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18.100B Definitions

1 Real Numbers

1. A *field* is a set F equipped with operations $+$ and $\times$ such that
   - $(F, +)$ and $(F \setminus \{0\}, \times)$ are Abelian groups
   - $x(y+z) = xy + xz$ for all $x, y, z \in F$. (Distributivity)
2. A field F is *ordered* if there exists a relation $<$ on F (with $x > y$ meaning $y < x$, $x \leq y$ meaning $x < y$ or $x = y$, etc) such that for all $x, y, z \in F$,
   - Exactly one of $x = y$, $x < y$, $x > y$ holds. (Trichotomy)
   - $x < y$ and $y < z$ implies $x < z$. (Transitivity)
   - $x < y$ implies $x + z < y + z$. (Additivity)
   - $x < y$ and $z > 0$ implies $xz < yz$. (Multiplicativity)

   We define $P = \{x \in F : x > 0\}$.
3. Let F be an ordered field.
   - $u \in F$ is an *upper bound* for a subset $S \subseteq F$ if $x \leq u$ for all $x \in S$. If an upper bound for S exists, we say S is *bounded above*.
   - $\ell \in F$ is a *lower bound* for a subset $S \subseteq F$ if $x \geq \ell$ for all $x \in S$. If an upper bound for S exists, we say S is *bounded below*.
   - If $S \subseteq F$ is bounded above and below, we say that it is *bounded*.
   - $u \in F$ is the *maximum* of S, denoted $\max S$, if u is an upper bound and $u \in S$.
   - $\ell \in F$ is the *minimum* of S, denoted $\min S$, if ℓ is a lower bound and $\ell \in S$.
   - $u \in F$ is the *supremum* of S, denoted $\sup S$, if it is the least upper bound for S. More precisely, we say that S has supremum
     $$\sup S = \min\{x \in F : x \text{ is an upper bound for } S\} \qquad \text{if it exists.}$$
   - $\ell \in F$ is the *infimum* of S, denoted $\inf S$, if it is the greatest lower bound for S. More precisely, we say that S has infimum
     $$\sup S = \max\{x \in F : x \text{ is an lower bound for } S\} \qquad \text{if it exists.}$$
   - By convention, $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$. If S is unbounded above (below) we say $\sup S = \infty$ ($\inf S = -\infty$).
   - We say that F is *complete* if it satisfies the *completeness axiom:* Every nonempty subset of F that is bounded above has a supremum.

2 Sequences

1. The absolute value function is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

2. A *sequence* $\{x_n\}_{n\in\mathbb{N}} = \{x_0, x_1, \cdots\}$ is an ordered list of real numbers. Explicitly, we have a function $x : \mathbb{N} \to \mathbb{R}$ and we denoted $x_n = x(n)$.

3. Let $\{x_n\}_{n\in\mathbb{N}}$ is said to *converge* to $\ell \in \mathbb{R}$ if

$$(\forall \varepsilon > 0)\,(\exists N \in \mathbb{N})\,(\forall n \geq N)\,(|x_n - \ell| < \varepsilon)$$

If this is true, we write $\lim\limits_{n\to\infty} x_n = \ell$.

4. $\{x_n\}_{n\in\mathbb{N}}$ is *bounded* if $\exists M \in \mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

5. $\{x_n\}_{n\in\mathbb{N}}$ is said to *diverge to* ∞, written as $x_n \to \infty$, if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$. The case $x_n \to -\infty$ is analogous.

6. $\{x_n\}_{n\in\mathbb{N}}$ is *monotone* if it is either nonincreasing ($x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$) or nondecreasing ($x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$)

7. A *subsequence* of $\{x_n\}_{n\in\mathbb{N}}$ is any ordered infinite subset. Precisely, it is some $\{x_{n_j}\}_{j\in\mathbb{N}}$ where $n_0 < n_1 < n_2 < \cdots$ are natural numbers.

8. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is *Cauchy* if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)(|x_n - x_m| < \varepsilon)$$

9. The *limit superior* and *limit inferior* of $\{x_n\}_{n\in\mathbb{N}}$ are defined by

$$\limsup x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right), \qquad \liminf x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right)$$

3 Series

1. Given a sequence $\{x_n\}_{n\in\mathbb{N}}$, we define the series

$$\sum_{k=0}^{n} x_k = x_0 + x_1 + \cdots + x_n \qquad \text{and} \qquad \sum_{k=0}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=0}^{n} x_k \text{ if it converges.}$$

2. The series $\sum_{k=0}^{\infty} a_k$ *converges absolutely* if $\sum_{k=0}^{\infty} |a_k|$ converges.

3. The *exponential function* is defined as

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

4. A series $\sum_{k=0}^{\infty} x_k$ is *unconditionally convergent* if any reordering of the x_k gives a series converging to the same number.

4 Topology of $\mathbb{R}$

1. - An *open interval* of $\mathbb{R}$ is $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$.
   - A *closed interval* of $\mathbb{R}$ is $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

   For a given set $E \subseteq \mathbb{R}$, we say that $p \in E$ is

   - an *interior point* of E if there exists $a < p < b$ such that $(a, b) \subseteq E$.
   - an *isolated point* of E if there exists $a < p < b$ such that $(a, b) \subseteq E = \{p\}$.
   - a *boundary point* if for all $a < p < b$, (a, b) intersects both E and E^c.
   - a *limit point* (or accumulation point) if for all $a < p < b$, $(a, b) \cap E$ is infinite.

   and we say E is

   - *open* if every $p \in E$ is an interior point of E.
   - *closed* if E contains all limit points of E.

2. - The *interior* of E, denoted $\mathring{E}$ or $\text{int}(E)$, is the set of its interior points.
   - The *closure* of E, denoted $\overline{E}$, is the union of E and its limit points.

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4. A set S is *countable* if there exists a surjection $f : \mathbb{N} \to S$.

5. - An *open cover* U of $E \subseteq \mathbb{R}$ is a collection of open sets $\{O_\alpha\}_{\alpha \in I}$ such that such that $E \subseteq \bigcup_{\alpha \in I} O_\alpha$.
   - $K \subseteq \mathbb{R}$ is (covering) *compact* if every open cover of K admits a finite subcover.
   - $K \subseteq \mathbb{R}$ is *sequentially compact* if every sequence in K admits a converging subsequence in K.

5 Metric Spaces

1. A *metric space* (X, d) is a set X equipped with a *metric* d, which is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that for all $x, y, z \in X$,

   - $d(x, y) = 0 \Leftrightarrow x = y$
   - $d(x, y) = d(y, x)$ (Symmetry)
   - $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

2. - Convergence: $(\forall \varepsilon > 0)\,(\exists N \in \mathbb{N})\,(\forall n \geq N)\,(d(x_n, \ell) < \varepsilon)$.
   - Cauchy sequence: $(\forall \varepsilon > 0)\,(\exists N \in \mathbb{N})\,(\forall m, n \geq N)\,(d(x_n, x_m) < \varepsilon)$.
   - Open/Closed balls: $\mathcal{B}(x, r) = \{y : d(x, y) < r\}$, $\overline{\mathcal{B}}(x, r) = \{y : d(x, y) \leq r\}$.
   - Open set: $(\forall x \in E)\,(\exists r > 0)\,(\mathcal{B}(x, r) \subseteq E)$. Closed set: E^c is open.
   - Neighborhood of $x \in X$: Any open set containing x.
   - Diameter of E: $\operatorname{diam}(E) = \sup\{d(x, y) : x, y \in E\}$. Bounded set: $\operatorname{diam}(E) < \infty$.
   - Limit point of E: Any neighborhood of it intersects E infinitely much.
   - Isolated point of E: Exists some neighbourhood that intersects E at only itself.
   - Closure of E: $\overline{E} = E \cup \{\text{limit points of } E\}$.
   - Interior of E: $\mathring{E} = \{x \in E : \text{exists neighborhood of } x \text{ contained in } E\}$.
   - E is *dense* in F if $F \subseteq \overline{E}$. (Equivalently, all neighborhoods of all points in F must intersect E.)
   - $K \subseteq X$ is *compact* if every open cover of K admits a finite subcover.
   - $K \subseteq X$ is *totally bounded* if $(\forall \varepsilon > 0)\,(\exists x_1, \cdots, x_n)\,(K \subseteq \mathcal{B}(x_1, \varepsilon) \cup \cdots \cup \mathcal{B}(x_n, \varepsilon))$.
   - $K \subseteq X$ is *complete* if every Cauchy sequence converges.
   - $K \subseteq X$ is *separable* if it has a countable dense subset.

6 Continuous Functions

1. - Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say $f : X \to Y$ is *continuous at* $x \in X$ if for every $x_n \to x$ we have $f(x_n) \to f(x)$.
   - $f : X \to Y$ is *continuous* if it is continuous at every $x \in X$.

2. $f : X \to Y$ is *uniformly continuous* if

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall d_X(x, y) < \delta)\,(d_Y(f(x), f(y)) < \varepsilon)\,.$$

   Remark: Here δ does not depend on x!

3. If X is compact, we define the *uniform metric* on $\mathcal{C}(X) = \{f : X \to \mathbb{R} \text{ continuous}\}$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$

4. Let $\{f_n : X \to \mathbb{R}\}_{n \in N}$ be a sequence of continuous functions.
   - We say f_n *converges pointwise* to f if $f_n(x) \to f(x)$ for all $x \in X$.
   - We say f_n *converges uniformly* to f if $\sup\limits_{x \in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

     This is equivalent to f_n converging in $(\mathcal{C}(X), d)$, so we can write $f_n \xrightarrow{d} f$.

5. - A set $K \subseteq \mathcal{C}(X)$ is *uniformly bounded* if there exists an $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $f \in K$ and $x \in X$.
   - A set $K \subseteq \mathcal{C}(X)$ is *(uniformly) equicontinuous* if

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall f \in K, d_X(x, y) < \delta)\,(d_Y(f(x), f(y)) < \varepsilon)\,.$$

7 Derivatives

1.
   - Let $f : I \to \mathbb{R}$ where $I \subseteq R$. Then we say $\lim_{x \to x_0} f(x) = \ell$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in I$ with $0 < |x - x_0| < \delta$.
   - Let I be an open interval. We say that $f : I \to \mathbb{R}$ is *differentiable at* x_0 if
   $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\delta \to 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta} \in \mathbb{R}$$
   exists, in which case we denote the limit by $f'(x_0)$, called the *derivative* at x_0. We say f is *differentiable* if f is differentiable at all points in I.
   - $\dfrac{f(x) - f(x_0)}{x - x_0}$ is called the *difference quotient* and represents the slope.
2. $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to have *directional derivative at* $x_0 \in \Omega$ *in direction* $v \in \mathbb{R}^n$ if
   $$Df(x_0)[v] := \lim_{\delta \to 0} \frac{f(x_0 + \delta v) - f(x_0)}{\delta}$$
   exists. We say f is *differentiable at* x_0 if $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.
3.
   - A function $f : I \to \mathbb{R}$ is *convex* if for all $x_1 < x_2$ in I and any $0 < \alpha < 1$,
   $$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2).$$
   We say that f is *strictly convex* if the inequality is always strict.
   - A function $f : I \to \mathbb{R}$ is *concave* if for all $x_1 < x_2$ in I and any $0 < \alpha < 1$,
   $$f(\alpha x_1 + (1 - \alpha)x_2) \geq \alpha f(x_1) + (1 - \alpha)f(x_2).$$
   We say that f is *strictly concave* if the inequality is always strict.
   - Define the *right and left derivative*
   $$f'_+(x_0) = \lim_{\delta \to 0^+} \frac{f(x_0 + \delta) - f(x_0)}{\delta}, \qquad f'_-(x_0) = \lim_{\delta \to 0^-} \frac{f(x_0 + \delta) - f(x_0)}{\delta}$$
4.
   - A function $f : I \to \mathbb{R}$ is *in* C^1 if it is differentiable and f' is continuous.
   - If $f'(x_0) = 0$, we say x_0 is a *critical point* and $f(x_0)$ is a *critical value.*
   - We say $y \in \mathbb{R}$ is a *regular value* if it is not a critical value.
   - A set $S \subseteq \mathbb{R}$ has *measure zero* if for all $\varepsilon > 0$ there exists countably many intervals that (i) covers S and (ii) have total combined length $< \varepsilon$.

8 Riemann Integral

1. - A *partition* of $[a, b]$ is a finite set of points $\sigma = \{a = x_0 < \cdots < x_N = b\}$.
   - The *size* $|\sigma|$ of σ is $\max_{1 \le i \le N} |x_i - x_{i-1}|$.
   - A partition σ' is a *refinement* of σ if $\sigma' \supseteq \sigma$.
   - Given a bounded $f : [a, b] \to \mathbb{R}$ and a partition σ of $[a, b]$,
     - The *upper (Riemann) sum* is $S(f, \sigma) = \displaystyle\sum_{i=1}^{N} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x)$.
     - The *lower (Riemann) sum* is $s(f, \sigma) = \displaystyle\sum_{i=1}^{N} (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$.
   - Given a bounded $f : [a, b] \to \mathbb{R}$,
     - The *upper (Riemann) integral* is $\mathcal{I}^+(f) = \inf\limits_{\forall \sigma} S(f, \sigma)$.
     - The *lower (Riemann) integral* is $\mathcal{I}^-(f) = \sup\limits_{\forall \sigma} s(f, \sigma)$.
   - A bounded $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* if $\mathcal{I}^-(f) = \mathcal{I}^+(f) := \displaystyle\int_a^b f(x)\, \mathrm{d}x$. Denote by $\mathcal{R}(a, b)$ the set of all Riemann integrable functions on $[a, b]$.
   - Given $f : [a, b] \in \mathbb{R}$ and $I \subseteq [a, b]$ an interval, define $\operatorname*{osc}\limits_I f = \sup\limits_I f - \inf\limits_I f$.
2. The *oscillation of f at point x* is $\operatorname{osc}(f, x) = \lim\limits_{\delta \to 0^+} \operatorname*{osc}\limits_{[x-\delta, x+\delta]} f \ge 0$
3. An *ordinary differential equation* (ODE) is a problem in the form

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0$$

   where $y(x)$ is a differentiable function from $\mathbb{R} \to \mathbb{R}^n$ to be solved.