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# Volterra Methods for Constructing Structural Dynamic Observables for Nonlinear Systems: an Extended Calculation.

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**Abstract.** Although a great deal of work has been carried out on structural dynamic systems under random excitation, there has been a comparatively small amount of this work concentrating on the calculation of the quantities commonly measured in structural dynamic tests. Among the existing work, the Volterra series, a means of predicting nonlinear system response for weakly nonlinear systems, has allowed the computation of various measurable quantities of interest for structural dynamics, including: auto- and cross-spectra, FRFs, coherences and higher-order spectra. These calculations are quite intensive and are typically only possible using computer algebra. A previous calculation by the authors for the coherence for a Duffing oscillator yielded results which showed some qualitative disagreement with numerical simulation; the object of the current paper is simply to extend the calculation in order to see if better agreement can be achieved.

## 1. Introduction

The Volterra series is well-known as a means of predicting nonlinear system response for weakly nonlinear systems. An attractive feature of the series is that it allows the calculation of closed-form expressions (albeit approximations based on the truncated series) for various quantities of interest to structural dynamicists; these include a number of objects commonly encountered in dynamic testing. The research of the current authors began with [1], in which the Frequency Response Function (FRF) for a Single-Degree-of-Freedom (SDOF) system – the randomly excited Duffing oscillator – was computed up to a certain order of approximation i.e. to a series order quadratic in  $P$ , the white noise intensity. The results were interesting; the computed FRF showed good qualitative agreement with an FRF calculated by numerical simulation in that the ‘resonance’ frequency increased with increasing levels of excitation. The approximation also shed light on the known observation that the Hilbert transform applied to a nonlinear system FRF under random excitation does not produce distortions. The analysis was extended to a Multi-DOF system in [2]. The calculations were very demanding and could only be carried out with the aid of computer algebra. In [3], the Volterra methodology was applied to the approximate computation of the coherence function, an often-used tool in structural dynamics for diagnosing nonlinearity. In this case, a Padé or rational approximation in  $P$  was used rather than the polynomial expansions used in [1] and [2]. Two systems were considered in [3]: a symmetric Duffing oscillator with cubic stiffness only and an asymmetric oscillator with quadratic and

cubic terms. In the case of the asymmetric oscillator with quadratic term dominant, a (1,1) approximation (ratio of linear functions) gave good quantitative and qualitative agreement with the results of numerical solution. However, even when a (2,2) approximation (ratio of quadratics) was computed for the symmetric oscillator, the expression under-predicted the results of numerical simulation and also failed to show some of the appropriate qualitative behaviour. The objective of the current paper is simply to extend the analysis of the coherence to a higher order in order to see if closer agreement with numerical simulation can be obtained. Recent developments in the use of the Volterra series in structural dynamics include [4] and [5] where analytical approximations to the bispectrum are obtained and applied.

The layout of the paper is as follows. Section Two reviews the necessary background on the Volterra series in general, and in the particular case of the SDOF Duffing oscillator. Section Three discusses how the Padé approximations are formed from the various cross-spectra involved and also describes the extended approximation for the symmetric case of the Duffing system. The paper concludes with some discussion in Section Four.

## 2. FRFs and Cross-Spectra

The coherence for a linear system is constructed as the ratio of certain cross-spectra and auto-spectra, or alternatively in terms of the FRF and the auto-spectra. Before proceeding to the construction of the coherence function for a nonlinear system in terms of a Volterra series expansion, it is useful to recall the definitions of the FRFs and cross-spectra for such systems in terms of the Volterra series.

It is well-known that many nonlinear systems or input-output processes  $x(t) \longrightarrow y(t)$  can be realised as a mapping [6,7],

$$y(t) = y_1(t) + y_2(t) + y_3(t) + \dots + y_n(t) + \dots \quad (1)$$

where

$$y_n(t) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d\tau_1 \dots d\tau_n h_n(\tau_1, \dots, \tau_n) x(t - \tau_1) \dots x(t - \tau_n) \quad (2)$$

Among the conditions for such a series to apply are time-invariance of the system of interest and polynomial nature for the nonlinearity.

This is the *Volterra series* and the functions  $h_n$  are the *Volterra kernels*. The dual frequency-domain representation is based on the *Higher-order FRFs* (HFRFs) or *Volterra kernel transforms*,  $H_n(\omega_1, \dots, \omega_n)$ ,  $n = 1, \dots, \infty$ , which are defined as the multi-dimensional Fourier transforms of the kernels.

$$H_n(\omega_1, \dots, \omega_n) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d\tau_1 \dots d\tau_n h_n(\tau_1, \dots, \tau_n) e^{-i(\omega_1 \tau_1 + \dots + \omega_n \tau_n)} \quad (3)$$

The definition of the FRF of a linear system based on the input/output cross-spectrum,  $S_{yx}(\omega)$ , and input autospectrum,  $S_{xx}(\omega)$ , is also well-known,

$$H(\omega) = H_1(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} \quad (4)$$

where the spectral quantities are defined as expectations i.e.  $S_{yx}(\omega) = E[\bar{Y}(\omega)X(\omega)]$ . The expectations are obtained by averaging discrete or fast Fourier transforms from neighbouring time segments.

The *composite FRF*  $\Lambda_r(\omega)$ , of a nonlinear system under random excitation, is defined similarly,

$$\Lambda_r(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} \quad (5)$$

Using the Volterra series representation in (1) results in the expression

$$\Lambda_r(\omega) = \frac{S_{y_1x}(\omega) + S_{y_2x}(\omega) + \dots + S_{y_nx}(\omega) + \dots}{S_{xx}(\omega)} \quad (6)$$

$\Lambda_r(\omega)$  was approximated in [1] by obtaining expressions for the various cross-spectra between the input and the individual output components. The general term used was,

$$S_{y_{2n-1}x}(\omega) = \frac{(2n)!S_{xx}(\omega)}{n!2^n(2\pi)^{n-1}} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d\omega_1 \dots d\omega_{n-1} \times \\ H_{2n-1}(\omega_1, -\omega_1, \dots, \omega_{n-1}, -\omega_{n-1}, \omega) S_{xx}(\omega_1) \dots S_{xx}(\omega_{n-1}) \quad (7)$$

Now, given that the input autospectrum is constant over all frequencies for a white noise input (i.e.  $S_{xx}(\omega) = P$ ), the composite FRF for random excitation follows. Substituting the constant  $S_{xx}$  into (7) gives

$$\Lambda_r(\omega) = \sum_{n=1}^{\infty} \frac{(2n)!P^{n-1}}{n!2^n(2\pi)^{n-1}} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d\omega_1 \dots d\omega_{n-1} H_{2n-1}(\omega_1, -\omega_1, \dots, \omega_{n-1}, -\omega_{n-1}, \omega) \quad (8)$$

The system of interest here is going to be the basic symmetrical Duffing oscillator with equation of motion,

$$m\ddot{y} + c\dot{y} + ky + k_3y^3 = x(t) \quad (9)$$

where  $m$  is the mass and  $c$  is the damping constant.  $k$  and  $k_3$  are the linear and cubic stiffnesses respectively. This system is time-invariant as required for the application of the Volterra series.

The relevant higher-order FRFs of this system for the forthcoming analysis are:

$$H_1(\omega) = \frac{1}{-m\omega^2 + ic\omega + k} \quad (10)$$

$$H_3(\omega_1, \omega_2, \omega_3) = -k_3 H_1(\omega_1) H_1(\omega_2) H_1(\omega_3) H_1(\omega_1 + \omega_2 + \omega_3) \quad (11)$$

$$H_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = \\ \frac{3}{10} k_3^2 H_1(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5) H_1(\omega_1) H_1(\omega_2) H_1(\omega_3) H_1(\omega_4) H_1(\omega_5) \times \\ \{H_1(\omega_1 + \omega_2 + \omega_3) + H_1(\omega_1 + \omega_2 + \omega_4) + H_1(\omega_1 + \omega_2 + \omega_5) + H_1(\omega_1 + \omega_3 + \omega_4) + H_1(\omega_1 + \omega_3 + \omega_5) + \\ H_1(\omega_1 + \omega_4 + \omega_5) + H_1(\omega_2 + \omega_3 + \omega_4) + H_1(\omega_2 + \omega_3 + \omega_5) + H_1(\omega_2 + \omega_4 + \omega_5) + H_1(\omega_3 + \omega_4 + \omega_5)\} \quad (12)$$

or, alternatively,

$$H_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = \frac{3}{10} k_3^2 H_1(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5) H_1(\omega_1) H_1(\omega_2) H_1(\omega_3) H_1(\omega_4) H_1(\omega_5) \times \{H_1(\omega_1 + \omega_2 + \omega_3) + 9 \text{ more combinations of three frequencies from five}\} \quad (13)$$

which were all needed for the calculation in [3]. Finally, the extended calculation requires,

$$H_7(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = -\frac{3}{70} k_3^3 H_1(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7) \times H_1(\omega_1) H_1(\omega_2) H_1(\omega_3) H_1(\omega_4) H_1(\omega_5) H_1(\omega_6) H_1(\omega_7) \times \{H_1(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5) H_1(\omega_1 + \omega_2 + \omega_3) + 219 \text{ other } H_1(.) H_1(.) \text{ products} + H_1(\omega_1 + \omega_2 + \omega_3) H_1(\omega_4 + \omega_5 + \omega_6) + 69 \text{ other } H_1(.) H_1(.) \text{ products}\} \quad (14)$$

The last term shows that the extended calculation here will be many times more demanding than those previously carried out. For the symmetric Duffing oscillator, all even-order kernels and their transforms are easily shown to vanish. The first stage in the calculation is to form the higher-order Padé approximation for the coherence.

### 3. Padé Approximations for the Coherence

The well-known expression for the coherence of a linear system is,

$$\gamma^2(\omega) = \frac{S_{yx}(\omega) S_{xy}(\omega)}{S_{xx}(\omega) S_{yy}(\omega)} = \frac{|H(\omega)|^2 S_{xx}(\omega)}{S_{yy}(\omega)} \quad (15)$$

where the interpretation is that the coherence is the ratio of the power in the output *linearly correlated with the measured input* divided by the power in the output. For a linear system, one expects this ratio to be unity. It is a textbook calculation [6], to show that the coherence is always less than unity if measurement noise is present on the input or output. It is also well-known that the coherence falls below unity if the system in question is nonlinear, although this belief is considerably less well-supported theoretically.

Assuming that the coherence is computed from the auto and cross-spectra as in the linear case, the expression in the nonlinear case is simply,

$$\gamma^2(\omega) = \frac{S_{yx}(\omega) S_{xy}(\omega)}{S_{xx}(\omega) S_{yy}(\omega)} = \frac{|\Lambda_r(\omega)|^2 S_{xx}(\omega)}{S_{yy}(\omega)} \quad (16)$$

with reference to equation (15). In terms of the individual Volterra components of the signal  $y$ , this becomes,

$$\gamma^2(\omega) = \frac{(S_{y_1x}(\omega) + S_{y_3x}(\omega) + S_{y_5x}(\omega) + \dots)(S_{xy_1}(\omega) + S_{xy_3}(\omega) + S_{xy_5}(\omega) + \dots)}{S_{xx}(\omega)(S_{y_1y_1}(\omega) + S_{y_1y_3}(\omega) + S_{y_3y_1}(\omega) + S_{y_2y_2} \dots)} \quad (17)$$

and this expression takes into account the fact that terms of the form  $S_{y_n x}(\omega)$  can be shown to vanish identically when  $n$  is even as do terms  $S_{y_m y_n}(\omega)$  where  $m+n$  is odd.

In terms of the dependence on the input level  $P$ , (with a slight abuse of notation) this looks like,

$$\gamma^2(\omega) = \frac{(O(P) + O(P^2) + O(P^3) + \dots)(O(P) + O(P^2) + O(P^3) + \dots)}{P(O(P) + O(P^2) + O(P^3) + \dots)} \quad (18)$$

so both the numerator and denominator are expanded in powers of  $P$ . This means that an approximation to the coherence will have the general Padé form,

$$\gamma_{(m,n)}^2 = \frac{\sum_{i=0}^m a_i P^i}{\sum_{i=0}^n b_i P^i} \quad (19)$$

where  $m$  and  $n$  are the orders of approximation of the numerator and denominator respectively. For the purposes of this paper it will be assumed that  $m = n$ .

To order (0,0), the expression for the coherence is,

$$\gamma_{(0,0)}^2(\omega) = \frac{S_{y_1 x}(\omega) S_{xy_1}(\omega)}{PS_{y_1 y_1}(\omega)} = \frac{PS_{y_1 y_1}(\omega)}{PS_{y_1 y_1}(\omega)} = 1 \quad (20)$$

on using the first lemma of Appendix A in [3]. This shows that to this order the coherence is unaffected by the nonlinearity. To order (1,1), one obtains from equation (17) (suppressing the argument  $\omega$ ),

$$\gamma_{(1,1)}^2 = \frac{S_{y_1 x} S_{xy_1} + S_{y_1 x} S_{xy_3} + S_{y_3 x} S_{xy_1}}{P(S_{y_1 y_1} + S_{y_1 y_3} + S_{y_3 y_1} + S_{y_2 y_2})} = \frac{S_{y_1 x} S_{xy_1} + 2\text{Re}S_{y_1 x} S_{xy_3}}{P(S_{y_1 y_1} + 2\text{Re}S_{y_1 y_3} + S_{y_2 y_2})} \quad (21)$$

and using the first lemma, from Appendix A in [3], this becomes,

$$\gamma_{(1,1)}^2 = \frac{S_{y_1 x} S_{xy_1} + 2\text{Re}S_{y_1 x} S_{xy_3}}{S_{y_1 x} S_{xy_1} + 2\text{Re}S_{y_1 x} S_{xy_3} + PS_{y_2 y_2}} = \frac{A_{11}}{A_{11} + B_{11}} \quad (22)$$

where,

$$A_{11} = S_{y_1 x} S_{xy_1} + 2\text{Re}S_{y_1 x} S_{xy_3} \quad (23)$$

and,

$$B_{11} = PS_{y_2 y_2} \quad (24)$$

One can immediately observe that if the Duffing oscillator is symmetric, then  $y_2 = 0$ ,  $S_{y_2 y_2} = B_{11} = 0$  and the expression for the coherence to this order collapses to  $\gamma_{(1,1)}^2 = 1$ . So for the symmetric system, the coherence is still insensitive to the nonlinearity to this order. If the oscillator is asymmetric, one would expect some distortion to occur and this is shown in [3]. In the symmetric case, higher-order approximations are required. The (2,2) Padé approximation will be considered next,

$$\gamma_{(2,2)}^2 = \frac{S_{y_1x}S_{xy_1} + S_{y_3x}S_{xy_3} + 2\text{Re}S_{y_1x}S_{xy_3} + 2\text{Re}S_{y_1x}S_{xy_5}}{P(S_{y_1y_1} + S_{y_1y_3} + S_{y_3y_1} + S_{y_3y_3} + S_{y_1y_5} + S_{y_5y_1})} \quad (25)$$

Analysis, based on the appendices in [3], eventually yields,

$$\gamma_{(2,2)}^2 = \frac{A_{22}}{A_{22} + B_{22}} \quad (26)$$

where,

$$A_{22} = |S_{y_1x}|^2 + |S_{y_3x}|^2 + 2\text{Re}S_{y_1x}S_{xy_3} + 2\text{Re}S_{y_1x}S_{xy_5} \quad (27)$$

and,

$$B_{22} = \frac{6P^4}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 |H_3(\omega_1, \omega_2, -\omega - \omega_1 - \omega_2)|^2 \quad (28)$$

This shows that to (2,2) order, the coherence for the symmetric Duffing oscillator finally shows some sensitivity to the nonlinearity. The relevant integrals for the (2,2) Padé approximation to the coherence were computed in [3]. For coefficient values of  $m=1$ ,  $c=20$ ,  $k=10^4$  and  $k_3=5 \times 10^9$ , the results for  $P=0.005$ ,  $0.01$  and  $0.02$  are shown in Figure 1.

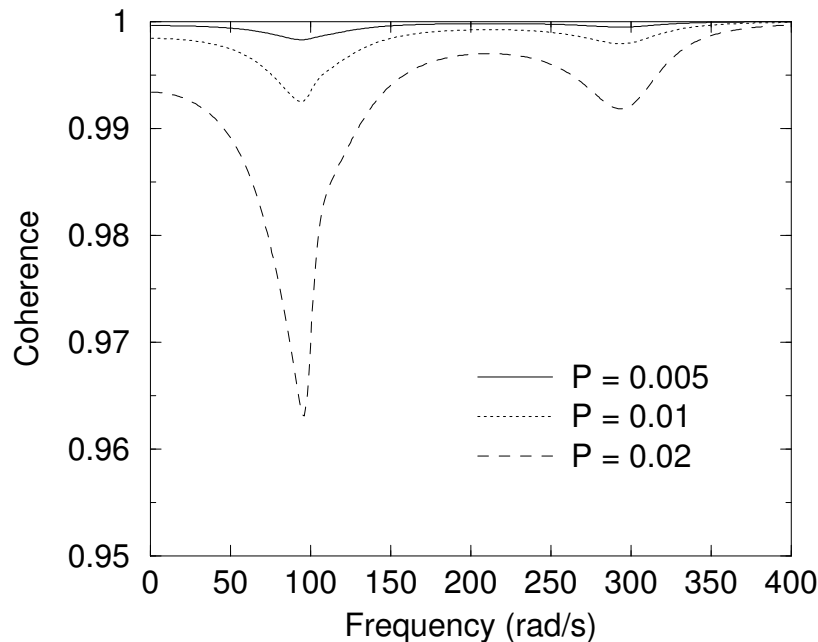


Figure 1. Coherence from (2,2) Padé approximation for symmetric Duffing oscillator.

A comparison with numerical simulation is made in Figure 2; the equation of motion (9) was integrated numerically using a Fourth-order Runge-Kutta scheme. The coefficient values were the same as in the Padé approximation. The excitation was a Gaussian white noise sequence with RMS values chosen to correspond to the  $P$  values above. This in itself was not trivial as the Gaussian sequence was filtered into the interval  $[0,200]$  Hz in order to ensure correct behaviour of the Runge-Kutta routine. This lowered the RMS from the required value and the final simulation used a corrected RMS to account for this. 190000 points of data for each case were taken and the coherence was

estimated from a 512-line FFT; this gave 185 averages. It is seen that the Volterra approximation considerably under-predicts the coherence distortion. The Padé approximation generates some of the correct qualitative features, i.e. is less than unity in the neighbourhood of the natural frequency and its third harmonic. However, it fails to predict the shift upwards in these dips which is shown in the numerical results. This was assumed to be a higher-order effect.

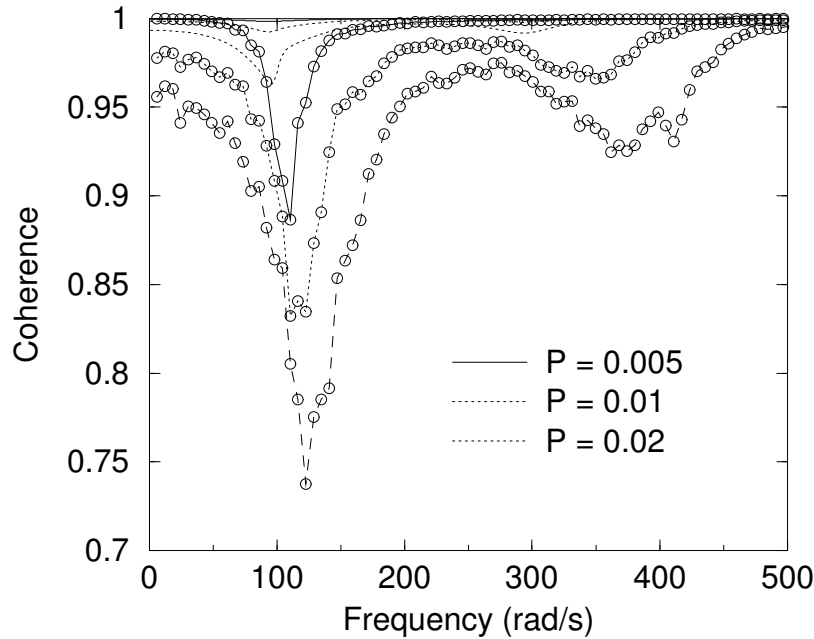


Figure 2. Coherence from (2,2) Padé approximation for symmetric Duffing oscillator. Comparison with numerical simulation (lines with circles).

The (3,3) approximation demands considerably more work. For the sake of brevity, only the final results will be given here. As before, the approximation takes the form,

$$\gamma_{(3,3)}^2 = \frac{A_{33}}{A_{33} + B_{33}} \quad (29)$$

where,

$$\begin{aligned} A_{33}(\omega) = & 1 - \frac{3k_3 P}{\pi} \operatorname{Re}\{H_1(\omega)I_1\} + \frac{9k_3^2 P^2}{4\pi^2} \operatorname{Re}\{|H_1(\omega)|^2 I_1^2 + 2H_1(\omega)^2 I_1^2 + 4H_1(\omega)[I_1 I_2 + I_3]\} \\ & - \frac{27k_3^3 P^3}{4\pi^3} \operatorname{Re}\{H_1(\omega)^3 I_1^3 + |H_1(\omega)|^2 [2I_1^2 I_2 + H_1(\omega)I_1^3 + I_1 I_3] + 4H_1(\omega)^2 [I_1^2 I_2 + I_1 I_3] \\ & + H_1(\omega)[2I_1^2 I_4 + I_1^2 I_5 + 4I_1 I_2^2 + 4I_1 I_6 + 4I_1 I_7 + 4I_8 + 2I_1 I_9 + 8I_{10} + 4I_{11} + 4I_{12} + \frac{2}{3}I_{13}]\} \end{aligned} \quad (30)$$

and,

$$B_{33}(\omega) = \frac{3k_3^2 P^2}{2\pi^2} I_{14} - \frac{9k_3^3 P^3}{2\pi^3} \operatorname{Re}\{|H_1(\omega)|^2 I_1 I_{14} + 2I_1 I_{15} + 4I_{16} + I_1 I_{17} + 2I_{18}\} \quad (31)$$



The integrals  $I_1$  to  $I_{18}$  are listed in the Appendix. Each integral was computed using the calculus of residues within the Maple computer algebra system [8]. The closed-form expressions for the integrals are far too long to give here, and in any case, do not appear to yield a great deal of insight into the problem.

The (3,3) approximation for the coherence is shown in Figure 3 for various values of  $P$ .

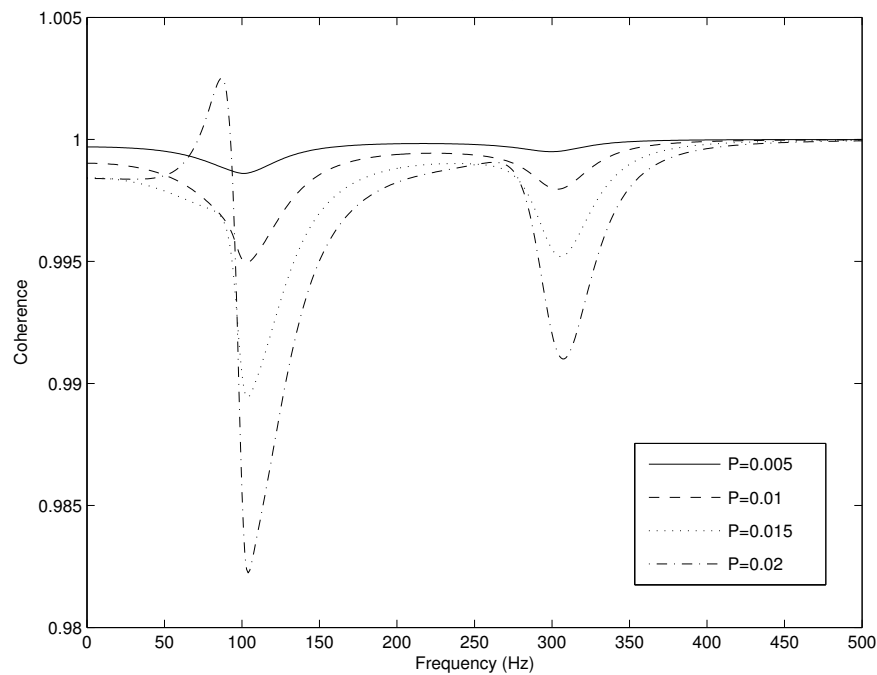


Figure 3. Coherence from (3,3) Padé approximation for symmetric Duffing oscillator.

The results are interesting, if a little disappointing in some respects. The maximum predicted drop in the coherence (at  $P = 0.02$ ) is now of the order of 0.2, a decrease on the (2,2) approximation, which gave about 0.4. The higher-order result is actually moving away from the result of the numerical simulation (Figure 2). A positive result of the calculation is that the higher-order approximation is now predicting an upward frequency shift in the coherence minima with increasing white-noise intensity. However, this shift is markedly lower than that observed in the numerical simulation. One aspect of the approximation which requires comment is the fact that the predicted coherence goes above unity, something which cannot occur; this is presumably due to the fact that the series is truncated, one assumes that still higher-order terms would correct this effect.

#### 4. Discussion and Conclusions

The results of the paper are summarised in the transition between Figures 1 and 3. In Figure 1, the (2,2) Padé approximation to the coherence function for a randomly-excited Duffing oscillator is given, as computed in a previous paper. The calculation has been extended to give the (3,3) Padé approximation here, and the result is given in Figure 3. The (2,2) approximation under-predicted the drops in coherence observed from numerical simulations at the resonance frequency and third-harmonic. In defiance of expectations, the (3,3) approximation under-predicts more severely. One possible explanation for this is that the Volterra series is taking an alternating form, converging on its

ultimate limit through a sequence of under and over-estimates. If this is the case, then the limit is presumably somewhere between the envelope provided by the (2,2) and (3,3) approximations and will itself be an under-estimate of the observed (simulated) coherence. While disappointing, this is not entirely unexpected as the calculation for the FRF in [1] showed such an underestimate of frequency shift with increasing excitation that one might conclude that the Volterra series only reproduces qualitative behaviour with confidence. In qualitative terms, the current calculation may be considered a success as the new approximation to the coherence does reproduce the shifts in the minima which appear in the numerical simulation. However, like the frequency shifts in the FRF, the shifts in the coherence are under-estimated by the Volterra approximation.

The current calculation is the most demanding that the authors have so far attempted, and the apparent conclusion seems to be that qualitative prediction accuracy with the Volterra series is rather elusive. However, it should be noted that the approximate coherence for the asymmetric Duffing oscillator in [3] did give good quantitative agreement with numerical simulation. This suggests that any immediate further work in this area should be directed towards trying to systematically determine the circumstances under which the Volterra series might be expected to give quantitative predictions rather than pursuing even higher-order calculations.

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## Appendix A.

(The limits of all integrals here are from  $-\infty$  to  $\infty$ .)

$$\begin{aligned}
 I_1 &= \int d\omega |H_1(\omega)|^2 \\
 I_2 &= \int d\omega H_1(\omega)^2 H_1(-\omega) \\
 I_3(\omega) &= \iint d\omega_1 d\omega_2 H_1(\omega + \omega_1 + \omega_2) |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \\
 I_4 &= \int d\omega H_1(\omega)^3 H_1(-\omega)
 \end{aligned}$$

$$\begin{aligned}
 I_5 &= \int d\omega |H_1(\omega)|^4 \\
 I_6(\omega) &= \iint d\omega_1 d\omega_2 H_1(\omega + \omega_1 + \omega_2) |H_1(\omega_1)|^2 |H_1(\omega_1)| |H_1(\omega_2)|^2 \\
 I_7(\omega) &= \iint d\omega_1 d\omega_2 H_1(\omega + \omega_1 + \omega_2) |H_1(\omega_1)|^2 |H_1(-\omega_1)| |H_1(\omega_2)|^2 \\
 I_8 &= \iiint d\omega_1 d\omega_2 d\omega_3 H_1(\omega_1 + \omega_2 + \omega_3) |H_1(\omega_1)|^2 |H_1(\omega_1)| |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 \\
 I_9(\omega) &= \iint d\omega_1 d\omega_2 H_1(\omega + \omega_1 + \omega_2)^2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \\
 I_{10}(\omega) &= \iiint d\omega_1 d\omega_2 d\omega_3 H_1(\omega + \omega_1 + \omega_2) H_1(\omega + \omega_1 + \omega_3) |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 \\
 I_{11}(\omega) &= \iiint d\omega_1 d\omega_2 d\omega_3 H_1(\omega + \omega_1 + \omega_2) H_1(\omega_1 + \omega_2 + \omega_3) |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 \\
 I_{12}(\omega) &= \iiint d\omega_1 d\omega_2 d\omega_3 H_1(\omega + \omega_1 + \omega_2) H_1(-\omega_1 - \omega_2 + \omega_3) |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 \\
 I_{13} &= \iiint d\omega_1 d\omega_2 d\omega_3 |H_1(\omega_1 + \omega_2 + \omega_3)|^2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 \\
 I_{14}(\omega) &= \iint d\omega_1 d\omega_2 |H_1(\omega - \omega_1 - \omega_2)|^2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \\
 I_{15}(\omega) &= \iint d\omega_1 d\omega_2 |H_1(\omega - \omega_1 - \omega_2)|^2 |H_1(\omega_1)|^2 |H_1(\omega_1)| |H_1(\omega_2)|^2 \\
 I_{16}(\omega) &= \iiint d\omega_1 d\omega_2 d\omega_3 H_1(\omega + \omega_1 - \omega_2) |H_1(\omega - \omega_2 - \omega_3)|^2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 \\
 I_{17}(\omega) &= \iint d\omega_1 d\omega_2 |H_1(\omega - \omega_1 - \omega_2)|^2 |H_1(\omega - \omega_1 - \omega_2)| |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \\
 I_{18}(\omega) &= \iiint d\omega_1 d\omega_2 d\omega_3 H_1(\omega_1 + \omega_2 + \omega_3) |H_1(\omega - \omega_2 - \omega_3)|^2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2
 \end{aligned}$$