 3.2: For any $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$:

$$\begin{aligned} \mathscr{L}_f(F_c \circ F_d) &= \mathscr{L}_f(F_c) \circ \mathscr{L}_f(F_d) \\ \mathscr{B}_f(c \circ d) &= \mathscr{B}_f(c) \circ \mathscr{B}_f(d). \end{aligned}$$

Proof: The proof is straightforward. For any well defined F_c and F_d,

$$\begin{aligned} \mathscr{L}_f(F_c \circ F_d) &= \mathscr{L}_f(F_{c\circ d}) = c \circ d \\ &= \mathscr{L}_f(F_c) \circ \mathscr{L}_f(F_d). \end{aligned}$$

Conversely, for any locally convergent c and d,

$$\begin{aligned} \mathscr{B}_f(c \circ d) &= F_{c\circ d} = F_c \circ F_d \\ &= \mathscr{B}_f(c) \circ \mathscr{B}_f(d). \end{aligned}$$

■

Example 3.1: Consider the linear time-invariant system $y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau$, where h is analytic at $t = 0$. Then $y = F_c[u]$ with $(c, x_0^k x_1) = h^{(k)}(0)$, $k \geq 0$ and zero otherwise. Letting $u(t) = \sum_{k\geq 0}(c_u, x_0^k)\, t^k/k!$ then it follows from Theorem 3.1 that $y(t) = \sum_{n\geq 0}(c_y, x_0^n)\, t^n/n!$, where

$$\begin{aligned} c_y &= c \circ c_u \\ &= \sum_{k\geq 0}(c, x_0^k x_1)\, x_0^k x_1 \circ c_u \\ &= \sum_{k\geq 0}(c, x_0^k x_1)\, x_0^{k+1} c_u. \end{aligned}$$

Therefore,

$$(c_y, x_0^n) = \sum_{k=0}^{n-1}(c, x_0^k x_1)\,(c_u, x_0^{n-1-k}), \quad n \geq 1,$$

which is just the conventional convolution sum. □

Example 3.2: Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$. It is easily verified for $n \geq 1$ that

$$x_i^n \circ c = \frac{1}{n!}(x_0 c_i)^{\sqcup\!\sqcup\, n}, \quad i = 1, 2, \cdots, m, \tag{2}$$

where $(\cdot)^{\sqcup\!\sqcup\, n}$ denotes the shuffle power. Applying the formal Laplace transform to both sides gives

$$\begin{aligned} \mathscr{B}_f\left[x_i^n \circ c\right] &= \mathscr{B}_f\left[\frac{1}{n!}(x_0 c_i)^{\sqcup\!\sqcup\, n}\right] \\ &= \frac{1}{n!}\left[\mathscr{B}_f[x_0 c_i]\right]^n \\ &= \frac{1}{n!}\left[\int_0^t F_{c_i}[u](\tau)\,d\tau\right]^n. \end{aligned}$$

□

Example 3.3: Consider a simple Wiener system as shown in Fig. 1 where $z(0) = 0$.

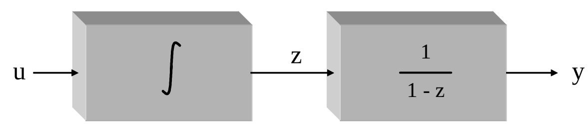

Fig. 1. A simple Wiener system.

The mapping $u \mapsto y$ can be written as

$$y(t) = \sum_{n=0}^{\infty}(E_{x_1}[u](t))^n = \sum_{n=0}^{\infty} E_{x_1^{\sqcup\!\sqcup\, n}}[u](t) = \sum_{n=0}^{\infty} n! E_{x_1^n}[u](t).$$

Therefore $y = F_c[u]$ where $c = \sum_{n\geq 0} n!\, x_1^n$. When $u(t) = t^m/m!$, for example, the formal Laplace transform of u is $c_u = x_0^m$, and from Theorem 3.1 and (2) it follows that

$$\begin{aligned} c_y &= \sum_{n=0}^{\infty} n!\, x_1^n \circ x_0^m = \sum_{n=0}^{\infty}(x_0^{m+1})^{\sqcup\!\sqcup\, n} \\ &= \sum_{n=0}^{\infty}\frac{((m+1)n)!}{((m+1)!)^n} x_0^{(m+1)n}. \end{aligned}$$

Consequently, as expected,

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty}\frac{((m+1)n)!}{((m+1)!)^n}\frac{t^{(m+1)n}}{((m+1)n)!} \\ &= \sum_{n=0}^{\infty}\frac{t^{(m+1)n}}{((m+1)!)^n} = \frac{1}{1 - \frac{t^{m+1}}{(m+1)!}}. \end{aligned}$$

□

Example 3.4: First consider the linear ordinary differential equation

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{n-1} b_i \frac{d^i u(t)}{dt^i}$$

with zero initial conditions. Integrate both sides of the equation n times and assume there exists a $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ such that $y(t) = F_c[u](t)$. Then after applying the formal Laplace transform, the equation becomes

$$\left(\delta + \sum_{i=0}^{n-1} a_i\, x_0^{n-1-i} x_1\right) \circ c = \sum_{i=0}^{n-1} b_i\, x_0^{n-1-i} x_1$$

$$\left(1 + \sum_{i=0}^{n-1} a_i\, x_0^{n-i}\right) c = \sum_{i=0}^{n-1} b_i\, x_0^{n-1-i} x_1.$$

Therefore,

$$c = \left(1 + \sum_{i=0}^{n-1} a_i\, x_0^{n-i}\right)^{-1} \sum_{i=0}^{n-1} b_i\, x_0^{n-1-i} x_1.$$

Rephrased in the language of the integral Laplace transform, this is equivalent to

$$\begin{aligned} Y(s) &= \left(1 + \sum_{i=0}^{n-1} a_i \frac{1}{s^{n-i}}\right)^{-1} \left(\sum_{i=0}^{n-1} b_i \frac{1}{s^{n-i}}\right) U(s) \\ &= \left(s^n + \sum_{i=0}^{n-1} a_i s^i\right)^{-1} \left(\sum_{i=0}^{n-1} b_i s^i\right) U(s). \end{aligned}$$

Now consider the nonlinear differential equation

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} + \sum_{i=2}^{k} p_i\, u(t) y^i(t) = \sum_{i=0}^{n-1} b_i \frac{d^i u(t)}{dt^i}$$

with zero initial conditions. Again integrate both side of the equation n times and assume $y(t) = F_c[u](t)$. Applying the formal Laplace transform gives

$$\begin{aligned} &\left(\delta + \sum_{i=0}^{n-1} a_i\, x_0^{n-1-i} x_1\right) \circ c + \sum_{i=2}^{k} p_i\, x_0^{n-1} x_1 (c^{\sqcup\!\sqcup\, i}) \\ &\quad = \sum_{i=0}^{n-1} b_i\, x_0^{n-1-i} x_1 \\ &\left(1 + \sum_{i=0}^{n-1} a_i\, x_0^{n-i}\right) c + \sum_{i=2}^{k} p_i\, x_0^{n-1} x_1 (c^{\sqcup\!\sqcup\, i}) \\ &\quad = \sum_{i=0}^{n-1} b_i\, x_0^{n-1-i} x_1. \end{aligned}$$

As in [6], a recursive procedure can be applied to solve the algebraic equation so that

$$c = c_1 + c_2 + \cdots$$

with

$$c_1 = \left(1 + \sum_{i=0}^{n-1} a_i\, x_0^{n-i}\right)^{-1} \sum_{i=0}^{n-1} b_i\, x_0^{n-1-i} x_1$$

and for $n \geq 2$

$$\begin{aligned} c_n &= \left(1 + \sum_{i=0}^{n-1} a_i\, x_0^{n-i}\right)^{-1} x_0^{n-1} x_1 \sum_{j=2}^{k} p_j \\ &\quad \sum_{\substack{\nu_1 \geq 1, \ldots, \nu_j \geq 1 \\ \nu_1 + \nu_2 + \cdots + \nu_j = n}} c_{\nu_1} \sqcup\!\sqcup\, c_{\nu_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_{\nu_j}. \end{aligned}$$

□

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