Combinatorial Optimization

Dozent: Stephan Held

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Contents

0 Organization

- Prerequisites
  - Basic knowledge of graph algorithms
  - Linear Programming (LP Duality)
  - Programming skills in C++
- Exam
  - Qualification requires 50% of the points in theoretical & programming exercises
  - Oral exam
- Books
  - "Combinatorial Optimization", Korte & Vygen
  - "Understanding & Using Linear Programming", B. Gärtner, J. Matouset
  - Skript (theorems & definitions)
  - Further book recommendations are on the website

1 Matchings

1.1 Introduction

Definition 1.1.

1. A *matching* M in a graph $G = (V, E)$ is a set of pairwise disjointed edges, i.e. they don't have a common endpoint.

   $\nu(G) :=$ max. cardinality of a matching in G

2. An *edge cover* C of a graph $G = (V, E)$ is a subset of E s.t. $V = \bigcup_{e \in C} e$.

   $\zeta(G) :=$ min. cardinality of an edge cover in G

3. A matching is called *perfect* (or *1-factor*) if it is an edge cover

4. $v \in V$ with $v \in e \in M$ is called *M-covered*

5. $v \in V$ is called *M-exposed* if it is not M-covered

Definition 1.2.

1. A *stable set* (independent set) S is a set of pairwise non-adjacent vertices.

   $\alpha(G) :=$ max. cardinality of a stable set

2. A *vertex cover* C is a subset of V s.t. $E = \bigcup_{\{x,y\}\in E,\ x\in G}\{x,y\}$
   $\tau(G) :=$ min. cardinality of a vertex cover

Lemma 1.3.

1. $\alpha(G) + \tau(G) = |V|$
2. $\nu(G) + \zeta(G) = |V|$ *if* G *has no isolated vertices*
3. $\zeta(G) = \alpha(G)$ *if* G *is bipartite and has no isolated vertices*

Problem. Cardinalty Matching Problem
Input: Graph $G = (V, E)$
Task: Find a maximum cardinality matching

Problem. Maximum Weight Matching Problem (MWMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a matching M maximizing $c(M)$

Problem. Minimum Weight Perfect Matching (MWPMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a perfect matching of minimum weight or decide that no perfect matching exists in G

Lemma 1.4. *The MWMP is equivalent to the MWPMP (i.e. there exists a transformation with linear complexity)*

Proof. Given a MWPMP instance (G, c), define $c' := K - c$ ($K := 1 + \sum_{e\in E} |c(e)|$).
$\Rightarrow$ Any maximum weight matching is a maximum cardinality matching

Given a MVMP instance (G, c), define G' as 2 copies of G where the 2 copies of a vertex are joined by an edge.
$\Rightarrow$ G' has a perfect matching. Define:

$$c'(e) := \begin{cases} -c(e) & \text{if } e \text{ is in the first copy} \\ 0 & \text{else} \end{cases}$$

A minimum weight perfect matching in G' gives us a maximum weight matching in G. □

Definition 1.5. Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching in G. A path P is *M-alternating* if its edges are alternatingly in and not in M. If both end points of this path are M-exposed, P is an *M-augmenting* path.

Lemma 1.6. *Given a matching M in G and an inclusion-wise maximal M-alternating path P,*

$$M\Delta P := M \setminus P \cup P \setminus M$$

is a matching. If P is M-augmenting, then $|M\Delta P| = |M| + 1$.

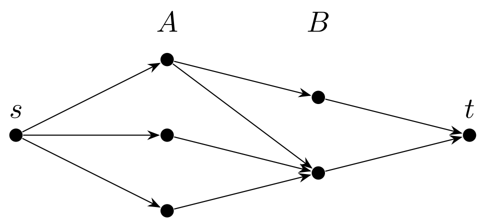

Figure 1: Example of the construction in Theorem 1.8

Theorem 1.7 (Petersen 1891, Berge 1957)**.** *Augmenting Path Theorem Given a graph $G = (V, E)$ and a matching M in G:*

$$|M| = \nu(G) \Leftrightarrow \nexists M\text{-augmenting path } P \text{ in } G$$

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": Assume there exists a matching M' with $|M'| > |M|$. Let $G' := (V, M\Delta M')$.
$\Rightarrow |\delta_{G'}(v)| \leq 2 \ \forall v \in V$
$\Rightarrow G'$ is the union of disjoint circuits and paths
$\Rightarrow$ all circuits are even and have the same number of edges from M and M'
$\Rightarrow \exists$ a path P in G' starting and ending with an edge in M'
$\Rightarrow P$ is an alternating path

□

1.2 Bipartite Matching

Theorem 1.8 (König 1931)**.** *If G is bipartite, then $\nu(G) = \tau(G)$*

Proof. Add vertices s and t edges between them to all vertices of the respective partition. Direct all edges from s to t. Then $\nu(G)$ is maximum number of disjoint s-t-paths. Menger $\Rightarrow$ This is equal to the minimum number of vertices that disconnect t from s. □

Theorem 1.9 (Hall 1935)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph. Then:*

$$G \text{ has a matching covering } A \Leftrightarrow |\Gamma(X)| \geq |X| \quad \forall X \subseteq A$$

Corollary 1.10. *Marriage Theorem*

$$|\Gamma(X)| \geq |X| \ \forall X \subseteq A \text{ and } |A| = |B| \Leftrightarrow G \text{ has a perfect matching}$$

Definition 1.12. The MWPMP for bipartite graphs is called *Assignment Problem.*

Theorem 1.13. *The Assignment Problem can be solved in time* $O(nm + n^2 \log m)$.

Proof. Use the Successive Shortest Paths algorithm in an auxiliary graph. $\square$

1.3 The Tutte Matrix & Randomized Matching

Definition 1.14. Let G be a simple, undirected graph. Let G' be an orientation of G and $(X_e)_{e\in E(G)}$. The *Tutte matrix* is defined as

$$T_G(X) := (T^*_{vw})_{v,w\in V(G)}$$

where

$$t^*_{vw} := \begin{cases} X_{\{v,w\}} & \text{if } (v,w) \in E(G) \\ -X_{\{v,w\}} & \text{if } -(v,w) \in E(G) \\ 0 & \text{else} \end{cases}$$

Remark 1.15. $T_G(X)$ is shew-symmetric (i.e. $T_G(X) = -(T_G(X))^t$). $\operatorname{rank}(T_G(X))$ is independent of the orientation of G. $\det(T_G(X))$ is a polyomial in X.

Theorem 1.16 (Tutte)**.** *A simple graph G has a perfect matching* $\Leftrightarrow \det(T_G(X)) \neq 0$

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and S_n be the permutation group.

$$\det T_G(X) = \sum_{\pi\in S_n} \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} t^*_{v_i,v_{\pi(i)}}$$

Let $S'_n := \{\pi \in S_n \mid \prod_{i=1}^n t^*_{v_i,v_{\pi_i}} \neq 0\}$. Each $\pi \in S_n$ corresponds to a digraph $H_\pi := (V(G), \{(v_i, v_{\pi(i)}) \mid i \in [n]^1\})$. We have $|\delta^+(v) = 1 = |\delta^-(v)|| \ \forall v \in V(H_\pi) \Rightarrow H_\pi$ is the union of disjoint circuits. If $\pi \in S'_n$, then $H_\pi \subset \overset{\Leftrightarrow}{G'}$.

If there exists $\pi \in S'_n$ s.t. H_π is a collection of even circuits, then this immediately yields a perfect matching in G (using every second edge of each circuit).

Otherwise, $\forall \pi \in S'_n$, H_π contains an odd circuit. Let $r(\pi) \in S'_n$ arise from π by reversing edges on the unique odd circuit containing a vertex with minimum index $\Rightarrow r(r(\pi)) = \pi$ and $\operatorname{sgn}(\pi) = \operatorname{sgn}(r(\pi))$. The second part is true since for reversing an odd cycle, we need an even number of swaps. Let $v_{i_1}, \ldots, v_{i_{2k+1}}$ be the "first" odd circuit. Then $r(\pi)$ is attained by $2k$ swaps: For $j = 1, \ldots, k$ swap $(\pi(i_{2j-1}), \pi(i_{2k}))$ and $(\pi(i_{2j}), \pi(i_{2k+1}))$.

[1] This is an abbreviation for $\{1, \ldots, n\}$.

$\prod_{i=1}^{n} t^*_{v_i v_{\pi(i)}} = -\prod_{i=1}^{n} t^*_{v_i v_{r(\pi(i))}}$ since there is an odd number of sign changes to t^*. $\Rightarrow \det(T_G(X)) = 0$. We have shown that if G has no perfect matching, then $\det T_G(X) = 0$.

Assume that G has a perfect matching M. Define π as $\pi(i) = j, \pi(j) = i$ where $\{i, j\} \in M$. $\Rightarrow \prod_{i=1}^{n} t^*_{v_i v_{\pi(i)}} = \prod_{e \in M} -X_e^2$ cannot be canceled out. In particular, $\det T_G(X) \neq 0$. $\square$

Remark 1.17. Picking $X' \in [0,1]^{E(G)}$ at random, we almost surely have (since the zero set of a non-zero polynomial is a set of measure zero):

$$\det T_G(X') \neq 0 \Leftrightarrow G \text{ has a perfect matching}$$

Theorem 1.18 (Lovász 1979)**.** *Let G be a simple graph and $X \in [0,1]^{E(G)}$ chosen randomly. Then almost surely* $\operatorname{rank}(T_G(X)) = 2\nu(G)$.

1.4 Tutte's Matching Theorem

Let $X \subseteq V(G)$. $G - X$ consists of even and odd (in terms of the number of vertices) connected components. We define $q_G(X)$ to be the number of odd components in $G - X$.

Definition 1.19. A graph G satisfies the *Tutte Condition* if $q_G(X) \leq |X|$ for all $X \subseteq V(G)$. $\emptyset \neq X \subseteq V(G)$ is called *barrier* if $q_G(X) = |X|$.

Proposition 1.20. *For any graph G and any $X \subseteq V(G)$:*

$$q_G(X) - |X| \equiv |V(G)| \mod 2$$

Definition 1.21. A graph G is *factor-critical* if $G - v$ has a perfect matching for all $v \in V(G)$. A matching is called *near-perfect* if it covers $|V(G)| - 1$ vertices.

Proposition 1.22. *If G is factor-critical, then it is connected.*

Theorem 1.23 (Tutte 1947)**.** *A graph G has a perfect matching $\Leftrightarrow$ Tutte Condition holds (i.e. $q_G(X) \leq |X| \ \forall X \subseteq V(G)$)*

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": We proceed by induction on $|V(G)|$. The case $|V(G)| = 2$ is clear.

Generally, if the Tutte Condition holds, then $|V(G)|$ must be even (pick $X = \emptyset$). Proposition 1.20 $\Rightarrow q_G(X) - |X|$ is even. Every $x \in V(G)$ induces a barrier $\{x\}$. Let X be a maximum barrier. Then $G - X$ doesn't have any even components (since otherwise a single vertex of such a component could be added to X).

Claim: Each odd component is factor-critical.
Let C be an odd component in $G - X$, $v \in V(C)$. Assume that $C - v$ does not have a perfect matching. Induction Hypothesis $\Rightarrow$ $C - v$ violates Tutte Condition.
$\Rightarrow \exists Y \subseteq V(C - v) : q_{C-v}(Y) > |Y|$
$\overset{1.20}{\Rightarrow} q_{C-v}(Y) \geq |Y| + 2$
Observe $X \cap \{v\} = Y \cap \{v\} = X \cap Y = \emptyset$:

$$\begin{aligned} q_G(X \cup Y \cup \{v\}) &= q_G(X) - 1 + q_C(Y \cup \{v\}) \\ &= |X| - 1 + q_{C-v}(Y) \\ &\geq |X| - 1 + |Y| + 2 \\ &= |X \cup Y| + 1 \\ &= |X \cup Y \cup \{v\}| \end{aligned}$$

$\Rightarrow X \cup Y \cup \{v\}$ is a barrier
$\Rightarrow$ Claim

Let G' arise from G by contracting each odd component into a single vertex. We have $V(G') = X \dot\cup Z$ and G' is bipartite. We have to show that G' has a perfect matching. If not, then $\exists A \subseteq Z : |\Gamma_{G'}(A)| < |A|$ $\Rightarrow q_G(\Gamma_{G'}(A)) \geq |A| > |\Gamma_{G'}(A)|$ which contradicts the Tutte Condition.

□

Theorem 1.24 (Berge 1958)**.**

$$|V(G)| = 2\nu(G) + \max_{X \subseteq V(G)} (q_G(X) - |X|)$$

Proof. For $X \subseteq V(G)$, any matching has at least $q_G(X) - |X|$ uncovered vertices, so "$\geq$" holds.

For the other inequality, add $k := \max_{X \subseteq V(G)}(q_G(X) - |X|)$ new vertices and connect them to all existing vertices, yielding a new graph H.

We claim that H has a perfect matching. This then implies:

$$2\nu(G) + k \geq 2\nu(H) - k = |V(H)| - k = |V(G)|$$

Assume that H does not have a perfect matching. Then by Tutte's Theorem, there exists $Y \subseteq V(H)$ with $q_H(Y) > |Y|$. By 1.20, $k \equiv |V(G)| \mod 2$, therefore $|V(H)|$ is even, so $Y \neq \emptyset$. Y must contain all new vertices, otherwise $H - Y$ would be connected[2] and $q_H(Y) \leq 1 \leq |Y|$.

$$\Rightarrow q_G(Y \cap |V(G)|) = q_H(Y) > |Y| = |Y \cap V(G)| + k$$

which is a contradiction to the choice of k. □

[2] Note that Y cannot contain all old vertices, since otherwise $q_H(Y) < |Y|$.

1.5 Ear Decompositions of Factor-Critical Graphs

Definition 1.25. Let G be a graph. An *ear decomposition* of G is a sequence $r, P_1, \ldots, P_k$ with $G = (r, \emptyset) + P_1 + \ldots + P_k$ such that each P_i is either a path with exactly the endpoints located in $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$ or a circuit where exactly one of the vertices belongs to $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$.

$P_1, \ldots, P_k$ are called *ears*. If $|V(P_1)| \geq 3$ and $P_2, \ldots, P_k$ are paths we call it a *proper* ear decomposition.

Theorem 1.27 (Whitney 1932)**.** *Let G be an undirected graph. Then:*

$$G \text{ 2-connected} \Leftrightarrow G \text{ has a proper ear decomposition}$$

Definition 1.28. An ear decomposition is *odd* if every ear has odd length (in terms of the number of edges).

Theorem 1.29. *Let G be an undirected graph. Then*

$$G \text{ factor-critical} \Leftrightarrow G \text{ has an odd ear decomposition}$$

The first vertex r of the ear decomposition can be chosen arbitrarily.

Proof.

"$\Leftarrow$": Let G be a graph with an odd ear decomposition $r, P_1, \ldots, P_k$. P_1 is an odd circuit, so it is factor-critical. We use induction on the number of ears. Let P be the last ear and G' be G before adding P. By the induction hypothesis, G' is factor-critical. Given $v \in V(G)$, we have to show that $G - v$ has a perfect matching.

Case 1: $v \in V(G')$. Then $G' - v$ has a perfect matching. Adding every second edge of P (excluding the endpoints) to it, yields a perfect matching of $G - v$.

Case 2: $v \in V(G) \setminus V(G')$. Let x, y be the endpoints of P. Without loss of generality let $P_{[v,x]}$ be even. There exists a perfect matching in $G' - x$. Together with every second edge of $P_{[v,y]}$ and $P_{[v,x]}$ this is a perfect matching in $G - v$.

"$\Rightarrow$": Let $r \in V(G)$ be any vertex. Let M be a perfect matching in $G - r$. Suppose we have an odd ear decomposition for $G' \subseteq G$ with $r \in V(G')$ and $M \cap E(G')$ is a near-perfect matching in G' (i.e. all vertices in G' except for r are matched with other vertices in G').

If $G' \neq G$, there is an edge $\{x, y\} \in E(G) \setminus E(G')$ with $x \in V(G')$ (by Proposition 1.22). If $y \in V(G')$, then $\{x, y\}$ can be chosen as the next ear. Otherwise, construct an M-alternating odd ear, starting with $\{x, y\}$. Let N be a matching in $G - y$. $M \Delta N$ contains a y-r-path P. Let w be the first vertex in $P \cap V(G')$. w is M-exposed in $P_{[y,w]}$, y is

N-exposed in $P_{[y,w]}$. Therefore $P_{[y,w]}$ is even and together with $\{x, y\}$ it forms an M-alternating odd ear.

Inductively, this argument yields an odd ear decomposition.

□

Definition 1.30. Let G be factor-critical and M a near-perfect matching. An M-alternating ear decomposition is an odd ear decomposition such that each ear is an M-alternating path or circuit C with:

$$|E(C) \cap M| = |E(C) \setminus M| - 1$$

Corollary 1.31. *For any factor-critical graph G and any near-perfect matching M in G, there exists in M-alternating ear decomposition of G.*

Definition 1.32. Let G be factor-critical, M a near-perfect matching and $r, P_1, \ldots, P_k$ an M-alternating ear decomposition of G. $\mu, \varphi : V(G) \to V(G)$ are *associated with the ear decomposition* if:

- $\{x, y\} \in M \Rightarrow \mu(x) = y$
- $\{x, y\} \in E(P_i) \setminus M$ and $x \notin \{r\} \cup \bigcup_{j \in [i]} V(P_j)$
  $\Rightarrow \varphi(x) = y$
- $\mu(r) = \varphi(r) = r$

Proposition 1.33. *Let G be a factor-critical graph and μ, φ functions associated with an M-alternating ear decomposition. Then this ear decomposition is unique up to the order of the ears. The Ear-Decomposition-Algorithm correctly determines an explicit list of the ears in linear time.*

Proof. Step 3 determines ears uniquely. The algorithm clearly runs in linear time. □

Lemma 1.34. *Let G be factor-critical and μ, φ associated with an M-alternating ear decomposition. Then the maximal path given by the initial sequence*

$$x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \ldots \tag{1}$$

defines an M-alternating x-r-path of even length.

Proof. We proceed by induction on the number of ears. Let $x \in V(G) \setminus \{r\}$ and P_i be the ear containing x. A subsequence of (1) is a subpath Q of P_i from x to $y \in \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$. Q starts with a matching edge and ends with a non-matching edge, so it has even length. If $y = r$, we are done, otherwise the statement follows from the induction hypothesis. □

1.6 Edmond's Matching Algorithm

Definition 1.35. Let G be a graph, M a matching in G. A *blossom* in G with respect to M is a factor-critical subgraph of B of G such that $|M \cap E(B)| = \frac{|V(B)|-1}{2}$. The vertex $r \in V(B)$ that is exposed by M is called the *base* of B.

Definition 1.36. Let G be a graph, M a matching in G, B a blossom and Q a M-alternating v-r-path of even length from $v \in V(G)$ that is M-exposed to the base r of B. Additionally, let $E(Q) \cap E(B) = \emptyset$. $B + Q$ is called a M-flower.

Lemma 1.37. *Let G be a graph, M a matching in G. Suppose there is a M-flower $B+Q$. Let G', M' result from G and M by contracting $V(B)$ into a single vertex. Then:*

$$M \text{ maximum matching in } G \Leftrightarrow M \text{ maximum matching in } G'$$

Proof.

"$\Leftarrow$": Assume that M is not maximum in G. $N := M \Delta E(Q)$ is a matching with $|N| = |M|$.
$\Rightarrow \exists N$-augmenting path P in G. At least one endpoint x of P is in $V(B)$. If P and B are disjoint, let y be the other endpoint of P. Otherwise, let y be the first vertex on P in B. $P' := P_{[x,y]}$ is an N'-augmenting path in G', so $|N'| = |M'| < \mu(G')$.

"$\Rightarrow$": Assume that M' is not maximum in G', so there exists a matching N' in G' with $|N'| > |M'|$. Let N_0 arise from N' in G, then N_0 contains ≤ 1 vertex from $V(B)$. Since B is factor-critical, N_0 can be extended by $k := \frac{|V(G)|-1}{2}$ edges to a matching N in G. We have

$$|N| = |N_0| + k = \left|N'\right| + k > \left|M'\right| + k = |M|$$

so M is not maximum.

$\square$

Lemma 1.39. *Let G be a graph, M a matching in G. $X \subseteq V(G)$ is the set of exposed vertices. We can find a shortest M-alternating X-X-walk of positive length in $O(|E(G)|)$ time.*

Proof. Define $D := (V(G), A)$ where:

$$A := \{(u,v) \mid \exists x \in V(G) : \{u,x\} \in E(G), \{x,v\} \in M\}$$

A shortest $X - \Gamma_G(X)$-path in D corresponds to a shortest X-X-walk in G. $\square$

Theorem 1.40. *Let $P = v_0, \dots, v_t$ be a shortest M-alternating X-X-walk in G. Then either*

- *P is an M-augmenting path* or
- *$v_0, \dots, v_j$ is an M-flower for some $j \le t$.*

Proof. If P is not a path, choose $i < j$ such that $v_i = v_j$ and j minimal. Then $v_0, \dots, v_{j-1}$ are distinct vertices. If $j - i$ is even, deleting $v_{i-1}, \dots, v_j$ from P yields a shorter walk, so $j - i$ is odd.

Case 1: j is even. Then i is odd and therefore $v_{i+1} = v_{j-1}$ must be the matching mate of $V_i = v_j$ which contradicts the minimality of j.

Case 2: j is odd. Then i is even, so $v_0, \dots, v_i$ is an M-alternating path of even length and $v_i, \dots, v_j$ is an M-alternating odd circuit, i.e. a blossom.

□

Algorithm 1: Edmond's Augmenting Path Search

Input: Graph G, matching M
Output: An M-augmenting path (if one exists)

$X :=$ set of exposed vertices
if *$\exists M$-alternating X-X-walk of positive length* **then**
    $P = v_0, \dots, v_t :=$ a shortest such walk
    if *P is a path* **then**
        return P
    else
        Choose j as in Theorem 1.40
        $v_0, \dots, v_j$ is an M-flower with blossom B
        Recurse on G/B
        Augment an M/B-augmenting path in G/B to an M-augmenting path P' in G
        return P'
else
    $\nexists M$-augmenting path

Theorem 1.41. *Given a graph G, a maximum cardinality matching can be found in time $O(n^2 m)$ where $n := |V(G)|, m := |E(G)|$*

Proof. Start with $M = \emptyset$ and iteratively find M-augmenting path P, set $M := M \Delta E(P)$. If no such path exists, then M is maximum. P can be found in time $O(mn)^3$. Since M is maximum after at most $\frac{n}{2}$ augmentation, we have total running time $O(n^2 m)$. $\square$

1.6.1 Growing forest - $O(n^3)$

Definition 1.42. Let G be a graph, M a matching in G. An *alternating forest* with respect to M in G is a forest F in G where:

- $V(F)$ contains all M-exposed vertices, each tree of F contains exactly one exposed vertex, its *root*.
- We call $v \in V(G)$ an outer (inner) vertex if it has even (odd) distance from the root of its component.
- $\forall v \in V(F)$ the unique path from v to the root of its component is M-alternating.
- $v \in V(G) \setminus V(F)$ is called *out-of-forest*.

Clearly, inner vertices always have degree 2.

Proposition 1.43. *In any alternating forest, the number of outer vertices that are not the root equals the number of inner vertices.*

Proof. For all outer vertices, there exists exactly one inner vertex on its path to the root. $\square$

[3] Here, m is the time required for finding a walk and the recursion depth is bounded by n.