A new symbolic calculus for the response of nonlinear systems *

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We present a new operational calculus for computing the response of nonlinear systems to various deterministic excitations. The use of a new tool: noncommutative generating power series, allows us to derive, by simple algebraic manipulations, the Volterra functional series of the solution of a large class of nonlinear forced differential equations. The symbolic calculus introduced appears as a natural generalization to the nonlinear domain, of the well known Heaviside operational calculus. Moreover, this method has the advantage of allowing the use of a computer.

Keywords: Nonlinear systems, Volterra series, Symbolic calculus.

1. Introduction

Volterra functional series have been widely used in the analysis of nonlinear circuits and systems [1–3]. This approach involves describing the input/output behaviour of a nonlinear system in terms of the functional expansion

$$y(t)=\int_{-\infty}^{+\infty} h_1(\tau_1)u(t-\tau_1)\,\mathrm{d}\tau_1+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2(\tau_1,\tau_2)\,u(t-\tau_1)u(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$
$$+\cdots+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n(\tau_1,\tau_2,\ldots,\tau_n)\,u(t-\tau_1)\cdots u(t-\tau_n)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n+\cdots \tag{1}$$

where $y(t)$ is the system output and $u(t)$ is the system input. This functional series may be regarded as representing a nonlinear system as a parallel bank of nonlinear subsystems. Each of these subsystems is specified by an impulse response $h_n(t_1,\ldots,t_n)$ known as the n^{th}-order Volterra kernel. If the Volterra kernels are known for a system, then the output $y(t)$, for a given input $u(t)$, can be obtained from (1). However, as for linear systems, where the use of the Laplace transform allows to develop a powerful operational calculus, one can introduce here the multiple Laplace transform. Indeed, let us consider the n^{th}-order output

$$y_n(t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n(\tau_1,\tau_2,\ldots,\tau_n)\,u(t-\tau_1)\cdots u(t-\tau_n)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n$$

and introduce a set of artificial variables $t_1, t_2,\ldots,t_n$ so that

$$y_{(n)}(t_1,t_2,\ldots,t_n)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n(\tau_1,\tau_2,\ldots,\tau_n)\,u(t_1-\tau_1)\cdots u(t_n-\tau_n)\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n \tag{2}$$

and

$$y_n(t)=y_{(n)}(t_1,t_2,\ldots,t_n)\big|_{t_1=t_2=\cdots=t_n=t}.$$

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Then, taking the n^{th}-order Laplace transform of both sides of (2) gives

$$Y_{(n)}(s_1,\dots,s_n) = H_n(s_1,\dots,s_n)\prod_{i=1}^{n} U(s_i)$$

where $U(s)$ is the usual first order Laplace transform of the input. Thus, as in the linear case, the convolution in the time domain corresponds to the multiplication in the frequency domain.

Now, assume that the n^{th} order Laplace transform of $y_{(n)}(t_1, t_2,\dots,t_n)$, $Y_{(n)}(s_1, s_2,\dots,s_n)$ is given and $y_n(t)$ is desired. Obviously, one can perform the n^{th}-order inverse Laplace transform of $Y_{(n)}(s_1, s_2,\dots,s_n)$ and set $t_1 = t_2 = \cdots = t_n = t$. However, this computation is often unwieldy. In order to bypass this difficulty, George [4] developed a method whereby the variables t_i could be set equal or 'associated' without leaving the transform domain, leading to a one-dimensional Laplace transform $Y_n(s)$. Then, only a single variable inverse Laplace transform is required to find the output signal $y_n(t)$. The procedure for computing $Y_n(s)$ from $Y_{(n)}(s_1, s_2,\dots,s_n)$ is called *association of variables*. Although explicit formulas for performing the associating operation in a wide class of Laplace transforms have been derived in the literature [5–7], this technique has not been often used. The main reason for this situation seems to be the tedious manipulations involved and the difficulty to implement them on a computer.

In this paper, we present a new operational calculus for computing the response of nonlinear systems to various deterministic inputs (steps, slopes, harmonics, etc). We use noncommutative generating power series introduced by Fliess [8] to derive the *closed form* Volterra functional series of the solution of a large class of nonlinear differential equations. A symbolic integration method is then introduced to evaluate this series for a specified set of functions $u(t)$. This symbolic calculus appears as a natural generalization, to the nonlinear domain, of the well known Heaviside calculus and can be implemented easily on a digital computer with the aid of some symbolic computation systems, like REDUCE or MACSYMA. Our approach is illustrated with a simple example.

2. A symbolic representation of Volterra series

Let us consider a system described by the Volterra series

$$y(t) = h_0(t) + \int_0^t h_1(t,\tau_1)\,u(\tau_1)\mathrm{d}\tau_1 + \cdots + \int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_2} h_n(t,\tau_n,\dots,\tau_1)\prod_{i=1}^{n} u(\tau_i)\mathrm{d}\tau_i + \cdots \tag{3}$$

where the kernels h_i, in triangular form, are assumed to be analytic in the neighborhood of the origin (this is the case for a large class of nonlinear systems [1]). Their Taylor expansion may be written

$$h_n(t,\tau_n,\dots,\tau_1) = \sum_{i_0,i_1,\dots,i_n \geq 0} h_n^{(i_0,i_1,\dots,i_n)} \frac{(t-\tau_n)^{i_n}(\tau_n-\tau_{n-1})^{i_{n-1}}\cdots(\tau_2-\tau_1)^{i_1}\tau_1^{i_0}}{i_n!\cdots i_1! i_0!}$$

with respect to the new variables $\tau_1, \tau_2-\tau_1,\dots,t-\tau_n$.

Each n-dimensional integral

$$\int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_2}\frac{(t-\tau_n)^{i_n}\cdots(\tau_2-\tau_1)^{i_1}\tau_1^{i_0}}{i_n!\cdots i_1! i_0!}u(\tau_n)\cdots u(\tau_1)\,\mathrm{d}\tau_n\cdots\mathrm{d}\tau_1$$

can then be shown to be equal to the iterated integral

$$\underbrace{x_0\cdots x_0}_{i_n}x_1\underbrace{x_0\cdots x_0}_{i_{n-1}}x_1\cdots x_1\underbrace{x_0\cdots x_0}_{i_0}$$

where the letter x_0 denotes the integration with respect to time and the letter x_1 the integration with respect to time after multiplying by the input u. This is a generalization of the well known formula

$$\int_0^t \frac{(t-\tau)^n}{n!} u(\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \cdots \int_0^{\tau_2} \mathrm{d}\tau_1 \int_0^{\tau_1} u(\tau_0)\,\mathrm{d}\tau_0.$$

This allows to write (3) symbolically in the form

$$\mathfrak{g} = \sum_{n \geq 0} \sum_{i_0, i_1, \ldots, i_n \geq 0} h_n^{(i_0, i_1, \ldots, i_n)} x_0^{i_n} x_1 \cdots x_0^{i_1} x_1 x_0^{i_0}. \tag{4}$$

$\mathfrak{g}$ is called the *noncommutative generating power series* associated with the system. This power series can, as we shall see in the next section, be derived directly from the nonlinear differential equations governing the dynamics of the system. Of course this is a noncommutative series because

$$\int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} u(\tau_2)\,\mathrm{d}\tau_2 \neq \int_0^t u(\tau_1)\,\mathrm{d}\tau_1 \int_0^{\tau_2} \mathrm{d}\tau_2,$$

that is, $x_0 x_1 \neq x_1 x_0$.

3. Derivation of noncommutative generating power series

In the following, we define some operations corresponding to various combinations of nonlinear systems. The quantities y_1 and y_2 denote the output of two systems being combined, while $\mathfrak{g}_1$ and $\mathfrak{g}_2$ denote their corresponding generating power series. The output of the composite system is denoted by y and its generating power series by $\mathfrak{g}$.

2.1. Addition

The generating power series associated with the system combination shown in Fig. 1(a) is the sum (see Appendix) of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

2.2. Multiplication

For the system combination shown in Fig. 1(b), it can be shown [8] that the generating power series is the shuffle product (see Appendix) of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

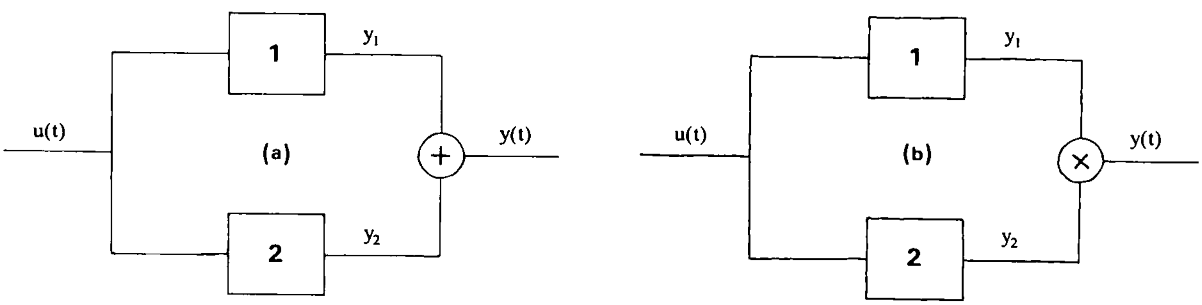

Fig. 1. System combination resulting from (a) the addition operation: $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, and (b) the multiplication operation: $\mathfrak{g} = \mathfrak{g}_1 ш \mathfrak{g}_2$.

2.3. Cascade systems

Consider the cascade connection of Fig. 2. The generating power series of the overall system is the composite (see Appendix) of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

Fig. 2. System resulting from the cascade operation: $\mathfrak{g} = \mathfrak{g}_1 \circ \mathfrak{g}_2$.

The previous rules allow us to obtain the generating power series representing the solution of a forced differential system directly from the governing equations. The equation we are going to consider is

$$Ly(t) + \sum_{i=2}^{m} p_i y^i(t) = u(t), \qquad L = \sum_{i=0}^{n} l_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \quad (l_n = 1), \tag{5}$$

or, in its integral form,

$$\begin{aligned} &y(t) + l_{n-1}\int_0^t y(\tau_1)\,\mathrm{d}\tau_1 + l_{n-2}\int_0^t \mathrm{d}\tau_2 \int_0^{\tau_2} y(\tau_1)\,\mathrm{d}\tau_1 + \cdots + l_0 \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \cdots \mathrm{d}\tau_2 \int_0^{\tau_2} y(\tau_1)\mathrm{d}\tau_1 \\ &\quad + \sum_{i=2}^{m} p_i \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \cdots \mathrm{d}\tau_2 \int_0^{\tau_2} y^i(\tau_1)\,\mathrm{d}\tau_1 = \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \cdots \mathrm{d}\tau_2 \int_0^{\tau_2} u(\tau_1)\,\mathrm{d}\tau_1 \end{aligned} \tag{6}$$

where we assume zero initial conditions.

Let $\mathfrak{g}$ denote the generating power series associated with $y(t)$, then (6) can be written symbolically

$$\left(1 + \sum_{j=0}^{n-1} l_j x_0^{n-1}\right) \mathfrak{g} + x_0^n \sum_{i=1}^{m} p_i \underbrace{\mathfrak{g} \sqcup\!\sqcup \mathfrak{g} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathfrak{g}}_{i \text{ times}} = x_0^{n-1} x_1$$

where $\mathfrak{g} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathfrak{g}$ corresponds, according to the previous rule, to the nonlinear functional $y^i(t)$.

This algebraic equation can be solved iteratively, following the recursive scheme

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \cdots + \mathfrak{g}_n + \cdots \tag{7}$$

with

$$\mathfrak{g}_1 = \left(1 + \sum_{i=0}^{n-1} l_i x_0^{n-i}\right)^{-1} x_0^{n-1} x_1$$

and

$$\mathfrak{g}_n = -\left(1 + \sum_{i=0}^{n-1} l_i x_0^{n-1}\right)^{-1} x_0^n \sum_{i=2}^{m} p_i \sum_{\nu_1 + \nu_2 + \cdots + \nu_i = n} \mathfrak{g}_{\nu_1} \sqcup\!\sqcup \mathfrak{g}_{\nu_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathfrak{g}_{\nu_i}.$$

To have the closed-form expression of $\mathfrak{g}_i$, one only needs to compute the shuffle product of noncommutative power series of the form

$$(1 - a_0 x_0)^{-1} x_{i_1} (1 - a_1 x_0)^{-1} x_{i_2} \cdots x_{i_p} (1 - a_p x_0)^{-1}; \quad i_1, i_2, \ldots, i_p \in \{0,1\}.$$

This results from the following proposition:

Proposition 1 [9]. *Given two formal power series*

$$\mathfrak{g}_1^p=(1-a_0x_x)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}\cdots x_{i_p}(1-a_px_0)^{-1}=\mathfrak{g}_1^{p-1}x_{i_p}(1-a_px_0)^{-1}$$

and

$$\mathfrak{g}_2^q=(1-b_0x_0)^{-1}x_{j_1}(1-b_1x_0)^{-1}x_{j_2}\cdots x_{j_q}(1-b_qx_o)^{-1}=\mathfrak{g}_2^{q-1}x_{j_q}(1-b_qx_0)^{-1}$$

where p and q belong to $\mathbb{N}$, the subscripts $i_1,\ldots,i_p,j_1,\ldots,j_q$ to $\{0,1\}$ and a_i, b_j to $\mathbb{C}$; the shuffle product is given by induction on the length by

$$\mathfrak{g}_1^p \sqcup\!\sqcup\, \mathfrak{g}_2^q=(\mathfrak{g}_1^p \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1})x_{j_q}[1-(a_p+b_q)x_0]^{-1}+(\mathfrak{g}_1^{p-1} \sqcup\!\sqcup\, \mathfrak{g}_2^q)x_{i_p}[1-(a_p+b_q)x_0]^{-1}$$

with

$$(1-ax_0)^{-1} \sqcup\!\sqcup\, (1-bx_0)^{-1}=[1-(a+b)x_0]^{-1}.$$

Using this proposition, $\mathfrak{g}_i$ is obtained as a finite sum of expressions of the form

$$(1-a_0x_0)^{-p_0}x_1(1-a_1x_0)^{-p_1}x_1\cdots x_1(1-a_ix_0)^{-p_i}. \tag{8}$$

Remark. The expansion (7) is 'equivalent' to the Volterra series expansion (3) of $y(t)$. The algebraic closed form expression of triangular Volterra kernels can be easily deduced from it since the expression (8) is a symbolic representation of the n-dimensional integral [9]

$$\int_0^t\int_0^{\tau_i}\cdots\int_0^{\tau_2} f_{a_0}^{p_0}(t-\tau_i)\cdots f_{a_{i-1}}^{p_{i-1}}(\tau_2-\tau_1)f_{a_i}^{p_i}(\tau_1)u(\tau_1)\cdots u(\tau_i)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_i \tag{9}$$

where

$$f_1^p(t)=\left(\sum_{j=0}^{p-1}\frac{\binom{j}{p-1}}{j!}a^jt^j\right)\mathrm{e}^{at}$$

and $\binom{j}{p-1}$ denotes the binomial coefficient.

Example. Let us consider the simple nonlinear circuit shown in Fig. 3.

The nonlinear differential equation relating the current excitation $i(t)$ and the voltage $v(t)$ across the capacitor is

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\alpha v+\beta v^2=i(t)$$

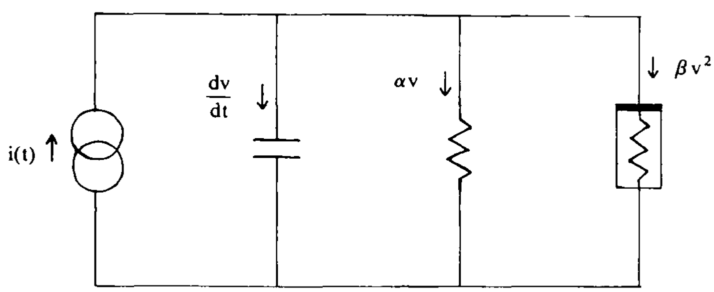

Fig. 3.

or in its integral form

$$v(t)+\alpha\int_0^t v(\tau)\,\mathrm{d}\tau+\beta\int_0^t v^2(\tau)\,\mathrm{d}\tau=\int_0^t i(\tau)\,\mathrm{d}\tau$$

where we assume a zero initial condition.

Thus for the specified circuit, the generating power series is the solution of the algebraic equation

$$\mathfrak{g}+\alpha x_0\mathfrak{g}+\beta x_0\mathfrak{g}\sqcup\!\sqcup\,\mathfrak{g}=x_1$$

or

$$\mathfrak{g}=-\beta(1+\alpha x_0)^{-1}x_0[\mathfrak{g}\sqcup\!\sqcup\,\mathfrak{g}]+(1+\alpha x_0)^{-1}x_1.$$

This equation is solved iteratively by a computer program. We obtain

$$\begin{aligned}\mathfrak{g}=\quad & 1\ {}_1x_{1\,0}\\ & -2\beta\ {}_1x_{0\,2}x_{1\,1}x_{1\,0}\\ & +4\beta^2\ {}_1x_{0\,2}x_{1\,1}x_{0\,2}x_{1\,1}x_{1\,0}\\ & +12\beta^2\ {}_1x_{0\,2}x_{0\,3}x_{1\,2}x_{1\,1}x_{1\,0}\\ & -8\beta^3\ {}_1x_{0\,2}x_{1\,1}x_{0\,2}x_{1\,1}x_{0\,2}x_{1\,1}x_{1\,0}\\ & -24\beta^3\ {}_1x_{0\,2}x_{0\,3}x_{1\,2}x_{1\,1}x_{0\,2}x_{1\,1}x_{1\,0}\\ & -72\beta^3\ {}_1x_{0\,2}x_{0\,3}x_{1\,2}x_{0\,3}x_{1\,2}x_{1\,1}x_{1\,0}\\ & -24\beta^3\ {}_1x_{0\,2}x_{1\,1}x_{0\,2}x_{0\,3}x_{1\,2}x_{1\,1}x_{1\,0}\\ & -144\beta^3\ {}_1x_{0\,2}x_{0\,3}x_{0\,4}x_{1\,3}x_{1\,2}x_{1\,1}x_{1\,0}\\ & \vdots\end{aligned}$$

where, for example, the symbolic notation

$$-2\beta\ {}_1x_{0\,2}x_{1\,1}x_{1\,0}$$

stands for

$$-2\beta(1+\alpha x_0)^{-1}x_0(1+2\alpha x_0)^{-1}x_1(1+\alpha x_0)^{-1}x_1.$$

4. A symbolic calculus of the response

Using noncommutative generating power series, we develop in this section a symbolic calculus for computing the response of nonlinear systems described by (5) to various deterministic inputs (steps, slopes, harmonics, etc.). We show, to this end, that integrals of the form (9) can be expressed in terms of 'elementary functions' for a specified set of functions $u(t)$. These results are to be compared with those of George [4] on association of variables.

Let us consider an analytic function $u(t)$ in the neighborhood of the origin

$$u(t)=\sum_{n\geq 0}u_n\frac{t^n}{n!}$$

and define the transform (known as the Laplace–Borel transform) [10]

$$\mathfrak{g}_u=\sum_{n\geq 0}u_nx_0^n.$$

Note that this series may be regarded as the generating power series associated with $u(t)$ since

$$\frac{t^n}{n!}=\int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} \mathrm{d}\tau_1 .$$

Then we can state the following result:

Proposition 2 [11]. *The integral* (9) *defines an analytic function, the Laplace–Borel transform of which is given by*

$$(1-a_0x_0)^{-p_0}x_0\left\{\mathfrak{g}_u ⧢ (1-a_1x_0)^{-p_1}x_0\left[\mathfrak{g}_u ⧢ \cdots x_0\left[\mathfrak{g}_u ⧢ (1-a_ix_0)^{-p_i}\right]\right]\right\}. \tag{10}$$

Thus, it is simply obtained by replacing each indeterminate x_1 *in* (8) *by the operator* $x_0\,[\mathfrak{g}_u ⧢ \cdot]$.

This results directly from the definition of the iterated integral and from the rule on the multiplication of functionals.

So we obtain a single-variable transform, i.e. the Laplace–Borel transform and thus, the shuffle product appears as an operation which implicitly takes into account the technique of association of variables.

Now, assume that $\mathfrak{g}_u$ is the rational fraction

$$(1-ax_0)^{-p},$$

that is, $u(t)$ is an exponential polynomial of the form

$$u(t)=\left(\sum_{j=0}^{p-1}\binom{j}{p-1}\frac{a^jt^j}{j!}\right)\mathrm{e}^{at}.$$

Then, the simple identity

$$(1-ax_0)^{-p}=\sum_{j=0}^{p-1}\binom{j}{p-1}a^j(1-ax_0)^{-1}\underbrace{x_0(1-ax_0)^{-1}x_0\cdots x_0(1-ax_0)^{-1}}_{j\text{ times}}$$

and Proposition 1 allow to derive a closed form expression for (10) as a rational fraction. The corresponding time function, that is, the value of the integral (9), results then from its decomposition into partial fractions. The same technique applies when $\mathfrak{g}_u$ is a general rational function, regular at the origin. Let us note here that to compute the response, one only needs to know the Laplace–Borel transform of some usual functions; see Table 1.

Application. Let us, for example, compute the response of the previous nonlinear circuit to the unit step

$$u(t)=1,\quad t\geqslant 0.$$

Table 1
Laplace–Borel transforms of some usual functions

$u(t)$	$\mathfrak{g}_u$
unit step	1
$\frac{t^n}{n!}$	x_0^n
$\left(\sum_{i=0}^{n-1}\frac{\binom{i}{n-1}}{i!}a^it^i\right)\mathrm{e}^{at}$	$(1-ax_0)^{-n}$
$\cos(\omega t)$	$(1+\omega^2x_0^2)^{-1}$

As the Laplace–Borel transform of the unit step is 1, the neutral element for the shuffle product, the Laplace–Borel transform of $y(t)$ is given simply by replacing each variable x_1 in the generating power series $\mathfrak{g}$ by the variable x_0:

$$
\begin{aligned}
&1 \quad {}_1 x_{0\,0}\\
&-2\beta \; {}_1 x_{0\,2} x_{0\,1} x_{0\,0}\\
&+4\beta^2 \, {}_1 x_{0\,2} x_{0\,1} x_{0\,2} x_{0\,1} x_{0\,0}\\
&+12\beta^2 \, {}_1 x_{0\,2} x_{0\,3} x_{0\,2} x_{0\,1} x_{0\,0}\\
&-8\beta^3 \, {}_1 x_{0\,2} x_{0\,1} x_{0\,2} x_{0\,1} x_{0\,2} x_{0\,1} x_{0\,0}\\
&-24\beta^3 \, {}_1 x_{0\,2} x_{0\,3} x_{0\,2} x_{0\,1} x_{0\,2} x_{0\,1} x_{0\,0}\\
&-72\beta^3 \, {}_1 x_{0\,2} x_{0\,3} x_{0\,2} x_{0\,3} x_{0\,2} x_{0\,1} x_{0\,0}\\
&-24\beta^3 \, {}_1 x_{0\,2} x_{0\,1} x_{0\,2} x_{0\,3} x_{0\,2} x_{0\,1} x_{0\,0}\\
&-144\beta^3 \, {}_1 x_{0\,2} x_{0\,3} x_{0\,4} x_{0\,3} x_{0\,2} x_{0\,1} x_{0\,0}
\end{aligned}
$$

Then decomposing into partial fractions, we get the original function:

$$
\begin{aligned}
y(t) = {} & \frac{1}{\alpha}(1-\mathrm{e}^{-\alpha t}) - \frac{\beta}{\alpha^3}(1 - 2\alpha t\,\mathrm{e}^{-\alpha t} - \mathrm{e}^{-2\alpha t})\\
& + \frac{\beta^2}{\alpha^5}\left[2 + (1 - 2\alpha t - 2\alpha^2 t^2)\,\mathrm{e}^{-\alpha t} - 2(1+2at)\,\mathrm{e}^{-2\alpha t} - \mathrm{e}^{-3\alpha t}\right] + \cdots .
\end{aligned}
$$

5. Conclusion

The calculus developed in this paper and based on noncommutative variables, seems more suitable for computing the response of nonlinear systems described by Volterra series than the technique of association of variables. Moreover, this method has the advantage of allowing the use of a computer. This becomes necessary, as soon as one tries to obtain high order terms.

Appendix

Let $X = \{x_0, x_1\}$ be a finite alphabet and X^* the free monoid generated by X. An element of X^* is a word, i.e. a finite sequence $x_{j_\nu} \cdots x_{j_0}$ of letters of the alphabet.

The product of two words $x_{j_\nu} \cdots x_{j_0}$ and $x_{k_\mu} \cdots x_{k_0}$ is the concatenation $x_{j_\nu} \cdots x_{j_0} x_{k_\mu} \cdots x_{k_0}$. This operation is noncommutative. The neutral element is called the empty word and written as 1.

The shuffle product of two words is defined by induction on the length by

$$
\begin{aligned}
&1 \sqcup\!\sqcup 1 = 1, \qquad \forall x \in X, \quad 1 \sqcup\!\sqcup x = x \sqcup\!\sqcup 1 = x,\\
&\forall x, x' \in X,\; w, w' \in X^*, \quad (xw) \sqcup\!\sqcup (x'w') = x[w \sqcup\!\sqcup (x'w')] + x'[(xw) \sqcup\!\sqcup w'].
\end{aligned}
$$

This operation consists of mixing the letters of the two words keeping the order of each one.

Example

$$
\begin{aligned}
x_0 x_1 \sqcup\!\sqcup x_1 x_0 &= x_0 x_1 x_1 x_0 + x_0 x_1 x_1 x_0 + x_0 x_1 x_0 x_1 + x_1 x_0 x_1 x_0 + x_1 x_0 x_0 x_1 + x_1 x_0 x_0 x_1\\
&= 2 x_0 x_1^2 x_0 + x_0 x_1 x_0 x_1 + x_1 x_0 x_1 x_0 + 2 x_1 x_0^2 x_1 .
\end{aligned}
$$

A formal power series with real or complex coefficients is written as a formal sum

$$g = \sum_{w \in X^*} (g,w)w, \quad (g,w) \in \mathbb{R} \text{ or } \mathbb{C}.$$

Let s_1 and s_2 be two formal power series; the following operations are defined:

Addition:

$$g_1 + g_2 = \sum_{x \in X^*} [(g_1,w) + (g_2,w)]\, w.$$

Shuffle product: The shuffle product is extended to formal power series by

$$g_1 ш g_2 = \sum_{w_1, w_2 \in X^*} [(g_1, w_1) \times (g_2, x_2)]\, w_1 ш w_2.$$

Composite of two series:

$$g_2 \circ g_1.$$

This operation is performed by replacing each variable x_1 in the formal power series g_2 by the group $x_0[g_1 ш \cdot]$.

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