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Topology Notes

Based on Pieloch's lecture notes for 18.901 Fall '23

1 Topological Spaces

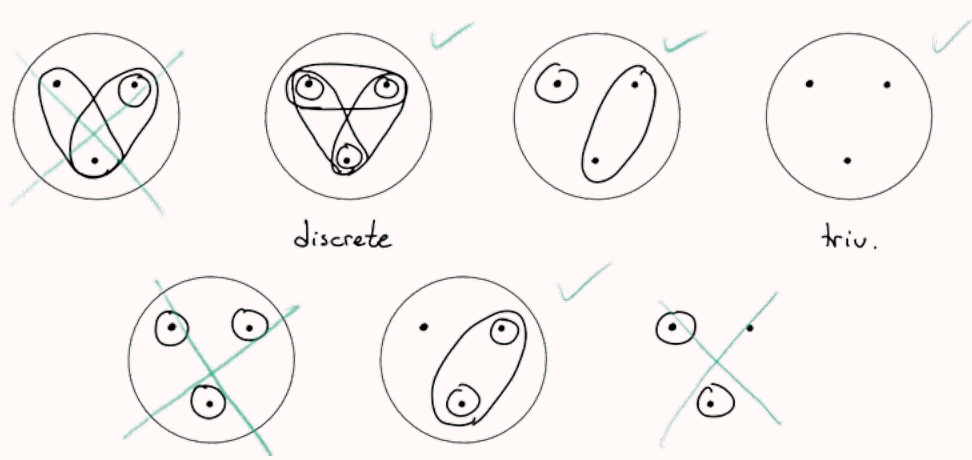
Definition 1.1.

1. A ***topology*** on a set X is a set $\mathscr{T}$ of subsets of X called ***open sets*** such that
   - $\varnothing, X \in \mathscr{T}$
   - $\mathscr{T}' \subseteq \mathscr{T} \implies \bigcup_{U \in \mathscr{T}'} U \in \mathscr{T}$. (Preserved under arbitrary unions)
   - $U_1, \cdots, U_n \in \mathscr{T} \implies \bigcap_{i=1}^{n} U_i \in \mathscr{T}$. (Preserved under finite intersections)

   $(X, \mathscr{T})$ – or just X when $\mathscr{T}$ is understood – is a ***(topological) space***.
2. Suppose $\mathscr{T}, \mathscr{T}'$ are two topologies on X with $\mathscr{T} \subseteq \mathscr{T}'$. We say $\mathscr{T}'$ is ***finer*** than $\mathscr{T}$ and $\mathscr{T}$ is ***coarser*** than $\mathscr{T}'$.
3. $A \subseteq X$ is ***closed*** if $X \backslash A$ is open. Hence $\varnothing, X$ are closed, and closedness is preserved under finite unions and arbitrary intersections.

Example 1.1.

1. The ***discrete topology*** on X is $\mathscr{T} = \mathscr{P}(X)$.
2. The ***trivial topology*** on X is $\mathscr{T} = \{\varnothing, X\}$.
3. $X = \{1, 2, 3\}$:



Definition 1.2. A set $\mathscr{B}$ of subsets of X is a ***basis*** if

- $X = \bigcup_{B \in \mathscr{B}} B$
- $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathscr{B} \implies (\exists B \in \mathscr{B})\,(x \in B \subseteq B_1 \cap B_2)$

Theorem 1.1. A basis $\mathscr{B}$ generates a topology $\mathscr{T}$ via

$$U \in \mathscr{T} \quad \Leftrightarrow \quad (\forall x \in U)\,(\exists B \in \mathscr{B})\,(x \in B \subseteq U)\,.$$

Proof. $\varnothing \in \mathscr{T}$ (vacuously) and $X \in \mathscr{T}$ since $\mathscr{B}$ covers X. We then verify the union and intersection properties:

- Suppose $U_\alpha \subseteq X$ are open, then $\bigcup_\alpha U_\alpha$ is open because

$$x \in \bigcup_\alpha U_\alpha \implies x \in U_\alpha \text{ for some } \alpha \implies x \in B_\alpha \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha$$

- Suppose U_1, U_2 are open, then $U_1 \cap U_2$ is open because

$$x \in U_1 \cap U_2 \implies \begin{cases} x \in B_1 \subseteq U_1 \text{ for some } B_1 \in \mathscr{B} \\ x \in B_2 \subseteq U_2 \text{ for some } B_2 \in \mathscr{B} \end{cases} \implies x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

  for some $B \in \mathscr{B}$. By induction, any finite intersection of open sets is open. ■

Example 1.2. Let $X = \mathbb{R}$. We can construct three topologies via the bases:

1. $\{(a, b) : a, b \in \mathbb{R}\}$ (the ***standard topology*** on $\mathbb{R}$)
2. $\{[a, b) : a, b \in \mathbb{R}\}$
3. $\{U \subseteq \mathbb{R} : U = \mathbb{R} \setminus \{x_1, \cdots, x_n\} \text{ for some } x_1, \cdots, x_n \in \mathbb{R}\}$

Note, (2) is finer than (1), and (1) is finer than (3).

Remark.

1. Infinite intersections may not be open. E.g. $\bigcap_n (-1/n, 1/n) = \{0\}$ is not open in the standard topology on $\mathbb{R}$.
2. Different bases could generate the same topology. E.g. For $X = \mathbb{R}^2$, open balls generate the same topology as open squares do.

Definition 1.3. Let X be a space, and $A \subseteq X$.

1. $\text{int}(A) = \bigcup\{U \subseteq A : U \text{ is open}\}$ is the ***interior*** of A.
2. $\overline{A} = \bigcap\{C \supseteq A : C \text{ is closed}\}$ is the ***closure*** of A.
3. A is ***dense*** if $\overline{A} = X$.

Example 1.3.

1. $\text{int}(A) = \overline{A} = A$ in the discrete topology.
2. $\text{int}(A) = \varnothing; \overline{A} = X$ in the trivial topology for any $A \neq \varnothing, X$.
3. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Warning. A, B dense does not imply $A \cap B$ dense, e.g. take $\mathbb{Q}$ and $\mathbb{Q} + \sqrt{2}$.

Theorem 1.2.

1. A open $\Leftrightarrow$ $A = \text{int}(A)$
2. A closed $\Leftrightarrow$ $A = \overline{A}$

Definition 1.4.

1. A ***neighborhood of $x \in X$*** is an open set that contains x.
2. $x \in X$ is a ***limit point*** of A if $(\forall x \in U \in \mathscr{T})\,(A \cap U \setminus \{x\} \neq \varnothing.)$
3. $x \in X$ is an ***adherent point*** of A if $(\forall x \in U \in \mathscr{T})\,(A \cap U \neq \varnothing.)$
4. The ***boundary*** of A is $\partial A = \{x \in X : x \text{ adh pt of } A \text{ and } X \setminus A\} = \overline{A} \cap \overline{X \setminus A}$.

Theorem 1.3.

1. $\overline{A} = \{\text{adherent pts of } A\} = A \cup \{\text{limit pts of } A\} = \text{int}(A) \sqcup \partial A$.
2. $X = \text{int}(A) \sqcup \partial A \sqcup \text{int}(X \setminus A)$.

Theorem 1.4. If U_1, U_2 are dense and open, then $U_1 \cap U_2$ is dense and open.

Proof. Suppose $x \in X$. We want to show that for any $x \in U$ open we have $U \cap (U_1 \cap U_2) \neq \varnothing$.

Since U_1 is dense, $U \cap U_1 \neq \varnothing$. Since U_2 is also dense, $U \cap U_1 \cap U_2 \neq \varnothing$. ■

2 Metric Spaces

Definition 2.1.

1. A ***metric*** on a set X is a function $d : X^2 \to \mathbb{R}$ such that
   - $d(x, y) \geqslant 0$ and equality holds if and only if $x = y$
   - $d(x, y) = d(y, x)$
   - $d(x, y) + d(y, z) \geqslant d(x, z)$

   The set $B_x(\varepsilon) = \{y : d(x, y) < \varepsilon\}$ is the (open) ***ε-ball centered at x***.
2. The ***metric topology*** on (X, d) is the topology generated by the basis

$$\mathscr{B} = \{B_x(r) : x \in X, r > 0\}$$

Example 2.1. The ***euclidean metric*** d on $\mathbb{R}^n$ is $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i (x_i - y_i)^2}$.

3 Subspace Spaces

Definition 3.1. Let $(X, \mathscr{T})$ be a space and $A \subseteq X$. The ***subspace topology*** on A (with respect to X) is

$$\mathscr{T}_A = \{A \cap U : U \in \mathscr{T}\}.$$

We call A with this topology a ***subspace*** of X.

Theorem 3.1. A basis $\mathscr{B}$ for $\mathscr{T}$ defines a basis $\mathscr{B}_A$ for $\mathscr{T}_A$ via

$$\mathscr{B}_A = \{A \cap B : B \in \mathscr{B}\}.$$

Remark. If (X, d) is a metric space and $A \subseteq X$ then (A, d_A) is a metric space where $d_A(a_1, a_2) = d(a_1, a_2)$.

Theorem 3.2. Let (X, d) be a metric space. Then the metric topology on $A \subseteq X$ agrees with the subspace topology of $A \subseteq X$.

Proof. The subspace topology on A has basis $\mathscr{B}_S = \{A \cap B_x(r)\}_{x \in X}$ whereas the metric topology on A has basis $\mathscr{B}_M = \left\{B_x^A(r)\right\} = \{A \cap B_x(r)\}_{x \in A} \subseteq \mathscr{B}_S$. On the other hand, given any open U in the subspace topology and $x \in U \subseteq A$, we have $x \in A \cap B_x(r) \subseteq U$ for some $r > 0$, but this is just $x \in B_x^A(r) \subseteq U$. Since $x \in U$ was arbitrary, U is open in the metric topology too. ■

Definition 3.2. $A \subseteq X$ (space) is discrete if its subspace topology is discrete.

Example 3.1. Is $X = \{0\} \cup_n \{1/n\}$ discrete in $\mathbb{R}$? No. $\{0\}$ is not open in X. If it were, then $\exists (a, b)$ such that $(a, b) \cap X = \{0\}$, but $1/n < b$ for large n.

Warning. $B = A = \mathbb{R} \times \{0\} \subseteq X = \mathbb{R}^2$ are examples for the following statements:

1. B open in A does not imply B open in X.
2. Suppose $A \subseteq Y \subseteq X$, then the $\operatorname{int}(A)$ in Y may not be $Y \cap \operatorname{int}(A)$.

But these versions are true:

Theorem 3.3.

1. B open in A, and A open in X, then B open in X.
2. Suppose $A \subseteq Y \subseteq X$, the closure of A in Y is $Y \cap$ (closure of A in X).

4 Product Spaces

Definition 4.1. Let $\{X_\alpha\}_\alpha$ be a collection of spaces.

1. The ***product topology*** on $X_1 \times \cdots \times X_n$ is generated by the basis
$$\mathscr{B} = \{Y_1 \times \cdots \times Y_n : Y_1, \cdots, Y_n \text{ open}\}$$
2. More generally, the ***product topology*** on $\prod_\alpha X_\alpha$ is generated by the basis
$$\mathscr{B} = \{\textstyle\prod_\alpha Y_\alpha : Y_\alpha \text{ open for all } \alpha, \text{ and only finitely many } Y_\alpha \neq X_\alpha\}$$

Theorem 4.1.

1. If $A \subseteq X; B \subseteq Y$ are subspaces, then the subspace topology and product topology on $A \times B$ agree.
2. The metric topology on $\mathbb{R}^n$ agrees with the product topology on $\mathbb{R}^n$.

5 Quotient Space

Definition 5.1.

- Let X be a space, Y be a set, and $q : X \to Y$ be surjective. The ***quotient topology*** on Y induced by the ***quotient map*** q is given by

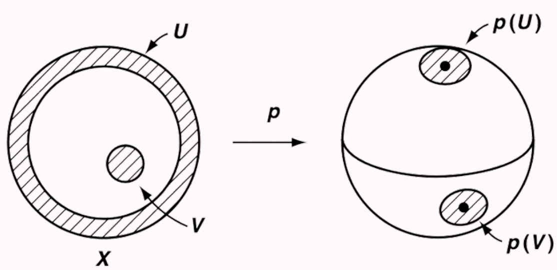
$$\mathscr{B} = \left\{U \subseteq Y : q^{-1}(U) \text{ open in } X\right\}$$

- Let $A \subseteq X$ be a subset and define $x \overset{A}{\sim} y \iff x = y$ or $x, y \in A$. We denote X/A the space on $X/\overset{A}{\sim}$ with quotient topology induced by the canonical map $q : X \twoheadrightarrow X/\overset{A}{\sim}$.

Remark. An equivalence relation $\sim$ on X determines the surjective ***canonical map*** $q : X \twoheadrightarrow X/\sim$ defined by $q(x) =$ equivalence class of x.

Example 5.1.

1. Consider the unit 2-disk $X = D^2 = \{x \times y : x^2 + y^2 \leqslant 1\}$. If we identify together all points on the boundary ∂D^2, we get the quotient space $D^2/\partial D^2$ that is homeomorphic with the subspace of $\mathbb{R}^3$ called the unit 2-sphere $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$.



2. We can construct a *torus* $S^1 \times S^1$ from the rectangle $[0,1] \times [0,1]$.
3. We can patch two disks $D^2 \sqcup D^2$ along their boundaries to obtain S^2. Formally, given a homeomorphism $\varphi : \partial D_1^2 \to D_2^2$, we have $(D_1^2 \sqcup D_2^2)/\sim\; = S^2$ where $x \sim y \iff x = y$ or $x \in \partial D_1^2, y \in \partial D_2^2, \varphi(x) = y$.

6 Continuous Functions

Definition 6.1. Let X, Y be spaces. A function $f : X \to Y$ is

- ***continuous at $x \in X$*** if $f^{-1}(V)$ is open in X for all neighborhoods V of $f(x)$.
- ***continuous*** if $f^{-1}(V)$ is open in X for all V open in Y.
- a ***homeomorphism*** if f is bijective, and f and f^{-1} are continuous.

Theorem 6.1.

1. Let $\mathscr{B}$ be a basis of X. The map $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for all $B \in \mathscr{B}$.
2. A composition of continuous functions is continuous.
3. Let $A \subseteq X$ be a subspace and $f : X \to Y$ be continuous. Then $f|_A$ is continuous.
4. Let $f : Z \to X \times Y$ where $f = f_X \times f_Y$. Then f is continuous if and only if f_X, f_Y are continuous.
5. Any quotient map is continuous. Given a quotient map $q : X \to Y$, $f : Y \to Z$ is continuous if and only if $g = f \circ q$ is continuous.

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

6. The following are equivalent to $f : X \to Y$ being continuous:

   (1) $f^{-1}(C)$ is closed for all closed $C \subseteq Y$.

   (2) Given any $x \in X$ and $f(x) \subseteq V$ open, there exists open U with $f(U) \subseteq V$.

   (3) $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

Proof of (6).

- Continuity is equivalent to (1) by taking complements.
- For (2), say f is continuous, then $U = f^{-1}(V)$ works. Conversely, say (2) is true. Then for any open $V \subseteq Y$, any $v \in V$ admits a neighborhood within V, which has an open preimage $U_v \subseteq X$. Then $f^{-1}(V) = \bigcup_{v \in V} U_v$ is open, and thus f is continuous.
- (1) $\Rightarrow$ (3). Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\overline{f(A)}\right)$ which is closed, we have $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$ and thus $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.
- (3) $\Rightarrow$ (1). Let $C \subseteq Y$ be closed. Then $f\left(\overline{f^{-1}(C)}\right) = \overline{f\left(f^{-1}(C)\right)} \subseteq \overline{C} = C$ and hence $\overline{f^{-1}(C)} \subseteq f^{-1}f\left(\overline{f^{-1}(C)}\right) \subseteq f^{-1}(C)$ and thus $f^{-1}(C)$ is closed. ■

Corollary 6.1. Say X, Y are metric spaces. $f : X \to Y$ is continuous if and only if

$$(\forall x \in X, \varepsilon > 0)\,(\exists \delta > 0)\,(\forall d_X(x,y) < \delta)\,(d_Y(f(x), f(y)) < \varepsilon)\,.$$

Theorem 6.2. (Pasting Lemma) Let $X = A \cup B$ be a space where A, B are closed. If $f_A : A \to Y$ and $f_B : B \to Y$ are continuous and $f_A(x) = f_B(x)$ for all $x \in A \cap B$, then $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_A(x) & x \in A \\ f_B(x) & x \in B \end{cases}$$

is continuous.

7 Limits and Continuity

Definition 7.1. $\{x_n\}_{n \in \mathbb{N}}$ in X ***converges*** to $x \in X$ if any neighborhood of x contains all but finitely many x_n. Write $x_n \to x$.

Warning. Limits may not be unique:

1. In the trivial topology, any sequence converges to all points.
2. In $\mathbb{R}_1 \sqcup \mathbb{R}_2/\sim$ where $x \sim y \iff x \in \mathbb{R}_1, y \in \mathbb{R}_2, x = y \neq 0$, we have

$$1/n \to 0_1 \quad \text{and} \quad 1/n \to 0_2 \quad (\textbf{\textit{fat point}})$$

Theorem 7.1. If $x_n \to x$, then $x \in \overline{\{x_n\}_n}$.

Definition 7.2. A space X is ***first-countable*** if for any $x \in X$, there exists a countable number of neighborhoods $U_1, U_2, \cdots$ such that any neighborhood of x contains some U_i. The $\{U_i\}$ is called a ***neighborhood basis*** of x.

Theorem 7.2. If X is first-countable,

1. $x \in \overline{A} \implies \exists x_1, x_2, \cdots \in A$ such that $x_n \to x$.
2. $f : X \to Y$ is continuous if and only if $(x_n \to x) \implies (f(x_n) \to f(x))$.

8 Connectedness

Definition 8.1. A space X is ***connected*** if there is no nontrivial clopen (closed and open) set $A \subseteq X$.

Example 8.1. The subspace $(0,1) \cup (2,3)$ of $\mathbb{R}$ is not connected.

Theorem 8.1. $[a,b] \subseteq \mathbb{R}$ is connected.

Proof. Suppose the contrary, that $[a,b] = A \sqcup B$ where A, B are closed and non-empty. WLOG Assume $b \in B$. Then $s = \sup A < b$. If $s \in A$, since A is also open, there exists $(s-\varepsilon, s+\varepsilon) \subseteq A \implies \sup A \geqslant s+\varepsilon$, a contradiction. Hence $s \in B$ instead. Since B is open, there exists $(s-\varepsilon, s+\varepsilon) \subseteq B$ and thus $\sup A \leqslant s-\varepsilon$, a contradiction. ∎

Definition 8.2. A space X is ***path-connected*** if every pair $x, y \in X$ can be joined by a *path* in X: a continuous map $\gamma : I = [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example 8.2.

1. $\mathbb{R}^n$ is path-connected. Use the path $\gamma(t) = t\mathbf{x} + (1-t)\mathbf{y}$.
2. S^n is path-connected. Use the path $\gamma(t) = \dfrac{t\mathbf{x} + (1-t)\mathbf{y}}{|t\mathbf{x} + (1-t)\mathbf{y}|}$.
3. A torus is path-connected: Start with a path in I^2 and then take the quotient.

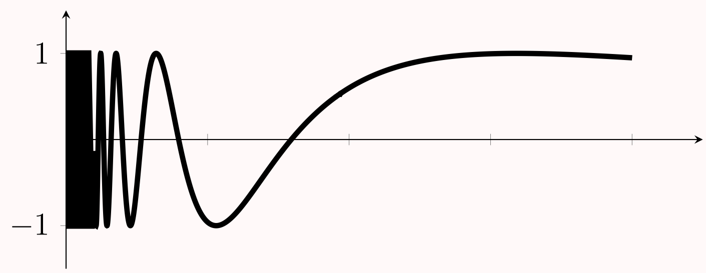
Theorem 8.2.

1. Any path-connected space is connected.
2. If $f : X \to Y$ is continuous and surjective,
   - X connected $\implies$ Y connected.
   - X path-connected $\implies$ Y path-connected.
3. Quotients of a (path-)connected space is (path-)connected.
4. A product of (path-)connected spaces is (path-)connected.

Example 8.3. The ***topologist's sine curve*** defined by

$$X = \{(x \times \sin(1/x)) : x > 0\} \cup \{0\} \times [-1, 1]$$

is connected but not path-connected.



Definition 8.3. The equivalence relation $x \sim y$ where there is a (path-)connected subspace containing both x, y partitions the space into (path-)connected ***components***.

9 Compactness

Definition 9.1.

1. An ***open cover*** of X is a collection of open sets that cover X. A space X is ***compact*** if every open cover of X admits a finite subcover.
2. A space X is ***sequentially compact*** if every sequence of points in X admits a convergent subsequence.

Theorem 9.1. 1st-countable + compact $\implies$ sequentially compact.

Proof. Suppose $\{x_n\}_n$ does not have a convergent subsequence. Let $x \in X$, then there exists a countable neighborhood basis $U_1, U_2, \cdots$. We can safely let $U_1 \supseteq U_2 \supseteq \cdots$ by taking successive intersections. Since there is no subsequence that converges to x, only finitely many x_n lie in U_n for some sufficiently large n. Hence, every $x \in X$ has a neighborhood U_x that intersects $\{x_n\}_n$ at a finite number of points. Taking the union of all U_x and applying compactness shows that $\{x_n\}_n$ is finite, so we can conclude by the pigeonhole principle. ■

Theorem 9.2.

1. Every closed subspace of a compact space is compact.
2. A continuous function maps compact spaces to a compact image.
3. Suppose X is compact and $C_1 \supseteq C_2 \supseteq \cdots$ is a sequence of closed and non-empty sets. Then $\bigcup_n C_n$ is non-empty.
4. A product of compact spaces is compact (Infinite case is hard: Tychonoff's Thm)
5. $[a, b]$ is compact.

Proof of (4). Suppose $[a, b] = \bigcup_\alpha U_\alpha$. Then

$$S = \{x \in [a, b] : [a, b] \text{ can be covered by finitely many } U_\alpha\}$$

contains $a \in S$ and is bounded above by b. Hence S has a supremum s.

Claim. $s \in S$.
Proof. Let $s \in U_\beta$ for some β, so there exists $(s - \varepsilon, s + \varepsilon) \subseteq U_\beta$. If $s \notin S$, just add U_β to the finite subcover of $[a, s - \varepsilon/2]$. □

Claim. $s = b$.
Proof. If not, then similarly, just add U_β to the finite subcover of $[a, s]$. □

Therefore $[a, b]$ can be covered by finitely many U_α. ■

Theorem 9.3. (Heine-Borel)
A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

- ($\Leftarrow$) $X \subseteq [-M, M]^n$ is a closed subset of a compact space, so X is compact.

- ($\Rightarrow$) Compactness on the open cover $\{B_0(r)\}_{r>0}$ shows X is bounded. We then show any limit pt x of X is in X: For all $n \in \mathbb{N}^*$, $C_n := \overline{B_x 1/n} \cap X \neq \varnothing$, and thus $\bigcap_n C_n = X \cap \{x\}$ is non-empty. ■

10 Hausdorff Spaces

Definition 10.1. A space X is ***Hausdorff*** if for any distinct $x, y \in X$ there exists disjoint neighborhoods $x \in U, y \in V$.

Example 10.1.

1. The trivial topology is not Hausdorff. The discrete topology is.
2. Metric spaces are Hausdorff.
3. The finite complement topology on $\mathbb{R}$ is not Hausdorff.
4. The space $\mathbb{R}_1 \sqcup \mathbb{R}_2/\sim$ containing the fat point is not Hausdorff.

Theorem 10.1. *X is Hausdorff if and only if $\Delta = \{(x \times x) : x \in X\} \subseteq X^2$ is closed.*

Proof.

- ($\Rightarrow$) If X is Hausdorff, for any $x \neq y$ there exists disjoint neighborhoods U, V of x, y respectively. Then $U \times V$ is a neighborhood of $(x \times y) \in X \times Y$ disjoint from Δ. Taking the union over all $(x \times y)$ implies Δ is closed.
- ($\Leftarrow$) If Δ is closed, given any $x \neq y$ there exists a basis neighborhood $U \times V$ of $(x \times y)$ disjoint from Δ. Then U, V are the desired neighborhoods. ■

Theorem 10.2.

1. In a Hausdorff space, a sequence of points converge to at most one point.
2. One-point sets in a Hausdorff space are closed.
3. A subspace of a Hausdorff space is Hausdorff.
4. A finite product of Hausdorff spaces is Hausdorff.
5. A compact subspace of a Hausdorff space is closed.

Warning. A quotient of a Hausdorff space may not be Hausdorff.

11 Normal Spaces

Definition 11.1.

1. X is $\boldsymbol{T_1}$ if one-point sets are closed.
2. A space is ***normal*** if it is T_1, and, for any pair of disjoint closed sets $A, B \subseteq X$ there exists disjoint open sets $U, V \subseteq X$ such that $A \subseteq U, B \subseteq V$.

Remark.

1. Normal $\implies$ Hausdorff $\implies$ T_1.
2. A quotient, subspace, or product of normal space(s) need not be normal.

Example 11.1.

1. The fat point $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$ is T_1 but not Hausdorff.
2. The ***K-topology*** on $\mathbb{R}$ generated by $\{(a,b)\} \cup \{(a,b) \backslash \bigcup_n \{1/n\}\}$ is Hausdorff but not normal.
3. The topology $\mathbb{R}_\ell$ on $\mathbb{R}$ generated by $\{[a,b)\}$ is normal, but $\mathbb{R}_\ell^2$ is not normal.

Theorem 11.1.

1. A closed subspace A of a normal space X is normal.
2. Compact + Hausdorff $\implies$ Normal.

Proof of (2). Suppose $A, B \subseteq X$ are disjoint and closed. Fix $a \in A$. Then for each $b \in B$ there exists disjoint neighborhoods $a \in U_b, b \in V_b$. Since B is also compact, there exists finitely many V_b that cover B. The union of such finitely many V_b and the intersection of their corresponding U_b form disjoint open sets containing a and B respectively. Repeat the same procedure for every $a \in A$ and then apply compactness of A. ∎

Theorem 11.2. Metric spaces are normal.

Proof. We can show that, for any subset $A \subseteq X$, the *point-to-set distance* $d(-, A) : X \to \mathbb{R}$ given by $d(x, A) = \inf\limits_{a \in A} d(x, a)$ is continuous. For disjoint closed sets A, B, the open sets

$$U = \{x : d(x, A) < d(x, B)\}, \qquad V = \{x : d(x, A) > d(x, B)\}$$

contain A, B respectively and are disjoint. ∎

Theorem 11.3. X is normal if and only if for any closed A and open U such that $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Theorem 11.4. (Urysohn's Lemma)
Let X be normal and A, B be disjoint closed sets of X. There exists a continuous map

$$f : X \to I$$

such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. Define open sets U_p for each $p \in \mathbb{Q} \cap [0,1]$ as follows: Enumerate $\mathbb{Q} \cap [0,1]$ such that 1 and 0 are the first two elements. Define $U_1 = X - B$ and by normality pick U_0 such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction, say we defined U_p for a finite number of p's and let r be the next rational in the enumeration. We must have $p < r < q$ where U_p, U_q are already defined. By normality we pick U_r such that $\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$.

Additionally, we let $U_p = \varnothing$ for all rationals $p < 0$ and $U_p = X$ for all rationals $p > 1$. Hence,

$$p < q \;\Rightarrow\; \overline{U_p} \subseteq U_q.$$

We then define $f(x) = \inf\{p : x \in U_p\}$. It is easy to see $f(A) = \{0\}$ and $f(B) = \{1\}$. We show that f is continuous.

Lemma 1. $x \in \overline{U_r} \;\Rightarrow\; f(x) \leqslant r$
Proof. If $x \in \overline{U_r}$, then $x \in U_s$ for every $s > r$. Hence $f(x) \leqslant r$. □

Lemma 2. $x \notin \overline{U_r} \;\Rightarrow\; f(x) \geqslant r$.
Proof. If $x \notin \overline{U_r}$, then $x \notin U_s$ for any $s < r$. Hence $f(x) \geqslant r$. □

Given a ball $I = (f(x) - \delta, f(x) + \delta)$, we wish to find a neighborhood U of x such that $f(U) \subseteq I$. First we choose rational numbers $p, q \in I$ such that $p < f(x) < q$. Then the open set $U_q \backslash \overline{U_p}$ is the desired neighborhood using the lemmas above. ■

Theorem 11.5. (Tietze Extension Theorem)
Let A be closed in a normal space X. Any continuous map from A to I can be extended to a continuous map from X to I. True also for $\mathbb{R}$ instead of I.

Proof. We show for $[-1,1]$ instead of I, and then for $(-1,1)$ instead of $\mathbb{R}$.

Lemma. If $f : A \to [-\varepsilon, \varepsilon]$ is continuous, there exists continuous $g : X \to \mathbb{R}$ with $g(X) \subseteq [-\varepsilon/3, \varepsilon/3]$ and $(g - f)(A) \subseteq [-2\varepsilon/3, 2\varepsilon/3]$.

Proof. Applying the Urysohn Lemma on the disjoint closed sets $L = f^{-1}([-\varepsilon, -\varepsilon/3])$ and $R = f^{-1}([\varepsilon/3, \varepsilon])$, there exists $g : X \to [-\varepsilon/3, \varepsilon/3]$ such that $g(L) = \{-\varepsilon/3\}$ and $g(R) = \{\varepsilon/3\}$. This g works. □

Now let $f : A \to [-1,1]$ be continuous. Then we can find $g_1 : X \to [-1/3, 1/3]$ such that $|f(a) - g_1(a)| \leqslant 2/3$ for all $a \in A$. Then we apply the Lemma on $f - g_1$ again, so we get $g_2 : X \to [-2/9, 2/9]$ such that $|f(a) - g_1(a) - g_2(a)| \leqslant 4/9$. Recursively, we get a sequence of functions g_n such that $g_{n+1} : X \to [-(2/3)^n/3, (2/3)^n/3]$ and

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leqslant \left(\frac{2}{3}\right)^{n+1}.$$

By the Weierstrass M-test, $g(x) = \displaystyle\sum_{n=1}^{\infty} g_n(x)$ converges to the desired function (Exercise).

To show the $(-1,1)$ version, take g from the $[-1,1]$ case. Apply the Urysohn Lemma to the disjoint closed sets A and $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$ to get a continuous $\varphi : X \to [0,1]$ so that $\varphi(D) = \{0\}$ and $\varphi(A) = \{1\}$. Then $h(x) = \varphi(x)g(x)$ works ($|h(x)| < 1$). ■

Urysohn Metrization Theorem

Definition 11.2.

1. A space is ***second-countable*** if it has a countable basis.
2. A space is ***metrizable*** if it is homeomorphic to a metric space.

Theorem 11.6. (Urysohn Metrization Theorem)
2nd countable + Normal $\implies$ Metrizable.

Proof. We first note that $I^\omega = \{\mathbf{x} = (x_1, x_2, \cdots) : x_i \in I\}$ with the metric

$$d(\mathbf{x}, \mathbf{y}) = \sup_n \frac{|x_n - y_n|}{n}.$$

is a metric space. Let X be normal with a countable basis $\mathscr{B}$. We will embed X into I^ω.

Lemma. There exists a collection $\{f_n : X \to I\}_{n\in\mathbb{N}}$ of continuous functions such that given any $x \in X$ and any neighborhood U, there exists some f_n that is positive at x but vanishes outside U.

Proof. For each $B, C \in \mathscr{B}$ with $\overline{B} \subseteq C$, apply the Urysohn Lemma to construct a continuous function $g_{B,C} : X \to I$ such that $g_{B,C}\left(\overline{B}\right) = \{1\}$ and $g_{B,C}(X\backslash C) = \{0\}$. $\left\{g_{B,C} : \overline{B} \subseteq C\right\}$ is the desired collection. It is countable because $\mathscr{B} \times \mathscr{B}$ is countable, and given any x with neighborhood U, we can choose by Theorem 11.3 the sequence of open sets $x \in B \subseteq \overline{B} \subseteq C \subseteq U$, and then use $g_{B,C}$. □

Using $\{f_n\}_{n\in\mathbb{N}}$ from the Lemma, define $F : X \to I^\omega$ such that

$$F(x) = (f_0(x), f_1(x), f_2(x), \cdots)$$

- F is injective because given $x \neq y$, there exists some $f_n(x) > 0 = f_n(y)$ (Hausdorff!).
- F is continuous: Let $B_x(\varepsilon) \subseteq I^\omega$. Fix an integer $N > 2/\varepsilon$. Since each f_n is continuous, for each $1 \leqslant n \leqslant N$ there exists a neighborhood $x \in U_n$ such that $y \in U_n \implies |f_n(x) - f_n(y)| \leqslant \varepsilon/2$. Hence for any $y \in U_1 \cap \cdots \cap U_N$,

$$\begin{aligned} d\left(F(x), F(y)\right) &= \sup_n \frac{|f_n(x) - f_n(y)|}{n} \\ &\leqslant \max\left(\sup_{1\leqslant n\leqslant N} \frac{|f_n(x) - f_n(y)|}{n}, \sup_{n>N} \frac{|f_n(x) - f_n(y)|}{n}\right) \\ &\leqslant \max\left(\frac{\varepsilon}{2}, \frac{1}{N+1}\right) < \varepsilon. \end{aligned}$$

- For each open set U in X, $F(U)$ is open in $F(X)$: Let $x \in U$ and $f(x) = z$. Choose a f_N that is positive at x but vanishes outside U. Let

$$W = F(X) \cap \pi_N^{-1}\left((0, 1]\right)$$

be open in $F(X)$. We claim that $z \in W \subseteq F(U)$. Firstly, we have $z = F(x) \in W$ because $f_N(x) > 0$. Secondly, given any $F(y) \in W$, we must have $f_N(y) > 0$. Since f_N vanishes outside U, y must be in U, so $F(y) \in F(U)$.

Therefore, X is homeomorphic to its image under F, a subspace of the metric space I^ω, which is also a metric space. ■

12 Manifolds

Definition 12.1. An ***n-manifold*** is a 2nd countable Hausdorff space X such that each $x \in X$ has a neighborhood homeomorphic with an open subset of $\mathbb{R}^n$. We also write $X = X^n$. A 1-manifold is a ***curve***, and a 2-manifold is a ***surface***.

Theorem 12.1. $X^n \times Y^m$ is an $(n+m)$-manifold.

Proof. Hausdorffness and 2nd Countability follow immediately. Fix $(x \times y) \in X \times Y$, then there exists neighborhoods U, V of x, y homeomorphic to $\mathbb{R}^n, \mathbb{R}^m$ respectively. Then $U \times V$ is a neighborhood of $(x \times y)$ homeomorphic to $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$. ■

Example 12.1.

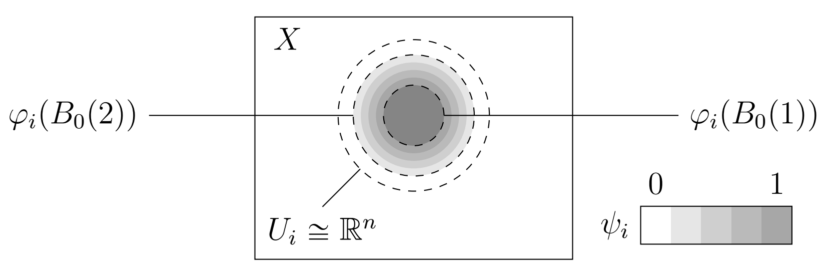
1. $\mathbb{R}^n$ is an n-manifold.
2. S^n is an n-manifold. (Write $S^n = e_1^n \cup e_2^n$ where $e^n = \operatorname{int}(D^n) \cong \mathbb{R}^n$).
3. The ***real projective space*** $\mathbb{RP}^n = S^n/\sim$ (where $x \sim y \iff x = \pm y$) is an n-manifold.
4. $T^n = \underbrace{S^1 \times \cdots S^1}_{n}$ is an n-manifold. T^2 is a ***torus***.
5. *Fact:* Every connected curve is homeomorphic to either $\mathbb{R}$ and S^1.

Theorem 12.2. A compact n-manifold X can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Proof. Each $x \in X$ admits a neighborhood U^x with a homeo $\varphi^x : \mathbb{R}^n \to U^x$. We can choose a basis $x \in B^x \subseteq \varphi^x(B_0(1))$, and hence by compactness of X via the B^x there exists $U_1, \cdots, U_m$ with homeos $\varphi_i : \mathbb{R}^n \to U_i$ and $X \subseteq \bigcup_i \varphi_i(B_0(1))$

By Urysohn's Lemma, there exists $\rho_i : X \to I$ such that $\rho_i\left(\overline{\varphi_i(B_0(1))}\right) = \{1\}$ and $\rho_i(X \backslash \varphi_i(B_0(2))) = \{0\}$. Via the pasting lemma, let $\psi_i : X \to \mathbb{R}^n$ be the continuous function

$$\psi_i(x) = \begin{cases} \rho_i(x)\varphi_i^{-1}(x) & x \in U_i \\ (0, \cdots, 0) & \text{otherwise} \end{cases}.$$



Then $F(x) = (\rho_1(x), \cdots, \rho_m(x), \psi_1(x), \cdots, \psi_m(x))$ embeds X into $\mathbb{R}^{m(n+1)}$. ■

13 Paracompactness

Definition 13.1.

- An open cover $\{U_\alpha\}_\alpha$ of X is ***locally finite*** if every $x \in X$ has a neighborhood that intersects only finitely many U_α.
- A ***refinement*** of an open cover $\{U_\alpha\}_\alpha$ of X is an open cover $\{V_\beta\}_\beta$ such that each V_β is contained in some U_α (depends on β).
- A space X is ***paracompact*** if it is Hausdorff, and, every open cover of X admits a locally finite refinement.

Warning.

1. Some sources do not require Hausdorffness in the definition.
2. Quotient/Subspace/Product of paracompact space(s) may not be paracompact.

Example 13.1. $\mathbb{R}^n$ is paracompact. Let $B(r)$ be the open ball of radius r centered at the origin. Given any open covering $\mathscr{A}$, for each $n \in \mathbb{N}^*$ we can pick a finite number of elements of $\mathscr{A}$ that covers $\overline{B(n)}$. Intersect them with $\mathbb{R}^n \backslash \overline{B(n-1)}$. The union of these open sets is a desired locally finite refinement.

Theorem 13.1.

1. A closed subspace of a paracompact space is paracompact.
2. Compact + Hausdorff $\implies$ Paracompact
3. Metric space $\implies$ Paracompact.
4. Paracompact $\implies$ Normal.

Proof of (4). Let A, B be closed and disjoint. We first prove the case when $A = \{a\}$. For each $b \in B$ pick disjoint neighborhoods $a \in U_b, v \in V_b$. Since $(X\backslash B) \cup_b V_b$ is an open cover of X, by paracompactness there exists a locally finite refinement of V_α's that cover B. Also, x has a neighborhood W that intersects only finitely many V_α, say $V_{b_1}, \cdots, V_{b_n}$. Then the open sets $U = U_{b_1} \cap \cdots \cap U_{b_n}$ and $V = V_{b_1} \cap \cdots \cap V_{b_n}$ form a desired pair.

For the general case, we update the notation so that for each $a \in A$ there exists disjoint open sets $a \in U_a, B \subseteq V_a$. Let $\{U_\alpha\}$ be a locally finite refinement that covers A, so $b \in B$ admits a neighborhood W_b that intersects finitely many U_α, say $U_{a_1}, \cdots, U_{a_n}$. We then let

$V_b = W_b \cap_i V_{a_i}$. Then $U = \bigcup_\alpha U_\alpha$ and $V = \bigcup_{b\in B} V_b$ give the desired separation. ■

Definition 13.2. A ***partition of unity*** on X for a locally finite open cover $\{U_\alpha\}_\alpha$ is a collection of continuous $\rho_\alpha : X \to I$ such that

- $\rho_\alpha(x) > 0 \implies x \in U_\alpha$
- $\sum_\alpha \rho_\alpha(x) = 1$ (well-defined due to local finiteness)

Theorem 13.2. Every cover of a paracompact space admits a refinement that has a partition of unity.

Proof. Let $\{U_\alpha\}$ be a cover of X. For each $x \in X$ there is an $x \in U_{\alpha_x}$ and hence we can pick $x \in W_x \subseteq \overline{W_x} \subseteq U_{\alpha_x}$ by normality. Let $\{V_\beta\}$ be a locally finite refinement of $\{W_x\}$. By Urysohn's Lemma, there exists $\psi_\beta : X \to I$ such that $\psi\left(\overline{V_\beta}\right) = \{1\}$ and $\psi\left(X \backslash U_{\alpha_\beta}\right) = \{0\}$. Then $\rho_\beta(x) = \psi_\beta(x)/\sum_\gamma \psi_\gamma(x)$ is a desired partition of unity. ■

Theorem 13.3. Manifold $\implies$ Paracompact.

Proof. We first prove that a manifold X can be a limit of increasing compact sets.

Lemma. $\exists K_1, K_2, \cdots$ compact with $K_n \subseteq \operatorname{int}(K_{n+1})$ and $X = \bigcup_n \operatorname{int}(K_n)$.
Proof. Let U_i with homeos $\varphi_i : \mathbb{R}^n \to U_i$ such that $\{\varphi_i(B_0(1))\}$ covers X. Then take the compact spaces $K_n = \bigcup_{i=1}^n \bigcup_{j=1}^n \varphi_i\left(\overline{B_0(j)}\right)$ for $n \in \mathbb{N}^*$. □

Let $X = \bigcup_\alpha U_\alpha$. Then for each n there exists $U_1^n, \cdots, U_{t_n}^n$ that cover the compact space K_n. Then $V_j^n = U_j^n \backslash K_{n-1}$ form a locally finite refinement: Any $x \in X$ is contained within some $\operatorname{int}(K_n)$, which means it can only be in the sets V_j^m $(1 \leqslant j \leqslant t_m)(1 \leqslant m \leqslant n)$. This is similar to Example 13.1. ■

14 Covering Dimension

Definition 14.1.

1. The ***covering dimension*** of a space X is the infimum over $n \in \mathbb{N}$ such that

$$(\forall \text{ open cover } \{U_\alpha\})\,(\exists \text{ refinement } \{V_\beta\})\,(\forall x \in X)\,(x \text{ is in} \leqslant n+1 \text{ of the } V_\beta)$$

or equivalently

$$\dim X = \max_{\mathscr{A} \text{ open cover } X} \left[\min_{\mathscr{B} \text{ refmt of } \mathscr{A}} \underbrace{\left(\max_{x \in X} |\{B \in \mathscr{B} : x \in B\}| \right)}_{\textbf{\textit{order}} \text{ of } \mathscr{B}} \right] - 1$$

2. A ***Lebesgue number*** for an open cover $\{U_\alpha\}$ of a compact metric space is a real $\delta > 0$ such that any subset of X of diameter $< \delta$ is contained within some U_α.

Theorem 14.1. (Lebesgue's Covering Lemma)
Any open cover $\{U_\alpha\}$ of a compact metric space (X, d) has a Lebesgue number.

Proof. Since X is compact, assume $\{U_\alpha\} = \{U_1, \cdots, U_n\}$. The map $f(x) = \max_{1 \leqslant i \leqslant n} d(x, X \backslash U_i) > 0$ is continuous on a compact space and thus $f(X)$ has a minimum $\delta > 0$. ■

Example 14.1.

1. Any compact subspace of $\mathbb{R}$ has dimension at most 1.

   Proof. Note that $\mathscr{C} = \left\{(n, n+1), \left(n - \frac{1}{2}, n + \frac{1}{2}\right) : n \in \mathbb{Z}\right\}$ has order 2. Let $\mathscr{A}$ be any open covering of a compact subspace X of $\mathbb{R}$, with some Lebesgue number $\delta > 0$. The image $\mathscr{I}$ of $\mathscr{C}$ under $f : x \mapsto \delta x/2$ is an open covering whose elements have diameter $\delta/2 < \delta$, and hence is an open refinement subcover of $\mathscr{A}$. Hence

$$\begin{aligned} \dim X &= \max_{\mathscr{A} \text{ open cover } X} \left[\min_{\substack{\mathscr{B} \text{ open refinement} \\ \text{subcover of } \mathscr{A}}} (\text{order of } \mathscr{B}) \right] - 1 \\ &\leqslant \max_{\mathscr{A} \text{ open cover } X} [2] - 1 = 1. \end{aligned}$$

2. $\dim I = 1$.

   Proof. We show that there is some open covering $\mathscr{A}$ such that any open refinement subcover of $\mathscr{A}$ has order at least 2. Let $\mathscr{A} = \{[0,1),(0,1]\}$ and let $\mathscr{B}$ be any open refinement subcovering. Since 0 and 1 cannot belong to the same refinement, $\mathscr{B}$ has at least two elements. Partition $\mathscr{B}$ into two nonempty parts $\mathscr{B}_1$ and $\mathscr{B}_2$. If $\mathscr{B}$ had order 1 then $\bigcup \mathscr{B}_1$ and $\bigcup \mathscr{B}_2$ disconnect $[0,1]$, a contradiction.

3. *Fact:* $\dim I^n = n$, and every compact subspace of $\mathbb{R}^n$ has dimension $\leqslant n$.

Theorem 14.2.

- If Y is a closed subspace of a finite dimensional space X, then $\dim Y \leqslant \dim X$.
- If $X = Y \cup Z$ where Y, Z are closed finite dimensional subspaces of X, then $\dim X = \max(\dim Y, \dim Z)$.
- Every compact subspace of $\mathbb{R}^N$ has dimension at most N.

Tangent: Baire's Theorem, Function Spaces and Geometry

Definition 14.2. Let X be a compact metric space.

1. $\mathcal{C}(X, \mathbb{R}^n) = \{f : X \to \mathbb{R}^n \text{ cts}\}$ is the metric space equipped with the uniform metric $d(f,g) = \sup_x |f(x) - g(x)|$.
2. For $A \subseteq X$, $\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y)$.
3. $\Delta(f) = \sup \{\operatorname{diam}(f^{-1}\{z\}) : z \in f(X)\}$ (Deviation of f from injectivity).

Remark. $\displaystyle\bigcap_n U_{1/n} = \{f : \Delta(f) = 0\} = \{f \text{ injective}\}$.

Theorem 14.3. (Baire's Theorem)
Let $\{U_n\}$ be a countable collection of dense open sets in a compact Hausdorff space X. Then $\bigcap_n U_n$ is dense in X.

Proof. Let W_1 be an open set. We want to show $W_1 \cap_n U_n \neq \varnothing$.

- Since U_1 is dense and open, there exists $x_1 \in W_1 \cap U_1$ open.
- Inductively, since X is normal, there exists $x_n \in W_n \subseteq \overline{W_n} \subseteq W_{n-1} \cap U_{n-1}$.

Since X is compact and $\overline{W_1} \supseteq \overline{W_2} \supseteq \cdots$, we have

$$\varnothing \neq \bigcap_n \overline{W_n} \subseteq \bigcap_n (U_n \cap W_n) \subseteq W \cap_n U_n. \qquad \blacksquare$$

Definition 14.3.

1. $\{z_0, \cdots, z_m\} \subseteq \mathbb{R}^n$ are ***geometrically independent*** if

$$\lambda_0 z_0 + \cdots + \lambda_m z_m = \mathbf{0},\ \lambda_0 + \cdots + \lambda_m = 0 \implies \lambda_0 = \cdots = \lambda_m = 0$$

2. $A \subseteq \mathbb{R}^n$ is in ***general position*** if any subset of size $n+1$ are geom. ind.

Theorem 14.4. Given $\{z_0, \cdots, z_m\} \subseteq \mathbb{R}^n$ and $\delta > 0$, there exists $\{y_0, \cdots, y_m\} \subseteq \mathbb{R}^n$ that is in general position such that all $|z_i - y_i| < \delta$.

Back to dimension theory

Theorem 14.5. (Embedding Compact Metric Spaces)
Every compact metric space X of dimension n can be embedded in $\mathbb{R}^{2n+1}$.

Define $U_\varepsilon = \{f \in \mathcal{C}(X, \mathbb{R}^{2n+1}) : \Delta(f) < \varepsilon\}$.

Claim. U_ε is open.

Proof. Let $f \in U_\varepsilon$, we want to show $\exists B_f(\delta) \subseteq U_\varepsilon$. Pick $\varepsilon < b < \Delta(f)$ and define

$$A = \{(x \times y) : d(x, y) \geqslant b\} \subseteq X^2$$

Note that $f(x) = f(y) \implies d(x, y) \leqslant \Delta(f) < b \implies (x \times y) \notin A$. Hence $|f(x) - f(y)|$ has a positive minimum 2δ on A. Now if $g \in B_f(\delta)$, then for any $(x \times y) \in A$,

$$|f(x) - g(x)| < \delta, \quad |f(y) - g(y)| < \delta, \quad |f(x) - f(y)| \geqslant 2\delta$$

so $g(x) \neq g(y)$. In other words, $g(x) = g(y) \implies d(x, y) < b \implies \Delta g \leqslant b < \varepsilon. \qquad \square$

Claim. U_ε is dense. (Difficult!)

Proof. Let $f \in \mathcal{C}\left(X, \mathbb{R}^{2n+1}\right)$ and $\delta > 0$, we want to find a $g \in B_f(\delta) \cap U_\varepsilon$. Firstly, we cover X with $V_1, \cdots, V_m$ such that

(1) $\operatorname{diam}(V_i) < \varepsilon/2$

(2) $\operatorname{diam}(f(V_i)) < \delta/2$

(3) Each $x \in X$ is in at most $n+1$ of the V_i.

To do this, pick a Lebesgue number $0 < \kappa < \varepsilon/4$ such that any $B_x(\kappa) \subseteq f^{-1}\left(B_y(\delta/4)\right)$ for some y. Since $\dim X \leqslant n$, there exists a refinement $\{V_\beta\}_\beta$ of $\{B_x(\kappa)\}_x$ such that (3) holds. Since $V_\beta B_{x(\beta)}(\kappa)$ for some $x(\beta)$, (1) and (2) also hold. By compactness, we can find a finite cover using V_i.

Let $\varphi_i : X \to \mathbb{R}$ be a partition of unity associated to the U_i. Also, fix $x_i \in U_i$ and $z_i \in \mathbb{R}^{2n+1}$ such that $|f(x_i) - z_i| < \delta/2$ and $\{z_i\}$ is in general position. Define

$$g(x) = \sum_i \varphi_i(x) z_i.$$

Then $d(f,g) < \delta$ because

$$|g(x) - f(x)| = \left|\sum_i \varphi_i(x)(z_i - f(x_i)) + \sum_i \varphi_i(x)(f(x_i) - f(x))\right| < \sum_i \varphi_i(x)\left(\frac{\delta}{2} + \frac{\delta}{2}\right) = \delta.$$

and $g \in U_\varepsilon$ because $g(x) = g(y) \implies \sum_i \left(\varphi_i(x) - \varphi_i(y)\right) z_i = \mathbf{0} \implies \varphi_i(x) = \varphi_i(y)\ \forall i$ since x, y are in $\leqslant 2(n+1)$ of the U_i. Since $\varphi_i(x) > 0$ for some i, we have $x, y \in U_i \implies d(x,y) < \varepsilon/2$. Therefore $\Delta(g) \leqslant \varepsilon/2 < \varepsilon$. $\square$

By Baire's theorem, $\bigcap_n U_{1/n}$ is dense and hence non-empty, i.e. there is a continuous injective $f : X \to \mathbb{R}^{2n+1}$. Also since X is compact and $f(X)$ is Hausdorff, f sends closed sets to closed sets (i.e. is closed). Hence f embeds X into $\mathbb{R}^{2n+1}$. $\blacksquare$

Theorem 14.6. (Embedding Manifolds)
Every manifold can embedded in some $\mathbb{R}^N$.

Proof. Let X be an m-manifold.

Lemma 1. Let $f : X \to \mathbb{R}^N$ such that $f^{-1}(\text{compact}) = \text{compact}$. Then f is closed (sends closed sets to closed sets).

Proof. Let $C \subseteq X$ be closed. Suppose $y \in \mathbb{R}^N \backslash f(C)$. By Heine-Borel, $\overline{B_y(\varepsilon)}$ is compact and hence $K = C \cap f^{-1}\left(\overline{B_y(\varepsilon)}\right)$ is compact $\implies f(K) \subseteq f(C)$ is compact $\implies V = B_y(\varepsilon) \backslash f(K)$ is a neighborhood of y. Note that

$$\begin{aligned} z \in V \cap f(C) &\implies \exists x \in f^{-1}\left(B_y(\varepsilon)\right) \cap C \subseteq K \text{ with } f(x) = z \\ &\implies z \in f(K) \implies V \cap f(C) = \varnothing \end{aligned}$$

and thus $f(C)$ is closed. □

Lemma 2. There exists continuous $f : X \to \mathbb{R}$ such that $f^{-1}(\text{compact}) = \text{compact}$.

Proof. Using the Lemma from Theorem 13.3, we can write X as a limit of increasing compact sets $\bigcup_n K_n$ where $K_n \subseteq \text{int}(K_{n+1})$. Since manifold $\implies$ paracompact $\implies$ normal, we can use Urysohn's Lemma to construct continuous maps $\varphi_n : X \to I$ such that $\varphi_n(K_n) \equiv 0$ and $\varphi_n\left(\overline{X \backslash K_{n+1}}\right) \equiv 1$. Then we define $f : X \to \mathbb{R}$ by $f = \sum_{n=1}^{\infty} \varphi_n$.

- $x \in K_n \implies \varphi_n(x) = \varphi_{n+1}(x) = \cdots = 0$ and hence f is well-defined.
- $x \notin K_n \implies \varphi_{n-1}(x) = \varphi_{n-2}(x) = \cdots = 1 \implies f(x) \geqslant n-1$.
- f is continuous: Given any $(a,b) \subseteq \mathbb{R}$, $f^{-1}((a,b)) \subseteq K_{\lceil b+2 \rceil}$ and hence $f^{-1}((a,b))$ is the preimage of (a,b) under $\sum_{n=1}^{\lceil b+1 \rceil} \varphi_n$ (a continuous map) which is open.
- $f^{-1}(C)$ is compact for any compact $C \subseteq \mathbb{R}$: Since C is closed and bounded, $f^{-1}(C)$ is closed and contained within some K_N (compact), and hence $f^{-1}(C)$ is compact (closed subspace of a compact space). □

Take K_n and f from Lemma 2, and denote $R_n = K_n \backslash \text{int}(K_{n-1})$ and $U_n = \text{int}(K_{n+1}) \backslash K_{n-2}$. By Urysohn's Lemma again, construct $\rho_n : X \to \mathbb{R}$ with $\rho_n(R_n) \equiv 1, \rho_n\left(X \backslash U_n\right) \equiv 0$.

Since $D_n = K_{n+1} \backslash \text{int}(K_{n-2})$ is compact and metrizable (normal and 2nd countable), there exists a cts closed inj $f_n : D_n \hookrightarrow \mathbb{R}^{2m+1}$. Then define $\psi_n : X \to \mathbb{R}^{2m+1}, \psi : X \to \mathbb{R}^{4m+3}$ as

$$\psi_n(x) = \begin{cases} \rho_n(x) f_n(x) & x \in U_n \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad \psi(x) = \left(\sum_{\text{even } n} \psi_n(x), \sum_{\text{odd } n} \psi_n(x), f(x) \right).$$

ψ is injective (Exercise: $f(x) = f(y) \implies x, y \in R_\ell$, and $\sum_{i \equiv_2 \ell} \psi_i(x) = \psi_\ell(x) = f_\ell(x) = f_\ell(y) \implies x = y$) and closed (for any compact $K \subseteq \mathbb{R}^N, \psi^{-1}(K)$ is closed and contained within the compact $f^{-1}(\pi_N(K))$). Thus ψ embeds X into $\mathbb{R}^{4m+3}$. ■

15 Homotopies

From now on, assume all 'maps' are continuous.

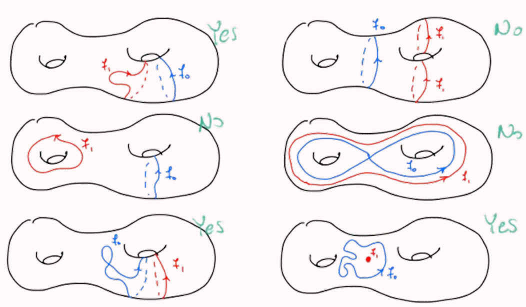
Definition 15.1.

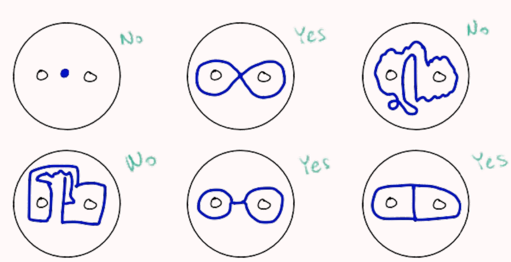
1. Given $f_0, f_1 : X \to Y$, a ***homotopy*** from f_0 to f_1 is $H : X \times I \to Y$ such that $f_0(x) = H(x, 0), f_1(x) = H(x, 1)$. We sometimes write $H(x, t) = f_t(x)$. If such homotopy exists, we say f_0, f_1 are ***homotopic*** ($f_0 \simeq f_1$).
2. A ***homotopy relative to*** $\boldsymbol{A} \subseteq X$ (homotopy rel A) is a homotopy $H : X \times I \to Y$ such that $H(a, t) = H(a, 0)$ for all $a \in A$.
3. A ***reparameterization*** of $\alpha : I \to X$ is a map $\beta : I \to X$ such that $\beta = \alpha \circ r$ where $r : I \to I$ satisfies $r(0) = 0, r(1) = 1$.
4. X, Y are ***homotopy equivalent*** ($X \simeq Y$) if there exists $f : X \to Y, g : Y \to X$ (called homotopy equivalences) such that $f \circ g \simeq \mathbf{1}_Y$ and $g \circ f \simeq \mathbf{1}_X$.
5. X is ***contractible*** if $X \simeq$ point. $f : X \to Y$ is ***nullhomotopic*** if $f \simeq$ constant.
6. A ***retraction*** of X onto $A \subseteq X$ is a map $r : X \to X$ with $r|_A = \mathbf{1}_A, r(X) = A$. If it exists, A is a ***retract*** of X.
7. A ***deformation retraction*** of X onto $A \subseteq X$ is a homotopy rel A from the identity on X to a retraction of X onto A. If it exists, A is a ***deformation retract*** of X.

Example 15.1.

(L) Which paths $f : S^1 \to T^2 \# T^2$ are homotopic?

(R) $D^2 \backslash \{x_0, x_1\}$ deformation retracts to which blue sets?





Remark.

1. If β is a reparameterization of α then $\alpha \simeq \beta$ rel $\{0,1\}$.
2. $X \cong Y \implies X \simeq Y$ but not converse, e.g. Möbius band $\simeq S^1 \simeq$ Band $S^1 \times I$.
3. *Fact:* $X \simeq Y \iff \exists Z$ that deformation retracts to both X and Y.

16 CW Complexes

Definition 16.1. A ***CW complex / cell complex*** is a space X built as such:

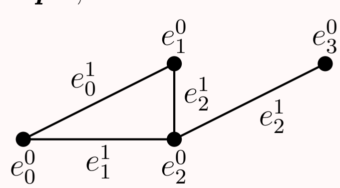
1. Start with a discrete set X^0, whose points are ***0-cells***.
2. Let D^n_α be n-balls (with $\partial D^n_\alpha = S^{n-1}_\alpha$). Inductively, form the ***n-skeleton*** X^n as the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ by identifying $x \sim \varphi_\alpha(x)$ where $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ are the ***attaching maps***. This makes $X^n = X^{n-1} \sqcup_\alpha \operatorname{int}(D^n_\alpha)$ as a set. The $e^n_\alpha = \operatorname{int}(D^n_\alpha)$ are called ***n-cells***.
3. One can stop after finite n, setting $X = X^n$. Or one can set $X = \bigcup_{n=0}^\infty X^n$, giving it the *weak topology*: $U \subseteq X$ is open $\Leftrightarrow U \cap X^n$ is open in X^n for all n.

The ***characteristic map*** of a cell e^n_α is the map

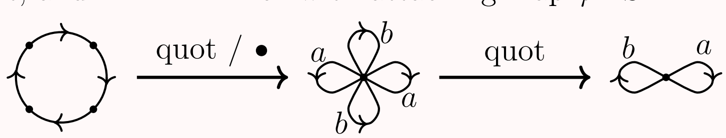
$$\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \sqcup_\beta D^n_\beta \xrightarrow{\text{quot}} X^n \hookrightarrow X$$

Example 16.1.

1. A 1-dim CW complex is a ***graph***, whose 0-cells are ***nodes*** and 1-cells are ***edges***.



2. $X = T^2$ is a CW complex, with $X^0 = \{e^0_0\}$, $X_1 = X^0 \sqcup e^0_a \sqcup e^0_b$ where $\varphi_a \equiv \varphi_b \equiv e^0_0$ being constant, and $X^2 = X^1 \sqcup e^2$ with attaching map $\varphi : S^1 \to X^1$ given by



Note: If we swap the direction of two adjacent leaves in the middle step, we get a ***Klein bottle***. Attaching maps matter!

3. The n-sphere S^n is a cell complex with two cells e^0 and e^n, with the attaching map $S^{n-1} \to e^0$. Or, we can inductively attach two n-cells to the equator S^{n-1}.

4. $\mathbb{RP}^n \cong S^n/(v \sim -v) \cong D^n/(v \sim -v : v \in \partial D^n)$ is a cell complex by attaching an n-cell to $\mathbb{RP}^{n-1}$ via the map $S^{n-1} \twoheadrightarrow \mathbb{RP}^{n-1}$. We can also have $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$.

Definition 16.2. A ***subcomplex*** of a CW complex X is a closed subspace $A \subseteq X$ that is a union of cells of X. The pair (X, A) is a ***CW pair***.

Example 16.2.

1. $\mathbb{RP}^k \subseteq \mathbb{RP}^n$ is a subcomplex ($k \leqslant n$).

2. $S^k \subseteq S^n$ is not a subcomplex with the two-cell structure, but is a subcomplex using the recursive CW structure.

Theorem 16.1.

- If X, Y are cell complexes, then $X \times Y$ is a cell complex, whose cells are $e^m_\alpha \times e^n_\beta$ where e^m_α, e^n_β are cells of X, Y respectively.

- If (X, A) is a CW pair, then the quotient space X/A is a cell complex, whose cells are the cells of $X \backslash A$, and one new 0-cell: the image of A in X/A.

Definition 16.3. $A \subseteq X$ has the ***homotopy extension property*** if given any map $f_0 : X \to Y$ and a homotopy $f_t|_A : A \to Y$ of $f_0|_A$, we can extend $f_t|_A$ to a homotopy f_t on X. Equivalently, given any maps $H_1 : X \times \{0\} \to Y$ and $H_2 : A \times I \to Y$ that agree on $A \times \{0\}$, there exists a map $H : X \times I \to Y$ such that H agrees with both H_1, H_2 where their domains meet.

Theorem 16.2. $A \subseteq X$ has the homotopy extension property if and only if

$$X \times \{0\} \cup A \times [0, 1] \text{ is a retract of } X \times [0, 1].$$

Proof. Let $Z = X \times \{0\} \cup A \times [0, 1]$.

- If $A \subseteq X$ has h.e.p then given the maps $H_1 : X \times \{0\} \to Z$ and $H_2 : A \times I \to Z$ with

$$H_1(x, 0) = (x, 0) \qquad \text{and} \qquad H_2(a, t) = (a, t)$$

  we can get an extension $H : X \times I \to Z$ constant on Z. Hence H is the retraction.

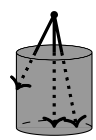
- The converse is easy if we assume A is closed. Say $r : X \times I \to Z$ is a retraction. Given any H_1, H_2 as in the definition, we can combine them via the Pasting Lemma to get $H_3 : Z \to Y$. Then $H_3 \circ r : X \times I \to Y$ is the required homotopy. For the full proof where A is not necessarily closed, see appendix of [Hatcher]. ■

Theorem 16.3. If (X, A) is a CW pair, A has the homotopy extension property.

Proof. To prove $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, we first prove

Lemma. $D^n \times \{0\} \cup \partial D^n \times I$ is a deformation retract of $D^n \times I$.

Proof. Consider radial projection r from $(\mathbf{0}, 2) \in D^n \times \mathbb{R}$:



Then $f_t = t \cdot r + (1 - t) \cdot \mathbf{1}$ is a deformation retract. □

Applying the deformation retraction to every D^n attached to X^{n-1} that is not in A^n, we get a deformation retraction H_n from $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$. Note that concatenating adjacent H_n and H_{n+1} gives a deformation retraction

$$\begin{aligned} X^{n+1} \times I &\xrightarrow{H_{n+1}} X^{n+1} \times \{0\} \cup \left(X^n \cup A^{n+1}\right) \times I \\ &\xrightarrow{H_n} X^{n+1} \times \{0\} \cup \left(\left(X^n \times \{0\} \cup \left(X^{n-1} \cup A^n\right) \times I\right) \cup \left(A^{n+1} \times I\right)\right) \\ &= X^{n+1} \times \{0\} \cup \left(X^{n-1} \cup A^{n+1}\right) \times I \end{aligned}$$

and thus by concatenating all $H_0, H_1, \cdots$ into $[1/4, 1/2], [1/8, 1/4], \cdots$ we get a deformation retract from $X \times I$ onto $X \times \{0\} \cup A \times I$. (In the infinite case, there is no continuity problem at $t = 0$ since X is given the weak topology). ■

Theorem 16.4. If (X, A) is a CW pair and A is contractible, then the quotient map $X \twoheadrightarrow X/A$ is a homotopy equivalence.

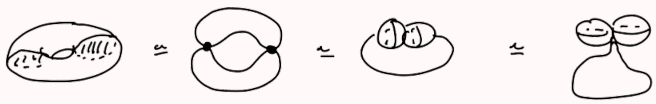
Proof. Let $f_t : X \to X$ be a homotopy extension of the contraction of A with $f_0 = \mathbf{1}_X$. Since $f_t(A) \subseteq A$ and $f_1(A) = \text{pt}$, we can construct well-defined maps $\overline{f_t}, g$ satisfying

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_t}"'] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \\ X/A \arrow[ur, "g"'] & \end{tikzcd}$$

Then $g \circ q = f_1 \simeq f_0 = \mathbf{1}_X$ and $q(g([x])) = q(g(q(x)) = q(f_1(x)) = \overline{f_1}(q(x)) = \overline{f_1}([x])$ and hence $q \circ g = \overline{f_1} \simeq \overline{f_0} = \mathbf{1}_{X/A}$, so g, q are homotopy equivalences.

Example 16.3.

1. 

2. 

17 Fundamental Groups

Definition 17.1.

1. A ***path*** on X is $\alpha : I \to X$. Define $\Omega_{x_0}(X) = \{\text{path } \alpha \mid \alpha(0) = \alpha(1) = x_0\}$.

2. Given paths $\alpha, \beta \in \Omega_{x_0}(X)$, define the ***concatenation*** $\alpha \cdot \beta \in \Omega_{x_0}(X)$ by

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s) & 0 \leqslant s \leqslant 0.5 \\ \beta(2s-1) & 0.5 \leqslant s \leqslant 1. \end{cases}$$

3. Given a path $\gamma \in \Omega_{x_0}(X)$, define the ***reversed path*** $\overline{\gamma}(t) = \gamma(1-t)$.

4. The ***fundamental group*** of X based at x_0 is the group

$$\pi_1(X, x_0) = \Omega_{x_0}(X)/\sim$$

where $\alpha \sim \beta \Leftrightarrow \alpha \simeq \beta \text{ rel } \{0,1\}$, with group law $[\alpha][\beta] = [\alpha \cdot \beta]$ and $[\gamma]^{-1} = \overline{\gamma}$.

Theorem 17.1. Let γ be a path from x_0 to x_1. The map $\Phi_\gamma : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $\Phi([\alpha]) = [\gamma \cdot \alpha \cdot \overline{\gamma}]$ is an isomorphism.

Corollary. If X is path-connected, $\pi_1(X, x)$ are isomorphic over all $x \in X$ (say $\pi_1(X)$).

Theorem 17.2. If X, Y are path-connected, $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

Definition 17.2. X is ***simply connected*** if X is path-connected and $\pi_1(X)$ is trivial.

Definition 17.3.

1. Write $f : (X, x_0) \to (Y, y_0)$ if $f : X \to Y$ and $f(x_0) = y_0$.
2. The ***homomorphism induced*** by $f : (X, x_0) \to (Y, y_0)$ is the homomorphism
$$f_* : \pi_1(X, x_0) \to \pi_1(X, y_0)$$
given by $f_*([\alpha]) = [f \circ \alpha]$.

Theorem 17.3.

1. $(f \circ g)_* = f_* \circ g_*$.
2. If $f, g : X \to Y$ are homotopic rel x_0, then $f_* = g_*$.
3. If $f : X \to Y$ is a homotopy equivalence, then f_* is an isomorphism.

Theorem 17.4. $\pi_1(S^1) = \mathbb{Z}$.

Proof. Let $p : \mathbb{R} \to S^1$ given by $p(\lambda) = (\cos(2\pi\lambda), \sin(2\pi\lambda))$. The following two facts will be proven in the Covering Spaces chapter.

1. Given any path γ of S^1, there exists a unique path $\tilde{\gamma}$ of $\mathbb{R}$ such that $\tilde{\gamma}(0) = 0$ and $\gamma = p \circ \tilde{\gamma}$.
2. Given any homotopy $f_t : I \to S^1$, there exists a unique homotopy $\tilde{f}_t : I \to \mathbb{R}$ such that $f_t = p \circ \tilde{f}_t$

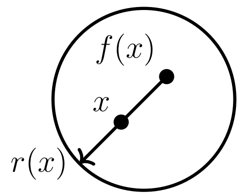
The map $\Phi([\gamma]) = \tilde{\gamma}(1) \in \mathbb{Z}$ is then a well-defined isomorphism. ∎

Theorem 17.5. If A is a retract of X, then the inclusion $i : A \hookrightarrow X$ induces an injective homomorphism i_*. If A is a defo retract of X, then i_* is an isomorphism.

Proof. Let $r : X \twoheadrightarrow A$ be a retraction. Then $r \circ i = \mathbf{1} \implies r_* \circ i_* = \mathbf{1} \implies i_*$ injective. If there is a deformation retraction, then i is a homotopy equivalence and hence i_* is an isomorphism. ∎

Theorem 17.6. (Brouwer's Fixed Point Theorem)
$f : D^2 \to D^2 \implies f(x) = x$ for some $x \in D^2$.

Proof. Otherwise, the map r defined by



is a retract from D^2 to S^1, so $i : S^1 \to D^2$ induces an injective $i_* : \mathbb{Z} \to \{0\}$, contradiction. ■

Theorem 17.7. (Fundamental Theorem of Algebra)
Every complex polynomial of positive degree has a root.

Proof. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $n > 0$. Assume f has no roots. Then

$$F(s,t,r) = \left(re^{2\pi is}\right)^n + a_{n-1}\left(re^{2\pi is}\right)^{n-1} t + \cdots + a_0 t^n \neq 0 \quad \forall s,t,r.$$

Then the homotopy

$$\frac{F(s,0,1)}{|F(s,0,1)|} \overset{t}{\simeq} \frac{F(s,1,1)}{|F(s,1,1)|} \overset{r}{\simeq} \frac{F(s,1,0)}{|F(s,1,0)|}$$

brings the path $e^{2\pi isn}$, that loops around the circle n times, to the trivial path 1. This is a contradiction since they correspond to different elements in $\pi_1(S^1)$. ■

18 Van Kampen's Theorem

Definition 18.1. Let $i_1 : H \hookrightarrow G_1$ and $i_2 : H \hookrightarrow G_2$ be homomorphisms. The ***amalgamated free product*** of G_1 and G_2 along H, denoted as $G = G_1 *_H G_2$, is the unique group (up to isomorphism) that satisfies

(1) There exists homomorphisms $\varphi_i : G_i \to G$ with $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$.

(2) For any other homomorphisms $\psi_i : G_i \to K$ with $\psi_1 \circ i_1 = \psi_2 \circ i_2$, there exists a unique homomorphism $\psi : G \to K$ with $\psi \circ \varphi_i = \psi_i$.

$$\begin{tikzcd}
H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "\varphi_1"] \arrow[ddr, bend left, "\psi_1"] & \\
G_2 \arrow[r, "\varphi_2"'] \arrow[drr, bend right, "\psi_2"'] & G \arrow[dr, dotted, "\psi", "\exists!"'] & \\
 & & K
\end{tikzcd}$$

If $H = \{0\}$, then $G_1 * G_2 = G_1 *_H G_2$ is just the ***free product*** of G_1 and G_2.

Remark.

1. Such a group always exists, e.g. if $G_i = \langle S_i \mid R_i \rangle$ then
$$G_1 *_H G_2 = \left\langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \cup \left\{ i_1(h) i_2(h^{-1}) : h \in H \right\} \right\rangle.$$
Uniqueness follows from the uniqueness of ψ between two such possible groups.

2. Think of $G_1 *_H G_2$ by first treating H as a common subgroup of G_1, G_2, then construct all possible words of finite length with letters from $G_1 \cup G_2$. When two adjacent letters in a word both come from the same G_i, or if they both belong to H, we can further simplify the word.

Example 18.1.

1. The free group with n letters is simply $F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}$.

2. The free product of $\mathbb{Z}_2 = \{1, a, a^2 = 1\}$ and itself $\mathbb{Z}_2 = \{1, b, b^2 = 1\}$ is
$$\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, a, b, ab, ba, aba, bab, \cdots\}$$
(This is the semi-direct product of $\mathbb{Z} = \langle c := ab \rangle, \mathbb{Z}_2 = \langle a \rangle$ with $ac = c^{-1}a$, sometimes called the *infinite dihedral group*.)

3. If we embed $H = \mathbb{Z}_2$ into the two $\mathbb{Z}_2$'s above by $h \mapsto a$ and $h \mapsto b$, then the free product collapses into
$$\mathbb{Z}_2 *_H \mathbb{Z}_2 = \left\{1, h, h^2 = 1\right\} = \mathbb{Z}_2$$

Theorem 18.1. (Van Kampen's Theorem, two-set version)
Suppose $X = U \cup V$ where $U, V, U \cap V$ are open and path-connected, then for $x_0 \in U \cap V$ we have $\pi_1(X, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$ (with $\pi_1(U \cap V, x_0) \hookrightarrow \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \hookrightarrow \pi_1(V, x_0)$ being the maps induced by the inclusions $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ respectively).

Example 18.2. $\pi_1(S^n) = \{0\}$ for $n \geqslant 2$ (*high-dim spheres are simply connected*).

S^n is the union of open neighborhoods of the north and south hemisphere, intersecting at the equator $\simeq S^{n-1}$. Hence $\pi_1(S^n) = \pi_1(e^n) *_{\pi_1(S^{n-1})} \pi_1(e^n) = \{0\} *_{\pi_1(S^{n-1})} \{0\} = \{0\}$.

Definition 18.2. Suppose $x_0 \in X, y_0 \in Y$. The ***wedge sum*** $(X, x_0) \vee (Y, y_0)$ is the space $(X \sqcup Y)/\{x_0, y_0\}$ (gluing X and Y together at x_0, y_0). Lazy: $X \vee Y$.

Example 18.3. $S^1 \vee S^1$ is the figure-eight, homemorphic to the shape ∞.

Theorem 18.2. If $\exists$ neighborhoods $x_0 \in U, y_0 \in V$ in X, Y such that $\{x_0\}, \{y_0\}$ are deformation retracts of U, V respectively, then $\pi_1(X \vee Y) = \pi_1(X) \times \pi_1(Y)$.

Proof. Let $H_t : U \to U, G_t : V \to V$ be deformation retracts onto x_0, y_0 respectively.

- We can define $\overline{G_t} : X \vee V \to X \vee V$ by

$$\begin{tikzcd} X \sqcup V \arrow[r, "\mathbf{1} \sqcup G_t"] \arrow[d, "q"'] & X \sqcup V \arrow[d, "q"] \\ X \vee V \arrow[r, "\overline{G_t}"'] & X \vee V \end{tikzcd}$$

  which is a deformation retraction of $X \vee V$ onto $X \vee \{y_0\} \cong X$. Hence $X \vee V$ deformation retracts onto X and (similarly) $U \vee Y$ deformation retracts onto Y.

- We claim that $U \vee V \subseteq X \vee Y$ is contractible. The map $F_t : U \vee V \to U \vee V$ defined by

$$\begin{tikzcd} U \sqcup V \arrow[r, "H_t \sqcup G_t"] \arrow[d, "q"'] & U \sqcup V \arrow[d, "q"] \\ U \vee V \arrow[r, "F_t"'] & U \vee V \end{tikzcd}$$

  is a deformation retraction onto $x_0 \in U \vee V$.

- By Van Kampen, $\pi_1(X \vee Y) = \pi_1(X \vee V) *_{\pi_1(U \vee V)} \pi_1(U \vee Y) = \pi_1(X) * \pi_1(Y)$. ■

Corollary 18.2. $\pi_1\left(\bigvee_{i=1}^n S^1\right) = F_n$.

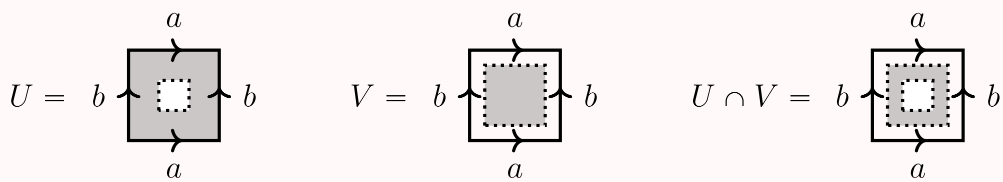
Theorem 18.3. If Γ is a connected graph, then $\pi_1(\Gamma) = F_{1-\chi(\Gamma)}$ where $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$ is the ***Euler characteristic*** of Γ.

Proof. Let T be a spanning tree of Γ, which is contractible. Then by collapsing T, the graph $\Gamma/T \simeq \Gamma$ is a wedge sum of $|E(\Gamma - T)|$ circles. Hence $\pi_1(\Gamma) = F_n$ where

$$n = |E(\Gamma)| - |E(T)| = |E(\Gamma)| - (|V(T)| - 1) = 1 - \chi(T).$$ ■

Theorem 18.4. If $i : H \to G = \langle S \mid R \rangle$, then $G *_H \{0\} = \langle S \mid R \cup i(H) \rangle$

Example 18.4. We can compute $\pi_1(T^2)$ as follows:



$$\pi_1(T^2) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \langle a, b \rangle *_{\mathbb{Z}} \{0\} = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle = \mathbb{Z}^2.$$

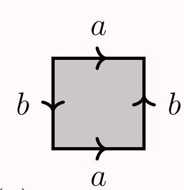
Fundamental Group of CW Complexes

Theorem 18.5.

1. Let X^2 be a CW complex obtained from X^1 by attaching 2-cells e^2_α via $\varphi_\alpha : \partial D^2_\alpha \to X^1$. For each α, let γ_α be a path on X^1 from x_0 to a point $z_\alpha \in \partial D^2_\alpha$.
$$\pi_1(X^2, x_0) = \pi_1(X^1, x_0)/N$$
where N is the normal closure of the subgroup of $\pi_1(X^1, x_0)$ generated by paths $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}]$ (treating φ_α as a closed path based at z_α).
2. Attaching n-cells ($n \geqslant 3$) does not change the fundamental group, i.e.
$$\pi_1(X, x_0) = \pi_1(X^2, x_0)$$

Example 18.5.

1. For the Klein bottle K, we have $\pi_1(K) = \langle a, b \rangle / N$ where N is generated by $aba^{-1}b$, so $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$.

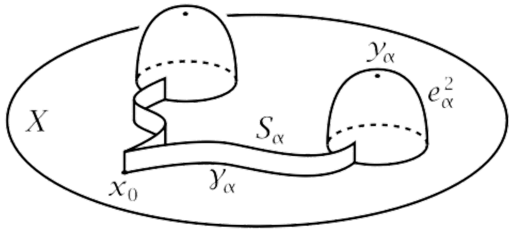


2. If X is obtained by attaching a single 2-cell to a circle $\mathbb{C}^\times$ via $\varphi(z) = z^n$, then $\pi_1(X) = \langle x \mid x^n \rangle = \mathbb{Z}_n$. In particular, $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$.

Corollary 18.5. Given any group G, there exists a space X with $\pi_1(X) = G$.

Proof. Write $G = \langle S \mid R \rangle$ and attach 2-cells (according to R) to the wedge sum $\bigvee_{s \in S} S^1_s$. ■

Proof of Theorem 18.5. First expand X^2 by bulging up the e_α^2's and then adding strips $S_\alpha = I \times I$ along each γ_α. Pick a $y_\alpha \in e_\alpha^2$ that is not on the strip. Call this larger space Z.



We then slice this space along half the height of the S_α's, and consider an open neighborhood of the top and bottom parts U, V respectively (e.g. $U = Z \backslash X^1$ and $V = Z \backslash \bigcup_\alpha \{y_\alpha\}$). U is contractible while V deformation retracts to X^1. Hence

$$\pi_1(X^2, x_0) = \pi_1(Z, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) = \{0\} *_{\pi_1(U \cap V, x_0)} \pi_1(X^1, x_0).$$

So it remains to show that $\pi_1(U \cap V, x_0)$ is generated by the $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}]$: We can apply Van Kampen again on $U \cap V$ by covering it with the open sets $A_\alpha = U \cap V \backslash \bigcup_{\beta \neq \alpha} D_\beta^2$ which deformation retract to a circle and hence is generated by $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}$. This shows (1).

To show (2), we perform the same procedure. However, in the last step, the A_α deformation retract to spheres, which are simply connected. The finite X^n case follows from induction. If X is infinite-dimensional, any closed loop at x_0 is compact and hence is contained in some finite X^n anyway. ■

Definition 18.3.

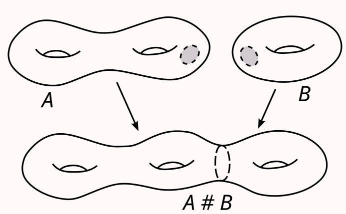
1. Let Σ, Σ' be surfaces. The ***connect sum***, $\Sigma \# \Sigma'$ is defined by

$$\left(\Sigma \backslash \mathrm{int}\left(D^2\right)\right) \sqcup \left(\Sigma' \backslash \mathrm{int}\left(D^2\right)\right) / \sim$$

where $\sim$ identifies boundary points.

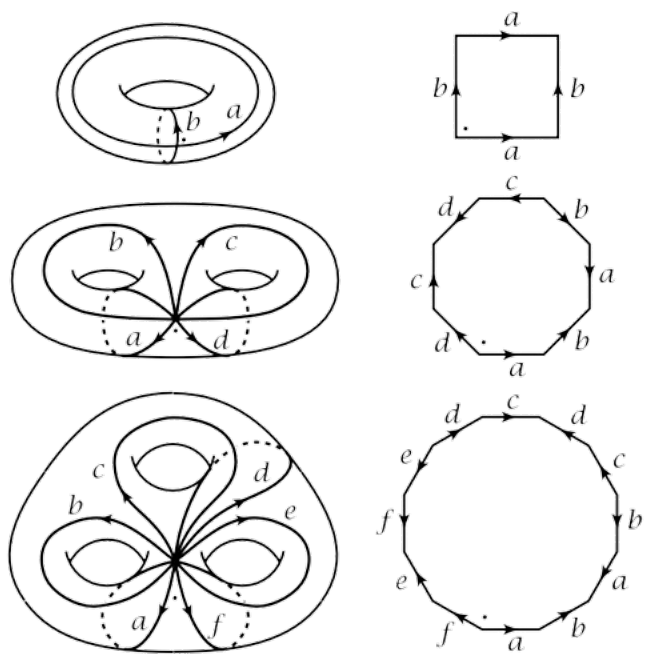
2. The ***surface of genus g*** is $\Sigma_g = \underbrace{T^2 \# \cdots \# T^2}_{n} \# S^2$ (The g-holed torus).

Example 18.6.



Theorem 18.6. $\pi_1(\Sigma_g) = \left\langle a_1, b_1, a_2, b_2, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \right\rangle$

Diagram.



19 Covering Spaces

Definition 19.1.

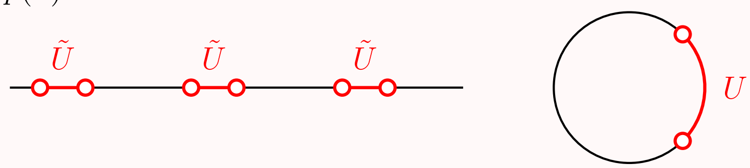
1. A ***covering space*** of X is a space $\tilde{X}$ with a map $p : \tilde{X} \to X$ such that every $x \in X$ admits a neighborhood U such that $f^{-1}(U) = \bigsqcup_\alpha \tilde{U}_\alpha$ (a disjoint union of open sets) where each $p\,|_{\tilde{U}_\alpha}$ is a homeomorphism. We say that U is ***evenly covered*** by the ***sheets*** $\tilde{U}_\alpha$.

2. A ***lift*** of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ with $f = p \circ \tilde{f}$.

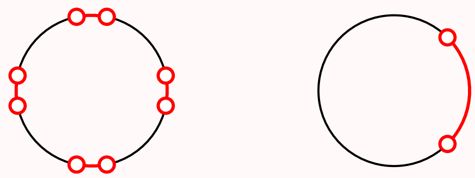
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

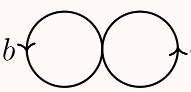
Example 19.1.

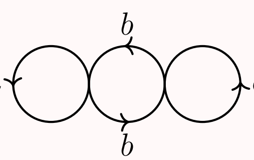
1. $p : \mathbb{R} \to S^1$, $p(\lambda) = e^{2\pi i \lambda}$.

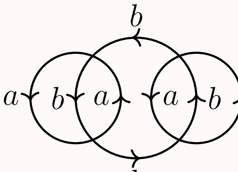


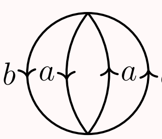
2. $p_n : S^1 \to S^1$, $p_n(z) = z^n$.

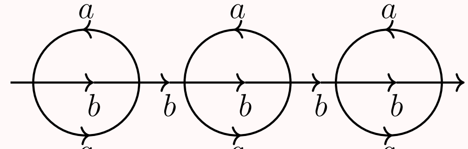


3. A few covering spaces of $S^1 \vee S^1$, as  :









Theorem 19.1. Let Y be a connected space and $f : Y \to X$. If two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ agree at some point $y \in Y$, then $\tilde{f}_1 = \tilde{f}_2$.

Proof. For any $z \in Y$, there is a neighborhood U of $f(z)$ that is evenly covered by sheets $\tilde{U}_\alpha$.

- $\left\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\right\}$ is open: Suppose $\tilde{f}_1(z) = \tilde{f}_2(z) \in \tilde{U}_\beta$, then by continuity there exists a neighborhood $z \in V$ with $\tilde{f}_1(V), \tilde{f}_2(V) \subseteq \tilde{U}_\beta$. Then

$$\tilde{f}_1|_V = p\,|_{\tilde{U}_\beta}^{-1} \circ p\,|_{\tilde{U}_\beta} \circ \tilde{f}_1\,|_V = p\,|_{\tilde{U}_\beta}^{-1} \circ f\,|_V = \tilde{f}_2|_V.$$

- $\left\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\right\}$ is closed: $\tilde{f}_1(z) \neq \tilde{f}_2(z) \implies \tilde{f}_1(z) \in \tilde{U}_{\beta_1}, \tilde{f}_2(z) \in \tilde{U}_{\beta_2}$ $(\beta_1 \neq \beta_2)$. By continuity there exists a neighborhood $z \in V$ with $\tilde{f}_i(V) \subseteq \tilde{U}_{\beta_i}$.

Since Y is connected, $Y = \left\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\right\}$. ■

Theorem 19.2. (Homotopy Lifting Property)
Given a homotopy $f_t : Y \to X$ and a lift $\tilde{f}_0 : Y \to \tilde{X}$ of f_0, there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of f_t that agrees with $\tilde{f}_0$.

Remark. The two facts used in Theorem 17.4 follow from the case $Y = \text{pt}$ and $Y = I$.

Proof. Use the $H(x,t) = f_t(x)$ notation.

Lemma. For any $y \in Y$, there exists open $y \in V$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each $H(V \times [t_i, t_{i+1}])$ is contained in some evenly covered U_i.

Proof. Fix y. For each $t \in I$ there exists a neighborhood U_t of $H(y,t)$ that is evenly covered, and there exists a basis $V_t \times W_t \subseteq Y \times I$ with $(y,t) \in V_t \times W_t \subseteq H^{-1}(U_t)$. Since the W_t cover I which is compact, we have a finite subcover $W_{s_0}, \cdots, W_{s_m}$ of I and hence we can take $V = V_{s_0} \cap \cdots \cap V_{s_m}$ and t_i the endpoints of all W_{s_k}.

We first prove the theorem by fixing y and restricting f_t on a neighborhood $V \subseteq Y$. By induction, suppose $\tilde{H}$ has been constructed over $V \times [0, t_i]$. Let $U \supseteq H(V \times [t_i, t_{i+1}])$ be evenly covered by sheets $\tilde{U}_\alpha$. Let $\tilde{H}(y, t_i) \in \tilde{U}_\beta$, then by the pasting lemma we can construct $\tilde{H}\mid_{V' \times [0, t_{i+1}]}$ by composing H with $p\mid_{\tilde{U}_\beta}^{-1}$, after restricting V to V' by intersecting the preimage of $\tilde{U}_\beta$. Relabelling V' as V, after a finite number of steps, we constructed $\tilde{f}_t$ on a neighborhood V of y. Note that such $\tilde{f}_t$ is unique at each $y \in V$ since $\{y\} \times I$ is connected.

To construct $\tilde{f}_t$ on the entire Y, we construct a unique $\tilde{f}$ on a neighborhood V_y at every $y \in Y$. By uniqueness on each $\{y\} \times I$, the $\tilde{f}$'s agree on the overlaps, so it is well-defined. By the same uniqueness, the entire $\tilde{f}$ is unique. ■

Theorem 19.3. Let $p_* : \pi_1\left(\tilde{X}, \tilde{x}_0\right) \to \pi_1(X, x_0)$ be induced by p.

1. p_* is injective.
2. $\mathrm{Im}(p_*) = \{[\alpha] \in \pi_1(X, x_0) : \tilde{\alpha}(0) = \tilde{\alpha}(1) = \tilde{x}_0\}$.

Proof. Suppose $p_*([\beta]) = 0$. Then $p \circ \beta \simeq$ constant rel $\{0,1\}$. This nullhomotopy has a unique lift in $\tilde{X}$, which gives a nullhomotopy for β, i.e. $[\beta] = 0$. This proves (1).

For (2), we have $p_*([\beta]) = [p \circ \beta] = \left[\tilde{\beta}\right]$. ■

Theorem 19.4. If $X, \tilde{X}$ are path-connected, then $|p^{-1}(x_0)| = [\pi_1(X, x_0) : \mathrm{Im}(p_*)]$ (There is a bijection between each preimage of x_0 and each coset of $\mathrm{Im}(p_*)$).

Definition 19.2. A space Y is ***locally*** **[insert property]** if for all $y \in Y$ and any neighborhood $y \in U$, there exists a neighborhood $y \in V \subseteq U$ that has the property.

Theorem 19.5. Say $f : (Y, y_0) \to (X, x_0)$ where Y is path-conn and locally path-conn.

$$\exists \tilde{f} : (Y, y_0) \to \left(\tilde{X}, \tilde{x}_0\right) \iff \mathrm{Im}\,(f_*) \subseteq \mathrm{Im}\,(p_*)$$

Proof. For $(\Rightarrow)$, $f = p \circ \tilde{f} \implies f_* = p_* \circ \tilde{f}_*$. For $(\Leftarrow)$, suppose $\mathrm{Im}\,(f_*) \subseteq \mathrm{Im}\,(p_*)$. Pick for each $y \in Y$ a path γ_y from y_0 to y, then define $\tilde{f}(y) = \widetilde{f \circ \gamma_y}(1)$ where $\widetilde{f \circ \gamma_y}(0) = \tilde{x}_0$.

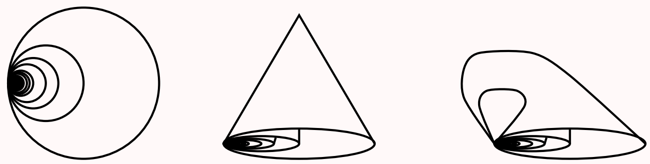
- $\tilde{f}$ is well-defined: Let γ_1, γ_2 be two paths from y_0 to y. Then $[\gamma_1 \cdot \overline{\gamma_2}] \in \pi_1(Y, y_0)$ and hence exists $[\alpha] \in \pi_1\left(\tilde{X}, \tilde{x}_0\right)$ with $p \circ \alpha \simeq f \circ \gamma_1 \cdot \overline{\gamma_2}$ rel $\{0, 1\}$. This homotopy H_t has a unique lift $\tilde{H}_t$ with $\tilde{H}_0 = \alpha$. Since $H_1 = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$, by uniqueness we have $\tilde{H}_1 = \widetilde{f \circ \gamma_1} \cdot \widetilde{f \circ \gamma_2}$, and thus $\widetilde{f \circ \gamma_1}(1) = \tilde{H}_1(0.5) = \widetilde{f \circ \gamma_2}(1)$.
- $\tilde{f}$ is a lift: $p \circ \tilde{f}(y) = p \circ \widetilde{f \circ \gamma_y}(1) = f \circ \gamma_y(1) = f(y)$.
- $\tilde{f}$ is continuous: Let W be a neighborhood of $\tilde{f}(y)$. Let $f(y) \in U$ be evenly covered by $\tilde{U}_\alpha$, and $\tilde{f}(y) \in \tilde{U}$. Since f is continuous, $\exists$ neighborhood $y \in V'$ with $f(V') \subseteq p\left(\tilde{U} \cap W\right)$. By local path-connectedness, let $y \in V \subseteq V'$ be a path-connected neighborhood. Then any path from y_0 to y can be extended to a path from y_0 to any $z \in V$. This eventually shows $\tilde{f}(V) \subseteq \tilde{U} \cap W$. ■

Definition 19.3.

1. A space X is ***semilocally simply connected*** if for all $x \in X$ there exists a neighborhood $x \in U$ where $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.
2. X is ***'nice'*** if it is path-conn, locally path-conn and semilocally simply conn.
3. $p : \tilde{X} \to X$ is a ***universal cover*** of X if X is path-conn and $\tilde{X}$ is simply conn.
4. Two covering spaces $p_i : \tilde{X}_i \to X$ are ***isomorphic*** if there exists a homeomorphism $\varphi : \tilde{X}_1 \to \tilde{X}_2$ with $p_2 \circ \varphi = p_1$.

Theorem 19.6. CW complexes are locally contractible and locally path-connected.

Example 19.2. The *Hawaiian Earring* H consisting of the union of all circles in $\mathbb{R}^2$ with center $(1/n, 0)$ and radius $1/n$ for each $n \in \mathbb{N}^*$ is not semilocally simply connected because any neighborhood of $(0,0)$ contains a full circle, which contains loops that cannot be nullhomotopic in H.



However, if we consider the *cone* obtained from H, defined by $CH = (H \times I)/(H \times \{0\})$, which is simply connected since it is contractible, then CH is semilocally simply connected. If we join the tip of the cone to the limit point at the base, then we form a space that is not simply connected (it is homotopy equivalent to S^1) but still semilocally simply connected.

Theorem 19.7.

1. If X is nice, then X has a universal cover.
2. If X is nice, for any subgroup $H \subseteq \pi_1(X, x_0)$ there exists a covering space $p_H : \tilde{X}_H \to X$ such that $\operatorname{Im}(p_{H*}) = H$.

Proof of (1). We use a Lemma and 5 steps.

Lemma. $\mathscr{B} = \{U \subseteq X : U \text{ open, path-conn, } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$ is a basis for the topology of a nice X.

Proof. $\mathscr{B}$ covers X since X is nice. Suppose $x \in U \cap V$ where $U, V \in \mathscr{B}$. Since X is locally path-conn, there exists path-connected $x \in W \subseteq U \cap V$, which means $\pi_1(W, x) \to \pi_1(U, x) \xrightarrow{\text{triv}} \pi_1(X, x) \implies W \in \mathscr{B}$. Hence $\mathscr{B}$ is a basis. To prove the second part, given any open $x \in W$, choose open $x \in V$ via semilocal simply connectedness, then pick a open and path-connected $U \subseteq V \cap W$ via local path-connectedness. Then $U \in \mathscr{B}$ with $U \subseteq W$. ■

1. Define $\tilde{X} = \{[\gamma] \mid \gamma : I \to X, \gamma(0) = x_0\}$ with $[\gamma] = [\delta] \iff \gamma \simeq \delta \text{ rel } \{0, 1\}$. Define the covering map $p : \tilde{X} \to X$ by $p([\gamma]) = \gamma(1)$.

2. Define the topology on $\tilde{X}$ as follows: Given $U \in \mathscr{B}$ and $[\gamma] \in \tilde{X}$ with $\gamma(1) \in U$, define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta : I \to U \text{ with } \eta(0) = \gamma(1)\}$$

Then $\{U_{[\gamma]}\}_{[\gamma]}$ is a basis (Exercise; $(*)$ Note that $[\delta] \in U_{[\gamma]} \implies U_{[\delta]} = U_{[\gamma]}$), and we generate the topology from it.

3. Claim: $p : U_{[\gamma]} \to U$ is a homeomorphism.
   - Surjective: U path-conn $\implies \forall x \in U, \exists$path η from $\gamma(1)$ to $x \implies p([\gamma \cdot \eta]) = x$.
   - Injective: $p([\gamma \cdot \eta]) = p([\gamma \cdot \eta']) \implies \eta(1) = \eta'(1)$. Since $\pi_1(U) \to \pi_1(X)$ is trivial, $\eta \simeq \eta' \implies [\gamma \cdot \eta] = [\gamma \cdot \eta']$.
   - Homeo: Note that $\{B \cap U\}_{B \in \mathscr{B}}$ and $\left\{B_{[\delta]} \cap U_{[\gamma]}\right\}_{B \in \mathscr{B}}$ are bases for U and $U_{[\gamma]}$ respectively, and (1) $p\left(B_{[\delta]} \cap U_{[\gamma]}\right) = B \cap U$; and (2) $p^{-1}(B \cap U) \cap U_{[\gamma]} = B_{[\delta]} \cap U_{[\gamma]}$ for any $[\delta] \in U_{[\gamma]}$ with $\delta(1) \in B$ since $B_{[\delta]} \subseteq U_{[\delta]} \overset{(*)}{=} U_{[\gamma]}$ and $p\,|_{V_{[\delta]}}$ is bijective.
4. p is a covering map: Given any $U \in \mathscr{B}$, $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$ which is a disjoint union of equivalence classes due to $(*)$.
5. $\tilde{X}$ is simply connected: Let $[x_0]$ = class of constant paths. Given any $[\gamma] \in \tilde{X}$, the path $\gamma_t(s) = \gamma(\min(s,t))$ form $\gamma(0)$ to $\gamma(t)$ brings $[x_0]$ to $[\gamma]$, and thus $\tilde{X}$ is path-connected. To show $\pi_1\left(\tilde{X}, [x_0]\right) = 0$, we show $\mathrm{Im}(p_*) = \{0\}$:
$$[\alpha] \in \mathrm{Im}(p_*) \implies [x_0] = \tilde{\alpha}(0) = \tilde{\alpha}(1) = [\alpha] \implies [\alpha] = 0. \qquad \blacksquare$$

Proof of (2). Consider the equivalence relation $\sim$ on the universal cover $\tilde{X}$ defined by
$$[\gamma] \sim [\delta] \iff \gamma(1) = \delta(1) \text{ and } [\gamma \cdot \overline{\delta}] \in H.$$
Then $X_H = \tilde{X}/\sim$ is a covering space of X. To show $\mathrm{Im}\,(p_{H*}) = H$: For any $[\alpha] \in \pi_1(X, x_0)$, we have a lift $\tilde{\alpha} = \alpha_t$ from $[\tilde{x_0}]$ to $[\alpha]$, so
$$[\alpha] \in H \iff [\tilde{x}_0] \sim [\alpha] \iff \tilde{\alpha}(0) = \tilde{\alpha}(1) \iff [\alpha] \in \mathrm{Im}(p_{H*})$$

Theorem 19.8. Let $\tilde{X}$ be path-connected. Then $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$ is conjugate to $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_1\right)\right)$ for any $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$.

Proof. Conjugate using any path from $\tilde{x}_0$ to $\tilde{x}_1$. $\blacksquare$

Theorem 19.9. (Classification Theorem of Covering Spaces)
If X is nice, there is a bijective ***Galois correspondence*** between
$$\left\{\begin{matrix}\text{conj. classes of} \\ \text{subgrps of } \pi_1(X, x_0)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{isom. classes of path-} \\ \text{conn. covers } \tilde{X} \to X\end{matrix}\right\}$$

Proof. We prove that $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic if and only if $\mathrm{Im}(p_{1*})$ and $\mathrm{Im}(p_{2*})$ are conjugate in $\pi_1(X)$.

- Assume $p_1 \cong p_2$. Then $\text{Im}(p_{1*}) = \text{Im}(p_{2*} \circ \varphi_*) = p_{2*}\left(\pi_1\left(\tilde{X}_2, \varphi\left(\tilde{x}_1\right)\right)\right)$ which is conjugate to $p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{x}_2\right)\right)$ since $p_2 \circ \varphi\left(\tilde{x}_1\right) = p_1\left(\tilde{x}_1\right) = x_0 = p_2\left(\tilde{x}_2\right)$.

- Suppose $\text{Im}(p_{1*}) = [\alpha]^{-1}\text{Im}(p_{2*})[\alpha]$. Let $\tilde{\alpha} : I \to \tilde{X}_2$ be a lift of α based at $\tilde{x}_2$. Then
$$p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{\alpha}(1)\right)\right) = p_{1*}\left(\pi_1\left(\tilde{X}_1, \tilde{x}_1\right)\right)$$
so there exists lifts $\tilde{p}_1 : \left(\tilde{X}_1, \tilde{x}_1\right) \to \left(\tilde{X}_2, \tilde{\alpha}(1)\right), \tilde{p}_1 : \left(\tilde{X}_2, \tilde{\alpha}(1)\right) \to \left(\tilde{X}_1, \tilde{x}_1\right)$ with
$$\begin{tikzcd}
 & \tilde{X}_2 \arrow[d, "p_2"] \arrow[dl, bend left, "\tilde{p}_2"] \\
\tilde{X}_1 \arrow[ur, bend left, "\tilde{p}_1"] \arrow[r, "p_1"'] & X
\end{tikzcd}$$
Since $\tilde{p}_2 \circ \tilde{p}_1$ and $\mathbf{1}$ agree at $\tilde{x}_1$ and ar both lifts of $p_1 : \tilde{X}_1 \to X$ to $\tilde{X}_1$, we have $\tilde{p}_2 \circ \tilde{p}_1 = \mathbf{1}$ and similarly $\tilde{p}_1 \circ \tilde{p}_2 = \mathbf{1}$. Thus $p_1 \cong p_2$. ∎

20 Regular Coverings

Definition 20.1.

1. A ***deck transformation*** is a self-isomorphism $\tilde{X} \to \tilde{X}$ of a covering space $p : \tilde{X} \to X$. The group of deck transformations is denoted $\text{Aut}(\tilde{X})$.
2. A covering space is ***regular*** if for each $x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, there exists $\varphi \in \text{Aut}(\tilde{X})$ with $\varphi(\tilde{x}) = \tilde{x}'$.

Example 20.1. $\text{Aut}(\mathbb{R}) = \mathbb{Z}$.

Theorem 20.1. $\varphi \in \text{Aut}(\tilde{X})$ is completely determined by $\varphi(\tilde{x}_0)$ when $\tilde{X}$ is path-connected and locally path-connected.

Proof. φ is a lift to $\tilde{X}$, which is uniquely determined by where it sends some point. ∎

Theorem 20.2. Suppose X is nice. Then $p : \tilde{X} \to X$ is regular if and only if $\text{Im}(p_*)$ is normal. When this is true, $\text{Aut}(\tilde{X}) = \pi_1(X, x_0)/\text{Im}(p_*)$.

Proof. p is regular $\Leftrightarrow$ for any $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there exists a lift φ where

$$\begin{tikzcd}
& \left(\tilde{X}, \tilde{x}'\right) \arrow[d, "p"] \\
\left(\tilde{X}, \tilde{x}\right) \arrow[ur, "\varphi"] \arrow[r, "p"'] & X
\end{tikzcd}$$

which is equivalent to $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}\right)\right) = p_*\left(\pi_1\left(\tilde{X}, \tilde{x}'\right)\right)\ \forall \tilde{x}, \tilde{x}'$, i.e. $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}\right)\right)$ is normal. To prove the second part, consider the map $\pi_1(X, x_0) \to \operatorname{Aut}(\tilde{X})$ is $p_*(\pi_1(X, x_0))$ given by $[\alpha] \mapsto \varphi_\alpha$ where $\varphi_\alpha(\tilde{\alpha}(0) = \tilde{x}_0) = \tilde{\alpha}(1)$.

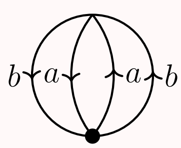
- It is a homomorphism: Since a lift of $\alpha \cdot \beta$ is $\tilde{\alpha} \cdot \varphi_\alpha(\tilde{\beta})$, so

$$[\alpha][\beta] \mapsto \varphi_{\alpha\cdot\beta}(\tilde{x}_0) = \varphi_\beta \circ \varphi_\alpha(\tilde{x}_0) \implies \varphi_{\alpha\cdot\beta} = \varphi_\beta\varphi_\alpha.$$

- It is injective: $[\alpha] \mapsto 0 \ \Leftrightarrow\ [\alpha] \in p_*\left(\pi_1\left(\tilde{X}, \tilde{x}\right)\right)$.

- It is surjective: Fix a path γ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1 \in p^{-1}(x_0)$. Then $[p \circ \gamma] \in \pi_1(X, x_0)$ has lift γ. ■

Example 20.2. A covering space of $S^1 \vee S^1$:



$\operatorname{Im}(p_*) = \langle a^2, ab^{-1}, ab\rangle \subseteq \langle a, b\rangle = \pi_1(S^1 \vee S^1, x_0)$ is normal, hence

$$\operatorname{Aut}(\tilde{X}) = \left\langle a, b \mid a^2, ab^{-1}, ab\right\rangle = \mathbb{Z}_2.$$