Techniques for High School Mathematics Contests

Tristan Chaang

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Contents

* *are nonessential topics*

Preface

This book is designed for students who already have a grasp in the Malaysian secondary school syllabus of maths (SPM) and want to prepare for mathematics competitions or Mathematical Olympiads in Malaysia. These include:

1. The Hua Luo Geng cup (HLG)
2. The Chen Jing Run cup (CJR)
3. The National Mathematical Olympiad (*Olimpiad Matematik Kebangsaan*, OMK)
4. The IMO National Selection Test (IMONST)

among others. However, if you are not from Malaysia, this book is still suitable depending on your country's standard on various mathematics contests. Please take note that this book is definitely not enough if you want to prepare for large, international contests such as the APMO or even the IMO.

There will be many facts and ideas to be grasped in this book. We would divide them into theorems, lemmas, corollaries, propositions etc. There is a slight difference in their definitions, which should be known by the reader:

Proposition	A mathematical statement
Lemma	A proposition paving the way for proving a theorem
Theorem	An important (and normally famous) proven proposition
Corollary	A proposition derived from a theorem

In order to not break up the discussion, some of the proofs of the proposed facts will not be written directly after it, but instead be arranged at the end of the book.

Tristan Chaang,
Malaysian IMO team member (2018 -)

Notations, Acronyms and Abbreviations

In this book, several special notations, acronyms, and abbreviations are used:

HLG	Hua Luo Geng cup
CJR	Chen Jing Run cup
OMK	National Maths Olympiad (*Olimpiad Matematik Kebangsaan*)
NST	National Selection Test
BIMO	IMO Camp (*Bengkel IMO*)
$\mathbb{N}$	Set of natural numbers (0 is not included)
$\mathbb{N}_0$	Set of natural numbers including 0
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{C}$	Set of complex numbers
$\lvert S\rvert$	The number of elements (cardinality) of set S. In some books $n(S)$ is used.
$\square$	Fact demonstrated / End of proof (*quod erat demonstrandum*, QED)
$\gcd(a,b)$	Greatest common divisor / Highest common factor of a and b
$\operatorname{lcm}(a,b)$	Lowest common multiple of a and b
LHS	Left hand side (of a relation comparing two expressions)
RHS	Right hand side (of a relation comparing two expressions)
$a \mid b$	a divides b (i.e. b/a is an integer)
$a \nmid b$	a does not divide b (i.e. b/a is not an integer)
$\overline{abc}$	Integer formed by digits a, b, c from left to right in base-10.
(ABC)	Unique circle passing through A, B, C.
$[S]$	Area of shape S.
$P \sim Q$	Shapes P and Q are similar.
$P \cong Q$	Shapes P and Q are congruent.
$\parallel$	Parallel.
WLOG	Without loss of generality.
$\binom{n}{r}$	The choose function, sometimes written as ${}_nC_r$.
$[n]$	The set $\{1, 2, \ldots, n\}$.
$n!$	Factorial function: $n! = n(n-1)\cdots 2 \cdot 1$.

Table 1: Notations, Acronyms, and Abbreviations

Chapter 1

Numbers

In Olympiad level, you only have to know about natural numbers, integers, rationals, irrationals, reals and complex numbers (complex numbers should be new to some people). We shall not define these sets rigorously in high-school level.

1.1 Natural Numbers $\mathbb{N}$

In some books, 0 is included in $\mathbb{N}$. However, we will define

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

1.2 Integers $\mathbb{Z}$

The notation of $\mathbb{Z}$ originates from the German word *Zahlen*. This set includes natural numbers, their negative versions and 0.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

1.3 Rationals $\mathbb{Q}$

The notation of $\mathbb{Q}$ originates from the Italian word *Quoziente*. It is the set of all fractions where numerators and denominators are integers.

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}, q \neq 0 \right\}$$

1.4 Reals $\mathbb{R}$

The real numbers are roughly described by 'all numbers that you can find on the number line'. For example, the numbers $0.123, \pi, e$ are all in $\mathbb{R}$.

1.5 Irrationals $\mathbb{R} \setminus \mathbb{Q}$

The irrationals are basically every other real number excluding the rationals. In other words, they cannot be expressed as a fraction of integers. The first irrational number formally discovered was $\sqrt{2}$ by Pythagoras. He reasoned why $\sqrt{2}$ is irrational by the following *proof by contradiction* method:

1. Prove that $\sqrt{2}$ is irrational.

Solution. Assume $\sqrt{2} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z} \setminus \{0\}$ and p, q are coprime. Then

$$2q^2 = p^2 \quad \Rightarrow \quad p \text{ is even.}$$

Let $p = 2k$, then

$$q^2 = 2k^2 \quad \Rightarrow \quad q \text{ is even.}$$

However, this contradicts the fact that p, q are irreducible. Hence the initial assumption is wrong, i.e. $\sqrt{2}$ is irrational. $\square$

Afterwards, many numbers were proven to be irrational, for example $\pi, e, \log 2$.

2. Prove that $\log 2$ is irrational.

Solution. Assume $\log 2 = \dfrac{p}{q}$ where $p, q \in \mathbb{Z} \setminus \{0\}$ and p, q are coprime. Then

$$\begin{aligned} 2^q &= 10^p \\ 2^{q-p} &= 5^p \\ \therefore (p, q) &= (0, 0) \end{aligned}$$

This is absurd. $\square$

3. (Hard) Prove that $e = \dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots$ is irrational.

Solution. Assume $e = \dfrac{p}{q}$ where $p, q \in \mathbb{Z} \setminus \{0\}$ and p, q are coprime. Then

$$\begin{aligned} \frac{q!}{0!} + \frac{q!}{1!} + \frac{q!}{2!} + \cdots &= \frac{p}{q} \cdot q! \\ \therefore \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \cdots &\in \mathbb{Z} \end{aligned}$$

However,

$$\begin{aligned} 0 &< \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \cdots \\ &< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots \\ &= \frac{1}{q} \leq 1 \end{aligned}$$

but there are no integers between 0 and 1! Contradiction. $\square$

1.6 Complex Numbers $\mathbb{C}$

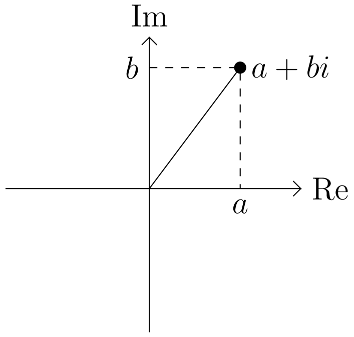
We introduce a new number i that satisfies

$$i^2 = -1.$$

In that case, we define the set of complex numbers as

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

The complex numbers cannot be defined on our real number line. It is an extension to the 2D plane: The number plane. Hence, we can plot every unique complex number on a plane just like this:



However, in this book we will seldom talk about complex numbers unless stated. Let's take a look at the equation

$$x^3 = 1 \Leftrightarrow (x-1)(x^2+x+1) = 0.$$

In the real numbers, only $x = 1$ is a solution. However, there are two more complex roots that satisfy $x^2 + x + 1$. They are

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

Mathematicians normally denote $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ as the **primitive 3rd root of unity**. Some amazing properties are

$$\omega^3 = 1 \text{ and } \omega^2 + \omega + 1 = 0 \text{ and } (x-\omega)(x-\omega^2) = x^2 + x + 1.$$

In fact, this property can be generalised. (Remember, proofs are compiled at the back.)

Theorem 1. *Let p be prime. If ω is a complex nonreal root of $x^p = 1$, then $1 + \omega + \cdots + \omega^{p-1} = 0$ and $(x-\omega)\cdots(x-\omega^{p-1}) = 1 + x + \cdots + x^{p-1}$.*

4. (HLG2019) If $\omega^7 = 1$ and $\omega \neq 1$, find the value of $(2+\omega)(2+\omega^2)\cdots(2+\omega^6)$.

Solution. Use the fact above:

$$\begin{aligned} (x-\omega)\cdots(x-\omega^6) &= 1 + x + \cdots + x^6 \\ (-2-\omega)\cdots(-2-\omega^6) &= 1 + (-2) + \cdots + (-2)^6 = 43 \\ \therefore (2+\omega)\cdots(2+\omega^6) &= 43. \end{aligned}$$

□

5. If x_1, x_2, x_3, x_4 are distinct roots of $x^5 = 1$ where $x_i \neq 1$ $(i = 1, 2, 3, 4)$, find the value of $x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$.

Solution. The given expression is equal to the coefficient of x^2 in

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

By Theorem 1,

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4) = x^4 + x^3 + x^2 + x + 1$$

hence the answer is 1. □

Here are some definitions worth mentioning. Let $z = a + bi$ where $a, b \in \mathbb{R}$.

Definition 1. The complex conjugate of z is $\overline{z} = a - bi$.

Definition 2. The magnitude (or size) of z is $|z| = a^2 + b^2 = z\overline{z}$.

Definition 3. The argument $\arg z$ of z is the angle θ subtended by z from the positive real axis, $-\pi < \theta \leq \pi$.

We also introduce a famous theorem due to Euler:

Theorem 2. *(Euler's Formula) Denote $|z| = r$ and $\arg z = \theta$. Then*

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Corollary 1. *(De Moivre's Law) If n is an integer,*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

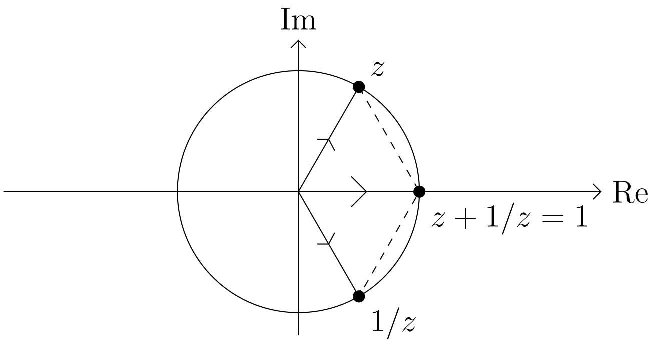
6. $z + \dfrac{1}{z} = 1$. Find the value of $z^{100} + \dfrac{1}{z^{100}}$.

Solution. We use De Moivre's Law to simplify the problem.

$$\begin{aligned}
z^2 - z + 1 &= 0 \\
\Rightarrow z &= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \\
&= \cos 60^\circ \pm i\sin 60^\circ \\
z^{100} &= \cos 6000^\circ \pm i\sin 6000^\circ \\
&= \cos 240^\circ \pm i\sin 240^\circ \\
z^{-100} &= \cos -6000^\circ \pm i\sin -6000^\circ \\
&= \cos 240^\circ \mp i\sin 240^\circ \\
\therefore z^{100} + z^{-100} &= 2\cos 240^\circ = -1.
\end{aligned}$$

□

Something worth to note: When z lies on the unit circle, $1/z = \overline{z}$. Therefore the expression $z + 1/z = 1$ can be visualised as a sum of two vectors:



7. Solve $|z-1| = |z+1|$.

Solution 1. Let $z = a + bi$. Then the equation becomes

$$\begin{aligned} |z-1| &= |z+1| \\ |(a-1)+bi| &= |(a+1)+bi| \\ \sqrt{(a-1)^2+b^2} &= \sqrt{(a+1)^2+b^2} \\ a &= 0 \end{aligned}$$

Thus any complex number bi where $b \in \mathbb{R}$ works. □

Solution 2. $|z-a|$ is equal to the distance between z and a on the complex plane. Thus the solutions to the equation is precisely the locus of points on the plane which is equidistant to 1 and -1. This is just the perpendicular bisector between them, which is the pure imaginary axis, hence z can be any complex number bi where $b \in \mathbb{R}$. □

1.7 Problems

1. Must a sum of rational numbers be rational?

2. Must a sum of irrational numbers be irrational?

3. Must a product of rational numbers be rational?

4. Must a product of irrational numbers be irrational?

5. (CJR2020) Let $\omega^{101} = 1$ but $\omega \neq 1$. Find the value of
$$(\omega-1)(\omega^2-1)\cdots(\omega^{100}-1).$$

6. (HLG2017) Given that ω is a complex number, $\omega^7 = 1, \omega \neq 1$, find the value of
$$\omega + \omega^2 + \omega^3 + \cdots + \omega^6.$$

7. Given that $r^{22} + r^{21} + \cdots + 1 = 0$, evaluate
$$(1+r)(1+r^2)\cdots(1+r^{22}).$$

8. Find the number of complex numbers z that satisfy
$$\begin{cases} |z-i| = 2 \\ |z-1| = 1. \end{cases}$$

9. A ring is a set that is closed under addition, subtraction and multiplication, whereas a field is a set that is closed under addition, subtraction, multiplication and division except by zero. Classify $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, (\mathbb{R} \setminus \mathbb{Q}), \mathbb{C}$ into
   - Rings;
   - Fields;
   - None of the above.
10. Prove that conjugation is closed under addition, subtraction, multiplication and division, i.e. $\overline{x * y} = \overline{x} * \overline{y}$ for $* = +, -, \times, /$.
11. A real number is *algebraic* if it is a root of a polynomial with integer coefficients. The numbers e and π are proven to be not algebraic, hence they are called *transcendental* numbers. We will not prove this. Let $\mathbb{A}$ be the set of algebraic numbers. Prove that $\mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$, and prove that $\sqrt{2 + \sqrt{3} + \sqrt[3]{5}}$ is algebraic.

Chapter 2

Integers

2.1 Prime Checking

How to determine if a number n s prime? One might say we should check all primes up to n and see if any one of them divides n. However, there is a much quicker way to this. In fact, if all the primes up to $\sqrt{n}$ do not divide n, then the factors of n must be larger than $\sqrt{n}$. If n is composite, then $n > \sqrt{n} \cdot \sqrt{n} = n$ which is a contradiction. Hence

Proposition 1. *n is prime if and only if all primes up to $\sqrt{n}$ does not divide n.*

1. Is 101 prime?

Solution. $\sqrt{101} < 11$, hence we only need to check $2, 3, 5$ and 7. Yes, 101 is prime. □

2. Is 119 prime?

Solution. Again, we check $2, 3, 5$ and 7. Indeed, $7 \mid 119$, so $119 = 7 \times 17$ is not prime. □

2.2 GCD and LCM

The GCD (also known as HCF) of two integers is the greatest common divisor of them, whereas the LCM of two integers is the lowest common multiple of them. If the prime factorisations of n and m are $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots$ and $m = p_1^{\beta_1} p_2^{\beta_2} \ldots$, then

$$\begin{aligned} \gcd(n, m) &= p_1^{\min(\alpha_1, \beta_1)} p_2^{\min(\alpha_2, \beta_2)} \ldots \\ \operatorname{lcm}(n, m) &= p_1^{\max(\alpha_1, \beta_1)} p_2^{\max(\alpha_2, \beta_2)} \ldots \end{aligned}$$

and this leads to

Proposition 2. *For any positive integers n and m, $\gcd(n, m)\operatorname{lcm}(n, m) = nm$.*

3. Given $\gcd(n, m) = 3, \operatorname{lcm}(n, m) = 36, m = 9$, find n.

Solution. If such an n exists, then by Proposition 2

$$3 \times 36 = 9 \times n$$
$$n = 12$$

and indeed, $n = 12$ works. $\square$

4. Given $\gcd(n, m) = 3, \operatorname{lcm}(n, m) = 121, m = 11$, find n.

Solution. If such an n exists, then by Proposition 2

$$3 \times 121 = 11 \times n$$
$$n = 33.$$

However, $n = 33$ does not satisfy $\gcd(n, m) = 3$. Thus **no such n exists**. $\square$

5. Find all positive integers a, b such that $\gcd(a, b) = 12, \operatorname{lcm}(a, b) = 216$.

Solution. $12 = 2^2 \times 3$, $216 = 2^3 \times 3^3$. Therefore $a = 2^{\alpha_1} \times 3^{\alpha_2}, b = 2^{\beta_1} \times 3^{\beta_1}$ where $\{\alpha_1, \beta_1\} = \{2, 3\}$ and $\{\alpha_2, \beta_2\} = \{1, 3\}$. Hence

$$\begin{cases} a = 2^2 \times 3^1 \\ b = 2^3 \times 3^3 \end{cases} \quad \begin{cases} a = 2^2 \times 3^3 \\ b = 2^3 \times 3^1 \end{cases} \quad \begin{cases} a = 2^3 \times 3^1 \\ b = 2^2 \times 3^3 \end{cases} \quad \begin{cases} a = 2^3 \times 3^3 \\ b = 2^2 \times 3^1 \end{cases}$$

are all solutions, i.e. $(a, b) = (12, 216), (108, 24), (24, 108), (216, 12)$. $\square$

2.3 Euclidean Algorithm

The Euclidean Algorithm is exceptionally useful in computing GCDs. In fact, computers do it this way. It stems from a simple fact, which is

Theorem 1. *(Euclid) For any integers a, b, we have $\gcd(a, b) = \gcd(a, b - a)$.*

This allows us to repeatedly subtract a as well. Hence $\gcd(a, b) = \gcd(a, b - ka)$ for any integer k. For example, $\gcd(102, 251) = \gcd(102, 47) = \gcd(8, 47) = \gcd(8, -1) = 1$.

6. Find $\gcd(12345, 67890)$.

Solution. $\gcd(12345, 67890) = \gcd(12345, 6165) = \gcd(15, 6165) = \gcd(15, 0) = 15$. $\square$

7. What are the possible values of $\gcd(n^2 + 1, n^3 + 7n^2 + 2n + 1)$?

Solution. We reduce the powers one-by-one.

$$\begin{aligned}\gcd(n^2+1, n^3+7n^2+2n+1) &= \gcd(n^2+1, n^3+7n^2+2n+1-n(n^2+1))\\ &= \gcd(n^2+1, 7n^2+n+1)\\ &= \gcd(n^2+1, 7n^2+n+1-7(n^2+1))\\ &= \gcd(n^2+1, n-6)\\ &= \gcd(n^2+1-n(n-6), n-6)\\ &= \gcd(6n+1, n-6)\\ &= \gcd(6n+1-6(n-6), n-6)\\ &= \gcd(37, n-6)\\ &= 1 \text{ or } 37\end{aligned}$$

since both 1 and 37 can be obtained using $n = 2$ and 6 respectively. $\square$

8. (IMO1959) Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible.

Solution. $\gcd(21n+4, 14n+3) = \gcd(7n+1, 14n+3) = \gcd(7n+1, 1) = 1.$ $\square$

9. (Hard) Find some integers a, b such that $101a + 73b = 1$

Solution. We perform the Euclidean Algorithm explicitly, recording each step:

$$\begin{aligned}101 &= 73(1) + 28\\ 73 &= 28(2) + 17\\ 28 &= 17(1) + 11\\ 17 &= 11(1) + 6\\ 11 &= 6(1) + 5\\ 6 &= 5(1) + 1\end{aligned}$$

And then reverse the order, starting from 1 and rebuild the large numbers:

$$\begin{aligned}1 &= 6 - 5\\ &= 6 - (11 - 6)\\ &= 6(2) - 11\\ &= (17 - 11)(2) - 11\\ &= 17(2) - 11(3)\\ &= 17(2) - (28 - 17)(3)\\ &= 17(5) - 28(3)\\ &= (73 - 28(2))(5) - 28(3)\\ &= 73(5) - 28(13)\\ &= 73(5) - (101 - 73)(13)\\ &= 101(-13) + 73(18)\end{aligned}$$

and thus $(a, b) = (-13, 18)$ is a solution. $\square$

2.4 p-adic Valuation

The p-adic valuation of n, denoted by $v_p(n)$, is the power of a prime p in the prime factorisation of n. For example, $12 = 2^2 \times 3$, hence

$$v_2(12) = 2 \qquad v_3(12) = 1 \qquad v_5(12) = 0.$$

Some properties of $v_p(n)$ can be summarised here. Property 2 uses the Floor Function discussed in Chapter 6. Can you see why it is true?

Property 1. $v_p(ab) = v_p(a) + v_p(b)$.

Property 2. $v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$.

10. Find $v_7(99!)$.

Solution. $v_7(99!) = \left\lfloor \frac{99}{7} \right\rfloor + \left\lfloor \frac{99}{49} \right\rfloor = 14 + 2 = 16.$ □

11. Find $v_5(1 \times 3 \times 5 \times \cdots \times 1001)$.

Solution 1. We first turn $1 \times 3 \times \cdots \times 1001$ into an expression involving factorials.

$$\begin{aligned}
&v_5(1 \times 3 \times 5 \times \cdots \times 1001)\\
&= v_5\left(\frac{1001!}{2 \times 4 \times \cdots \times 1000}\right)\\
&= v_5\left(\frac{1001!}{2^{500} \times 500!}\right)\\
&= v_5(1001!) - v_5(2^{500} \times 500!)\\
&= v_5(1001!) - v_5(500!)\\
&= \left(\left\lfloor \frac{1001}{5} \right\rfloor + \left\lfloor \frac{1001}{25} \right\rfloor + \left\lfloor \frac{1001}{125} \right\rfloor + \left\lfloor \frac{1001}{625} \right\rfloor\right) - \left(\left\lfloor \frac{500}{5} \right\rfloor + \left\lfloor \frac{500}{25} \right\rfloor + \left\lfloor \frac{500}{125} \right\rfloor\right)\\
&= (200 + 40 + 8 + 1) - (100 + 20 + 4)\\
&= 125.
\end{aligned}$$

□

Solution 2.

$$\begin{aligned}
&v_5(1 \times 3 \times 5 \times \cdots \times 1001)\\
&= v_5(5 \times 15 \times 25 \times \cdots \times 995)\\
&= v_5(5^{100} \times 1 \times 3 \times 5 \times \cdots \times 199)\\
&= 100 + v_5(1 \times 3 \times 5 \times \cdots \times 199)\\
&= 100 + v_5(5 \times 15 \times 25 \times \cdots \times 195)\\
&= 100 + v_5(5^{20} \times 1 \times 3 \times 5 \times \cdots \times 39)\\
&= 100 + 20 + v_5(5 \times 15 \times 25 \times 35)\\
&= 100 + 20 + 1 + 1 + 2 + 1\\
&= 125.
\end{aligned}$$

□

12. Find the number of ending zeroes in 101!.

To count the number of ending zeroes, we want the highest power of 10 that divides it. For example, 50100 has two ending zeroes because 10^2 divides it. However, we clearly know that the prime factor 2 is way more frequent than the prime factor 5 in the prime decomposition of 101!. Therefore, we just need to find $v_5(101!)$ only.

Solution. $v_5(101!) = \left\lfloor \frac{101}{5} \right\rfloor + \left\lfloor \frac{101}{25} \right\rfloor + \left\lfloor \frac{101}{125} \right\rfloor + \cdots = 20 + 4 = 24.$ □

2.5 Functions $d(n), \sigma(n), \phi(n)$

The Number of Positive Divisors $d(n)$

This function calculates the number of positive factors, for example $d(12) = 6$ since it has $1, 2, 3, 4, 6, 12$ as factors. In fact,

Proposition 3. *The number of positive factors of* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ *is*

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots$$

13. Find the number of positive divisors of 120.

Solution. $d(2^3 \times 3 \times 5) = 4 \times 2 \times 2 = 16.$ □

14. Classify positive integers n that have 34 positive factors.

Solution. Since $34 = 2 \times 17$, $n = p^{33}$ or $n = pq^{16}$ where $p \neq q$ are prime factors. □

The Sum of Positive Divisors $\sigma(n)$

This function calculates the sum of positive factors, for example $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. In fact,

Proposition 4. *The sum of positive factors of* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ *is*

$$\begin{aligned}\sigma(n) &= (1 + p_1 + \cdots + p_1^{\alpha_1})(1 + p_2 + \cdots + p_2^{\alpha_2}) \cdots \\ &= \frac{(p_1^{\alpha_1+1} - 1)(p_2^{\alpha_2+1} - 1) \cdots}{(p_1 - 1)(p_2 - 1) \cdots}\end{aligned}$$

15. Find the sum of positive divisors of 120.

Solution. $\sigma(2^3 \times 3 \times 5) = (1 + 2 + 4 + 8)(1 + 3)(1 + 5) = 360.$ □

16. Find n such that it has 11 positive divisors and sum of factors equal to 2047.

Solution. The first condition implies $n = p^{10}$ for some prime p. Then, we solve

$$1 + p + \cdots + p^{10} = 2047$$

which yields $p = 2$. Therefore $n = 2^{10} = 1024$. □

The Number of Coprime Integers $\phi(n)$

This function calculates the number of positive integers $\leq n$ but coprime (relatively prime) to n, for example $\phi(12) = 4$ since only $1, 5, 7, 11$ are coprime to 12. In fact,

Proposition 5. *The number of integers $0 < k \leq n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ coprime to n is*

$$\begin{aligned}\phi(n) &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots \\ &= (p_1^{\alpha_1} - p_1^{\alpha_1 - 1})(p_2^{\alpha_2} - p_2^{\alpha_2 - 1})\cdots\end{aligned}$$

17. Find the number of positive integers $k < 120$ such that $\gcd(k, 120) = 1$.

Solution. $\phi(2^3 \times 3 \times 5) = (8 - 4)(3 - 1)(5 - 1) = 32$. □

18. Find the number of positive integers $k < 250$ such that $\gcd(k, 120) = 1$.

Solution. As from the previous example, $[0, 120)$ contains 32 desired numbers. Similarly, $[120, 240)$ contains 32 desired numbers too, since $\gcd(x, 120) = \gcd(x - 120, 120)$. There are 2 desired numbers in $[240, 250)$ which are 241 and 247. Summing up, there are $32 + 32 + 2 = 66$ numbers. □

2.6 Problems

1. (CJR2018) How many prime numbers are between 990 and 1000?
2. (HLG2017) Find the sum of all the positive factors of 840.
3. (CJR2018) How many positive integers less than 132 are relatively prime to 132?
4. (HLG2018) How many positive integers less than 200 have exactly 6 positive factors?
5. (HLG2019) Find the number of positive factors of 2019.
6. (HLG2019) Find all positive integers n such that $\sqrt{n^2 + 124n}$ is an integer.
7. (HLG2019) Among the positive integers less than 1200, how many of them are relatively prime to 60?
8. (HLG2019) If 500! is divisible by 6^k, find the largest possible value of k.
9. (CJR2018) How many pairs of digits (a, b) satisfy $12 \mid \overline{a789b}$ where $a \neq 0$?
10. (CJR2018) How many pairs of positive integers a, b are such that the greatest common divisor of a and b is $2 \times 3 \times 5 \times 7$, and the least common multiple of a and b is $2^2 \times 3^2 \times 5^2 \times 7$?

Chapter 3

Modular Arithmetic

3.1 Congruence

Two integers a and b are said to be congruent mod m if their remainders are equal when divided by m, we write this as

$$a \equiv b \pmod{m}$$

For example,

$$16 \equiv 9 \equiv 2 \equiv -5 \equiv -12 \pmod{7}.$$

3.2 Properties of Congruence

Below are a few properties that should be recognised in a problem:

Property 1. Congruence is additive, for example $7 + 8 \equiv 3 + 0 \equiv 3 \pmod{4}$.

Property 2. Congruence is multiplicative, for example $7 \times 9 \equiv 3 \times 1 \equiv 3 \pmod{4}$.

Property 3. If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$.

Property 4. If $ak \equiv bk \pmod{m}$, then $a \equiv b \pmod{\frac{m}{\gcd(k, m)}}$.

Property 4 basically means if we want to divide both sides by a number, the modulo must be divided as much as possible too.

1. Find the remainder of $1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15$ when divided by 4.

We can first use our knowledge to reduce each term to smaller numbers. For example, we can do the following:

Solution 1.

$$\begin{aligned}
&1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \\
&\equiv 1 \times 3 \times 1 \times 3 \times 1 \times 3 \times 1 \times 3 \\
&\equiv 81 \\
&\equiv 1 \pmod{4}
\end{aligned}$$

and hence the remainder is 1. □

However, we can be a bit smarter: We see that $-1 \equiv 3 \pmod 4$, hence we can do:

Solution 2.

$$
\begin{aligned}
&1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \\
&\equiv 1 \times -1 \times 1 \times -1 \times 1 \times -1 \times 1 \times -1 \\
&\equiv 1 \pmod 4
\end{aligned}
$$

and hence the remainder is 1. □

2. Solve $7x \equiv 3 \pmod 9$.

In this example, we want to use property 4 to reduce the 7, therefore we can try changing the RHS.

Solution.

$$
\begin{aligned}
7x &\equiv 3 \equiv 12 \equiv 21 \pmod 9 \\
\therefore x &\equiv 3 \pmod 9
\end{aligned}
$$

hence $x = \cdots, -15, -6, 3, 12, \cdots$. □

3. Solve $6x \equiv 7 \pmod 9$.

Solution. The LHS is a multiple of 3, which must leave a remainder of one of $0, 3, 6$ when divided by 9. Hence x has no integer solutions. □

4. Solve $3x \equiv 6 \pmod 9$.

Solution.

$$
\begin{aligned}
3x &\equiv 6 \pmod 9 \\
\frac{3x}{3} &\equiv \frac{6}{3} \pmod{\frac{9}{3}} \\
x &\equiv 2 \pmod 3
\end{aligned}
$$

hence $x = \cdots, -4, -1, 2, 5, \cdots$. □

5. Solve $6x \equiv 3 \pmod 9$.

Solution.

$$
\begin{aligned}
6x &\equiv 3 \pmod 9 \\
6x &\equiv 12 \pmod 9 \\
\frac{6x}{6} &\equiv \frac{12}{6} \pmod{\frac{9}{3}} && (6 \nmid 9, \text{ so take GCD, } 3) \\
x &\equiv 2 \pmod 3
\end{aligned}
$$

hence $x = \cdots, -4, -1, 2, 5, \cdots$. □

3.3 Large powers

We have come to one of the most famous problems in contests. In this section, the most important property is Property 3. Why? Let's take a look at some examples.

6. Find the remainder when 2^{10000} is divided by 127.

Solution. Note that $2^7 \equiv 1 \pmod{127}$, hence

$$2^{10000} \equiv (2^7)^{1428}(2^4) \equiv 1^{1428} \cdot 16 \equiv 16 \pmod{127}.$$

and we are done. □

In short, by reducing the base into a simpler number (for example 1 or -1), the answer becomes extremely obvious. In fact, this property can be translated into:

Property 5. If $x^k \equiv 1 \pmod{m}$, then $x^n \equiv x^{(n \bmod k)} \pmod{m}$.

Definition 1. The **order** of x modulo m is the smallest positive integer k such that $x^k \equiv 1 \pmod{m}$.

In other words, we can always try to find the order, and then simplify. Let's read more examples.

7. Find the remainder when 4^{1000} is divided by 13.

Solution. Note that $4^3 \equiv 64 \equiv -1 \pmod{13}$, hence $4^6 \equiv 1 \pmod{13}$. Therefore the order is 6. Consequently, we have $4^{1000} \equiv 4^4 \equiv 9 \pmod{13}$. □

8. Find the remainder when $2019^{2019^{2019}}$ is divided by 7.

Solution. $2019^{2019^{2019}} \equiv 3^{2019^{2019}} \pmod{7}$. Since $3^6 \equiv 1 \pmod{7}$,

$$2019^{2019^{2019}} \equiv 3^{2019^{2019}} \equiv 3^{(2019^{2019} \bmod 6)} \equiv 3^3 \equiv 6 \pmod{7}$$

and we are done. □

3.4 Chinese Remainder Theorem (CRT)

During the Han dynasty, the famous general Han Xin was once asked how many soldiers he had but there were too many of them to count. From above the tower, he roughly knew that there were around 1000 soldiers. He then ordered everyone to divide into groups of 3, in the end 2 soldiers were left without a group. Dividing into groups of 5, in the end 4 soldiers were left. Dividing into groups of 7, in the end 3 soldiers were left. Before anyone asked why he did so, Han Xin had already answered: '1004.'

What's the logic behind this story? We can first write the number of soldiers x into three modular relations:

$$\begin{cases} x \equiv 2 \pmod{3} \\ x \equiv 4 \pmod{5} \\ x \equiv 3 \pmod{7} \end{cases}$$

Now we combine these three relations into one. How? We can tackle them one by one.

$$\begin{cases} x \equiv 2 \pmod 3 \\ x \equiv 4 \pmod 5 \end{cases} \tag{1}$$

$$\Leftrightarrow \begin{cases} x \equiv 2, 5, 8, 11, 14 \pmod{15} \\ x \equiv 4, 9, 14 \pmod{15} \end{cases}$$

$$\therefore x \equiv 14 \pmod{15}$$

and now we tackle

$$\begin{cases} x \equiv 14 \pmod{15} \\ x \equiv 3 \pmod 7 \end{cases} \tag{2}$$

$$\Leftrightarrow \begin{cases} x \equiv 14, 29, 44, 59, 74, 89, 104 \pmod{105} \\ x \equiv 3, 10, 17, 24, 31, 38, 45, 52, 59, 66, 73, 80, 87, 94, 101 \pmod{105} \end{cases}$$

$$\therefore x \equiv 59 \pmod{105}.$$

As such, we conclude that $x = 59, 164, 269, \cdots$. Accordingly, the one closest to 1000 is

$$x = 59 + 9 \times 105 = 1004.$$

From above we see the method of solving simultaneous congruences, but it seems to be too brute forced. Can we find a way to simplify it? Here we introduce the Chinese Remainder Theorem which will speed things up.

Theorem 1. *(Chinese Remainder Theorem) A set of simultaneous congruences*

$$x \equiv a_i \pmod{M_i} \quad (i = 1, \ldots, n)$$

where M_i are pairwise coprime, corresponds to a unique result

$$x \equiv A \pmod{M_1 M_2 \cdots M_n}.$$

Back to the Han Xin problem. Now we can compute (1) easily because

$$\begin{cases} x \equiv 2 \equiv -1 \pmod 3 \\ x \equiv 4 \equiv -1 \pmod 5 \end{cases}$$

$$\therefore x \equiv -1 \pmod{15}$$

and we do not need to check if there are any other solutions due to CRT. Next, we solve (2) by using a better method. By substituting $x \equiv -1 \pmod{15}$ into $x \equiv 3 \pmod 7$,

$$\begin{aligned} 15k - 1 &\equiv 3 \pmod 7 \\ k &\equiv 4 \pmod 7 \\ \therefore x &= 15k - 1 \\ &= 15(7m + 4) - 1 \\ &= 105m + 59 \end{aligned}$$

and then we are done. $\square$

9. Solve

$$\begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 4 \pmod 5 \\ x \equiv 2 \pmod 7 \end{cases}$$

Solution.

$$\begin{aligned} \begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 4 \pmod 5 \end{cases} &\Leftrightarrow \quad x \equiv 4 \pmod{15} \\ \begin{cases} x \equiv 4 \pmod{15} \\ x \equiv 2 \pmod 7 \end{cases} &\Leftrightarrow \quad 15k + 4 \equiv 2 \pmod 7 \\ &\Leftrightarrow \quad k \equiv 5 \pmod 7 \\ &\Leftrightarrow \quad x = 15(7m+5) + 4 \\ &\Leftrightarrow \quad x = 105m + 79 \end{aligned}$$

and hence $x \equiv 79 \pmod{105}$. □

10. Solve

$$\begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 4 \pmod{15} \\ x \equiv 2 \pmod 7 \end{cases}$$

Solution.

$$\begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 4 \pmod{15} \\ x \equiv 2 \pmod 7 \end{cases} \Leftrightarrow \begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 1 \pmod 3 \\ x \equiv 4 \pmod 5 \\ x \equiv 2 \pmod 7 \end{cases} \Leftrightarrow \begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 4 \pmod 5 \\ x \equiv 2 \pmod 7 \end{cases}$$

and this becomes Example 9. □

11. Solve

$$\begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 5 \pmod{15} \\ x \equiv 2 \pmod 7 \end{cases}$$

Solution.

$$\begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 5 \pmod{15} \\ x \equiv 2 \pmod 7 \end{cases} \Leftrightarrow \begin{cases} x \equiv 1 \pmod 3 \\ x \equiv 2 \pmod 3 \\ x \equiv 0 \pmod 5 \\ x \equiv 2 \pmod 7 \end{cases}$$

which is a contradiction. Therefore, there are no integer solutions. □

12. Find the last two digits of $7^{7^{7^7}}$.

Solution. We want modulo 100, hence by CRT we only need modulo 4 and 25.

$$7^{7^{7^7}} \equiv (-1)^{\text{odd}} \equiv 3 \pmod 4$$

For modulo 25, note that the order is 4 since $7^2 \equiv -1 \pmod{25}$. Hence

$$7^{7^{7^7}} \equiv 7^{(7^{7^7} \bmod 4)} \equiv 7^3 \equiv 18 \pmod{25}.$$

Therefore, $7^{7^{7^7}} \equiv 43 \pmod{100}$, which has last two digits 43. □

13. Given $n \in \mathbb{N}$, is it always possible to have n consecutive integers, each divisible by a square greater than 1? For example, if $n = 3$, then we can, for example, take the numbers $124, 125, 126$, which are divisible by $4, 25, 9$ respectively.

This is one of the harder problems in this chapter. We see that we want to establish some congruences of $x+1, x+2, \ldots, x+n$ respectively. Since the modulus can be any square greater than 1, we just choose them such that it satisfies CRT.

Solution. Let $m_1^2, \ldots, m_n^2$ be pairwise coprime integers. The answer is yes if we can solve

$$\begin{cases} x+1 \equiv 0 \pmod{m_1^2} \\ x+2 \equiv 0 \pmod{m_2^2} \\ \quad\vdots \\ x+n \equiv 0 \pmod{m_n^2} \end{cases} \Leftrightarrow \begin{cases} x \equiv -1 \pmod{m_1^2} \\ x \equiv -2 \pmod{m_2^2} \\ \quad\vdots \\ x \equiv -n \pmod{m_n^2} \end{cases}$$

which we can by CRT. Hence, yes it is possible. □

3.5 Fermat's Little Theorem

This theorem is extremely useful when we want to find n such that $a^n \equiv 1 \pmod p$ where p is prime. In fact, the statement is so elegant:

Theorem 2. *(Fermat's Little Theorem) If p is prime and x is not a multiple of p, then $x^{p-1} \equiv 1 \pmod p$. If p^n is a prime power, then $x^{p^n - p^{n-1}} \equiv 1 \pmod{p^n}$.*

But what if the modulus is not prime? That theorem would be called Euler's Theorem but it is a bit more complicated than Fermat's Little Theorem and it is less often used, so we would not state it here (Those interested can search it up now!).

14. Find the last two digits of $9^{9^{9^9}}$.

Solution. We want modulo 100, hence by CRT we only need modulo 4 and 25.

$$9^{9^{9^9}} \equiv 1 \pmod 4$$

For modulo 25, since $9^{25-5} \equiv 1 \pmod{25}$,

$$9^{9^{9^9}} \equiv 9^{9^{9^9} \bmod 20} \equiv 9^9 \equiv 81^4 \cdot 9 \equiv 6^4 \cdot 9 \equiv \cdots \equiv 14 \pmod{25}.$$

Therefore, $9^{9^{9^9}} \equiv 89 \pmod{100}$. □

15. (CJR2017) (Hard) Find the number of positive integers $n < 1000$ such that $66 \mid n^n+1$.

<u>*Solution*</u>. We start by seeing

$$\begin{cases} n^n + 1 \equiv 0 \pmod 2 & (1) \\ n^n + 1 \equiv 0 \pmod 3 & (2) \\ n^n + 1 \equiv 0 \pmod{11} & (3) \end{cases}$$

(1) $\Leftrightarrow n$ is odd.
(2) $\Leftrightarrow n \equiv 2 \pmod 3$ since the power is odd and clearly $n \not\equiv 0, 1 \pmod 3$.
Now we come to the hard part, (3). $n \equiv -1 \pmod{11}$ clearly works. $n \equiv 0, 1$ clearly does not. Now we are left with $n \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{11}$.
Case 1: $n \equiv \pm 2 \pmod{11}$. We see $(\pm 2)^2 \equiv 4; (\pm 2)^3 \equiv \pm 8; (\pm 2)^4 \equiv 5; (\pm 2)^5 \equiv \mp 1$. Therefore, only $n \equiv 2 \pmod{11}$ works and $n = 5, 15, 25, \cdots$. In other words,

$$n \equiv 35 \pmod{110}.$$

Case 2: $n \equiv \pm 3 \pmod{11}$. We see $(\pm 3)^2 \equiv 9; (\pm 3)^3 \equiv \pm 5; (\pm 3)^4 \equiv 4; (\pm 3)^5 \equiv \pm 1$. Therefore, only $n \equiv -3 \pmod{11}$ works and $n = 5, 15, 25, \cdots$. In other words,

$$n \equiv 85 \pmod{110}.$$

Case 3: $n \equiv \pm 4 \pmod{11}$. We see $(\pm 4)^2 \equiv 5; (\pm 4)^3 \equiv \pm 9; (\pm 4)^4 \equiv 3; (\pm 4)^5 \equiv \pm 1$. Therefore, only $n \equiv -4 \pmod{11}$ works and $n = 5, 15, 25, \cdots$. In other words,

$$n \equiv 95 \pmod{110}.$$

Case 4: $n \equiv \pm 5 \pmod{11}$. We see $(\pm 5)^2 \equiv 3; (\pm 5)^3 \equiv \pm 4; (\pm 5)^4 \equiv 9; (\pm 5)^5 \equiv \pm 1$. Therefore, only $n \equiv -5 \pmod{11}$ works and $n = 5, 15, 25, \cdots$. In other words,

$$n \equiv 105 \pmod{110}.$$

We don't need to consider $n \equiv 1 \pmod 2$ anymore since it is already embedded in $\pmod{110}$. Hence,

$$\begin{cases} n \equiv 1 \pmod 2 \\ n \equiv 2 \pmod 3 \\ n \equiv -1 \pmod{11} \end{cases} \Rightarrow \quad n \equiv -1 \pmod{66}.$$

or

$$\begin{cases} n \equiv 2 \pmod 3 \\ n \equiv 35, 85, 95, 105 \pmod{110} \end{cases} \Rightarrow \quad n \equiv 35, 95, 215, 305 \pmod{330}.$$

Therefore, $n \equiv 35, 65, 95, 131, 197, 215, 263, 305, 329 \pmod{330}$.

Since $1000 = 330 \times 3 + 10$, there are $9 \times 3 = 27$ such n. $\square$

3.6 Divisibility

Here are some rules that are well known and useful. Can you figure out why?

- For mod 2, 5, consider the last digit, e.g. $1234 \equiv 4 \equiv 0 \pmod{2}$.
- For mod 4, consider the last two digits, e.g. $1234 \equiv 34 \equiv 2 \pmod{4}$.
- For mod 8, consider the last three digits, e.g. $1234 \equiv 234 \equiv 2 \pmod{8}$.
- For mod 3, 9, take the sum of digits, e.g. $456 \equiv 4+5+6 \equiv 15 \equiv 1+5 \equiv 6 \pmod{9}$.
- For mod 11, consider the alternating sum of digits where the last digit is positive, e.g. $1234 \equiv -1+2-3+4 \equiv 2 \pmod{11}$.

3.7 Problems

1. (HLG2016) Find the units digit of $2013^{2013} \times 2017^{2017}$.
2. (HLG2017) Find the units digit of $2017! - 9$.
3. (HLG2018) Find the units digit of $9^9 + 90^{90} + 901^{901} + 9018^{9018}$.
4. Find the last two digits of $17^{17^{17^{17}}}$.
5. Find the smallest $n \in \mathbb{N}$ such that $43 \mid (5x - 11)$.
6. Find the smallest $n \in \mathbb{N}$ such that $111 \mid (321x - 75)$.
7. Prove that $15 \nmid n^2 + n + 2$.
8. (CJR2017) Find the last three digits of $9 \times 99 \times \cdots \times \underbrace{9 \cdots 9}_{2017\ 9's}$.
9. (HLG2014) Find the last two digits of $7^{2013}11^{2014}13^{2015}$.
10. (HLG2015) Find the sum of the last two digits of $1! + 3! + \cdots + 2015!$.
11. (HLG2016) Find the remainder when $\underbrace{8 \cdots 8}_{2016\ 8's} - \underbrace{5 \cdots 5}_{2016\ 5's}$ is divided by 11.
12. (HLG2017) How many two-digit number pairs (m, n) are there such that $m - n = 16$ and $100 \mid m^2 - n^2$?
13. (HLG2018) How many four-digit numbers $\overline{abcd}$ are there such that $\overline{abcd}$ and $\overline{dbca}$ are both multiples of 7?
14. (Germany 2004) A student writes down the numbers $1, 2, \cdots, 2004$ on a whiteboard. Each step he selects a group of numbers on the whiteboard, erases them, then he writes down the remainder when the sum of the selected numbers is divided by 11. After many steps, two numbers remain on the whiteboard. One of them is 1000. Find the other number.
15. n is a given positive integer. Is it possible to have n consecutive integers such that none of them is a prime power?

Chapter 4

Polynomials

A polynomial consists of variables and coefficients, that involves only addition, subtraction, multiplication, and non-negative integer exponents of variables. In other words,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial where n is a non-negative integer. If $a_n \neq 0$, then $P(x)$ is said to be of *degree* n (i.e. the highest exponent of x). For example,

1. $x^2 - 6x + 7$ is a polynomial of degree 2.
2. $x^9 + 7$ is a polynomial of degree 9.
3. 10 is a polynomial of degree 0.
4. 0 is the special **zero polynomial**: It is normally said to be of degree $-\infty$.

4.1 Factorisation

Factorisation is a process of writing a polynomial into products of several polynomials of smaller degree. For example $x^2 - 7x + 6 = (x - 6)(x - 1)$. In high school, you are normally taught that there are some polynomials that cannot be factorised (for example, $x^2 + 1$. Apparently, that is true if we are dealing with real numbers only. Here, we will expand into the complex numbers. In that case, many theorems will become consistent. In the field of complex numbers,

Theorem 1. *Any polynomial of degree n can be factored into n linear polynomials.*

1. Factorise $x^2 + 6x + 10$ completely.

Solution. By the Quadratic Formula, the two roots are

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 10}}{2} = -3 \pm i.$$

By comparing coefficients, we immediately know

$$x^2 + 6x + 10 = (x + 3 + i)(x + 3 - i)$$

□

4.2 Roots of a Polynomial

The roots of $P(x)$ are the numbers r such that $P(r) = 0$. For example, the roots of $x^2 - 1$ are 1 and -1. Sometimes, a polynomial can have two equal roots, for example $x^3 - 5x^2 + 3x + 9 = (x-3)^2(x+1)$ has roots 3, 3, -1. However, we still say it has 3 roots (counted with multiplicity). Consequently, each polynomial of degree n has exactly n roots by Theorem 1. The following definition is useful before introducing a theorem:

Definition 1. The **k-th Elementary Symmetric Sum** S_k of a collection of numbers $\{a_1, \ldots, a_n\}$ is the sum of all possible products of k distinct elements in S.

E.g. the 3rd ESS of $\{5, 6, 6, 8\}$ is $S_3 = 5 \cdot 6 \cdot 6 + 5 \cdot 6 \cdot 8 + 5 \cdot 6 \cdot 8 + 6 \cdot 6 \cdot 8 = 1066$.

Theorem 2. *(Vieta) If S is the collection of roots of $a_n x^n + \cdots + a_0$, then*

$$\begin{aligned} S_1 &= -\frac{a_{n-1}}{a_n} \\ S_2 &= \frac{a_{n-2}}{a_n} \\ &\vdots \\ S_n &= (-1)^n \frac{a_0}{a_n} \end{aligned}$$

In other words, the number $(-1)^k \dfrac{a_{n-k}}{a_n}$ is the k-th ESS of the roots.

This may look complicated, but actually you had learnt it before in high school. When $n = 2$, it is just the SOR-POR identity: $x^2 - (SOR)x + (POR) = 0$.

2. Let r_1, r_2, r_3 be the roots of $x^3 - 2x^2 + 5x - 1$. Compute

(a) $r_1 + r_2 + r_3$

(b) $r_1 r_2 + r_2 r_3 + r_1 r_3$

(c) $r_1 r_2 r_3$

(d) $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$

Solution. By Vieta's Theorem,

(a) $r_1 + r_2 + r_3 = 2$

(b) $r_1 r_2 + r_2 r_3 + r_1 r_3 = 5$

(c) $r_1 r_2 r_3 = 1$

(d) $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{r_1 r_2 + r_2 r_3 + r_1 r_3}{r_1 r_2 r_3} = 5$ $\square$

3. Solve the following simultaneous equations:

$$\begin{cases} a+b+c=1 \\ ab+bc+ca=-17 \\ abc=15 \end{cases}$$

Solution. By Vieta's Theorem, a, b, c are the roots of the polynomial

$$\begin{aligned} x^3-x^2-17x-15 &= (x+1)(x^2-2x-15) \\ &= (x+1)(x+3)(x-5) \end{aligned}$$

Hence $(a, b, c) = (-1, -3, 5)$ and all of its permutations. □

4.3 Division Algorithm

When we divide 16 by 3, a quotient 5 and a remainder 1 is obtained, hence we write $16 = 5 \times 3 + 1$. Note that the remainder is always smaller than the dividend 3. Similarly, when we divide a polynomial $P(x)$ by another polynomial $Q(x)$, a quotient $M(x)$ and remainder $R(x)$ is obtained. In other words,

Theorem 3. *For any nonzero polynomials $P(x), Q(x)$, there are unique polynomials $M(x), R(x)$ such that*

$$P(x) = Q(x)M(x) + R(x)$$

where the degree of $R(x)$ is less than the degree of $Q(x)$.

4. Divide $P(x) = x^4 + 3x + 1$ by $x^2 + 1$. Write them according to the division algorithm.

Solution. We can do long division, or

$$\begin{aligned} x^4+3x+1 &= x^2(x^2+1) - x^2 + 3x + 1 \\ &= x^2(x^2+1) - (x^2+1) + 3x + 2 \\ &= (x^2-1)(x^2+1) + 3x + 2 \end{aligned}$$

and we are done since $3x + 2$ has smaller degree than $x^2 + 1$. □

5. If $P(x)$ leaves a remainder of $3x^3+1$ when divided by $(x^2+3x+1)^2$, find the remainder when $P(x)$ is divided by $x^2 + 3x + 1$.

Solution. We first write

$$\begin{aligned} P(x) &= (x^2+3x+1)^2 M_1(x) + 3x^3 + 1 \\ P(x) &= (x^2+3x+1) M_2(x) + R(x). \end{aligned}$$

Can we take $R(x) = 3x^3 + 1$ straightaway? If its degree is lower than $x^2 + 3x + 1$ then yes, but in this case it is not. Therefore we need to reduce $3x^3 + 1$. Note that

$$\begin{aligned} 3x^3+1 &= (3x-9)(x^2+3x+1) + 24x + 10 \\ \therefore P(x) &= (x^2+3x+1)^2 M_1(x) + 3x^3 + 1 \\ &= (x^2+3x+1)^2 M_1(x) + (3x-9)(x^2+3x+1) + 24x + 10 \\ &= (x^2+3x+1)((x^2+3x+1)M_1(x) + 3x - 9) + 24x + 10 \end{aligned}$$

and thus the remainder is $24x + 10$. □

6. If $P(x)$ leaves a remainder of $x^2 + x + 1$ when divided by $(x^2 - 1)(3x + 2)$, find the remainder when $P(x)$ is divided by $x - 1$.

Solution. We first write

$$P(x) = (x^2 - 1)(3x + 2)M_1(x) + x^2 + x + 1$$

Again we reduce $x^2 + x + 1$,

$$\begin{aligned} x^2 + x + 1 &= (x - 1)(x + 2) + 3 \\ \therefore P(x) &= (x^2 - 1)(3x + 2)M_1(x) + x^2 + x + 1 \\ &= (x - 1)((x + 1)(3x + 2)M_1(x) + x + 2) + 3 \end{aligned}$$

and thus the remainder is 3. □

4.4 Polynomial Manipulation

7. If $x^2 - 7x + 1 = 0$, find the value of $x^4 - 8x^3 + 7x^2 + 6x + 10$.

Can we find a way to evaluate it without finding what x is? Our main goal here is to simplify everything into small degrees, then we can easily find the answer. Normally, in competitions, most terms cancel out in the end.

Solution. Since $x^2 = 7x - 1$, we can infer

$$\begin{aligned} x^3 &= x(7x - 1) = 7x^2 - x = 7(7x - 1) - x = 48x - 7 \\ x^4 &= x(48x - 7) = 48x^2 - 7x = 48(7x - 1) - 7x = 329x - 48 \end{aligned}$$

and therefore

$$\begin{aligned} &x^4 - 8x^3 + 7x^2 + 6x + 10 \\ &= (329x - 48) - 8(48x - 7) + 7(7x - 1) + 6x + 10 \\ &= 11 \end{aligned}$$

and we are done without even finding what x is. □

8. If $x^2 - 7x + 1 = 0$, find the value of $\dfrac{1}{x^2} - \dfrac{5}{x} + 2x$.

In this case we have negative powers (not a polynomial). We again change everything into simple powers, such as degree 1.

Solution. Since $1 = 7x - x^2$, we can infer

$$\begin{aligned} \frac{1}{x} &= 7 - x \\ \frac{1}{x^2} &= \frac{7}{x} - 1 = 7(7 - x) - 1 = 48 - 7x \end{aligned}$$

and therefore

$$\begin{aligned} &\frac{1}{x^2} - \frac{5}{x} + 2x \\ &= (48 - 7x) - 5(7 - x) + 2x \\ &= 13 \end{aligned}$$

and we are done. □

9. (HLG2017) If $x^2 - 8x + 1 = 0$, find the value of $x^4 + \dfrac{1}{x^4}$.

Here we encounter another common and well-known technique. When we are dealing with $x^n + \dfrac{1}{x^n}$, we only need to start from $x + \dfrac{1}{x}$.

<u>*Solution.*</u> Note that

$$\begin{aligned} x + \frac{1}{x} &= 8 \\ x^2 + \frac{1}{x^2} + 2x \cdot \frac{1}{x} &= 64 \\ x^2 + \frac{1}{x^2} &= 62 \\ x^4 + \frac{1}{x^4} + 2x^2 \cdot \frac{1}{x^2} &= 62^2 \\ x^4 + \frac{1}{x^4} &= 3842 \end{aligned}$$

and we are done again. □

10. (HLG2015) If $x - \dfrac{1}{x} = 1$, find the value of $x^5 - \dfrac{1}{x^5}$.

<u>*Solution.*</u> Note that

$$\begin{aligned} x^2 + \frac{1}{x^2} &= 3 & \left(x^2 + \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right) &= 3 \\ x^4 + \frac{1}{x^4} &= 7 & x^3 - \frac{1}{x^3} - x + \frac{1}{x} &= 3 \\ \left(x^4 + \frac{1}{x^4}\right)\left(x - \frac{1}{x}\right) &= 7 & x^3 - \frac{1}{x^3} &= 4 \\ x^5 - \frac{1}{x^5} - x^3 + \frac{1}{x^3} &= 7 & & \\ \Rightarrow x^5 - \frac{1}{x^5} &= 11 & & \end{aligned}$$

and we are done again. □

4.5 Special Techniques

Let $P(x) = a_n x^n + \cdots + a_0$. Try and think why the following are true:

Property 1. The constant term of $P(x)$ is $P(0)$.

Property 2. The sum of coefficients of $P(x)$ is $P(1)$.

Property 3. The sum of even-indexed coefficients of $P(x)$ is $\dfrac{P(1) + P(-1)}{2}$.

Property 4. The sum of odd-indexed coefficients of $P(x)$ is $\dfrac{P(1) - P(-1)}{2}$.

4.6 Problems

1. (CJR2017) Let x, y be nonzero reals such that $x \neq y$. If $x + \dfrac{9}{x} = y + \dfrac{9}{y}$, find xy.

2. (CJR2019) Given that the polynomial $P(x)$ leaves a remainder of $4x - 7$ when it is divided by $2x^2 - 3x - 2$, find the remainder when $P(x)$ is divided by $2x + 1$.

3. (CJR2018) Given that $2x^2 - 7x + 4 = 0$, find the value of $41x - 4x^3$.

4. (CJR2018) Given that a polynomial $P(x)$ leaves a remainder of $2x+3$ when divided by $x^2 + 1$, and leaves a remainder of $-4x + 13$ when divided by $x^2 - 1$. If $P(x)$ leaves a remainder of $r(x)$ when divided by $x^4 - 1$, find the value of $r(-3)$.

5. Must a polynomial $f(x)$ that attains integer values at all integers x have only integer coefficients?

6. (HLG2018) If $(1-2x+3x^2)^{10} = a_0+a_1x+\cdots+a_{20}x^{20}$, find the value of $a_1+\cdots+a_{20}$.

7. (HLG2018) If $x + y = 7$ and $xy = 11$, find the value of $x^6 + y^6$.

8. (HLG2018) If a, b are real numbers such that $x^2 - x + 2$ is a factor of $ax^5 + bx^4 + 16$, find the value of a.

9. (HLG2018) If α, β, γ are the three solutions of $(x - 59)^3 + (2x - 64)^3 = (3x - 123)^3$, find the value of $\alpha + \beta + \gamma$.

10. A rational point is a point (x, y) on the plane such that x, y are both rational. For any line ℓ connecting two rational points on the curve $y^2 = x^3 + 3x^2 + 1$, prove that if ℓ meets the curve at a third point, this point must be a rational point.

Chapter 5

Sums and Products

5.1 Summation and Product Notation

In this chapter (or more generally, the whole book) we will use frequently use these two symbols: Capital Sigma (Σ) and Capital Pi (Π). The operators associated to them are the summation and product respectively. In mathematical terms,

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + \cdots + f(b)$$

$$\prod_{k=a}^{b} f(k) = f(a) \times f(a+1) \times \cdots \times f(b)$$

For example,

$$\sum_{k=1}^{5} \log k = \log 1 + \log 2 + \log 3 + \log 4 + \log 5$$

$$\prod_{k=1}^{5} (k+3) = (1+3)(2+3)(3+3)(4+3)(5+3)$$

In this case, $\log k$ is called the **summand**, and $(k+3)$ is called the **multiplier/multiplicand**. Besides, k is called the **index** while 1 and 5 are called the **bounds**.

We can also write **condition(s)** instead of bounds, e.g. if $S = \{1, 4, 5, 8, 9\}$,

$$\sum_{k \in S} \log k = \log 1 + \log 4 + \log 5 + \log 8 + \log 9 \qquad \prod_{\substack{k \in S \\ k \text{ even}}} k = 4 \times 8 = 32.$$

In many cases, a summation or product may not be easily computed (e.g. the examples above!). However, some summations and products come in neater variations, which we can find a method to easily compute even though the bounds are arbitrarily large.

Note that these properties hold:

$$\sum (f(x) + g(x)) = \sum f(x) + \sum g(x)$$

$$\sum cf(x) = c \sum f(x)$$

$$\prod (f(x)g(x)) = \prod f(x) \prod g(x).$$

where c is independent of the index.

5.2 Sum Formulae

These three formulae are very indispensable in contest maths. They are:

Proposition 1. $\displaystyle\sum_{k=1}^{n} k = \frac{k(k+1)}{2}$

Proposition 2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}$

Proposition 3. $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{k(k+1)}{2}\right]^2$

1. Compute $4^2 + 7^2 + \cdots + 100^2$.

We recognise this as a series of numbers with obvious pattern. Our first step is to find out the summand, which is $(3k+1)^2$. Then we can start to use the sum formulae.

Solution.

$$\begin{aligned}\sum_{k=1}^{33}(3k+1)^2 &= \sum_{k=1}^{33}(9k^2+6k+1)\\ &= 9\sum_{k=1}^{33}k^2 + 6\sum_{k=1}^{33}k + \sum_{k=1}^{33}1\\ &= 9\cdot\frac{33\cdot 34\cdot 67}{6} + 6\cdot\frac{33\cdot 34}{2} + 33\\ &= 116160.\end{aligned}$$

Note that $\sum 1$ is not 1, but instead repeated additions of 1, in this case 33. □

5.3 Telescoping Sum and Products

These types of sums and products are a bit harder. We first look at a simple example:

2. Compute $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)$.

Solution. $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) = \left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = \frac{n+1}{2}$. □

This solution is elegant! We managed to cancel off many terms in between. This ‘term-cancelling’ is the main aim of solving telescoping sums or products. In fact, the term ‘telescoping’ is used because the sum/product can be described as closing a telescope: Making a long expression collapse into something short, which can be summarised by the following two propositions:

Proposition 4. *(Telescoping Sum)*

$$\sum_{k=1}^{n} f(k+1) - f(k) = f(n+1) - f(1).$$

Proposition 5. *(Telescoping Product)*

$$\prod_{k=1}^{n} \frac{f(k+1)}{f(k)} = \frac{f(n+1)}{f(1)}.$$

3. Compute $\dfrac{1}{1\times 2} + \dfrac{1}{2\times 3} + \cdots + \dfrac{1}{(n-1)n}$.

Solution.

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n-1} \frac{(k+1)-k}{k(k+1)} \\
&= \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 1 - \frac{1}{n}
\end{aligned}$$

□

4. Compute $\dfrac{1}{1\times 2\times 3} + \dfrac{1}{2\times 3\times 4} + \cdots + \dfrac{1}{(n-2)(n-1)n}$.

Solution.

$$\begin{aligned}
\sum_{k=1}^{n-2} \frac{1}{k(k+1)(k+2)} &= \sum_{k=1}^{n-2} \frac{1}{k+1}\left(\frac{1}{k(k+2)}\right) \\
&= \sum_{k=1}^{n-2} \frac{1}{k+1} \cdot \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) \\
&= \frac{1}{2}\sum_{k=1}^{n-2} \left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right) \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n-1)n}\right)
\end{aligned}$$

□

5. (CJR2018) Find the coefficient of x^{18} in $(x+1)(x+2)\cdots(x+20)$.

Solution 1. Multiplying eighteen x's together, there are always 2 numbers left to be multiplied. Hence the coefficient is the 2nd elementary symmetric sum (see Polynomials chapter) of $\{1,\ldots,20\}$. That would be

$$\begin{aligned}
&\quad 1\times 2+1\times 3+\cdots+1\times 20\\
&+2\times 3+2\times 4+\cdots+2\times 20\\
&+\cdots+19\times 20\\
&=\sum_{n=1}^{19} n((n+1)+\cdots+20)\\
&=\sum_{n=1}^{19} n\frac{(20-n)(20+n+1)}{2}\\
&=\frac{1}{2}\sum_{n=1}^{19}(-n^3-n^2+420n)\\
&=\frac{1}{2}\left(-\left[\frac{19\times 20}{2}\right]^2-\frac{19\times 20\times 39}{6}+420\cdot\frac{19\times 20}{2}\right)\\
&=20615.
\end{aligned}$$

□

Solution 2. The 2nd elementary symmetric sum of $\{1,\ldots,20\}$ is also

$$\frac{(1+2+\cdots+20)^2-(1^2+\cdots+20^2)}{2}=\frac{(20\times 21/2)^2+20\times 21\times 41/6}{2}=20615.$$

□

5.4 Gaussian Pairing

In the 1700s, Carl Friedrich Gauss, as an elementary school student, answered the question $1+\cdots+100$ in seconds. He amazed everyone in class especially the teacher since no one expected anyone to compute such a large sum. How did he know what the answer is? He used a very elegant trick, of course. Here, we will introduce the method Gauss used, the now-called **Gaussian pairing**.

6. Evaluate $1+2+\cdots+99+100$.

The trick is to pair up elements from both sides, i.e. $\{1,100\},\{2,99\},\cdots$. These pairs all sum up to a constant number which is 101. How many pairs are there? 50 pairs!

Solution. $50\times 101=5050$. □

7. p is an odd prime. Find the remainder of $\displaystyle\sum_{k=0}^{p-1}\frac{k^p-k}{p}$ when divided by p.

We know that each term is an integer by Fermat's Little Theorem, but that does not help at all. What we can do is to do Gaussian pairing and simplify them. Note that $k = 0$ can be omitted. In this solution, the binomial theorem is involved:

Solution.

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{k^p - k}{p} &= \sum_{k=1}^{(p-1)/2} \left(\frac{k^p - k}{p} + \frac{(p-k)^p - (p-k)}{p} \right) \\
&= \sum_{k=1}^{(p-1)/2} \left(\frac{k^p + (p-k)^p}{p} - 1 \right) \\
&= \sum_{k=1}^{(p-1)/2} \left(\frac{k^p + (\cdots)p^2 + p \cdot pk^{p-1} - k^p}{p} - 1 \right) && \text{(Binomial Theorem)} \\
&= \sum_{k=1}^{(p-1)/2} ((\cdots)p - 1) \equiv \sum_{k=1}^{(p-1)/2} (-1) \equiv \frac{p+1}{2} \pmod{p}
\end{aligned}$$

and hence the remainder is $\dfrac{p+1}{2}$. □

5.5 Infinite Series

The notion of an infinite series might be hard to understand. How can one evaluate a sum which has an infinite number of terms?

Infinite series are formally studied in undergraduate Analysis courses. However, we shall keep it simple: There are two types of infinite series. Say

$$S = \sum_{k=1}^{\infty} a_k.$$

Firstly, S is said to be **convergent** if and only if the sequence $S_n = \displaystyle\sum_{k=1}^{n} a_k$ has a limit L, and we define the 'answer' to S as L. For example, to check if

$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

is convergent, we first simplify

$$\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$$

and note that $1 - \dfrac{1}{2^n}$ converges to 1. Hence we say $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots = 1$.

Secondly, S is said to be **divergent** if and only if the sequence $S_n = \displaystyle\sum_{k=1}^{n} a_k$ does not have a limit. For example,

$$1 - 1 + 1 - 1 + 1 + \cdots + (-1)^{n+1} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

and thus $1-1+1-1+\cdots$ is divergent - it has no answer. Another example would be

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

which diverges to infinity, hence $1+2+3+4+\cdots$ is divergent.

Incorrect Example. Let $S=1+2+4+8+\cdots$. Then

$$2S=2+4+8+16+\cdots=(1+2+4+8+\cdots)-1=S-1$$

and hence $S=-1$. $\square$

This is incorrect because no tests are taken to determine whether S is convergent or divergent. **We can manipulate S algebraically if and only if S is convergent**, otherwise anything we do to S is mathematically meaningless. Indeed, $1+2+4+\cdots+2^{n-1}=2^n-1$ has no limit, hence S diverges.

However, sadly, in high-school level, the problems involving infinite series are always assumed to be convergent, that is, you do not have to ensure whether the given sum is divergent or convergent - you just have to find the answer. Therefore, formally speaking, the solutions to the examples below are **logically invalid** because no tests are done to determine convergence, but it is what you need to do in contests.

8. Evaluate $S=\dfrac{1}{1}+\dfrac{2}{2}+\dfrac{3}{4}+\dfrac{4}{8}+\dfrac{5}{16}+\cdots$.

<u>Solution</u>. The denominators form a GP, hence multiplying by 2 will only shift the values:

$$\frac{1}{2}S=\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$$

and now we subtract term by term according to their denominators.

$$\begin{aligned}S-\frac{1}{2}S&=\frac{1}{1}+\frac{2-1}{2}+\frac{3-2}{4}+\frac{4-3}{8}+\frac{5-4}{16}+\cdots\\&=\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=2\end{aligned}$$

and therefore $S=4$. $\square$

9. If $S=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots$, write $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots$ in terms of S.

<u>Solution</u>. Here we don't have to find the exact value of S (for your information, $S=\pi^2/6$).

$$\begin{aligned}\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots&=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)\\&=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)-\frac{1}{2^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)\\&=S-\frac{1}{4}S=\frac{3}{4}S.\end{aligned}$$

$\square$

5.6 Problems

1. (HLG2015) Find $\displaystyle\sum_{n=1}^{33} \frac{12}{(3n-2)(3n+1)}$.

2. $n \geq 3$ is an integer and α_n, β_n are the roots of $x^2 + (n^2-3)x + 3n$. Find the value of $\dfrac{1}{(\alpha_4-3)(\beta_4-3)} + \dfrac{1}{(\alpha_5-3)(\beta_5-3)} + \cdots + \dfrac{1}{(\alpha_{99}-3)(\beta_{99}-3)}$.

3. (HLG2017) Simplify $\sqrt{1+\dfrac{1}{1^2}+\dfrac{1}{2^2}} + \sqrt{1+\dfrac{1}{2^2}+\dfrac{1}{3^2}} + \cdots + \sqrt{1+\dfrac{1}{99^2}+\dfrac{1}{100^2}}$.

4. (HLG2017) Evaluate $\dfrac{1}{2}+\dfrac{4}{4}+\dfrac{9}{8}+\dfrac{16}{16}+\dfrac{25}{32}+\dfrac{36}{64}+\cdots$.

5. (HLG2016) Evaluate $1-\dfrac{2}{3}+\dfrac{3}{3^2}-\dfrac{4}{3^3}+\dfrac{5}{3^4}-\dfrac{6}{3^5}+\cdots$.

6. (HLG2016) If $S = \dfrac{1}{1^3}+\dfrac{1}{2^3}+\dfrac{1}{3^3}+\cdots$, write $\dfrac{1}{1^3}+\dfrac{1}{3^3}+\dfrac{1}{5^3}+\cdots$ in terms of S.

7. Simplify $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n!$. Hint: $k\cdot k! = (k+1-1)k!$.

8. (HLG2015) Find n if

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{n-1}+\sqrt{n}} = 10.$$

9. (HLG2014) Find the value of

$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\frac{5}{3!+4!+5!}+\cdots+\frac{50}{48!+49!+50!}.$$

10. (HLG2013) Let $f(x) = \dfrac{9^x}{9^x+27}$. Find the value of

$$f\left(\frac{1}{9}\right)+f\left(\frac{2}{9}\right)+f\left(\frac{3}{9}\right)+\cdots+f\left(\frac{26}{9}\right).$$

Chapter 6

The Floor Function

6.1 The Floor Function

The notation of the floor function is invented by Carl Friedrich Gauss. It is an interesting, most of the time confusing, type of function that we should know prior to joining contests.

The **floor** of x, denoted by $\lfloor x \rfloor$, is the largest integer not more than x. For example,

$$\begin{aligned} \lfloor 0.3 \rfloor &= 0 \\ \lfloor 5 \rfloor &= 5 \\ \lfloor \pi \rfloor &= 3 \\ \lfloor -3.2 \rfloor &= -4. \end{aligned}$$

In contests, the most useful inequality in dealing with floor functions is

Proposition 1. $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.

1. Evaluate $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{10000} \rfloor$.

We wish to analyse how the frequencies of values behave in this expression. For example, 1 appears three times ($\lfloor \sqrt{1} \rfloor = \lfloor \sqrt{2} \rfloor = \lfloor \sqrt{3} \rfloor = 1$), 2 appears five times ($\lfloor \sqrt{4} \rfloor = \lfloor \sqrt{5} \rfloor = \lfloor \sqrt{6} \rfloor = \lfloor \sqrt{7} \rfloor = \lfloor \sqrt{8} \rfloor = 2$) etc. Generally, how many times does k appear?

Solution. The last term is 100 obviously. For each $k = 1, 2, \ldots, 99$, we find the number of n that satisfies $\lfloor \sqrt{n} \rfloor = k$:

$$\begin{aligned} k \le \sqrt{n} &< k + 1 \\ k^2 \le n &< k^2 + 2k + 1 \\ k^2 \le n &\le k^2 + 2k \end{aligned}$$

hence there are $2k + 1$ terms that are equal to k. Therefore the original expression is,

$$\begin{aligned} \sum_{k=1}^{99} k(2k+1) + 100 &= 2\sum_{k=1}^{99} k^2 + \sum_{k=1}^{99} k + 100 \\ &= 2 \cdot \frac{99 \cdot 100 \cdot 199}{6} + \frac{99 \cdot 100}{2} + 100 \\ &= 661750. \end{aligned}$$

$\square$

2. Solve for positive reals, $4x^2 - 40\lfloor x \rfloor + 51 = 0$.

<u>*Solution*</u>.

$$\begin{cases} 4x^2 - 40x + 51 \le 0 \\ 4x^2 - 40(x-1) + 51 > 0 \end{cases} \Rightarrow \begin{cases} \dfrac{3}{2} \le x \le \dfrac{17}{2} \\ x < \dfrac{7}{2} \text{ or } x > \dfrac{13}{2} \end{cases} \Rightarrow x \in \left[\frac{3}{2}, \frac{7}{2}\right) \cup \left(\frac{13}{2}, \frac{17}{2}\right]$$

In that case, $\lfloor x \rfloor = 1, 2, 3, 6, 7, 8$. Substituting back,

$\lfloor x \rfloor$	$4x^2$	x	$\lfloor x \rfloor$
1	-11	$-$	$-$
2	29	$\sqrt{29}/2$	2
3	69	$\sqrt{69}/2$	4
6	189	$\sqrt{189}/2$	6
7	229	$\sqrt{229}/2$	7
8	269	$\sqrt{269}/2$	8

$$\therefore x = \frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}.$$

$\square$

3. Evaluate $\left\lfloor \dfrac{19999^3 - 2 \times 19999}{19998} \right\rfloor$.

<u>*Solution*</u>. Let $x = 19999$. Then

$$\begin{aligned} \left\lfloor \frac{x^3 - 2x}{x-1} \right\rfloor &= \left\lfloor \frac{(x-1)(x^2+x-1) - 1}{x-1} \right\rfloor \\ &= \left\lfloor (x^2 + x - 1) - \frac{1}{x-1} \right\rfloor \\ &= x^2 + x - 2 \\ &= x(x+1) - 2 \\ &= 19999 \times 20000 - 2 \\ &= 399979998. \end{aligned}$$

$\square$

4. If $r \in \mathbb{R}$ such that $\left\lfloor r + \dfrac{19}{100} \right\rfloor + \left\lfloor r + \dfrac{20}{100} \right\rfloor + \cdots + \left\lfloor r + \dfrac{91}{100} \right\rfloor = 546$, find $\lfloor 100r \rfloor$.

<u>*Solution*</u>. Let $n = \lfloor r \rfloor$, then all the terms above consist of n's and $(n+1)'s$. Say there are k number of n's. Then there are $73 - k$ number of $(n+1)$'s, i.e.

$$\begin{aligned} nk + (n+1)(73-k) &= 546 \\ 73n - k &= 473 \\ n &= 6 + \frac{35+k}{73} \\ \therefore (k, n) &= (38, 7). \end{aligned}$$

Hence, $\left\lfloor r+\frac{56}{100}\right\rfloor = 7$, $\left\lfloor r+\frac{57}{100}\right\rfloor = 8$. Translating into inequalities,

$$r+\frac{56}{100} < 8 \le r+\frac{57}{100}$$
$$743 \le 100r < 744.$$

Therefore $\lfloor 100r\rfloor = 743$. $\square$

6.2 The Ceiling and Rounding Function

The **ceiling** function, denoted by $\lceil x\rceil$, is the least integer not less than x, hence

$$\lceil x\rceil - 1 < x \le \lceil x\rceil.$$

The **rounding** function, denoted by $\lceil x\rfloor$, is the nearest integer from x, hence

$$\lceil x\rfloor - \frac{1}{2} \le x < \lceil x\rfloor + \frac{1}{2}.$$

and the problem solving techniques are still analogous.

6.3 Problems

1. (CJR2019) Given that $x \in \mathbb{Z}$ and $\left\lfloor \frac{x-128}{7}\right\rfloor = -3$, find the largest possible x.

2. (CJR2018) Evaluate $\left\lfloor \frac{3^2}{1}\right\rfloor + \left\lfloor \frac{4^2}{2}\right\rfloor + \left\lfloor \frac{5^2}{3}\right\rfloor + \cdots + \left\lfloor \frac{42^2}{40}\right\rfloor$

3. (HLG2015) Evaluate $\lfloor \log_2 1\rfloor + \lfloor \log_2 2\rfloor + \lfloor \log_2 3\rfloor + \cdots + \lfloor \log_2 128\rfloor$.

4. (China 1993) Find the last two digits of $\left\lfloor \frac{10^{93}}{10^{31}+3}\right\rfloor$.

5. (China 1990) Find the value of $\left\lfloor \frac{2+\sqrt{2}}{2}\right\rfloor + \left\lfloor \frac{3+\sqrt{3}}{3}\right\rfloor + \cdots + \left\lfloor \frac{1990+\sqrt{1990}}{1990}\right\rfloor$.

6. (China 1986) Let $k = \frac{305}{503}$. Find the value of $\lfloor k\rfloor + \lfloor 2k\rfloor + \cdots + \lfloor 502k\rfloor$.

7. (Russia 1980) How many different numbers are there in $\left\lfloor \frac{1^2}{1980}\right\rfloor, \left\lfloor \frac{2^2}{1980}\right\rfloor, \ldots, \left\lfloor \frac{1980^2}{1980}\right\rfloor$?

8. (Russia 1991) Solve for positive reals, $\lfloor x\rfloor\{x\}+x = 2\{x\}+10$, where $\{x\} = x-\lfloor x\rfloor$.

9. Let $x = \lfloor x\rfloor + \{x\}$. Prove for any nonsquare positive integer n that

$$\frac{1}{2\sqrt{n}} < \{\sqrt{n}\} < 1.$$

10. Prove that

$$\left\lfloor \frac{\lfloor x\rfloor}{n}\right\rfloor = \left\lfloor \frac{x}{n}\right\rfloor.$$

Chapter 7

Recursion

A **recursive function** is a function that is defined in terms of itself. One of the most famous examples is the Fibonacci number sequence:

$$\begin{cases} F(n) = F(n-1) + F(n-2) \\ F(0) = 0, F(1) = 1 \end{cases}$$

However, most of them can be explicitly expressed as a normal function, i.e. the **general form**. Let's start from the easy examples.

7.1 First Order Recurrence

Let's define whatever's in the title of this section. A **first order recurrence** is a recurrence in which a_n is only expressed in terms of a_{n-1}. A higher order recurrence might have a_n expressed in terms of a_{n-1}, a_{n-2} etc.

1. Find the general form of $a_n = a_{n-1} + 3, a_0 = 2$.

Solution. We have $a_n = a_{n-1} + 3 = a_{n-2} + 2 \times 3 = \cdots = a_0 + n \times 3 = 3n + 2$. $\square$

2. Find the general form of $a_n = 4a_{n-1}, a_0 = 7$.

Solution. We have $a_n = 4a_{n-1} = 4^2 a_{n-2} = \cdots = 4^n a_0 = 7 \cdot 4^n$. $\square$

3. Find the general form of $a_n = 4a_{n-1} - 3, a_0 = 2$.

Solution. Note that $a_n - 1 = 4(a_{n-1} - 1)$. Hence let $b_n = a_n - 1$ to get

$$b_n = 4b_{n-1} = 4^2 b_{n-2} = \cdots = 4^n b_0 = 4^n(a_0 - 1) = 4^n.$$

And hence $a_n = 1 + 4^n$. $\square$

4. Find the general form of $a_n = 5a_{n-1} + 3, a_0 = 6$.

Solution. Let $a_n + p = 5(a_{n-1} + p) \;\Leftrightarrow\; a_n = 5a_{n-1} + 4p$. Thus $p = 3/4$. Hence let $b_n = a_n + 3/4$, then $b_n = 5b_{n-1} \;\Rightarrow\; b_n = (27/4)5^n$, yielding

$$a_n = \frac{27 \cdot 5^n - 3}{4}.$$

$\square$

Let's end this section with the everything-case.

5. Find the general form of $a_n = Aa_{n-1} + B, a_0 = C$ where $A \neq 1$.

Solution. Let $a_n + p = A(a_{n-1} + p) \Leftrightarrow a_n = Aa_{n-1} + (A-1)p$. Thus $p = B/(A-1)$. Hence let $b_n = a_n + B/(A-1)$, then

$$\begin{aligned} b_n &= Ab_{n-1} \\ \Rightarrow b_n &= \left(C + \frac{B}{A-1}\right) A^n \\ \therefore a_n &= \left(C + \frac{B}{A-1}\right) A^n - \frac{B}{A-1}. \end{aligned}$$

□

7.2 Second or Higher Order Recurrence

We introduce a theorem, but the examples below will make it more easily fathomable:

Theorem 1. *If $a_n = p_1 a_{n-1} + p_2 a_{n-2} + \cdots + p_k a_{n-k}$, then*

$$a_n = C_1 r_1^n + C_2 r_2^n + \cdots + C_k r_k^n$$

where $r_1, r_2, \ldots, r_k$ are the roots of $x^k - p_1 x^{k-1} - p_2 x^{k-2} - \cdots - p_k$, also known as the ***characteristic polynomial*** *of this recurrence. However, if repeated roots (say $r_1 = r_2 = \cdots = r_m = R$) occur, then the associated terms are replaced with*

$$\cdots + (C_1 + C_2 n + \cdots + C_m n^{m-1}) R^n + \cdots$$

instead of $\cdots + C_1 r_1^n + C_2 r_2^n + \cdots + C_m r_m^n + \cdots$. Here, C_i, p_i are all constants.

6. Find the general form of $a_n = 4a_{n-1} - 3a_{n-2}, a_0 = 2, a_1 = 3$.

Solution. The roots of the characteristic polynomial satisfy $x^2 = 4x - 3 \Rightarrow x = 1, 3$, thus

$$a_n = C_1 \cdot 1^n + C_2 \cdot 3^n.$$

By taking $a_0 = 2, a_1 = 3$, we just need to solve C_1, C_2 by simultaneous equations:

$$\begin{cases} C_1 + C_2 = 2 \\ C_1 + 3C_2 = 3 \end{cases} \Rightarrow \begin{cases} C_1 = 3/2 \\ C_2 = 1/2 \end{cases}$$

and hence $a_n = \dfrac{3 + 3^n}{2}$. □

7. Find the general form of $a_n = 6a_{n-1} - 9a_{n-2}, a_0 = 2, a_1 = 3$.

Solution. The characteristic polynomial is $x^2 = 6x - 9 \Rightarrow x = 3, 3$. Hence

$$a_n = (C_1 + C_2 n) \cdot 3^n.$$

By taking $a_0 = 2, a_1 = 3$, we just need to solve C_1, C_2 by simultaneous equations:

$$\begin{cases} C_1 = 2 \\ 3(C_1 + C_2) = 3 \end{cases} \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = -1 \end{cases}$$

and hence $a_n = (2 - n) \cdot 3^n$. □

8. Find the general form of

$$\begin{cases} a_n = 12a_{n-1} - 53a_{n-2} + 102a_{n-3} - 72a_{n-4} \\ a_0 = 2, a_1 = 21, a_2 = 109, a_3 = 471 \end{cases}$$

Solution. The characteristic polynomial is $x^4 = 12x^3 - 53x^2 + 102x - 72 \Rightarrow x = 2, 3, 3, 4$. Therefore

$$a_n = C_1 \cdot 2^n + (C_2 + nC_3) \cdot 3^n + C_4 \cdot 4^n$$

By taking $a_0 = 2, a_1 = 3$, we just need to solve C_1, C_2 by simultaneous equations:

$$\begin{cases} C_1 + C_2 + C_4 & = 2 \\ 2C_1 + 3C_2 + 3C_3 + 4C_4 & = 21 \\ 4C_1 + 9C_2 + 18C_3 + 16C_4 & = 109 \\ 8C_1 + 27C_2 + 81C_3 + 64C_4 & = 471 \end{cases} \Rightarrow \begin{cases} C_1 = -1 \\ C_2 = 1 \\ C_3 = 4 \\ C_4 = 2 \end{cases}$$

and hence $a_n = -2^n + (4n + 1) \cdot 3^n + 2^{2n+1}$. □

9. Find the general form of $a_n = 6a_{n-1} - 5a_{n-2} + 6, a_0 = 2, a_1 = 3$.

We have to remove the 6 before finding the characteristic polynomial.

Solution. From $a_n = 6a_{n-1} - 5a_{n-2} + 6$, we also have $a_{n-1} = 6a_{n-2} - 5a_{n-3} + 6$. Subtracting,

$$a_n = 7a_{n-1} - 11a_{n-2} + 5a_{n-3}.$$

The characteristic polynomial is $x^3 = 7x^2 - 11x + 5 \Rightarrow x = 1, 1, 5$. Hence

$$a_n = (C_1 + C_2 n) \cdot 1^n + C_3 \cdot 5^n.$$

By taking $a_0 = 2, a_1 = 3, a_2 = 14$, we just need to solve

$$\begin{cases} C_1 + C_3 = 2 \\ C_1 + C_2 + 5C_3 = 3 \\ C_1 + 2C_2 + 25C_3 = 14 \end{cases} \Rightarrow \begin{cases} C_1 = 11/8 \\ C_2 = -3/2 \\ C_3 = 5/8 \end{cases}$$

and hence $a_n = \dfrac{11 - 12n + 5^{n+1}}{8}$. □

10. Find the **recursion** form of $a_n = (3 + \sqrt{7})^n + (3 - \sqrt{7})^n$.

Solution. The characteristic polynomial with roots $3 \pm \sqrt{7}$ should be $x^2 - (SOR)x + (POR) = 0 \Rightarrow x^2 - 6x + 2 = 0$. Hence, $a_n = 6a_{n-1} - 2a_{n-2}$ and also $a_0 = 2, a_1 = 6$. □

7.3 Non-Homogeneous Recurrence

Unlike homogeneous recurrences, non-homogeneous recurrences have variable coefficients. However, it can normally be transformed into homogeneous recurrences quickly.

11. Find the general form of $a_n = 4a_{n-1} + 3n, a_0 = 2$.

Solution. From $a_n = 4a_{n-1} + 3n$ we also have $a_{n-1} = 4a_{n-2} + 3(n-1)$. Subtracting,

$$a_n = 5a_{n-1} - 4a_{n-2} + 3$$

and then similarly $a_{n-1} = 5a_{n-2} - 4a_{n-3} + 3$. Subtracting again,

$$a_n = 6a_{n-1} - 9a_{n-2} + 4a_{n-3}$$

and we are back on track. Remaining parts are left to the reader. □

12. Find the general form of $a_n = 12a_{n-1} - 3^{n+1}, a_0 = 2$.

Solution. Divide both sides by 3^{n+1},

$$\begin{aligned}\frac{a_n}{3^{n+1}} &= \frac{12a_{n-1}}{3^{n+1}} - 1\\ \frac{a_n}{3^n} &= 4\frac{a_{n-1}}{3^{n-1}} - 3\\ b_n &= 4b_{n-1} - 3 \ (b_0 = 2).\end{aligned}$$

and we are back to Example 3. Hence $b_n = 1 + 4^n \Rightarrow a_n = 3^n + 12^n$. □

7.4 Value of $\lfloor r^n \rfloor$

13. Find the value of $\lfloor (4+\sqrt{13})^5 \rfloor$.

Solution. Note that $0 < 4 - \sqrt{13} < 1$ and $(4+\sqrt{13})^5 + (4-\sqrt{13})^5$ is an integer by the Binomial Theorem (terms with $\sqrt{13}$ cancel out). Therefore,

$$\lfloor (4+\sqrt{13})^5 \rfloor = (4+\sqrt{13})^5 + (4-\sqrt{13})^5 - 1.$$

Let $a_n = (4+\sqrt{13})^n + (4-\sqrt{13})^n$. Then $a_0 = 2, a_1 = 8$ and $a_n = 8a_{n-1} - 3a_{n-2}$. Therefore

$$\begin{aligned}a_2 &= 8(8) - 3(2) = 58\\ a_3 &= 8(58) - 3(8) = 440\\ a_4 &= 8(440) - 3(58) = 3346\\ a_5 &= 8(3346) - 3(440) = 25448\end{aligned}$$

hence $\lfloor (4+\sqrt{13})^5 \rfloor = 25447$. □

14. Find the units digit of $\lfloor (7+\sqrt{40})^{1000} \rfloor$.

Solution. Similarly, we want $a_{1000} - 1 \pmod{10}$ where $a_0 = 2, a_1 = 14, a_n = 14a_{n-1} - 9a_{n-2}$. In fact, we can take everything modulo 10,

$$\begin{aligned} a_n &\equiv 4a_{n-1} + a_{n-2} \\ a_0 &\equiv 2, a_1 \equiv 4 \\ \therefore (a_n) &\equiv (2, 4, 8, 6, 2, 4, 8, \cdots) \end{aligned}$$

and hence the sequence is periodic after every 4 terms. Therefore we can conclude

$$a_{1000} \equiv a_0 \equiv 2 \pmod{10}$$

or $\lfloor (7 + \sqrt{40})^{1000} \rfloor \equiv 1 \pmod{10}$. $\square$

7.5 Special Methods

15. Find the general form of $a_n = 2a_{n-1}^2 - 1, a_1 = \sqrt{3}/2$.

Solution. We can see $|a_n| \leq 1$ so we can let $a_n = \cos\theta_n$. Then $\theta_1 = \pi/6$

$$\cos\theta_n = 2\cos^2\theta_{n-1} - 1 = \cos 2\theta_{n-1} \quad \Rightarrow \quad \theta_n = 2^{n-1}\theta_1 = \frac{2^{n-2}\pi}{3}$$

and hence $a_n = \arccos \dfrac{2^{n-2}\pi}{3}$. $\square$

16. Find the general form of $a_n = \dfrac{1 + a_{n-1}}{1 - a_{n-1}}, a_1 = \sqrt{3}$.

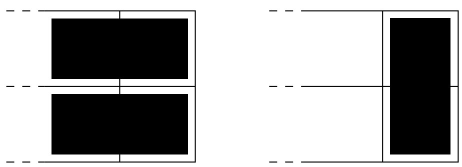
Solution. Let $a_n = \tan\theta_n$. Hence $\theta_1 = \pi/3$ and

$$\tan\theta_n = \frac{1 + \tan\theta_{n-1}}{1 - \tan\theta_{n-1}} = \tan\left(\theta_{n-1} + \frac{\pi}{4}\right)$$

and again similarly we have $a_n = \arctan \dfrac{(3n+1)\pi}{12}$. $\square$

17. In how many ways can a 2×10 grid of squares be tiled with dominoes? (A domino covers two adjacent squares in the grid, and a tiling is a way of placing the dominoes so that every square is covered and no two dominoes overlap).

Solution. Let the number of ways in a $2 \times n$ grid be a_n. Near the end, the dominos must configure in one of the following two ways:



which leaves us with a $2 \times (n-2)$ and $2 \times (n-1)$ grid respectively. $\therefore a_n = a_{n-1} + a_{n-2}$. Since $a_1 = 1, a_2 = 2$, we have $(a_n) = (1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots)$. Hence 89. $\square$

7.6 Problems

1. (CJR2017) a, b, x, y are real numbers such that

$$\begin{cases} ax + by = 2 \\ ax^2 + by^2 = 3 \\ ax^3 + by^3 = 4 \\ ax^4 + by^4 = 7. \end{cases}$$

   Find the value of $ax^5 + by^5$.

2. (HLG2013) Find the value of $\lfloor (2 + 2\sqrt{2})^5 \rfloor$.

3. (HLG2017) Given that the first term of a sequence (a_n) is $a_1 = 5$, and $a_{n+1} = \dfrac{1 + \sqrt{3}a_n}{\sqrt{3} - a_n}$ for all $n \geq 1$. Find the value of a_{100}.

4. How many 8-digit numbers $\overline{d_1 \cdots d_8}$ satisfy the conditions that $d_i \in \{1, 2, 3\}(1 \leq i \leq 8)$ and there are no adjacent 2s? Hint: Let the answer be a_8. Find the recursive form of a_n and start from small values of n.

5. There is a flight of stairs with 10 steps. In each move a person can walk up 1 step, 2 steps or 3 steps. How many ways are there to reach the top?

6. (CJR2019) Given that (a_n) is a sequence defined by $a_1 = 7$, and $a_{n+1} = \dfrac{12a_n}{37 - a_n^2}$ for all $n \leq 1$. If $S = a_1 + 2a_2 + \cdots + 2019a_{2019}$, find the sum of the digits of S.

7. Consider a staircase with 10 steps, and every time we are allowed to proceed 1 step, 2 steps or 3 steps. How many different ways are there to reach the top?

8. Divide a circle into n sectors $S_1, \ldots, S_n$. We have $m \geq 2$ colours. Find the number of ways to colour the sectors such that each sector is coloured exactly once and adjacent sectors must have different colours. (The circle is fixed in position)

   Attempt problems 9 and 10 after reading Chapter 11:

9. (HLG2017) Given that (x_n) is a sequence defined by $x_0 = 1$ and

$$x_n = -\frac{225}{n}(x_0 + x_1 + \cdots + x_{n-1})$$

   for $n \geq 1$. Find the value of $x_0 + 2x_1 + 2^2x_2 + \cdots + 2^{225}x_{225}$.

10. Prove that

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k}.$$

   Hint: Use the identity $\dbinom{n}{k} = \dbinom{n-1}{k} + \dbinom{n-1}{k-1}$.

Chapter 8

Geometry

8.1 Analytic Geometry

Analytic geometry in contests consists of Cartesian coordinates, trigonometry and complex numbers. However, we will focus on coordinates and trigonometry only.

8.1.1 Cartesian Coordinates

The most important equations in Cartesian Coordinates:

Proposition 1. *Lines are uniquely expressed as $y = mx + c$ (m, c are constants).*

Proposition 2. *The circle centred at (a, b) with radius r is $(x-a)^2+(y-b)^2 = r^2$.*

Theorem 1. *(Point-to-point) The distance between two points $(x_1, y_1), (x_2, y_2)$ is*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Theorem 2. *(Point-to-line) The distance from (x_1, y_1) to line $ax + by + c = 0$ is*

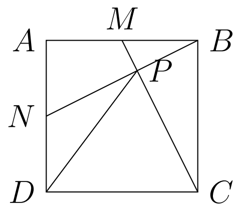
$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|.$$

Theorem 3. *The area of the triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is*

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_2)$$

1. (HLG2018) $ABCD$ is a square with side length 100 and the midpoints of AB and DA are M, N respectively. CM and BN intersect at P. Find the length of DP.

We can plot this onto a graph with $D(0, 0)$ and $A(0, 100)$.

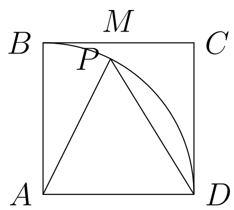


Solution.

$BN: \quad y = \frac{1}{2}x + 50. \quad CM: \quad y = -2x + 200$
$\Rightarrow P(60, 80)$
$\Rightarrow DP = \sqrt{(60-0)^2 + (80-0)^2} = 100.$

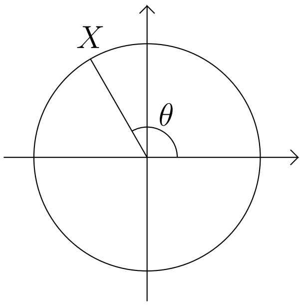
2. $ABCD$ is a square with side length 1 and M is the midpoint of BC. AM intersects quadrant ABD at P. Find the area of APD.

We can plot this onto a graph with $A(0, 0)$ and $B(0, 1)$.



Solution.
$AM: \quad y = 2x.$ arc $AB: \quad x^2 + y^2 = 1.$
$\Rightarrow P(\sqrt{5}/5, 2\sqrt{5}/5)$
$\Rightarrow [ADP] = \sqrt{5}/5.$

8.1.2 Trigonometry



If $X = (x, y)$ subtends an angle θ on the unit circle, then $x = \cos\theta, y = \sin\theta$. Using this definition, many identities such as $\cos(-x) = \cos x$, $\sin(90°-x) = \cos x$ do not need to be stubbornly memorised. However, a lot more formulae are useful to be memorised:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$

$2 \sin A \cos B = \sin(A + B) + \cos(A - B).$

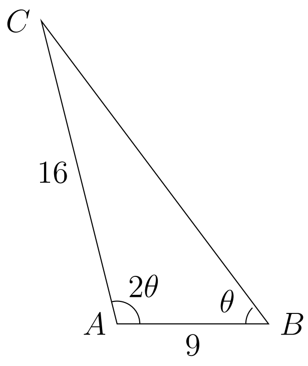
$2 \sin A \sin B = -\cos(A + B) + \cos(A - B).$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B).$

Theorem 4. *(Law of Sines) In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, where R is the circumradius of ABC.*

Theorem 5. *(Law of Cosines) In any triangle ABC, $c^2 = a^2 + b^2 - 2ab \cos C$.*

3. (HLG2017) ABC is a triangle such that $\angle A = 2\angle B$, $b = 16, c = 9$. Find a.



Solution.

$$\frac{9}{16} = \frac{\sin(180^\circ - 3\theta)}{\sin\theta} = \frac{\sin 3\theta}{\sin\theta} = 4\cos^2\theta - 1$$
$$\Rightarrow \cos\theta = \frac{5}{8} \Rightarrow \cos 2\theta = -\frac{7}{32}$$
$$\Rightarrow BC = \sqrt{9^2 + 16^2 - 2\cdot 9\cdot 16\cos 2\theta} = 20.$$

8.2 Synthetic Geometry

8.2.1 Similar and Congruent Triangles

Two triangles are similar ($ABC \sim XYZ$) if and only if their corresponding sides form a fixed ratio. These statements are equivalent to $ABC \sim XYZ$:

- $AB/XY = BC/YZ = AC/XZ$.
- $\angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z$.
- $\angle A = \angle X, AB/XY = AC/XZ$.

Next, two triangles are congruent ($ABC \cong XYZ$) if and only if their corresponding sides are equal. Furthermore, two triangles are congruent if and only if they are similar AND one pair of corresponding sides is equal.

8.2.2 Angle Chasing

Proposition 3. *The sum of interior angles and exterior angles of an n-gon are equal to $180^\circ(n-2)$ and 360° respectively.*

Proposition 4. *In a triangle ABC, ext.$\angle A = \angle B + \angle C$.*

Theorem 6. *$ABCD$ is cyclic $\Leftrightarrow$ ext.$\angle A = \angle C \Leftrightarrow \angle ABD = \angle ACD$.*

8.2.3 Length and Area Chasing

Proposition 5. *If BCD is a line and A is any point, $[ABC]/[ACD] = BC/CD$.*

Proposition 6. *If $\ell \parallel BC$ and A is any point on ℓ, then $[ABC]$ is constant.*

Theorem 7. *(Angle Bisector) If D lies on side BC of a triangle ABC, $\dfrac{AB}{AC} = \dfrac{DB}{DC}$.*

Theorem 8. *(Menelaus) Let ABC be a triangle and line ℓ intersects BC, AC, AB at X, Y, Z respectively. Then $\dfrac{BX}{XC}\cdot\dfrac{CY}{YA}\cdot\dfrac{AZ}{ZB} = 1$.*

8.2.4 Centres of a Triangle

There are a few important centres of triangles ABC that should be known:

Circumcentre: The intersection of perpendicular bisectors.
Incentre: The intersection of angle bisectors.
Orthocentre: The intersection of altitudes.
Centroid: The intersection of medians.

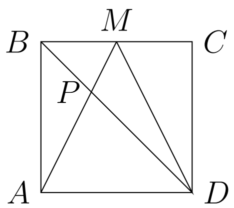
8.2.5 Circles

Let A, B, C, D be points on a circle. The following are properties concerning the shape formed by these points. Note that Theorem 10 is just Theorem 9 where $C = D$.

Theorem 9. *If AB and CD intersect at P (regardless of the point being inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.*

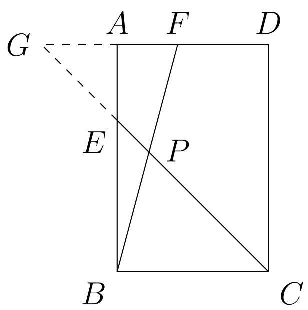
Theorem 10. *If AB intersect the tangent at C at Q, then $PA \cdot PB = PQ^2$.*

4. $ABCD$ is a square with area 20 and M is the midpoint of BC. AM intersects BD at P. Find the area of PMD.



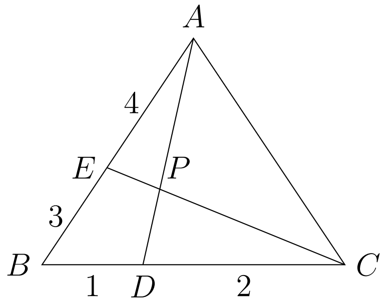
Solution.
$\because BPM \sim DAP,\ BP/PD = 1/2.$
$$\Rightarrow [PMD] = \frac{2}{3}[BMD] = \frac{2}{3} \times \frac{1}{4}[ABCD] = \frac{10}{3}.$$

5. $ABCD$ is a rectangle. E, F are on AB, AD such that $AE : EB = 1 : 2, AF : FD = 2 : 3$. BF and CE intersect at P. Find $BP : PF$.



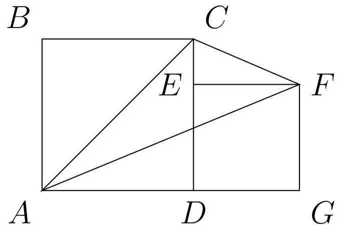
Solution.
$$\frac{BP}{PF} = \frac{BC}{GF} = \frac{BC}{GA + AF} = \frac{1}{\frac{GA}{BC} + \frac{AF}{BC}}$$
$$= \frac{1}{\frac{AE}{EB} + \frac{AF}{AD}} = \frac{1}{\frac{1}{2} + \frac{2}{5}} = \frac{10}{9}.$$

6. ABC is a triangle. D, E are on BC, AC such that $BD : DC = 1 : 2, AE : EC = 4 : 3$. AD and BE intersect at P. Find $DP : PA$ and $EP : PC$.



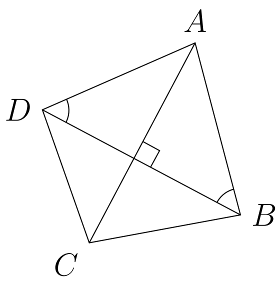
Solution.
$$\because \frac{BC}{CD}\cdot\frac{DP}{PA}\cdot\frac{AE}{EB}=1,\ \frac{DP}{PA}=\frac{1}{2}.$$
$$\because \frac{BA}{AE}\cdot\frac{EP}{PC}\cdot\frac{CD}{DB}=1,\ \frac{EP}{PC}=\frac{2}{7}.$$

7. $ABCD$ and $DEFG$ are non-overlapping squares such that E lies on segment CD. Find the value of $[CFA]/[ABCD]$.



Solution.
$$\because CA \parallel DF,\ [CAF]=[CAD].$$
$$\therefore \frac{[CFA]}{[ABCD]}=\frac{[CAD]}{[ABCD]}=\frac{1}{2}.$$

8. $\angle ABD = 55°, \angle ADB = 45°, \angle CDB = 35°, AC \perp BD$. Find $\angle DBC$.



Solution.
$$\because CA \parallel DF,\ [CAF]=[CAD].$$
$$\therefore \frac{[CFA]}{[ABCD]}=\frac{[CAD]}{[ABCD]}=\frac{1}{2}.$$

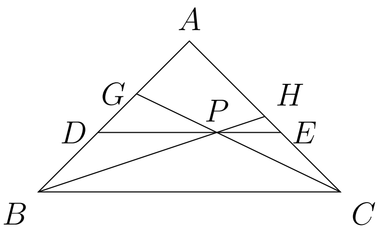
8.3 Problems

1. (CJR2018) In ΔABC, D lies on segment BC such that $BD : CD = 2 : 1$, $AB = 20$, $AC = 13$, $BC = 18$. Find AD^2.

2. (CJR2019) $ABCD$ is a trapezium with $AB \parallel CD$, $AB > CD$, $\angle DAB = 90°$, and M is the midpoint of BC. If $AB + CD + AD = 62$, $AM = 25$, find $[ABCD]$.

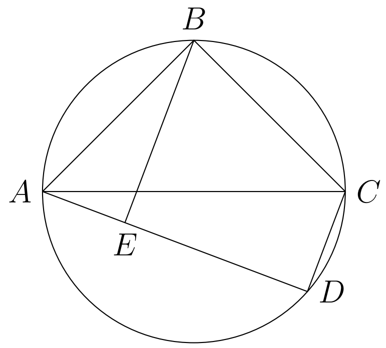
3. (HLG2018) In triangle ABC, E is a point on segment BC, D, F are points on AB such that $DE \parallel AC$, $EF \parallel CD$ and CD bisects $\angle ACB$. Given that $BC = 54, AC = 18$ and $AF = 14$, find the length of BF.

4. In the figure below, $AB = AC = 1$, $DE \parallel BC$ and $DE : BC = 2 : 3$. P is a point on the line segment DE. The extensions of the line CP and the line BP intersect AB and AC respectively at G and H. Find $\dfrac{120}{BG}+\dfrac{120}{CH}$.



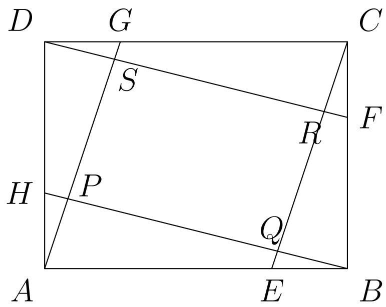
5. (HLG2018) $ABCD$ is a parallelogram. W, X, Y, Z are respectively the midpoints of AB, BC, CD and DA. P is a point on the line segment WX. Find $[PYZ]/[ABCD]$.

6. (HLG2016) In the figure below, AC is a diameter of the circle, BE is perpendicular to AD at E. Given that $AB = BC$, and $[ABCD] = 144$, find BE.

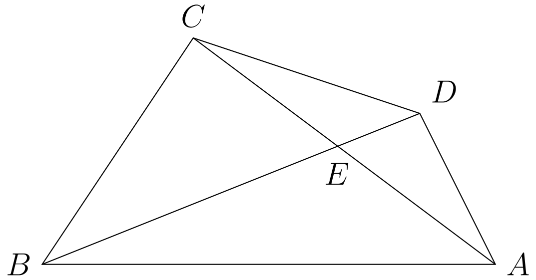


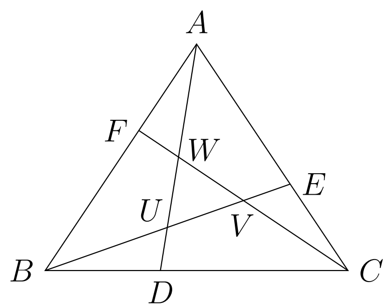
7. (HLG2013) Two circles C_1, C_2 with radii 9 and 7 respectively are externally tangent to each other at point C. The common tangent of the two circles touches C_1, C_2 at A, B respectively. Find the length of BC.

8. (CJR2020) In the figure below, $ABCD$ is a rectangle, $AG \parallel CE$, $BH \parallel DF$, $AH : AD = 7 : 85$, $BE : BA = 3 : 11$. Find $[PQRS]/[ABCD]$.



9. (HLG2018) $ABCD$ is a convex quadrilateral, AC and BD intersect at point E. Given that $AB = 12, BC = 8, CD = 7, DA = 5, \tan\angle AED = 2$, find $[ABCD]$.





10. (CJR2019) In ΔABC, D, E, F are respectively points on BC, CA and AB. $AD \cap BE = U$, $BE \cap CF = V$, and $CF \cap AD = W$. Given that $BU = UE$, $CV = VF$, $AW = WD$, and $[ABC] = 12$, find $[UVW]$ in the form $x - \sqrt{y}$.

Chapter 9

Inequalities

9.1 The Trivial Inequality

Before we encounter the advanced inequalities, we must first introduce the most rudimentary inequality in real numbers: If $x \in \mathbb{R}$, then

$$x^2 \geq 0.$$

9.2 The AM-GM and Cauchy-Schwarz Inequality

AM-GM and Cauchy are among the most useful inequalities used in Olympiads.

Theorem 1. *(AM-GM) If $a_1, \ldots, a_n$ are positive real numbers, then*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

with equality if and only if $a_1 = \cdots = a_n$.

Theorem 2. *(Cauchy) If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real numbers, then*

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2$$

with equality if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$.

Now, how do we apply these two inequalities in competition problems? Let's look at some examples.

1. Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Let's analyse the problem. We see a bunch of squares in the LHS, and some 'combined' terms in the RHS. That looks similar to AM-GM, but how can we have three terms produced? We can try backtracking to see what produces ab. Yes! $(a^2 + b^2)/2 \geq ab$. Using the same expressions for b, c and c, a, we immediately have our solution.

Solution 1. By AM-GM, we have

$$\begin{aligned}\frac{a^2+b^2}{2} &\geq ab\\ \frac{b^2+c^2}{2} &\geq bc\\ \frac{c^2+a^2}{2} &\geq ca.\end{aligned}$$

and adding these three inequalities together yields the desired result. □

Can we try Example 1.2.3 with Cauchy? Let's see. The RHS looks similar to Cauchy too, what if adapt Cauchy into this context? Aha!

Solution 2. By Cauchy,

$$\begin{aligned}(a^2+b^2+c^2)(b^2+c^2+a^2) &\geq (ab+bc+ca)^2\\ a^2+b^2+c^2 &\geq ab+bc+ca\end{aligned}$$

and we are done. □

2. (HLG2017) Find the minimal value of a^2+b^2 if $3a+b=1$.

Now we see squares and non-squares. Here is how we can do it.

Solution.

$$\begin{aligned}(a^2+b^2)(3^2+1^2) &\geq (3a+b)^2\\ a^2+b^2 &\geq \frac{1}{10}.\end{aligned}$$

and equality holds if $(a,b)=(3/10,1/10)$. □

Note that in order for a value to achieve its minimum, the equality must be able to be achieved. Otherwise, it might not be said as its minimum. Here is a trap example.

3. (BIMO2018) $a,b \in \mathbb{R}^+$ such that $a+b \leq 1$. Find the minimum value of $ab+\frac{1}{ab}$.

Hmm... If we apply AM-GM we are immediately done, aren't we? However, equality can only hold if $ab=1/(ab)$, but we can never find such a,b that satisfy $a+b\leq 1$ too! Here is a technique: We first guess what will happen, and then try and prove it. What's a nice guess? Probably $a=b=0.5$. In that case, $ab+1/(ab)=4.25$, where $ab=0.25$ and $1/(ab)=4$. But they are unequal! Let's split them into a bunch of equal terms then.

Solution.

$$\begin{aligned}ab+\frac{1}{ab} &= ab+\frac{1}{16ab}+\frac{1}{16ab}+\cdots+\frac{1}{16ab}\\ &\geq 2\sqrt{ab\cdot\frac{1}{16ab}}+\frac{15}{16ab}\\ &\geq \frac{1}{2}+\frac{15}{16((a+b)/2)^2}\\ &\geq \frac{1}{2}+\frac{15}{16(1/2)^2}=\frac{17}{4}.\end{aligned}$$

and equality can be achieved when $a=b=0.5$. □

4. (CJR2017) $a+b+c=6$, find the maximum value of a^3b^2c.

This is another tricky one. How can we suddenly get powers of 3 or 2 by using AM-GM? Suddenly something strikes: We can switch $a = a/3+a/3+a/3$ and $b = b/2+b/2$. In that case, three a's and two b's are multiplied together.

Solution.

$$\frac{\frac{a}{3}+\frac{a}{3}+\frac{a}{3}+\frac{b}{2}+\frac{b}{2}+c}{6} \geq \sqrt[6]{\frac{a}{3}\cdot\frac{a}{3}\cdot\frac{a}{3}\cdot\frac{b}{2}\cdot\frac{b}{2}\cdot c}$$
$$a^3b^2c \leq 108$$

and equality holds when $(a,b,c)=(3,2,1)$. □

9.3 Trigonometric Inequalities

In this section, you must be familiar with how trigonometric functions work. Normally, we can substitute a trigonometric parameter if a condition allows.

5. If $(x+5)^2+(y-12)^2=1$, find the range of x^2+y^2.

Solution. Substitute $(x,y)=(\sin\theta-5,\cos\theta+12)$. Then

$$\begin{aligned} x^2+y^2 &= -10\sin\theta+24\cos\theta+170 \\ &= 26\cos\left(\theta+\arctan\frac{5}{12}\right)+170 \in [144,196] \end{aligned}$$

and the equalities can clearly be achieved. □

6. If $x^2+4y^2=9$, find the range of xy.

Solution. Substitute $(x,y)=\left(3\sin\theta,\frac{3}{2}\cos\theta\right)$. Then

$$xy=\frac{9}{2}\sin\theta\cos\theta=\frac{9}{4}\sin 2\theta \in \left[-\frac{9}{4},\frac{9}{4}\right]$$

and the equalities can clearly be achieved. □

9.4 Geometric Inequalities

In this section you need to have some basic knowledge of 2D graphs. If we see something like the distance formula popping out in the problem, we might probably use this method.

Theorem 3. *(Point-to-point) The distance between two points* $(x_1,y_1),(x_2,y_2)$ *is*

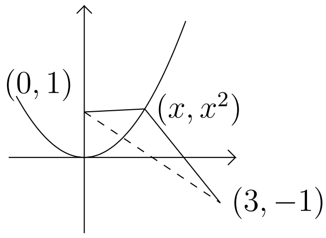
$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

Theorem 4. *(Point-to-line) The distance from* (x_1,y_1) *to line* $ax+by+c=0$ *is*

$$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|.$$

7. Find the minimum value of $\sqrt{x^4 - x^2 + 1} + \sqrt{x^4 + 3x^2 - 6x + 10}$.

This is a bunch of square roots, which immediately motivates us to use the geometry method with the point-to-point distance formula. We complete the square to obtain $\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{(x-3)^2 + (x^2+1)^2}$, hence this is just the distance $(0,1) \to (x, x^2) \to (3, -1)$.



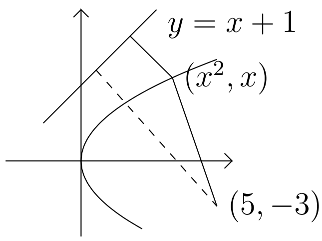
Solution.

$$\begin{aligned}
&\sqrt{x^4 - x^2 + 1} + \sqrt{x^4 + 3x^2 - 6x + 10}\\
&= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{(x-3)^2 + (x^2+1)^2}\\
&\geq \sqrt{3^2 + 2^2} = \sqrt{13}
\end{aligned}$$

and equality can be achieved clearly according to the diagram. □

8. Find the minimal value of $x^2 - x + 1 + \sqrt{2x^4 - 18x^2 + 12x + 68}$.

Here we have a non-squareroot part. We can associate it with the point-to-line distance formula. In the end, it is just the distance from $(5, -3) \to (x^2, x) \to y = x + 1$.



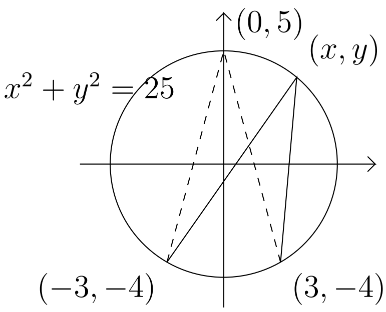
Solution.

$$\begin{aligned}
&x^2 - x + 1 + \sqrt{2x^4 - 18x^2 + 12x + 68}\\
&= \sqrt{2}\left(\left|\frac{x^2 - x + 1}{\sqrt{1^2 + 1^2}}\right| + \sqrt{(x^2-5)^2 + (x+3)^2}\right)\\
&\geq \sqrt{2}\left|\frac{5 - (-3) + 1}{\sqrt{2}}\right| = 8
\end{aligned}$$

and equality can be achieved clearly according to the diagram. □

9. If $x^2 + y^2 = 25$, find the maximal value of $\sqrt{8y - 6x + 50} + \sqrt{8y + 6x + 50}$.

There are no squares inside the square root, but we can always use the condition to produce some squares. In fact, another point-to-point distance formula arises.



Solution.

$$\begin{aligned}
&\sqrt{8y - 6x + 50} + \sqrt{8y + 6x + 50}\\
&= \sqrt{x^2 + y^2 + 8y - 6x + 25}\\
&\quad + \sqrt{x^2 + y^2 + 8y + 6x + 25}\\
&= \sqrt{(x-3)^2 + (y+4)^2}\\
&\quad + \sqrt{(x+3)^2 + (y+4)^2}\\
&\leq 6\sqrt{10}
\end{aligned}$$

and equality can be achieved clearly according to the diagram. □

9.5 Problems

Assume a, b, c are positive reals.

1. Prove $\dfrac{a+b}{2} \geq \sqrt{ab}$.

2. Prove $2x^2 + 3y^2 - 2xy - 2x - 4y + 3 \geq 0$.

3. Prove $(a+b)(b+c)(c+a) \geq 8abc$.

4. a, b, c are sides of a triangle. Prove $abc \geq (a+b-c)(b+c-a)(c+a-b)$.

5. (CJR2019) Find the minimal value of $303 - \dfrac{200x^2}{x^4+16}$.

6. (CJR2019) Find the minimal value of $\sqrt{2x-1} + \sqrt{243-2x}$.

7. (CJR2017) $ABCD$ is a rectangle inscribed in the ellipse $x^2/9 + y^2/4 = 1$ such that its sides are parallel to the axes. Find the largest possible area of rectangle $ABCD$.

8. (HLG2018) Find the minimal value of $\dfrac{x^2}{2} + \dfrac{162}{x^2}$.

9. (HLG2017) If $a, b, c \in \mathbb{R}^+$, find the minimal value of $(a+b+c)\left(\dfrac{1}{a} + \dfrac{9}{b} + \dfrac{25}{c}\right)$.

10. (HLG2017) If x, y are positive numbers such that $2x + 3y = 2016$ and xy has the maximum value, find the value of $x - y$.

11. (HLG2015) If x, y are real numbers, find the minimum possible value of $\sqrt{(x-1)^2 + (y+3)^2} + \sqrt{(x+11)^2 + (y-2)^2}$.

12. (HLG2016) If x, y, z are positive real numbers such that $xy + 2yz + 3zx = 7$, find the smallest possible value of $4x^2 + 3y^2 + 5z^2$.

13. (HLG2014) Given that $x^2 + y^2 = 8x - 6y + 144$, find the smallest value of $x^2 + y^2$.

14. Find the maximal value of $\dfrac{\sqrt{1+x^2+x^4} - \sqrt{1+x^4}}{x}$ $(x \neq 0)$.

15. Find the range of $\sqrt{x+1} + \sqrt{4-2x}$.

Chapter 10

Diophantine Equations

Diophantine was a Greek Mathematician in the 2nd Century AD. A Diophantine equation is an equation solving for (positive) integer solutions. Unfortunately, these equations are sometimes very hard to solve as equations could appear in unpredictable ways (see Fermat's Last Theorem!). In this chapter, we will be looking at some famous techniques that might help you in many contests.

10.1 Linear Diophantine

These are the simplest of them all. Methods include **parameterising** and **tabulating**.

> **Theorem 1.** *If $ax + by = c$ and $\gcd(a, b) = 1$, we have the parametric solutions*
>
> $$\begin{cases} x = x_0 + bt \\ y = y_0 - at \end{cases} \quad (t \in \mathbb{Z})$$
>
> *where (x_0, y_0) is any solution of the problem.*

1. How many pairs of positive integers (x, y) satisfy $6x + 7y = 1001$?

Solution. $(x, y) = (0, 143)$ is one of the solutions. Hence

$$\begin{cases} x = 7t \\ y = 143 - 6t \end{cases} \quad (t \in \mathbb{Z})$$

Since $7t, 143 - 6t > 0 \Rightarrow 1 \leq t \leq 23$, there are 23 positive integer solutions. □

2. How many pairs of positive integers (x, y) satisfy $6x + 8y = 1001$?

Solution. LHS is even but RHS is not. No integer solutions. □

3. How many triples of positive integers (x, y, z) satisfy $2x + 4y + 5z = 120$?

Solution. First, $z = 2k$. Substituting into the equation yields

$$\begin{aligned} x + 2y + 5k &= 60 \\ 2y + 5k &= 60 - x \end{aligned}$$

Hence we are finding the number of (y, k) such that $2y + 5k \leq 59$. We can tabulate:

k	$y \le$	k	$y \le$
1	27	7	12
2	24	8	9
3	22	9	7
4	19	10	4
5	17	11	2
6	14	12	−1

and the number of solutions is $27+24+22+19+17+14+12+9+7+4+2 = 157$. $\square$

10.2 Simon's Favourite Factoring Trick (SFFT)

This is extremely important! When we see expressions like $ab + 6a - 7b + 10$, SFFT is a must-use! (sometimes, you'll see people using the term 'Completing the Rectangle')

4. Solve in positive integers, $xy + 2x - 4y - 12 = 0$.

Solution. We use SFFT, look at how it works:

$$\begin{aligned} xy + 2x - 4y - 12 &= 0 \\ xy + 2x - 4y &= 12 \\ (x-4)(y+2) &= 12 - 8 \\ (x-4)(y+2) &= 4. \end{aligned}$$

But $4 = 1 \times 4 = 2 \times 2 = 4 \times 1$, comparing yields $(x, y) = (5, 2)$ only. $\square$

5. Solve in positive integers, $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$.

Solution. Rearrange the equation, it is another SFFT expression again!

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} &= \frac{1}{2} \\ ab - 2a - 2b &= 0 \\ (a-2)(b-2) &= 4 \end{aligned}$$

But $4 = 1 \times 4 = 2 \times 2 = 4 \times 1$, hence $(a, b) = (3, 6), (4, 4), (6, 3)$ only. $\square$

6. Solve in positive integers, $4ab + a + b - 31 = 0$.

Oh dear, there is coefficient in front of ab. What do we do?

Solution.

$$\begin{aligned} 4ab + a + b &= 31 \\ ab + \frac{1}{4}a + \frac{1}{4}b &= \frac{31}{4} \\ \left(a + \frac{1}{4}\right)\left(b + \frac{1}{4}\right) &= \frac{31}{4} + \frac{1}{16} \\ (4a+1)(4b+1) &= 125 \end{aligned}$$

$125 = 1 \times 125 = 5 \times 25 = 25 \times 5 = 125 \times 1$ yields $(a, b) = (1, 6), (6, 1)$ only. $\square$

10.3 Bounding

When it comes to non-linear Diophantine equations, things might become very complicated. One of the most important avenues is **bounding**.

7. (HLG2017) $a, b, c \in \mathbb{N}$. Solve $a + b + c = \dfrac{abc}{2}$.

Solution. First we see that the order of a, b, c does not matter, hence without loss of generality (WLOG) we let $a \le b \le c$. In such a case, since $3a \le a + b + c \le 3c$,

$$\begin{aligned} 3a \le \frac{abc}{2} &\le 3c \\ ab \le 6 &\le bc \end{aligned}$$

$bc \ge 6$ doesn't tell much. $ab \le 6$ is the key: It means $a^2 \le 6 \Rightarrow a = 1, 2$ only.

$$\begin{aligned} 1 + b + c &= \frac{bc}{2} & 2 + b + c &= bc \\ bc - 2b - 2c - 2 &= 0 & bc - b - c - 2 &= 0 \\ \therefore (a, b, c) &= (1, 3, 8), (1, 4, 5) & (a, b, c) &= (2, 2, 4) \qquad \text{(SFFT)} \end{aligned}$$

Thus, $(a, b, c) = (1, 3, 8), (1, 4, 5), (2, 2, 4)$ and its **permutations** (important!). $\square$

8. How many $(a, b, c) \in \mathbb{N}^3$ satisfy $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 1$?

Solution. WLOG $a \le b \le c$, then $\dfrac{3}{a} \ge 1$, but $a \ne 1 \Rightarrow a = 2, 3$.

$$\begin{aligned} \frac{1}{b} + \frac{1}{c} &= \frac{1}{2} & \frac{1}{b} + \frac{1}{c} &= \frac{2}{3} \\ bc - 2b - 2c &= 0 & 2bc - 3b - 3c &= 0 \\ (b - 2)(c - 2) &= 4 & (2b - 3)(2c - 3) &= 9 \\ (a, b, c) &= (2, 3, 6), (2, 4, 4) & (a, b, c) &= (3, 2, 6) \text{ (rejected)} \end{aligned}$$

and hence there are $3! + \dfrac{3!}{2!} = 9$ solutions. $\square$

10.4 Perfect Squares

9. Find the least positive integer n such that $50500n$ is a perfect square.

Solution. Since $50500 = 2^2 \times 5^3 \times 101$, the only primes to compensate for a perfect square is 5 and 101. Hence $n = 505k^2$ $(k \in \mathbb{Z}) \Rightarrow n_{\min} = 505$. $\square$

10. Solve in integers, $\sqrt{x} + \sqrt{y} = \sqrt{24}$.

How can we handle all the square roots? Some squaring must be done.

Solution.

$$\begin{aligned}\sqrt{x} &= \sqrt{2^3 \times 3} - \sqrt{y} \\ x &= 5 \times 401 - 2\sqrt{2^3 \times 3 \times y} + y \\ \therefore y &= 6y_1^2 \quad (\text{similarly } x = 6x_1^2)\end{aligned}$$

Substituting back, we have

$$\begin{aligned}\sqrt{6x_1^2} + \sqrt{6y_1^2} &= \sqrt{24} \\ x_1 + y_1 &= 2 \\ \therefore (x_1, y_1) &= (2,0), (1,1), (0,2).\end{aligned}$$

This immediately gives $(x, y) = (24, 0), (6, 6), (0, 24)$. □

11. Find the number of integer solutions for $x(x+1) + 1 = y^2$.

This problem can be handled by **completing the square**.

Solution. First $x(x+1)$ must be even, hence $y = 2y_1 + 1$,

$$\begin{aligned}4x(x+1) + 4 &= 4y^2 \\ (2x+1)^2 + 3 &= (2y)^2.\end{aligned}$$

Are there perfect squares that differ by 3? Only 1 and 4. Hence $(x, y) = (0, 1)$. □

12. Find the number of integer solutions for $x(x+1) = y^2$.

Solution 1. Complete the square again.

$$\begin{aligned}4x(x+1) &= 4y^2 \\ (2x+1)^2 - 1 &= (2y)^2.\end{aligned}$$

and hence $(x, y) = (0, 0)$. □

Solution 2. Since $\gcd(x, x+1) = 1$, they must both be perfect squares (think in terms of prime factorisation). In other words, x and $x+1$ are perfect squares that differ by 1. The only possible pair is $(0, 1)$ which means $x = 0 \Rightarrow (x, y) = (0, 0)$. □

10.5 Factorising

Factorising is also very important in solving non-linear Diophantine equations.

13. Solve in positive integers, $2x^2 + 5y^2 = 11(xy - 11)$.

Solution. This is factorised into $(2x - y)(x - 5y) = -121$. Hence we are solving

$$\begin{cases} 2x - y = -121, -11, -1, 1, 11, 121 \\ x - 5y = 1, 11, 121, -121, -11, -1 \end{cases}$$

which leads to $(x, y) = (14, 27)$. □

14. Solve in positive integers, $x^4 + 4 = p$ where p is prime.

Solution. Factorising,

$$p = x^4 + 4 = (x^2 + 2)^2 - 4x^2 = (x^2 + 2 + 2x)(x^2 + 2 - 2x).$$

However, p is prime, hence $x^2 + 2 - 2x = 1 \Rightarrow x = 1 \Rightarrow (x, p) = (1, 5)$. $\square$

15. Solve in positive integers, $x^2 - 1 = 4y^k$ where $k \geq 2$.

Solution. Immediately we know $x = 2x_1 + 1$, hence

$$\begin{aligned} 4x_1^2 + 4x_1 &= 4y^k \\ x_1(x_1 + 1) &= y^k. \end{aligned}$$

Since $\gcd(x_1, x_1 + 1) = 1$, $(x_1, x_1 + 1) = (a^k, b^k)$. Hence $x_1 = 0 \Rightarrow (x, y) = (1, 0)$. $\square$

16. Find the largest k such that 3^{11} can be written as a sum of k consecutive integers.

Solution. Let $(n + 1) + \cdots + (n + k) = 3^{11}$, then

$$k(2n + k + 1) = 2 \times 3^{11}$$

Since $k < 2n + k + 1$, we want to split 2×3^{11} into the closest numbers, i.e. 2×3^5 and 3^6. Indeed, there is such a solution $(k, n) = (2 \times 3^5, (3^5 - 1)/2)$. Therefore $k_{\max} = 486$. $\square$

10.6 Pythagorean Triples

By Pythagoras Theorem, right-angled triangles satisfy $a^2 + b^2 = c^2$. However, there is a parameterised form of (a, b, c) that is sometimes extremely useful:

Theorem 2. *If a, b, c satisfies Pythagoras's Equation, then*

$$\begin{cases} a = (x^2 - y^2)k \\ b = (2xy)k \\ c = (x^2 + y^2)k \end{cases} \quad \text{or} \quad \begin{cases} a = (2xy)k \\ b = (x^2 - y^2)k \\ c = (x^2 + y^2)k \end{cases}$$

17. (OMK2015) Find all positive integer triples (a, b, c) such that they form a right triangle with equal perimeter and area.

Solution. Let $a = (x^2 - y^2)k, b = (2xy)k, c = (x^2 + y^2)k$ and substitute into $\frac{ab}{2} = a + b + c$,

$$\begin{aligned} (x^2 - y^2)(2xy)k^2 &= 2(x^2 - y^2)k + 2(2xy)k + 2(x^2 + y^2)k \\ (x - y)yk &= 2 \\ \therefore (x - y, y, k) &= (2, 1, 1), (1, 2, 1), (1, 1, 2) \end{aligned}$$

which implies $(a, b, c) = (6, 8, 10), (5, 12, 13)$ and its permutations. $\square$

10.7 Problems

1. (HLG2016) Find the number of $(x, y) \in \mathbb{N}^2$ that satisfy $3x + 4y = 101$.
2. (HLG2018) Prime numbers p, q satisfy $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{86}{534^2 - 533^2}$. Find $p + q$.
3. (HLG2018) How many pairs of $(x, y) \in \mathbb{N}^2$ satisfy $6x + 14y = 2018$?
4. (HLG2018) How many pairs of integers (x, y) satisfy $x^2 + y^2 = 3(x + y) + xy$?
5. (HLG2018) If p, q and r are primes with $pqr = 11(p+q+r)$, find the smallest value of $p + q + r$.
6. (HLG2017) How many triples of positive integers (x, y, z) satisfy the equation $19x + 20y + 21z = 399$?
7. (HLG2015) Given that a is a positive integer such that $n = 2a^2 - 7a + 3$ is a prime number, find the value of n.
8. (CJR2018) Let a, b, c be pairwise relatively prime positive integers that satisfy
$$\frac{8a}{b + 2c} = \frac{4b}{c + 3a} = \frac{3c}{5a + 3b}.$$
Find the value of b.
9. (Slovenia) Solve in integers, $\sqrt{x} + \sqrt{y} = \sqrt{2004}$.
10. (HLG2018) How many triples of positive integers (x, y, z) satisfy
$$\begin{cases} xy + 3xz = 144 \\ 2xz - yz = 63? \end{cases}$$
Bonus: How about just integers?
11. (CJR2018) How many $(a, b, c, d) \in \mathbb{N}^4$ satisfy $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} + \dfrac{1}{d} = 1$?
12. Find the number of elements, in terms of n, in the set
$$\{\{a, b, c\} \subset \mathbb{N}_0 : a + b + c = 2n \quad (n \in \mathbb{N}_0)\}$$
Note that a, b, c are pairwise distinct and unordered (i.e. $\{1, 2, 3\} = \{1, 3, 2\}$)

Chapter 11

Counting

11.1 Bijections

Say we have a theatre with 300 seats, and we want to know how many people there are in the audience when all seats are seated. The obvious answer is there would be 300 people, and this logic involves what we call **bijections** in mathematics.

We observe a relation between each person and each seat. Each seat exactly accommodates one person whereas each person seats on one chair. That means, there is a **one-to-one correspondence (bijection)** from the set of seats and the set of people. Therefore, **if there exists a bijection between two finite sets A and B, then** $|A| = |B|$. However, one should remember that in order for a correspondence to be a bijection, the correspondence must be bidirectional, i.e. each element in A corresponds to one element in B AND each element in B corresponds to one element in A.

1. Are these two values the same?

(a) The number of ways to choose three numbers $a < b < c$ from $[n] = \{1, 2, \ldots, n\}$ such that $a + c = 2b$.

(b) The number of ways to choose two distinct even numbers plus the number of ways to choose two distinct odd numbers from $[n]$.

Solution. Yes. ((a) $\Rightarrow$ (b)) Since $a + c$ is an even number, either a, c are both even or both odd, satisfying (b). ((b) $\Rightarrow$ (a)) If two distinct numbers a, c with equal parity (even/oddness) are chosen, then they must sum up to an even number, hence $b = \frac{a+c}{2}$ exists uniquely, therefore $a < b < c$ is chosen, satisfying (a). $\square$

Note that this is actually a big deal! The value for (a) has a weird condition, but we know that the value for (a) is just equal to the answer for (b), hence it only suffices to solve (b) (which is easier)! Let's see another example:

2. (OMK2014) A subset of $S = \{1, 2, \ldots, 2014\}$ is called *good* if

(a) It does not contain both 1 and 2 (it can contain a 1 or 2, but not both)

(b) The sum of elements is divisible by 3.

E.g. $\{1, 23, 456, 789\}$ is good. Find the number of good subsets of S.

The condition here is very bizarre. However, it turns out to be not hard if we 'biject' this problem into something else.

Solution. We prove that the number of good subsets of S is exactly equal to the number of subsets (no condition!) of $S' = \{3, 4, \ldots, 2014\}$. We prove in both directions,

(i) Let A be any subset of S'. Then if the sum of elements in A is divisible by 3, then immediately A is a good subset. If the sum of elements in A is 1 (mod 3), then $A \cup \{2\}$ is a good subset. If the sum of elements in A is 2 (mod 3), then $A \cup \{1\}$ is a good subset. Hence any subset of S' corresponds to a good subset of S.

(ii) Let A be a good subset of S, then $S \setminus \{1, 2\}$ is a subset of S'. Hence any good subset of S corresponds to a subset of S'.

Therefore, the answer is just the number of subsets of S', which is just 2^{2011}. □

11.2 The Choose Function

The notation of 'n choose k' indicates the number of ways to choose r **unordered** distinct elements from n distinct elements. This number is denoted as any of the following:

$$\binom{n}{k}, \quad C^n_k, \quad {}_nC_k, \quad C_{n,k}$$

In this book, we will use the first notation listed above. Now let's compute it. Say the n elements are $a_1, a_2, \ldots, a_n$. We can choose the first element in n ways, leaving $n-1$ elements: Then the second element can be chosen in $n-1$ ways, leaving $n-2$ elements and so on. In the end, we will have $n(n-1)\ldots(n-k+1)$ ways to choose a k-element tuple. However, there are $k!$ permutations for each selection of k elements. As such,

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

3. Compute the number of ways to choose 4 kids from 10 kids.

Solution. $\binom{10}{4} = \dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$ □

4. Compute the number of ways to choose 2 spades from a deck of 52 cards.

Solution. $\binom{13}{2} = \dfrac{13 \times 12}{2 \times 1} = 78.$ □

5. How many ways are there to arrange the letters A, B, C, C, C, D?

We use the idea of how the choose function was invented. Treating the three C's as separate variables, there are $6!$ ways to arrange them. However, this time the overall order does matter, except the three C's, which has $3!$ ways to arrange – hence $3!$ duplicates for the same arrangement. Therefore, there are $6!/3!$ ways.

Solution. $\dfrac{6!}{3!} = 120.$ □

6. How many ways are there to arrange the letters A, B, C, C, C, D, D?

By the same idea as above, there are 3!2! ways to rearrange the duplicated parts.

<u>*Solution*</u>. $\dfrac{7!}{3!2!} = 420.$ $\square$

7. How many ways are there to choose 3 letters from $A, B, C, C, C, D, E, F, G$?

The choose function does not directly help here as we can choose more than one C. Hence, we divide into cases.

<u>*Solution*</u>.

Case	Ways
At most one C	$\binom{7}{3} = 35$
Two C's	$\binom{6}{1} = 6$
Three C's	1

$\therefore 35 + 6 + 1 = 42.$ $\square$

8. How many ways are there to form two groups of k and $n-k$ people from n people?

Choose k people first, the remaining $n-k$ will automatically become a group.

<u>*Solution*</u>. $\dbinom{n}{k}.$ $\square$

9. How many ways are there to form two groups of k and $n-k$ people from n people and put them into two classes?

Similar to Example 8, after which we have 2 ways to arrange them into classes.

<u>*Solution*</u>. $2\dbinom{n}{k}.$ $\square$

10. What is the coefficient of x^6 in $(2+x)^{10}$?

Expanding $(2+x)^{10}$, we see that in order to have six x's multiplied together, it must multiply together with the remaining four 2's. There are also $\dbinom{10}{6}$ ways to choose such a configuration. Summing up, we get the coefficient of x^6.

<u>*Solution*</u>. $2^4\dbinom{10}{6} = 3360.$ $\square$

11. Find the number of positive integer triples (a, b, c) such that $a + b + c = 100$?

This involves a nice trick. Assume we have 100 balls lined up in a row. We choose 2 separators two separate them into three parts, then we can associate this with the solution (a, b, c). There are 99 spaces to place 2 separators. Hence,

<u>*Solution*</u>. $\dbinom{99}{2} = 4851.$ $\square$

12. Find the number of non-negative integer triples (a, b, c) such that $a + b + c = 100$?

Can we use the separator trick now? Unfortunately not, since this time two separators can be placed in the same space (when 0 occurs). So what do we do?

<u>*Solution*</u>. $a + b + c = 100 \Leftrightarrow (a+1) + (b+1) + (c+1) = 103$. In this case, the terms $a+1, b+1, c+1$ become positive integers. Hence again $\dbinom{102}{2} = 5151.$ $\square$

11.3 Properties of the Choose Function

Choosing k wanted items bijects to removing $n-k$ unwanted items:

Property 1. $\binom{n}{k} = \binom{n}{n-k}$.

The total number of ways to choose any number of objects must be 2^n since each element can either be chosen or not chosen (2 ways):

Property 2. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

To choose k elements, you can either first choose a_1 then choose $k-1$ elements from the remaining $n-1$ elements; or you can refuse a_1 and choose all k elements from the remaining $n-1$ elements:

Property 3. $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

(Slightly harder) To choose m students from a selected set of k students from n students, we can first choose the m students directly from all of the n students, then count the multiplicity. Each such move will repeat itself for $\binom{n-m}{k-m}$ times:

Property 4. $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

Let's look at some examples:

13. Compute $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$.

There is one gap between each term, so how do we deal with this? We can use Property 3 here to 'fill in the gaps'.

Solution.

$$
\begin{aligned}
&\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots \\
&= \binom{n-1}{0} + \left[\binom{n-1}{1} + \binom{n-1}{2}\right] + \left[\binom{n-1}{3} + \binom{n-1}{4}\right] + \cdots \\
&= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4} + \cdots \\
&= 2^{n-1}.
\end{aligned}
$$

□

14. Compute $1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots$.

How can we handle the coefficients? There is a way to remove them, by reducing the choose function:

Solution.

$$\begin{aligned}
&1\cdot\binom{n}{1}+2\cdot\binom{n}{2}+3\cdot\binom{n}{3}+\cdots\\
&=1\cdot\frac{n}{1}\binom{n-1}{0}+2\cdot\frac{n}{2}\binom{n-1}{1}+3\cdot\frac{n}{3}\binom{n-1}{2}+\cdots\\
&=n\binom{n-1}{0}+n\binom{n-1}{1}+n\binom{n-1}{2}+\cdots\\
&=n2^{n-1}.
\end{aligned}$$

□

11.4 Binomial Theorem

The binomial theorem is a theorem involving the expansion of expressions such as $(a+b)^n$.

Theorem 1. $(a+b)^n=\displaystyle\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$

Using the binomial theorem, many combinatorial sums can be tackled accordingly, especially when $a=1$.

15. Simplify $\binom{n}{0}+7\binom{n}{1}+7^2\binom{n}{2}+\cdots+7^n\binom{n}{n}$.

The coefficients are powers, which motivates us to use the binomial theorem!

Solution.

$$\binom{n}{0}+7\binom{n}{1}+7^2\binom{n}{2}\cdots+7^n\binom{n}{n}=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}7^k=(1+7)^n=8^n.$$

□

16. Compute $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}$.

Solution.

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=\sum_{k=0}^{n}\binom{n}{k}(-1)^k=(1-1)^n=0.$$

□

11.5 De Morgan's Law

Theorem 2. *If $A_1, \dots, A_n$ are sets, then*

$$(A_1 \cap \cdots \cap A_n)' = A_1' \cup \cdots \cup A_n'$$
$$(A_1 \cup \cdots \cup A_n)' = A_1' \cap \cdots \cap A_n'$$

where S' denotes the complement of S, i.e. the set of elements not in S.

11.6 Principle of Inclusion and Exclusion

This regards the cardinality of a union of sets:

Theorem 3. *(P.I.E) If $A_1, \dots, A_n$ are sets, then*

$$\begin{aligned}|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\ - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|\end{aligned}$$

or in concised notation,

$$\left| \bigcup_{k=1}^{n} A_i \right| = \sum_{\varnothing \ne I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|$$

For example, for the cases $n = 2, 3, 4$,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| \\ &\quad - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| \\ &\quad + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|\end{aligned}$$

Often, De Morgan's Law works together with P.I.E. Let's apply them.

17. (CJR2018) 3 students, each gives a gift to their teacher. The teacher then randomly distributes the three gifts to these three students, each student one. Find the probability that no student gets back his own gift.

Solution. There are $3! = 6$ ways to distribute the gifts. Let A_i be the event in which the i-th student gets his own gift (note that there are no restrictions for the other students). Hence A_i' is the event in which the i-th student does not get his own gift. Therefore the number of desired ways is

$$
\begin{aligned}
&|A_1' \cap A_2' \cap A_3'| \\
&= |(A_1 \cup A_2 \cup A_3)'| \\
&= 6 - |A_1 \cup A_2 \cup A_3| \\
&= 6 - (|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|) \\
&= 6 - (2! + 2! + 2! - 1! - 1! - 1! + 0!) = 2.
\end{aligned}
$$

and thus the probability is $\frac{2}{6} = \frac{1}{3}$. □

18. (Derangements) Define D_n as the number of ways to permute n elements such that none of the elements gets put back into its original place. Prove that

$$D_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right).$$

Solution. There are $n! = 6$ ways to permute the elements. Let A_i be the event in which the i-th element is fixed at its place. Hence A_i' is the event in which the i-th element does not get put back to its original place. Therefore the number of desired ways is

$$|A_1' \cap \cdots \cap A_n'| = |(A_1 \cup \cdots \cup A_n)'| = n! - |A_1 \cup \cdots \cup A_n|$$

but

$$
\begin{aligned}
|A_1 \cup \cdots \cup A_n| &= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
&\quad - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| \\
&= \sum_{1 \le i \le n} (n-1)! - \sum_{1 \le i_1 < i_2 \le n} (n-2)! + \sum_{1 \le i_1 < i_2 < i_3 \le n} (n-3)! \\
&\quad - \cdots + (-1)^{n+1} 0! \\
&= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{n+1}\binom{n}{n} 0! \\
&= \frac{n!}{1!(n-1)!} \cdot (n-1)! - \frac{n!}{2!(n-2)!} \cdot (n-2)! + \frac{n!}{3!(n-3)!} \cdot (n-3)! \\
&\quad - \cdots + (-1)^{n+1} \frac{n!}{n!0!} \cdot 0! \\
&= n!\left(\frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right)
\end{aligned}
$$

and thus

$$D_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right).$$

□

11.7 Problems

1. (CJR2017) How many different ways can the letters in the English word COMPETITION be rearranged in such a way that the first five letters are all vowels?

2. (HLG2017) Given that $S = \{1, 2, 3, \ldots, 199, 200\}$ and $H = \{(a, b, c) \mid a, b, c \in S, a < c, a + c = 2b\}$. Find the number of elements of H.

3. (CJR2019) Simplify $\binom{37}{1} - \binom{37}{2} + \binom{37}{3} - \binom{37}{4} + \cdots - \binom{37}{18}$.

4. (CJR2017) How many positive integer solutions satisfy $x_1 + x_2 + x_3 = 32$?

5. (HLG2019) How many 5-digit positive integers $\overline{x_1x_2x_3x_4x_5}$ are there such that $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$?

6. (HLG2018) Let $A = \{1, 2, 3, \cdots, 30\}$, $P = \{a_1, a_2, a_3\}$ is a subset of A such that $a_1 + 6 \leq a_2 + 4 \leq a_3$. How many such subsets does the set A have?

7. (CJR2020) A school has four clubs, A, B, C, D, whose members are students in this school. Every two clubs have 227 common members. Every three clubs have 117 common members. There are exactly 17 students that join all four clubs. At least how many students does club A have?

8. (China 1998) How many ways are there to choose three distinct elements from $\{0, \ldots, 9\}$ such that they sum up to an even number not less than 10?

9. How many integer solutions are there to
$$x_1 + \cdots + x_6 = 20, \quad 0 \leq x_i \leq 8?$$

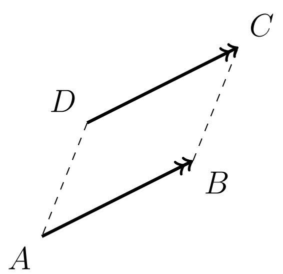
10. Find the value of
$$\sum_{k=p}^{n} (-1)^k \binom{n}{k} \binom{k}{p}$$
in terms of n and p.

Chapter 12

* Vector Geometry

12.1 Vectors in n-dimensional space

A vector is a mathematical object that has **magnitude** and **direction**, but **no location**. In n-dimensional space (we will only cover $n = 2$ and 3), we can associate two points A and B with two choices of vectors: the vector $\overrightarrow{AB}$ (from A to B), or the vector $\overrightarrow{BA}$ (from B to A). However, the endpoints of vector $\overrightarrow{AB}$ does not necessarily be at A and B, as a vector does not carry 'location' in its meaning. For example, if $ABCD$ is a parallelogram (figure below), then $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are exactly the same vector, so we say $\overrightarrow{AB} = \overrightarrow{DC}$.



In this chapter, we assume that the reader has basic knowledge of vector addition and scaling. In this book, we will use the following few notations and definitions:

Definition 1. The vector from the origin to the point $(a_1, \ldots, a_n)$ is denoted as $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, or sometimes for neatness purposes, $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}^T$.

Definition 2. The **length** of a vector $\vec{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is $|\vec{v}| = \sqrt{a_1^2 + \cdots + a_n^2}$.

Please note that $|\vec{a} + \vec{b}|$ is not necessarily equal to $|\vec{a}| + |\vec{b}|$, but:

Theorem 1. *(Triangle Inequality)* $|\vec{a}| + |\vec{b}| \geq |\vec{a} + \vec{b}|$, *with equality iff* $\vec{a} \parallel \vec{b}$.

We also introduce an operation called the **dot product**:

Definition 3. The **dot product** of two vectors $\vec{a} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}^T$ and $\vec{b} = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}^T$ is the scalar

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n.$$

We notice a few obvious properties:

Property 1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

Property 2. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

Property 3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

Using these properties we find a nice relation:

Theorem 2. *$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.*

Corollary. $\vec{u} \perp \vec{v} \Leftrightarrow \vec{u} \cdot \vec{v} = 0$.

1. Let $O(0,0), A(3,4), B(1,-3)$. Find $\sin\angle AOB$.

Solution. Let $\theta = \angle AOB$ which is the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\cos\theta = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|} = \frac{(3)(1) + (4)(-3)}{\sqrt{3^2 + 4^2}\sqrt{1^2 + (-3)^2}} = -\frac{9}{5\sqrt{10}}$$

and thus $\sin\theta = \sqrt{1 - \left(-\frac{9}{5\sqrt{10}}\right)^2} = \frac{13}{50}\sqrt{10}$. □

2. Let $\vec{u}, \vec{v}$ be perpendicular vectors such that $|\vec{u}| = 3, |\vec{v}| = 1$. Find $(3\vec{u} + 2\vec{v}) \cdot (\vec{u} + 5\vec{v})$.

Solution. $(3\vec{u} + 2\vec{v}) \cdot (\vec{u} + 5\vec{v}) = 3|\vec{u}|^2 + 15\vec{u} \cdot \vec{v} + 2\vec{v} \cdot \vec{u} + 10|\vec{v}|^2 = 3(3^2) + 10(1^2) = 37$. □

12.2 Linear Independence

Definition 4. A **linear combination** of vectors $\vec{v}_1, \ldots, \vec{v}_n$ is a sum

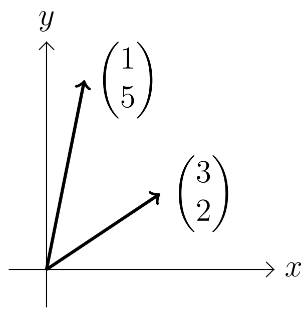
$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

for some real numbers (scalars) $c_1, \ldots, c_n$.

E.g. $2\vec{u} - 7\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

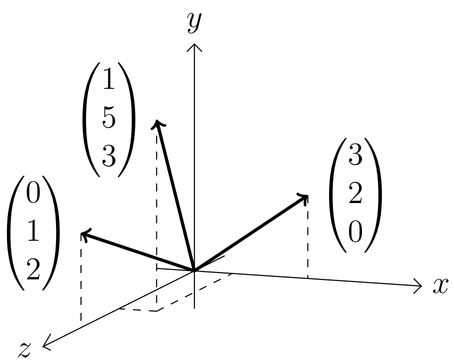
Definition 5. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is said to be **linearly independent** if none of the vectors can be written as a linear combination of the remaining vectors in the set, otherwise they are called linearly dependent.

Let's see what this means in 2D and 3D space. In 2D space, the set must contain at most 2 vectors and they cannot lie on the same line, such as $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\5\end{pmatrix}$:



This is because $\begin{pmatrix}3\\2\end{pmatrix}$ cannot be written as a scaled $\begin{pmatrix}1\\5\end{pmatrix}$ and vice versa.

In 3D space, the set can only contain at most three vectors and they cannot lie on the same plane, such as $\begin{pmatrix}3\\2\\0\end{pmatrix}$, $\begin{pmatrix}1\\5\\3\end{pmatrix}$ and $\begin{pmatrix}0\\2\\5\end{pmatrix}$:



Theorem 3. *If a set of linearly independent vectors* $\{\vec{v}_1, \ldots, \vec{v}_n\}$ *satisfies*

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0,$$

then

$$c_1 = \cdots = c_n = 0.$$

We would want to see if an arbitrary vector can be written as a linear combination of some fixed vectors. For example, let us fix $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\5\end{pmatrix}$. Then we see the vectors $\begin{pmatrix}9\\19\end{pmatrix}$ and $\begin{pmatrix}5\\12\end{pmatrix}$ can be written as

$$\begin{pmatrix}9\\19\end{pmatrix} = 2\begin{pmatrix}3\\2\end{pmatrix} + 3\begin{pmatrix}1\\5\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}5\\12\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} + 2\begin{pmatrix}1\\5\end{pmatrix}$$

respectively. The question is, can any 2D vector be represented as a linear combination of $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\5\end{pmatrix}$? The answer is yes! We see that the two fixed vectors are 'pointing in different directions', so geometrically, we can visualise that the actions of scaling and adding can definitely 'lead' us to any point on the plane. We say that $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\5\end{pmatrix}$ **span** the entire 2D plane. If two vectors in 2D plane point towards the same direction, then they are not linearly independent, such as $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}6\\4\end{pmatrix}$: They span only a line, and this is why we are not interested in linearly dependent vectors, because in this case one vector $\begin{pmatrix}3\\2\end{pmatrix}$ can already span that line, i.e. $\begin{pmatrix}6\\4\end{pmatrix}$ is 'useless'.

Similarly, in 3D space, if 3 vectors lie on the same plane, say $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}$, then they only span one plane, which can already be spanned by only just two vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$. In conclusion (plus more examples):

Linearly independent sets

- $\left\{\begin{pmatrix}3\\2\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}3\\2\\0\end{pmatrix}, \begin{pmatrix}1\\5\\3\end{pmatrix}, \begin{pmatrix}0\\2\\5\end{pmatrix}\right\}$

Linearly dependent sets

- $\left\{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}, \text{more vectors}\right\}$
- $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}\right\}$

Definition 6. A set of n linearly independent vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ in n-dimensions is called a **basis**. The vectors $\vec{v}_i$ are called **basis vectors**.

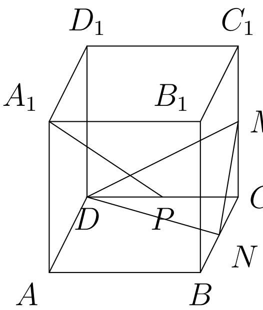
What can a set of basis vectors do? Apparently, we can now 'reach' every point in space, that is, any n-dimensional vector can now be expressed as a linear combination of basis vectors. In fact, the representation is unique!

Theorem 4. *If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis, then any n-dimensional vector $\vec{v}$ can be uniquely expressed as*

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n.$$

Now, we can write ANY vector in terms of basis vectors. This is extremely useful!

3. $ABCD - A'B'C'D'$ is a cube and M, N, P are the midpoints of CC_1, BC, CD. Prove that $A_1P \perp DMN$.



<u>*Solution.*</u>

Let $\vec{x} = \overrightarrow{AB}, \vec{y} = \overrightarrow{AD}, \vec{z} = \overrightarrow{AA_1}$ be a basis.

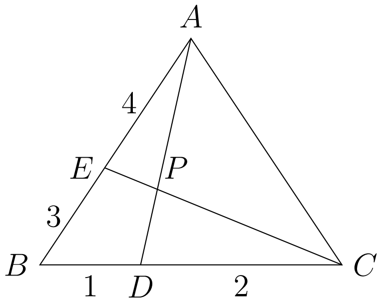
$\overrightarrow{A_1P} = \frac{1}{2}\vec{x} + \vec{y} - \vec{z},\ \overrightarrow{DN} = \vec{x} - \frac{1}{2}\vec{y},\ \overrightarrow{DM} = \vec{x} + \frac{1}{2}\vec{z}.$

$\overrightarrow{A_1P} \cdot \overrightarrow{DN} = \frac{1}{2} - \frac{1}{2} + 0 = 0$

$\overrightarrow{A_1P} \cdot \overrightarrow{DM} = \frac{1}{2} + 0 - \frac{1}{2} = 0$

Since A_1P is perpendicular to both DM, DN, it is perpendicular to plane DMN. $\square$

4. ABC is a triangle. D, E are on BC, AC such that $BD : DC = 1 : 2, AE : EC = 4 : 3$. AD and BE intersect at P. Find $DP : PA$ and $EP : PC$.



<u>*Solution.*</u>

Let $\vec{a} = \overrightarrow{BA}$, $\vec{c} = \overrightarrow{BC}$ be a 2D basis, and let $\overrightarrow{EC} = k\overrightarrow{EP}$.

$\overrightarrow{AD} = -\vec{a} + \frac{1}{3}\vec{c}.$

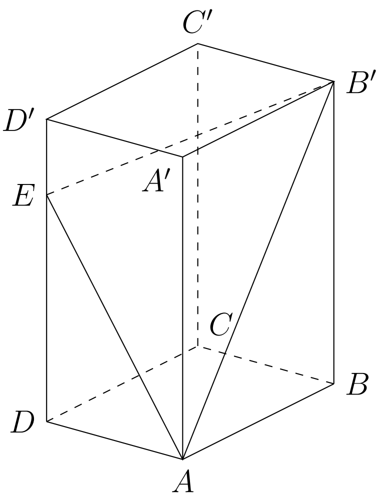
$\overrightarrow{AP} = -\frac{4k+3}{7k}\vec{a} + \frac{1}{k}\vec{c}.$

$\therefore \frac{4k+3}{7k} : \frac{1}{k} = 1 : 3 \Rightarrow k = \frac{9}{2}.$

Therefore, $EP : PC = 2 : 7$. $\square$

5. $ABCD - A'B'C'D'$ is a cuboid such that $AB = AD = 1, BB' = 4$. E lies on DD' such that $DE = 3$. Find the acute angle between planes $ABCD$ and AEB'.

When we are dealing with angles between planes, we can consider a vector perpendicular to each of the planes, and then evaluate the angle using the dot product rule. Again, we need to choose a basis – we can choose $\overrightarrow{AD}, \overrightarrow{AB}$ and the unit vector of $\overrightarrow{AA'}$.



<u>*Solution.*</u>

Let $\vec{k}$ be the upwards unit vector, $\vec{b} = \overrightarrow{AB}$, $\vec{d} = \overrightarrow{AD}$.

Let $\vec{u} = x\vec{b} + y\vec{d} + \vec{k}$ be perpendicular to plane ABE'.

$$\begin{cases} \vec{u} \perp \overrightarrow{AE} \Leftrightarrow (x\vec{b} + y\vec{d} + \vec{k}) \cdot (\vec{d} + 3\vec{k}) = 0 \Leftrightarrow y = -3 \\ \vec{u} \perp \overrightarrow{AB'} \Leftrightarrow (x\vec{b} + y\vec{d} + \vec{k}) \cdot (\vec{b} + 4\vec{k}) = 0 \Leftrightarrow x = -4 \end{cases}$$

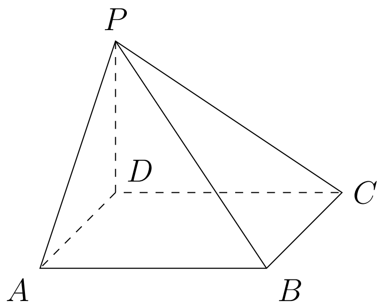
$\therefore \vec{u} = -4\vec{b} - 3\vec{d} + \vec{k}.$

$$\therefore \cos\theta = \frac{\vec{u} \cdot \vec{k}}{|\vec{u}||\vec{k}|} = \frac{0 + 0 + 1}{\sqrt{(-4)^2 + (-3)^2 + 1}} = \frac{\sqrt{26}}{26}.$$

Therefore, the angle is $\arccos \frac{\sqrt{26}}{26}$ (for your information, about $78.69°$). $\square$

One may ask, why can we assume the coefficient of $\vec{k}$ in $\vec{u}$ to be 1? This is because there are infinitely many vectors perpendicular to plane ABE' up to scaling, hence we can just scale a vector until its coefficient of $\vec{k}$ is 1.

6. $ABCD-P$ is a tetrahedron such that $ABCD$ is a rectangle, $PD \perp ABCD$, and $PD = AD = k$. Can PBA and PBC be perpendicular?



Solution.

Let $\vec{x} = \overrightarrow{DC}, \vec{y} = \overrightarrow{DA}, \vec{z} = \overrightarrow{DP}$ be a basis, and let $PBA \perp \vec{u} = a\vec{x} + b\vec{y} + \vec{z}$, $PBC \perp \vec{v} = c\vec{x} + d\vec{y} + \vec{z}$.

$$\begin{cases} \vec{u} \perp \overrightarrow{AB} \Leftrightarrow (a\vec{x} + b\vec{y} + \vec{z}) \cdot \vec{x} = 0 \Leftrightarrow a = 0 \\ \vec{u} \perp \overrightarrow{AP} \Leftrightarrow (b\vec{y} + \vec{z}) \cdot (-\vec{y} + \vec{z}) = 0 \Leftrightarrow b = 1 \end{cases}$$

$$\begin{cases} \vec{v} \perp \overrightarrow{CB} \Leftrightarrow (c\vec{x} + d\vec{y} + \vec{z}) \cdot \vec{y} = 0 \Leftrightarrow d = 0 \\ \vec{v} \perp \overrightarrow{PC} \Leftrightarrow (c\vec{x} + \vec{z}) \cdot (-\vec{x} + \vec{z}) = 0 \Leftrightarrow c = k^2/|\vec{x}|^2 \end{cases}$$

$\therefore \vec{u} \cdot \vec{v} = (\vec{y} + \vec{z}) \cdot (c\vec{x} + \vec{z}) = |\vec{z}|^2 \neq 0.$

Therefore, the two planes can never be perpendicular. □

12.3 Problems

1. Let $O(0,0,0), A(1,5,3), B(3,-2,1)$ be points in space. Find $\cos\angle AOB$.
2. (CJR2018) D lies on side BC of the triangle ABC. We have $BD : CD = 2 : 1, AB = 20, AC = 13, BC = 18$. Find AD^2.
3. Let $\mathcal{P}$ be a plane and $\vec{n}$ is a vector perpendicular to $\mathcal{P}$. Prove that the perpendicular distance from a point X to $\mathcal{P}$ is $d = \dfrac{|\vec{n} \cdot XY|}{|\vec{n}|}$, where Y is any point on $\mathcal{P}$.
4. $ABC - A_1B_1C_1$ is a triangular prism with $\angle ACB = 90°, \angle BAC = 30°, BC = 1, AA_1 = \sqrt{6}$. Let M be the midpoint of CC_1. Prove $AB_1 \perp A_1M$.
5. Let $ABCD$ be a square an P be a point not on the plane $ABCD$. Points M and N lie on PA and BD respectively such that $PM : MA = BN : ND = 5 : 8$. Prove that MN is parallel to plane PBC.
6. $ABCD - A'B'C'D'$ is a cuboid, M and E are the midpoints of DD' and BM respectively. Point N lies on AC such that $AN : NC = 2 : 1$. Prove that A', E, N are collinear.
7. $ABCD - A'B'C'D'$ is a cube, E is the midpoint of CC'. Prove that planes $A'BD, EBD$ are perpendicular.
8. Prove the Menelaus Theorem (Chapter 8).

Chapter 13

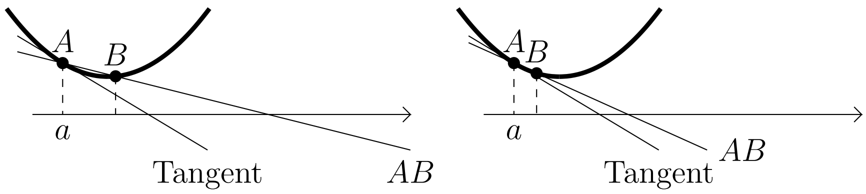
* Calculus

Think about what you have learnt in high-school. $(x^n)' = nx^{n-1}$? $(uv)' = uv' + u'v$? The area under a curve can be solved by integration? Integration is anti-differentiation?

The explanations of all these ideas are possibly not given (if not vaguely mentioned) in the high-school syllabus. However, if we want to use calculus with certainty, we have to understand these concepts at a deeper level.

13.1 Differentiation

Say we want to find the gradient of the tangent of a curve $y = f(x)$ at $x = a$. Fix the point $A(a, f(a))$ and consider the point $B(a + h, f(a + h))$ where h is very, very small. The line AB is hence very close to being the tangent at A. In other words,



$$\text{gradient}_{AB} \approx \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

We want h to very, very small, so this is where limits come up.

Definition 1. The gradient of the tangent of the curve $y = f(x)$ at $x = a$ is

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If such a limit exists, $f(x)$ is said to be differentiable at $x = a$. If f is differentiable at all points of its domain, f is said to be differentiable.

Treating $\dfrac{dy}{dx}$ as a function f' on any $x \in \mathbb{R}$, we can say

$$\frac{dy}{dx}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

.

1. Prove that $\dfrac{d(x^n)}{dx} = nx^{n-1}$ for any positive integer n.

<u>*Solution*</u>.

$$\begin{aligned}
\frac{d(x^n)}{dx} &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n\right) - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n}{h} \qquad (*) \\
&= \lim_{h\to 0} \left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}\right) \\
&= nx^{n-1}.
\end{aligned}$$

□

One notable thing is why we can cancel off the two h's in $(*)$. When we say $h \to 0$, we actually mean that h is approaching 0, but *never* 0.

2. Prove that $\dfrac{d(\sqrt{x})}{dx} = \dfrac{1}{2\sqrt{x}}$ for any positive integer n.

<u>*Solution*</u>.

$$\begin{aligned}
\frac{d(\sqrt{x})}{dx} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

□

3. Prove that $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + \dfrac{du}{dx}v$ for any positive integer n.

<u>*Solution*</u>.

$$\begin{aligned}
\frac{d(uv)}{dx} &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{u(x+h)v(x+h) - u(x+h)v(x)}{h} + \frac{u(x+h)v(x) - u(x)v(x)}{h}\right) \\
&= \lim_{h\to 0} \left(u(x+h)\left(\frac{v(x+h) - v(x)}{h}\right) + \left(\frac{u(x+h) - u(x)}{h}\right)v(x)\right) \\
&= u\frac{dv}{dx} + \frac{du}{dx}v.
\end{aligned}$$

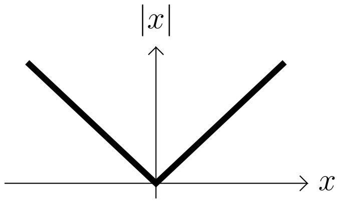
□

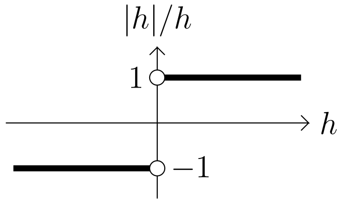
4. Is $|x|$ ($x \in \mathbb{R}$) differentiable?

<u>*Solution*</u>. No. We will prove this by proving that $|x|$ is not differentiable at $x = 0$. Assume the contrary, then its derivative at $x = 0$ is

$$\lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

but this limit doesn't exist since the left limit (-1) and right limit (1) are different.





□

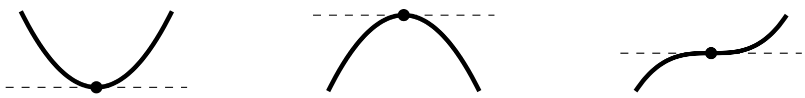
13.1.1 Higher order derivatives

Treating $f'(x)$ as a function, then we can define $f''(x)$ if f' is differentiable. Inductively,

$$f^{(n)}(x) = f \underbrace{''\cdots''}_{n \text{ times}}(x)$$

13.1.2 Local and Global Extrema

By using differentiation, we can find **stationary** points on a curve. These are points at which the first derivative is 0. They can be classified into three types:



Local minimum Local maximum Inflection point

Say $f'(a) = 0$, then

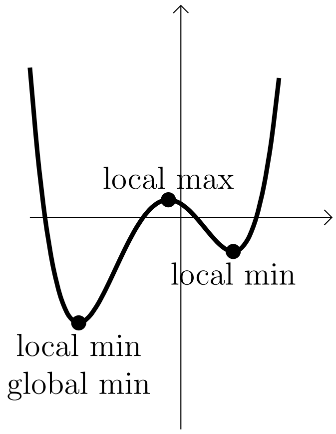
- if $f''(a) < 0$, then the curve $f'(x)$ at $x = a$ is decreasing, so $f'(x)$ is positive on the left of $x = a$, and negative on the right. Therefore, $f(x)$ is decreasing to the left of $x = a$, and increasing to the right. Thus $(a, f(a))$ is a *local minimum*.

- if $f''(a) > 0$, then the curve $f'(x)$ at $x = a$ is increasing, so $f'(x)$ is negative on the left of $x = a$, and positive on the right. Therefore, $f(x)$ is increasing on the left of $x = a$, and decreasing on the right. Thus $(a, f(a))$ is a *local maximum*.

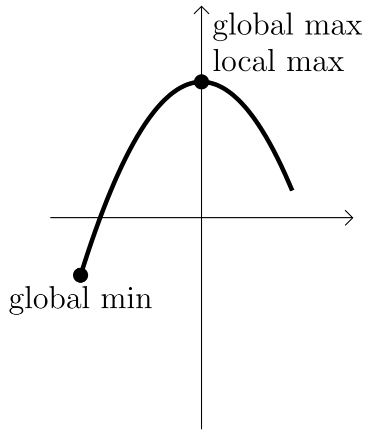
- if $f''(a) = 0$, we cannot conclude about which type the point is. For example,

    1. $f(x) = x^4$ has a local minimum at $x = 0$, and $f'(0) = f''(0) = 0$.
    2. $g(x) = -x^4$ has a local maximum at $x = 0$, and $g'(0) = g''(0) = 0$.
    3. $h(x) = x^3$ has an inflection point at $x = 0$, and $h'(0) = h''(0) = 0$.

Note that a local minimum/maximum (or, for simplicity, *extremum*) may not be a **global extremum**. A global minimum/maximum is a point that attains the minimal/maximal value on the entire domain.



$50x^4 + 55x^3 - 83x^2 - 31x + 9$
$(x \in \mathbb{R})$



$1.8 - x^2$
$(-1.6 \leq x \leq 1.2)$

5. (HLG2014) Find the range of the function $f(x) = x^2 + 2\sqrt{4 - x^2}$ $(-2 \leq x \leq 2)$.

Solution 1. We first find the stationary point(s).

$$\begin{aligned} f'(x) = 2x + 2\left(-\frac{2x}{2\sqrt{4-x^2}}\right) &= 0 \\ \frac{x(\sqrt{4-x^2}-1)}{\sqrt{4-x^2}} &= 0 \\ x &= 0, \pm\sqrt{3} \end{aligned}$$

Now we examine $f(-2), f(-\sqrt{3}), f(0), f(\sqrt{3}), f(2)$ because $-2, 2$ are the endpoints of the domain, which could potentially be global extrema.

$$f(-2) = 4, \qquad f(-\sqrt{3}) = 5, \qquad f(0) = 4, \qquad f(\sqrt{3}) = 5, \qquad f(2) = 4.$$

Thus the range of $f(x)$ is $[4, 5]$. □

Solution 2. Parameterise $x = 2\sin\theta$ $(-\pi/2 \leq \theta \leq \pi/2)$.

$$\begin{aligned} f(x) &= 4\sin^2\theta + 2\sqrt{4 - 4\sin^2\theta} \\ &= 4\sin^2\theta + 4\cos\theta \\ &= 4(1 + \cos\theta - \cos^2\theta) \\ &= -4\left(\cos\theta - \frac{1}{2}\right)^2 + 5 \end{aligned}$$

Since $0 \leq \cos\theta \leq 1$, we have $0 \leq \left(\cos\theta - \frac{1}{2}\right)^2 \leq \frac{1}{4}$. Hence $4 \leq f(x) \leq 5$. □

6. Find the minimum value of $3\sin^3 x - \sin x - 4$.

Solution. Let $y = \sin x$. Now we analyse $f(y) = 3y^3 - y - 4$ where $y \in [-1, 1]$.

$$f'(y) = 9y^2 - 1 = 0 \qquad \Rightarrow \qquad y = \pm\frac{1}{3}.$$

Now we just see $f(-1), f(-1/3), f(1/3), f(1)$:

$$f(-1) = -6, \qquad f\left(-\frac{1}{3}\right) = -\frac{38}{9}, \qquad f\left(\frac{1}{3}\right) = -\frac{34}{9}, \qquad f(1) = -2$$

Therefore the minimum value is -6. □

13.1.3 Handling multiple variables

To find the range of multivariate functions, we have to slowly decrease the number of variables by differentiating with respect to one variable, holding all other variables constant. By doing this, we are taking a **partial derivative**. Note that the notation is a bit different.

7. Find the maximum value of $f(x,y) = (x+y)^2(30-y)(30+y-2x)$, where $0 \le x, y \le 30$.

Solution. Let's handle the boundary points first:

$$\begin{aligned}
x = 0 : f(0,y) &= y^2(30-y^2) \le 15^2 = 225 \\
x = 30 : f(30,y) &= -(900-y^2)^2 \le 0 \\
y = 0 : f(x,0) &= 30x^2(30-2x) \le 30\left(\frac{x+x+(30-2x)}{3}\right)^3 = 30000 \\
y = 30 : f(x,30) &= 0
\end{aligned}$$

Now we focus on the interior stationary points. First keep y constant, then

$$\frac{\partial f(x,y)}{\partial x} = (30-y)\left[2(x+y)(30+y-2x) + (x+y)^2(-2)\right] = 0 \quad \Rightarrow x = 10.$$

Now we have $f(10,y) = (10+y)^3(30-y)$ as a stationary point for all $0 < y < 30$. Since there is one variable left, we can do normal differentiation:

$$\frac{df(10,y)}{dy} = 3(10+y)^2(30-y) + (10+y)^3(-1) = 0 \quad \Rightarrow y = 20.$$

At this stationary point, we have $f(10,20) = 90000$, which is higher than all the boundary values. Thus the maximal value of $f(x,y)$ is 90000. □

8. Find the maximum value of $f(x,\theta) = (1-x)^2 \sin 2\theta + 4x(1-x)\cos\theta$, where $0 < \theta < \frac{\pi}{2}$.

Solution. We first expand $f(x,\theta) = (\sin 2\theta - 4\cos\theta)x^2 + (-2\sin 2\theta + 4\cos\theta)x + \sin 2\theta$. The leading coefficient is $2\cos\theta(\sin\theta - 2) < 0$, hence f always has a global maximum for all values of θ. Differentiating,

$$\frac{\partial f}{\partial x} = 2(\sin 2\theta - 4\cos\theta)x + (-2\sin 2\theta + 4\cos\theta) = 0 \quad \Rightarrow x = \frac{1-\sin\theta}{2-\sin\theta}$$

Substituting x back into f, we have

$$\begin{aligned}
f &= \left(1 - \frac{1-\sin\theta}{2-\sin\theta}\right)^2 \sin 2\theta + 4\left(\frac{1-\sin\theta}{2-\sin\theta}\right)\left(1 - \frac{1-\sin\theta}{2-\sin\theta}\right)\cos\theta \\
&= \frac{2\cos\theta}{2-\sin\theta} \\
\therefore f' &= \frac{(2-\sin\theta)(-2\sin\theta) - (-\cos\theta)(2\cos\theta)}{(2-\sin\theta)^2} = 0 \quad \Rightarrow \sin\theta = \frac{1}{2}
\end{aligned}$$

and thus $\theta = 30°$. This gives $f_{\max} = \dfrac{2(\sqrt{3}/2)}{2-1/2} = \dfrac{2\sqrt{3}}{3}$. □

13.1.4 L'Hôpital's Rule

Assume $f(c) = g(c) = 0$ and $g'(c) \neq 0$. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\dfrac{f(x)-f(c)}{x-c}}{\dfrac{g(x)-g(0)}{x-c}} = \frac{\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}}{\lim\limits_{x\to c} \dfrac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)}.$$

This is a powerful tool! If the numerator and denominator both give 0 at $x = c$, then its limit at $x = c$ (if it exists) is the same as if we differentiated the numerator and the denominator. However, there is a generalised rule which we will not prove in this book because it involves advanced knowledge in analysis (Sorry!).

Theorem 1. *(L'Hôpital's Rule) Let f and g be differentiable functions. If $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ or $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) \to \pm\infty$, we have*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

9. Find $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$.

Solution. Since the fraction at $x = 0$ is $\dfrac{0}{0}$, we can apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = 0.$$

□

10. Find $\lim\limits_{x\to 0} xe^{1/x}$.

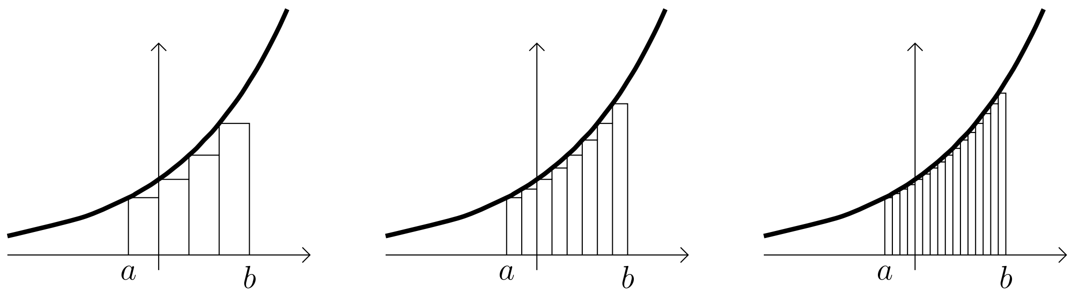
Solution. Since the fraction $\dfrac{e^{1/x}}{1/x}$ at $x = 0$ is $\dfrac{\infty}{\infty}$, we can apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{e^{1/x}}{1/x} = \lim_{x\to 0} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0} e^{1/x} \to \pm\infty.$$

The limit does not exist. □

13.2 Integration

We start by defining integration as the **area under a curve**. This can be done by computing a limit of a sum of areas of thin rectangles:



As we divide the interval $[a, b]$ into n parts, each subinterval will have length $\frac{b-a}{n}$. The height of the k-th rectangle is thus $f\left(a + k \cdot \frac{b-a}{n}\right)$.

Definition 2. The area from $x = a$ to $x = b$ under a curve $f(x)$ is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + k \cdot \frac{b-a}{n}\right).$$

If such a limit exists for all a, b, we say that f is integrable.

11. Find $\displaystyle\int_0^1 x^2\ dx$ using Definition 2.

<u>*Solution*</u>.

$$\begin{aligned}
\int_0^1 x^2\ dx &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} \\
&= \lim_{n\to\infty} \frac{2n^3 - 3n^2 + n}{6n^3} = \frac{1}{3}.
\end{aligned}$$

□

We now build a connection between differentiation and integration, called the **Fundamental Theorem of Calculus**. Do read the proof at the back of the book!

Theorem 2. *For any integrable function $f(x)$ and constant a,*

$$\frac{d}{dX} \int_a^X f(x)\ dx = f(X).$$

Replacing $f(x)$ with $f'(x)$, we have

$$\begin{aligned}
\frac{d}{dX} \int_a^X f'(x)\ dx &= f'(X) \\
\frac{d}{dX} \left(-f(X) + \int_a^X f'(x)\ dx\right) &= 0 \\
-f(X) + \int_a^X f'(x)\ dx &= c
\end{aligned}$$

Substituting $X = \alpha$ and $X = \beta$ and cancelling c yields the following corollary:

Corollary 1. *For any differentiable function $f(x)$ and constants α, β,*

$$\int_\alpha^\beta f'(x)\ dx = f(\beta) - f(\alpha).$$

Therefore, when computing integrals $\int_a^b f(x)\ dx$, we want to try and find a function $F(x)$ such that $F'(x) = f(x)$. In fact, if there is one such $F(x)$, then all $F(x) + c$ work too. Using one of them is enough.

12. Find $\displaystyle\int_0^1 x^2\ dx$ using Corollary 1.

<u>*Solution*</u>. A function f whose derivative is x^2 is $f(x) = x^3/3$. Hence by Corollary 1,

$$\int_0^1 x^2\ dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

□

The shortcut for writing the set of functions whose derivative is $f(x)$ is simply $\int f(x)\ dx$, called the **antiderivative** or **indefinite integral**. For example,

$$\int x^2\ dx = \frac{x^3}{3} + c.$$

However, this notation is very misleading! The equality above isn't a true equality - The RHS is not an element but a set of functions. The above notation is only a way of saying

$$\{f(x) \mid f'(x) = x^2\} = \left\{ \frac{x^3}{3} + c \;\middle|\; c \in \mathbb{R} \right\}.$$

13. (HLG2018) Given that $F(x) = \displaystyle\int_{x^2}^1 \sqrt{1 + 3t^3}\ dt$, find $F'(1)$.

<u>*Solution*</u>. Let $G(x) = \displaystyle\int_1^x \sqrt{1 + 3t^3}\ dt$, then $F(x) = -G(x^2)$. By Theorem 1,

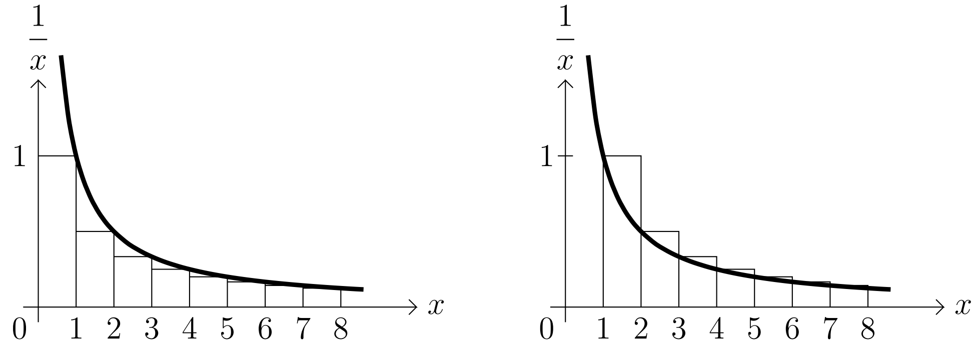
$$\begin{aligned} \frac{d}{dx}G(x) &= \sqrt{1 + 3x^3} \\ \therefore \frac{d}{dx}F(x) = -\frac{d}{dx}G(x^2) &= -\frac{dx^2}{dx} \cdot \frac{d}{dx^2}G(x^2) = -2x\sqrt{1 + 3x^6} \\ \therefore F'(1) &= -2(1)\sqrt{1 + 3(1)^6} = -4. \end{aligned}$$

□

13.2.1 Using integrals for inequalities

14. Prove that $\ln(n + 1) < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \ln n + 1$.

<u>*Solution*</u>. We consider the graph of $y = 1/x$.



From the diagram we know that

$$
\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} &< 1+\int_1^n \frac{1}{x}\,dx = 1+\ln n \\
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} &> \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)
\end{aligned}
$$

□

Note that the gap $(\ln n+1)-\ln(n+1)=\ln\left(\frac{n}{n+1}\right)+1<1$, so $\ln(n+1)$ is a good approximation for this series! Furthermore, since $\ln(n+1)\to\infty$ when $n\to\infty$, we have also indirectly showed that this series diverges to infinity.

15. (HLG2018) Find $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$.

Solution. From the same diagram as the previous problem we have

$$
\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\approx\int_{n+1}^{2n}\frac{1}{x}\,dx=\ln\left(\frac{2n}{n+1}\right)
$$

As $n\to\infty$, the sum converges to $\ln 2$. □

13.3 Problems

1. Prove the chain rule: $(f\circ g)'(x)=f'(g(x))\cdot g'(x)$. Hint:

$$
\lim_{h\to 0}\frac{f(g(x)+h)-f(g(x))}{h}=\lim_{h\to 0}\frac{f(g(x)+g(x+h)-g(x))-f(g(x))}{g(x+h)-g(x)}
$$

2. Prove the quotient rule: $f(x)=\frac{u(x)}{v(x)}\Rightarrow f'(x)=\frac{u'(x)v(x)-u(x)v'(x)}{v(x)^2}$. (Here we replaced h with $g(x+h)-g(x)$ because it approaches 0 too)

3. (HLG2013) Given that $f(x)=\begin{cases}\frac{x}{e^x-1} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$, find $f'(0)$. Recall $(e^x)'=e^x$.

4. (HLG2015) Find $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x}-x\right)$.

5. (HLG2015) Given that $f(x)$ is a differentiable function such that $3f'(x)+f(x)=0$ and $f(0)=2$, find the value of $\int_0^1 f(x)\,dx+3f(1)$.

6. Try as many Chapter 9 problems as possible using calculus.

Chapter 14

* Generating Functions

14.1 Generating Functions

Definition 1. $C_k[f]$ is the coefficient of x^k in a polynomial $f(x)$.

Proposition 1. *Let f be a polynomial. Then*

$$C_n[x^{n+1}f] = 0$$

Proposition 2. *Let f and g be two polynomials. Then*

$$C_n[f+g] = C_n[f] + C_n[g]$$

Proposition 3. *Let f and g be two polynomials. Then*

$$C_n[fg] = \sum_{k=0}^{n} C_k[f] \cdot C_{n-k}[g]$$

Theorem 1. *(Binomial Theorem) For $n \in \mathbb{N}$, we have*

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

Theorem 2. *(Extended Binomial) For $n \in \mathbb{N}$, we have*

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k$$

Normally, if the top part of the choose function involves k, then we use **Theorem 2** (Exceptions: Example 3), otherwise we use **Theorem 1**. What can generating functions do? A **generating function** is a method of encoding an infinite sequence of numbers a_n by treating them as the coefficients of a polynomial.

1. Prove that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$.

<u>Solution</u>.

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} &= \sum_{k=0}^{n} C_k[(1+x)^n]\cdot C_{n-k}[(1+x)^n]\\ &= C_n[(1+x)^n(1+x)^n]\\ &= C_n[(1+x)^{2n}] = \binom{2n}{n}.\end{aligned}$$

□

2. Prove that $\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\binom{n+1}{k}\binom{2n-2k}{n} = n+1$.

Here, we use a variation of Proposition 3.

<u>Solution</u>.

$$\begin{aligned}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\binom{n+1}{k}\binom{2n-2k}{n} &= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} C_{2k}[(1-x^2)^{n+1}]\cdot C_{n-2k}\left[\frac{1}{(1-x)^{n+1}}\right]\\ &= C_n\left[\frac{(1-x^2)^{n+1}}{(1-x)^{n+1}}\right]\\ &= C_n[(1+x)^{n+1}] = n+1\end{aligned}$$

□

3. Prove that

$$\sum_{k=p}^{n}(-1)^k\binom{n}{k}\binom{k}{p} = \begin{cases} 0 & \text{if } p\neq n\\ (-1)^n & \text{if } p=n\end{cases}$$

In this case, we don't use Theorem 2 for $\binom{k}{p}$, but we use Theorem 1.

<u>Solution</u>.

$$\begin{aligned}\sum_{k=p}^{n}(-1)^k\binom{n}{k}\binom{k}{p} &= \sum_{k=p}^{n}(-1)^k\binom{n}{k}C_p[(1+x)^k]\\ &= C_p\left[\sum_{k=p}^{n}(-1)^k\binom{n}{k}(1+x)^k\right]\\ &= C_p[(1-(1+x))^n]\\ &= C_p[(-x)^n] = \begin{cases}(-1)^n & \text{if } p\neq n\\ 0 & \text{if } p=n\end{cases}\end{aligned}$$

□

4. Prove that $\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k+1}\binom{n}{k} = \frac{n}{n+1}$.

Solution.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k+1}\binom{n}{k} x^{k+1} &= x - \int_0^x \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^k dX \\
&= x - \int_0^x (1-X)^n dX \\
&= x + \frac{1}{n+1}(1-X)^{n+1}\bigg|_0^x \\
&= \frac{n}{n+1} \quad \text{(we let } x = 1\text{)}
\end{aligned}$$

□

5. Prove that $\displaystyle\sum_{k=m}^{n} \binom{k}{m} = \frac{n+1}{m+1}$.

Solution.

$$\begin{aligned}
\sum_{k=m}^{n} \binom{k}{m} &= \sum_{k=0}^{n-m} (1) \binom{m+k}{m} \\
&= \sum_{k=0}^{n-m} C_{n-m-k}[1 + x + x^2 + \cdots] \cdot C_k \left[\frac{1}{(1-x)^{m+1}}\right] \\
&= \sum_{k=0}^{n-m} C_{n-m-k} \left[\frac{1}{1-x}\right] \cdot C_k \left[\frac{1}{(1-x)^{m+1}}\right] \\
&= C_{n-m} \left[\frac{1}{(1-x)^{m+2}}\right] = \binom{n+1}{m+1}
\end{aligned}$$

□

6. n is a positive integer. How many polynomials $P(x)$ are there such that its coefficients are in $\{0, 1, 2, 3\}$ and $P(2) = n$?

Solution.

$$\begin{aligned}
C_n \left[\prod_{k=0}^{\infty} (1 + x^{2^k} + x^{2\cdot 2^k} + x^{3\cdot 2^k})\right] &= C_n \left[\prod_{k=0}^{\infty} \frac{1 - x^{2^{k+2}}}{1 - x^{2^k}}\right] \\
&= C_n \left[\frac{1}{(1-x)(1-x^2)}\right] \\
&= C_n \left[\frac{1}{4} \cdot \frac{1}{1+x} - \frac{1}{4} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{(1-x)^2}\right] \\
&= \frac{(-1)^n}{4} + \frac{1}{4} + \frac{n+1}{2} = 1 + \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}$$

□

7. Prove that the number of partitions of n into distinct positive integers is equal to the number of partitions of n into odd terms.

<u>*Solution.*</u> It suffices to show that

$$\begin{aligned}
\prod_{\substack{k=1\\k\text{ odd}}}^{\infty}(1+x^k+x^{2k}+\cdots) &= \prod_{k=1}^{\infty}(1+x^k)\\
\Leftrightarrow \prod_{\substack{k=1\\k\text{ odd}}}^{\infty}\frac{1}{1-x^k} &= \prod_{k=1}^{\infty}\frac{1-x^{2k}}{1-x^k}\\
\Leftrightarrow \prod_{\substack{k=1\\k\text{ odd}}}^{\infty}\frac{1}{1-x^k} &= \prod_{\substack{k=2\\k\text{ even}}}^{\infty}(1-x^k)\prod_{k=1}^{\infty}\frac{1}{1-x^k}\\
\Leftrightarrow \prod_{\substack{k=1\\k\text{ odd}}}^{\infty}\frac{1}{1-x^k}\prod_{\substack{k=2\\k\text{ even}}}^{\infty}\frac{1}{1-x^k} &= \prod_{k=1}^{\infty}\frac{1}{1-x^k}\\
\Leftrightarrow \prod_{k=1}^{\infty}\frac{1}{1-x^k} &= \prod_{k=1}^{\infty}\frac{1}{1-x^k}
\end{aligned}$$

□

8. Let $\alpha(n)$ be the number of ways to write n as an ordered summation of 1s and 2s (must include one of each). Let $\beta(n)$ be the number of ways to write n as an ordered summation of distinct positive integers greater than 1. Prove that $\alpha(n)=\beta(n+2)$.

<u>*Solution.*</u>

$$\begin{aligned}
\sum_{k=1}^{\infty}\alpha(k)x^k &= \sum_{k=1}^{\infty}(x+x^2)^k = \frac{1}{1-x-x^2}-1\\
\sum_{k=1}^{\infty}\beta(k)x^k &= \sum_{k=1}^{\infty}(x^2+x^3+\dots)^k\\
&= \sum_{k=1}^{\infty}\left(\frac{x^2}{1-x}\right)^k\\
&= \frac{1}{1-\frac{x^2}{1-x}}-1\\
&= \frac{x^2}{1-x-x^2}\\
&= x^2+x^2\sum_{k=1}^{\infty}\alpha(k)x^k\\
\therefore \alpha(n) &= \beta(n+2)
\end{aligned}$$

□

9. Find the number of 4-digit positive integers with digit sum 12.

Solution.

$$
\begin{aligned}
& C_{12}\left[\sum_{k=1}^{9} x^k \sum_{k=0}^{9} x^k \sum_{k=0}^{9} x^k \sum_{k=0}^{9} x^k\right] \\
&= C_{12}\left[\frac{x-x^{10}}{1-x}\left(\frac{1-x^{10}}{1-x}\right)^3\right] \\
&= C_{12}\left[\left(\frac{1-x^{10}}{1-x}\right)^4 - \left(\frac{1-x^{10}}{1-x}\right)^3\right] \\
&= C_{12}\left[(1-4x^{10})\sum_{k=0}^{\infty}\binom{k+3}{3}x^k\right] - C_{12}\left[(1-3x^{10})\sum_{k=0}^{\infty}\binom{k+2}{2}x^k\right] \\
&= \left[\binom{12+3}{3} - 4\binom{2+3}{3}\right] - \left[\binom{12+2}{2} - 3\binom{2+2}{2}\right] \\
&= 342
\end{aligned}
$$

$\square$

10. Find the general formula for a_n where $a_0 = 2, a_1 = 4, a_n = 4a_{n-1} - 3a_{n-2}$.

Solution. Let

$$
\begin{cases}
f(x) = \sum_{k=0}^{\infty} a_k x^k \\
xf(x) = \sum_{k=1}^{\infty} a_{k-1} x^k \\
x^2 f(x) = \sum_{k=2}^{\infty} a_{k-2} x^k
\end{cases}
$$

then we have

$$
\begin{aligned}
(1-4x+3x^2)f(x) &= 2-4x \\
f(x) &= \frac{2(1-2x)}{(1-3x)(1-x)} \\
&= \frac{1}{1-3x} + \frac{1}{1-x} \\
&= \sum_{k=0}^{\infty}(3^k+1)x^k
\end{aligned}
$$

and hence $a_n = 3^n + 1$. $\square$

11. Find the value for the infinite sum

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Solution. Consider the series

$$1 - x + x^2 - \cdots = \frac{1}{1+x}$$

If we integrate both sides from 0 to t, we get

$$\begin{aligned}
\int_0^t (1 - x + x^2 - \cdots)dx &= \int_0^t \frac{dx}{1+x} \\
\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\right)\Big|_0^t &= (\ln(1+x))|_0^t \\
t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \cdots &= \ln(1+t)
\end{aligned}$$

Let $t \to 1$, and we have

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \ (\approx 0.693).$$

□

14.2 Problems

1. Prove that

$$\sum_{k=p}^{n} \binom{n}{k}\binom{k}{p} = 2^{n-p}\binom{n}{p}.$$

2. Prove that

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k}\binom{2n-2k}{n} = 2^n.$$

3. Prove that

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \binom{n}{n-2k}\binom{2k}{k} = \binom{2n}{n}.$$

4. Prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k}\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \binom{2n+1}{n}.$$

5. Prove that

$$\sum_{k=0}^{n-m} \binom{n}{m+k}\binom{m+k}{m} = 2^{n-m}\binom{n}{m}.$$

6. Prove that

$$\sum_{k=0}^{n} \binom{a+k}{b+k} = \binom{a+n+1}{b+n} - \binom{a}{b-1}.$$

7. Simplify into a generating function:

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \binom{2k}{k} x^k.$$

8. Prove that

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} (1 - (1-x)^k) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}.$$

9. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0} 2^k \binom{n}{k}^2.$$

Appendix A

Mathematical Reasoning

A statement, proposition or assertion is a sentence which is either true or false (not both, otherwise see [8]). The state of being true or false is the **truth value** of the statement.

Negation

In mathematics it is important to determine what the opposite of a given mathematical statement is. The **negation** S' of a statement S is a statement such that the truth value of S' is opposite of S.

For example, the negation of

'X is a boy'	is	'X is not a boy'

and the negation of

'All apples are red'	is	'Not all apples are red'

instead of 'All apples are not red'.

Implication

Say A and B are two statements. When we have

'If statement A is true, then statement B is true',

we say that 'A **implies** B', or 'A is **sufficient** for B', or 'B is **necessary** for A', denoted by $A \Rightarrow B$. However, this does NOT mean $B \Rightarrow A$. For example, an apple is a fruit, however a fruit might not be an apple. Note that an implication of two statements is also a statement.

Logical AND

The **conjunction** of statements A, B is a statement which gives a truth value of 'true' if both A and B are true, and gives 'false' otherwise. We normally say the conjunction of A, B as A AND B.

Logical OR

The **disjunction** of statements A, B is a statement which gives a truth value of 'true' if any of A or B is true, and gives 'false' otherwise. We normally say the conjunction of A, B as A OR B.

Equivalence

Say A and B are two statements. When we have

$$\text{`}A \Rightarrow B\text{' AND `}B \Rightarrow A\text{'}$$

we say that 'A is **equivalent** to B', or 'A **if and only if (iff)** B', or 'A is **sufficient and necessary** for B', denoted as $A \Leftrightarrow B$. E.g.

$$\begin{aligned} \text{`X is an equilateral triangle'} \quad &\Leftrightarrow \quad \text{`X is a triangle with angles } 60^\circ \text{ only'} \\ \text{`}a = b\text{'} \quad &\Leftrightarrow \quad \text{`}a + c = b + c\text{'} \end{aligned}$$

Contrapositive Property

Previously we have mentioned that $A \Rightarrow B$ does not mean $B \Rightarrow A$. However the useful **contrapositive property** says that

$$\text{`}A \Rightarrow B\text{'} \qquad \Leftrightarrow \qquad \text{`}B' \Rightarrow A'\text{'}.$$

For example, 'an apple is a fruit' exactly means 'a non-fruit is a non-apple'.

Proof

A proof is a sequence of implications, which explains why a statement is true. Once a theorem is proven true, it is true forever, hence we say proofs are *absolute*. Sometimes, we may use various previously established theorems in a proof. However, all of these theorems are also derived from more fundamental facts. The most fundamental facts are the *axioms*, for example the Peano Axioms and the Euclidean Axioms, which are set and cannot be proven. In a proof, we may use well-known notations, such as $+$ means addition etc. However, we are certainly allowed to define our own notation, as long as you state it in your proof.

Note that a proof must be directed at the desired statement. Say a problem wants you to prove that if a condition C is satisfied, then n is even. Note that you CANNOT assume that n is even and use it imply C, as this will prove the backwards direction instead of the forward direction, which is wrong (as stated in '$A \Rightarrow B$ does not imply $B \Rightarrow A$')!

Also, you cannot prove for a specific case of a statement only. Say a problem wants you to prove that 'for any rectangle R, the condition C must be satisfied'. Then you CANNOT assume R is a square! This is because there are other rectangles you have not proven for!

Although you are not allowed to assume the result, you are still allowed to assume the negation of the result, as seen in II. *Proof by Contradiction* in Appendix B. This uses the contrapositive property.

There are many methods of proof, and the most famous ones will be stated in Appendix B which is in the next page.

Appendix B

Methods of Proof

I. *Direct Proof*

'$A \Rightarrow B$'

Most of the proofs in this book are direct.

E.g. Chapter 2 Theorem 1 is proven directly.

II. *Proof by Contradiction or Contraposition*

'A, not $B \Rightarrow$ Absurd result' or 'not $B \Rightarrow$ not A'

Sometimes, assuming the contrary would be a faster way to tackle the problem, especially when there are too many cases when we prove it directly.

1. If $a_1, \ldots, a_n$ are integers such that the product $a_1 \ldots a_n$ is odd, then all a_i are odd.

Proof: Assume the contrary, then <u>at least one</u> (not all!) a_i is even. This a_i will contribute a factor of 2, causing $a_1 \ldots a_n$ to be even, which is a contradiction. Thus all a_i are odd. $\square$

E.g. Chapter 10 Theorem 1 is proven using contradiction.

III. *Proof by Induction*

This is normally used in problems dealing with integers. Let $P(n)$ be the statement for n.

(a) Basic Induction: $P(n) \Rightarrow P(n+1)$ <u>and</u> $P(1)$ is true.
(b) Strong Induction: $P(1), \ldots, P(n) \Rightarrow P(n+1)$ <u>and</u> $P(1)$ is true.
(c) Forward-Backward Induction: $P(n) \Rightarrow P(2n)$ <u>and</u> $P(n) \Rightarrow P(n-1)$ <u>and</u> $P(1)$ is true.
(d) Others: For example $P(n) \Rightarrow P(n+3)$ <u>and</u> $P(1), P(2), P(3)$ are true.

2. Prove that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Proof: For $n = 1$, $LHS = RHS = 1$, which is true. Assume $1 + \cdots + k = \dfrac{k(k+1)}{2}$. Then $1 + 2 + \cdots + k + (k+1) = \dfrac{k(k+1)}{2} + (k+1) = \dfrac{(k+1)(k+2)}{2}$ which completes the problem statement. Therefore, the statement is true for all $n \in N$. $\square$

3. (IMONST2020) Consider the following one-person game: A player starts with score 0 and writes the number 20 on an empty whiteboard. At each step, she may erase any one integer (call it a) and writes two positive integers (call them b and c) such that $b+c=a$. The player then adds $b \times c$ to her score. She repeats the step several times until she ends up with all 1s on the whiteboard. Then the game is over, and the final score is calculated. Let M, m be the maximum and minimum final score that can be possibly obtained respectively. Find $M-m$.

Answer: 0.
Proof: Generalise 20 to any positive integer n. We prove that the final score is always the same, which is $\binom{n}{2}$. For $n=1$, the final score is always 0. Assume the statement is true for $n=1,\ldots,k-1$. Then for $n=k$, the player first splits k into a and b, hence ab is added to the score. By inductive hypothesis, a and b will always end up with scores $\binom{a}{2}$ and $\binom{b}{2}$ respectively. Therefore the final score is

$$ab+\binom{a}{2}+\binom{b}{2}=\frac{2ab+a^2-a+b^2-b}{2}=\frac{(a+b)(a+b-1)}{2}=\binom{k}{2}.$$

This completes the induction, hence $M=m$. $\square$

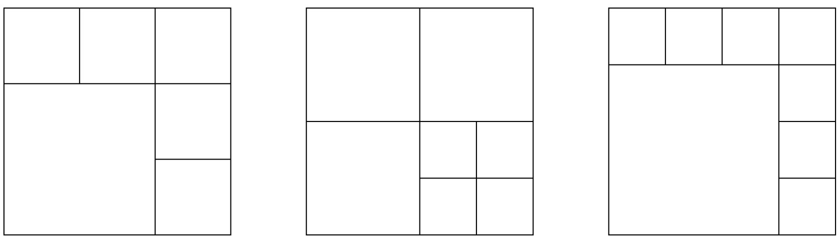
E.g. Chapter 9 Theorem 1 is proven using forward-backward induction.

IV. *Proof by Construction*

This is normally used in problems proving existence. To prove existence, we just need to show that an example exists. However, this does not show uniqueness. The problem below mixes construction and induction:

4. Prove that a square can be divided into any number n ($n \geq 6$) of disjoint squares.

Proof: We first prove for $n=6,7,8$. They are described below:

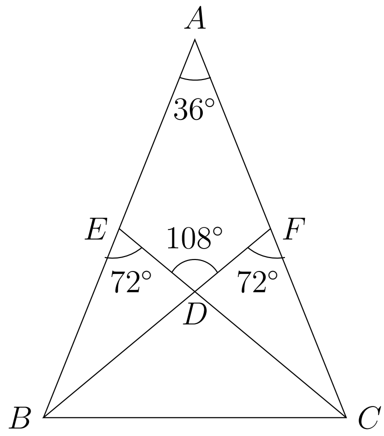


Assume n can. Then choose any square and divide it into 4 equally sized pieces, hence $n+3$ can. Together with the 3 base cases, we infer that all $n \geq 6$ can. $\square$

E.g. Chapter 3 Theorem 1 is proven using construction.

Fallacy: Circular Reasoning

Circular reasoning is one of the most common mistakes in doing mathematics. Let's take a look at an example.



The question is, is DBC isosceles? From the look of the diagram, it may seem so. Many would take the symmetry of the diagram as a fact and say

$$\angle DBC = \frac{180° - 108°}{2} = 36°$$

which concludes that DBC is isosceles(?).

Why is this proof invalid? Here, one assumed that 'DBC is isosceles' and then derived 'DBC is isosceles' again, i.e showing that $A \Leftrightarrow A$ (the reflexivity property of logical equivalence, which is by all means trivial).

One often forgets that 'the result to be shown' must be deemed 'uncertain' before a flow of logic is established to ensure the result is true. Indeed, counterexamples can be easily found, showing that DBC is not necessarily isosceles.

Please keep this logical fallacy in mind., **always remember that the result remains unproven until you manage to do so.**

Sketch of Proofs

These are the proofs of each theorem stated in the book, some of which have advanced proofs, and I will provide links for further information in the bibliography if necessary.

Chapter 1

Theorem 1: Since $1, \omega, \omega^2, \ldots, \omega^{p-1}$ are distinct roots of $x^p - 1 = 0$, we have

$$x^p - 1 = (x-1)(x-\omega)\cdots(x-\omega^{p-1}).$$

By computing the sum of roots using Vieta's theorem,

$$1 + \omega + \omega^2 + \cdots + \omega^{p-1} = 0.$$

Theorem 2: We use the identity (Taylor Series, see [6]):

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

hence

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

Chapter 2

Theorem 1: Let $d = \gcd(a, b)$, $D = \gcd(a, b-a)$. Then we have

$$\begin{aligned} d \mid a, b \ \Rightarrow\ d \mid a, b-a \ \Rightarrow\ d \mid D. \\ D \mid a, b-a \ \Rightarrow\ d \mid a, (b-a)+a \ \Rightarrow\ D \mid d. \end{aligned}$$

Therefore $d = D$.

Property 1: Prime factorise a and b. For any prime, $p^u \cdot p^v = p^{u+v}$, hence

$$v_p(ab) = v_p(a) + v_p(b).$$

Property 2: There are $\lfloor n/p \rfloor$ multiples of p not exceeding n. However, those multiples with p^2 has 2 factors of p. Hence we add $\lfloor n/p^2 \rfloor$ to compensate. Then we also consider multiples of $p^3, p^4, \ldots$ and we get the desired sum, which is

$$v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

Proposition 3: Since the factors of n are in the form $p_1^{x_1} p_2^{x_2} \ldots$ where $0 \leq x_i \leq \alpha_i$ (hence $\alpha_i + 1$ options for x_i), the total number of factors is

$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots$$

Proposition 4: Expanding $(1 + p_1 + \cdots + p_1^{\alpha_1})(1 + p_2 + \cdots + p_2^{\alpha_2}) \cdots$, we find that every term obtained is precisely a factor, in other words,

$$\sum_{0 \leq x_i \leq \alpha_i} p_1^{x_1} p_2^{x_2} \cdots = \sum_{x_1=0}^{\alpha_1} p_1^{x_1} \sum_{x_2=0}^{\alpha_2} p_2^{x_2} \ldots$$

Proposition 5: This is an application of the Principle of Inclusion and Exclusion. Let P_i be the set of multiples of p_i not exceeding n. We know

$$\begin{aligned}
\left| \bigcup_{i=1}^{N} P_i \right| &= \sum_{1 \leq i \leq N} |P_i| - \sum_{1 \leq i < j \leq N} |P_i \cap P_j| + \sum_{1 \leq i < j < k \leq N} |P_i \cap P_j \cap P_k| - \cdots \\
&= \sum_{1 \leq i \leq N} \frac{n}{p_i} - \sum_{1 \leq i < j \leq N} \frac{n}{p_i p_j} + \sum_{1 \leq i < j < k \leq N} \frac{n}{p_i p_j p_k} - \cdots \\
\therefore \phi(n) &= n - \left| \bigcup_{i=1}^{N} P_i \right| \\
&= n - \sum_{1 \leq i \leq N} \frac{n}{p_i} + \sum_{1 \leq i < j \leq N} \frac{n}{p_i p_j} - \sum_{1 \leq i < j < k \leq N} \frac{n}{p_i p_j p_k} + \cdots \\
&= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots .
\end{aligned}$$

Chapter 3

Theorem 1: If $x \equiv A \pmod{M_1 \cdots M_n}$, we take $a_i = A \pmod{M_i}$ for each i. Conversely, if $x \equiv a_i \pmod{M_i}$ for each i, we let

$$\begin{aligned}
A &= c_1 M_2 \cdots M_n + M_1 c_2 M_3 \cdots M_n + \cdots + M_1 \cdots M_{n-1} c_n \\
\Rightarrow A &\equiv M_1 \cdots M_{i-1} c_i M_{i+1} \cdots M_n \pmod{M_i}
\end{aligned}$$

Recall the fact that there always exists a solution a (called the inverse of b modulo N) to $ab \equiv 1 \pmod{N}$ if b and N are coprime. For each $1 \leq i \leq n$, we can indeed choose c_i to be a_i multiplied by the inverse of $M_1 \cdots M_{i-1} M_{i+1} \cdots M_n$ modulo M_i. This is possible since all M_i are mutually coprime. Hence $x \equiv A \pmod{M_i}$ is the desired result.

Theorem 2. Let $S = \{0, 1, \ldots, p^n - 1\} \setminus \{0, p, 2p, \ldots, p(p^{n-1} - 1)\}$. Multiply each element in S by x to get xS. However, each term in xS is still coprime to p^n and if

$xi \equiv xj \pmod{p^n}$, then $i \equiv j \pmod{p^n}$. Hence xS is just a permutation of S, modulo p^n. Therefore their products of elements are congruent, i.e.

$$\prod_{k\in S}(xk) \equiv \prod_{k\in S} k \pmod{p^n}$$

Cancel off identical terms to get $x^{p^n-p^{n-1}} \equiv 1 \pmod{p^n}$.

Divisibility for 2,4,8: $10^k a + b \equiv b \pmod{2^k}$ $(k = 1, 2, 3)$.

Divisibility for 9: $10^n a_n + 10^{n-1}a_{n-1} + \cdots + a_0 \equiv a_n + a_{n-1} + \cdots + a_0 \pmod 9$.

Divisibility for 11: Same method as 9.

Chapter 4

Theorem 1: This is called the Fundamental Theorem of Algebra (see [4]). It suffices to prove that $P(x)$ must have one root r, then $P(x) = (x-r)Q(x)$ and we can proceed to induct on the degree of the polynomial.

Theorem 2: Expand

$$a_n x^n + \cdots + a_0 = a_n(x-r_1)\cdots(x-r_n)$$

and compare each coefficient to get the desired result.

Theorem 3: Existence is shown by long division. To prove uniqueness, assume there are two ways to express $P(x)$,

$$\begin{aligned} P(x) &= Q(x)M_1(x) + R_1(x) \\ P(x) &= Q(x)M_2(x) + R_2(x) \end{aligned}$$

Subtracting yields $0 = Q(x)(M_1(x) - M_2(x)) + (R_1(x) - R_2(x))$. Since $\deg Q > \deg R_i$, $M_1(x) - M_2(x)$ must be the zero polynomial, which derives a contradiction.

Chapter 5

Proposition 1: $2\sum_{k=1}^{n} k = \sum_{k=1}^{n} k + \sum_{k=1}^{n}(n+1-k) = \sum_{k=1}^{n}(n+1) = n(n+1)$.

$$\Rightarrow \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proposition 2: $3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \sum_{k=1}^{n}((k+1)^3 - k^3) = (n+1)^3 - 1$.

$$\Rightarrow \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proposition 3: $4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \sum_{k=1}^{n}((k+1)^4 - k^3) = (n+1)^4 - 1.$

$$\Rightarrow \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Proposition 4: $[f(n+1)-f(n)]+[f(n)-f(n-1)]+\cdots+[f(2)-f(1)] = f(n+1)-f(1).$

Proposition 5: $f(n+1)/f(n) \times f(n)/f(n-1) \times \cdots \times f(2)/f(1) = f(n+1)/f(1).$

Chapter 6

Proposition 1: Comes from the definition.

Chapter 7

Theorem 1: We prove that a_n satisfies the recursion form. Since $P(r_i) = 0$,

$$\begin{aligned} a_n - \sum_{j=1}^{k} p_{n-j} a_j &= \sum_{i=1}^{k} C_i r_i^n - \sum_{j=1}^{k} p_{n-j} \sum_{i=1}^{k} C_i r_i^{n-j} \\ &= \sum_{i=1}^{k} C_i (r_i^n - \sum_{j=1}^{k} p_{n-j} r_i^{n-j}) \\ &= \sum_{i=1}^{k} C_i \cdot P(r_i) = 0 \end{aligned}$$

Denote $x^{\underline{m}} = x(x-1)\cdots(x-m)$. If there are m repeated roots R, we first rewrite

$$a_n = \cdots + (C_1 + C_2 n + C_3 n^2 \cdots + C_m n^{m-1}) R^n + \cdots$$

as

$$= \cdots + (c_1 + c_2 n + c_3 n^{\underline{2}} \cdots + c_m n^{\underline{m-1}}) R^n + \cdots \qquad (*)$$

for some new constants c_i. Recall that if $P(x)$ has m repeated roots R, then R is also a root of the $d(\leq m-1)$-th derivative $P^{(d)}(x)$. Focusing on these terms,

$$\begin{aligned} & R^n \sum_{i=1}^{m} c_i n^{\underline{i-1}} - \sum_{j=1}^{k} p_{n-j} R^j \sum_{i=1}^{m} c_i j^{\underline{i-1}} \\ &= \sum_{i=1}^{m} R^{i-2} c_i \left(n^{\underline{i-1}} R^{n-i+2} - \sum_{j=1}^{k} p_{n-j} j^{\underline{i-1}} R^{j-i+2} \right) \\ &= \sum_{i=1}^{m} R^{i-2} c_i \cdot P^{(i-1)}(R) = 0. \end{aligned}$$

Chapter 8

Proposition 1: Definition of a line.

Proposition 2, Theorem 1: Pythagorean Theorem.

Theorem 2: If (x_1, y_1) lies on the line then we are done. Otherwise, project (x_1, y_1) onto the line $ax + by + c = 0$ at (x_0, y_0). Let $x_1 - x_0 = \Delta x, y_1 - y_0 = \Delta y$,

$$\begin{cases} \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{b}{a} \\ ax_0 + by_0 + c = 0. \end{cases} \quad \Leftrightarrow \begin{cases} b\Delta x = a\Delta y \\ a\Delta x + b\Delta y = ax_1 + by_1 + c. \end{cases}$$

The distance would be

$$\begin{aligned} &\sqrt{(\Delta x)^2 + (\Delta y)^2} \\ =& \sqrt{\frac{a^2(\Delta x)^2 + b^2(\Delta x)^2 + a^2(\Delta y)^2 + b^2(\Delta y)^2}{a^2 + b^2}} \\ =& \sqrt{\frac{a^2(\Delta x)^2 + ab\Delta x\Delta y + ab\Delta x\Delta y + b^2(\Delta y)^2}{a^2 + b^2}} \\ =& \sqrt{\frac{(a\Delta x + b\Delta y)^2}{a^2 + b^2}} \\ =& \left|\frac{a\Delta x + b\Delta y}{\sqrt{a^2 + b^2}}\right| = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| \end{aligned}$$

Theorem 3: Note that for any $A(a_1, a_2), B(b_1, b_2)$, the signed area (positive if anticlockwise, otherwise negative) of OAB is

$$\frac{1}{2}b_1b_2 + \frac{1}{2}(a_2 + b_2)(a_1 - b_1) - \frac{1}{2}a_1a_2 = \frac{1}{2}(a_1b_2 - a_2b_1) = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Therefore the signed area of $A_1A_2A_3$ is

$$\begin{aligned} [A_1A_2A_3] &= [OA_1A_2] - [OA_1A_3] + [OA_2A_3] \\ &= \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}\right) \\ &= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} \end{aligned}$$

Theorem 4: Let the circumcentre be O. In triangle OBC,

$$R\sin A = R\sin\frac{\angle BOC}{2} = \frac{BC}{2} = \frac{a}{2} \Rightarrow \frac{a}{\sin A} = 2R.$$

Theorem 5: Let the altitude from A be h. Then

$$\begin{aligned} c^2 &= h^2 + (b - a\cos C)^2 \\ &= h^2 + b^2 - 2ab\cos C + a^2\cos^2 C \\ &= h^2 + b^2 - 2ab\cos C + a^2 - a^2\sin^2 C \\ &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

Proposition 3: In a triangle, $A + B + C = 180°$ since the rotation of the sides about the interior angles only cause the sides rotate by half a circle (Take a pen and align them with the sides, and then rotate such that they turn into the next side etc). Then, triangulate an arbitrary n-gon into $(n-2)$ triangles, hence the sum of interior angles is $(n-2)180°$.

Proposition 4: Corollary from $A + B + C = 180°$.

Theorem 6: If $ABCD$ is a cyclic quadrilateral, then

$$\angle ABD = (1/2)\angle AOD = \angle ACD$$
$$ext.\angle A = 180° - \angle DAB = \frac{1}{2}(180° - \angle DOB) = \angle C$$

Conversely, If any of the bottom two conditions hold, consider the circumcircle of ABC and let $CD \cap (ABC) = D'$, but D' satisfies these two conditions too, so $D = D'$.

Proposition 5: Equal height, hence ratio of areas equals ratio the base lengths.

Proposition 6: Equal height, equal base lengths, hence equal area.

Theorem 7: $\dfrac{AB}{BD} = \dfrac{\sin ADB}{\sin BAD} = \dfrac{\sin ADC}{\sin CAD} = \dfrac{AC}{CD}$.

Theorem 8: This is equivalent to Monge's Theorem (see [5]), stating that the pairwise enlargement centers of three circles in the plane are collinear. This can be proven by erecting three spheres on the plane and consider the intersection of the two tangential planes which is a line.

Theorem 9,10: We use the similarity between triangle PAC and PBD.

$$\begin{cases} \angle PAC = \angle PDB \\ \angle PCA = \angle PBD \end{cases} \Rightarrow PAC \sim PDB \Rightarrow \frac{PA}{PD} = \frac{PC}{PB} \Rightarrow PA \cdot PB = PC \cdot PD.$$

Chapter 9

Theorem 1: Let the statement for n be $P(n)$. To prove $P(2)$:

$$\frac{a+b}{2} \geq \sqrt{ab} \Leftrightarrow (\sqrt{a} - \sqrt{b})^2 \geq 0.$$

Note that substituting two $P(n)$ cases into the places of a and b of $P(2)$ yields $P(2n)$ and hence $P(2^k)$ is true for any $k \in \mathbb{N}$. Assume $P(n)$ is true, then

$$\frac{a_1 + \cdots + a_{n-1} + \frac{a_1+\cdots+a_{n-1}}{n-1}}{n} \geq \left(a_1 \cdots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}\right)^{1/n}$$
$$\left(\frac{a_1 + \cdots + a_{n-1}}{n-1}\right)^n \geq a_1 \cdots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}$$
$$\frac{a_1 + \cdots + a_{n-1}}{n-1} \geq (a_1 \cdots a_{n-1})^{1/(n-1)}$$

and hence $P(n)$ is true for all $n \in \mathbb{N}$.

Theorem 2: The polynomial $\sum(a_i x - b_i)^2$ is nonnegative, hence the discriminant $\Delta = 4(\sum a_i b_i)^2 - 4(\sum a_i)(\sum b_i)$ must be nonpositive (at most one root).

Theorem 3,4: Already covered in Chapter 8.

Chapter 10

Theorem 1: Assume there is another $(x, y) \notin S = \{(x_0 + bt, y_0 - at)\}$ which satisfies the equation. Then $a(x - x_0) + b(y - y_0) = 0$ and hence $b \mid x - x_0$ and $a \mid y - y_0$ which means $(x, y) = (x_0 + bk + bt, y_0 - ak - at) \in S$, a contradiction.

Theorem 2: Without loss of generality assume $\gcd(a, b) = 1$, then we can ignore the k by cancellation on both sides. Note that a, b cannot both be odd, otherwise $c^2 \equiv 1 + 1 \equiv 2 \pmod 4$. Therefore say a is even and b is odd. Then

$$\left(\frac{a}{2}\right)^2 = \left(\frac{c+b}{2}\right)\left(\frac{c-b}{2}\right).$$

But $\gcd\left(\frac{c+b}{2}, \frac{c-b}{2}\right) = \gcd\left(\frac{c+b}{2}, b\right) = 1$. Hence $\frac{c+b}{2}$ and $\frac{c-b}{2}$ are both squares, say x^2 and y^2 respectively, then

$$\begin{aligned} a &= 2xy \\ b &= x^2 - y^2 \\ c &= x^2 + y^2 \end{aligned}$$

Chapter 11

Theorem 1: After expansion, each term is in the form $a^{n-k}b^k$. There are $\binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$ ways to select a term $a^{n-k}b^k$, hence the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$.

Theorem 2: We prove by induction. For $n = 2$,

$$\begin{aligned} x \in (A_1 \cap A_2)' &\Rightarrow x \notin A_1 \cap A_2 \Rightarrow x \notin A_1 \text{ or } x \notin A_2 \Rightarrow x \in A_1' \cup A_2' \\ x \notin (A_1 \cap A_2)' &\Rightarrow x \in A_1 \cap A_2 \Rightarrow x \in A_1 \text{ and } x \in A_2 \\ &\Rightarrow x \notin A_1' \text{ and } x \notin A_2' \Rightarrow x \notin A_1' \cup A_2' \\ \therefore (A_1 \cap A_2)' &= A_1' \cup A_2' \end{aligned}$$

and similarly we can prove $(A_1 \cup A_2) = A_1' \cap A_2'$. For $n \geq 2$, assume it is true for n. Then

$$\begin{aligned} (A_1 \cap \cdots \cap A_{n+1})' &= ((A_1 \cap \cdots \cap A_n) \cap A_{n+1})' = (A_1 \cap \cdots \cap A_n)' \cup A_{n+1}' \\ &= (A_1' \cup \cdots \cup A_n') \cup A_{n+1}' = A_1' \cup \cdots \cup A_{n+1}' \end{aligned}$$

and similarly we can prove $(A_1 \cup \cdots \cup A_{n+1})' = A_1' \cap \cdots \cap A_{n+1}'$.

Theorem 3: Let ξ be the union of all A_i. For any A_i, define $A_i' = \xi \setminus A_i$ and $f_i(x) = \begin{cases} 1 & (x \in A_i) \\ 0 & (x \notin A_i) \end{cases}$. We construct the following function on ξ:

$$F(x) = \begin{cases} 1 & (x \in \bigcap_{i=1}^n A_i') \\ 0 & (x \notin \bigcap_{i=1}^n A_i') \end{cases}$$

Using $f_i(x)$ to describe $F(x)$:

$$\begin{aligned} F(x) &= \prod_{i=1}^{n} (1 - f_i(x)) \\ &= \sum_{I \subseteq [n]} (-1)^{|I|} \prod_{i \in I} f_i(x) \\ &= 1 - \sum_{\varnothing \neq I \subseteq [n]} (-1)^{|I|+1} \prod_{i \in I} f_i(x). \end{aligned}$$

Since $\bigcap\limits_{i=1}^{n} A_i'$ is the empty set,

$$\begin{aligned} 0 = \left| \bigcap_{i=1}^{n} A_i' \right| &= \sum_{x \in \xi} F(x) \\ &= \sum_{x \in \xi} \left(1 - \sum_{\varnothing \neq I \subseteq [n]} (-1)^{|I|+1} \prod_{i \in I} f_i(x) \right) \\ &= |\xi| - \sum_{\varnothing \neq I \subseteq [n]} (-1)^{|I|+1} \sum_{x \in \xi} \prod_{i \in I} f_i(x) \\ &= |\xi| - \sum_{\varnothing \neq I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| \\ \therefore |\xi| &= \sum_{\varnothing \neq I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|. \end{aligned}$$

Chapter 12

Theorem 1: The shortest path between two points is a straight line segment.

Theorem 2: By the cosine rule, $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos\theta \Rightarrow \vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$.

Theorem 3: If some $c_i \neq 0$, then v_i can be expressed as a linear combination of the remaining vectors, a contradiction.

Theorem 4: If there are two ways $\vec{v} = \sum b_i v_i = \sum c_i v_i$, then by subtraction we get $0 = \sum (b_i - c_i) v_i$. By Theorem 3, $b_i = c_i$ for all i, which is a contradiction.

Chapter 13

Theorem 1: see [9] **Theorem 2**:

$$\begin{aligned}
\frac{d}{dX}\int_a^X f(x)\ dx &= \lim_{h\to 0}\frac{\int_a^{X+h} f'(x)\ dx - \int_a^X f'(x)\ dx}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\int_X^{X+h} f(x)\ dx \\
&= \lim_{h\to 0}\frac{1}{h}\left[\lim_{n\to\infty}\frac{h}{n}\sum_{k=0}^{n-1} f\left(X + k\cdot\frac{h}{n}\right)\right] \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(X) = f(X).
\end{aligned}$$

Chapter 14

Theorem 1: Proven in Chapter 12.

Theorem 2: The extended binomial can be proven by using the Taylor Series (see [6]).

Solutions

The solutions below are only *meant as a guide*. Tiresome computations will be omitted. Furthermore, these are not the only methods! The answers are also underlined.

Chapter 1

1. Yes. E.g. $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$.

2. No. For example, $\sqrt{2}+(2-\sqrt{2})=2$. If $2-\sqrt{2}$ is rational, then $\sqrt{2}=2-(2-\sqrt{2})$ is rational which is a contradiction. Hence $2-\sqrt{2}$ is irrational, and two irrational numbers may indeed sum up to a rational number.

3. Yes. E.g. $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}$.

4. No. E.g. $\sqrt{2}\times\sqrt{2}=2$.

5. $(\omega-1)\cdots(\omega^{100}-1)=(1-\omega)\cdots(1-\omega^{100})=1+(1)+\cdots+(1)^{100}=\underline{101}$.

6. $\omega+\cdots+\omega^6=0-1=\underline{-1}$.

7. Apply Theorem 1 and let $x=-1$, we get
$$(-1-r)(-1-r^2)\cdots(-1-r^{22})=(-1)^{22}+(-1)^{21}+\cdots+1=1.$$
Therefore, $(1+r)(1+r^2)\cdots(1+r^{22})=\underline{1}$.

8. Interpret the complex plane as the coordinate plane. The first condition describes z lying on the circle with centre $(0,1)$ and radius 2; whereas the second condition describes z lying on the circle with centre $(1,0)$ and radius 1. There are exactly two intersection points between these two circles, hence $\underline{2}$ values for z.

9. Rings: $\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$. Fields: $\mathbb{Q},\mathbb{R},\mathbb{C}$.

10. We analyse the four cases:
$$(a-bi)+(c-di)=(a+c)-(b+d)i$$
$$(a-bi)-(c-di)=(a-c)-(b-d)i$$
$$(a-bi)(c-di)=(ac-bd)-(ad+bc)i=\overline{(ac-bd)+(ad+bc)i}=\overline{(a+bi)(c+di)}$$
$$\frac{a-bi}{c-di}=\frac{(a-bi)(c-di)}{c^2+d^2}=\frac{\overline{(a+bi)(c+di)}}{c^2+d^2}=\overline{\left(\frac{(a+bi)(c+di)}{c^2+d^2}\right)}=\overline{\left(\frac{a+bi}{c+di}\right)}$$

11. All rational numbers are algebraic because a/b is a root of the polynomial $bx - a$, hence $\mathbb{Q} \subset \mathbb{A}$. Since algebraic numbers are defined among the real numbers, every algebraic number is automatically a real number, hence $\mathbb{A} \subset \mathbb{R}$. To prove that $\sqrt{2 + \sqrt{3} + \sqrt[3]{5}}$ is algebraic, we must find a polynomial that has it as a root. This can be done through manipulation:

$$\begin{aligned} x &= \sqrt{2 + \sqrt{3} + \sqrt[3]{5}} \\ x^2 - 2 - \sqrt{3} &= \sqrt[3]{5} \\ (x^2 - 2 - \sqrt{3})^3 &= 5 \\ x^6 - 3(2 + \sqrt{3})x^4 + 3(2 + \sqrt{3})^2 x^2 - (2 + \sqrt{3})^3 &= 5 \\ x^6 - 6x^4 + 21x^2 - 31 &= (3x^4 - 6x^2 + 15)\sqrt{3}. \end{aligned}$$

After squaring both sides, we successfully get a polynomial with integer coefficients that has x as a root (we do not need to explicitly expand them now, as it is obvious we have got the desired result)

Chapter 2

1. Multiples of 2,3,5 are ruled out easily. 991 and 997 are left to check. Since $\sqrt{1000} < 32$, we only need to check prime factors up to 31. $\underline{\text{991 and 997 are indeed prime}}$.

2. $\sigma(840) = \sigma(2^3 \cdot 3 \cdot 5 \cdot 7) = (1 + 2 + 4 + 8)(1 + 3)(1 + 5)(1 + 7) = \underline{2880}$.

3. $\phi(132) = \phi(2^2 \cdot 3 \cdot 11) = (2^2 - 2)(3 - 1)(11 - 1) = \underline{40}$.

4. $d(n) = 6 = 3 \times 2 \Rightarrow n = p^5$ or $p^2 q$.

   In the first case, $p^5 < 200 \Leftrightarrow p = 2$.

   In the second case, $2p^2 \leq p^2 q \leq 200 \Rightarrow p = 2, 3, 5, 7$. If $p = 2$, $q = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$.

   If $p = 3$, $q = 2, 5, 7, 11, 13, 17, 19$.

   If $p = 5$, $q = 2, 3, 7$.

   If $p = 7$, $q = 2, 3$.

   In conclusion there are $\underline{27 \text{ such } n}$.

5. $d(2019) = d(3 \cdot 673) = (1 + 1)(1 + 1) = \underline{4 \text{ factors}}$.

6. $n^2 + 124n = (n + 62)^2 - 62^2 = k^2 \Rightarrow (n + 62 + k)(n + 62 - k) = 2^2 \cdot 31^2$. Since $n + 62 + k$ and $n + 62 - k$ differ by an even number, they have equal parity, so we must have $\left(\frac{n + 62 + k}{2}\right)\left(\frac{n + 62 - k}{2}\right) = 31^2 \Rightarrow \left(\frac{n + 62 + k}{2}, \frac{n + 62 - k}{2}\right) = (961, 1) \Rightarrow \underline{n = 900}$.

7. Since 60 and 1200 have the same prime factors, we just need $\phi(1200) = \phi(2^4 \cdot 3 \cdot 5^2) = (16-8)(3-1)(25-5) = \underline{320}$.

8. The number of prime factors of 2 must be much higher than that of 3, hence we want $v_3(500!) = 166 + 55 + 18 + 6 + 2 = \underline{247}$.

9. $4 \mid \overline{9b} \Rightarrow b = 2, 6$. Also $3 \mid a + 7 + 8 + 9 + b \Rightarrow 3 \mid a + b$ hence $(a, b) = (1, 2), (4, 2), (7, 2), (3, 6), (6, 6), (9, 6)$. There are $\underline{\text{6 solutions}}$.

10. The exponents of 2,3,5 must be 1 or 2 (two choices); the exponent of 7 must be 1 (one choice). Hence there are $\underline{2 \cdot 2 \cdot 2 \cdot 1 = 8 \text{ pairs}}$ of solutions.

Chapter 3

1. $3^{2013} \cdot 7^{2017} \equiv 3 \times 7 \equiv \underline{1} \pmod{10}$.

2. $0 - 9 \equiv \underline{1} \pmod{10}$.

3. $-1 + 0 + 1 + 8^{9018} \equiv \underline{4} \pmod{10}$.

4. Mod 4, $17^{17^{17^{17}}} \equiv 1 \pmod 4$. To find mod 25, we know by Fermat's Theorem that $17^{20} \equiv 1 \pmod{25}$, hence we want to find $17^{17^{17}} \bmod 20$. To find this,

$$\begin{cases} 17^{17^{17}} \equiv 1 \pmod 4 \\ 17^{17^{17}} \equiv 2^{17^{17}} \equiv 2 \pmod 5 \end{cases} \Rightarrow 17^{17^{17}} \equiv 17 \pmod{20}$$
$$\Rightarrow 17^{17^{17^{17}}} \equiv 17^{17} \equiv 14^8 \cdot 17 \equiv (-4)^4 \cdot 17 \equiv 6 \times 17 \equiv 2 \pmod{25}.$$

Combining mod 4 and mod 25, we have $17^{17^{17^{17}}} \equiv \underline{77} \pmod{100}$.

5. $x \equiv \dfrac{11}{5} \equiv \dfrac{140}{5} \equiv \underline{28} \pmod{43}$.

6. $x \equiv \dfrac{25}{107} \equiv -\dfrac{25}{4} \equiv \dfrac{12}{4} \equiv \underline{3} \pmod{37}$.

7. Testing $n = 0, 1, 2$, we find $n^2 + n + 2 \equiv 2, 1, 2 \not\equiv 0 \pmod 3$.

8. $9 \times 99 \times (-1)^{2015} \equiv \underline{109} \pmod{1000}$.

9. Mod 4, $(-1)(1)(1) \equiv 3$. Mod 25, $7^{13} \cdot 11^{14} \cdot 13^{15} \equiv (-9)(-1)^6(-4)^7(-6)^7 \equiv 9 \pmod{25}$. Hence $\underline{59} \pmod{100}$.

10. $1! + 3! + 5! + 7! + 9! + 0 \equiv 1 + 6 + 20 + 40 + 80 \equiv 47 \pmod{100}$. $\therefore 4 + 7 = \underline{11}$.

11. $0 - 0 \equiv 0 \pmod{11}$. $\underline{\text{No remainder}}$.

12. $100 \mid 16(m+n) \Rightarrow 25 \mid m + n \Rightarrow m + n = 25, 50, 75, 100, 125, 150, 175$, together with $m - n = 16$ yields $(m, n) = (33, 17), (58, 42), (83, 67)$. $\underline{\text{3 solutions}}$.

13. Subtracting we get $a \equiv d \pmod 7$. Hence there are 13 solutions for $(a, d) = (1, 1), \ldots, (9, 9), (1, 8), (2, 9), (8, 1), (9, 2)$. Also $10b + c$ is a multiple of 7. There are 15 two-digit multiples of 7. Hence there are $13 \times 15 = \underline{\text{195 solutions}}$.

14. The sum of numbers on the board mod 11 is invariant (does not change). Hence $x \equiv (1+\cdots+2004) - 1000 \equiv 4 \pmod{11}$. Since $1000 > 11$, the other number must be a remainder and hence < 11, therefore $\underline{x = 4}$.

15. Yes. By CRT, $x + i \equiv 0 \pmod{p_i q_i}$ for $i = 1, \ldots, n$ is solvable. Choose p_i and q_i to ALL be distinct primes. Then none of $x + i$ is a pure prime power.

Chapter 4

1. x, y are distinct roots of the equation $a + 1/a = k \Leftrightarrow a^2 - ka + 1 \equiv 0$, hence $\underline{xy = 1}$.

2. $2x^2 - 3x - 2 = (2x+1)(x-2)$. Hence the remainder is $4x - 7 \pmod{2x+1} = \underline{-9}$.

3. $41x - 4x^3 = -x(4x^2 - 41) = -x(14x - 49) = -7(2x^2 - 7x) = -7(-4) = \underline{28}$.

4. $P(x) = Q(x)(x^2+1) + 2x + 3$. Let $x = \pm 1$, then $Q(1) = 2, Q(-1) = 8 \Rightarrow Q(x) = Q_2(x)(x^2-1) - 3x + 5 \Rightarrow P(x) = Q_2(x)(x^4-1) - 3x^3 + 5x^2 - x + 8 \Rightarrow \underline{r(-3) = 137}$.

5. No. E.g. The polynomial $P(x) = \dfrac{x(x+1)}{2}$ always attains integer values.

6. $(1 - 2(1) + 3(1)^2)^{10} - 1 = \underline{1023}$.

7. $x^6 + y^6 = (x^2+y^2)(x^4 - x^2y^2 + y^4) = (7^2 - 2\cdot 11)((7^2 - 2\cdot 11)^2 - 3(11)^2) = \underline{9882}$.

8. Long division. The remainder would be $(-3b - a)x + (16 + 6a + 2b)$ which must be 0, hence $\underline{(a, b) = (-3, 1)}$.

9. $18x^3 - 2376x^2 + \cdots = 0$. By Vieta, $\alpha + \beta + \gamma = \dfrac{2376}{18} = \underline{132}$.

10. Let $\ell : y = mx + c$. Note that m, c are rational since ℓ connects two rational points. Then the three discussed points are the solutions of $(mx+c)^2 = x^3 + 3x^2 + 1 \Leftrightarrow x^3 - (m^2 - 3)x^2 + \cdots = 0$. By Vieta's Theorem, the three roots sum up to $m^2 - 3$ which is a rational number. Since the x-coordinate of the previous two points are rational, the third point must have rational x-coordinate. From $y = mx + c$, we know that this point must have rational y-coordinate too.

Chapter 5

1. $4\sum\left(\dfrac{1}{3n-2} - \dfrac{1}{3n+1}\right) = 4\left(1 - \dfrac{1}{100}\right) = \underline{\dfrac{99}{25}}$.

2. $\sum \dfrac{1}{(\alpha_n - 3)(\beta_n - 3)} = \sum \dfrac{1}{(3n) - 3(3 - n^2) + 9} = \sum \dfrac{1}{3n(n+1)} = \underline{\dfrac{2}{25}}$.

3. $\sum \sqrt{1 + \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2}} = \sum \dfrac{n^2+n+1}{n(n+1)} = 99 + \sum \dfrac{1}{n(n+1)} = \underline{100 - \dfrac{1}{100}}$.

4. $S = 2S - S = 1 + \dfrac{3}{2} + \dfrac{5}{4} + \dfrac{7}{8} + \cdots, \therefore 2S - S = 2 + 2 + \dfrac{2}{2} + \dfrac{2}{4} + \dfrac{2}{8} + \cdots = 2 + 2(2) = \underline{6}$.

5. $-3S - S = -3 + 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \cdots = -\frac{9}{4} \Rightarrow S = \underline{\frac{9}{16}}$.

6. $S - \frac{1}{8}S = \underline{\frac{7}{8}S}$.

7. $k \cdot k! = (k+1)! - k!$. $\therefore (2! - 1!) + (3! - 2!) + \cdots + ((n+1)! - n!) = \underline{(n+1)! - 1}$.

8. $1/(\sqrt{n-1} + \sqrt{n}) = \sqrt{n} - \sqrt{n-1}$. Hence $10 = \sqrt{n} - \sqrt{1} \Rightarrow \underline{n = 121}$.

9. $\frac{k}{(k-2)! + (k-1)! + k!} = \frac{k-1}{k!} = \frac{1}{(k-1)!} - \frac{1}{k!}$. Hence the answer is $\underline{\frac{1}{2} - \frac{1}{50!}}$.

10. Gaussian pairing: $f(k) + f(3-k) = \frac{9^x}{9^x + 27} + \frac{9^{3-x}}{9^{3-x} + 27} = \frac{9^x}{9^x + 27} + \frac{27}{27 + 9^x} = 1$.
Hence $\left(f(\frac{1}{9}) + f(\frac{26}{9})\right) + \left(f(\frac{2}{9}) + f(\frac{25}{9})\right) + \cdots + \left(f(\frac{13}{9}) + f(\frac{14}{9})\right) = \underline{13}$.

Chapter 6

1. $-3 \le \frac{x - 128}{7} < 3 \Leftrightarrow 107 \le x < 149 \Rightarrow \underline{x_{\max} = 148}$.

2. $\left\lfloor \frac{(k+2)^2}{k} \right\rfloor = k + 4 + \left\lfloor \frac{4}{k} \right\rfloor$. Note that $\left\lfloor \frac{4}{k} \right\rfloor = 0$ when $k \ge 5$. Hence the answer is $(1 + \cdots + 40) + 4 \times 40 + 4 + 2 + 1 + 1 = \underline{988}$.

3. $k \le \log_2 n < k + 1 \Leftrightarrow 2^k \le n \le 2^{k+1} - 1$. Thus $\sum_{k=1}^{6} k \cdot 2^k + 7 = \underline{649}$.

4. $\left\lfloor \frac{x^3}{x+3} \right\rfloor = x^2 - 3x + 9 + \left\lfloor -\frac{27}{x+3} \right\rfloor = x^2 - 3x + 8 \equiv \underline{8} \pmod{100}$.

5. $\left\lfloor \frac{x + \sqrt{x}}{x} \right\rfloor = 1 + \left\lfloor \frac{1}{\sqrt{x}} \right\rfloor = 1$. Hence the sum is $\underline{1989}$.

6. Gaussian pairing: $\lfloor ik \rfloor + \lfloor (503 - i)k \rfloor = 304 \Rightarrow \text{Ans} = 251 \times 304 = \underline{76304}$.

7. $k = \left\lfloor \frac{x^2}{1980} \right\rfloor \Leftrightarrow 1980k \le x^2 \le 1980k + 1979$. For $x \le 989$, the squares differ not more than 1979 hence there must exist a unique k for each x. In this case $1 \le k \le 494$ are all achievable. For $x \ge 990$, the squares differ by more than 1979, hence no two x's can produce the same k. Since $990 \le x \le 1980$, there are exactly 991 such k. In total, there are $494 + 991 = \underline{1485 \text{ solutions}}$.

8. Let $n = \lfloor x \rfloor, r = \{x\}$. Then $nr + n + r = 2r + 10 \Rightarrow n(r+1) = r + 10 \Rightarrow n = 1 + \frac{9}{r+1} \in [6, 10]$. Tabulating the values $n = 6, 7, 8, 9, 10$, we find the corresponding r are $\frac{4}{5}, \frac{1}{2}, \frac{2}{7}, \frac{1}{8}, 0$. Hence $\underline{x = 6\frac{4}{5}, 7\frac{1}{2}, 8\frac{2}{7}, 9\frac{1}{8}, 10}$.

9. $1 > \{\sqrt{n}\} = \sqrt{n} - \lfloor \sqrt{n} \rfloor = \frac{n - \lfloor \sqrt{n} \rfloor^2}{\sqrt{n} + \lfloor \sqrt{n} \rfloor} \ge \frac{1}{\sqrt{n} + \lfloor \sqrt{n} \rfloor} > \frac{1}{2\sqrt{n}}$.

10. Note that $n\left\lfloor\frac{x}{n}\right\rfloor \le n\cdot\frac{x}{n} = x$ but since the LHS is an integer, we can strengthen it to $n\left\lfloor\frac{x}{n}\right\rfloor \le \lfloor x\rfloor$. Now since $\left\lfloor\frac{\lfloor x\rfloor}{n}\right\rfloor \le \left\lfloor\frac{x}{n}\right\rfloor$ and $\left\lfloor\frac{\lfloor x\rfloor}{n}\right\rfloor > \frac{\lfloor x\rfloor}{n} - 1 \ge \left\lfloor\frac{x}{n}\right\rfloor - 1$,

$$\left\lfloor\frac{x}{n}\right\rfloor - 1 < \left\lfloor\frac{\lfloor x\rfloor}{n}\right\rfloor \le \left\lfloor\frac{x}{n}\right\rfloor \quad\Rightarrow\quad \left\lfloor\frac{\lfloor x\rfloor}{n}\right\rfloor = \left\lfloor\frac{x}{n}\right\rfloor.$$

Chapter 7

1. The five numbers form an order-2 recursive relation, $(2,3,4,7,?)$. Solving $2a+3b = 4, 3a+4b = 7$ we get $(a,b) = (5.-2)$. Hence $? = 4(5)+7(-2) = \underline{6}$.

2. Let $a_n = (2+2\sqrt{2})^n + (2-2\sqrt{2})^n$. Then $a_n = 4a_{n-1} + 4a_{n-2}$. Starting from $a_0 = 2, a_1 = 4$, we have $a_5 = 232$. Hence $\lfloor(2+2\sqrt{2})^5\rfloor = a_5 = \underline{232}$.

3. Let $a_n = \tan\theta_n$. Then $\theta_n = \theta_{n-1} + 30°$. Hence $\theta_{100} = \theta_1 + 99\cdot 30° = \theta_1 + 90°$, i.e. $a_{100} = \tan(\theta_1+90) = -\frac{1}{a_1} = \underline{-\frac{1}{5}}$.

4. If the first digit is 1 or 3, then the remaining $n-1$ digits obey the same rules. If the first digit is 2, then the second digit is 1 or 3, and the remaining $n-2$ digits obey the same rules. Hence $a_n = 2a_{n-1} + 2a_{n-2}$ which implies $(a_1,\ldots,a_8) = (3,8,22,60,164,448,1224,3344)$. The answer is $a_8 = \underline{3344}$.

5. By induction it is easy to prove that $(a_1,\cdots) = (7,-7,7,-7,\cdots)$. The answer is $7 - 2\cdot 7 + 3\cdot 7 - \cdots + 2019\cdot 7 = 7070 \equiv \underline{70} \pmod{100}$.

6. Let a_n be the number of ways to reach the n-th step. We see that $a_1 = 1, a_2 = 2, a_3 = 4$. Note that we can always reach the n-th step ($n \ge 4$) from the $(n-1), (n-2)$, or $(n-3)$-th step. Hence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Therefore,

$$(a_i)_{i=1}^{10} = (1,2,4,7,13,24,44,81,149,274) \quad\Rightarrow\quad \underline{274 \text{ ways}}.$$

7. Let the number of ways be a_n. There are m ways to colour S_1, $m-1$ ways to colour S_2, then S_3, etc until S_{n-1} (not guaranteeing S_1 and S_n have distinct colour). This gives $m(m-1)^{n-1}$. Now we subtract the number of ways where S_1 and S_n have equal colour. In this case, combine S_1 and S_n into just one sector and the configuration turns into the $n-1$ sector case, hence there are a_{n-1} ways. In total,

$$a_n = m(m-1)^{n-1} - a_{n-1}, \quad a_2 = m(m-1).$$

Solving yields $a_n = \underline{(m-1)^n + (-1)^n(m-1) \text{ ways}}$ to colour the sectors.

8. By subtracting $-nx_n = 225(x_0+\cdots+x_{n-1})$ from $-(n-1)x_{n-1} = 225(x_0+\cdots+x_{n-2})$ we get $\frac{x_n}{x_{n-1}} = \frac{226-n}{-n} \Rightarrow x_n = (-1)^n\binom{225}{n}$. Substituting,

$$\sum_{i=0}^{225} 2^i a_i = \sum_{i=0}^{225} (-2)^i\binom{225}{i} = (1-2)^{225} = \underline{-1}.$$

In the last part we used the binomial theorem.

9. Let the LHS be a_n.

$$\begin{aligned} a_n &= \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n-1}{k} + \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n-1}{k-1} \\ &= \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n-1}{k} - \sum_{k=1}^{n}(-1)^{k}\frac{1}{n}\binom{n}{k} \\ &= a_{n-1} - \frac{1}{n}\left(0 - \binom{n}{0}\right) = a_{n-1} + \frac{1}{n}. \end{aligned}$$

Since $a_1 = 1$, recursively we get $a_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k}$.

Chapter 8

1. $AD^2 = 20^2 + 12^2 - 2\cdot 20\cdot 12\cos B = 544 - 480\cdot\dfrac{20^2+18^2-13^2}{2\cdot 20\cdot 18} = \underline{174}$.

2. By Pythagoras,

$$\begin{aligned} 25^2 &= \left(\frac{CD+AB}{2}\right)^2 + \left(\frac{AD}{2}\right)^2 = \left(\frac{CD+AB+AD}{2}\right)^2 - 2\left(\frac{CD+AB}{2}\right)\left(\frac{AD}{2}\right) \\ &= 31^2 - 2\left(\frac{CD+AB}{2}\right)\left(\frac{AD}{2}\right) \Rightarrow \text{Area} = \frac{(CD+AB)AD}{2} = \underline{336}. \end{aligned}$$

3. $\dfrac{BF}{FD} = \dfrac{BE}{EC} = \dfrac{BD}{DA} = \dfrac{BC}{CA} = 3 \Rightarrow BF : FD : DA = 9:3:4 \Rightarrow BF = \dfrac{9}{7}\cdot 14 = \underline{18}$.

4. $\begin{cases} \dfrac{BF}{BG} = \dfrac{BC-PF}{BC} \\ \dfrac{CE}{CH} = \dfrac{BC-PE}{BC} \end{cases} \Rightarrow \dfrac{1/3}{BG} + \dfrac{1/3}{CH} = 2 - \dfrac{FE}{BC} = \dfrac{4}{3} \Rightarrow \underline{\dfrac{120}{BG} + \dfrac{120}{CH} = 480}.$

5. $\dfrac{[PYZ]}{[ABCD]} = \dfrac{1}{2}\times\dfrac{[WXYZ]}{[ABCD]} = \underline{\dfrac{1}{4}}$.

6. Rotate ΔBAE 90° anticlockwise about B to get $\Delta BCE'$. Then $BEDE'$ is a square, hence $BE = \sqrt{[BEDE']} = \sqrt{[ABCD]} = \underline{12}$.

7. Let the centres of C_1, C_2 be O_1, O_2 respectively, then O_1, C, O_2 are collinear. Let the line parallel to AB intersect AO_1 at D. We have

$$\begin{aligned} BC &= \sqrt{7^2+7^2-2\cdot 7\cdot 7\cos\angle CO_2B} \\ &= \sqrt{98-98\cos(90^\circ+\angle DO_1O_2)} \\ &= \sqrt{98+98\sin\angle DO_1O_2} \\ &= \sqrt{98+98\cdot\frac{9-7}{9+7}} = \underline{\frac{21}{2}}. \end{aligned}$$

8. Let $BH \cap CD = Q$.

$$\begin{aligned}[PQRS] &= \frac{PS}{AG}[AGCE] = \left(1 - \frac{AP}{AG} - \frac{SG}{AG}\right)\frac{AE}{AB}[ABCD] \\ &= \left(1 - \frac{AB}{AB+GD+DQ} - \frac{EB}{EB+CD+DQ}\right)\frac{8}{11}[ABCD] \\ &= \left(1 - \frac{11}{11+3+11(78/7)} - \frac{3}{3+11+11(78/7)}\right)\frac{8}{11}[ABCD].\end{aligned}$$

which simplifies to $\underline{\dfrac{[PQRS]}{[ABCD]} = \dfrac{156}{239}}$.

9. Using the Cosine Rule,

$$\begin{cases} EA^2 + EB^2 &= 2EA \cdot EB \cos\angle AEB + 12^2 \\ EB^2 + EC^2 &= 2EB \cdot EC \cos\angle BEC + 8^2 \\ EC^2 + ED^2 &= 2EC \cdot ED \cos\angle CED + 7^2 \\ ED^2 + EA^2 &= 2ED \cdot EA \cos\angle DEA + 5^2 \end{cases}$$

Since $\cos\angle AEB = \cos\angle CED = -\cos\angle BEC = -\cos\angle DEA$, combining equations 1,3 and 2,4 above gives the following rearranged form:

$$(EA \cdot EB + EC \cdot CD + EB \cdot EC + ED \cdot EA)\cos\angle AED = 52$$
$$\Rightarrow (EA + EC)(EB + ED)\cos\angle AED = 52$$
$$\therefore [ABCD] = \frac{AC \cdot BD \sin AED}{2} = \frac{52 \sin AED}{2\cos AED} = 26 \times 2 = \underline{52}.$$

10. From Menelaus, $\dfrac{BC}{CD} = \dfrac{FB}{AF}, \dfrac{AB}{BF} = \dfrac{EA}{CE}, \dfrac{CA}{AE} = \dfrac{CD}{DB}$ hence

$$\frac{BC}{CD} = \frac{1}{\frac{AB}{BF} - 1} = \frac{1}{\frac{1}{\frac{CA}{AE}-1} - 1} = \frac{1}{\frac{1}{\frac{CD}{DB}-1} - 1} = \frac{1}{\frac{1}{\frac{1}{\frac{BC}{CD}-1}-1} - 1}.$$

$\therefore \dfrac{BC}{CD} = \dfrac{1+\sqrt{5}}{2} := \phi$. Similarly $\dfrac{AB}{BF} = \dfrac{CA}{AE} = \phi$. Repeatedly using Menelaus again we have $\dfrac{UV}{UE} = \dfrac{1}{\phi}, \dfrac{WU}{UA} = \dfrac{1}{1+\phi}$. Hence $[WUV] = \dfrac{UV}{UE} \cdot \dfrac{WU}{UA} \cdot [UAE] = \dfrac{1}{\phi} \cdot \dfrac{1}{1+\phi} \cdot \dfrac{1}{2} \cdot [BAE] = \dfrac{1}{\phi} \cdot \dfrac{1}{1+\phi} \cdot \dfrac{1}{2} \cdot \dfrac{1}{\phi} \cdot (12) = \underline{21 - \sqrt{405}}$.

Chapter 9

1. $\Leftrightarrow (\sqrt{a} - \sqrt{b})^2 \geq 0$.

2. $\Leftrightarrow (x-y)^2 + (x-1)^2 + 2(y-1)^2 \geq 0$.

3. Multiply $a + b \geq 2\sqrt{ab}$ and its equivalents.

4. Multiply $(a+b-c) + (a+c-b) \geq 2\sqrt{(a+b-c)(a+c-b)}$ and its equivalents.

5. $303 - \dfrac{200x^2}{16+x^4} \geq 303 - \dfrac{200x^2}{2\sqrt{16x^4}} = \underline{278}$. Equality iff $x = 2$.

6. Cauchy, $\sqrt{2x-1}+\sqrt{243-2x} \leq \sqrt{(2x-1+243-2x)(1+1)} = \underline{22}$. Equality iff $x = 61$.

7. Let $(3\cos\theta, 2\sin\theta) \in \mathbb{R}^+$ be on the ellipse. Then the area is $4(3\cos\theta)(2\sin\theta) = 12\sin 2\theta \leq \underline{12}$. Equality iff $\theta = 45°$.

8. $\geq 2\sqrt{\frac{x^2}{2}\cdot\frac{162}{x^2}} = \underline{18}$. Equality iff $x = \pm\sqrt{18}$.

9. Cauchy, $\geq (1+3+5)^2 = \underline{81}$. Equality iff $(a,b,c) = (n, 3n, 5n)$.

10. Equality iff $2x = 3y = \frac{2016}{2} \Rightarrow x - y = 504 - 336 = \underline{168}$.

11. $\text{dist}((1,-3) \leftrightarrow (x,y) \leftrightarrow (-11,2)) \geq \sqrt{(1+11)^2+(-3-2)^2} = \underline{13}$.

12. $4x^2+3y^2+5z^2 = (x^2+y^2)+2(y^2+z^2)+3(x^2+z^2) \geq 2xy+4yz+6xz = \underline{14}$. Equality iff $x = y = z = \pm\sqrt{\frac{7}{6}}$.

13. Since $(x-4)^2+(y+3)^2 = 13^2$, let $(x,y) = (13\cos\theta+4, 13\sin\theta-3)$. Then $x^2+y^2 = 194+104\cos\theta-78\sin\theta = 194+130\cos(\theta-\alpha) \geq \underline{64}$.

14. $= \sqrt{x^2+\frac{1}{x^2}+1} - \sqrt{x^2+\frac{1}{x^2}} := \sqrt{u+1}-\sqrt{u} = \frac{1}{\sqrt{u+1}+\sqrt{u}} \leq \frac{1}{\sqrt{3}+\sqrt{2}} = \underline{\sqrt{3}-\sqrt{2}}$ where $u \geq 2$ due to AM-GM.

15. Let $u = \sqrt{x+1}, v = \sqrt{4-2x} (u, v \geq 0)$. Then $u^2/3 + v^2/6 = 1$ hence the point lies on this ellipse. The line $u + v = k$ is a line with gradient -1. To achieve maximum k, the line is tangent to the ellipse, hence we solve (u,v) by discriminant $= 0$ to get $(u,v) = (1,2) \Rightarrow k = 3$. On the other hand, to achieve minimum k, the line must pass through $(\sqrt{3}, 0)$, otherwise there will be no more intersections with $u, v \geq 0$ if k is even lower. Hence $(u,v) = (\sqrt{3},0) \Rightarrow k = \sqrt{3}$. Therefore, $\underline{\sqrt{3} \leq \sqrt{x+1}+\sqrt{4-2x} \leq 3}$.

Chapter 10

1. $(x.y) = (-1+4t, 26-3t)$ hence there are <u>8 solutions</u>.

2. $1067(q-p) = 86pq$ Hence $p, q \mid 1067 = 11 \times 97 \Rightarrow (p,q) = (11, 97)$. Therefore $\underline{p+q = 108}$.

3. $3x+7y = 1009$. $(x,y) = (334-7t, 1+3t)$. Hence <u>48 solutions</u>.

4. $(x+y)^2 = 3(x+y+xy)$ and hence we infer $3 \mid x+y, xy \Rightarrow x = 3X, y = 3Y$.

$$\begin{gathered} X+Y+XY = X^2+Y^2 \geq 2|XY| \\ 0 \geq 2|XY| - XY - X - Y \geq |XY| - |X| - |Y| \\ (|X|-1)(|Y|-1) \leq 1 \\ \therefore X, Y \in \{-2,-1,0,1,2\}. \end{gathered}$$

Testing yields $(X,Y) = (0,0),(0,1),(1,0),(1,2),(2,1)$ which gives the solutions $\underline{(x,y) = (0,0),(0,3),(3,0),(3,6),(6,3)}$.

5. WLOG let $p \leq q \leq r$. Then $pqr \leq 33r \Rightarrow p \leq \sqrt{33} \Rightarrow p = 2, 3, 5$.
If $p = 2$, then $(2q-11)(2r-11) = 165 \Rightarrow (q, r) = (11, 13)$.
If $p = 3$, then $(3q-11)(3r-11) = 220 \Rightarrow (q, r) = (7, 11)$.
If $p = 5$, then $5 \leq q \leq 33/5 \Rightarrow q = 5 \Rightarrow r \notin \mathbb{N}$.
Hence the largest $p + q + r = \max(2 + 11 + 13, 3 + 7 + 11) = \underline{26}$.

6. $z \leq (399-19-20)/21 \Rightarrow z \leq 17$. Taking mod 20, we have $x \equiv z+1 \pmod{20}$. Since $0 < z+1 < 20$, we have $x = 20k+z+1$ $(k \geq 0)$. Hence $19(20k+z+1)+20y+21z = 399 \Rightarrow 2z + y = 19(1-k) > 0$. Therefore $k = 0$ and we solve $2z + y = 19, x = z+1$ to get $(x, y, z) = (2, 17, 1), (3, 15, 2), (4, 13, 3), \ldots, (10, 1, 7)$, which are $\underline{\text{8 solutions}}$.

7. $n = (2a-1)(a-3)$ is prime, hence $2a-1 = 1$ (leads to $n = -2$) or $a-3 = 1 \Rightarrow n = 7$. Therefore $\underline{n = 7}$.

8. $k = \dfrac{8a}{b+2c} = \dfrac{4b}{c+3a} = \dfrac{3c}{5a+3b} = \dfrac{8a+4b+3c}{b+2c+c+3a+5a+3b} = 1$. Hence

$$\begin{cases} 8a = b + 2c \\ 4b = c + 3a \\ 3c = 5a + 3b \end{cases} \Rightarrow a : b : c = 9 : 14 : 29 \Rightarrow (a, b, c) = (9, 14, 29) \Rightarrow \underline{b = 14}.$$

9. $\sqrt{y} = \sqrt{2004} - \sqrt{x} \Rightarrow y = 2004 + x - 2\sqrt{2^2 \cdot 3 \cdot 167x} \Rightarrow x = 501a^2$. Similarly, $y = 501b^2$. The equation becomes $a + b = 2 \Rightarrow (a, b) = (0, 2), (1, 1), (2, 0) \Rightarrow \underline{(x, y) = (0, 2004), (501, 501), (2004, 0)}$.

10. We solve the bonus problem instead: Note that $(x, y, z) \Leftrightarrow (-x, -y, -z)$, so WLOG $y > 0$. Also we have $2xy + 3yz = 99$. Hence we can tabulate y:

y		63	(x, z)
1	$2x + 3z = 99$	$2xz - z$	$\varnothing$
3	$2x + 3z = 33$	$2xz - 3z$	$(6, 7), (12, 3)$
9	$2x + 3z = 11$	$2xz - 9z$	$\varnothing$
11	$2x + 3z = 9$	$2xz - 11z$	$\varnothing$
33	$2x + 3z = 3$	$2xz - 33z$	$(6, -3), (12, -7)$
99	$2x + 3z = 1$	$2xz - 99z$	$\varnothing$

Including their opposites, we have $2 \times 4 = \underline{\text{8 solutions}}$.

11. Again WLOG $a \leq b \leq c \leq d$. Then $a = 2, 3, 4$. If $a = 2$, then $2 < b \leq 6$, hence

b	Final equation	(a, b, c, d)
3	$(c-6)(d-6) = 2^2 \cdot 3^2$	4 tuples of distinct numbers + $(2, 3, 12, 12)$
4	$(c-4)(d-4) = 2^4$	2 tuples of distinct numbers + $(2, 4, 8, 8)$
5	$(3c-10)(3d-10) = 2^2 \cdot 5^2$	$(2, 5, 5, 10)$
6	$(c-3)(d-3) = 3^2$	$(2, 6, 6, 6)$

If $a = 3$, then $3 \leq b \leq 4$, hence

b	Final equation	(a, b, c, d)
3	$(c-6)(d-6) = 2^2 \cdot 3^2$	$(3, 3, 4, 12), (3, 3, 6, 6)$
4	$(c-4)(d-4) = 2^4$	$(3, 4, 4, 6)$

If $a = 4$, then $b = 4$ and hence $(a, b, c, d) = (4, 4, 4, 4)$.

Now, counting all permutations we have $6 \times 4! + 5 \times \frac{4!}{2!} + 1 \times \frac{4!}{2! \cdot 2!} + 1 \times \frac{4!}{3!} + 1 \times \frac{4!}{4!} =$ $\underline{215 \text{ solutions}}$.

12. Note that the total number of ordered solutions (a, b, c) to $a + b + c = 2n$ is $\binom{2n+2}{2}$. Let the answer be $f(n)$, and let $g(n)$ be the number of solutions to $a + a + b = 2n,\ a \neq b$. If $3 \nmid n$, then $(a, b) = (0, 2n), (1, 2n - 2), \ldots, (n, 0)$ are solutions, hence $g(n) = n + 1$. Since (a, a, b) can be permuted in 3 ways,

$$\binom{2n+2}{2} = 3g(n) + 3!f(n) \Rightarrow f(n) = \frac{n^2 - 1}{3}.$$

If $3 \mid n$, then $(a, b) = (0, 2n), (1, 2n - 2), \ldots, (n, 0)$ except $(2n/3, 2n/3)$ are solutions, hence $g(n) = n$. Now, considering $\{2n/3, 2n/3, 2n/3\}$ is a new solution,

$$\binom{2n+2}{2} = 1 + 3g(n) + 3!f(n) \Rightarrow f(n) = \frac{n^2}{3}.$$

In conclusion,

$$f(n) = \begin{cases} \dfrac{n^2 - 1}{3} & \text{if } 3 \nmid n \\ \dfrac{n^2}{3} & \text{if } 3 \mid n \end{cases} \qquad \text{or} \qquad \underline{f(n) = \left\lfloor \frac{n^2}{3} \right\rfloor}$$

Chapter 11

1. Vowels: O, E, I, I, O. Consonants: C, M, P, T, T, N. Total number $= \frac{5!}{2!} \times \frac{6!}{2!} =$ $\underline{21600 \text{ solutions}}$.

2. Each $(a, b, c) \in H$ one-to-one corresponds to $\{a, c\} \in S$ such that a, c are either both even or both odd. Since there are 100 even numbers and 100 odd numbers, the total number is $\binom{100}{2} + \binom{100}{2} = \underline{9900 \text{ solutions}}$.

3. Using $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, we get a telescoping series which ends up with

$$\left(\binom{36}{0} + \binom{36}{1}\right) - \left(\binom{36}{1} + \binom{36}{2}\right) + \cdots - \left(\binom{36}{17} + \binom{36}{18}\right) = \underline{1 - \binom{36}{18}}.$$

4. $\binom{31}{2} = \underline{465 \text{ solutions}}$.

5. For every desired 5-tuple $X = (x_1, x_2, x_3, x_4, x_5)$, we consider the 5-tuple $Y = (y_1, y_2, y_3, y_4, y_5) = (x_1, x_2 + 1, x_3 + 2, x_4 + 3, x_5 + 4)$. Then $1 \leq y_1 < y_2 < y_3 < y_4 < y_5 \leq 9 + 4 = 13$. This is a one-to-one correspondence, hence the number of X is equal to the number of Y, which is $\binom{13}{5} = \underline{1287 \text{ solutions}}$.

6. We have $a_2 \geq a_1 + 2$ and $a_3 \geq a_2 + 4$, so we consider the triple $(a_1, a_2 - 1, a_3 - 3)$. Then $1 \leq a_1 < a_2 - 1 < a_3 - 3 \leq 27$ and this is a one-to-one correspondence, hence the total number is $\binom{27}{3} = \underline{2925 \text{ solutions}}$.

7. Let A, B, C, D be respectively the set of students in the club A, B, C, D.

$$\begin{aligned}|A \cap (B \cup C \cup D)| &= |A \cap B| + |A \cap C| + |A \cap D| - |A \cap B \cap C| - |A \cap B \cap D| \\ &\quad - |A \cap C \cap D| + |A \cap B \cap C \cap D| \\ &= 3 \times 227 - 3 \times 117 + 17 = 347.\end{aligned}$$

Therefore, club A has at least $\underline{347 \text{ students}}$.

8. Ignoring the constraint ≥ 10, we have either $\{\text{even}, \text{even}, \text{even}\}$ or $\{\text{even}, \text{odd}, \text{odd}\}$, hence $\binom{5}{3} + \binom{5}{2}\binom{5}{1} = 60$ ways. The triples that sum up to an even number less than 10 are: $\{0, 1, \text{"3 or 5 or 7"}\}, \{0, 2, \text{"4 or 6"}\}, \{0, 3, 5\}, \{1, 2, \text{"3 or 5"}\}, \{1, 3, 4\}$. Therefore the total number would be $\underline{60 - 9 = 51 \text{ ways}}$.

9. Let A_i be the set of solutions where $x_i \geq 9$. The total number of solutions without the constraint is $\binom{20+5}{5} = 53130$. By P.I.E, there are a total number of

$$\begin{aligned}53130 - \binom{6}{1}|A_1| + \binom{6}{2}|A_1 \cap A_2| &= 53130 - 6\binom{11+5}{5} + 15\binom{2+5}{5} \\ &= \underline{27237 \text{ solutions}}.\end{aligned}$$

10.

$$\begin{aligned}\sum_{k=p}^{n}(-1)^k\binom{n}{k}\binom{k}{p} &= \sum_{k=p}^{n}(-1)^k\binom{n}{p}\binom{n-p}{k-p} \\ &= \binom{n}{p}\sum_{k=p}^{n}(-1)^k\binom{n-p}{k-p} \\ &= \binom{n}{p}\sum_{k=0}^{n-p}(-1)^{k+p}\binom{n-p}{k} \\ &= \underline{\begin{cases}(-1)^n & (n = p) \\ 0 & (n \neq p)\end{cases}}\end{aligned}$$

Chapter 12

1. $\cos\angle AOB = \dfrac{(1)(3) + (5)(-2) + (3)(1)}{\sqrt{1^2 + 5^2 + 3^2} \cdot \sqrt{3^2 + 2^2 + 1^2}} = \underline{-\dfrac{2\sqrt{10}}{35}}$.

2. Let $\vec{x} = \overrightarrow{AB}, \vec{y} = \overrightarrow{AC}$. Then

$$\begin{aligned}
|\overrightarrow{AD}|^2 &= \left|\frac{1}{3}\vec{x} + \frac{2}{3}\vec{y}\right|^2 \\
&= \frac{|\vec{x}|^2}{9} + \frac{4|\vec{y}|^2}{9} + \frac{4}{9}\vec{x}\cdot\vec{y} \\
&= \frac{|\vec{x}|^2}{9} + \frac{4|\vec{y}|^2}{9} + \frac{4}{9}|\vec{x}||\vec{y}|\cos\theta \\
&= \frac{20^2}{9} + \frac{4(13)^2}{9} + \frac{4}{9}(20)(13)\frac{20^2+13^2-18^2}{2(20)(13)} \\
&= \underline{174}.
\end{aligned}$$

3. Let θ be the angle between XY and a vector perpendicular to $\mathcal{P}$. Then

$$d = |\overrightarrow{XY}|\cos\theta = |\overrightarrow{XY}|\frac{\overrightarrow{XY}\cdot\vec{n}}{|\overrightarrow{XY}||\vec{n}|} = \frac{\overrightarrow{XY}\cdot\vec{n}}{|\vec{n}|}.$$

4. Let $\vec{x} = \overrightarrow{CA}, \vec{y} = \overrightarrow{CB}, \vec{z} = \overrightarrow{AA_1}$. Then

$$\overrightarrow{AB_1}\cdot\overrightarrow{A_1M} = (-\vec{x}+\vec{y}+\vec{z})\cdot\left(-\vec{x}-\frac{1}{2}\vec{z}\right) = |\vec{x}|^2 - \frac{1}{2}|\vec{z}|^2 = 3 - \frac{1}{2}(6) = 0.$$

5. $\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = \frac{8}{13}\overrightarrow{PA} + \frac{5}{13}\overrightarrow{AD} + \frac{8}{13}\overrightarrow{AB} = \frac{8}{13}\overrightarrow{PB} + \frac{5}{13}\overrightarrow{BC}$ which is a linear combination on the plane PBC.

6. Let $\vec{x} = \overrightarrow{AB}, \vec{y} = \overrightarrow{AD}, \vec{z} = \overrightarrow{AA'}$. We find that $\overrightarrow{A'E} = \frac{1}{2}\vec{x} + \frac{1}{2}\vec{y} - \frac{3}{4}\vec{z}, \overrightarrow{A'N} = \frac{2}{3}\vec{x} + \frac{2}{3}\vec{y} - \vec{z}$. Hence $4\overrightarrow{A'E} = 3\overrightarrow{A'N}$, i.e. $A'EN$ are collinear.

7. Let $\vec{x} = \overrightarrow{AB}, \vec{y} = \overrightarrow{AD}, \vec{z} = \overrightarrow{AA'}$. Denote vectors $A'BD \perp \vec{u} = a\vec{x} + b\vec{y} + \vec{z}$ and $EBD \perp \vec{v} = c\vec{x} + d\vec{y} + \vec{z}$.

$$\begin{cases} \vec{u}\cdot\overrightarrow{A'B} = (a\vec{x}+b\vec{y}+\vec{z})\cdot(\vec{x}-\vec{z}) = a-1 = 0 \\ \vec{u}\cdot\overrightarrow{BD} = (a\vec{x}+b\vec{y}+\vec{z})\cdot(\vec{x}-\vec{y}) = a-b = 0 \end{cases}$$

$$\begin{cases} \vec{v}\cdot\overrightarrow{BE} = (c\vec{x}+d\vec{y}+\vec{z})\cdot\left(\vec{y}+\frac{1}{2}\vec{z}\right) = d+\frac{1}{2} = 0 \\ \vec{v}\cdot\overrightarrow{BD} = (c\vec{x}+d\vec{y}+\vec{z})\cdot(\vec{x}-\vec{y}) = c-d = 0 \end{cases}$$

$$\therefore \vec{u}\cdot\vec{v} = (\vec{x}+\vec{y}+\vec{z})\cdot\left(-\frac{1}{2}\vec{x}-\frac{1}{2}\vec{y}+\vec{z}\right) = -\frac{1}{2}-\frac{1}{2}+1 = 0.$$

8. We prove using signed lengths. Let $\vec{x} = \overrightarrow{AB}, \vec{y} = \overrightarrow{AC}, \dfrac{AP}{PB} = p, \dfrac{BQ}{QC} = q, \dfrac{CR}{RA} = r$.

$$\begin{aligned}
\overrightarrow{PQ} &= \lambda\overrightarrow{QR} \\
\overrightarrow{PB} + \overrightarrow{BQ} &= \lambda(\overrightarrow{QC} + \overrightarrow{CR}) \\
\frac{PB}{AB}\vec{x} + \frac{BQ}{BC}(\vec{y} - \vec{x}) &= \lambda\left(\frac{QC}{BC}(\vec{y} - \vec{x}) + \frac{CR}{AC}\vec{y}\right) \\
\frac{1}{1+p}\vec{x} + \frac{1}{1+1/q}(\vec{y} - \vec{x}) &= \lambda\left(\frac{1}{1+q}(\vec{y} - \vec{x}) + \frac{1}{-1-1/r}\vec{y}\right) \\
\left(\frac{1}{1+p} - \frac{1}{1+1/q}\right)\vec{x} + \frac{1}{1+1/q}\vec{y} &= -\frac{\lambda}{1+q}\vec{x} + \left(\frac{\lambda}{1+q} - \frac{\lambda}{1+1/r}\right)\vec{y} \\
\therefore \left(\frac{1}{1+p} - \frac{1}{1+1/q}\right) : \frac{1}{1+1/q} &= -\frac{1}{1+q} : \left(\frac{1}{1+q} - \frac{1}{1+1/r}\right) \\
&\Rightarrow pqr = -1
\end{aligned}$$

Taking absolute values, $\dfrac{AP}{PB} \cdot \dfrac{BQ}{QC} \cdot \dfrac{CR}{RA} = 1$.

NOTE: In problem 8 we used signed lengths (also known as directed lengths). These are lengths that carry a sign (plus or minus) according to which direction they are pointing at, in that case, $XY + YZ = XZ$ no matter where the points X, Y, Z are on a line. That's why signed lengths are useful in dealing with ambiguous configurations. Using signed lengths, the Menelaus Theorem would be written as

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RA} = -1$$

instead of 1.

Chapter 13

1.

$$\begin{aligned}
f'(g(x)) \cdot g'(x) &= \lim_{h\to 0} \frac{f(g(x)+h) - f(g(x))}{h} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(g(x) + g(x+h) - g(x)) - f(g(x))}{g(x+h) - g(x)} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = (f \circ g)'(x).
\end{aligned}$$

2. We can apply the product and chain rule:

$$\begin{aligned}
f'(x) &= u(x)\left(\frac{1}{v(x)}\right)' + u'(x) \cdot \frac{1}{v(x)} \\
&= u(x)\left(-\frac{v'(x)}{v(x)^2}\right) + u'(x) \cdot \frac{1}{v(x)} \\
&= \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}.
\end{aligned}$$

3. Apply L'Hôpital's Rule twice:

$$f'(0) = \lim_{x\to 0}\left(\frac{x}{e^x-1}\right)' = \lim_{x\to 0}\frac{(e^x-1)(1)-(e^x)(x)}{(e^x-1)^2}$$
$$\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{e^x-(e^x+e^x x)}{2(e^x-1)e^x} = \lim_{x\to 0}\frac{-x}{2(e^x-1)} \overset{\text{L'H}}{=} \frac{-1}{2e^x} = \underline{-\frac{1}{2}}.$$

4.

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x}-x\right) = \lim_{x\to\infty}\frac{3x}{\sqrt{x^2+3x}+x} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{3}{\dfrac{2x+3}{2\sqrt{x^2+3x}}+1}$$
$$= \lim_{x\to\infty}\frac{3}{\dfrac{2+3/x}{2\sqrt{1+3/x}}+1} = \underline{\frac{3}{2}}.$$

5. $\displaystyle\int_0^1 f(x)\ dx + 3f(1) = -3\int_0^1 f'(x)\ dx + 3f(1) = -3(f(1)-f(0)) + 3f(1) = \underline{6}.$

Chapter 14

1.

$$\sum_{k=p}^{n}\binom{n}{k}\binom{k}{p} = \sum_{k=p}^{n}\binom{n}{k}C_p[(1+x)^k] = C_p\left[\sum_{k=p}^{n}\binom{n}{k}(1+x)^k\right]$$
$$= C_p\left[\sum_{k=0}^{n}\binom{n}{k}(1+x)^k\right] = C_p[(1+(1+x))^n] = 2^{n-p}\binom{n}{p}.$$

2.

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n} = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}C_{2k}[(1-x^2)^n]C_{n-2k}\left[\frac{1}{(1-x)^{n+1}}\right]$$
$$= C_n\left[\frac{(1-x^2)^n}{(1-x)^{n+1}}\right] = C_n\left[\frac{(1+x)^n}{1-x}\right] = C_n\left[\frac{(2-(1-x))^n}{1-x}\right]$$
$$= C_n\left[\sum_{k=0}^{n}2^{n-k}(-1)^k(1-x)^{k-1}\right] = C_n\left[\frac{2^n}{1-x}\right] = 2^n.$$

3.

$$\binom{2n}{n} = C_n[(1+x)^{2n}] = C_n[(1+x^2+2x)^n] = C_n\left[\sum_{k=0}^{n}\binom{n}{k}(2x)^{n-k}(1+x^2)^k\right]$$
$$= C_n\left[\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(2x)^{n-2k}(1+x^2)^{2k}\right] = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}2^{n-2k}C_{2k}\left[(1+x^2)^{2k}\right]$$
$$= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}2^{n-2k}\binom{2k}{k} = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}2^{n-2k}\binom{n}{n-2k}\binom{2k}{k}.$$

4.

$$\begin{aligned}
&\sum_{k=0}^{n} 2^k \binom{n}{k}\binom{n-k}{\lfloor \frac{n-k}{2}\rfloor} = \sum_{k=0}^{n} 2^{n-k}\binom{n}{k}\binom{k}{\lfloor \frac{k}{2}\rfloor}\\
&= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} 2^{n-2k}\binom{n}{2k}\binom{2k}{k} + \sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor} 2^{n-2k-1}\binom{n}{2k+1}\binom{2k+1}{k}\\
&= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} 2^{n-2k}\binom{n}{2k} C_{2k}[(1+x^2)^{2k}] + \sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor} 2^{n-2k-1}\binom{n}{2k+1} C_{2k}[(1+x^2)^{2k+1}]\\
&= C_n\left[(2x+1+x^2)^n\right] + C_{n-1}\left[(2x+1+x^2)^n\right]\\
&= \binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n}.
\end{aligned}$$

5.

$$\begin{aligned}
&\sum_{k=0}^{n-m}\binom{n}{m+k}\binom{m+k}{m} = \sum_{k=0}^{n-m}\binom{n}{m+k} C_m\left[(1+x)^{m+k}\right]\\
&= C_m\left[\sum_{k=0}^{n-m}\binom{n}{m+k}(1+x)^{m+k}\right] = C_m\left[\sum_{k=0}^{n}\binom{n}{k}(1+x)^k\right]\\
&= C_m\left[(1+(1+x))^n\right] = 2^{n-m}\binom{n}{m}.
\end{aligned}$$

6.

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{a+k}{b+k} = \sum_{k=0}^{n}\binom{a+k}{a-b} = \sum_{k=0}^{n+b}\binom{a-b+k}{a-b} - \sum_{k=0}^{b-1}\binom{a-b+k}{a-b}\\
&= \sum_{k=0}^{n+b} C_{n+b-k}\left[\frac{1}{1-x}\right] C_k\left[\frac{1}{(1-x)^{a-b+1}}\right] - \sum_{k=0}^{b-1} C_{b-1-k}\left[\frac{1}{1-x}\right] C_k\left[\frac{1}{(1-x)^{a-b+1}}\right]\\
&= C_{n+b}\left[\frac{1}{(1-x)^{a-b+2}}\right] - C_{b-1}\left[\frac{1}{(1-x)^{a-b+2}}\right] = \binom{a+n+1}{b+n} - \binom{a}{b-1}.
\end{aligned}$$

7.

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{2^k}\binom{2k}{k}x^k &= \sum_{k=0}^{\infty}\frac{(2k-1)(2k-3)\dots(1)}{k!}x^k\\
&= \sum_{k=0}^{\infty}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\dots\cdot\frac{2k-1}{2}}{k!}(2x)^k\\
&= \sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\dots\left(-\frac{1}{2}-(k-1)\right)}{k!}(-2x)^k\\
&= \underline{(1-2x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-2x}}}.
\end{aligned}$$

8.

$$
\begin{aligned}
&\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k}(1-(1-x)^k)\\
&= \int_0^x \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k}(k(1-X)^{k-1})dX\\
&= \int_0^x \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(1-X)^{k-1}dX\\
&= -\int_0^x \frac{1}{1-X}\sum_{k=1}^{n}(-1)^k\binom{n}{k}(1-X)^k dX\\
&= -\int_0^x \frac{1}{1-X}\left[(1-(1-X))^n-1\right]dX\\
&= \int_0^x \frac{1-X^n}{1-X}dX = \int_0^x (1+X+\cdots+X^{n-1})dX\\
&= x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}.
\end{aligned}
$$

9.

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} &= \sum_{k=0}^{n}\binom{n}{k}C_n[(1+x)^{n+k}]\\
&= C_n\left[(1+x)^n\sum_{k=0}^{n}\binom{n}{k}(1+x)^k\right]\\
&= C_n\left[(1+x)^n(1+1+x)^n\right]\\
&= \sum_{k=0}^{n} C_{n-k}\left[(1+x)^n\right]C_k\left[(2+x)^n\right]\\
&= \sum_{k=0}^{n} 2^k\binom{n}{k}^2.
\end{aligned}
$$

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