Logic in Computer Science

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Chapter 1

2015-09-01

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What is logic about?

- Rules for correct reasoning.
- Foundation of mathematics.
- Mathematics about mathematics (metamathematics)

History:

- Babylonian mathematics $\sim$ 2000 BC. Had no notion of proof, but gave very general examples. Could calculate $\sqrt{2}$ to 7 decimals.
- Pythagoras 580-495 BC. Were shocked by that the $\sqrt{2}$ was irrational.
- **Aristotle** 384-322 BC Called his rules for **Syllogism**, and had **24** of them.

  Example: All dogs have four legs Carlo is a dog. => Carlo has four legs
- Euclid $\sim$ 300 BC. Geometry. Wrote Euclids elements, which used axioms. Used for a long time in schools.
- Archimedes 287-212 BC.
- Islamic Logic: Avicerna (ibn-suna) 980-1037 AD.
- Medival logic: 1200-1600 BC.

Scientific revolution:

- Galileo 1564-1642. The first one to conduct proper experiments. Science, not logic nor mathemathics.
- Descartes 1596-1650. Started an intellectual revolution. Was Both a philosopher and a mathematician. Rationalism, "Cogito ergo sum". He wanted to write everything we know from the start.
- **Leibniz** 1646-1716. Also invented differential calculus. Wanted to construct a langugae in which any argument could be represented and checked to be correct or not.
- Newton 1646-1727.

Newton, Leibniz and Descartes were all convinced that god existed, and use that as a basis for some of their proofs, even though it is unproven.

Empiricism:

- Hume 1711 - 1776. Was a bit secular, and was interested in what it was possible to know.
- Boole 1815 - 1864 Laws of thought. Propositional logic. $\wedge\vee, \rightarrow, \neg$
- **Gottlieb Frege** 1848 - 1925 Wrote a very important text about logic Begrüffnschift (1879) Predicate logic, proposistional logic plus quantifiers: $\forall, \exists$

  Grundgesetsse 1893, 1902 If P(x) is a property, then we can form the set $\{x|P(x)\}$
- Bertrand Russel. Found the Russel paradox, i.e.: According to Frege, you can form the set
  $$A = \{x|x \notin x\}$$
  Assume that $A \in A$ (1) According to the definition we then get that $A \notin A$ which contradicts the assumption which hence must be false, i.e. $A \notin A$, but then (2) follow, by the definition of A, that $A \in A$.
- Cantor $\sim$ 1870 Set Theory. Potential infinity $0, 1, 2, \dots, n, n+1, \dots$ Actual infinity: $\{0, 1, 2, \dots\}$
- Zemelo 1908 Given a set G, then we can form $\{x \in G|P(x)\}$

- Brouwer All mathematical objects must be constructed by us. A proposition is true if and only if we can prove it (complete). $A \vee \neg A$.

- Constructivism: Only things we can construct exists.
- Platonism: Objects exists independent of us.
- Formalism: Things can only be derived.

Chapter 2

2015-09-04

Missed

Definition ($\vdash$ (derives))**.** Syntatic implication. If $S \vdash \psi$, it means that ψ can be derived from the formulas in S.

Definition ($\models$ (models))**.** Semantic implication. $A \models B$ means that B is true in every model in which A is true.

Chapter 3

2015-09-08

3.1 Inductive definitions

The set N of natural numbers is inductively defined by

(i) 0 is a natural number

(ii) if n is a natrual number, then $\mathrm{succ}(n)$ is a natural number.

Expressed by rules:

(i) $0 \in \mathbb{N}$

(ii) $\frac{n \in \mathbb{N}}{\mathrm{succ}(n) \in \mathbb{N}}$

Example of a recursively defined function, $n!$

$$\begin{cases} \mathrm{fac}(0) & = 1 \\ \mathrm{fac}(\mathrm{succ}(n)) & = \mathrm{succ}(n) \cdot \mathrm{fac}(n) \end{cases}$$

The set $\mathcal{F}$ of proposistional formulas is inductively defined by

(i) atoms, including $\perp$, are formulas

(ii) if $\emptyset$ and ψ are formulas, so are $(\emptyset \wedge \psi), (\emptyset \vee \psi), (\emptyset \to \psi)$ and $(\neg \emptyset)$

Example. Define a function par which computes the number of parentheses in a formula. Done by recursion.

(i) $\mathrm{par}(\emptyset) = 0$ when $\emptyset$ is an atom.

(ii)

$$\begin{aligned} \mathrm{par}((\emptyset \wedge \psi)) &= \mathrm{par}(\emptyset) + \mathrm{par}(\psi) + 2 \\ \mathrm{par}((\emptyset \vee \psi)) &= \mathrm{par}(\emptyset) + \mathrm{par}(\psi) + 2 \\ &\vdots \\ \mathrm{par}((\neg\emptyset)) &= \mathrm{par}(\emptyset) + 2 \end{aligned}$$

Truth tables

$\emptyset$	ψ	$\emptyset \wedge \psi$	$\emptyset \vee \psi$	$\emptyset \to \psi$	$\neg\emptyset$
T	T	T	T	T	F
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	F	T	T

Two kind of semantics

(i) in terms of truth values "The meaning of a formula is its truth value" (Frege)

(ii) In terms of proofs. Constructive semantics.

Definition (1.28, p 37)**.** A **valuation** v is a function from the set of atoms to the set of truth values $v : \{p, q, r, p_1, p_2, p_3, \ldots\} \to \{T, F\}$ v can be extended to the set of proposistional formulas by recursion.

(i) v is already defined on the atoms ($v(\bot) = F$)

(ii)

$$v(\emptyset \wedge \psi) = \begin{cases} T & \text{if } v(\psi) = v(\emptyset) = T \\ F & \text{Otherwise} \end{cases}$$

$$v(\emptyset \vee \psi) = \begin{cases} T & \text{if } v(\psi) = T \text{ or } v(\emptyset) = T \\ F & \text{Otherwise} \end{cases}$$

$$v(\emptyset \to \psi) = \begin{cases} F & \text{if } v(\emptyset) = T \text{ and } v(\psi) = F \\ T & \text{Otherwise} \end{cases}$$

$$v(\not\emptyset) = \begin{cases} T & \text{if } v(\emptyset) = F \\ F & \text{Otherwise} \end{cases}$$

Definition (1.34, p 46)**.** If for all valuations in which all $\emptyset_1, \ldots, \emptyset_n$ evaluates to T, also ψ evaluates to T then

$$\emptyset_1, \ldots, \emptyset_n \models \psi$$

holds.

Example. $p \wedge q \models p$ holds. Nota $p \wedge q \vdash$ with a completely different argument, namely the derivation

1. $p \wedge q$ premise
2. p $\wedge e_1, 1$

Example.

$$p, q \models p \wedge q$$

Theorem (Soundness (1.35))**.**

$$\emptyset_1, \ldots, \emptyset_n \vdash \psi \Rightarrow \emptyset_1, \ldots, \emptyset_n \models \psi$$

Proof. Proof by induction on the derivation

$$\dfrac{\begin{matrix}\vdots & \vdots \\ \psi_1 & \psi_2\end{matrix}}{\psi_1 \wedge \psi_2} \wedge i$$

$$\dfrac{\begin{matrix}\vdots \\ \psi_1 \wedge \psi_2\end{matrix}}{\psi_1} \wedge e_i$$

$$\dfrac{\begin{matrix}\psi_1 \\ \vdots \\ \psi_2\end{matrix}}{\psi_1 \to \psi_2} \to i$$

Q.E.D

Theorem (1.4.4 p 49. Completeness)**.**

$$\emptyset_1, \ldots, \emptyset_n \models \psi \to \emptyset_1, \ldots, \emptyset_n \vdash \psi$$

Proof by kalmar 1930. The construction of Kalmar's derivation is exponential in the number of atoms of the formulas $\sim 2^n$*, where* n *is the number of atoms.*

Definition. If $\models \emptyset$ hodls, we say that $\emptyset$ is a **tautology**

$\emptyset$ is true for all valuations

$[\![\emptyset]\!]_r = T$ for all v

$$\emptyset_1, \ldots, \emptyset_n \models \psi \Leftrightarrow \models (\emptyset_1 \wedge \cdots \wedge \emptyset_n) \to \psi$$

3.2 Constructive semantics

Brouwer, Heyting, Kolmogorov. 1930.

A proposistion is defined by laying down what counts as a proof of it.

To prove	we must	Type
$\emptyset \wedge \psi$	prove $\emptyset$ and prove ψ	$A \times B$
$\emptyset \vee \psi$	prove $\emptyset$ or prove ψ	$A + B$
$\emptyset \to \psi$	give a method which to each proof of $\emptyset$ gives a proof of ψ	$A \to B$
$\perp$	nothing counts as a proof of $\perp$	$\perp$

Bishop 1967. Foundations of constructive analysis, a great work in constructive semantics.

Thierry Coquand was given a prize for his work in logic.

$$\wedge i : \quad \frac{\emptyset \quad \psi}{\emptyset \wedge \psi} \qquad \text{x-induction} : \quad \frac{a \in A \quad b \in B}{\langle a, b\rangle \in A \times B} \times i$$

$$\wedge e_1 : \quad \frac{\emptyset \wedge \psi}{\emptyset}$$

$$\text{x-elimination} : \quad \frac{c \in A \times B}{\mathrm{fst}(c) \in A} \times e_1$$

$$\wedge e_2 : \quad \frac{\emptyset \wedge \psi}{\psi}$$

$$\text{x-elimination} : \quad \frac{c \in A \times B}{\mathrm{snd}(c) \in B} \times e_2$$

Where $\mathrm{fst}(\langle a, b\rangle) = a$ and $\mathrm{snd}(\langle a, b\rangle) = b$.

Curry 1958. - noticed implications
Howard 1969 - extended to predicate logic.
Per Marten-Löf 1970. Type theory.