RESEARCH REPORT SERIES
(*Statistics #2001-01*)

Multi-Way Survey Stratification and Sampling

William E. Winkler

Statistical Research Division
U.S. Bureau of the Census
Washington D.C. 20233

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William E. Winkler, william.e.winkler@census.gov 1/ 010803

ABSTRACT

For some populations there may be two or more stratifying criteria that are desirable in the sample design. Multi-way stratification allows increased precision of estimates of each of the variables whose precision is increased by typical univariate estimators corresponding to single-criteria designs. The sample need not be allocated to every multi-way population cell induced by a set of single-criteria designs. As a special case, a solution to a problem posed in Neyman (1934) is presented.

Key words and phrases. Iterative Proportional Fitting; Convex Constraint; Controlled Allocation; Iterated Integer-LP Algorithm.

1.   INTRODUCTION

   This paper provides a methodology for multi-way stratification and sampling that builds on techniques introduced by Bryant, Hartley, and Jessen (1960, hereafter denoted BHJ). The 2-way designs of BHJ require restrictions on the distributions of the variables in the population while the multi-way designs of this paper hold for general distributions. In some situations, it may be natural or desirable to stratify a population to be sampled by two or more alternative criteria. For instance, a population of households can be classified by regions or type of community such as rural, urban, or metropolitan. A population of retail fuel oil dealers might be stratified by measures of sizes that are correlated with desired amounts of sales of specific fuels. Three sets of size ranges of specific fuel types (such as gasoline, number 2 fuel oil, and residual fuel) give three stratifications.

   To illustrate more specifically, let the first stratification criteria give R strata and have variable X for which we estimate a population total. Let the second criteria have C strata and variable Y. If both stratification criteria are used to stratify the population, there are RC 2-way cells. If variances are needed, then standard single-criteria sampling techniques require that sample size be at least 2RC.

   BHJ presented a 2-way stratification design and formulas that, for samples less than 2RC, permits unbiased estimation of the population totals and variances associated with variables X and Y. For restricted distributions and pairs of single-criteria designs, within a very small correction factor, variances under the 2-way design for the estimators of totals for X and Y are less than or equal the variances under the single-criteria designs. The basic validity of the BHJ design depends on two restrictions. The single-criteria designs must allocate samples according to the proportion of the population in a stratum. The proportion of the population in 2-way cells must equal the product of the proportions in the corresponding single-criteria cells. For minor non-proportionality, BHJ adjust 2-way cell allocations to correspond more closely to 2-way population proportions. The adjustments reduce overall variance by decreasing the dominant, between-cell variance component. Within-cell variances remain roughly constant because they depend primarily on the fixed stratification. Variance is no longer necessarily less than variance under the single-criteria designs.

This paper presents a multi-way (or n-way) stratification design that holds for arbitrary populations and sets of single-criteria designs. Similar to BHJ, we try to reduce or minimize between-cell variance while keeping sample size constant. For the most highly restricted 2-way situations considered by BHJ, the n-way technique yields results that are roughly equivalent to theirs. For more general populations and sets of single-criteria designs, we provide a systematic method of accounting for finite population corrections, n-way population cells having zero members, extreme deviations from proportionality, and extreme variation of within-cell population totals and variances.

Total variation, the square root of the sum of squares of the coefficients of variation (*cv*) associated with the n variables covered by a design, is used to compare the n-way stratification technique with combinations of single-criteria techniques. For 2-way designs having the same sample size as a pair of single-criteria designs, we provide an example showing that total variation can be reduced by more than a factor of two. For 3-way designs, another example shows that total variation can be reduced by more than a factor of four.

The n-way stratification technique is computationally intensive for two reasons. The first is that it uses Dykstra's (1985a) Generalized Iterative Fitting Procedure (GIFP) to allocate single-criteria samples to n-way cells. The fitted solution can contain fractional components. The second reason is an algorithm that creates and solves a series of integer LP problems. The algorithm allows the solution from Dykstra's GIFP to be represented as a convex sum of nonnegative integer arrays each having the same margins as the fitted solution. The algorithm generalizes 2-way controlled rounding results of Causey, Cox, and Ernst (1985, hereafter CCE).

Sitter and Skinner (1994, hereafter SS) used LP methods of optimization based on ideas introduced by Rao and Nigam (1990, 1992). They showed how their method contrasts with the method of CCE. The method of this paper also generalizes the method of SS. For 2-dimensional examples of moderate size and any example in three or more dimensions, it provides a computationally feasible alternative. As noted by SS, their LP method does not scale well. We will show further considerations about why their LP method will not scale well. The SS method is more straightforward in concept. It may be easier to apply in small 2-dimensional situations. Because the method of this paper provides much greater control over the sampling design, we are able to provide much more accurate variance estimates under weaker assumptions than those made by SS (section 5).

The outline of this paper is as follows. In the second section of this paper, we provide background, notation, and an example. The third section contains the n-way stratification and the method for allocating n single-criteria samples to n-way cells. An example illustrates how the classical method of iterative proportional fitting for loglinear models fails while Dykstra's GIFP provides proper limiting solutions. In the fourth section, we provide the sample design. The probabilistic structure for the design depends on the specific convex-sum representation of the GIFP-fitted result. In the fifth section, an application of a theorem of J.N.K. Rao (1975) provides formulas for unbiased estimators of population totals and their variances. The discussion in the sixth section consists of three parts. The first covers the problem posed in Neyman (1934). In the second, 2-way designs using the techniques of this paper are related to the designs of BHJ. The third describes relationships to Sitter and Skinner (1994) and computational issues. The seventh section is a summary.

2. NOTATION AND PRELIMINARIES

   This section contains definitions and an example illustrating the controlled allocations. A complicated example of a controlled allocation associated with a 2-way design is presented in Section 4.

2.1. Definitions

   Let I be a finite index set and A_i, $i \in I$, represent an array of nonnegative real numbers. Each A_i will represent a single entry in the array. In some applications the arrays will be matrices and $i=(i,j)$ or $i=(i,j,k)$. We say that m is a *margin count* for array A_i, $i \in I$, if there exists a subset I_j of I such that

$$\sum_{i \in I_j} A_i = m.$$

   Let N_i, $i \in I$, denote the array of counts associated with a population and n_i, $i \in I$, to denote a sample associated with a population. Let N denote the total population size and n denote the sample size. If $i=(i,j,k)$ and n_i, $i \in I$, is the sample induced by three single-criteria samples, then margins of the form

$$\sum_{j,k} n_{(i,j,k)} = m_i$$

correspond to the sample sizes in single-criteria strata.

   Let A be an arbitrary nonnegative matrix having a set of integer margins. We say that a nonnegative integer matrix B is a *controlled allocation* of matrix A if B has the same margins and integer cell values in A agree with the corresponding cell values in B. If, further, each cell value in B is within one of the corresponding cell value in A, then we call B a *controlled rounding* of A. For two-dimensional arrays having nonnegative integer margins, controlled roundings can always be found using standard transportation theory (see e.g. Dantzig, 1963; Gass, 1986, pp. 321-326). For three or more dimensions, controlled roundings do not generally exist (see e.g. Gass, 1986, p. 419; Causey, Cox, and Ernst, 1985, p. 907).

   For n-dimensional allocations, Ernst (1989) has shown that if cell deviations are allowed to vary by as much as 2^{n-2} (i.e., to base 2^{n-2}), then controlled allocations always exist. In n-dimensions, we will also refer to a *controlled rounding to base 2^{n-2}*.

2.2 Example

   Example 2.1. Example of a Controlled Allocation and of a Controlled Rounding

   Matrix A is a 5x5 array with one integer entry, cell $(1,3) = 1.0$, and six zeros. The remaining 18 cells are nonintegers (Table 1). The marginal totals for the first variable are given in the last column and the marginal totals for the second variable are given in the last row. They both sum to 10. For the allocation problem, we have 9 units to distribute to 18 cells if we remove the fixed integer cell. Matrix B is a controlled allocation of matrix A (Table 1). We observe that cell $(2,2)$ in B deviates by more than one from the corresponding cell in A. Matrix C is a controlled rounding of A (Table 1). The cells in C differ by at most one from the corresponding cells in A. Neither matrix B nor matrix C is unique.

2.3 An Additional Relationship and Extension

   In the procedure of this paper, we determine univariate stratifications using conventional methods such as given by Dalenius and Hodges or Ekman (see e.g., Winkler 2000). If the

sample were drawn using conventional stratified sampling, then each variable would satisfy a priori specified constraints on their cvs. This would mean that integer marginal constraints of the form $m_i = \Sigma_{i,j} \ n_{(i,j,k)}$ would need to be satisfied. Although we will not be using simple stratified random sampling in the final design, this additional restriction may better allow the final cvs (under the more complicated design) to also approximately satisfy various univariate restraints. Alternatively, we could use a design where $n_i = n \ N_i/N$ where n is the total sample size and N_i/N is the population proportion in cell **i**. This type of design will generally yield fractional sample sizes in the margins and not, without further restrictions, satisfy univariate cv restraints in moderately skewed population distributions. The advantage of the design of the form $n_i = n \ N_i/N$ is that it is straightforward. The advantage of the method of this paper is that we can take designs in which integer marginal constraints are allowed. The iterative fitting method gives a method of determining individual cell allocations n_i when integer marginal constraints are given. The method assures that individual cell values n_i are less than the corresponding population proportions N_i/N which cannot be done with classical iterative proportional fitting. The methods of sections 4 and 5 in which a probabilistic structure is given for multi-way sampling and estimation hold for arbitrary n_i whether they are given as n N_i/N, by the method of section 3, or some other method.

3.   GENERALIZED ITERATIVE PROPORTIONAL FITTING

   To allocate samples satisfying n single-criteria stratifications to n-way population cells, we use Dykstra's (1985a,b, 1987, also Winkler 1990) Generalized Iterative Fitting Procedure (GIFP). The initial array is the n-way array of population proportions and the marginal values are the sample allocations to the n sets of single-criteria strata. Deviations of sample proportions from population proportions are minimized on the average (except for rounding) by the systematic fit provided by the Kullback-Leibler measure. If only linear marginal constraints need be satisfied, Dykstra's GIFP provides answers identical to those provided by classical IPF. Because Dykstra's GIFP allows convex constraints, we bound sample allocations to n-way cells to be less than their population values. Whereas BHJ required that their 2-way populations have proportions that are products of 1-way population proportions (i.e., independent), we allow fitting to satisfy general interaction restraints in n-way tables (e.g., Bishop, Fienberg, and Holland 1975).

   Example 3.1. Breakdown of Classical IPF in Sampling Problem

   Using ordinary IPF with initial matrix **N** (Table 2) and fixed marginal constraints yields matrix **A** (Table 3). The entry in cell (1,1) exceeds the population value of 2 units. Applying Dykstra's GIFP with cell (1,1) constrained to be less than or equal to two yields matrix **B** (Table 4).

   Winkler (1990) provides another example in which two cells in the matrix obtained by ordinary IPF exceed populations values. After fitting with Dykstra's GIFP with the two cells restrained to be less than the population value, one of the two fitted cells is less than its upper bound. This demonstrates that Dykstra's GIFP is a generalization of ordinary IPF even for the simple types of n-way allocations of this paper. The obvious extension of ordinary IPF is to fix the values in the cells that exceed population bounds to be equal to the population bound. The obvious extension will not yield the correct fitted solution.

4.   N-WAY DESIGN

   In the previous section, we addressed the issue of n-way stratification in a manner that generalizes that of BHJ. In this section, we provide a method for allocating the sample of the

multi-way stratification to n-way cells. With the relatively un-skewed distributions of their paper, BHJ were able to get some cancellation of within-cell variances due to a latin-squares aspect of their design. The intent is to realize some of the benefits that were achieved by BHJ with skewed population distributions.

   The basic idea of this section is to extend the 2-way design of Causey, Cox, and Ernst (1985) to n-way designs. The basic method of proof is similar to that of CCE but depends additionally on the existence on roundings to base 2^{n-2} as shown by Ernst (1989). There is considerable flexibility in the 1-way stratifications and designs. If the 1-way designs are varied slightly, then it is often possible to get solutions in which the roundings are to base 1 rather than base 2^{n-2}.

4.1. Convex Sum of Nonnegative Integer Matrices

   Let stratum sample counts m_j, j = 1, 2, ... , s, be determined by a set of single-criteria designs. Let N_i, $i \in I$, be the n-way population counts.   Let g_i, $i \in I$, be the solution given by Dykstra's GIFP. The array g_i, $i \in I$, satisfies

$$\sum_{ij \in I_j} g_{ij} = m_j, \ j = 1, ..., s, \quad \text{and} \quad g_i \leq N_i, \ i \in I, \tag{4.1}$$

where I_j is the subset of I associated with marginal constraint m_j, j = 1, ...,s.

   We wish to find a sequence of arrays M_{ik}, $i \in I$, k = 1, ..., t, each having nonnegative integer entries and margins m_j, j = 1, ..., s, and positive constants p_k, k = 1, ..., t, such that

$$\sum_{k=1}^{t} p_k = 1 \quad \text{and} \quad \sum_{k=1}^{t} p_k M_{ik} = g_i, \ i \in I. \tag{4.2}$$

   Representation (4.2) yields the n-way design and a random nonnegative integer matrix having expected value g_i, $i \in I$. We merely sample an array M_{il}, $i \in I$, with probability proportional to size p_l, l = 1, 2, ···, t, and then choose a simple random sample of size M_{il} in cell i for all $i \in I$. The first stage of sampling is called the *allocation step* and the second stage the *selection step*. The proof of (4.2) is in the appendix.

4.2. Example of Convex Sum

   Example 4.1. Representation for 2-Way Design

   The algorithm giving representation (4.2) is applied to the fitted 5x5 matrix of Example 3.1 (Table 5). The first matrix M_0 is the fitted matrix derived using Dykstra's GIFP. The next fifteen matrices M_k, k = 1, ···, 15, are the nonnegative integer matrices obtained by the iterative integer LP procedure. The final matrix M_{16} is the convex sum (with the entries in the column headed by p_k used as the coefficients) of the integer matrices M_k, k = 1, ···, 15.

4.3 Relationship to CCE and Computational Considerations

   Representation (4.2) generalizes the results of CCE to n-dimensions. The general existence of the individual matrices M_{il} depends on a result of Ernst (1989) about controlled allocations is n-dimensions to base 2^{n-2}. In some situations, we can vary the univariate stratifications (i.e., the integer marginal constraints) in such a manner that all of the M_{il} are controlled roundings (i..e, to base 1). In all situations, the algorithm reduces the problem of finding the matrices M_{il} of representation (4.2) to a set of capacitated LP problems where the capacity (the distance of the rounding) is determined by the base 2^{n-2}. Even for 3-dimensional designs in skewed populations, the software runs in less than 90 seconds. A key feature of the approach of CCE

and of this paper is that the algorithm only finds a very small proportion of all individual matrices M_{il}. Rather than find all of the matrices M that are controlled roundings or controlled allocations, the algorithm finds a very small number that yield representation (4.2).

5.   ESTIMATION METHODOLOGY

A key facet of the work in this paper is the ability to estimate variances. In the framework of this paper, it is possible to apply a general result of Rao (1975). Alternatives such as bootstrap (e.g., Winkler 1986) can also be used for the n-way designs. CCE did not provide a method of estimating variances. For the 2-way situation, the variance-estimation method of BHJ can be shown to yield very similar results to the estimation methods of this paper. In certain situations, there will be only one unit allocated to an n-way cell. In those situations, within-cell variances can be approximated by collapsing as it is in single-criteria (i.e., univariate) sampling.

5.1. Estimation of the Population Total and Its Variance

Let Y be a population total and let Y_i be the value for cell i. Let $Y_i^\wedge$ be any unbiased estimator of Y_i, $i\epsilon I$. Let $M_i^\sim$, $i\epsilon I$, be the random nonnegative integer matrix that takes value M_{ij}, $i\epsilon I$, with probability proportional to size p_j (see formula 4.2). Let $E^{(1)}$ denote allocation step expectation and $E^{(2)}$ selection step expectation. Note that $E^{(1)}(M_i^\sim) = g_i$, $i\epsilon I$. Given a fixed realization of matrix $M_i^\sim$ selection step samples are independent. For cell i, let σ_i^2 be the variance of the population mean and let $\sigma_i^{\wedge 2}$ be an estimator such that $E^{(2)}(\sigma_i^{\wedge 2}) = \sigma_i^2$. In the selection step, any unbiased estimate of Y_i can be used. For the empirical examples of sections 5.2, 5.3, and 5.4 we use

$$Y_i^\wedge = (N_i / M_i^\sim) \Sigma_j y_{ij} \cdot 1_{ij},$$

where y_{ij} is the quantitative value assigned to unit ij in cell i and 1_{ij} is the indicator that unit ij is sampled.

THEOREM. Let g_i, $i\epsilon I$, be the array obtained using Dykstra's GIFP. Let p_j, $j = 1, ..., t$, be nonnegative constants and M_{ij}, $i\epsilon I$, $j = 1, ..., t$, be nonnegative integer matrices such that

$$\sum_{j=1}^{t} p_j = 1 \quad \text{and} \quad \sum_{j=1}^{t} p_j M_{ij} = g_i, i\epsilon I.$$

Let $M_i^\sim$, $i\epsilon I$, be the random matrix of sample allocations. An unbiased estimator of the population total is

$$T^\wedge = \sum_{i\epsilon I} (M_i^\sim / g_i) \cdot Y_i^\wedge. \tag{5.1}$$

An unbiased estimator of its variance is

$$v^\wedge = \sum_{i\epsilon I} (M_i^\sim/g_i - 1)^2 \cdot Y_i^{\wedge 2} +$$

$$\sum_{i \neq j \in I} (M_i^{\sim}/g_i - 1) \cdot (M_j^{\sim}/g_j - 1) \cdot \hat{Y_i} \cdot \hat{Y_j} + \qquad (5.2)$$

$$\sum_{i \in I} (2 \cdot M_i^{\sim}/g_i - 1) \cdot N_i \cdot (N_i - M_i^{\sim}) \cdot \hat{\sigma_i}^2/M_i.$$

The summations are over $g_i > 0$. Cells $i \in I$ for which $g_i = 0$ have no n-way population members. The proofs of (5.1) and (5.2) are in the appendix.

5.2. Example of Variance Estimation for a 2-Way Design

Example 5.1 shows that, for fixed size samples, estimators under the multi-way design can have lower total variation for two variables than total variation under judiciously chosen single-criteria designs.

Example 5.1. Estimation Under a Two-Way Design

We use the database of Example 4.1. Each record contains volumetric data representing the sales of distillate fuel oil in residential and nonresidential end-use sectors. A generalized version of the Dalenius-Hodges method (Cochran, 1977, pp. 127-129, Winkler 2000) is used for single-criteria stratification. The single-criteria designs are highly non-proportional. The magnitudes of the nonzero values range from 72,000 to 1 for the first variable and from 85,000 to 1 for the second. The second variable only takes nonzero values for 60 percent of the records.

To reduce *cv*s, each single-criteria design allocates the same 13 records with certainty. The remaining 1251 records are stratified into two different sets of five strata having a total sample allocation of 37. Each design has strata sample allocations of 6, 6, 7, 8, and 10. The *cv*s are 0.044 and 0.028, respectively. The first 13 records are also allocated with certainty under the 2-way design. Single-criteria strata boundaries apportion the remaining 1251 records into a 5x5 array.

The results of the two single-criteria stratifications and the corresponding 2-way design are compared with two optimal single-criteria designs (Table 6). Given the sample size of 50, the optimal designs yield minimal *cv*s for exactly one variable. To do this, the optimal designs allocate less than 13 records with certainty. Sufficiently many noncertainty strata are created so that only two sample elements are allocated in each.

The first set of *four cvs* is for optimal univariate designs in which the stratifying variable agrees with one of the variables being estimated. Diagonal elements are low (0.012 and 0.009). Off-diagonal elements are dramatically higher (0.334 and 0.407) because the stratifying variables are not highly correlated with the variables for which the *cv*s are computed. Regression using the two variables yields an R-square value less than 0.2. If, however, we apply standard contingency table techniques (Bishop, Fienberg, and Holland, 1975) to the underlying population matrix **N** (Table 2), we reject independence at the 95 percent level of confidence.

The second set of numbers is for single-criteria designs used in creating the 2-way stratification. Diagonal elements (0.044 and 0.028) are higher than the diagonal elements in the first matrix. Off-diagonal entries, 0.289 and 0.282, are lower. The final set consists of the *cv*s 0.104 and 0.041 for the 2-way design. They are also higher than the diagonal entries in the first matrix and approximately twice as high the diagonal entries in the second matrix. They are substantially lower than the highest of the off-diagonal entries for the respective variables (0.334 and 0.407). Using the stratification given by the first row of the first matrix, we have total variation 0.334 while the n-way design yields total variation 0.112. Ignoring the finite

population correction (*fpc*), sample size must increase by a factor greater than 6 to equal the total variation and *cv*s of the n-way case.

Five hundred independent samples were drawn to evaluate the empirical performance of the estimators based on the 2-way design. The relative empirical biases were 0.008 and 0.000, and the empirical *cv*s were 0.100 and 0.036, respectively. The biases of 0.04 and 0.05 in the estimated *cv*s follow because we must approximate within-cell variances of cells having sample allocations of one. The approximation involved substituting the minimum of the within-row and within-column variances for each such cell. The approximation generally overestimates the within-cell variance.

5.3. Three-Way Variance Example

This example involves a 3-way design using three single-criteria stratifications. The database and procedures are similar to those in Example 5.1. The single-criteria designs had the same 15 units allocated with certainty and a remaining 50 units allocated in 4, 4, and 2 strata that were determined by three different measures of size, respectively. Thus, the 3-way design array is 4x4x2.

Total variation ranged from 0.466 (the best in the case of standard single-criteria stratification techniques) to 0.115 (multi-way) (Table 7). Ignoring the *fpc*, we would have to increase the sample size by a factor greater than 16 to equal the total variation and *cv*s given for the multi-way design. Based on 200 independent samples, the empirical biases of the multi-way estimators of the three variables were -0.004, 0.000, and -0.004, respectively. The empirical *cv*s were 0.070, 0.056, and 0.73, and the biases of the estimates of the *cv*s were 0.001, 0.001, and -0.002, respectively.

5.4. Components of Variance

For the 3-way design example of section 5.3, the within-cell variance component (last summation in equation (5.2)) contributes an average of 67, 83, and 86 percent to the total variance for variables 1, 2, and 3, respectively (Table 8). The average is based on fifteen samples. The extremes are 41 and 100 percent, 27 and 100 percent, and 45 and 100 percent for variables 1, 2, and 3, respectively.

6.   **DISCUSSION**

The main strategy of multi-way stratification is in two steps. First, develop single-criteria designs that are not much worse than optimal single-criteria designs. Second, assure that the single-criteria designs can be allocated to n-way cells in a manner that is most consistent with the population proportions in those cells. The method of BHJ, SS, and this paper all provide a representation of the form (4.2) that, in turn, allows the association of a probabilistic model with a design n_i.

6.1. Problem Posed in Neyman Paper

Neyman (1934, p. 579) observed that stratification can increase the precision of an estimator. Bowley (Neyman 1934, p. 608), in his discussion, noted that the stratification and selection of units for one purpose is not necessarily the best for other purposes. This paper presents n-way designs that allow moderate regulation of the precision of estimators of totals for sets of non-correlated variables.

6.2. Relationship to the Design of BHJ

For some populations and pairs of designs BHJ could show (1960, section 9) that, within a very small correction factor, variances under the 2-way design for the estimators of totals were less than or equal the variances under the single-criteria designs. If we use the convex

representation of BHJ for such a situation, then estimator (5.1) gives the same total as estimator (10) of BHJ (when appropriately modified for totals instead of means). The probability structures of the 2-way designs agree and estimator (5.2) gives the same theoretical variance as the BHJ variance (1960, equation 12) (also when properly modified for totals).

Generally, the probabilistic structure associated with the estimators depends on the convex sum. The convex sum of BHJ contains all possible roundings of the sample allocation matrix. Representation (4.2) contains a subset of all possible roundings and is not unique.

6.3. Relationship to the work of Sitter and Skinner

Sitter and Skinner (1994) extend ideas from Rao and Nigam (1990, 1992). In this section, we will indicate a more plausible reason why their method improved over that of BHJ for the type of simulated data that they considered. We will also demonstrate why the methods of this paper are more general and computationally more tractable in large situations.

SS explicitly wanted to restrict their two-way designs to the form:

$$\sum_{s \in S_n} n_{ij}(s)\, p(s) = nP_{ij} \quad \text{for } i = 1, \ldots, R \text{ and } j = 1, \ldots, C. \tag{6.1}$$

Here $P_{ij} = N_{ij}/N$, n is the sample size, and S_n are designs having sample size n. Representation (6.1) corresponds to the representation (4.2) of the Theorem of this paper. A key difference and difficulty is that SS needed to know all of the integer arrays $n_{ij}(s)$ prior to running their LP algorithms. The integer arrays $n_{ij}(s)$ are part of the restraints of the LP problem. As SS indicated, when there are more constraints, equation (6.1) is difficult to solve using the LP method of their paper but should be easier to solve using a method similar to the method of CCE. The method of this paper solves the equivalent situation for higher dimensional situations and for populations of variables that are reasonably highly skewed. It should be far faster than the LP method of SS. The method of CCE was not extended beyond two dimensions. It is not clear how individuals could find all integer arrays $n_{ij}(s)$ corresponding to a design n_i in three or more dimensions. If one does not have all of the integer arrays $n_{ij}(s)$ prior to solving (6.1) using the LP method of SS, then one cannot generally solve (6.1). Thus, the challenges go beyond the straightforward computational difficulties pointed out by SS.

SS explicitly minimized a loss function of the form:

$$w(s) = \sum_i (n_i.(s) - nP_i.)^2 - \sum_j (n_{.j}(s) - nP_{.j})^2. \tag{6.2}$$

As SS indicated, there are designs for which the minimal solution of (6.2) is zero. The designs of this paper (even in higher dimension) yield minimal solution of zero for equation (6.2). SS (section 2.3) indicated how their LP method could be extended to higher dimensions using a squared-error loss function for which different variables could be weighted by various factors. Their method does not generally yield a minimal solution of zero.

SS indicated that their method is better at controlling the covariances $Cov(n_{ij}, n_{i'j'})$ which directly affect the between-cell variance. A more plausible explanation, as shown in this paper, is that their method was better at controlling the allocated proportions n_{ij} to be close to nP_{ij} than the method of BHJ.

The designs of this paper had restraints in which the univariate stratifications all had integer allocations that were greater than 1. As an approximation for the estimate of the within-cell variance S_i^2, we can use the minimum of the appropriate 1-dimensional variances. If $\mathbf{i} = (i,j)$, then we can use the minimum of the estimates from the sample values associated with row i or

with column j.   SS assume that the variance $S_\mathbf{i}^2$ are constant across cells **i.**   They also assume that $S_\mathbf{i}^2$ can be estimated by the values in a cell having allocated sample greater than 1. With moderately skewed population distributions, their assumptions would yield inappropriate estimates of $S_\mathbf{i}^2$. Their within-cell variance estimates could be strongly biased upwards or downwards. The methods of this paper easily extend to three or more dimensions.

7.   **SUMMARY**

   This paper presents a methodology for using two or more single-criteria stratifications to produce a multi-way stratification of the population. If each single-criteria design allows control over the precision of estimators of totals for one variable, then multi-way stratification, with no increase in sample size, can yield estimators having only moderate loss of precision for all variables.

1/ This paper reports the results of research and analysis undertaken by Census Bureau staff. It has undergone a Census Bureau review more limited in scope than that given to official Census Bureau publications. This report is released to inform interested parties of research and to encourage discussion.

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APPENDIX

To prepare for the proof of equation (4.2), we need more formal definitions and notation. We consider arrays in n dimensions. Let R_i, $i \in I$, be a array of nonnegative integers, A_i, $i \in I$, be an array of nonnegative reals, and m_j, $j = 1, ...,s$, be a set of positive integer marginal restraints. We say that a nonnegative integer array M_i, $i \in I$, is a *controlled allocation with restraint array* R_i, $i \in I$, for array A_i, $i \in I$, if A_i, $i \in I$, and M_i, $i \in I$, each have margins m_j, $j = 1, ..., s$, and

$|A_i - M_i| < R_i$ if A_i is not an integer and

$A_i = M_i$ if A_i is an integer or zero.

Array R_i, $i \in I$, controls how far the allocated array M_i, $i \in I$, can deviate from the original array A_i, $i \in I$. We observe that R_i must necessarily take positive values if A_i is not an integer. R_i can take any nonnegative integer value if A_i is an integer. We will say that nonnegative real array A_i, $i \in I$, has a *controlled allocation* if we can find nonnegative integer arrays M_i, $i \in I$, and R_i, $i \in I$, such that M_i, $i \in I$, is a controlled allocation with restraint array R_i, $i \in I$, for A_i, $i \in I$. If the restraint array R_i, $i \in I$, consists of zeros and ones and has the property that if A_i is an integer then R_i is zero, we say that M_i, $i \in I$, is a *controlled rounding* of A_i, $i \in I$. See Ernst (1989) for more information.

Restraint arrays need not be unique. For convenience, we will generally only consider the smallest restraint array controlling the allocation of an arbitrary array A_i, $i \in I$, to a nonnegative integer array M_i, $i \in I$. By smallest, we mean that no other nonnegative integer array having entries that are less than or equal the corresponding entries in R_i, $i \in I$, and having one entry that is strictly less can serve as the restraint array. The smallest restraint array is unique. The allocated array M_i, $i \in I$, is generally not unique even if the restraint array is the smallest. In the proof of (4.2) the restraint matrix R_i, $i \in I$, will be fixed for all controlled allocations. The restraint matrix will not be referred to because the proof does not depend on its explicit form.

Proof of (4.2). We define sequence of arrays A_{ik}, $i \in I$, $k = 1, ..., t$, M_{ik}, $i \in I$, $k = 1, ..., t$, and positive constants p_k, $k = 1, ..., t$, inductively. For $k = 1$, let $A_{i1} = g_i$, $i \in I$, and let M_{i1}, $i \in I$, be any controlled allocation of A_{i1}, $i \in I$. Define B_{i1}, $i \in I$ as follows: if $A_{i1} - M_{i1} > 0$ then B_{i1} is the smallest integer greater than A_{i1}, if $M_{i1} - A_{i1} > 0$ then B_{i1} is the largest integer smaller than A_{i1}, and if $M_{i1} = A_{i1}$ then $B_{i1} = 0$.

Let $I_1 = \{i \in I: A_{i1} \neq M_{i1}\}$,

$$e_1 = \max_{i \in I_1} |(A_{i1} - M_{i1})/(M_{i1} - B_{i1})|,$$

$p_1 = (1 - e_1)$, and

$$A_{i2} = M_{i1} + (A_{i1} - M_{i1}) / e_1, \ i \in I.$$

For $k > 1$, let M_{ik}, $i \in I$, be any controlled allocation of A_{ik}, $i \in I$. Define B_{ik}, $i \in I$ as follows: if $A_{ik} - M_{ik} > 0$ then B_{ik} is the smallest integer greater than A_{ik}, if $M_{ik} - A_{ik} > 0$ then B_{ik} is the largest integer smaller than A_{ik}, and if $M_{ik} = A_{ik}$ then $B_{ik} = 0$.

Let $I_k = \{i \in I: Ai_k \neq M_{ik}\}$,

$$e_k \equiv \max_{i \in I_k} |(A_{ik} - M_{ik})/(M_{ik} - B_{ik})|, \tag{A.1}$$

$$p_k = (1 - \sum_{j=1}^{k-1} p_j) (1 - e_k), \tag{A.2}$$

$$A_{ik+1} = M_{ik} + (A_{ik} - M_{ik}) / e_k, \ i \in I. \tag{A.3}$$

Observe that $A_{ik} \geq 0$ for $i \in I$ and all k. If A_{ik} becomes an integer for some k, then it remains an integer for subsequent values of k. At each step k, one additional cell of A_{ik+1} (the one associated with the normalized maximum deviation e_k) is set to an integer value. As I is finite, by (A.1), there exists a smallest integer t such that $e_t = 0$ if and only if controlled allocations M_{ik} exist for $k = 1, 2, \cdots, t$.

We now show that $\sum_{k=1}^{t} p_k M_{ik} = A_i$, $i \in I$.

Solving (A.3) for M_{ik} and multiplying by p_k, we obtain

$$p_k M_{ik} = p_k (A_{ik} - e_k A_{ik+1})/(1 - e_k)$$

which together with (A.1) yields

$$p_1 M_{i1} = A_{i1} + (p_1 - 1)/ A_{i2} \ \text{ and,}$$

for $2 \leq k \leq t$,

$$p_k M_{ik} = (1 - \sum_{j=1}^{k-1} p_j) A_{ik} + (\sum_{j=1}^{k} p_j - 1)) A_{ik}+1.$$

Consequently,

$$\sum_{k=1}^{t-1} p_k M_{ik} = A_{i1} + (\sum_{j=1}^{t-1} p_j - 1) A_{it}$$

$$= A_{i1} + (-p_t) A_{it}, \ i \in I,$$

which, in turn, yields the desired result. $\square$

Proof of (5.1) and (5.2). We have that

$$E(T^{\wedge}) = E^{(1)}E^{(2)}(\sum_{i \in I} (M_i^{\sim} / g_i) \cdot Y_i^{\wedge})$$

$$= E^{(1)}(\sum_{i \in I} (M_i^{\sim} / g_i) \cdot Y_i)$$

$$= \sum_{i \in I} Y_i = Y,$$

where Y is the true population total.

We observe that $E^{(1)}(M_i^{\sim}/g_i) = 1$ and that, conditional on the allocation step, sampling under the selection step is independent across cells. Now,

$$T^{\wedge} = \sum_{i \in I} (M_i^{\sim} / g_i) \cdot Y_i^{\wedge}$$

$$= \sum_{i \in I} (M_i^{\sim}/g_i) \cdot Y_i + \sum_{i \in I} (M_i^{\sim}/g_i) \cdot (Y_i^{\wedge} - Y_i) \equiv A1 + A2.$$

As in BHJ (1960, pp. 113-114), it is straightforward to show that A1 and A2 are uncorrelated under the selection step and that

$$f(Y) \equiv \sum_{i \in I} (M_i^{\sim}/g_i - 1)^2 \cdot Y_i^2 +$$

$$\sum_{i\neq j\in I} (M_i^\sim/g_i-1)\cdot(M_j^\sim/g_j-1)\cdot Y_i\cdot Y_j$$

is an unbiased estimator of $Var(A_1)$.

Consequently, we may apply Theorem of Rao (1975) to obtain (5.2). Specifically, a_{is}, b_{is}, and d_{ijs} in Rao's formulas are replaced by $M_i^\sim/g_i$, $(M_i^\sim/g_i - 1)^2$, and $(M_i^\sim/g_i - 1)\cdot(M_j^\sim/g_j - 1)$, respectively. $\square$

Table 1. Examples of Matrices Illustrating
          Controlled Allocations

Matrix **A**

```
1.1  0.7  1.0  0.5  0.7  |  4
1.1  1.5  0.0  0.1  0.3  |  3
0.0  0.5  0.4  0.1  0.0  |  1
0.2  0.1  0.6  0.1  0.0  |  1
0.6  0.2  0.0  0.2  0.0  |  1
----------------------------
 3    3    2    1    1   | 10
```

Matrix **B**

```
2.0  0.0  1.0  0.0  1.0  |  4
0.0  3.0  0.0  0.0  0.0  |  3
0.0  0.0  1.0  0.0  0.0  |  1
0.0  0.0  0.0  1.0  0.0  |  1
1.0  0.0  0.0  0.0  0.0  |  1
----------------------------
 3    3    2    1    1   | 10
```

Matrix **C**

```
2.0  0.0  1.0  0.0  1.0  |  4
1.0  2.0  0.0  0.0  0.0  |  3
0.0  1.0  0.0  0.0  0.0  |  1
0.0  0.0  1.0  0.0  0.0  |  1
0.0  0.0  0.0  1.0  0.0  |  1
----------------------------
 3    3    2    1    1   | 10
```

```
Table 2.   Population Counts N Induced by Two
           Single-Criteria Stratifications  1/

Var 1                 Variable 2 Strata
Strata       1    2    3    4      5   Total

   1         2    7    4    1     11 |   25
   2         3    5    7   17     31 |   63
   3         0   10   16   47     85 |  158
   4         2    3   10   78    257 |  350
   5         3    5   29   67    551 |  655
             ----------------------------
            10   30   66  210    935 | 1251


1/  The quantitative measures of size associated with
the variables, by strata, are:
Variable 1- 1 : 20589-8116, 2 : 8115-3501, 3 : 3500-1501,
            4 : 1500-501, and 5 : 500-0; and
Variable 2- 1 : 16423-5195, 2 : 5194-2260, 3 : 2259-648,
            4 : 647-152, and 5 : 150-0.
```

```
Table 3.  Fitted Sample Matrix A Obtained by Classical IPF
          Marginal Totals are Fixed

Variable 1             Variable 2 Strata
Strata        1       2       3       4       5      Total

    1       2.172   2.393   0.997   0.097   0.341 |   6.
    2       2.098   1.101   1.124   1.059   0.618 |   6.
    3       0.0     1.641   1.914   2.182   1.263 |   7.
    4       0.820   0.387   0.941   2.848   3.004 |   8.
    5       0.910   0.478   2.024   1.814   4.774 |  10.
            ---------------------------------------------
            6.      6.      7.      8.     10.     |  37.
```

Table 4. Fitted Sample Matrix B Obtained by Dykstra's GIFP
          Marginal Totals are Fixed

Variable 1 Strata	Variable 2 Strata					Total
	1	2	3	4	5	
1	2.000	2.483	1.052	0.103	0.362	6.
2	2.182	1.061	1.101	1.046	0.610	6.
3	0.0	1.614	1.914	2.200	1.272	7.
4	0.860	0.377	0.930	2.840	2.993	8.
5	0.958	0.465	2.003	1.811	4.763	10.
	6.	6.	7.	8.	10.	37.

Table 5. Example of Iterative Integer LP Procedure

Cell of Matrix M_k

k	p_k	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(2,1)	(2,2)	(2,3)
0	0.000	2.000	2.483	1.052	0.103	0.362	2.182	1.061	1.101
1	0.140	2.000	3.000	1.000	0.000	0.000	3.000	1.000	1.000
2	0.042	2.000	3.000	1.000	0.000	0.000	3.000	1.000	1.000
3	0.007	2.000	2.000	1.000	0.000	1.000	2.000	2.000	1.000
4	0.054	2.000	3.000	1.000	0.000	0.000	2.000	2.000	1.000
5	0.029	2.000	3.000	1.000	0.000	0.000	2.000	1.000	2.000
6	0.114	2.000	3.000	1.000	0.000	0.000	2.000	1.000	1.000
7	0.104	2.000	3.000	1.000	0.000	0.000	2.000	1.000	1.000
8	0.017	2.000	2.000	2.000	0.000	0.000	2.000	1.000	1.000
9	0.035	2.000	2.000	2.000	0.000	0.000	2.000	1.000	1.000
10	0.003	2.000	2.000	1.000	0.000	1.000	2.000	1.000	1.000
11	0.032	2.000	2.000	1.000	0.000	1.000	2.000	1.000	2.000
12	0.043	2.000	2.000	1.000	0.000	1.000	2.000	1.000	1.000
13	0.237	2.000	2.000	1.000	0.000	1.000	2.000	1.000	1.000
14	0.040	2.000	2.000	1.000	0.000	1.000	2.000	1.000	2.000
15	0.103	2.000	2.000	1.000	1.000	0.000	2.000	1.000	1.000
16	1.000	2.000	2.483	1.052	0.103	0.362	2.182	1.061	1.101

Cell of Matrix M_k

k	(2,4)	(2,5)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(4,1)	(4,2)
0	1.046	0.610	0.000	1.614	1.914	2.200	1.272	0.860	0.377
1	1.000	0.000	0.000	1.000	2.000	2.000	2.000	0.000	1.000
2	1.000	0.000	0.000	1.000	2.000	2.000	2.000	1.000	0.000
3	1.000	0.000	0.000	1.000	2.000	2.000	2.000	1.000	1.000
4	1.000	0.000	0.000	1.000	2.000	2.000	2.000	1.000	0.000
5	1.000	0.000	0.000	1.000	1.000	3.000	2.000	1.000	1.000
6	1.000	1.000	0.000	1.000	2.000	3.000	1.000	1.000	1.000
7	1.000	1.000	0.000	2.000	2.000	2.000	1.000	1.000	0.000
8	1.000	1.000	0.000	2.000	1.000	3.000	1.000	1.000	1.000
9	1.000	1.000	0.000	2.000	2.000	2.000	1.000	1.000	1.000
10	2.000	0.000	0.000	2.000	2.000	2.000	1.000	1.000	1.000
11	1.000	0.000	0.000	2.000	2.000	2.000	1.000	1.000	1.000
12	2.000	0.000	0.000	2.000	2.000	2.000	1.000	1.000	0.000
13	1.000	1.000	0.000	2.000	2.000	2.000	1.000	1.000	0.000
14	1.000	0.000	0.000	2.000	1.000	3.000	1.000	1.000	0.000
15	1.000	1.000	0.000	2.000	2.000	2.000	1.000	1.000	0.000
16	1.046	0.610	0.000	1.614	1.914	2.200	1.272	0.860	0.377

Table 5. Example of Iterative Integer LP Procedure (continued)

Cell of Matrix M_k

k	(4,3)	(4,4)	(4,5)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)
0	0.930	2.840	2.993	0.958	0.465	2.003	1.811	4.763
1	1.000	3.000	3.000	1.000	0.000	2.000	2.000	5.000
2	1.000	3.000	3.000	0.000	1.000	2.000	2.000	5.000
3	1.000	3.000	2.000	1.000	0.000	2.000	2.000	5.000
4	1.000	3.000	3.000	1.000	0.000	2.000	2.000	5.000
5	1.000	2.000	3.000	1.000	0.000	2.000	2.000	5.000
6	1.000	2.000	3.000	1.000	0.000	2.000	2.000	5.000
7	1.000	3.000	3.000	1.000	0.000	2.000	2.000	5.000
8	1.000	2.000	3.000	1.000	0.000	2.000	2.000	5.000
9	0.000	3.000	3.000	1.000	0.000	2.000	2.000	5.000
10	0.000	3.000	3.000	1.000	0.000	3.000	1.000	5.000
11	0.000	3.000	3.000	1.000	0.000	2.000	2.000	5.000
12	1.000	3.000	3.000	1.000	1.000	2.000	1.000	5.000
13	1.000	3.000	3.000	1.000	1.000	2.000	2.000	4.000
14	1.000	3.000	3.000	1.000	1.000	2.000	1.000	5.000
15	1.000	3.000	3.000	1.000	1.000	2.000	1.000	5.000
16	0.930	2.840	2.993	0.958	0.465	2.003	1.811	4.763

Table 6. CVs for Single-Criteria Designs and 2-Way Design 1/

	CVs		Total Variation
	Var 1	Var 2	2/
===	===	===	===
Optimal Single-Criteria			
Stratifying Var 1	.012	.334	.334
Stratifying Var 2	.407	.009	.407
Single-Criteria Designs for Multi-Way Design			
Stratifying Var 1	.044	.289	.292
Stratifying Var 2	.282	.028	.283
Multi-Way Design	.104	.041	.112

1/ Each of the second set of single-criteria designs and
   the 2-way design allocate the same 13 population members
   with certainty and 37 randomly to the strata containing
   the remaining 1251 population members.
2/ Square root of the sum of squares of two CV columns.

Table 7. CVs for Single-Criteria Designs and 3-Way Design 1/

	CVs			Total Variation 2/
	Var 1	Var 2	Var 3	
Optimal Single-Criteria				
Stratifying Var 1	.001	.490	.372	.615
Stratifying Var 2	.306	.001	.351	.466
Stratifying Var 3	.604	.884	.001	1.071
Single-Criteria Designs for Multi-Way Design				
Stratifying Var 1	.047	.291	.221	.368
Stratifying Var 2	.178	.043	.194	.267
Stratifying Var 3	.207	.272	.065	.349
Multi-Way Design	.071	.055	.071	.115

1/ Each of the second set of single-criteria designs and the
   3-way design allocate the same 15 population members
   with certainty and 50 randomly to the strata containing
   the remaining population members.
2/ Square root of the sum of squares of three *cv* columns.

```
     Table 8.  Components of Variance for 3-Way Design Example
                          Fifteen Samples

            |     Estimated     |         CV         |     Within-Cell
            |     Total  1/     |                    |  Variance Proportion
            |     Variable      |     Variable       |       Variable
            |  1  |  2  |  3    |  1  |  2  |  3     |  1   |  2   |  3
       -------------------------------------------------------------------
          1,423  1,195  1,684   .101  .056  .083     .48    .99    .75
          1,384  1,297  1,576   .059  .056  .042    1.00    .92    .89
          1,337  1,269  1,450   .083  .050  .021     .54    .78    .80
          1,278  1,246  1,469   .060  .045  .035     .82    .81    .97
          1,387  1,246  1,696   .048  .059  .090     .60    .96   1.00
          1,345  1,119  1,630   .056  .074  .038     .92    .83    .95
          1,211  1,194  1,554   .061  .045  .046     .91    .84    .92
          1,460  1,226  1,636   .104  .049  .061     .50    .79    .98
          1,355  1,191  1,540   .057  .039  .052     .42   1.00    .72
          1,428  1,250  1,843   .100  .069  .098     .41    .98    .99
          1,345  1,259  1,845   .071  .087  .083     .51    .71    .99
          1,271  1,229  1,808   .063  .075  .089     .76    .62    .86
          1,424  1,336  1,550   .047  .067  .043     .95    .27    .82
          1,436  1,115  1,637   .085  .029  .054     .49    .98    .45
          1,497  1,285  1,833   .051  .066  .095     .81    .97    .88
       -------------------------------------------------------------------
  AVER    1,372  1,230  1,650   .073  .060  .070     .67    .83    .86
       -------------------------------------------------------------------
```

1/   True values are 1354, 1254, and 1654.