ALGEBRAIC COMPUTATION OF THE SOLUTION OF SOME NONLINEAR DIFFERENTIAL EQUATIONS

F. LAMNABHI-LAGARRIGUE
M. LAMNABHI

Laboratoire des Signaux et Systèmes
C.N.R.S. - E.S.E.
Plateau du Moulon
91190 - GIF SUR YVETTE, France.

1. INTRODUCTION

In this paper, we are concerned with solving algebraically the forced non linear differential equation

$$Ly + \sum a_i y^i = u(t)$$

where L is a linear differential operator with constant coefficients. Several papers have been written about the use of multidimensional Fourier or Laplace transforms for the solution of such a differential equation (see for example, Lubbock et Bansal [1]); but the computations based on these are often tedious and seem difficult to implement on a computer.

The use of a new tool : non commutative generating power series [2] allows us to derive, by simple algebraic manipulations, a functional expansion (i.e. the Volterra series) of the solution. The symbolic calculus introduced appears as a natural generalization, to the nonlinear domain, of the well known Heaviside's operational calculus. Moreover, by using symbolic computation systems, like REDUCE or MACSYMA, it leads to an easy implementation on computer; this becomes necessary as soon as high order approximations are desired.

2. A SIMPLE INTRODUCTIVE EXAMPLE.

Let us consider the nonlinear differential equation

$$\dot{y}(t) = A_0 y(t) + A_1 y(t)\, u(t) \tag{1}$$

where $y(t)$ is an n-dimensional vector and A_0, A_1 are square matrices (equations of this form are called, in control theory, bilinear). Both sides of the differential equation (1) can be integrated to obtain

$$y(t) = y(0) + A_0 \int_0^t y(\sigma_1) d\sigma_1 + A_1 \int_0^t u(\sigma_1)\, y(\sigma_1)\, d\sigma_1 \tag{2}$$

This expression can be solved by repeated substitutions. For example,

$$y(\sigma_1) = y(0) + A_0 y(0) \int_0^{\sigma_1} d\sigma_2 + A_1 y(0) \int_0^{\sigma_1} u(\sigma_2) d\sigma_2$$

can be substitute into (2) to obtain

$$y(t) = y(0) + A_0 y(0) \int_0^t d\sigma_1 + A_0^2 y(0) \int_0^t d\sigma_1 \int_0^{\sigma_1} d\sigma_2 + A_0 A_1 y(0) \int_0^t d\sigma_1 \int_0^{\sigma_1} u(\sigma_2) d\sigma_2$$

$$+ A_1 y(0) \int_0^t u(\sigma_1) d\sigma_1 + A_1 A_0 y(0) \int_0^t u(\sigma_1) d\sigma_1 \int_0^{\sigma_1} d\sigma_2 + A_1^2 y(0) \int_0^t u(\sigma_1) d\sigma_1 \int_0^{\sigma_1} u(\sigma_2) d\sigma_2$$

Repeating this process indefinitely yields the well known Peano-Baker formula :

$$y(t) = y(0) + \sum_{\nu \geq 0} \sum_{j_0, j_1, \ldots, j_\nu = 0}^{1} A_{j_\nu} \ldots A_{j_1} A_{j_0} y(0) \int_0^t d\xi_{j_\nu} \ldots d\xi_{j_0} \qquad (3)$$

where the iterated integral $\int_0^t d\xi_{j_\nu} \ldots d\xi_{j_0}$ is defined by induction on the length by

$$\xi_0(t) = t \qquad\qquad \xi_1(t) = \int_0^t u(\tau) d\tau$$

$$\int_0^t d\xi_{j_\nu} \ldots d\xi_{j_0} = \int_0^t d\xi_{j_\nu}(\tau) \int_0^\tau d\xi_{j_{\nu-1}} \ldots d\xi_{j_0}$$

If we denote by the letter x_0 the integration with respect to time and by the letter x_1 the integration with respect to time after multiplying by the function $u(t)$, (3) can be written symbolically in the form :

$$g = y(0) + \sum_{\nu \geq 0} \sum_{j_0, j_1, \ldots, j_\nu = 0}^{1} A_{j_\nu} \ldots A_{j_1} A_{j_0} y(0) x_{j_\nu} \ldots x_{j_0}$$

g is called the non-commutative generating power series associated with $y(t)$. Of course, this is a non-commutative series because

$$\int_0^t d\tau_1 \int_0^{\tau_1} u(\tau_2) d\tau_2 \neq \int_0^t u(\tau_1)\, d\tau_1 \int_0^{\tau_2} d\tau_2$$

that is, $x_0 x_1 \neq x_1 x_0$.

3. NON COMMUTATIVE GENERATING POWER SERIES

Let $X = \{x_0, x_1\}$ be a finite alphabet and X^* the monoid generated by X. An element of X^* is a word, i.e. a sequence $x_{j_\nu} \ldots x_{j_0}$ of letters of the alphabet. The product of two words $x_{j_\nu} \ldots x_{j_0}$ and $x_{k_\mu} \ldots x_{k_0}$ is the concatenation $x_{j_\nu} \ldots x_{j_0} x_{k_\mu} \ldots x_{k_0}$. The neutral element is called the empty word and denoted by 1. A formal power series with real or complex coefficients is written as a formal sum

$$g = \sum_{w \in X^*} (g, w) w, \quad (g, w) \in \mathbb{R} \text{ or } \mathbb{C}$$

Let g_1 and g_2 be two formal power series, the following operations are defined :

<u>Addition</u> $$g_1 + g_2 = \sum_{w\in X^*} [(g_1,w) + (g_2,w)]w$$

<u>Cauchy product</u> $$g_1 \cdot g_2 = \sum_{w\in X^*} [\sum_{w_1w_2=w} (g_1,w_1)\ (g_2,w_2)]w$$

A series g is invertible if and only if, its constant term, $(g,1)$ is not zero.

<u>Shuffle product</u> $$g_1 \omega g_2 = \sum_{w_1,w_2\in X^*} (g_1,w_1)\ (g_2,w_2) w_1 \omega w_2$$

The shuffle product of two words is defined by induction on the length by

$$1\omega 1 = 1$$

$$\forall\ x\in X, \qquad 1\omega x = x\omega 1 = x$$

$$\forall x,\ x'\in X,\ \forall w,\ w'\in X^*,\ (xw)\ \omega\ (x'\ w') = x[w\omega(x'\ w')] + x'[(xw)\omega w']$$

This operation consists of mixing the letters of the two words keeping the order of each one. For example

$$x_ox_1\ x_1x_o = x_ox_1x_1x_o + x_ox_1x_1x_o + x_ox_1x_ox_1 + x_1x_ox_1x_o + x_1x_ox_ox_1 + x_1x_ox_ox_1 =$$

$$2x_ox^2x_o + x_ox_1x_ox_1 + x_1x_ox_1x_o + 2x_1x_o^2x_1$$

Let us consider a non-commutative power series g; given a function $u(t)$, g will define a functional $y\{t,u(t)\}$ if we replace, following the previous section, the word $x_{j_\nu}\ldots x_{j_o}$ in g by the corresponding iterated integral $\int_o^t d\xi_{j_\nu}\ldots d\xi_{j_o}$. Thus, the numerical value is

$$y\{t,u(t)\} = (g,1) + \sum_{\nu\geq o}\ \sum_{j_o,\ldots,j_\nu=o}^{1} (g,x_{j_\nu}\ldots x_{j_o}) \int_o^t d\xi_{j_\nu}\ldots d\xi_{j_o} \qquad (4)$$

g is known as the non-commutative generating power series associated with the functional y. Now, one can state the important result [2] :

<u>*Theorem*</u> : The product of two nonanticipative functionals of the form (4) is a functional of the same kind, the generating power series of which is the shuffle product of the two generating power series.

4. <u>DERIVATION OF GENERATING POWER SERIES</u>

In this section, we describe an algorithm for finding algebraically the generating power series associated with the solution of a nonlinear forced differential equation. The equation we are going to consider is

$$Ly(t) + \sum_{i=2}^{m} a_i y^i(t) = u(t)$$

$$L = \sum_{i=o}^{n} \ell_i \frac{d^i}{dt^i}, \quad (\ell_n = 1) \tag{5}$$

or, in its integral form

$$y(t) + \ell_{n-1}\int_o^t y(\tau_1)d\tau_1 + \ell_{n-2}\int_o^t d\tau_2 \int_o^{\tau_2} y(\tau_1)d\tau_1 + \ldots + \ell_o \int_o^t d\tau_n \int_o^{\tau_n} d\tau_{n-1} \ldots$$
$$\ldots d\tau_2 \int_o^{\tau_2} y(\tau_1)d\tau_1 + \sum_{i=2}^{m} a_i \int_o^t d\tau_n \int_o^{\tau_n} d\tau_{n-1}\ldots d\tau_2 \int_o^{\tau_2} y^i(\tau_1)d\tau_1 =$$
$$= \int_o^t d\tau_n \int_o^{\tau_n} d\tau_{n-1}\ldots d\tau_2 \int_o^{\tau_2} u(\tau_1)d\tau_1 \tag{6}$$

Here we assume, for simplicity's sake, zero initial conditions.

Let g denotes the generating power series associated with y(t), then (6) can be written symbolically

$$g + \sum_{j=o}^{n-1} \ell_j x_o^{n-i} g + x_o^n \sum_{i=1}^{m} a_i \underbrace{g\omega\ldots\omega g}_{i \text{ times}} = x_o^{n-1}x_1$$

where $\underbrace{g\omega\ldots\omega g}_{i \text{ times}}$ corresponds, according to the previous theorem, to the nonlinear functional $y^i(t)$.

This algebraic equation can be solved iteratively, following the recursive scheme

$$g = g_1 + g_2 + \ldots + g_n + \ldots$$

with

$$g_1 = (1 + \sum_{i=o}^{n-1} \ell_i x_o^{n-i})^{-1} x_o^{n-1}x_1$$

and

$$g_n = -(1 + \sum_{i=o}^{n-1} \ell_i x_o^{n-i})^{-1} x_o^n \sum_{i=2}^{m} a_i \sum_{\nu_1+\nu_2+\ldots+\nu_i=n} g_{\nu_1}\omega g_{\nu_2}\omega\ldots\omega g_{\nu_i}$$

To have the closed form expression of g_i, one only need to compute the shuffle product of non-commutative power series of the form

$$(1-a_o x_o)^{-1}x_{i_1}(1-a_1x_o)^{-1}x_{i_2}\ldots x_{i_p}(1-a_p x_o)^{-1}; \quad i_1,i_2,\ldots,i_p \in \{0,1\}.$$

This results from the following proposition [3].

Proposition 1 : Given two formal power series

$$g_1^p = (1-a_ox_o)^{-1}x_{i_1}(1-a_1x_o)^{-1}x_{i_2}\dots x_{i_p}(1-a_px_o)^{-1} = g_1^{p-1}x_{i_p}(1-a_px_o)^{-1}$$

and

$$g_2^q = (1-b_ox_o)^{-1}x_{j_1}(1-b_1x_o)^{-1}x_{j_2}\dots x_{j_q}(1-b_qx_o)^{-1} = g_2^{q-1}x_{j_q}(1-b_qx_o)^{-1}$$

where p and q belongs to $\mathbb{N}$, the subscripts $i_1,\dots,i_p,j_1,\dots,j_q$ to $\{0,1\}$ and a_i,b_j to $\mathbb{C}$; the shuffle product is given by induction on the length by

$$g_1^p \omega g_2^q = (g_1^p \omega g_2^{q-1})\, x_{j_q}\, [1-(a_p+b_q)x_o]^{-1}$$

$$+ g_1^{p-1}\omega g_2^q\, x_{i_p}\, [1-(a_p+b_q)x_o]^{-1}$$

with $\quad (1-ax_o)^{-1}\,\omega(1-bx_o)^{-1} = [1-(a+b)x_o]^{-1}$.

Using this proposition, g_i is obtained as a finite sum of expressions of the form

$$(1-a_ox_o)^{-p_o}x_1(1-a_1x_o)^{-p_1}x_1\dots x_1(1-a_ix_o)^{-p_i} \tag{7}$$

Now, it can be shown [3] that this expression is a symbolic representation of the n-dimensional integral

$$\int_o^t\int_o^{\tau_i}\dots\int_o^{\tau_2} f_{a_o}^{p_o}(t-\tau_i)\dots f_{a_{i-1}}^{p_{i-1}}(\tau_2-\tau_1)f_{a_i}^{p_i}(\tau_1)u(\tau_1)u(\tau_2)\dots u(\tau_i)d\tau_1d\tau_2\dots d\tau_i \tag{8}$$

where

$$f_a^p(t) = \left(\sum_{j=o}^{p-1}\frac{\binom{j}{p-1}}{j!}a^jt^j\right)e^{at}$$

Thus, what has been shown to this point is that for the nonlinear forced differential equation (5), it is possible to derive algebraically a functional expansion of the solution y(t). (Functionals of this form are known, in control theory, as Volterra series). In the next section, we show that integrals of the form (8) can be expressed in terms of "elementary functions" for a specified set of function u(t).

5. A SYMBOLIC INTEGRATION METHOD.

Let us consider an analytic function u(t)

$$u(t) = \sum_{n\geq o} u_n\frac{t^n}{n!}$$

and define the transform (known as the Laplace-Borel transform)

$$g_u = \sum_{n\geq o} u_n x_o^n$$

(This series may be regarded as the generating power series associated with u(t) since

$$\frac{t^n}{n!} = \int_o^t d\tau_n \int_o^{\tau_n} d\tau_{n-1} \int_o^{\tau_{n-1}} \cdots \int_o^{\tau_2} d\tau_1)$$

Then, we can state the following result [4] :

Proposition 2 : The integral (8) defines an analytic function, the Laplace-Borel transform of which is given by

$$(1-a_o x_o)^{-p_o} x_o\{g_u \omega (1-a_1 x_o)^{-p_1} x_o[g_u \omega \ldots x_o[g_u \omega (1-a_i x_o)^{-p_i}] \ldots]\} \qquad (9)$$

Thus, it is simply obtained, by replacing each indeterminate x_1 in (7) by the operator $x_o[g_u \omega .]$.

Now, assume that g_u is the rational fraction

$$(1 - ax_o)^{-p}$$

That is, u(t) is an exponential polynomial of the form

$$u(t) = (\sum_{j=o}^{p-1} \binom{j}{p-1} \frac{a^j t^j}{j!}) e^{at}$$

then, the simple identity

$$(1-ax_o)^{-p} = \sum_{j=o}^{p-1} \binom{j}{p-1} a^j (1-ax_o)^{-1} \underbrace{x_o (1-ax_o)^{-1} x_o \ldots x_o}_{j \text{ times}} (1-ax_o)^{-1}$$

and the proposition 1 allow to derive a closed for (9) as a rational fraction. The corresponding time function, that is the value of the integral (8), results then from its decomposition into partial fractions. The same technique applies when g_u is a general rational fraction, regular at the origin.

6. EXAMPLE.

Consider the nonlinear differential equation

$$\frac{dy}{dt} + \alpha y + \beta y^2 = u(t)$$

or its integral form

$$y(t) + \alpha \int_0^t y(\tau)d\tau + \beta \int_0^t y^2(\tau)d\tau = \int_0^t u(\tau)d\tau$$

where we assume a zero initial condition.

Thus, the generating power series is simply the solution of

$$g + \alpha x_0 g + \beta x_0 g \omega g = x_1$$

or

$$g = -\beta(1 + \alpha x_0)^{-1} x_0 g \omega g + (1 + \alpha x_0)^{-1} x_1$$

This equation is solved iteratively by a computer program. We obtain

$$
\begin{array}{rrll}
g = & 1 & & {}_1 x_1 {}_0 \\
 & -2 & \beta & {}_1 x_0 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +4 & \beta^2 & {}_1 x_0 {}_2 x \ {}_1 x_0 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +12 & \beta^2 & {}_1 x_0 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_1 {}_0 \\
 & -8 & \beta^3 & {}_1 x_0 {}_2 x_1 {}_1 x_0 {}_2 x_1 {}_1 x_0 {}_2 x_1 {}_1 x_1 {}_0 \\
 & -24 & \beta^3 & {}_1 x_0 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_0 {}_2 x_1 {}_1 x_1 {}_0 \\
 & -72 & \beta^3 & {}_1 x_0 {}_2 x_0 {}_3 x_1 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_1 {}_0 \\
 & -24 & \beta^3 & {}_1 x_0 {}_2 x_1 {}_1 x_0 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_1 {}_0 \\
 & -144 & \beta^3 & {}_1 x_0 {}_2 x_0 {}_3 x_0 {}_4 x_1 {}_3 x_1 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +48 & \beta^4 & {}_1 x_0 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_0 {}_2 x_1 {}_1 x_0 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +144 & \beta^4 & {}_1 x_0 {}_2 x_0 {}_3 x_1 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_0 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +432 & \beta^4 & {}_1 x_0 {}_2 x_0 {}_3 x_1 {}_2 x_0 {}_3 x_1 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +288 & \beta^4 & {}_1 x_0 {}_2 x_0 {}_3 x_0 {}_4 x_1 {}_3 x_1 {}_2 x_1 {}_1 x_0 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +864 & \beta^4 & {}_1 x_0 {}_2 x_0 {}_3 x_0 {}_4 x_1 {}_3 x_1 {}_2 x_0 {}_3 x_1 {}_2 x_1 {}_1 x_1 {}_0 \\
 & +1728 & \beta^4 & {}_1 x_0 {}_2 x_0 {}_3 x_0 {}_4 x_1 {}_3 x_0 {}_4 x_1 {}_3 x_1 {}_2 x_1 {}_1 x_1 {}_0 \\
 & \vdots & &
\end{array}
$$

where, for example, the symbolic notation

$$-2\beta \quad {}_1 x_0 {}_2 x_1 \ {}_1 x_1 {}_0$$

stands for

$$-2\beta(1 + \alpha x_0)^{-1}x_0(1 + 2\alpha x_0)^{-1}x_1(1 + \alpha x_0)^{-1}x_1$$

Now let us, for example, compute the solution y(t) when u(t) is the unit step

$$u(t) = 1 \quad t \geq 0$$

As the Laplace-Borel transform of the unit step is 1, the neutral element for the shuffle product, the Laplace-Borel transform of y(t) is given simply by replacing each variable x_1 in the generating power series g by the variable x_0.

Then decomposing into partial fractions, we get the original function :

$$\begin{aligned} y(t) = {} & \frac{1}{\alpha}(1 - e^{-\alpha t}) \\ & - \frac{\beta}{\alpha^3}(1 - 2\alpha t e^{-\alpha t} - e^{-2\alpha t}) \\ & + \frac{\beta^2}{\alpha^5}[2 + (1-2\alpha t-2\alpha^2 t^2)e^{-\alpha t} - 2(1+2\alpha t)e^{-2\alpha t} - e^{-3\alpha t}] + \ldots \end{aligned}$$

REFERENCES

[1] J.K. LUBBOCK and V.S. BANSAL, Multidimensional Laplace transforms for solution of nonlinear equations. Proc. IEEE, 166, 1969, p. 2075-2082.

[2] M. FLIESS, Fonctionnelles causales non linéaires et indéterminées non commutatives, Bull. Soc. Math. France, 109, 1981, p. 3-40.

[3] F. LAMNABHI-LAGARRIGUE, M. LAMNABHI, Détermination algébrique des noyaux de Volterra associés à certains systèmes non linéaires, Ricerche di Automatica, 10, 1979, p. 17-26.

[4] M. LAMNABHI, Séries de Volterra et séries génératrices non commutatives, thèse de Docteur-Ingénieur, Université Paris XI, Orsay, 1980.