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# Application of a new functional expansion to the cubic anharmonic oscillator

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(Received 27 May 1981; accepted for publication 16 October 1981)

A new representation of causal functionals is introduced which makes use of noncommutative generating power series and iterated integrals. This technique allows the solutions of nonlinear differential equations with forcing terms to be obtained in a simple and natural way. It generalizes some properties of Fourier and Laplace transforms to nonlinear systems and leads to effective computations of various perturbative expansions. Illustrations by means of the cubic anharmonic oscillator are given in both the deterministic and the stochastic cases.

PACS numbers: 02.30. – k

## INTRODUCTION

Recently a new approach to causal functionals was proposed using noncommutative variables and iterated integrals.[1] This algebraic viewpoint enables us to obtain in closed form solutions of nonlinear differential equations with forcing terms. This can be done in a very simple and natural way using the vector fields connected with the equation. The rules for manipulating noncommutative variables, where the product is replaced by the shuffle, generalize Heaviside symbolic calculus to the nonlinear domain, i.e., noncommutative variables allow us to extend some properties of Laplace and Fourier transforms to nonlinear systems.

The aim of this paper is to illustrate this theory, which has appeared in engineering, by some physical examples. After some necessary recapitulation, we compare the fundamental formula giving the solution of a nonlinear differential equation with some recent attempts due to Uzes,[2] Jouvet and Phythian,[3] and Langouche *et al.*[4] Morton and Corrsin[5] used Fourier transforms for giving the solution of the cubic anharmonic oscillator, commonly known as the Duffing equation. Their computations, which had only an heuristic value, are completely justified with our noncommutative variables.

The last section is devoted to the study of statistical properties of the output of the cubic anharmonic oscillator driven by a Gaussian white noise. Noncommutative variables give a systematic understanding of the derivation of the first perturbative terms of the moments and lead to an easy implementation on computers.[6]

## I. NONCOMMUTATIVE GENERATING POWER SERIES

### A. Free monoid and noncommutative formal power series

Let $X^*$ be the *free monoid*[7] generated by a finite set $X = \{x_0, ..., x_n\}$ called the *alphabet*. Every element of $X^*$ is a *word* and consists of a finite sequence $x_{j_\nu} \cdots x_{j_0}$ of letters of the alphabet. The product of two words $x_{j_\nu} \cdots x_{j_0}$ and $x_{k_\mu} \cdots x_{k_0}$ is the concatenation $x_{j_\nu} \cdots x_{j_0} x_{k_\mu} \cdots x_{k_0}$. This operation is noncommutative. The neutral element is called the *empty word* and written with 1.

Let $\mathbb{R}\langle X \rangle$ and $\mathbb{R}\langle\langle X \rangle\rangle$ be the $\mathbb{R}$-algebras of formal polynomials and power series (ps) with real coefficients and noncommutative variables $x_j \in X$. An element $s \in \mathbb{R}\langle\langle X \rangle\rangle$ is written as a formal sum

$$s = \sum \{(s, w)w \mid w \in X^*\}, \quad \text{where } (s, w) \in \mathbb{R}.$$

Addition and (Cauchy) multiplication are defined by[8]

$$s_1 + s_2 = \sum \{[(s_1, w) + (s_2, w)]w \mid w \in X^*\},$$

$$s_1 s_2 = \sum \left\{ \left[ \sum_{w_1 w_2 = w} (s_1, w_1)(s_2, w_2) \right] w \mid w \in X^* \right\}.$$

### B. Iterated integrals and analytic causal functionals

Let $\xi_0, \xi_1, ..., \xi_n : [0, T] \to \mathbb{R}$ be $n + 1$ continuous functions with bounded variations. We define the *iterated integral*[9] $\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}$ $(0 \leqslant t \leqslant T)$ by induction on the length

$$\int_0^t d\xi_j = \xi_j(t) - \xi_j(0) \quad (j = 0, 1, ..., n),$$

$$\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0} = \int_0^t d\xi_{j_\nu}(\tau) \int_0^\tau d\xi_{j_{\nu-1}} \cdots d\xi_{j_0},$$

where the last integral is a Stieltjes integral.

To the inputs $u_1, ..., u_n : [0, T] \to \mathbb{R}$, which are assumed to be piecewise continuous, one associates the iterated integral

$$\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}, \text{ where } \xi_0(\tau) = \tau, \ \xi_i(\tau) = \int_0^\tau u_i(\sigma) d\sigma \quad (i = 1, ..., n).$$

Now consider a noncommutative ps $g \in \mathbb{R}\langle\langle X \rangle\rangle$. It defines a causal, or nonanticipative, functional[10] of the inputs $u_i$ if we replace the word $x_{j_\nu} \cdots x_{j_0}$ by the corresponding iterated integral $\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}$. Thus, the numerical value[11] is

$$y(t; u_1, ..., u_n) = (g, 1) + \sum_{\nu \geqslant 0} \sum_{j_0, ..., j_\nu = 0}^{n} (g, x_{j_\nu} \cdots x_{j_0}) \times \int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}. \quad (1)$$

Such a causal functional is said to be *analytic* with the *generating ps* g.

### C. Fundamental formula

Consider the following differential system, which is assumed to be of first order without loss of generality,

$$\begin{cases} \dot{q}(t)(= dq/dt) = A_0(q) + \sum_{i=1}^{n} u_i(t) A_i(q), \\ y(t) = h(q). \end{cases} \quad (2)$$

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The state $q$ belongs to a real analytic manifold $Q$ [the initial state $q(0)$ is given]; the vector fields $A_0, A_1, ..., A_n$ and the function $h: Q \to \mathbb{R}$ are analytic. The inputs (or controls) $u_1, ..., u_n: [0, T] \to \mathbb{R}$ are often forces.

Take some local coordinates chart, where $q = (q^1, ..., q^N)$ and

$$A_j = \sum_{k=1}^{N} \theta_j^k (q^1, ..., q^N) \frac{\partial}{\partial q^k}$$

(the $\theta_j^k$ are analytic functions of $q^1, ..., q^N$). Recall then that the first line of (2) is equivalent to

$$\dot{q}^k(t) = \theta_0^k + \sum_{i=1}^{n} u_i(t) \theta_i^k \quad (k = 1, ..., N).$$

One can prove that the output $y$ of (2) is an analytic causal functional of $u_1, ..., u_n$ defined by the generating ps[12]

$$\mathfrak{g} = h|_{q(0)} + \sum_{\nu \geqslant 0} \sum_{j_0, ..., j_\nu = 0}^{n} A_{j_0} \cdots A_{j_\nu} h|_{q(0)} x_{j_\nu} \cdots x_{j_0} \tag{3}$$

[the notation $|_{q(0)}$ means the value at $q(0)$].

The formula (3) and its proof generalize Gröbner's work[13] on Lie series and free differential equations of the form $\dot{q}(t) = A(q)$.

Uzes[2] tried also to extend Gröbner's theory to get the solution of forced nonlinear differential equations, using Gâteaux–Fréchet's functional derivatives.[14] The latest expansions are really useful if the time $t$ is fixed once and for all. In the dynamic case, where time varies, they lead to a more complex formulation than (3). On the other hand, Jouvet and Phythian[3] and Langouche, Roekaerts, and Tirapegui[4] used a formalized operator which does not give the generating functional in a simple form. These comparisons, and others we can make with engineering attempts,[15–17] lead us to think that for causal functionals the natural expansion is done with noncommutative generating power series.

### D. Volterra series

Volterra series are until now the functional expansions most commonly used.[2,4,15–17] With only one input, one obtains

$$\begin{aligned} y(t; u_1) &= w_0(t) + \int_0^t w_1(t, \tau_1) u_1(\tau_1) d\tau_1 \\ &+ \int_0^t \int_0^{\tau_2} w_2(t, \tau_2, \tau_1) u_1(\tau_2) u_1(\tau_1) d\tau_2 d\tau_1 \\ &+ \cdots + \int_0^t \int_0^{\tau_s} \cdots \int_0^{\tau_2} w_s(t, \tau_s, ..., \tau_1) u_1(\tau_s) \cdots u_1(\tau_1) \\ &\times d\tau_s \cdots d\tau_1 + \cdots. \end{aligned} \tag{4}$$

Kernels here are in a triangular form; hence $t \geqslant \tau_s \geqslant \cdots \geqslant \tau_1 \geqslant 0$. One can also use the symmetric form

$$\begin{aligned} y(t; u_1) &= w'_0(t) + \int_0^t w'_1(t, \tau_1) u_1(\tau_1) d\tau_1 + \cdots \\ &+ \int_0^t \cdots \int_0^t w'_s(t, \tau_s, ..., \tau_1) u_1(\tau_s) \cdots u_1(\tau_1) \\ &\times d\tau_s \cdots d\tau_1 + \cdots, \end{aligned}$$

where the $w'_s$ are symmetric functions of the variables $\tau_s, ..., \tau_1$. In each case the kernels are uniquely defined up to a set of measure zero.

In these expansions, the linear, quadratic, cubic, etc.,... contributions are separated.

There is, in fact, a strong relationship between Volterra series and noncommutative generating ps. One can show that a Volterra series defines an analytic causal functional if, and only if, for all $s \geqslant 0$, the kernel $w_s(t, \tau_s, ..., \tau_1)$ is an analytic function of $t, \tau_s, ..., \tau_1$.

Consider the differential system

$$\begin{cases} \dot{q}(t) = A_0(q) + u_1(t) A_1(q) \\ y(t) = h(q) \end{cases}$$

of the form (2), with only a single input. From (3), we can get the output $y$ as a Volterra series (4), where the kernels are given by[18]

$$w_0(t) = \sum_{\nu \geqslant 0} A_0^{\nu} h|_{q(0)} \frac{t^\nu}{\nu!} = e^{tA_0} h|_{q(0)},$$

$$\begin{aligned} w_1(t, \tau_1) &= \sum_{\nu_0, \nu_1 \geqslant 0} A_0^{\nu_0} A_1 A_0^{\nu_1} h|_{q(0)} \frac{(t - \tau_1)^{\nu_1} \tau_1^{\nu_0}}{\nu_1! \nu_0!} \\ &= e^{\tau_1 A_0} A_1 e^{(t - \tau_1) A_0} h|_{q(0)}, \end{aligned}$$

$$\begin{aligned} &w_2(t, \tau_2, \tau_1) \\ &= \sum_{\nu_0, \nu_1, \nu_2 \geqslant 0} A_0^{\nu_0} A_1 A_0^{\nu_1} A_1 A_0^{\nu_2} h|_{q(0)} \frac{(t - \tau_2)^{\nu_2} (\tau_2 - \tau_1)^{\nu_1} \tau_1^{\nu_0}}{\nu_2! \nu_1! \nu_0!} \\ &= e^{\tau_1 A_0} A_1 e^{(\tau_2 - \tau_1) A_0} A_1 e^{(t - \tau_2) A_0} h|_{q(0)}, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} w_s(t, \tau_s, ..., \tau_1) &= \sum_{\nu_0, ..., \nu_s \geqslant 0} A_0^{\nu_0} A_1 \cdots A_1 A_0^{\nu_s} h|_{q(0)} \frac{(t - \tau_s)^{\nu_s} \cdots \tau_1^{\nu_0}}{\nu_s! \cdots \nu_0!} \\ &= e^{\tau_1 A_0} A_1 \cdots A_1 e^{(t - \tau_s) A_0} h|_{q(0)}. \end{aligned}$$

## II. A NONCOMMUTATIVE SYMBOLIC CALCULUS

### A. Presentation

The generating ps representing the solution of a forced differential system can be obtained by a noncommutative symbolic calculus which generalizes Heaviside symbolic, or operational, calculus. Rather than a general formulation,[1] we apply the method to the cubic anharmonic oscillator, i.e., the Duffing equation

$$\ddot{y}(t) + \alpha \dot{y}(t) + y(t) + \beta y^3(t) = u_1(t). \tag{5}$$

To account for the cubic term, we introduce a new operation on generating ps: we define the *shuffle product* by induction on the length of words

$$1 ш 1 = 1,$$

$$\forall x \in X, \quad x ш 1 = 1 ш x = x,$$

$$\forall x, x' \in X, \quad \forall w, w' \in X^*,$$

$$(xw) ш (x'w') = x[w ш (x'w')] + x'[(xw) ш w']. \tag{6}$$

So the shuffle product of two words is a homogeneous polynomial, the degree of which is the sum of the length of the words. For example

$$x_0 x_1 ш x_1 x_0 = 2x_0 x_1^2 x_0 + x_0 x_1 x_0 x_1 + x_1 x_0 x_1 x_0 + 2x_1 x_0^2 x_1.$$

The shuffle product of two generating ps $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathbb{R}\langle\langle X \rangle\rangle$ is given by

$$\mathfrak{g}_1 ш \mathfrak{g}_2 = \sum \{(\mathfrak{g}_1, w_1)(\mathfrak{g}_2, w_2) w_1 ш w_2 \mid w_1, w_2 \in X^*\}.$$

Consider now the product of two iterated integrals

$$\left(\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}\right)\left(\int_0^t d\xi_{k_\mu} \cdots d\xi_{k_0}\right).$$

An integration by parts gives

$$\int_0^t d\xi_{j_\nu}(\tau)\left[\left(\int_0^\tau d\xi_{j_{\nu-1}} \cdots d\xi_{j_0}\right)\left(\int_0^\tau d\xi_{k_\mu} \cdots d\xi_{k_0}\right)\right]$$
$$+\int_0^t d\xi_{k_\mu}(\tau)\left[\left(\int_0^\tau d\xi_{j_\nu} \cdots d\xi_{j_0}\right)\left(\int_0^\tau d\xi_{k_{\mu-1}} \cdots d\xi_{k_0}\right)\right].$$

This last formula is similar to the definition (6) of the shuffle product.[19] We then have the following important result.

**Theorem**[20]: The product of two analytic causal functionals is a functional of the same kind, the generating power series of which is the shuffle product of the two generating power series.

Consider again (5) in the following integral form:

$$y(t) + \alpha\int_0^t y(\tau)d\tau + \int_0^t d\tau \int_0^\tau y(\sigma)d\sigma + \beta\int_0^t d\tau\int_0^\tau y^3(\sigma)d\sigma$$
$$= \int_0^t d\tau\int_0^\tau u_1(\sigma)d\sigma + at + b, \qquad (7)$$

where $a = \dot{y}(0) + \alpha y(0)$ and $b = y(0)$.

The previous theorem and the relationship between iterated integrals and noncommutative indeterminates allow us to write (7) in the following form:

$$\mathfrak{g} + \alpha x_0 \mathfrak{g} + x_0^2 \mathfrak{g} + \beta x_0^2 \mathfrak{g} ш \mathfrak{g} ш \mathfrak{g} = x_0 x_1 + a x_1 + b. \qquad (8)$$

The algebraic equation can be solved iteratively with the fixed point theorem, according to the scheme

$$\mathfrak{g}_{i+1} + \alpha x_0 \mathfrak{g}_i + x_0^2 \mathfrak{g}_i + \beta x_0^2 \mathfrak{g}_i ш \mathfrak{g}_i ш \mathfrak{g}_i = x_0 x_1 + a x_1 + b.$$

Noncommutative variables extend some properties of Laplace–Fourier transforms to the nonlinear domain. Indeed, with the linear differential equation

$$\ddot{y} + y = u_1(t)$$

we associate for $y(0) = \dot{y}(0) = 0$, by the method described above, the generating ps

$$\mathfrak{g} + x_0^2 \mathfrak{g} = x_0 x_1,$$
$$\mathfrak{g} = (1 + x_0^2)^{-1} x_0 x_1,$$

which is analogous to the classical transfer function $1/(p^2 + 1)$.

### B. An example of functional expansion

Consider again Eq. (5), seeking for small $\beta$ a perturbative expansion of the form

$$y(t) = y_0(t) + \beta y_1(t) + \beta^2 y_2(t) + \cdots.$$

In a quoted paper, Morton and Corrsin,[5] to this end, used the Fourier transform. This is heuristic because there is no simple relationship between the transform of a product and a product of transforms. After briefly reviewing their work, we show that our noncommutative symbolic calculus justifies it rigorously.

Making use of harmonic analysis, the authors write $y$ and $u_1$ in the form

$$y(t) = \sum_\omega Y(\omega) e^{i\omega t} \quad \text{with } Y(\omega) = \frac{1}{2T}\int_{-T}^{+T} y(t) e^{-i\omega t} dt,$$
$$u_1(t) = \sum_\omega U_1(\omega) e^{i\omega t} \quad \text{with } U_1(\omega) = \frac{1}{2T}\int_{-T}^{+T} u_1(\omega) e^{-i\omega t} dt,$$

where $[-T, +T]$ can be very large. Equation (5) becomes

$$(1 - \omega^2 + i\alpha\omega)Y(\omega) + \beta\sum_{\omega'}\sum_{\omega''} Y(\omega - \omega')Y(\omega' - \omega'')Y(\omega'')$$
$$= U_1(\omega)$$

or

$$(1 - \omega^2 + i\alpha\omega)Y(\omega) + \beta \sum_{\omega_1 + \omega_2 + \omega_3 = \omega} Y(\omega_1)Y(\omega_2)Y(\omega_3)$$
$$= U_1(\omega).$$

Terms of the perturbative expansion

$$Y(\omega) = Y_0(\omega) + \beta Y_1(\omega) + \beta^2 Y_2(\omega) + \cdots$$

are then

$$Y_0(\omega) = (1 - \omega^2 + i\alpha\omega)^{-1} U_1(\omega) = -S(\omega)U_1(\omega),$$
$$Y_1(\omega) = S(\omega) \sum_{\omega_1 + \omega_2 + \omega_3 = \omega} Y_0(\omega_1)Y_0(\omega_2)Y_0(\omega_3),$$
$$Y_2(\omega) = 3S(\omega) \sum_{\omega_1 + \omega_2 + \omega_3 = \omega} Y_0(\omega_1)Y_0(\omega_2)Y_1(\omega_3).$$

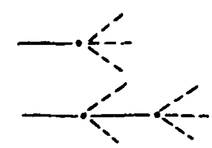
These expressions become more and more complicated. The use of Feynman type diagrams in which

a straight line (——) represents $S(\omega)$,
a dot (.) represents $\beta$,
a dashed line (---) represents $Y_0(\omega)$,

allows us to simplify the manipulations. We deduce the following representations:

$Y_1 \longleftrightarrow$

$Y_2 \longleftrightarrow 3$



The one-to-one correspondance between $Y_k$ and these diagrams obeys the following rules:

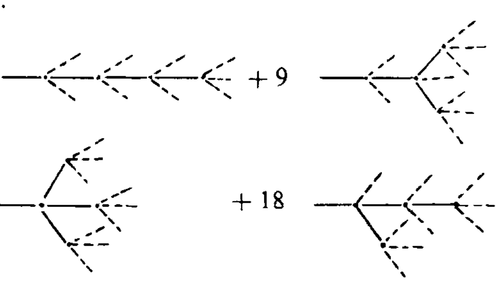
Four elements are joined at each vertex, at least one of which is a straight line.

There is a factor of 3 associated with every vertex having one or two dashed lines entering it.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

So to draw $Y_k$ one must take $k$ straight lines and $k$ vertices, combining them in all possible ways consistent with the above rules and adding the necessary $(2k+1)Y_0$'s (dashed lines).

$Y_4 \leftrightarrow 27$ $+\,9$ $+$ $+\,18$



Let us write the solution $\mathfrak{g}$ of (8) in the perturbative form

$$\mathfrak{g} = \mathfrak{g}_0 + \beta\mathfrak{g}_1 + \beta^2\mathfrak{g}_2 + \cdots.$$

Hence,

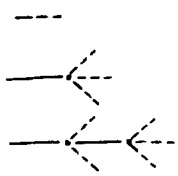
$$\mathfrak{g}_0 = (1 + \alpha x_0 + x_0^2)^{-1}(x_0 x_1 + a x_0 + b) = -S(x_0)(x_0 x_1 + a x_0 + b),$$
$$\mathfrak{g}_1 = S(x_0)\mathfrak{g}_0 ш \mathfrak{g}_0 ш \mathfrak{g}_0,$$
$$\mathfrak{g}_2 = 3S(x_0)\mathfrak{g}_0 ш \mathfrak{g}_0 ш \mathfrak{g}_1,$$

which represent them as $Y_0$, $Y_1$, $Y_2$ by

$\mathfrak{g}_0 \leftrightarrow$

$\mathfrak{g}_1 \leftrightarrow$

$\mathfrak{g}_2 \leftrightarrow$



The connection between three branches corresponds to the shuffle of three series.[21]

## III. SYSTEMS DRIVEN BY WHITE GAUSSIAN NOISES

### A. Generalities

A classical problem of convergence of functional expansions arises when the inputs are white Gaussian noises. This happens with generating ps as well as with the other techniques.

As we will see in the following, it is, however, instructive to use noncommutative variables. To this end, we must first give a meaning to stochastic iterated integrals $\int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}$, where the $\xi_i(t) = b_i(t)$ are Wiener processes, or Brownian motions which, for simplicity's sake, are supposed to be mutually independent and standard, i.e., $\langle b_i(t)\rangle = 0$, $\langle b_i^2(t)\rangle = |t|$. To keep the rules of ordinary calculus and taking account of approximation properties,[22] we use Stratonovich integrals.[23,24] If $\xi_0(t) = t$, $\xi_i(t) = b_i(t)$, we set

$$\int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0} = \int_0^t d\xi_{j_\nu}(\tau)\int_0^\tau d\xi_{j_{\nu-1}}\cdots d\xi_{j_0},$$

where for $j_\nu \neq 0$, this last integral is a Strotonovich integral.

It is also necessary to compute the average $\langle \int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}\rangle$ of iterated integrals. The following proposition can be compared with Wick's theorem.

*Proposition*: The moment $\langle \int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}\rangle$ of the iterated integral $\int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}$ is given by induction on the length by

$$\left\langle \int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}\right\rangle = \begin{cases} \int_0^t d\tau \left\langle \int_0^\tau d\xi_{j_{\nu-1}}\cdots d\xi_{j_0}\right\rangle & \text{if } j_\nu = 0, \\ \int_0^t \frac{d\tau}{2}\left\langle \int_0^\tau d\xi_{j_{\nu-2}}\cdots d\xi_{j_0}\right\rangle & \text{if } j_\nu = j_{\nu-1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof*: The iterated integral

$$B_{j_\nu\cdots j_0} = \int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}$$

satisfies the Stratonovich stochastic differential equation

$$dB_{j_\nu\cdots j_0} = B_{j_{\nu-1}\cdots j_0}\circ d\xi_{j\nu}.$$

(For $j_\nu = 0$, i.e., $d\xi_{j\nu} = dt$, the result is trivial). This Stratonovich stochastic differential is related to that of Itô by

$$dB_{j_\nu\cdots j_0} = B_{j_{\nu-1}\cdots j_0}\cdot d\xi_{j\nu} + \tfrac{1}{2}dB_{j_{\nu-1}\cdots j_0}\cdot d\xi_{j\nu},$$

where the symbol . denotes the differential in the Itô's sense. Hence,

$$dB_{j_\nu\cdots j_0} = B_{j_{\nu-1}\cdots j_0}\cdot d\xi_{j_\nu} + \tfrac{1}{2}\left[B_{j_{\nu-2}\cdots j_0}\cdot d\xi_{j_{\nu-1}} + \tfrac{1}{2}dB_{j_{\nu-2}\cdots j_0}\cdot d\xi_{j_{\nu-1}}\right]\cdot d\xi_{j_\nu}.$$

From the definition of the Itô stochastic differentials, we have

$$\langle B_{j_{\nu-1}\cdots j_0}\cdot d\xi_{j_\nu}\rangle = 0.$$

Finally, the classical rules of stochastic calculus,

$$\begin{cases} db.db \simeq dt, \\ db.dt \simeq 0, \\ dt.dt \simeq 0, \end{cases}$$

lead to

$$\langle dB_{j_\nu\cdots j_0}\rangle = \begin{cases} \frac{dt}{2}\cdot\langle B_{j_{\nu-2}\cdots j_0}\rangle & \text{if } j_\nu = j_{\nu-1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

### B. Statistics of the solutions of stochastic differential equations

Consider again the system (2); here in Stratonovich stochastic form

$$\begin{cases} dq = A_0(q)dt + \sum_{i=1}^n A_i(q)db_i, \\ y(t) = h(q). \end{cases}$$

$b_1, b_2, \ldots, b_n$ are standard Wiener processes, which are mutually independent [the initial state $q(0)$ is given]. Applying the previous rules to the fundamental formula, we get[25]

$$\langle y(t)\rangle = h|_{q(0)} + \sum_{\nu\geqslant 0}\frac{t^\nu}{\nu!}\left(A_0 + \frac{1}{2}\sum_{i=1}^n A_i^2\right)^\nu h|_{q(0)} = \left[\exp t\left(A_0 + \frac{1}{2}\sum_{i=1}^n A_i^2\right)\right]h|_{q(0)}.$$

*Example*: The following system is described by

$$\begin{cases} dq = \left(B_0 + \sum_{i=1}^n B_i db_i\right)q(t), \\ y(t) = \lambda q(t). \end{cases}$$

The state $q$ belongs to $\mathbb{R}^N$; $B_j$ ($j = 0, \ldots, n$) and $\lambda$ are, respectively, square matrices and row vectors of order $N$ (systems of this form are known, in control theory, as *regular* or *bilinear* systems). We have

$$y(t) = \lambda\left(1 + \sum_{\nu\geqslant 0}\sum_{j_0,\ldots,j_\nu=0}^n B_{j_\nu}\cdots B_{j_0}\int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}\right)q(0),$$

hence

$$\langle y(t)\rangle = \lambda\left[\exp t\left(B_0 + \frac{1}{2}\sum_{i=1}^n B_i^2\right)\right]q(0). \tag{9}$$

In this particular case, we see that we have convergence and the formula (9) is then rigorous.[26]

Figure 1 gives the time expansion up to orders 8 and 12 of the moment $\langle q(t)\rangle$ where

$$\ddot q + \dot q + q + 0, 2q^3 = \dot b(t),$$

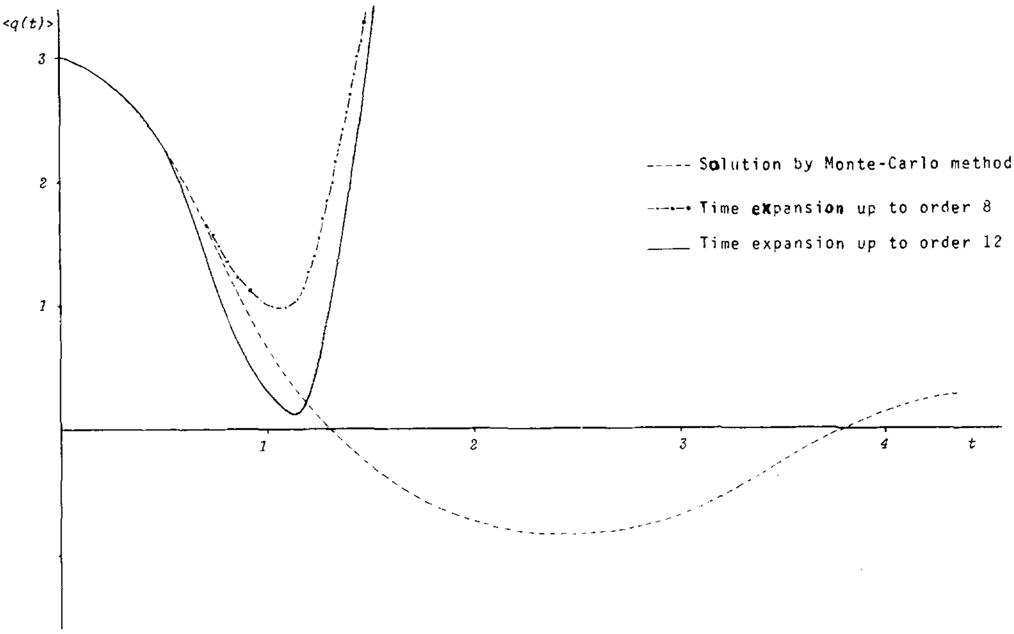



FIG. 1. First moment of the solution of the equation $\ddot{q} + \dot{q} + q + 0.2q^3 = \dot{b}(t)$ with $\sigma^2 = 5$, $q(0) = 3$, $\dot{q}(0) = 0$.

with $q(0) = 3$, $\dot{q}(0) = 0$.[27] The symbol $\dot{b}$ is the formal derivative of a Wiener process $b$, i.e., $\dot{b}$ is a Gaussian white noise. Here $\langle b(t)\rangle = 0$, $\langle b^2(t)\rangle = 5|t|$.

In the following we study perturbative expansions from which we can expect better results.

### C. Perturbative expansions with respect to nonlinearity

Consider the nonlinear differential equation

$$Ly + \beta P(y) = \dot{b}(t) \quad (y(0), \dot{y}(0), \ldots, \text{are given}),$$

where $L$ is a differential operator with constant coefficients, $P$ a polynomial, and $\beta$ a small parameter. Here we seek a perturbative expansion for the solution $y(t)$,

$$y(t) = y_0(t) + \beta y_1(t) + \beta^2 y_2(t) + \cdots. \tag{10}$$

Techniques using noncommutative variables, shown in the Appendix, give the ps $\mathfrak{g}_i$ corresponding to the $y_i$:

$$\mathfrak{g} = \mathfrak{g}_0 + \beta\mathfrak{g}_1 + \beta^2\mathfrak{g}_2 + \cdots.$$

$\mathfrak{g}$ is the generating ps associated to $y$. From a result analogous to the previous proposition, it is possible to derive the first terms of the perturbative expansion of $\langle y(t)\rangle$ and more generally of $\langle y^n(t)\rangle$.

*Application*: We refer again to the anharmonic oscillator

$$\ddot{y} + \alpha\dot{y} + y + \beta y^3 = \dot{b}(t),$$

for which we compare our results with those of Morton and Corrsin[5] (Fig. 2). The generating ps associated with the solution $y$ verifies the algebraic equation

$$\mathfrak{g} = -\beta(1 + \alpha x_0 + x_0^2)^{-1}\mathfrak{g}\,\text{ш}\,\mathfrak{g}\,\text{ш}\,\mathfrak{g} + (1 + \alpha x_0 + x_0^2)^{-1}(x_0 x_1 + a x_0 + b).$$

Setting

$$(1 + \alpha x_0 + x_0^2) = (1 - a_1 x_0)(1 - a_2 x_0)$$

and

$$(1 + \alpha x_0 + x_0^2)^{-1}(ax + b) = A_1(1 - a_1 x_0)^{-1} + A_2(1 - a_2 x_0)^{-1},$$

we obtain[28]

$$\mathfrak{g}_0 = \begin{array}{|ccc|} \hline 1 & 1 & \\ & 1 & 0 \\ & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccc|} \hline 1 & 1 & \\ & 0 & 0 \\ & 1 & 0 \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline 1 & X & Y & \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ \hline \end{array},$$

$$\mathfrak{g}_1 = \begin{array}{|cccc|} \hline 1 & X & Y & \\ & 1 & 0 & 3 \\ & 0 & 1 & 0 \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline 3 & X & X & \\ & 1 & 0 & 2 \\ & 0 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline 3 & X & X & \\ & 1 & 0 & 1 \\ & 0 & 1 & 2 \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline 1 & X & X & \\ & 1 & 0 & 0 \\ & 0 & 1 & 3 \\ \hline \end{array}$$

$$\begin{array}{|cccccc|} \hline 3 & X & X & X & Y & \\ & 1 & 0 & 3 & 2 & 2 \\ & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|} \hline 6 & X & X & X & Y & \\ & 1 & 0 & 2 & 1 & 1 \\ & 0 & 1 & 1 & 2 & 1 \\ \hline \end{array}$$

$$\begin{array}{|cccccc|}\hline 3 & X & X & X & Y & \\ 1 & 0 & 1 & 0 & 0 & \\ 0 & 1 & 2 & 3 & 2 & \\\hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 6 & X & X & X & Y & X & Y & \\ & 1 & 0 & 3 & 2 & 2 & 1 & 1\\ & 0 & 1 & 0 & 1 & 0 & 1 & 0\\\hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 12 & X & X & X & X & Y & Y & \\ & 1 & 0 & 3 & 2 & 1 & 1 & 1\\ & 0 & 1 & 0 & 1 & 2 & 1 & 0\\\hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 6 & X & X & X & Y & X & Y & \\ & 1 & 0 & 2 & 1 & 1 & 0 & 0\\ & 0 & 1 & 1 & 2 & 1 & 2 & 1\\\hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 12 & X & X & X & X & Y & Y & \\ & 1 & 0 & 2 & 1 & 0 & 0 & 0\\ & 0 & 1 & 1 & 2 & 2 & 2 & 1\\\hline\end{array}$$

For the first moment, we have then

$$\langle \mathfrak{g}_0\rangle = \begin{array}{|ccc|}\hline 1 & 1 & \\ & 1 & 0\\ & 0 & 0\\\hline\end{array} \quad \begin{array}{|ccc|}\hline 1 & 1 & \\ & 0 & 0\\ & 1 & 0\\\hline\end{array},$$

$$\langle \mathfrak{g}_1\rangle = \begin{array}{|cccc|}\hline 1 & X & X & \\ & 1 & 0 & 3\\ & 0 & 1 & 0\\\hline\end{array} \quad \begin{array}{|cccc|}\hline 3 & X & X & \\ & 1 & 0 & 2\\ & 0 & 1 & 1\\\hline\end{array} \quad \begin{array}{|cccc|}\hline 3 & X & X & \\ & 1 & 0 & 1\\ & 0 & 1 & 2\\\hline\end{array} \quad \begin{array}{|cccc|}\hline 1 & X & X & \\ & 1 & 0 & 0\\ & 0 & 1 & 3\\\hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 12\left(\frac{\sigma^2}{2}\right) & X & X & X & X & X & \\ & 1 & 0 & 3 & 2 & 1 & 1\\ & 0 & 1 & 0 & 1 & 2 & 0\\\hline\end{array} \quad \begin{array}{|ccccccc|}\hline 12\frac{\sigma^2}{2} & X & X & X & X & X & \\ & 1 & 0 & 2 & 1 & 0 & 0\\ & 0 & 1 & 1 & 1 & 2 & 1\\\hline\end{array}$$

Figure 2 gives the perturbative expansion up to order 2 of steady-state moment $\langle y^2\rangle$.

## IV. CONCLUSION

The functional methods proposed here are mathematically rigorously correct in the deterministic case, where they clarify various former attempts. In the stochastic case, their algebraic nature simplifies the computations of perturbative expansions.

## APPENDIX

Consider the differential equation

$$Ly + \beta P(y) + \dot{b}(t)$$

with

$$L = \sum_{i=0}^{n} l_i \frac{d}{dt_i} \quad (l_n = 1)$$

and

$$P(x) = \sum_{j=1}^{m} p_j x^j.$$

As previously (Sec. IIA), the generating ps associated with $y$ is given by

$$\left(\sum_{i=0}^{n} l_i x_0^{n-i}\right)\mathfrak{g} + x_0^n \beta \sum_{j=1}^{m} p_j \mathfrak{g}^{\text{ш} j} = x_0^{n-1}x_1 + \sum_{i=0}^{n-1} \delta_i x_0^i$$

or

$$\mathfrak{g} = -\left(\sum_{i=0}^{n} l_i x_0^{n-i}\right)^{-1} x_0^n \beta \sum_{j=1}^{m} p_j \mathfrak{g}^{\text{ш} j} + \left(\sum_{i=0}^{n} l_i x_0^{n-i}\right)^{-1}\left(x_0^{n-1}x_1 + \sum_{i=0}^{n-1} \delta_i x_0^i\right),$$

where $\delta_i (i = 0,...,n-1)$ are constants depending on the initial conditions.

The expansion (10) is "equivalent" to that of $\mathfrak{g}$ in powers of $\beta$:

$$\mathfrak{g} = \mathfrak{g}_0 + \beta\mathfrak{g}_1 + \beta^2\mathfrak{g}_2 + \cdots$$

with

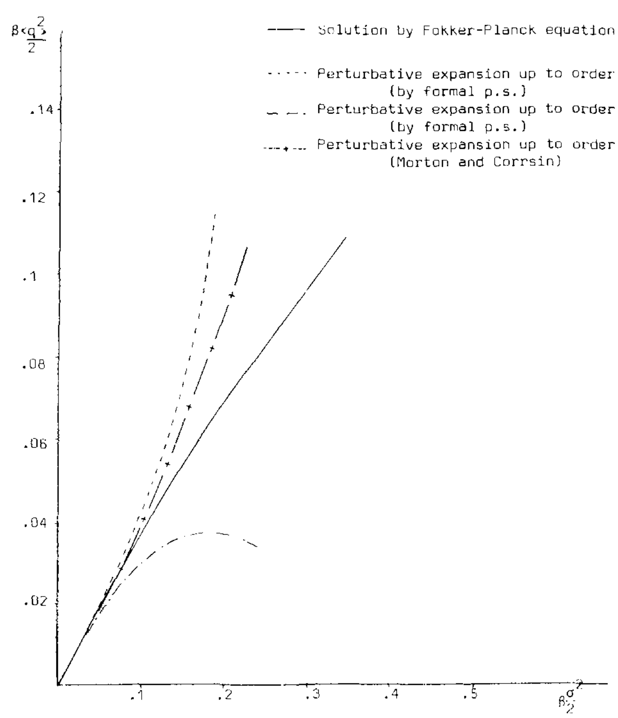



FIG. 2. Second steady-state moment of the solution of the equation $\ddot{q} + \dot{q} + q + \beta q^3 = \dot{b}(t)$, $\langle b^2(t)\rangle = \sigma^2|t|$.

$$\mathfrak{g}_0=\left(\sum_{i=0}^{n} l_i x_0^{n-i}\right)^{-1}\left(x_0^{n-1}x_1+\sum_{i=0}^{n-1}\delta_i x_0^i\right)$$

and

$$\mathfrak{g}_k=-\left(\sum_{i=0}^{n} l_i x_0^{n-i}\right)^{-1} x_0^n \sum_{j=1}^{m} \sum_{\substack{k_1,\dots,k_j\\ k_1+\cdots+k_j=k}} p_{k_1}p_{k_2}\cdots$$

$$\times p_{k_j}\mathfrak{g}_{k_1}ш\mathfrak{g}_{k_2}ш\cdotsш\mathfrak{g}_{k_j}.$$

To have the rational expression of $g_i$, we need to compute the shuffle product of powers series of the form

$$(1-a_0x_0)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}\cdots(1-a_px_0)^{-1}. \qquad \text{(A1)}$$

*Proposition*[29]: Given two formal ps

$$S_1^p=(1-a_0x_0)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}\cdots x_{i_p}(1-a_px_0)^{-1}=S_1^{p-1}x_{i_p}(1-a_px_0)^{-1}$$

and

$$S_2^q=(1-b_0x_0)^{-1}x_{j_1}(1-b_1x_0)^{-1}x_{j_2}\cdots x_{j_q}(1-b_qx_0)^{-1}=S_2^{q-1}x_{j_q}(1-b_qx_0)^{-1},$$

where $p$ and $q$ belong to $\mathbb{N}$, the subscripts $i_1,\dots,i_p,j_1,\dots,j_q$ to $\{0,1\}$, and $a_i$, $b_j$ to $\mathbb{C}$; the shuffle product is given by induction on the length by

$$S_1^pшS_2^q=(S_1^pшS_2^{q-1})x_{j_q}[1-(a_p+b_q)x_0]^{-1}+(S_1^{p-1}шS_2^q)x_{i_p}[1-(a_p+b_q)x_0]^{-1}$$

with

$$(1-ax_0)^{-1}ш(1-bx_0)^{-1}=[1-(a+b)x_0]^{-1}.$$

This shows that $\mathfrak{g}_i(i\geqslant 0)$ is a finite sum of expressions of the form (11). To derive perturbative expansion of the first moment,

$$\langle\mathfrak{g}\rangle=\langle\mathfrak{g}_0\rangle+\beta\langle\mathfrak{g}_1\rangle+\beta^2\langle\mathfrak{g}_2\rangle+\cdots,$$

we should compute

$$\langle(1-a_0x_0)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}\cdots x_{i_p}(1-a_px_0)^{-1}\rangle.$$

This is given (see the proposition of Sec. IIIA) by induction on the length by

$$\langle(1-a_0x_0)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}\cdots x_{i_p}(1-a_px_0)^{-1}\rangle = \begin{cases}(1-a_0x_0)^{-1}x_0\langle(1-a_1x_0)^{-1}x_{i_2}\cdots x_{i_p}(1-a_px_0)^{-1}\rangle & \text{if } i_1=0,\\ \frac{1}{2}(1-a_0x_0)^{-1}x_0\langle(1-a_2x_0)^{-1}x_{i_3}\cdots x_{i_p}(1-a_px_0)^{-1}\rangle & \text{if } i_1=i_2=1,\\ 0 & \text{otherwise.}\end{cases}$$

$\mathfrak{g}_i$ is then a rational fraction in the only variable $x_0$. Its decomposition into partial fractions and the following lemma give its corresponding expression in time.

*Lemma*: The rational fraction $(1-ax_0)^{-p}$ corresponds to the exponential polynomial

$$\left(\sum_{j=0}^{p-1}\binom{j}{p-1}\frac{a^jt^j}{j!}\right)e^{at}.$$

This results easily from

$$(1-ax_0)^{-p}=(1+ax_0)^{p-1}ш(1-ax_0)^{-1}.$$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[7] Remember that the free monoid is an important subject of investigation in some questions resulting from theoretical computer science. One should cite here the name of M. P. Schützenberger. See, for example, S. Eilenberg, *Automata, Languages and Machines* (Academic, New York, 1974), Vol. A; G. Lallement, *Semigroups and Combinational Applications* (Wiley, New York, 1979).
[8] In the case $n=0$, one finds again the commutative algebras $\mathbb{R}[x_0]$ and $\mathbb{R}[[x_0]]$ of polynomials and power series in one variable.
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[10] A functional is said to be causal, or nonanticipative, if at time $t$ its value depends on the values of the $u_i(\tau)$ only for $\tau<t$.
[11] Equation (1) is supposed to be absolutely convergent for $t$ and $\max_{0,\tau,t}|u_i(\tau)|$ sufficiently small.
[12] It is worth noting that the order of subscripts in the sequences $A_{j_0}\cdots A_{j_1}h|_{q(0)}$ and $x_{j_1}\cdots x_{j_0}$ are inverted.
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the mathematical validity of the foregoing is not ensured. This formula could also be derived, in an heuristic way, from the Fokker–Planck equation by path integral techniques. See, for example, R. L. Stratonovich, Sel. Transl. Math. Stat. Prob. **10**, 273 (1971); and R. Graham, Z. Phys. B **26**, 281 (1977).

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[27]It should be remembered that, for this equation with an additive noise, Itô's and Stratonovich's interpretations are equivalent.

[28]The notation

$$\begin{array}{|ccccccc|} \hline C & & x_{i_1} & & x_{i_2} \quad \cdots & & x_{i_p} \\ & c_{10} & & c_{11} & & c_{1(p-1)} & c_{1p} \\ & c_{20} & & c_{21} & & c_{2(p-1)} & c_{2p} \\ \hline \end{array}$$

means

$$CA_1^{c_{1p}}A_2^{c_{2p}}[1-(c_{10}a_1+c_{20}a_2)x_0]^{-1}x_{i_1}\cdots x_{i_p}[1-(c_{1p}a_1+c_{2p}a_2)x_0]^{-1}.$$

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