Law of the Unconscious Statistician

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"This law is not a trivial result of definitions as it might at first appear, but rather must be proved." - Wikipedia

In this article I will prove the Law of the Unconscious Statistician (LOTUS).

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1 Background

A few days ago I was writing an article titled 'On Continuous Distributions', attempting to expand as much as possible the meaning of the distributions taught in A-level's Further Mathematics Syllabus. I tried to give a proof for every detail needed to derive the pdfs of the distributions, and one of them was the definition of expectation.

I noticed the textbook wrote, the definition of $E(X)$ is $\int_{-\infty}^{\infty} xf_X(x)\ dx$ whereas the definition of $E(g(X))$ for some function g is $\int_{-\infty}^{\infty} g(x)f_X(x)\ dx$. However, this got me wondering: According to the first definition, we can construct the pdf $f_{g(X)}(x)$ of $g(X)$ and then have $E(g(X)) = \int_{-\infty}^{\infty} xf_{g(X)}(x)\ dx$, but this is a different form from the second definition. Unless there is a clear reason why these two forms mean the same thing, I will not accept both definitions if they can potentially clash with each other.

Assume we take the second definition instead, then $E(X) = \int_{-\infty}^{\infty} xf_X(x)\ dx$ follows immediately, but with this we get that $E(g(X)) = \int_{-\infty}^{\infty} xf_{g(X)}(x)\ dx$ again by replacing X by $g(X)$.

Therefore, I feel there is a need to prove that the two expressions are equal. This situation is similar to how the scalar product is sometimes defined as $x_1y_1 + x_2y_2 + \cdots$, but is sometimes defined as $|X||Y|\cos\theta$. The difference is, the equivalence between these two statements is quite easy to prove.

After some attempts I couldn't manage to prove LOTUS completely because things get complicated when $g(X)$ is not bijective. Thus I went online and see if this was well-known, and after some digging, I still couldn't find anyone giving a complete proof. However, I did find out that this relation has name, which is the Law of the Unconscious Statistician. Wikipedia gives a so-called proof, but it assumes $g(x)$ is bijective and monotonic.

Therefore, I will try to take on the challenge to prove LOTUS completely in this article.

2 Preliminaries

1. The probability density function (pdf) $f_X(x)$ of a random variable X is the function satisfying

$$\mathcal{P}(a \leq x \leq b) = \int_a^b f_X(x)\ dx \qquad \text{for all } a \leq b.$$

2. At every differentiable point of $\mathcal{P}(X \leq x)$, the pdf satisfies

$$f_X(x) = \frac{d}{dx}\ \mathcal{P}(X \leq x).$$

3. The expectation of X is

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\ dx$$

4. The derivative of an invertible function $g(x)$ is

$$\frac{dg^{-1}}{dx} = \frac{1}{g'(g^{-1}(x))}$$

3 Law of the Unconscious Statistician

Given a (Riemann integrable) function $g : \mathbb{R} \to \mathbb{R}$, the expected value of $E(g(X))$, if it exists, is

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\ dx.$$

In other words,

$$\int_{-\infty}^{\infty} x f_{g(X)}(x)\ dx = \int_{-\infty}^{\infty} g(x) f_X(x)\ dx.$$

This theorem is incredibly useful because the second form of computing expectation is usually much easier to deal with, as we will see in the Example section.

4 The Proof on Wikipedia

Assume $g(x)$ has an inverse and is strictly increasing. Then

$$\begin{aligned}
E(g(X)) &= \int_{-\infty}^{\infty} x f_{g(X)}(x)\ dx \\
&= \int_{-\infty}^{\infty} x \left[\frac{d}{du}\mathcal{P}(g(X) \leq u)\right]_{u=x} dx \\
&= \int_{-\infty}^{\infty} x \left[\frac{d}{du}\mathcal{P}(X \leq g^{-1}(u))\right]_{u=x} dx \\
&= \int_{-\infty}^{\infty} x \left[\frac{d}{du}F(g^{-1}(u))\right]_{u=x} dx \\
&= \int_{-\infty}^{\infty} x \left[F'(g^{-1}(u)) \cdot \frac{1}{g'(g^{-1}(u))}\right]_{u=x} dx \\
&= \int_{-\infty}^{\infty} x \cdot f_X(g^{-1}(x)) \cdot \frac{1}{g'(g^{-1}(x))}\ dx
\end{aligned} \tag{1}$$

Applying the substitution $u = g^{-1}(x)$:

$$= \int_{-\infty}^{\infty} g(u) \cdot f_X(u)\ du.$$

This proof relies on the very wild assumption that $g(x)$ has an inverse and is strictly increasing. This allows (1) to be true. Without that assumption, we have to think further.

5 Full Proof

We will dissect $g(x)$ into sections where g is constant, strictly decreasing, or strictly increasing. Let $\cdots, I_{-1}, I_0, I_1, \cdots$ be disjoint open intervals such that $\sup I_i = \inf I_{i+1}$, and $g_i := g|_{I_i}$ is either constant or strictly monotone. Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} x f_{g(X)}(x)\ dx \\
&= \int_{-\infty}^{\infty} x \left[\frac{d}{du}\mathcal{P}(g_(X) \leq u)\right]_{u=x} dx \\
&= \int_{-\infty}^{\infty} x \left[\sum_{i=-\infty}^{\infty} \frac{d}{du}\mathcal{P}(g_i(X) \leq u)\right]_{u=x} dx
\end{aligned} \tag{2}$$

We will now analyse the value of $\frac{d}{du}\mathcal{P}(g_i(X) \leq u)$ depending on whether g_i is increasing, decreasing or constant.

If g_i is increasing and $u \in \mathrm{Im}(g_i)$,

$$\begin{aligned}\frac{d}{du}\mathcal{P}(g_i(X) \le u) &= \frac{d}{du}\mathcal{P}(X \le g_i^{-1}(u)) \\ &= f_X(g_i^{-1}(u)) \cdot (g_i^{-1})'(u) \\ &= \frac{f_X(g_i^{-1}(u))}{g_i'(g_i^{-1}(u))}.\end{aligned}$$

If g_i is decreasing and $u \in \mathrm{Im}(g_i)$,

$$\begin{aligned}\frac{d}{du}\mathcal{P}(g_i(X) \le u) &= \frac{d}{du}\mathcal{P}(X \ge g_i^{-1}(u)) \\ &= -f_X(g_i^{-1}(u)) \cdot (g_i^{-1})'(u) \\ &= -\frac{f_X(g_i^{-1}(u))}{g_i'(g_i^{-1}(u))}.\end{aligned}$$

If $u \notin \mathrm{Im}(g_i)$ or g_i is constant, the value is either 0 or 1, thus

$$\frac{d}{du}\mathcal{P}(g_i(X) \le u) = 0.$$

Denoting $I_i = (a_i, b_i)$ and $\mu_i = \begin{cases} 1 & \text{if } g_i \text{ is increasing;} \\ -1 & \text{if } g_i \text{ is decreasing;} \\ 0 & \text{if } g_i \text{ constant.} \end{cases}$, the expression in (2) is

$$\begin{aligned}&\int_{-\infty}^{\infty} x \left[\sum_{i=-\infty}^{\infty} \frac{d}{du}\mathcal{P}(g_i(X) \le u) \right]_{u=x} dx \\ &= \sum_{i=-\infty}^{\infty} \mu_i \int_{\min(g(a_i),g(b_i))}^{\max(g(a_i),g(b_i))} x \cdot \frac{f_X(g_i^{-1}(x))}{g_i'(g_i^{-1}(x))}\ dx\end{aligned}$$

Applying the substitution $v = g_i^{-1}(x)$ and noting how μ_i swaps the bounds, the above is equal to

$$\sum_{i=-\infty}^{\infty} \int_{a_i}^{b_i} g_i(v) \cdot f_X(v)\ dv = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\ dx,$$

as desired. $\square$

6 An Example

Assume $g(x) = x^2$ and $f_X(x) = \frac{1}{100}$ for $-20 \le x \le 80$ and $f_X(x) = 0$ otherwose. Then

$$f_{X^2}(x) = \frac{d}{dx}\mathcal{P}(X^2 \le x).$$

If $x \leq 0$, then $f_{X^2}(x) = 0$. If $x > 0$,

$$
\begin{aligned}
&\frac{d}{dx}\mathcal{P}(X^2 \leq x) \\
&= \frac{d}{dx}\mathcal{P}\left(-\sqrt{x} \leq X \leq \sqrt{x}\right) \\
&= \frac{d}{dx}\left(\mathcal{P}\left(X \leq \sqrt{x}\right) - \mathcal{P}\left(X \leq -\sqrt{x}\right)\right) \\
&= f_X(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} - f_X(-\sqrt{x}) \cdot \left(-\frac{1}{2\sqrt{x}}\right) \\
&= \begin{cases} \dfrac{1}{100\sqrt{x}} & \text{if } 0 < x \leq 400; \\ \dfrac{1}{200\sqrt{x}} & \text{if } 400 < x \leq 6400; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

We have two ways to compute $E(X^2)$. One,

$$
\begin{aligned}
E(X^2) &= \int_0^{400} x \cdot \frac{1}{100\sqrt{x}}\ dx + \int_{400}^{6400} x \cdot \frac{1}{200\sqrt{x}}\ dx \\
&= \frac{1}{150} \cdot 400^{3/2} + \frac{1}{300} \cdot 6400^{3/2} - \frac{1}{300} \cdot 400^{3/2} \\
&= \frac{5200}{3}.
\end{aligned}
$$

Two,

$$
\begin{aligned}
E(X^2) &= \int_{-20}^{80} x^2 \cdot \frac{1}{100}\ dx \\
&= \frac{1}{100}\left(\frac{80^3}{3} - \frac{(-20)^3}{3}\right) \\
&= \frac{5200}{3}.
\end{aligned}
$$

We see that the answers are consistent. This is exactly what we were looking for. $\square$