18.100B Definitions

1 Real Numbers

1. A *field* is a set F equipped with operations $+$ and $\times$ such that

   - $(F, +)$ and $(F \setminus \{0\}, \times)$ are Abelian groups
   - $x(y + z) = xy + xz$ for all $x, y, z \in F$.   (Distributivity)

2. A field F is *ordered* if there exists a relation $<$ on F (with $x > y$ meaning $y < x$, $x \leq y$ meaning $x < y$ or $x = y$, etc) such that for all $x, y, z \in F$,

   - Exactly one of $x = y$, $x < y$, $x > y$ holds.   (Trichotomy)
   - $x < y$ and $y < z$ implies $x < z$.   (Transitivity)
   - $x < y$ implies $x + z < y + z$.   (Additivity)
   - $x < y$ and $z > 0$ implies $xz < yz$.   (Multiplicativity)

   We define $P = \{x \in F : x > 0\}$.

3. Let F be an ordered field.

   - $u \in F$ is an *upper bound* for a subset $S \subseteq F$ if $x \leq u$ for all $x \in S$. If an upper bound for S exists, we say S is *bounded above.*
   - $\ell \in F$ is a *lower bound* for a subset $S \subseteq F$ if $x \geq \ell$ for all $x \in S$. If an upper bound for S exists, we say S is *bounded below.*
   - If $S \subseteq F$ is bounded above and below, we say that it is *bounded.*
   - $u \in F$ is the *maximum* of S, denoted $\max S$, if u is an upper bound and $u \in S$.
   - $\ell \in F$ is the *minimum* of S, denoted $\min S$, if ℓ is a lower bound and $\ell \in S$.
   - $u \in F$ is the *supremum* of S, denoted $\sup S$, if it is the least upper bound for S. More precisely, we say that S has supremum

     $$\sup S = \min\{x \in F : x \text{ is an upper bound for } S\} \qquad \text{if it exists.}$$

   - $\ell \in F$ is the *infimum* of S, denoted $\inf S$, if it is the greatest lower bound for S. More precisely, we say that S has infimum

     $$\sup S = \max\{x \in F : x \text{ is an lower bound for } S\} \qquad \text{if it exists.}$$

   - By convention, $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$. If S is unbounded above (below) we say $\sup S = \infty$ ($\inf S = -\infty$).
   - We say that F is *complete* if it satisfies the *completeness axiom:* Every nonempty subset of F that is bounded above has a supremum.

2 Sequences

1. The absolute value function is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

2. A *sequence* $\{x_n\}_{n \in \mathbb{N}} = \{x_0, x_1, \cdots\}$ is an ordered list of real numbers. Explicitly, we have a function $x : \mathbb{N} \to \mathbb{R}$ and we denoted $x_n = x(n)$.

3. Let $\{x_n\}_{n \in \mathbb{N}}$ is said to *converge* to $\ell \in \mathbb{R}$ if

$$(\forall \varepsilon > 0) \, (\exists N \in \mathbb{N}) \, (\forall n \geq N) \, (|x_n - \ell| < \varepsilon)$$

If this is true, we write $\lim_{n \to \infty} x_n = \ell$.

4. $\{x_n\}_{n \in \mathbb{N}}$ is *bounded* if $\exists M \in \mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

5. $\{x_n\}_{n \in \mathbb{N}}$ is said to *diverge to* ∞, written as $x_n \to \infty$, if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$. The case $x_n \to -\infty$ is analogous.

6. $\{x_n\}_{n \in \mathbb{N}}$ is *monotone* if it is either nonincreasing ($x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$) or nondecreasing ($x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$)

7. A *subsequence* of $\{x_n\}_{n \in \mathbb{N}}$ is any ordered infinite subset. Precisely, it is some $\{x_{n_j}\}_{j \in \mathbb{N}}$ where $n_0 < n_1 < n_2 < \cdots$ are natural numbers.

8. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is *Cauchy* if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)(|x_n - x_m| < \varepsilon)$$

9. The *limit superior* and *limit inferior* of $\{x_n\}_{n \in \mathbb{N}}$ are defined by

$$\limsup x_n = \lim_{n \to \infty} \left(\sup_{k \geq n} x_k \right), \qquad \liminf x_n = \lim_{n \to \infty} \left(\inf_{k \geq n} x_k \right)$$

3   Series

1. Given a sequence $\{x_n\}_{n\in\mathbb{N}}$, we define the series

$$\sum_{k=0}^{n} x_k = x_0 + x_1 + \cdots + x_n \qquad \text{and} \qquad \sum_{k=0}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=0}^{n} x_k \text{ if it converges.}$$

2. The series $\displaystyle\sum_{k=0}^{\infty} a_k$ *converges absolutely* if $\displaystyle\sum_{k=0}^{\infty} |a_k|$ converges.

3. The *exponential function* is defined as

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

4. A series $\displaystyle\sum_{k=0}^{\infty} x_k$ is *unconditionally convergent* if any reordering of the x_k gives a series converging to the same number.

4   Topology of $\mathbb{R}$

1.   • An *open interval* of $\mathbb{R}$ is $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

   • A *closed interval* of $\mathbb{R}$ is $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

   For a given set $E \subseteq \mathbb{R}$, we say that $p \in E$ is

   • an *interior point* of E if there exists $a < p < b$ such that $(a, b) \subseteq E$.

   • an *isolated point* of E if there exists $a < p < b$ such that $(a, b) \subseteq E = \{p\}$.

   • a *boundary point* if for all $a < p < b$, (a, b) intersects both E and E^c.

   • a *limit point* (or accumulation point) if for all $a < p < b$, $(a, b) \cap E$ is infinite.

   and we say E is

   • *open* if every $p \in E$ is an interior point of E.

   • *closed* if E contains all limit points of E.

2.   • The *interior* of E, denoted $\mathring{E}$ or $\text{int}(E)$, is the set of its interior points.

   • The *closure* of E, denoted $\overline{E}$, is the union of E and its limit points.

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4. A set S is *countable* if there exists a surjection $f : \mathbb{N} \to S$.

5.   • An *open cover* U of $E \subseteq \mathbb{R}$ is a collection of open sets $\{O_\alpha\}_{\alpha \in I}$ such that such that $E \subseteq \bigcup_{\alpha \in I} O_\alpha$.

   • $K \subseteq \mathbb{R}$ is (covering) *compact* if every open cover of K admits a finite subcover.

   • $K \subseteq \mathbb{R}$ is *sequentially compact* if every sequence in K admits a converging subsequence in K.

5   Metric Spaces

1. A *metric space* (X, d) is a set X equipped with a *metric d*, which is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that for all $x, y, z \in X$,

   - $d(x, y) = 0 \Leftrightarrow x = y$
   - $d(x, y) = d(y, x)$ (Symmetry)
   - $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

2. - Convergence: $(\forall \varepsilon > 0)\, (\exists N \in \mathbb{N})\, (\forall n \geq N)\, (d(x_n, \ell) < \varepsilon)$.
    - Cauchy sequence: $(\forall \varepsilon > 0)\, (\exists N \in \mathbb{N})\, (\forall m, n \geq N)\, (d(x_n, x_m) < \varepsilon)$.
    - Open/Closed balls: $\mathcal{B}(x, r) = \{y : d(x, y) < r\}$, $\overline{\mathcal{B}}(x, r) = \{y : d(x, y) \leq r\}$.
    - Open set: $(\forall x \in E)\, (\exists r > 0)\, (\mathcal{B}(x, r) \subseteq E)$. Closed set: E^c is open.
    - Neighborhood of $x \in X$: Any open set containing x.
    - Diameter of E: $\mathrm{diam}(E) = \sup \{d(x, y) : x, y \in E\}$. Bounded set: $\mathrm{diam}(E) < \infty$.
    - Limit point of E: Any neighborhood of it intersects E infinitely much.
    - Isolated point of E: Exists some neighbourhood that intersects E at only itself.
    - Closure of E: $\overline{E} = E \cup \{\text{limit points of } E\}$.
    - Interior of E: $\mathring{E} = \{x \in E : \text{exists neighborhood of } x \text{ contained in } E\}$.
    - E is *dense* in F if $F \subseteq \overline{E}$. (Equivalently, all neighborhoods of all points in F must intersect E.)
    - $K \subseteq X$ is *compact* if every open cover of K admits a finite subcover.
    - $K \subseteq X$ is *totally bounded* if $(\forall \varepsilon > 0)\, (\exists x_1, \cdots, x_n)\, (K \subseteq \mathcal{B}(x_1, \varepsilon) \cup \cdots \cup \mathcal{B}(x_n, \varepsilon))$.
    - $K \subseteq X$ is *complete* if every Cauchy sequence converges.
    - $K \subseteq X$ is *separable* if it has a countable dense subset.

6 Continuous Functions

1. • Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say $f : X \to Y$ is *continuous at $x \in X$* if for every $x_n \to x$ we have $f(x_n) \to f(x)$.

   • $f : X \to Y$ is *continuous* if it is continuous at every $x \in X$.

2. $f : X \to Y$ is *uniformly continuous* if

$$(\forall \varepsilon > 0) \, (\exists \delta > 0) \, (\forall d_X(x, y) < \delta) \, (d_Y(f(x), f(y)) < \varepsilon).$$

   Remark: Here δ does not depend on x!

3. If X is compact, we define the *uniform metric* on $\mathcal{C}(X) = \{f : X \to \mathbb{R} \text{ continuous}\}$:

$$d(f, g) = \sup \{|f(x) - g(x)| : x \in X\}$$

4. Let $\{f_n : X \to \mathbb{R}\}_{n \in N}$ be a sequence of continuous functions.

   • We say f_n *converges pointwise* to f if $f_n(x) \to f(x)$ for all $x \in X$.

   • We say f_n *converges uniformly* to f if $\sup\limits_{x \in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

   This is equivalent to f_n converging in $(\mathcal{C}(X), d)$, so we can write $f_n \xrightarrow{d} f$.

5. • A set $K \subseteq \mathcal{C}(X)$ is *uniformly bounded* if there exists an $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $f \in K$ and $x \in X$.

   • A set $K \subseteq \mathcal{C}(X)$ is *(uniformly) equicontinuous* if

$$(\forall \varepsilon > 0) \, (\exists \delta > 0) \, (\forall f \in K, d_X(x, y) < \delta) \, (d_Y(f(x), f(y)) < \varepsilon).$$

7   Derivatives

1. • Let $f : I \to \mathbb{R}$ where $I \subseteq R$. Then we say $\lim\limits_{x \to x_0} f(x) = \ell$ if for all $\varepsilon > 0$, there
    exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in I$ with $0 < |x - x_0| < \delta$.

    • Let I be an open interval. We say that $f : I \to \mathbb{R}$ is *differentiable at x_0* if

    $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\delta \to 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta} \in \mathbb{R}$$

    exists, in which case we denote the limit by $f'(x_0)$, called the *derivative* at x_0.
    We say f is *differentiable* if f is differentiable at all points in I.

    • $\dfrac{f(x) - f(x_0)}{x - x_0}$ is called the *difference quotient* and represents the slope.

2. $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to have *directional derivative at $x_0 \in \Omega$ in direction $v \in \mathbb{R}^n$* if

    $$Df(x_0)[v] := \lim_{\delta \to 0} \frac{f(x_0 + \delta v) - f(x_0)}{\delta}$$

    exists. We say f is *differentiable at x_0* if $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.

3. • A function $f : I \to \mathbb{R}$ is *convex* if for all $x_1 < x_2$ in I and any $0 < \alpha < 1$,

    $$f(\alpha x_1 + (1 - \alpha)x_2) \le \alpha f(x_1) + (1 - \alpha)f(x_2).$$

    We say that f is *strictly convex* if the inequality is always strict.

    • A function $f : I \to \mathbb{R}$ is *concave* if for all $x_1 < x_2$ in I and any $0 < \alpha < 1$,

    $$f(\alpha x_1 + (1 - \alpha)x_2) \ge \alpha f(x_1) + (1 - \alpha)f(x_2).$$

    We say that f is *strictly concave* if the inequality is always strict.

    • Define the *right and left derivative*

    $$f'_+(x_0) = \lim_{\delta \to 0^+} \frac{f(x_0 + \delta) - f(x_0)}{\delta}, \qquad f'_-(x_0) = \lim_{\delta \to 0^-} \frac{f(x_0 + \delta) - f(x_0)}{\delta}$$

4. • A function $f : I \to \mathbb{R}$ is *in C^1* if it is differentiable and f' is continuous.

    • If $f'(x_0) = 0$, we say x_0 is a *critical point* and $f(x_0)$ is a *critical value*.

    • We say $y \in \mathbb{R}$ is a *regular value* if it is not a critical value.

    • A set $S \subseteq \mathbb{R}$ has *measure zero* if for all $\varepsilon > 0$ there exists countably many intervals
    that (i) covers S and (ii) have total combined length $< \varepsilon$.

8   Riemann Integral

1. - A *partition* of $[a, b]$ is a finite set of points $\sigma = \{a = x_0 < \cdots < x_N = b\}$.
    - The *size* $|\sigma|$ of σ is $\max_{1 \le i \le N} |x_i - x_{i-1}|$.
    - A partition σ' is a *refinement* of σ if $\sigma' \supseteq \sigma$.
    - Given a bounded $f : [a, b] \to \mathbb{R}$ and a partition σ of $[a, b]$,
        - The *upper (Riemann) sum* is $S(f, \sigma) = \sum_{i=1}^{N} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x)$.
        - The *lower (Riemann) sum* is $s(f, \sigma) = \sum_{i=1}^{N} (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$.
    - Given a bounded $f : [a, b] \to \mathbb{R}$,
        - The *upper (Riemann) integral* is $\mathcal{I}^+(f) = \inf_{\forall \sigma} S(f, \sigma)$.
        - The *lower (Riemann) integral* is $\mathcal{I}^-(f) = \sup_{\forall \sigma} s(f, \sigma)$.
    - A bounded $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* if $\mathcal{I}^-(f) = \mathcal{I}^+(f) := \int_a^b f(x) \, dx$. Denote by $\mathcal{R}(a, b)$ the set of all Riemann integrable functions on $[a, b]$.
    - Given $f : [a, b] \in \mathbb{R}$ and $I \subseteq [a, b]$ an interval, define $\operatorname*{osc}_I f = \sup_I f - \inf_I f$.

2. The *oscillation of f at point x* is $\operatorname{osc}(f, x) = \lim_{\delta \to 0^+} \operatorname*{osc}_{[x-\delta, x+\delta]} f \ge 0$

3. An *ordinary differential equation* (ODE) is a problem in the form

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0$$

   where $y(x)$ is a differentiable function from $\mathbb{R} \to \mathbb{R}^n$ to be solved.

4. - Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence and $c \in \mathbb{R}$. A *power series* is a series in x of the form

$$\sum_{k=0}^{\infty} a_k (x - c)^k.$$

    For each $x \in \mathbb{R}$ for which the series converges we get a function $f(x)$.
    - The *radius of convergence* of a power series $\sum_{k=0}^{\infty} a_k (x - c)^k$ is

$$R = \frac{1}{\limsup_{k \to \infty} |a_k|^{1/k}} \in \mathbb{R}_{\ge 0} \cup \{\infty\}.$$

5. - A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is *infinitely differentiable* ($f \in \mathcal{C}^{\infty}(I)$) if the n-th derivative $f^{(n)}$ exists for all $n \in \mathbb{N}$.

   - A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is *analytic* if there exists a power series that is equal to $f(x)$ for all $x \in I$.

   - Given a function $f \in \mathcal{C}^{\infty}$, the associated *Taylor series* of f at $c \in \mathbb{R}$ is

   $$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$$