

18.901 Notes

1 Topological Spaces

Definition 1.1. A **topology** on a set X is a collection \mathcal{T} of subsets of X such that:

- $\emptyset, X \in \mathcal{T}$.
- The union of (possibly uncountably many) sets in \mathcal{T} is in \mathcal{T} .
- The intersection of finitely many sets in \mathcal{T} is in \mathcal{T} .

A set X for which a topology has been specified is called a **topological space**. Elements of \mathcal{T} are called **open sets** of X . The complements of open sets of X are called **closed sets** of X . Sets that are both open and closed in X are called **clopen** in X . For any $x \in X$, a **neighborhood** of x is an open set of X containing x .

Example 1.1.

1. The **discrete topology** on X is $\mathcal{P}(X)$, i.e. all subsets of X are open in X .
2. The **indiscrete (or trivial) topology** on X is $\{\emptyset, X\}$.
3. The **finite complement topology** on X is $\{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$. The **countable complement topology** on X is $\{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is countable}\}$.

Exercise. Show that the above are valid topologies.

Definition 1.2. Suppose \mathcal{T} and \mathcal{T}' are topologies on X .

- We say \mathcal{T} is **finer** than \mathcal{T}' if $\mathcal{T} \supseteq \mathcal{T}'$. (Draw a diagram to visualize)
- We say \mathcal{T} is **coarser** than \mathcal{T}' if $\mathcal{T} \subseteq \mathcal{T}'$.
- We say \mathcal{T} and \mathcal{T}' are **incomparable** if neither $\mathcal{T} \supseteq \mathcal{T}'$ nor $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 1.3. A (topological) **basis** on X is a collection \mathcal{B} of subsets of X where

- $\bigcup_{B \in \mathcal{B}} B = X$. (\mathcal{B} **covers** X , i.e. every $x \in X$ belongs to some $B \in \mathcal{B}$)
- Any x in two basis elements B_1, B_2 belongs to some basis element $B_3 \subseteq B_1 \cap B_2$.

The **topology generated by** \mathcal{B} consists of all possible *unions* of elements in \mathcal{B} .

Theorem 1.1. The topology \mathcal{T} generated by a basis \mathcal{B} is in fact a topology on X .

Proof. Firstly, $\emptyset \in \mathcal{T}$ since it is the empty union. Also, $X \in \mathcal{T}$ because \mathcal{B} covers X . We then verify the union and intersection properties:

- Given any $\{U_\alpha\}_\alpha \subseteq \mathcal{T}$, each U_α is a union of elements in \mathcal{B} , so $\bigcup_\alpha U_\alpha$ is also a union of elements in \mathcal{B} , and hence is in \mathcal{T} .
- Given any $U_1, U_2 \in \mathcal{T}$, we show that $U_1 \cap U_2 \in \mathcal{T}$: Let $x \in U_1 \cap U_2$, then x belongs to some basis element $B_1 \subseteq U_1$ and some basis element $B_2 \subseteq U_2$. Hence x belongs to some basis element $B(x) \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Now, $U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B(x)$ and thus $U_1 \cap U_2 \in \mathcal{T}$. The general finite intersection case follows from induction. ■

Corollary 1.1. Suppose we have a topology with some basis \mathcal{B} . Given any open set U and an element $x \in U$, there exists some basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

We can often reduce a problem about open sets to one about just the basis elements:

Theorem 1.2. Let \mathcal{T} and \mathcal{T}' be topologies on X with bases \mathcal{B} and \mathcal{B}' respectively. Then \mathcal{T}' is finer than \mathcal{T} if and only if for every possible $x \in B \in \mathcal{B}$ there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. (We can always get a smaller one in \mathcal{B}')

Definition 1.4.

- The **standard topology** on \mathbb{R} is the topology generated by the collection of open intervals $(a, b) = \{x : a < x < b\}$. This is the default topology we assume.
- The **lower-limit topology** on \mathbb{R} is the topology generated by the collection of half-open intervals $[a, b) = \{x : a \leq x < b\}$. We write $\mathbb{R} = \mathbb{R}_\ell$ in this case.
- The **K-topology** on \mathbb{R} is the topology generated by all open intervals (a, b) and sets of the form $(a, b) \setminus \{1/n : n \in \mathbb{N}^*\}$. We write $\mathbb{R} = \mathbb{R}_K$ in this case.

By Theorem 1.2, we can show that \mathbb{R}_ℓ and \mathbb{R}_K are strictly finer than \mathbb{R} , but are incomparable.

Generalizing from Definition 1.3, we can generate a topology from any collection of subsets that covers X (i.e. without necessarily having the second condition).

Definition 1.5. Let \mathcal{S} be a collection of subsets of X that covers X . The **topology generated by \mathcal{S}** is the collection of all possible unions of all possible *finite intersections* of elements in \mathcal{S} . We say that \mathcal{S} is a **subbasis** of this topology.

2 Separation Axioms

Definition 2.1. A topological space X is said to satisfy the T_1 *Axiom* if all singleton sets are closed. Assuming X is T_1 , we say that it could satisfy

- **T_2 (Hausdorff):** For each pair of distinct points $x_1, x_2 \in X$, there exist neighborhoods U_1, U_2 respectively that are disjoint.
- **T_3 (Regular):** For any $x \in X$ and closed set A of X not containing x , there exist open sets U_1, U_2 containing $\{x\}, A$ respectively and are disjoint.
- **T_4 (Normal):** For each pair of disjoint closed sets A, B of X , there exist open sets U_1, U_2 containing A, B respectively and are disjoint.

3 Order, Product, and Subspace Topology

Definition 3.1. Let X have a simple order relation $<$. The *intervals* of X are:

$$\begin{aligned} (a, b) &= \{x \in X : a < x < b\} & [a, b) &= \{x \in X : a \leq x < b\} \\ (a, b] &= \{x \in X : a < x \leq b\} & [a, b] &= \{x \in X : a \leq x \leq b\} \end{aligned}$$

and the *rays* of X are:

$$\begin{aligned} (a, +\infty) &= \{x \in X : x > a\} & (-\infty, a) &= \{x \in X : x < a\} \\ [a, +\infty) &= \{x \in X : x \geq a\} & (-\infty, a] &= \{x \in X : x \leq a\} \end{aligned}$$

The *order topology* on an ordered set X is the topology generated by the basis consisting of open intervals (a, b) and open rays $(a, +\infty), (-\infty, a)$.

Example 3.1.

1. The order topology on \mathbb{R} is the standard topology.
2. The order topology on \mathbb{Z} is the discrete topology.
3. The order topology on $\{1, 2\} \times \mathbb{N}^*$ is *not* the discrete topology. Which one-point set is not open?

Definition 3.2. Let $X = \prod_{\alpha \in J} X_\alpha$ be a cartesian product of sets $\{X_\alpha\}_{\alpha \in J}$. The *projection* of X onto index β is the function $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$.

Definition 3.3. Let X, Y be topological spaces. The **box/product topology** on $X \times Y$ is generated by the basis $\{U \times V : U, V \text{ are open in } X, Y \text{ respectively}\}$

This definition can generalize to finite cartesian products. However, for infinite cartesian products, there is a distinction between box and product topologies:

Definition 3.4. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces.

- The **box topology** on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis $\{\prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \text{ for all } \alpha\}$.
- The **product topology** on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis $\{\prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \text{ for all } \alpha, \text{ but only finitely many } \alpha \text{ satisfy } U_\alpha \neq X_\alpha\}$.

For reasons to be seen, the product topology is the most preferred one.

Theorem 3.1. $\bigcup_{\beta \in J} \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$ is a subbasis for the product topology.

Definition 3.5. Let Y be a subset of a topological space X . The **subspace topology** on Y (with respect to X) consists of sets of the form $Y \cap U$ where U is open in X , i.e.

$$V \text{ open in } Y \Leftrightarrow V = Y \cap U \text{ where } U \text{ open in } X$$

Example 3.2. $[0, 1]^2$ as a subspace of the dictionary ordered \mathbb{R}^2 and as its own dictionary order topology are different! The set $\{0.5\} \times (0.5, 1]$ is open in the former topology but not in the latter. The latter topology is called the **ordered square** I_o^2 .

4 Limit Points

Definition 4.1. Let A be a subset of a topological space X .

- The **interior** of A , denoted $\text{Int}(A)$ or $\overset{\circ}{A}$, is the largest open set contained in A .
- The **closure** of A , denoted \bar{A} , is the smallest closed set that contains A .

Definition 4.2. Let A be a subset of a topological space X . We say that $x \in X$ is a **limit point** of A if every neighborhood of x intersects $A \setminus \{x\}$.

Theorem 4.1. $\overline{A} = A \cup \{\text{limit points of } A\}$.

Proof.

- (\subseteq) We prove that if $a \in \overline{A} \setminus A$, then a is a limit point of A . Let U be any neighborhood of a . If $U \cap A = \emptyset$, then $X \setminus U$ is a closed set containing A , so $\overline{A} \cap (X \setminus U)$ is a smaller (no a) closed set than \overline{A} containing A , a contradiction. Hence $U \cap A \neq \emptyset$.
- (\supseteq) Since $A \subseteq \overline{A}$, it suffices to show that all limit points of A are in \overline{A} . Let a be a limit point of A . If $a \notin \overline{A}$, then a lies in the open set $X \setminus \overline{A}$ and hence has a neighborhood $U \subseteq X \setminus \overline{A}$. This means $U \cap \overline{A} = \emptyset \Rightarrow U \cap A = \emptyset$, contradiction. ■

Corollary 4.1. A set is closed if and only if it contains all its limit points.

5 Continuous Functions

Definition 5.1. Let X, Y be topological spaces. A function $f : X \rightarrow Y$ is

- **continuous at $x \in X$** if $f^{-1}(V)$ is open in X for all neighborhoods V of $f(x)$.
- **continuous** if $f^{-1}(V)$ is open in X for all V open in Y .
- a **homeomorphism** if f is bijective, and f and f^{-1} are continuous.

Theorem 5.1.

- In both box/product topologies, if all X_α are Hausdorff, $\prod_{\alpha \in J} X_\alpha$ is Hausdorff.
- In both box/product topologies, $\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}$.
- In just the product topology, $f = (f_\alpha)_{\alpha \in J} : A \rightarrow \prod_{\alpha \in J} X_\alpha$ is continuous if and only if each f_α is continuous.

6 Metric Topology

Definition 6.1. A *metric* on X is a function $d : X^2 \rightarrow \mathbb{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$ and equality holds if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

$d(x, y)$ is called the ***distance*** between x and y . The set $B_d(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ is called the (open) ***ε -ball centered at x*** .

Definition 6.2. The *metric topology* induced by a metric d on X is the topology generated by the collection of all open balls. A topological space X is said to be ***metrizable*** if there exists some metric that induces the topology of X .

Theorem 6.1. Given a metric d on X , the metric $\bar{d}(x, y) = \min\{d(x, y), 1\}$ induces the same topology as d does.

Example 6.1.

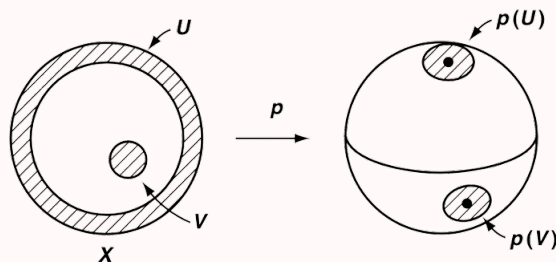
1. The ***euclidean metric*** d on \mathbb{R}^n is $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.
2. The ***standard uniform metric*** \bar{d} on \mathbb{R} is $\bar{d}(x, y) = \min\{|x - y|, 1\}$.
3. The ***uniform metric*** $\bar{\rho}$ on \mathbb{R}^J (for any index set J) is $\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha)$.
4. The metric $D(\mathbf{x}, \mathbf{y}) = \sup_i \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$ induces the product topology on \mathbb{R}^ω .

7 Quotient Topology

Definition 7.1.

- Let X, Y be topological spaces. A surjective map $p : X \rightarrow Y$ is a **quotient map** if ‘ U is open in Y if and only if $p^{-1}(U)$ is open in X ’. In other words, p is a quotient map if and only if p is continuous and maps **saturated** (some union of $p^{-1}(\{y\})$) open sets of X to open sets of Y .
- Let X be a space, A be a set, and $p : X \rightarrow A$ be surjective. The **quotient topology** induced by p is the unique topology on A where p is a quotient map.
- Let X be a space and X^* be a partition of X . X^* is a **quotient space** of X , under the quotient topology induced by the natural mapping $p : X \rightarrow X^*$.

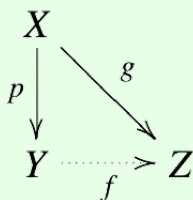
Example 7.1. Consider the unit 2-disk $X = D_2 = \{x \times y : x^2 + y^2 \leq 1\}$. We let X^* be the partition of X consisting (i) the boundary $S^1 = \{x \times y : x^2 + y^2 = 1\}$, and (ii) the singleton sets $\{x \times y\}$ for all interior points $x^2 + y^2 < 1$. Then we can show that the quotient space X^* is homeomorphic with the subspace of \mathbb{R}^3 called the unit 2-sphere $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$. (See diagram below)



Using similar ideas, we can construct a *torus* from the rectangle $[0, 1] \times [0, 1]$.

Theorem 7.1. Let $p : X \rightarrow Y$ be a quotient map. Let Z be a space and let $g : X \rightarrow Z$ be a map that is constant on each one-point preimage $p^{-1}(\{y\})$. Therefore g induces a map $f : Y \rightarrow Z$ where $f \circ p = g$. Then

- f is continuous if and only if g is continuous.
- f is a quotient map if and only if g is a quotient map.



8 Connectedness

Definition 8.1. A topological space X is said to be **connected** if there is no nontrivial clopen set A . (Equivalently, there is no $A \neq \emptyset, X$ where A and $X \setminus A$ are both open.)

Example 8.1. The subspace $(0, 1) \cup (2, 3)$ of \mathbb{R} is not connected.

Theorem 8.1.

- Let C be clopen in X . Any connected subspace of X lies within either C or $X \setminus C$.
- The union of connected subspaces that have a common point is connected.
- If A is a connected subspace of X and $A \subseteq B \subseteq \overline{A}$ then B is connected.

- A continuous function maps a connected space to a connected image.
- A cartesian product of connected spaces (in the product topology) is connected.

Definition 8.2. A simply ordered set L with more than one element is a **linear continuum** if (i) L has the least upper bound property, and (ii) for any $x < y$ there exists $x < z < y$.

Theorem 8.2. In an order topology, linear continuums, and intervals and rays in a linear continuum are connected. (Hence intervals in \mathbb{R} are connected.)

Theorem 8.3. (Intermediate Value Theorem)

Let $f : X \rightarrow Y$ be continuous, X is connected, and Y is ordered. If $a, b \in X$ and r is a point lying between $f(a), f(b)$, then there exists $c \in X$ such that $f(c) = r$.

Path-Connectedness

Definition 8.3. A space is *path-connected* if every pair $x, y \in X$ can be joined by a *path* in X : a continuous map $f : [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Example 8.2. All path-connected spaces are connected. The converse is not true, e.g. the ordered square and the topologist's sine curve

Components and Path Components

Definition 8.4. The equivalence relation $x \sim y$ where there is a (path-)connected subspace containing both x, y partitions the space into (path-)connected *components*.

Local Connectedness

Definition 8.5. A space is *locally (path-)connected at x* if for every neighborhood U of x , there is a (path-)connected neighborhood V of x contained in U .

9 Compactness

Definition 9.1. An *open covering* of X is a collection of open sets that cover X . A space X is *compact* if every open covering of X admits a finite subcovering.

Theorem 9.1.

- Every closed subspace of a compact space is compact.
- If Y is a compact subspace of a Hausdorff space and $x_0 \notin Y$, there exists disjoint open sets U, V containing $\{x_0\}$ and Y respectively.
- Every compact subspace of a Hausdorff space is closed.
- A continuous function maps compact spaces to a compact image.

Theorem 9.2. (Tychonoff Theorem)

A cartesian product of compact spaces is compact.

Compactness via Closed Sets

Definition 9.2. A collection \mathcal{C} of subsets of X has the *finite intersection property* if every finite subcollection has nonempty intersection.

Theorem 9.3. X is compact if and only if for every collection \mathcal{C} of closed sets having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty.

Compactness on \mathbb{R}^n

Theorem 9.4. Let X be a simply ordered set having the least upper bound property. Every closed interval in X is compact.

Theorem 9.5. (Heine-Borel)

A subspace A of \mathbb{R}^n is compact if and only if it is closed and bounded (in either the euclidean metric or square metric).

Functions on Compact Spaces**Theorem 9.6. (Extreme Value Theorem)**

Let $f : X \rightarrow Y$ be continuous, X is compact, and Y is ordered. There exists $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

Theorem 9.7. (Lebesgue Number Lemma)

Let \mathcal{A} be an open covering of a metric space (X, d) . If X is compact, there exists $\delta > 0$ such that any subset of X with diameter (supremum of pairwise distances) less than δ admits an element of \mathcal{A} containing it. Here δ is a *Lebesgue number* for \mathcal{A} .

Definition 9.3. A function f from metric space (X, d_X) to metric space (Y, d_Y) is *uniformly continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Theorem 9.8. (Heine-Cantor)

A continuous function on a compact space is uniformly continuous.

Theorem 9.9. A nonempty compact Hausdorff space with no isolated points (where $\{x\}$ is open) is uncountable. (Hence $[0, 1]$ is uncountable)

Proof. Let $\{x_n\}_{n \in \mathbb{N}^*}$ be an enumeration of X . Let $V_0 = X$. Given the nonempty open set V_{n-1} , choose V_n to be a nonempty open set such that $V_n \subseteq V_{n-1}$ and $x_n \notin \overline{V_n}$: We can choose this by using the Hausdorff condition on x_n and some other point in V_{n-1} , and then take the intersection of V_{n-1} and the neighborhood around this other point. The nested sequence

$$\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$$

satisfies the finite intersection property and hence by compactness there exists $x \in \bigcap_n \overline{V_n}$. Such an x cannot be any x_n , because $x_n \notin \overline{V_n}$ but $x \in \overline{V_n}$. ■

10 Countability Axioms

Definition 10.1.

- **1st Countability Axiom:** There is a countable basis at every $x \in X$, i.e. for each x there exists countably many neighborhoods $\{U_n\}_{n \in \mathbb{N}}$ so that any neighborhood of x contains some U_n .
- **2nd Countability Axiom:** There is a countable basis for X .