Multivariate Calculus

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Contents

1 Non-Cartesian Coordinate Systems

A coordinate system assigns a unique n-tuple of real numbers to each point in $n-$dimensional Euclidean space. In n dimensions, the most familiar coordinate system is the Cartesian system. However, depending on the context, it can be more useful to use other systems.

1.1 Polar Coordinates (2D)

Each point (except the origin/pole) on the 2D plane is uniquely represented by (r, θ) where $r > 0$ and θ is mod 2π. The point (r, θ) in polar coordinates is the point $(r\cos\theta, r\sin\theta)$ in Cartesian.

Conversion from Polar to Cartesian:

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

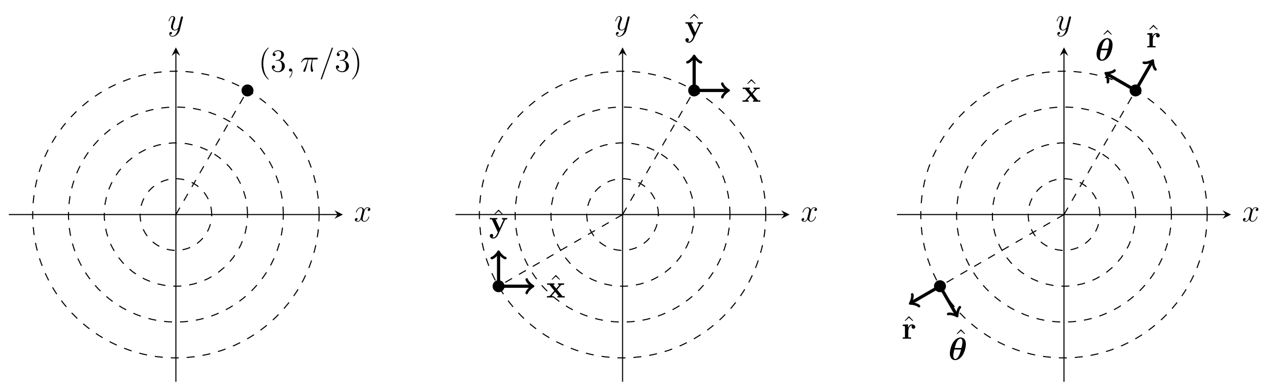


Figure 1: Polar Coordinates.

In any coordinate system, sometimes we would like to analyse what happens under a small change of one parameter, keeping other parameters constant. In polar, the parameters are r and θ. If we increase r slightly, the point moves outwards from the origin slightly. If we increase θ slightly, the point moves anticlockwise slightly. In light of these alterations, we define at every point (r, θ) the unit vectors

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

As you will learn later, $\hat{\boldsymbol{\theta}} = \mathrm{d}\hat{\mathbf{r}}/\mathrm{d}\theta$. Note that unlike $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the directions of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ depend on the point. Furthermore, we define

$$\mathbf{r} = r\hat{\mathbf{r}}$$

as the position vector of the point. This will be similarly defined for all other coordinate systems.

Note that the origin does not have a unique coordinate ($(0, \theta)$ for any θ) – it is said to be *singular*.

We also define the polar base matrix

$$\mathbf{B}_p = (\hat{\mathbf{r}}\ \hat{\boldsymbol{\theta}}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Note that $\mathbf{B}_p$ is orthogonal since $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are. Thus $\mathbf{B}_p^{-1} = \mathbf{B}_p^T$.

1.2 Spherical Coordinates (3D)

In this system, the first coordinate is again the distance r from the origin. For the next two parameters, we view the system as a globe with radius r. The second coordinate θ is the so-called latitude as measured from the north pole. The third coordinate ϕ is the longitude as measured from the positive x-axis, just as in polar. The point (r, θ, ϕ) in spherical is thus the point $(r \sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ in Cartesian[1].

Conversion from Spherical to Cartesian:

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

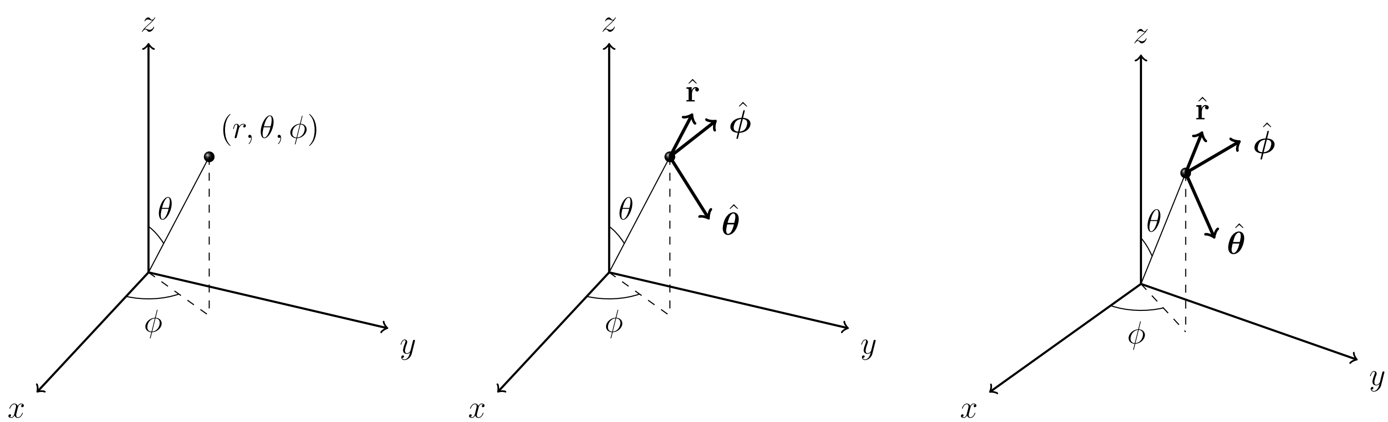


Figure 2: Spherical Coordinates

Again, we define, as shown in Figure 2, the unit vectors

$$
\hat{\mathbf{r}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \hat{\boldsymbol{\theta}} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad \hat{\boldsymbol{\phi}} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}
$$

as the directions of change of $\mathbf{r}$ when r, θ, ϕ are slightly increased. Note that $\hat{\boldsymbol{\theta}} = \mathrm{d}\hat{\mathbf{r}}/\mathrm{d}\theta$ and $\hat{\boldsymbol{\phi}}\sin\theta = \mathrm{d}\hat{\mathbf{r}}/\mathrm{d}\phi$ as you will learn later. In Figure 2, I presented two diagrams of the same configuration from different angles so that the reader can get the gist of the 3-dimensionality.

The singular points in spherical coordinates are all the points on the z-axis.

Similarly, we define the spherical base matrix

$$
\mathbf{B}_s = (\hat{\mathbf{r}}\ \hat{\boldsymbol{\theta}}\ \hat{\boldsymbol{\phi}}) = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}.
$$

Again $\mathbf{B}_s^{-1} = \mathbf{B}_s^T$.

[1]Notice the convention we have for the symbols θ and ϕ: θ is the latitute because the axis of rotation is horizontal (hence the use of the symbol θ), whereas ϕ is the longitude because the axis of rotation is vertical (hence ϕ).

1.3 Cylindrical Coordinates (3D)

In this system, we retain the z-coordinate of every point, but the x and y coordinates are transformed to polar. Therefore, the point (ρ, ϕ, z) in cylindrical (note that we used ρ instead of r because it doesn't denote the distance from the origin, but instead the perpendicular distance to the z-axis) is the point $(\rho\cos\phi, \rho\sin\phi, z)$ in Cartesian.

Conversion from Cylindrical to Cartesian:

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}$$

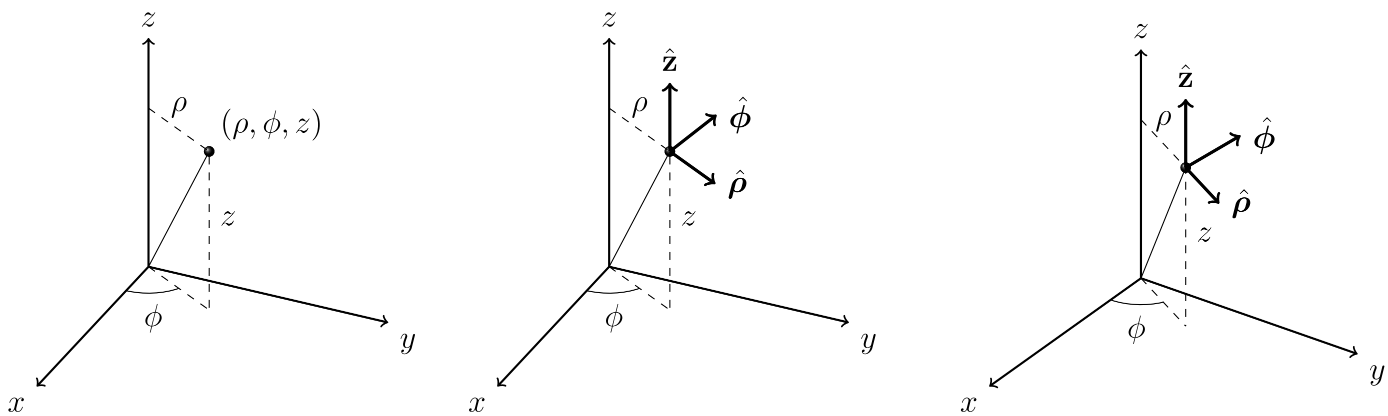


Figure 3: Cylindrical Coordinates

Similarly, we define

$$\hat{\boldsymbol{\rho}} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{\phi}} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\mathbf{B}_c = (\hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\phi}}\ \hat{\mathbf{z}}) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and also $\mathbf{B}_c^{-1} = \mathbf{B}_c^T$.

Exercise 1.1. Prove that

$$\hat{\mathbf{r}} = \frac{\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$$

Exercise 1.2. Find the conversion of spherical coordinates to cylindrical coordinates. Answer:

$$\begin{aligned} \rho &= r\sin\theta \\ \phi &= \phi \\ z &= r\cos\theta. \end{aligned}$$

2 Vector Fields

A function f is said to be a *scalar function/field* if the image of f are scalars. E.g.

$$f(x) = x^2$$
$$f(x, y) = x + y$$
$$f(x, y, z) = \sin(x \sin(y \sin z)))$$

are scalar functions. On the other hand, a function $\mathbf{f}$ is said to be a *vector field* if the image of $\mathbf{f}$ are vectors. E.g.

$$\mathbf{f}(x) = \begin{pmatrix} 3 \\ x \end{pmatrix}$$
$$\mathbf{f}(x, y) = \begin{pmatrix} x^2 + y^2 \\ 2xy \end{pmatrix}$$
$$\mathbf{f}(x, y, z) = \begin{pmatrix} xyz \\ x \end{pmatrix}$$
$$\mathbf{f}(x, y, z) = \begin{pmatrix} x^2 - y^2 \\ yz \\ xy \\ x + z \end{pmatrix}$$

are vector fields. In this handout we will usually handle vector fields with 2- or 3- dimensional vector images. For neatness, we will denote scalar functions by lowercase letters $(f, g, \cdots)$ whereas vector fields bold letters $(\mathbf{f}, \mathbf{g}, \cdots)$.

Vector fields are tricky to visualise: They assign a vector to each point on the plane / in space. Some people describe a vector field using *field lines*, which are multiple curves so that the vector associated to each point is tangent to the direction of the curve. Here are a few examples:

$\mathbf{f}(x, y) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$: $\mathbf{f}(x, y) = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$: $\mathbf{f}(x, y) = (x^2 + y^2)\hat{\mathbf{x}} + y^2\hat{\mathbf{y}}$: $\mathbf{f}(x, y) = \hat{\mathbf{x}} + (y/x)\hat{\mathbf{y}}$:

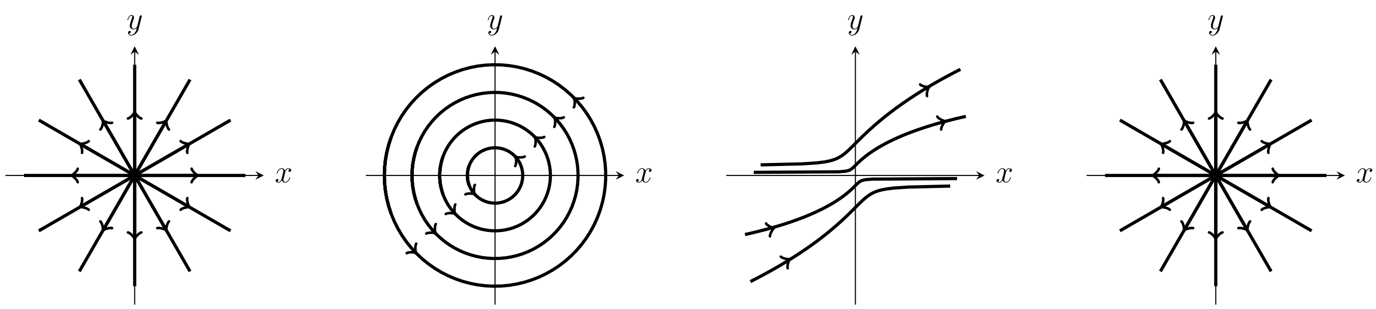


Figure 4: Vector Field Lines

The disadvantage of field lines is that it does not show the magnitude of the vector associated to every point – only the direction.

One of the methods of finding the set of field lines $(x(t), y(t))$ for a given vector field is to make the tangent vector $(\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t)$ parallel to the vector field:

$$\mathbf{f}(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \parallel \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix}$$
$$\Leftrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} f_2(x, y) = \frac{\mathrm{d}y}{\mathrm{d}t} f_1(x, y) \qquad \text{or} \qquad f_2(x, y) = \frac{\mathrm{d}y}{\mathrm{d}x} f_1(x, y)$$

For the four vector fields in Figure 4, we derive the field lines:

1. $\mathbf{f}(x, y) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$.

$$\begin{aligned}
y &= x\frac{\mathrm{d}y}{\mathrm{d}x} \\
\int x^{-1}\mathrm{d}x &= \int y^{-1}\mathrm{d}y \\
\ln|x| &= \ln|y| + c \\
x &= Cy
\end{aligned}$$

2. $\mathbf{f}(x, y) = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$.

$$\begin{aligned}
-x &= y\frac{\mathrm{d}y}{\mathrm{d}x} \\
-\int 2x\mathrm{d}x &= \int 2y\mathrm{d}y \\
-x^2 &= y^2 + c \\
x^2 + y^2 &= C
\end{aligned}$$

3. $\mathbf{f}(x, y) = (x^2 + y^2)\hat{\mathbf{x}} + y^2\hat{\mathbf{y}}$. (This one's a little trickier)

$$\begin{aligned}
y^2 &= (x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= \left(1 + \frac{x^2}{y^2}\right)
\end{aligned}$$

Apply the substitution $u = x/y$:

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}y} &= \frac{y\ \mathrm{d}x/\mathrm{d}y - x}{y^2} \\
&= \frac{1}{y}\left(\frac{\mathrm{d}x}{\mathrm{d}y} - u\right) \\
&= \frac{1}{y}(1 + u^2 - u) \\
\int \frac{\mathrm{d}y}{y} &= \int \frac{\mathrm{d}u}{u^2 - u + 1} \\
\ln|y| + C &= \int \frac{\mathrm{d}u}{(u - 1/2)^2 + 3/4} \\
\ln|y| + C &= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(u - \frac{1}{2}\right)\right) \\
\ln|y| + C &= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\frac{x}{y} - \frac{1}{2}\right)\right)
\end{aligned}$$

We can write x in terms of y but we will just stop here.

4. $\mathbf{f}(x, y) = \hat{\mathbf{x}} + (y/x)\hat{\mathbf{y}}$

$$\frac{y}{x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

and we are back to the first case.

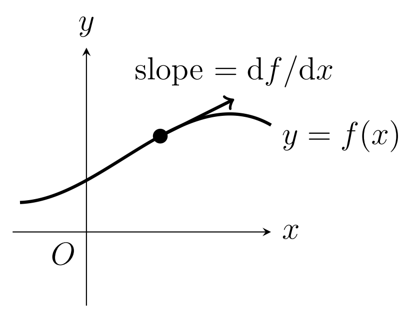
3 Differentiation

From now on, a function is said to be n-dimensional if it has n inputs, i.e. $f(x_1, x_2, \cdots, x_n)$. We only focus on functions f that are continuously differentiable in all directions.

In 1 dimension, the derivative of a function $f(x)$ is fairly easy to understand:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Intuitively, when we increase x by a small amount, f changes slightly, and the ratio of these changes is the derivative.



In 2 dimensions, things get a little trickier. Given a surface $z = f(x, y)$ and a point P on it, how can we define a derivative at that point? We will present two ways: Partial derivatives and directional derivatives.

3.1 Partial Derivatives

At P, we can draw two planes Π_x and Π_y, representing constant x- and y-coordinate respectively. If we fix x (or y), we get a cross section of the surface through P, and we can then take the derivative on that curve. In Figure 4(b), y is fixed and x is increased slightly, so the resulting slope is written as

$$\left.\frac{\partial f}{\partial x}\right|_y$$

i.e. the *partial derivative* with respect to x where y is fixed.

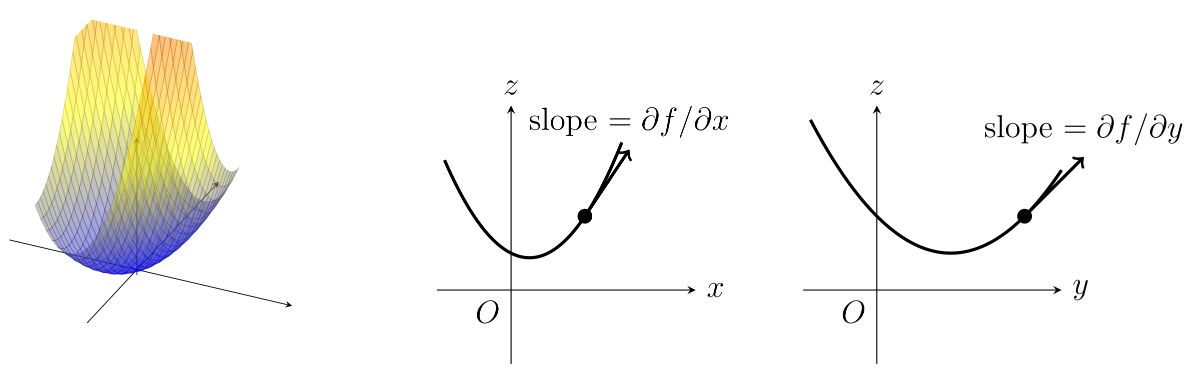


Figure 5: (a) The surface $z = f(x, y)$. (b) The plane $y = 2$. (c) The plane $x = 1$.

Example. If $f(x, y) = x^2 + xy + 2y^2$, then $\partial f/\partial x|_y = 2x + y$, and $\partial f/\partial y|_x = x + 4y$. □

Sometimes you want to instead take the cross section sliced by other vertical plane. In that case, we may not be keeping x or y constant. E.g. if we take the plane defined by the line $x + 2y = 0$, then if we increase y slightly, x will decrease twice as much. f will then change slightly, and the ratio of (change in f)/(change in y) is

$$\left.\frac{\partial f}{\partial y}\right|_{x+2y}$$

Example. Again taking $f(x, y) = x^2 + xy + 2y^2$, we want to change y and keep $x + 2y$ constant, so use the substitution $u = x + 2y$ to write f in terms of u and y instead.

$$\begin{aligned} f(x, y) &= x^2 + xy + 2y^2 \\ &= (u - 2y)^2 + (u - 2y)y + 2y^2 \\ &= 4y^2 - 3uy + u^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \left.\frac{\partial f}{\partial y}\right|_{x+2y} &= \left.\frac{\partial f}{\partial y}\right|_{u} \\ &= 8y - 3u \\ &= 8y - 3(x + 2y) \\ &= -3x + 2y. \end{aligned}$$

□

Notice from the two examples above that

$$\left.\frac{\partial f}{\partial y}\right|_{x} \qquad \text{and} \qquad \left.\frac{\partial f}{\partial y}\right|_{x+2y}$$

are obviously different. It is therefore important to know what is being kept constant when computing a partial derivative. If the context is clear, e.g. when f is written as $f(x, y)$, then we can write

$$\frac{\partial f}{\partial y} = \left.\frac{\partial f}{\partial y}\right|_{x}$$

signifying that we are keeping that other input (x) constant. In formal terms,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h} \\ \left.\frac{\partial f}{\partial y}\right|_{x+2y} &= \lim_{h\to 0} \frac{f(x - 2h, y + h) - f(x, y)}{h} \end{aligned}$$

but it is better to stick to the intuitive way of understanding it.

Exercise 3.1. Find a formal expression for $\partial f/\partial y|_{ax+by}$ where $a \neq 0$. *Answer:*

$$\left.\frac{\partial f}{\partial y}\right|_{ax+by} = \lim_{h\to 0} \frac{f(x - bh/a, y + h) - f(x, y)}{h}.$$

Why is the partial derivative undefined when $a = 0$?

Assume we are at point $P(x, y, z)$ on a surface $z = f(x, y)$. If we increase x by a small amount $\mathrm{d}x$, then f increases[2] by $(\partial f/\partial x)\mathrm{d}x$. If we increase y by a small amount $\mathrm{d}y$, then f increases by $(\partial f/\partial y)\mathrm{d}y$. That means, if we increase both x and y by $\mathrm{d}x$ and $\mathrm{d}y$ respectively, then f increases by $(\partial f/\partial x)\mathrm{d}x + (\partial f/\partial y)\mathrm{d}y$:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \left(= \left.\frac{\partial f}{\partial x}\right|_y \mathrm{d}x + \left.\frac{\partial f}{\partial y}\right|_x \mathrm{d}y \right).$$

This is the *chain rule* for partial derivatives. Of course, $\mathrm{d}f$ here depends on how much $\mathrm{d}x$ and $\mathrm{d}y$ are. If this seems a bit too casual to you, we can write the above in terms of actual partial derivatives:

$$\left.\frac{\partial f}{\partial t}\right|_u = \left.\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\right|_u + \left.\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right|_u.$$

Example. $f(x, y) = (2x + 3y)^{50}(x - 2y)^{40}$. Find $\left.\dfrac{\partial f}{\partial(x+y)}\right|_{x-y}$.

Solution 1. Let $u = x + y, v = x - y$, then

$$\begin{aligned}
f(x, y) &= \left(u + v + 3\frac{u-v}{2}\right)^{50}\left(\frac{u+v}{2} - (u - v)\right)^{40} \\
&= \frac{1}{2^{90}}(5u - v)^{50}(-u + 3v)^{40} \\
\therefore \left.\frac{\partial f}{\partial u}\right|_v &= \frac{250}{2^{90}}(5u - v)^{49}(-u + 3v)^{40} - \frac{40}{2^{90}}(5u - v)^{50}(-u + 3v)^{39} \\
&= 125(2x + 3y)^{49}(x - 2y)^{40} - 20(2x + 3y)^{50}(x - 2y)^{39} \\
&= (2x + 3y)^{49}(x - 2y)^{39}(85x - 310y).
\end{aligned}$$

□

Solution 2. It's easy to find $\left.\dfrac{\partial(2x+3y)}{\partial(x+y)}\right|_{x-y} = \dfrac{5}{2}$ and $\left.\dfrac{\partial(x-2y)}{\partial(x+y)}\right|_{x-y} = -\dfrac{1}{2}$, so by the Chain Rule,

$$\begin{aligned}
\left.\frac{\partial f}{\partial(x+y)}\right|_{x-y} &= \frac{\partial f}{\partial(2x+3y)}\left.\frac{\partial(2x+3y)}{\partial(x+y)}\right|_{x-y} + \frac{\partial f}{\partial(x-2y)}\left.\frac{\partial(x-2y)}{\partial(x+y)}\right|_{x-y} \\
&= \frac{5}{2}\frac{\partial f}{\partial(2x+3y)} - \frac{1}{2}\frac{\partial f}{\partial(x-2y)} \\
&= \frac{5}{2}\cdot 50(2x + 3y)^{49}(x - 2y)^{40} - \frac{1}{2}\cdot 40(2x + 3y)^{50}(x - 2y)^{39}. \\
&= (2x + 3y)^{49}(x - 2y)^{39}(85x - 310y).
\end{aligned}$$

□

Before we move on to 3 dimensions, let's present another analogy for the partial derivative. In Figure 4(a) we have the surface $z = f(x, y)$, and it is coloured. If we view the surface from above vertically, we will observe the xy plane with each point coloured according to its z-value. The partial derivative, is then the change in colour over the change in small x or small y.

Why is this useful? Because in 3 dimensions, we can't visually imagine the hypersurface $w = f(x, y, z)$ anymore (we need a 4D graph). However we can think of it as colouring the 3D space with a colour corresponding to a particular w-value. The concept of partial derivatives follow naturally. (Of course, if we formally define what a partial derivative is, then we can even generalise what a partial derivative means for n-dimensions, but I don't want to go that far)

For a 3 dimensional graph, we need to fix two parameters to define a cross-sectional plane. Everything we've learnt so far generalises quite easily:

[2] of course, it may *decrease* or stay constant, but that is taken care of by the sign of $\partial f/\partial x$.

Example. If $f(x, y, z) = x^3 + 2yz$, then $\partial f/\partial x = 3x^2$, $\partial f/\partial y = 2z$, and $\partial f/\partial z = 2y$. $\square$

Chain Rule:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

or equivalently

$$\left.\frac{\partial f}{\partial t}\right|_u = \frac{\partial f}{\partial x}\left.\frac{\partial x}{\partial t}\right|_u + \frac{\partial f}{\partial y}\left.\frac{\partial y}{\partial t}\right|_u + \frac{\partial f}{\partial z}\left.\frac{\partial z}{\partial t}\right|_u$$

3.2 Directional Derivatives

Here we view the coordinates in terms of vectors. Instead of $f(x, y)$ we can say $f(\mathbf{r})$. To calculate the derivative at a point P on the surface $z = f(\mathbf{r})$, we again first have to choose a direction. This direction can be represented by a vector $\mathbf{v}$. The directional derivative is defined as

$$D_{\mathbf{v}}(f) = \lim_{h \to 0} \frac{f(\mathbf{r} + h\mathbf{v}) - f(\mathbf{r})}{h}.$$

Intuitively, this is the rate of change of f if we move at point $\mathbf{r}$ with velocity $\mathbf{v}$. Therefore,

- When $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $D_{\mathbf{v}}(f) = \left.\frac{\partial f}{\partial x}\right|_y$.
- When $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $D_{\mathbf{v}}(f) = \left.\frac{\partial f}{\partial y}\right|_x$.

Say we take the cross-section plane defined by the line $ax + by = c$. If we were travelling in the y direction with speed 1, then we would travelling with velocity

$$\mathbf{v} = \begin{pmatrix} -b/a \\ 1 \end{pmatrix}$$

on that line[3]. Therefore,

$$\begin{aligned}\left.\frac{\partial f}{\partial y}\right|_{ax+by} &= D_{\mathbf{v}}(f) \\ &= \lim_{h \to 0} \frac{f((x, y) + (-b/a, 1)h) - f(x, y)}{h}. \\ &= \lim_{h \to 0} \frac{f(x - bh/a, y + h) - f(x, y)}{h}\end{aligned}$$

and this is consistent with Exercise 3.1.

Again, directional derivatives can be easily generalised to 3 dimensions, with the same definition.

[3]Imagine an xz plane travelling in the y-direction, and look at the intersection of this plane with that line.

4 Gradient

Theorem 4.1. At every point on $z = f(x, y)$, there exists a unique vector ∇f such that

$$D_{\mathbf{v}}(f) = \mathbf{v} \cdot \nabla f$$

for all vectors $\mathbf{v}$.

Proof. Existence: Let $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

$$\begin{aligned}
D_{\mathbf{v}}(f) &= \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} \\
&= \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x, y+bh)}{h} + \lim_{h \to 0} \frac{f(x, y+bh) - f(x,y)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h, y+(b/a)h) - f(x, y+(b/a)h)}{h/a} + \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h/b} \\
&= a \lim_{h \to 0} \left[\frac{\partial f}{\partial x}(x, y+(b/a)h) + o(h) \right] + b \frac{\partial f}{\partial y}(x,y) \\
&= a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = \mathbf{v} \cdot \begin{pmatrix} f_x \\ f_y \end{pmatrix}
\end{aligned}$$

where f_x and f_y denote the partial derivatives. (This notation will be used later on)

Uniqueness: If there exist two such vectors $\nabla_1 f$ and $\nabla_2 f$, then by subtraction $0 = \mathbf{v} \cdot (\nabla_1 f - \nabla_2 f)$ for all vectors $\mathbf{v}$. Hence $\nabla_1 f - \nabla_2 f = \mathbf{0}$. □

This vector ∇f is called the *gradient* of f at that point. In Cartesian coordinates,

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}, \qquad \text{(2 dimensions)}$$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \text{(3 dimensions)}$$

Before we state a few properties of ∇f, we introduce the so-called contour map of a 2D function.

The graph of a 2D function $z = f(x, y)$ is a surface, and thus if we slice the surface with the horizontal plane $z = c$ for some c, then we get a curve (not always, but we will just focus on such a situation). This is called a *contour curve* of f. Picking suitable cs, we can plot on the xy plane a number of contour curves corresponding to different cs. This is the *contour map* of f.

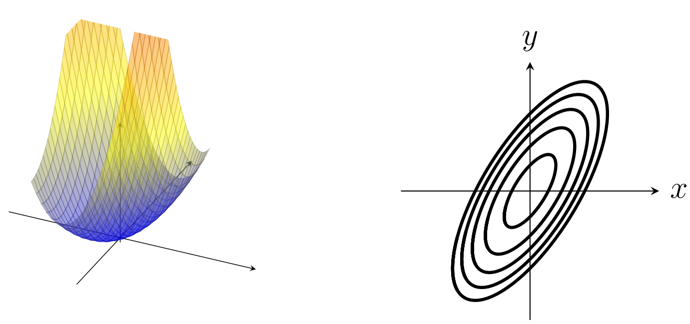


Figure 6: Contour Map of $f(x, y)$ where $c = 1, 2, 3, 4, 5$.

Theorem. Let P be a point on the xy plane and $\mathcal{C}$ a contour curve of f passing through P. Then ∇f at P is normal to $\mathcal{C}$.

Proof. Pick a vector $\mathbf{v}$ tangent to $\mathcal{C}$. Then $D_{\mathbf{v}}(f) = 0$ since f does not change on $\mathcal{C}$. Therefore, $\mathbf{v} \cdot \nabla f = 0$, i.e. ∇f is perpendicular to $\mathbf{v}$, and thus to $\mathcal{C}$.

Example. Find a vector normal to the curve $x^3 - 2x^2y + 7y^2 - 6xy = 13$ at the point $(1, 2)$.

Solution. Let $f(x, y) = x^3 - 2x^2y + 7y^2 - 6xy$. Then the curve is a contour curve of f. Directly,

$$\nabla f = \begin{pmatrix} 3x^2 - 4xy - 6y \\ -2x^2 + 14y - 6x \end{pmatrix} = \begin{pmatrix} -17 \\ 20 \end{pmatrix}$$

at $(1, 2)$. Therefore $-17\hat{\mathbf{x}} + 20\hat{\mathbf{y}}$ is an answer. □

The gradient generalises to 3 dimensions similarly. Instead of contour curves, we have contour surfaces.

Example. Find a vector normal to the surface $x^2 + y^2 + 2z^2 = 5$ at the point $(2, 1, 0)$.

Solution. Let $f(x, y, z) = x^2 + y^2 + 2z^2$. Then the surface is a contour surface of f. Directly,

$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 4z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

at $(2, 1, 0)$. Therefore $2\hat{\mathbf{x}} + 1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$ is an answer. □

4.1 Polar Coordinates

We want to write ∇f in terms of $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, \hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ instead.

Theorem 4.2. In Polar Coordinates,

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$$

This is more succinctly written as

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta}$$

Proof. From $(x, y) = (r\cos\theta, r\sin\theta)$, we know

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Hence by the Chain Rule,

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

which can be written more neatly as

$$
\begin{aligned}
\begin{pmatrix} f_r \\ f_\theta \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \mathbf{B}_p^T \begin{pmatrix} f_x \\ f_y \end{pmatrix}
\end{aligned}
$$

so the gradient is

$$
\begin{aligned}
\nabla f &= \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\
&= \mathbf{B}_p \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \end{pmatrix} \\
&= (\hat{\mathbf{r}}\ \hat{\boldsymbol{\theta}}) \begin{pmatrix} f_r \\ r^{-1} f_\theta \end{pmatrix} \\
&= f_r \hat{\mathbf{r}} + \frac{1}{r} f_\theta \hat{\boldsymbol{\theta}}
\end{aligned}
$$

□

4.2 Spherical Coordinates

Theorem 4.3. In Spherical Coordinates,

$$
\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta} \hat{\boldsymbol{\phi}} \frac{\partial}{\partial \phi}
$$

Proof. By the Chain Rule,

$$
\begin{aligned}
\begin{pmatrix} f_r \\ f_\theta \\ f_\phi \end{pmatrix} &= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix} \mathbf{B}_s^T \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \\
\Rightarrow \nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} &= \mathbf{B}_s \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-1} & 0 \\ 0 & 0 & (r\sin\theta)^{-1} \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \\ f_\phi \end{pmatrix} \\
&= (\hat{\mathbf{r}}\ \hat{\boldsymbol{\theta}}\ \hat{\boldsymbol{\phi}}) \begin{pmatrix} f_r \\ r^{-1} f_\theta \\ (r\sin\theta)^{-1} f_\phi \end{pmatrix} \\
&= f_r \hat{\mathbf{r}} + \frac{1}{r} f_\theta \hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta} f_\phi \hat{\boldsymbol{\phi}}
\end{aligned}
$$

□

Example. In physics, a conservative force is the negative gradient of its potential: $\mathbf{F} = -\nabla V$. For the gravitational force,

$$
\mathbf{F} = -G\frac{mM}{r^2}\hat{\mathbf{r}} = \alpha \frac{1}{r^2}\hat{\mathbf{r}} + 0\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\phi}}.
$$

To find a potential for this force, we seek to find solutions to

$$\begin{aligned}\frac{\partial V}{\partial r} &= -\alpha\frac{1}{r^2}\\ \frac{\partial V}{\partial \theta} &= 0\\ \frac{\partial V}{\partial \phi} &= 0.\end{aligned}$$

The last two indicate V is independent of θ, ϕ. Hence $V = V(r)$ satisfies

$$\frac{\mathrm{d}V}{\mathrm{d}r} = -\alpha\frac{1}{r^2} \qquad \Rightarrow \qquad V = -\int \alpha r^{-2}\mathrm{d}r = \alpha\frac{1}{r} + C.$$

By convention, we let $V = 0$ at infinity, so

$$V = -G\frac{mM}{r}.$$

□

4.3 Cylindrical Coordinates

Theorem 4.4. In Cylindrical Coordinates,

$$\nabla = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Proof. By the Chain Rule,

$$\begin{aligned}\begin{pmatrix} f_\rho \\ f_\phi \\ f_z \end{pmatrix} &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -r\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}\\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{B}_c^T \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}\\ \Rightarrow \nabla f &= \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \mathbf{B}_c \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_\rho \\ f_\phi \\ f_z \end{pmatrix}\\ &= (\hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\phi}}\ \hat{\mathbf{z}}) \begin{pmatrix} f_\rho \\ r^{-1} f_\phi \\ f_z \end{pmatrix}\\ &= f_\rho\hat{\boldsymbol{\rho}} + \frac{1}{r} f_\phi \hat{\boldsymbol{\phi}} + f_z\hat{\mathbf{z}}\end{aligned}$$

□

5 Multivariate Integration

In 1 dimension, an integral is 'the area under the curve'

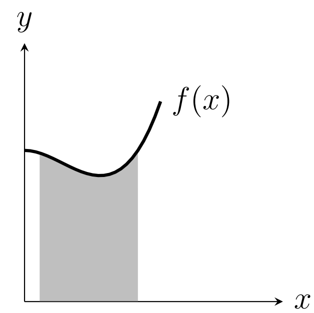


Figure 7: $\int_a^b f(x)\ \mathrm{d}x = \int_{[a,b]} f(x)\ \mathrm{d}x$

Another way of describing an integral is that it is a sum of many $f(x)\ \mathrm{d}x$ terms, where $\mathrm{d}x$ is a small difference in x and $f(x)$ is the value of the function at that particular x. This way of understanding an integral is crucial for vector integration in the next section.

In this section, we will be focusing on integrals of multivariate functions $f(x, y, \cdots)$. For a 2D function $f(x, y)$, we have a surface. Instead of integrating on intervals on a line (like $[a, b]$ above), we can now integrate on a region on the plane!

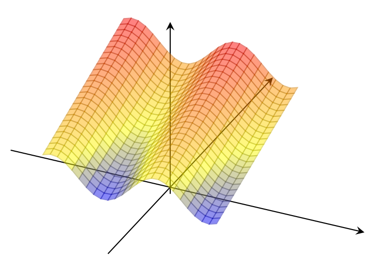


Figure 8: $f(x, y) = x + \sin y + 1$

Say we integrate on the region $R:\ x^2 + y^2 \leq 1$ in Figure 8, i.e. the unit disc. What we would get is the volume of a 'cylinder' with a distorted upper surface. Then we would write the integral as

$$\iint_R (x + \sin y + 1)\ \mathrm{d}x\ \mathrm{d}y. \qquad (*)$$

The two integral signs highlight that we are integrating on a 2D surface (though it is still okay to just write one sign). We will explore the methods to computer such integrals later.

5.1 Parametrising the Region

To compute an integral on a region, we first have to use intervals to describe the region. We can do this in any coordinate system.

Example. For the region $x^2 + y^2 \le 1$, here are two ways we can do it:

$$\begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{cases} \qquad \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$

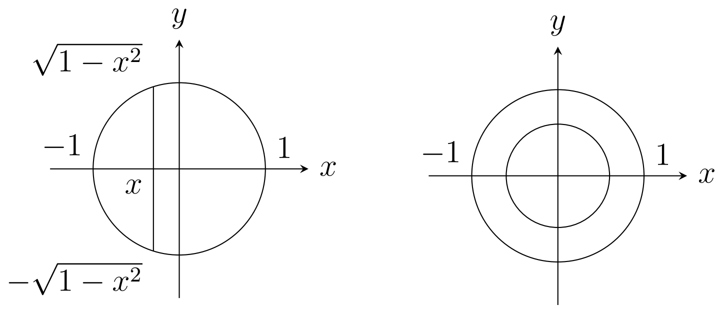


Figure 9: $x^2 + y^2 \le 1$

Example. For the triangle with vertices $(0,0),(1,0),(0,1)$, here is a way:

$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 - x \end{cases}$$

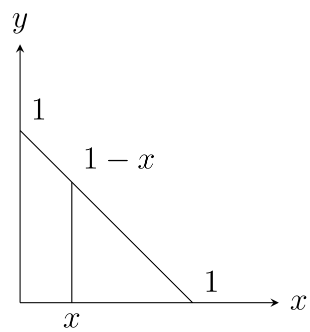


Figure 10: $x^2 + y^2 \le 1$

5.2 Computing the Integral in Cartesian Coordinates

If the region is parametrised by $a \le x \le b, c(x) \le y \le d(x)$, then

$$\iint_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ \mathrm{d}y\ \mathrm{d}x.$$

Example. The integral in $(*)$ is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x + \sin y + 1)\ \mathrm{d}y\ \mathrm{d}x = \int_{-1}^{1} 2(x+1)\sqrt{1-x^2}\ \mathrm{d}x = 2\pi$$

5.3 Computing the Integral under a Change in Coordinates

Assume $x = au + bv$ and $y = cu + dv$. Then under a change of dx and dy, the area of du dv is always actually

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

times larger than dx dy, but when we compute the integral

$$\iint f(x,y) \; \mathrm{d}u \; \mathrm{d}v,$$

each du dv is treated as having the same area as dx dy. Therefore, the correction is

$$\iint f(x,y) \; \mathrm{d}x \; \mathrm{d}y = \iint f(x,y) \begin{vmatrix} a & b \\ c & d \end{vmatrix} \; \mathrm{d}u \; \mathrm{d}v.$$

In general, when we have a change in coordinates $u(x,y)$ and $v(x,y)$, then we have to write

$$\mathrm{d}x \; \mathrm{d}y = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u \; \mathrm{d}v = \frac{\partial(x,y)}{\partial(u,v)} \mathrm{d}u \; \mathrm{d}v$$

The determinant $\dfrac{\partial(x,y)}{\partial(u,v)}$ here is called the *Jacobian* under this change of coordinates. If you forget which side to multiply the Jacobian with, just look at the 'denominators' $\partial u, \partial v$.

Example. Compute the integral $\displaystyle\iint_R (x+y)^{100}(x-y)^{50} \; \mathrm{d}x \; \mathrm{d}y$ where R is the square connecting the four points $(\pm 1, \pm 1)$.

Solution. The region R is the intersection of the two regions $-1 \le x + y \le 1$ and $-1 \le x - y \le 1$. Therefore we let $u = x + y, v = x - y$, i.e. $x = (u+v)/2, y = (u-v)/2$ then $\frac{\partial(x,y)}{\partial(u,v)} = -1/2$. We'll just take the absolute value since we are just concerned about the size of the scale factor.

$$\begin{aligned} \iint_R (x+y)^{100}(x-y)^{50} \mathrm{d}x \; \mathrm{d}y &= \frac{1}{2} \iint_R u^{100} v^{50} \; \mathrm{d}u \; \mathrm{d}v \\ &= \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} u^{100} v^{50} \; \mathrm{d}u \; \mathrm{d}v \\ &= \frac{1}{2} \int_{-1}^{1} \frac{2}{101} v^{50} \; \mathrm{d}v \\ &= \frac{1}{2} \cdot \frac{2}{101} \cdot \frac{2}{51} = \frac{2}{5151}. \end{aligned}$$

5.4 Polar Coordinates

In Polar Coordinates, the Jacobian changes according to location, but that is still okay as we are working under the integral:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$
$$\Rightarrow \mathrm{d}x \; \mathrm{d}y = r \; \mathrm{d}r \; \mathrm{d}\theta.$$

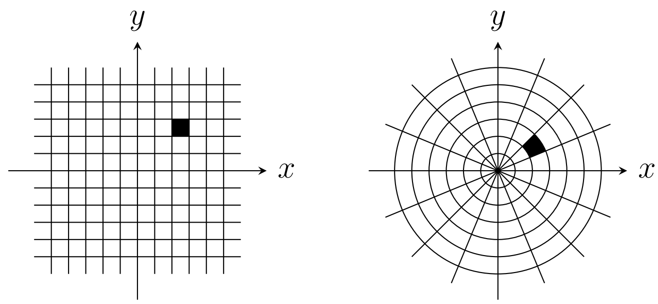


Figure 11: $\mathrm{d}x\ \mathrm{d}y$ vs $\mathrm{d}r\ \mathrm{d}\theta$

Example. Look back at the example in section 5.2:

$$\begin{aligned}
\iint_R (x+\sin y+1)\ \mathrm{d}x\mathrm{d}y &= \int_0^{2\pi}\int_0^1 (r\cos\theta+\sin(r\sin\theta)+1)r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 (r^2\cos\theta+r\sin(r\sin\theta)+r)\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\frac{1}{3}\cos\theta-\frac{\cos(\sin\theta)}{\sin\theta}+\frac{\sin(\sin\theta)}{\sin^2\theta}+1\right)\ \mathrm{d}\theta \qquad (1)\\
&= 0-0+0+\int_0^{2\pi}\mathrm{d}\theta = 2\pi \qquad (2)
\end{aligned}$$

where in (1) we used integration by parts

$$\begin{aligned}
\int_0^1 r\sin(cr)\ \mathrm{d}r &= -\frac{1}{c}\int_0^1 r\frac{\mathrm{d}}{\mathrm{d}r}\cos(cr)\ \mathrm{d}r \\
&= -\frac{1}{c}\left((r\cos(cr))_0^1-\int_0^1\cos(cr)\ \mathrm{d}r\right) \\
&= -\frac{1}{c}\left(\cos(c)-\frac{\sin(c)}{c}\right)
\end{aligned}$$

and in (2) we notice

$$\frac{\cos(\sin(\pi+\theta))}{\sin(\pi+\theta)}=\frac{\cos(\sin(\pi-\theta))}{\sin(\pi-\theta)} \qquad \text{and} \qquad \frac{\sin(\sin(\pi+\theta))}{\sin^2(\pi+\theta)}=-\frac{\sin(\sin(\pi-\theta))}{\sin^2(\pi-\theta)}$$

and thus their integrals cancel off when θ goes from 0 to 2π.

Example. The Gaussian Integral $I=\displaystyle\int_{-\infty}^{\infty}e^{-x^2}\ \mathrm{d}x$:

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty}e^{-x^2}\ \mathrm{d}x\int_{-\infty}^{\infty}e^{-y^2}\ \mathrm{d}y \\
&= \iint_{\forall\mathbb{R}^2}e^{-x^2-y^2}\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_0^{\infty}\int_0^{2\pi}e^{-r^2}r\ \mathrm{d}\theta\ \mathrm{d}r \\
&= 2\pi\int_0^{\infty}e^{-r^2}r\ \mathrm{d}r \\
&= -\pi\int_0^{\infty}e^{-r^2}\mathrm{d}(-r^2)=\pi.
\end{aligned}$$

Since $I \geq 0$, $I = \sqrt{\pi}$. $\square$

Exercise. A swimming pool is circular with 40 feet diameter. The depth is constant along the east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool when full. (*Answer:* 1800π).

5.5 Spherical and Cylindrical Coordinates

Exercise. Prove that the Jacobian for Spherical is $r^2 \sin\theta$ and the Jacobian for Cylindrical is ρ.

> Digression: **Centre of Mass and Moment of Inertia**
>
> In physics, if we have a solid object X of mass M, the *centre of mass* of X is
>
> $$\langle \mathbf{r} \rangle = \frac{1}{M} \iiint_X \mathbf{r} \, \mathrm{d}m = \frac{1}{M} \iiint_X \begin{pmatrix} x \\ y \\ z \end{pmatrix} \varrho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$
>
> where ϱ (a slightly more fancy ρ in order to distinguish it from the Cylindrical coordinate) is the density at every point. Don't be intimidated by the vectors; we are just integrating on 3 components.
>
> If we have a solid object X and a fixed axis of rotation ℓ, the *moment of inertia* of X with respect to ℓ is
>
> $$I = \iiint_X d^2 \, \mathrm{d}m = \iiint_X d^2 \varrho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$
>
> where d is the perpendicular distance to ℓ.

Example. Find the location of the centre of mass of a uniform cone with height H and base radius R.

Solution. We can describe this cone in Cylindrical Coordinates easily:

$$\begin{gathered} 0 \leq z \leq H \\ 0 \leq \rho \leq (R/H)z \\ 0 \leq \phi \leq 2\pi. \end{gathered}$$

Obviously by symmetry the centre of mass lies on the z-axis. Hence we just want to find the z-coordinate of it. Since the cone is uniform, we might as well assume $\varrho = 1, M = \pi R^2 H/3$. Thus the z-coordinate is

$$\begin{aligned} &\frac{3}{\pi R^2 H} \int_0^H \int_0^{2\pi} \int_0^{Rz/H} z\rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}z \\ &= \frac{3}{\pi R^2 H} \int_0^H \int_0^{2\pi} z \frac{R^2 z^2}{2H^2} \, \mathrm{d}\phi \, \mathrm{d}z \\ &= \frac{3 \cdot 2\pi}{\pi R^2 H} \int_0^H \frac{R^2 z^3}{2H^2} \, \mathrm{d}z \\ &= \frac{3 \cdot 2\pi}{\pi R^2 H} \frac{R^2}{2H^2} \frac{H^4}{4} = \frac{3}{4} H. \end{aligned}$$

Therefore the centre of mass is $0.75H$ vertically below the vertex. $\square$

Note. There is a quicker method to find $3H/4$. If we have n bodies with masses $m_1, \cdots, m_n$ and their centre of masses lie on the straight line, then the location of the combined centre of mass is the weighted average of their positions. Therefore, we can slice the cone into thin discs of volume $\pi(Rz/H)^2 \, \mathrm{d}z$ where z runs from 0 to H. The combined centre of mass is thus at

$$\frac{3}{\pi R^2 H} \int_0^H z \cdot \pi \frac{R^2 z^2}{H^2} \, \mathrm{d}z = \frac{3}{4} H.$$

Example. Find the location of the centre of mass of a uniform hemisphere with radius R.

Solution. We can describe this cone in Spherical Coordinates easily:

$$\begin{gathered} 0 \leq r \leq R \\ 0 \leq \theta \leq \pi/2 \\ 0 \leq \phi \leq 2\pi. \end{gathered}$$

Again the centre of mass obviously lies on the z-axis. Hence we just want to find the z-coordinate of it. We assume $\varrho = 1, M = 2\pi R^3/3$. Thus the z-coordinate is

$$\begin{aligned} &\frac{3}{2\pi R^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R z r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \frac{3}{2\pi R^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R r^3 \cos\theta \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \frac{3}{2\pi R^3} \frac{R^4}{4} \cdot 2\pi \int_0^{\pi/2} \frac{1}{2} \sin 2\theta \, \mathrm{d}\theta \\ &= \frac{3}{2\pi R^3} \frac{R^4}{4} \cdot 2\pi \cdot \frac{1}{2} = \frac{3}{8} R \end{aligned}$$

Therefore the centre of mass is $0.375R$ away from the centre. □

Example. Find the moment of inertia of a uniform ball of mass M with radius R with respect to a line ℓ passing through the centre.

Solution. Let ℓ be the z-axis. In Spherical Coordinates, $d = r \sin\theta$. Also, $\varrho = M/(4\pi R^3/3)$

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi} \int_0^R (r \sin\theta)^2 \frac{3M}{4\pi R^3} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \frac{3M}{4\pi R^3} \int_0^{2\pi} \int_0^{\pi} \int_0^R r^4 \sin^3\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= -\frac{3M}{4\pi R^3} \frac{R^5}{5} \cdot 2\pi \cdot 2 \int_0^{\pi/2} (1 - \cos^2\theta) \, \mathrm{d}(\cos\theta) \\ &= -\frac{3M}{4\pi R^3} \frac{R^5}{5} \cdot 2\pi \cdot 2 \cdot \left(-\frac{2}{3}\right) = \frac{2}{5} M R^2 \end{aligned}$$

□

6 Vector Integration

7 Divergence and Curl

The gradient operator only operates on scalar fields (the image is a vector field). Here, we introduce two operators that operate on vector fields (the first one gives back a scalar field, whereas the next one gives back a vector field).

Example. Consider a radial vector field $\mathbf{f} = r^n \hat{\mathbf{r}}$.

References

[1] Differential and Integral Equations by Collins