Fliess Operators in Cascade and Feedback Systems

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Abstract — **Given two nonlinear input-output systems represented as Fliess operators, this paper considers properties of the cascade and feedback interconnected systems. The cascade system is characterized via the *composition product* for formal power series. In the multivariable setting this product is shown to be generally well defined and continuous in its first argument. Then a condition is introduced under which the composition product preserves rationality and local convergence. In preparation for the feedback analysis, it is also shown that the composition product produces a contractive mapping on the set of all formal power series using the familiar ultrametric. Next, the feedback connection is considered in the special case where inputs are supplied from an exosystem which is itself a Fliess operator. In particular, a sufficient condition is given under which a unique solution to the feedback equation is known to exist. Then the closed-loop system is characterized as the output of new Fliess operator when a certain series factorization property is available. This leads to an implicit characterization of the *feedback product* for formal power series.**

I. Introduction

Let $I = \{0, 1, \ldots, m\}$ denote an alphabet and I^* the set of all words over I. A formal power series is any mapping of the form $I^* \mapsto \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle I\rangle\rangle$. For each $c \in \mathbb{R}^\ell\langle\langle I\rangle\rangle$, one can formally associate a corresponding m-input, ℓ-output operator F_c in the following manner. Let $p \geq 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_p$ norm. With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define inductively for each $\eta \in I^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\phi = 1$, and

$$E_{i_k i_{k-1}\cdots i_1}[u](t) = \int_{t_0}^{t} u_{i_k}(\tau) E_{i_{k-1}\cdots i_1}[u](\tau, t_0)\, d\tau,$$

where $u_0(t) \equiv 1$. The input-output operator corresponding to c is then

$$F_c[u](t) = \sum_{\eta\in I^*} (c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator*. In the classical literature where these operators first appeared [4, 5, 6, 9, 10, 11], it is normally assumed that there exists real numbers $K > 0$ and $M \geq 1$ such that

$$|(c, \eta)| \leq KM^{|\eta|}|\eta|!, \qquad \forall\, \eta \in I^*, \tag{1}$$

where $|z| = \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, and $|\eta|$ denotes the number of symbols in η. These growth conditions on the coefficients of c insure that there exist positive real numbers R and T_0 such that for all piecewise continuous u with $\|u\|_\infty \leq R$ and $T \leq T_0$, the series defining F_c converges uniformly and absolutely on $[t_0, t_0 + T]$. Under such conditions the power series c is said to be *locally convergent*. More recently, it was shown in [7] that the growth condition (1) also implies that F_c constitutes a well defined operator from $B_p^m(R)[t_0,\, t_0 + T]$ into $B_q^\ell(S)[t_0,\, t_0 + T]$ for sufficiently small $R, S, T > 0$, where the numbers $p, q \in [1, \infty]$ are conjugate exponents, i.e., $1/p + 1/q = 1$ with $(1, \infty)$ being a conjugate pair by convention.

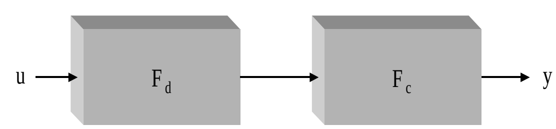

Fig. 1: Two Fliess operators connected in cascade.

In many applications input-output systems are interconnected in a variety of ways. Given two Fliess operators, F_c and F_d with $c, d \in \mathbb{R}^m\langle\langle I\rangle\rangle$, Figure 1 shows a cascade connection where $y = F_c[F_d[u]]$. It was shown in [3] for the SISO case (i.e., $\ell = m = 1$) that there always exists a series $c \circ d$ such that $y = F_{c\circ d}[u]$, but a general analysis of this *composition product* is apparently not available in the literature. So in Section II the composition product is first defined in the multivariable setting and various fundamental properties are presented. Then a condition is introduced under which the composition product preserves rationality and local convergence. In Section III, in preparation for the feedback analysis, it is next shown that the composition product produces a contractive mapping on the set of all formal power series using the familiar ultrametric.

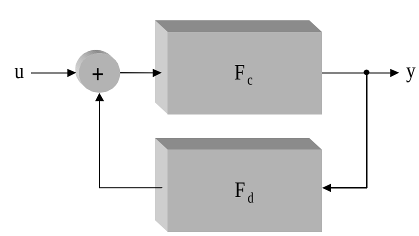

Fig. 2: Two Fliess operators connected in a feedback configuration.

Figure 2 shows a feedback connection with the corresponding feedback equation

$$y = F_c[u + F_d[y]]. \tag{2}$$

Such a feedback system is said to be *well-posed* if the support of c and d each contain at least one word having a nonzero symbol. If, for example, F_c is a linear operator then formally the solution to equation (2) is

$$y = F_c[u] + F_c \circ F_d \circ F_c[u] + \cdots \tag{3}$$

But it is not immediately clear that this series converges in any manner, and in particular, converges to another Fliess operator, say $F_{c@d}$ for some $c@d \in \mathbb{R}^m\langle\langle I \rangle\rangle$. When F_c is nonlinear, the problem is further complicated by the fact that a simple series representation is not apparent. In Section IV the feedback connection is considered in the special case where the inputs are supplied from an exosystem which is itself a Fliess operator as shown in Figure 3. In particular, a sufficient condition is given under which a unique solution y of the feedback equation (2) is known to exist if $u = F_b[v]$. Then y is characterized as the output of a new Fliess operator when a certain series factorization property is available. This leads to an implicit characterization of the *feedback product* $c@d$. The section is concluded with various examples.

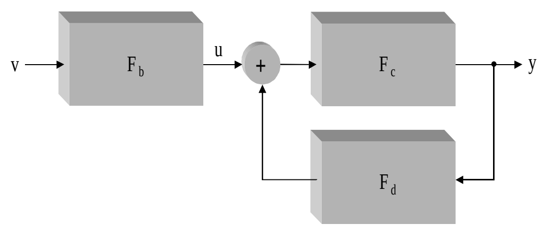

Fig. 3: A feedback configuration with a Fliess operator exosystem providing the inputs.

II. The Composition Product

The composition product of two series over an alphabet $X = \{x_0, x_1\}$ is defined recursively in terms of the shuffle product [2, 3]. For any $\eta \in X^*$ and $d \in \mathbb{R}\langle\langle X \rangle\rangle$, let

$$\eta \circ d := \begin{cases} \eta & : \quad |\eta|_{x_1} = 0 \\ x_0^{k+1}[d \sqcup\!\sqcup (\eta' \circ d)] & : \quad \eta = x_0^k x_1 \eta', \quad k \geq 0, \end{cases} \tag{4}$$

where $|\eta|_{x_1}$ denotes the number of symbols in η equivalent to x_1. For $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ the definition is extended to

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \eta \circ d. \tag{5}$$

For SISO systems it is easily shown that $F_c \circ F_d = F_{c \circ d}$. For the multivariable case it is necessary to consider power series of the form $d : X^* \mapsto \mathbb{R}^m$, where $X = \{x_0, x_1, \ldots, x_m\}$ is an arbitrary alphabet with $m+1$ letters. In which case, the defining equation (4), which preserves the key property $F_c \circ F_d = F_{c \circ d}$ (see [8]), becomes

$$\eta \circ d = \begin{cases} \eta & : \quad |\eta|_{x_i} = 0, \quad \forall i \neq 0 \\ x_0^{k+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \quad \eta = x_0^k x_i \eta', \quad k \geq 0, \quad i \neq 0, \end{cases}$$

where $d_i : \xi \mapsto ((d, \xi))_i$, the i-th component of (d, ξ). Observe that in general for

$$\eta = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \cdots x_0^{n_1} x_{i_1} x_0^{n_0}, \tag{6}$$

where $i_j \neq 0$ for $j = 1, \ldots, k$, it follows that

$$\eta \circ d = x_0^{n_k+1}[d_{i_k} \sqcup\!\sqcup x_0^{n_{k-1}+1}[d_{i_{k-1}} \sqcup\!\sqcup \cdots x_0^{n_1+1}[d_{i_1} \sqcup\!\sqcup x_0^{n_0}] \cdots]].$$

The following theorem guarantees that the composition product is well defined by insuring that the series (5) is summable.

Theorem II.1 *Given a fixed $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$, the family of series $\{\eta \circ d : \eta \in X^*\}$ is locally finite, and therefore summable.*

Proof: Given an arbitrary $\eta \in X^*$ expressed in the form (6), it follows directly that

$$\begin{aligned} \mathrm{ord}(\eta \circ d) &= n_0 + k + \sum_{j=1}^{k} n_j + \mathrm{ord}(d_{i_j}) \\ &= |\eta| + \sum_{j=1}^{|\eta| - |\eta|_{x_0}} \mathrm{ord}(d_{i_j}), \end{aligned} \tag{7}$$

where the *order* of c is defined by

$$\mathrm{ord}(c) = \begin{cases} \inf\{|\eta| : \eta \in \mathrm{supp}(c)\} & : \quad c \neq 0 \\ \infty & : \quad c = 0, \end{cases}$$

and $\mathrm{supp}(c) := \{\eta \in X^* : (c, \eta) \neq 0\}$ denotes the support of c. Hence, for any $\xi \in X^*$

$$\begin{aligned} I_d(\xi) &:= \{\eta \in X^* : (\eta \circ d, \xi) \neq 0\} \\ &\subset \{\eta \in X^* : \mathrm{ord}(\eta \circ d) \leq |\xi|\} \\ &= \{\eta \in X^* : |\eta| + \sum_{j=1}^{|\eta| - |\eta|_{x_0}} \mathrm{ord}(d_{i_j}) \leq |\xi|\}. \end{aligned}$$

Clearly this latter set is finite, and thus $I_d(\xi)$ is finite for all $\xi \in X^*$. This fact implies summability [1]. ■

From the definition, it is easily verified that for any series $c, d, e \in \mathbb{R}^m\langle\langle X \rangle\rangle$,

$$(c + d) \circ e = c \circ e + d \circ e,$$

but in general $c \circ (d + e) \neq c \circ d + c \circ e$. A special exception are *linear series*. A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is called linear if

$$\mathrm{supp}(c) \subseteq \{\eta = x_0^{n_1} x_i x_0^{n_0}, \quad i \in \{1, 2, \ldots, m\}, \quad n_1, n_0 \geq 0\}.$$

Since the shuffle product distributes over addition, given any $\eta = x_0^{n_1} x_i x_0^{n_0}$:

$$\begin{aligned} \eta \circ (d + e) &= x_0^{n_1+1}[(d + e)_i \sqcup\!\sqcup x_0^{n_0}] \\ &= x_0^{n_1+1}(d_i \sqcup\!\sqcup x_0^{n_0}) + x_0^{n_1+1}(e_i \sqcup\!\sqcup x_0^{n_0}) \\ &= \eta \circ d + \eta \circ e. \end{aligned}$$

Therefore,

$$\begin{aligned} c \circ (d + e) &= \sum_{\eta \in I^*} (c, \eta)\, \eta \circ (d + e) \\ &= \sum_{\eta \in I^*} (c, \eta)\, \eta \circ d + (c, \eta)\, \eta \circ e \\ &= c \circ d + c \circ e. \end{aligned}$$

A linear series should not be confused with a rational series. The series

$$c = \sum_{k=0}^{\infty} k!\, x_0^k x_1$$

is linear but not rational, while the bilinear series

$$c=\sum_{k=0}^{\infty}\sum_{i_1,\ldots,i_k=0}^{\infty}(CN_{i_k}\cdots N_{i_1}z_0)\,x_{i_k}\cdots x_{i_1}$$

with $N_i\in\mathbb{R}^{n\times n}$ and $C^T, z_0\in\mathbb{R}^{n\times 1}$ is rational but not linear.

The set $\mathbb{R}\langle\langle X\rangle\rangle$ forms a metric space under the ultrametric

$$\begin{array}{rcl}\mathrm{dist} & : & \mathbb{R}\langle\langle X\rangle\rangle\times\mathbb{R}\langle\langle X\rangle\rangle\mapsto\mathbb{R}^+\cup\{0\}\\ & : & (c,d)\mapsto\sigma^{\mathrm{ord}(c-d)},\end{array}$$

where $\sigma\in(0,1)$ is arbitrary. The following theorem states that the composition product on $\mathbb{R}\langle\langle X\rangle\rangle\times\mathbb{R}\langle\langle X\rangle\rangle$ is at least continuous in its first argument. The result extends naturally to vector-valued series in a componentwise fashion.

Theorem II.2 *Let $\{c_i\}_{i\geq 1}$ be a sequence in $\mathbb{R}\langle\langle X\rangle\rangle$ with $\lim_{i\to\infty}c_i=c$. Then $\lim_{i\to\infty}(c_i\circ d)=c\circ d$.*

Proof: Define the sequence $n_i=\mathrm{ord}(c_i-c)$ for $i\geq 1$. Since c is the limit of the sequence $\{c_i\}_{i\geq 1}$, $\{n_i\}_{i\geq 1}$ must have a monotone increasing subsequence $\{n_{i_j}\}$. Now observe that

$$\mathrm{dist}(c_i\circ d,c\circ d)=\sigma^{\mathrm{ord}((c_i-c)\,\circ d)}$$

and

$$\begin{aligned}\mathrm{ord}((c_{i_j}-c)\circ d) &= \mathrm{ord}\left(\sum_{\eta\in\mathrm{supp}(c_{i_j}-c)}(c_{i_j}-c,\eta)\,\eta\circ d\right)\\ &\geq \inf_{\eta\in\mathrm{supp}(c_{i_j}-c)}\mathrm{ord}(\eta\circ d)\\ &= \inf_{\eta\in\mathrm{supp}(c_{i_j}-c)}|\eta|+(|\eta|-|\eta|_{x_0})\,\mathrm{ord}(d)\\ &\geq n_{i_j}.\end{aligned}$$

Thus, $\mathrm{dist}(c_{i_j}\circ d,c\circ d)\leq\sigma^{n_{i_j}}$ for all $j\geq 1$, and the theorem is proven. ■

It is shown in [3] by counter example that the composition product is *not* a rational operation. That is, the composition of two rational series does not in general produce a rational series. But in [2], it is shown in the SISO case that special classes of rational series produce rational series when the composition product is applied. The following definition is the multivariable extention of this essential property, and the corresponding rationality proof is not significantly different.

Definition II.1 *A series $c\in\mathbb{R}\langle\langle X\rangle\rangle$ is **limited relative to x_i** if there exists an integer $N_i\geq 0$ such that*

$$\sup_{\eta\in\mathrm{supp}(c)}|\eta|_{x_i}\leq N_i.$$

*If c is limited relative to x_i for every $i=1,2,\ldots,m$ then c is **input-limited**. In such cases, let $N_c:=\sum_i N_i$. A series $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is input-limited if each component series, c_j, is input-limited for $j=1,2,\ldots,\ell$. In this case, $N_c:=\max_j N_{c_j}$.*

It is shown next that the composition product will preserve local convergence if its first argument is input-limited.

Theorem II.3 *Suppose $c,d\in\mathbb{R}^m\langle\langle X\rangle\rangle$ are locally convergent series with growth constants K_c,M_c and K_d,M_d, respectively. If c is input-limited then $c\circ d$ is locally convergent with*

$$|(c\circ d,\nu)|<K_cK_d^{N_c}(N_c(N_c+1)M)^{|\nu|}|\nu|!,\quad\forall\nu\in X^*,\tag{8}$$

where $M=\max(M_c,M_d)$.

The proof of this result requires the following two lemmas.

Lemma II.1 *[12] The relations below hold for arbitrary $c,d\in\mathbb{R}\langle\langle X\rangle\rangle$ and $\nu\in X^*$:*

$$\begin{aligned}\textbf{(a)}\quad(c\sqcup\!\sqcup\, d,\nu) &= \sum_{\eta,\xi\in X^*}(c,\eta)(d,\xi)(\eta\sqcup\!\sqcup\,\xi,\nu)\\ &= \sum_{i=0}^{|\nu|}\sum_{\substack{\eta\in X^i,\\ \xi\in X^{|\nu|-i}}}(c,\eta)(d,\xi)(\eta\sqcup\!\sqcup\,\xi,\nu)\end{aligned}$$

$$\textbf{(b)}\quad\sum_{\substack{\eta\in X^i,\\ \xi\in X^{|\nu|-i}}}(\eta\sqcup\!\sqcup\,\xi,\nu)\leq\binom{|\nu|}{i},$$

where X^i denotes the set of all words in X^* of length i. When $c,d\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$, these relations hold componentwise where the shuffle product is defined as $(c\sqcup\!\sqcup\, d)_i=c_i\sqcup\!\sqcup\, d_i$ for $i=1,2,\ldots,\ell$.

Lemma II.2 *[12] Suppose $c,d\in\mathbb{R}\langle\langle X\rangle\rangle$ are locally convergent series with growth constants K_c,M_c and K_d,M_d, respectively. Then $c\sqcup\!\sqcup\, d$ is locally convergent with*

$$|(c\sqcup\!\sqcup\, d,\nu)|<K_cK_d(2M)^{|\nu|}|\nu|!,\quad\forall\nu\in X^*,\tag{9}$$

where $M=\max(M_c,M_d)$.

In regards to this last lemma, the result in [12] is actually slightly stronger, i.e., it is shown there that

$$|(c\sqcup\!\sqcup\, d,\nu)|<K_cK_dM^{|\nu|}(|\nu|+1)!,\quad\forall\nu\in X^*.$$

But in the present context, the bound in (9) is more convenient and simply follows from the fact that $n+1\leq 2^n$ for all $n\geq 0$.

Proof of Theorem II.3: The theorem can be proven by making several key observations. Only the SISO case is considered here for brevity. First, it follows using Lemma II.1 that for any $n_0\geq 0$ and $\nu\in X^*$:

$$\begin{aligned}|(d\sqcup\!\sqcup\, x_0^{n_0},\nu)| &= \left|\sum_{i=0}^{|\nu|}\sum_{\substack{\eta\in X^i,\\ \xi\in X^{|\nu|-i}}}(x_0^{n_0},\eta)(d,\xi)(\eta\sqcup\!\sqcup\,\xi,\nu)\right|\\ &\leq \sum_{\xi\in X^{|\nu|-n_0}}(K_dM_d^{|\xi|}|\xi|!)(x_0^{n_0}\sqcup\!\sqcup\,\xi,\nu)\\ &\leq K_dM_d^{|\nu|-n_0}(|\nu|-n_0)!\binom{|\nu|}{n_0}\\ &= \left(\frac{K_dM_d^{-n_0}}{n_0!}\right)M_d^{|\nu|}|\nu|!.\end{aligned}$$

Similarly observe that for any $n_i\geq 0$:

$$\begin{aligned}|(x_0^{n_i+1}d,\nu)| &= \begin{cases}|(d,(x_0^{n_i+1})^{-1}(\nu))| & :\ \nu=x_0^{n_i+1}\nu'\\ 0 & :\ \text{otherwise}\end{cases}\\ &\leq K_dM_d^{|\nu|-(n_i+1)}(|\nu|-(n_i+1))!\\ &\leq \left(\frac{K_dM_d^{-(n_i+1)}}{(n_i+1)!}\right)M_d^{|\nu|}|\nu|!.\end{aligned}$$

Thus, given an arbitrary word η expressed in the form of (6), it follows inductively with the aid of Lemma II.2 (precisely, its extension to $k-1$ shuffle products) that

$$|(\eta \circ d, \nu)| \leq \left(K_d^k \frac{M_d^{-|\eta|}}{n_0!(n_1+1)!\cdots(n_k+1)!} \right) (kM_d)^{|\nu|}|\nu|!.$$

Now if c is input-limited the theorem is proven as follow:

$$\begin{aligned}
&|(c \circ d, \nu)| \\
&= \left| \sum_{\eta \in \operatorname{supp}(c) \cap I_d(\nu)} (c, \eta)(\eta \circ d, \nu) \right| \\
&\leq \sum_{i=0}^{|\nu|} \sum_{\substack{\eta \in \operatorname{supp}(c) \\ n_0+n_1+\ldots+n_k+k=i}} \left[K_c M_c^{|\eta|} |\eta|! \right] \cdot \\
&\quad \left[\left(K_d^k \frac{M_d^{-|\eta|}}{n_0!(n_1+1)!\cdots(n_k+1)!} \right) (kM_d)^{|\nu|}|\nu|! \right] \\
&\leq K_c K_d^{N_c} (N_c M)^{|\nu|} |\nu|! \cdot \\
&\quad \sum_{i=0}^{|\nu|} \sum_{\substack{\eta \in \operatorname{supp}(c) \\ n_0+n_1+\ldots+n_k+k=i}} \left(\frac{|\eta|!}{n_0!(n_1+1)!\cdots(n_k+1)!} \right) \\
&\leq K_c K_d^{N_c} (N_c M)^{|\nu|} (N_c+1)^{|\nu|} |\nu|!.
\end{aligned}$$

The final step uses the fact that the terms in the summation are the multinomials $(n_0, n_1+1, \ldots, n_k+1)!$ and $k \leq N_c$. ■

When c and d are linear series (i.e., $N_c = 1$), the growth condition (8) is known to be conservative. Using the fact that $\sum_{k=0}^{n} \binom{n}{k}^{-1} < 3$ for any $n \geq 0$, it can be shown that a tighter bound is

$$|(c \circ d, \nu)| < K_c K_d M^{|\nu|} |\nu|!, \quad \forall \nu \in X^*. \tag{10}$$

It is conjectured that at least the bound

$$|(c \circ d, \nu)| < K_c K_d^{N_c} (N_c M)^{|\nu|} |\nu|!, \quad \forall \nu \in X^* \tag{11}$$

applies in the general case, and perhaps even the stronger bound

$$|(c \circ d, \nu)| < K_c K_d^{N_c} M^{|\nu|} |\nu|!, \quad \forall \nu \in X^*. \tag{12}$$

III. Contractive Mappings on $\mathbb{R}\langle\langle X \rangle\rangle$

The metric space $(\mathbb{R}\langle\langle X \rangle\rangle, \operatorname{dist})$ is known to be complete [1]. Given a fixed $c \in \mathbb{R}\langle\langle X \rangle\rangle$, consider the mapping $\mathbb{R}\langle\langle X \rangle\rangle \mapsto \mathbb{R}\langle\langle X \rangle\rangle : d \mapsto c \circ d$. It is proven in this section that this mapping is always a contraction on $\mathbb{R}\langle\langle X \rangle\rangle$, i.e.,

$$\operatorname{dist}(c \circ d, c \circ e) < \operatorname{dist}(d, e), \quad \forall d, e \in \mathbb{R}\langle\langle X \rangle\rangle.$$

The focus is on the SISO case ($X = \{x_0, x_1\}$) where any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ can be written unambiguously in the form

$$c = c_0 + c_1 + \cdots,$$

where $c_k \in \mathbb{R}\langle\langle X \rangle\rangle$ has the property that $\eta \in \operatorname{supp}(c_k)$ only if $|\eta|_{x_1} = k$. Some of the series c_k may be the zero series. When $c_0 = 0$, c is referred to as being *homogeneous*. When $c_k = 0$ for $k = 0, 1, \ldots, l-1$ and $c_l \neq 0$ then c is called *homogeneous of order* l. In this setting consider the following theorem.

Theorem III.1 *For any $c_k \in \mathbb{R}\langle\langle X \rangle\rangle$ with $X = \{x_0, x_1\}$*

$$\operatorname{dist}(c_k \circ d, c_k \circ e) \leq \sigma^k \cdot \operatorname{dist}(d, e), \quad \forall d, e \in \mathbb{R}\langle\langle X \rangle\rangle.$$

Proof: The proof is by induction for the nontrivial case where $c_k \neq 0$. First suppose $k = 0$. From the definition of the composition product it follows directly that $\eta \circ d = \eta$ for all $\eta \in \operatorname{supp}(c_0)$. Therefore,

$$c_0 \circ d = \sum_{\eta \in \operatorname{supp}(c_0)} (c_0, \eta)\, \eta \circ d = \sum_{\eta \in \operatorname{supp}(c_0)} (c_0, \eta)\, \eta = c_0,$$

and

$$\begin{aligned}
\operatorname{dist}(c_0 \circ d, c_0 \circ e) &= \operatorname{dist}(c_0, c_0) = 0 \\
&\leq \sigma^0 \cdot \operatorname{dist}(d, e).
\end{aligned}$$

Now fix any $k \geq 0$ and assume the claim is true for all $c_0, c_1, \ldots, c_k$. In particular, this implies that

$$\operatorname{ord}(c_k \circ d - c_k \circ e) \geq k + \operatorname{ord}(d - e). \tag{13}$$

For any $j \geq 0$, words in $\operatorname{supp}(c_j)$ have the form $\eta_j = x_0^{n_j} x_1 x_0^{n_{j-1}} x_1 \cdots x_0^{n_0}$ and $\eta_{j+1} = x_0^{n_{j+1}} x_1 \eta_j$. Observe then that

$$\begin{aligned}
&c_{k+1} \circ d - c_{k+1} \circ e \\
&= \sum_{\eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1})\, \eta_{k+1} \circ d - (c_{k+1}, \eta_{k+1})\, \eta_{k+1} \circ e \\
&= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1}) \; \big[{x_0}^{n_{k+1}+1} [d \sqcup\!\sqcup [\eta_k \circ d]] - \\
&\qquad {x_0}^{n_{k+1}+1} [e \sqcup\!\sqcup [\eta_k \circ e]] \big] \\
&= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1}) \; \big[{x_0}^{n_{k+1}+1} [d \sqcup\!\sqcup [\eta_k \circ d]] - \\
&\qquad {x_0}^{n_{k+1}+1} [d \sqcup\!\sqcup [\eta_k \circ e]] \big] + \big[{x_0}^{n_{k+1}+1} [d \sqcup\!\sqcup [\eta_k \circ e]] - \\
&\qquad {x_0}^{n_{k+1}+1} [e \sqcup\!\sqcup [\eta_k \circ e]] \big] \\
&= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1}) \; \big[{x_0}^{n_{k+1}+1} [d \sqcup\!\sqcup [\eta_k \circ d - \eta_k \circ e]] + \\
&\qquad {x_0}^{n_{k+1}+1} [(d - e) \sqcup\!\sqcup [\eta_k \circ e]] \big]
\end{aligned}$$

using the fact that the shuffle product distributes over addition. Next, applying the identity (7) and the inequality (13) with $c_k = \eta_k$, it follows that

$$\begin{aligned}
&\operatorname{ord}(c_{k+1} \circ d - c_{k+1} \circ e) \\
&\geq \min \Big\{ \inf_{\eta_{k+1} \in \operatorname{supp}(c_{k+1})} n_{k+1} + 1 + \operatorname{ord}(d) + k + \\
&\qquad \operatorname{ord}(d - e), \inf_{\eta_{k+1} \in \operatorname{supp}(c_{k+1})} n_{k+1} + 1 + \operatorname{ord}(d - e) \\
&\qquad + |\eta_k| + k \cdot \operatorname{ord}(e) \Big\} \\
&\geq k + 1 + \operatorname{ord}(d - e),
\end{aligned}$$

and thus,

$$\operatorname{dist}(c_{k+1} \circ d, c_{k+1} \circ e) \leq \sigma^{k+1} \cdot \operatorname{dist}(d, e).$$

Hence, $\operatorname{dist}(c_k \circ d, c_k \circ e) \leq \sigma^k \cdot \operatorname{dist}(d, e)$ holds for any $k \geq 0$. ■

Applying the above theorem leads to following result.

Theorem III.2 *If $c \in \mathbb{R}\langle\langle X \rangle\rangle$ with $X = \{x_0, x_1\}$ then for any series c_0'*

$$\mathrm{dist}((c_0'+c)\circ d, (c_0'+c)\circ e) = \mathrm{dist}(c\circ d, c\circ e), \quad \forall d, e \in \mathbb{R}\langle\langle X \rangle\rangle. \tag{14}$$

If c is homogeneous of order $l \geq 1$ then

$$\mathrm{dist}(c \circ d, c \circ e) \leq \sigma^l \cdot \mathrm{dist}(d, e), \quad \forall d, e \in \mathbb{R}\langle\langle X \rangle\rangle. \tag{15}$$

Proof: The equality is proven first. Since the metric dist is shift-invariant:

$$\begin{aligned}
&\mathrm{dist}((c_0' + c) \circ d, (c_0' + c) \circ e) \\
&= \mathrm{dist}\left(c_0' \circ d + c \circ d, c_0' \circ e + c \circ e\right) \\
&= \mathrm{dist}\left(c_0' + c \circ d, c_0' + c \circ e\right) \\
&= \mathrm{dist}\left(c \circ d, c \circ e\right).
\end{aligned}$$

The inequality is proven next by first selecting any fixed $l \geq 1$ and showing inductively that it holds for any partial sum $\sum_{i=l}^{l+k} c_i$ where $k \geq 0$. When $k = 0$ Theorem III.1 implies that

$$\mathrm{dist}(c_l \circ d, c_l \circ e) \leq \sigma^l \cdot \mathrm{dist}(d, e).$$

If the result is true for partial sums up to any fixed k then using the ultrametric property

$$\mathrm{dist}(d, e) \leq \max\{\mathrm{dist}(d, f), \mathrm{dist}(f, e)\}, \quad \forall d, e, f \in \mathbb{R}\langle\langle X \rangle\rangle,$$

it follows that

$$\begin{aligned}
&\mathrm{dist}\left(\left(\sum_{i=l}^{l+k+1} c_i\right) \circ d, \left(\sum_{i=l}^{l+k+1} c_i\right) \circ e\right) \\
&= \mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ d, \left(\sum_{i=l}^{l+k} c_i\right) \circ e + c_{l+k+1} \circ e\right) \\
&\leq \max\left\{\mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ d, \left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ e\right), \mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ e, \left(\sum_{i=l}^{l+k} c_i\right) \circ e + c_{l+k+1} \circ e\right)\right\} \\
&= \max\left\{\mathrm{dist}(c_{l+k+1} \circ d, c_{l+k+1} \circ e), \mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d, \left(\sum_{i=l}^{l+k} c_i\right) \circ e\right)\right\} \\
&\leq \max\left\{\sigma^{l+k+1} \cdot \mathrm{dist}(d, e), \sigma^l \cdot \mathrm{dist}(d, e)\right\} \\
&\leq \sigma^l \cdot \mathrm{dist}(d, e).
\end{aligned}$$

Hence, the result holds for all $k \geq 0$. Finally the theorem is proven by noting that $c = lim_{k\to\infty} \sum_{i=l}^{l+k} c_i$ and using the continuity of the composition product proven in Theorem II.2 and the metric dist. ■

The main result of this section is given below.

Theorem III.3 *For any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ with $X = \{x_0, x_1\}$, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}\langle\langle X \rangle\rangle$.*

Proof: Choose any series $d, e \in \mathbb{R}\langle\langle X \rangle\rangle$. If c is homogeneous of order $l \geq 1$ then the result follows directly from equation (15). Otherwise, observe that via equation (14):

$$\begin{aligned}
\mathrm{dist}(c \circ d, c \circ e) &= \mathrm{dist}\left(\left(\sum_{l=1}^{\infty} c_i\right) \circ d, \left(\sum_{l=1}^{\infty} c_i\right) \circ e\right) \\
&\leq \sigma \cdot \mathrm{dist}(d, e) \\
&< \mathrm{dist}(d, e).
\end{aligned}$$

■

IV. The Feedback Connection with Fliess Exosystem

The general goal is to determine when there exists a y which satisfies the feedback equation (2), and in particular, when does there exist a generating series e so that $y = F_e[u]$ over some appropriate input set. In the latter case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d\circ e}[u]],$$

and the *feedback product* is defined by $c@d = e$. To make the analysis simpler, it is assumed throughout that $u = F_b(v)$ for some locally convergent series b and that all feedback systems under consideration are well-posed. In which case, the following theorem is possible.

Theorem IV.1 *Let $b, c, d \in \mathbb{R}\langle\langle I \rangle\rangle$ with $I = \{0, 1\}$. Then:*

1. *The mapping*

$$\begin{aligned}
S \quad &: \quad \mathbb{R}\langle\langle I \rangle\rangle \mapsto \mathbb{R}\langle\langle I \rangle\rangle \\
&: \quad \tilde{e}_i \mapsto \tilde{e}_{i+1} = c \circ (b + d \circ \tilde{e}_i)
\end{aligned}$$

   has a unique fixed point $\tilde{e}$.

2. *If b, c, d and $\tilde{e}$ are locally convergent then the feedback equation (2) has the unique solution $y = F_{\tilde{e}}[v]$ for any admissible v.*
3. *If $\tilde{e} = e \circ b$ for some locally convergent series e then $c@d = e$.*

Proof:
1. The mapping S is a contraction since via Theorem III.3:

$$\begin{aligned}
\mathrm{dist}(S(\tilde{e}_i), S(\tilde{e}_j)) &< \mathrm{dist}(b + d \circ \tilde{e}_i, b + d \circ \tilde{e}_j) \\
&= \mathrm{dist}(d \circ \tilde{e}_i, d \circ \tilde{e}_j) \\
&< \mathrm{dist}(\tilde{e}_i, \tilde{e}_j).
\end{aligned}$$

Therefore, the mapping S has a unique fixed point $\tilde{e}$, that is,

$$\tilde{e} = c \circ (b + d \circ \tilde{e}).$$

2. From the stated assumptions concerning b, c, d and $\tilde{e}$ it follows that

$$\begin{aligned}
F_{\tilde{e}}[v] &= F_{c\circ(b+d\circ\tilde{e})}[v] \\
&= F_c[F_b[v] + F_d[F_{\tilde{e}}[v]]]
\end{aligned}$$

for any admissible v. Therefore equation (2) has the unique solution $y = F_{\tilde{e}}[v]$.

3. Since e is locally convergent

$$y = F_{\tilde{e}}[v] = F_e[F_b[v]] = F_e[u],$$

thus $c@d = e$. ■

This last result suggests several (open) problems. Are there conditions on b, c, and d alone which will insure that $\tilde{e}$ above is locally convergent ? In light of Theorem II.3, requiring c and d to be input-limited as well as locally convergent appears to be a good start, but it is not known if this condition is necessary. Next, when does there exist a factorization of the form $\tilde{e} = e \circ b$, where e is locally convergent ? Finally, can the theorem be generalized to the case where the inputs are simply from an L_p space and not filtered through a Fliess operator a priori ? Some insight into these questions is provide by the following examples.

Example IV.1 Suppose c and d are locally convergent. If c is also a linear series, one can formally write using equation (3)

$$c@d = c + \sum_{k=1}^{\infty} (c \circ d)^{\circ k} \circ c, \tag{16}$$

where $c^{\circ k}$ denotes k copies of c composed $k-1$ times. In light of Theorem II.2, $c@d$ is well defined as long as the family of series $\{(c \circ d)^{\circ k} : k \geq 1\}$ is summable, and it is easily verified that

$$((c \circ d)^{\circ k}, \nu) = 0, \quad \forall k > |\nu|.$$

So for this special case an application of Theorem IV.1 can be avoided for concluding that $c@d$ is well defined. Now from Theorem II.3 $c \circ d$ is locally convergent. If the (conjectured) growth condition (12) holds then it follows immediately that

$$|((c \circ d)^{\circ k}, \nu)| \leq K_{c\circ d}^k M^{|\nu|} |\nu|!, \quad \forall \nu \in X^*,$$

and thus,

$$\begin{aligned} |(c@d, \nu)| &\leq K_c M_c^{|\nu|} |\nu|! + \sum_{k=1}^{|\nu|} K_c K_{c\circ d}^k M^{|\nu|} |\nu|! \\ &\leq K_c \left(\sum_{k=0}^{|\nu|} K_{c\circ d}^k \right) M^{|\nu|} |\nu|!, \quad \forall \nu \in X^*. \end{aligned}$$

If, for example, $K_{c\circ d} > 1$ then

$$|(c@d, \nu)| \leq K_c \frac{K_{c\circ d}}{K_{c\circ d} - 1} (K_{c\circ d} M)^{|\nu|} |\nu|!, \quad \forall \nu \in X^*.$$

Thus, for a linear series c, the closed-loop system can be described by the Fliess operator $F_{c@d}$ with $c@d$ given by (16), if $c \circ d$ satisfies the growth condition (12). If, in addition, d is linear then $c \circ d$ always satisfies the bound (12) (c.f. (10)) and in fact $K_{c\circ d} = K_c K_d$. □

Example IV.2 Consider a *generalized series* δ with the defining property that δ is the identity element for the composition product, i.e., $c \circ \delta = \delta \circ c = c$ for any $c \in \mathbb{R}\langle\langle X \rangle\rangle$. Then $F_\delta[u] = u$ for any u, and a unity feedback system has the generating series $c@\delta$. Setting $b = 0$ in Figure 3 (or effectively setting $u \equiv 0$), a self-exciting feedback loop is described by $F_{c@\delta}[0]$, where

$$c@\delta = \lim_{k \to \infty} c^{\circ k}. \tag{17}$$

From Theorem IV.1, part 1, one may conclude that the sequence converges in general. Now suppose that c is the rational series $\sum_{j=0}^{\infty} x_1^j$. This series is clearly locally convergent, but not input-limited. When $k = 2$ in equation (17), a direct calculation (see [3]) gives

$$(c \circ c, x_0^j x_1^j) = (c^{\sqcup\!\sqcup\, j}, x_1^j) = j^j < (2j)!, \quad j > 0.$$

Therefore $c \circ c$ is no longer rational, but it is still locally convergent as far as these particular coefficients are concerned. (If this holds for all words in $\mathrm{supp}(c \circ c)$, it would show that the property of input-limited assumed in Theorem II.3 is not a necessary condition for local convergence under the composition product.) Now for composition powers $k > 2$ it is an open question as to whether local convergence is preserved. It is almost certain that the growth rate increases as k increases, as was the case from $k = 1$ to $k = 2$, but the order of $c^{\circ k}$ also increases as a function of k. At present it remains to be determined whether this unity feedback system has a Fliess operator representation or not. □

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