

## 18.100B Definitions

## 1 Real Numbers

1. A *field* is a set  $F$  equipped with operations  $+$  and  $\times$  such that
  - $(F, +)$  and  $(F \setminus \{0\}, \times)$  are Abelian groups
  - $x(y + z) = xy + xz$  for all  $x, y, z \in F$ . (Distributivity)
2. A field  $F$  is *ordered* if there exists a relation  $<$  on  $F$  (with  $x > y$  meaning  $y < x$ ,  $x \leq y$  meaning  $x < y$  or  $x = y$ , etc) such that for all  $x, y, z \in F$ ,
  - Exactly one of  $x = y$ ,  $x < y$ ,  $x > y$  holds. (Trichotomy)
  - $x < y$  and  $y < z$  implies  $x < z$ . (Transitivity)
  - $x < y$  implies  $x + z < y + z$ . (Additivity)
  - $x < y$  and  $z > 0$  implies  $xz < yz$ . (Multiplicativity)

We define  $P = \{x \in F : x > 0\}$ .

3. Let  $F$  be an ordered field.
  - $u \in F$  is an *upper bound* for a subset  $S \subseteq F$  if  $x \leq u$  for all  $x \in S$ . If an upper bound for  $S$  exists, we say  $S$  is *bounded above*.
  - $\ell \in F$  is a *lower bound* for a subset  $S \subseteq F$  if  $x \geq \ell$  for all  $x \in S$ . If an upper bound for  $S$  exists, we say  $S$  is *bounded below*.
  - If  $S \subseteq F$  is bounded above and below, we say that it is *bounded*.
  - $u \in F$  is the *maximum* of  $S$ , denoted  $\max S$ , if  $u$  is an upper bound and  $u \in S$ .
  - $\ell \in F$  is the *minimum* of  $S$ , denoted  $\min S$ , if  $\ell$  is a lower bound and  $\ell \in S$ .
  - $u \in F$  is the *supremum* of  $S$ , denoted  $\sup S$ , if it is the least upper bound for  $S$ . More precisely, we say that  $S$  has supremum

$$\sup S = \min\{x \in F : x \text{ is an upper bound for } S\} \quad \text{if it exists.}$$

- $\ell \in F$  is the *infimum* of  $S$ , denoted  $\inf S$ , if it is the greatest lower bound for  $S$ . More precisely, we say that  $S$  has infimum

$$\inf S = \max\{x \in F : x \text{ is a lower bound for } S\} \quad \text{if it exists.}$$

- By convention,  $\inf \emptyset = \infty$  and  $\sup \emptyset = -\infty$ . If  $S$  is unbounded above (below) we say  $\sup S = \infty$  ( $\inf S = -\infty$ ).
- We say that  $F$  is *complete* if it satisfies the *completeness axiom*: Every nonempty subset of  $F$  that is bounded above has a supremum.

## 2 Sequences

1. The absolute value function is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

2. A *sequence*  $\{x_n\}_{n \in \mathbb{N}} = \{x_0, x_1, \dots\}$  is an ordered list of real numbers. Explicitly, we have a function  $x : \mathbb{N} \rightarrow \mathbb{R}$  and we denoted  $x_n = x(n)$ .
3. Let  $\{x_n\}_{n \in \mathbb{N}}$  is said to *converge* to  $\ell \in \mathbb{R}$  if

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall n \geq N) (|x_n - \ell| < \varepsilon)$$

If this is true, we write  $\lim_{n \rightarrow \infty} x_n = \ell$ .

4.  $\{x_n\}_{n \in \mathbb{N}}$  is *bounded* if  $\exists M \in \mathbb{R}$  such that  $|x_n| < M$  for all  $n \in \mathbb{N}$ .
5.  $\{x_n\}_{n \in \mathbb{N}}$  is said to *diverge to*  $\infty$ , written as  $x_n \rightarrow \infty$ , if for all  $M \in \mathbb{R}$  there exists  $N \in \mathbb{N}$  such that  $x_n \geq M$  for all  $n \geq N$ . The case  $x_n \rightarrow -\infty$  is analogous.
6.  $\{x_n\}_{n \in \mathbb{N}}$  is *monotone* if it is either nonincreasing ( $x_n \geq x_{n+1}$  for all  $n \in \mathbb{N}$ ) or nondecreasing ( $x_n \leq x_{n+1}$  for all  $n \in \mathbb{N}$ ).
7. A *subsequence* of  $\{x_n\}_{n \in \mathbb{N}}$  is any ordered infinite subset. Precisely, it is some  $\{x_{n_j}\}_{j \in \mathbb{N}}$  where  $n_0 < n_1 < n_2 < \dots$  are natural numbers.
8. A sequence  $\{x_n\}_{n \in \mathbb{N}}$  is *Cauchy* if

$$(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall m, n \geq N) (|x_n - x_m| < \varepsilon)$$

9. The *limit superior* and *limit inferior* of  $\{x_n\}_{n \in \mathbb{N}}$  are defined by

$$\limsup x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right), \quad \liminf x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right)$$

### 3 Series

1. Given a sequence  $\{x_n\}_{n \in \mathbb{N}}$ , we define the series

$$\sum_{k=0}^n x_k = x_0 + x_1 + \cdots + x_n \quad \text{and} \quad \sum_{k=0}^{\infty} x_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n x_k \text{ if it converges.}$$

2. The series  $\sum_{k=0}^{\infty} a_k$  *converges absolutely* if  $\sum_{k=0}^{\infty} |a_k|$  converges.

3. The *exponential function* is defined as

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

4. A series  $\sum_{k=0}^{\infty} x_k$  is *unconditionally convergent* if any reordering of the  $x_k$  gives a series converging to the same number.

## 4 Topology of $\mathbb{R}$

1.
  - An *open interval* of  $\mathbb{R}$  is  $(a, b) = \{x \in \mathbb{R} : a < x < b\}$  for some  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ .
  - A *closed interval* of  $\mathbb{R}$  is  $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$  for some  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ .

For a given set  $E \subseteq \mathbb{R}$ , we say that  $p \in E$  is

- an *interior point* of  $E$  if there exists  $a < p < b$  such that  $(a, b) \subseteq E$ .
- an *isolated point* of  $E$  if there exists  $a < p < b$  such that  $(a, b) \subseteq E = \{p\}$ .
- a *boundary point* if for all  $a < p < b$ ,  $(a, b)$  intersects both  $E$  and  $E^c$ .
- a *limit point* (or accumulation point) if for all  $a < p < b$ ,  $(a, b) \cap E$  is infinite.

and we say  $E$  is

- *open* if every  $p \in E$  is an interior point of  $E$ .
  - *closed* if  $E$  contains all limit points of  $E$ .
2.
    - The *interior* of  $E$ , denoted  $\overset{\circ}{E}$  or  $\text{int}(E)$ , is the set of its interior points.
    - The *closure* of  $E$ , denoted  $\overline{E}$ , is the union of  $E$  and its limit points.
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  4. A set  $S$  is *countable* if there exists a surjection  $f : \mathbb{N} \rightarrow S$ .
  5.
    - An *open cover*  $U$  of  $E \subseteq \mathbb{R}$  is a collection of open sets  $\{O_\alpha\}_{\alpha \in I}$  such that such that  $E \subseteq \bigcup_{\alpha \in I} O_\alpha$ .
    - $K \subseteq \mathbb{R}$  is (covering) *compact* if every open cover of  $K$  admits a finite subcover.
    - $K \subseteq \mathbb{R}$  is *sequentially compact* if every sequence in  $K$  admits a converging subsequence in  $K$ .

## 5 Metric Spaces

1. A *metric space*  $(X, d)$  is a set  $X$  equipped with a *metric*  $d$ , which is a function  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $x, y, z \in X$ ,
  - $d(x, y) = 0 \Leftrightarrow x = y$
  - $d(x, y) = d(y, x)$  (Symmetry)
  - $d(x, z) \leq d(x, y) + d(y, z)$  (Triangle Inequality)
2.
  - Convergence:  $(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall n \geq N) (d(x_n, \ell) < \varepsilon)$ .
  - Cauchy sequence:  $(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall m, n \geq N) (d(x_n, x_m) < \varepsilon)$ .
  - Open/Closed balls:  $\mathcal{B}(x, r) = \{y : d(x, y) < r\}$ ,  $\overline{\mathcal{B}}(x, r) = \{y : d(x, y) \leq r\}$ .
  - Open set:  $(\forall x \in E) (\exists r > 0) (\mathcal{B}(x, r) \subseteq E)$ . Closed set:  $E^c$  is open.
  - Neighborhood of  $x \in X$ : Any open set containing  $x$ .
  - Diameter of  $E$ :  $\text{diam}(E) = \sup \{d(x, y) : x, y \in E\}$ . Bounded set:  $\text{diam}(E) < \infty$ .
  - Limit point of  $E$ : Any neighborhood of it intersects  $E$  infinitely much.
  - Isolated point of  $E$ : Exists some neighbourhood that intersects  $E$  at only itself.
  - Closure of  $E$ :  $\overline{E} = E \cup \{\text{limit points of } E\}$ .
  - Interior of  $E$ :  $\overset{\circ}{E} = \{x \in E : \text{exists neighborhood of } x \text{ contained in } E\}$ .
  - $E$  is *dense* in  $F$  if  $F \subseteq \overline{E}$ . (Equivalently, all neighborhoods of all points in  $F$  must intersect  $E$ .)
  - $K \subseteq X$  is *compact* if every open cover of  $K$  admits a finite subcover.
  - $K \subseteq X$  is *totally bounded* if  $(\forall \varepsilon > 0) (\exists x_1, \dots, x_n) (K \subseteq \mathcal{B}(x_1, \varepsilon) \cup \dots \cup \mathcal{B}(x_n, \varepsilon))$ .
  - $K \subseteq X$  is *complete* if every Cauchy sequence converges.
  - $K \subseteq X$  is *separable* if it has a countable dense subset.

## 6 Continuous Functions

1.
  - Let  $(X, d_X), (Y, d_Y)$  be metric spaces. We say  $f : X \rightarrow Y$  is *continuous at*  $x \in X$  if for every  $x_n \rightarrow x$  we have  $f(x_n) \rightarrow f(x)$ .
  - $f : X \rightarrow Y$  is *continuous* if it is continuous at every  $x \in X$ .
2.  $f : X \rightarrow Y$  is *uniformly continuous* if

$$(\forall \varepsilon > 0) (\exists \delta > 0) (\forall d_X(x, y) < \delta) (d_Y(f(x), f(y)) < \varepsilon).$$

Remark: Here  $\delta$  does not depend on  $x$ !

3. If  $X$  is compact, we define the *uniform metric* on  $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{R} \text{ continuous}\}$ :

$$d(f, g) = \sup \{|f(x) - g(x)| : x \in X\}$$

4. Let  $\{f_n : X \rightarrow \mathbb{R}\}_{n \in \mathbb{N}}$  be a sequence of continuous functions.

- We say  $f_n$  *converges pointwise* to  $f$  if  $f_n(x) \rightarrow f(x)$  for all  $x \in X$ .
- We say  $f_n$  *converges uniformly* to  $f$  if  $\sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

This is equivalent to  $f_n$  converging in  $(\mathcal{C}(X), d)$ , so we can write  $f_n \xrightarrow{d} f$ .

5.
  - A set  $K \subseteq \mathcal{C}(X)$  is *uniformly bounded* if there exists an  $M \in \mathbb{R}$  such that  $f(x) \leq M$  for all  $f \in K$  and  $x \in X$ .
  - A set  $K \subseteq \mathcal{C}(X)$  is *(uniformly) equicontinuous* if

$$(\forall \varepsilon > 0) (\exists \delta > 0) (\forall f \in K, d_X(x, y) < \delta) (d_Y(f(x), f(y)) < \varepsilon).$$

## 7 Derivatives

- Let  $f : I \rightarrow \mathbb{R}$  where  $I \subseteq \mathbb{R}$ . Then we say  $\lim_{x \rightarrow x_0} f(x) = \ell$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - \ell| < \varepsilon$  for all  $x \in I$  with  $0 < |x - x_0| < \delta$ .
  - Let  $I$  be an open interval. We say that  $f : I \rightarrow \mathbb{R}$  is *differentiable at  $x_0$*  if

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\delta \rightarrow 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta} \in \mathbb{R}$$

exists, in which case we denote the limit by  $f'(x_0)$ , called the *derivative* at  $x_0$ . We say  $f$  is *differentiable* if  $f$  is differentiable at all points in  $I$ .

- $\frac{f(x) - f(x_0)}{x - x_0}$  is called the *difference quotient* and represents the slope.
- $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to have *directional derivative at  $x_0 \in \Omega$  in direction  $v \in \mathbb{R}^n$*  if

$$Df(x_0)[v] := \lim_{\delta \rightarrow 0} \frac{f(x_0 + \delta v) - f(x_0)}{\delta}$$

exists. We say  $f$  is *differentiable at  $x_0$*  if  $Df(x_0) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a linear map.

- A function  $f : I \rightarrow \mathbb{R}$  is *convex* if for all  $x_1 < x_2$  in  $I$  and any  $0 < \alpha < 1$ ,

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2).$$

We say that  $f$  is *strictly convex* if the inequality is always strict.

- A function  $f : I \rightarrow \mathbb{R}$  is *concave* if for all  $x_1 < x_2$  in  $I$  and any  $0 < \alpha < 1$ ,

$$f(\alpha x_1 + (1 - \alpha)x_2) \geq \alpha f(x_1) + (1 - \alpha)f(x_2).$$

We say that  $f$  is *strictly concave* if the inequality is always strict.

- Define the *right and left derivative*

$$f'_+(x_0) = \lim_{\delta \rightarrow 0^+} \frac{f(x_0 + \delta) - f(x_0)}{\delta}, \quad f'_-(x_0) = \lim_{\delta \rightarrow 0^-} \frac{f(x_0 + \delta) - f(x_0)}{\delta}$$

- A function  $f : I \rightarrow \mathbb{R}$  is in  $C^1$  if it is differentiable and  $f'$  is continuous.
  - If  $f'(x_0) = 0$ , we say  $x_0$  is a *critical point* and  $f(x_0)$  is a *critical value*.
  - We say  $y \in \mathbb{R}$  is a *regular value* if it is not a critical value.
  - A set  $S \subseteq \mathbb{R}$  has *measure zero* if for all  $\varepsilon > 0$  there exists countably many intervals that (i) covers  $S$  and (ii) have total combined length  $< \varepsilon$ .

## 8 Riemann Integral

1.
  - A *partition* of  $[a, b]$  is a finite set of points  $\sigma = \{a = x_0 < \cdots < x_N = b\}$ .
  - The *size*  $|\sigma|$  of  $\sigma$  is  $\max_{1 \leq i \leq N} |x_i - x_{i-1}|$ .
  - A partition  $\sigma'$  is a *refinement* of  $\sigma$  if  $\sigma' \supseteq \sigma$ .
  - Given a bounded  $f : [a, b] \rightarrow \mathbb{R}$  and a partition  $\sigma$  of  $[a, b]$ ,
    - The *upper (Riemann) sum* is  $S(f, \sigma) = \sum_{i=1}^N (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x)$ .
    - The *lower (Riemann) sum* is  $s(f, \sigma) = \sum_{i=1}^N (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$ .
  - Given a bounded  $f : [a, b] \rightarrow \mathbb{R}$ ,
    - The *upper (Riemann) integral* is  $\mathcal{I}^+(f) = \inf_{\forall \sigma} S(f, \sigma)$ .
    - The *lower (Riemann) integral* is  $\mathcal{I}^-(f) = \sup_{\forall \sigma} s(f, \sigma)$ .
  - A bounded  $f : [a, b] \rightarrow \mathbb{R}$  is *Riemann integrable* if  $\mathcal{I}^-(f) = \mathcal{I}^+(f) := \int_a^b f(x) \, dx$ .  
Denote by  $\mathcal{R}(a, b)$  the set of all Riemann integrable functions on  $[a, b]$ .
  - Given  $f : [a, b] \rightarrow \mathbb{R}$  and  $I \subseteq [a, b]$  an interval, define  $\operatorname{osc}_I f = \sup_I f - \inf_I f$ .
  - The *oscillation of  $f$  at point  $x$*  is  $\operatorname{osc}(f, x) = \lim_{\delta \rightarrow 0^+} \operatorname{osc}_{[x-\delta, x+\delta]} f \geq 0$ .
  - An *ordinary differential equation* (ODE) is a problem in the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0$$

where  $y(x)$  is a differentiable function from  $\mathbb{R} \rightarrow \mathbb{R}^n$  to be solved.