Multivariate Calculus

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1 Line and Surface Integration

1.1 Line Integrals

In 1 dimension, we can integrate along any interval on the x-axis, where each step we move forward by $\mathrm{d}x$. In 2 or 3 dimensions, we can first view the function as a function of the position vector: $f(x, y, \cdots) = f(\mathbf{r})$. Next, instead of integrating along a fixed line (the x-axis previously), we can now choose any smooth curve $\mathcal{C}$ and integrate along it, where each step we move forward by $\mathrm{d}\mathbf{r}$ on the curve. Formally, it is defined as such:

Definition. Given a curve $\mathcal{C}$ (with direction) with some parametrisation $(x(t), y(t), \cdots)$ where $a \leq t \leq b$,

$$\int_{\mathcal{C}} f(\mathbf{r})\ \mathrm{d}\mathbf{r} = \int_a^b f(\mathbf{r})\ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\ \mathrm{d}t.$$

This type of integral is called a *line integral*[1]. The good thing is that $\int_{\mathcal{C}} f(\mathbf{r})\ \mathrm{d}\mathbf{r}$ is independent of the parametrisation of $\mathcal{C}$. We shall not prove that here. Let's look at a few examples:

Example. Consider the surface $f(x, y) = 1$. If we integrate anticlockwise along the quarter circle with radius R centred at the origin ($\mathcal{C}$), we can first parametrise it as $(R\cos\theta, R\sin\theta)$ where $0 \leq \theta \leq \pi/2$, so

$$\begin{aligned}
\int_{\mathcal{C}} f(\mathbf{r})\ \mathrm{d}\mathbf{r} &= \int_0^{\pi/2} \frac{\mathrm{d}}{\mathrm{d}\theta} \begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix}\ \mathrm{d}\theta \\
&= RH \int_0^{\pi/2} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\ \mathrm{d}\theta \\
&= \begin{pmatrix} -R \\ R \end{pmatrix}
\end{aligned}$$

which makes sense because summing up all the little vectors in Figure 12 gives the vector $\begin{pmatrix} -R \\ R \end{pmatrix}$.

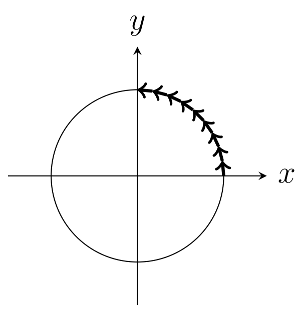

Figure 1: Caption

Instead of summing up $f\ \mathrm{d}\mathbf{r}$ where f is a scalar function, we can also sum up $\mathbf{f} \cdot \mathrm{d}\mathbf{r}$ or $\mathbf{f} \times \mathrm{d}\mathbf{r}$ where $\mathbf{f}$ is a vector field.

Example. If $\mathcal{C}$ is the unit circle anticlockwise, parametrised by $(R\cos\theta, R\sin\theta)$ where $0 \leq \theta \leq 2\pi$,

[1] Might as well call it a *curve integral*.

then

$$
\begin{aligned}
&\int_{\mathcal{C}} \begin{pmatrix} -y/\sqrt{x^2+y^2} \\ x/\sqrt{x^2+y^2} \end{pmatrix} \cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta} \begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix} \; \mathrm{d}\theta \\
&= R \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \; \mathrm{d}\theta \\
&= 2\pi R
\end{aligned}
$$

which makes sense because $\begin{pmatrix} -y/\sqrt{x^2+y^2} \\ x/\sqrt{x^2+y^2} \end{pmatrix}$ is a circular vector field with length 1 everywhere, and we are just integrating the dot product along the circle, so we get the circumference of it.

Note. If the curve $\mathcal{C}$ is a closed curve like in the example above, we can write $\oint_{\mathcal{C}}$ instead of $\int_{\mathcal{C}}$. This just emphasises it is a closed curve. It is optional.

Example. Consider $\mathcal{C}$ as the line connecting $(0,0,0)$ to $(1,0,0)$ and consider the uniform vector field $\mathbf{f}(\mathbf{r}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ everywhere. Then

$$
\int_{\mathcal{C}} \mathbf{f}(\mathbf{r}) \times \mathrm{d}\mathbf{r} = \int_0^1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \; \mathrm{d}t = \int_0^1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; \mathrm{d}t = \int_0^1 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \; \mathrm{d}t = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.
$$

1.2 Surface Integrals

We can do line integrals in both 2 and 3 dimensions. However, *surface integrals* only work in 3 dimensions. For a given surface $\mathcal{S}$, we can first give it an orientation, i.e. which side is the 'positive' side and which side is the 'negative side'. **Note: For closed surfaces, we define by convention the positive side to be the outer side.** Then, for any infinitesimal area $\mathrm{d}A$ on the surface at a point, we will define the vector $\mathrm{d}\mathbf{A}$ as the vector originating from the point, pointing outwards from the 'positive' side, with length $\mathrm{d}A$. We can sometimes write this as $\mathrm{d}\mathbf{A} = \hat{\mathbf{n}}\; \mathrm{d}A$.

We can use a parametrisation definition for the surface $\mathcal{S}$ as we did for line integrals. However, for our purposes, we just need to know the concept of it:

Example. Let $\mathbf{f}(\mathbf{r})$ be the radial vector field of length 1 everywhere:

$$\mathbf{f}(\mathbf{r}) = \hat{\mathbf{r}}.$$

Then if we let $\mathcal{S}$ be the surface of the sphere of radius R centred at the origin, we know $\hat{\mathbf{r}} = \hat{\mathbf{n}}$ everywhere on $\mathcal{S}$. Hence

$$
\begin{aligned}
\oint_{\mathcal{S}} \mathbf{f}(\mathbf{r}) \cdot \mathrm{d}\mathbf{A} &= \oint_{\mathcal{S}} \hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{A} \\
&= \oint_{\mathcal{S}} \hat{\mathbf{r}} \cdot \hat{\mathbf{n}}\; \mathrm{d}A \\
&= \oint_{\mathcal{S}} \mathrm{d}A \\
&= \text{surface area of } S \\
&= 4\pi R^2.
\end{aligned}
$$

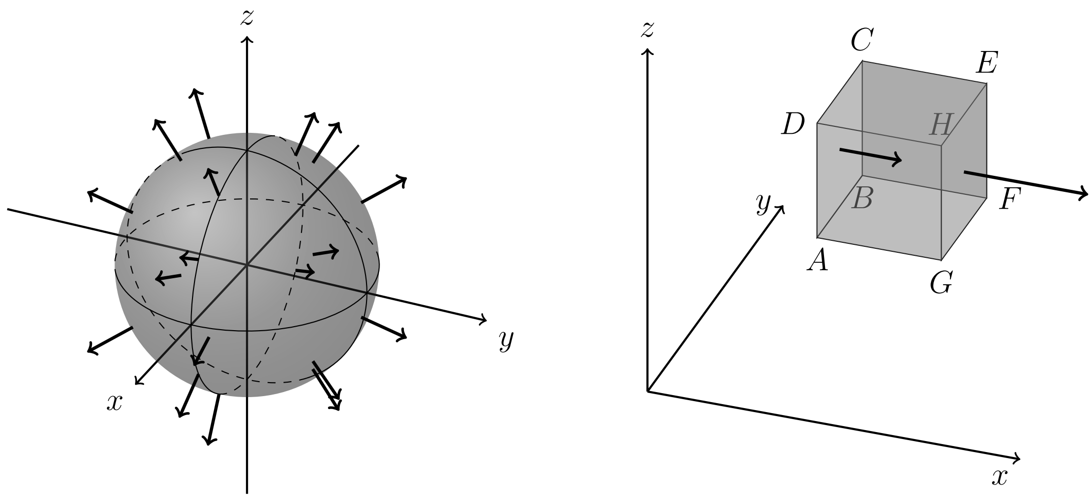

Figure 2: Example

Example. Say $\mathbf{f}(x,y,z) = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$. Then if we set $\mathcal{S}$ to be the cube with centre $(1.5, 1.5, 1.5)$ with side length 1, then

$$\oint_{\mathcal{S}} \mathbf{f} \times \mathrm{d}\mathbf{A} = \int_{ABCD} + \int_{EFGH} + \int_{CDHE} + \cdots$$

The integrals for faces $ABCD$ and $EFGH$ are zero because $\hat{\mathbf{n}} \parallel \mathbf{f}$. The integrals for faces $CDHE$ and $ABFG$ are of opposite signs since the $\hat{\mathbf{n}}$s for both faces are directly opposite while $\mathbf{f}$ is the same. Same goes for faces $DHGA$ and $CEFB$. Therefore

$$\oint_{\mathcal{S}} \mathbf{f} \times \mathrm{d}\mathbf{A} = 0.$$

On the other hand,

$$\oint_{\mathcal{S}} \mathbf{f} \cdot \mathrm{d}\mathbf{A} = \int_{ABCD} + \int_{EFGH} = -1 \cdot 1 + 2 \cdot 1 = 1.$$

We will normally deal with vector fields that are allow massive simplification on the surface in discussion, so we don't have to worry much about complicated examples. Those are normally helped by computers.

2 Divergence and Curl

The gradient operator only operates on scalar fields (the image is a vector field). Here, we introduce two operators that operate on vector fields (the first one gives back a scalar field, whereas the next one gives back a vector field). Henceforth, for any *solid* object X, we denote ∂X to be the *boundary* of X. E.g. the boundary of a disc is a circle, the boundary of a ball is a sphere. Also, $|X|$ will denote volume (or area, or hypervolume, depending on the dimension).

2.1 Divergence

Example. Consider a radial vector field $\mathbf{f} = r\hat{\mathbf{r}}$. If we take the ball $\mathcal{V}$ centred at the origin with radius R,

$$\frac{1}{|\mathcal{V}|}\oint_{\partial\mathcal{V}} \mathbf{f}\cdot \mathrm{d}\mathbf{A} = \frac{3}{4\pi R^3}R(\text{surface area}) = 3 > 0.$$

If we take $\mathcal{V}$ to be very small (we will write this as $\mathcal{V} \to 0$ even though $\mathcal{V}$ is a solid), the integral is still $3 > 0$. This roughly means, locally at the origin, there is a net outward flow of vectors.

Example. Let $\mathbf{f} = \begin{pmatrix} x^2 \\ 0 \\ 0 \end{pmatrix}$. If we take the solid cube $\mathcal{V}$ centred at $(1,1,1)$ with side length s,

$$\frac{1}{|\mathcal{V}|}\oint_{\partial\mathcal{V}} \mathbf{f}\cdot \mathrm{d}\mathbf{A} = \frac{1}{s^3}(-s^2(1-s/2)^2 + s^2(1+s/2)^2) = 2 > 0.$$

If we take $\mathcal{V} \to 0$ again, the integral is still $2 > 0$. This roughly means, locally at $(1,1,1)$, there is a net outward flow of vectors again.

The quantity $\lim_{\mathcal{V}\to 0}\frac{1}{|\mathcal{V}|}\oint_{\partial\mathcal{V}} \mathbf{f}\cdot \mathrm{d}\mathbf{A} = \lim_{\mathcal{V}\to 0}\frac{1}{|\mathcal{V}|}\oint_{\partial\mathcal{V}} \hat{\mathbf{n}}\cdot \mathbf{f}\ \mathrm{d}A$ is known to be the divergence of $\mathbf{f}$ at that point. There is a very simple formula for divergence:

Consider $\mathcal{V}$ to be a cuboid with small side lengths $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ respectively, and let $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$. Then in the x-direction, $\int \mathbf{f}\cdot \mathrm{d}\mathbf{A} = f_1(x+\mathrm{d}x, y, z)\cdot \mathrm{d}y\ \mathrm{d}z - f_1(x,y,z)\cdot \mathrm{d}y\ \mathrm{d}z$ (See Figure below). Similarly, we obtain the other terms in the y- and z-direction.

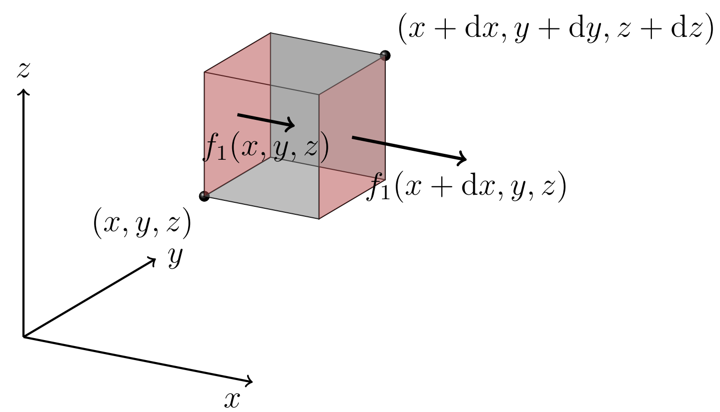

Figure 3: $\int \mathbf{f}\cdot \mathrm{d}\mathbf{A}$ in the x-direction

In the end,

$$\begin{aligned}
\frac{1}{|\mathcal{V}|}\oint_{\partial\mathcal{V}} \mathbf{f}\cdot \mathrm{d}\mathbf{A} &= \frac{1}{\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z}(f_1(x+\mathrm{d}x,y,z)\cdot \mathrm{d}y\ \mathrm{d}z - f_1(x,y,z)\cdot \mathrm{d}y\ \mathrm{d}z + \cdots) \\
&= \frac{f_1(x+\mathrm{d}x,y,z)-f_1(x,y,z)}{\mathrm{d}x} + \cdots \\
&= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.
\end{aligned}$$

This is denoted as $\nabla\cdot\mathbf{f}$. It works for 2 or 3 dimensions.

Example. $\mathbf{f} = \begin{pmatrix} x^2+xy+y^2 \\ 2x+y^2 \end{pmatrix} \Rightarrow \nabla\cdot\mathbf{f} = \frac{\partial}{\partial x}(x^2+xy+y^2) + \frac{\partial}{\partial y}(2x+y^2) = 2x+y+2y = 2x+3y.$

Any shape $\mathcal{V}$ containing the point (x,y,z) can be decomposed into small cuboids $\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$ in the limit, so it doesn't matter what $\mathcal{V}$.

The following theorem is quite useful in many areas. It interchanges a volume integral with a surface integral, or in 2D, an area integral with a line integral:

Divergence Theorem. $\displaystyle\int_{\mathcal{V}} \nabla\cdot\mathbf{f}\ \mathrm{d}V = \oint_{\partial\mathcal{V}} \hat{\mathbf{n}}\cdot\mathbf{f}\ \mathrm{d}A.$

Sketch of Proof. Here is a way to understand the theorem. Given any solid $\mathcal{V}$, we can always divide it into infinitesimal volumes $\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$. By 'summing' up all the divergences in $\mathcal{V}$, notice that the contribution by two neighbouring infinitesimal volumes will cancel out where they attach, because we are adding divergences in two orientations, in and out. In the end, the resultant volume integral only gives the contributions of $\hat{\mathbf{n}}\cdot\mathbf{f}$ by the surface of $\mathcal{V}$. The 2D version is analogous. □

To see this theorem in action, let's see a few examples.

Example. Consider a ball centred at the origin with radius R and the vector field $\mathbf{f} = r\hat{\mathbf{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The divergence is $\nabla\cdot\mathbf{f} = 1+1+1 = 3$ everywhere. Therefore, $\int \nabla\cdot\mathbf{f}\ \mathrm{d}V = 3(4\pi R^3/3) = 4\pi R^3$ in the ball. On the other hand, $\oint \hat{\mathbf{n}}\cdot\mathbf{f}\ \mathrm{d}A = R(\text{surface area}) = 4\pi R^3$ too. □

2.2 Curl

The curl operator only works in 3 dimensions as it involves a cross product. Instead of measuring net outflow, this operator measures net rotation about a point (anticlockwise being positive):

$$\lim_{\mathcal{V}\to 0}\frac{1}{|\mathcal{V}|}\oint_{\partial\mathcal{V}} \hat{\mathbf{n}}\times\mathbf{f}\ \mathrm{d}A = -\lim_{\mathcal{V}\to 0}\frac{1}{|\mathcal{V}|}\oint_{\partial\mathcal{V}} \mathbf{f}\times \mathrm{d}\mathbf{A}.$$

Notice there is a negative sign in the RHS[2]. Let's look at one example before we move on to show the simple formula for curl.

Example. Consider the whirlpool vector field $\mathbf{f}(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$, and take $\mathcal{V}$ to be the cylinder centred at the origin with radius s and z-height s:

[2]By convention, the $\hat{\mathbf{n}}$ is on the left of $\mathbf{f}$ in the cross product, so that the result points according to the right hand rule. Some authors resolve this problem by always writing integrals as $\int \mathrm{d}\mathbf{A}\cdots$ instead of $\int\cdots\mathrm{d}\mathbf{A}$. I won't do that here.

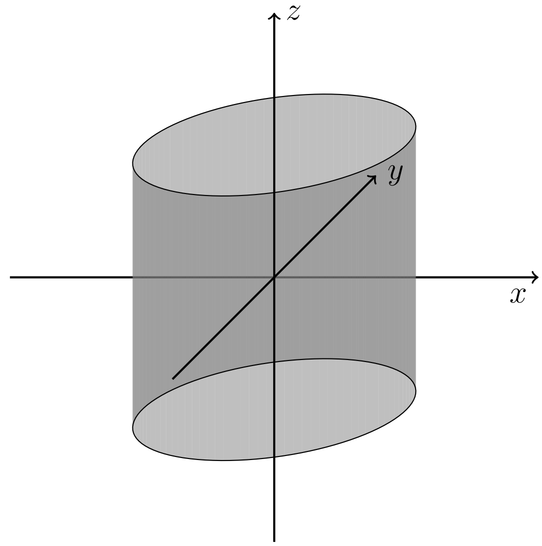

Figure 4: Cylinder at the origin

The contribution to the curl by the two flat circular faces cancel off as the top and bottom normal vectors $\hat{\mathbf{n}}$ are directly opposite. On the curved face, $\hat{\mathbf{n}} \times \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$, thus the contribution to the curl by curved face is

$$\frac{1}{|\mathcal{V}|} \int \hat{\mathbf{n}} \times \mathbf{f} \, \mathrm{d}A = \frac{1}{\pi s^3} \int \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} \, \mathrm{d}A = \frac{1}{\pi s^3} \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} \cdot (2\pi s)(s) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

which is in the direction we would expect, by the right-hand rule.

Now the formula for curl, we apply the same concept we did for divergence, taking a small cuboid of length $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ and then compute the integral on its surface. Let's skip the details here as an exercise: The result is

$$\begin{pmatrix} \partial f_3/\partial y - \partial f_2/\partial z \\ \partial f_1/\partial z - \partial f_3/\partial x \\ \partial f_2/\partial x - \partial f_1/\partial y \end{pmatrix} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

The expression on the RHS is obviously sloppy notation, but it is definitely neater. The expression above is denoted as $\nabla \times \mathbf{f}$. As a recap, let's restate $\nabla f, \nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$, using the sloppy notation again on the rightmost expression:

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} f \tag{Gradient}$$

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \mathbf{f} \tag{Divergence}$$

$$\nabla \times \mathbf{f} = \begin{pmatrix} \partial f_3/\partial y - \partial f_2/\partial z \\ \partial f_1/\partial z - \partial f_3/\partial x \\ \partial f_2/\partial x - \partial f_1/\partial y \end{pmatrix} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \mathbf{f} \tag{Curl}$$

That is the reason for the notations! It perfectly fits if $\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$, so now you have a method to memorise all of them.

Just as the divergence theorem, we can derive an analogous theorem for curl:

Curl Theorem / Stokes' Theorem. $\displaystyle\int_{\mathcal{V}} \nabla \times \mathbf{f} \; \mathrm{d}V = \oint_{\partial\mathcal{V}} \hat{\mathbf{n}} \times \mathbf{f} \; \mathrm{d}A.$

References

[1] Differential and Integral Equations by Collins