Techniques for High School Mathematics Contests

Tristan Chaang

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Contents

Preface

This book is designed for students who already have a grasp in the Malaysian secondary school syllabus of maths and want to prepare for mathematics competitions or Mathematical Olympiads in Malaysia. These include:

1. The Hua Luo Geng cup (HLG)
2. The Chen Jing Run cup (CJR)
3. The National Mathematical Olympiad (*Olimpiad Matematik Kebangsaan*, OMK)
4. The IMO National Selection Test (IMONST)

among others. However, if you are not from Malaysia, this book is still suitable depending on your country's standard on various mathematics competitions. Please take note that this book is definitely not enough if you want to prepare for large, international contests such as the APMO or even the IMO.

There will be many facts and ideas to be grasped in this book. We would divide them into theorems, lemmas, corollaries, propositions etc. There is a slight difference in their definitions, but in most cases they are interchangeable:

Proposition	A mathematical statement
Lemma	A proposition paving the way for proving a theorem
Theorem	An important (and normally famous) proven proposition
Corollary	A proposition derived from a theorem

For simplicity, the proof of all the proposed facts will not be squeezed between the contents of the book, but instead be arranged at the section at the back of the book.

Tristan Chaang,
Malaysian IMO team member (2018 -)

Notations, Acronyms and Abbreviations

In this book, several special notations, acronyms, and abbreviations are used:

HLG	Hua Luo Geng cup
CJR	Chen Jing Run cup
OMK	National Maths Olympiad (*Olimpiad Matematik Kebangsaan*)
NST	National Selection Test
BIMO	IMO Camp (*Bengkel IMO*)
$\mathbb{N}$	Set of natural numbers (0 is not included)
$\mathbb{N}_0$	Set of natural numbers including 0
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{C}$	Set of complex numbers
$\lvert S\rvert$	The number of elements (cardinality) of set S. In some books $n(S)$ is used.
$\square$	Fact demonstrated / End of proof (*quod erat demonstrandum*, QED)
$\gcd(a, b)$	Greatest common divisor / Highest common factor of a and b
$\operatorname{lcm}(a, b)$	Lowest common multiple of a and b
LHS	Left hand side (of a relation comparing two expressions)
RHS	Right hand side (of a relation comparing two expressions)
$a \mid b$	a divides b (i.e. b/a is an integer)
$a \nmid b$	a does not divide b (i.e. b/a is not an integer)
$\overline{abc}$	Integer formed by digits a, b, c from left to right in base-10.
(ABC)	Unique circle passing through A, B, C.
$[S]$	Area of shape S.
$P \sim Q$	Shapes P and Q are similar.
$P \cong Q$	Shapes P and Q are congruent.
$\parallel$	Parallel.
WLOG	Without loss of generality.
$\binom{n}{r}$	The choose function, sometimes written as ${}_nC_r$.
$[n]$	The set $\{1, 2, \ldots, n\}$.

Table 1: Notations, Acronyms, and Abbreviations

Chapter 1

Numbers

In Olympiad level, you only have to know about natural numbers, integers, rationals, irrationals, reals and complex numbers (complex numbers should be new to some people).

1.1 Natural Numbers $\mathbb{N}$

In some countries in the world, 0 is included in $\mathbb{N}$. However, we will define

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

1.2 Integers $\mathbb{Z}$

The notation of $\mathbb{Z}$ originates from the German word *Zahlen*. This set includes natural numbers, their negative versions and 0.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

1.3 Rationals $\mathbb{Q}$

The notation of $\mathbb{Q}$ originates from the Italian word *Quoziente*. It is the set of all fractions where numerators and denominators are integers.

$$\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$$

1.4 Reals $\mathbb{R}$

The real numbers are roughly described by 'all numbers that you can find on the number line'. For example, the numbers $0.123, \pi, e$ are all in $\mathbb{R}$.

1.5 Irrationals $\mathbb{R} \setminus \mathbb{Q}$

The irrationals are basically every other real number excluding the rationals. In other words, they cannot be expressed as a fraction of integers. The first irrational number formally discovered was $\sqrt{2}$ by Pythagoras. He reasoned why $\sqrt{2}$ is irrational by the

Chapter 2

Integers

2.1 Prime Checking

How to determine if a number n is prime? One might say we should check all primes up to n and see if any one of them divides n. However, there is a much quicker way to this. In fact, if all the primes up to $\sqrt{n}$ do not divide n, then the factors of n must be larger than $\sqrt{n}$. If n is composite, then $n > \sqrt{n} \cdot \sqrt{n} = n$ which is a contradiction. Hence

> **Proposition 1.** *n is prime if and only if all primes up to $\sqrt{n}$ does not divide n.*

1. Is 101 prime?

Solution. $\sqrt{101} < 11$, hence we only need to check $2, 3, 5$ and 7. Yes, 101 is prime. $\square$

2.2 GCD and LCM

The GCD (also known as HCF) of two integers is the greatest common divisor of them, whereas the LCM of two integers is the lowest common multiple of them. If the prime factorisations of n and m are $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots$ and $m = p_1^{\beta_1} p_2^{\beta_2} \dots$, then

$$\begin{aligned}\gcd(n,m) &= p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \dots \\ \operatorname{lcm}(n,m) &= p_1^{\max(\alpha_1,\beta_1)} p_2^{\max(\alpha_2,\beta_2)} \dots\end{aligned}$$

and this leads to

> **Proposition 2.** *For any positive integers n and m,* $\gcd(n,m)\operatorname{lcm}(n,m) = nm$.

2. Find all positive integers a, b such that $\gcd(a,b) = 12, \operatorname{lcm}(a,b) = 216$.

Solution. $12 = 2^2 \times 3$, $216 = 2^3 \times 3^3$. Therefore $a = 2^{\alpha_1} \times 3^{\alpha_2}, b = 2^{\beta_1} \times 3^{\beta_1}$ where $\{\alpha_1, \beta_1\} = \{2, 3\}$ and $\{\alpha_2, \beta_2\} = \{1, 3\}$. Hence

$$\begin{cases} a = 2^2 \times 3^1 \\ b = 2^3 \times 3^3 \end{cases} \quad \begin{cases} a = 2^2 \times 3^3 \\ b = 2^3 \times 3^1 \end{cases} \quad \begin{cases} a = 2^3 \times 3^1 \\ b = 2^2 \times 3^3 \end{cases} \quad \begin{cases} a = 2^3 \times 3^3 \\ b = 2^2 \times 3^1 \end{cases}$$

are all solutions, i.e. $(a, b) = (12, 216), (108, 24), (24, 108), (216, 12)$. $\square$

Chapter 3

Modular Arithmetic

3.1 Congruence

Two integers a and b are said to be congruent mod m if their remainders are equal when divided by m, we write this as

$$a \equiv b \pmod{m}$$

For example,

$$16 \equiv 9 \equiv 2 \equiv -5 \equiv -12 \pmod{7}.$$

3.2 Properties of Congruence

Below are a few properties that should be recognised in a problem:

Property 1. Congruence is additive, for example $7 + 8 \equiv 3 + 0 \equiv 3 \pmod{4}$.

Property 2. Congruence is multiplicative, for example $7 \times 9 \equiv 3 \times 1 \equiv 3 \pmod{4}$.

Property 3. If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$.

Property 4. If $ak \equiv bk \pmod{m}$, then $a \equiv b \pmod{\frac{m}{\gcd(k,m)}}$.

Property 4 basically means if we want to divide both sides by a number, the modulus must be divided as much as possible too.

1. Find the remainder of $1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15$ when divided by 4.

We can first use our knowledge to reduce each term to smaller numbers. For example, we can do

Solution 1.

$$\begin{aligned}
&1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \\
&\equiv 1 \times 3 \times 1 \times 3 \times 1 \times 3 \times 1 \times 3 \\
&\equiv 81 \\
&\equiv 1 \pmod{4}
\end{aligned}$$

and hence the remainder is 1. □

Chapter 4

Polynomials

A polynomial consists of variables and coefficients, that involves only addition, subtraction, multiplication, and non-negative integer exponents of variables. In other words,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial where n is a non-negative integer. If $a_n \neq 0$, then $P(x)$ is said to be of *degree* n (i.e. the highest exponent of x). For example,

1. $x^2 - 6x + 7$ is a polynomial of degree 2.
2. $x^9 + 7$ is a polynomial of degree 9.
3. 10 is a polynomial of degree 0.
4. 0 is the special **zero polynomial**: It is normally said to be of degree $-\infty$.

4.1 Factorisation

Factorisation is a process of writing a polynomial into products of several polynomials of smaller degree. For example $x^2 - 7x + 6 = (x-6)(x-1)$. In high school, you are normally taught that there are some polynomials that cannot be factorised (for example, $x^2 + 1$. Apparently, that is true if we are dealing with real numbers only. Here, we will expand into the complex numbers. In that case, many theorems will become consistent. In the field of complex numbers,

Theorem 1. *Any polynomial of degree n can be factored into n linear polynomials.*

1. Factorise $x^2 + 6x + 10$ completely.

Solution. By the Quadratic Formula, the two roots are

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 10}}{2} = -3 \pm i.$$

By comparing coefficients, we immediately know

$$x^2 + 6x + 10 = (x + 3 + i)(x + 3 - i)$$

□

Chapter 5

Sums and Products

5.1 Summation and Product Notation

In this chapter (or more generally, the whole book) we will use frequently use these two symbols: Capital Sigma (Σ) and Capital Pi (Π). The operators associated to them are the summation and product respectively. In mathematical terms,

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + \cdots + f(b)$$

$$\prod_{k=a}^{b} f(k) = f(a) \times f(a+1) \times \cdots \times f(b)$$

For example,

$$\sum_{k=1}^{5} \log k = \log 1 + \log 2 + \log 3 + \log 4 + \log 5$$

$$\prod_{k=1}^{5} (k+3) = (1+3)(2+3)(3+3)(4+3)(5+3)$$

In this case, $\log k$ is called the **summand**, and $(k+3)$ is called the **multiplier/multiplicand**. Besides, k is called the **index** while 1 and 5 are called the **bounds**.

We can also write **condition(s)** instead of bounds, e.g. if $S = \{1, 4, 5, 8, 9\}$,

$$\sum_{k \in S} \log k = \log 1 + \log 4 + \log 5 + \log 8 + \log 9 \qquad \prod_{\substack{k \in S \\ k \text{ even}}} k = 4 \times 8 = 32.$$

In many cases, a summation or product may not be easily computed (e.g. the examples above!). However, some summations and products come in neater variations, which we can find a method to easily compute even though the bounds are arbitrarily large.

Note that these properties hold:

$$\sum (f(x) + g(x)) = \sum f(x) + \sum g(x)$$

$$\sum c f(x) = c \sum f(x)$$

$$\prod (f(x) g(x)) = \prod f(x) \prod g(x).$$

where c is independent of the index.

Chapter 6

The Floor Function

6.1 The Floor Function

The notation of the floor function is invented by Carl Friedrich Gauss. It is an interesting, most of the time confusing, type of function that we should know prior to joining contests.

The **floor** of x, denoted by $\lfloor x \rfloor$, is the largest integer not more than x. For example,

$$\begin{aligned}\lfloor 0.3 \rfloor &= 0\\ \lfloor 5 \rfloor &= 5\\ \lfloor \pi \rfloor &= 3\\ \lfloor -3.2 \rfloor &= -4.\end{aligned}$$

In contests, the most useful inequality in dealing with floor functions is

Proposition 1. $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

1. Evaluate $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{10000} \rfloor$.

We wish to analyse how the frequencies of values behave in this expression. For example, 1 appears three times ($\lfloor \sqrt{1} \rfloor = \lfloor \sqrt{2} \rfloor = \lfloor \sqrt{3} \rfloor = 1$), 2 appears five times ($\lfloor \sqrt{4} \rfloor = \lfloor \sqrt{5} \rfloor = \lfloor \sqrt{6} \rfloor = \lfloor \sqrt{7} \rfloor = \lfloor \sqrt{8} \rfloor = 2$) etc. Generally, how many times does k appear?

Solution. The last term is 100 obviously. For each $k = 1, 2, \ldots, 99$, we find the number of n that satisfies $\lfloor \sqrt{n} \rfloor = k$:

$$\begin{aligned}k \leq \sqrt{n} &< k + 1\\ k^2 \leq n &< k^2 + 2k + 1\\ k^2 \leq n &\leq k^2 + 2k\end{aligned}$$

hence there are $2k + 1$ terms that are equal to k. Therefore the original expression is,

$$\begin{aligned}\sum_{k=1}^{99} k(2k+1) + 100 &= 2\sum_{k=1}^{99} k^2 + \sum_{k=1}^{99} k + 100\\ &= 2 \cdot \frac{99 \cdot 100 \cdot 199}{6} + \frac{99 \cdot 100}{2} + 100\\ &= 661750.\end{aligned}$$

Chapter 7

Recursion

A **recursive function** is a function that is defined in terms of itself. One of the most famous examples is the Fibonacci number sequence:

$$\begin{cases} F(n) = F(n-1) + F(n-2) \\ F(0) = 0, F(1) = 1 \end{cases}$$

However, most of them can be explicitly expressed as a normal function, i.e. the **general form**. Let's start from the easy examples.

7.1 First Order Recurrence

Let's define whatever's in the title of this section. A **first order recurrence** is a recurrence in which a_n is only expressed in terms of a_{n-1}. A higher order recurrence might have a_n expressed in terms of a_{n-1}, a_{n-2} etc.

1. Find the general form of $a_n = a_{n-1} + 3, a_0 = 2$.

Solution. We have $a_n = a_{n-1} + 3 = a_{n-2} + 2 \times 3 = \cdots = a_0 + n \times 3 = 3n + 2$. $\square$

2. Find the general form of $a_n = 4a_{n-1}, a_0 = 7$.

Solution. We have $a_n = 4a_{n-1} = 4^2 a_{n-2} = \cdots = 4^n a_0 = 7 \cdot 4^n$. $\square$

3. Find the general form of $a_n = 4a_{n-1} - 3, a_0 = 2$.

Solution. Note that $a_n - 1 = 4(a_{n-1} - 1)$. Hence let $b_n = a_n - 1$ to get

$$b_n = 4b_{n-1} = 4^2 b_{n-2} = \cdots = 4^n b_0 = 4^n(a_0 - 1) = 4^n.$$

And hence $a_n = 1 + 4^n$. $\square$

4. Find the general form of $a_n = 5a_{n-1} + 3, a_0 = 6$.

Solution. Let $a_n + p = 5(a_{n-1} + p) \iff a_n = 5a_{n-1} + 4p$. Thus $p = 3/4$. Hence let $b_n = a_n + 3/4$, then $b_n = 5b_{n-1} \Rightarrow b_n = (27/4)5^n$, yielding

$$a_n = \frac{27 \cdot 5^n - 3}{4}.$$

$\square$

Chapter 8

Geometry

8.1 Analytic Geometry

Analytic geometry in contests consists of Cartesian coordinates, trigonometry and complex numbers. However, we will focus on coordinates and trigonometry only.

8.1.1 Cartesian Coordinates

The most important equations in Cartesian Coordinates:

Proposition 1. *Lines are uniquely expressed as $y = mx + c$ (m, c are constants).*

Proposition 2. *The circle centred at (a, b) with radius r is $(x-a)^2+(y-b)^2 = r^2$.*

Theorem 1. *(Point-to-point) The distance between two points $(x_1, y_1), (x_2, y_2)$ is*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Theorem 2. *(Point-to-line) The distance from (x_1, y_1) to line $ax + by + c = 0$ is*

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|.$$

Theorem 3. *The area of the triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is*

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_2)$$

1. (HLG2018) $ABCD$ is a square with side length 100 and the midpoints of AB and DA are M, N respectively. CM and BN intersect at P. Find the length of DP.

We can plot this onto a graph with $D(0, 0)$ and $A(0, 100)$.

Chapter 9

Inequalities

9.1 The Trivial Inequality

Before we encounter the advanced inequalities, we must first introduce the most rudimentary inequality in real numbers: If $x \in \mathbb{R}$, then

$$x^2 \geq 0.$$

9.2 The AM-GM and Cauchy-Schwarz Inequality

AM-GM and Cauchy are among the most useful inequalities used in Olympiads.

Theorem 1. *(AM-GM) If $a_1, \ldots, a_n$ are positive real numbers, then*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

with equality if and only if $a_1 = \cdots = a_n$.

Theorem 2. *(Cauchy) If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real numbers, then*

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2$$

with equality if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$.

Now, how do we apply these two inequalities in competition problems? Let's look at some examples.

1. Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Let's analyse the problem. We see a bunch of squares in the LHS, and some 'combined' terms in the RHS. That looks similar to AM-GM, but how can we have three terms produced? We can try backtracking to see what produces ab. Yes! $(a^2 + b^2)/2 \geq ab$. Using the same expressions for b, c and c, a, we immediately have our solution.

Chapter 10

Diophantine Equations

Diophantine was a Greek Mathematician in the 2nd Century AD. A Diophantine equation is an equation solving for (positive) integer solutions. These equations are sometimes very hard to solve as equations could appear in unpredictable ways. In this chapter, we will be looking at some famous techniques that might help you in most contests.

10.1 Linear Diophantine

These are the simplest of them all. Methods include **parameterising** and **tabulating**.

Theorem 1. *If* $ax + by = c$ *and* $\gcd(a, b) = 1$, *we have the parametric solutions*

$$\begin{cases} x = x_0 + bt \\ y = y_0 - at \end{cases} \quad (t \in \mathbb{Z})$$

where (x_0, y_0) *is any solution of the problem.*

1. How many pairs of positive integers (x, y) satisfy $6x + 7y = 1001$?

Solution. $(x, y) = (0, 143)$ is one of the solutions. Hence

$$\begin{cases} x = 7t \\ y = 143 - 6t \end{cases} \quad (t \in \mathbb{Z})$$

Since $7t, 143 - 6t > 0 \Rightarrow 1 \leq t \leq 23$, there are 23 positive integer solutions. □

2. How many pairs of positive integers (x, y) satisfy $6x + 8y = 1001$?

Solution. LHS is even but RHS is not. No integer solutions. □

3. How many triples of positive integers (x, y, z) satisfy $2x + 4y + 5z = 120$?

Solution. First, $z = 2k$. Substituting into the equation yields

$$\begin{aligned} x + 2y + 5k &= 60 \\ 2y + 5k &= 60 - x \end{aligned}$$

Hence we are finding the number of (y, k) such that $2y + 5k \leq 59$. We can tabulate:

Chapter 11

Counting

11.1 Bijection

Say we have a theatre with 300 seats, and we want to know how many people there are in the audience. The thing is, all seats are seated! The obvious answer is there would be 300 people, and this logic involves what we call **bijections** in mathematics.

We observe a relation between each person and each seat. Each seat exactly accommodates one person whereas each person seats on one chair. That means, there is a **one-to-one correspondence (bijection)** from the set of seats and the set of people. Therefore, **if there exists a bijection between two finite sets A and B, then** $|A| = |B|$. However, one should remember that in order for a correspondence to be a bijection, the correspondence must be bidirectional, i.e. each element in A corresponds to one element in B AND each element in B corresponds to one element in A.

1. Are these two values the same?

(a) The number of ways to choose three numbers $a < b < c$ from $[n] = \{1, 2, \ldots, n\}$ such that $a + c = 2b$.

(b) The number of ways to choose two distinct even numbers plus the number of ways to choose two distinct odd numbers from $[n]$.

Solution. Yes. ((a) $\Rightarrow$ (b)) Since $a + c$ is an even number, either a, c are both even or both odd, satisfying (b). ((b) $\Rightarrow$ (a)) If two distinct numbers a, c with equal parity (even/oddness) are chosen, then they must sum up to an even number, hence $b = \frac{a+c}{2}$ exists uniquely, therefore $a < b < c$ is chosen, satisfying (a). $\square$

Note that this is actually a big deal! The value for (a) has a weird condition, but we know that the value for (a) is just equal to the answer for (b), hence it only suffices to solve (b) (which is easier)! Let's see another example:

2. (OMK2014) A subset of $S = \{1, 2, \ldots, 2014\}$ is called *good* if

(a) It does not contain both 1 and 2 (it can contain a 1 or 2, but not both)

(b) The sum of elements is divisible by 3.

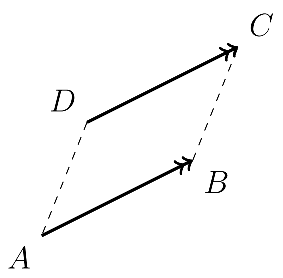
E.g. $\{1, 23, 456, 789\}$ is good. Find the number of good subsets of S.

Chapter 12

* Vector Geometry

12.1 Vectors in n-dimensional space

A vector is a mathematical object that has **magnitude** and **direction**, but **no location**. In n-dimensional space (we will only cover $n = 2$ and 3), we can associate two points A and B with two choices of vectors: the vector $\overrightarrow{AB}$ (from A to B), or the vector $\overrightarrow{BA}$ (from B to A). However, the endpoints of vector $\overrightarrow{AB}$ does not necessarily be at A and B, as a vector does not carry 'location' in its meaning. For example, if $ABCD$ is a parallelogram (figure below), then $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are exactly the same vector, so we say $\overrightarrow{AB} = \overrightarrow{DC}$.



In this chapter, we assume that the reader has basic knowledge of vector addition and scaling. In this book, we will use the following few notations and definitions:

Definition 1. The vector from the origin to the point $(a_1, \ldots, a_n)$ is denoted as $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, or sometimes for neatness purposes, $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}^T$.

Definition 2. The **length** of a vector $\vec{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is $|\vec{v}| = \sqrt{a_1^2 + \cdots + a_n^2}$.

Please note that $|\vec{a} + \vec{b}|$ is not necessarily equal to $|\vec{a}| + |\vec{b}|$, but:

Theorem 1. *(Triangle Inequality)* $|\vec{a}| + |\vec{b}| \geq |\vec{a} + \vec{b}|$, *with equality iff* $\vec{a} \parallel \vec{b}$.

Chapter 13

* Generating Functions

13.1 Generating Functions

Definition 1. $C_k[f]$ is the coefficient of x^k in a polynomial $f(x)$.

Proposition 1. *Let f be a polynomial. Then*

$$C_n[x^{n+1}f] = 0$$

Proposition 2. *Let f and g be two polynomials. Then*

$$C_n[f+g] = C_n[f] + C_n[g]$$

Proposition 3. *Let f and g be two polynomials. Then*

$$C_n[fg] = \sum_{k=0}^{n} C_k[f] \cdot C_{n-k}[g]$$

Theorem 1. *(Binomial Theorem) For $n \in \mathbb{N}$, we have*

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

Theorem 2. *(Extended Binomial) For $n \in \mathbb{N}$, we have*

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k$$

Normally, if the top part of the choose function involves k, then we use **Theorem 2** (Exceptions: Example 3), otherwise we use **Theorem 1**. What can generating functions do? A **generating function** is a method of encoding an infinite sequence of numbers a_n by treating them as the coefficients of a polynomial.

Appendix A

Mathematical Reasoning

A statement, proposition or assertion is a sentence which is either true or false. The state of being true or false is the **truth value** of the statement.

Negation

In mathematics it is important to determine what the opposite of a given mathematical statement is. The **negation** S' of a statement S is a statement such that the truth value of S' is opposite of S.

For example, the negation of

'X is a boy'	is	'X is not a boy'

and the negation of

'All apples are red'	is	'Not all apples are red'

instead of 'All apples are not red'.

Implication

Say A and B are two statements. When we have

'If statement A is true, then statement B is true',

we say that 'A **implies** B', or 'A is **sufficient** for B', or 'B is **necessary** for A', denoted by $A \Rightarrow B$. However, this does NOT mean $B \Rightarrow A$. For example, an apple is a fruit, however a fruit might not be an apple. Note that an implication of two statements is also a statement.

Logical AND

The **conjunction** of statements A, B is a statement which gives a truth value of 'true' if both A and B are true, and gives 'false' otherwise. We normally say the conjunction of A, B as A AND B.

Logical OR

The **disjunction** of statements A, B is a statement which gives a truth value of 'true' if any of A or B is true, and gives 'false' otherwise. We normally say the conjunction of A, B as A OR B.

Equivalence

Say A and B are two statements. When we have

$$\text{`}A \Rightarrow B\text{' AND `}B \Rightarrow A\text{'}$$

we say that 'A is **equivalent** to B', or 'A **if and only if (iff)** B', or 'A is **sufficient and necessary** for B', denoted as $A \Leftrightarrow B$. E.g.

$$\begin{aligned} \text{`X is an equilateral triangle'} \quad &\Leftrightarrow \quad \text{`X is a triangle with angles } 60^\circ \text{ only'} \\ \text{`}a = b\text{'} \quad &\Leftrightarrow \quad \text{`}a + c = b + c\text{'} \end{aligned}$$

Contrapositive Property

Previously we have mentioned that $A \Rightarrow B$ does not mean $B \Rightarrow A$. However the useful **contrapositive property** says that

$$\text{`}A \Rightarrow B\text{'} \qquad \Leftrightarrow \qquad \text{`}B' \Rightarrow A'\text{'}.$$

For example, 'an apple is a fruit' exactly means 'a non-fruit is a non-apple'.

Proof

A proof is a sequence of implications, which explains why a statement is true. Once a theorem is proven true, it is true forever, hence we say proofs are *absolute*. Sometimes, we may use various previously established theorems in a proof. However, all of these theorems are also derived from more fundamental facts. The most fundamental facts are the *axioms*, for example the Peano Axioms and the Euclidean Axioms, which are set and cannot be proven. In a proof, we may use well-known notations, such as $+$ means addition etc. However, we are certainly allowed to define our own notation, as long as you state it in your proof.

Note that a proof must be directed at the desired statement. Say a problem wants you to prove that if a condition C is satisfied, then n is even. Note that you CANNOT assume that n is even and use it imply C, as this will prove the backwards direction instead of the forward direction, which is wrong (as stated in '$A \Rightarrow B$ does not imply $B \Rightarrow A$')!

Also, you cannot prove for a specific case of a statement only. Say a problem wants you to prove that 'for any rectangle R, the condition C must be satisfied'. Then you CANNOT assume R is a square! This is because there are other rectangles you have not proven for!

Although you are not allowed to assume the result, you are still allowed to assume the negation of the result, as seen in II. *Proof by Contradiction* in Appendix B. This uses the contrapositive property.

There are many methods of proof, and the most famous ones will be stated in Appendix B which is in the next page.

Appendix B

Methods of Proof

I. *Direct Proof*

'True fact $T \Rightarrow$ Result R'

Most of the proofs in this book are direct.

E.g. Chapter 2 Theorem 1 is proven directly.

II. *Proof by Contradiction*

'$R' \Rightarrow$ Absurd result F'

Sometimes, assuming the contrary would be a faster way to tackle the problem, especially when there are too many cases when we prove it directly.

1. If $a_1, \ldots, a_n$ are integers such that $a_1 \ldots a_n$ is odd, then all a_i are odd.

Proof: Assume the contrary, then <u>at least one</u> (not all!) a_i is even. This a_i will contribute a factor of 2, causing $a_1 \ldots a_n$ to be even, which is a contradiction. Thus all a_i are odd. $\square$

E.g. Chapter 10 Theorem 1 is proven using contradiction.

III. *Proof by Induction*

This is normally used in problems dealing with integers. Let $P(n)$ be the statement for n.

(a) Basic Induction: $P(n) \Rightarrow P(n+1)$ <u>and</u> $P(1)$ is true.
(b) Strong Induction: $P(1), \ldots, P(n) \Rightarrow P(n+1)$ <u>and</u> $P(1)$ is true.
(c) Forward-Backward Induction: $P(n) \Rightarrow P(2n)$ <u>and</u> $P(n) \Rightarrow P(n-1)$ <u>and</u> $P(1)$ is true.
(d) Others: For example $P(n) \Rightarrow P(n+3)$ <u>and</u> $P(1), P(2), P(3)$ are true.

2. Prove that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Proof: For $n = 1$, $LHS = RHS = 1$, which is true. Assume $1 + \cdots + k = \dfrac{k(k+1)}{2}$. Then $1 + 2 + \cdots + k + (k+1) = \dfrac{k(k+1)}{2} + (k+1) = \dfrac{(k+1)(k+2)}{2}$ which completes the problem statement. Therefore, the statement is true for all $n \in N$. $\square$

3. (IMONST2020) Consider the following one-person game: A player starts with score 0 and writes the number 20 on an empty whiteboard. At each step, she may erase any one integer (call it a) and writes two positive integers (call them b and c) such that $b + c = a$. The player then adds $b \times c$ to her score. She repeats the step several times until she ends up with all 1s on the whiteboard. Then the game is over, and the final score is calculated. Let M, m be the maximum and minimum final score that can be possibly obtained respectively. Find $M - m$.

Answer: 0.
Proof: Generalise 20 to any positive integer n. We prove that the final score is always the same, which is $\binom{n}{2}$. For $n = 1$, the final score is always 0. Assume the statement is true for $n = 1, \ldots, k - 1$. Then for $n = k$, the player first splits k into a and b, hence ab is added to the score. By inductive hypothesis, a and b will always end up with scores $\binom{a}{2}$ and $\binom{b}{2}$ respectively. Therefore the final score is

$$ab + \binom{a}{2} + \binom{b}{2} = \frac{2ab + a^2 - a + b^2 - b}{2} = \frac{(a+b)(a+b-1)}{2} = \binom{k}{2}.$$

This completes the induction, hence $M = m$. □

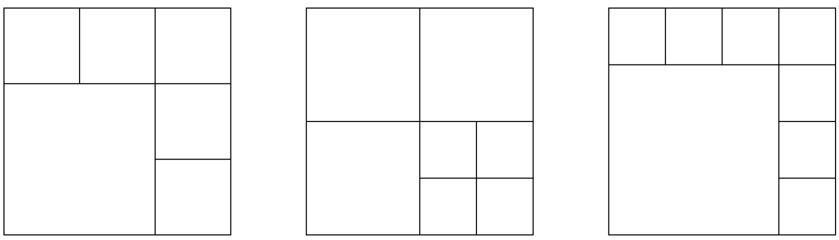
E.g. Chapter 9 Theorem 1 is proven using forward-backward induction.

IV. *Proof by Construction*

This is normally used in problems proving existence. To prove existence, we just need to show that an example exists. However, this does not show uniqueness. The problem below mixes construction and induction:

4. Prove that a square can be divided into any number n ($n \geq 6$) of disjoint squares.

Proof: We first prove for $n = 6, 7, 8$. They are described below:



Assume n can. Then choose any square and divide it into 4 equally sized pieces, hence $n + 3$ can. Together with the 3 base cases, we infer that all $n \geq 6$ can. □

E.g. Chapter 3 Theorem 1 is proven using construction.