ALGEBRAIC COMPUTATION OF THE STATISTICS OF THE SOLUTION OF SOME NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

F. LAMNABHI-LAGARRIGUE
Laboratoire des Signaux et Systèmes
E.S.E., Plateau du Moulon
91190 Gif sur Yvette

M. LAMNABHI
Laboratoire d'Electronique et de Physique Appliquée
3, Avenue Descartes
94450 Limeil-Brevannes

This paper presents an algebraic method for computing the statistics of the solution of some stochastic non-linear differential equations by mean of the Volterra functional expansion. The symbolic calculus introduced, based on noncommutative variables and iterated integrals has the advantage of allowing easily the use of symbolic computation systems, like REDUCE or MACSYMA, to perform the manipulations. This becomes necessary as soon as one tries to get high order terms.

I. INTRODUCTION

Volterra or Wiener functional series has been widely used in the study of the statistical properties of the output of nonlinear systems with a random input. They have been introduced by Wiener in 1942 [10] for nonlinear circuits analysis. Since Wiener's early work, many authors have dealt with the subject. Among them, the papers by Barrett [2] and Bedrosian and Rice [3] are of a particulary importance. But the difficulties involved in obtaining the terms of the series and in performing the required integrations reduce the interest of the method.

Recently, a new approach of causal functionals was proposed, using noncommutative variables and iterated integrals [4]. In this approach, the input/output behaviour of a nonlinear system is represented in termes of a formal power series, called the generating power series. There is, in fact, a strong relationship between Volterra series and noncommutative generating power series. The present paper shows how to use this series to determine the statistics of the output of some nonlinear systems driven by a white Gaussian noise. It is a sequel to an earlier paper [7] where we described an algebraic algorithm for computing the response of a nonlinear system to deterministic inputs. The symbolic calculus introduced has the advantage of allowing easily the use of symbolic computation systems, like REDUCE or MACSYMA, to perform the manipulations. This becomes necessary as soon as one tries to get high order terms.

II. VOLTERRA SERIES

Let us consider a system described by the Volterra series [8],[9]

$$y(t) = h_o(t)+\int_0^t h_1(t,\tau_1)u(\tau_1)d\tau_1+\ldots+\int_0^t\int_0^{\tau_n}\ldots\int_0^{\tau_2} h_n(t,\tau_n,\ldots,\tau_1)u(\tau_1)\ldots u(\tau_n)d\tau_1\ldots d\tau_n+\ldots \tag{1}$$

where y(t) is the system output and u(t) is the system input. The kernels h_i are assumed to be analytic. Their Taylor expansion may be written

$$h_n(t,\tau_n,\ldots,\tau_1) = \sum_{i_o,i_1,\ldots,i_n\geq 0} h_n^{(i_o,i_1,\ldots,i_n)} \times \frac{(t-\tau_n)^{i_n}(\tau_n-\tau_{n-1})^{i_{n-1}}\ldots(\tau_2-\tau_1)^{i_1}\tau_1^{i_o}}{i_n!\ldots i_1!\ldots i_o!}$$

with respect to the new variables $\tau_1,\tau_2-\tau_1,\ldots,t-\tau_n$.

Each n-dimensional integral

$$\int_0^t\int_0^{\tau_n}\ldots\int_0^{\tau_2} \frac{(t-\tau_n)^{i_n}(\tau_n-\tau_{n-1})^{i_{n-1}}\ldots(\tau_2-\tau_1)^{i_1}\tau_1^{i_o}}{i_n!i_{n-1}!\ldots i_o!} u(\tau_1)\ldots u(\tau_n)d\tau_1\ldots d\tau_n$$

can be shown to be equal to the iterated integral

$$\underbrace{x_o\ldots x_o}_{i_n}\ x_1\ \underbrace{x_o\ldots x_o}_{i_{n-1}}\ x_1\ldots\ldots\ldots\ldots x_1\ \underbrace{x_o\ldots x_o}_{i_o}$$

where the lettre x_o denotes the integration with respect to time and the letter x_1 the integration with respect to time after multiplying by the input u. This is a generalization of the well known formula

$$\int_0^t \frac{(t-\tau)^n}{n!} u(\tau)d\tau = \int_0^t d\tau_n \int_0^{\tau_n} \cdots \int_0^{\tau_2} d\tau_1 \int_0^{\tau_1} u(\tau_o)d\tau_o$$

This allow to write (1) symbolically in the form

$$g = \sum_{n\geq 0} \sum_{i_o,i_1,\ldots,i_n\geq 0} h_n^{(i_o,i_1,\ldots,i_n)} x_o^{i_n} x_1 \ldots x_o^{i_1} x_1 x_o^{i_o}$$

g is called the *noncommutative generating power series* associated with the system. This power series can, as we shall see in a next section, be derived directly from the nonlinear differential equations governing the dynamic of a system.

Of course this is a noncommutative series because

$$\int_0^t d\tau_1 \int_0^{\tau_1} u(\tau_2)d\tau_2 \neq \int_0^t u(\tau_1)d\tau_1 \int_0^{\tau_2} d\tau_2$$

that is $x_o x_1 \neq x_1 x_o$.

III. <u>NONCOMMUTATIVE GENERATING POWER SERIES</u>

Let $X = \{x_o,x_1\}$ be a finite alphabet and X^* the monoid generated by X. An element of X^* is a word, i.e. a sequence $x_{j_\nu}\ldots x_{j_o}$ of letters of the alphabet. The product of two words $x_{j_\nu}\ldots x_{j_o}$ and $x_{k_\mu}\ldots x_{k_o}$ is the concatenation $x_{j_\nu}\ldots x_{j_o}x_{k_\mu}\ldots x_{k_o}$. The neutral element is called the empty word and denoted by 1. A formal power series with real or complex coefficients is written as a formal sum

$$g = \sum_{w\in X^*}(g,w)w \ , \quad (g,w) \in \mathbb{R} \text{ or } \mathbb{C}.$$

Let g_1 and g_2 be two formal power series, the following operations are defined :

<u>Addition</u> $$g_1 + g_2 = \sum_{w\in X^*} \Big[(g_1,w) + (g_2,w)\Big]w$$

<u>Cauchy product</u> $$g_1 \cdot g_2 = \sum_{w\in X^*} \Big[\sum_{w_1w_2=w} (g_1,w_1)(g_2,w_2)\Big]w$$

<u>Shuffle product</u> $$g_1 \,\omega\, g_2 = \sum_{w_1,w_2\in X^*} (g_1,w_1)(g_2,w_2)w_1\omega w_2$$

The shuffle product of two words consists of mixing the letters of the two words keeping the order of each one. For example

$$x_o x_1 ш x_1 x_p = x_o x_1 x_1 x_o + x_o x_1 x_1 x_o + x_o x_1 x_o x_1 + x_1 x_o x_1 x_o$$

$$+ x_1 x_o x_o x_1 + x_1 x_o x_o x_1$$

$$= 2x_o x_1^2 x_o + x_o x_1 x_o x_1 + x_1 x_o x_1 x_o + x_1 x_o^2 x_1$$

Let us now consider two generating power series g_1 and g_2 associated respectively with the output systems $y_1\{t,u(t)\}$ and $y_2\{t,u(t)\}$. It can be shown [4] that the generating power series associated with the product

$$y_1\{t,u(t)\} \times y_2\{t,u(t)\}$$

is the shuffle product of the generating power series g_1 and g_2.

IV. DERIVATION OF GENERATING POWER SERIES

In this section, we describe an algorithm for finding algebraically the generating power series associated with the solution of a nonlinear forced differential equation. The equation we are going to consider is

$$Ly(t) + \sum_{i=2}^{m} a_i y^i(t) = u(t)$$

$$L = \sum_{i=0}^{n} \ell_i \frac{d^i}{dt^i}, \qquad (\ell_n = 1)$$

or, in its integral form

$$y(t) + \ell_{n-1} \int_0^t y(\tau_1)d\tau_1 + \ell_{n-2} \int_0^t d\tau_2 \int_0^{\tau_2} y(\tau_1)d\tau_1$$

$$+ \dots + \ell_o \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots d\tau_2 \int_0^{\tau_2} y(\tau_1)d\tau_1$$

$$+ \sum_{i=2}^{m} a_i \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots d\tau_2 \int_0^{\tau_2} y^i(r_1)d\tau_1 =$$

$$= \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots d\tau_2 \int_0^{\tau_2} u(\tau_1)d\tau_1 \qquad (2)$$

Here we assume, for simplicity's sake, zero initial conditions.

Let g denotes the generating power series associated with y(t), then (2) can be written symbolically

$$g + \sum_{j=0}^{n-1} \ell_j x_o^{n-i} g + x_o^n \sum_{i=1}^{m} a_i \underbrace{g\omega\ldots\omega g}_{\text{i times}} = x_o^{n-1} x_1$$

where $\underbrace{g\omega\ldots\omega g}_{\text{i times}}$ corresponds, according to the previous theorem, to the nonlinear functional $y^i(t)$.

This algebraic equation can be solved iteratively, following the recursive scheme

$$g = g_1 + g_2 + \ldots + g_n + \ldots$$

with

$$g_1 = \left(1 + \sum_{i=0}^{n-1} \ell_i x_o^{n-i}\right)^{-1} x_o^{n-1} x_1$$

and

$$g_n = -\left(1 + \sum_{i=0}^{n-1} \ell_i x_o^{n-i}\right)^{-1} x_o^n \sum_{i=2}^{m} a_i \sum_{\nu_1+\nu_2+\ldots+\nu_i=n} g_{\nu_1} \omega g_{\nu_2} \ldots \omega g_{\nu_i}$$

To have the closed form expression of g_i, one only need to compute the shuffle product of noncommutative power series of the form

$$\left(1-a_o x_o\right)^{-1} x_{i_1} \left(1-a_1 x_o\right)^{-1} x_{i_2} \ldots x_{i_p} \left(1-a_p x_o\right)^{-1} \quad ; \; i_1, i_2, \ldots, i_p \in \{0,1\}.$$

This results from the following proposition [6] :

Proposition 1 : Given two formal power series

$$g_1^p = \left(1-a_o x_o\right)^{-1} x_{i_1} \left(1-a_1 x_o\right)^{-1} x_{i_2} \ldots x_{i_p} \left(1-a_p x_o\right)^{-1} = g_1^{p-1} x_{i_p} \left(1-a_p x_o\right)^{-1}$$

and

$$g_2^q = \left(1-b_o x_o\right)^{-1} x_{j_1} \left(1-b_1 x_o\right)^{-1} x_{j_2} \ldots x_{j_q} \left(1-b_q x_o\right)^{-1} = g_2^{q-1} x_{j_q} \left(1-b_q x_o\right)^{-1}$$

where p and q belongs to $\mathbb{N}$, the subscripts $i_1, \ldots, i_p, j_1, \ldots, j_q$ to $\{0,1\}$ and a_i, b_j to $\mathbb{C}$; the shuffle product is given by induction on the length by

$$g_1^p \omega g_2^q = \left(g_1^p \omega g_2^{q-1}\right) x_{j_q} \left[1 - (a_p + b_q) x_o\right]^{-1} + \left(g_1^{p-1} \omega g_2^q\right) x_{i_p} \left[1 - (a_p + b_q) x_o\right]^{-1}$$

with $\left(1 - a x_o\right)^{-1} \omega \left(1 - b x_o\right)^{-1} = \left[1 - (a+b) x_o\right]^{-1}$.

Using this proposition, g_i is obtained as a finite sum of expressions of the form :

$$\left(1-a_0x_0\right)^{-1} x_{i_1}\left(1-a_1x_0\right)^{-1} x_{i_2}\cdots\cdots x_{i_n}\left(1-a_nx_0\right)^{-1} \quad ; \; i_1,\ldots,i_n\in\{0,1\} \tag{3}$$

<u>Example 1</u> :

The technique presented above is now used to compute the generating power series associated with the nonlinear differential equation

$$\dot{y}(t) + k_1y(t) + k_2y^2(t) = u(t) \tag{4}$$

Its integral form is :

$$y(t) + k_1\int_0^t y(\tau)\,d\tau + k_2\int_0^t y^2(\tau)\,d\tau = \int_0^t u(\tau)\,d\tau$$

where we assume a zero initial condition.

Thus, the generating power series g is simply the solution of

$$g + k_1x_0g + k_2x_0g\omega g = x_1$$

This equation is solved iteratively by a computer program (table 1).

$$
\begin{array}{rrll}
g = & 1 & & \begin{array}{ccc} & x_1 & \\ k_1 & & 0 \end{array} \\
& -2 & k_2 & \begin{array}{ccccccc} & x_0 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& +4 & k_2^2 & \begin{array}{ccccccccccc} & x_0 & & x_1 & & x_0 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& +12 & k_2^2 & \begin{array}{ccccccccccc} & x_0 & & x_0 & & x_1 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & 3k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& -8 & k_2^3 & \begin{array}{ccccccccccccccc} & x_0 & & x_1 & & x_0 & & x_1 & & x_0 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & k_1 & & 2k_1 & & k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& -24 & k_2^3 & \begin{array}{ccccccccccccccc} & x_0 & & x_0 & & x_1 & & x_1 & & x_0 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & 3k_1 & & 2k_1 & & k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& -72 & k_2^3 & \begin{array}{ccccccccccccccc} & x_0 & & x_0 & & x_1 & & x_0 & & x_1 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & 3k_1 & & 2k_1 & & 3k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& -24 & k_2^3 & \begin{array}{ccccccccccccccc} & x_0 & & x_1 & & x_0 & & x_0 & & x_1 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & k_1 & & 2k_1 & & 3k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& -144 & k_2^3 & \begin{array}{ccccccccccccccc} & x_0 & & x_0 & & x_0 & & x_1 & & x_1 & & x_1 & & x_1 & \\ k_1 & & 2k_1 & & 3k_1 & & 4k_1 & & 3k_1 & & 2k_1 & & k_1 & & 0 \end{array} \\
& \vdots & &
\end{array}
$$

<u>table 1</u>

where the symbolic notation

$$\begin{array}{ccccccccc} & x_{i_1} & & x_{i_2} & \cdots\cdots & & x_{i_n} & & \\ a_0 & & a_1 & & & a_{n-1} & & a_n \end{array} \quad ; \; i_1,\ldots,i_n \in \{0,1\}$$

stands for $\left(1+a_o x_o\right)^{-1} x_{i_1}\left(1+a_1 x_o\right)^{-1} x_{i_2}\ldots\left(1+a_{n-1}x_o\right)^{-1} x_{i_n}\left(1+a_n x_o\right)^{-1}$

Remark : The expansion, in table 1, is equivalent to the Volterra series expansion of the solution up to order 5. The algebraic closed form expression of triangular Volterra kernels can be easily deduced from it [6], since the expression (3) is the symbolic representation of the n-dimensional integral

$$\int_o^t\int_o^{\tau_n}\ldots\int_o^{\tau_2} e^{a_o(t-\tau_n)} e^{a_1(\tau_n-\tau_{n-1})}\ldots e^{a_{n-1}(\tau_2-\tau_1)} e^{a_n\tau_1} u_{i_n}(\tau_1)\ldots u_{i_2}(\tau_{n-1}) u_{i_1}(\tau_n)\, d\tau_1\ldots d\tau_n$$

where

$$\{i_1,\ldots,i_n\} \in \{0,1\}, \quad u_o(\tau) = 1 \quad \text{and} \quad u_1(\tau) = u(\tau),$$

which we shall denote later by

$$\left[\left(1 - a_o x_o\right)^{-1} x_{i_1}\left(1 - a_1 x_o\right)^{-1} x_{i_2}\ldots x_{i_n}\left(1 - a_n x_o\right)^{-1}\right]_o^t$$

V. A SYMBOLIC CALCULUS FOR THE OUTPUT STATISTICS

In the previous section, we used noncommutative generating power series to derive, by simple algebraic manipulations, a functional expansion (i.e. the Volterra series) of the solution of some nonlinear differential equations. In the following the derived series is used to obtain closed form expressions for the *moments* and *correlations* of the output of a Volterra system driven by Gaussian white noise.

Output moments

Let us consider a zero-mean Gaussian process u(t) with correlation function

$$E[u(t)u(\tau)] = \sigma^2\delta(t-\tau)$$

where δ is the Dirac delta function (a process of this kind is usually called a white Gaussian noise) [1]. The first-order moment of the output $E[y(t)]$ is then obtained by a simple rule on each term (3) of the functional expansion of y(t). This rule results [5] from the classical rules of stochastic differential calculus and is given by induction on the length by

$$E\left[\int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_2} e^{a_o(t-\tau_n)} u_{i_1}(\tau_n) e^{a_1(\tau_n-\tau_{n-1})} u_{i_2}(\tau_{n-1})\cdots u_{i_n}(\tau_1) e^{a_n\tau_1} d\tau_1\cdots d\tau_n\right]$$

$$= \begin{cases} \int_0^t e^{a_o(t-\tau_n)} d\tau_n E\left[\int_0^{\tau_n} e^{a_1(\tau_n-\tau_{n-1})} u_{i_2}(\tau_{n-1})\cdots u_{i_n}(\tau_1) e^{a_n\tau_1} d\tau_1\cdots d\tau_{n-1}\right] & \underline{\text{if } i_1 = 0} \\ \left(\frac{\sigma^2}{2}\right)\int_0^t e^{a_o(t-\tau_n)} d\tau_n E\left[\int_0^{\tau_n} e^{a_2(\tau_n-\tau_{n-2})} u_{i_3}(\tau_{n-2})\cdots u_{i_n}(\tau_1) e^{a_n\tau_1} d\tau_1\cdots d\tau_{n-2}\right] & \underline{\text{if } i_1 = i_2 = 1} \\ 0 \quad \underline{\text{otherwise}} & \end{cases}$$

In fact, the moment of the output function $y(t)$ can be obtained directly from its associated generating power series by the following algebraic rule which is a symbolic representation of the previous one :

$$<\left(1 - a_o x_o\right)^{-1} x_{i_1}\left(1 - a_1 x_o\right)^{-1} x_{i_2}\cdots x_{i_n}\left(1 - a_n x_o\right)^{-1}> \tag{5}$$

$$= \begin{cases} \left(1-a_o x_o\right)^{-1} x_o<\left(1-a_1 x_o\right)^{-1} x_{i_2}\cdots x_{i_n}\left(1-a_n x_o\right)^{-1}> & \underline{\text{if } i_1 = 0} \\ \left(\frac{\sigma^2}{2}\right)\left(1-a_o x_o\right)^{-1} x_o<\left(1-a_2 x_o\right)^{-1} x_{i_3}\cdots x_{i_n}\left(1-a_n x_o\right)^{-1}> & \underline{\text{if } i_1 = i_2 = 1} \\ 0 \quad \underline{\text{otherwise}} & \end{cases}$$

This results in a rational fraction in the only variable x_o. Its decomposition into partial fractions and the following lemma give its corresponding expression in time.

<u>Lemma</u> : The rational fraction

$$\left(1 - a x_o\right)^{-p}$$

corresponds to the exponential polynomial

$$\left\{\sum_{j=0}^{p-1} \binom{j}{p-1} \frac{a^j t^j}{j!}\right\} e^{at} .$$

In order to illustrate the use of this rule, consider again the nonlinear differential equation (4). For the first-order moment of $y(t)$ we have then

$$\langle g \rangle \;=\; \begin{array}{l} -2(\frac{\sigma^2}{2})k_2 \quad k_1 \; x_0 \; 2k_1 \; x_0 \; 0 \\ -24(\frac{\sigma^2}{2})^2 k_2^3 \quad k_1 \; x_0 \; 2k_1 \; x_0 \; 3k_1 \; x_0 \; k_1 \; x_0 \; 2k_1 \; x_0 \; 0 \\ -144(\frac{\sigma^2}{2})^2 k_2^3 \quad k_1 \; x_0 \; 2k_1 \; x_0 \; 3k_1 \; x_0 \; 4k_1 \; x_0 \; 2k_1 \; x_0 \; 0 \\ \vdots \end{array}$$

By decomposing into partial fractions, we get the corresponding time function :

$$\begin{aligned} E[y(t)] = {} & 2\left(\frac{\sigma^2}{2}\right)\frac{1}{k_1^2}\left(e^{-k_1 t} - \frac{3}{4} - \frac{1}{4}k_1 t\; e^{-2k_1 t}\right) \\ & + 24\left(\frac{\sigma^2}{2}\right)^2 \frac{1}{k_1^5}\left[\left(\frac{1}{4}+\frac{1}{2}k_1 t\right)e^{-k_1 t} - \left(\frac{3}{4} - \frac{1}{4}k_1 t\right)e^{-2k_1 t} + \frac{1}{12}e^{-3k_1 t} + \frac{1}{12}\right] \\ & + 144\left(\frac{\sigma^2}{2}\right)^2 \frac{1}{k_1^5}\left[\frac{1}{6}e^{-k_1 t} - \frac{1}{4}k_1 t e^{-2k_1 t} - \frac{1}{6}e^{-3k_1 t} + \frac{1}{48}e^{-4k_1 t} - \frac{1}{48}\right] \\ & + \ldots \end{aligned}$$

High order moments $E[y^n(t)]$ result in the same way from the series $\underbrace{g\omega g \ldots \omega g}_{n \text{ times}}$.

<u>Output autocorrelation</u>

In this section, we are interested in computing the output autocorrelation function defined by

$$R_{yy}(t_1,t_2) \;=\; E[y(t_1)y(t_2)]$$

where y(t) is the functional expansion derived previously. As y(t) is a sum of expressions of the form (3), we only need to compute the partial autocorrelation functions :

$$\begin{aligned} E\Bigg[& \left(\int_0^{t_1} \cdots \int_0^{\tau_2} e^{a_o(t_1-\tau_p)} u_{i_1}(\tau_p)\ldots u_{i_p}(\tau_1) e^{a_p \tau_1} d\tau_1 \ldots d\tau_p\right) \\ & \left(\int_0^{t_2} \cdots \int_0^{\tau_2} e^{b_o(t_2-\tau_q)} u_{j_1}(\tau_q)\ldots u_{j_q}(\tau_1) e^{b_q \tau_1} d\tau_1 \ldots d\tau_q\right)\Bigg] \end{aligned} \qquad (6)$$

Let us assume $t_2 > t_1$; then the second integral may be decomposed as follows :

$$\int_0^{t_2}\int_0^{\tau_q}\cdots\int_0^{\tau_2} e^{b_o(t_2-\tau_q)} u_{j_1}(\tau_q)\cdots u_{j_q}(\tau_1) e^{b_q\tau_1} d\tau_1\cdots d\tau_q =$$

$$\left(\int_0^{t_1}\int_0^{\tau_q}\cdots\int_0^{\tau_2} e^{b_o(t_2-\tau_q)} u_{j_1}(\tau_q)\cdots u_{j_q}(\tau_1) e^{b_q\tau_1} d\tau_1\cdots d\tau_q\right)\left(\int_{t_1}^{t_2} e^{b_o\tau_1} d\tau_1\right)$$

$$+\left(\int_0^{t_1}\int_0^{\tau_{q-1}}\cdots\int_0^{\tau_2} e^{b_1(t_2-\tau_{q-1})} u_{j_2}(\tau_{q-1})\cdots u_{j_q}(\tau_1) e^{b_q\tau_1} d\tau_1\cdots d\tau_{q-1}\right)$$

$$\times\left(\int_{t_1}^{t_2}\int_{t_1}^{\tau_2} e^{b_o(t_2-\tau_2)} u_{j_1}(\tau_1) e^{b_1\tau_1} d\tau_1 d\tau_2\right)$$

$$+\ \cdots$$

$$+\left(\int_0^{t_1} e^{b_q\tau_1} d\tau_1\right)\left(\int_{t_1}^{t_2}\int_{t_1}^{\tau_q}\cdots\int_{t_1}^{\tau_2} e^{b_o(t_2-\tau_q)} u_{j_1}(\tau_q)\cdots u_{j_q}(\tau_1) e^{b_q\tau_1} d\tau_1\cdots d\tau_q\right)$$

and this leads for the expression (6) to

$$E\Big[\left(\int_0^{t_1}\cdots\int_0^{\tau_2} e^{a_o(t_1-\tau_p)} u_{i_1}(\tau_p)\cdots u_{i_p}(\tau_1) e^{a_p\tau_1} d\ _1\cdots d\tau_p\right)$$

$$\left(\int_{t_1}^{t_2}\cdots\int_{t_1}^{\tau_2} e^{b_o(t_2-\tau_q)} u_{j_1}(\tau_q)\cdots u_{j_q}(\tau_1) e^{b_q\tau_1} d\tau_1\cdots d\tau_q\right)\Big] E\left[\int_{t_1}^{t_2} e^{b_o\tau_1} d\tau_1\right]$$

$$+\ \cdots$$

$$+\ E\Big[\left(\int_0^{t_1}\cdots\int_0^{\tau_2} e^{a_o(t_1-\tau_p)} u_{i_1}(\tau_p)\cdots u_{i_p}(\tau_1) e^{a_p\tau_1} d\tau_1\cdots d\tau_p\right)\left(\int_0^{t_1} e^{b_q\tau_1} d\tau_1\right)\Big]$$

$$\times\ E\left[\int_{t_1}^{t_2}\cdots\int_{t_1}^{\tau_2} e^{b_o(t_2-\tau_q)} u_{j_1}(\tau_q)\cdots u_{j_q}(\tau_1) e^{b_q\tau_1} d\tau_1\cdots d\tau_q\right]$$

where we used statistical independance between integrals over $[0,t_1]$ and $[t_1,t_2]$.

Now recalling that the product of two iterated integrals over the same interval corresponds to the shuffle product of their generating power series, we derive the following symbolic rule :

$$<\left[\left(1-a_o x_o\right)^{-1} x_{i_1} \cdots x_{i_p}\left(1-a_p x_o\right)^{-1}\right]_o^{t_1} \times \left[\left(1-b_o x_o\right)^{-1} x_{j_1} \cdots x_{j_q}\left(1-b_q x_o\right)^{-1}\right]_o^{t_2}>$$

$$= \left[<\left\{\left(1-a_o x_o\right.^{-1} x_{i_1} \cdots x_{i_p}\left(1-a_p x_o\right)^{-1}\right\} \omega \left\{\left(1-b_o x_o\right)^{-1} x_{j_1} \cdots x_{j_q}\left(1-b_q x_o\right)^{-1}\right\}>\right]_o^{t_1} \times$$
$$\left[<\left(1-b_o x_o\right)^{-1}>\right]_{t_1}^{t_2} \qquad (7)$$

$$+ \left[<\left\{\left(1-a_o x_o\right)^{-1} x_{i_1} \cdots x_{i_p}\left(1-a_p x_o\right)^{-1}\right\} \omega \left\{\left(1-b_1 x_o\right)^{-1} x_{j_2} \cdots x_{j_q}\left(1-b_q x_o\right)^{-1}\right\}>\right]_o^{t_1} \times$$
$$\left[<\left(1-b_o x_o\right)^{-1} x_{j_1}\left(1-b_1 x_o\right)^{-1}>\right]_{t_1}^{t_2}$$

$$+ \ldots$$
$$+ \left[<\left\{\left(1-a_o x_o\right)^{-1} x_{i_1} \cdots x_{i_p}\left(1-a_p x_o\right)^{-1}\right\} \omega \left(1-b\ x_o\right)^{-1}>\right]_o^{t_1} \times$$
$$\left[<\left(1-b_o x_o\right)^{-1} x_{j_1} \cdots x_{j_q}\left(1-b_q x_o\right)^{-1}>\right]_{t_1}^{t_2}$$

Example 2 : Let us consider the linear stochastic differential equation

$$\dot{v} = -\alpha v + u(t), \quad v(0) = c$$

where $u(t)$ is a white Gaussian process and c is assumed independent of $u(t)$. The generating power series associated with $v(t)$ is given by

$$<g> = \left(1 + \alpha x_o\right)^{-1} x_1 + c\left(1 + \alpha x_o\right)^{-1} .$$

In accordance with (5) and (7), $v(t)$ has mean value

$$<g> = E(c)\left(1 + \alpha x_o\right)^{-1}$$

that is $$E\left(v(t)\right) = E(c)e^{-\alpha t}$$

and covariance

$$<[g]_o^{t_1}[g]_o^{t_2}> = <\left[\left(1+\alpha x_o\right)^{-1}x_1\right]_o^{t_1}\left[\left(1+\alpha x_o\right)^{-1}x_1\right]_o^{t_2}>$$

$$+\left\{<c\left[\left(1+\alpha x_o\right)^{-1}x_1\right]_o^{t_1}\left[\left(1+\alpha x_o\right)^{-1}\right]_o^{t_2}>=0\right\}$$

$$+\left\{<c\left[\left(1+\alpha x_o\right)^{-1}\right]_o^{t_1}\left[\left(1+\alpha x_o\right)^{-1}x_1\right]_o^{t_2}>=0\right\}$$

$$+ <c^2\left[\left(1+\alpha x_o\right)^{-1}\right]_o^{t_1}\left[\left(1+\alpha x_o\right)^{-1}\right]_o^{t_2}> =$$

$$= \left[<\left(1+\alpha x_o\right)^{-1}x_1\,\omega\,\left(1+\alpha x_o\right)^{-1}x_1>\right]_o^{t_1}\left[\left(1+\alpha x_o\right)^{-1}\right]_{t_1}^{t_2} +$$

$$\left\{<\left[\left(1+\alpha x_o\right)^{-1}x_1\right]_o^{t_1}\left[\left(1+\alpha x_o\right)^{-1}x_1\right]_o^{t_2}>=0\right\} + <c^2\left[\left(1+\alpha x_o\right)^{-1}\right]_o^{t_1}\left[\left(1+\alpha x_o\right)^{-1}\right]_o^{t_2}>$$

that is $E\left(v(t_1)v(t_2)\right) = \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha t_1}\right)e^{-\alpha(t_2-t_1)} + E(c^2)e^{-\alpha(t_1+t_2)}$

If we begin with an $N\left(o,\frac{\sigma^2}{2\alpha}\right)$-distributed c then v(t) is a stationary Gaussian process (sometimes called a colored noise) such that

$$E\left(v(t)\right) = 0 \quad \text{and} \quad E\left(v(t_1)v(t_2)\right) = \frac{\sigma^2}{2\alpha}e^{-\alpha|t_2-t_1|}$$

<u>Remark</u> : The techniques presented in this paper still apply for a rationally correlated noise input ; for example considering example 1 in conjonction with example 2 allows to deal with the nonlinear system

$$\dot{y}(t) + k_1y(t) + k_2y^2(t) = v(t)$$

where v(t) is a Gaussian colored noise. The following equivalent nonlinear system with a white Gaussian noise input u(t) is then considered :

$$\begin{cases} \dot{v} = -\ v + u(t) \\ \dot{y} = -\ k_1y - k_2y^2 + v(t). \end{cases}$$

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