Proving $d = 5$ Works.

Since $-5 \equiv 3 \pmod 4$, $\mathbb{Z}\left[\sqrt{-5}\right]$ is the ring of integers of $\mathbb{Q}\left[\sqrt{-5}\right]$. Let $p \mid k^2 + 5$.

Claim 1. The ideal (p) can be decomposed as $\mathfrak{p}\overline{\mathfrak{p}}$ for some ideal $\mathfrak{p} \neq \overline{\mathfrak{p}}$.

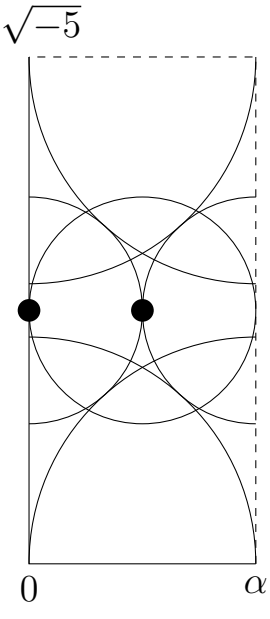
Proof. Let $\mathfrak{p} = \left(p, k + \sqrt{-5}\right)$. Then $\mathfrak{p}\overline{\mathfrak{p}} = \left(p^2, pk \pm p\sqrt{-5}, k^2 + 5\right)$. All the generators are divisible by p, hence $\mathfrak{p}\overline{\mathfrak{p}} \subseteq (p)$. However, the gcd of p^2 and $\left(pk + p\sqrt{-5}\right) + \left(pk - p\sqrt{-5}\right)$ is p, thus $(p) \subseteq \mathfrak{p}\overline{\mathfrak{p}}$. Assume $\mathfrak{p} = \overline{\mathfrak{p}}$, then $Ap + B\left(k - \sqrt{-5}\right) = k + \sqrt{-5}$ for some $A, B \in \mathbb{Z}\left[\sqrt{-5}\right]$. Write $A = a_1 + a_2\sqrt{-5}$ and $B = b_1 + b_2\sqrt{-5}$ and thus

$$\begin{cases} a_1 p + k b_1 + 5 b_2 = k \\ a_2 p - b_1 + k b_2 = 1 \end{cases} \quad \Rightarrow \quad (a_1 + k a_2)p + (5 + k^2) b_2 = 2k$$

has solutions for integers a_1, a_2, b_1, b_2. This is impossible as $p \mid$ LHS but $p \nmid$ RHS. $\square$

Claim 2. The class group of $\mathbb{Z}\left[\sqrt{-5}\right]$ is C_2.

Proof. It suffices to prove that there are only two types of sublattices in $\mathbb{Z}\left[\sqrt{-5}\right]$ up to orientation-preserving transformations. Let $\mathcal{L}$ be a sublattice of $\mathbb{Z}\left[\sqrt{-5}\right]$ and α be nonzero with minimal norm. Therefore $\mathcal{L}$ contains the sublattice $\mathcal{A}$ spanned by $\left(\alpha, \alpha\sqrt{-5}\right)$. If $\mathcal{L} = \mathcal{A}$ then this ideal is just (α), otherwise let $\beta \in \mathcal{L} \setminus \mathcal{A}$ be situated in the parallelogram $x\alpha + y\alpha\sqrt{-5}$ where $0 \leq x, y < 1$. Note that β cannot lie inside the four quarter circles as shown on the right due to minimality of α. For the remaining region, any β lying there, multiplied by two, will be $< |\alpha|$ distance away from some point in $\mathcal{A}$ (Verified by applying an origin-homothety with scale 2 onto the circle and the two semicircles). Therefore, $2\beta \in \mathcal{A}$, i.e. $\beta = \frac{\sqrt{-5}}{2}\alpha$ or $\frac{1+\sqrt{-5}}{2}\alpha$ (The two points labelled in the diagram). The former implies $-\frac{5}{2}\alpha \in \mathcal{L} \Rightarrow \frac{1}{2}\alpha \in \mathcal{L}$, contradicting minimality of α. Thus $\beta = \frac{1+\sqrt{-5}}{2}\alpha$. Therefore any ideal is in the form (α) or $\left(\alpha, \frac{1+\sqrt{-5}}{2}\alpha\right)$. $\square$



By claim 2, the product of any two ideals in the same ideal class belongs to the unit ideal class, i.e. is a principal ideal. Therefore $\mathfrak{p}\mathfrak{p} = (x)$ for some $x \in \mathbb{Z}\left[\sqrt{-5}\right]$. We know $x \neq p$ otherwise $\mathfrak{p} = \overline{\mathfrak{p}}$ by the cancellation law, hence

$$(p^2) = (p)(p) = \mathfrak{p}\overline{\mathfrak{p}}\mathfrak{p}\overline{\mathfrak{p}} = \mathfrak{p}\mathfrak{p}\overline{\mathfrak{p}}\overline{\mathfrak{p}} = (x)(\overline{x}) = (x\overline{x})$$

i.e $p^2 = x\overline{x} = \left(m + n\sqrt{-5}\right)\left(m - n\sqrt{-5}\right) = m^2 + 5n^2$ for some $(m, n) \neq (p, 0)$. $\square$

Generalising.

The ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}\left[\sqrt{-d}\right]$ is

$$\begin{cases} \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} & \text{if } -d \equiv 2, 3 \pmod 4 \\ \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z} \text{ or } a + 0.5, b + 0.5 \in \mathbb{Z}\} & \text{if } -d \equiv 1 \pmod 4 \end{cases}$$

I will analyse the case $-d \equiv 2, 3 \pmod 4$ for simplicity. We see that

$$\begin{aligned} &\text{The ideal } (p) \text{ is prime in } \mathbb{Z}\left[\sqrt{-d}\right] \\ \iff &\mathbb{Z}\left[\sqrt{-d}\right]/(p) \text{ is an integral domain} \\ \iff &\mathbb{F}_p\left[\sqrt{-d}\right] \text{ is an integral domain} \\ \iff &\mathbb{F}_p[x]/(x^2 + d) \text{ is an integral domain} \\ \iff &\text{The ideal } (x^2 + d) \text{ is prime in } \mathbb{F}_p[x] \end{aligned}$$

Since $-d$ has a quadratic residue mod p, the ideal (p) is not prime. Therefore $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$. The Dedekind-Kummer Theorem gives

$$p \text{ ramifies} \iff p \mid \Delta(x^2 + d) = -4d \qquad (*)$$

Claim. $p^2 = m^2 + dn^2$ $(n \neq 0)$ for all integer primes $p \nmid 2d$ iff the class group of $\mathcal{O}_K$ is

$$C_2 \times C_2 \times \cdots \times C_2.$$

Proof. ($\Leftarrow$) The order of every class is 1 and 2, thus $\mathfrak{p}\overline{\mathfrak{p}} = (x)$ for some x. Since p does not ramify, $\mathfrak{p} \neq \overline{\mathfrak{p}}$ and hence $x \neq p$. Therefore $(p^2) = (p)(p) = \mathfrak{p}\mathfrak{p}\overline{\mathfrak{p}}\overline{\mathfrak{p}} = (x)(\overline{x}) \Rightarrow p^2 = x\overline{x}$. ($\Rightarrow$) Assume some ideal class $\langle\mathfrak{a}\rangle$ has order > 2. Then $\mathfrak{a}\mathfrak{a}$ is not principal. Decomposing $\mathfrak{a}$ into prime ideals, there must exist some prime ideal $\mathfrak{p}$ where $\mathfrak{p}\mathfrak{p}$ is not principal. Let $\mathfrak{p}\overline{\mathfrak{p}} = (p) \Rightarrow (p^2) = \mathfrak{p}\mathfrak{p}\overline{\mathfrak{p}}\overline{\mathfrak{p}}$ is not expressible as a product of conjugate principal ideals.□

Therefore, the problem statement after changing 5 to d works if and only if the class group is $C_2 \times C_2 \times \cdots \times C_2$. (From Internet:) The values of d for which the class group is C_2 are $5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427$. The values of d for which the class group is C_1 are $1, 2, 3, 7, 11, 19, 43, 67, 163$. Picking those with $-2, -3 \pmod 4$, we have $d = 1, 2, 5, 6, 10, 13, 22, 37, 58$. Also, $\mathbb{Z}\left[\sqrt{-21}\right]$ has class group $C_2 \times C_2$, so $d = 21$ works too. Therefore,

$$d = 1, 2, 5, 6, 10, 13, 21, 22, 37, 58$$

all work. There might be others. □