Tristan Chaang May 13, 2024

18.100B Theorems

1 Real Numbers

1. The set $\mathbb{R}$ of real numbers is the unique complete ordered field.
2. **(Existence of $\sqrt{2}$)**
   There exists $r \in \mathbb{R}$ with $r^2 = 2$.
3. **(Archimedean Property)**
   Let x, y be reals. Then
   A) $y > 0 \implies \exists\, n \in \mathbb{N}$ such that $ny > x$.
   B) $x < y \implies \exists\, q \in \mathbb{Q}$ such that $x < q < y$. ($\mathbb{Q}$ is dense in $\mathbb{R}$)
4. **(Principle of Induction)**
   For a property $P(n)$ ($n \in \mathbb{N}$), if $P(0)$ and $P(n) \implies P(n+1)$ ($n \in \mathbb{N}$) are true, then $P(n)$ is true for all $n \in \mathbb{N}$.

2 Sequences

1. **(Triangle Inequality)**
   $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.
2. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to both ℓ and ℓ', then $\ell = \ell'$.
3. If $\lim\limits_{n\to\infty} x_n = \ell$ and $\lim\limits_{n\to\infty} y_n = \ell'$, then
   - $\lim\limits_{n\to\infty} (x_n + y_n) = \ell + \ell'$
   - $\lim\limits_{n\to\infty} (x_n y_n) = \ell\ell'$
   - if $\ell \neq 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} (x_n + y_n) = 1/\ell$
4. **(Squeeze Theorem)**
   If $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = \ell$ and $x_n \leq z_n \leq y_n$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} z_n = \ell$.
5. **(Monotone Convergence Theorem)**
   If $\{x_n\}_{n\in\mathbb{N}}$ is nondecreasing and bounded above, then it converges. Similarly, if it is nonincreasing and bounded below, then it converges.
6. Every sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a monotone subsequence.
7. **(Bolzano-Weierstrass)**
   Every bounded sequence has a convergent subsequence.
8. In $\mathbb{R}$, a sequence converges if and only if it is Cauchy.
9. $\{x_n\}_{n\in\mathbb{N}}$ converges if and only if $\limsup x_n = \liminf x_n \in \mathbb{R}$.

3 Series

1. **(Comparison Test)**

   If $|a_k| \leq b_k$ for all $k \geq N_0$ and $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.

2. **(Alternating Series Test)**

   If $x_k \geq 0$ is non-increasing and $x_k \to 0$, then $\sum_{k=0}^{\infty} (-1)^k x_k$ converges.

3. **(Ratio Test)**

   If all $x_k \neq 0$ and $\lim_{n\to\infty} \left|\frac{x_{k+1}}{x_k}\right| < 1$, then $\sum_{k=0}^{\infty} x_k$ converges.

4. $e := \exp(1)$ is irrational.

5. $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ for all $x \in \mathbb{R}$.

6. **(Products of Series)**

   If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge absolutely, then $\sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell}\right) = \sum_{k=0}^{\infty} a_k \sum_{k=0}^{\infty} b_k$.

7. **(Dirichlet)**

   If $\sum_{k=0}^{\infty} x_k$ is absolutely convergent, it is unconditionally convergent.

8. **(Riemann)**

   If $\sum_{k=0}^{\infty} x_k$ converges but not absolutely, then for any $\ell \in \mathbb{R}$ or $\ell = \pm\infty$ there exists some rearrangement σ such that $\sum_{k=0}^{\infty} x_{\sigma(k)} = \ell$.

4 Topology of $\mathbb{R}$

1. $\mathbb{R}$ is not countable (*uncountable*).

2. Every open set of $\mathbb{R}$ is a countable union of disjoint open intervals.

3. Let $K \subseteq \mathbb{R}$. The following are equivalent:

   (a) K is compact.

   (b) K is sequentially compact.

   (c) K is closed and bounded.

4. **(Cantor's Intersection Theorem)**
   Let $\{K_n\}_{n\in N}$ be a sequence of nonempty compact sets in $\mathbb{R}$ such that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$. Then $K = \bigcap_{n\in\mathbb{N}}$ is compact and nonempty.

5 Metric Spaces

1. Let $K \subseteq \mathbb{R}$. The following are equivalent:
   (a) K is compact.
   (b) K is sequentially compact.
   (c) K is complete and totally bounded.

2. **(Baire)**
   Let (X, d) be a complete metric space and O_n is open and dense in X for all $n \in \mathbb{N}$. Then $O = \bigcup_{n\in\mathbb{N}} O_n$ is dense in X.

6 Continuous Functions

1. $f : X \to Y$ is continuous at x if and only if

$$(\forall\varepsilon > 0)\,(\exists\delta > 0)\,(\forall d_X(x, y) < \delta)\,(d_Y(f(x), f(y)) < \varepsilon)\,.$$

2. $f : X \to Y$ is continuous if and only if for all open sets U in Y, $f^{-1}(U)$ is open in X.

3. **(Banach Fixed Point Theorem)**
   Let (X, d) be complete and $f : X \to X$ be α-Lipschitz for some $0 < \alpha < 1$ (such functions are called *contractions*). Then f has a unique fixed point: $f(a) = a$.

4. If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.

5. **(Heine-Cantor)**
   If X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.

6. If X is compact, $f : X \to \mathbb{R}$ is continuous, then $f(X)$ has a maximum and minimum.

7. **(Intermediate Value Theorem)**
   If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < \mu < f(b)$, there exists $c \in [a, b]$ with $f(c) = \mu$.

8. $(\mathcal{C}(X), d)$ is complete.

9. **(Arzelà-Ascoli)**
   Let X be compact. $K \subseteq \mathcal{C}(X)$ is *relatively compact* (i.e. $\overline{K}$ is compact) if and only if it is uniformly bounded and uniformly equicontinuous.

7 Derivatives

1. If f is differentiable at x_0, then it is continuous at x_0.

2. **(Chain Rule)**
   If f, g are differentiable at x_0, then $f \circ g$ is differentiable at x_0, with
   $$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

3. If $f : [a, b] \to \mathbb{R}$ is differentiable, then the maximum of f occurs at either a, b or a point x_0 with $f'(x_0) = 0$. *Note:* Maximum exists since $[a, b]$ is compact.

4. **(Rolle's)**
   If $f : [a, b] \to \mathbb{R}$ is continuous, f is differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ with $f'(c) = 0$.

5. **(Mean Value Theorem)**
   If $f : [a, b] \to \mathbb{R}$ is continuous, f is differentiable on (a, b), then there exists $c \in (a, b)$ with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

6. **(L'Hôpital's Rule)**
   Let f, g be differentiable on I, and let $x_0 \in I$ such that $f(x_0) = g(x_0) = 0$, and $g'(x) = 0$ on some $\mathcal{B}(x_0, \varepsilon)$, and $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ exists.
   $$\text{Then} \qquad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

7. Say f is convex on I. Then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ for all $x < y$ in I.

8. If f is convex, f' exists except at countably many points.

9. **(Sard's Theorem)**
   Let $f : \mathbb{R} \to \mathbb{R}$ be in $\mathcal{C}^1$. Then $\{\text{critical values of } f\} \subseteq \mathbb{R}$ has measure zero.

10. Any regular value of $f : [a, b] \to \mathbb{R}$ in $\mathcal{C}^1$ has a finite pre-image.

8 Riemann Integral

1. The following are equivalent:

   - $f \in \mathcal{R}(a, b)$.
   - $(\forall \varepsilon > 0)\,(\exists \sigma)\,(S(f, \sigma) - s(f, \sigma) < \varepsilon)$.

- $(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall |\sigma| < \delta)\,(S(f,\sigma) - s(f,\sigma) < \varepsilon)$.
- $(\forall \varepsilon > 0)\,(\exists N > 0)\,(\forall n \geq N)\,(S(f,\sigma_n) - s(f,\sigma_n) < \varepsilon)$ where

$$\sigma_n = \left\{ a + \frac{k}{n}(b-a) : 0 \leq k \leq n \right\} \quad \text{(equipartition)}$$

- $(\exists \mathcal{I} \in \mathbb{R})\,(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall |\sigma| < \delta)\,(\forall \xi_i \in [x_{i-1}, x_i])$:

$$\left| \sum_{i=1}^{N} (x_i - x_{i-1}) f(\xi_i) - \mathcal{I} \right| < \varepsilon.$$

2. Continuous functions are Riemann integrable.

3. **(Fundamental Theorem of Calculus / FTC)**
If $f : [a,b] \to \mathbb{R}$ is continuous, then $F(x) = \int_a^x f$ is differentiable with $F' = f$.

4. **(Integral Form of FTC)**
If $F : [a,b] \to \mathbb{R}$ is in $\mathcal{C}^1$, then $\int_a^b F' = F(b) - F(a)$.

5. **(Integration by Parts)**
If $f, g : [a,b] \to \mathbb{R}$ are in $\mathcal{C}^1$, then $\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$.

6. **(Characterization of Riemann Integrability)**
$f \in \mathcal{R}(a,b)$ if and only if
   - f is bounded, and
   - The set of points of discontinuity of f has measure zero.

7. **(Picard-Lindelöf/Cauchy-Lipschitz)**
Let $D \subseteq \mathbb{R}^2$ be open and $(x_0, y_0) \in D$. Let $f : D \to \mathbb{R}$ be L-Lipschitz in the second variable (namely $|f(x,y_1) - f(x,y_2)| \leq L\,|y_1 - y_2|$). Then for some $\varepsilon > 0$ there exists a unique solution $y : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ to the ODE

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0.$$

8.
   - If $R = 0$, the series converges only at $x = c$.
   - If $R = \infty$, the series converges absolutely for all $x \in \mathbb{R}$.
   - If $0 < R < \infty$, the series converges absolutely for $|x - c| < R$ and does not converge for $|x - c| > R$.

9. Let $f(x) = \displaystyle\sum_{k=0}^{\infty} a_k x^k$ be a power series and let $|x| < R$. Then the partial sums $f_n = \displaystyle\sum_{k=0}^{n} a_k x^k$ converge uniformly to f on any compact interval $[a,b] \subseteq (-R, R)$.

10. Given $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-R, R)$ we have that f is differentiable on $(-R, R)$ with

$$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}.$$

11. Let $f \in \mathcal{C}^n((-R, R))$ for some $R > 0$ and $p_n(x)$ be its *n-th Taylor polynomial*

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

Then $\lim_{x \to 0} \dfrac{|f(x) - p_n(x)|}{|x|^n} = 0$. (We also write this as $f(x) = p_n(x) + o(x^n)$.)

12. **(Weierstrass Approximation)**
For all $f \in \mathcal{C}([a, b])$ there exists a sequence of polynomials p_n such that $p_n \to f$ uniformly. In other words, $\{\text{polynomials}\}$ is dense in $\mathcal{C}([a, b])$.