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18.901 Notes

1 Topological Spaces

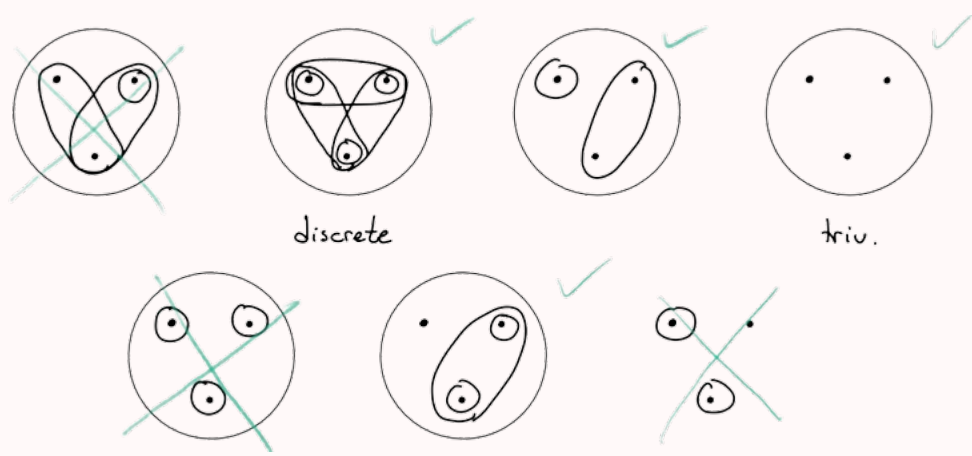
Definition 1.1.

1. A ***topology*** on a set X is a set $\mathscr{T}$ of subsets of X called ***open sets*** such that
   - $\varnothing, X \in \mathscr{T}$
   - $\mathscr{T}' \subseteq \mathscr{T} \implies \bigcup_{U \in \mathscr{T}'} U \in \mathscr{T}$. (Preserved under arbitrary unions)
   - $U_1, \cdots, U_n \in \mathscr{T} \implies \bigcap_{i=1}^{n} U_i \in \mathscr{T}$. (Preserved under finite intersections)

   $(X, \mathscr{T})$ – or just X when $\mathscr{T}$ is understood – is a ***(topological) space***.
2. Suppose $\mathscr{T}, \mathscr{T}'$ are two topologies on X with $\mathscr{T} \subseteq \mathscr{T}'$. We say $\mathscr{T}'$ is ***finer*** than $\mathscr{T}$ and $\mathscr{T}$ is ***coarser*** than $\mathscr{T}'$.
3. $A \subseteq X$ is ***closed*** if $X \backslash A$ is open. Hence $\varnothing, X$ are closed, and closedness is preserved under finite unions and arbitrary intersections.

Example 1.1.

1. The ***discrete topology*** on X is $\mathscr{T} = \mathscr{P}(X)$.
2. The ***trivial topology*** on X is $\mathscr{T} = \{\varnothing, X\}$.
3. $X = \{1, 2, 3\}$:



Definition 1.2. A set $\mathscr{B}$ of subsets of X is a ***basis*** if

- $X = \bigcup_{B\in\mathscr{B}} B$
- $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathscr{B} \implies (\exists B \in \mathscr{B})\,(x \in B \subseteq B_1 \cap B_2)$

Theorem 1.1. A basis $\mathscr{B}$ generates a topology $\mathscr{T}$ via

$$U \in \mathscr{T} \;\Leftrightarrow\; (\forall x \in U)\,(\exists B \in \mathscr{B})\,(x \in B \subseteq U)\,.$$

Proof. $\varnothing \in \mathscr{T}$ (vacuously) and $X \in \mathscr{T}$ since $\mathscr{B}$ covers X. We then verify the union and intersection properties:

- Suppose $U_\alpha \subseteq X$ are open, then $\bigcup_\alpha U_\alpha$ is open because

$$x \in \bigcup_\alpha U_\alpha \implies x \in U_\alpha \text{ for some } \alpha \implies x \in B_\alpha \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha$$

- Suppose U_1, U_2 are open, then $U_1 \cap U_2$ is open because

$$x \in U_1 \cap U_2 \implies \begin{cases} x \in B_1 \subseteq U_1 \text{ for some } B_1 \in \mathscr{B} \\ x \in B_2 \subseteq U_2 \text{ for some } B_2 \in \mathscr{B} \end{cases} \implies x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

  for some $B \in \mathscr{B}$. By induction, any finite intersection of open sets is open. ∎

Example 1.2. Let $X = \mathbb{R}$. We can construct three topologies via the bases:

1. $\{(a, b) : a, b \in \mathbb{R}\}$ (the ***standard topology*** on $\mathbb{R}$)
2. $\{[a, b) : a, b \in \mathbb{R}\}$
3. $\{U \subseteq \mathbb{R} : U = \mathbb{R}\backslash \{x_1, \cdots, x_n\} \text{ for some } x_1, \cdots, x_n \in \mathbb{R}\}$

Note, (2) is finer than (1), and (1) is finer than (3).

Remark.

1. Uncountable intersections may not be open. E.g. $\bigcap_n(-1/n, 1/n) = \{0\}$ is not open in the standard topology on $\mathbb{R}$.
2. Different bases could generate the same topology. E.g. For $X = \mathbb{R}^2$, open balls generate the same topology as open squares do.

Definition 1.3. Let X be a space, and $A \subseteq X$.

1. $\text{int}(A) = \bigcup\{U \subseteq A : U \text{ is open}\}$ is the ***interior*** of A.
2. $\overline{A} = \bigcap\{C \supseteq A : C \text{ is closed}\}$ is the ***closure*** of A.
3. A is ***dense*** if $\overline{A} = X$.

Example 1.3.

1. $\text{int}(A) = \overline{A} = A$ in the discrete topology.
2. $\text{int}(A) = \varnothing; \overline{A} = X$ in the trivial topology for any $A \neq \varnothing, X$.
3. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Warning. A, B dense does not imply $A \cap B$ dense, e.g. take $\mathbb{Q}$ and $\mathbb{Q} + \sqrt{2}$.

Theorem 1.2.

1. A open $\Leftrightarrow$ $A = \text{int}(A)$
2. A closed $\Leftrightarrow$ $A = \overline{A}$

Definition 1.4.

1. A ***neighborhood of $x \in X$*** is an open set that contains x.
2. $x \in X$ is a ***limit point*** of A if $(\forall x \in U \in \mathscr{T})\,(A \cap U \backslash \{x\} \neq \varnothing.)$
3. $x \in X$ is an ***adherent point*** of A if $(\forall x \in U \in \mathscr{T})\,(A \cap U \neq \varnothing.)$
4. $\partial A = \{x \in X : x \text{ limit pt of } A \text{ and } X \backslash A\}$

Theorem 1.3.

1. $\overline{A} = \{\text{adherent pts of } A\} = A \cup \{\text{limit pts of } A\} = \text{int}(A) \sqcup \partial A$.
2. $X = \text{int}(A) \sqcup \partial A \sqcup \text{int}(X \backslash A)$.

Theorem 1.4. If U_1, U_2 are dense and open, then $U_1 \cap U_2$ is dense and open.

Proof. Suppose $x \in X$. We want to show that for any $x \in U$ open we have $U \cap (U_1 \cap U_2) \neq \varnothing$.

Since U_1 is dense, $U \cap U_1 \neq \varnothing$. Since U_2 is also dense, $U \cap U_1 \cap U_2 \neq \varnothing$. ■

2 Metric Spaces

Definition 2.1.

1. A ***metric*** on a set X is a function $d : X^2 \to \mathbb{R}$ such that
   - $d(x, y) \geqslant 0$ and equality holds if and only if $x = y$
   - $d(x, y) = d(y, x)$
   - $d(x, y) + d(y, z) \geqslant d(x, z)$

   The set $B_x(\varepsilon) = \{y : d(x, y) < \varepsilon\}$ is the (open) ***ε-ball centered at x***.
2. The ***metric topology*** on (X, d) is the topology generated by the basis

$$\mathscr{B} = \{B_x(r) : x \in X, r > 0\}$$

Example 2.1. The ***euclidean metric*** d on $\mathbb{R}^n$ is $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i (x_i - y_i)^2}$.

3 Subspace Spaces

Definition 3.1. Let $(X, \mathscr{T})$ be a space and $A \subseteq X$. The ***subspace topology*** on A (with respect to X) is

$$\mathscr{T}_A = \{A \cap U : U \in \mathscr{T}\}.$$

We call A with this topology a ***subspace*** of X.

Theorem 3.1. A basis $\mathscr{B}$ for $\mathscr{T}$ defines a basis $\mathscr{B}_A$ for $\mathscr{T}_A$ via

$$\mathscr{B}_A = \{A \cap B : B \in \mathscr{B}\}.$$

Remark. If (X, d) is a metric space and $A \subseteq X$ then (A, d_A) is a metric space where $d_A(a_1, a_2) = d(a_1, a_2)$.

Theorem 3.2. Let (X, d) be a metric space. Then the metric topology on $A \subseteq X$ agrees with the subspace topology of $A \subseteq X$.

Proof. The subspace topology on A has basis $\mathscr{B}_S = \{A \cap B_x(r)\}_{x \in X}$ whereas the metric topology on A has basis $\mathscr{B}_M = \left\{B_x^A(r)\right\} = \{A \cap B_x(r)\}_{x \in A} \subseteq \mathscr{B}_S$. On the other hand, given any open U in the subspace topology and $x \in U \subseteq A$, we have $x \in A \cap B_x(r) \subseteq U$ for some $r > 0$, but this is just $x \in B_x^A(r) \subseteq U$. Since $x \in U$ was arbitrary, U is open in the metric topology too. ∎

Definition 3.2. $A \subseteq X$ (space) is discrete if its subspace topology is discrete.

Example 3.1. Is $X = \{0\} \cup_n \{1/n\}$ discrete in $\mathbb{R}$? No. $\{0\}$ is not open in X. If it were, then $\exists (a, b)$ such that $(a, b) \cap X = \{0\}$, but $1/n < b$ for large n.

Warning. $B = A = \mathbb{R} \times \{0\} \subseteq X = \mathbb{R}^2$ are examples for the following statements:

1. B open in A does not imply B open in X.
2. Suppose $A \subseteq Y \subseteq X$, then the $\mathrm{int}(A)$ in Y may not be $Y \cap \mathrm{int}(A)$.

But these versions are true:

Theorem 3.3.

1. B open in A, and A open in X, then B open in X.
2. Suppose $A \subseteq Y \subseteq X$, the closure of A in Y is $Y \cap$ (closure of A in X).

4 Product Spaces

Definition 4.1. Let $\{X_\alpha\}_\alpha$ be a collection of spaces.

1. The ***product topology*** on $X_1 \times \cdots \times X_n$ is generated by the basis
$$\mathscr{B} = \{Y_1 \times \cdots \times Y_n : Y_1, \cdots, Y_n \text{ open}\}$$
2. More generally, the ***product topology*** on $\prod_\alpha X_\alpha$ is generated by the basis
$$\mathscr{B} = \{\textstyle\prod_\alpha Y_\alpha : Y_\alpha \text{ open for all } \alpha, \text{ and only finitely many } Y_\alpha \neq X_\alpha\}$$

Theorem 4.1.

1. If $A \subseteq X; B \subseteq Y$ are subspaces, then the subspace topology and product topology on $A \times B$ agree.
2. The metric topology on $\mathbb{R}^n$ agrees with the product topology on $\mathbb{R}^n$.

5 Quotient Space

Definition 5.1.

- Let X be a space, Y be a set, and $q : X \to Y$ be surjective. The ***quotient topology*** on Y induced by the ***quotient map*** q is given by

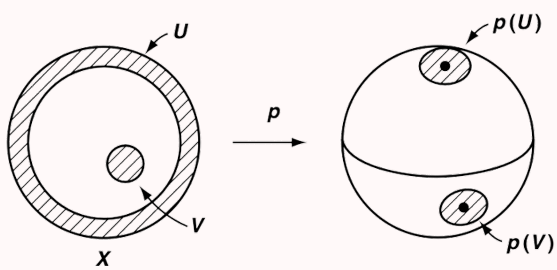
$$\mathscr{B} = \left\{U \subseteq Y : q^{-1}(U) \text{ open in } X\right\}$$

- Let $A \subseteq X$ be a subset and define $x \overset{A}{\sim} y \quad \Leftrightarrow \quad x = y$ or $x, y \in A$. We denote X/A the space on $X/\overset{A}{\sim}$ with quotient topology induced by the canonical map $q : X \twoheadrightarrow X/\overset{A}{\sim}$.

Remark. An equivalence relation $\sim$ on X determines the surjective ***canonical map*** $q : X \twoheadrightarrow X/\sim$ defined by $q(x)$ = equivalence class of x.

Example 5.1.

1. Consider the unit 2-disk $X = D^2 = \{x \times y : x^2 + y^2 \leqslant 1\}$. If we identify together all points on the boundary ∂D^2, we get the quotient space $D^2/\partial D^2$ that is homeomorphic with the subspace of $\mathbb{R}^3$ called the unit 2-sphere $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$.



2. We can construct a *torus* $S^1 \times S^1$ from the rectangle $[0,1] \times [0,1]$.

3. We can patch two disks $D^2 \sqcup D^2$ along their boundaries to obtain S^2. Formally, given a homeomorphism $\varphi : \partial D_1^2 \to D_2^2$, we have $(D_1^2 \sqcup D_2^2)/\sim\, = S^2$ where $x \sim y \iff x = y$ or $x \in \partial D_1^2, y \in \partial D_2^2, \varphi(x) = y$.

6 Continuous Functions

Definition 6.1. Let X, Y be spaces. A function $f : X \to Y$ is

- ***continuous at $x \in X$*** if $f^{-1}(V)$ is open in X for all neighborhoods V of $f(x)$.
- ***continuous*** if $f^{-1}(V)$ is open in X for all V open in Y.
- a ***homeomorphism*** if f is bijective, and f and f^{-1} are continuous.

Theorem 6.1.

1. Let $\mathscr{B}$ be a basis of X. The map $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for all $B \in \mathscr{B}$.

2. A composition of continuous functions is continuous.

3. Let $A \subseteq X$ be a subspace and $f : X \to Y$ be continuous. Then $f\,|_A$ is continuous.

4. Let $f : Z \to X \times Y$ where $f = f_X \times f_Y$. Then f is continuous if and only if f_X, f_Y are continuous.

5. Any quotient map is continuous. Given a quotient map $p : X \to Y$, $f : Y \to Z$ is continuous if and only if $g = f \circ p$ is continuous.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

6. The following are equivalent to $f : X \to Y$ being continuous:

   (1) $f^{-1}(C)$ is closed for all closed $C \subseteq Y$.

   (2) Given any $x \in X$ and $f(x) \subseteq V$ open, there exists open U with $f(U) \subseteq V$.

   (3) $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

Proof of (6).

- Continuity is equivalent to (1) by taking complements.
- For (2), say f is continuous, then $U = f^{-1}(V)$ works. Conversely, say (2) is true. Then for any open $V \subseteq Y$, any $v \in V$ admits a neighborhood within V, which has an open preimage $U_v \subseteq X$. Then $f^{-1}(V) = \bigcup_{v \in V} U_v$ is open, and thus f is continuous.
- (1) $\Rightarrow$ (3). Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\overline{f(A)}\right)$ which is closed, we have $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$ and thus $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.
- (3) $\Rightarrow$ (1). Let $C \subseteq Y$ be closed. Then $f\left(\overline{f^{-1}(C)}\right) = \overline{f\left(f^{-1}(C)\right)} \subseteq \overline{C} = C$ and hence $\overline{f^{-1}(C)} \subseteq f^{-1}f\left(\overline{f^{-1}(C)}\right) \subseteq f^{-1}(C)$ and thus $f^{-1}(C)$ is closed. ■

Corollary 6.1. Say X, Y are metric spaces. $f : X \to Y$ is continuous if and only if

$$(\forall x \in X, \varepsilon > 0)\,(\exists \delta > 0)\,(\forall d_X(x, y) < \delta)\,(d_Y\,(f(x), f(y)) < \varepsilon)\,.$$

Theorem 6.2. (Pasting Lemma) Let $X = A \cup B$ be a space where A, B are closed. If $f_A : A \to Y$ and $f_B : B \to Y$ are continuous and $f_A(x) = f_B(x)$ for all $x \in A \cap B$, then $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_A(x) & x \in A \\ f_B(x) & x \in B \end{cases}$$

is continuous.

7 Limits and Continuity

Definition 7.1. $\{x_n\}_{n \in \mathbb{N}}$ in X ***converges*** to $x \in X$ if any neighborhood of x contains all but finitely many x_n. Write $x_n \to x$.

Warning. Limits may not be unique:

1. In the trivial topology, any sequence converges to all points.
2. In $\mathbb{R}_1 \sqcup \mathbb{R}_2/\sim$ where $x \sim y \iff x \in \mathbb{R}_1, y \in \mathbb{R}_2, x = y \neq 0$, we have

$$1/n \to 0_1 \quad \text{and} \quad 1/n \to 0_2 \quad (\textbf{\textit{fat point}})$$

Theorem 7.1. If $x_n \to x$, then $x \in \overline{\{x_n\}_n}$.

Definition 7.2. A space X is ***first-countable*** if for any $x \in X$, there exists a countable number of neighborhoods $U_1, U_2, \cdots$ such that any neighborhood of x contains some U_i. The $\{U_i\}$ is called a ***neighborhood basis*** of x.

Theorem 7.2. If X is first-countable,

1. $x \in \overline{A} \implies \exists x_1, x_2, \cdots \in A$ such that $x_n \to x$.
2. $f : X \to Y$ is continuous if and only if $(x_n \to x) \implies (f(x_n) \to f(x))$.

8 Connectedness

Definition 8.1. A space X is ***connected*** if there is no nontrivial clopen (closed and open) set $A \subseteq X$.

Example 8.1. The subspace $(0, 1) \cup (2, 3)$ of $\mathbb{R}$ is not connected.

Theorem 8.1. $[a, b] \subseteq \mathbb{R}$ is connected.

Proof. Suppose the contrary, that $[a, b] = A \sqcup B$ where A, B are closed and non-empty. WLOG Assume $b \in B$. Then $s = \sup A < b$. If $s \in A$, since A is also open, there exists $(s - \varepsilon, s + \varepsilon) \subseteq A \implies \sup A \geqslant s + \varepsilon$, a contradiction. Hence $s \in B$ instead. Since B is open, there exists $(s - \varepsilon, s + \varepsilon) \subseteq B$ and thus $\sup A \leqslant s - \varepsilon$, a contradiction. ■

Definition 8.2. A space X is ***path-connected*** if every pair $x, y \in X$ can be joined by a *path* in X: a continuous map $\gamma : I = [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example 8.2.

1. $\mathbb{R}^n$ is path-connected. Use the path $\gamma(t) = t\mathbf{x} + (1 - t)\mathbf{y}$.
2. S^n is path-connected. Use the path $\gamma(t) = \dfrac{t\mathbf{x} + (1 - t)\mathbf{y}}{|t\mathbf{x} + (1 - t)\mathbf{y}|}$.
3. A torus is path-connected: Start with a path in I^2 and then take the quotient.

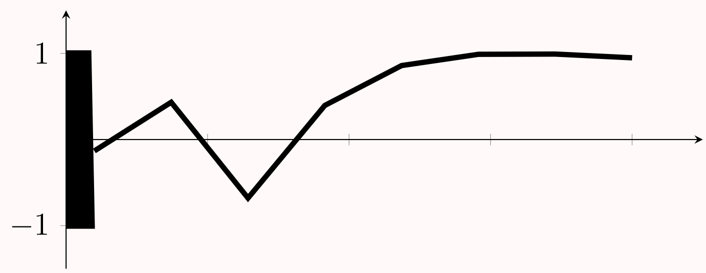
Theorem 8.2.

1. Any path-connected space is connected.
2. If $f : X \to Y$ is continuous and surjective,
   - X connected $\implies$ Y connected.
   - X path-connected $\implies$ Y path-connected.
3. Quotients of a (path-)connected space is (path-)connected.
4. A product of (path-)connected spaces is (path-)connected.

Example 8.3. The ***topologist's sine curve*** defined by

$$X = \{(x \times \sin(1/x)) : x > 0\} \cup \{0\} \times [-1, 1]$$

is connected but not path-connected.



Definition 8.3. The equivalence relation $x \sim y$ where there is a (path-)connected subspace containing both x, y partitions the space into (path-)connected ***components***.

9 Compactness

Definition 9.1.

1. An ***open cover*** of X is a collection of open sets that cover X. A space X is ***compact*** if every open cover of X admits a finite subcover.
2. A space X is ***sequentially compact*** if every sequence of points in X admits a convergent subsequence.

Theorem 9.1. 1st-countable + compact $\implies$ sequentially compact.

Proof. Suppose $\{x_n\}_n$ does not have a convergent subsequence. Let $x \in X$, then there exists a countable neighborhood basis $U_1, U_2, \cdots$. We can safely let $U_1 \supseteq U_2 \supseteq \cdots$ by taking successive intersections. Since there is no subsequence that converges to x, only finitely many x_n lie in U_n for some sufficiently large n. Hence, every $x \in X$ has a neighborhood U_x that intersects $\{x_n\}_n$ at a finite number of points. Taking the union of all U_x and applying compactness shows that $\{x_n\}_n$ is finite, so we can conclude by the pigeonhole principle. ■

Theorem 9.2.

1. Every closed subspace of a compact space is compact.
2. A continuous function maps compact spaces to a compact image.
3. Suppose X is compact and $C_1 \supseteq C_2 \supseteq \cdots$ is a sequence of closed and non-empty sets. Then $\bigcup_n C_n$ is non-empty.
4. A product of compact spaces is compact (Infinite case is hard: Tychonoff's Thm)
5. $[a, b]$ is compact.

Proof of (4). Suppose $[a, b] = \bigcup_\alpha U_\alpha$. Then

$$S = \{x \in [a, b] : [a, b] \text{ can be covered by finitely many } U_\alpha\}$$

contains $a \in S$ and is bounded above by b. Hence S has a supremum s.

Claim. $s \in S$.
Proof. Let $s \in U_\beta$ for some β, so there exists $(s - \varepsilon, s + \varepsilon) \subseteq U_\beta$. If $s \notin S$, just add U_β to the finite subcover of $[a, s - \varepsilon/2]$. □

Claim. $s = b$.
Proof. If not, then similarly, just add U_β to the finite subcover of $[a, s]$. □

Therefore $[a, b]$ can be covered by finitely many U_α. ■

Theorem 9.3. (Heine-Borel)
A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

- $(\Leftarrow)$ $X \subseteq [-M, M]^n$ is a closed subset of a compact space, so X is compact.

- ($\Rightarrow$) Compactness on the open cover $\{B_0(r)\}_{r>0}$ shows X is bounded. We then show any limit pt x of X is in X: For all $n \in \mathbb{N}^*$, $C_n := \overline{B_x 1/n} \cap X \neq \varnothing$, and thus $\bigcap_n C_n = X \cap \{x\}$ is non-empty. ■

10 Hausdorff Spaces

Definition 10.1. A space X is ***Hausdorff*** if for any distinct $x, y \in X$ there exists disjoint neighborhoods $x \in U, y \in V$.

Example 10.1.

1. The trivial topology is not Hausdorff. The discrete topology is.
2. Metric spaces are Hausdorff.
3. The finite complement topology on $\mathbb{R}$ is not Hausdorff.
4. The space $\mathbb{R}_1 \sqcup \mathbb{R}_2/\sim$ containing the fat point is not Hausdorff.

Theorem 10.1. X is Hausdorff if and only if $\Delta = \{(x \times x) : x \in X\} \subseteq X^2$ is closed.

Proof.

- ($\Rightarrow$) If X is Hausdorff, for any $x \neq y$ there exists disjoint neighborhoods U, V of x, y respectively. Then $U \times V$ is a neighborhood of $(x \times y) \in X \times Y$ disjoint from Δ. Taking the union over all $(x \times y)$ implies Δ is closed.
- ($\Leftarrow$) If Δ is closed, given any $x \neq y$ there exists a basis neighborhood $U \times V$ of $(x \times y)$ disjoint from Δ. Then U, V are the desired neighborhoods. ■

Theorem 10.2.

1. In a Hausdorff space, a sequence of points converge to at most one point.
2. One-point sets in a Hausdorff space are closed.
3. A subspace of a Hausdorff space is Hausdorff.
4. A finite product of Hausdorff spaces is Hausdorff.
5. A compact subspace of a Hausdorff space is closed.

Warning. A quotient of a Hausdorff space may not be Hausdorff.

11 Normal Spaces

Definition 11.1.

1. X is ***T_1*** if one-point sets are closed.
2. A space is ***normal*** if it is T_1, and, for any pair of disjoint closed sets $A, B \subseteq X$ there exists disjoint open sets $U, V \subseteq X$ such that $A \subseteq U, B \subseteq V$.

Remark.

1. Normal $\implies$ Hausdorff $\implies$ T_1.
2. A quotient, subspace, or product of normal space(s) need not be normal.

Example 11.1.

1. The fat point $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$ is T_1 but not Hausdorff.
2. The ***K-topology*** on $\mathbb{R}$ generated by $\{(a,b)\} \cup \{(a,b) \backslash \bigcup_n \{1/n\}\}$ is Hausdorff but not normal.
3. The topology $\mathbb{R}_\ell$ on $\mathbb{R}$ generated by $\{[a,b)\}$ is normal, but $\mathbb{R}_\ell^2$ is not normal.

Theorem 11.1.

1. A closed subspace A of a normal space X is normal.
2. Compact + Hausdorff $\implies$ Normal.

Proof of (2). Suppose $A, B \subseteq X$ are disjoint and closed. Fix $a \in A$. Then for each $b \in B$ there exists disjoint neighborhoods $a \in U_b, b \in V_b$. Since B is also compact, there exists finitely many V_b that cover B. The union of such finitely many V_b and the intersection of their corresponding U_b form disjoint open sets containing a and B respectively. Repeat the same procedure for every $a \in A$ and then apply compactness of A. ∎

Theorem 11.2. Metric spaces are normal.

Proof. We can show that, for any subset $A \subseteq X$, the *point-to-set distance* $d(-, A) : X \to \mathbb{R}$ given by $d(x, A) = \inf_{a \in A} d(x, a)$ is continuous. For disjoint closed sets A, B, the open sets

$$U = \{x : d(x,A) < d(x,B)\}, \qquad V = \{x : d(x,A) > d(x,B)\}$$

contain A, B respectively and are disjoint. ∎

Theorem 11.3. X is normal if and only if for any closed A and open U such that $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Theorem 11.4. (Urysohn's Lemma)
Let X be normal and A, B be disjoint closed sets of X. There exists a continuous map

$$f : X \to I$$

such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. Define open sets U_p for each $p \in \mathbb{Q} \cap [0,1]$ as follows: Enumerate $\mathbb{Q} \cap [0,1]$ such that 1 and 0 are the first two elements. Define $U_1 = X - B$ and by normality pick U_0 such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction, say we defined U_p for a finite number of p's and let r be the next rational in the enumeration. We must have $p < r < q$ where U_p, U_q are already defined. By normality we pick U_r such that $\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$.

Additionally, we let $U_p = \varnothing$ for all rationals $p < 0$ and $U_p = X$ for all rationals $p > 1$. Hence,

$$p < q \;\Rightarrow\; \overline{U_p} \subseteq U_q.$$

We then define $f(x) = \inf\{p : x \in U_p\}$. It is easy to see $f(A) = \{0\}$ and $f(B) = \{1\}$. We show that f is continuous.

Lemma 1. $x \in \overline{U_r} \;\Rightarrow\; f(x) \leqslant r$
Proof. If $x \in \overline{U_r}$, then $x \in U_s$ for every $s > r$. Hence $f(x) \leqslant r$. □

Lemma 2. $x \notin \overline{U_r} \;\Rightarrow\; f(x) \geqslant r$.
Proof. If $x \notin \overline{U_r}$, then $x \notin U_s$ for any $s < r$. Hence $f(x) \geqslant r$. □

Given a ball $I = (f(x) - \delta, f(x) + \delta)$, we wish to find a neighborhood U of x such that $f(U) \subseteq I$. First we choose rational numbers $p, q \in I$ such that $p < f(x) < q$. Then the open set $U_q \backslash \overline{U_p}$ is the desired neighborhood using the lemmas above. ■

Theorem 11.5. (Tietze Extension Theorem)
Let A be closed in a normal space X. Any continuous map from A to I can be extended to a continuous map from X to I. True also for $\mathbb{R}$ instead of I.

Proof. We show for $[-1,1]$ instead of I, and then for $(-1,1)$ instead of $\mathbb{R}$.

Lemma. If $f : A \to [-\varepsilon, \varepsilon]$ is continuous, there exists continuous $g : X \to \mathbb{R}$ with $g(X) \subseteq [-\varepsilon/3, \varepsilon/3]$ and $(g - f)(A) \subseteq [-2\varepsilon/3, 2\varepsilon/3]$.

Proof. Applying the Urysohn Lemma on the disjoint closed sets $L = f^{-1}([-\varepsilon, -\varepsilon/3])$ and $R = f^{-1}([\varepsilon/3, \varepsilon])$, there exists $g : X \to [-\varepsilon/3, \varepsilon/3]$ such that $g(L) = \{-\varepsilon/3\}$ and $g(R) = \{\varepsilon/3\}$. This g works. □

Now let $f : A \to [-1,1]$ be continuous. Then we can find $g_1 : X \to [-1/3, 1/3]$ such that $|f(a) - g_1(a)| \leqslant 2/3$ for all $a \in A$. Then we apply the Lemma on $f - g_1$ again, so we get $g_2 : X \to [-2/9, 2/9]$ such that $|f(a) - g_1(a) - g_2(a)| \leqslant 4/9$. Recursively, we get a sequence of functions g_n such that $g_{n+1} : X \to [-(2/3)^n/3, (2/3)^n/3]$ and

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leqslant \left(\frac{2}{3}\right)^{n+1}.$$

By the Weierstrass M-test, $g(x) = \sum_{n=1}^{\infty} g_n(x)$ converges to the desired function (Exercise).

To show the $(-1,1)$ version, take g from the $[-1,1]$ case. Apply the Urysohn Lemma to the disjoint closed sets A and $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$ to get a continuous $\varphi : X \to [0,1]$ so that $\varphi(D) = \{0\}$ and $\varphi(A) = \{1\}$. Then $h(x) = \varphi(x)g(x)$ works ($|h(x)| < 1$). ■

Urysohn Metrization Theorem

Definition 11.2.

1. A space is ***second-countable*** if it has a countable basis.
2. A space is ***metrizable*** if it is homeomorphic to a metric space.

Theorem 11.6. (Urysohn Metrization Theorem)
2nd countable + Normal $\implies$ Metrizable.

Proof. We first note that $I^\omega = \{\mathbf{x} = (x_1, x_2, \cdots) : x_i \in I\}$ with the metric

$$d(\mathbf{x}, \mathbf{y}) = \sup_n \frac{|x_n - y_n|}{n}.$$

is a metric space. Let X be normal with a countable basis $\mathscr{B}$. We will embed X into I^ω.

Lemma. There exists a collection $\{f_n : X \to I\}_{n\in\mathbb{N}}$ of continuous functions such that given any $x \in X$ and any neighborhood U, there exists some f_n that is positive at x but vanishes outside U.

Proof. For each $B, C \in \mathscr{B}$ with $\overline{B} \subseteq C$, apply the Urysohn Lemma to construct a continuous function $g_{B,C} : X \to I$ such that $g_{B,C}\left(\overline{B}\right) = \{1\}$ and $g_{B,C}(X\backslash C) = \{0\}$. $\left\{g_{B,C} : \overline{B} \subseteq C\right\}$ is the desired collection. It is countable because $\mathscr{B} \times \mathscr{B}$ is countable, and given any x with neighborhood U, we can choose by Theorem 11.3 the sequence of open sets $x \in B \subseteq \overline{B} \subseteq C \subseteq U$, and then use $g_{B,C}$. $\square$

Using $\{f_n\}_{n\in\mathbb{N}}$ from the Lemma, define $F : X \to I^\omega$ such that

$$F(x) = (f_0(x), f_1(x), f_2(x), \cdots)$$

- F is injective because given $x \neq y$, there exists some $f_n(x) > 0 = f_n(y)$ (Hausdorff!).
- F is continuous: Let $B_x(\varepsilon) \subseteq I^\omega$. Fix an integer $N > 2/\varepsilon$. Since each f_n is continuous, for each $1 \leqslant n \leqslant N$ there exists a neighborhood $x \in U_n$ such that $y \in U_n \implies |f_n(x) - f_n(y)| \leqslant \varepsilon/2$. Hence for any $y \in U_1 \cap \cdots \cap U_N$,

$$\begin{aligned} d\left(F(x), F(y)\right) &= \sup_n \frac{|f_n(x) - f_n(y)|}{n} \\ &\leqslant \max\left(\sup_{1\leqslant n\leqslant N} \frac{|f_n(x) - f_n(y)|}{n}, \sup_{n>N} \frac{|f_n(x) - f_n(y)|}{n}\right) \\ &\leqslant \max\left(\frac{\varepsilon}{2}, \frac{1}{N+1}\right) < \varepsilon. \end{aligned}$$

- For each open set U in X, $F(U)$ is open in $F(X)$: Let $x \in U$ and $f(x) = z$. Choose a f_N that is positive at x but vanishes outside U. Let

$$W = F(X) \cap \pi_N^{-1}\left((0, 1]\right)$$

be open in $F(X)$. We claim that $z \in W \subseteq F(U)$. Firstly, we have $z = F(x) \in W$ because $f_N(x) > 0$. Secondly, given any $F(y) \in W$, we must have $f_N(y) > 0$. Since f_N vanishes outside U, y must be in U, so $F(y) \in F(U)$.

Therefore, X is homeomorphic to its image under F, a subspace of the metric space I^ω, which is also a metric space. $\blacksquare$

12 Manifolds

Definition 12.1. An ***n-manifold*** is a 2nd countable Hausdorff space X such that each $x \in X$ has a neighborhood homeomorphic with an open subset of $\mathbb{R}^n$. We also write $X = X^n$. A 1-manifold is a ***curve***, and a 2-manifold is a ***surface***.

Theorem 12.1. $X^n \times Y^m$ is an $(n+m)$-manifold.

Proof. Hausdorffness and 2nd Countability follow immediately. Fix $(x \times y) \in X \times Y$, then there exists neighborhoods U, V of x, y homeomorphic to $\mathbb{R}^n, \mathbb{R}^m$ respectively. Then $U \times V$ is a neighborhood of $(x \times y)$ homeomorphic to $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$. ∎

Example 12.1.

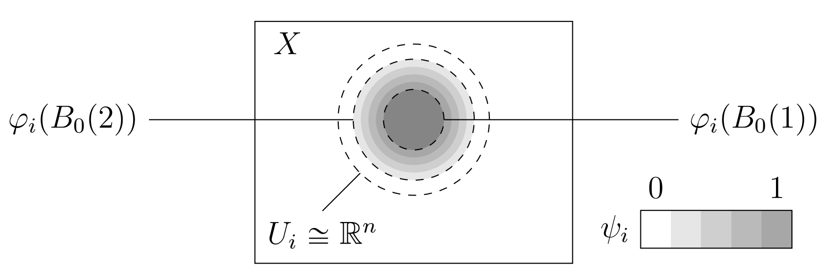
1. $\mathbb{R}^n$ is an n-manifold.
2. S^n is an n-manifold. (Write $S^n = e_1^n \cup e_2^n$ where $e^n = \operatorname{int}(D^n) \cong \mathbb{R}^n$).
3. The ***real projective space*** $\mathbb{RP}^n = S^n/\sim$ (where $x \sim y \Leftrightarrow x = \pm y$) is an n-manifold.
4. $T^n = \underbrace{S^1 \times \cdots S^1}_{n}$ is an n-manifold. T^2 is a ***torus***.
5. *Fact:* Every connected curve is homeomorphic to either $\mathbb{R}$ and S^1.

Theorem 12.2. A compact n-manifold X can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Proof. Each $x \in X$ admits a neighborhood U^x with a homeo $\varphi^x : \mathbb{R}^n \to U^x$. We can choose a basis $x \in B^x \subseteq \varphi^x(B_0(1))$, and hence by compactness of X via the B^x there exists $U_1, \cdots, U_m$ with homeos $\varphi_i : \mathbb{R}^n \to U_i$ and $X \subseteq \bigcup_i \varphi_i(B_0(1))$

By Urysohn's Lemma, there exists $\rho_i : X \to I$ such that $\rho_i\left(\overline{\varphi_i(B_0(1))}\right) = \{1\}$ and $\rho_i(X \backslash \varphi_i(B_0(2))) = \{0\}$. Via the pasting lemma, let $\psi_i : X \to \mathbb{R}^n$ be the continuous function

$$\psi_i(x) = \begin{cases} \rho_i(x)\varphi_i^{-1}(x) & x \in U_i \\ (0, \cdots, 0) & \text{otherwise} \end{cases}.$$



Then $F(x) = (\rho_1(x), \cdots, \rho_m(x), \psi_1(x), \cdots, \psi_m(x))$ embeds X into $\mathbb{R}^{m(n+1)}$. ∎

13 Paracompactness

Definition 13.1.

- An open cover $\{U_\alpha\}_\alpha$ of X is ***locally finite*** if every $x \in X$ has a neighborhood that intersects only finitely many U_α.
- A ***refinement*** of an open cover $\{U_\alpha\}_\alpha$ of X is an open cover $\{V_\beta\}_\beta$ such that each V_β is contained in some U_α (depends on β).
- A space X is ***paracompact*** if it is Hausdorff, and, every open cover of X admits a locally finite refinement.

Warning.

1. Some sources do not require Hausdorffness in the definition.
2. Quotient/Subspace/Product of paracompact space(s) may not be paracompact.

Example 13.1. $\mathbb{R}^n$ is paracompact. Let $B(r)$ be the open ball of radius r centered at the origin. Given any open covering $\mathscr{A}$, for each $n \in \mathbb{N}^*$ we can pick a finite number of elements of $\mathscr{A}$ that covers $\overline{B(n)}$. Intersect them with $\mathbb{R}^n \backslash \overline{B(n-1)}$. The union of these open sets is a desired locally finite refinement.

Theorem 13.1.

1. A closed subspace of a paracompact space is paracompact.
2. Compact + Hausdorff $\implies$ Paracompact
3. Metric space $\implies$ Paracompact.
4. Paracompact $\implies$ Normal.

Proof of (4). Let A, B be closed and disjoint. We first prove the case when $A = \{a\}$. For each $b \in B$ pick disjoint neighborhoods $a \in U_b, v \in V_b$. Since $(X\backslash B) \cup_b V_b$ is an open cover of X, by paracompactness there exists a locally finite refinement of V_α's that cover B. Also, x has a neighborhood W that intersects only finitely many V_α, say $V_{b_1}, \cdots, V_{b_n}$. Then the open sets $U = U_{b_1} \cap \cdots \cap U_{b_n}$ and $V = V_{b_1} \cap \cdots \cap V_{b_n}$ form a desired pair.

For the general case, we update the notation so that for each $a \in A$ there exists disjoint open sets $a \in U_a, B \subseteq V_a$. Let $\{U_\alpha\}$ be a locally finite refinement that covers A, so $b \in B$ admits a neighborhood W_b that intersects finitely many U_α, say $U_{a_1}, \cdots, U_{a_n}$. We then let

$V_b = W_b \cap_i V_{a_i}$. Then $U = \bigcup_\alpha U_\alpha$ and $V = \bigcup_{b \in B} V_b$ give the desired separation. ■

Definition 13.2. A ***partition of unity*** on X is a locally finite open cover $\{U_\alpha\}_\alpha$ equipped with continuous $\rho_\alpha : X \to I$ such that

- $\rho_\alpha(x) > 0 \implies x \in U_\alpha$
- $\sum_\alpha \rho_\alpha(x) = 1$ (well-defined due to local finiteness)

Theorem 13.2. Every cover of a paracompact space admits a refinement that has a partition of unity.

Proof. Let $\{U_\alpha\}$ be a cover of X. For each $x \in X$ there is an $x \in U_{\alpha_x}$ and hence we can pick $x \in W_x \subseteq \overline{W_x} \subseteq U_{\alpha_x}$ by normality. Let $\{V_\beta\}$ be a locally finite refinement of $\{W_x\}$. By Urysohn's Lemma, there exists $\psi_\beta : X \to I$ such that $\psi\left(\overline{V_\beta}\right) = \{1\}$ and $\psi\left(X \backslash U_{\alpha_\beta}\right) = \{0\}$. Then $\rho_\beta(x) = \psi_\beta(x) / \sum_\gamma \psi_\gamma(x)$ is a desired partition of unity. ■

Theorem 13.3. Manifold $\implies$ Paracompact.

Proof. Let X be a manifold.

Lemma. $\exists K_1, K_2, \cdots$ compact with $K_n \subseteq \operatorname{int}(K_{n+1})$ and $X = \bigcup_n \operatorname{int}(K_n)$.
Proof. Let U_i with homeos $\varphi_i : \mathbb{R}^n \to U_i$ such that $\{\varphi_i(B_0(1))\}$ covers X. Then take the compact space $K_n = \bigcup_{i=1}^n \bigcup_{j=1}^n \varphi_i\left(\overline{B_0(j)}\right)$. □

Let $X = \bigcup_\alpha U_\alpha$. Then for each n there exists $U_1^n, \cdots, U_{t_n}^n$ that cover the compact space K_n. Then $V_j^n = U_j^n \backslash K_{n-1}$ form a locally finite refinement. ■

14 Covering Dimension

Definition 14.1.

1. The ***covering dimension*** of a space X is the infimum over $n \in \mathbb{N}$ such that

$$(\forall \text{ open cover } \{U_\alpha\})\,(\exists \text{ refinement } \{V_\beta\})\,(\forall x \in X)\,(x \text{ is in} \leqslant n+1 \text{ of the } V_\beta)$$

   or equivalently

$$\dim X = \max_{\mathscr{A} \text{ open cover } X}\left[\min_{\mathscr{B} \text{ refmt of } \mathscr{A}} \underbrace{\left(\max_{x\in X} |\{B \in \mathscr{B} : x \in B\}|\right)}_{\textbf{\textit{order}} \text{ of } \mathscr{B}}\right] - 1$$

2. A ***Lebesgue number*** for an open cover $\{U_\alpha\}$ of a compact metric space is a real $\delta > 0$ such that any subset of X of diameter $< \delta$ is contained within some U_α.

Theorem 14.1. (Lebesgue's Covering Lemma)
Any open cover $\{U_\alpha\}$ of a compact metric space (X, d) has a Lebesgue number.

Proof. Since X is compact, assume $\{U_\alpha\} = \{U_1, \cdots, U_n\}$. The map $f(x) = \max_{1\leqslant i\leqslant n} d(x, X\backslash U_i) > 0$ is continuous on a compact space and thus $f(X)$ has a minimum $\delta > 0$. ■

Example 14.1.

1. Any compact subspace of $\mathbb{R}$ has dimension at most 1.

   Proof. Note that $\mathscr{C} = \left\{(n, n+1), \left(n - \frac{1}{2}, n + \frac{1}{2}\right) : n \in \mathbb{Z}\right\}$ has order 2. Let $\mathscr{A}$ be any open covering of a compact subspace X of $\mathbb{R}$, with some Lebesgue number $\delta > 0$. The image $\mathscr{I}$ of $\mathscr{C}$ under $f : x \mapsto \delta x/2$ is an open covering whose elements have diameter $\delta/2 < \delta$, and hence is an open refinement subcover of $\mathscr{A}$. Hence

$$\begin{aligned}\dim X &= \max_{\mathscr{A} \text{ open cover } X}\left[\min_{\substack{\mathscr{B} \text{ open refinement} \\ \text{subcover of } \mathscr{A}}} (\text{order of } \mathscr{B})\right] - 1 \\ &\leqslant \max_{\mathscr{A} \text{ open cover } X} [2] - 1 = 1.\end{aligned}$$

2. $\dim I = 1$.

   Proof. We show that there is some open covering $\mathscr{A}$ such that any open refinement subcover of $\mathscr{A}$ has order at least 2. Let $\mathscr{A} = \{[0,1),(0,1]\}$ and let $\mathscr{B}$ be any open refinement subcovering. Since 0 and 1 cannot belong to the same refinement, $\mathscr{B}$ has at least two elements. Partition $\mathscr{B}$ into two nonempty parts $\mathscr{B}_1$ and $\mathscr{B}_2$. If $\mathscr{B}$ had order 1 then $\bigcup \mathscr{B}_1$ and $\bigcup \mathscr{B}_2$ disconnect $[0,1]$, a contradiction.

3. *Fact:* $\dim I^n = n$, and every compact subspace of $\mathbb{R}^n$ has dimension $\leqslant n$.

Theorem 14.2.

- If Y is a closed subspace of a finite dimensional space X, then $\dim Y \leqslant \dim X$.
- If $X = Y \cup Z$ where Y, Z are closed finite dimensional subspaces of X, then $\dim X = \max(\dim Y, \dim Z)$.
- Every compact subspace of $\mathbb{R}^N$ has dimension at most N.

Theorem 14.3. (The embedding Theorem)
Every compact metrizable space X of dimension m can be embedded in $\mathbb{R}^{2m+1}$.

Definition 14.2. Let X be a compact metric space.

1. $\mathcal{C}(X, \mathbb{R}^n) = \{f : X \to \mathbb{R}^n \text{ cts}\}$ is the metric space equipped with the uniform metric $d(f,g) = \sup_x |f(x) - g(x)|$.
2. For $A \subseteq X$, $\operatorname{diam}(A) = \sup_{x,y\in A} d(x,y)$.
3. For $f \in \mathcal{C}(X, \mathbb{R}^n)$, $\operatorname{diam}(f) = \sup\{\operatorname{diam}(f^{-1}\{z\}) : z \in f(X)\}$.
4. $U_\varepsilon = \{f \in \mathcal{C}(X, \mathbb{R}^n) : \operatorname{diam}(f) < \varepsilon\}$.

Remark. $\bigcap_n U_{1/n} = \{f \in \mathcal{C}(X, \mathbb{R}^n) : f \text{ injective}\}$, and U_ε is open and dense.

15 Homotopies

From now on, we assume all 'maps' are continuous.

Definition 15.1.

- The ***mapping cylinder*** M_f of a map $f : X \to Y$ is the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by identifying each $(x, 1)$ with $f(x)$.
- A ***deformation retraction*** of a space X onto a subspace $A \subseteq X$ is a family of maps $f_t : X \to X, t \in [0, 1]$ such that $f_0 = \mathbf{1}, f_1(X) = A$, and $f_t |_A = \mathbf{1}$ for all t. The map $(x, t) \mapsto f_t(x)$ should also be continuous.
- A ***homotopy*** is a family of maps $f_t : X \to Y, t \in [0, 1]$ such that $(x, t) \mapsto f_t(x)$ is continuous. f_0, f_1 are ***homotopic***, written $f_0 \simeq f_1$, if there exists a homotopy f_t connecting them. If $f_t |_A$ is independent of t for some $A \subseteq X$, then f_t is a ***homotopy relative to*** A.
- A ***retraction*** of X onto $A \subseteq X$ is a map $r : X \to X$ such that $r(X) = A$ and $r |_A = \mathbf{1}$. A deformation retraction is just a homotopy from the identity map to a retraction of X onto A.
- $f : X \to Y$ is a ***homotopy equivalence*** if there exists $g : X \to Y$ such that $fg \simeq \mathbf{1}_Y, gf \simeq \mathbf{1}_X$. We then say X and Y are ***homotopy equivalent*** (written $X \simeq Y$) or have the same ***homotopy type***.
- X is ***contractible*** if $X \simeq \{0\}$. $f : X \to Y$ is ***nullhomotopic*** if $f \simeq$ constant.

16 CW Complexes

Definition 16.1. A ***CW complex / cell complex*** is a space X built as such:

1. Start with a discrete set X^0, whose points are ***0-cells***.
2. Let D^n_α be n-balls (with $\partial D^n_\alpha = S^{n-1}_\alpha$). Inductively, form the n***-skeleton*** (n-dimensional CW complex) X^n as the quotient space of $X^{n-1} \coprod_\alpha D^n_\alpha$ by identifying $x \sim \varphi_\alpha(x)$ where $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ are the ***attaching maps***. This makes $X^n = X^{n-1} \coprod_\alpha \operatorname{int}(D^n_\alpha)$ as a set. The e^n_α are called n***-cells***.
3. One can stop after finite n, setting $X = X^n$. Or one can set $X = \bigcup_{n=0}^\infty X^n$, giving it the *weak topology*: $U \subseteq X$ is open $\Leftrightarrow U \cap X^n$ is open in X^n for all n.

The ***characteristic map*** of a cell $e^n_\alpha = \operatorname{int}(D^n_\alpha)$ is the map

$$\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \coprod_\beta D^n_\beta \xrightarrow{\text{quot}} X^n \hookrightarrow X$$

Example 16.1.

1. A one-dimensional cell complex X^1 is a ***graph***, where nodes are 0-cells and edges are 1-cells.
2. The n-sphere S^n is a cell complex with two cells e^0 and e^n, with the attaching map $S^{n-1} \to e^0$. Or, we can inductively attach two n-cells to the equatorial S^{n-1}.
3. The ***real projective n-space*** $\mathbb{RP}^n \cong S^n/(v \sim -v) \cong D^n/(v \sim -v : v \in \partial D^n)$ is a cell complex by attaching an n-cell to $\mathbb{RP}^{n-1}$ via the map $S^{n-1} \twoheadrightarrow \mathbb{RP}^{n-1}$. We can also have $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$.

Definition 16.2. A ***subcomplex*** of a CW complex X is a closed subspace $A \subseteq X$ that is a union of cells of X. The pair (X, A) is a ***CW pair***.

Example 16.2.

1. $\mathbb{RP}^k \subseteq \mathbb{RP}^n$ is a subcomplex $(k \leqslant n)$.
2. $S^k \subseteq S^n$ is not a subcomplex with the two-cell structure, but is a subcomplex using the recursive CW structure.

Theorem 16.1.

- If X, Y are cell complexes, then $X \times Y$ is a cell complex, whose cells are $e^m_\alpha \times e^n_\beta$ where e^m_α, e^n_β are cells of X, Y respectively.
- If (X, A) is a CW pair, then the quotient space X/A is a cell complex, whose cells are the cells of $X \backslash A$, and one new 0-cell: the image of A in X/A.

Theorem 16.2. If (X, A) is a CW pair and A is contractible, then the quotient map $X \twoheadrightarrow X/A$ is a homotopy equivalence.

Definition 16.3. $A \subseteq X$ has the ***homotopy extension property*** if given any map $f_0 : X \to Y$ and a homotopy $f_t\,|_A : A \to Y$ of $f_0\,|_A$, we can extend $f_t\,|_A$ to a homotopy f_t on X.

Theorem 16.3. $A \subseteq X$ has the homotopy extension property if and only if $X \times \{0\} \cup A \times [0, 1]$ is a retract of $X \times [0, 1]$.

Theorem 16.4. If (X, A) is a CW pair, A has the homotopy extension property.