34. Steiner's Straight-edge Problem

Prove that every construction that can be done with compass and straight-edge can be done with straight-edge alone given a fixed circle in the plane.

As far back as 1759 Lambert had solved a whole series of geometric constructions with straight-edge alone in his book *Freie Perspektive*, published in Zürich that year. He is also the source of the term "straight-edge geometry". After Lambert, French mathematicians, primarily Poncelet and Brianchon, took up straight-edge geometry, particularly after the publication of Mascheroni's *Geometria del compasso* gave a new stimulus to these studies, and they attempted to find as many constructions as possible with straight-edge alone.

With a straight-edge alone, it is possible to represent only rational algebraic expressions (so not for example, $\sqrt{ab}$). This suggested to Poncelet that an additional fixed circle (with its center) must be given in order to draw with straight-edge alone all algebraic expressions that can be constructed with compass and straight-edge. This was confirmed by Jakob Steiner (1796-1863), the greatest geometer since the days of Appolonius, in his celebrated book *Die geometrischen Konstruktionen asgeführt mittels der geraden Linie und Eines festen Kreises* (Geometrical constructions carried out with straight lines and a fixed circle), Berlin, 1833.

The solution presented here is based on that in Steiner's book, except that we have eliminated everything that is not strictly essential for the purpose at hand, and we have also made it somewhat more elementary. Since in straight-edge geometry, the intersection of two straight lines is known directly, we need only demonstrate that the two fundamental problems II and III of No. 33 can be solved using a straight-edge and fixed circle.

As in the solution of Mascheroni's problem, we must first solve several preliminary problems, in this case five of them.

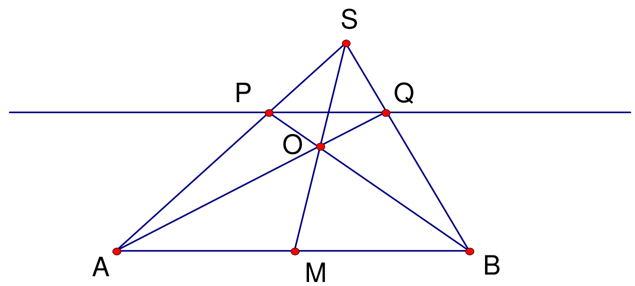
Prelim 1. Construct a line through point P parallel to a given line. (P is not on the given line.)

Solution. Steiner considers two cases:

1a. two points A and B and their midpoint M on the given line are known. We will call the straight line a "directed straight line" in this case.

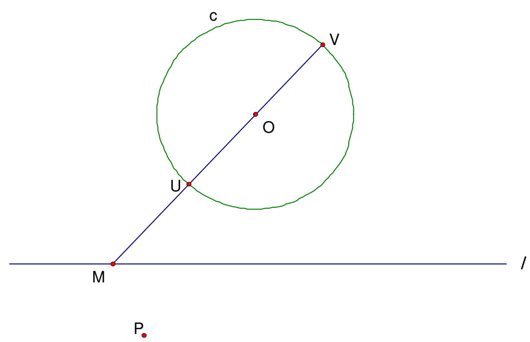
1b. the given straight line is arbitrary.

1a. Draw AP and let S be a point on AP extended. Connect S with M and B. Let O be the intersection point of BP and MS. Finally let line AO meet BS at Q.

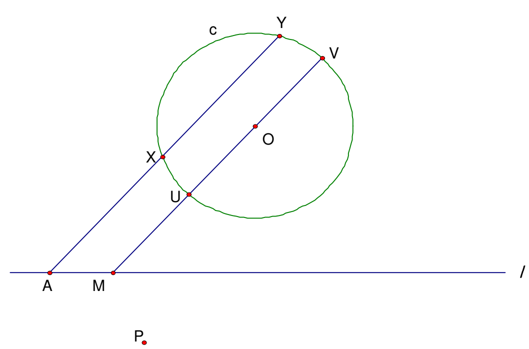


By Ceva's Theorem, $\frac{AM}{MB}\frac{BQ}{QS}\frac{SP}{PA} = 1$ from which it follows that $\frac{BQ}{QS} = \frac{AP}{PS}$ and then $\frac{BS}{QS} = \frac{AS}{PS}$. Thus $\triangle ABS \sim \triangle PQS$, and line PQ is parallel to line AB (since $\angle ABS = \angle PQS$).

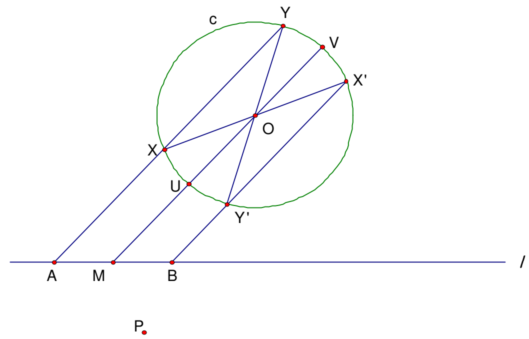
1b. Let l be the line, $c = c(O, r)$ be the fixed circle, and P a point off l.



Let line MO intersect c in points U and V, making MO a directed straight line. Use 1a to construct a line through a point A on l parallel to MO and intersecting c in points X and Y :



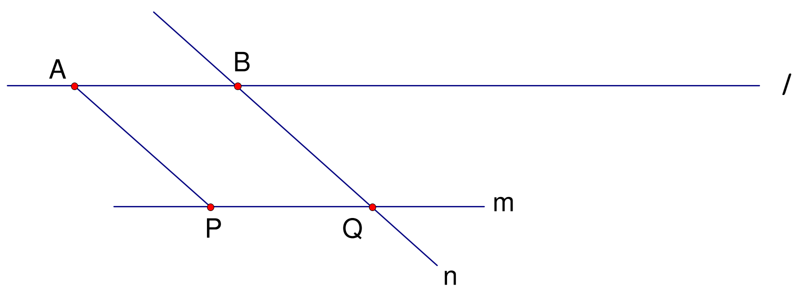
Let XOX' and YOY' be diameters of c, and let line $X'Y'$ meet l at B.



Then $AM = MB$ and l is a directed straight line. The parallel to l through P can then be constructed in accordance with 1a. □

Corollary. Shift AB parallel to itself so that one of its endpoints lies on a given point P (off line AB).

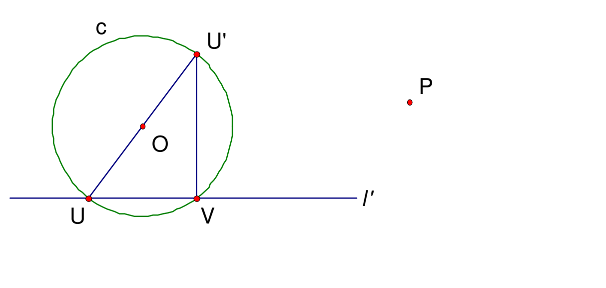
Solution. Let m be the parallel through P to AB, and n be the parallel through B to AP. Let $Q = m \cap n$. Then PQ is the desired line segment.



A similar construction shifts AB so that B lies on P. □

Prelim 2. Construct a perpendicular through a point P to a given line l.

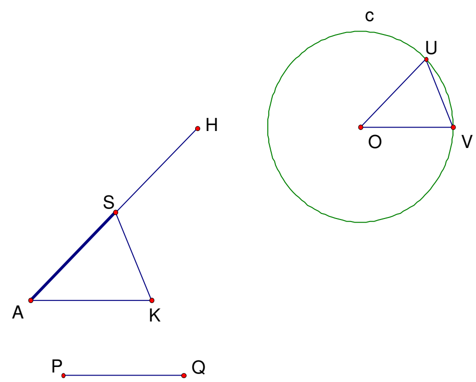
Solution. Draw l' parallel to l so that it cuts c at U and V. Draw the diameter UOU' and chord VU'.



$\angle UVU'$ is an inscribed angle in a semicircle, hence a right angle. Thus VU' is perpendicular to UV and l. Finally construct the parallel to VU' through P in accordance with 1; this parallel is the desired perpendicular. $\square$

Prelim 3. Construct a segment AS at a given point A of length PQ in a given direction.

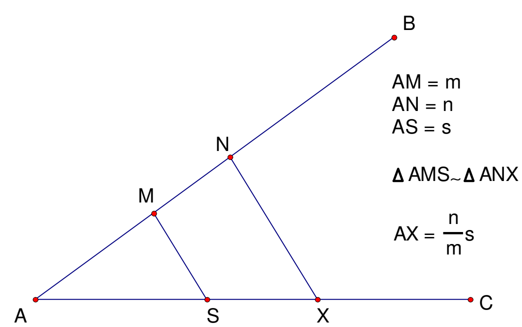
Solution. If necessary use a parallel shift by the Corollary to I above so that the given direction is vector $\overrightarrow{AH}$. Use it again to displace PQ parallel to itself to AK. Then draw two radii OU and OV of c in directions AH and AK.



Finally draw the parallel to UV through K; its intersection S with line AH is the desired point. $\square$

Prelim 4. Given segments of length n, m, s, construct a segment of length $x = \frac{n}{m}s$.

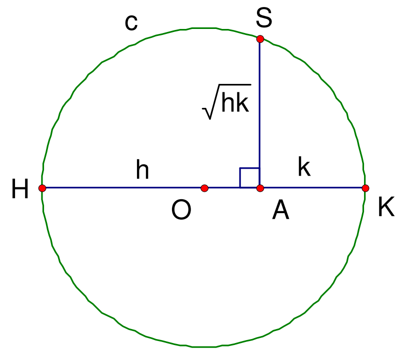
Solution. From any point A, draw two rays AB and AC, and (use Prelim 3) to mark off distances $AM = m$, $AN = n$ on AB and AS on AC. Let the parallel through N to MS intersect AC at X.



Then $AX = \frac{n}{m}s$. $\square$

Prelim 5. Given segments of length a and b, construct a segment of length $\sqrt{ab}$.

Solution. Let $x = \sqrt{ab}$, d be the diameter of fixed circle c and $t = a + b$. Note that t is constructible by Prelim 3. With $h = \frac{d}{t}a$, $k = \frac{d}{t}b$ and $s = \sqrt{hk}$, it follows that $x = \sqrt{\frac{th}{d}\frac{tk}{d}} = \frac{t}{d}s$. Note that $h + k = d$, so we can construct segment $HA = h$ on diameter HK of c; then $AK = k$. Use Prelim 2 to construct the perpendicular to HK through A, and call its intersection with c point S.

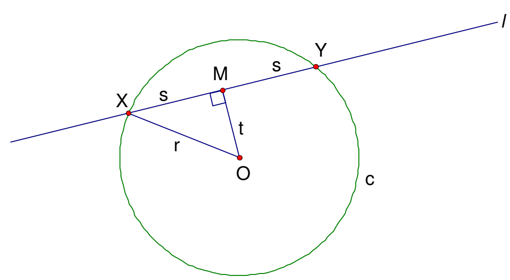


Then $x = \frac{t}{d}s$ is constructible by Prelim 4. $\square$

The solution to the two basic construction problems is now simple.

II. *Construct the point of intersection of a given line and a given circle.*

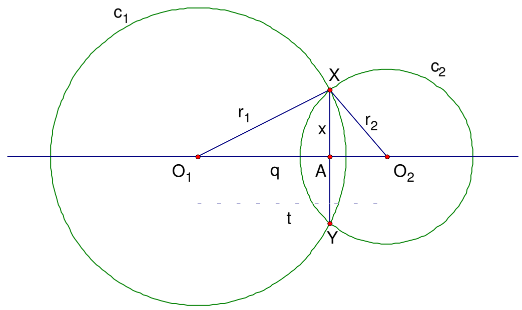
Solution. Let ℓ be the given line and $c(O,r)$ be the given circle. We must construct X and Y, the points of intersection of ℓ and $c(O,r)$. Let $2s$ be the length of chord XY, M be its midpoint and t be the distance OM.



From right triangle $\triangle OMX$, $s^2 = r^2 - t^2$ or $s = \sqrt{(r+t)(r-t)}$. t is constructible by Prelim 2, and $r \pm t$ by Prelim 3. s can then be constructed by Prelim 5, and by Prelim 3 again, X and Y can be found. $\square$

III. *Construct the points of intersection of two given circles.*

Solution. Let $c_1 = c(O_1, r_1)$ and $c_2 = c(O_2, r_2)$ be the two given circles, and X and Y be the points of intersection (to be constructed). Let A be the point of intersecton of XY and the line of the centers O_1O_2. Let t, q and x be the distances O_1O_2, O_1A, and XA resepectively.



We will show that q and x are constructible (in straight-edge geometry); then Prelim 3 will allow us to construct A (at a distance of q from O_1), Prelim 2 the perpendicular to O_1O_2 at A, and Prelim 3 X and Y (at a distance of x from A) on this perpendicular.

Finding q. Apply the law of cosines to $\triangle O_1O_2X$ to get

$$\begin{aligned} r_2^2 &= r_1^2 + t^2 - 2r_1 t \cos \angle XO_1O_2 \\ &= r_1^2 + t^2 - 2t(r_1 \cos \angle XO_1O_2) \\ &= r_1^2 + t^2 - 2tq \end{aligned}$$

Set $d = \sqrt{r_1^2 + t^2}$ so that $q = \frac{(d+r_2)(d-r_2)}{2t}$. d, the hypotenuse of a right triangle with legs of length r_1 and t, is constructible by Prelims 2 and 3, $n = d + r_2, m = 2t, s = d - r_2$ are constructible by Prelim 3, and $q = \frac{n}{m}s$ is constructible by Prelim 4.

Finding x. From (right) $\triangle AO_1X$, $x^2 = r_1^2 - q^2$, and $x = \sqrt{(r_1 + q)(r_1 - q)}$. Prelim 3 provides a construction of $h = r_1 + q$ and $k = r_1 - q$, and Prelim 5 a construction of $x = \sqrt{hk}$. $\square$