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An Algebraic Approach to Nonlinear Functional Expansions

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Abstract—**A new theory of functional expansion is presented which makes use of formal power series in several noncommutative variables and of iterated integrals. A simple closed-form expression for the solution of a nonlinear differential equation with forcing terms is derived, which enables us to give the corresponding Volterra kernels with utmost precision. The noncommutative variables give birth to a symbolic calculus which generalizes in a nonlinear setting many features of the Laplace and Fourier transforms and which is developed in order to simplify some computations, like the so-called association of variables, related to high-order transfer functions.**

Introduction

THE USE OF functional expansions in order to represent nonlinear systems has been energetically studied in the last thirty years, in engineering as well as in physics. Functional expansions have been applied in every branch of nonlinear system theory: identification and modelization, realization, stability, optimal control, stochastic differential equations and filtering, and so on. Until recently, almost all of the expansions used have been of the Volterra type or, in the stochastic case, of the Wiener type. There has been an enormous number of publications on these expansions. For simplicity let us mention here only the early works by Wiener [54], Barrett [2], and George [24], which are still worth reading, and the two recent books by Rugh [45] and Schetzen [48].

In this survey, we shall present an entirely new approach to functional expansions, using formal power series in several noncommutative variables. These formal powers series were introduced by Schützenberger [49] in computer science as a generalization of automata and formal languages. There were first used by [15] in 1973 to give a theory of realization for bilinear systems, which shows their relations to finite automata [17], [31], [44]. Since then,

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many papers have developed the theory in various directions. In this review, we shall show how to represent an input–output behavior by a noncommutative power series via the iterated integrals and we shall give the fundamental formula expressing the solution of a differential equation with forcing terms [19]. This simple formula completely explains the relationship between ordinary differential equations and functional expansions, something which has been sought for a long time. Using this formula, it is possible to determine the Taylor expansions of the corresponding Volterra kernels with utmost precision [19], [20].

Several methods for computing the Volterra functional representation corresponding to a state equation have been derived in the literature. Among them, the method of exponential inputs is particularly well known and allows the nonlinear transfer functions to be determined recursively. However, the computations involved are often unwieldy and seem difficult to implement on a computer. The use of noncommutative series allows us to derive the Volterra functional series of the solution of a large class of nonlinear forced differential equations [34], [35] by simple algebraic manipulations. In the last part, we present a new operational calculus for computing the response of nonlinear systems to various deterministic excitations. The calculus introduced compares advantageously with the method of association of variables [24]. It leads to a natural generalization of the well-known Heaviside symbolic calculus for time-invariant linear systems. This remarkable feature shows that we have introduced a kind of nonlinear Fourier–Laplace transform [19], [33]. We shall illustrate the computability of our methods by working out the case of a simple nonlinear electrical circuit. In this connection, we stress the fact that the algebraic and combinatorial nature of our approach fits very well with the symbolic computation methods which are now being developed in programming (see [35]). Some other applications related to physics have been studied elsewhere [22].

It is worth pointing out that our concepts are intimately related to the differential geometric methods which have become mainstays of nonlinear control theory, following the pioneering work by Hermann [29] and Lobry [38] (for the connections with Volterra series, see Brockett [6], Lesiak and Krener [36], and Crouch [11]). For a detailed explanation the reader is advised to consult the presentation of realization theory in [21].

I. Analytic Causal Functionals

A. Heuristic Introduction of the Noncommutative Variables

The following systems, called *bilinear* systems,[1] have been studied for some fifteen years:

$$\begin{cases} \dot{q}(t) = \left(M_0 + \sum_{i=1}^{m} u_i(t) M_i\right) q(t) \\ y(t) = \lambda q(t). \end{cases} \tag{I.1}$$

The state q belongs to a finite-dimensional $\boldsymbol{R}$-vector space Q; the initial state $q(0)$ is given. The mappings $M_0, M_1, \cdots, M_m$: $Q \to Q$, and λ: $Q \to \boldsymbol{R}$ are $\boldsymbol{R}$-linear. The inputs $u_1, \cdots, u_m$ are real-valued and, for the sake of simplicity, are assumed to be piece-wise continuous.

Thanks to the well-known Peano-Baker formula (cf. Gantmakher [23], p. 127), the output y of system (I.1) may be expressed in the following way:

$$y(t) = \lambda\left[1 + \sum_{\nu \geq 0} \sum_{j_0, \cdots, j_\nu = 0}^{m} M_{j_\nu} \cdots M_{j_0} \int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}\right] q(0). \tag{I.2}$$

The *iterated integral* $\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}$ is defined recursively on its length:

$$\xi_0(\tau) = \tau, \qquad \xi_i(\tau) = \int_0^\tau u_i(\sigma)\, d\sigma, \qquad i = 1, \cdots, m$$

$$\int_0^t d\xi_j = \xi_j(t), \qquad j = 0, 1, \cdots, m$$

$$\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0} = \int_0^t d\xi_{j_\nu}(\tau) \int_0^\tau d\xi_{j_{\nu-1}} \cdots d\xi_{j_0}$$

$$= \begin{cases} \int_0^t \left(\int_0^\tau d\xi_{j_{\nu-1}} \cdots d\xi_{j_0}\right) d\tau, & \text{if } j_\nu = 0 \\ \int_0^t \left(\int_0^\tau d\xi_{j_{\nu-1}} \cdots d\xi_{j_0}\right) u_i(\tau)\, d\tau, & \text{if } j_\nu = i. \end{cases}$$

Now introduce the finite set $X = \{x_0, x_1, \cdots, x_m\}$. Replacing each iterated integral $\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}$ in (I.2) by the corresponding sequence $x_{j_\nu} \cdots x_{j_0}$ we obtain the following expression:

$$g = \lambda q(0) + \sum_{\nu \geq 0} \sum_{j_0, \cdots, j_\nu = 0}^{m} \lambda M_{j_\nu} \cdots M_{j_0} q(0) x_{j_\nu} \cdots x_{j_0}. \tag{I.3}$$

This is a series in the noncommutative variables $x_j \in X$. The noncommutativity of the variables reflects the fact that the permutation of indices in an iterated integral changes in general its numerical value.

Example: Taking $u_1(t) = t^\alpha$ we find

$$\int_0^t d\xi_0\, d\xi_1 = \frac{t^{\alpha+2}}{(\alpha+2)(\alpha+1)} \quad \text{and} \quad \int_0^t d\xi_1\, d\xi_0 = \frac{t^{\alpha+2}}{\alpha+2}.$$

They differ iff $\alpha \neq 0$.

The series (I.3) characterizes the input–output behavior of system (I.1) in exactly the same way as do rational transfer functions for time-invariant linear systems.

B. Some Elementary Algebraic Structures

The algebraic structures that have appeared in the foregoing paragraph will now be defined more precisely.

The finite set $X = \{x_0, x_1, \cdots, x_m\}$ is called the *alphabet*. It generates the *free monoid* X^*, the elements of which are finite sequences $x_{j_\nu} \cdots x_{j_0}$ called *words*. The product is the *concatenation*:

$$(x_{j_\nu} \cdots x_{j_0})(x_{k_\mu} \cdots x_{k_0}) = x_{j_\nu} \cdots x_{j_0} x_{k_\mu} \cdots x_{k_0}$$

[1] They are also called *internally bilinear* systems, *regular* systems, or *affine* systems.

The neutral element is called the *empty word* and is denoted by 1.

The *length* $|w|$ of a word $w \in X$ is its number of letters: $|x_0 x_1^3 x_0| = 5$. The length of the empty word 1 is zero.

Let $\boldsymbol{R}\langle X \rangle$ and $\boldsymbol{R}\langle\langle X \rangle\rangle$ be the $\boldsymbol{R}$-algebras of formal polynomials and formal power series with real coefficients and in the noncommutative variables $x_j \in X$. A series $s \in \boldsymbol{R}\langle\langle X \rangle\rangle$ is noted

$$s = \sum \{(s,w)w | w \in X^*\}, \qquad \text{where } (s,w) \in \boldsymbol{R}.$$

A polynomial is a series with at most a finite number of nonzero coefficients (s,w). Sum and (Cauchy) product are defined by

$$s_1 + s_2 = \sum \{[(s_1,w) + (s_2,w)] w | w \in X^*\}$$

$$s_1 s_2 = \sum \left\{ \left[\sum_{uv = w} (s_1,u)(s_2,v) \right] w | w \in X^* \right\}.$$

C. Definition of the Functionals and of the Generating Series

Under some condition of convergence, we associate with each noncommutative series $g \in \boldsymbol{R}\langle\langle X \rangle\rangle$ a causal functional (i.e., an input–output behavior) by replacing each word $x_{j_\nu} \cdots x_{j_0}$ by the corresponding iterated integral $\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}$ (cf. Section I-A):

$$y(t; u_1, \cdots, u_m) = (g,1) + \sum_{\nu \geq 0} \sum_{j_0, \cdots, j_\nu = 0}^{m} (g, x_{j_\nu} \cdots x_{j_0}) \int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}. \tag{I.4}$$

We assume that the series (I.4) is absolutely convergent for t and $\max_{0 \leq \tau \leq t} |u_i(\tau)|$ $(i = 1, \cdots, m)$ sufficiently "small". This is verified if the coefficients satisfy the following growth condition: there exist $K, L > 0$ such that

$$|(g,w)| < K |w|! L^{|w|}. \tag{I.5}$$

It can be shown that to distinct series of $\boldsymbol{R}\langle\langle X \rangle\rangle$ there correspond distinct functionals. Therefore, the following definition is valid. A causal functional is said to be *analytic* if, and only if, it is defined as in formula (I.4) by a noncommutative formal power series which is called the *generating* series of the functional.

Remarks. (i) Causality means only that the value $y(t)$ of the functional depends of the inputs $u_i(\tau)$ for $\tau \leq t$.

(ii) The notion of an analytic functional has already been given various meanings in the literature (cf. Volterra [52], Hille and Philips [30], etc).

(iii) If the output is vector-valued in $\boldsymbol{R}^s$, one should take an s-uple of generating series.

The notion of analytic causal functional generalizes in some sense the notion of an analytic function. In order to illustrate this assertion, consider a generating series $g \in \boldsymbol{R}\langle\langle X \rangle\rangle$ which is *exchangeable*: if $\mathfrak{S}_\nu$ denotes the symmetric group acting on the set $\{0, 1, \cdots, \nu\}$, g is exchangeable iff for any $\sigma \in \mathfrak{S}_\nu$, $(g, x_{j_{\sigma(\nu)}} \cdots x_{j_{\sigma(0)}}) = (g, x_{j_\nu} \cdots x_{j_0})$, i.e., iff the coefficients do not depend on the order of the variables. A simple computation then shows that

$$\sum_{\sigma \in \mathfrak{S}_\nu} \int_0^t d\xi_{j_{\sigma(\nu)}} \cdots d\xi_{j_{\sigma(0)}} = \xi_{j_\nu}(t) \cdots \xi_{j_0}(t).$$

The value of the functional is given by

$$\begin{aligned} &\sum_{k_0, k_1, \cdots, k_m \geq 0} (g, x_0^{k_0} x_1^{k_1} \cdots x_m^{k_m}) \xi_0(t)^{k_0} \xi_1(t)^{k_1} \cdots \xi_m(t)^{k_m} \\ &= \sum_{k_0, k_1, \cdots, k_m \geq 0} (g, x_0^{k_0} x_1^{k_1} \cdots x_m^{k_m}) t^{k_0} \xi_1(t)^{k_1} \cdots \xi_m(t)^{k_m}. \end{aligned} \tag{I.6}$$

This is the classical Taylor expansion of an analytic function in the "ordinary" variables $\xi_0(t) = t, \xi_1(t), \cdots, \xi_m(t)$.

D. Rationality

A formal power series $s \in \boldsymbol{R}\langle\langle X \rangle\rangle$ is *invertible* if there exists $s^{-1} \in \boldsymbol{R}\langle\langle X \rangle\rangle$ such that $ss^{-1} = s^{-1}s = 1$. This is the case iff the *constant term* $(s,1)$ is not zero. We can then write

$$s = (s,1)[1 - s']$$

where $s' \in \boldsymbol{R}\langle\langle X \rangle\rangle$ and $(s',1) = 0$. We thus obtain

$$s^{-1} = \frac{1}{(s,1)}[1 - s']^{-1} = \frac{1}{(s,1)} \sum_{k \geq 0} s'^k.$$

A subalgebra R of $\boldsymbol{R}\langle\langle X \rangle\rangle$ is said to be *rationally closed* iff the inverse s^{-1} of any invertible $s \in R$ again belongs to R.

The $\boldsymbol{R}$-algebra $\boldsymbol{R}\langle\langle X \rangle\rangle$ of noncommutative *rational* power series is the least rationally closed subalgebra of $\boldsymbol{R}\langle\langle X \rangle\rangle$ which contains the polynomials $\boldsymbol{R}\langle X \rangle$.[2] This means that any $r \in \boldsymbol{R}\langle(X)\rangle$ is obtained from a finite set of polynomials of $\boldsymbol{R}\langle X \rangle$ by making a finite number of additions, (Cauchy) multiplications, and inversions in a given order.

The fundamental property of the rational power series is the so-called Kleene–Schützenberger theorem. If End(Q) denotes the set of endomorphisms (i.e., of $\boldsymbol{R}$-linear mappings $Q \to Q$) of a $\boldsymbol{R}$-vector space Q, a (linear) *representation* μ: $X^* \to \text{End}(Q)$ is a homomorphism of the free monoid X^* in the multiplicative monoid of End(Q), i.e., if $w, w' \in X^*$, $\mu(ww') = \mu(w)\mu(w')$.

Theorem I.1. (Kleene–Schützenberger, cf. [49], [16], [44]). A formal power series $r \in \boldsymbol{R}\langle\langle X \rangle\rangle$ is rational if, and only if, there exist a finite-dimensional $\boldsymbol{R}$-vector space Q, an element $\gamma \in Q$, and a $\boldsymbol{R}$-linear mapping d: $Q \to \boldsymbol{R}$ such that

$$r = \sum \{(\lambda \mu(w) \gamma) w | w \in X^*\}.$$

It is then obvious that the series (I.3) is rational and we can state:

Corollary I.2. A generating power series corresponds to a finite-dimensional bilinear system if, and only if, it is rational.

This correspondence is analogous to that between finite-dimensional time-invariant linear systems and rational

[2] The notion of noncommutative rational power series is due to Schützenberger [49].

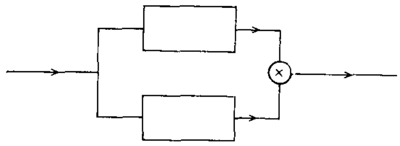

Fig. 1.

transfer functions. This gives a realization theory for bilinear systems [17], [31], [44] which makes use of an extended Hankel matrix [16].

E. Shuffle Product

Take two systems with the same inputs and consider the product of the outputs, i.e., the multiplicative parallel connection (Fig. 1). If the two systems may be defined by generating power series, is the same true of the composite system?

First we have to introduce a new and very important operation called the *shuffle product*, written $\sqcup\!\sqcup$. We define the shuffle product of two words of X recursively on the length:

$$1 \sqcup\!\sqcup 1 = 1, \qquad \forall w \in X^*, \qquad w \sqcup\!\sqcup 1 = 1 \sqcup\!\sqcup w = w$$
$$\forall x_j, x_{j'} \in X, \qquad \forall w, w' \in X^*$$
$$(x_j w) \sqcup\!\sqcup (x_j' w') = x_j\left[w \sqcup\!\sqcup (x_j' w')\right] + x_j'\left[(x_j w) \sqcup\!\sqcup w'\right]. \tag{I.7}$$

The shuffle product of two words gives a homogeneous polynomial of degree equal to the sum of the lengths of the words.

Examples: (i)

$$x^k \sqcup\!\sqcup x^{n-k} = \binom{n}{k} x^n \qquad (n \geqslant k).$$

(ii) $x_0 x_1 \sqcup\!\sqcup x_2 = x_0 x_1 x_2 + x_0 x_2 x_1 + x_2 x_0 x_1$.

The shuffle product of two series $s_1, s_2 \in \boldsymbol{R}\langle\langle X \rangle\rangle$ is defined by

$$s_1 \sqcup\!\sqcup s_2 = \sum \{(s_1, w_1)(s_2, w_2) w_1 \sqcup\!\sqcup w_2 | w_1, w_2 \in X^*\}.$$

With the addition and this new product, the sets $\boldsymbol{R}\langle X \rangle$ and $\boldsymbol{R}\langle\langle X \rangle\rangle$ become commutative, integral $\boldsymbol{R}$-algebras with unit 1.

Remark: For the reader who is more mathematically oriented, let us add that the shuffle product may be viewed as a very natural operation in the framework of *Hopf algebras* (cf. Sweedler [50]).

The well-known formula for integration by parts allows us to write the product of two iterated integrals in the following way:

$$\left(\int_0^t d\xi_{j_\nu} \cdots d\xi_{j_0}\right)\left(\int_0^t d\xi_{k_\mu} \cdots d\xi_{k_0}\right)$$
$$= \int_0^t d\xi_{j_\nu}(\tau)\left[\left(\int_0^\tau d\xi_{j_{\nu-1}} \cdots d\xi_{j_0}\right)\left(\int_0^\tau d\xi_{k_\mu} \cdots d\xi_{k_0}\right)\right]$$
$$+ \int_0^t d\xi_{k_\mu}(\tau)\left[\left(\int_0^\tau d\xi_{j_\nu} \cdots d\xi_{j_0}\right)\left(\int_0^\tau d\xi_{k_{\mu-1}} \cdots d\xi_{k_0}\right)\right].$$

This corresponds to (I.7) and we may state the following result, essentially due to Ree [43]:

Theorem I.3. The product of two analytic causal functionals is again an analytic causal functional the generating series of which is the shuffle product of the two generating series.

Remarks: (i) One can show that the shuffle product of two rational power series is again rational [19]. Therefore, the product in the foregoing sense of two bilinear systems is again bilinear.

(ii) The additive parallel connection of two systems described by generating series is obviously given by the sum of the series. The cascade connection may also be computed in terms of noncommutative variables (cf. Ferfera [13]).

II. Functional Expansions of the Solutions of Differential Equations with Forcing Terms

A. The Fundamental Formula

The search for a functional expansion of solutions of differential equations with forcing terms is a classical problem in engineering as well as in physics. Since previous attempts have almost exclusively used Volterra series, a brief analysis of the existing literature is postponed until the next chapter.

Consider the *control-linear* system

$$\begin{cases} \dot{q}(t) = A_0(q) + \sum_{i=1}^{m} u_i(t) A_i(q) \\ y(t) = h(q). \end{cases} \tag{II.1}$$

The state q belongs to a finite-dimensional $\boldsymbol{R}$-analytic *manifold* Q.[3] The vector fields $A_0, A_1, \cdots, A_m$: $Q \to TQ$ (TQ: tangent bundle), the output function h: $Q \to \boldsymbol{R}$ are analytic and defined in a neighborhood of the initial state $q(0)$. Readers who have not worked with vector fields will understand what this means in the language of a local coordinates chart $q = (q^1, \cdots, q^N)$. The vector field $A_j(q)$ is then a first-order linear differential operator

$$A_j(q) = \sum_{k=1}^{N} \theta_j^k(q^1, \cdots, q^N) \frac{\partial}{\partial q^k}, \qquad j = 0, 1, \cdots, m$$

where the θ_j^k: $\boldsymbol{R}^N \to \boldsymbol{R}$ are analytic in a neighborhood of $q^1(0), \cdots, q^N(0)$. The first line of (II.1) may be rewritten in the more classical form

$$\dot{q}^k(t) = \theta_0^k(q^1, \cdots, q^N) + \sum_{i=1}^{m} u_i(t) \theta_i^k(q^1, \cdots, q^N),$$
$$k = 1, \cdots, N. \tag{II.2}$$

The next theorem generalizes Gröbner's work [26], [27] on free differential equations.

[3] If the reader is not familiar with differential geometry, he will not lose very much of the meaning by assuming that the state belongs to some $\boldsymbol{R}^N$. But, since we do need the modern notion of *vector fields*, we refer to the numerous textbooks on differentiable geometry, for example, Arnold [1].

Theorem II.1. The output y of system (II.1) is an analytic causal functional of the inputs $u_1,\cdots,u_n$ given by the generating series

$$\boxed{\mathfrak{g} = h|_{q(0)} + \sum_{\nu\geq 0}\sum_{j_0,\cdots,j_\nu=0}^{m} A_{j_0}\cdots A_{j_\nu}h|_{q(0)}x_{j_\nu}\cdots x_{j_0}.} \tag{II.3}$$

The bar $|_{q(0)}$ indicates evaluation at $q(0)$. Taking into account the fact that vector fields are differential operators, the notation $A_{j_0}\cdots A_{j_\nu}h$ means the iterated derivative of h with respect to $A_{j_\nu},\cdots,A_{j_0}$.[4] The inverse order of the sequences $x_{j_\nu}\cdots x_{j_0}$ and $A_{j_0}\cdots A_{j_\nu}$ should be noted.

Because of its importance, we will call (II.3) the *fundamental formula*. Before giving its proof, we will illustrate it by some examples.

Remark: In this survey paper we consider only continuous-time systems. Normand–Cyrot [41] has recently introduced a discrete-time analog of the fundamental formula which gives birth to as rich a theory as in the continuous-time case.

B. Examples

(i) Free Differential Systems

With Gröbner [26], [27], consider a system without inputs

$$\begin{cases} \dot q(t) = A_0(q) \\ y(t) = h(q). \end{cases}$$

Using formula (II.3), we find

$$y(t) = \sum_{\alpha\geq 0} A_0^\alpha h|_{q(0)}\frac{t^\alpha}{\alpha!} = e^{tA_0}h|_{q(0)}.$$

This is a familiar formula in the case of a linear differential system.

(ii) Commutative Vector Fields

Suppose that in system (II.1) the vector fields commute, i.e., that the corresponding differential operators commute: $A_jA_{j'} = A_{j'}A_j$. This is usually expressed by writing that the *Lie brackets* $[A_j, A_{j'}] = A_jA_{j'} - A_{j'}A_j$ are zero. Formula (II.3) says that the generating series $\mathfrak{g}$ is exchangeable (cf. Section I.C) and (I.6) becomes

$$y(t) = \sum_{k_0,k_1,\cdots,k_m} A_0^{k_0}A_1^{k_1}\cdots A_m^{k_m}h|_{q(0)} \cdot \frac{t^{k_0}\xi_1(t)^{k_1}\cdots\xi_m(t)^{k_m}}{k_0!k_1!\cdots k_m!}$$

(cf. Gröbner [26], [27]). No use of iterated integrals is then required.

(iii) Time-Varying Systems

Consider the system

$$\begin{cases} \dot q(t) = B_0(t,q) + \sum_{i=1}^{m} u_i(t)B_i(t,q) \\ y(t) = b(t,q) \end{cases}$$

which is identical to (II.1) except that the vector fields $B_0, B_1,\cdots,B_m$, and the output function b may depend analytically on time. As usual we reduce it to an autonomous system by adding one dimension q^0

$$\begin{cases} \dot q^0(t) = 1, \qquad q^0(0) = 0 \\ \dot q(t) = B_0(q^0,q) + \sum_{i=1}^{m} u_i(t)B_i(q^0,q) \\ y(t) = b(q^0,q). \end{cases}$$

By applying (II.3), we are led to introduce the new vector fields

$$C_0(t,q) = \frac{\partial}{\partial t} + B_0(t,q)$$
$$C_i(t,q) = B_i(t,q), \qquad i = 1,\cdots,m.$$

The corresponding generating series is

$$\mathfrak{g} = b|_{\bar q(0)} + \sum_{\nu\geq 0}\sum_{j_0,\cdots,j_\nu=0}^{m} C_{j_0}\cdots C_{j_\nu}b|_{\bar q(0)}x_{j_\nu}\cdots x_{j_0}$$

where $\bar q(0) = (0, q(0))$.

(iv) Bilinear Systems

Consider again the bilinear system (I.1). We choose a basis for the finite-dimensional vector space Q where $q = (q^1,\cdots,q^N)$. The corresponding vector fields are

$$A_j(q) = (q^1,\cdots,q^N)\,{}^tM_j \begin{pmatrix} \dfrac{\partial}{\partial q^1} \\ \vdots \\ \dfrac{\partial}{\partial q^N} \end{pmatrix}, \qquad j = 0,1,\cdots,m$$

where ${}^tM_j$ is the transpose matrix. Writing $\lambda = (\lambda_1,\cdots,\lambda_N)$, the output function h is $h(q) = \lambda_1q^1 + \cdots + \lambda_Nq^N$. Because of the transposition, (II.3) gives once again the series (I.3).

C. Proof

The proof of the fundamental formula is essentially algebraic and generalizes that of Gröbner [26], [27] on free differential systems. We will only sketch it and omit the part concerning the convergence which uses the same majoring series as Gröbner. For more details, see [19].

First consider $m+1$ formal vector fields (i.e., formal first-order linear differential operators)

$$A_j = \sum_{k=1}^{N}\theta_j^k(q^1,\cdots,q^N)\frac{\partial}{\partial q^k}, \qquad j = 0,1,\cdots,m$$

where the $\theta_j^k \in \boldsymbol{R}[[q^1,\cdots,q^N]]$ are formal power series with real coefficients, in the commutative variables $q^1,\cdots,q^N$. The A_j's act in an obvious way on any element of $\boldsymbol{R}[[q^1,\cdots,q^N]]$. We must now introduce the $\boldsymbol{R}$-algebra $\boldsymbol{R}[[q^1,\cdots,q^N]]\langle\langle X\rangle\rangle$, i.e., the set of formal power series in the noncommutative variables $x_j \in X$, but with coefficients

[4]The derivative with respect to a vector field is called a *Lie derivative* in differential geometry.

in $\boldsymbol{R}[[q^1,\cdots,q^N]]$. Addition and shuffle product (and also the Cauchy product) are defined in a completely analogous way.

Define the application Λ: $\boldsymbol{R}[q^1,\cdots,q^N]]\to\boldsymbol{R}[[q^1,\cdots,q^N]]\langle\langle X\rangle\rangle$ by

$$h\mapsto\Lambda(h)=h+\sum_{\nu\geqslant 0}\sum_{j_0,\cdots,j_\nu=0}^{m}A_{j_0}\cdots A_{j_\nu}hx_{j_\nu}\cdots x_{j_0}$$

where the right-hand side is a kind of noncommutative *Lie series*.

Lemma II.2. If $h_1,h_2\in\boldsymbol{R}[[q^1,\cdots,q^N]]$, $\alpha_1,\alpha_2\in\boldsymbol{R}$ one can write

$$\Lambda(\alpha_1h_1+\alpha_2h_2)=\alpha_1\Lambda(h_1)+\alpha_2\Lambda(h_2)$$
$$\Lambda(h_1h_2)=\Lambda(h_1)ш\Lambda(h_2). \quad (II.4)$$

Proof: Only the equality (II.4) requires a proof. It follows from

$$A_{j_0}\cdots A_{j_\nu}(h_1h_2)=A_{j_0}\cdots A_{j_{\nu-1}}(h_1A_{j_\nu}h_2+h_2A_{j_\nu}h_1)$$

which reduces to (I.7) defining the shuffle product. ●

As an application, consider an element of $\boldsymbol{R}[[q^1,\cdots,q^N]]$

$$h=\sum_{\alpha_1,\cdots,\alpha_N\geqslant 0}h_{\alpha_1,\cdots,\alpha_n}(q^1)^{\alpha_1}\cdots(q^N)^{\alpha_N}.$$

We get

$$\Lambda(h)=\sum_{\alpha_1,\cdots,\alpha_N\geqslant 0}h_{\alpha_1,\cdots,\alpha_n}[\Lambda(q^1)]^{ш\alpha_1}ш\cdotsш[\Lambda(q^N)]^{ш\alpha_n} \quad (II.5)$$

where $ш\,\alpha_k$ means the power α_k with respect to the shuffle product.

Take local coordinates as in (II.2) and set $(k=1,\cdots,N)$

$$\dot{q}^k(t)=q^k(0)+\sum_{\nu\geqslant 0}\sum_{j_0,\cdots,j_\nu=0}^{m}A_{j_0}\cdots A_{j_\nu}q^k|_{q(0)}\cdot\int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0} \quad (II.6)$$

$$y(t)=h|_{q(0)}+\sum_{\nu\geqslant 0}\sum_{j_0,\cdots,j_\nu=0}^{m}A_{j_0}\cdots A_{j_\nu}h|_{q(0)}\int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}. \quad (II.7)$$

We have to show that the quantities q^k and y defined by (II.6) and (II.7) do satisfy the equations (II.1). Because of formula (II.5), it is clear that $y(t)=h(q^1(t),\cdots,q^k(t))$.

Consider now the derivative with respect to time of an iterated integral. We get

$$\frac{d}{dt}\int_0^t d\xi_{j_\nu}\cdots d\xi_{j_0}=\begin{cases}\int_0^t d\xi_{j_{\nu-1}}\cdots d\xi_{j_0}, & \text{if } j_0=0\\ u_i(t)\int_0^t d\xi_{j_{\nu-1}}\cdots d\xi_{j_0}, & \text{if } j_\nu=i\in\{1,\cdots,m\}.\end{cases}$$

We are now able to derive the two members of (II.6)

$$\dot{q}^k(t)=\left[\theta_0^k|_{q(0)}+\sum_{\alpha\geqslant 0}\sum_{j_0,\cdots,j_\alpha=0}^{m}A_{j_0}\cdots A_{j_\alpha}\theta_0^k|_{q(0)}\cdot\int_0^t d\xi_{j_\alpha}\cdots d\xi_{j_0}\right]$$
$$+\sum_{i=1}^{m}u_i(t)\left[\theta_i^k|_{q(0)}+\sum_{\alpha\geqslant 0}\sum_{j_0,\cdots,j_\alpha=0}^{m}A_{j_0}\cdots A_{j_\alpha}\theta_i^k|_{q(0)}\cdot\int_0^t d\xi_{j_\alpha}\cdots d\xi_{j_0}\right].$$

Once again, because of formula (II.5), we obtain

$$\dot{q}^k(t)=\theta_0^k(q^1,\cdots,q^N)+\sum_{i=1}^{m}u_i(t)\theta_i^k(q^1,\cdots,q^N).$$

This is the required expressions. ●

D. A Noncommutative Symbolic Calculus

The practical computation of functional expansions and, in particular of the generating series, is not a simple matter. We propose here a recursive determination of the coefficients by a fixed-point method.

Equations (II.2) are equivalent to

$$q^k(t)=q^k(0)+\int_0^t\theta_0^k(q^1(\tau),\cdots,q^N(\tau))\,d\tau+\sum_{i=1}^{m}\int_0^t u_i(\tau)\theta_i^k(q^1(\tau),\cdots,q^N(\tau))\,d\tau.$$

By hypothesis, the θ_j^k and the output function h are expandable in Taylor series around the initial state $q(0)$ which we may take as the origin of the coordinates. To a function such as

$$a(q^1,\cdots,q^N)=\sum_{\alpha_1,\cdots,\alpha_N\geqslant 0}a_{\alpha_1,\cdots,\alpha_N}(q^1)^{\alpha_1}\cdots(q^N)^{\alpha_N}$$

associate

$$\overset{ш}{a}(s^1,\cdots,s^N)=\sum_{\alpha_1,\cdots,\alpha_N\geqslant 0}a_{\alpha_1,\cdots,\alpha_N}(s^1)^{ш\alpha_1}ш\cdotsш(s^N)^{ш\alpha_N}$$

where $s^1,\cdots,s^N\in\boldsymbol{R}\langle\langle X\rangle\rangle$ are power series with constant terms such that $\Sigma a_{\alpha_1,\cdots,\alpha_N}(s^1,1)^{\alpha^1}\cdots(s^N,1)^{\alpha_N}$ is convergent.

The very meaning of the variables $x_0,x_1,\cdots,x_m$ with respect to the iterated integrals shows that system (II.1) is equivalent to

$$\begin{cases}\mathrm{g}^k=q^k(0)+x_0\overset{ш}{\theta_0^k}(\mathrm{g}^1,\cdots,\mathrm{g}^N)+\sum_{i=1}^{m}x_i\overset{ш}{\theta_i^k}(\mathrm{g}^1,\cdots,\mathrm{g}^N)\\ \mathrm{g}=\overset{ш}{h}(\mathrm{g}^1,\cdots,\mathrm{g}^N).\end{cases} \quad (II.8)$$

There exists one and only one $(N+1)$-uple $\mathrm{g}^1,\cdots,\mathrm{g}^N,\mathrm{g}$ solution of (II.8), which is obtained by iterations.[5]

[5]In the $\boldsymbol{R}$-algebra $\boldsymbol{R}\langle\langle X\rangle\rangle$, consider the two-sided ideal (X) of the power series with zero constant terms. The sequence $(X),(X)^2,\cdots$, defines a fundamental set of neighborhoods of zero and thus a topology where $\boldsymbol{R}\langle\langle X\rangle\rangle$ is a complete metrizable space. System (II.8) defines a contraction and it is then possible to apply the fixed-point theorem (for more details, see [19]).

The following examples will, we hope, illuminate these rather terse explanations. They will also be employed in Section IV.

Examples:

(i) Consider a linear stationary system

$$\begin{cases} \dot{q}(t) = Fq(t) + \sum_{i=1}^{m} u_i(t) G_i \\ y(t) = Hq(t). \end{cases}$$

The state q belongs to a finite-dimensional $\boldsymbol{R}$-vector space Q and $q(0) = 0$. The G_i are elements of Q; the mappings F: $Q \to Q$, H: $Q \to \boldsymbol{R}$ are $\boldsymbol{R}$-linear. With the foregoing notations, we get

$$q(t) = F\int_0^t q(\tau)\, d\tau + \sum_{i=1}^{m} G_i \int_0^t u_i(\tau)\, d\tau$$

and, if $q = (q^1, \cdots, q^N)$

$$\begin{cases} {}^t(\mathfrak{g}^1, \cdots, \mathfrak{g}^N) = Fx_0 {}^t(\mathfrak{g}^1, \cdots, \mathfrak{g}^N) + \sum_{i=1}^{m} G_i x_i \\ \mathfrak{g} = H^t(\mathfrak{g}^1, \cdots, \mathfrak{g}^N). \end{cases}$$

It follows

$$(1 - Fx_0)^t(\mathfrak{g}i^1, \cdots, \mathfrak{g}^N) = \sum_{i=1}^{m} G_i x_i$$

and

$$\mathfrak{g} = H(1 - Fx_0)^{-1}\left(\sum_{i=1}^{m} G_i x_i\right).$$

Since we know that the matrix transfer function is

$$H(p - F)^{-1} {}^t(G_1, \cdots, G_m)$$

we see that the generating series is equivalent to it up to an elementary change of variables.

(ii) Take again the bilinear system (I.1). We get

$$q(t) = q(0) + \int_0^t \left(M_0\, d\tau + \sum_{i=1}^{m} u_i(\tau) M_i\, d\tau \right) q(\tau)$$

whence

$$\begin{cases} {}^t(\mathfrak{g}^1, \cdots, \mathfrak{g}^N) = q(0) + \left(M_0 x_0 + \sum_{i=1}^{m} M_i x_i \right) {}^t(\mathfrak{g}^1, \cdots, \mathfrak{g}^N) \\ \mathfrak{g} = \lambda\, {}^t(\mathfrak{g}^1, \cdots, \mathfrak{g}^N) \end{cases}$$

and

$$\begin{aligned} \mathfrak{g} &= \lambda \left(1 - \sum_{j=0}^{m} M_j x_j \right) q(0) \\ &= \lambda q(0) + \sum_{\nu \geqslant 0} \sum_{j_0, \cdots, j_\nu = 0} \lambda M_{j_\nu} \cdots M_{j_0} q(0) x_{j_\nu} \cdots x_{j_0} \end{aligned}$$

which is again the series (I.3).

(iii) Consider the Duffing equation, which is well known in nonlinear mechanics

$$\ddot{y}(t) + a\dot{y}(t) + \omega^2 y(t) + by^3(t) = u_1(t).$$

The corresponding integral equation is

$$\begin{aligned} y(t) &+ a\int_0^t y(\tau)\, d\tau + \omega^2 \int_0^t d\tau \int_0^\tau y(\sigma)\, d\sigma \\ &+ b\int_0^t d\tau \int_0^\tau y^3(\sigma)\, d\sigma \\ &= \int_0^t d\tau \int_0^\tau u_1(\sigma)\, d\sigma + \beta t + \alpha \end{aligned}$$

where α and β are two constants depending on the initial conditions: $\alpha = y(0)$, $\beta = \dot{y}(0) + ay(0)$. Hence the generating series of y satisfies

$$\mathfrak{g} + ax_0 \mathfrak{g} + \omega^2 x_0^2 \mathfrak{g} + bx_0^2(\mathfrak{g} ш \mathfrak{g} ш \mathfrak{g}) = x_0 x_1 + \beta x_0 + \alpha.$$

Remarks: (i) Practical computations will be given in Section IV.

(ii) In [22], one will find an application of the noncommutative symbolic calculus to give an explanation of heuristic computations made by physicists by means of the Fourier transform in a nonlinear setting. This confirms the fact that the noncommutative variables do generalize some properties of the Fourier-Laplace transforms.

III. Volterra Series

A. Definition and Relationship with Generating Series

Volterra series are today the most popular tool when one deals with functional expansions. Recall that Volterra introduced at the end of the last century the following *regular* functional expansions (cf. Volterra [52], Lévy [37]):

$$\begin{aligned} k_0 &+ \int_a^b k_1(\tau_1) u(\tau_1)\, d\tau_1 \\ &+ \int_a^b \int_a^b k_2(\tau_2, \tau_1) u(\tau_2) u(\tau_1)\, d\tau_2\, d\tau_1 \\ &+ \cdots + \int_a^b \cdots \int_a^b k_s(\tau_s, \cdots, \tau_1) \\ &\cdot u(\tau_2) \cdots u(\tau_1)\, d\tau_s \cdots d\tau_1 + \cdots \end{aligned} \tag{III.1}$$

where the limits of integration are fixed once and for all.

In order to take into account the time evolution, a time-varying integration limit is introduced and one gets the so-called *Volterra series*

$$\begin{aligned} y(t; u_1) &= w_0(t) + \int_0^t w_1(t, \tau_1) u(\tau_1)\, d\tau_1 \\ &+ \int_0^t \int_0^{\tau_2} w_2(t, \tau_2, \tau_1) u(\tau_2) u(\tau_1)\, d\tau_2\, d\tau_1 \\ &+ \cdots + \int_0^t \int_0^{\tau_s} \cdots \int_0^{\tau_2} w_s(t, \tau_s, \cdots, \tau_1) \\ &\cdot u(\tau_s) \cdots u(\tau_1)\, d\tau_s \cdots d\tau_1 + \cdots . \end{aligned} \tag{III.2}$$

There are several ways to define the kernels w_s:

—In (III.2), we used the triangular version where $t \geqslant \tau_s \geqslant \cdots \geqslant \tau_2 \geqslant \tau_1 \geqslant 0$,

—A very current way is to use symmetric kernels

$$y(t;u_1)=w_0'(t)+\int_0^t w_1'(t,\tau_1)u(\tau_1)\,d\tau_1 + \int_0^t\int_0^t w_2'(t,\tau_2,\tau_1)u(\tau_2)u(\tau_1)\,d\tau_2\,d\tau_1 + \cdots + \int_0^t\int_0^t\cdots\int_0^t w_s'(t,\tau_s,\cdots,\tau_1) \cdot u(\tau_s)\cdots u(\tau_1)\,d\tau_s\cdots d\tau_1+\cdots$$

where w_s' is a symmetric function of the variables $\tau_1,\cdots,\tau_s$. The passage from one form to the other is classic.

Remark: As usual the input is assumed, for simplicity, to be one dimensional.

Roughly speaking the study of Volterra series may be divided into three main classes: multidimensional Laplace–Fourier transforms (see Sections III-D and IV for details and references), differential geometric methods (see Introduction), and approaches related to functional analysis (see, e.g., Halme [28], Gilbert [25], De Figueirodo and Dwyer [12], Sandberg [46], [47]). Among the numerous attempts to apply Volterra series let us quote the recent and original ones to nonlinear oscillations and Hopf bifurcations (Chua and Tang [10], Tang, Mees, and Chua [51]).

In fact there is a closed and perhaps unexpected relationship between Volterra series and generating series.

Theorem III.1. The triangular Volterra series (III.1) defines an analytic causal functional if, and only if, for any $s\geqslant 0$, the kernel $w_s(t,\tau_s,\cdots,\tau_1)$ is an analytic function of the variables $\tau_1,\cdots,\tau_s,t$ in a neighborhood of the origin, such that the radius of convergence of $w_0,w_1,\cdots,w_s,\cdots$, are bounded from below by a strictly positive quantity.

Proof: Expand $w_s(t,\tau_s,\cdots,\tau_1)$ not with respect to the variables $t,\tau_s,\cdots,\tau_1$ but $t-\tau_s,\tau_s-\tau_{s-1},\cdots,\tau_2-\tau_1,\tau_1$:

$$w_s(t,\tau_s,\cdots,\tau_1) = \sum_{\alpha_0,\alpha_1,\cdots,\alpha_s\geqslant 0} w_{s,\alpha_0,\cdots,\alpha_s}\frac{(t-\tau_s)^{\alpha_s}\cdots(\tau_2-\tau_1)^{\alpha_1}\tau_1^{\alpha_0}}{\alpha_s!\cdots\alpha_1!\alpha_0!}. \tag{III.3}$$

A simple computation shows that to the word $x_0^{\alpha_s}x_1\cdots x_0^{\alpha_1}x_1x_0^{\alpha_0}$ corresponds the multiple integral

$$\int_0^t\int_0^{\tau_s}\cdots\int_0^{\tau_2}\frac{(t-\tau_s)^{\alpha_s}u(\tau_s)\cdots(\tau_2-\tau_1)^{\alpha_1}u(\tau_1)\tau_1^{\alpha_0}}{\alpha_s!\cdots\alpha_1!\alpha_0!} \cdot d\tau_s\cdots d\tau_1.$$

Hence we can associate with (III.3) the generating series of $\boldsymbol{R}\langle\langle x_0,x_1\rangle\rangle$

$$\sum_{\alpha_0,\cdots,\alpha_s\geqslant 0} w_{s,\alpha_0,\cdots,\alpha_s}x_0^{\alpha_s}x_1\cdots x_0^{\alpha_1}x_1x_0^{\alpha_0}. \qquad \bullet$$

Remark: As Volterra had already noticed, series (III.1) may be considered as a functional generalization of Taylor expansions (cf. [52], [37]). On the other hand, a Volterra series (III.2) where t is not fixed once and for all is certainly by no means a Taylor expansion, although this has often been asserted in the literature. It is in fact a perturbative expansion with respect to the product of the inputs. In a future publication, it will be shown that for causal functionals, generating series have a natural interpretation as Taylor expansions.

B. Application of the Fundamental Formula

Consider again a system analogous to (II.1) with the exception that we take a one-dimensional input

$$\begin{cases}\dot q(t)=A_0(q)+u(t)A_1(q)\\ y(t)=h(q).\end{cases} \tag{III.4}$$

The determination of the Volterra series giving the output y has a long history beginning at the end of the fifties mainly with Barrett [2] and George [24] and culuminating with Lesiak and Krener [36] to which one should compare what follows (in this large time interval, let us mention for example Waddington and Fallside [53] and Parente [42] among papers we have not already quoted). Theorems II.1 and III.1 yield:

Proposition III.2. The output of system (III.4) may be expanded in a triangular Volterra series (III.2) where the kernels are analytic functions given by

$$w_0(t)=\sum_{\nu\geqslant 0}A_0^{\nu}h|_{q(0)}\frac{t^{\nu}}{\nu!}=e^{tA_0}h|_{q(0)}$$

$$w_1(t,\tau_1)=\sum_{\nu_0,\nu_1\geqslant 0}A_0^{\nu_0}A_1A_0^{\nu_1}h|_{q(0)}\frac{(t-\tau_1)^{\nu_1}\tau_1^{\nu_0}}{\nu_1!\nu_0!} = e^{\tau_1A_0}A_1e^{(t-\tau_1)A_0}h|_{q(0)}$$

$$w_2(t,\tau_2,\tau_1)=\sum_{\nu_0,\nu_1,\nu_2\geqslant 0}A_0^{\nu_0}A_1A_0^{\nu_1}A_1A_0^{\nu_2}h|_{q(0)}\frac{(t-\tau_2)^{\nu_2}(\tau_2-\tau_1)^{\nu_1}\tau_1^{\nu_0}}{\nu_2!\nu_1!\nu_0!} = e^{\tau_1A_0}A_1e^{(\tau_2-\tau_1)A_0}A_1e^{(t-\tau_2)A_0}h|_{q(0)}$$

$$w_s(t,\tau_s,\cdots,\tau_1)=\sum_{\nu_0,\cdots,\nu_s\geqslant 0}A_0^{\nu_0}A_1A_0^{\nu_1}\cdots A_1A_0^{\nu_s}h|_{q(0)}\frac{(t-\tau_s)^{\nu_s}\cdots\tau_1^{\nu_0}}{\nu_s!\cdots\nu_0!} = e^{\tau_1A_0}A_1e^{(\tau_2-\tau_1)A_0}A_1\cdots A_1e^{(t-\tau_s)A_0}h|_{q(0)}.$$

As for the fundamental formula, we will not dwell on the question of convergence. More details are to be found in Lesiak and Krener [36].

C. A Variant

If x and y are two noncommutative variables a well-known by-product of the so-called Baker–Campbell–Hausdorff formula gives (cf. Bourbaki [4], p. 59)

$$e^x y e^{-x} = \sum_{\nu \geq 0} \frac{1}{\nu!} ad_x^\nu y.$$

The operator ad_x^ν is defined recursively on ν

$ad_x^0 y = y$

$ad_x^1 y = [x, y]$, where $[x, y] = xy - yx$ is the Lie bracket

$ad_x^{\nu+1} y = [x, ad_x^\nu y]$.

With obvious notations, we may write

$$e^x y e^{-x} = e^{ad_x} y.$$

We deduce a variant of the formulas of the foregoing proposition which was stated in [20] in order to solve a singular optimal control problem.

Proposition III.3. The output of system (III.4) may be expanded in a triangular Volterra series (III.2) where the kernels are analytic functions given by

$$w_0(t) = e^{tA_0} h|_{q(0)}$$

$$w_1(t, \tau_1) = \sum_{\nu \geq 0} \frac{1}{\nu!} (ad^\nu_{\tau_1 A_0} A_1) e^{tA_0} h|_{q(0)}$$

$$= (e^{ad_{\tau_1 A_0}} A_1) e^{tA_0} h|_{q(0)}$$

$$w_2(t, \tau_2, \tau_1) = \sum_{\nu_0, \nu_1 \geq 0} \frac{1}{\nu_0! \nu_1!} (ad^{\nu_0}_{\tau_1 A_0} A_1)(ad^{\nu_1}_{\tau_2 A_0} A_1) e^{tA_0} h|_{q(0)}$$

$$= (e^{ad_{\tau_1 A_0}} A_1)(e^{ad_{\tau_2 A_0}} A_1) e^{tA_0} h|_{q(0)}$$

$$w_s(t, \tau_s, \cdots, \tau_1) = \sum_{\nu_0, \cdots, \nu_s \geq 0} \frac{1}{\nu_0! \cdots \nu_s!} (ad^{\nu_s}_{\tau_s A_0} A_1) \cdots (ad^{\nu_s}_{\tau_1 A_0} A_1) e^{tA_0} h|_{q(0)}$$

$$= \left(e^{ad_{\tau_1 A_0}} A_1\right) \cdots \left(e^{ad_{\tau_s A_0}} A_1\right) e^{tA_0} h|_{q(0)}.$$

Remarks: (i) In those formulas, the Volterra kernels w_s are expressed with respect to $\tau_1, \tau_2, \cdots, \tau_s, t$ and not, as before, with $\tau_1, \tau_2 - \tau_1, \cdots, t - \tau_s$. Notice that we do need Lie brackets to give the precise expression.

(ii) We know (Gröbner [26]) that $e^{tA_0} h|_{q(0)} = h|_{q_t}$, where q_t is the point reached at time t by q satisfying $\dot{q} = A_0(q)$ with the initial condition $q(0)$. Hence we arrive at the following formulas where the derivations are taken with respect to $q(0)$, i.e., w.r.t. the dependence of q_t on the initial condition

$$w_0(t) = h(q_t)|_{q(0)}$$

$$w_1(t, \tau_1) = (e^{ad_{\tau_1 A_0}} A_1) h(q_t)|_{q(0)}$$

$$w_2(t, \tau_2, \tau_1) = (e^{ad_{\tau_1 A_0}} A_1)(e^{ad_{\tau_2 A_0}} A_1) h(q_t)|_{q(0)}$$

$$w_s(t, \tau_s, \cdots, \tau_1) = (e^{ad_{\tau_1 A_0}} A_1) \cdots (e^{ad_{\tau_s A_0}} A_1) h(q_t)|_{q(0)}.$$

Obviously the same change could be made with the formulas of Theorem III.1.

D. Some Remarks on Transfer Functions

Consider first the linear stationary system

$$y(t) = \int_0^t w_1(t - \tau) u(\tau)\, d\tau$$

where

$$w_1(t - \tau) = \sum_{\nu \geq 0} a_\nu \frac{(t - \tau)^\nu}{\nu!}$$

is an analytic function. From Theorem III.1, the corresponding generating series is

$$\sum_{\nu \geq 0} a_\nu x_0^\nu x_1$$

when the transfer function is

$$\sum_{\nu \geq 0} \frac{a_\nu}{p^{\nu+1}}.$$

This confirms the relationship between Laplace transforms and noncommutative variables.

Consider now the homogeneous time-invariant system in triangular form

$$y(t) = \int_0^t \int_0^{\tau_s} \cdots \int_0^{\tau_2} w_s(t - \tau_s, \tau_s - \tau_{s-1}, \cdots, \tau_2 - \tau_1) \cdot u(\tau_s) \cdots u(\tau_1)\, d\tau_s \cdots d\tau_1.$$

With respect to $t - \tau_s, \tau_s - \tau_{s-1}, \cdots, \tau_2 - \tau_1, \tau_1$ time-invariance implies that w_s is independant of τ_1. Corresponding to the Taylor expansion

$$w_s = \sum_{\alpha_1, \cdots, \alpha_s \geq 0} a_{\alpha_1, \cdots, \alpha_s} \frac{(t - \tau_1)^{\alpha_s} \cdots (\tau_2 - \tau_1)^{\alpha_1}}{\alpha_s! \cdots \alpha_1!}$$

we get the generating series

$$\sum_{\alpha_1, \cdots, \alpha_s \geq 0} a_{\alpha_1, \cdots, \alpha_s} x_0^{\alpha_s} x_1 \cdots x_0^{\alpha_1} x_1.$$

It is equivalent up to an elementary change of variables to the *regular* transfer function introduced by Mitzel and Rugh [40] (see also Rugh [45]) which is here

$$\sum_{\alpha_1, \cdots, \alpha_s \geq 0} a_{\alpha_1, \cdots, \alpha_s} \frac{1}{p_s^{\alpha_s + 1}} \cdots \frac{1}{p_1^{\alpha_1 + 1}}.$$

Those transfer functions exhibit some remarkable properties: they correspond to a bilinear system iff they are *recognizable* (cf. [14]), i.e., of the form $P(p_1,\cdots,p_s)/Q_1(p_1)\cdots Q_s(p_s)$ where $P, Q_1,\cdots,Q_s$ are polynomials. As noted in [18], this property may be seen as a direct consequence of the theory of noncommutative rational power series.

Remark: There are at least three types of high-order transfer functions one can introduce:

—the symmetric ones corresponding to symmetric Volterra kernels (see Section IV),

—the regular ones due to Mitzel and Rugh and which have a very natural interpretation thanks to noncommutative variables,

—the multidimensional Laplace transforms of the triangular Volterra kernels with respect to the variables $\tau_1,\cdots,\tau_s, t$ (Section III-C). Until now they do not seem to have attracted any attention. One should certainly select the type of transfer function according to the problem one is interested in.

IV. Determination of Volterra Series

When the input–output behavior of a system described by state equations is of interest, a representation, generally the Volterra functional representation of the solution of the state equations, is needed. Several methods have been developed in the literature for determining the kernels or the associated transfer functions based on classical symbolic methods (cf. Brillant [5], George [24], Bedrosian and Rice [3], Bussgang, Ehrman and Graham [7], Chua and Ng [8], [9]). Among them, the method of exponential inputs is particularly used. After briefly reviewing this method, we describe a new technique based on generating power series. This approach has the advantage of allowing more easily the use of a computer. This becomes necessary as soon as one tries to obtain high-order terms. The two methods are illustrated by analyzing a simple nonlinear circuit.

A. Exponential Input Method

A convenient analysis method of evaluating the nonlinear transfer functions is the so-called "*exponential input*" method which is considered in this section (Bedrosian and Rice [3], Chua and Ng [8], [9]). Following those papers, we will consider, unlike Section III, stationary Volterra series where the time interval of integration is unbounded

$$\begin{aligned} y(t) &= \int_0^\infty h_1(\tau_1)u(t-\tau_1)\,d\tau_1 \\ &+ \int_0^\infty\int_0^\infty h_2(\tau_2,\tau_1)u(t-\tau_1)u(t-\tau_2)\,d\tau_1 d\tau_2 + \\ &\cdots + \int_0^\infty\int_0^\infty\cdots\int_0^\infty h_n(\tau_1,\tau_2,\cdots,\tau_n) \\ &\cdot\prod_{i=1}^n u(t-\tau_i)\,d\tau_1 d\tau_2\cdots d\tau_n + \cdots. \end{aligned} \tag{IV.1}$$

The input u is one-dimensional. It is well known that, without loss of generality, the kernels can be assumed to be symmetric. In fact any kernel $h_n(\tau_1,\cdots,\tau_n)$ in (IV.1) can be replaced by a symmetric one by setting

$$h_n^{\text{sym}}(\tau_1,\cdots,\tau_n) = \frac{1}{n!}\sum_{\substack{\text{all permutations}\\ \text{of } \tau_1,\cdots,\tau_n}} h_n(\tau_{i_1},\cdots,\tau_{i_n}).$$

The multiple Laplace transform of the nth-order Volterra kernel $n>0$

$$H_n(s_1,\cdots,s_n) = \int_0^\infty\cdots\int_0^\infty h_n(\tau_1,\cdots,\tau_n) \cdot\exp(-s_1\tau_1 - s_2\tau_1\cdots - s_n\tau_n)\,d\tau_1\cdots d\tau_n$$

is called the *nth-order transfer function*. Since $h_n(\tau_1,\cdots,\tau_n)$ is symmetric, so is $H_n(s_1,\cdots,s_n)$.

Thus the output of a nonlinear system in the form (IV.1) entails the determination of Volterra kernels $h_n(\tau_1,\cdots,\tau_n)$ or similarly the nonlinear transfer functions $H_n(s_1,\cdots,s_n)$. To this end, let the input $u(t)$ be a sum of exponentials

$$u(t) = e^{s_1 t} + e^{s_2 t} + \cdots + e^{s_k t}$$

where $s_1, s_2,\cdots,s_k$ are *rationally independent*.[6] Then (IV.1) becomes

$$y(t) = \sum_{n=1}^{+\infty}\left[\sum_{k_1=1}^{k}\sum_{k_2=1}^{k}\cdots\sum_{k_n=1}^{k} H_n(s_{k_1},s_{k_2},\cdots,s_{k_n}) \cdot\exp\left(\left(s_{k_1}+s_{k_2}+\cdots+s_{k_n}\right)t\right)\right]. \tag{IV.2}$$

If each s_i occurs in $(s_{k1},\cdots,s_{k_n})m_i$ times, then there are $n!/(m_1!m_2!\cdots m_k!)$ identical terms in the expression between brackets.

Thus (IV.2) can be written in the form

$$y(t) = \sum_{n=1}^{+\infty}\sum_m \frac{n!}{m_1!m_2!\cdots m_k!}H_n(s_{k_1},\cdots,s_{k_n}) \cdot\exp(s_{k_1}+\cdots+s_{k_n})t$$

where m under the summation sign indicates that the sum includes all the distinct vectors $(m_1,\cdots,m_k)$ such that

$$\sum_{i=1}^k m_i = n.$$

Note that if $m_1=m_2=\cdots=m_k=1$, then the amplitude associated with the exponential component $\exp(s_1+\cdots+s_k)t$ is simply $k!H_k(s_1,\cdots,s_k)$. This suggests a recursive procedure for determining all the nonlinear transfer functions from the equations defining the behavior of a system. Rather than a general formulation we apply the method to the simple nonlinear circuit (Bussgang, Ehrman, and Graham [7]) of Fig. 2 consisting of a capacitor, a linear resistor, and a nonlinear resistor in parallel with the current source $i(t)$.

The nonlinear differential equation relating the current excitation $i(t)$ and the voltage $v(t)$ across the capacitor is

[6] The number $s_1, s_2,\cdots,s_k$ are said to be rationally independent if there are no rational numbers $\alpha_1,\alpha_2,\cdots,\alpha_k$ such that the sum $\alpha_1 s_1+\alpha_2 s_2+\cdots+\alpha_k s_k$ is rational.

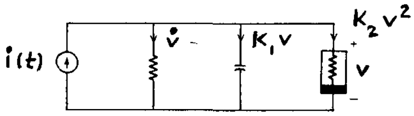

Fig. 2.

given by

$$\frac{dv}{dt}+k_1v(t)+k_2v^2(t)=i(t). \qquad \text{(IV.3)}$$

A single exponential excitation is first considered

$$i(t)=e^{st}.$$

Equating the coefficients of e^{st} on both sides of (IV.3) after the substitution of (IV.2) for $v(t)$ we get

$$(s+k_1)H_1(s)=1.$$

Thus for the specified circuit, the first-order Volterra transfer function is

$$H_1(s)=\frac{1}{s+k_1}.$$

To determine $H_2(s_1,s_2)$ let us take

$$i(t)=e^{s_1t}+e^{s_2t}$$

and identify the coefficient of the term $2!e^{(s_1+s_2)t}$, after the substitution of (IV.2) for $v(t)$, in both sides of (IV.3). We obtain

$$(s_1+s_2+k_1)H_2(s_1,s_2)+k_2H_1(s_1)H_2(s_2)=0.$$

This yields $H_2(s_1,s_2)$ in terms of $H_1(s)$

$$H_2(s_1,s_2)=-k_2H_1(s_1)H_1(s_2)H_1(s_1+s_2).$$

Similarly, the third-order nonlinear transfer function is obtained by injecting a mixture of three exponential inputs

$$i(t)=e^{s_1t}+e^{s_2t}+e^{s_3t}$$

and equating the coefficients of $3!e^{(s_1+s_2+s_3)t}$ on both sides of (IV.3)

$$H_3(s_1,s_2,s_3)=-\frac{2}{3}\left[H_2(s_1,s_2)H_1(s_3)+H_2(s_2,s_3)H_1(s_1)+H_2(s_1,s_3)H_1(s_2)\right]H_1(s_1+s_2+s_3).$$

Repeating this process indefinitely gives higher order nonlinear transfer functions in terms of lower order nonlinear transfer functions.

B. Generating Power Series Method

In Section III-A, we have derived the relationship between functional Volterra series and generating power series. Here, we describe an algorithm for finding algebraically the generating power series associated with the solution of a nonlinear forced differential equation. The equation we are going to consider is

$$Ly(t)+\sum_{i=2}^{m}p_iy^i(t)=u(t)$$

$$L=\sum_{i=0}^{n}l_i\frac{d^i}{dt^i}, \qquad l_n=1 \qquad \text{(IV.4)}$$

or, in its integral form

$$\begin{aligned}y(t)&+l_{n-1}\int_0^t y(\tau_1)\,d\tau_1+l_{n-2}\int_0^t d\tau_2\int_0^{\tau_2}y(\tau_1)\,d\tau_1\\&+\cdots+l_0\int_0^t d\tau_n\int_0^{\tau_n}d\tau_n\int_0^{\tau_n}d\tau_{n-1}\cdots d\tau_2\int_0^{\tau_2}y(\tau_1)\,d\tau_1\\&+\sum_{i=2}^{m}p_i\int_0^t d\tau_n\int_0^{\tau_n}d\tau_{n-1}\cdots d\tau_2\int_0^{\tau_2}y^i(\tau_1)\,d\tau_1\\&=\int_0^t d\tau_n\int_0^{\tau_n}d\tau_{n-1}\cdots d\tau_2\int_0^{\tau_2}u(\tau_1)\,d\tau_1\end{aligned} \qquad \text{(IV.5)}$$

if we assume zero initial conditions.

Let $\mathfrak{g}$ denotes the generating series associated with $y(t)$, then (IV.5) can be written symbolically as

$$\left(1+\sum_{j=0}^{n-1}l_jx_0^{n-i}\right)\mathfrak{g}+x_0^n\sum_{i=0}^{m}p_i\mathfrak{g}^{\sqcup\!\sqcup^i=x_0n-1}x_1$$

where $\mathfrak{g}\sqcup\!\sqcup^i$ corresponds to the nonlinear functional $y^i(t)$.

This algebraic equation can be solved iteratively, following the recursive scheme

$$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2+\cdots+\mathfrak{g}_n+\cdots \qquad \text{(IV.6)}$$

with

$$\mathfrak{g}_1=\left(1+\sum_{i=0}^{n-1}l_ix_0^{n-i}\right)^{-1}x_0^{n-1}x_1$$

and

$$\mathfrak{g}_n=\left(1+\sum_{i=0}^{n-1}l_ix_0^{n-i}\right)^{-1}x_0^n\cdot\sum_{i=2}^{m}p_i\sum_{\nu_1+\nu_2+\cdots+\nu_i=n}\mathfrak{g}_{\nu_1}\sqcup\!\sqcup\,\mathfrak{g}_{\nu_2}\sqcup\!\sqcup\cdots\sqcup\!\sqcup\,\mathfrak{g}_{\nu_i}.$$

This formula shows that the computation of $\mathfrak{g}_n$ requires the shuffle product of expressions of the form

$$R_1(x_0)x_1R_2(x_0)x_1\cdots x_1R_p(x_0)$$

where $R_j(x_0)$ $(j=1,\cdots,p)$ is a rational series in the single variable x_0. Decomposing this into partial fractions allows us to consider only the expressions

$$(1-a_0x_0)^{-p_0}x_1(1-a_1x_0)^{-p_1}x_1\cdots x_1(1-a_qx_0)^{-p_q}, \qquad p_i\in N, \quad (i=0,\cdots,q).$$

Finally, if we note that

$$(1-ax_0)^{-p}=(1-ax_0)^{-(p-1)}+a(1-ax_0)^{-1}x_0(1-ax_0)^{-(p-1)}$$

we only need to compute the shuffle product of noncommutative power series of the form

$$(1-a_0x_0)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}\cdots x_{i_p}(1-a_px_0)^{-1}, \qquad i_1,\cdots,i_p\in\{0,1\}.$$

This results from the following proposition.

Proposition IV.1. (cf. [34]). Given two noncommutative power series

$$\mathfrak{g}_1^p = (1-a_0x_0)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}\cdots x_{i_p}(1-a_px_0)^{-1}$$
$$= \mathfrak{g}_1^{p-1}x_{i_p}(1-a_px_0)^{-1}$$

and

$$\mathfrak{g}_2^q = (1-b_0x_0)^{-1}x_{j_1}(1-b_1x_0)^{-1}x_{j_2}\cdots x_{j_q}(1-b_qx_0)^{-1}$$
$$= \mathfrak{g}_2^{q-1}x_{j_q}(1-b_qx_0)^{-1}$$

where p and q belongs to $\boldsymbol{N}$, the subscripts $i_1,\cdots,i_p$, $j_1,\cdots,j_q$ to $\{0,1\}$ and a_i, b_j to $\boldsymbol{C}$; the shuffle product is given by induction on the length by

$$\mathfrak{g}_1^p \sqcup\!\sqcup\, \mathfrak{g}_2^q = (\mathfrak{g}_1^p \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1})x_{j_q}[1-(a_p+b_q)x_0]^{-1} + (\mathfrak{g}_1^{p-1} \sqcup\!\sqcup\, \mathfrak{g}_2^q)x_{i_p}[1-(a_p+b_q)x_0]^{-1}$$

with

$$(1-ax_0)^{-1} \sqcup\!\sqcup\, (1-bx_0)^{-1} = [1-(a+b)x_0]^{-1}.$$

For example, if we choose

$$\mathfrak{g}_1 = (1-a_0x_0)^{-1}x_{i_1}(1-a_1x_0)^{-1}x_{i_2}(1-a_2x_0)^{-1}$$

and

$$\mathfrak{g}_2 = (1-b_0x_0)^{-1}x_{j_1}(1-b_1x_0)^{-1}$$

then

$$\begin{aligned}\mathfrak{g}_1 \sqcup\!\sqcup\, \mathfrak{g}_2 = {} & [1-(a_0+b_0)x_0]^{-1}x_{i_1}[1-(a_1+b_0)x_0]^{-1}x_{i_2}\\ & \cdot[1-(a_2+b_0)x_0]^{-1}x_{j_1}[1-(a_2+b_1)x_0]^{-1}\\ & +[1-(a_0+b_0)x_0]^{-1}x_{i_1}[1-(a_1+b_0)x_0]^{-1}x_{j_1}\\ & \cdot[1-(a_1+b_1)x_0]^{-1}x_{i_2}[1-(a_2+b_1)x_0]^{-1}\\ & +[1-(a_0+b_0)x_0]^{-1}x_{j_1}[1-(a_0+b_1)x_0]^{-1}x_{i_1}\\ & \cdot[1-(a_1+b_1)x_0]^{-1}x_{i_2}[1-(a_2+b_1)x_0]^{-1}.\end{aligned}$$

Proof: Let us consider the formal power series

$$\mathfrak{g}_1^p = \mathfrak{g}_1^{p-1}x_{i_p}(1-a_px_0)^{-1}$$

and

$$\mathfrak{g}_2^q = \mathfrak{g}_2^{q-1}x_{j_p}(1-b_qx_0)^{-1}.$$

Their shuffle product is given by

$$\mathfrak{g}_1^{p-1}x_{i_p}\left[1+a_p(1-a_px_0)^{-1}x_0\right] \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1}x_{j_q} \cdot\left[1+b_q(1-b_qx_0)^{-1}x_0\right]$$

where we use the identity

$$(1-ax_0)^{-1} = 1+a(1-ax_0)^{-1}x_0. \qquad \text{(IV.7)}$$

Applying the definition of the shuffle product and equality (IV.7) gives

$$\begin{aligned}&\mathfrak{g}_1^p \sqcup\!\sqcup\, \mathfrak{g}_2^q\\ &= \left[\mathfrak{g}_1^{p-1}x_{i_p}(1-a_px_0)^{-1} \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1}\right]x_{j_q}\\ &\quad+\left[\mathfrak{g}_1^{p-1} \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1}x_{j_q}(1-b_qx_0)^{-1}\right]x_{i_p}\\ &\quad+\left[\mathfrak{g}_1^{p-1}x_{i_p}(1-a_px_0)^{-1} \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1}x_{j_q}(1-b_qx_0)^{-1}\right]a_px_0\\ &\quad+\left[\mathfrak{g}_1^{p-1}x_{i_p}(1-a_px_0)^{-1} \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1}x_{j_q}(1-b_qx_0)^{-1}\right]b_qx_0.\end{aligned}$$

Thus we obtain

$$(\mathfrak{g}_1^p \sqcup\!\sqcup\, \mathfrak{g}_2^q)[1-(a_p+b_q)x_0] = (\mathfrak{g}_1^p \sqcup\!\sqcup\, \mathfrak{g}_2^{q-1})x_{j_q}+(\mathfrak{g}_1^{p-1} \sqcup\!\sqcup\, \mathfrak{g}_2^q)x_{i_p}$$

Q.E.D.

Using this proposition, $\mathfrak{g}_i$ is obtained as a finite sum of expressions of the form

$$(1-a_0x_0)^{-p_0}x_1(1-a_1x_0)^{-p_1}x_1\cdots x_1(1-a_ix_0)^{-p_i}. \qquad \text{(IV.8)}$$

The expansion (IV.6) is "equivalent" to the Volterra series expansion of $y(t)$ (cf. Section III-A). The algebraic closed-form expression of triangular Volterra kernels can be easily deduced from it since the expression (IV.8) is a symbolic representation of the i-dimensional integral

$$\int_0^t\int_0^{\tau_i}\cdots\int_0^{\tau_2} f_{a_0}^{p_0}(t-\tau_i)\cdots f_{a_{i-1}}^{p_{i-1}}(\tau_2-\tau_1)f_{a_i}^{p_i}(\tau_1) \cdot u(\tau_i)\cdots u(\tau_1)\,d\tau_i\cdots d\tau_1 \qquad \text{(IV.9)}$$

where

$$f_a^p(t) = \left(\sum_{j=0}^{p-1}\frac{\binom{p-1}{j}}{j!}a^jt^j\right)e^{at}.$$

The technique presented above is now used to compute the generating power series associated with the previous nonlinear circuit. The integral form of (IV.3) is

$$v(t)+k_1\int_0^t v(\tau)\,d\tau + k_2\int_0^t v^2(\tau)\,d\tau = \int_0^t i(\tau)\,d\tau$$

where we assume a zero initial condition.

Thus for the specified circuit, the generating power series is the solution of the algebraic equation

$$\mathfrak{g} + k_1x_0\mathfrak{g} + k_2x_0\mathfrak{g} \sqcup\!\sqcup\, \mathfrak{g} = x_1$$

or

$$\mathfrak{g} = -k_2(1+k_1x_0)^{-1}x_0[\mathfrak{g} \sqcup\!\sqcup\, \mathfrak{g}]+(1+k_1x_0)^{-1}x_1.$$

This equation is solved iteratively by a computer program (cf. [32], [35]). We obtain Table I, where the symbolic notation

$$a_0 \overset{x_{i_1}}{\quad} a_1 \overset{x_{i_2}}{\quad} \cdots \overset{x_{i_n}}{\quad} a_n, \qquad i_1,\cdots,i_n \in \{0,1\}$$

TABLE I

1		$k_1 {}^{x_1} 0$
-2	k_2	$k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+4	k_2^2	$k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+12	k_2^2	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
-8	k_2^3	$k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
-24	k_2^3	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
-72	k_2^3	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
-24	k_2^3	$k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
-144	k_2^3	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_0} 4k_1 {}^{x_1} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+48	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+144	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+432	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+288	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_0} 4k_1 {}^{x_1} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+864	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_0} 4k_1 {}^{x_1} 3k_1 {}^{x_1} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+1728	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_0} 4k_1 {}^{x_1} 3k_1 {}^{x_0} 4k_1 {}^{x_1} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+864	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_0} 3k_1 {}^{x_0} 4k_1 {}^{x_1} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+144	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+2880	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_0} 3k_1 {}^{x_0} 4k_1 {}^{x_0} 5k_1 {}^{x_1} 4k_1 {}^{x_1} 3k_1 {}^{x_1} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
+16	k_2^4	$k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_0} 2k_1 {}^{x_1} k_1 {}^{x_1} 0$
$\vdots$		

stands for

$$(1+a_0x_0)^{-1}x_{i_1}(1+a_1x_0)^{-1}x_{i_2}\cdots (1+a_{n-1}x_0)^{-1}x_{i_n}(1+a_nx_0)^{-1}$$

Remarks: (i) The expansion in Table I is "equivalent" to the Volterra series expansion of the solution up to order 5.

(ii) Viennot has recently informed the authors that he has found a combinatorial interpretation of the previous computations which should make the programming much easier.

V. Symbolic Calculus for the Response of Nonlinear Systems

Our next objective will be to show how the Volterra series can be used to determine the output of a system subject to various deterministic excitations (steps, slopes, harmonics, etc.). In the linear case, Laplace and Fourier transforms are systematic and powerful tools of operational calculus. A direct generalization of these techniques, to the nonlinear domain, leads to multidimensional Laplace and Fourier transforms; but the computation based on these is often tedious, even for low-order Volterra kernels, and seems difficult to implement on a computer. An alternative method presented here, based on commutative variables and on the properties of iterated integrals, leads to a simple nonlinear generalization of Heaviside symbolic calculus and to an easy implementation on a computer. It is compared with the *association of variables* introduced by George [24], that we shall now briefly review.

A. Association of Variables

If the Volterra kernels are known for a system, then the output $y(t)$ for a given input $u(t)$ could be obtained. However, as for linear systems, where the use of Laplace transform allows one to develop a powerful operational calculus, one can introduce here the multiple Laplace transform. Indeed, let us consider the "nth-order output" corresponding to the Volterra series (IV.1)

$$y_n(t)=\int_0^\infty\int_0^\infty\cdots\int_0^\infty h_n(\tau_1,\tau_2,\cdots,\tau_n)\cdot u(t-\tau_1)\cdots u(t-\tau_n)\,d\tau_1\cdots d\tau_n.$$

Introduce a set of artificial variables $t_1,t_2,\cdots,t_n$ so that

$$y_{(n)}(t_1,t_2,\cdots,t_n)=\int_0^\infty\int_0^\infty\cdots\int_0^\infty h_0(\tau_1,\tau_2,\cdots,\tau_n)\cdot u(t_1-\tau_1)\cdots u(t_n-\tau_n)\,d\tau_1\cdots d\tau_n \quad \text{(V.1)}$$

and

$$y_n(t)=y_{(n)}(t_1,t_2,\cdots,t_n)|_{t_1=t_2=\cdots=t}.$$

Then, taking the nth-order Laplace transform of both sides of (V.1) gives

$$Y_{(n)}(s_1,\cdots,s_n)=H_n(s_1,\cdots,s_n)\prod_{i=1}^n U(s_i)$$

where $U(s_i)$ is the usual first-order Laplace transform of the input. Thus, as in the linear case, the convolution in the time domain corresponds to the multiplication in the frequency domain.

Now, assume that the nth-order Laplace transform of $y_{(n)}(t_1,t_2,\cdots,t_n)$, $Y_{(n)}(s_1,s_2,\cdots,s_n)$ is given and $y_n(t)$ is desired. Obviously, one can perform the nth-order inverse Laplace transform of $Y_{(n)}(s_1,s_2,\cdots,s_n)$

$$y_{(n)}(t_1,t_2,\cdots,t_n)=\frac{1}{(2\pi j)^n}\int_{\sigma_n-j\infty}^{\sigma_n+j\infty}\cdots\int_{\sigma_1-j\infty}^{\sigma_1+j\infty}Y_{(n)}(s_1,\cdots,s_n)\cdot e^{s_1t_1+\cdots+s_nt_n}\,ds_1\cdots ds_n \quad \text{(V.2)}$$

and set $t_1=t_2=\cdots=t_n=t$. However, this computation is often unwieldy. In order to bypass this difficulty, George [24] developed a method whereby the t_i variables could be set equal or *associated* without leaving the transform domain, leading to a one-dimensional Laplace transform $Y_n(s)$. Indeed, let us consider a two variables transform

$Y_{(2)}(s_1, s_2)$, setting $t_1 = t_2 = t$ in (V.2) yields

$$y_{(2)}(t_1, t_1) = \frac{1}{2\pi j}\int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty}\left[\frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} Y_{(2)}(s_1, s_2) e^{s_1 t}\, ds_1\right] e^{s_2 t}\, ds_2.$$

Changing the variable of integration s_1 to $s = s_1 + s_2$ gives

$$y_{(2)}(t_1, t_1) = \frac{1}{2\pi j}\int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty}\left[\frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} Y_{(2)}(s - s_1, s_2) e^{(s - s_2)t}\, ds\right] e^{s_2 t}\, ds_2$$

or, by interchanging the order of integration

$$y_{(2)}(t_1, t_1) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty}\left[\frac{1}{2\pi j}\int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} Y_{(2)}(s - s_2, s_2)\, ds_2\right] e^{st}\, ds.$$

Thus the associated transform $Y_2(s)$ is

$$Y_2(s) = \frac{1}{2\pi j}\int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} Y_{(2)}(s - s_2, s_2)\, ds_2. \qquad \text{(V.3)}$$

Similarly, a transform of any order can be reduced to a first-order transform by successive pairwise associations.

For example, let us consider the third-order term

$$\frac{1}{(s_1 + s_2 + s_3 + a)(s_1 + a)(s_2 + a)(s_3 + a)}.$$

Using (V.3) to associate the variables s_2 and s_3 yields

$$\frac{1}{(s_1 + s_2 + a)(s_1 + a)(s_2 + 2a)}.$$

Then associating s_1 and s_2 results in the first-order transform

$$\frac{1}{(s + a)(s + 3a)}.$$

The procedure for computing $Y_n(s)$ from $Y_{(n)}(s_1, s_2, \cdots, s_n)$ is called *association of variables*. Although an explicit formula for performing the associating operation in a wide class of Laplace transforms has been obtained in the literature (see Rugh [45] and the references herein), this technique has seldom been used. The main reason for this situation seems to be the tedious manipulations involved and the difficulty in decomposing them on a computer.

B. Noncommutative Symbolic Calculus

Using the noncommutative generating power series, we develop in this section a symbolic calculus for computing the response of nonlinear systems described by (IV.4) to various deterministic inputs (steps, slopes, harmonics, etc.). Such computations may be of interest in order to get nonlinear distortions. To this end we show that integrals of the form (IV.9) can be expressed in terms of "elementary functions" for a specified set of functions $u(t)$.

Let the input $u(t)$ be an analytic function

$$u(t) = \sum_{n \geq 0} u_{(n)} \frac{t^n}{n!}.$$

Define the transform (known as the Laplace–Borel transform) of $u(t)$

$$\mathfrak{g}_{u(t)} = \sum_{n \geq 0} u_{(n)} x_0^n.$$

(This series may be regarded as the generating power series associated with $u(t)$ since

$$\frac{t^n}{n!} = \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \int_0^{\tau_{n-1}} \cdots \int_0^{\tau} d\tau_1\Big).$$

Then we can state the following result [33]:

Proposition V.1. The integration (IV.9) defines an analytic function, the Laplace–Borel transform of which is given by

$$(1 - a_0 x_0)^{-p_0} x_0 \Big\{ \mathfrak{g}_{u(t)} ш (1 - a_1 x_0)^{-p_1} x_0 \cdot \big[\mathfrak{g}_{u(t)} ш \cdots x_0 \big[\mathfrak{g}_{u(t)} ш (1 - a_i x_0)^{-p_i} \big] \cdots \big] \Big\}. \qquad \text{(V.4)}$$

Thus it is simply obtained, by replacing each indeterminate x_1 in (IV.8) by the operator $x_0[\mathfrak{g}_{u(t)} ш \cdot]$.

This results in a single variable transform (i.e., the Laplace–Borel transform) and thus the shuffle product appears as an operation which implicitly takes into account the technique of association of variables.

Proof: Let $f_1(t)$ and $f_2(t)$ be two analytic functions, $\mathfrak{g}_{f_1(t)}$ and $\mathfrak{g}_{f_2(t)}$ their corresponding Laplace–Borel transforms. Based on the results in Section III-D, one can first state that

$$\mathfrak{g}_{\int_0^t f_1(\tau)\, d\tau} = x_0 \mathfrak{g}_{f_1(t)} \qquad \text{(V.5)}$$

and

$$\mathfrak{g}_{f_2(t) \times f_2(t)} = \mathfrak{g}_{f_1(t)} ш\, \mathfrak{g}_{f_2(t)}. \qquad \text{(V.6)}$$

Now, let us consider the expression

$$(1 - a_0 x_0)^{-p_0} x_1 (1 - a_1 x_0)^{-p_1} \qquad \text{(V.7)}$$

which is the symbolic representation of the integral series

$$\sum_{n \geq 0} \binom{p_0 + n - 1}{n} a_0^n \int_0^t d\tau_n \int_0^{\tau_n} \cdots \int_0^{\tau_2} d\tau_1 \int_0^{\tau_1} u(\tau_0) f_{a_1}^{p_1}(\tau_0)\, d\tau_0 \qquad \text{(V.8)}$$

where

$$f_a^p(t) = \sum_{n \geq 0} \binom{p + n - 1}{n} a^n \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1$$

is the inverse Laplace–Borel transform of $(1 - a x_0)^{-p}$. Following (V.5) and (V.6), the Laplace–Borel transform of (V.8) is

$$\sum_{n \geq 0} \left\{ \binom{p_0 + n - 1}{n} a_0^n x_0^n \right\} x_0 \big[\mathfrak{g}_{u(t)} ш (1 - a_1 x_0)^{-p_1} \big] = (1 - a_0 x_0)^{-p_0} x_0 \big[\mathfrak{g}_{u(t)} ш (1 - a_1 x_0)^{-p_1} \big].$$

Thus it is simply obtained by replacing the indeterminate x_1 in (V.7) by the operator $x_0[\mathfrak{g}_{u(t)} ш \cdot]$.

The proposition (V.1) results from repeated application of this rule to the expression

$$(1 - a_0 x_0)^{-p_0} x_1 (1 - a_1 x_0)^{-p_1} x_1 \cdots x_1 (1 - a_i x_0)^{-p_i}. \quad \blacksquare$$

TABLE II
LAPLACE–BOREL TRANSFORM OF SOME FUNCTIONS

$u(t)$	$g_{u(t)}$
unit step	1
$\frac{t^n}{n!}$	x_0^n
$\left(\sum_{i=0}^{n-1} \frac{\binom{i}{n-1}}{i!} a^i t^i\right) e^{at}$	$(1-ax_0)^{-n}$
$\cos(\omega t)$	$(1+\omega^2 x_0^2)^{-1}$

Now, assume that $g_{u(t)}$ is the rational fraction

$$(1-ax_0)^{-p}$$

that is, $u(t)$ is an exponential polynomial of the form

$$u(t)=\left(\sum_{j=0}^{p-1}\binom{p-1}{j}\frac{a^j t^j}{j!}\right)e^{at}$$

then, the simple identity

$$(1-ax_0)^{-p}=\sum_{j=0}^{p-1}\binom{p-1}{j}a^j(1-ax_0)^{-1}$$

$$\cdot \underbrace{x_0(1-ax_0)^{-1}x_0\cdots x_0(1-ax_0)^{-1}}_{j \text{ times}}$$

and Proposition (IV.1) allow to derive a closed-form expression for (V.4) as a rational fraction. The corresponding time function, that is, the value of the integral (IV.9), then results from the decomposition into partial fractions of this rational fraction. The same technique applies when $g_{u(t)}$ is a general rational function, regular at the origin. Let us note here, that to compute the response, one only needs to know the Laplace–Borel transform of some common functions (Table II).

In order to illustrate the use of this rule, consider again the nonlinear system shown in Fig. 1. Its response to an input $i(t)$ derives from Table I by

$$\begin{aligned}&(1+k_1x_0)^{-1}x_0\left[g_{i(t)}ш 1\right]\\&\quad-2k_2(1+k_1x_0)^{-1}x_0(1+2k_1x_0)^{-1}x_0\\&\quad\cdot\left[g_{i(t)}ш(1+k_1x_0)^{-1}x_0\left[g_{i(t)}ш 1\right]\right]\\&\quad+4k_2^2(1+k_1x_0)^{-1}x_0(1+2k_1x_0)^{-1}x_0\\&\quad\cdot\left\{g_{i(t)}ш(1+k_1x_0)^{-1}x_0(1+2k_1x_0)^{-1}x_0\right.\\&\quad\left.\cdot\left[g_{i(t)}ш(1+k_1x_0)^{-1}x_0\left[g_{i(t)}ш 1\right]\right]\right\}+\cdots \qquad \text{(V.9)}\end{aligned}$$

where $g_{i(t)}$ is the Laplace–Borel transform of $i(t)$.

Application.

1) Let us, for example, compute the response of the system to the unit step

$$i(t)=1, \qquad t\geqslant 0.$$

As the Laplace–Borel transform of the unit step is 1, the neutral element for the shuffle product, the Laplace–Borel transform of $v(t)$ is given simply by replacing each variable x_1 in the generating power series g by the variable x_0 (Table III). Then, by decomposing into partial fractions, we get the original function

$$\begin{aligned}v(t)=&\frac{1}{k_1}(1-e^{-k_1t})-\frac{k_2}{k_1^3}(1-2k_1te^{-k_1t}-e^{-2k_1t})\\&+\frac{k_2^2}{k_1^5}\left[2+(1-2k_1t-2k_1^2t^2)e^{-k_1t}\right.\\&\left.-2(1+2k_1t)e^{-2k_1t}-e^{-3k_1t}\right]+\cdots.\end{aligned}$$

TABLE III

1		k_1	x_0	0												
-2	k_2	k_1	x_0	$2k_1$	x_0	k_1	x_0	0								
+4	k_2^2	k_1	x_0	$2k_1$	x_0	k_1	x_0	$2k_1$	x_0	k_1	x_0	0				
+12	k_2^2	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1$	x_0	k_1	x_0	0				
-8	k_2^3	k_1	x_0	$2k_1$	x_0	k_1	x_0	$2k_1$	x_0	k_1	x_0	$2k_1$	x_0	k_1	x_0	0
-24	k_2^3	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1$	x_0	k_1	x_0	$2k_1$	x_0	k_1	x_0	0
-72	k_2^3	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1$	x_0	k_1	x_0	0
-24	k_2^3	k_1	x_0	$2k_1$	x_0	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1$	x_0	k_1	x_0	0
-144	k_2^3	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$4k_1$	x_0	$3k_1$	x_0	$2k_1$	x_0	k_1	x_0	0
⋮																

TABLE IV

$\frac{1}{2}$		k_1	x_0	$j\omega$								
$-\frac{1}{2}$	k_2	k_1	x_0	$2k_1$	x_0	$k_1+j\omega$	x_0	$2j\omega$				
$-\frac{1}{2}$	k_2	k_1	x_0	$2k_1$	x_0	$k_1+j\omega$	x_0	0				
$+\frac{1}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$k_1+j\omega$	x_0	$2k_1+j\omega$	x_0	$k_1+2j\omega$	x_0	$3j\omega$
$+\frac{1}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$k_1+j\omega$	x_0	$2k_1+j\omega$	x_0	$k_1+2j\omega$	x_0	$j\omega$
$+\frac{1}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$k_1+j\omega$	x_0	$2k_1+j\omega$	x_0	k_1	x_0	$j\omega$
$+\frac{1}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$k_1+j\omega$	x_0	$2k_1+j\omega$	x_0	k_1	x_0	$-j\omega$
$+\frac{3}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1+j\omega$	x_0	$k_1+2j\omega$	x_0	$3j\omega$
$+\frac{3}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1+j\omega$	x_0	$k_1+2j\omega$	x_0	$j\omega$
$+\frac{3}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1+j\omega$	x_0	k_1	x_0	$j\omega$
$+\frac{3}{2}$	k_2^2	k_1	x_0	$2k_1$	x_0	$3k_1$	x_0	$2k_1+j\omega$	x_0	k_1	x_0	$-j\omega$
+ complex conjugates												
+												

2) Let us now consider a harmonic input

$$i(t)=\cos\omega t=\frac{1}{2}\left[e^{j\omega t}+e^{-j\omega t}\right],\qquad t\geqslant 0.$$

Its Laplace–Borel transform is

$$\mathfrak{g}_{i(t)}=\frac{1}{2}\left[(1-j\omega x_0)^{-1}+(1+j\omega x_0)^{-1}\right].$$

Applying (V.9) gives (Table IV). The time-domain response is then

$$v(t)=\frac{1}{k_1^2+\omega^2}\left[-k_1e^{-k_1t}+k_1\cos\omega t+\omega\sin\omega t\right]$$
$$+\frac{k_2}{2(k_1^2+\omega^2)}\left[\frac{1}{k_1}e^{-2k_1t}+\frac{2}{\omega}\sin\omega te^{-k_1t}-\frac{1}{k_1}\right]$$
$$+\frac{k_2}{2}\left[\frac{4}{k_1(k_1^2+4\omega^2)}e^{-k_1t}+\frac{k_1^2-\omega^2}{k_1(k_1^2+\omega^2)^2}e^{-2k_1t}\right.$$
$$-\frac{2\left[2k_1\omega\cos\omega t-(k_1^2-\omega^2)\sin\omega t\right]}{\omega(k_1^2+\omega^2)^2}e^{-k_1t}$$
$$\left.\cdot\frac{\left[k_1(k_1^2-\omega^2)-4k_1\omega^2\right]\cos 2\omega t-\left[2\omega(k_1^2-\omega^2)+2k_1^2\omega\right]\sin 2\omega t}{(k_1^2+4\omega^2)(k_1^2+\omega^2)^2}\right]+\cdots.$$

The steady-state response may be obtained directly from Table III by considering only those terms which do not vanish for large t

$$v(t)=\frac{1}{2}\left.\frac{x_0}{1+k_1x_0}\right|_{x_0=-1/j\omega}e^{j\omega t}$$
$$-\frac{1}{2}k_2\left.\frac{x_0^3}{(1+k_1x_0)(1+2k_1x_0)[1+(k_1+j\omega)x_0]}\right|_{x_0=-1/2j\omega}e^{2j\omega t}$$
$$-\frac{1}{2}k_2^2\left.\frac{x_0^5}{(1+k_1x_0)(1+2k_1x_0)[1+(k_1+j\omega)x_0][1+(2k_1+j\omega)x_0][1+(k_1+2j\omega)x_0]}\right|_{x_0=-1/3j\omega}e^{3j\omega t}$$
$$-\frac{1}{2}k_2^2\left.\frac{x_0^5}{(1+k_1x_0)(1+2k_1x_0)[1+(k_1+j\omega)x_0][1+(2k_1+j\omega)x_0][1+(k_1+2j\omega)x_0]}\right|_{x_0=-1/j\omega}e^{j\omega t}$$
$$-\frac{1}{2}k_2^2\left.\frac{x_0^5}{(1+k_1x_0)(1+2k_1x_0)[1+(k_1+j\omega)x_0][1+(2k_1+j\omega)x_0](1+k_1x_0)}\right|_{x_0=-1/j\omega}e^{j\omega t}$$
$$-\frac{1}{2}k_2^2\left.\frac{x_0^5}{(1+k_1x_0)(1+2k_1x_0)[1+(k_1+j\omega)x_0][1+(2k_1+j\omega)x_0](1+k_1x_0)}\right|_{x_0=+1/j\omega}e^{-j\omega t}$$
$$+\text{complex conjugates}+\cdots.$$

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