Tristan Chaang June 4, 2024

18.901 Notes

1 Topological Spaces

Definition 1.1. A ***topology*** on a set X is a collection $\mathscr{T}$ of subsets of X such that:

- $\varnothing, X \in \mathscr{T}$.
- The union of (possibily uncountably many) sets in $\mathscr{T}$ is in $\mathscr{T}$.
- The intersection of finitely many sets in $\mathscr{T}$ is in $\mathscr{T}$.

A set X for which a topology has been specified is called a ***topological space***. Elements of $\mathscr{T}$ are called ***open sets*** of X. The complements of open sets of X are called ***closed sets*** of X. Sets that are both open and closed in X are called ***clopen*** in X. For any $x \in X$, a ***neighborhood*** of x is an open set of X containing x.

Example 1.1.

1. The ***discrete topology*** on X is $\mathscr{P}(X)$, i.e. all subsets of X are open in X.
2. The ***indiscrete (or trivial) topology*** on X is $\{\varnothing, X\}$.
3. The ***finite complement topology*** on X is $\{\varnothing\} \cup \{A \subseteq X : X \backslash A \text{ is finite}\}$. The ***countable complement topology*** on X is $\{\varnothing\} \cup \{A \subseteq X : X \backslash A \text{ is countable}\}$.

Exercise. Show that the above are valid topologies.

Definition 1.2. Suppose $\mathscr{T}$ and $\mathscr{T}'$ are topologies on X.

- We say $\mathscr{T}$ is ***finer*** than $\mathscr{T}'$ if $\mathscr{T} \supseteq \mathscr{T}'$. (Draw a diagram to visualize)
- We say $\mathscr{T}$ is ***coarser*** than $\mathscr{T}'$ if $\mathscr{T} \subseteq \mathscr{T}'$.
- We say $\mathscr{T}$ and $\mathscr{T}'$ are ***incomparable*** if neither $\mathscr{T} \supseteq \mathscr{T}'$ nor $\mathscr{T} \subseteq \mathscr{T}'$.

Definition 1.3. A (topological) ***basis*** on X is a collection $\mathscr{B}$ of subsets of X where

- $\displaystyle\bigcup_{B \in \mathscr{B}} B = X$. ($\mathscr{B}$ ***covers*** X, i.e. every $x \in X$ belongs to some $B \in \mathscr{B}$)
- Any x in two basis elements B_1, B_2 belongs to some basis element $B_3 \subseteq B_1 \cap B_2$.

The ***topology generated by*** $\mathscr{B}$ consists of all possible *unions* of elements in $\mathscr{B}$.

Theorem 1.1. The topology $\mathscr{T}$ generated by a basis $\mathscr{B}$ is in fact a topology on X.

Proof. Firstly, $\varnothing \in \mathscr{T}$ since it is the empty union. Also, $X \in \mathscr{T}$ because $\mathscr{B}$ covers X. We then verify the union and intersection properties:

- Given any $\{U_\alpha\}_\alpha \subseteq \mathscr{T}$, each U_α is a union of elements in $\mathscr{B}$, so $\bigcup_\alpha U_\alpha$ is also a union of elements in $\mathscr{B}$, and hence is in $\mathscr{T}$.

- Given any $U_1, U_2 \in \mathscr{B}$, we show that $U_1 \cap U_2 \in \mathscr{B}$: Let $x \in U_1 \cap U_2$, then x belongs to some basis element $B_1 \subseteq U_1$ and some basis element $B_2 \subseteq U_2$. Hence x belongs to some basis element $B(x) \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Now, $U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B(x)$ and thus $U_1 \cap U_2 \in \mathscr{T}$. The general finite intersection case follows from induction. ∎

Corollary 1.1. Suppose we have a topology with some basis $\mathscr{B}$. Given any open set U and an element $x \in U$, there exists some basis element $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

We can often reduce a problem about open sets to one about just the basis elements:

Theorem 1.2. Let $\mathscr{T}$ and $\mathscr{T}'$ be topologies on X with bases $\mathscr{B}$ and $\mathscr{B}'$ respectively. Then $\mathscr{T}'$ is finer than $\mathscr{T}$ if and only if for every possible $x \in B \in \mathscr{B}$ there exists $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$. *(We can always get a smaller one in $\mathscr{B}'$)*

Definition 1.4.

- The ***standard topology*** on $\mathbb{R}$ is the topology generated by the collection of open intervals $(a, b) = \{x : a < x < b\}$. This is the default topology we assume.

- The ***lower-limit topology*** on $\mathbb{R}$ is the topology generated by the collection of half-open intervals $[a, b) = \{x : a \leqslant x < b\}$. We write $\mathbb{R} = \mathbb{R}_\ell$ in this case.

- The ***K-topology*** on $\mathbb{R}$ is the topology generated by all open intervals (a, b) and sets of the form $(a, b)\setminus\{1/n : n \in \mathbb{N}^*\}$. We write $\mathbb{R} = \mathbb{R}_K$ in this case.

By Theorem 1.2, we can show that $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than $\mathbb{R}$, but are incomparable.

Generalizing from Definition 1.3, we can generate a topology from any collection of subsets that covers X (i.e. without necessarily having the second condition).

Definition 1.5. Let $\mathscr{S}$ be a collection of subsets of X that covers X. The ***topology generated by*** $\mathscr{S}$ is the collection of all possible unions of all possible *finite intersections* of elements in $\mathscr{S}$. We say that $\mathscr{S}$ is a ***subbasis*** of this topology.

2 Order, Product, and Subspace Topology

Definition 2.1. Let X have a simple order relation $<$. The ***intervals*** of X are:

$$(a,b) = \{x \in X : a < x < b\} \qquad [a,b) = \{x \in X : a \leqslant x < b\}$$
$$(a,b] = \{x \in X : a < x \leqslant b\} \qquad [a,b] = \{x \in X : a \leqslant x \leqslant b\}$$

and the ***rays*** of X are:

$$(a,+\infty) = \{x \in X : x > a\} \qquad (-\infty,a) = \{x \in X : x < a\}$$
$$[a,+\infty) = \{x \in X : x \geqslant a\} \qquad (-\infty,a] = \{x \in X : x \leqslant a\}$$

The ***order topology*** on an ordered set X is the topology generated by the basis consisting of open intervals (a,b) and open rays $(a,+\infty),(-\infty,a)$.

Example 2.1.

1. The order topology on $\mathbb{R}$ is the standard topology.
2. The order topology on $\mathbb{Z}$ is the discrete topology.
3. The order topology on $\{1,2\} \times \mathbb{N}^*$ is *not* the discrete topology. Which one-point set is not open?

Definition 2.2. Let $X = \prod_{\alpha \in J} X_\alpha$ be a cartesian product of sets $\{X_\alpha\}_{\alpha \in J}$. The ***projection*** of X onto index β is the function $\pi_\beta\left((x_\alpha)_{\alpha \in J}\right) = x_\beta$.

Definition 2.3. Let X, Y be topological spaces. The ***box/product topology*** on $X \times Y$ is generated by the basis $\{U \times V : U, V \text{ are open in } X, Y \text{ respectively}\}$

This definition can generalize to finite cartesian products. However, for infinite cartesian products, there is a distinction between box and product topologies:

Definition 2.4. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces.

- The ***box topology*** on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis $\{\prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \text{ for all } \alpha\}$.
- The ***product topology*** on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis $\{\prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \text{ for all } \alpha\text{, but only finitely many } \alpha \text{ satisfy } U_\alpha \neq X_\alpha\}$.

For reasons to be seen, the product topology is the most preferred one.

Theorem 2.1. $\bigcup_{\beta \in J} \left\{ \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta \right\}$ is a subbasis for the product topology.

Definition 2.5. Let Y be a subset of a topological space X. The ***subspace topology*** on Y (with respect to X) consists of sets of the form $Y \cap U$ where U is open in X, i.e.

$$V \text{ open in } Y \iff V = Y \cap U \text{ where } U \text{ open in } X$$

Example 2.2. $[0,1]^2$ as a subspace of the dictionary ordered $\mathbb{R}^2$ and as its own dictionary order topology are different! The set $\{0.5\} \times (0.5, 1]$ is open in the former topology but not in the latter. The latter topology is called the ***ordered square*** I_o^2.

3 Limit Points

Definition 3.1. Let A be a subset of a topological space X.

- The ***interior*** of A, denoted $\operatorname{Int}(A)$ or $\mathring{A}$, is the largest open set contained in A.
- The ***closure*** of A, denoted $\overline{A}$, is the smallest closed set that contains A.

Definition 3.2. Let A be a subset of a topological space X. We say that $x \in X$ is a ***limit point*** of A if every neighborhood of x intersects $A \backslash \{x\}$.

Theorem 3.1. $\overline{A} = A \cup \{\text{limit points of } A\}$.

Proof.

- $(\subseteq)$ We prove that if $a \in \overline{A} \backslash A$, then a is a limit point of A. Let U be any neighborhood of a. If $U \cap A = \varnothing$, then $X \backslash U$ is a closed set containing A, so $\overline{A} \cap (X \backslash U)$ is a smaller (no a) closed set than $\overline{A}$ containing A, a contradiction. Hence $U \cap A \neq \varnothing$.
- $(\supseteq)$ Since $A \subseteq \overline{A}$, it suffices to show that all limit points of A are in $\overline{A}$. Let a be a limit point of A. If $a \notin \overline{A}$, then a lies in the open set $X \backslash \overline{A}$ and hence has a neighborhood $U \subseteq X \backslash \overline{A}$. This means $U \cap \overline{A} = \varnothing \Rightarrow U \cap A = \varnothing$, contradiction. ■

Corollary 3.1. A set is closed if and only if it contains all its limit points.

Definition 3.3. A space X is ***Hausdorff*** if for each pair of distinct points $x_1, x_2 \in X$, there exist neighborhoods U_1, U_2 respectively that are disjoint.

Theorem 3.2.

- Every neighborhood of a limit point of a subset A of a Hausdorff space intersects A at *infinitely many* points.
- A sequence of points in a Hausdorff space converges to at most one point of X.

4 Continuous Functions

Definition 4.1. Let X, Y be topological spaces. A function $f : X \to Y$ is

- ***continuous at $x \in X$*** if $f^{-1}(V)$ is open in X for all neighborhoods V of $f(x)$.
- ***continuous*** if $f^{-1}(V)$ is open in X for all V open in Y.
- a ***homeomorphism*** if f is bijective, and f and f^{-1} are continuous.

Theorem 4.1.

- In both box/product topologies, if all X_α are Hausdorff, $\prod_{\alpha \in J} X_\alpha$ is Hausdorff.
- In both box/product topologies, $\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}$.
- In just the product topology, $f = (f_\alpha)_{\alpha \in J} : A \to \prod_{\alpha \in J} X_\alpha$ is continuous if and only if each f_α is continuous.

5 Metric Topology

Definition 5.1. A ***metric*** on X is a function $d : X^2 \to \mathbb{R}$ such that

- $d(x, y) \geqslant 0$ for all $x, y \in X$ and equality holds if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) + d(y, z) \geqslant d(x, z)$ for all $x, y, z \in X$.

$d(x, y)$ is called the ***distance*** between x and y. The set $B_d(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ is called the (open) ***ε-ball centered at x***.

Definition 5.2. The ***metric topology*** induced by a metric d on X is the topology generated by the collection of all open balls. A topological space X is said to be ***metrizable*** if there exists some metric that induces the topology of X.

Theorem 5.1. Given a metric d on X, the metric $\overline{d}(x,y) = \min\{d(x,y), 1\}$ induces the same topology as d does.

Example 5.1.

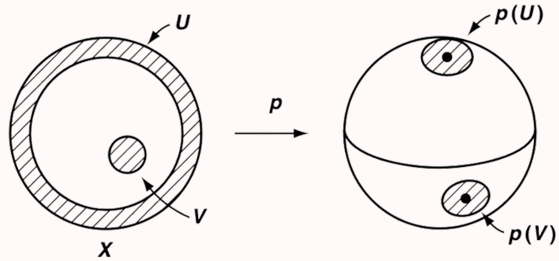
1. The ***euclidean metric*** d on $\mathbb{R}^n$ is $d(\mathbf{x},\mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.
2. The ***standard uniform metric*** $\overline{d}$ on $\mathbb{R}$ is $\overline{d}(x,y) = \min\{|x-y|, 1\}$.
3. The ***uniform metric*** $\overline{\rho}$ on $\mathbb{R}^J$ (for any index set J) is $\overline{\rho}(\mathbf{x},\mathbf{y}) = \sup\limits_{\alpha\in J} \overline{d}(x_\alpha, y_\alpha)$.
4. The metric $D(\mathbf{x},\mathbf{y}) = \sup_i \left\{\frac{\overline{d}(x_i,y_i)}{i}\right\}$ induces the product topology on $\mathbb{R}^\omega$.

6 Quotient Topology

Definition 6.1.

- Let X, Y be topological spaces. A surjective map $p : X \to Y$ is a ***quotient map*** if 'U is open in Y if and only if $p^{-1}(U)$ is open in X'. In other words, p is a quotient map if and only if p is continuous and maps ***saturated*** (some union of $p^{-1}(\{y\})$) open sets of X to open sets of Y.
- Let X be a space, A be a set, and $p : X \to A$ be surjective. The ***quotient topology*** induced by p is the unique topology on A where p is a quotient map.
- Let X be a space and X^* be a partition of X. X^* is a ***quotient space*** of X, under the quotient topology induced by the natural mapping $p : X \to X^*$.

Example 6.1. Consider the unit 2-disk $X = D_2 = \{x \times y : x^2 + y^2 \leqslant 1\}$. We let X^* be the partition of X consisting (i) the boundary $S^1 = \{x \times y : x^2 + y^2 = 1\}$, and (ii) the singleton sets $\{x \times y\}$ for all interior points $x^2 + y^2 < 1$. Then we can show that the quotient space X^* is homeomorphic with the subspace of $\mathbb{R}^3$ called the unit 2-sphere $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$. (See diagram below)



Using similar ideas, we can construct a *torus* from the rectangle $[0, 1] \times [0, 1]$.

Theorem 6.1. Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each one-point preimage $p^{-1}(\{y\})$. Therefore g induces a map $f : Y \to Z$ where $f \circ p = g$. Then

- f is continuous if and only if g is continuous.
- f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

7 Connectedness

Definition 7.1. A topological space X is said to be ***connected*** if there is no nontrivial clopen set A. (Equivalently, there is no $A \notin \{\varnothing, X\}$ where A and $X \backslash A$ are both open.)

Example 7.1. The subspace $(0, 1) \cup (2, 3)$ of $\mathbb{R}$ is not connected.

Theorem 7.1.

- Let C be clopen in X. Any connected subspace of X lies within either C or $X \backslash C$.
- The union of connected subspaces that have a common point is connected.
- If A is a connected subspace of X and $A \subseteq B \subseteq \overline{A}$ then B is connected.

- A continuous function maps a connected space to a connected image.
- A cartesian product of connected spaces (in the product topology) is connected.

Definition 7.2. A simply ordered set L with more than one element is a ***linear continuum*** if (i) L has the least upper bound property, and (ii) for any $x < y$ there exists $x < z < y$.

Theorem 7.2. In an order topology, linear continuums, and intervals and rays in a linear continuum are connected. (Hence intervals in $\mathbb{R}$ are connected.)

Theorem 7.3. (Intermediate Value Theorem)
Let $f : X \to Y$ be continuous, X is connected, and Y is ordered. If $a, b \in X$ and r is a point lying between $f(a), f(b)$, then there exists $c \in X$ such that $f(c) = r$.

Path-Connectedness

Definition 7.3. A space is ***path-connected*** if every pair $x, y \in X$ can be joined by a *path* in X: a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Example 7.2. All path-connected spaces are connected. The converse is not true, e.g. the ordered square and the topologist's sine curve

Components and Path Components

Definition 7.4. The equivalence relation $x \sim y$ where there is a (path-)connected subspace containing both x, y partitions the space into (path-)connected ***components***.

Local Connectedness

Definition 7.5. A space is ***locally (path-)connected at x*** if for every neighborhood U of x, there is a (path-)connected neighborhood V of x contained in U.

8 Compactness

Definition 8.1. An ***open covering*** of X is a collection of open sets that cover X. A space X is ***compact*** if every open covering of X admits a finite subcovering.

Theorem 8.1.

- Every closed subspace of a compact space is compact.
- If Y is a compact subspace of a Hausdorff space and $x_0 \notin Y$, there exists disjoint open sets U, V containing $\{x_0\}$ and Y respectively.
- Every compact subspace of a Hausdorff space is closed.
- A continuous function maps compact spaces to a compact image.

Theorem 8.2. (Tychonoff Theorem)
A cartesian product of compact spaces is compact.

Compactness via Closed Sets

Definition 8.2. A collection $\mathscr{C}$ of subsets of X has the ***finite intersection property*** if every finite subcollection has nonempty intersection.

Theorem 8.3. X is compact if and only if for every collection $\mathscr{C}$ of closed sets having the finite intersection property, the intersection $\bigcap_{C\in\mathscr{C}} C$ is nonempty.

Compactness on $\mathbb{R}^n$

Theorem 8.4. Let X be a simply ordered set having the least upper bound property. Every closed interval in X is compact.

Theorem 8.5. (Heine-Borel)
A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (in either the euclidean metric or square metric).

Functions on Compact Spaces

Theorem 8.6. (Extreme Value Theorem)
Let $f : X \to Y$ be continuous, X is compact, and Y is ordered. There exists $c, d \in X$ such that $f(c) \leqslant f(x) \leqslant f(d)$ for all $x \in X$.

Theorem 8.7. (Lebesgue Number Lemma)
Let $\mathscr{A}$ be an open covering of a metric space (X, d). If X is compact, there exists $\delta > 0$ such that any subset of X with diameter (supremum of pairwise distances) less than δ admits an element of $\mathscr{A}$ containing it. Here δ is a ***Lebesgue number*** for $\mathscr{A}$.

Definition 8.3. A function f from metric space (X, d_X) to metric space (Y, d_Y) is ***uniformly continuous*** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \;\Rightarrow\; d_Y(f(x), f(y)) < \varepsilon.$$

Theorem 8.8. (Heine-Cantor)
A continuous function on a compact space is uniformly continuous.

Theorem 8.9. A nonempty compact Hausdorff space with no isolated points (where $\{x\}$ is open) is uncountable. (Hence $[0, 1]$ is uncountable)

Proof. Let $\{x_n\}_{n \in \mathbb{N}*}$ be an enumeration of X. Let $V_0 = X$. Given the nonempty open set V_{n-1}, choose V_n to be a nonempty open set such that $V_n \subseteq V_{n-1}$ and $x_n \notin \overline{V_n}$: We can choose this by using the Hausdorff condition on x_n and some other point in V_{n-1}, and then take the intersection of V_{n-1} and the neighborhood around this other point. The nested sequence

$$\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$$

satisfies the finite intersection property and hence by compactness there exists $x \in \bigcap_n \overline{V_n}$. Such an x cannot be any x_n, because $x_n \notin \overline{V_n}$ but $x \in \overline{V_n}$. ■

Limit Point Compactness and Sequential Compactness

Definition 8.4.

- X is ***limit point compact*** if every infinite subset of X has a limit point.
- X is ***sequentially compact*** if every sequence of points in X admits a convergent subsequence.

Theorem 8.10. When X is metrizable, all three forms of compactness are equivalent.

Proof. (Compact $\Rightarrow$ Limit Point Compact) Let X be compact and say $A \subseteq X$ has no limit points; we prove that A is finite. Since A vacuously contains all limit points, A is closed. Plus, since no element of A is a limit point, every $a \in A$ admits a neighborhood U_a that only intersects A at a alone. $\{X - A\} \cup \{U_a\}_{a \in A}$ is now an open covering of X, and hence admits a finite subcovering. This shows that A is finite. (True even for non-metric spaces)

(Limit Point Compact $\Rightarrow$ Sequentially Compact) Let X be limit point compact and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X. If $\{x_n : n \in \mathbb{N}\}$ is finite, then some element repeats infinitely often, i.e. a trivial subsequence. If not, then $\{x_n : n \in \mathbb{N}\}$ has a limit point x. Repeatedly taking arbitrarily small ε-balls of x gives a subsequence converging to it.

(Sequentially Compact $\Rightarrow$ Compact) Let $\mathscr{A}$ be an open covering. We prove two properties:

- *X has a Lebesgue number*:
  Assume not, so for each $n \in \mathbb{N}^*$ there exists some C_n with diameter $< 1/n$ that is not contained in any element of $\mathscr{A}$. Choose an $x_n \in C_n$ for each n, and say some subsequence x_{n_i} converges to x. $\mathscr{A}$ contains some A that contains x, and we can pick some $B(x, 2/n_N) \subseteq A$. Pick $M \geqslant N$ such that $d(x_{n_M}, x) < 1/n_N$. Now since C_{n_M} has diameter $< 1/n_N$, any $c \in C_{n_M}$ satisfies $d(c,x) \leqslant d(c, x_{n_M}) + d(x_{n_M}, x) < 1/n_N + 1/n_N = 2/n_N$ and hence $C_{n_M} \subseteq A$, a contradiction.
- *X is **totally bounded*** (can be finitely covered by ε-balls for any $\varepsilon > 0$):
  Let $\varepsilon > 0$ and $x_0 \in X$. Assume X cannot be finitely covered by ε-balls. Then we can always pick x_{n+1} not in $B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$, giving a sequence where pairs of points differ by $\geqslant \varepsilon$, a contradiction.

Now let δ be a Lebesgue number for $\mathscr{A}$. Let $U_1, \cdots, U_k$ be a finite covering of $(\delta/3)$-balls. Then the diameter of each U_i is $< \delta$, so each U_i is contained within some $A_i \in \mathscr{A}$. Then $\{A_1, \cdots, A_k\}$ is a finite subcovering. Hence X is compact. ■

9 Countability Axioms

Definition 9.1.

- ***1st Countability Axiom:*** There is a countable basis at every $x \in X$, i.e. for each x there exists countably many neighborhoods $\{U_n\}_{n\in\mathbb{N}}$ so that any neighborhood of x contains some U_n.
- ***2nd Countability Axiom:*** There is a countable basis for X.
- ***Lindelöf***: Every open covering admits a *countable* subcovering.
- ***Separable***: There is a countable dense subset. (***Dense*** means the closure is X)

Theorem 9.1. 2nd Countability implies the other three.

10 Separation Axioms

Definition 10.1. A topological space X is said to satisfy the ***T_1 Axiom*** if all singleton sets are closed. Assuming X is T_1, we say that it could satisfy

- ***T_2 (Hausdorff):*** For each pair of distinct points $x_1, x_2 \in X$, there exist neighborhoods U_1, U_2 respectively that are disjoint.
- ***T_3 (Regular):*** For any $x \in X$ and closed set A of X not containing x, there exist open sets U_1, U_2 containing $\{x\}, A$ respectively and are disjoint.
- ***T_4 (Normal):*** For each pair of disjoint closed sets A, B of X, there exist open sets U_1, U_2 containing A, B respectively and are disjoint.

Theorem 10.1. Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff (recall T_1 is assumed)

Theorem 10.2.

- X is regular if and only if for any $x \in X$ and neighborhood U, there exists a neighborhood V such that $\overline{V} \subseteq U$.
- X is normal if and only if for any closed A and open $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.

Theorem 10.3. T_2 and T_3 are preserved under subspaces and products.

Example 10.1. $\mathbb{R}_\ell$ is normal but the ***Sorgengrey plane*** $\mathbb{R}_\ell^2$ is not normal.

Theorem 10.4.
Compact Hausdorff spaces, metrizable spaces, and well-ordered spaces are normal.

Urysohn Lemma

Theorem 10.5. (Urysohn Lemma)
Let X be normal and A, B be disjoint closed sets of X. There exists a continuous map

$$f : X \to [0, 1]$$

such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. Define open sets U_p for each $p \in \mathbb{Q} \cap [0, 1]$ as follows: Enumerate $\mathbb{Q} \cap [0, 1]$ such that 1 and 0 are the first two elements. Define $U_1 = X - B$ and by normality pick U_0 such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction, say we defined U_p for a finite number of p's and let r be the next rational in the enumeration. We must have $p < r < q$ where U_p, U_q are already defined. By normality we pick U_r such that $\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$.

Additionally, we let $U_p = \varnothing$ for all rationals $p < 0$ and $U_p = X$ for all rationals $p > 1$. Hence,

$$p < q \;\Rightarrow\; \overline{U_p} \subseteq U_q.$$

We then define $f(x) = \inf \{p : x \in U_p\}$. It is easy to see $f(A) = \{0\}$ and $f(B) = \{1\}$. We show that f is continuous.

Lemma 1. $x \in \overline{U_r} \;\Rightarrow\; f(x) \leqslant r$
Proof. If $x \in \overline{U_r}$, then $x \in U_s$ for every $s > r$. Hence $f(x) \leqslant r$. □

Lemma 2. $x \notin \overline{U_r} \;\Rightarrow\; f(x) \geqslant r$.
Proof. If $x \notin \overline{U_r}$, then $x \notin U_s$ for any $s < r$. Hence $f(x) \geqslant r$. □

Given a ball $I = (f(x) - \delta, f(x) + \delta)$, we wish to find a neighborhood U of x such that $f(U) \subseteq I$. First we choose rational numbers $p, q \in I$ such that $p < f(x) < q$. Then the open set $U_q \backslash \overline{U_p}$ is the desired neighborhood using the lemmas above. ■

Urysohn Metrization Theorem

Theorem 10.6. Every Lindelöf regular space is normal.

Proof. Let X be Lindelöf and regular, and let A and B be closed in X. Since B is closed, each point $a \in A$ has a neighborhood U'_a not intersecting B. Using regularity (Theorem 10.2), pick a neighborhood U_a whose closure lies in U'_a. Therefore $\{U_a\}_{a\in A}$ is an open covering of A whose closures do not intersect B. Since X is Lindelöf, there is a countable subcovering $\{U_n\}_{n\in\mathbb{N}}$. Similarly, choose a countable open covering $\{V_n\}_{n\in\mathbb{N}}$ of B where each $\overline{V_n}$ is disjoint from A.

The sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B respectively, but they may not be disjoint. We define for each $n \in \mathbb{N}$,

$$U_n^* = U_n \backslash \bigcup_{i=1}^{n} \overline{V_i} \qquad \text{and} \qquad V_n^* = V_n \backslash \bigcup_{i=1}^{n} \overline{U_i}.$$

(Exercise) Then $U^* = \bigcup U_n^*$ and $V^* = \bigcup V_n^*$ are disjoint open sets containing A and B respectively. ■

Theorem 10.7. (Urysohn Metrization Theorem)
Every 2nd countable regular space is metrizable.

Proof. Let X be regular with a countable basis $\mathscr{B}$. From Example 5.1, it suffices to imbed X into $\mathbb{R}^\omega$. We first prove a lemma:

Lemma. There exists a collection $\{f_n : X \to [0,1]\}_{n\in\mathbb{N}}$ of continuous functions such that given any $x \in X$ and any neighborhood U, there exists some f_n that is positive at x but vanishes outside U.

Proof. For each $B, C \in \mathscr{B}$ with $\overline{B} \subseteq C$, apply the Urysohn Lemma to construct a continuous function $g_{B,C} : X \to [0,1]$ such that $g_{B,C}(\overline{B}) = \{1\}$ and $g_{B,C}(X\backslash C) = \{0\}$. $\left\{g_{B,C} : \overline{B} \subseteq C\right\}$ is the desired collection. It is countable because $\mathscr{B} \times \mathscr{B}$ is countable, and given any x with neighborhood U, we can choose by regularity and definition of basis the sequence of open sets $x \in B \subseteq \overline{B} \subseteq C \subseteq U$, and then use $g_{B,C}$. □

Using $\{f_n\}_{n\in\mathbb{N}}$ from the Lemma, define $F : X \to \mathbb{R}^\omega$ such that

$$F(x) = (f_0(x), f_1(x), f_2(x), \cdots)$$

Firstly, F is continuous because each component is continuous and $\mathbb{R}^\omega$ has the *product* topology. Secondly, F is injective because given $x \neq y$, there exists some $f_n(x) > 0 = f_n(y)$ (X is Hausdorff!). It remains to show that for each open set U in X, $F(U)$ is open in $F(X)$.

Let $x \in U$ and $f(x) = z$. Choose a f_N that is positive at x but vanishes outside U. Let

$$W = F(X) \cap \pi_N^{-1}((0, +\infty))$$

be open in $F(X)$. We claim that $z \in W \subseteq F(U)$. Firstly, we have $z = F(x) \in W$ because $f_N(x) > 0$. Secondly, given any $F(y) \in W$, we must have $f_N(y) > 0$. Since f_N vanishes outside U, y must be in U, so $F(y) \in F(U)$. ■

Tietze Extension Theorem

Theorem 10.8. (Tietze Extension Theorem)
Let X be normal and A be closed in X. Any continuous map from A to $[-1, 1]$ can be extended to a continuous map from X to $[-1, 1]$. True also for $\mathbb{R}$ instead of $[-1, 1]$.

Proof.

Lemma. If $f : A \to [-\varepsilon, \varepsilon]$ is continuous, there exists continuous $g : X \to \mathbb{R}$ with $g(X) \subseteq [-\varepsilon/3, \varepsilon/3]$ and $(g - f)(A) \subseteq [-2\varepsilon/3, 2\varepsilon/3]$.

Proof. Applying the Urysohn Lemma on the disjoint closed sets $L = f^{-1}([-\varepsilon, -\varepsilon/3])$ and $R = f^{-1}([\varepsilon/3, \varepsilon])$, there exists $g : X \to [-\varepsilon/3, \varepsilon/3]$ such that $g(L) = \{-\varepsilon/3\}$ and $g(R) = \{\varepsilon/3\}$. This g works. □

Now let $f : A \to [-1, 1]$ be continuous. Then we can find $g_1 : X \to [-1/3, 1/3]$ such that $|f(a) - g_1(a)| \leqslant 2/3$ for all $a \in A$. Then we apply the Lemma on $f - g_1$ again, so we get $g_2 : X \to [-2/9, 2/9]$ such that $|f(a) - g_1(a) - g_2(a)| \leqslant 4/9$. Recursively, we get a sequence of functions g_n such that $g_{n+1} : X \to [-(2/3)^n/3, (2/3)^n/3]$ and

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leqslant \left(\frac{2}{3}\right)^{n+1}.$$

By the Weierstrass M-test, $g(x) = \sum_{n=1}^{\infty} g_n(x)$ converges to the desired function (Exercise).

To show the $\mathbb{R}$ version, it suffices to show the $(-1, 1)$ version since they are homeomorphic. Take g from the $[-1, 1]$ case. Apply the Urysohn Lemma to the disjoint closed sets A and $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$ to get a continuous $\varphi : X \to [0, 1]$ so that $\varphi(D) = \{0\}$ and $\varphi(A) = \{1\}$. Then $h(x) = \varphi(x)g(x)$ works. ■