

# 1 Topological Spaces

## Definition 1.1.

1. A **topology** on a set  $X$  is a set  $\mathcal{T}$  of subsets of  $X$  called **open sets** such that

- $\emptyset, X \in \mathcal{T}$
- $\mathcal{T}' \subseteq \mathcal{T} \implies \bigcup_{U \in \mathcal{T}'} U \in \mathcal{T}$ . (Preserved under arbitrary unions)
- $U_1, \dots, U_n \in \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$ . (Preserved under finite intersections)

$(X, \mathcal{T})$  – or just  $X$  when  $\mathcal{T}$  is understood – is a **(topological) space**.

2. Suppose  $\mathcal{T}, \mathcal{T}'$  are two topologies on  $X$  with  $\mathcal{T} \subseteq \mathcal{T}'$ . We say  $\mathcal{T}'$  is **finer** than  $\mathcal{T}$  and  $\mathcal{T}$  is **coarser** than  $\mathcal{T}'$ .

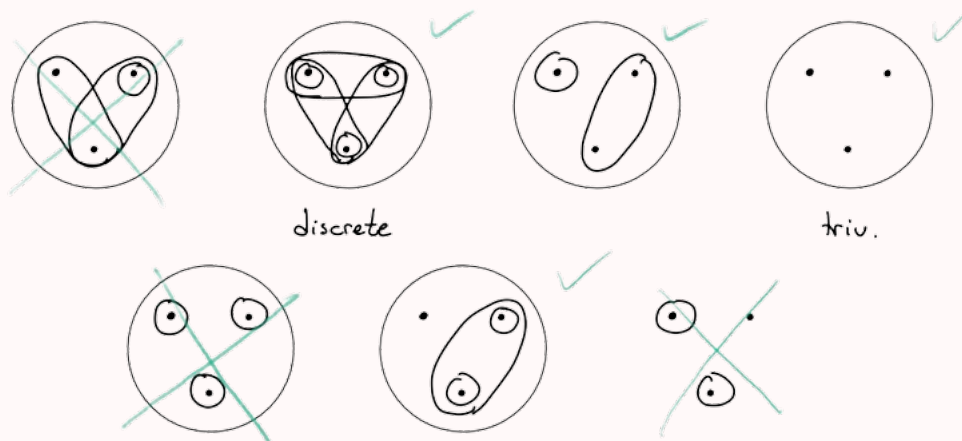
3.  $A \subseteq X$  is **closed** if  $X \setminus A$  is open. Hence  $\emptyset, X$  are closed, and closedness is preserved under finite unions and arbitrary intersections.

## Example 1.1.

1. The **discrete topology** on  $X$  is  $\mathcal{T} = \mathcal{P}(X)$ .

2. The **trivial topology** on  $X$  is  $\mathcal{T} = \{\emptyset, X\}$ .

3.  $X = \{1, 2, 3\}$ :



**Definition 1.2.** A set  $\mathcal{B}$  of subsets of  $X$  is a **basis** if

- $X = \bigcup_{B \in \mathcal{B}} B$
- $x \in B_1 \cap B_2$  with  $B_1, B_2 \in \mathcal{B} \implies (\exists B \in \mathcal{B}) (x \in B \subseteq B_1 \cap B_2)$

**Theorem 1.1.** A basis  $\mathcal{B}$  generates a topology  $\mathcal{T}$  via

$$U \in \mathcal{T} \iff (\forall x \in U) (\exists B \in \mathcal{B}) (x \in B \subseteq U).$$

*Proof.*  $\emptyset \in \mathcal{T}$  (vacuously) and  $X \in \mathcal{T}$  since  $\mathcal{B}$  covers  $X$ . We then verify the union and intersection properties:

- Suppose  $U_\alpha \subseteq X$  are open, then  $\bigcup_\alpha U_\alpha$  is open because

$$x \in \bigcup_\alpha U_\alpha \implies x \in U_\alpha \text{ for some } \alpha \implies x \in B_\alpha \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha$$

- Suppose  $U_1, U_2$  are open, then  $U_1 \cap U_2$  is open because

$$x \in U_1 \cap U_2 \implies \begin{cases} x \in B_1 \subseteq U_1 \text{ for some } B_1 \in \mathcal{B} \\ x \in B_2 \subseteq U_2 \text{ for some } B_2 \in \mathcal{B} \end{cases} \implies x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

for some  $B \in \mathcal{B}$ . By induction, any finite intersection of open sets is open. ■

**Example 1.2.** Let  $X = \mathbb{R}$ . We can construct three topologies via the bases:

1.  $\{(a, b) : a, b \in \mathbb{R}\}$  (the **standard topology** on  $\mathbb{R}$ )
2.  $\{[a, b) : a, b \in \mathbb{R}\}$
3.  $\{U \subseteq \mathbb{R} : U = \mathbb{R} \setminus \{x_1, \dots, x_n\} \text{ for some } x_1, \dots, x_n \in \mathbb{R}\}$

Note, (2) is finer than (1), and (1) is finer than (3).

**Remark.**

1. Infinite intersections may not be open. E.g.  $\bigcap_n (-1/n, 1/n) = \{0\}$  is not open in the standard topology on  $\mathbb{R}$ .
2. Different bases could generate the same topology. E.g. For  $X = \mathbb{R}^2$ , open balls generate the same topology as open squares do.

**Definition 1.3.** Let  $X$  be a space, and  $A \subseteq X$ .

1.  $\text{int}(A) = \bigcup \{U \subseteq A : U \text{ is open}\}$  is the **interior** of  $A$ .
2.  $\bar{A} = \bigcap \{C \supseteq A : C \text{ is closed}\}$  is the **closure** of  $A$ .
3.  $A$  is **dense** if  $\bar{A} = X$ .

**Example 1.3.**

1.  $\text{int}(A) = \bar{A} = A$  in the discrete topology.
2.  $\text{int}(A) = \emptyset; \bar{A} = X$  in the trivial topology for any  $A \neq \emptyset, X$ .
3.  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

**Warning.**  $A, B$  dense does not imply  $A \cap B$  dense, e.g. take  $\mathbb{Q}$  and  $\mathbb{Q} + \sqrt{2}$ .

**Theorem 1.2.**

1.  $A \text{ open} \Leftrightarrow A = \text{int}(A)$
2.  $A \text{ closed} \Leftrightarrow A = \bar{A}$

**Definition 1.4.**

1. A **neighborhood of  $x \in X$**  is an open set that contains  $x$ .
2.  $x \in X$  is a **limit point** of  $A$  if  $(\forall x \in U \in \mathcal{T}) (A \cap U \setminus \{x\} \neq \emptyset)$ .
3.  $x \in X$  is an **adherent point** of  $A$  if  $(\forall x \in U \in \mathcal{T}) (A \cap U \neq \emptyset)$ .
4. The **boundary** of  $A$  is  $\partial A = \{x \in X : x \text{ adh pt of } A \text{ and } X \setminus A\} = \bar{A} \cap \overline{X \setminus A}$ .

**Theorem 1.3.**

1.  $\bar{A} = \{\text{adherent pts of } A\} = A \cup \{\text{limit pts of } A\} = \text{int}(A) \sqcup \partial A$ .
2.  $X = \text{int}(A) \sqcup \partial A \sqcup \text{int}(X \setminus A)$ .

**Theorem 1.4.** If  $U_1, U_2$  are dense and open, then  $U_1 \cap U_2$  is dense and open.

*Proof.* Suppose  $x \in X$ . We want to show that for any  $U \in \mathcal{T}$  open we have  $U \cap (U_1 \cap U_2) \neq \emptyset$ .

Since  $U_1$  is dense,  $U \cap U_1 \neq \emptyset$ . Since  $U_2$  is also dense,  $U \cap U_1 \cap U_2 \neq \emptyset$ . ■

## 2 Metric Spaces

### Definition 2.1.

1. A **metric** on a set  $X$  is a function  $d : X^2 \rightarrow \mathbb{R}$  such that

- $d(x, y) \geq 0$  and equality holds if and only if  $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) + d(y, z) \geq d(x, z)$

The set  $B_x(\varepsilon) = \{y : d(x, y) < \varepsilon\}$  is the (open)  **$\varepsilon$ -ball centered at  $x$** .

2. The **metric topology** on  $(X, d)$  is the topology generated by the basis

$$\mathcal{B} = \{B_x(r) : x \in X, r > 0\}$$

**Example 2.1.** The **euclidean metric**  $d$  on  $\mathbb{R}^n$  is  $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i (x_i - y_i)^2}$ .

## 3 Subspace Spaces

**Definition 3.1.** Let  $(X, \mathcal{T})$  be a space and  $A \subseteq X$ . The **subspace topology** on  $A$  (with respect to  $X$ ) is

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}.$$

We call  $A$  with this topology a **subspace** of  $X$ .

**Theorem 3.1.** A basis  $\mathcal{B}$  for  $\mathcal{T}$  defines a basis  $\mathcal{B}_A$  for  $\mathcal{T}_A$  via

$$\mathcal{B}_A = \{A \cap B : B \in \mathcal{B}\}.$$

**Remark.** If  $(X, d)$  is a metric space and  $A \subseteq X$  then  $(A, d_A)$  is a metric space where  $d_A(a_1, a_2) = d(a_1, a_2)$ .

**Theorem 3.2.** Let  $(X, d)$  be a metric space. Then the metric topology on  $A \subseteq X$  agrees with the subspace topology of  $A \subseteq X$ .

*Proof.* The subspace topology on  $A$  has basis  $\mathcal{B}_S = \{A \cap B_x(r)\}_{x \in X}$  whereas the metric topology on  $A$  has basis  $\mathcal{B}_M = \{B_x^A(r)\} = \{A \cap B_x(r)\}_{x \in A} \subseteq \mathcal{B}_S$ . On the other hand, given any open  $U$  in the subspace topology and  $x \in U \subseteq A$ , we have  $x \in A \cap B_x(r) \subseteq U$  for some  $r > 0$ , but this is just  $x \in B_x^A(r) \subseteq U$ . Since  $x \in U$  was arbitrary,  $U$  is open in the metric topology too. ■

**Definition 3.2.**  $A \subseteq X$  (space) is discrete if its subspace topology is discrete.

**Example 3.1.** Is  $X = \{0\} \cup_n \{1/n\}$  discrete in  $\mathbb{R}$ ? No.  $\{0\}$  is not open in  $X$ . If it were, then  $\exists(a, b)$  such that  $(a, b) \cap X = \{0\}$ , but  $1/n < b$  for large  $n$ .

**Warning.**  $B = A = \mathbb{R} \times \{0\} \subseteq X = \mathbb{R}^2$  are examples for the following statements:

1.  $B$  open in  $A$  does not imply  $B$  open in  $X$ .
2. Suppose  $A \subseteq Y \subseteq X$ , then the  $\text{int}(A)$  in  $Y$  may not be  $Y \cap \text{int}(A)$ .

But these versions are true:

**Theorem 3.3.**

1.  $B$  open in  $A$ , and  $A$  open in  $X$ , then  $B$  open in  $X$ .
2. Suppose  $A \subseteq Y \subseteq X$ , the closure of  $A$  in  $Y$  is  $Y \cap (\text{closure of } A \text{ in } X)$ .

## 4 Product Spaces

**Definition 4.1.** Let  $\{X_\alpha\}_\alpha$  be a collection of spaces.

1. The **product topology** on  $X_1 \times \cdots \times X_n$  is generated by the basis

$$\mathcal{B} = \{Y_1 \times \cdots \times Y_n : Y_1, \dots, Y_n \text{ open}\}$$

2. More generally, the **product topology** on  $\prod_\alpha X_\alpha$  is generated by the basis

$$\mathcal{B} = \{\prod_\alpha Y_\alpha : Y_\alpha \text{ open for all } \alpha, \text{ and only finitely many } Y_\alpha \neq X_\alpha\}$$

**Theorem 4.1.**

1. If  $A \subseteq X; B \subseteq Y$  are subspaces, then the subspace topology and product topology on  $A \times B$  agree.
2. The metric topology on  $\mathbb{R}^n$  agrees with the product topology on  $\mathbb{R}^n$ .

## 5 Quotient Space

**Definition 5.1.**

- Let  $X$  be a space,  $Y$  be a set, and  $q : X \rightarrow Y$  be surjective. The **quotient topology** on  $Y$  induced by the **quotient map**  $q$  is given by

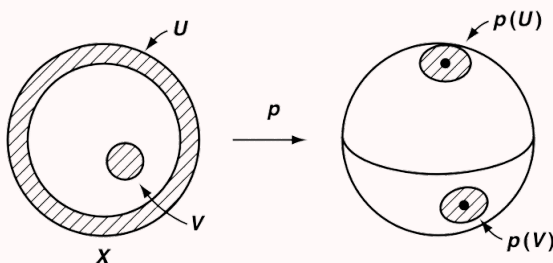
$$\mathcal{B} = \{U \subseteq Y : q^{-1}(U) \text{ open in } X\}$$

- Let  $A \subseteq X$  be a subset and define  $x \stackrel{A}{\sim} y \Leftrightarrow x = y \text{ or } x, y \in A$ . We denote  $X/A$  the space on  $X/\stackrel{A}{\sim}$  with quotient topology induced by the canonical map  $q : X \twoheadrightarrow X/\stackrel{A}{\sim}$ .

**Remark.** An equivalence relation  $\sim$  on  $X$  determines the surjective **canonical map**  $q : X \twoheadrightarrow X/\sim$  defined by  $q(x) = \text{equivalence class of } x$ .

**Example 5.1.**

1. Consider the unit 2-disk  $X = D^2 = \{x \times y : x^2 + y^2 \leq 1\}$ . If we identify together all points on the boundary  $\partial D^2$ , we get the quotient space  $D^2/\partial D^2$  that is homeomorphic with the subspace of  $\mathbb{R}^3$  called the unit 2-sphere  $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$ .



2. We can construct a *torus*  $S^1 \times S^1$  from the rectangle  $[0, 1] \times [0, 1]$ .
3. We can patch two disks  $D^2 \sqcup D^2$  along their boundaries to obtain  $S^2$ . Formally, given a homeomorphism  $\varphi : \partial D_1^2 \rightarrow D_2^2$ , we have  $(D_1^2 \sqcup D_2^2) / \sim = S^2$  where  $x \sim y \Leftrightarrow x = y$  or  $x \in \partial D_1^2, y \in \partial D_2^2, \varphi(x) = y$ .

## 6 Continuous Functions

**Definition 6.1.** Let  $X, Y$  be spaces. A function  $f : X \rightarrow Y$  is

- **continuous at  $x \in X$**  if  $f^{-1}(V)$  is open in  $X$  for all neighborhoods  $V$  of  $f(x)$ .
- **continuous** if  $f^{-1}(V)$  is open in  $X$  for all  $V$  open in  $Y$ .
- a **homeomorphism** if  $f$  is bijective, and  $f$  and  $f^{-1}$  are continuous.

**Theorem 6.1.**

1. Let  $\mathcal{B}$  be a basis of  $X$ . The map  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(B)$  is open for all  $B \in \mathcal{B}$ .
2. A composition of continuous functions is continuous.
3. Let  $A \subseteq X$  be a subspace and  $f : X \rightarrow Y$  be continuous. Then  $f|_A$  is continuous.
4. Let  $f : Z \rightarrow X \times Y$  where  $f = f_X \times f_Y$ . Then  $f$  is continuous if and only if  $f_X, f_Y$  are continuous.
5. Any quotient map is continuous. Given a quotient map  $q : X \rightarrow Y$ ,  $f : Y \rightarrow Z$  is continuous if and only if  $g = f \circ q$  is continuous.

$$\begin{array}{ccc}
 X & & \\
 q \downarrow & \searrow g & \\
 Y & \xrightarrow{\quad f \quad} & Z
 \end{array}$$

6. The following are equivalent to  $f : X \rightarrow Y$  being continuous:
  - (1)  $f^{-1}(C)$  is closed for all closed  $C \subseteq Y$ .
  - (2) Given any  $x \in X$  and  $f(x) \in V$  open, there exists open  $U$  with  $f(U) \subseteq V$ .
  - (3)  $f(\overline{A}) \subseteq \overline{f(A)}$  for all  $A \subseteq X$ .

*Proof of (6).*

- Continuity is equivalent to (1) by taking complements.
- For (2), say  $f$  is continuous, then  $U = f^{-1}(V)$  works. Conversely, say (2) is true. Then for any open  $V \subseteq Y$ , any  $v \in V$  admits a neighborhood within  $V$ , which has an open preimage  $U_v \subseteq X$ . Then  $f^{-1}(V) = \bigcup_{v \in V} U_v$  is open, and thus  $f$  is continuous.
- (1)  $\Rightarrow$  (3). Since  $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$  which is closed, we have  $\overline{A} \subseteq f^{-1}(\overline{f(A)})$  and thus  $f(\overline{A}) \subseteq \overline{f(A)}$ .
- (3)  $\Rightarrow$  (1). Let  $C \subseteq Y$  be closed. Then  $f(\overline{f^{-1}(C)}) = \overline{f(f^{-1}(C))} \subseteq \overline{C} = C$  and hence  $\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{f(f^{-1}(C))}) \subseteq f^{-1}(C)$  and thus  $f^{-1}(C)$  is closed.  $\blacksquare$

**Corollary 6.1.** Say  $X, Y$  are metric spaces.  $f : X \rightarrow Y$  is continuous if and only if

$$(\forall x \in X, \varepsilon > 0) (\exists \delta > 0) (\forall d_X(x, y) < \delta) (d_Y(f(x), f(y)) < \varepsilon).$$

**Theorem 6.2. (Pasting Lemma)** Let  $X = A \cup B$  be a space where  $A, B$  are closed. If  $f_A : A \rightarrow Y$  and  $f_B : B \rightarrow Y$  are continuous and  $f_A(x) = f_B(x)$  for all  $x \in A \cap B$ , then  $f : X \rightarrow Y$  defined by

$$f(x) = \begin{cases} f_A(x) & x \in A \\ f_B(x) & x \in B \end{cases}$$

is continuous.

## 7 Limits and Continuity

**Definition 7.1.**  $\{x_n\}_{n \in \mathbb{N}}$  in  $X$  **converges** to  $x \in X$  if any neighborhood of  $x$  contains all but finitely many  $x_n$ . Write  $x_n \rightarrow x$ .

**Warning.** Limits may not be unique:

1. In the trivial topology, any sequence converges to all points.
2. In  $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$  where  $x \sim y \Leftrightarrow x \in \mathbb{R}_1, y \in \mathbb{R}_2, x = y \neq 0$ , we have

$$1/n \rightarrow 0_1 \quad \text{and} \quad 1/n \rightarrow 0_2 \quad (\textbf{fat point})$$



**Theorem 7.1.** If  $x_n \rightarrow x$ , then  $x \in \overline{\{x_n\}_n}$ .

**Definition 7.2.** A space  $X$  is **first-countable** if for any  $x \in X$ , there exists a countable number of neighborhoods  $U_1, U_2, \dots$  such that any neighborhood of  $x$  contains some  $U_i$ . The  $\{U_i\}$  is called a **neighborhood basis** of  $x$ .

**Theorem 7.2.** If  $X$  is first-countable,

1.  $x \in \overline{A} \implies \exists x_1, x_2, \dots \in A$  such that  $x_n \rightarrow x$ .
2.  $f : X \rightarrow Y$  is continuous if and only if  $(x_n \rightarrow x) \implies (f(x_n) \rightarrow f(x))$ .

## 8 Connectedness

**Definition 8.1.** A space  $X$  is **connected** if there is no nontrivial clopen (closed and open) set  $A \subseteq X$ .

**Example 8.1.** The subspace  $(0, 1) \cup (2, 3)$  of  $\mathbb{R}$  is not connected.

**Theorem 8.1.**  $[a, b] \subseteq \mathbb{R}$  is connected.

*Proof.* Suppose the contrary, that  $[a, b] = A \sqcup B$  where  $A, B$  are closed and non-empty. WLOG Assume  $b \in B$ . Then  $s = \sup A < b$ . If  $s \in A$ , since  $A$  is also open, there exists  $(s - \varepsilon, s + \varepsilon) \subseteq A \implies \sup A \geq s + \varepsilon$ , a contradiction. Hence  $s \in B$  instead. Since  $B$  is open, there exists  $(s - \varepsilon, s + \varepsilon) \subseteq B$  and thus  $\sup A \leq s - \varepsilon$ , a contradiction. ■

**Definition 8.2.** A space  $X$  is **path-connected** if every pair  $x, y \in X$  can be joined by a *path* in  $X$ : a continuous map  $\gamma : I = [0, 1] \rightarrow X$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ .

**Example 8.2.**

1.  $\mathbb{R}^n$  is path-connected. Use the path  $\gamma(t) = t\mathbf{x} + (1 - t)\mathbf{y}$ .
2.  $S^n$  is path-connected. Use the path  $\gamma(t) = \frac{t\mathbf{x} + (1 - t)\mathbf{y}}{|t\mathbf{x} + (1 - t)\mathbf{y}|}$ .
3. A torus is path-connected: Start with a path in  $I^2$  and then take the quotient.

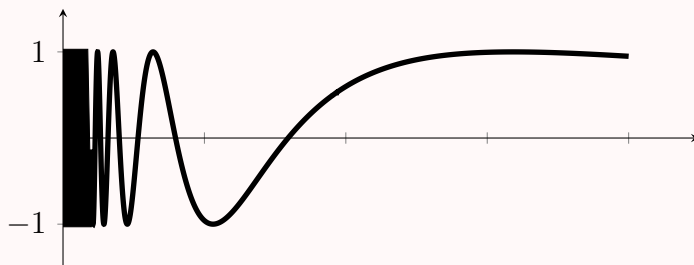
**Theorem 8.2.**

1. Any path-connected space is connected.
2. If  $f : X \rightarrow Y$  is continuous and surjective,
  - $X$  connected  $\implies Y$  connected.
  - $X$  path-connected  $\implies Y$  path-connected.
3. Quotients of a (path-)connected space is (path-)connected.
4. A product of (path-)connected spaces is (path-)connected.

**Example 8.3.** The *topologist's sine curve* defined by

$$X = \{(x \times \sin(1/x)) : x > 0\} \cup \{0\} \times [-1, 1]$$

is connected but not path-connected.



**Definition 8.3.** The equivalence relation  $x \sim y$  where there is a (path-)connected subspace containing both  $x, y$  partitions the space into (path-)connected **components**.

## 9 Compactness

**Definition 9.1.**

1. An **open cover** of  $X$  is a collection of open sets that cover  $X$ . A space  $X$  is **compact** if every open cover of  $X$  admits a finite subcover.
2. A space  $X$  is **sequentially compact** if every sequence of points in  $X$  admits a convergent subsequence.

**Theorem 9.1.** 1st-countable + compact  $\implies$  sequentially compact.

*Proof.* Suppose  $\{x_n\}_n$  does not have a convergent subsequence. Let  $x \in X$ , then there exists a countable neighborhood basis  $U_1, U_2, \dots$ . We can safely let  $U_1 \supseteq U_2 \supseteq \dots$  by taking successive intersections. Since there is no subsequence that converges to  $x$ , only finitely many  $x_n$  lie in  $U_n$  for some sufficiently large  $n$ . Hence, every  $x \in X$  has a neighborhood  $U_x$  that intersects  $\{x_n\}_n$  at a finite number of points. Taking the union of all  $U_x$  and applying compactness shows that  $\{x_n\}_n$  is finite, so we can conclude by the pigeonhole principle. ■

**Theorem 9.2.**

1. Every closed subspace of a compact space is compact.
2. A continuous function maps compact spaces to a compact image.
3. Suppose  $X$  is compact and  $C_1 \supseteq C_2 \supseteq \dots$  is a sequence of closed and non-empty sets. Then  $\bigcup_n C_n$  is non-empty.
4. A product of compact spaces is compact (Infinite case is hard: Tychonoff's Thm)
5.  $[a, b]$  is compact.

*Proof of (4).* Suppose  $[a, b] = \bigcup_\alpha U_\alpha$ . Then

$$S = \{x \in [a, b] : [a, x] \text{ can be covered by finitely many } U_\alpha\}$$

contains  $a \in S$  and is bounded above by  $b$ . Hence  $S$  has a supremum  $s$ .

**Claim.**  $s \in S$ .

*Proof.* Let  $s \in U_\beta$  for some  $\beta$ , so there exists  $(s - \varepsilon, s + \varepsilon) \subseteq U_\beta$ . If  $s \notin S$ , just add  $U_\beta$  to the finite subcover of  $[a, s - \varepsilon/2]$ . □

**Claim.**  $s = b$ .

*Proof.* If not, then similarly, just add  $U_\beta$  to the finite subcover of  $[a, s]$ . □

Therefore  $[a, b]$  can be covered by finitely many  $U_\alpha$ . ■

**Theorem 9.3. (Heine-Borel)**

A subspace  $A$  of  $\mathbb{R}^n$  is compact if and only if it is closed and bounded.

*Proof.*

- ( $\Leftarrow$ )  $X \subseteq [-M, M]^n$  is a closed subset of a compact space, so  $X$  is compact.

- ( $\Rightarrow$ ) Compactness on the open cover  $\{B_0(r)\}_{r>0}$  shows  $X$  is bounded. We then show any limit pt  $x$  of  $X$  is in  $X$ : For all  $n \in \mathbb{N}^*$ ,  $C_n := \overline{B_x 1/n} \cap X \neq \emptyset$ , and thus  $\bigcap_n C_n = X \cap \{x\}$  is non-empty. ■

## 10 Hausdorff Spaces

**Definition 10.1.** A space  $X$  is **Hausdorff** if for any distinct  $x, y \in X$  there exists disjoint neighborhoods  $x \in U, y \in V$ .

**Example 10.1.**

1. The trivial topology is not Hausdorff. The discrete topology is.
2. Metric spaces are Hausdorff.
3. The finite complement topology on  $\mathbb{R}$  is not Hausdorff.
4. The space  $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$  containing the fat point is not Hausdorff.

**Theorem 10.1.**  $X$  is Hausdorff if and only if  $\Delta = \{(x \times x) : x \in X\} \subseteq X^2$  is closed.

*Proof.*

- ( $\Rightarrow$ ) If  $X$  is Hausdorff, for any  $x \neq y$  there exists disjoint neighborhoods  $U, V$  of  $x, y$  respectively. Then  $U \times V$  is a neighborhood of  $(x \times y) \in X \times Y$  disjoint from  $\Delta$ . Taking the union over all  $(x \times y)$  implies  $\Delta$  is closed.
- ( $\Leftarrow$ ) If  $\Delta$  is closed, given any  $x \neq y$  there exists a basis neighborhood  $U \times V$  of  $(x \times y)$  disjoint from  $\Delta$ . Then  $U, V$  are the desired neighborhoods. ■

**Theorem 10.2.**

1. In a Hausdorff space, a sequence of points converge to at most one point.
2. One-point sets in a Hausdorff space are closed.
3. A subspace of a Hausdorff space is Hausdorff.
4. A finite product of Hausdorff spaces is Hausdorff.
5. A compact subspace of a Hausdorff space is closed.

**Warning.** A quotient of a Hausdorff space may not be Hausdorff.

## 11 Normal Spaces

### Definition 11.1.

1.  $X$  is  $\mathbf{T}_1$  if one-point sets are closed.
2. A space is **normal** if it is  $T_1$ , and, for any pair of disjoint closed sets  $A, B \subseteq X$  there exists disjoint open sets  $U, V \subseteq X$  such that  $A \subseteq U, B \subseteq V$ .

### Remark.

1. Normal  $\implies$  Hausdorff  $\implies T_1$ .
2. A quotient, subspace, or product of normal space(s) need not be normal.

### Example 11.1.

1. The fat point  $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$  is  $T_1$  but not Hausdorff.
2. The  **$K$ -topology** on  $\mathbb{R}$  generated by  $\{(a, b)\} \cup \{(a, b) \setminus \bigcup_n \{1/n\}\}$  is Hausdorff but not normal.
3. The topology  $\mathbb{R}_\ell$  on  $\mathbb{R}$  generated by  $\{[a, b)\}$  is normal, but  $\mathbb{R}_\ell^2$  is not normal.

### Theorem 11.1.

1. A closed subspace  $A$  of a normal space  $X$  is normal.
2. Compact + Hausdorff  $\implies$  Normal.

*Proof of (2).* Suppose  $A, B \subseteq X$  are disjoint and closed. Fix  $a \in A$ . Then for each  $b \in B$  there exists disjoint neighborhoods  $U_b, V_b$ . Since  $B$  is also compact, there exists finitely many  $V_b$  that cover  $B$ . The union of such finitely many  $V_b$  and the intersection of their corresponding  $U_b$  form disjoint open sets containing  $a$  and  $B$  respectively. Repeat the same procedure for every  $a \in A$  and then apply compactness of  $A$ . ■

### Theorem 11.2. Metric spaces are normal.

*Proof.* We can show that, for any subset  $A \subseteq X$ , the *point-to-set distance*  $d(-, A) : X \rightarrow \mathbb{R}$  given by  $d(x, A) = \inf_{a \in A} d(x, a)$  is continuous. For disjoint closed sets  $A, B$ , the open sets

$$U = \{x : d(x, A) < d(x, B)\}, \quad V = \{x : d(x, A) > d(x, B)\}$$

contain  $A, B$  respectively and are disjoint. ■

**Theorem 11.3.**  $X$  is normal if and only if for any closed  $A$  and open  $U$  such that  $A \subseteq U$ , there exists an open set  $V$  such that  $A \subseteq V \subseteq \overline{V} \subseteq U$ .

**Theorem 11.4. (Urysohn's Lemma)**

Let  $X$  be normal and  $A, B$  be disjoint closed sets of  $X$ . There exists a continuous map

$$f : X \rightarrow I$$

such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

*Proof.* Define open sets  $U_p$  for each  $p \in \mathbb{Q} \cap [0, 1]$  as follows: Enumerate  $\mathbb{Q} \cap [0, 1]$  such that 1 and 0 are the first two elements. Define  $U_1 = X - B$  and by normality pick  $U_0$  such that  $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$ . By induction, say we defined  $U_p$  for a finite number of  $p$ 's and let  $r$  be the next rational in the enumeration. We must have  $p < r < q$  where  $U_p, U_q$  are already defined. By normality we pick  $U_r$  such that  $\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$ .

Additionally, we let  $U_p = \emptyset$  for all rationals  $p < 0$  and  $U_p = X$  for all rationals  $p > 1$ . Hence,

$$p < q \Rightarrow \overline{U_p} \subseteq U_q.$$

We then define  $f(x) = \inf \{p : x \in U_p\}$ . It is easy to see  $f(A) = \{0\}$  and  $f(B) = \{1\}$ . We show that  $f$  is continuous.

**Lemma 1.**  $x \in \overline{U_r} \Rightarrow f(x) \leq r$

*Proof.* If  $x \in \overline{U_r}$ , then  $x \in U_s$  for every  $s > r$ . Hence  $f(x) \leq r$ . □

**Lemma 2.**  $x \notin \overline{U_r} \Rightarrow f(x) \geq r$ .

*Proof.* If  $x \notin \overline{U_r}$ , then  $x \notin U_s$  for any  $s < r$ . Hence  $f(x) \geq r$ . □

Given a ball  $I = (f(x) - \delta, f(x) + \delta)$ , we wish to find a neighborhood  $U$  of  $x$  such that  $f(U) \subseteq I$ . First we choose rational numbers  $p, q \in I$  such that  $p < f(x) < q$ . Then the open set  $U_q \setminus \overline{U_p}$  is the desired neighborhood using the lemmas above. ■

**Theorem 11.5. (Tietze Extension Theorem)**

Let  $A$  be closed in a normal space  $X$ . Any continuous map from  $A$  to  $I$  can be extended to a continuous map from  $X$  to  $I$ . True also for  $\mathbb{R}$  instead of  $I$ .

*Proof.* We show for  $[-1, 1]$  instead of  $I$ , and then for  $(-1, 1)$  instead of  $\mathbb{R}$ .

**Lemma.** If  $f : A \rightarrow [-\varepsilon, \varepsilon]$  is continuous, there exists continuous  $g : X \rightarrow \mathbb{R}$  with  $g(X) \subseteq [-\varepsilon/3, \varepsilon/3]$  and  $(g - f)(A) \subseteq [-2\varepsilon/3, 2\varepsilon/3]$ .

*Proof.* Applying the Urysohn Lemma on the disjoint closed sets  $L = f^{-1}([-\varepsilon, -\varepsilon/3])$  and  $R = f^{-1}([\varepsilon/3, \varepsilon])$ , there exists  $g : X \rightarrow [-\varepsilon/3, \varepsilon/3]$  such that  $g(L) = \{-\varepsilon/3\}$  and  $g(R) = \{\varepsilon/3\}$ . This  $g$  works.  $\square$

Now let  $f : A \rightarrow [-1, 1]$  be continuous. Then we can find  $g_1 : X \rightarrow [-1/3, 1/3]$  such that  $|f(a) - g_1(a)| \leq 2/3$  for all  $a \in A$ . Then we apply the Lemma on  $f - g_1$  again, so we get  $g_2 : X \rightarrow [-2/9, 2/9]$  such that  $|f(a) - g_1(a) - g_2(a)| \leq 4/9$ . Recursively, we get a sequence of functions  $g_n$  such that  $g_{n+1} : X \rightarrow [-(2/3)^n/3, (2/3)^n/3]$  and

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1}.$$

By the Weierstrass  $M$ -test,  $g(x) = \sum_{n=1}^{\infty} g_n(x)$  converges to the desired function (Exercise).

To show the  $(-1, 1)$  version, take  $g$  from the  $[-1, 1]$  case. Apply the Urysohn Lemma to the disjoint closed sets  $A$  and  $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$  to get a continuous  $\varphi : X \rightarrow [0, 1]$  so that  $\varphi(D) = \{0\}$  and  $\varphi(A) = \{1\}$ . Then  $h(x) = \varphi(x)g(x)$  works ( $|h(x)| < 1$ ).  $\blacksquare$

## Urysohn Metrization Theorem

### Definition 11.2.

1. A space is **second-countable** if it has a countable basis.
2. A space is **metrizable** if it is homeomorphic to a metric space.

### Theorem 11.6. (Urysohn Metrization Theorem)

2nd countable + Normal  $\implies$  Metrizable.

*Proof.* We first note that  $I^\omega = \{\mathbf{x} = (x_1, x_2, \dots) : x_i \in I\}$  with the metric

$$d(\mathbf{x}, \mathbf{y}) = \sup_n \frac{|x_n - y_n|}{n}.$$

is a metric space. Let  $X$  be normal with a countable basis  $\mathcal{B}$ . We will embed  $X$  into  $I^\omega$ .

**Lemma.** There exists a collection  $\{f_n : X \rightarrow I\}_{n \in \mathbb{N}}$  of continuous functions such that given any  $x \in X$  and any neighborhood  $U$ , there exists some  $f_n$  that is positive at  $x$  but vanishes outside  $U$ .

*Proof.* For each  $B, C \in \mathcal{B}$  with  $\overline{B} \subseteq C$ , apply the Urysohn Lemma to construct a continuous function  $g_{B,C} : X \rightarrow I$  such that  $g_{B,C}(\overline{B}) = \{1\}$  and  $g_{B,C}(X \setminus C) = \{0\}$ .  $\{g_{B,C} : \overline{B} \subseteq C\}$  is the desired collection. It is countable because  $\mathcal{B} \times \mathcal{B}$  is countable, and given any  $x$  with neighborhood  $U$ , we can choose by Theorem 11.3 the sequence of open sets  $x \in B \subseteq \overline{B} \subseteq C \subseteq U$ , and then use  $g_{B,C}$ .  $\square$

Using  $\{f_n\}_{n \in \mathbb{N}}$  from the Lemma, define  $F : X \rightarrow I^\omega$  such that

$$F(x) = (f_0(x), f_1(x), f_2(x), \dots)$$

- $F$  is injective because given  $x \neq y$ , there exists some  $f_n(x) > 0 = f_n(y)$  (Hausdorff!).
- $F$  is continuous: Let  $B_x(\varepsilon) \subseteq I^\omega$ . Fix an integer  $N > 2/\varepsilon$ . Since each  $f_n$  is continuous, for each  $1 \leq n \leq N$  there exists a neighborhood  $U_n$  such that  $y \in U_n \implies |f_n(x) - f_n(y)| \leq \varepsilon/2$ . Hence for any  $y \in U_1 \cap \dots \cap U_N$ ,

$$\begin{aligned} d(F(x), F(y)) &= \sup_n \frac{|f_n(x) - f_n(y)|}{n} \\ &\leq \max \left( \sup_{1 \leq n \leq N} \frac{|f_n(x) - f_n(y)|}{n}, \sup_{n > N} \frac{|f_n(x) - f_n(y)|}{n} \right) \\ &\leq \max \left( \frac{\varepsilon}{2}, \frac{1}{N+1} \right) < \varepsilon. \end{aligned}$$

- For each open set  $U$  in  $X$ ,  $F(U)$  is open in  $F(X)$ : Let  $x \in U$  and  $f(x) = z$ . Choose a  $f_N$  that is positive at  $x$  but vanishes outside  $U$ . Let

$$W = F(X) \cap \pi_N^{-1}((0, 1])$$

be open in  $F(X)$ . We claim that  $z \in W \subseteq F(U)$ . Firstly, we have  $z = F(x) \in W$  because  $f_N(x) > 0$ . Secondly, given any  $F(y) \in W$ , we must have  $f_N(y) > 0$ . Since  $f_N$  vanishes outside  $U$ ,  $y$  must be in  $U$ , so  $F(y) \in F(U)$ .

Therefore,  $X$  is homeomorphic to its image under  $F$ , a subspace of the metric space  $I^\omega$ , which is also a metric space.  $\blacksquare$

## 12 Manifolds

**Definition 12.1.** An ***n*-manifold** is a 2nd countable Hausdorff space  $X$  such that each  $x \in X$  has a neighborhood homeomorphic with an open subset of  $\mathbb{R}^n$ . We also write  $X = X^n$ . A 1-manifold is a ***curve***, and a 2-manifold is a ***surface***.



**Theorem 12.1.**  $X^n \times Y^m$  is an  $(n + m)$ -manifold.

*Proof.* Hausdorffness and 2nd Countability follow immediately. Fix  $(x \times y) \in X \times Y$ , then there exists neighborhoods  $U, V$  of  $x, y$  homeomorphic to  $\mathbb{R}^n, \mathbb{R}^m$  respectively. Then  $U \times V$  is a neighborhood of  $(x \times y)$  homeomorphic to  $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ . ■

**Example 12.1.**

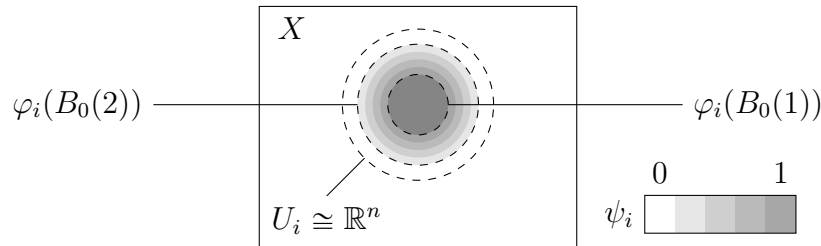
1.  $\mathbb{R}^n$  is an  $n$ -manifold.
2.  $S^n$  is an  $n$ -manifold. (Write  $S^n = e_1^n \cup e_2^n$  where  $e^n = \text{int}(D^n) \cong \mathbb{R}^n$ ).
3. The **real projective space**  $\mathbb{RP}^n = S^n / \sim$  (where  $x \sim y \Leftrightarrow x = \pm y$ ) is an  $n$ -manifold.
4.  $T^n = \underbrace{S^1 \times \cdots \times S^1}_n$  is an  $n$ -manifold.  $T^2$  is a **torus**.
5. *Fact:* Every connected curve is homeomorphic to either  $\mathbb{R}$  and  $S^1$ .

**Theorem 12.2.** A compact  $n$ -manifold  $X$  can be embedded in  $\mathbb{R}^N$  for some  $N \in \mathbb{N}$ .

*Proof.* Each  $x \in X$  admits a neighborhood  $U^x$  with a homeo  $\varphi^x : \mathbb{R}^n \rightarrow U^x$ . We can choose a basis  $x \in B^x \subseteq \varphi^x(B_0(1))$ , and hence by compactness of  $X$  via the  $B^x$  there exists  $U_1, \dots, U_m$  with homeos  $\varphi_i : \mathbb{R}^n \rightarrow U_i$  and  $X \subseteq \bigcup_i \varphi_i(B_0(1))$

By Urysohn's Lemma, there exists  $\rho_i : X \rightarrow I$  such that  $\rho_i(\overline{\varphi_i(B_0(1))}) = \{1\}$  and  $\rho_i(X \setminus \varphi_i(B_0(2))) = \{0\}$ . Via the pasting lemma, let  $\psi_i : X \rightarrow \mathbb{R}^n$  be the continuous function

$$\psi_i(x) = \begin{cases} \rho_i(x)\varphi_i^{-1}(x) & x \in U_i \\ (0, \dots, 0) & \text{otherwise} \end{cases}.$$



Then  $F(x) = (\rho_1(x), \dots, \rho_m(x), \psi_1(x), \dots, \psi_m(x))$  embeds  $X$  into  $\mathbb{R}^{m(n+1)}$ . ■

## 13 Paracompactness

### Definition 13.1.

- An open cover  $\{U_\alpha\}_\alpha$  of  $X$  is **locally finite** if every  $x \in X$  has a neighborhood that intersects only finitely many  $U_\alpha$ .
- A **refinement** of an open cover  $\{U_\alpha\}_\alpha$  of  $X$  is an open cover  $\{V_\beta\}_\beta$  such that each  $V_\beta$  is contained in some  $U_\alpha$  (depends on  $\beta$ ).
- A space  $X$  is **paracompact** if it is Hausdorff, and, every open cover of  $X$  admits a locally finite refinement.

### Warning.

1. Some sources do not require Hausdorffness in the definition.
2. Quotient/Subspace/Product of paracompact space(s) may not be paracompact.

**Example 13.1.**  $\mathbb{R}^n$  is paracompact. Let  $B(r)$  be the open ball of radius  $r$  centered at the origin. Given any open covering  $\mathcal{A}$ , for each  $n \in \mathbb{N}^*$  we can pick a finite number of elements of  $\mathcal{A}$  that covers  $\overline{B(n)}$ . Intersect them with  $\mathbb{R}^n \setminus \overline{B(n-1)}$ . The union of these open sets is a desired locally finite refinement.

### Theorem 13.1.

1. A closed subspace of a paracompact space is paracompact.
2. Compact + Hausdorff  $\implies$  Paracompact
3. Metric space  $\implies$  Paracompact.
4. Paracompact  $\implies$  Normal.

*Proof of (4).* Let  $A, B$  be closed and disjoint. We first prove the case when  $A = \{a\}$ . For each  $b \in B$  pick disjoint neighborhoods  $u \in U_b, v \in V_b$ . Since  $(X \setminus B) \cup_b V_b$  is an open cover of  $X$ , by paracompactness there exists a locally finite refinement of  $V_\alpha$ 's that cover  $B$ . Also,  $x$  has a neighborhood  $W$  that intersects only finitely many  $V_\alpha$ , say  $V_{b_1}, \dots, V_{b_n}$ . Then the open sets  $U = U_{b_1} \cap \dots \cap U_{b_n}$  and  $V = V_{b_1} \cap \dots \cap V_{b_n}$  form a desired pair.

For the general case, we update the notation so that for each  $a \in A$  there exists disjoint open sets  $u \in U_a, B \subseteq V_a$ . Let  $\{U_\alpha\}$  be a locally finite refinement that covers  $A$ , so  $b \in B$  admits a neighborhood  $W_b$  that intersects finitely many  $U_\alpha$ , say  $U_{a_1}, \dots, U_{a_n}$ . We then let

$V_b = W_b \cap_i V_{a_i}$ . Then  $U = \bigcup_{\alpha} U_{\alpha}$  and  $V = \bigcup_{b \in B} V_b$  give the desired separation. ■

**Definition 13.2.** A *partition of unity* on  $X$  for a locally finite open cover  $\{U_{\alpha}\}_{\alpha}$  is a collection of continuous  $\rho_{\alpha} : X \rightarrow I$  such that

- $\rho_{\alpha}(x) > 0 \implies x \in U_{\alpha}$
- $\sum_{\alpha} \rho_{\alpha}(x) = 1$  (well-defined due to local finiteness)

**Theorem 13.2.** Every cover of a paracompact space admits a refinement that has a partition of unity.

*Proof.* Let  $\{U_{\alpha}\}$  be a cover of  $X$ . For each  $x \in X$  there is an  $x \in U_{\alpha_x}$  and hence we can pick  $x \in W_x \subseteq \overline{W_x} \subseteq U_{\alpha_x}$  by normality. Let  $\{V_{\beta}\}$  be a locally finite refinement of  $\{W_x\}$ . By Urysohn's Lemma, there exists  $\psi_{\beta} : X \rightarrow I$  such that  $\psi_{\beta}(\overline{V_{\beta}}) = \{1\}$  and  $\psi_{\beta}(X \setminus U_{\alpha_{\beta}}) = \{0\}$ . Then  $\rho_{\beta}(x) = \psi_{\beta}(x) / \sum_{\gamma} \psi_{\gamma}(x)$  is a desired partition of unity. ■

**Theorem 13.3.** Manifold  $\implies$  Paracompact.

*Proof.* We first prove that a manifold  $X$  can be a limit of increasing compact sets.

**Lemma.**  $\exists K_1, K_2, \dots$  compact with  $K_n \subseteq \text{int}(K_{n+1})$  and  $X = \bigcup_n \text{int}(K_n)$ .

*Proof.* Let  $U_i$  with homeos  $\varphi_i : \mathbb{R}^n \rightarrow U_i$  such that  $\{\varphi_i(B_0(1))\}$  covers  $X$ . Then take the compact spaces  $K_n = \bigcup_{i=1}^n \bigcup_{j=1}^n \varphi_i(\overline{B_0(j)})$  for  $n \in \mathbb{N}^*$ . □

Let  $X = \bigcup_{\alpha} U_{\alpha}$ . Then for each  $n$  there exists  $U_1^n, \dots, U_{t_n}^n$  that cover the compact space  $K_n$ . Then  $V_j^n = U_j^n \setminus K_{n-1}$  form a locally finite refinement: Any  $x \in X$  is contained within some  $\text{int}(K_n)$ , which means it can only be in the sets  $V_j^m$  ( $1 \leq j \leq t_m$ ) ( $1 \leq m \leq n$ ). This is similar to Example 13.1. ■

## 14 Covering Dimension

### Definition 14.1.

1. The **covering dimension** of a space  $X$  is the infimum over  $n \in \mathbb{N}$  such that

$$(\forall \text{ open cover } \{U_\alpha\}) (\exists \text{ refinement } \{V_\beta\}) (\forall x \in X) (x \text{ is in } \leq n + 1 \text{ of the } V_\beta)$$

or equivalently

$$\dim X = \max_{\mathcal{A} \text{ open cover } X} \left[ \min_{\mathcal{B} \text{ refint of } \mathcal{A}} \underbrace{\left( \max_{x \in X} |\{B \in \mathcal{B} : x \in B\}| \right)}_{\text{order of } \mathcal{B}} \right] - 1$$

2. A **Lebesgue number** for an open cover  $\{U_\alpha\}$  of a compact metric space is a real  $\delta > 0$  such that any subset of  $X$  of diameter  $< \delta$  is contained within some  $U_\alpha$ .

### Theorem 14.1. (Lebesgue's Covering Lemma)

Any open cover  $\{U_\alpha\}$  of a compact metric space  $(X, d)$  has a Lebesgue number.

*Proof.* Since  $X$  is compact, assume  $\{U_\alpha\} = \{U_1, \dots, U_n\}$ . The map  $f(x) = \max_{1 \leq i \leq n} d(x, X \setminus U_i) > 0$  is continuous on a compact space and thus  $f(X)$  has a minimum  $\delta > 0$ . ■

### Example 14.1.

1. Any compact subspace of  $\mathbb{R}$  has dimension at most 1.

*Proof.* Note that  $\mathcal{C} = \{(n, n+1), (n - \frac{1}{2}, n + \frac{1}{2}) : n \in \mathbb{Z}\}$  has order 2. Let  $\mathcal{A}$  be any open covering of a compact subspace  $X$  of  $\mathbb{R}$ , with some Lebesgue number  $\delta > 0$ . The image  $\mathcal{J}$  of  $\mathcal{C}$  under  $f : x \mapsto \delta x/2$  is an open covering whose elements have diameter  $\delta/2 < \delta$ , and hence is an open refinement subcover of  $\mathcal{A}$ . Hence

$$\begin{aligned} \dim X &= \max_{\mathcal{A} \text{ open cover } X} \left[ \min_{\substack{\mathcal{B} \text{ open refinement} \\ \text{subcover of } \mathcal{A}}} (\text{order of } \mathcal{B}) \right] - 1 \\ &\leq \max_{\mathcal{A} \text{ open cover } X} [2] - 1 = 1. \end{aligned}$$

2.  $\dim I = 1$ .

*Proof.* We show that there is some open covering  $\mathcal{A}$  such that any open refinement subcover of  $\mathcal{A}$  has order at least 2. Let  $\mathcal{A} = \{[0, 1), (0, 1]\}$  and let  $\mathcal{B}$  be any open refinement subcovering. Since 0 and 1 cannot belong to the same refinement,  $\mathcal{B}$  has at least two elements. Partition  $\mathcal{B}$  into two nonempty parts  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . If  $\mathcal{B}$  had order 1 then  $\bigcup \mathcal{B}_1$  and  $\bigcup \mathcal{B}_2$  disconnect  $[0, 1]$ , a contradiction.

3. *Fact:*  $\dim I^n = n$ , and every compact subspace of  $\mathbb{R}^n$  has dimension  $\leq n$ .

### Theorem 14.2.

- If  $Y$  is a closed subspace of a finite dimensional space  $X$ , then  $\dim Y \leq \dim X$ .
- If  $X = Y \cup Z$  where  $Y, Z$  are closed finite dimensional subspaces of  $X$ , then  $\dim X = \max(\dim Y, \dim Z)$ .
- Every compact subspace of  $\mathbb{R}^N$  has dimension at most  $N$ .

## Tangent: Baire's Theorem, Function Spaces and Geometry

**Definition 14.2.** Let  $X$  be a compact metric space.

1.  $\mathcal{C}(X, \mathbb{R}^n) = \{f : X \rightarrow \mathbb{R}^n \text{ cts}\}$  is the metric space equipped with the uniform metric  $d(f, g) = \sup_x |f(x) - g(x)|$ .
2. For  $A \subseteq X$ ,  $\text{diam}(A) = \sup_{x, y \in A} d(x, y)$ .
3.  $\Delta(f) = \sup \{\text{diam}(f^{-1}\{z\}) : z \in f(X)\}$  (Deviation of  $f$  from injectivity).

**Remark.**  $\bigcap_n U_{1/n} = \{f : \Delta(f) = 0\} = \{f \text{ injective}\}.$

### Theorem 14.3. (Baire's Theorem)

Let  $\{U_n\}$  be a countable collection of dense open sets in a compact Hausdorff space  $X$ . Then  $\bigcap_n U_n$  is dense in  $X$ .

*Proof.* Let  $W_1$  be an open set. We want to show  $W_1 \cap_n U_n \neq \emptyset$ .

- Since  $U_1$  is dense and open, there exists  $x_1 \in W_1 \cap U_1$  open.
- Inductively, since  $X$  is normal, there exists  $x_n \in W_n \subseteq \overline{W_n} \subseteq W_{n-1} \cap U_{n-1}$ .

Since  $X$  is compact and  $\overline{W_1} \supseteq \overline{W_2} \supseteq \cdots$ , we have

$$\emptyset \neq \bigcap_n \overline{W_n} \subseteq \bigcap_n (U_n \cap W_n) \subseteq W \cap_n U_n. \quad \blacksquare$$

### Definition 14.3.

1.  $\{z_0, \dots, z_m\} \subseteq \mathbb{R}^n$  are **geometrically independent** if

$$\lambda_0 z_0 + \cdots + \lambda_m z_m = \mathbf{0}, \quad \lambda_0 + \cdots + \lambda_m = 0 \implies \lambda_0 = \cdots = \lambda_m = 0$$

2.  $A \subseteq \mathbb{R}^n$  is in **general position** if any subset of size  $n + 1$  are geom. ind.

**Theorem 14.4.** Given  $\{z_0, \dots, z_m\} \subseteq \mathbb{R}^n$  and  $\delta > 0$ , there exists  $\{y_0, \dots, y_m\} \subseteq \mathbb{R}^n$  that is in general position such that all  $|z_i - y_i| < \delta$ .

## Back to dimension theory

### Theorem 14.5. (Embedding Compact Metric Spaces)

Every compact metric space  $X$  of dimension  $n$  can be embedded in  $\mathbb{R}^{2n+1}$ .

Define  $U_\varepsilon = \{f \in \mathcal{C}(X, \mathbb{R}^{2n+1}) : \Delta(f) < \varepsilon\}$ .

**Claim.**  $U_\varepsilon$  is open.

*Proof.* Let  $f \in U_\varepsilon$ , we want to show  $\exists B_f(\delta) \subseteq U_\varepsilon$ . Pick  $\varepsilon < b < \Delta(f)$  and define

$$A = \{(x \times y) : d(x, y) \geq b\} \subseteq X^2$$

Note that  $f(x) = f(y) \implies d(x, y) \leq \Delta(f) < b \implies (x \times y) \notin A$ . Hence  $|f(x) - f(y)|$  has a positive minimum  $2\delta$  on  $A$ . Now if  $g \in B_f(\delta)$ , then for any  $(x \times y) \in A$ ,

$$|f(x) - g(x)| < \delta, \quad |f(y) - g(y)| < \delta, \quad |f(x) - f(y)| \geq 2\delta$$

so  $g(x) \neq g(y)$ . In other words,  $g(x) = g(y) \implies d(x, y) < b \implies \Delta g \leq b < \varepsilon$ .  $\square$

**Claim.**  $U_\varepsilon$  is dense. (Difficult!)

*Proof.* Let  $f \in \mathcal{C}(X, \mathbb{R}^{2n+1})$  and  $\delta > 0$ , we want to find a  $g \in B_f(\delta) \cap U_\varepsilon$ . Firstly, we cover  $X$  with  $V_1, \dots, V_m$  such that

- (1)  $\text{diam}(V_i) < \varepsilon/2$
- (2)  $\text{diam}(f(V_i)) < \delta/2$
- (3) Each  $x \in X$  is in at most  $n+1$  of the  $V_i$ .

To do this, pick a Lebesgue number  $0 < \kappa < \varepsilon/4$  such that any  $B_x(\kappa) \subseteq f^{-1}(B_y(\delta/4))$  for some  $y$ . Since  $\dim X \leq n$ , there exists a refinement  $\{V_\beta\}_\beta$  of  $\{B_x(\kappa)\}_x$  such that (3) holds. Since  $V_\beta \subseteq B_{x(\beta)}(\kappa)$  for some  $x(\beta)$ , (1) and (2) also hold. By compactness, we can find a finite cover using  $V_i$ .

Let  $\varphi_i : X \rightarrow \mathbb{R}$  be a partition of unity associated to the  $U_i$ . Also, fix  $x_i \in U_i$  and  $z_i \in \mathbb{R}^{2n+1}$  such that  $|f(x_i) - z_i| < \delta/2$  and  $\{z_i\}$  is in general position. Define

$$g(x) = \sum_i \varphi_i(x) z_i.$$

Then  $d(f, g) < \delta$  because

$$|g(x) - f(x)| = \left| \sum_i \varphi_i(x)(z_i - f(x_i)) + \sum_i \varphi_i(x)(f(x_i) - f(x)) \right| < \sum_i \varphi_i(x) \left( \frac{\delta}{2} + \frac{\delta}{2} \right) = \delta.$$

and  $g \in U_\varepsilon$  because  $g(x) = g(y) \implies \sum_i (\varphi_i(x) - \varphi_i(y)) z_i = \mathbf{0} \implies \varphi_i(x) = \varphi_i(y) \forall i$  since  $x, y$  are in  $\leq 2(n+1)$  of the  $U_i$ . Since  $\varphi_i(x) > 0$  for some  $i$ , we have  $x, y \in U_i \implies d(x, y) < \varepsilon/2$ . Therefore  $\Delta(g) \leq \varepsilon/2 < \varepsilon$ .  $\square$

By Baire's theorem,  $\bigcap_n U_{1/n}$  is dense and hence non-empty, i.e. there is a continuous injective  $f : X \rightarrow \mathbb{R}^{2n+1}$ . Also since  $X$  is compact and  $f(X)$  is Hausdorff,  $f$  sends closed sets to closed sets (i.e. is closed). Hence  $f$  embeds  $X$  into  $\mathbb{R}^{2n+1}$ .  $\blacksquare$

#### **Theorem 14.6. (Embedding Manifolds)**

Every manifold can be embedded in some  $\mathbb{R}^N$ .

*Proof.* Let  $X$  be an  $m$ -manifold.

**Lemma 1.** Let  $f : X \rightarrow \mathbb{R}^N$  such that  $f^{-1}(\text{compact}) = \text{compact}$ . Then  $f$  is closed (sends closed sets to closed sets).

*Proof.* Let  $C \subseteq X$  be closed. Suppose  $y \in \mathbb{R}^N \setminus f(C)$ . By Heine-Borel,  $\overline{B_y(\varepsilon)}$  is compact and hence  $K = C \cap f^{-1}(\overline{B_y(\varepsilon)})$  is compact  $\implies f(K) \subseteq f(C)$  is compact  $\implies V = B_y(\varepsilon) \setminus f(K)$  is a neighborhood of  $y$ . Note that

$$\begin{aligned} z \in V \cap f(C) &\implies \exists x \in f^{-1}(B_y(\varepsilon)) \cap C \subseteq K \text{ with } f(x) = z \\ &\implies z \in f(K) \implies V \cap f(C) = \emptyset \end{aligned}$$

and thus  $f(C)$  is closed.  $\square$

**Lemma 2.** There exists continuous  $f : X \rightarrow \mathbb{R}$  such that  $f^{-1}(\text{compact}) = \text{compact}$ .

*Proof.* Using the Lemma from Theorem 13.3, we can write  $X$  as a limit of increasing compact sets  $\bigcup_n K_n$  where  $K_n \subseteq \text{int}(K_{n+1})$ . Since manifold  $\implies$  paracompact  $\implies$  normal, we can use Urysohn's Lemma to construct continuous maps  $\varphi_n : X \rightarrow I$  such that  $\varphi_n(K_n) \equiv 0$  and  $\varphi_n(\overline{X \setminus K_{n+1}}) \equiv 1$ . Then we define  $f : X \rightarrow \mathbb{R}$  by  $f = \sum_{n=1}^{\infty} \varphi_n$ .

- $x \in K_n \implies \varphi_n(x) = \varphi_{n+1}(x) = \dots = 0$  and hence  $f$  is well-defined.
- $x \notin K_n \implies \varphi_{n-1}(x) = \varphi_{n-2}(x) = \dots = 1 \implies f(x) \geq n - 1$ .
- $f$  is continuous: Given any  $(a, b) \subseteq \mathbb{R}$ ,  $f^{-1}((a, b)) \subseteq K_{[b+2]}$  and hence  $f^{-1}((a, b))$  is the preimage of  $(a, b)$  under  $\sum_{n=1}^{[b+1]} \varphi_n$  (a continuous map) which is open.
- $f^{-1}(C)$  is compact for any compact  $C \subseteq \mathbb{R}$ : Since  $C$  is closed and bounded,  $f^{-1}(C)$  is closed and contained within some  $K_N$  (compact), and hence  $f^{-1}(C)$  is compact (closed subspace of a compact space).  $\square$

Take  $K_n$  and  $f$  from Lemma 2, and denote  $R_n = K_n \setminus \text{int}(K_{n-1})$  and  $U_n = \text{int}(K_{n+1}) \setminus K_{n-2}$ . By Urysohn's Lemma again, construct  $\rho_n : X \rightarrow \mathbb{R}$  with  $\rho_n(R_n) \equiv 1, \rho_n(X \setminus U_n) \equiv 0$ .

Since  $D_n = K_{n+1} \setminus \text{int}(K_{n-2})$  is compact and metrizable (normal and 2nd countable), there exists a cts closed inj  $f_n : D_n \hookrightarrow \mathbb{R}^{2m+1}$ . Then define  $\psi_n : X \rightarrow \mathbb{R}^{2m+1}, \psi : X \rightarrow \mathbb{R}^{4m+3}$  as

$$\psi_n(x) = \begin{cases} \rho_n(x)f_n(x) & x \in U_n \\ \mathbf{0} & \text{otherwise} \end{cases} \quad \psi(x) = \left( \sum_{\text{even } n} \psi_n(x), \sum_{\text{odd } n} \psi_n(x), f(x) \right).$$

$\psi$  is injective (Exercise:  $f(x) = f(y) \implies x, y \in R_\ell$ , and  $\sum_{i=2\ell} \psi_i(x) = \psi_\ell(x) = f_\ell(x) = f_\ell(y) \implies x = y$ ) and closed (for any compact  $K \subseteq \mathbb{R}^N$ ,  $\psi^{-1}(K)$  is closed and contained within the compact  $f^{-1}(\pi_N(K))$ ). Thus  $\psi$  embeds  $X$  into  $\mathbb{R}^{4m+3}$ .  $\blacksquare$



## 15 Homotopies

From now on, assume all ‘maps’ are continuous.

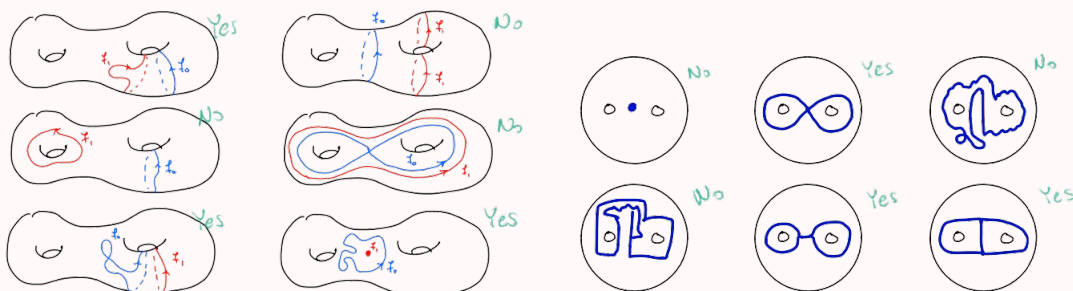
### Definition 15.1.

1. Given  $f_0, f_1 : X \rightarrow Y$ , a **homotopy** from  $f_0$  to  $f_1$  is  $H : X \times I \rightarrow Y$  such that  $f_0(x) = H(x, 0)$ ,  $f_1(x) = H(x, 1)$ . We sometimes write  $H(x, t) = f_t(x)$ . If such homotopy exists, we say  $f_0, f_1$  are **homotopic** ( $f_0 \simeq f_1$ ).
2. A **homotopy relative to  $A \subseteq X$**  (homotopy rel  $A$ ) is a homotopy  $H : X \times I \rightarrow Y$  such that  $H(a, t) = H(a, 0)$  for all  $a \in A$ .
3. A **reparameterization** of  $\alpha : I \rightarrow X$  is a map  $\beta : I \rightarrow X$  such that  $\beta = \alpha \circ r$  where  $r : I \rightarrow I$  satisfies  $r(0) = 0, r(1) = 1$ .
4.  $X, Y$  are **homotopy equivalent** ( $X \simeq Y$ ) if there exists  $f : X \rightarrow Y, g : Y \rightarrow X$  (called homotopy equivalences) such that  $f \circ g \simeq \mathbf{1}_Y$  and  $g \circ f \simeq \mathbf{1}_X$ .
5.  $X$  is **contractible** if  $X \simeq \text{point}$ .  $f : X \rightarrow Y$  is **nullhomotopic** if  $f \simeq \text{constant}$ .
6. A **retraction** of  $X$  onto  $A \subseteq X$  is a map  $r : X \rightarrow X$  with  $r|_A = \mathbf{1}_A, r(X) = A$ . If it exists,  $A$  is a **retract** of  $X$ .
7. A **deformation retraction** of  $X$  onto  $A \subseteq X$  is a homotopy rel  $A$  from the identity on  $X$  to a retraction of  $X$  onto  $A$ . If it exists,  $A$  is a **deformation retract** of  $X$ .

### Example 15.1.

(L) Which paths  $f : S^1 \rightarrow T^2 \# T^2$  are homotopic?

(R)  $D^2 \setminus \{x_0, x_1\}$  deformation retracts to which blue sets?



**Remark.**

1. If  $\beta$  is a reparameterization of  $\alpha$  then  $\alpha \simeq \beta \text{ rel } \{0, 1\}$ .
2.  $X \cong Y \implies X \simeq Y$  but not converse, e.g. Möbius band  $\simeq S^1 \simeq \text{Band } S^1 \times I$ .
3. *Fact:*  $X \simeq Y \iff \exists Z$  that deformation retracts to both  $X$  and  $Y$ .

## 16 CW Complexes

**Definition 16.1.** A *CW complex / cell complex* is a space  $X$  built as such:

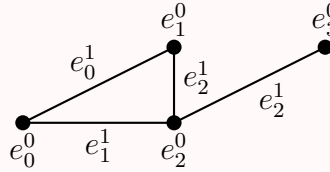
1. Start with a discrete set  $X^0$ , whose points are **0-cells**.
2. Let  $D_\alpha^n$  be  $n$ -balls (with  $\partial D_\alpha^n = S_\alpha^{n-1}$ ). Inductively, form the  **$n$ -skeleton**  $X^n$  as the quotient space of  $X^{n-1} \sqcup_\alpha D_\alpha^n$  by identifying  $x \sim \varphi_\alpha(x)$  where  $\varphi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$  are the **attaching maps**. This makes  $X^n = X^{n-1} \sqcup_\alpha \text{int}(D_\alpha^n)$  as a set. The  $e_\alpha^n = \text{int}(D_\alpha^n)$  are called  **$n$ -cells**.
3. One can stop after finite  $n$ , setting  $X = X^n$ . Or one can set  $X = \bigcup_{n=0}^\infty X^n$ , giving it the *weak topology*:  $U \subseteq X$  is open  $\iff U \cap X^n$  is open in  $X^n$  for all  $n$ .

The **characteristic map** of a cell  $e_\alpha^n$  is the map

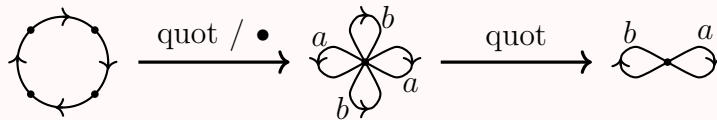
$$\Phi_\alpha : D_\alpha^n \hookrightarrow X^{n-1} \sqcup_\beta D_\beta^n \xrightarrow{\text{quot}} X^n \hookrightarrow X$$

**Example 16.1.**

1. A 1-dim CW complex is a **graph**, whose 0-cells are **nodes** and 1-cells are **edges**.



2.  $X = T^2$  is a CW complex, with  $X^0 = \{e_0^0\}$ ,  $X_1 = X^0 \sqcup e_a^0 \sqcup e_b^0$  where  $\varphi_a \equiv \varphi_b \equiv e_0^0$  being constant, and  $X^2 = X^1 \sqcup e^2$  with attaching map  $\varphi : S^1 \rightarrow X^1$  given by



*Note:* If we swap the direction of two adjacent leaves in the middle step, we get a **Klein bottle**. Attaching maps matter!

3. The  $n$ -sphere  $S^n$  is a cell complex with two cells  $e^0$  and  $e^n$ , with the attaching map  $S^{n-1} \rightarrow e^0$ . Or, we can inductively attach two  $n$ -cells to the equator  $S^{n-1}$ .
4.  $\mathbb{RP}^n \cong S^n/(v \sim -v) \cong D^n/(v \sim -v : v \in \partial D^n)$  is a cell complex by attaching an  $n$ -cell to  $\mathbb{RP}^{n-1}$  via the map  $S^{n-1} \rightarrow \mathbb{RP}^{n-1}$ . We can also have  $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$ .

**Definition 16.2.** A *subcomplex* of a CW complex  $X$  is a closed subspace  $A \subseteq X$  that is a union of cells of  $X$ . The pair  $(X, A)$  is a *CW pair*.

**Example 16.2.**

1.  $\mathbb{RP}^k \subseteq \mathbb{RP}^n$  is a subcomplex ( $k \leq n$ ).
2.  $S^k \subseteq S^n$  is not a subcomplex with the two-cell structure, but is a subcomplex using the recursive CW structure.

**Theorem 16.1.**

- If  $X, Y$  are cell complexes, then  $X \times Y$  is a cell complex, whose cells are  $e_\alpha^m \times e_\beta^n$  where  $e_\alpha^m, e_\beta^n$  are cells of  $X, Y$  respectively.
- If  $(X, A)$  is a CW pair, then the quotient space  $X/A$  is a cell complex, whose cells are the cells of  $X \setminus A$ , and one new 0-cell: the image of  $A$  in  $X/A$ .

**Definition 16.3.**  $A \subseteq X$  has the *homotopy extension property* if given any map  $f_0 : X \rightarrow Y$  and a homotopy  $f_t|_A : A \rightarrow Y$  of  $f_0|_A$ , we can extend  $f_t|_A$  to a homotopy  $f_t$  on  $X$ . Equivalently, given any maps  $H_1 : X \times \{0\} \rightarrow Y$  and  $H_2 : A \times I \rightarrow Y$  that agree on  $A \times \{0\}$ , there exists a map  $H : X \times I \rightarrow Y$  such that  $H$  agrees with both  $H_1, H_2$  where their domains meet.

**Theorem 16.2.**  $A \subseteq X$  has the homotopy extension property if and only if

$$X \times \{0\} \cup A \times [0, 1] \text{ is a retract of } X \times [0, 1].$$

*Proof.* Let  $Z = X \times \{0\} \cup A \times [0, 1]$ .

- If  $A \subseteq X$  has h.e.p then given the maps  $H_1 : X \times \{0\} \rightarrow Z$  and  $H_2 : A \times I \rightarrow Z$  with

$$H_1(x, 0) = (x, 0) \quad \text{and} \quad H_2(a, t) = (a, t)$$

we can get an extension  $H : X \times I \rightarrow Z$  constant on  $Z$ . Hence  $H$  is the retraction.

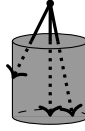
- The converse is easy if we assume  $A$  is closed. Say  $r : X \times I \rightarrow Z$  is a retraction. Given any  $H_1, H_2$  as in the definition, we can combine them via the Pasting Lemma to get  $H_3 : Z \rightarrow Y$ . Then  $H_3 \circ r : X \times I \rightarrow Y$  is the required homotopy. For the full proof where  $A$  is not necessarily closed, see appendix of [Hatcher]. ■

**Theorem 16.3.** If  $(X, A)$  is a CW pair,  $A$  has the homotopy extension property.

*Proof.* To prove  $X \times \{0\} \cup A \times I$  is a retract of  $X \times I$ , we first prove

**Lemma.**  $D^n \times \{0\} \cup \partial D^n \times I$  is a deformation retract of  $D^n \times I$ .

*Proof.* Consider radial projection  $r$  from  $(0, 2) \in D^n \times \mathbb{R}$ :



Then  $f_t = t \cdot r + (1 - t) \cdot \mathbf{1}$  is a deformation retract. □

Applying the deformation retraction to every  $D^n$  attached to  $X^{n-1}$  that is not in  $A^n$ , we get a deformation retraction  $H_n$  from  $X^n \times I$  onto  $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ . Note that concatenating adjacent  $H_n$  and  $H_{n+1}$  gives a deformation retraction

$$\begin{aligned} X^{n+1} \times I &\xrightarrow{H_{n+1}} X^{n+1} \times \{0\} \cup (X^n \cup A^{n+1}) \times I \\ &\xrightarrow{H_n} X^{n+1} \times \{0\} \cup ((X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I) \cup (A^{n+1} \times I)) \\ &= X^{n+1} \times \{0\} \cup (X^{n-1} \cup A^{n+1}) \times I \end{aligned}$$

and thus by concatenating all  $H_0, H_1, \dots$  into  $[1/4, 1/2], [1/8, 1/4], \dots$  we get a deformation retract from  $X \times I$  onto  $X \times \{0\} \cup A \times I$ . (In the infinite case, there is no continuity problem at  $t = 0$  since  $X$  is given the weak topology). ■

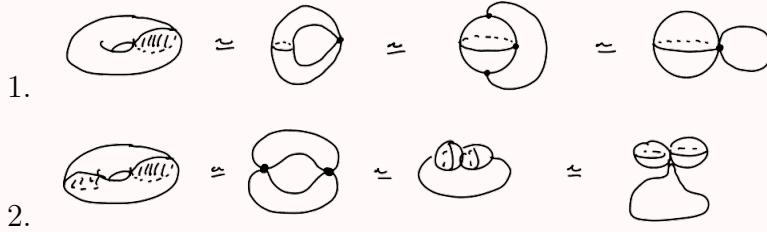
**Theorem 16.4.** If  $(X, A)$  is a CW pair and  $A$  is contractible, then the quotient map  $X \twoheadrightarrow X/A$  is a homotopy equivalence.

*Proof.* Let  $f_t : X \rightarrow X$  be a homotopy extension of the contraction of  $A$  with  $f_0 = \mathbf{1}_X$ . Since  $f_t(A) \subseteq A$  and  $f_1(A) = \text{pt}$ , we can construct well-defined maps  $\bar{f}_t, g$  satisfying

$$\begin{array}{ccc} X & \xrightarrow{f_t} & X \\ q \downarrow & & \downarrow q \\ X/A & \xrightarrow{\bar{f}_t} & X/A \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f_1} & X \\ q \downarrow & \nearrow g & \\ X/A & & \end{array}$$

Then  $g \circ q = \underline{f_1} \simeq \underline{f_0} = \mathbf{1}_X$  and  $q(g([x])) = q(g(q(x))) = q(f_1(x)) = \overline{f_1}(q(x)) = \overline{f_1}([x])$  and hence  $q \circ g = \overline{f_1} \simeq \overline{f_0} = \mathbf{1}_{X/A}$ , so  $g, q$  are homotopy equivalences.

**Example 16.3.**



## 17 Fundamental Groups

**Definition 17.1.**

1. A **path** on  $X$  is  $\alpha : I \rightarrow X$ . Define  $\Omega_{x_0}(X) = \{\text{path } \alpha \mid \alpha(0) = \alpha(1) = x_0\}$ .
2. Given paths  $\alpha, \beta \in \Omega_{x_0}(X)$ , define the **concatenation**  $\alpha \cdot \beta \in \Omega_{x_0}(X)$  by

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq 0.5 \\ \beta(2s - 1) & 0.5 \leq s \leq 1. \end{cases}$$

3. Given a path  $\gamma \in \Omega_{x_0}(X)$ , define the **reversed path**  $\bar{\gamma}(t) = \gamma(1 - t)$ .
4. The **fundamental group** of  $X$  based at  $x_0$  is the group

$$\pi_1(X, x_0) = \Omega_{x_0}(X) / \sim$$

where  $\alpha \sim \beta \Leftrightarrow \alpha \simeq \beta \text{ rel } \{0, 1\}$ , with group law  $[\alpha][\beta] = [\alpha \cdot \beta]$  and  $[\gamma]^{-1} = [\bar{\gamma}]$ .

**Theorem 17.1.** Let  $\gamma$  be a path from  $x_0$  to  $x_1$ . The map  $\Phi_\gamma : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  by  $\Phi([\alpha]) = [\gamma \cdot \alpha \cdot \bar{\gamma}]$  is an isomorphism.

**Corollary.** If  $X$  is path-connected,  $\pi_1(X, x)$  are isomorphic over all  $x \in X$  (say  $\pi_1(X)$ ).

**Theorem 17.2.** If  $X, Y$  are path-connected,  $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ .

**Definition 17.2.**  $X$  is **simply connected** if  $X$  is path-connected and  $\pi_1(X)$  is trivial.

**Definition 17.3.**

1. Write  $f : (X, x_0) \rightarrow (Y, y_0)$  if  $f : X \rightarrow Y$  and  $f(x_0) = y_0$ .
2. The **homomorphism induced** by  $f : (X, x_0) \rightarrow (Y, y_0)$  is the homomorphism

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

given by  $f_*([\alpha]) = [f \circ \alpha]$ .

**Theorem 17.3.**

1.  $(f \circ g)_* = f_* \circ g_*$ .
2. If  $f, g : X \rightarrow Y$  are homotopic rel  $x_0$ , then  $f_* = g_*$ .
3. If  $f : X \rightarrow Y$  is a homotopy equivalence, then  $f_*$  is an isomorphism.

**Theorem 17.4.**  $\pi_1(S^1) = \mathbb{Z}$ .

*Proof.* Let  $p : \mathbb{R} \rightarrow S^1$  given by  $p(\lambda) = (\cos(2\pi\lambda), \sin(2\pi\lambda))$ . The following two facts will be proven in the Covering Spaces chapter.

1. Given any path  $\gamma$  of  $S^1$ , there exists a unique path  $\tilde{\gamma}$  of  $\mathbb{R}$  such that  $\tilde{\gamma}(0) = 0$  and  $\gamma = p \circ \tilde{\gamma}$ .
2. Given any homotopy  $f_t : I \rightarrow S^1$ , there exists a unique homotopy  $\tilde{f}_t : I \rightarrow \mathbb{R}$  such that  $f_t = p \circ \tilde{f}_t$ .

The map  $\Phi([\gamma]) = \tilde{\gamma}(1) \in \mathbb{Z}$  is then a well-defined isomorphism. ■

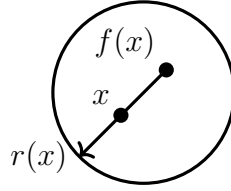
**Theorem 17.5.** If  $A$  is a retract of  $X$ , then the inclusion  $i : A \hookrightarrow X$  induces an injective homomorphism  $i_*$ . If  $A$  is a deformation retract of  $X$ , then  $i_*$  is an isomorphism.

*Proof.* Let  $r : X \rightarrow A$  be a retraction. Then  $r \circ i = \mathbf{1} \implies r_* \circ i_* = \mathbf{1} \implies i_*$  injective. If there is a deformation retraction, then  $i$  is a homotopy equivalence and hence  $i_*$  is an isomorphism. ■

**Theorem 17.6. (Brouwer's Fixed Point Theorem)**

$f : D^2 \rightarrow D^2 \implies f(x) = x$  for some  $x \in D^2$ .

*Proof.* Otherwise, the map  $r$  defined by



is a retract from  $D^2$  to  $S^1$ , so  $i : S^1 \rightarrow D^2$  induces an injective  $i_* : \mathbb{Z} \rightarrow \{0\}$ , contradiction. ■

### Theorem 17.7. (Fundamental Theorem of Algebra)

Every complex polynomial of positive degree has a root.

*Proof.* Let  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$  where  $n > 0$ . Assume  $f$  has no roots. Then

$$F(s, t, r) = (re^{2\pi is})^n + a_{n-1} (re^{2\pi is})^{n-1} t + \cdots + a_0 t^n \neq 0 \quad \forall s, t, r.$$

Then the homotopy

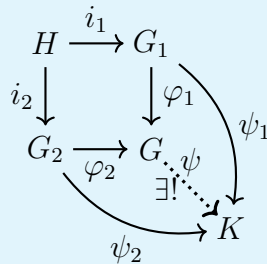
$$\frac{F(s, 0, 1)}{|F(s, 0, 1)|} \stackrel{t}{\simeq} \frac{F(s, 1, 1)}{|F(s, 1, 1)|} \stackrel{r}{\simeq} \frac{F(s, 1, 0)}{|F(s, 1, 0)|}$$

brings the path  $e^{2\pi isn}$ , that loops around the circle  $n$  times, to the trivial path 1. This is a contradiction since they correspond to different elements in  $\pi_1(S^1)$ . ■

## 18 Van Kampen's Theorem

**Definition 18.1.** Let  $i_1 : H \hookrightarrow G_1$  and  $i_2 : H \hookrightarrow G_2$  be homomorphisms. The **amalgamated free product** of  $G_1$  and  $G_2$  along  $H$ , denoted as  $G = G_1 *_H G_2$ , is the unique group (up to isomorphism) that satisfies

- (1) There exists homomorphisms  $\varphi_i : G_i \rightarrow G$  with  $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$ .
- (2) For any other homomorphisms  $\psi_i : G_i \rightarrow K$  with  $\psi_1 \circ i_1 = \psi_2 \circ i_2$ , there exists a unique homomorphism  $\psi : G \rightarrow K$  with  $\psi \circ \varphi_i = \psi_i$ .



If  $H = \{0\}$ , then  $G_1 * G_2 = G_1 *_H G_2$  is just the **free product** of  $G_1$  and  $G_2$ .

**Remark.**

1. Such a group always exists, e.g. if  $G_i = \langle S_i \mid R_i \rangle$  then

$$G_1 *_H G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \cup \{i_1(h)i_2(h^{-1}) : h \in H\} \rangle.$$

Uniqueness follows from the uniqueness of  $\psi$  between two such possible groups.

2. Think of  $G_1 *_H G_2$  by first treating  $H$  as a common subgroup of  $G_1, G_2$ , then construct all possible words of finite length with letters from  $G_1 \cup G_2$ . When two adjacent letters in a word both come from the same  $G_i$ , or if they both belong to  $H$ , we can further simplify the word.

**Example 18.1.**

1. The free group with  $n$  letters is simply  $F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_n$ .

2. The free product of  $\mathbb{Z}_2 = \{1, a, a^2 = 1\}$  and itself  $\mathbb{Z}_2 = \{1, b, b^2 = 1\}$  is

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, a, b, ab, ba, aba, bab, \dots\}$$

(This is the semi-direct product of  $\mathbb{Z} = \langle c := ab \rangle, \mathbb{Z}_2 = \langle a \rangle$  with  $ac = c^{-1}a$ , sometimes called the *infinite dihedral group*.)

3. If we embed  $H = \mathbb{Z}_2$  into the two  $\mathbb{Z}_2$ 's above by  $h \mapsto a$  and  $h \mapsto b$ , then the free product collapses into

$$\mathbb{Z}_2 *_H \mathbb{Z}_2 = \{1, h, h^2 = 1\} = \mathbb{Z}_2$$

**Theorem 18.1. (Van Kampen's Theorem, two-set version)**

Suppose  $X = U \cup V$  where  $U, V, U \cap V$  are open and path-connected, then for  $x_0 \in U \cap V$  we have  $\pi_1(X, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$  (with  $\pi_1(U \cap V, x_0) \hookrightarrow \pi_1(U, x_0)$  and  $\pi_1(U \cap V, x_0) \hookrightarrow \pi_1(V, x_0)$  being the maps induced by the inclusions  $U \cap V \hookrightarrow U$  and  $U \cap V \hookrightarrow V$  respectively).

**Example 18.2.**  $\pi_1(S^n) = \{0\}$  for  $n \geq 2$  (*high-dim spheres are simply connected*).

$S^n$  is the union of open neighborhoods of the north and south hemisphere, intersecting at the equator  $\simeq S^{n-1}$ . Hence  $\pi_1(S^n) = \pi_1(e^n) *_{\pi_1(S^{n-1})} \pi_1(e^n) = \{0\} *_{\pi_1(S^{n-1})} \{0\} = \{0\}$ .



**Definition 18.2.** Suppose  $x_0 \in X, y_0 \in Y$ . The **wedge sum**  $(X, x_0) \vee (Y, y_0)$  is the space  $(X \sqcup Y)/\{x_0, y_0\}$  (gluing  $X$  and  $Y$  together at  $x_0, y_0$ ). Lazy:  $X \vee Y$ .

**Example 18.3.**  $S^1 \vee S^1$  is the figure-eight, homemorphic to the shape  $\infty$ .

**Theorem 18.2.** If  $\exists$  neighborhoods  $x_0 \in U, y_0 \in V$  in  $X, Y$  such that  $\{x_0\}, \{y_0\}$  are deformation retracts of  $U, V$  respectively, then  $\pi_1(X \vee Y) = \pi_1(X) \times \pi_1(Y)$ .

*Proof.* Let  $H_t : U \rightarrow U, G_t : V \rightarrow V$  be deformation retracts onto  $x_0, y_0$  respectively.

- We can define  $\overline{G}_t : X \vee V \rightarrow X \vee V$  by

$$\begin{array}{ccc} X \sqcup V & \xrightarrow{1 \sqcup G_t} & X \sqcup V \\ q \downarrow & & \downarrow q \\ X \vee V & \xrightarrow{\overline{G}_t} & X \vee V \end{array}$$

which is a deformation retraction of  $X \vee V$  onto  $X \vee \{y_0\} \cong X$ . Hence  $X \vee V$  deformation retracts onto  $X$  and (similarly)  $U \vee Y$  deformation retracts onto  $Y$ .

- We claim that  $U \vee V \subseteq X \vee Y$  is contractible. The map  $F_t : U \vee V \rightarrow U \vee V$  defined by

$$\begin{array}{ccc} U \sqcup V & \xrightarrow{H_t \sqcup G_t} & U \sqcup V \\ q \downarrow & & \downarrow q \\ U \vee V & \xrightarrow{F_t} & U \vee V \end{array}$$

is a deformation retraction onto  $x_0 \in U \vee V$ .

- By Van Kampen,  $\pi_1(X \vee Y) = \pi_1(X \vee V) *_{\pi_1(U \vee V)} \pi_1(U \vee Y) = \pi_1(X) * \pi_1(Y)$ . ■

**Corollary 18.2.**  $\pi_1(\bigvee_{i=1}^n S^1) = F_n$ .

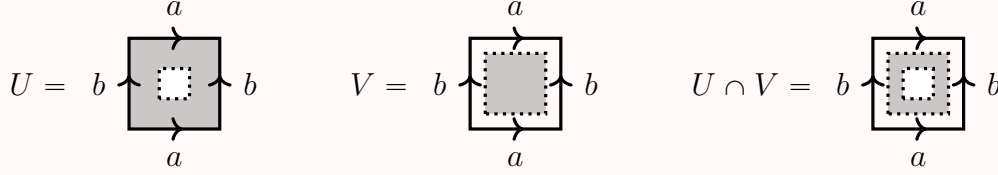
**Theorem 18.3.** If  $\Gamma$  is a connected graph, then  $\pi_1(\Gamma) = F_{1-\chi(\Gamma)}$  where  $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$  is the **Euler characteristic** of  $\Gamma$ .

*Proof.* Let  $T$  be a spanning tree of  $\Gamma$ , which is contractible. Then by collapsing  $T$ , the graph  $\Gamma/T \simeq \Gamma$  is a wedge sum of  $|E(\Gamma - T)|$  circles. Hence  $\pi_1(\Gamma) = F_n$  where

$$n = |E(\Gamma)| - |E(T)| = |E(\Gamma)| - (|V(T)| - 1) = 1 - \chi(T). \quad \blacksquare$$

**Theorem 18.4.** If  $i : H \rightarrow G = \langle S \mid R \rangle$ , then  $G *_H \{0\} = \langle S \mid R \cup i(H) \rangle$

**Example 18.4.** We can compute  $\pi_1(T^2)$  as follows:



$$\pi_1(T^2) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \langle a, b \rangle *_{\mathbb{Z}} \{0\} = \langle a, b \mid aba^{-1}b^{-1} \rangle = \mathbb{Z}^2.$$

## Fundamental Group of CW Complexes

**Theorem 18.5.**

1. Let  $X^2$  be a CW complex obtained from  $X^1$  by attaching 2-cells  $e_\alpha^2$  via  $\varphi_\alpha : \partial D_\alpha^2 \rightarrow X^1$ . For each  $\alpha$ , let  $\gamma_\alpha$  be a path on  $X^1$  from  $x_0$  to a point  $z_\alpha \in \partial D_\alpha^2$ .

$$\pi_1(X^2, x_0) = \pi_1(X^1, x_0) / N$$

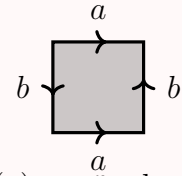
where  $N$  is the normal closure of the subgroup of  $\pi_1(X^1, x_0)$  generated by paths  $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha]$  (treating  $\varphi_\alpha$  as a closed path based at  $z_\alpha$ ).

2. Attaching  $n$ -cells ( $n \geq 3$ ) does not change the fundamental group, i.e.

$$\pi_1(X, x_0) = \pi_1(X^2, x_0)$$

**Example 18.5.**

1. For the Klein bottle  $K$ , we have  $\pi_1(K) = \langle a, b \rangle / N$  where  $N$  is generated by  $aba^{-1}b$ , so  $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$ .

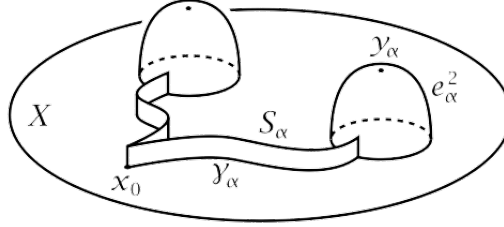


2. If  $X$  is obtained by attaching a single 2-cell to a circle  $\mathbb{C}^\times$  via  $\varphi(z) = z^n$ , then  $\pi_1(X) = \langle x \mid x^n \rangle = \mathbb{Z}_n$ . In particular,  $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ .

**Corollary 18.5.** Given any group  $G$ , there exists a space  $X$  with  $\pi_1(X) = G$ .

*Proof.* Write  $G = \langle S \mid R \rangle$  and attach 2-cells (according to  $R$ ) to the wedge sum  $\bigvee_{s \in S} S_s^1$ . ■

*Proof of Theorem 18.5.* First expand  $X^2$  by bulging up the  $e_\alpha^2$ 's and then adding strips  $S_\alpha = I \times I$  along each  $\gamma_\alpha$ . Pick a  $y_\alpha \in e_\alpha^2$  that is not on the strip. Call this larger space  $Z$ .



We then slice this space along half the height of the  $S_\alpha$ 's, and consider an open neighborhood of the top and bottom parts  $U, V$  respectively (e.g.  $U = Z \setminus X^1$  and  $V = Z \setminus \bigcup_\alpha \{y_\alpha\}$ ).  $U$  is contractible while  $V$  deformation retracts to  $X^1$ . Hence

$$\pi_1(X^2, x_0) = \pi_1(Z, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) = \{0\} *_{\pi_1(U \cap V, x_0)} \pi_1(X^1, x_0).$$

So it remains to show that  $\pi_1(U \cap V, x_0)$  is generated by the  $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}]$ : We can apply Van Kampen again on  $U \cap V$  by covering it with the open sets  $A_\alpha = U \cap V \setminus \bigcup_{\beta \neq \alpha} D_\beta^2$  which deformation retract to a circle and hence is generated by  $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}$ . This shows (1).

To show (2), we perform the same procedure. However, in the last step, the  $A_\alpha$  deformation retract to spheres, which are simply connected. The finite  $X^n$  case follows from induction. If  $X$  is infinite-dimensional, any closed loop at  $x_0$  is compact and hence is contained in some finite  $X^n$  anyway. ■

### Definition 18.3.

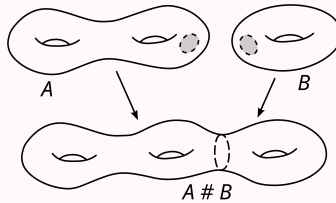
1. Let  $\Sigma, \Sigma'$  be surfaces. The **connect sum**,  $\Sigma \# \Sigma'$  is defined by

$$(\Sigma \setminus \text{int}(D^2)) \sqcup (\Sigma' \setminus \text{int}(D^2)) / \sim$$

where  $\sim$  identifies boundary points.

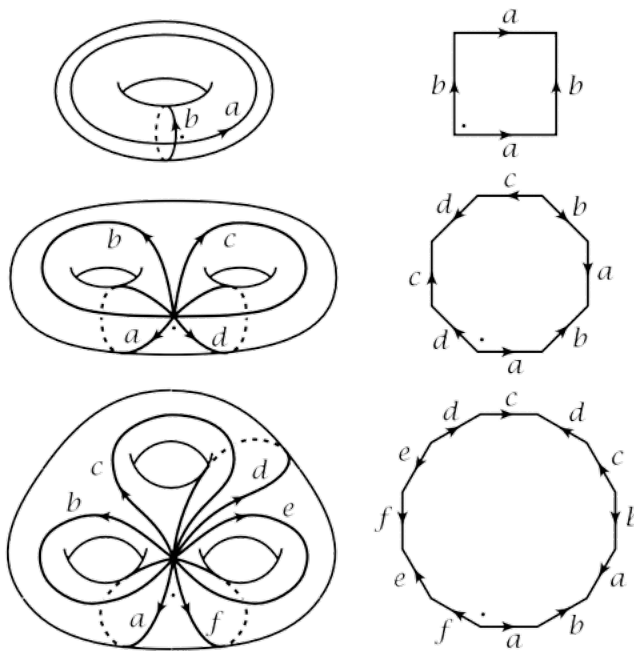
2. The **surface of genus  $g$**  is  $\Sigma_g = \underbrace{T^2 \# \cdots \# T^2}_n \# S^2$  (The  $g$ -holed torus).

### Example 18.6.



**Theorem 18.6.**  $\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} \rangle$

Diagram.



## 19 Covering Spaces

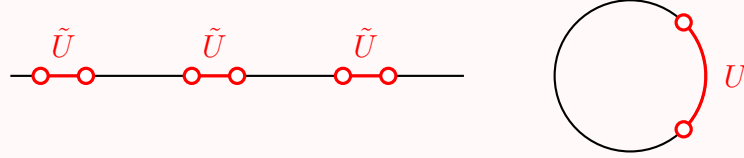
**Definition 19.1.**

1. A **covering space** of  $X$  is a space  $\tilde{X}$  with a map  $p : \tilde{X} \rightarrow X$  such that every  $x \in X$  admits a neighborhood  $U$  such that  $f^{-1}(U) = \bigsqcup_{\alpha} \tilde{U}_{\alpha}$  (a disjoint union of open sets) where each  $p|_{\tilde{U}_{\alpha}}$  is a homeomorphism. We say that  $U$  is **evenly covered** by the **sheets**  $\tilde{U}_{\alpha}$ .
2. A **lift** of a map  $f : Y \rightarrow X$  is a map  $\tilde{f} : Y \rightarrow \tilde{X}$  with  $f = p \circ \tilde{f}$ .

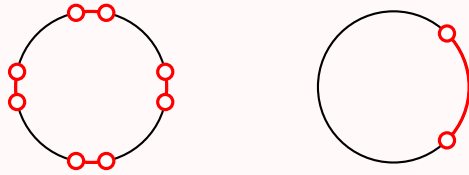
$$\begin{array}{ccc}
 & \tilde{f} \nearrow & \tilde{X} \\
 Y & \xrightarrow{f} & X \\
 & & \downarrow p
 \end{array}$$

**Example 19.1.**

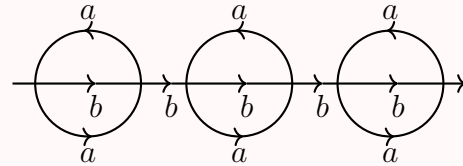
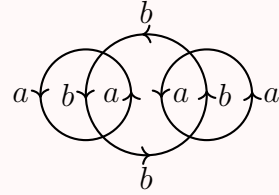
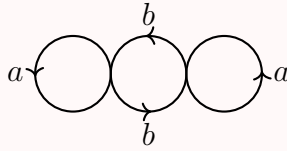
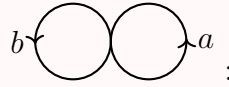
1.  $p : \mathbb{R} \rightarrow S^1, p(\lambda) = e^{2\pi i \lambda}$ .



2.  $p_n : S^1 \rightarrow S^1, p_n(z) = z^n$ .



3. A few covering spaces of  $S^1 \vee S^1$ , as



**Theorem 19.1.** Let  $Y$  be a connected space and  $f : Y \rightarrow X$ . If two lifts  $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$  agree at some point  $y \in Y$ , then  $\tilde{f}_1 = \tilde{f}_2$ .

*Proof.* Let  $U$  be a neighborhood of  $f(y)$  that is evenly covered, by sheets  $\tilde{U}_\alpha$ .

- $\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\}$  is open: Suppose  $\tilde{f}_1(z) = \tilde{f}_2(z) \in \tilde{U}_\beta$ , then by continuity there exists a neighborhood  $z \in V$  with  $\tilde{f}_1(V), \tilde{f}_2(V) \subseteq \tilde{U}_\beta$ . Then

$$\tilde{f}_1|_V = p|_{\tilde{U}_\beta}^{-1} \circ p|_{\tilde{U}_\beta} \circ \tilde{f}_1|_V = p|_{\tilde{U}_\beta}^{-1} \circ f|_V = \tilde{f}_2|_V.$$

- $\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\}$  is closed:  $\tilde{f}_1(z) \neq \tilde{f}_2(z) \implies \tilde{f}_1(z) \in \tilde{U}_{\beta_1}, \tilde{f}_2(z) \in \tilde{U}_{\beta_2} \ (\beta_1 \neq \beta_2)$ . By continuity there exists a neighborhood  $z \in V$  with  $\tilde{f}_i(V) \subseteq \tilde{U}_{\beta_i}$ .

Since  $Y$  is connected,  $Y = \{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\}$ . ■

**Theorem 19.2. (Homotopy Lifting Property)**

Given a homotopy  $f_t : Y \rightarrow X$  and a lift  $\tilde{f}_0 : Y \rightarrow \tilde{X}$  of  $f_0$ , there exists a unique homotopy  $\tilde{f}_t : Y \rightarrow \tilde{X}$  of  $f_t$  that agrees with  $\tilde{f}_0$ .

**Remark.** The two facts used in Theorem 17.4 follow from the case  $Y = \text{pt}$  and  $Y = I$ .

*Proof.* Use the  $H(x, t) = f_t(x)$  notation.

**Lemma.** For any  $y \in Y$ , there exists open  $y \in V$  and  $0 = t_0 < t_1 < \dots < t_n = 1$  such that  $H(V \times [t_i, t_{i+1}]) \subseteq U_i$  for all  $i$ .

*Proof.* Fix  $y$ . For each  $t \in I$  there exists a neighborhood  $U_t$  of  $H(y, t)$  that is evenly covered, and there exists a basis  $V_t \times W_t \subseteq Y \times I$  with  $(y, t) \in V_t \times W_t \subseteq H^{-1}(U_t)$ . Since the  $W_t$  cover  $I$  which is compact, we have a finite subcover  $W_{s_0}, \dots, W_{s_m}$  of  $I$  and hence we can take  $V = V_{s_0} \cap \dots \cap V_{s_m}$  and  $t_i$  the endpoints of all  $W_{s_k}$ .

We first prove the theorem by fixing  $y$  and restricting  $f_t$  on a neighborhood  $V \subseteq Y$ . By induction, suppose  $\tilde{H}$  has been constructed over  $V \times [0, t_i]$ . Let  $U \supseteq H(V \times [t_i, t_{i+1}])$  be evenly covered by sheets  $\tilde{U}_\alpha$ . Let  $\tilde{H}(y, t_i) \in \tilde{U}_\beta$ , then by the pasting lemma we can construct  $\tilde{H}|_{V' \times [0, t_i]}$ , where  $V'$  is a smaller open set than  $V$  by restricting to the pre-image within  $\tilde{U}_\beta$ . Relabelling  $V'$  as  $V$ , after a finite number of steps, we constructed  $\tilde{f}_t$  on a neighborhood  $V$  of  $y$ . Note that such  $\tilde{f}_t$  is unique by the previous Theorem on the common point  $(v, t)$  for each  $v \in V$ .

To construct  $\tilde{f}_t$  on the entire  $Y$ , we construct a unique  $\tilde{f}$  on a neighborhood  $V_y$  at every  $y \in Y$ . By uniqueness on each  $\{y\} \times I$ , the  $\tilde{f}$ 's agree on the overlaps. By the same uniqueness, the entire  $\tilde{f}$  is unique. ■

**Theorem 19.3.** Let  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  be induced by  $p$ .

1.  $p_*$  is injective.
2.  $\text{Im}(p_*) = \{[\alpha] \in \pi_1(X, x_0) : \tilde{\alpha}(0) = \tilde{\alpha}(1) = \tilde{x}_0\}$ .

*Proof.* Suppose  $p_*([\beta]) = 0$ . Then  $p \circ \beta \simeq \text{constant rel } \{0, 1\}$ . This nullhomotopy has a unique lift in  $\tilde{X}$ , which gives a nullhomotopy for  $\beta$ , i.e.  $[\beta] = 0$ . This proves (1).

For (2), we have  $p_*([\beta]) = [p \circ \beta] = [\tilde{\beta}]$ . ■

**Theorem 19.4.** If  $X, \tilde{X}$  are path-connected, then  $|p^{-1}(x_0)| = [\pi_1(X, x_0) : \text{Im}(p_*)]$  (There is a bijection between each preimage of  $x_0$  and each coset of  $\text{Im}(p_*)$ ).

**Definition 19.2.** A space  $Y$  is **locally path-connected** if for all  $y \in Y$  and any neighborhood  $y \in U$ , there exists a neighborhood  $y \in V \subseteq U$  that is path-connected.

**Theorem 19.5.** Say  $f : (Y, y_0) \rightarrow (X, x_0)$  where  $Y$  is path-connn and locally path-connn.

$$\exists \tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0) \iff \text{Im}(f_*) \subseteq \text{Im}(p_*)$$

*Proof.* For  $(\Rightarrow)$ ,  $f = p \circ \tilde{f} \implies f_* = p_* \circ \tilde{f}_*$ . For  $(\Leftarrow)$ , suppose  $\text{Im}(f_*) \subseteq \text{Im}(p_*)$ . Pick for each  $y \in Y$  a path  $\gamma_y$  from  $y_0$  to  $y$ , then define  $\tilde{f}(y) = \tilde{g}_y(1)$  where  $g_y = f \circ \gamma_y$  and  $\tilde{g}_y(0) = \tilde{x}_0$ .

- $\tilde{f}$  is well-defined: Let  $\gamma_1, \gamma_2$  be two paths from  $y_0$  to  $y$ . Then  $[\gamma_1 \cdot \overline{\gamma_2}] \in \pi_1(Y, y_0)$  and hence exists  $[\alpha] \in \pi_1(\tilde{X}, \tilde{x}_0)$  with  $p \circ \alpha \simeq f \circ \gamma_1 \cdot \overline{\gamma_2} \text{ rel } \{0, 1\}$ . This homotopy  $H_t$  has a unique lift  $\tilde{H}_t$  with  $\tilde{H}_0 = \alpha$ . Since  $H_1 = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ , by uniqueness if we have  $g_1 = f \circ \gamma_1, g_2 = f \circ \gamma_2$  then  $\tilde{H}_1 = \tilde{g}_1 \cdot \tilde{g}_2$ , and thus  $\tilde{g}_1(1) = \tilde{H}_1(0.5) = \tilde{g}_2(1)$ .
- $\tilde{f}$  is a lift:  $p \circ \tilde{f}(y) = p \circ \tilde{g}_y(1) = g_y(1) = f(y)$ .
- $\tilde{f}$  is continuous: Let  $W$  be a neighborhood of  $\tilde{f}(y)$ . Let  $f(y) \in U$  be evenly covered by  $\tilde{U}_\alpha$ , and  $\tilde{f}(y) \in \tilde{U}$ . Since  $f$  is continuous,  $\exists$  neighborhood  $y \in V'$  with  $f(V') \subseteq p(\tilde{U} \cap W)$ . By local path-connectedness, let  $y \in V \subseteq V'$  be a path-connected neighborhood. Then any path from  $y_0$  to  $y$  can be extended to a path from  $y_0$  to any  $z \in V$ . This eventually shows  $\tilde{f}(V) \subseteq \tilde{U} \cap W$ . ■

**Definition 19.3.**

1. A space  $X$  is **locally contractible** if  $\forall x \in X, \exists$  neighborhood  $x \in U \simeq \text{pt}$ .
2. A space  $X$  is **locally simply connected** if  $\forall x \in X, \exists$  neighborhood  $x \in U$  that is simply connected.
3. A space  $X$  is **semilocally simply connected** if for all  $x \in X$  there exists a neighborhood  $x \in U$  where  $\pi_1(U, x) \rightarrow \pi_1(X, x)$  is trivial.
4.  $X$  is **'nice'** if it is path-connn, locally path-connn and semilocally simply conn.
5.  $p : \tilde{X} \rightarrow X$  is a **universal cover** of  $X$  if  $X$  is path-connn and  $\tilde{X}$  is simply conn.
6. Two covering spaces  $p_i : \tilde{X}_i \rightarrow X$  are **isomorphic** if there exists a homeomorphism  $\varphi : \tilde{X}_1 \rightarrow \tilde{X}_2$  with  $p_2 \circ \varphi = p_1$ .

**Theorem 19.6.** CW complexes are locally contractible and locally path-connected.

**Theorem 19.7.**

1. If  $X$  is nice, then  $X$  has a universal cover.
2. If  $X$  is nice, for any subgroup  $H \subseteq \pi_1(X, x_0)$  there exists a covering space  $p_H : \tilde{X}_H \rightarrow X$  such that  $\text{Im}(p_{H*}) = H$ .

*Proof of (1).* We use a Lemma and 5 steps.

**Lemma.**  $\mathcal{B} = \{U \subseteq X : U \text{ open, path-conn, } \pi_1(U) \rightarrow \pi_1(X) \text{ trivial}\}$  is a basis for the topology of a nice  $X$ .

*Proof.*  $\mathcal{B}$  covers  $X$  since  $X$  is nice. Suppose  $x \in U \cap V$  where  $U, V \in \mathcal{B}$ . Since  $X$  is locally path-conn, there exists path-connected  $x \in W \subseteq U \cap V$ , which means  $\pi_1(W, x) \rightarrow \pi_1(U, x) \xrightarrow{\text{triv}} \pi_1(X, x) \implies W \in \mathcal{B}$ . Hence  $\mathcal{B}$  is a basis. To prove the second part, given any open  $x \in W$ , choose open  $x \in V$  via semilocal simply connectedness, then pick a open and path-connected  $U \subseteq V \cap W$  via local path-connectedness. Then  $U \in \mathcal{B}$  with  $U \subseteq W$ . ■

1. Define  $\tilde{X} = \{[\gamma] \mid \gamma : I \rightarrow X, \gamma(0) = x_0\}$  with  $[\gamma] = [\delta] \iff \gamma \simeq \delta \text{ rel } \{0, 1\}$ . Define the covering map  $p : \tilde{X} \rightarrow X$  by  $p([\gamma]) = \gamma(1)$ .
2. Define the topology on  $\tilde{X}$  as follows: Given  $U \in \mathcal{B}$  and  $[\gamma] \in \tilde{X}$  with  $\gamma(1) \in U$ , define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta : I \rightarrow U \text{ with } \eta(0) = \gamma(1)\}$$

Then  $\{U_{[\gamma]}\}_{[\gamma]}$  is a basis (Exercise; (\*) Note that  $[\delta] \in U_{[\gamma]} \implies U_{[\delta]} = U_{[\gamma]}$ ), and we generate the topology from it.

3. Claim:  $p : U_{[\gamma]} \rightarrow U$  is a homeomorphism.
  - Surjective:  $U \text{ path-conn} \implies \forall x \in U, \exists \text{ path } \eta \text{ from } \gamma(1) \text{ to } x \implies p([\gamma \cdot \eta]) = x$ .
  - Injective:  $p([\gamma \cdot \eta]) = p([\gamma \cdot \eta']) \implies \eta(1) = \eta'(1)$ . Since  $\pi_1(U) \rightarrow \pi_1(X)$  is trivial,  $\eta \simeq \eta' \implies [\gamma \cdot \eta] = [\gamma \cdot \eta']$ .
  - Homeo: Note that  $\{B \cap U\}_{B \in \mathcal{B}}$  and  $\{B_{[\delta]} \cap U_{[\gamma]}\}_{B \in \mathcal{B}}$  are bases for  $U$  and  $U_{[\gamma]}$  respectively, and (1)  $p(B_{[\delta]} \cap U_{[\gamma]}) = B \cap U$ ; and (2)  $p^{-1}(B \cap U) \cap U_{[\gamma]} = B_{[\delta]} \cap U_{[\gamma]}$  for any  $[\delta] \in U_{[\gamma]}$  with  $\delta(1) \in B$  since  $B_{[\delta]} \subseteq U_{[\delta]} \stackrel{(*)}{=} U_{[\gamma]}$  and  $p|_{U_{[\delta]}}$  is bijective.
4.  $p$  is a covering map: Given any  $U \in \mathcal{B}$ ,  $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$  which is a disjoint union of equivalence classes due to (\*).



5.  $\tilde{X}$  is simply connected: Let  $[x_0]$  = class of constant paths. Given any  $[\gamma] \in \tilde{X}$ , the path  $\gamma_t(s) = \gamma(\min(s, t))$  from  $\gamma(0)$  to  $\gamma(t)$  brings  $[x_0]$  to  $[\gamma]$ , and thus  $\tilde{X}$  is path-connected. To show  $\pi_1(\tilde{X}, [x_0]) = 0$ , we show  $\text{Im}(p_*) = \{0\}$ :

$$[\alpha] \in \text{Im}(p_*) \implies [x_0] = \tilde{\alpha}(0) = \tilde{\alpha}(1) = [\alpha] \implies [\alpha] = 0. \quad \blacksquare$$

*Proof of (2).* Consider the equivalence relation  $\sim$  on the universal cover  $\tilde{X}$  defined by

$$[\gamma] \sim [\delta] \iff \gamma(1) = \delta(1) \text{ and } [\gamma \cdot \bar{\delta}] \in H.$$

Then  $X_H = \tilde{X}/\sim$  is a covering space of  $X$ . To show  $\text{Im}(p_{H*}) = H$ : For any  $[\alpha] \in \pi_1(X, x_0)$ , we have a lift  $\tilde{\alpha} = \alpha_t$  from  $[\tilde{x}_0]$  to  $[\alpha]$ , so

$$[\alpha] \in H \iff [\tilde{x}_0] \sim [\alpha] \iff \tilde{\alpha}(0) = \tilde{\alpha}(1) \iff [\alpha] \in \text{Im}(p_{H*})$$

**Theorem 19.8.** Let  $\tilde{X}$  be path-connected. Then  $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$  is conjugate to  $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_1\right)\right)$  for any  $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ .

*Proof.* Conjugate using any path from  $\tilde{x}_0$  to  $\tilde{x}_1$ . \blacksquare

**Theorem 19.9. (Classification Theorem of Covering Spaces)**

If  $X$  is nice, there is a bijective ***Galois correspondence*** between

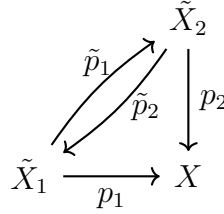
$$\left\{ \begin{array}{l} \text{conj. classes of} \\ \text{subgrps of } \pi_1(X, x_0) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isom. classes of path-} \\ \text{conn. covers } \tilde{X} \rightarrow X \end{array} \right\}$$

*Proof.* We prove that  $p_1 : \tilde{X}_1 \rightarrow X$  and  $p_2 : \tilde{X}_2 \rightarrow X$  are isomorphic if and only if  $\text{Im}(p_{1*})$  and  $\text{Im}(p_{2*})$  are conjugate in  $\pi_1(X)$ .

- Assume  $p_1 \cong p_2$ . Then  $\text{Im}(p_{1*}) = \text{Im}(p_{2*} \circ \varphi_*) = p_{2*}\left(\pi_1\left(\tilde{X}_2, \varphi(\tilde{x}_1)\right)\right)$  which is conjugate to  $p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{x}_2\right)\right)$  since  $p_2 \circ \varphi(\tilde{x}_1) = p_1(\tilde{x}_1) = x_0 = p_2(\tilde{x}_2)$ .
- Suppose  $\text{Im}(p_{1*}) = [\alpha]^{-1}\text{Im}(p_{2*})[\alpha]$ . Let  $\tilde{\alpha} : I \rightarrow \tilde{X}_2$  be a lift of  $\alpha$  based at  $\tilde{x}_2$ . Then

$$p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{\alpha}(1)\right)\right) = p_{1*}\left(\pi_1\left(\tilde{X}_1, \tilde{x}_1\right)\right)$$

so there exists lifts  $\tilde{p}_1 : \left(\tilde{X}_1, \tilde{x}_1\right) \rightarrow \left(\tilde{X}_2, \tilde{\alpha}(1)\right), \tilde{p}_1 : \left(\tilde{X}_2, \tilde{\alpha}(1)\right) \rightarrow \left(\tilde{X}_1, \tilde{x}_1\right)$  with



Since  $\tilde{p}_2 \circ \tilde{p}_1$  and  $\mathbf{1}$  agree at  $\tilde{x}_1$  and are both lifts of  $p_1 : \tilde{X}_1 \rightarrow X$  to  $\tilde{X}_1$ , we have  $\tilde{p}_2 \circ \tilde{p}_1 = \mathbf{1}$  and similarly  $\tilde{p}_1 \circ \tilde{p}_2 = \mathbf{1}$ . Thus  $p_1 \cong p_2$ . ■

## 20 Regular Coverings

### Definition 20.1.

1. A **deck transformation** is a self-isomorphism  $\tilde{X} \rightarrow \tilde{X}$  of a covering space  $p : \tilde{X} \rightarrow X$ . The group of deck transformations is denoted  $\text{Aut}(\tilde{X})$ .
2. A covering space is **regular** if for each  $x \in X$  and  $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ , there exists  $\varphi \in \text{Aut}(\tilde{X})$  with  $\varphi(\tilde{x}) = \tilde{x}'$ .

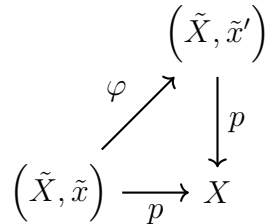
**Example 20.1.**  $\text{Aut}(\mathbb{R}) = \mathbb{Z}$ .

**Theorem 20.1.**  $\varphi \in \text{Aut}(\tilde{X})$  is completely determined by  $\varphi(\tilde{x}_0)$  when  $\tilde{X}$  is path-connected and locally path-connected.

*Proof.*  $\varphi$  is a lift to  $\tilde{X}$ , which is uniquely determined by where it sends some point. ■

**Theorem 20.2.** Suppose  $X$  is nice. Then  $p : \tilde{X} \rightarrow X$  is regular if and only if  $\text{Im}(p_*)$  is normal. When this is true,  $\text{Aut}(\tilde{X}) = \pi_1(X, x_0)/\text{Im}(p_*)$ .

*Proof.*  $p$  is regular  $\Leftrightarrow$  for any  $\tilde{x}, \tilde{x}' \in p^{-1}(x)$  there exists a lift  $\varphi$  where



which is equivalent to  $p_* \left( \pi_1 \left( \tilde{X}, \tilde{x} \right) \right) = p_* \left( \pi_1 \left( \tilde{X}, \tilde{x}' \right) \right) \quad \forall \tilde{x}, \tilde{x}'$ , i.e.  $p_* \left( \pi_1 \left( \tilde{X}, \tilde{x} \right) \right)$  is normal. To prove the second part, consider the map  $\pi_1(X, x_0) \rightarrow \text{Aut}(\tilde{X})$  is  $p_*(\pi_1(X, x_0))$  given by  $[\alpha] \mapsto \varphi_\alpha$  where  $\varphi_\alpha(\tilde{\alpha}(0) = \tilde{x}_0) = \tilde{\alpha}(1)$ .

- It is a homomorphism: Since a lift of  $\alpha \cdot \beta$  is  $\tilde{\alpha} \cdot \varphi_\alpha(\tilde{\beta})$ , so

$$[\alpha][\beta] \mapsto \varphi_{\alpha \cdot \beta}(\tilde{x}_0) = \varphi_\beta \circ \varphi_\alpha(\tilde{x}_0) \implies \varphi_{\alpha \cdot \beta} = \varphi_\beta \varphi_\alpha.$$

- It is injective:  $[\alpha] \mapsto 0 \iff [\alpha] \in p_* \left( \pi_1 \left( \tilde{X}, \tilde{x} \right) \right)$ .
- It is surjective: Fix a path  $\gamma$  in  $\tilde{X}$  from  $\tilde{x}_0$  to  $\tilde{x}_1 \in p^{-1}(x_0)$ . Then  $[p \circ \gamma] \in \pi_1(X, x_0)$  has lift  $\gamma$ . ■