Tristan Chaang July 2, 2024

Topology Notes

1 Topological Spaces

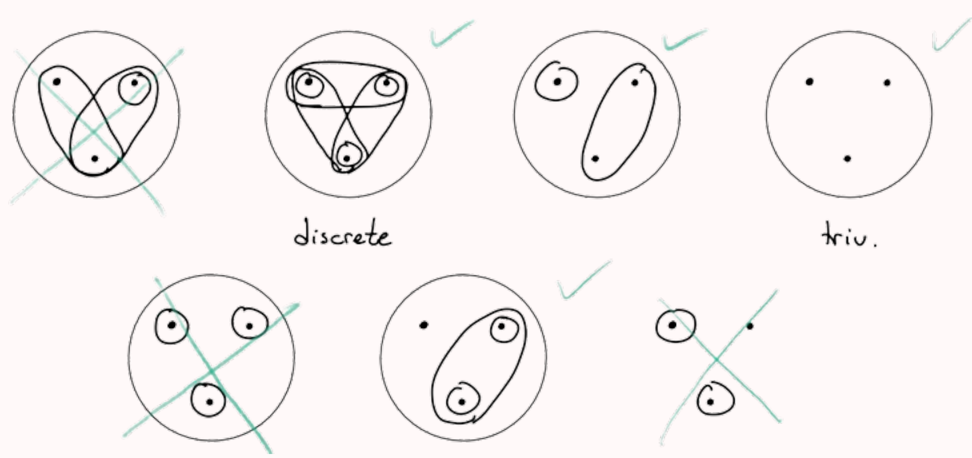
Definition 1.1.

1. A ***topology*** on a set X is a set $\mathscr{T}$ of subsets of X called ***open sets*** such that
   - $\varnothing, X \in \mathscr{T}$
   - $\mathscr{T}' \subseteq \mathscr{T} \implies \bigcup_{U \in \mathscr{T}'} U \in \mathscr{T}$. (Preserved under arbitrary unions)
   - $U_1, \cdots, U_n \in \mathscr{T} \implies \bigcap_{i=1}^{n} U_i \in \mathscr{T}$. (Preserved under finite intersections)

   $(X, \mathscr{T})$ – or just X when $\mathscr{T}$ is understood – is a ***(topological) space***.
2. Suppose $\mathscr{T}, \mathscr{T}'$ are two topologies on X with $\mathscr{T} \subseteq \mathscr{T}'$. We say $\mathscr{T}'$ is ***finer*** than $\mathscr{T}$ and $\mathscr{T}$ is ***coarser*** than $\mathscr{T}'$.
3. $A \subseteq X$ is ***closed*** if $X \backslash A$ is open. Hence $\varnothing, X$ are closed, and closedness is preserved under finite unions and arbitrary intersections.

Example 1.1.

1. The ***discrete topology*** on X is $\mathscr{T} = \mathscr{P}(X)$.
2. The ***trivial topology*** on X is $\mathscr{T} = \{\varnothing, X\}$.
3. $X = \{1, 2, 3\}$:



Definition 1.2. A set $\mathscr{B}$ of subsets of X is a ***basis*** if

- $X = \bigcup_{B \in \mathscr{B}} B$
- $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathscr{B} \implies (\exists B \in \mathscr{B})\,(x \in B \subseteq B_1 \cap B_2)$

Theorem 1.1. A basis $\mathscr{B}$ generates a topology $\mathscr{T}$ via

$$U \in \mathscr{T} \quad \Leftrightarrow \quad (\forall x \in U)\,(\exists B \in \mathscr{B})\,(x \in B \subseteq U)\,.$$

Proof. $\varnothing \in \mathscr{T}$ (vacuously) and $X \in \mathscr{T}$ since $\mathscr{B}$ covers X. We then verify the union and intersection properties:

- Suppose $U_\alpha \subseteq X$ are open, then $\bigcup_\alpha U_\alpha$ is open because

$$x \in \bigcup_\alpha U_\alpha \implies x \in U_\alpha \text{ for some } \alpha \implies x \in B_\alpha \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha$$

- Suppose U_1, U_2 are open, then $U_1 \cap U_2$ is open because

$$x \in U_1 \cap U_2 \implies \begin{cases} x \in B_1 \subseteq U_1 \text{ for some } B_1 \in \mathscr{B} \\ x \in B_2 \subseteq U_2 \text{ for some } B_2 \in \mathscr{B} \end{cases} \implies x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

  for some $B \in \mathscr{B}$. By induction, any finite intersection of open sets is open. ■

Example 1.2. Let $X = \mathbb{R}$. We can construct three topologies via the bases:

1. $\{(a, b) : a, b \in \mathbb{R}\}$ (the ***standard topology*** on $\mathbb{R}$)
2. $\{[a, b) : a, b \in \mathbb{R}\}$
3. $\{U \subseteq \mathbb{R} : U = \mathbb{R} \backslash \{x_1, \cdots, x_n\} \text{ for some } x_1, \cdots, x_n \in \mathbb{R}\}$

Note, (2) is finer than (1), and (1) is finer than (3).

Remark.

1. Uncountable intersections may not be open. E.g. $\bigcap_n (-1/n, 1/n) = \{0\}$ is not open in the standard topology on $\mathbb{R}$.
2. Different bases could generate the same topology. E.g. For $X = \mathbb{R}^2$, open balls generate the same topology as open squares do.

Definition 1.3. Let X be a space, and $A \subseteq X$.

1. $\text{int}(A) = \bigcup\{U \subseteq A : U \text{ is open}\}$ is the ***interior*** of A.
2. $\overline{A} = \bigcap\{C \supseteq A : C \text{ is closed}\}$ is the ***closure*** of A.
3. A is ***dense*** if $\overline{A} = X$.

Example 1.3.

1. $\text{int}(A) = \overline{A} = A$ in the discrete topology.
2. $\text{int}(A) = \varnothing; \overline{A} = X$ in the trivial topology for any $A \neq \varnothing, X$.
3. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Warning. A, B dense does not imply $A \cap B$ dense, e.g. take $\mathbb{Q}$ and $\mathbb{Q} + \sqrt{2}$.

Theorem 1.2.

1. A open $\Leftrightarrow$ $A = \text{int}(A)$
2. A closed $\Leftrightarrow$ $A = \overline{A}$

Definition 1.4.

1. A ***neighborhood of $x \in X$*** is an open set that contains x.
2. $x \in X$ is a ***limit point*** of A if $(\forall x \in U \in \mathscr{T})\,(A \cap U \backslash \{x\} \neq \varnothing.)$
3. $x \in X$ is an ***adherent point*** of A if $(\forall x \in U \in \mathscr{T})\,(A \cap U \neq \varnothing.)$
4. The ***boundary*** of A is $\partial A = \{x \in X : x \text{ adh pt of } A \text{ and } X\backslash A\} = \overline{A} \cap \overline{X\backslash A}$.

Theorem 1.3.

1. $\overline{A} = \{\text{adherent pts of } A\} = A \cup \{\text{limit pts of } A\} = \text{int}(A) \sqcup \partial A$.
2. $X = \text{int}(A) \sqcup \partial A \sqcup \text{int}(X\backslash A)$.

Theorem 1.4. If U_1, U_2 are dense and open, then $U_1 \cap U_2$ is dense and open.

Proof. Suppose $x \in X$. We want to show that for any $x \in U$ open we have $U \cap (U_1 \cap U_2) \neq \varnothing$.

Since U_1 is dense, $U \cap U_1 \neq \varnothing$. Since U_2 is also dense, $U \cap U_1 \cap U_2 \neq \varnothing$. ■

2 Metric Spaces

Definition 2.1.

1. A ***metric*** on a set X is a function $d : X^2 \to \mathbb{R}$ such that
   - $d(x, y) \geqslant 0$ and equality holds if and only if $x = y$
   - $d(x, y) = d(y, x)$
   - $d(x, y) + d(y, z) \geqslant d(x, z)$

   The set $B_x(\varepsilon) = \{y : d(x, y) < \varepsilon\}$ is the (open) ***ε-ball centered at x***.
2. The ***metric topology*** on (X, d) is the topology generated by the basis

$$\mathscr{B} = \{B_x(r) : x \in X, r > 0\}$$

Example 2.1. The ***euclidean metric*** d on $\mathbb{R}^n$ is $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i (x_i - y_i)^2}$.

3 Subspace Spaces

Definition 3.1. Let $(X, \mathscr{T})$ be a space and $A \subseteq X$. The ***subspace topology*** on A (with respect to X) is

$$\mathscr{T}_A = \{A \cap U : U \in \mathscr{T}\}.$$

We call A with this topology a ***subspace*** of X.

Theorem 3.1. A basis $\mathscr{B}$ for $\mathscr{T}$ defines a basis $\mathscr{B}_A$ for $\mathscr{T}_A$ via

$$\mathscr{B}_A = \{A \cap B : B \in \mathscr{B}\}.$$

Remark. If (X, d) is a metric space and $A \subseteq X$ then (A, d_A) is a metric space where $d_A(a_1, a_2) = d(a_1, a_2)$.

Theorem 3.2. Let (X, d) be a metric space. Then the metric topology on $A \subseteq X$ agrees with the subspace topology of $A \subseteq X$.

Proof. The subspace topology on A has basis $\mathscr{B}_S = \{A \cap B_x(r)\}_{x \in X}$ whereas the metric topology on A has basis $\mathscr{B}_M = \left\{B_x^A(r)\right\} = \{A \cap B_x(r)\}_{x \in A} \subseteq \mathscr{B}_S$. On the other hand, given any open U in the subspace topology and $x \in U \subseteq A$, we have $x \in A \cap B_x(r) \subseteq U$ for some $r > 0$, but this is just $x \in B_x^A(r) \subseteq U$. Since $x \in U$ was arbitrary, U is open in the metric topology too. ■

Definition 3.2. $A \subseteq X$ (space) is discrete if its subspace topology is discrete.

Example 3.1. Is $X = \{0\} \cup_n \{1/n\}$ discrete in $\mathbb{R}$? No. $\{0\}$ is not open in X. If it were, then $\exists (a, b)$ such that $(a, b) \cap X = \{0\}$, but $1/n < b$ for large n.

Warning. $B = A = \mathbb{R} \times \{0\} \subseteq X = \mathbb{R}^2$ are examples for the following statements:

1. B open in A does not imply B open in X.
2. Suppose $A \subseteq Y \subseteq X$, then the $\mathrm{int}(A)$ in Y may not be $Y \cap \mathrm{int}(A)$.

But these versions are true:

Theorem 3.3.

1. B open in A, and A open in X, then B open in X.
2. Suppose $A \subseteq Y \subseteq X$, the closure of A in Y is $Y \cap$ (closure of A in X).

4 Product Spaces

Definition 4.1. Let $\{X_\alpha\}_\alpha$ be a collection of spaces.

1. The ***product topology*** on $X_1 \times \cdots \times X_n$ is generated by the basis
$$\mathscr{B} = \{Y_1 \times \cdots \times Y_n : Y_1, \cdots, Y_n \text{ open}\}$$
2. More generally, the ***product topology*** on $\prod_\alpha X_\alpha$ is generated by the basis
$$\mathscr{B} = \{\textstyle\prod_\alpha Y_\alpha : Y_\alpha \text{ open for all } \alpha, \text{ and only finitely many } Y_\alpha \neq X_\alpha\}$$

Theorem 4.1.

1. If $A \subseteq X; B \subseteq Y$ are subspaces, then the subspace topology and product topology on $A \times B$ agree.
2. The metric topology on $\mathbb{R}^n$ agrees with the product topology on $\mathbb{R}^n$.

5 Quotient Space

Definition 5.1.

- Let X be a space, Y be a set, and $q : X \to Y$ be surjective. The ***quotient topology*** on Y induced by the ***quotient map*** q is given by

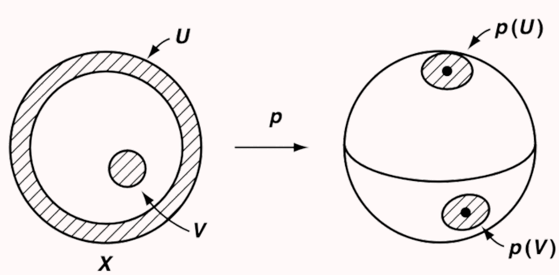
$$\mathscr{B} = \left\{U \subseteq Y : q^{-1}(U) \text{ open in } X\right\}$$

- Let $A \subseteq X$ be a subset and define $x \overset{A}{\sim} y \iff x = y$ or $x, y \in A$. We denote X/A the space on $X/\overset{A}{\sim}$ with quotient topology induced by the canonical map $q : X \twoheadrightarrow X/\overset{A}{\sim}$.

Remark. An equivalence relation $\sim$ on X determines the surjective ***canonical map*** $q : X \twoheadrightarrow X/\sim$ defined by $q(x) =$ equivalence class of x.

Example 5.1.

1. Consider the unit 2-disk $X = D^2 = \{x \times y : x^2 + y^2 \leqslant 1\}$. If we identify together all points on the boundary ∂D^2, we get the quotient space $D^2/\partial D^2$ that is homeomorphic with the subspace of $\mathbb{R}^3$ called the unit 2-sphere $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$.



2. We can construct a *torus* $S^1 \times S^1$ from the rectangle $[0,1] \times [0,1]$.

3. We can patch two disks $D^2 \sqcup D^2$ along their boundaries to obtain S^2. Formally, given a homeomorphism $\varphi : \partial D_1^2 \to D_2^2$, we have $(D_1^2 \sqcup D_2^2)/\sim\, = S^2$ where $x \sim y \iff x = y$ or $x \in \partial D_1^2, y \in \partial D_2^2, \varphi(x) = y$.

6 Continuous Functions

Definition 6.1. Let X, Y be spaces. A function $f : X \to Y$ is

- ***continuous at $x \in X$*** if $f^{-1}(V)$ is open in X for all neighborhoods V of $f(x)$.
- ***continuous*** if $f^{-1}(V)$ is open in X for all V open in Y.
- a ***homeomorphism*** if f is bijective, and f and f^{-1} are continuous.

Theorem 6.1.

1. Let $\mathscr{B}$ be a basis of X. The map $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for all $B \in \mathscr{B}$.

2. A composition of continuous functions is continuous.

3. Let $A \subseteq X$ be a subspace and $f : X \to Y$ be continuous. Then $f\,|_A$ is continuous.

4. Let $f : Z \to X \times Y$ where $f = f_X \times f_Y$. Then f is continuous if and only if f_X, f_Y are continuous.

5. Any quotient map is continuous. Given a quotient map $q : X \to Y$, $f : Y \to Z$ is continuous if and only if $g = f \circ q$ is continuous.

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

6. The following are equivalent to $f : X \to Y$ being continuous:

   (1) $f^{-1}(C)$ is closed for all closed $C \subseteq Y$.

   (2) Given any $x \in X$ and $f(x) \subseteq V$ open, there exists open U with $f(U) \subseteq V$.

   (3) $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

Proof of (6).

- Continuity is equivalent to (1) by taking complements.
- For (2), say f is continuous, then $U = f^{-1}(V)$ works. Conversely, say (2) is true. Then for any open $V \subseteq Y$, any $v \in V$ admits a neighborhood within V, which has an open preimage $U_v \subseteq X$. Then $f^{-1}(V) = \bigcup_{v \in V} U_v$ is open, and thus f is continuous.
- (1) $\Rightarrow$ (3). Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\overline{f(A)}\right)$ which is closed, we have $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$ and thus $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.
- (3) $\Rightarrow$ (1). Let $C \subseteq Y$ be closed. Then $f\left(\overline{f^{-1}(C)}\right) = \overline{f\left(f^{-1}(C)\right)} \subseteq \overline{C} = C$ and hence $\overline{f^{-1}(C)} \subseteq f^{-1}f\left(\overline{f^{-1}(C)}\right) \subseteq f^{-1}(C)$ and thus $f^{-1}(C)$ is closed. ■

Corollary 6.1. Say X, Y are metric spaces. $f : X \to Y$ is continuous if and only if

$$(\forall x \in X, \varepsilon > 0)\,(\exists \delta > 0)\,(\forall d_X(x, y) < \delta)\,(d_Y\left(f(x), f(y)\right) < \varepsilon)\,.$$

Theorem 6.2. (Pasting Lemma) Let $X = A \cup B$ be a space where A, B are closed. If $f_A : A \to Y$ and $f_B : B \to Y$ are continuous and $f_A(x) = f_B(x)$ for all $x \in A \cap B$, then $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_A(x) & x \in A \\ f_B(x) & x \in B \end{cases}$$

is continuous.

7 Limits and Continuity

Definition 7.1. $\{x_n\}_{n \in \mathbb{N}}$ in X ***converges*** to $x \in X$ if any neighborhood of x contains all but finitely many x_n. Write $x_n \to x$.

Warning. Limits may not be unique:

1. In the trivial topology, any sequence converges to all points.
2. In $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$ where $x \sim y \iff x \in \mathbb{R}_1, y \in \mathbb{R}_2, x = y \neq 0$, we have

$$1/n \to 0_1 \quad \text{and} \quad 1/n \to 0_2 \quad (\textbf{\textit{fat point}})$$

Theorem 7.1. If $x_n \to x$, then $x \in \overline{\{x_n\}_n}$.

Definition 7.2. A space X is ***first-countable*** if for any $x \in X$, there exists a countable number of neighborhoods $U_1, U_2, \cdots$ such that any neighborhood of x contains some U_i. The $\{U_i\}$ is called a ***neighborhood basis*** of x.

Theorem 7.2. If X is first-countable,

1. $x \in \overline{A} \implies \exists x_1, x_2, \cdots \in A$ such that $x_n \to x$.
2. $f : X \to Y$ is continuous if and only if $(x_n \to x) \implies (f(x_n) \to f(x))$.

8 Connectedness

Definition 8.1. A space X is ***connected*** if there is no nontrivial clopen (closed and open) set $A \subseteq X$.

Example 8.1. The subspace $(0,1) \cup (2,3)$ of $\mathbb{R}$ is not connected.

Theorem 8.1. $[a,b] \subseteq \mathbb{R}$ is connected.

Proof. Suppose the contrary, that $[a,b] = A \sqcup B$ where A, B are closed and non-empty. WLOG Assume $b \in B$. Then $s = \sup A < b$. If $s \in A$, since A is also open, there exists $(s-\varepsilon, s+\varepsilon) \subseteq A \implies \sup A \geqslant s+\varepsilon$, a contradiction. Hence $s \in B$ instead. Since B is open, there exists $(s-\varepsilon, s+\varepsilon) \subseteq B$ and thus $\sup A \leqslant s - \varepsilon$, a contradiction. ∎

Definition 8.2. A space X is ***path-connected*** if every pair $x, y \in X$ can be joined by a *path* in X: a continuous map $\gamma : I = [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example 8.2.

1. $\mathbb{R}^n$ is path-connected. Use the path $\gamma(t) = t\mathbf{x} + (1-t)\mathbf{y}$.
2. S^n is path-connected. Use the path $\gamma(t) = \dfrac{t\mathbf{x} + (1-t)\mathbf{y}}{|t\mathbf{x} + (1-t)\mathbf{y}|}$.
3. A torus is path-connected: Start with a path in I^2 and then take the quotient.

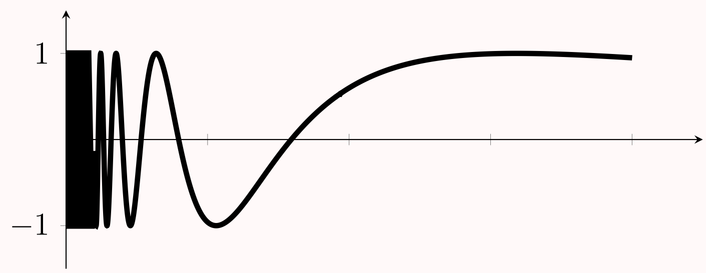
Theorem 8.2.

1. Any path-connected space is connected.
2. If $f : X \to Y$ is continuous and surjective,
   - X connected $\implies$ Y connected.
   - X path-connected $\implies$ Y path-connected.
3. Quotients of a (path-)connected space is (path-)connected.
4. A product of (path-)connected spaces is (path-)connected.

Example 8.3. The ***topologist's sine curve*** defined by

$$X = \{(x \times \sin(1/x)) : x > 0\} \cup \{0\} \times [-1, 1]$$

is connected but not path-connected.



Definition 8.3. The equivalence relation $x \sim y$ where there is a (path-)connected subspace containing both x, y partitions the space into (path-)connected ***components***.

9 Compactness

Definition 9.1.

1. An ***open cover*** of X is a collection of open sets that cover X. A space X is ***compact*** if every open cover of X admits a finite subcover.
2. A space X is ***sequentially compact*** if every sequence of points in X admits a convergent subsequence.

Theorem 9.1. 1st-countable + compact $\implies$ sequentially compact.

Proof. Suppose $\{x_n\}_n$ does not have a convergent subsequence. Let $x \in X$, then there exists a countable neighborhood basis $U_1, U_2, \cdots$. We can safely let $U_1 \supseteq U_2 \supseteq \cdots$ by taking successive intersections. Since there is no subsequence that converges to x, only finitely many x_n lie in U_n for some sufficiently large n. Hence, every $x \in X$ has a neighborhood U_x that intersects $\{x_n\}_n$ at a finite number of points. Taking the union of all U_x and applying compactness shows that $\{x_n\}_n$ is finite, so we can conclude by the pigeonhole principle. ■

Theorem 9.2.

1. Every closed subspace of a compact space is compact.
2. A continuous function maps compact spaces to a compact image.
3. Suppose X is compact and $C_1 \supseteq C_2 \supseteq \cdots$ is a sequence of closed and non-empty sets. Then $\bigcup_n C_n$ is non-empty.
4. A product of compact spaces is compact (Infinite case is hard: Tychonoff's Thm)
5. $[a, b]$ is compact.

Proof of (4). Suppose $[a, b] = \bigcup_\alpha U_\alpha$. Then

$$S = \{x \in [a, b] : [a, b] \text{ can be covered by finitely many } U_\alpha\}$$

contains $a \in S$ and is bounded above by b. Hence S has a supremum s.

Claim. $s \in S$.
Proof. Let $s \in U_\beta$ for some β, so there exists $(s - \varepsilon, s + \varepsilon) \subseteq U_\beta$. If $s \notin S$, just add U_β to the finite subcover of $[a, s - \varepsilon/2]$. □

Claim. $s = b$.
Proof. If not, then similarly, just add U_β to the finite subcover of $[a, s]$. □

Therefore $[a, b]$ can be covered by finitely many U_α. ■

Theorem 9.3. (Heine-Borel)
A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

- $(\Leftarrow)$ $X \subseteq [-M, M]^n$ is a closed subset of a compact space, so X is compact.

- ($\Rightarrow$) Compactness on the open cover $\{B_0(r)\}_{r>0}$ shows X is bounded. We then show any limit pt x of X is in X: For all $n \in \mathbb{N}^*$, $C_n := \overline{B_x 1/n} \cap X \neq \varnothing$, and thus $\bigcap_n C_n = X \cap \{x\}$ is non-empty. ■

10 Hausdorff Spaces

Definition 10.1. A space X is ***Hausdorff*** if for any distinct $x, y \in X$ there exists disjoint neighborhoods $x \in U, y \in V$.

Example 10.1.

1. The trivial topology is not Hausdorff. The discrete topology is.
2. Metric spaces are Hausdorff.
3. The finite complement topology on $\mathbb{R}$ is not Hausdorff.
4. The space $\mathbb{R}_1 \sqcup \mathbb{R}_2/\sim$ containing the fat point is not Hausdorff.

Theorem 10.1. X is Hausdorff if and only if $\Delta = \{(x \times x) : x \in X\} \subseteq X^2$ is closed.

Proof.

- ($\Rightarrow$) If X is Hausdorff, for any $x \neq y$ there exists disjoint neighborhoods U, V of x, y respectively. Then $U \times V$ is a neighborhood of $(x \times y) \in X \times Y$ disjoint from Δ. Taking the union over all $(x \times y)$ implies Δ is closed.
- ($\Leftarrow$) If Δ is closed, given any $x \neq y$ there exists a basis neighborhood $U \times V$ of $(x \times y)$ disjoint from Δ. Then U, V are the desired neighborhoods. ■

Theorem 10.2.

1. In a Hausdorff space, a sequence of points converge to at most one point.
2. One-point sets in a Hausdorff space are closed.
3. A subspace of a Hausdorff space is Hausdorff.
4. A finite product of Hausdorff spaces is Hausdorff.
5. A compact subspace of a Hausdorff space is closed.

Warning. A quotient of a Hausdorff space may not be Hausdorff.

11 Normal Spaces

Definition 11.1.

1. X is $\boldsymbol{T_1}$ if one-point sets are closed.
2. A space is ***normal*** if it is T_1, and, for any pair of disjoint closed sets $A, B \subseteq X$ there exists disjoint open sets $U, V \subseteq X$ such that $A \subseteq U, B \subseteq V$.

Remark.

1. Normal $\implies$ Hausdorff $\implies$ T_1.
2. A quotient, subspace, or product of normal space(s) need not be normal.

Example 11.1.

1. The fat point $\mathbb{R}_1 \sqcup \mathbb{R}_2/\sim$ is T_1 but not Hausdorff.
2. The ***K-topology*** on $\mathbb{R}$ generated by $\{(a,b)\} \cup \{(a,b)\backslash \bigcup_n \{1/n\}\}$ is Hausdorff but not normal.
3. The topology $\mathbb{R}_\ell$ on $\mathbb{R}$ generated by $\{[a,b)\}$ is normal, but $\mathbb{R}_\ell^2$ is not normal.

Theorem 11.1.

1. A closed subspace A of a normal space X is normal.
2. Compact + Hausdorff $\implies$ Normal.

Proof of (2). Suppose $A, B \subseteq X$ are disjoint and closed. Fix $a \in A$. Then for each $b \in B$ there exists disjoint neighborhoods $a \in U_b, b \in V_b$. Since B is also compact, there exists finitely many V_b that cover B. The union of such finitely many V_b and the intersection of their corresponding U_b form disjoint open sets containing a and B respectively. Repeat the same procedure for every $a \in A$ and then apply compactness of A. ■

Theorem 11.2. Metric spaces are normal.

Proof. We can show that, for any subset $A \subseteq X$, the *point-to-set distance* $d(-, A) : X \to \mathbb{R}$ given by $d(x, A) = \inf_{a \in A} d(x, a)$ is continuous. For disjoint closed sets A, B, the open sets

$$U = \{x : d(x,A) < d(x,B)\}, \qquad V = \{x : d(x,A) > d(x,B)\}$$

contain A, B respectively and are disjoint. ■

Theorem 11.3. X is normal if and only if for any closed A and open U such that $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Theorem 11.4. (Urysohn's Lemma)
Let X be normal and A, B be disjoint closed sets of X. There exists a continuous map

$$f : X \to I$$

such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. Define open sets U_p for each $p \in \mathbb{Q} \cap [0,1]$ as follows: Enumerate $\mathbb{Q} \cap [0,1]$ such that 1 and 0 are the first two elements. Define $U_1 = X - B$ and by normality pick U_0 such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction, say we defined U_p for a finite number of p's and let r be the next rational in the enumeration. We must have $p < r < q$ where U_p, U_q are already defined. By normality we pick U_r such that $\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$.

Additionally, we let $U_p = \varnothing$ for all rationals $p < 0$ and $U_p = X$ for all rationals $p > 1$. Hence,

$$p < q \ \Rightarrow\ \overline{U_p} \subseteq U_q.$$

We then define $f(x) = \inf \{p : x \in U_p\}$. It is easy to see $f(A) = \{0\}$ and $f(B) = \{1\}$. We show that f is continuous.

Lemma 1. $x \in \overline{U_r} \ \Rightarrow\ f(x) \leqslant r$
Proof. If $x \in \overline{U_r}$, then $x \in U_s$ for every $s > r$. Hence $f(x) \leqslant r$. □

Lemma 2. $x \notin \overline{U_r} \ \Rightarrow\ f(x) \geqslant r$.
Proof. If $x \notin \overline{U_r}$, then $x \notin U_s$ for any $s < r$. Hence $f(x) \geqslant r$. □

Given a ball $I = (f(x) - \delta, f(x) + \delta)$, we wish to find a neighborhood U of x such that $f(U) \subseteq I$. First we choose rational numbers $p, q \in I$ such that $p < f(x) < q$. Then the open set $U_q \backslash \overline{U_p}$ is the desired neighborhood using the lemmas above. ■

Theorem 11.5. (Tietze Extension Theorem)
Let A be closed in a normal space X. Any continuous map from A to I can be extended to a continuous map from X to I. True also for $\mathbb{R}$ instead of I.

Proof. We show for $[-1,1]$ instead of I, and then for $(-1,1)$ instead of $\mathbb{R}$.

Lemma. If $f : A \to [-\varepsilon, \varepsilon]$ is continuous, there exists continuous $g : X \to \mathbb{R}$ with $g(X) \subseteq [-\varepsilon/3, \varepsilon/3]$ and $(g - f)(A) \subseteq [-2\varepsilon/3, 2\varepsilon/3]$.

Proof. Applying the Urysohn Lemma on the disjoint closed sets $L = f^{-1}([-\varepsilon, -\varepsilon/3])$ and $R = f^{-1}([\varepsilon/3, \varepsilon])$, there exists $g : X \to [-\varepsilon/3, \varepsilon/3]$ such that $g(L) = \{-\varepsilon/3\}$ and $g(R) = \{\varepsilon/3\}$. This g works. □

Now let $f : A \to [-1,1]$ be continuous. Then we can find $g_1 : X \to [-1/3, 1/3]$ such that $|f(a) - g_1(a)| \leqslant 2/3$ for all $a \in A$. Then we apply the Lemma on $f - g_1$ again, so we get $g_2 : X \to [-2/9, 2/9]$ such that $|f(a) - g_1(a) - g_2(a)| \leqslant 4/9$. Recursively, we get a sequence of functions g_n such that $g_{n+1} : X \to [-(2/3)^n/3, (2/3)^n/3]$ and

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leqslant \left(\frac{2}{3}\right)^{n+1}.$$

By the Weierstrass M-test, $g(x) = \sum_{n=1}^{\infty} g_n(x)$ converges to the desired function (Exercise).

To show the $(-1,1)$ version, take g from the $[-1,1]$ case. Apply the Urysohn Lemma to the disjoint closed sets A and $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$ to get a continuous $\varphi : X \to [0,1]$ so that $\varphi(D) = \{0\}$ and $\varphi(A) = \{1\}$. Then $h(x) = \varphi(x)g(x)$ works ($|h(x)| < 1$). ■

Urysohn Metrization Theorem

Definition 11.2.

1. A space is ***second-countable*** if it has a countable basis.
2. A space is ***metrizable*** if it is homeomorphic to a metric space.

Theorem 11.6. (Urysohn Metrization Theorem)
2nd countable + Normal $\implies$ Metrizable.

Proof. We first note that $I^\omega = \{\mathbf{x} = (x_1, x_2, \cdots) : x_i \in I\}$ with the metric

$$d(\mathbf{x}, \mathbf{y}) = \sup_n \frac{|x_n - y_n|}{n}.$$

is a metric space. Let X be normal with a countable basis $\mathscr{B}$. We will embed X into I^ω.

Lemma. There exists a collection $\{f_n : X \to I\}_{n\in\mathbb{N}}$ of continuous functions such that given any $x \in X$ and any neighborhood U, there exists some f_n that is positive at x but vanishes outside U.

Proof. For each $B, C \in \mathscr{B}$ with $\overline{B} \subseteq C$, apply the Urysohn Lemma to construct a continuous function $g_{B,C} : X \to I$ such that $g_{B,C}\left(\overline{B}\right) = \{1\}$ and $g_{B,C}(X\backslash C) = \{0\}$. $\left\{g_{B,C} : \overline{B} \subseteq C\right\}$ is the desired collection. It is countable because $\mathscr{B} \times \mathscr{B}$ is countable, and given any x with neighborhood U, we can choose by Theorem 11.3 the sequence of open sets $x \in B \subseteq \overline{B} \subseteq C \subseteq U$, and then use $g_{B,C}$. □

Using $\{f_n\}_{n\in\mathbb{N}}$ from the Lemma, define $F : X \to I^\omega$ such that

$$F(x) = (f_0(x), f_1(x), f_2(x), \cdots)$$

- F is injective because given $x \neq y$, there exists some $f_n(x) > 0 = f_n(y)$ (Hausdorff!).
- F is continuous: Let $B_x(\varepsilon) \subseteq I^\omega$. Fix an integer $N > 2/\varepsilon$. Since each f_n is continuous, for each $1 \leqslant n \leqslant N$ there exists a neighborhood $x \in U_n$ such that $y \in U_n \implies |f_n(x) - f_n(y)| \leqslant \varepsilon/2$. Hence for any $y \in U_1 \cap \cdots \cap U_N$,

$$\begin{aligned} d\left(F(x), F(y)\right) &= \sup_n \frac{|f_n(x) - f_n(y)|}{n} \\ &\leqslant \max\left(\sup_{1\leqslant n\leqslant N} \frac{|f_n(x) - f_n(y)|}{n}, \sup_{n>N} \frac{|f_n(x) - f_n(y)|}{n}\right) \\ &\leqslant \max\left(\frac{\varepsilon}{2}, \frac{1}{N+1}\right) < \varepsilon. \end{aligned}$$

- For each open set U in X, $F(U)$ is open in $F(X)$: Let $x \in U$ and $f(x) = z$. Choose a f_N that is positive at x but vanishes outside U. Let

$$W = F(X) \cap \pi_N^{-1}\left((0, 1]\right)$$

be open in $F(X)$. We claim that $z \in W \subseteq F(U)$. Firstly, we have $z = F(x) \in W$ because $f_N(x) > 0$. Secondly, given any $F(y) \in W$, we must have $f_N(y) > 0$. Since f_N vanishes outside U, y must be in U, so $F(y) \in F(U)$.

Therefore, X is homeomorphic to its image under F, a subspace of the metric space I^ω, which is also a metric space. ■

12 Manifolds

Definition 12.1. An ***n-manifold*** is a 2nd countable Hausdorff space X such that each $x \in X$ has a neighborhood homeomorphic with an open subset of $\mathbb{R}^n$. We also write $X = X^n$. A 1-manifold is a ***curve***, and a 2-manifold is a ***surface***.

Theorem 12.1. $X^n \times Y^m$ is an $(n+m)$-manifold.

Proof. Hausdorffness and 2nd Countability follow immediately. Fix $(x \times y) \in X \times Y$, then there exists neighborhoods U, V of x, y homeomorphic to $\mathbb{R}^n, \mathbb{R}^m$ respectively. Then $U \times V$ is a neighborhood of $(x \times y)$ homeomorphic to $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$. ■

Example 12.1.

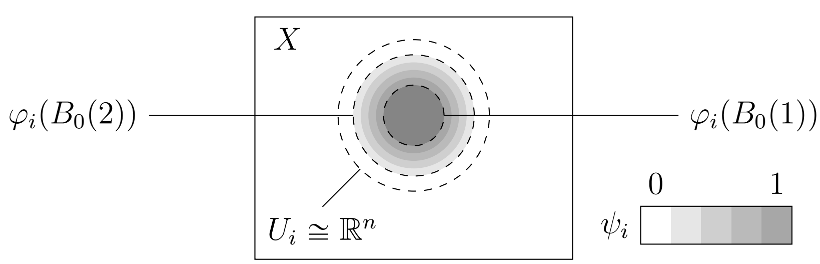
1. $\mathbb{R}^n$ is an n-manifold.
2. S^n is an n-manifold. (Write $S^n = e_1^n \cup e_2^n$ where $e^n = \text{int}(D^n) \cong \mathbb{R}^n$).
3. The ***real projective space*** $\mathbb{RP}^n = S^n / \sim$ (where $x \sim y \iff x = \pm y$) is an n-manifold.
4. $T^n = \underbrace{S^1 \times \cdots S^1}_{n}$ is an n-manifold. T^2 is a ***torus***.
5. *Fact:* Every connected curve is homeomorphic to either $\mathbb{R}$ and S^1.

Theorem 12.2. A compact n-manifold X can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Proof. Each $x \in X$ admits a neighborhood U^x with a homeo $\varphi^x : \mathbb{R}^n \to U^x$. We can choose a basis $x \in B^x \subseteq \varphi^x(B_0(1))$, and hence by compactness of X via the B^x there exists $U_1, \cdots, U_m$ with homeos $\varphi_i : \mathbb{R}^n \to U_i$ and $X \subseteq \bigcup_i \varphi_i(B_0(1))$

By Urysohn's Lemma, there exists $\rho_i : X \to I$ such that $\rho_i\left(\overline{\varphi_i(B_0(1))}\right) = \{1\}$ and $\rho_i(X \backslash \varphi_i(B_0(2))) = \{0\}$. Via the pasting lemma, let $\psi_i : X \to \mathbb{R}^n$ be the continuous function

$$\psi_i(x) = \begin{cases} \rho_i(x)\varphi_i^{-1}(x) & x \in U_i \\ (0, \cdots, 0) & \text{otherwise} \end{cases}.$$



Then $F(x) = (\rho_1(x), \cdots, \rho_m(x), \psi_1(x), \cdots, \psi_m(x))$ embeds X into $\mathbb{R}^{m(n+1)}$. ■

13 Paracompactness

Definition 13.1.

- An open cover $\{U_\alpha\}_\alpha$ of X is ***locally finite*** if every $x \in X$ has a neighborhood that intersects only finitely many U_α.
- A ***refinement*** of an open cover $\{U_\alpha\}_\alpha$ of X is an open cover $\{V_\beta\}_\beta$ such that each V_β is contained in some U_α (depends on β).
- A space X is ***paracompact*** if it is Hausdorff, and, every open cover of X admits a locally finite refinement.

Warning.

1. Some sources do not require Hausdorffness in the definition.
2. Quotient/Subspace/Product of paracompact space(s) may not be paracompact.

Example 13.1. $\mathbb{R}^n$ is paracompact. Let $B(r)$ be the open ball of radius r centered at the origin. Given any open covering $\mathscr{A}$, for each $n \in \mathbb{N}^*$ we can pick a finite number of elements of $\mathscr{A}$ that covers $\overline{B(n)}$. Intersect them with $\mathbb{R}^n \backslash \overline{B(n-1)}$. The union of these open sets is a desired locally finite refinement.

Theorem 13.1.

1. A closed subspace of a paracompact space is paracompact.
2. Compact + Hausdorff $\implies$ Paracompact
3. Metric space $\implies$ Paracompact.
4. Paracompact $\implies$ Normal.

Proof of (4). Let A, B be closed and disjoint. We first prove the case when $A = \{a\}$. For each $b \in B$ pick disjoint neighborhoods $a \in U_b, v \in V_b$. Since $(X\backslash B) \cup_b V_b$ is an open cover of X, by paracompactness there exists a locally finite refinement of V_α's that cover B. Also, x has a neighborhood W that intersects only finitely many V_α, say $V_{b_1}, \cdots, V_{b_n}$. Then the open sets $U = U_{b_1} \cap \cdots \cap U_{b_n}$ and $V = V_{b_1} \cap \cdots \cap V_{b_n}$ form a desired pair.

For the general case, we update the notation so that for each $a \in A$ there exists disjoint open sets $a \in U_a, B \subseteq V_a$. Let $\{U_\alpha\}$ be a locally finite refinement that covers A, so $b \in B$ admits a neighborhood W_b that intersects finitely many U_α, say $U_{a_1}, \cdots, U_{a_n}$. We then let

$V_b = W_b \cap_i V_{a_i}$. Then $U = \bigcup_\alpha U_\alpha$ and $V = \bigcup_{b \in B} V_b$ give the desired separation. ■

Definition 13.2. A ***partition of unity*** on X for a locally finite open cover $\{U_\alpha\}_\alpha$ is a collection of continuous $\rho_\alpha : X \to I$ such that

- $\rho_\alpha(x) > 0 \implies x \in U_\alpha$
- $\sum_\alpha \rho_\alpha(x) = 1$ (well-defined due to local finiteness)

Theorem 13.2. Every cover of a paracompact space admits a refinement that has a partition of unity.

Proof. Let $\{U_\alpha\}$ be a cover of X. For each $x \in X$ there is an $x \in U_{\alpha_x}$ and hence we can pick $x \in W_x \subseteq \overline{W_x} \subseteq U_{\alpha_x}$ by normality. Let $\{V_\beta\}$ be a locally finite refinement of $\{W_x\}$. By Urysohn's Lemma, there exists $\psi_\beta : X \to I$ such that $\psi\left(\overline{V_\beta}\right) = \{1\}$ and $\psi\left(X \backslash U_{\alpha_\beta}\right) = \{0\}$. Then $\rho_\beta(x) = \psi_\beta(x)/\sum_\gamma \psi_\gamma(x)$ is a desired partition of unity. ■

Theorem 13.3. Manifold $\implies$ Paracompact.

Proof. We first prove that a manifold X can be a limit of increasing compact sets.

Lemma. $\exists K_1, K_2, \cdots$ compact with $K_n \subseteq \operatorname{int}(K_{n+1})$ and $X = \bigcup_n \operatorname{int}(K_n)$.
Proof. Let U_i with homeos $\varphi_i : \mathbb{R}^n \to U_i$ such that $\{\varphi_i(B_0(1))\}$ covers X. Then take the compact spaces $K_n = \bigcup_{i=1}^n \bigcup_{j=1}^n \varphi_i\left(\overline{B_0(j)}\right)$ for $n \in \mathbb{N}^*$. □

Let $X = \bigcup_\alpha U_\alpha$. Then for each n there exists $U_1^n, \cdots, U_{t_n}^n$ that cover the compact space K_n. Then $V_j^n = U_j^n \backslash K_{n-1}$ form a locally finite refinement: Any $x \in X$ is contained within some $\operatorname{int}(K_n)$, which means it can only be in the sets V_j^m $(1 \leqslant j \leqslant t_m)(1 \leqslant m \leqslant n)$. This is similar to Example 13.1. ■

14 Covering Dimension

Definition 14.1.

1. The ***covering dimension*** of a space X is the infimum over $n \in \mathbb{N}$ such that

$$(\forall \text{ open cover } \{U_\alpha\})\,(\exists \text{ refinement } \{V_\beta\})\,(\forall x \in X)\,(x \text{ is in} \leqslant n+1 \text{ of the } V_\beta)$$

or equivalently

$$\dim X = \max_{\mathscr{A} \text{ open cover } X}\left[\min_{\mathscr{B} \text{ refmt of } \mathscr{A}} \underbrace{\left(\max_{x \in X} |\{B \in \mathscr{B} : x \in B\}|\right)}_{\textbf{\textit{order}} \text{ of } \mathscr{B}}\right] - 1$$

2. A ***Lebesgue number*** for an open cover $\{U_\alpha\}$ of a compact metric space is a real $\delta > 0$ such that any subset of X of diameter $< \delta$ is contained within some U_α.

Theorem 14.1. (Lebesgue's Covering Lemma)
Any open cover $\{U_\alpha\}$ of a compact metric space (X, d) has a Lebesgue number.

Proof. Since X is compact, assume $\{U_\alpha\} = \{U_1, \cdots, U_n\}$. The map $f(x) = \max\limits_{1 \leqslant i \leqslant n} d(x, X \backslash U_i) > 0$ is continuous on a compact space and thus $f(X)$ has a minimum $\delta > 0$. ∎

Example 14.1.

1. Any compact subspace of $\mathbb{R}$ has dimension at most 1.

   Proof. Note that $\mathscr{C} = \left\{(n, n+1), \left(n - \frac{1}{2}, n + \frac{1}{2}\right) : n \in \mathbb{Z}\right\}$ has order 2. Let $\mathscr{A}$ be any open covering of a compact subspace X of $\mathbb{R}$, with some Lebesgue number $\delta > 0$. The image $\mathscr{I}$ of $\mathscr{C}$ under $f : x \mapsto \delta x/2$ is an open covering whose elements have diameter $\delta/2 < \delta$, and hence is an open refinement subcover of $\mathscr{A}$. Hence

$$\begin{aligned}\dim X &= \max_{\mathscr{A} \text{ open cover } X}\left[\min_{\substack{\mathscr{B} \text{ open refinement} \\ \text{subcover of } \mathscr{A}}} (\text{order of } \mathscr{B})\right] - 1 \\ &\leqslant \max_{\mathscr{A} \text{ open cover } X} [2] - 1 = 1.\end{aligned}$$

2. $\dim I = 1$.

   Proof. We show that there is some open covering $\mathscr{A}$ such that any open refinement subcover of $\mathscr{A}$ has order at least 2. Let $\mathscr{A} = \{[0,1),(0,1]\}$ and let $\mathscr{B}$ be any open refinement subcovering. Since 0 and 1 cannot belong to the same refinement, $\mathscr{B}$ has at least two elements. Partition $\mathscr{B}$ into two nonempty parts $\mathscr{B}_1$ and $\mathscr{B}_2$. If $\mathscr{B}$ had order 1 then $\bigcup \mathscr{B}_1$ and $\bigcup \mathscr{B}_2$ disconnect $[0,1]$, a contradiction.

3. *Fact:* $\dim I^n = n$, and every compact subspace of $\mathbb{R}^n$ has dimension $\leqslant n$.

Theorem 14.2.

- If Y is a closed subspace of a finite dimensional space X, then $\dim Y \leqslant \dim X$.
- If $X = Y \cup Z$ where Y, Z are closed finite dimensional subspaces of X, then $\dim X = \max(\dim Y, \dim Z)$.
- Every compact subspace of $\mathbb{R}^N$ has dimension at most N.

Tangent: Baire's Theorem, Function Spaces and Geometry

Definition 14.2. Let X be a compact metric space.

1. $\mathcal{C}(X, \mathbb{R}^n) = \{f : X \to \mathbb{R}^n \text{ cts}\}$ is the metric space equipped with the uniform metric $d(f,g) = \sup_x |f(x) - g(x)|$.
2. For $A \subseteq X$, $\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y)$.
3. $\Delta(f) = \sup\{\operatorname{diam}(f^{-1}\{z\}) : z \in f(X)\}$ (Deviation of f from injectivity).

Remark. $\bigcap_n U_{1/n} = \{f : \Delta(f) = 0\} = \{f \text{ injective}\}$.

Theorem 14.3. (Baire's Theorem)
Let $\{U_n\}$ be a countable collection of dense open sets in a compact Hausdorff space X. Then $\bigcap_n U_n$ is dense in X.

Proof. Let W_1 be an open set. We want to show $W_1 \cap_n U_n \neq \varnothing$.

- Since U_1 is dense and open, there exists $x_1 \in W_1 \cap U_1$ open.
- Inductively, since X is normal, there exists $x_n \in W_n \subseteq \overline{W_n} \subseteq W_{n-1} \cap U_{n-1}$.

Since X is compact and $\overline{W_1} \supseteq \overline{W_2} \supseteq \cdots$, we have

$$\varnothing \neq \bigcap_n \overline{W_n} \subseteq \bigcap_n (U_n \cap W_n) \subseteq W \cap_n U_n. \qquad \blacksquare$$

Definition 14.3.

1. $\{z_0, \cdots, z_m\} \subseteq \mathbb{R}^n$ are ***geometrically independent*** if

$$\lambda_0 z_0 + \cdots + \lambda_m z_m = \mathbf{0},\ \lambda_0 + \cdots + \lambda_m = 0 \implies \lambda_0 = \cdots = \lambda_m = 0$$

2. $A \subseteq \mathbb{R}^n$ is in ***general position*** if any subset of size $n+1$ are geom. ind.

Theorem 14.4. Given $\{z_0, \cdots, z_m\} \subseteq \mathbb{R}^n$ and $\delta > 0$, there exists $\{y_0, \cdots, y_m\} \subseteq \mathbb{R}^n$ that is in general position such that all $|z_i - y_i| < \delta$.

Back to dimension theory

Theorem 14.5. (Embedding Compact Metric Spaces)
Every compact metric space X of dimension n can be embedded in $\mathbb{R}^{2n+1}$.

Define $U_\varepsilon = \{f \in \mathcal{C}(X, \mathbb{R}^{2n+1}) : \Delta(f) < \varepsilon\}$.

Claim. U_ε is open.

Proof. Let $f \in U_\varepsilon$, we want to show $\exists B_f(\delta) \subseteq U_\varepsilon$. Pick $\varepsilon < b < \Delta(f)$ and define

$$A = \{(x \times y) : d(x,y) \geqslant b\} \subseteq X^2$$

Note that $f(x) = f(y) \implies d(x,y) \leqslant \Delta(f) < b \implies (x \times y) \notin A$. Hence $|f(x) - f(y)|$ has a positive minimum 2δ on A. Now if $g \in B_f(\delta)$, then for any $(x \times y) \in A$,

$$|f(x) - g(x)| < \delta, \quad |f(y) - g(y)| < \delta, \quad |f(x) - f(y)| \geqslant 2\delta$$

so $g(x) \neq g(y)$. In other words, $g(x) = g(y) \implies d(x,y) < b \implies \Delta g \leqslant b < \varepsilon.$ $\square$

Claim. U_ε is dense. (Difficult!)

Proof. Let $f \in \mathcal{C}(X, \mathbb{R}^{2n+1})$ and $\delta > 0$, we want to find a $g \in B_f(\delta) \cap U_\varepsilon$. Firstly, we cover X with $V_1, \cdots, V_m$ such that

(1) $\operatorname{diam}(V_i) < \varepsilon/2$

(2) $\operatorname{diam}(f(V_i)) < \delta/2$

(3) Each $x \in X$ is in at most $n+1$ of the V_i.

To do this, pick a Lebesgue number $0 < \kappa < \varepsilon/4$ such that any $B_x(\kappa) \subseteq f^{-1}(B_y(\delta/4))$ for some y. Since $\dim X \leqslant n$, there exists a refinement $\{V_\beta\}_\beta$ of $\{B_x(\kappa)\}_x$ such that (3) holds. Since $V_\beta B_{x(\beta)}(\kappa)$ for some $x(\beta)$, (1) and (2) also hold. By compactness, we can find a finite cover using V_i.

Let $\varphi_i : X \to \mathbb{R}$ be a partition of unity associated to the U_i. Also, fix $x_i \in U_i$ and $z_i \in \mathbb{R}^{2n+1}$ such that $|f(x_i) - z_i| < \delta/2$ and $\{z_i\}$ is in general position. Define

$$g(x) = \sum_i \varphi_i(x) z_i.$$

Then $d(f,g) < \delta$ because

$$|g(x) - f(x)| = \left|\sum_i \varphi_i(x)(z_i - f(x_i)) + \sum_i \varphi_i(x)(f(x_i) - f(x))\right| < \sum_i \varphi_i(x)\left(\frac{\delta}{2} + \frac{\delta}{2}\right) = \delta.$$

and $g \in U_\varepsilon$ because $g(x) = g(y) \implies \sum_i (\varphi_i(x) - \varphi_i(y)) z_i = \mathbf{0} \implies \varphi_i(x) = \varphi_i(y)\ \forall i$ since x, y are in $\leqslant 2(n+1)$ of the U_i. Since $\varphi_i(x) > 0$ for some i, we have $x, y \in U_i \implies d(x,y) < \varepsilon/2$. Therefore $\Delta(g) \leqslant \varepsilon/2 < \varepsilon$. □

By Baire's theorem, $\bigcap_n U_{1/n}$ is dense and hence non-empty, i.e. there is a continuous injective $f : X \to \mathbb{R}^{2n+1}$. Also since X is compact and $f(X)$ is Hausdorff, f sends closed sets to closed sets (i.e. is closed). Hence f embeds X into $\mathbb{R}^{2n+1}$. ■

Theorem 14.6. (Embedding Manifolds)
Every manifold can embedded in some $\mathbb{R}^N$.

Proof. Let X be an m-manifold.

Lemma 1. Let $f : X \to \mathbb{R}^N$ such that $f^{-1}(\text{compact}) = \text{compact}$. Then f is closed (sends closed sets to closed sets).

Proof. Let $C \subseteq X$ be closed. Suppose $y \in \mathbb{R}^N \backslash f(C)$. By Heine-Borel, $\overline{B_y(\varepsilon)}$ is compact and hence $K = C \cap f^{-1}\left(\overline{B_y(\varepsilon)}\right)$ is compact $\implies f(K) \subseteq f(C)$ is compact $\implies V = B_y(\varepsilon) \backslash f(K)$ is a neighborhood of y. Note that

$$\begin{aligned} z \in V \cap f(C) &\implies \exists x \in f^{-1}\left(B_y(\varepsilon)\right) \cap C \subseteq K \text{ with } f(x) = z \\ &\implies z \in f(K) \implies V \cap f(C) = \varnothing \end{aligned}$$

and thus $f(C)$ is closed. □

Lemma 2. There exists continuous $f : X \to \mathbb{R}$ such that $f^{-1}(\text{compact}) = \text{compact}$.

Proof. Using the Lemma from Theorem 13.3, we can write X as a limit of increasing compact sets $\bigcup_n K_n$ where $K_n \subseteq \text{int}(K_{n+1})$. Since manifold $\implies$ paracompact $\implies$ normal, we can use Urysohn's Lemma to construct continuous maps $\varphi_n : X \to I$ such that $\varphi_n(K_n) \equiv 0$ and $\varphi_n\left(\overline{X \backslash K_{n+1}}\right) \equiv 1$. Then we define $f : X \to \mathbb{R}$ by $f = \sum_{n=1}^{\infty} \varphi_n$.

- $x \in K_n \implies \varphi_n(x) = \varphi_{n+1}(x) = \cdots = 0$ and hence f is well-defined.
- $x \notin K_n \implies \varphi_{n-1}(x) = \varphi_{n-2}(x) = \cdots = 1 \implies f(x) \geqslant n - 1$.
- f is continuous: Given any $(a, b) \subseteq \mathbb{R}$, $f^{-1}((a, b)) \subseteq K_{\lceil b+2 \rceil}$ and hence $f^{-1}((a, b))$ is the preimage of (a, b) under $\sum_{n=1}^{\lceil b+1 \rceil} \varphi_n$ (a continuous map) which is open.
- $f^{-1}(C)$ is compact for any compact $C \subseteq \mathbb{R}$: Since C is closed and bounded, $f^{-1}(C)$ is closed and contained within some K_N (compact), and hence $f^{-1}(C)$ is compact (closed subspace of a compact space). □

Take K_n and f from Lemma 2, and denote $R_n = K_n \backslash \text{int}(K_{n-1})$ and $U_n = \text{int}(K_{n+1}) \backslash K_{n-2}$. By Urysohn's Lemma again, construct $\rho_n : X \to \mathbb{R}$ with $\rho_n(R_n) \equiv 1, \rho_n(X \backslash U_n) \equiv 0$.

Since $D_n = K_{n+1} \backslash \text{int}(K_{n-2})$ is compact and metrizable (normal and 2nd countable), there exists a cts closed inj $f_n : D_n \hookrightarrow \mathbb{R}^{2m+1}$. Then define $\psi_n : X \to \mathbb{R}^{2m+1}, \psi : X \to \mathbb{R}^{4m+3}$ as

$$\psi_n(x) = \begin{cases} \rho_n(x) f_n(x) & x \in U_n \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad \psi(x) = \left(\sum_{\text{even } n} \psi_n(x), \sum_{\text{odd } n} \psi_n(x), f(x) \right).$$

ψ is injective (Exercise: $f(x) = f(y) \implies x, y \in R_\ell$, and $\sum_{i \equiv_2 \ell} \psi_i(x) = \psi_\ell(x) = f_\ell(x) = f_\ell(y) \implies x = y$) and closed (for any compact $K \subseteq \mathbb{R}^N, \psi^{-1}(K)$ is closed and contained within the compact $f^{-1}(\pi_N(K))$). Thus ψ embeds X into $\mathbb{R}^{4m+3}$. ■

15 Homotopies

From now on, assume all 'maps' are continuous.

Definition 15.1.

1. Given $f_0, f_1 : X \to Y$, a ***homotopy*** from f_0 to f_1 is $H : X \times I \to Y$ such that $f_0(x) = H(x, 0), f_1(x) = H(x, 1)$. We sometimes write $H(x,t) = f_t(x)$. If such homotopy exists, we say f_0, f_1 are ***homotopic*** ($f_0 \simeq f_1$).

2. A ***homotopy relative to A*** $\subseteq X$ (homotopy rel A) is a homotopy $H : X \times I \to Y$ such that $H(a,t) = H(a,0)$ for all $a \in A$.

3. A ***reparameterization*** of $\alpha : I \to X$ is a map $\beta : I \to X$ such that $\beta = \alpha \circ r$ where $r : I \to I$ satisfies $r(0) = 0, r(1) = 1$.

4. X, Y are ***homotopy equivalent*** ($X \simeq Y$) if there exists $f : X \to Y, g : Y \to X$ (called homotopy equivalences) such that $f \circ g \simeq \mathbf{1}_Y$ and $g \circ f \simeq \mathbf{1}_X$.

5. X is ***contractible*** if $X \simeq$ point. $f : X \to Y$ is ***nullhomotopic*** if $f \simeq$ constant.

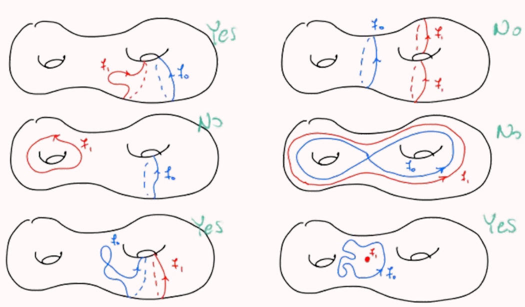
6. A ***retraction*** of X onto $A \subseteq X$ is a map $r : X \to X$ with $r|_A = \mathbf{1}_A, r(X) = A$. If it exists, A is a ***retract*** of X.

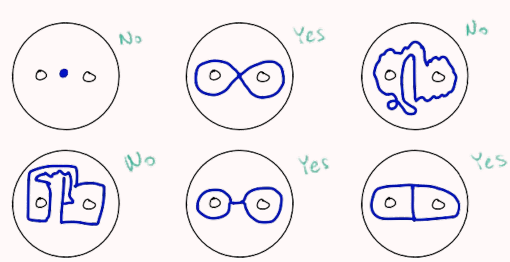
7. A ***deformation retraction*** of X onto $A \subseteq X$ is a homotopy rel A from the identity on X to a retraction of X onto A. If it exists, A is a ***deformation retract*** of X.

Example 15.1.

(L) Which paths $f : S^1 \to T^2 \# T^2$ are homotopic?

(R) $D^2 \backslash \{x_0, x_1\}$ deformation retracts to which blue sets?





Remark.

1. If β is a reparam of α then $\alpha \simeq \beta$ rel $\{0,1\}$.
2. $X \cong Y \implies X \simeq Y$ but not converse, e.g. Möbius band $\simeq S^1 \simeq$ Band $S^1 \times I$.
3. *Fact:* $X \simeq Y \iff \exists Z$ that deformation retracts to both X and Y.

16 CW Complexes

Definition 16.1. A ***CW complex / cell complex*** is a space X built as such:

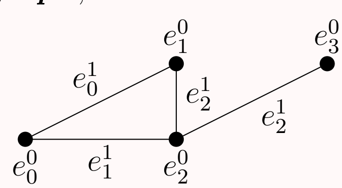
1. Start with a discrete set X^0, whose points are ***0-cells***.
2. Let D^n_α be n-balls (with $\partial D^n_\alpha = S^{n-1}_\alpha$). Inductively, form the ***n-skeleton*** X^n as the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ by identifying $x \sim \varphi_\alpha(x)$ where $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ are the ***attaching maps***. This makes $X^n = X^{n-1} \sqcup_\alpha \operatorname{int}(D^n_\alpha)$ as a set. The $e^n_\alpha = \operatorname{int}(D^n_\alpha)$ are called ***n-cells***.
3. One can stop after finite n, setting $X = X^n$. Or one can set $X = \bigcup_{n=0}^\infty X^n$, giving it the *weak topology*: $U \subseteq X$ is open $\Leftrightarrow U \cap X^n$ is open in X^n for all n.

The ***characteristic map*** of a cell e^n_α is the map

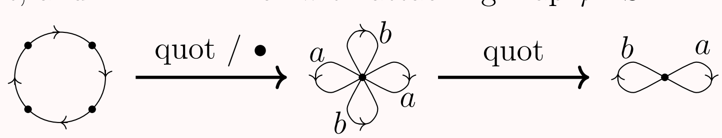
$$\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \sqcup_\beta D^n_\beta \xrightarrow{\text{quot}} X^n \hookrightarrow X$$

Example 16.1.

1. A 1-dim CW complex is a ***graph***, whose 0-cells are ***nodes*** and 1-cells are ***edges***.



2. $X = T^2$ is a CW complex, with $X^0 = \{e^0_0\}$, $X_1 = X^0 \sqcup e^0_a \sqcup e^0_b$ where $\varphi_a \equiv \varphi_b \equiv e^0_0$ being constant, and $X^2 = X^1 \sqcup e^2$ with attaching map $\varphi : S^1 \to X^1$ given by



Note: If we swap the direction of two adjacent leaves in the middle step, we get a ***Klein bottle***. Attaching maps matter!

3. The n-sphere S^n is a cell complex with two cells e^0 and e^n, with the attaching map $S^{n-1} \to e^0$. Or, we can inductively attach two n-cells to the equator S^{n-1}.
4. $\mathbb{RP}^n \cong S^n/(v \sim -v) \cong D^n/(v \sim -v : v \in \partial D^n)$ is a cell complex by attaching an n-cell to $\mathbb{RP}^{n-1}$ via the map $S^{n-1} \twoheadrightarrow \mathbb{RP}^{n-1}$. We can also have $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$.

Definition 16.2. A ***subcomplex*** of a CW complex X is a closed subspace $A \subseteq X$ that is a union of cells of X. The pair (X, A) is a ***CW pair***.

Example 16.2.

1. $\mathbb{RP}^k \subseteq \mathbb{RP}^n$ is a subcomplex $(k \leqslant n)$.
2. $S^k \subseteq S^n$ is not a subcomplex with the two-cell structure, but is a subcomplex using the recursive CW structure.

Theorem 16.1.

- If X, Y are cell complexes, then $X \times Y$ is a cell complex, whose cells are $e^m_\alpha \times e^n_\beta$ where e^m_α, e^n_β are cells of X, Y respectively.
- If (X, A) is a CW pair, then the quotient space X/A is a cell complex, whose cells are the cells of $X \backslash A$, and one new 0-cell: the image of A in X/A.

Definition 16.3. $A \subseteq X$ has the ***homotopy extension property*** if given any map $f_0 : X \to Y$ and a homotopy $f_t|_A : A \to Y$ of $f_0|_A$, we can extend $f_t|_A$ to a homotopy f_t on X. Equivalently, given any maps $H_1 : X \times \{0\} \to Y$ and $H_2 : A \times I \to Y$ that agree on $A \times \{0\}$, there exists a map $H : X \times I \to Y$ such that H agrees with both H_1, H_2 where their domains meet.

Theorem 16.2. $A \subseteq X$ has the homotopy extension property if and only if

$$X \times \{0\} \cup A \times [0, 1] \text{ is a retract of } X \times [0, 1].$$

Proof. Let $Z = X \times \{0\} \cup A \times [0, 1]$.

- If $A \subseteq X$ has h.e.p then given the maps $H_1 : X \times \{0\} \to Z$ and $H_2 : A \times I \to Z$ with

$$H_1(x, 0) = (x, 0) \qquad \text{and} \qquad H_2(a, t) = (a, t)$$

we can get an extension $H : X \times I \to Z$ constant on Z. Hence H is the retraction.

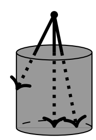
- The converse is easy if we assume A is closed. Say $r : X \times I \to Z$ is a retraction. Given any H_1, H_2 as in the definition, we can combine them via the Pasting Lemma to get $H_3 : Z \to Y$. Then $H_3 \circ r : X \times I \to Y$ is the required homotopy. For the full proof where A is not necessarily closed, see appendix of [Hatcher]. ■

Theorem 16.3. If (X, A) is a CW pair, A has the homotopy extension property.

Proof. To prove $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, we first prove

Lemma. $D^n \times \{0\} \cup \partial D^n \times I$ is a deformation retract of $D^n \times I$.

Proof. Consider radial projection r from $(\mathbf{0}, 2) \in D^n \times \mathbb{R}$:



Then $f_t = t \cdot r + (1 - t) \cdot \mathbf{1}$ is a deformation retract. □

Applying the deformation retraction to every D^n attached to X^{n-1} that is not in A^n, we get a deformation retraction H_n from $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$. Note that concatenating adjacent H_n and H_{n+1} gives a deformation retraction

$$\begin{aligned} X^{n+1} \times I &\xrightarrow{H_{n+1}} X^{n+1} \times \{0\} \cup \left(X^n \cup A^{n+1}\right) \times I \\ &\xrightarrow{H_n} X^{n+1} \times \{0\} \cup \left(\left(X^n \times \{0\} \cup \left(X^{n-1} \cup A^n\right) \times I\right) \cup \left(A^{n+1} \times I\right)\right) \\ &= X^{n+1} \times \{0\} \cup \left(X^{n-1} \cup A^{n+1}\right) \times I \end{aligned}$$

and thus by concatenating all $H_0, H_1, \cdots$ into $[1/4, 1/2], [1/8, 1/4], \cdots$ we get a deformation retract from $X \times I$ onto $X \times \{0\} \cup A \times I$. (In the infinite case, there is no continuity problem at $t = 0$ since X is given the weak topology). ■

Theorem 16.4. If (X, A) is a CW pair and A is contractible, then the quotient map $X \twoheadrightarrow X/A$ is a homotopy equivalence.

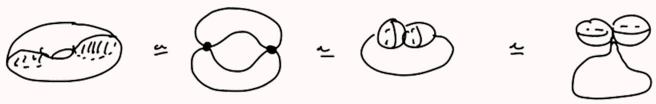
Proof. Let $f_t : X \to X$ be a homotopy extension of the contraction of A with $f_0 = \mathbf{1}_X$. Since $f_t(A) \subseteq A$ and $f_1(A) = \text{pt}$, we can construct well-defined maps $\overline{f_t}, g$ satisfying

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_t}"'] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \\ X/A \arrow[ur, "g"'] & \end{tikzcd}$$

Then $g \circ q = f_1 \simeq f_0 = \mathbf{1}_X$ and $q(g([x])) = q(g(q(x)) = q(f_1(x)) = \overline{f_1}(q(x)) = \overline{f_1}([x])$ and hence $q \circ g = \overline{f_1} \simeq \overline{f_0} = \mathbf{1}_{X/A}$, so g, q are homotopy equivalences.

Example 16.3.

1. 

2. 

17 Fundamental Groups

Definition 17.1.

1. A ***path*** on X is $\alpha : I \to X$. Define $\Omega_{x_0}(X) = \{\text{path } \alpha \mid \alpha(0) = \alpha(1) = x_0\}$.
2. Given paths $\alpha, \beta \in \Omega_{x_0}(X)$, define the ***concatenation*** $\alpha \cdot \beta \in \Omega_{x_0}(X)$ by
$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s) & 0 \leqslant s \leqslant 0.5 \\ \beta(2s-1) & 0.5 \leqslant s \leqslant 1. \end{cases}$$
3. Given a path $\gamma \in \Omega_{x_0}(X)$, define the ***reversed path*** $\overline{\gamma}(t) = \gamma(1-t)$.
4. The ***fundamental group*** of X based at x_0 is the group
$$\pi_1(X, x_0) = \Omega_{x_0}(X)/\sim$$
where $\alpha \sim \beta \iff \alpha \simeq \beta \text{ rel } \{0,1\}$, with group law $[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$ and $[\gamma]^{-1} = \overline{\gamma}$.

Theorem 17.1. Let γ be a path from x_0 to x_1. The map $\Phi_\gamma : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $\Phi([\alpha]) = [\gamma \cdot \alpha \cdot \overline{\gamma}]$ is an isomorphism.

Corollary. If X is path-connected, $\pi_1(X, x)$ are isomorphic over all $x \in X$ (say $\pi_1(X)$).

Theorem 17.2. If X, Y are path-connected, $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

Definition 17.2. X is ***simply connected*** if X is path-connected and $\pi_1(X)$ is trivial.

Definition 17.3.

1. Write $f : (X, x_0) \to (Y, y_0)$ if $f : X \to Y$ and $f(x_0) = y_0$.
2. The ***homomorphism induced*** by $f : (X, x_0) \to (Y, y_0)$ is the homomorphism
$$f_* : \pi_1(X, x_0) \to \pi_1(X, y_0)$$
given by $f_*([\alpha]) = [f \circ \alpha]$.

Theorem 17.3.

1. $(f \circ g)_* = f_* \circ g_*$.
2. If $f, g : X \to Y$ are homotopic rel x_0, then $f_* = g_*$.
3. If $f : X \to Y$ is a homotopy equivalence, then f_* is an isomorphism.

Theorem 17.4. $\pi_1(S^1) = \mathbb{Z}$.

Proof. Let $p : \mathbb{R} \to S^1$ given by $p(\lambda) = (\cos(2\pi\lambda), \sin(2\pi\lambda))$. The following two facts will be proven in the Covering Spaces chapter.

1. Given any path γ of S^1, there exists a unique path $\tilde{\gamma}$ of $\mathbb{R}$ such that $\tilde{\gamma}(0) = 0$ and $\gamma = p \circ \tilde{\gamma}$.
2. Given any homotopy $f_t : I \to S^1$, there exists a unique homotopy $\tilde{f}_t : I \to \mathbb{R}$ such that $f_t = p \circ \tilde{f}_t$

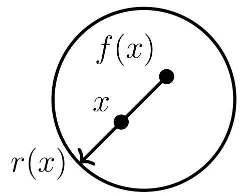
The map $\Phi([\gamma]) = \tilde{\gamma}(1) \in \mathbb{Z}$ is then a well-defined isomorphism. ∎

Theorem 17.5. If A is a retract of X, then the inclusion $i : A \hookrightarrow X$ induces an injective homomorphism i_*. If A is a defo retract of X, then i_* is an isomorphism.

Proof. Let $r : X \twoheadrightarrow A$ be a retraction. Then $r \circ i = \mathbf{1} \implies r_* \circ i_* = \mathbf{1} \implies i_*$ injective. If there is a deformation retraction, then i is a homotopy equivalence and hence i_* is an isomorphism. ∎

Theorem 17.6. (Brouwer's Fixed Point Theorem)
$f : D^2 \to D^2 \implies f(x) = x$ for some $x \in D^2$.

Proof. Otherwise, the map r defined by



is a retract from D^2 to S^1, so $i : S^1 \to D^2$ induces an injective $i_* : \mathbb{Z} \to \{0\}$, contradiction. ■

> **Theorem 17.7. (Fundamental Theorem of Algebra)**
> Every complex polynomial of positive degree has a root.

Proof. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $n > 0$. Assume f has no roots. Then

$$\gamma_t(s) = \frac{f(t \cdot e^{2\pi is})}{|f(t \cdot e^{2\pi is})|}$$

form a homotopy between γ_1 and the trivial loop γ_0. Hence $[\gamma_1] = 0 \in \mathbb{Z}$. However,

$$\delta_t(s) = \frac{F_t(e^{2\pi is})}{|F_t(e^{2\pi is})|}$$

with $F_t(x) = x^n + a_{n-1}x^{n-1}t + \cdots + a_0t^n$ is a homotopy between $\delta_1 = \gamma_1$ and the path $\delta_0(s) = e^{2\pi ins}$ that loops around the circle $n > 0$ times, and hence $[\gamma_1] = n \neq 0$. ■

18 Van Kampen's Theorem

> **Definition 18.1.** Let $i_1 : H \hookrightarrow G_1$ and $i_2 : H \hookrightarrow G_2$ be homomorphisms. The ***amalgamated free product*** of G_1 and G_2 along H, denoted as $G = G_1 *_H G_2$, is the unique group (up to isomorphism) that satisfies
>
> (1) There exists homomorphisms $\varphi_i : G_i \to G$ with $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$.
>
> (2) For any other homomorphisms $\psi_i : G_i \to K$ with $\psi_1 \circ i_1 = \psi_2 \circ i_2$, there exists a unique homomorphism $\psi : G \to K$ with $\psi \circ \varphi_i = \psi_i$.
>
> $$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "\varphi_1"] \arrow[ddr, bend left, "\psi_1"] & \\ G_2 \arrow[r, "\varphi_2"'] \arrow[drr, bend right, "\psi_2"'] & G \arrow[dr, dotted, "\psi", "\exists!"'] & \\ & & K \end{tikzcd}$$
>
> If $H = \{0\}$, then $G_1 * G_2 = G_1 *_H G_2$ is just the ***free product*** of G_1 and G_2.

Remark.

1. Such a group always exists, e.g. if $G_i = \langle S_i \mid R_i \rangle$ then

$$G_1 *_H G_2 = \left\langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \cup \left\{ i_1(h) i_2(h^{-1}) : h \in H \right\} \right\rangle.$$

   Uniqueness follows from the uniqueness of ψ between two such possible groups.

2. Think of $G_1 *_H G_2$ by first treating H as a common subgroup of G_1, G_2, then construct all possible words of finite length with letters from $G_1 \cup G_2$. When two adjacent letters in a word both come from the same G_i, or if they both belong to H, we can further simplify the word.

Example 18.1.

1. The free group with n letters is simply $F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}$.

2. The free product of $\mathbb{Z}_2 = \{1, a, a^2 = 1\}$ and itself $\mathbb{Z}_2 = \{1, b, b^2 = 1\}$ is

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, a, b, ab, ba, aba, bab, \cdots\}$$

   (This is the semi-direct product of $\mathbb{Z} = \langle c := ab \rangle, \mathbb{Z}_2 = \langle a \rangle$ with $ac = c^{-1}a$, sometimes called the *infinite dihedral group*.)

3. If we embed $H = \mathbb{Z}_2$ into the two $\mathbb{Z}_2$'s above by $h \mapsto a$ and $h \mapsto b$, then the free product collapses into

$$\mathbb{Z}_2 *_H \mathbb{Z}_2 = \left\{1, h, h^2 = 1\right\} = \mathbb{Z}_2$$

Theorem 18.1. (Van Kampen's Theorem, two-set version)
Suppose $X = U \cup V$ where $U, V, U \cap V$ are open and path-connected, then for $x_0 \in U \cap V$ we have $\pi_1(X, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$ (with $\pi_1(U \cap V, x_0) \hookrightarrow \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \hookrightarrow \pi_1(V, x_0)$ being the maps induced by the inclusions $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ respectively).

Example 18.2. $\pi_1(S^n) = \{0\}$ for $n \geqslant 2$ (*high-dim spheres are simply connected*).

S^n is the union of open neighborhoods of the north and south hemisphere, intersecting at the equator $\simeq S^{n-1}$. Hence $\pi_1(S^n) = \pi_1(e^n) *_{\pi_1(S^{n-1})} \pi_1(e^n) = \{0\} *_{\pi_1(S^{n-1})} \{0\} = \{0\}$.

Definition 18.2. Suppose $x_0 \in X, y_0 \in Y$. The ***wedge sum*** $(X, x_0) \vee (Y, y_0)$ is the space $(X \sqcup Y)/\{x_0, y_0\}$ (gluing X and Y together at x_0, y_0). Lazy: $X \vee Y$.

Example 18.3. $S^1 \vee S^1$ is the figure-eight, homemorphic to the shape ∞.

Theorem 18.2. If $\exists$ neighborhoods $x_0 \in U, y_0 \in V$ in X, Y such that $\{x_0\}, \{y_0\}$ are deformation retracts of U, V respectively, then $\pi_1(X \vee Y) = \pi_1(X) \times \pi_1(Y)$.

Proof. Let $H_t : U \to U, G_t : V \to V$ be deformation retracts onto x_0, y_0 respectively.

- We can define $\overline{G_t} : X \vee V \to X \vee V$ by

$$\begin{tikzcd} X \sqcup V \arrow[r, "\mathbf{1} \sqcup G_t"] \arrow[d, "q"'] & X \sqcup V \arrow[d, "q"] \\ X \vee V \arrow[r, "\overline{G_t}"'] & X \vee V \end{tikzcd}$$

  which is a deformation retraction of $X \vee V$ onto $X \vee \{y_0\} \cong X$. Hence $X \vee V$ deformation retracts onto X and (similarly) $U \vee Y$ deformation retracts onto Y.

- We claim that $U \vee V \subseteq X \vee Y$ is contractible. The map $F_t : U \vee V \to U \vee V$ defined by

$$\begin{tikzcd} U \sqcup V \arrow[r, "H_t \sqcup G_t"] \arrow[d, "q"'] & U \sqcup V \arrow[d, "q"] \\ U \vee V \arrow[r, "F_t"'] & U \vee V \end{tikzcd}$$

  is a deformation retraction onto $x_0 \in U \vee V$.

- By Van Kampen, $\pi_1(X \vee Y) = \pi_1(X \vee V) *_{\pi_1(U \vee V)} \pi_1(U \vee Y) = \pi_1(X) * \pi_1(Y)$. ■

Corollary 18.2. $\pi_1\left(\bigvee_{i=1}^n S^1\right) = F_n$.

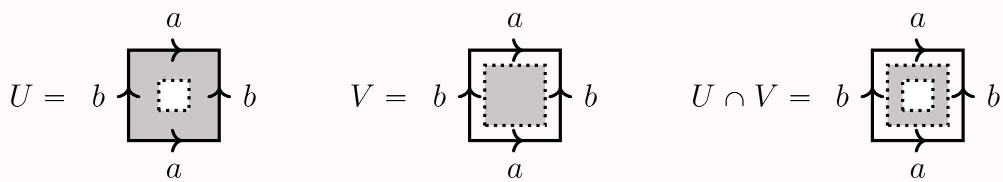
Theorem 18.3. If Γ is a connected graph, then $\pi_1(\Gamma) = F_{1-\chi(\Gamma)}$ where $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$ is the ***Euler characteristic*** of Γ.

Proof. Let T be a spanning tree of Γ, which is contractible. Then by collapsing T, the graph $\Gamma/T \simeq \Gamma$ is a wedge sum of $|E(\Gamma - T)|$ circles. Hence $\pi_1(\Gamma) = F_n$ where

$$n = |E(\Gamma)| - |E(T)| = |E(\Gamma)| - (|V(T)| - 1) = 1 - \chi(T).$$ ■

Theorem 18.4. If $i : H \to G = \langle S \mid R \rangle$, then $G *_H \{0\} = \langle S \mid R \cup i(H) \rangle$

Example 18.4. We can compute $\pi_1(T^2)$ as follows:



$$\pi_1(T^2) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \langle a, b \rangle *_{\mathbb{Z}} \{0\} = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle = \mathbb{Z}^2.$$

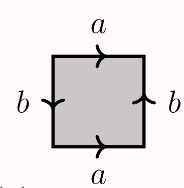
Fundamental Group of CW Complexes

Theorem 18.5.

1. Let X^2 be a CW complex obtained from X^1 by attaching 2-cells e^2_α via $\varphi_\alpha : \partial D^2_\alpha \to X^1$. For each α, let γ_α be a path on X^1 from x_0 to a point $z_\alpha \in \partial D^2_\alpha$.
$$\pi_1(X^2, x_0) = \pi_1(X^1, x_0)/N$$
where N is the normal closure of the subgroup of $\pi_1(X^1, x_0)$ generated by paths $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}]$ (treating φ_α as a closed path based at z_α).
2. Attaching n-cells ($n \geqslant 3$) does not change the fundamental group, i.e.
$$\pi_1(X, x_0) = \pi_1(X^2, x_0)$$

Example 18.5.

1. For the Klein bottle K, we have $\pi_1(K) = \langle a, b \rangle / N$ where N is generated by $aba^{-1}b$, so $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$.

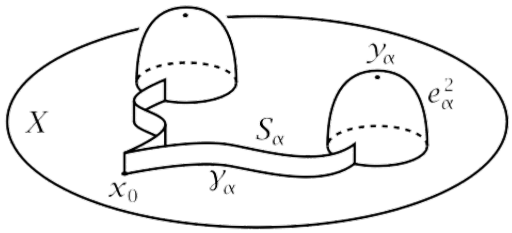


2. If X is obtained by attaching a single 2-cell to a circle $\mathbb{C}^\times$ via $\varphi(z) = z^n$, then $\pi_1(X) = \langle x \mid x^n \rangle = \mathbb{Z}_n$. In particular, $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$.

Corollary 18.5. Given any group G, there exists a space X with $\pi_1(X) = G$.

Proof. Write $G = \langle S \mid R \rangle$ and attach 2-cells (according to R) to the wedge sum $\bigvee_{s \in S} S^1_s$. ■

Proof of Theorem 18.5. First expand X^2 by bulging up the e^2_α's and then adding strips $S_\alpha = I \times I$ along each γ_α. Pick a $y_\alpha \in e^2_\alpha$ that is not on the strip. Call this larger space Z.



We then slice this space along half the height of the S_α's, and consider an open neighborhood of the top and bottom parts U, V respectively (e.g. $U = Z \backslash X^1$ and $V = Z \backslash \bigcup_\alpha \{y_\alpha\}$). U is contractible while V deformation retracts to X^1. Hence

$$\pi_1(X^2, x_0) = \pi_1(Z, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) = \{0\} *_{\pi_1(U \cap V, x_0)} \pi_1(X^1, x_0).$$

So it remains to show that $\pi_1(U \cap V, x_0)$ is generated by the $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}]$: We can apply Van Kampen again on $U \cap V$ by covering it with the open sets $A_\alpha = U \cap V \backslash \bigcup_{\beta \neq \alpha} D^2_\beta$ which deformation retract to a circle and hence is generated by $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}$. This shows (1).

To show (2), we perform the same procedure. However, in the last step, the A_α deformation retract to spheres, which are simply connected. The finite X^n case follows from induction. If X is infinite-dimensional, any closed loop at x_0 is compact and hence is contained in some finite X^n anyway. ∎

Definition 18.3.

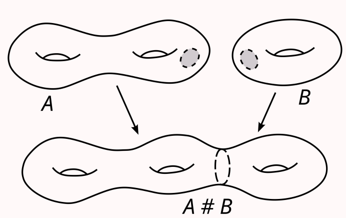
1. Let Σ, Σ' be surfaces. The ***connect sum***, $\Sigma \# \Sigma'$ is defined by

$$\left(\Sigma \backslash \operatorname{int}\left(D^2\right)\right) \sqcup \left(\Sigma' \backslash \operatorname{int}\left(D^2\right)\right) / \sim$$

   where $\sim$ identifies boundary points.

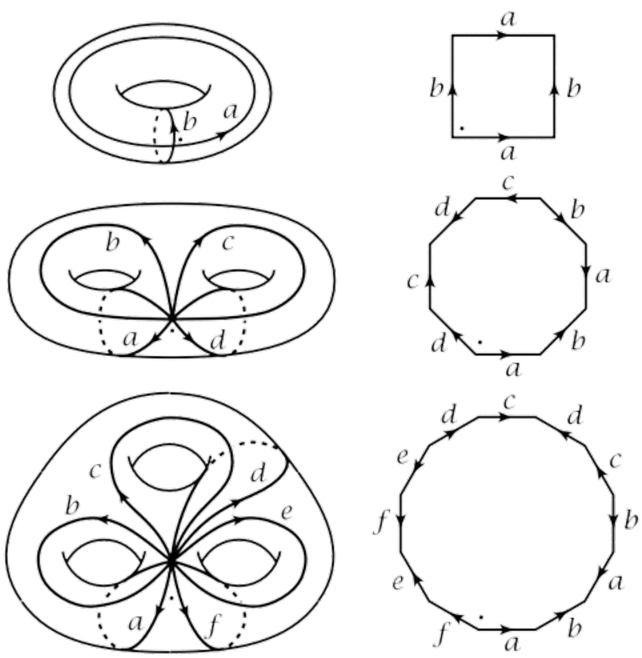
2. The ***surface of genus g*** is $\Sigma_g = \underbrace{T^2 \# \cdots \# T^2}_{n} \# S^2$ (The g-holed torus).

Example 18.6.



Theorem 18.6. $\pi_1(\Sigma_g) = \left\langle a_1, b_1, a_2, b_2, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \right\rangle$

Diagram.



19 Covering Spaces

Definition 19.1.

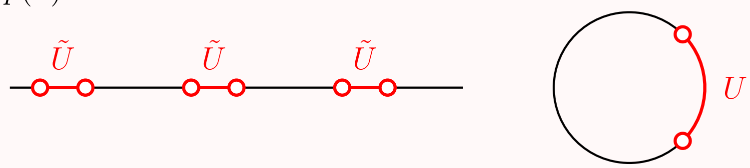
1. A ***covering space*** of X is a space $\tilde{X}$ with a map $p : \tilde{X} \to X$ such that every $x \in X$ admits a neighborhood U such that $f^{-1}(U) = \bigsqcup_\alpha \tilde{U}_\alpha$ (a disjoint union of open sets) where each $p|_{\tilde{U}_\alpha}$ is a homeomorphism. We say that U is ***evenly covered*** by the ***sheets*** $\tilde{U}_\alpha$.

2. A ***lift*** of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ with $f = p \circ \tilde{f}$.

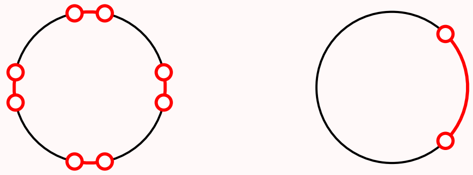
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

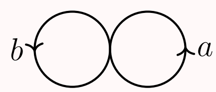
Example 19.1.

1. $p : \mathbb{R} \to S^1$, $p(\lambda) = e^{2\pi i\lambda}$.



2. $p_n : S^1 \to S^1$, $p(z) = z^n$.



3. A few covering spaces of $S^1 \vee S^1$, as 

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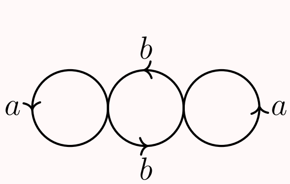

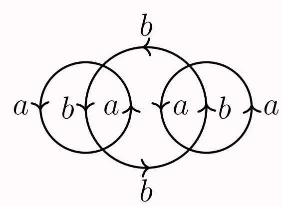

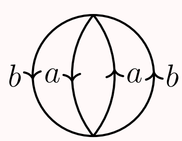

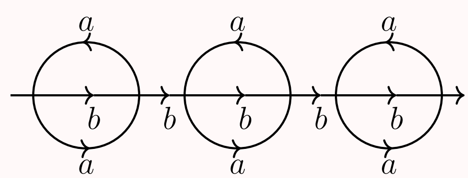

Theorem 19.1. Let Y be a connected space and $f : Y \to X$. If two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ agree at some point $y \in Y$, then $\tilde{f}_1 = \tilde{f}_2$.

Proof. Let U be a neighborhood of $f(y)$ that is evenly covered, by sheets $\tilde{U}_\alpha$.

- $\left\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\right\}$ is open: Suppose $\tilde{f}_1(z) = \tilde{f}_2(z) \in \tilde{U}_\beta$, then by continuity there exists a neighborhood $z \in V$ with $\tilde{f}_1(V), \tilde{f}_2(V) \subseteq \tilde{U}_\beta$. Then

$$\tilde{f}_1|_V = p\,|_{\tilde{U}_\beta}^{-1} \circ p\,|_{\tilde{U}_\beta} \circ \tilde{f}_1\,|_V = p\,|_{\tilde{U}_\beta}^{-1} \circ f\,|_V = \tilde{f}_2|_V.$$

- $\left\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\right\}$ is closed: $\tilde{f}_1(z) \neq \tilde{f}_2(z) \implies \tilde{f}_1(z) \in \tilde{U}_{\beta_1}, \tilde{f}_2(z) \in \tilde{U}_{\beta_2}$ ($\beta_1 \neq \beta_2$). By continuity there exists a neighborhood $z \in V$ with $\tilde{f}_i(V) \subseteq \tilde{U}_{\beta_i}$.

Since Y is connected, $Y = \left\{z \in Y : \tilde{f}_1(z) = \tilde{f}_2(z)\right\}$. ■

Theorem 19.2. (Homotopy Lifting Property)
Given a homotopy $f_t : Y \to X$ and a lift $\tilde{f}_0 : Y \to \tilde{X}$ of f_0, there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of f_t that agrees with $\tilde{f}_0$.

Remark. The two facts used in Theorem 17.4 follow from the case $Y = \text{pt}$ and $Y = I$.

Proof. Use the $H(x,t) = f_t(x)$ notation.

Lemma. For any $y \in Y$, there exists open $y \in V$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $H\left(V \times [t_i, t_{i+1}]\right) \subseteq U_i$ for all i.

Proof. Fix y. For each $t \in I$ there exists a neighborhood U_t of $H(y,t)$ that is evenly covered, and there exists a basis $V_t \times W_t \subseteq Y \times I$ with $(y,t) \in V_t \times W_t \subseteq H^{-1}(U_t)$. Since the W_t cover I which is compact, we have a finite subcover $W_{s_0}, \cdots, W_{s_m}$ of I and hence we can take $V = V_{s_0} \cap \cdots \cap V_{s_m}$ and t_i the endpoints of all W_{s_k}.

We first prove the theorem by fixing y and restricting f_t on a neighborhood $V \subseteq Y$. By induction, suppose $\tilde{H}$ has been constructed over $V \times [0, t_i]$. Let $U \supseteq H(V \times [t_i, t_{i+1}])$ be evenly covered by sheets $\tilde{U}_\alpha$. Let $\tilde{H}(y, t_i) \in \tilde{U}_\beta$, then by the pasting lemma we can construct $\tilde{H}\mid_{V' \times [0,t_i]}$, where V' is a smaller open set than V by restricting to the pre-image within $\tilde{U}_\beta$. Relabelling V' as V, after a finite number of steps, we constructed $\tilde{f}_t$ on a neighborhood V of y. Note that such $\tilde{f}_t$ is unique by the previous Theorem on the common point (v,t) for each $v \in V$.

To construct $\tilde{f}_t$ on the entire Y, we construct a unique $\tilde{f}$ on a neighborhood V_y at every $y \in Y$. By uniqueness on each $\{y\} \times I$, the $\tilde{f}$'s agree on the overlaps. By the same uniqueness, the entire $\tilde{f}$ is unique. ■

Theorem 19.3. Let $p_* : \pi_1\left(\tilde{X}, \tilde{x}_0\right) \to \pi_1(X, x_0)$ be induced by p.

1. p_* is injective.
2. $\text{Im}(p_*) = \{[\alpha] \in \pi_1(X, x_0) : \tilde{\alpha}(0) = \tilde{\alpha}(1) = \tilde{x}_0\}$.

Proof. Suppose $p_*([\beta]) = 0$. Then $p \circ \beta \simeq$ constant rel $\{0,1\}$. This nullhomotopy has a unique lift in $\tilde{X}$, which gives a nullhomotopy for β, i.e. $[\beta] = 0$. This proves (1).

For (2), we have $p_*([\beta]) = [p \circ \beta] = \left[\tilde{\beta}\right]$. ■

Theorem 19.4. If $X, \tilde{X}$ are path-connected, then $|p^{-1}(x_0)| = [\pi_1(X, x_0) : \mathrm{Im}(p_*)]$ (There is a bijection between each preimage of x_0 and each coset of $\mathrm{Im}(p_*)$).

Definition 19.2. A space Y is ***locally path-connected*** if for all $y \in Y$ and any neighborhood $y \in U$, there exists a neighborhood $y \in V \subseteq U$ that is path-connected.

Theorem 19.5. Say $f : (Y, y_0) \to (X, x_0)$ where Y is path-conn and locally path-conn.

$$\exists \tilde{f} : (Y, y_0) \to \left(\tilde{X}, \tilde{x}_0\right) \iff \mathrm{Im}\,(f_*) \subseteq \mathrm{Im}\,(p_*)$$

Proof. For ($\Rightarrow$), $f = p \circ \tilde{f} \implies f_* = p_* \circ \tilde{f}_*$. For ($\Leftarrow$), suppose $\mathrm{Im}\,(f_*) \subseteq \mathrm{Im}\,(p_*)$. Pick for each $y \in Y$ a path γ_y from y_0 to y, then define $\tilde{f}(y) = \tilde{g}_y(1)$ where $g_y = f \circ \gamma_y$ and $\tilde{g}_y(0) = \tilde{x}_0$.

- $\tilde{f}$ is well-defined: Let γ_1, γ_2 be two paths from y_0 to y. Then $[\gamma_1 \cdot \overline{\gamma_2}] \in \pi_1(Y, y_0)$ and hence exists $[\alpha] \in \pi_1\left(\tilde{X}, \tilde{x}_0\right)$ with $p \circ \alpha \simeq f \circ \gamma_1 \cdot \overline{\gamma_2}$ rel $\{0, 1\}$. This homotopy H_t has a unique lift $\tilde{H}_t$ with $\tilde{H}_0 = \alpha$. Since $H_1 = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$, by uniqueness if we have $g_1 = f \circ \gamma_1, g_2 = f \circ \gamma_2$ then $\tilde{H}_1 = \tilde{g}_1 \cdot \tilde{\overline{g}}_2$, and thus $\tilde{g}_1(1) = \tilde{H}_1(0.5) = \tilde{g}_2(1)$.
- $\tilde{f}$ is a lift: $p \circ \tilde{f}(y) = p \circ \tilde{g}_y(1) = g_y(1) = f(y)$.
- $\tilde{f}$ is continuous: Let W be a neighborhood of $\tilde{f}(y)$. Let $f(y) \in U$ be evenly covered by $\tilde{U}_\alpha$, and $\tilde{f}(y) \in \tilde{U}$. Since f is continuous, $\exists$ neighborhood $y \in V'$ with $f(V') \subseteq p\left(\tilde{U} \cap W\right)$. By local path-connectedness, let $y \in V \subseteq V'$ be a path-connected neighborhood. Then any path from y_0 to y can be extended to a path from y_0 to any $z \in V$. This eventually shows $\tilde{f}(V) \subseteq \tilde{U} \cap W$. ■

Definition 19.3.

1. A space X is ***locally contractible*** if $\forall x \in X$, $\exists$ neighborhood $x \in U \simeq \mathrm{pt}$.
2. A space X is ***locally simply connected*** if $\forall x \in X$, $\exists$ neighborhood $x \in U$ that is simply connected.
3. A space X is ***semilocally simply connected*** if for all $x \in X$ there exists a neighborhood $x \in U$ where $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.
4. X is ***'nice'*** if it is path-conn, locally path-conn and semilocally simply conn.
5. $p : \tilde{X} \to X$ is a ***universal cover*** of X if X is path-conn and $\tilde{X}$ is simply conn.
6. Two covering spaces $p_i : \tilde{X}_i \to X$ are ***isomorphic*** if there exists a homeomorphism $\varphi : \tilde{X}_1 \to \tilde{X}_2$ with $p_2 \circ \varphi = p_1$.

Theorem 19.6. CW complexes are locally contractible and locally path-connected.

Theorem 19.7.

1. If X is nice, then X has a universal cover.
2. If X is nice, for any subgroup $H \subseteq \pi_1(X, x_0)$ there exists a covering space $p_H : \tilde{X}_H \to X$ such that $\mathrm{Im}\,(p_{H*}) = H$.

Proof of (1). We use a Lemma and 5 steps.

Lemma. $\mathscr{B} = \{U \subseteq X : U \text{ open, path-conn, } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$ is a basis for the topology of a nice X.

Proof. $\mathscr{B}$ covers X since X is nice. Suppose $x \in U \cap V$ where $U, V \in \mathscr{B}$. Since X is locally path-conn, there exists path-connected $x \in W \subseteq U \cap V$, which means $\pi_1(W, x) \to \pi_1(U, x) \xrightarrow{\text{triv}} \pi_1(X, x) \implies W \in \mathscr{B}$. Hence $\mathscr{B}$ is a basis. To prove the second part, given any open $x \in W$, choose open $x \in V$ via semilocal simply connectedness, then pick a open and path-connected $U \subseteq V \cap W$ via local path-connectedness. Then $U \in \mathscr{B}$ with $U \subseteq W$. ■

1. Define $\tilde{X} = \{[\gamma] \mid \gamma : I \to X, \gamma(0) = x_0\}$ with $[\gamma] = [\delta] \iff \gamma \simeq \delta \text{ rel } \{0,1\}$. Define the covering map $p : \tilde{X} \to X$ by $p([\gamma]) = \gamma(1)$.
2. Define the topology on $\tilde{X}$ as follows: Given $U \in \mathscr{B}$ and $[\gamma] \in \tilde{X}$ with $\gamma(1) \in U$, define
$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta : I \to U \text{ with } \eta(0) = \gamma(1)\}$$
Then $\{U_{[\gamma]}\}_{[\gamma]}$ is a basis (Exercise; $(*)$ Note that $[\delta] \in U_{[\gamma]} \implies U_{[\delta]} = U_{[\gamma]}$), and we generate the topology from it.
3. Claim: $p : U_{[\gamma]} \to U$ is a homeomorphism.
   - Surjective: U path-conn $\implies \forall x \in U, \exists \text{path } \eta$ from $\gamma(1)$ to $x \implies p([\gamma \cdot \eta]) = x$.
   - Injective: $p([\gamma \cdot \eta]) = p([\gamma \cdot \eta']) \implies \eta(1) = \eta'(1)$. Since $\pi_1(U) \to \pi_1(X)$ is trivial, $\eta \simeq \eta' \implies [\gamma \cdot \eta] = [\gamma \cdot \eta']$.
   - Homeo: Note that $\{B \cap U\}_{B \in \mathscr{B}}$ and $\{B_{[\delta]} \cap U_{[\gamma]}\}_{B \in \mathscr{B}}$ are bases for U and $U_{[\gamma]}$ respectively, and (1) $p\left(B_{[\delta]} \cap U_{[\gamma]}\right) = B \cap U$; and (2) $p^{-1}(B \cap U) \cap U_{[\gamma]} = B_{[\delta]} \cap U_{[\gamma]}$ for any $[\delta] \in U_{[\gamma]}$ with $\delta(1) \in B$ since $B_{[\delta]} \subseteq U_{[\delta]} \overset{(*)}{=} U_{[\gamma]}$ and $p|_{V_{[\delta]}}$ is bijective.
4. p is a covering map: Given any $U \in \mathscr{B}$, $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$ which is a disjoint union of equivalence classes due to $(*)$.

5. $\tilde{X}$ is simply connected: Let $[x_0]$ = class of constant paths. Given any $[\gamma] \in \tilde{X}$, the path $\gamma_t(s) = \gamma(\min(s,t))$ form $\gamma(0)$ to $\gamma(t)$ brings $[x_0]$ to $[\gamma]$, and thus $\tilde{X}$ is path-connected. To show $\pi_1\left(\tilde{X}, [x_0]\right) = 0$, we show $\mathrm{Im}(p_*) = \{0\}$:

$$[\alpha] \in \mathrm{Im}(p_*) \implies [x_0] = \tilde{\alpha}(0) = \tilde{\alpha}(1) = [\alpha] \implies [\alpha] = 0.$$ ■

Proof of (2). Consider the equivalence relation $\sim$ on the universal cover $\tilde{X}$ defined by

$$[\gamma] \sim [\delta] \iff \gamma(1) = \delta(1) \text{ and } [\gamma \cdot \overline{\delta}] \in H.$$

Then $X_H = \tilde{X}/\sim$ is a covering space of X. To show $\mathrm{Im}\,(p_{H*}) = H$: For any $[\alpha] \in \pi_1(X, x_0)$, we have a lift $\tilde{\alpha} = \alpha_t$ from $[\tilde{x_0}]$ to $[\alpha]$, so

$$[\alpha] \in H \iff [\tilde{x}_0] \sim [\alpha] \iff \tilde{\alpha}(0) = \tilde{\alpha}(1) \iff [\alpha] \in \mathrm{Im}(p_{H*})$$

Theorem 19.8. Let $\tilde{X}$ be path-connected. Then $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$ is conjugate to $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_1\right)\right)$ for any $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$.

Proof. Conjugate using any path from $\tilde{x}_0$ to $\tilde{x}_1$. ■

Theorem 19.9. (Classification Theorem of Covering Spaces)
If X is nice, there is a bijective ***Galois correspondence*** between

$$\left\{\begin{matrix}\text{conj. classes of} \\ \text{subgrps of } \pi_1(X, x_0)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{isom. classes of path-} \\ \text{conn. covers } \tilde{X} \to X\end{matrix}\right\}$$

Proof. We prove that $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic if and only if $\mathrm{Im}(p_{1*})$ and $\mathrm{Im}(p_{2*})$ are conjugate in $\pi_1(X)$.

- Assume $p_1 \cong p_2$. Then $\mathrm{Im}(p_{1*}) = \mathrm{Im}(p_{2*} \circ \varphi_*) = p_{2*}\left(\pi_1\left(\tilde{X}_2, \varphi\left(\tilde{x}_1\right)\right)\right)$ which is conjugate to $p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{x}_2\right)\right)$ since $p_2 \circ \varphi\left(\tilde{x}_1\right) = p_1\left(\tilde{x}_1\right) = x_0 = p_2\left(\tilde{x}_2\right)$.

- Suppose $\mathrm{Im}(p_{1*}) = [\alpha]^{-1}\mathrm{Im}(p_{2*})[\alpha]$. Let $\tilde{\alpha} : I \to \tilde{X}_2$ be a lift of α based at $\tilde{x}_2$. Then

  $$p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{\alpha}(1)\right)\right) = p_{1*}\left(\pi_1\left(\tilde{X}_1, \tilde{x}_1\right)\right)$$

  so there exists lifts $\tilde{p}_1 : \left(\tilde{X}_1, \tilde{x}_1\right) \to \left(\tilde{X}_2, \tilde{\alpha}(1)\right), \tilde{p}_1 : \left(\tilde{X}_2, \tilde{\alpha}(1)\right) \to \left(\tilde{X}_1, \tilde{x}_1\right)$ with

$$\begin{tikzcd} & \tilde{X}_2 \arrow[dl, bend left, "\tilde{p}_2"] \arrow[d, "p_2"] \\ \tilde{X}_1 \arrow[ur, bend left, "\tilde{p}_1"] \arrow[r, "p_1"'] & X \end{tikzcd}$$

Since $\tilde{p}_2 \circ \tilde{p}_1$ and $\mathbf{1}$ agree at $\tilde{x}_1$ and ar both lifts of $p_1 : \tilde{X}_1 \to X$ to $\tilde{X}_1$, we have $\tilde{p}_2 \circ \tilde{p}_1 = \mathbf{1}$ and similarly $\tilde{p}_1 \circ \tilde{p}_2 = \mathbf{1}$. Thus $p_1 \cong p_2$. ■

20 Regular Coverings

Definition 20.1.

1. A ***deck transformation*** is a self-isomorphism $\tilde{X} \to \tilde{X}$ of a covering space $p : \tilde{X} \to X$. The group of deck transformations is denoted $\text{Aut}(\tilde{X})$.
2. A covering space is ***regular*** if for each $x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, there exists $\varphi \in \text{Aut}(\tilde{X})$ with $\varphi(\tilde{x}) = \tilde{x}'$.

Example 20.1. $\text{Aut}(\mathbb{R}) = \mathbb{Z}$.

Theorem 20.1. $\varphi \in \text{Aut}(\tilde{X})$ is completely determined by $\varphi(\tilde{x}_0)$ when $\tilde{X}$ is path-connected and locally path-connected.

Proof. φ is a lift to $\tilde{X}$, which is uniquely determined by where it sends some point. ■

Theorem 20.2. Suppose X is nice. Then $p : \tilde{X} \to X$ is regular if and only if $\text{Im}(p_*)$ is normal. When this is true, $\text{Aut}(\tilde{X}) = \pi_1(X, x_0)/\text{Im}(p_*)$.

Proof. p is regular $\Leftrightarrow$ for any $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there exists a lift φ where

$$\begin{tikzcd} & \left(\tilde{X}, \tilde{x}'\right) \arrow[d, "p"] \\ \left(\tilde{X}, \tilde{x}\right) \arrow[ur, "\varphi"] \arrow[r, "p"'] & X \end{tikzcd}$$

which is equivalent to $p_*\left(\pi_1\left(\tilde{X},\tilde{x}\right)\right) = p_*\left(\pi_1\left(\tilde{X},\tilde{x}'\right)\right)\ \forall \tilde{x},\tilde{x}'$, i.e. $p_*\left(\pi_1\left(\tilde{X},\tilde{x}\right)\right)$ is normal. To prove the second part, consider the map $\pi_1(X,x_0) \to \mathrm{Aut}(\tilde{X})$ is $p_*(\pi_1(X,x_0))$ given by $[\alpha] \mapsto \varphi_\alpha$ where $\varphi_\alpha(\tilde{\alpha}(0) = \tilde{x}_0) = \tilde{\alpha}(1)$.

- It is a homomorphism: Since a lift of $\alpha \cdot \beta$ is $\tilde{\alpha} \cdot \varphi_\alpha(\tilde{\beta})$, so

$$[\alpha][\beta] \mapsto \varphi_{\alpha\cdot\beta}(\tilde{x}_0) = \varphi_\beta \circ \varphi_\alpha(\tilde{x}_0) \implies \varphi_{\alpha\cdot\beta} = \varphi_\beta\varphi_\alpha.$$

- It is injective: $[\alpha] \mapsto 0 \ \Leftrightarrow\ [\alpha] \in p_*\left(\pi_1\left(\tilde{X},\tilde{x}\right)\right)$.

- It is surjective: Fix a path γ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1 \in p^{-1}(x_0)$. Then $[p \circ \gamma] \in \pi_1(X,x_0)$ has lift γ. ■