Functional analysis of nonlinear circuits: a generating power series approach

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Abstract: In the paper, an algorithm, by means of which a large class of nonlinear electronic circuits can be analysed, is described. This algorithm is based on a recent algebraic approach to Volterra functional expansions using noncommutative generating power series; this approach allows a natural generalisation, to the nonlinear domain, of the symbolic operational calculus of Heaviside, widely used in linear system theory. Moreover, it has the advantage, compared with the method using the multidimensional Fourier or Laplace transforms, of allowing an easy implementation on a computer. Some examples using a developed program are presented.

1 Introduction

The input/output behaviour of a large class of nonlinear circuits can be described by the Volterra functional expansion [12, 13]

$$y(t) = \int_0^t h_1(t, \tau_1)u(\tau_1)\,d\tau_1 + \int_0^t \int_0^{\tau_2} h_2(t, \tau_2, \tau_1)u(\tau_1)u(\tau_2)\,d\tau_1\,d\tau_2 + \cdots \tag{1}$$

where $y(t)$ is the circuit output and $u(t)$ is the circuit input (assumed to be scalar for simplicity sake). h_n is the nth-order Volterra kernel. This expansion is a generalisation of the well-known convolution integral

$$y(t) = \int_0^t h_1(t, \tau_1)u(\tau_1)\,d\tau_1$$

used in linear system theory.

Volterra's work on functional expansions dates from the beginning of the century and is summarised in his book [14]. The original application of Volterra functionals to the analysis of nonlinear circuits is due to Wiener [16]. Since Wiener's early work, a considerable number of publications have been devoted to the subject. For an overview, the reader is referred to References 2, 3 and 15 and the references therein. Although the Volterra series has been successfully used in many applications, it has not received a great deal of attention from engineers and designers. The reason for this seems to be the tedious computations involved in the determination of Volterra kernels. Moreover, it is often difficult to obtain the response for a given input, even when the Volterra kernels are known.

The algebraic approach to nonlinear functional expansions reviewed in Reference 6 and based on noncommutative generating power series, offers a powerful and systematic tool for analysing a large class of nonlinear systems. Following this approach, the aim of this paper is to develop general algorithms for analysing practical circuits described in terms of elementary components such as nonlinear resistors, capacitors, inductors and controlled sources. A formulation for describing the equations of practical circuits is given, and general conditions for the existence of their solution in term of a Volterra functional expansion are derived. This formulation has the advantage of allowing the use of a computer. The manipulations involved in each step of the proposed analysis have been automatised on a computer using the Lisp language (List Programming) [17] which is well adapted to symbolic and algebraic manipulations. Note that such a systematisation of the computations has been carried out by some authors [1, 4] in the case of harmonic stationary analysis using multidimensional Laplace transforms. But, even in this particular case, the derivation of the functional expansion is done numerically, so that one needs to repeat the whole computations for each set of input frequencies; this is not the case in our approach. Moreover, the method proposed in this paper allows, for the first time in our knowledge, the automatic derivation of the transitory behaviour from the Volterra functional expansion. Therefore, it offers an alternative to the technique of association of variables [8] which seems difficult to implement on a computer.

In Sections 1 and 2, we briefly review the concept of generating power series and their links with Volterra series. Section 3 is devoted to the description of the algorithm which allows a convenient set of nonlinear integro-differential and algebraic equations describing the behaviour of general nonlinear circuits to be obtained. Then we show, in Section 4, how to derive the generating power series from this set of equations. Finally, we describe, in Section 5, how one can derive, from the generating power series, the analytic response, in term of exponential polynomials, to typical inputs like Dirac functions, steps, harmonics etc.

2 Introduction to noncommutative generating power series

Let us consider the nonlinear differential equation

$$\dot{y}(t) = A_0\,y(t) + A_1 y(t)u(t) \tag{2}$$

where $y(t)$ is an n-dimensional vector and A_0, A_1 are square matrices (equations of this form are called, in control theory, bilinear). Both sides of the differential eqn. 2 can be integrated to obtain

$$y(t) = y(0) + A_0 \int_0^t y(\sigma_1)\,d\sigma_1 + A_1 \int_0^t u(\sigma_1)y(\sigma_1)\,d\sigma_1 \tag{3}$$

This expression can be solved by repeated substitutions. For example,

$$y(\sigma_1) = y(0) + A_0\,y(0) \int_0^{\sigma_1} d\sigma_2 + A_1 y(0) \int_0^{\sigma_1} u(\sigma_2)\,d\sigma_2$$

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can be substituted into eqn. 3 to obtain

$$y(t) = y(0) + A_0 y(0) \int_0^t d\sigma_1 + A_0^2 y(0) \int_0^t d\sigma_1 \times \int_0^{\sigma_1} d\sigma_2 + A_0 A_1 y(0) \int_0^t d\sigma_1 \int_0^{\sigma_1} u(\sigma_2)\, d\sigma_2 + A_1 y(0) \int_0^t u(\sigma_1)\, d\sigma_1 + A_1 A_0 y(0) \int_0^t u(\sigma_1)\, d\sigma_1 \times \int_0^{\sigma_1} d\sigma_2 + A_1^2 y(0) \int_0^t u(\sigma_1)\, d\sigma_1 \int_0^{\sigma_1} u(\sigma_2)\, d\sigma_2$$

Repeating this process indefinitely yields the well-known Peano-Baker formula:

$$y(t) = y(0) + \sum_{v \geq 0} \sum_{j_0, j_1, \ldots, j_v = 0}^{l} A_{j_v}, \ldots, A_{j_1} A_{j_0} y(0) \times \int_0^t d\xi_{j_v}, \ldots, d\xi_{j_0} \tag{4}$$

where the iterated integral $\int d\xi_{j_v}, \ldots, d\xi_{j_0}$ is defined by induction on the length by

$$\xi_0(t) = t, \qquad \xi_1(t) = \int_0^t u(\tau)\, d\tau,$$

$$\int_0^t d\xi_{j_v}, \ldots, d\xi_{j_0} = \int_0^t d\xi_{j_v}(\tau) \int_0^\tau d\xi_{j_{v-1}}, \ldots, d\xi_{j_0}$$

If we denote by the letter x_0 the integration with respect to time and by the letter x_1 the integration with respect to time after multiplying by the function $u(t)$, eqn. 4 can be written symbolically in the form:

$$g = y(0) + \sum_{v \geq 0} \sum_{j_0, j_1, \ldots, j_v = 0}^{l} A_{j_v}, \ldots, A_{j_1} A_{j_0} y(0) x_{j_v}, \ldots, x_{j_0}$$

g is called the noncommutative generating power series (GPS) associated with $y(t)$. Of course, this is a noncommutative series because

$$\int_0^t d\tau_1 \int_0^{\tau_1} u(\tau_2)\, d\tau_2 \neq \int_0^t u(\tau_1)\, d\tau_1 \int_0^{\tau_1} d\tau_2$$

that is, $x_0 x_1 \neq x_1 x_0$.

3 Link between Volterra series and noncommutative generating power series

In this Section we briefly recall the relation between Volterra series and generating power series. More details concerning this question and related subjects may be found in Reference 6.

Let us consider a nonlinear circuit described by the Volterra series, eqn. 1. For the sake of simplicity, we consider here circuits with only a single input. For most practical nonlinear circuit with memory, the kernels h_n, $n \in N$, are analytic functions. Their Taylor expansion may be written

$$h_n(t, \tau_n, \ldots, \tau_1) = \sum_{i_0, i_1, \ldots, i_n \geq 0} c_{i_0, i_1, \ldots, i_n} \times \frac{(t-\tau_n)^{i_n}(\tau_n - \tau_{n-1})^{i_{n-1}}, \ldots, (\tau_2 - \tau_1)^{i_1} \tau_1^{i_0}}{i_n! \ldots, i_1! \, i_0!}$$

with respect to the variables τ_1, $\tau_2 - \tau_1$, $\ldots$, $\tau_n - \tau_{n-1}$, $t - \tau_n$. On the other hand, each n-dimensional integral

$$\int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} \frac{(t-\tau_n)^{i_n}, \ldots, (\tau_2 - \tau_1)^{i_1} \tau_1^{i_0}}{i_n! \ldots, i_1! \, i_0!} \times u(\tau_1), \ldots, u(\tau_n)\, d\tau_1, \ldots, d\tau_n$$

can be shown to be equal to the iterated integral

$$\int_0^t \underbrace{d\xi_0, \ldots, d\xi_0}_{i_n} d\xi_1 \underbrace{d\xi_0, \ldots, d\xi_0}_{i_{n-1}} d\xi_1, \ldots, d\xi_1 \underbrace{d\xi_0, \ldots, d\xi_0}_{i_0} \tag{5}$$

This is a generalisation of the formula

$$\int_0^t \frac{(t-\tau)^n}{n!} u(\tau)\, d\tau = \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \times \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} d\tau_1 \int_0^{\tau_1} u(\tau_0)\, d\tau_0$$

Now, if we denote, as previously, by the letter x_0 the integration with respect to time, and by the letter x_1 the integration with respect to time after multiplying by the function $u(t)$, eqn. 1 can be written symbolically in the form:

$$g = g_1 + g_2 + \cdots + g_n + \cdots \tag{6}$$

where

$$g_n = \sum_{i_0, i_1, \ldots, i_n \geq 0} c_{i_0, \ldots, i_n} x_0^{i_n} x_1 x_0^{i_{n-1}}, \ldots, x_1 x_0^{i_0}$$

g is called the noncommutative generating power series associated with $y(t)$.

The expansion, eqn. 6, of the GPS associated with $y(t)$ is 'equivalent' to the Volterra series expansion of $y(t)$: g_n represents the contribution of the nth-order Volterra kernel; indeed, one can note that each term of g_n contains exactly n occurrences of the letter x_l representing the input $u(t)$.

Conversely, let us consider the noncommutative GPS eqn. 6. Given a function $u(t)$, g will define a functional $y(t)$ if we replace the word $x_0^{i_n} x_1 x_0^{i_{n-1}}, \ldots, x_0^{i_0}$ in g by the corresponding iterated integral, eqn. 5. Thus, the numerical value associated with g is

$$y(t) = y_1(t) + y_2(t) + \cdots + y_n(t) + \cdots$$

$$y_n(t) = \sum_{i_0, i_1, \ldots, i_n \geq 0} c_{i_0, \ldots, i_n} \times \int_0^t d\xi_0^{i_n}\, d\xi_1\, d\xi_0^{i_{n-1}}, \ldots, d\xi_1\, d\xi_0^{i_0} \quad n \geq 1$$

One can now state the important result [5] (also recalled in Reference 6):

Theorem: The product of two analytic causal functionals of the form of eqn. 1 is a functional of the same kind, the GPS of which is the shuffle product (see Appendix) of the two GPSs.

4 Description of nonlinear circuits

4.1 Assumptions on nonlinear circuits

Most of nonlinear electronic circuits encountered can be described in terms of elementary nonlinear components such as nonlinear resistors, capacitors, inductors and controlled and independent sources:

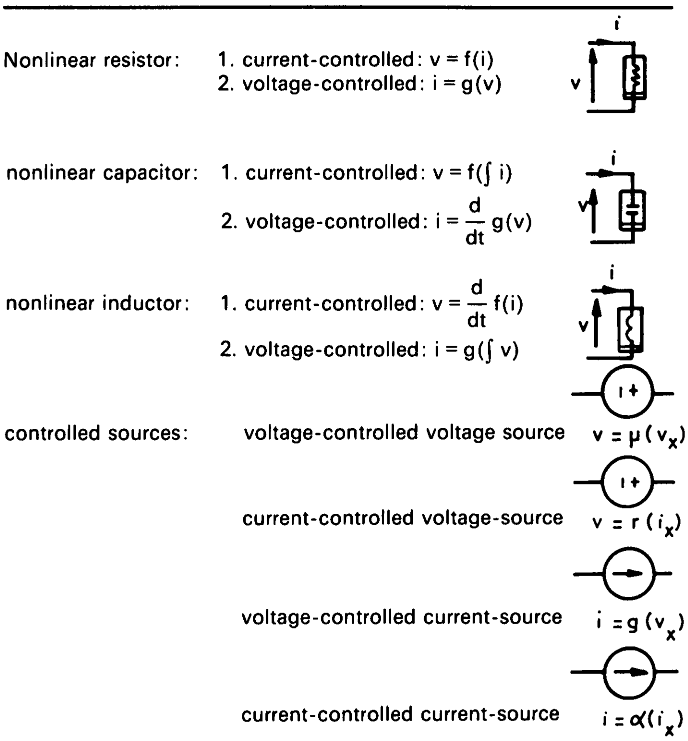

Fig. 1 *Representation of lumped electronic nonlinear elements*

where v and i denote, respectively, the voltage across a branch of the circuit and the current flowing in it; v_x and i_x are, respectively, a voltage and a current controlling variable. Representations 1 and 2 correspond, respectively, to impedance and admittance descriptions of the nonlinear element. Note that elements that operate in a monotonic region of their characteristic possess both representations. These components are generally operated in a region where their behaviour is described by a power-series expansion about their quiescient or DC points. These expansions can be expressed in one of the following general forms which correspond to the Taylor expansions of the functions f, g, μ, r and α:

$$
\begin{aligned}
w(t) &= \sum_{n\geq 1} a_n z^n(t) \\
w(t) &= \sum_{n\geq 1} b_n \left[\int_0^t z(\tau)\, d\tau \right]^n \\
w(t) &= \frac{d}{dt}\left[\sum_{n\geq 1} c_n z^n(t) \right]
\end{aligned} \tag{7}
$$

Depending on the nonlinear element considered, and on its representation (impedance or admittance), w and z may represent either a current or a voltage incremental variable. z is called the controlling variable and w the controlled one.

Note: even when both representations 1 and 2 exist for an element, it may be preferable to use the one, the power series expansion, eqn. 7, of which is more rapidly convergent.

Separating the summations in eqn. 7 into a linear part plus second- and higher-order terms suggests that each nonlinear element may be seen as a parallel (if w is a current) or a cascade (if w is a voltage) combination of a linear element ($n = 1$) and a strictly nonlinear element ($n \geq 2$); this leads to an equivalent representation of the nonlinear elements given in Fig. 2.

Let us first consider these strictly nonlinear elements as independent sources and modify the circuit by embedding

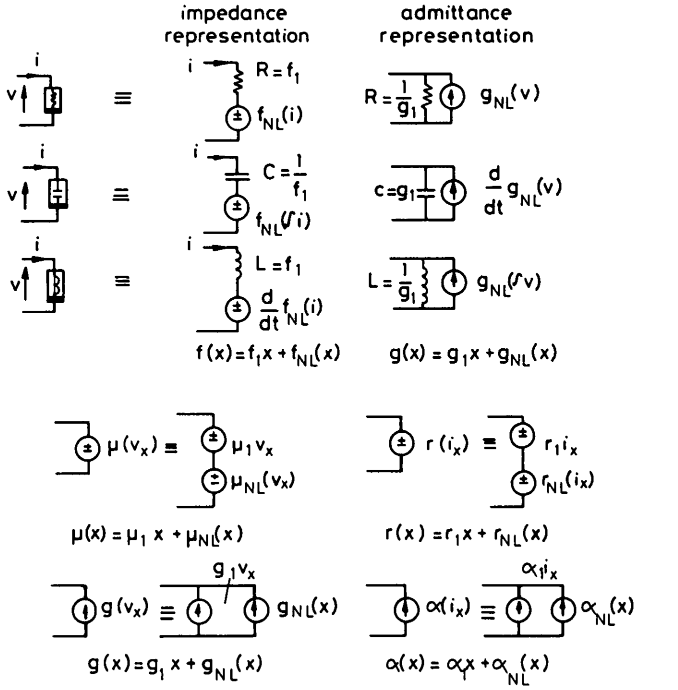

Fig. 2

the linear component of each nonlinear element into the linear circuit. This results in a linear circuit called the modified linear circuit. Using Kirschhoff's current and voltage laws, a standard linear analysis can be carried out.

To avoid dealing with certain types of networks whose functional representation may fail to exist, we shall assume that the networks meet the following requirements:

(*a*) *H1:* (*H2:*) Consider each nonlinear capacitor (inductor) described by an admittance (impedance) representation and its associated nonlinear independent current (voltage) source. Let i and v denote, respectively, the source current and its branch voltage. Assume that all the other independent current (voltage) sources (inputs and sources associated with the other nonlinear elements) are open circuited (short circuited) and that all independent voltage (current) sources (inputs and sources associated with nonlinear elements) are short circuited (open circuited); then the linear transfer function linking i and v and associated with the resulting linear circuit must be strictly proper*.

Circuits which do not satisfy H1 or H2 depend on an infinite number of higher-order derivatives of some inputs. This is illustrated by the following example:

Consider the nonlinear circuit, Fig. 3. For the modified

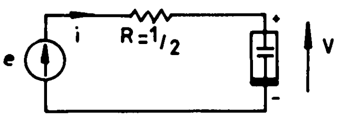

Fig. 3 $i = \frac{d}{dt} v^2$

linear circuit associated with this circuit see Fig. 4 where the nonlinear capacitor, which consists only of a strictly nonlinear element, has been replaced by an independent current source. In order to show that the hypothesis H1 is not verified (the capacitor being described in an admittance form), the independent voltage source is short circuited and the linear transfer function, linking the current i

* In linear system theory, a rational function $G(s)$ is said to be strictly proper if $G(\infty) = 0$.

through the current source and the voltage across it, must be searched for: here $(v/i) = R$, which is obviously not a strictly proper rational function.

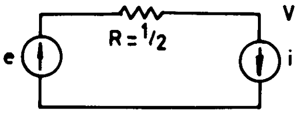

Fig. 4 $i = \frac{d}{dt} v^2$

Now, the nonlinear differential equation describing the behaviour of this circuit is

$$e + v\frac{dv}{dt} = v \tag{8}$$

Eqn. 8 can be solved iteratively following the Picard iterative scheme

$$v_0 = e$$

$$v_n = e + v_{n-1}\frac{dv_{n-1}}{dt} \qquad n \geqslant 1$$

to yield

$$v = e + e\frac{de}{dt} + 2e\left(\frac{de}{dt}\right)^2 + e^2\frac{d^2e}{dt^2} + \cdots \tag{9}$$

Expr. 9 makes explicit the dependency of v on the derivatives of the voltage input e. On the other hand, eqn. 8 is equivalent to the differential equation

$$\frac{dt}{dv} = \frac{v}{v - e}$$

which can be solved analytically, at least for a constant input voltage e. This gives

$$v + eLn\left(1 - \frac{v}{e}\right) = t$$

which must be considered only for $t \geqslant 0$.

This formula shows that the solution has a nondefined first-order derivative at zero which is a sufficient condition for the nonexistence of a Volterra analytical functional expansion of the solution $v(t)$.

(*b*) *H3*: The modified linear circuit defined by embedding the linear component of each nonlinear element into the linear circuit and considering the strictly nonlinear part as an independent source must be 'well-behaved'.

This means that the modified linear circuit possesses a unique defined solution and that, in particular, no circuit variable tends to infinity with the input frequency as for the following example.

Let us consider the nonlinear circuit of Fig. 5, and its modified linear associated circuit of Fig. 6. Instead of

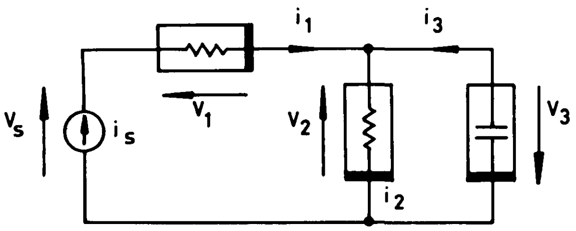

Fig. 5
$v_1 = f(i_1)$
$v_2 = h(i_2)$
$v_3 = \Gamma(\int i_3)$

carrying a standard linear analysis for testing the hypothesis H3, an easy topological way is to use the superposition principle. Indeed, consider the four linear circuits of Fig. 7

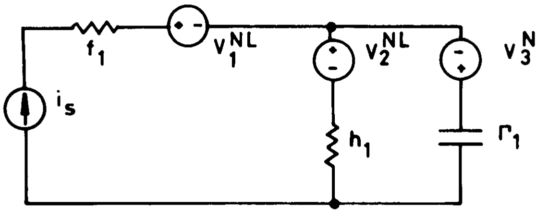

Fig. 6 *Circuit obtained from that of Fig. 5 by imbedding the linear part of the nonlinear elements into the linear circuit*

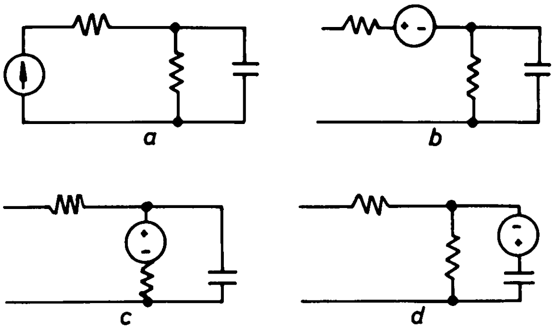

Fig. 7 *Linear circuits obtained by applying the superposition principle to the circuit of Fig. 6*

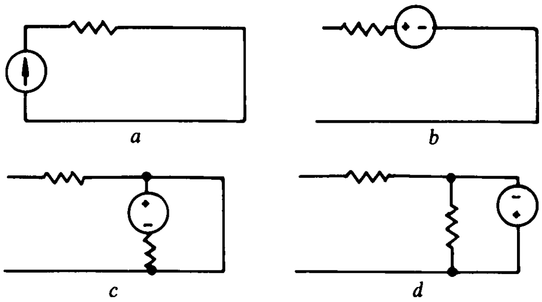

Fig. 8 *Circuits equivalent at high frequencies to those of Fig. 7*

and, respectively, the equivalent (at high frequencies) four linear circuits obtained by short-circuiting capacitors and open-circuiting inductors, Fig. 8.

Then it is easy to see that all circuit variables have defined values except for circuits Fig. 7*c* and *d* in the case of $h_1 = 0$. Indeed, in that case, the current in the nonlinear resistor branch 2 is infinite.

Note: in practical circuits, H1, H2 and H3 are very general hypothesis which are generally fulfilled.

4.2 Descriptive equations

Any lumped circuit obeys three basic laws: the Kirchhoff's voltage law (KVL), the Kirchhoff's current law (KCL), and the elements' law (branch characteristics); Let each passive nonlinear element (resistor, capacitor, inductor) be described by its controlling variable: current, if it has an impedance representation, or voltage for the admittance. Let the current sources be described by the voltages across their branches and the voltage sources by the currents flowing in their branches. For a nonlinear circuit containing p branches and n nodes, one may then write $n - 1$ KCL equations and $p - (n - 1)$ KVL equations. If one keeps in these equations only the descriptive variables using the branch characteristics, one gets p equations linking the p unknown variables. These equations are of three types:

(a) *E1: Dynamical equations:* These are generally integro-differential equations linking a set, as reduced as possible, of variables of the circuit, which allows the behaviour of the circuit to be described totally.

(b) *E2: Output equations:* These are functions connecting variables described by the dynamical equations to the remaining variables.

(c) *E3: Reduction equations.* These equations are linear; they allow the number of unknowns in the previous set of equations to be reduced. They correspond to

(i) KCL at nodes joining only passive elements described in an admittance form or independent current input sources or dependent voltage sources if the current flowing through their branch appears as a controlling variable of another element.

(ii) KVL for loops containing only passive elements described in an impedance form or independent voltage input sources or dependent current sources if the voltage across their branch appears as a controlling variable of an other element.

Example: Let us consider again the nonlinear circuit of Fig. 5. This circuit is described by the following set of equations derived as shown previously:

$$E1: h(i_2) + \Gamma\left(\int i_3\right) = 0$$

$$E2: v_s = f(i_1) + h(i_2)$$

$$E3: \begin{cases} i_s = i_1 \\ i_1 + i_3 = i_2 \end{cases}$$

Using $E3$, $E1$ and $E2$ can be written:

$$\begin{cases} h(i_2) + \Gamma\left(\int (i_2 - i_s)\right) = 0 \\ v_s = h(i_2) + f(i_s) \end{cases} \tag{10}$$

The analysis proposed here, which is, in the general case, a mixed variable analysis (i.e. the unknown variables may be currents and/or voltages), does not differ fundamentally from the graph analysis method presented in Reference 16, p. 147, except that the graph method seems to us less suitable for an automatic computer derivation of the descriptive equations. Indeed it assumes the existence of a special proper tree in the circuit, a condition which could not be satisfied even for very simple circuits. Moreover, when the circuit comprises only nonlinear passive elements described in the admittance form and current sources, our approach leads to a nodal analysis. Conversely, in the case of circuits containing only nonlinear elements described in the impedance form and voltages sources, it is equivalent to a loop analysis.

5 Derivation of the GPS associated with nonlinear circuits

In this Section, we show how to derive the GPS associated with the solutions of the set of integro-differential equations $E1$, $E2$ and $E3$. Consider first the simple nonlinear circuit of Fig. 5.

Let

$$f(i_2) = \sum_{n \geq 1} f_n i_1^n \qquad h(i_2) = \sum_{n \geq 1} h_n i_2^n$$

and

$$\Gamma\left(\int i_3\right) = \sum_{n \geq 1} \Gamma_n \left(\int i_3\right)^n$$

Recalling (see Section I or, for further details, Reference 6) that x_0 denotes the integration with respect to time and x_1 denotes the integration with respect to time after multiplying by the input $i_s(t)$, allows us to write eqn. 10 symbolically as follows:

$$\left.\begin{aligned} &\sum_{n \geq 1} h_n g_2^{\sqcup n} + \sum_{n \geq 1} \Gamma_n (x_0 g_2 - x_1)^{\sqcup n} = 0 \\ &g_s = \sum_{n \geq 1} h_n g_2^{\sqcup n} + \sum_{n \geq 1} f_n g_s^{\sqcup n} \end{aligned}\right\} \tag{11}$$

where g_2 is the GPS associated with i_2, g_s is the GPS associated with i_s† and $g_2^{\sqcup n} = g_2 \sqcup, \ldots, \sqcup g_2$ (n-times) is the GPS associated with i_2^n (Theorem 1, Section 2). Then, the GPS g_2 and g_s are the solutions of the algebraic eqns. 11. Separating the summation in eqn. 10 into linear and strictly nonlinear parts gives for the first equation of eqns. 11

$$\begin{aligned} (h_1 + \Gamma_1 x_0) g_2 = \Gamma_1 x_1 &- \sum_{n \geq 2} h_n g_2^{\sqcup n} \\ &- \sum_{n \geq 2} \Gamma_n (x_0 g_2 - x_1)^{\sqcup n} \end{aligned}$$

More generally, one can show that the dynamical equations $E1$ can be written in a symbolic way as

$$M(x_0)[g] = [L] + [NL]$$

or

$$[g] = M(x_0)^{-1}([L] + [NL]) \tag{12}$$

where $M(x_0)$ is a square matrix depending on the variable x_0, $[g]$ is the GPS associated with the dynamical variables, and $[L]$ and $[NL]$ represent, respectively, the independent input sources and the nonlinear sources associated with the nonlinear elements; for the previous example

$$M(x_0) = h_1 + \Gamma_1 x_0$$

$$[L] = \Gamma_1 x_1$$

and

$$[NL] = -\sum_{n \geq 1} h_n g_2^{\sqcup n} - \sum_{n \geq 2} \Gamma_n (x_0 g_2 - x_1)^{\sqcup n}$$

In order to obtain the same expansion as eqn. 6, equivalent to the Volterra series (see Section 2), eqn. 12 is solved iteratively following the scheme

$$[g] = [g]_1 + [g]_2 + \cdots + [g]_n + \cdots \tag{13}$$

where

$$[g] = M(x_0)^{-1}[L] \tag{14}$$

and $[g]_n$ collects all terms in $M(x_0)^{-1}$ $[N \cdot L]$ depending on $[g]_1, [g]_2, \ldots, [g]_{n-1}$ and containing n-occurrences of the letter x_1.

Example: For the circuit of Fig. 5, we obtain:

$$[g_2]_1 = \frac{\Gamma_1}{h_1}\left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1$$

† The GPS associated with an analytic function $i(t) = \sum_{n \geq 0} i_n(t^n/n!)$ is defined by $\sum_{n \geq 0} i_n x_0^n$ (see Section 5)

$$
\begin{aligned}
[g_2]_m = \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} \Bigg\{ & - \sum_{n \geqslant 2} \frac{h_n}{h_1} \sum_{\substack{v_1, \ldots, v_n \geqslant 1 \\ v_1 + \cdots + v_n = m}} [g_2]_{v_1} \sqcup\!\sqcup \\
& \times [g_2]_{v_2} \sqcup\!\sqcup, \ldots, \sqcup\!\sqcup [g_2]_{v_n} \\
& - \sum_{n \geqslant 2} \frac{\Gamma_n}{h_1} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \\
& \times \sum_{\substack{v_1, \ldots, v_n \geqslant 1 \\ v_1 + \cdots + v_n = m-n+i}} (x_0 [g_2]_{v_1}) \sqcup\!\sqcup, \ldots, \sqcup\!\sqcup (x_0 [g_2]_{v_i}) \\
& \times \sqcup\!\sqcup (\underbrace{x_1 \sqcup\!\sqcup, \ldots, \sqcup\!\sqcup x_1}_{n-i \text{ times}}) \Bigg\}
\end{aligned}
$$

$m \geqslant 2$

It is now easy to see that the computation of $[g]_n$ requires two things: The formal inversion of the matrix $M(x_0)$ and the computation of the shuffle product of rational expressions in the two noncommutative variables x_0 and x_1. We will see that such computations involve only rational expressions of the following special form

$$
(1 - a_0 x_0)^{-1} x_{i_1} (1 - a_1 x_0)^{-1} \times x_{i_2} (1 - a_2 x_0)^{-1} \ldots, x_{i_p} (1 - a_p x_0)^{-1} \quad (15)
$$

$i_1, \ldots, i_p \in \{0, 1\}$, $a_0, a_1, \ldots, a_p \in \mathbb{R}$ or $\mathbb{C}$.

5.1 Computation of the inverse of the matrix $M(x_0)$

In order to compute the inverse of the formal matrix $M(x_0)$, we first give an equivalent form of eqn. 12 which is more suitable for analysing complex (M has a large dimension) circuits. Assuming $H1$, $H2$ and $H3$, eqn. 12 may always be written as

$$
([A] + x_0 [B])[g]' = [L]' + [NL]' \quad (16)
$$

where $[A]$ and $[B]$ are square matrices and, moreover, $[A]$ is nonsingular and where $[g]'$ may be an augmented (dimension) version of the vector $[g]$, as for the following example.

Consider the nonlinear circuit in Fig. 9.

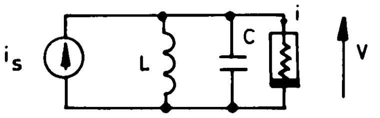

Fig. 9 $i = f(v)$

It is described by the nonlinear equation

$$
\frac{1}{LC} \int v + \frac{dv}{dt} + \frac{1}{C} \sum_{i=1}^{\infty} f_i v^i = \frac{1}{C} i_s \quad (17)
$$

Applying the symbolic rules, the equation giving the GPS associated with $v(t)$ is

$$
M(x_0) g_r = \left[\frac{1}{C} x_1\right] + \left[-\frac{1}{C} x_0 \sum_{i=2}^{\infty} f_i g_v^{\sqcup\!\sqcup i}\right]
$$

where

$$
M(x_0) = \left(1 + \frac{1}{C} x_0 + \frac{1}{LC} x_0^2\right)
$$

Now, eqn. 17 can also be written in the equivalent form

$$
\begin{cases}
\dfrac{dv}{dt} = w \\[2mm]
\dfrac{1}{LC} \displaystyle\int v + w + \frac{1}{C} \sum_{i=1}^{\infty} f_i v^i = \frac{1}{C} i_s + \frac{1}{C} \int \frac{di_s}{dt}
\end{cases}
$$

resulting for the modified GPS $[g]' = ([g_v], [g_w])^T$ in the following equation:

$$
\begin{cases}
g_v = x_0 g_w \\[2mm]
\dfrac{1}{LC} x_0 g_v + g_w + \dfrac{f_1}{C} g_v = \dfrac{1}{C} x_1 - \dfrac{1}{C} \displaystyle\sum_{i=2}^{\infty} f_i g_v^{\sqcup\!\sqcup i}
\end{cases} \quad (18)
$$

where the letter x_1 corresponds, this time, to the derivative of the input: di_s/dt. Eqn. 18 can then be put in a form similar to eqn. 16

$$
\left[\begin{pmatrix} 1 & 0 \\ \dfrac{f_1}{C} & 1 \end{pmatrix} + x_0 \begin{pmatrix} 0 & -1 \\ \dfrac{1}{LC} & 0 \end{pmatrix}\right] \begin{bmatrix} g_v \\ g_w \end{bmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{C} x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -\dfrac{1}{C} \displaystyle\sum_{i \geqslant 2} f_i g_v{}^i \end{pmatrix}
$$

The matrix $[A]$ being nonsingular eqn. 16 can also be written

$$
(I + x_0 [A]^{-1} [B])[g]' = [A]^{-1} ([L]' + [NL]')
$$

where I is the identity matrix. Let J be the Jordan canonical form of the matrix $[A]^{-1}[B]$, i.e.

$$
J = \begin{bmatrix}
\begin{matrix} J_1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & J_1 \end{matrix} & \\
& \ddots & \\
& & \begin{matrix} J_n & 1 & 0 \\ & \ddots & 1 \\ 0 & & J_n \end{matrix}
\end{bmatrix}
$$

with $[A]^{-1}[B] = TJT^{-1}$.

Let $[s] = T^{-1}[g]'$. Then using the transformations T and T^{-1}, one obtains

$$
(I + x_0 J)[s] = T^{-1} [A]^{-1} ([L] + [N \cdot L]) \quad (19)
$$

Now, the formal inverse of $M(x_0) = I + x_0 J$ is easily given by

$$
\begin{bmatrix}
\begin{matrix}
(1 + J_1 x_0)^{-1} & -(1 + J_1 x_0)^{-1} x_0 (1 + J_1 x_0)^{-1} & +(1 + J_1 x_0)^{-1} x_0 (1 + J_1 x_0)^{-1} x_0 (1 + J_1 x_0)^{-1} \cdots \\
0 & (1 + J_1 x_0)^{-1} & -(1 + J_1 x_0)^{-1} x_0 (1 + J_1 x_0)^{-1} \cdots \\
0 & 0 & (1 + J_1 x_0)^{-1} \cdots
\end{matrix} & & 0 \\
& \ddots & \\
0 & & \begin{matrix}
(1 + J_n x_0)^{-1} & -1(+J_n x_0)^{-1} x_0 (1 + J_n x_0)^{-1} & +(1 + J_n x_0)^{-1} x_0 (1 + J_n x_0)^{-1} x_0 (1 + J_n x_0)^{-1} \cdots \\
0 & (1 + J_n x_0)^{-1} & -(1 + J_n x_0)^{-1} x_0 (1 + J_n x_0)^{-1} \cdots \\
0 & 0 & (1 + J_n x_0)^{-1} \cdots
\end{matrix}
\end{bmatrix}
$$

This derives from known results on functions of matrices (see for example Reference 7, p. 100 with $f(\lambda) = (1 + \lambda x_0)^{-1}$). Note that every term of this matrix is of the form of eqn. 15. Eqn. 19 is solved according to the scheme presented in eqns. 13 and 14

$$[s] = [s]_1 + [s]_2 + \cdots \tag{20}$$

with

$$[s]_1 = (I + x_0 J)^{-1} T^{-1} [A]^{-1} [L]'$$

yielding

$$[g]'_n = T[s]_n$$

5.2 Shuffle product of expressions in the form of eqn. 12

From eqns. 19 and 20 (see also the previous example) it is readily seen that the computation of $[g]$ requires the computation of the shuffle product of expressions of the special simple form of eqn. 15. Indeed, $[g]_1$ has the form of eqn. 15 and the following proposition [9] shows that the structure of noncommutative expressions of the form of eqn. 15 is preserved under the shuffle product.

Proposition 1: Given two noncommutative power series

$$g_1^p = (1 - a_0 x_0)^{-1} x_{i_1} (1 - a_1 x_0)^{-1} x_{i_2}, \ldots, x_{i_p} (1 - a_p x_0)^{-1}$$
$$= g_1^{p-1} x_{i_p} (1 - a_p x_0)^{-1}$$

and

$$g_2^q = (1 - b_0 x_0)^{-1} x_{j_1} (1 - b_1 x_0)^{-1} x_{j_2}, \ldots, x_{j_q} (1 - b_q x_0)^{-1}$$
$$= g_2^{q-1} x_{j_q} (1 - b_q x_0)^{-1}$$

where p and q belong to $\mathbb{N}$, the subscripts $i_1, \ldots, i_p, j_1, \ldots, j_q$ to $\{0, 1, \ldots, m\}$ and $a_i, b_j \in \mathbb{C}$; the shuffle product of g_1^p and g_2^q is given by induction on the length by

$$g_1^p \sqcup\!\sqcup g_2^q = (g_1^p \sqcup\!\sqcup g_2^{q-1}) x_{j_q} [1 - (a_p + b_q) x_0]^{-1}$$
$$+ (g_1^{p-1} \sqcup\!\sqcup g_2^q) x_{i_p} [1 - (a_p + b_q) x_0]^{-1}$$

with

$$(1 - a x_0)^{-1} \sqcup\!\sqcup (1 - b x_0)^{-1} = [1 - (a + b) x_0]^{-1}$$

Let us consider again the circuit of Fig. 5. It follows that

$$[g]_1 = \frac{\Gamma_1}{h_1}\left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1}$$

$$[g]_2 = \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} \left\{ -2 \frac{\Gamma_2}{h_1}\left(\frac{\Gamma_1}{h_1}\right)^2 \right.$$
$$\times x_0 \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1 x_0 \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1$$
$$- 4 \frac{\Gamma_2}{h_1}\left(\frac{\Gamma_1}{h_1}\right)^2 x_0 \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_0$$
$$\times \left(1 + 2\frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1 \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1}$$
$$+ 2 \frac{\Gamma_2}{h_1}\frac{\Gamma_1}{h_1} x_0 \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1^2$$
$$+ 2 \frac{\Gamma_2}{h_1}\frac{\Gamma_1}{h_1} x_0 \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1$$
$$\times \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1 + 2 \frac{\Gamma_2}{h_1}\frac{\Gamma_1}{h_1} x_0 x_1$$
$$\times \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} - \frac{\Gamma_2}{h_1}\left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1^2$$
$$- 2 \frac{h_2}{h_1}\left(\frac{\Gamma_1}{h_1}\right)^2 \left(1 + \frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1$$
$$\left. \times \left(1 + 2\frac{\Gamma_1}{h_1} x_0\right)^{-1} x_1 \right\}$$

This proposition offers an easy mean of systematising on a computer the manipulations involved in the derivation of the GPS. These manipulations have been entirely implemented on a computer [11] using Lisp [17].

6 Response to typical inputs (Dirac, steps, harmonics, ...)

In the linear case, Laplace and Fourier transforms are systematic and powerful tools of operational calculus. A direct generalisation of these techniques, to the nonlinear domain, leads to multidimensional Laplace and Fourier transforms and the technique of association of variables [8, 11]. However, the computations based on these are often tedious and seem difficult to implement on a computer.

In this Section we present a symbolic calculus, based on noncommutative variables, which seems very suitable for the analysis of nonlinear circuits.

Let us consider the input $u(t)$, assumed to be analytic

$$u(t) = \sum_{n \geq 0} u_n \frac{t^n}{n!}$$

and let us define the following transform, known as the Laplace-Borel transform of $u(t)$:

$$g_u = \sum_{n \geq 0} u_n x_0^n$$

(This series may be regarded as the GPS associated with $u(t)$ since

$$\frac{t^n}{n!} = \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1$$

Table 1 gives the Laplace-Borel transform of some usual functions:

Table 1: Laplace-Borel transforms of some usual functions

$u(t)$	g_u
unit step	1
$\frac{t^n}{n!}$	x_0^n
$\left(\sum_{i=0}^{n-1} \frac{\binom{i}{n-1}}{i!} a^i t^i\right) e^{at}$	$(1 - a x_0)^{-n}$
$\cos(\omega t)$	$(1 + \omega^2 x_0^2)^{-1}$

Proposition 2 [10]: The Laplace-Borel transform of the response of a nonlinear circuit to the inputs $u(t)$, is obtained by replacing each variable x_1 in the GPS $[g]$ associated with the nonlinear circuit by the operator $x_0[g_u \sqcup\!\sqcup]$.

We have seen in Proposition 1 that the GPS $[g]$ is obtained as a finite sum of expressions of the form of eqn. 15. The response associated with this term to a Dirac input function $u(t) = \delta(t)$ results from the following proposition:

Proposition 3: The Laplace-Borel transform of the response associated with the term

$$S_n = (1 + a_0 x_0)^{-1} x_{i_1} (1 + a_1 x_0)^{-1} x_{i_2}, \ldots,$$
$$(1 + a_{n-1} x_0)^{-1} x_{i_n} (1 + a_n x_0)^{-1} = (1 + a_0 x_0)^{-1} x_{i_1} S_{n-1}$$

to a Dirac input function $u(t) = \delta(t)$ is given by induction on the length by

$$Resp(S_n) = \begin{cases} (1 + a_0 x_0)^{-1} x_0 Resp(S_{n-1}) & \text{if } i_1 = 0 \\ \dfrac{1}{n!} \quad (1 + a_0 x_0)^{-1} & \text{if } i_1 = 1 \text{ and } \\ & i_2 = \cdots = i_n = 1 \\ 0 & \text{if } i_1 = 1 \text{ and } \\ & \exists p,\ p > 1,\ i_p \neq 1 \end{cases}$$

Applying Propositions 1, 2 and 3 to the GPS $[g]$ results in a series of rational fractions in the only variable x_0. Decomposing them into partial fractions and taking the inverse Laplace-Borel gives the corresponding time response in term of exponential polynomials [6, 10].

Example: Let us consider the nonlinear circuit of Fig. 10

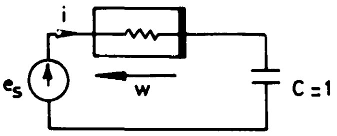

Fig. 10 *Simple nonlinear circuit the GPS of which is to be determined*
$w = i - \dfrac{i^3}{3}$

the behaviour of which is described by the nonlinear integral equation:

$$\int i + i - \tfrac{1}{3} i^3 = e_s \tag{21}$$

It can be shown that the circuit has an asymptotically stable behaviour only for the range $|e_S(t)| < \frac{2}{3}$, $t \geqslant 0$, of the input voltage and that, within this range, the Volterra functional series is uniformly convergent over the infinite time interval.

Let us first compute the response of this circuit to a step function input $e_S(t) = V$, $t \geqslant 0$. The equation for the GPS associated with $i(t)$ is

$$g = (1 + x_0)^{-1} x_1 + \tfrac{1}{3} g^{ш3} - \tfrac{1}{3}(1 + x_0)^{-1} x_0 g^{ш3}$$

where x_1 corresponds to the input de/dt. Here the identity

$$(1 + x_0)^{-1} = 1 - (1 + x_0)^{-1} x_0$$

is used to obtain in the iterative process eqns. 13–14 only expressions of the form of eqn. 15. The terms of the expansion 13 of the GPS g, up to order 5 are listed in Table 2.

Now, the response of this circuit to various input signals can be derived directly from the expansion of the GPS g, by using Propositions 2 and 3. For example, for the step function $e_S(t) = V$, $t \geqslant 0$, we obtain the time response by using first Proposition 3 for $u(t) = (de/dt) = V\ \delta(t)$:

$$\begin{aligned}
&(1.0 * (1 + 1.0 * X0)^{(-1)}) \\
&+ (0.3333333 * (1 + 3.0 * X0)^{(-1)}) \\
&+ ((-0.3333333) * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 3.0 * X0)^{(-1)}) \\
&+ (0.3333332 * (1 + 5.0 * X0)^{(-1)}) \\
&+ ((-0.3333332) * (1 + 3.0 * X0)^{(-1)} * X0 * (1 + 5.0 * X0)^{(-1)}) \\
&+ ((-0.3333332) * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 5.0 * X0)^{(-1)}) \\
&+ (0.3333332 * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 3.0 * X0)^{(-1)} * X0 * (1 + 5.0 * X0)^{(-1)})
\end{aligned} \tag{22}$$

and then, taking the inverse Laplace-Borel transform of rational terms in eqn. 22

$$\begin{aligned}
&\text{(order 1)} \\
&(1.0 * e^{((-1.0)*t)}) \\
&\text{(order 3)} \\
&+ (0.4999999 * e^{((-3.0)*t)}) \\
&+ ((-0.1666666) * e^{((-1.0)*t)}) \\
&\text{(order 5)} \\
&+ (0.6249998 * e^{((-5.0)*t)}) \\
&+ ((-0.2499999) * e^{((-3.0)*t)}) \\
&+ ((-0.416666) * e^{((-1.0)*t)}) \\
&\text{(order 7)} \\
&+ (1.020833 * e^{((-7.0)*t)}) \\
&+ ((-0.5208331) * e^{((-5.0)*t)}) \\
&+ ((-0.0208333) * e^{((-3.0)*t)}) \\
&+ ((-0.0347222) * e^{((-1.0)*t)})
\end{aligned}$$

Approximations up to order 1, 3, 5, 7 for $V = 0.63$ are plotted in Fig. 11 and compared with the exact solution.

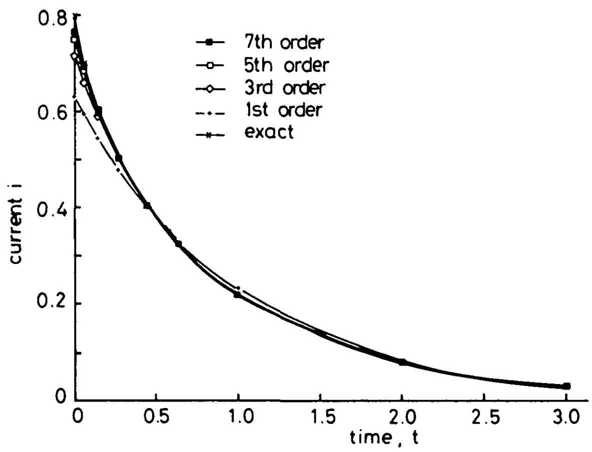

Fig. 11 *Step time domain response of the circuit of Fig. 10*

Note that, even for a large value of the input signal (close to the limit value for the convergence of the series), only a few terms of the series are sufficient to ensure a good approximation of the solution.

7 Comparison with the transfer function approach

To compare our algorithm for computing the response of nonlinear circuits with the transfer function approach based on multidimensional Laplace transforms [8, 11], let us consider again the above example. First, in this method the kernels of the Volterra series associated with the response of the nonlinear circuit need to be determined.

Table 2: Terms of expansion 13 of the GPS g up to order 5

$(1.0 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ (1.9999997 * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ ((-1.9999997) * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1$
$* (1 + 1.0 * X0)^{(-1)} * X1)$
$+ (39.999992 * (1 + 5.0 * X0)^{(-1)} * X1 * (1 + 4.0 * X0)^{(-1)} * X1 * (1 + 3.0 * X0)^{(-1)} * X1$
$* (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ ((-39.999992) * (1 + 3.0 * X0)^{(-1)} * X0 * (1 + 5.0 * X0)^{(-1)} * X1 * (1 + 4.0 * X0)^{(-1)} * X1$
$* (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ ((-15.9999968) * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X0 * (1 + 4.0 * X0)^{(-1)} * X1$
$* (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ ((-3.9999992) * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X0$
$* (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ ((-39.999992) * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 5.0 * X0)^{(-1)} * X1 * (1 + 4.0 * X0)^{(-1)} * X1$
$* (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ (39.999992 * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 3.0 * X0)^{(-1)} * X0 * (1 + 5.0 * X0)^{(-1)} * X1$
$* (1 + 4.0 * X0)^{(-1)} * X1 * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ (15.9999968 * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X0$
$* (1 + 4.0 * X0)^{(-1)} * X1 * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$
$+ (3.9999992 * (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1$
$* (1 + 1.0 * X0)^{(-1)} * X0 * (1 + 3.0 * X0)^{(-1)} * X1 * (1 + 2.0 * X0)^{(-1)} * X1 * (1 + 1.0 * X0)^{(-1)} * X1)$

This is generally done using the exponential input method. The determination of the nth order Volterra kernel is carried out by considering a harmonic input $\sum_{i=1}^{n} e^{p_i t}$ (p_i not being harmonically related), substituting it into the governing equation, eqn. 21, together with the Volterra series corresponding to this input

$$i(t) = \int_{-\infty}^{+\infty} h_i(\tau) \sum_{i=1}^{n} e^{p_i(t-\tau)}\, d\tau + \iint_{-\infty}^{+\infty} h_2(\tau_1, \tau_2) \times \sum_{i=1}^{n} \sum_{j=1}^{n} e^{p_i(t-\tau_1)} e^{p_j(t-\tau_2)}\, d\tau_1\, d\tau_2 + \cdots$$

$$= \sum_{i=1}^{n} H_1(p_i) e^{p_i t} + \sum_{i=1}^{n} \sum_{j=1}^{n} H_2(p_i, p_j) e^{(p_i + p_j)t} + \cdots$$

where H_n is the n-dimensional Laplace transform of the kernel h_n, finally, equating the coefficients of the exponential term $\sum_{i=1}^{n} e^{p_i t}$ in the two sides of the resulting equation. For example, the first-order Volterra kernel is determined using the input $e_S(t) = e^{p_1 t}$; this gives

$$\frac{1}{p_1} H_1(p_1) + H_1(p_1) = 1$$

that is

$$H_1(p_1) = \frac{p_1}{1 + p_1}$$

Similarly, the third-order Volterra kernel is derived using the input

$$e_S(t) = e^{p_1 t} + e^{p_2 t} + e^{p_3 t}$$

$$\frac{1}{p_1 + p_2 + p_3} H_3(p_1, p_2, p_3) + H_3(p_1, p_2, p_3) - \tfrac{1}{3} H_1(p_1) H_1(p_2) H_1(p_3) = 0$$

that is

$$H_3(p_1, p_2, p_3) = \tfrac{1}{3} H_1(p_1) H_1(p_2) H_1(p_3) H_1(p_1 + p_2 + p_3)$$

Repeating this process allows us to determine recursively higher-order kernels. However, as can be noticed from the evaluation of the fifth- and seventh-order Volterra kernels, the computations rapidly become too unwieldy to carry out by hand.

Now, given the Laplace transform of the input $E_S(p)$, the temporal response may be computed using a generalised Heaviside calculus, or, more precisely, the technique of association of variables [8]

$$I(p) = H_1(p) E_S(p) + [H_3(p_1, p_2, p_3) E_s(p_1) \times E_S(p_2) E_S(p_3)]_{ass}(p) + \cdots$$

where $[\cdot]_{ass}$ represents the monodimensional Laplace transform resulting from the association of the variables p_1, p_2, p_3. For example, for the step input $E_S(p) = (1/p)$ considered above, we have:

$$H_1(p) E_S(p) = \frac{1}{1 + p}$$

and

$$[H_3(p_1, p_2, p_3) E_S(p_1) E_S(p_2) E_S(p_3)]_{ass}(p)$$

$$= \left[\frac{1}{3} \frac{p_1 + p_2 + p_3}{1 + p_1 + p_2 + p_3} \frac{1}{1 + p_1} \frac{1}{1 + p_2} \frac{1}{1 + p_3}\right]_{ass}(p)$$

$$= \frac{1}{3}\left[\frac{1}{p + 3} - \frac{1}{(p + 1)(p + 3)}\right]$$

where the last associated transform has been derived using the table given in Reference 11. Finally, the temporal response is obtained using the monodimensional inverse Laplace transform:

$$i(t) = e^{-t} + (\tfrac{1}{2}e^{-3t} - \tfrac{1}{6}e^{-t}) + \cdots$$

It is important to point out that a general formula for performing the association of variables does not exist. Several rules corresponding to different associations have been derived in the literature. However, no attempt has been made to automatise these computations, which become unwieldy for high-order terms of the series, the reason for this lies essentially in the lack of generality of the method. Our approach offers this generality, but at the expense of an exponentially growing number of terms in the Volterra series, since the method deals with noncommuting variables. However, this does not limit its interest, since the involved algorithms are easily programmable on a computer.

8 References

1 BOUVILLE, C., and DUBOIS, J.L.: 'Analyse des réseaux faiblements non linéaires par les développements en series de Volterra. Application aux montages à transistors bipolaires', *Ann. Telecom.*, 1978, **33**, pp. 213–224

2 CHUA, L.O., and NG, C.Y.: 'Frequency-domain analysis of nonlinear systems: general theory', *Electronic Circuits Syst.*, 1979, **3**, pp. 165–185

3 CHUA, L.O., and NG, C.Y.: 'Frequency-domain analysis of nonlinear systems: formulation of transfer function', *Electronic Circuits Syst.*, 1979, **3**, pp. 257–269

4 CLASSEN, IR. T.A.C.M., and DE BEER, C.A.F.M.: 'Nonlinear network analysis program'. Nat. Lab. Technical Note No 72/73, November 1973

5 FLIESS, M.: 'Fonctionnelles causales non linéaires et indéterminées non commutatives', *Bull. Soc. Math. France*, 1981, **109**, pp. 3–40

6 FLIESS, M., LAMNABHI, M., and LAMNABHI-LAGARRIGUE, F.: 'Algebraic approach to nonlinear functional expansions', *IEEE Trans.*, 1983, **CAS-30**, pp. 554–570

7 GANTMACHER, F.R.: 'The theory of matrices' (Chelsea Publishing Company, New York, 1960)

8 GEORGE, D.: 'Continuous nonlinear systems', MIT RLE Techn. Rep. 355, 1959

9 LAMNABHI-LAGARRIGUE, F., and LAMNABHI, M.: 'Détermination algébrique des noyaux de Volterra associés à certain systèmes non linéaires', *Ric. Autom.*, 1979, **10**, pp. 17–26

10 LAMNABHI, M.: 'A new symbolic response of nonlinear systems', *Systems & Control Lett.*, 1982, **2**, pp. 154–162

11 LUBOCK, J.K., and BANSAL, U.S.: 'Multidimensional Laplace transforms for solutions of nonlinear equations', *Proc. IEE*, 1969, **116**, pp. 2075–2083

12 RUGH, W.J.: 'Nonlinear system theory' (Johns Hopkins University Press, 1981)

13 SCHETZEN, M.: 'The Volterra and Wiener theories of nonlinear systems' (Wiley, 1980)

14 VOLTERRA, V.: 'Theory of functionals' (translated from the Spanish) (Dover, New York, 1959)

15 WEINER, D.D., and SPINA, J.: 'Sinusoidal analysis and modeling of weakly nonlinear circuits' (Van Nostrand, 1980)

16 WIENER, N.: 'Nonlinear problems in random theory' (Wiley, 1958)

17 WINSTON, P.H., and HORN, B.K.P.: 'LISP' (Addison-Wesley, 1984)

9 Appendix

Let $X = \{x_0, \ldots, x_m\}$ be a finite alphabet and X^* the monoid generated by X; an element of X^* is a word, i.e. a sequence $x_{j_\nu}, \ldots, x_{j_0}$ of letters of the alphabet. The product of two words $x_{j_\nu}, \ldots, x_{j_0}$ and $x_{k_\mu}, \ldots, x_{k_0}$ is the concatenation $x_{j_\nu}, \ldots, x_{j_0} x_{k_\mu}, \ldots, x_{k_0}$. The neutral element is called the empty word and is denoted by 1. A formal power series with real or complex coefficients is written as a formal sum

$$g = (g, 1) + \sum_{\nu \geqslant 0} \sum_{j_0, \ldots, j_\nu = 0}^{m} (g, x_{j_\nu}, \ldots, x_{j_0})$$

$$\times\, x_{j_\nu}, \ldots, x_{j_0} \qquad (g, x_{j_\nu}, \ldots, x_{j_0}) \in \mathbb{R} \text{ or } \mathbb{C}$$

Let g_1 and g_2 be two formal power series, the following operations are defined:

Addition

$$g_1 + g_2 = \sum_{w \in X^*} [(g_1, w) + (g_2, w)]w$$

Cauchy product

$$g_1 \cdot g_2 = \sum_{w \in X^*} \left[\sum_{w_1 w_2 = w} (g_1, w_1)(g_2, w_2) \right] w$$

Shuffle product

$$g_1 \sqcup\!\sqcup\, g_2 = \sum_{w_1, w_2 \in X^*} (g_1, w_1)(g_2, w_2) w_1 \sqcup\!\sqcup\, w_2$$

The shuffle product of two words is defined by induction on the length by

$$1 \sqcup\!\sqcup\, 1 = 1$$

$$\forall x \in X,\ 1 \sqcup\!\sqcup\, x = x \sqcup\!\sqcup\, 1 = x$$

$$\forall x, x' \in X,\ \forall w, w' \in X^*,\ (xw) \sqcup\!\sqcup\, (x'w') = x[w \sqcup\!\sqcup\, (x'w')] + x'[(xw) \sqcup\!\sqcup\, w']$$

This operation consists of mixing the letters of the two words keeping the order of each one. For example

$$\begin{aligned} x_0 x_1 \sqcup\!\sqcup\, x_1 x_0 &= x_0 x_1 x_1 x_0 + x_0 x_1 x_1 x_0 + x_0 x_1 x_0 x_1 \\ &\quad + x_1 x_0 x_1 x_0 + x_1 x_0 x_0 x_1 + x_1 x_0 x_0 x_1 \\ &= 2x_0 x^2 x_0 + x_0 x_1 x_0 x_1 + x_1 x_0 x_1 x_0 \\ &\quad + 2x_1 x_0^2 x_1 \end{aligned}$$