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Generating Series for Nonlinear Cascade and Feedback Systems

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Abstract

Given two analytic nonlinear input-output systems represented as Fliess operators, four system interconnections are considered in a unified setting: the parallel connection, product connection, cascade connection and feedback connection. In each case, the corresponding generating series is produced, when one exists, and conditions for convergence of the corresponding Fliess operator are given. In the process, an existing notion of a *composition product* for formal power series is generalized to the multivariable setting, and its set of known properties is expanded. In addition, the notion of a *feedback product* for formal power series is introduced and characterized.

1 Introduction

Let $I = \{0, 1, \ldots, m\}$ denote an alphabet and I^* the set of all words over I. A formal power series in I is any mapping of the form $I^* \mapsto \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell \langle\langle I \rangle\rangle$. For each $c \in \mathbb{R}^\ell\langle\langle I \rangle\rangle$, one can formally associate a corresponding m-input, ℓ-output operator F_c in the following manner. Let $p \geq 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_p$ norm and $B_p^m(R)[a, b] := \{u \in L_p^m[a, b] : \|u\|_p \leq R\}$. With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define inductively for each $\eta \in I^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to C[t_0, t_0 + T]$ by $E_\phi = 1$, and

$$E_{i_k i_{k-1} \cdots i_1}[u](t, t_0) = \int_{t_0}^{t} u_{i_k}(\tau) E_{i_{k-1} \cdots i_1}[u](\tau, t_0)\, d\tau,$$

where $u_0(t) \equiv 1$. The input-output operator corresponding to c is then

$$F_c[u](t) = \sum_{\eta \in I^*} (c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator*. In the classical literature where these operators first appeared [4, 5, 6, 9, 10, 11], it is normally assumed that there exists real numbers $K > 0$ and $M \geq 1$ such that

$$|(c, \eta)| \leq K M^{|\eta|} |\eta|!, \qquad \forall \eta \in I^*, \tag{1}$$

where $|z| = \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, and $|\eta|$ denotes the number of symbols in η. This growth condition on the coefficients of c insures that there exist positive real numbers R and T_0 such that for all piecewise continuous u with $\|u\|_\infty \leq R$ and $T \leq T_0$, the series defining F_c converges uniformly and absolutely on $[t_0, t_0 + T]$. Under such conditions the power series c is said to be *locally convergent*. More recently, it was shown in [8] that the growth condition (1) also implies that F_c constitutes a well defined operator from $B_p^m(R)[t_0, t_0 + T]$ into $B_q^\ell(S)[t_0, t_0 + T]$ for sufficiently small $R, S, T > 0$, where the numbers $p, q \in [1, \infty]$ are conjugate exponents, i.e., $1/p + 1/q = 1$ with $(1, \infty)$ being a conjugate pair by convention.

In many applications input-output systems are interconnected in a variety of ways. Given two Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell\langle\langle I \rangle\rangle$ are locally convergent, Figure 1 shows four elementary interconnections. In the case of the cascade and feedback connections it is assumed that $\ell = m$. The general goal of this paper is to describe in a unified manner the generating series for each of the four interconnections shown in Figure 1 and conditions under which it is locally convergent. The parallel connection is the trivial case, and the product connection was analyzed in [12]. They are included for completeness and some of the analysis is applicable to the other two interconnections. It was shown in [3] for the SISO case (i.e., $\ell = m = 1$) that there always exists a series $c \circ d$ such that $y = F_c[F_d[u]] = F_{c \circ d}[u]$, but a multivariable analysis of this *composition product* is apparently not available in the literature, nor are any results about local convergence. So in Section 2 the composition product is first investigated independent of the interconnection problem. This material is a continuation of that which appeared in [7]. In Section 3, the three *nonrecursive* connections: the parallel, product and cascade connections are analyzed primarily by applying the results of Section 2. In Section 4 the feedback connection is considered in the case where the system inputs are being generated by an exosystem which is itself a Fliess operator. This leads to an implicit characterization of the *feedback product*, $c@d$, for formal power series.

2 The Composition Product

The composition product of two series $c, d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ over an alphabet $X = \{x_0, x_1, \ldots, x_m\}$ is defined recursively in terms of the shuffle product. For

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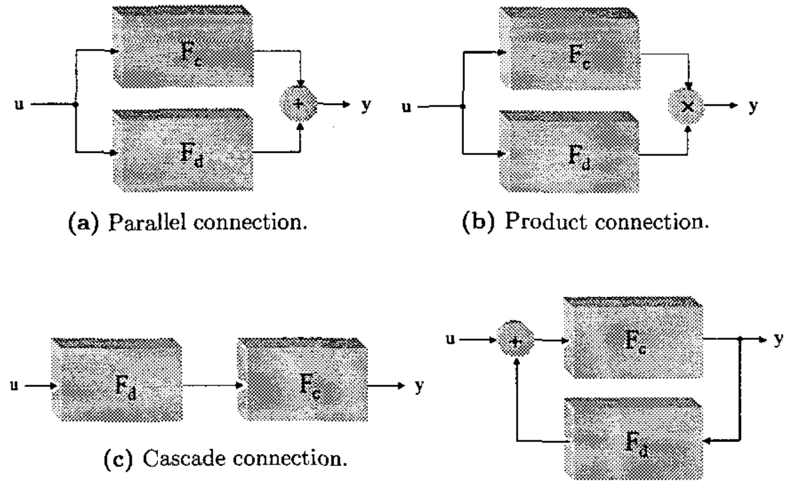

(a) Parallel connection. **(b)** Product connection.

(c) Cascade connection. **(d)** Feedback connection.

Figure 1: Elementary system interconnections.

any $\eta \in X^*$ let

$$\eta \circ d = \begin{cases} \eta & : |\eta|_{x_i} = 0,\ \forall i \neq 0 \\ x_0^{k+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \eta = x_0^k x_i \eta',\ k \geq 0,\ i \neq 0, \end{cases}$$

where $|\eta|_{x_i}$ denotes the number of symbols in η equivalent to x_i and $d_i : \xi \mapsto (d,\xi)_i$, the i-th component of (d,ξ). Consequently, if

$$\eta = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \cdots x_0^{n_1} x_{i_1} x_0^{n_0}, \qquad (2)$$

where $i_j \neq 0$ for $j = 1, \ldots, k$, then it follows that

$$\eta \circ d = \\ x_0^{n_k+1}[d_{i_k} \sqcup\!\sqcup x_0^{n_{k-1}+1}[d_{i_{k-1}} \sqcup\!\sqcup \cdots x_0^{n_1+1}[d_{i_1} \sqcup\!\sqcup x_0^{n_0}] \cdots]].$$

The next theorem insures that the composition product of two series is well defined.

Theorem 2.1 *Given a fixed $d \in \mathbb{R}^m \ll X \gg$, the family of series $\{\eta \circ d : \eta \in X^*\}$ is locally finite, and therefore summable.*

Proof: Given an arbitrary $\eta \in X^*$ expressed in the form (2), it follows directly that

$$\begin{aligned} \operatorname{ord}(\eta \circ d) &= n_0 + k + \sum_{j=1}^{k} n_j + \operatorname{ord}(d_{i_j}) \\ &= |\eta| + \sum_{j=1}^{|\eta|-|\eta|_{x_0}} \operatorname{ord}(d_{i_j}), \end{aligned}$$

where the *order* of c is defined as

$$\operatorname{ord}(c) = \begin{cases} \inf\{|\eta| : \eta \in \operatorname{supp}(c)\} & : c \neq 0 \\ \infty & : c = 0, \end{cases}$$

and $\operatorname{supp}(c) := \{\eta \in X^* : (c,\eta) \neq 0\}$ denotes the support of c. Hence, for any $\xi \in X^*$

$$\begin{aligned} I_d(\xi) &:= \{\eta \in X^* : (\eta \circ d, \xi) \neq 0\} \\ &\subset \{\eta \in X^* : \operatorname{ord}(\eta \circ d) \leq |\xi|\} \\ &= \{\eta \in X^* : |\eta| + \sum_{j=1}^{|\eta|-|\eta|_{x_0}} \operatorname{ord}(d_{i_j}) \leq |\xi|\}. \end{aligned}$$

Clearly this latter set is finite, and thus $I_d(\xi)$ is finite for all $\xi \in X^*$. This fact implies summability [1]. ∎

For $c, d \in \mathbb{R}^m \ll X \gg$ the composition product is defined as

$$c \circ d = \sum_{\eta \in X^*} (c,\eta)\, \eta \circ d.$$

It is easily verified for any series $c, d, e \in \mathbb{R}^m \ll X \gg$ that

$$(c + d) \circ e = c \circ e + d \circ e,$$

but in general $c \circ (d + e) \neq c \circ d + c \circ e$. A special exception are *linear series.* A series $c \in \mathbb{R}^\ell \ll X \gg$ is called linear if

$$\operatorname{supp}(c) \subseteq \{\eta \in X^* : \eta = x_0^{n_1} x_i x_0^{n_0}, \\ i \in \{1, 2, \ldots, m\},\ n_1, n_0 \geq 0\}.$$

The set $\mathbb{R} \ll X \gg$ forms a metric space under the ultrametric

$$\begin{aligned} \operatorname{dist} &: \mathbb{R} \ll X \gg \times\ \mathbb{R} \ll X \gg\ \mapsto \mathbb{R}^+ \cup \{0\} \\ &: (c, d) \mapsto \sigma^{\operatorname{ord}(c-d)}, \end{aligned}$$

where $\sigma \in (0,1)$ is arbitrary. The following theorem states that the composition product on $\mathbb{R} \ll X \gg \times \mathbb{R} \ll X \gg$ is at least continuous in its first argument. The result extends naturally to vector-valued series in a componentwise fashion.

Theorem 2.2 *[7] Let $\{c_i\}_{i\geq 1}$ be a sequence in $\mathbb{R}\langle\langle X\rangle\rangle$ with $\lim_{i\to\infty} c_i = c$. Then $\lim_{i\to\infty}(c_i \circ d) = c \circ d$.*

The metric space $(\mathbb{R}\langle\langle X\rangle\rangle, \mathrm{dist})$ is known to be complete [1]. Given a fixed $c \in \mathbb{R}\langle\langle X\rangle\rangle$, consider the mapping $\mathbb{R}\langle\langle X\rangle\rangle \mapsto \mathbb{R}\langle\langle X\rangle\rangle : d \mapsto c \circ d$.

Theorem 2.3 *[7] For any $c \in \mathbb{R}\langle\langle X\rangle\rangle$ with $X = \{x_0, x_1\}$, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}\langle\langle X\rangle\rangle$.*

It is shown in [3] by counter example that the composition product is *not* a rational operation. That is, the composition of two rational series does not in general produce a rational series. But in [2], it is shown in the SISO case that a special class of rational series produce rational series when the composition product is applied. The following definition is the multivariable extention of the defining property of this class, and the corresponding rationality proof is not significantly different.

Definition 2.1 *A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is **limited relative to x_i** if there exists an integer $N_i \geq 0$ such that*

$$\sup_{\eta \in \mathrm{supp}(c)} |\eta|_{x_i} \leq N_i.$$

*If c is limited relative to x_i for every $i = 1, 2, \ldots, m$ then c is **input-limited**. In such cases, let $N_c := \sum_i N_i$. A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is input-limited if each component series, c_j, is input-limited for $j = 1, 2, \ldots, \ell$. In this case, $N_c := \max_j N_{c_j}$.*

The property of local convergence is next considered in this setting.

Theorem 2.4 *[7] Suppose $c, d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are locally convergent series with growth constants K_c, M_c and K_d, M_d, respectively. If c is input-limited with $N_c \geq 1$ then $c \circ d$ is locally convergent with*

$$|(c \circ d, \nu)| < K_c K_d^{N_c}(N_c + 1)\,(M(N_c + 1))^{|\nu|}|\nu|!,$$

for all $\nu \in X^*$ and where $M = \max\{M_c, M_d\}$.

It was demonstrated by example in [7] that input-limited is only sufficient for local convergence. But removing this assumption introduces significant complexity into the analysis. This issue is considered further here. The main tools employed are the basic properties of shuffle product given below.

Theorem 2.5 *[12] For arbitrary $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ and any $\nu \in X^*$:*

1. $$(c \sqcup\!\sqcup\, d, \nu) = \sum_{\xi, \bar{\xi} \in X^*} (c, \xi)(d, \bar{\xi})(\xi \sqcup\!\sqcup\, \bar{\xi}, \nu) = \sum_{i=0}^{|\nu|} \sum_{\substack{\xi \in X^i \\ \bar{\xi} \in X^{|\nu|-i}}} (c, \xi)(d, \bar{\xi})(\xi \sqcup\!\sqcup\, \bar{\xi}, \nu)$$

2. $$\sum_{\substack{\xi \in X^i \\ \bar{\xi} \in X^{|\nu|-i}}} (\xi \sqcup\!\sqcup\, \bar{\xi}, \nu) \leq \binom{|\nu|}{i}, \quad 0 \leq i \leq |\nu|,$$

where $X^i := \{\xi \in X^* : |\xi| = i\}$.

Now for any $\eta \in X^*$ with $X = \{x_0, x_1\}$ one can uniquely define a set of integers $n_j \geq 0$ and a set of subwords $\eta_j \in X^*$, $j \geq 0$, by the following iteration

$$\eta_{j+1} = x_0^{n_{j+1}} x_1 \eta_j, \quad \eta_0 = x_0^{n_0}, \tag{3}$$

so that $\eta = \eta_k$ where $k = |\eta|_{x_1}$ (cf. equation (2)). Using the set of subwords $\{\eta_0, \eta_1, \ldots, \eta_k\}$ define a family of corresponding functions:

$$\begin{aligned}
S_{\eta_0}(n) &= \frac{1}{|\eta_0|!}, \quad n \geq 0 \\
S_{\eta_1}(n) &= \frac{1}{(n)_{n_1+1}} S_{\eta_0}(n), \quad n \geq |\eta_1| \\
S_{\eta_j}(n) &= \frac{1}{(n)_{n_j+1}} \sum_{i=0}^{n-|\eta_j|} S_{\eta_{j-1}}(n - (n_j + 1) - i), \\
&\qquad n \geq |\eta_j|, \quad 2 \leq j \leq k,
\end{aligned}$$

where $(n)_i = n(n-1)\cdots(n-i+1)$ denotes the falling factorial. Using these functions the following theorem provides a bound for $|(\eta_j \circ d, \nu)|$ using *only* the coefficient bounds for d, the triangle inequality, and the inequality in Lemma 2.5. It therefore produces a least upperbound on the growth of the coefficients of $\eta_j \circ d$ in the general case.

Theorem 2.6 *For $X = \{x_0, x_1\}$ let $d \in \mathbb{R}\langle\langle X\rangle\rangle$ be locally convergent with growth constants K_d and M_d. For any $\eta \in X^*$ with subwords $\{\eta_0, \eta_1, \ldots, \eta_k\}$ as defined in equation (3) it follows that*

$$|(\eta_j \circ d, \nu)| \leq K_d^j M_d^{-|\eta_j|} M_d^{|\nu|} |\nu|!\, S_{\eta_j}(|\nu|), \; \forall \nu \in X^*,$$

where $j = 0, 1, \ldots, k$. (Since $(\eta_j \circ d, \nu) = 0$ for $|\nu| < |\eta_j|$, the value of $S_{\eta_j}(|\nu|)$ when $|\nu| < |\eta_j|$ can assigned to be any nonnegative value.)

Proof: The proof is by induction. The case $j = 0$ is trivial. When $j = 1$ observe that employing the left-shift operator [12], denoted by $(\cdot)^{-n}$, gives

$$\begin{aligned}
|(\eta_1 \circ d, \nu)| &= \left|(x_0^{n_1+1}(d \sqcup\!\sqcup\, x_0^{n_0}), \nu)\right| \\
&= \left|(d \sqcup\!\sqcup\, x_0^{n_0}, \underbrace{x_0^{-(n_1+1)}(\nu)}_{\theta})\right| \\
&= \left|\sum_{\xi \in X^{|\theta|-n_0}} (d, \xi)(\xi \sqcup\!\sqcup\, x_0^{n_0}, \theta)\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{\xi\in X^{|\theta|-n_0}} (K_d M_d^{|\xi|}|\xi|!)\,(\xi \sqcup x_0^{n_0},\theta) \\
&\le K_d M_d^{|\theta|-n_0}(|\theta|-n_0)!\binom{|\theta|}{n_0} \\
&= K_d M_d^{-|\eta_1|} M_d^{|\nu|} S_{\eta_1}(|\nu|)\,|\nu|!.
\end{aligned}
$$

Now assume that the result holds up to some fixed $j \ge 1$. Then in a similar fashion

$$
\begin{aligned}
|(\eta_{j+1}\circ d,\nu)| &= \left|(d \sqcup (\eta_j\circ d), \underbrace{x_0^{-(n_j+1)}(\nu)}_{\theta})\right| \\
&= \left|\sum_{i=0}^{|\theta|}\sum_{\substack{\xi\in X^i\\ \bar\xi\in X^{|\theta|-i}}} (d,\xi)(\eta_j\circ d,\bar\xi)(\xi\sqcup\bar\xi,\nu)\right|.
\end{aligned}
$$

Since $(\eta_j\circ d,\bar\xi)=0$ for $|\bar\xi|<|\eta_j|$ it follows that

$$
\begin{aligned}
|(\eta_{j+1}\circ d,\nu)| &\le \sum_{i=0}^{|\theta|-|\eta_j|}\sum_{\substack{\xi\in X^i\\ \bar\xi\in X^{|\theta|-i}}} (K_d M_d^{|\xi|}|\xi|!)\cdot \\
&\quad \left(K_d^j M_d^{-|\eta_j|} M_d^{|\bar\xi|}|\bar\xi|!\, S_{\eta_j}(|\bar\xi|)\right)(\xi\sqcup\bar\xi,\theta) \\
&\le K_d^{j+1} M_d^{-|\eta_{j+1}|} M_d^{|\nu|}\cdot \\
&\quad \sum_{i=0}^{|\theta|-|\eta_j|} i!\,(|\theta|-i)!\, S_{\eta_j}(|\theta|-i)\binom{|\theta|}{i} \\
&= K_d^{j+1} M_d^{-|\eta_{j+1}|} M_d^{|\nu|}|\nu|!\frac{1}{(|\nu|)_{n_{j+1}+1}}\cdot \\
&\quad \sum_{i=0}^{|\theta|-|\eta_j|} S_{\eta_j}(|\nu|-(n_{j+1}+1)-i) \\
&= K_d^{j+1} M_d^{-|\eta_{j+1}|} M_d^{|\nu|}|\nu|!\, S_{\eta_{j+1}}(|\nu|).
\end{aligned}
$$

Hence, the claim is true for all $j \ge 0$. ■

The main application of this result is below.

Theorem 2.7 *For $X=\{x_0,x_1\}$ suppose $c,d\in\mathbb{R}\langle\langle X\rangle\rangle$ are locally convergent series with growth constants K_c, M_c and K_d, M_d, respectively. If $K_d\ge 1$ then*

$$
|(c\circ d,\nu)| \le K_c (K_d M)^{|\nu|}|\nu|!\,\psi(|\nu|),\ \ \forall \nu\in X^*,
$$

where

$$
\psi(n) := \sum_{i,j=0}^{n}\sum_{\eta_j\in X^i} |\eta_j|!\, S_{\eta_j}(n),\ \ n\ge 0
$$

and $M=\max\{M_c,M_d\}$.

Proof: Since $\eta\in I_d(\nu)$ only if $|\eta|\le|\nu|$, it follows that

$$
\begin{aligned}
|(c\circ d,\nu)| &= \left|\sum_{i,j=0}^{|\nu|}\sum_{\eta_j\in X^i}(c,\eta_j)(\eta_j\circ d,\nu)\right| \\
&\le \sum_{i,j=0}^{|\nu|}\sum_{\eta_j\in X^i}(K_c M^{|\eta_j|}|\eta_j|!)\cdot \\
&\quad (K_d^j M^{-|\eta_j|} M^{|\nu|}|\nu|!\, S_{\eta_j}(|\nu|)) \\
&\le K_c (K_d M)^{|\nu|}|\nu|!\,\psi(|\nu|).
\end{aligned}
$$

■

In order for $c\circ d$ to be locally convergent in general, it is necessary that $\psi(n)\le\beta\alpha^n$ for some $\alpha,\beta\in\mathbb{R}$. At present this is an open problem. It appears that some form of combinatoric analysis is required. When c and d are both linear series, it can be shown directly that

$$
|(c\circ d,\nu)| < K_c K_d M^{|\nu|}|\nu|!,\ \ \forall\nu\in X^*.
$$

3 The Nonrecursive Connections

In this section the generating series are produced for the three nonrecursive interconnections and their local convergence is characterized.

Theorem 3.1 *Suppose $c,d\in\mathbb{R}^\ell\langle\langle I\rangle\rangle$ are locally convergent power series. Then each nonrecursive interconnected input-output system shown in Figure 1 (a)-(c) has a Fliess operator representation generated by a locally convergent series as indicated:*

1. *$F_c + F_d = F_{c+d}$*
2. *$F_c\cdot F_d = F_{c\,\sqcup\, d}$*
3. *$F_c\circ F_d = F_{c\circ d}$, where $\ell=m$, and c is input-limited.*

Proof:

1. Observe that

$$
\begin{aligned}
F_c[u](t)+F_d[u](t) &= \sum_{\eta\in I^*}[(c,\eta)+(d,\eta)]\,E_\eta[u](t,t_0) \\
&= F_{c+d}[u](t).
\end{aligned}
$$

Since c and d are locally convergent, define $M=\max\{M_c,M_d\}$. Then for any $\eta\in I^*$ it follows that

$$
\begin{aligned}
|(c+d,\eta)| &= |(c,\eta)+(d,\eta)| \\
&\le (K_c+K_d)M^{|\eta|}|\eta|!
\end{aligned}
$$

or $c+d$ is locally convergent.

2. In light of the componentwise definition of the shuffle product, it can be assumed here without loss of generality that $\ell=1$. Thus,

$$
\begin{aligned}
&F_c[u](t)F_d[u](t) \\
&\quad = \sum_{\eta\in I^*}(c,\eta)E_\eta[u](t,t_0)\sum_{\xi\in I^*}(c,\xi)E_\xi[u](t,t_0)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\eta,\xi\in I^*} (c,\eta)(d,\xi)\; E_\eta[u](t,t_0)E_\xi[u](t,t_0) \\
&= \sum_{\eta,\xi\in I^*} (c,\eta)(d,\xi)\; E_{\eta \sqcup\!\sqcup\, \xi}[u](t,t_0) \\
&= F_{c \sqcup\!\sqcup\, d}[u](t).
\end{aligned}
$$

Applying Lemma 2.2.7 from [12] and using the fact that $n+1 \leq 2^n$, $n \geq 0$, gives

$$|(c \sqcup\!\sqcup\, d, \eta)| \leq K_c K_d (2M)^{|\eta|}|\eta|!.$$

Thus, $c \sqcup\!\sqcup\, d$ is locally convergent.

3. For any $\eta \in I^*$ and $d \in \mathbb{R}^m \ll I \gg$ the corresponding Fliess operators are

$$
\begin{aligned}
F_\eta[u](t) &= E_\eta[u](t,t_0) \\
F_d[u](t) &= \sum_{\xi\in I^*}(d,\xi)E_\xi[u](t,t_0).
\end{aligned}
$$

Therefore,

$$(F_\eta \circ F_d[u])(t) = E_\eta\left[F_d[u]\right](t,t_0).$$

If $|\eta| = |\eta|_{x_0}$ then

$$
\begin{aligned}
(F_\eta \circ F_d[u])(t) &= E_\eta[u](t,t_0) = F_\eta[u](t) \\
&= F_{\eta\circ d}[u](t).
\end{aligned}
$$

If, on the other hand, $\eta = \underbrace{0\cdots 0}_{k \text{ times}} i\eta'$ with $i \neq 0$ then

$$
\begin{aligned}
&(F_\eta \circ F_d[u])(t) \\
&\quad = E_{\underbrace{0\cdots 0}_{k \text{ times}} i\eta'}[F_d[u]](t,t_0) \\
&\quad = \underbrace{\int_{t_0}^{t}\cdots\int_{t_0}^{\tau_2}}_{k+1 \text{ times}} F_{d_i}[u](\tau)E_{\eta'}[F_d[u]](\tau,t_0)\cdot \\
&\qquad\qquad d\tau_1\ldots d\tau_{k+1} \\
&\quad = \underbrace{\int_{t_0}^{t}\cdots\int_{t_0}^{\tau_2}}_{k+1 \text{ times}} F_{d_i \sqcup\!\sqcup\, (\eta'\circ d)}[u](\tau,t_0)\, d\tau_1\ldots d\tau_{k+1} \\
&\quad = F_{\underbrace{0\cdots 0}_{k+1 \text{ times}}[d_i \sqcup\!\sqcup\, (\eta'\circ d)]}[u](t) \\
&\quad = F_{\eta\circ d}[u](t).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(F_c \circ F_d[u])(t) \\
&\quad = \sum_{\eta\in I^*}(c,\eta)E_\eta[F_d[u]](t,t_0) \\
&\quad = \sum_{\eta\in I^*}(c,\eta)F_{\eta\circ d}[u](t) \\
&\quad = \sum_{\eta\in I^*}(c,\eta)\left[\sum_{\nu\in I^*}(\eta\circ d,\nu)E_\nu[u](t,t_0)\right] \\
&\quad = \sum_{\nu\in I^*}\left[\sum_{\eta\in I^*}(c,\eta)(\eta\circ d,\nu)\right]E_\nu[u](t,t_0) \\
&\quad = \sum_{\nu\in I^*}(c\circ d,\nu)E_\nu(t,t_0) \\
&\quad = F_{c\circ d}[u](t).
\end{aligned}
$$

Local convergence of $c \circ d$ under the stated conditions follows from Theorem 2.4. ∎

c being input-limited is only a *sufficient* condition for the composition product to produce a locally convergent series. If in Theorem 2.4 it is instead assumed that both c and d have finite Lie rank, then the mappings F_c and F_d each have a finite dimensional analytic state space realization, and therefore so does the mapping $F_c \circ F_d$. The classical literature then provides that the generating series $c \circ d$ must be locally convergent [11]. An example of this situation is given below.

Example 3.1 Consider the state space system

$$
\begin{aligned}
\dot{z} &= z^2 u, \;\; z(0) = 1 \\
y &= z.
\end{aligned}
$$

It is easily verified that $y = F_c[u]$ where the only nonzero coefficients are $(c, \underbrace{1\cdots 1}_{k}) = k!$, $k \geq 0$. So c is not input-limited. But the mapping $F_{c\circ c}$ clearly has the analytic state space realization,

$$
\begin{aligned}
\dot{z}_1 &= z_1^2 u, \;\; z_1(0) = 1 \\
\dot{z}_2 &= z_2^2 z_1, \;\; z_2(0) = 1 \\
y &= z_2,
\end{aligned}
$$

and therefore the generating series $c \circ c$ must be locally convergent. □

4 The Feedback Connection

The general goal of this section is to determine when there exists a y which satisfies the feedback equation corresponding to Figure 1(d). In particular, when does there exist a generating series e so that $y = F_e[u]$ and

$$F_e[u] = F_c[u + F_{d\circ e}[u]].$$

The *feedback product* is then defined by $c@d = e$. To make the analysis simpler, it is assumed through out that the inputs u are supplied by an exosystem which is itself a Fliess operator as shown in Figure 2. That

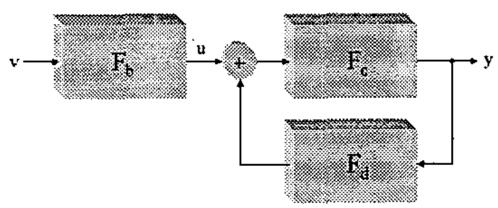

Figure 2: A feedback configuration with a Fliess operator exosystem providing the inputs.

is, $u = F_b(v)$ for some locally convergent series b. In this setting, a sufficient condition is given under which a unique solution y of the feedback equation is known to exist, and y is characterized as the output of a new Fliess operator when a certain series factorization property is available. This leads to an implicit characterization of the feedback product $c@d$.

Theorem 4.1 *Let $b, c, d \in \mathbb{R}\langle\langle I\rangle\rangle$ with $I = \{0, 1\}$. Then:*

1. *The mapping*

$$S \;:\; \mathbb{R}\langle\langle I\rangle\rangle \mapsto \mathbb{R}\langle\langle I\rangle\rangle$$
$$: \; \tilde{e}_i \mapsto \tilde{e}_{i+1} = c \circ (b + d \circ \tilde{e}_i)$$

has a unique fixed point $\tilde{e}$.

2. *If b, c, d and $\tilde{e}$ are locally convergent then the feedback equation has the unique solution $y = F_{\tilde{e}}[v]$ for any admissible v.*

3. *If $\tilde{e} = e \circ b$ for some locally convergent series e then $c@d = e$.*

Proof:

1. The mapping S is a contraction since via Theorem 2.3:

$$\begin{aligned}\mathrm{dist}(S(\tilde{e}_i), S(\tilde{e}_j)) &< \mathrm{dist}(b + d \circ \tilde{e}_i, b + d \circ \tilde{e}_j) \\ &= \mathrm{dist}(d \circ \tilde{e}_i, d \circ \tilde{e}_j) \\ &< \mathrm{dist}(\tilde{e}_i, \tilde{e}_j).\end{aligned}$$

Therefore, the mapping S has a unique fixed point $\tilde{e}$, that is,

$$\tilde{e} = c \circ (b + d \circ \tilde{e}).$$

2. From the stated assumptions concerning b, c, d and $\tilde{e}$ it follows that

$$\begin{aligned}F_{\tilde{e}}[v] &= F_{c\circ(b+d\circ\tilde{e})}[v] \\ &= F_c[F_b[v] + F_d[F_{\tilde{e}}[v]]]\end{aligned}$$

for any admissible v. Therefore the feedback equation has the unique solution $y = F_{\tilde{e}}[v]$.

3. Since e is locally convergent

$$y = F_{\tilde{e}}[v] = F_e[F_b[v]] = F_e[u],$$

thus $c@d = e$. ∎

This result suggests several open problems. Are there conditions on b, c, and d alone which will insure that $\tilde{e}$ above is locally convergent ? When does there exist a factorization of the form $\tilde{e} = e \circ b$, where e is locally convergent ? Can the theorem be generalized to the case where the inputs are simply from an L_p space and not filtered through a Fliess operator a priori ?

Example 4.1 Consider a *generalized series* δ with the defining property that δ is the identity element for the composition product, i.e., $c \circ \delta = \delta \circ c = c$ for any $c \in \mathbb{R}\langle\langle X\rangle\rangle$. Then (mapping the symbols $x_i \mapsto i$) $F_\delta[u] = u$ for any u, and a unity feedback system has the generating series $c@\delta$. Setting $b = 0$ in Figure 2 (or effectively setting $u \equiv 0$), a self-excited feedback loop is described by $F_{\tilde{e}}[v]$, where $\tilde{e} = c \circ \tilde{e}$ and $\tilde{e} = e \circ 0$. In this case $\tilde{e} = \lim_{k\to\infty} c^{\circ k}$ and $c@d|_{d=\delta} = e = \tilde{e} = \lim_{k\to\infty} c^{\circ k}$. From Theorem 4.1, $c@\delta$ is well defined in general. Analogous to the situation with the composition product, if c has finite Lie rank then $c@\delta$ will always be locally convergent. For example, when $c = 1 + x_1$ it is easy verified that $c@\delta = \sum_{k\geq 0} x_0^k$ so that $F_{c@\delta}[0](t) = e^t$ for $t \geq 0$. When $c = 1 + 2x_1 + 2x_1^2$ it follows that $c@\delta = \sum_{k\geq 0}(k+1)!\,x_0^k$ and $F_{c@\delta}[0](t) = 1/(1-t)^2$ for $0 \leq t < 1$. □

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