

# Techniques for High School Mathematics Contests

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# Contents

<b>Preface</b>	<b>7</b>
<b>1 Numbers</b>	<b>9</b>
1.1 Natural Numbers $\mathbb{N}$	9
1.2 Integers $\mathbb{Z}$	9
1.3 Rationals $\mathbb{Q}$	9
1.4 Reals $\mathbb{R}$	9
1.5 Irrationals $\mathbb{R} \setminus \mathbb{Q}$	9
1.6 Complex Numbers $\mathbb{C}$	11
1.7 Problems	13
<b>2 Integers</b>	<b>15</b>
2.1 Prime Checking	15
2.2 GCD and LCM	15
2.3 Euclidean Algorithm	16
2.4 $p$ -adic Valuation	18
2.5 Functions $d(n), \sigma(n), \phi(n)$	19
2.6 Problems	20
<b>3 Modular Arithmetic</b>	<b>21</b>
3.1 Congruence	21
3.2 Properties of Congruence	21
3.3 Large powers	23
3.4 Chinese Remainder Theorem (CRT)	23
3.5 Fermat's Little Theorem	26
3.6 Divisibility	28
3.7 Problems	28
<b>4 Polynomials</b>	<b>29</b>
4.1 Factorisation	29
4.2 Roots of a Polynomial	30
4.3 Division Algorithm	31
4.4 Polynomial Manipulation	32
4.5 Special Techniques	33
4.6 Problems	34

<b>5</b>	<b>Sums and Products</b>	<b>35</b>
5.1	Summation and Product Notation . . . . .	35
5.2	Sum Formulae . . . . .	36
5.3	Telescoping Sum and Products . . . . .	36
5.4	Gaussian Pairing . . . . .	38
5.5	Infinite Series . . . . .	39
5.6	Problems . . . . .	41
<b>6</b>	<b>The Floor Function</b>	<b>43</b>
6.1	The Floor Function . . . . .	43
6.2	The Ceiling and Rounding Function . . . . .	45
6.3	Problems . . . . .	45
<b>7</b>	<b>Recursion</b>	<b>47</b>
7.1	First Order Recurrence . . . . .	47
7.2	Second or Higher Order Recurrence . . . . .	48
7.3	Non-Homogeneous Recurrence . . . . .	50
7.4	Value of $\lfloor r^n \rfloor$ . . . . .	50
7.5	Special Methods . . . . .	51
7.6	Problems . . . . .	52
<b>8</b>	<b>Geometry</b>	<b>53</b>
8.1	Analytic Geometry . . . . .	53
8.1.1	Cartesian Coordinates . . . . .	53
8.1.2	Trigonometry . . . . .	54
8.2	Synthetic Geometry . . . . .	55
8.2.1	Similar and Congruent Triangles . . . . .	55
8.2.2	Angle Chasing . . . . .	55
8.2.3	Length and Area Chasing . . . . .	55
8.2.4	Centres of a Triangle . . . . .	56
8.2.5	Circles . . . . .	56
8.3	Problems . . . . .	57
<b>9</b>	<b>Inequalities</b>	<b>59</b>
9.1	The Trivial Inequality . . . . .	59
9.2	The AM-GM and Cauchy-Schwarz Inequality . . . . .	59
9.3	Trigonometric Inequalities . . . . .	61
9.4	Geometric Inequalities . . . . .	61
9.5	Problems . . . . .	63
<b>10</b>	<b>Diophantine Equations</b>	<b>65</b>
10.1	Linear Diophantine . . . . .	65
10.2	Simon's Favourite Factoring Trick (SFFT) . . . . .	66
10.3	Bounding . . . . .	67
10.4	Perfect Squares . . . . .	67
10.5	Factorising . . . . .	68
10.6	Pythagorean Triples . . . . .	69
10.7	Problems . . . . .	70

<b>11 Counting</b>	<b>71</b>
11.1 Bijections . . . . .	71
11.2 The Choose Function . . . . .	72
11.3 Properties of the Choose Function . . . . .	74
11.4 Binomial Theorem . . . . .	75
11.5 De Morgan's Law . . . . .	76
11.6 Principle of Inclusion and Exclusion . . . . .	76
11.7 Problems . . . . .	78
<b>12 * Vector Geometry</b>	<b>79</b>
12.1 Vectors in $n$ -dimensional space . . . . .	79
12.2 Linear Independence . . . . .	80
12.3 Problems . . . . .	84
<b>13 * Calculus</b>	<b>85</b>
13.1 Differentiation . . . . .	85
13.1.1 Higher order derivatives . . . . .	87
13.1.2 Local and Global Extrema . . . . .	87
13.1.3 Handling multiple variables . . . . .	89
13.1.4 L'Hôpital's Rule . . . . .	90
13.2 Integration . . . . .	90
13.2.1 Using integrals for inequalities . . . . .	92
13.3 Problems . . . . .	93
<b>14 * Generating Functions</b>	<b>95</b>
14.1 Generating Functions . . . . .	95
14.2 Problems . . . . .	100
<b>Appendix A: Mathematical Reasoning</b>	<b>103</b>
<b>Appendix B: Methods of Proofs</b>	<b>105</b>
<b>Sketch of Proofs</b>	<b>109</b>
<b>Solutions</b>	<b>119</b>
<b>Bibliography</b>	<b>137</b>

*\* are nonessential topics*



# Preface

This book is designed for students who already have a grasp in the Malaysian secondary school syllabus of maths (SPM) and want to prepare for mathematics competitions or Mathematical Olympiads in Malaysia. These include:

1. The Hua Luo Geng cup (HLG)
2. The Chen Jing Run cup (CJR)
3. The National Mathematical Olympiad (*Olimpiad Matematik Kebangsaan*, OMK)
4. The IMO National Selection Test (IMONST)

among others. However, if you are not from Malaysia, this book is still suitable depending on your country's standard on various mathematics contests. Please take note that this book is definitely **not enough** if you want to prepare for large, international contests such as the Asian Pacific Mathematical Olympiad (APMO) or even the International Mathematical Olympiad (IMO).

There will be many facts and ideas to be grasped in this book. We would divide them into theorems, lemmas, corollaries, propositions etc. There is a slight difference in their definitions, which should be known by the reader:

<b>Proposition</b>	A mathematical statement
<b>Lemma</b>	A proposition paving the way for proving a theorem
<b>Theorem</b>	An important (and normally famous) proven proposition
<b>Corollary</b>	A result derived from a theorem

In order to not break up the discussion, some of the proofs of the proposed facts will not be written directly after it, but instead be arranged at the end of the book.

Tristan Chaang,  
Malaysian IMO team member (2018 -)



# Notations, Acronyms and Abbreviations

In this book, several special notations, acronyms, and abbreviations are used:

<b>HLG</b>	Hua Luo Geng cup
<b>CJR</b>	Chen Jing Run cup
<b>OMK</b>	National Maths Olympiad ( <i>Olimpiad Matematik Kebangsaan</i> )
<b>NST</b>	National Selection Test
<b>BIMO</b>	IMO Camp ( <i>Bengkel IMO</i> )
$\mathbb{N}$	Set of natural numbers (0 is not included)
$\mathbb{N}_0$	Set of natural numbers including 0
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{C}$	Set of complex numbers
$ S $	The number of elements (cardinality) of set $S$ . In some books $n(S)$ is used.
$\square$	Fact demonstrated / End of proof ( <i>quod erat demonstrandum</i> , QED)
$\gcd(a, b)$	Greatest common divisor / Highest common factor of $a$ and $b$
$\text{lcm}(a, b)$	Lowest common multiple of $a$ and $b$
LHS	Left hand side (of a relation comparing two expressions)
RHS	Right hand side (of a relation comparing two expressions)
$a \mid b$	$a$ divides $b$ (i.e. $b/a$ is an integer)
$a \nmid b$	$a$ does not divide $b$ (i.e. $b/a$ is not an integer)
$\overline{abc}$	Integer formed by digits $a, b, c$ from left to right in base-10.
$(ABC)$	Unique circle passing through $A, B, C$ .
$[S]$	Area of shape $S$ .
$P \sim Q$	Shapes $P$ and $Q$ are similar.
$P \cong Q$	Shapes $P$ and $Q$ are congruent.
$\parallel$	Parallel.
<b>WLOG</b>	Without loss of generality.
$\binom{n}{r}$	The choose function, sometimes written as ${}_nC_r$ .
$[n]$	The set $\{1, 2, \dots, n\}$ .
$n!$	Factorial function: $n! = n(n-1) \cdots 2 \cdot 1$ .

Table 1: Notations, Acronyms, and Abbreviations

# Chapter 1

## Numbers

In Olympiad level, you only have to know about natural numbers, integers, rationals, irrationals, reals and complex numbers (complex numbers should be new to some people). We shall not define these sets rigorously in high-school level.

### 1.1 Natural Numbers $\mathbb{N}$

In some books, 0 is included in  $\mathbb{N}$ . However, we will define

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

### 1.2 Integers $\mathbb{Z}$

The notation of  $\mathbb{Z}$  originates from the German word *Zahlen*. This set includes natural numbers, their negative versions and 0.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

### 1.3 Rationals $\mathbb{Q}$

The notation of  $\mathbb{Q}$  originates from the Italian word *Quoziente*. It is the set of all fractions where numerators and denominators are integers.

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

### 1.4 Reals $\mathbb{R}$

The real numbers are roughly described by ‘all numbers that you can find on the number line’. For example, the numbers 0.123,  $\pi$ ,  $e$  are all in  $\mathbb{R}$ .

### 1.5 Irrationals $\mathbb{R} \setminus \mathbb{Q}$

The irrationals are basically every other real number excluding the rationals. In other words, they cannot be expressed as a fraction of integers. The first irrational number

# Chapter 2

## Integers

### 2.1 Prime Checking

How to determine if a number  $n$  is prime? One might say we should check all primes up to  $n$  and see if any one of them divides  $n$ . However, there is a much quicker way to this. In fact, if all the primes up to  $\sqrt{n}$  do not divide  $n$ , then the factors of  $n$  must be larger than  $\sqrt{n}$ . If  $n$  is composite, then  $n > \sqrt{n} \cdot \sqrt{n} = n$  which is a contradiction. Hence

**Proposition 1.**  *$n$  is prime if and only if all primes up to  $\sqrt{n}$  does not divide  $n$ .*

1. Is 101 prime?

Solution.  $\sqrt{101} < 11$ , hence we only need to check 2, 3, 5 and 7. Yes, 101 is prime.  $\square$

2. Is 119 prime?

Solution. Again, we check 2, 3, 5 and 7. Indeed,  $7 \mid 119$ , so  $119 = 7 \times 17$  is not prime.  $\square$

### 2.2 GCD and LCM

The GCD (also known as HCF) of two integers is the greatest common divisor of them, whereas the LCM of two integers is the lowest common multiple of them. If the prime factorisations of  $n$  and  $m$  are  $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots$  and  $m = p_1^{\beta_1} p_2^{\beta_2} \dots$ , then

$$\begin{aligned}\gcd(n, m) &= p_1^{\min(\alpha_1, \beta_1)} p_2^{\min(\alpha_2, \beta_2)} \dots \\ \text{lcm}(n, m) &= p_1^{\max(\alpha_1, \beta_1)} p_2^{\max(\alpha_2, \beta_2)} \dots\end{aligned}$$

and this leads to

**Proposition 2.** *For any positive integers  $n$  and  $m$ ,  $\gcd(n, m)\text{lcm}(n, m) = nm$ .*

# Chapter 3

## Modular Arithmetic

### 3.1 Congruence

Two integers  $a$  and  $b$  are said to be congruent mod  $m$  if their remainders are equal when divided by  $m$ , we write this as

$$a \equiv b \pmod{m}$$

For example,

$$16 \equiv 9 \equiv 2 \equiv -5 \equiv -12 \pmod{7}.$$

### 3.2 Properties of Congruence

Below are a few properties that should be recognised in a problem:

**Property 1.** Congruence is additive, for example  $7 + 8 \equiv 3 + 0 \equiv 3 \pmod{4}$ .

**Property 2.** Congruence is multiplicative, for example  $7 \times 9 \equiv 3 \times 1 \equiv 3 \pmod{4}$ .

**Property 3.** If  $a \equiv b \pmod{m}$ , then  $a^n \equiv b^n \pmod{m}$ .

**Property 4.** If  $ak \equiv bk \pmod{m}$ , then  $a \equiv b \pmod{\frac{m}{\gcd(k, m)}}$ .

Property 4 basically means if we want to divide both sides by a number, the modulo must be divided as much as possible too.

1. Find the remainder of  $1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15$  when divided by 4.

We can first use our knowledge to reduce each term to smaller numbers. For example, we can do the following:

Solution 1.

$$\begin{aligned} &1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \\ &\equiv 1 \times 3 \times 1 \times 3 \times 1 \times 3 \times 1 \times 3 \\ &\equiv 81 \\ &\equiv 1 \pmod{4} \end{aligned}$$

and hence the remainder is 1. □

# Chapter 4

## Polynomials

A polynomial consists of variables and coefficients, that involves only addition, subtraction, multiplication, and non-negative integer exponents of variables. In other words,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial where  $n$  is a non-negative integer. If  $a_n \neq 0$ , then  $P(x)$  is said to be of *degree  $n$*  (i.e. the highest exponent of  $x$ ). For example,

1.  $x^2 - 6x + 7$  is a polynomial of degree 2.
2.  $x^9 + 7$  is a polynomial of degree 9.
3. 10 is a polynomial of degree 0.
4. 0 is the special **zero polynomial**: It is normally said to be of degree  $-\infty$ .

### 4.1 Factorisation

Factorisation is a process of writing a polynomial into products of several polynomials of smaller degree. For example  $x^2 - 7x + 6 = (x - 6)(x - 1)$ . In high school, you are normally taught that there are some polynomials that cannot be factorised (for example,  $x^2 + 1$ ). Apparently, that is true if we are dealing with real numbers only. Here, we will expand into the complex numbers. In that case, many theorems will become consistent. In the field of complex numbers,

**Theorem 1.** *Any polynomial of degree  $n$  can be factored into  $n$  linear polynomials.*

1. Factorise  $x^2 + 6x + 10$  completely.

Solution. By the Quadratic Formula, the two roots are

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 10}}{2} = -3 \pm i.$$

By comparing coefficients, we immediately know

$$x^2 + 6x + 10 = (x + 3 + i)(x + 3 - i)$$

□

# Chapter 5

## Sums and Products

### 5.1 Summation and Product Notation

In this chapter (or more generally, the whole book) we will use frequently use these two symbols: Capital Sigma ( $\Sigma$ ) and Capital Pi ( $\Pi$ ). The operators associated to them are the summation and product respectively. In mathematical terms,

$$\sum_{k=a}^b f(k) = f(a) + f(a+1) + \cdots + f(b)$$
$$\prod_{k=a}^b f(k) = f(a) \times f(a+1) \times \cdots \times f(b)$$

For example,

$$\sum_{k=1}^5 \log k = \log 1 + \log 2 + \log 3 + \log 4 + \log 5$$
$$\prod_{k=1}^5 (k+3) = (1+3)(2+3)(3+3)(4+3)(5+3)$$

In this case,  $\log k$  is called the **summand**, and  $(k+3)$  is called the **multiplier/multiplicand**. Besides,  $k$  is called the **index** while 1 and 5 are called the **bounds**.

We can also write **condition(s)** instead of bounds, e.g. if  $S = \{1, 4, 5, 8, 9\}$ ,

$$\sum_{k \in S} \log k = \log 1 + \log 4 + \log 5 + \log 8 + \log 9 \qquad \prod_{\substack{k \in S \\ k \text{ even}}} k = 4 \times 8 = 32.$$

In many cases, a summation or product may not be easily computed (e.g. the examples above!). However, some summations and products come in neater variations, which we can find a method to easily compute even though the bounds are arbitrarily large.

Note that these properties hold:

$$\sum (f(x) + g(x)) = \sum f(x) + \sum g(x)$$
$$\sum cf(x) = c \sum f(x)$$
$$\prod (f(x)g(x)) = \prod f(x) \prod g(x).$$

where  $c$  is independent of the index.

# Chapter 6

## The Floor Function

### 6.1 The Floor Function

The notation of the floor function is invented by Carl Friedrich Gauss. It is an interesting, most of the time confusing, type of function that we should know prior to joining contests.

The **floor** of  $x$ , denoted by  $\lfloor x \rfloor$ , is the largest integer not more than  $x$ . For example,

$$\begin{aligned}\lfloor 0.3 \rfloor &= 0 \\ \lfloor 5 \rfloor &= 5 \\ \lfloor \pi \rfloor &= 3 \\ \lfloor -3.2 \rfloor &= -4.\end{aligned}$$

In contests, the most useful inequality in dealing with floor functions is

**Proposition 1.**  $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ .

1. Evaluate  $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{10000} \rfloor$ .

We wish to analyse how the frequencies of values behave in this expression. For example, 1 appears three times ( $\lfloor \sqrt{1} \rfloor = \lfloor \sqrt{2} \rfloor = \lfloor \sqrt{3} \rfloor = 1$ ), 2 appears five times ( $\lfloor \sqrt{4} \rfloor = \lfloor \sqrt{5} \rfloor = \lfloor \sqrt{6} \rfloor = \lfloor \sqrt{7} \rfloor = \lfloor \sqrt{8} \rfloor = 2$ ) etc. Generally, how many times does  $k$  appear?

*Solution.* The last term is 100 obviously. For each  $k = 1, 2, \dots, 99$ , we find the number of  $n$  that satisfies  $\lfloor \sqrt{n} \rfloor = k$ :

$$\begin{aligned}k &\leq \sqrt{n} < k+1 \\ k^2 &\leq n < k^2 + 2k + 1 \\ k^2 &\leq n \leq k^2 + 2k\end{aligned}$$

hence there are  $2k + 1$  terms that are equal to  $k$ . Therefore the original expression is,

$$\begin{aligned}\sum_{k=1}^{99} k(2k+1) + 100 &= 2 \sum_{k=1}^{99} k^2 + \sum_{k=1}^{99} k + 100 \\ &= 2 \cdot \frac{99 \cdot 100 \cdot 199}{6} + \frac{99 \cdot 100}{2} + 100 \\ &= 661750.\end{aligned}$$

# Chapter 7

## Recursion

A **recursive function** is a function that is defined in terms of itself. One of the most famous examples is the Fibonacci number sequence:

$$\begin{cases} F(n) = F(n-1) + F(n-2) \\ F(0) = 0, F(1) = 1 \end{cases}$$

However, most of them can be explicitly expressed as a normal function, i.e. the **general form**. Let's start from the easy examples.

### 7.1 First Order Recurrence

Let's define whatever's in the title of this section. A **first order recurrence** is a recurrence in which  $a_n$  is only expressed in terms of  $a_{n-1}$ . A higher order recurrence might have  $a_n$  expressed in terms of  $a_{n-1}, a_{n-2}$  etc.

1. Find the general form of  $a_n = a_{n-1} + 3, a_0 = 2$ .

Solution. We have  $a_n = a_{n-1} + 3 = a_{n-2} + 2 \times 3 = \dots = a_0 + n \times 3 = 3n + 2$ . □

2. Find the general form of  $a_n = 4a_{n-1}, a_0 = 7$ .

Solution. We have  $a_n = 4a_{n-1} = 4^2a_{n-2} = \dots = 4^n a_0 = 7 \cdot 4^n$ . □

3. Find the general form of  $a_n = 4a_{n-1} - 3, a_0 = 2$ .

Solution. Note that  $a_n - 1 = 4(a_{n-1} - 1)$ . Hence let  $b_n = a_n - 1$  to get

$$b_n = 4b_{n-1} = 4^2b_{n-2} = \dots = 4^n b_0 = 4^n(a_0 - 1) = 4^n.$$

And hence  $a_n = 1 + 4^n$ . □

4. Find the general form of  $a_n = 5a_{n-1} + 3, a_0 = 6$ .

Solution. Let  $a_n + p = 5(a_{n-1} + p) \Leftrightarrow a_n = 5a_{n-1} + 4p$ . Thus  $p = 3/4$ . Hence let  $b_n = a_n + 3/4$ , then  $b_n = 5b_{n-1} \Rightarrow b_n = (27/4)5^n$ , yielding

$$a_n = \frac{27 \cdot 5^n - 3}{4}.$$

□



# Chapter 8

## Geometry

### 8.1 Analytic Geometry

Analytic geometry in contests consists of Cartesian coordinates, trigonometry and complex numbers. However, we will focus on coordinates and trigonometry only.

#### 8.1.1 Cartesian Coordinates

The most important equations in Cartesian Coordinates:

**Proposition 1.** *Lines are uniquely expressed as  $y = mx + c$  ( $m, c$  are constants).*

**Proposition 2.** *The circle centred at  $(a, b)$  with radius  $r$  is  $(x - a)^2 + (y - b)^2 = r^2$ .*

**Theorem 1.** *(Point-to-point) The distance between two points  $(x_1, y_1), (x_2, y_2)$  is*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

**Theorem 2.** *(Point-to-line) The distance from  $(x_1, y_1)$  to line  $ax + by + c = 0$  is*

$$\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|.$$

**Theorem 3.** *The area of the triangle with vertices  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$  is*

$$\frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = \frac{1}{2} (x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3)$$

1. (HLG2018)  $ABCD$  is a square with side length 100 and the midpoints of  $AB$  and  $DA$  are  $M, N$  respectively.  $CM$  and  $BN$  intersect at  $P$ . Find the length of  $DP$ .

We can plot this onto a graph with  $D(0, 0)$  and  $A(0, 100)$ .

# Chapter 9

## Inequalities

### 9.1 The Trivial Inequality

Before we encounter the advanced inequalities, we must first introduce the most rudimentary inequality in real numbers: If  $x \in \mathbb{R}$ , then

$$x^2 \geq 0.$$

### 9.2 The AM-GM and Cauchy-Schwarz Inequality

AM-GM and Cauchy are among the most useful inequalities used in Olympiads.

**Theorem 1.** (AM-GM) If  $a_1, \dots, a_n$  are positive real numbers, then

$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n}$$

with equality if and only if  $a_1 = \dots = a_n$ .

**Theorem 2.** (Cauchy) If  $a_1, \dots, a_n, b_1, \dots, b_n$  are real numbers, then

$$(a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) \geq (a_1 b_1 + \dots + a_n b_n)^2$$

with equality if and only if  $\frac{a_1}{b_1} = \dots = \frac{a_n}{b_n}$ .

Now, how do we apply these two inequalities in competition problems? Let's look at some examples.

1. Prove that  $a^2 + b^2 + c^2 \geq ab + bc + ca$ .

Let's analyse the problem. We see a bunch of squares in the LHS, and some 'combined' terms in the RHS. That looks similar to AM-GM, but how can we have three terms produced? We can try backtracking to see what produces  $ab$ . Yes!  $(a^2 + b^2)/2 \geq ab$ . Using the same expressions for  $b, c$  and  $c, a$ , we immediately have our solution.

# Chapter 10

## Diophantine Equations

Diophantine was a Greek Mathematician in the 2nd Century AD. A Diophantine equation is an equation solving for (positive) integer solutions. Unfortunately, these equations are sometimes very hard to solve as equations could appear in unpredictable ways (see Fermat's Last Theorem!). In this chapter, we will be looking at some famous techniques that might help you in many contests.

### 10.1 Linear Diophantine

These are the simplest of them all. Methods include **parameterising** and **tabulating**.

**Theorem 1.** *If  $ax + by = c$  and  $\gcd(a, b) = 1$ , we have the parametric solutions*

$$\begin{cases} x = x_0 + bt \\ y = y_0 - at \end{cases} \quad (t \in \mathbb{Z})$$

*where  $(x_0, y_0)$  is any solution of the problem.*

1. How many pairs of positive integers  $(x, y)$  satisfy  $6x + 7y = 1001$ ?

Solution.  $(x, y) = (0, 143)$  is one of the solutions. Hence

$$\begin{cases} x = 7t \\ y = 143 - 6t \end{cases} \quad (t \in \mathbb{Z})$$

Since  $7t, 143 - 6t > 0 \Rightarrow 1 \leq t \leq 23$ , there are 23 positive integer solutions. □

2. How many pairs of positive integers  $(x, y)$  satisfy  $6x + 8y = 1001$ ?

Solution. LHS is even but RHS is not. No integer solutions. □

3. How many triples of positive integers  $(x, y, z)$  satisfy  $2x + 4y + 5z = 120$ ?

Solution. First,  $z = 2k$ . Substituting into the equation yields

$$\begin{aligned} x + 2y + 5k &= 60 \\ 2y + 5k &= 60 - x \end{aligned}$$

Hence we are finding the number of  $(y, k)$  such that  $2y + 5k \leq 59$ . We can tabulate:

# Chapter 11

## Counting

### 11.1 Bijections

Say we have a theatre with 300 seats, and we want to know how many people there are in the audience when all seats are seated. The obvious answer is there would be 300 people, and this logic involves what we call **bijections** in mathematics.

We observe a relation between each person and each seat. Each seat exactly accommodates one person whereas each person seats on one chair. That means, there is a **one-to-one correspondence (bijection)** from the set of seats and the set of people. Therefore, **if there exists a bijection between two finite sets  $A$  and  $B$ , then  $|A| = |B|$ .** However, one should remember that in order for a correspondence to be a bijection, the correspondence must be bidirectional, i.e. each element in  $A$  corresponds to one element in  $B$  AND each element in  $B$  corresponds to one element in  $A$ .

1. Are these two values the same?

- (a) The number of ways to choose three numbers  $a < b < c$  from  $[n] = \{1, 2, \dots, n\}$  such that  $a + c = 2b$ .
- (b) The number of ways to choose two distinct even numbers plus the number of ways to choose two distinct odd numbers from  $[n]$ .

*Solution.* Yes.  $((a) \Rightarrow (b))$  Since  $a + c$  is an even number, either  $a, c$  are both even or both odd, satisfying (b).  $((b) \Rightarrow (a))$  If two distinct numbers  $a, c$  with equal parity (even/oddness) are chosen, then they must sum up to an even number, hence  $b = \frac{a+c}{2}$  exists uniquely, therefore  $a < b < c$  is chosen, satisfying (a).  $\square$

Note that this is actually a big deal! The value for (a) has a weird condition, but we know that the value for (a) is just equal to the answer for (b), hence it only suffices to solve (b) (which is easier)! Let's see another example:

2. (OMK2014) A subset of  $S = \{1, 2, \dots, 2014\}$  is called *good* if

- (a) It does not contain both 1 and 2 (it can contain a 1 or 2, but not both)
- (b) The sum of elements is divisible by 3.

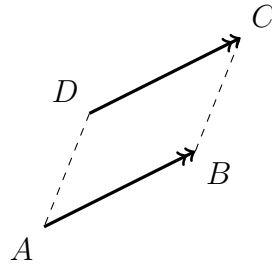
E.g.  $\{1, 23, 456, 789\}$  is good. Find the number of good subsets of  $S$ .

# Chapter 12

## \* Vector Geometry

### 12.1 Vectors in $n$ -dimensional space

A vector is a mathematical object that has **magnitude** and **direction**, but **no location**. In  $n$ -dimensional space (we will only cover  $n = 2$  and  $3$ ), we can associate two points  $A$  and  $B$  with two choices of vectors: the vector  $\overrightarrow{AB}$  (from  $A$  to  $B$ ), or the vector  $\overrightarrow{BA}$  (from  $B$  to  $A$ ). However, the endpoints of vector  $\overrightarrow{AB}$  does not necessarily be at  $A$  and  $B$ , as a vector does not carry ‘location’ in its meaning. For example, if  $ABCD$  is a parallelogram (figure below), then  $\overrightarrow{AB}$  and  $\overrightarrow{DC}$  are exactly the same vector, so we say  $\overrightarrow{AB} = \overrightarrow{DC}$ .



In this chapter, we assume that the reader has basic knowledge of vector addition and scaling. In this book, we will use the following few notations and definitions:

**Definition 1.** The vector from the origin to the point  $(a_1, \dots, a_n)$  is denoted as  $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ , or sometimes for neatness purposes,  $(a_1 \ \cdots \ a_n)^T$ .

**Definition 2.** The **length** of a vector  $\vec{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$  is  $|\vec{v}| = \sqrt{a_1^2 + \cdots + a_n^2}$ .

Please note that  $|\vec{a} + \vec{b}|$  is not necessarily equal to  $|\vec{a}| + |\vec{b}|$ , but:

**Theorem 1.** (*Triangle Inequality*)  $|\vec{a}| + |\vec{b}| \geq |\vec{a} + \vec{b}|$ , with equality iff  $\vec{a} \parallel \vec{b}$ .

# Chapter 13

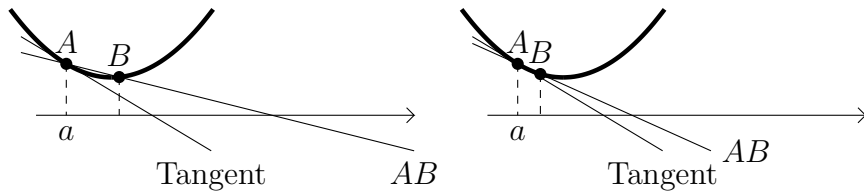
## \* Calculus

Think about what you have learnt in high-school.  $(x^n)' = nx^{n-1}$ ?  $(uv)' = uv' + u'v$ ? The area under a curve can be solved by integration? Integration is anti-differentiation?

The explanations of all these ideas are possibly not given (if not vaguely mentioned) in the high-school syllabus. However, if we want to use calculus with certainty, we have to understand these concepts at a deeper level.

### 13.1 Differentiation

Say we want to find the gradient of the tangent of a curve  $y = f(x)$  at  $x = a$ . Fix the point  $A(a, f(a))$  and consider the point  $B(a + h, f(a + h))$  where  $h$  is very, very small. The line  $AB$  is hence very close to being the tangent at  $A$ . In other words,



$$\text{gradient}_{AB} \approx \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

We want  $h$  to very, very small, so this is where limits come up.

**Definition 1.** The gradient of the tangent of the curve  $y = f(x)$  at  $x = a$  is

$$\left. \frac{dy}{dx} \right|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

If such a limit exists,  $f(x)$  is said to be differentiable at  $x = a$ . If  $f$  is differentiable at all points of its domain,  $f$  is said to be differentiable.

Treating  $\frac{dy}{dx}$  as a function  $f'$  on any  $x \in \mathbb{R}$ , we can say

$$\frac{dy}{dx}(x) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

# Chapter 14

## \* Generating Functions

### 14.1 Generating Functions

**Definition 1.**  $C_k[f]$  is the coefficient of  $x^k$  in a polynomial  $f(x)$ .

**Proposition 1.** Let  $f$  be a polynomial. Then

$$C_n[x^{n+1}f] = 0$$

**Proposition 2.** Let  $f$  and  $g$  be two polynomials. Then

$$C_n[f + g] = C_n[f] + C_n[g]$$

**Proposition 3.** Let  $f$  and  $g$  be two polynomials. Then

$$C_n[fg] = \sum_{k=0}^n C_k[f] \cdot C_{n-k}[g]$$

**Theorem 1.** (Binomial Theorem) For  $n \in \mathbb{N}$ , we have

$$(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

**Theorem 2.** (Extended Binomial) For  $n \in \mathbb{N}$ , we have

$$\frac{1}{(1 - x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n + k}{k} x^k$$

Normally, if the top part of the choose function involves  $k$ , then we use **Theorem 2** (Exceptions: Example 3), otherwise we use **Theorem 1**. What can generating functions do? A **generating function** is a method of encoding an infinite sequence of numbers  $a_n$  by treating them as the coefficients of a polynomial.

# Appendix A

# Mathematical Reasoning

A statement, proposition or assertion is a sentence which is either true or false (not both, otherwise see [8]). The state of being true or false is the **truth value** of the statement.

## Negation

In mathematics it is important to determine what the opposite of a given mathematical statement is. The **negation**  $S'$  of a statement  $S$  is a statement such that the truth value of  $S'$  is opposite of  $S$ .

For example, the negation of

'X is a boy'                      is                      'X is not a boy'

and the negation of

‘All apples are red’                      is                      ‘Not all apples are red’

instead of ‘All apples are not red’.

## Implication

Say  $A$  and  $B$  are two statements. When we have

‘If statement  $A$  is true, then statement  $B$  is true’,

we say that ‘ $A$  **implies**  $B$ ’, or ‘ $A$  is **sufficient** for  $B$ ’, or ‘ $B$  is **necessary** for  $A$ ’, denoted by  $A \Rightarrow B$ . However, this does NOT mean  $B \Rightarrow A$ . For example, an apple is a fruit, however a fruit might not be an apple. Note that an implication of two statements is also a statement.

## Logical AND

The **conjunction** of statements  $A, B$  is a statement which gives a truth value of ‘true’ if both  $A$  and  $B$  are true, and gives ‘false’ otherwise. We normally say the conjunction of  $A, B$  as  $A$  AND  $B$ .

## Logical OR

The **disjunction** of statements  $A, B$  is a statement which gives a truth value of ‘true’ if any of  $A$  or  $B$  is true, and gives ‘false’ otherwise. We normally say the conjunction of  $A, B$  as  $A$  OR  $B$ .



# Appendix B

## Methods of Proof

### I. *Direct Proof*

‘ $A \Rightarrow B$ ’

Most of the proofs in this book are direct.

E.g. Chapter 2 Theorem 1 is proven directly.

### II. *Proof by Contradiction or Contraposition*

‘ $A$ , not  $B \Rightarrow$  Absurd result’ or ‘not  $B \Rightarrow$  not  $A$ ’

Sometimes, assuming the contrary would be a faster way to tackle the problem, especially when there are too many cases when we prove it directly.

1. If  $a_1, \dots, a_n$  are integers such that the product  $a_1 \dots a_n$  is odd, then all  $a_i$  are odd.

**Proof:** Assume the contrary, then at least one (not all!)  $a_i$  is even. This  $a_i$  will contribute a factor of 2, causing  $a_1 \dots a_n$  to be even, which is a contradiction. Thus all  $a_i$  are odd.  $\square$

E.g. Chapter 10 Theorem 1 is proven using contradiction.

### III. *Proof by Induction*

This is normally used in problems dealing with integers. Let  $P(n)$  be the statement for  $n$ .

- (a) Basic Induction:  $P(n) \Rightarrow P(n+1)$  and  $P(1)$  is true.
- (b) Strong Induction:  $P(1), \dots, P(n) \Rightarrow P(n+1)$  and  $P(1)$  is true.
- (c) Forward-Backward Induction:  $P(n) \Rightarrow P(2n)$  and  $P(n) \Rightarrow P(n-1)$  and  $P(1)$  is true.
- (d) Others: For example  $P(n) \Rightarrow P(n+3)$  and  $P(1), P(2), P(3)$  are true.

2. Prove that  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ .

**Proof:** For  $n = 1$ ,  $LHS = RHS = 1$ , which is true. Assume  $1 + \dots + k = \frac{k(k+1)}{2}$ .

Then  $1 + 2 + \dots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$  which completes the problem statement. Therefore, the statement is true for all  $n \in N$ .  $\square$

# Sketch of Proofs

These are the proofs of each theorem stated in the book, some of which have advanced proofs, and I will provide links for further information in the bibliography if necessary.

## Chapter 1

**Theorem 1:** Since  $1, \omega, \omega^2, \dots, \omega^{p-1}$  are distinct roots of  $x^p - 1 = 0$ , we have

$$x^p - 1 = (x - 1)(x - \omega) \cdots (x - \omega^{p-1}).$$

By computing the sum of roots using Vieta's theorem,

$$1 + \omega + \omega^2 + \cdots + \omega^{p-1} = 0.$$

**Theorem 2:** We use the identity (Taylor Series, see [6]):

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

hence

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

## Chapter 2

**Theorem 1:** Let  $d = \gcd(a, b)$ ,  $D = \gcd(a, b - a)$ . Then we have

$$\begin{aligned} d \mid a, b &\Rightarrow d \mid a, b - a \Rightarrow d \mid D. \\ D \mid a, b - a &\Rightarrow d \mid a, (b - a) + a \Rightarrow D \mid d. \end{aligned}$$

Therefore  $d = D$ .

**Property 1:** Prime factorise  $a$  and  $b$ . For any prime,  $p^u \cdot p^v = p^{u+v}$ , hence

$$v_p(ab) = v_p(a) + v_p(b).$$

# Solutions

The solutions below are only *meant as a guide*. Tiresome computations will be omitted. Furthermore, these are not the only methods! The answers are also underlined.

## Chapter 1

1. Yes. E.g.  $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ .
2. No. For example,  $\sqrt{2} + (2 - \sqrt{2}) = 2$ . If  $2 - \sqrt{2}$  is rational, then  $\sqrt{2} = 2 - (2 - \sqrt{2})$  is rational which is a contradiction. Hence  $2 - \sqrt{2}$  is irrational, and two irrational numbers may indeed sum up to a rational number.
3. Yes. E.g.  $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ .
4. No. E.g.  $\sqrt{2} \times \sqrt{2} = 2$ .
5.  $(\omega - 1) \cdots (\omega^{100} - 1) = (1 - \omega) \cdots (1 - \omega^{100}) = 1 + (1) + \cdots + (1)^{100} = \underline{101}$ .
6.  $\omega + \cdots + \omega^6 = 0 - 1 = \underline{-1}$ .
7. Apply Theorem 1 and let  $x = -1$ , we get

$$(-1 - r)(-1 - r^2) \cdots (-1 - r^{22}) = (-1)^{22} + (-1)^{21} + \cdots + 1 = 1.$$

Therefore,  $(1 + r)(1 + r^2) \cdots (1 + r^{22}) = \underline{1}$ .

8. Interpret the complex plane as the coordinate plane. The first condition describes  $z$  lying on the circle with centre  $(0, 1)$  and radius 2; whereas the second condition describes  $z$  lying on the circle with centre  $(1, 0)$  and radius 1. There are exactly two intersection points between these two circles, hence 2 values for  $z$ .
9. Rings:  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ . Fields:  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ .
10. We analyse the four cases:

$$\begin{aligned} (a - bi) + (c - di) &= (a + c) - (b + d)i \\ (a - bi) - (c - di) &= (a - c) - (b - d)i \\ (a - bi)(c - di) &= (ac - bd) - (ad + bc)i = \overline{(ac - bd) + (ad + bc)i} = \overline{(a + bi)(c + di)} \\ \frac{a - bi}{c - di} &= \frac{(a - bi)(c - di)}{c^2 + d^2} = \frac{\overline{(a + bi)(c + di)}}{c^2 + d^2} = \overline{\left( \frac{(a + bi)(c + di)}{c^2 + d^2} \right)} = \overline{\left( \frac{a + bi}{c + di} \right)} \end{aligned}$$

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