Combinatorial Optimization

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Contents

0 Organization

- Prerequisites
  - Basic knowledge of graph algorithms
  - Linear Programming (LP Duality)
  - Programming skills in C++
- Exam
  - Qualification requires 50% of the points in theoretical & programming exercises
  - Oral exam
- Books
  - "Combinatorial Optimization", Korte & Vygen
  - "Understanding & Using Linear Programming", B. Gärtner, J. Matouset
  - Skript (theorems & definitions)
  - Further book recommendations are on the website

1 Matchings

1.1 Introduction

Definition 1.1.

1. A *matching* M in a graph $G = (V, E)$ is a set of pairwise disjointed edges, i.e. they don't have a common endpoint.

   $\nu(G) :=$ max. cardinality of a matching in G

2. An *edge cover* C of a graph $G = (V, E)$ is a subset of E s.t. $V = \bigcup_{e \in C} e$.

   $\zeta(G) :=$ min. cardinality of an edge cover in G

3. A matching is called *perfect* (or *1-factor*) if it is an edge cover
4. $v \in V$ with $v \in e \in M$ is called *M-covered*
5. $v \in V$ is called *M-exposed* if it is not M-covered

Definition 1.2.

1. A *stable set* (independent set) S is a set of pairwise non-adjacent vertices.

$\alpha(G) :=$ max. cardinality of a stable set

2. A *vertex cover* C is a subset of V s.t. $E = \bigcup_{\{x,y\}\in E,\ x\in G}\{x, y\}$

   $\tau(G) :=$ min. cardinality of a vertex cover

Lemma 1.3.

1. *$\alpha(G) + \tau(G) = |V|$*
2. *$\nu(G) + \zeta(G) = |V|$ if G has no isolated vertices*
3. *$\zeta(G) = \alpha(G)$ if G is bipartite and has no isolated vertices*

Problem. Cardinality Matching Problem
Input: Graph $G = (V, E)$
Task: Find a maximum cardinality matching

Problem. Maximum Weight Matching Problem (MWMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a matching M maximizing $c(M)$

Problem. Minimum Weight Perfect Matching (MWPMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a perfect matching of minimum weight or decide that no perfect matching exists in G

Lemma 1.4. *The MWMP is equivalent to the MWPMP (i.e. there exists a transformation with linear complexity)*

Proof. Given a MWPMP instance (G, c), define $c' := K - c$ ($K := 1 + \sum_{e\in E} |c(e)|$).
$\Rightarrow$ Any maximum weight matching is a maximum cardinality matching

Given a MWMP instance (G, c), define G' as 2 copies of G where the 2 copies of a vertex are joined by an edge.
$\Rightarrow$ G' has a perfect matching. Define:

$$c'(e) := \begin{cases} -c(e) & \text{if } e \text{ is in the first copy} \\ 0 & \text{else} \end{cases}$$

A minimum weight perfect matching in G' gives us a maximum weight matching in G. □

Definition 1.5. Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching in G. A path P is *M-alternating* if its edges are alternatingly in and not in M. If both end points of this path are M-exposed, P is an *M-augmenting* path.

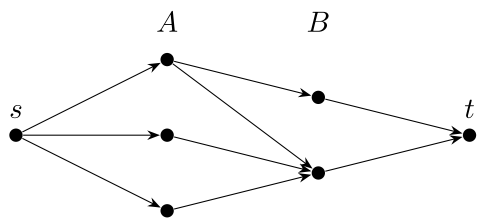


Figure 1: Example of the construction in Theorem 1.8

Lemma 1.6. *Given a matching M in G and an inclusion-wise maximal M-alternating path P,*

$$M \Delta P := M \setminus P \cup P \setminus M$$

is a matching. If P is M-augmenting, then $|M \Delta P| = |M| + 1$.

Theorem 1.7 (Petersen 1891, Berge 1957)**.** *Augmenting Path Theorem Given a graph $G = (V, E)$ and a matching M in G:*

$$|M| = \nu(G) \Leftrightarrow \nexists M\text{-augmenting path } P \text{ in } G$$

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": Assume there exists a matching M' with $|M'| > |M|$. Let $G' := (V, M \Delta M')$.
$\Rightarrow |\delta_{G'}(v)| \leq 2 \ \forall v \in V$
$\Rightarrow G'$ is the union of disjoint circuits and paths
$\Rightarrow$ all circuits are even and have the same number of edges from M and M'
$\Rightarrow \exists$ a path P in G' starting and ending with an edge in M'
$\Rightarrow P$ is an alternating path

□

1.2 Bipartite Matching

Theorem 1.8 (König 1931)**.** *If G is bipartite, then $\nu(G) = \tau(G)$*

Proof. Add vertices s and t edges between them to all vertices of the respective partition. Direct all edges from s to t. Then $\nu(G)$ is maximum number of disjoint s-t-paths. Menger $\Rightarrow$ This is equal to the minimum number of vertices that disconnect t from s. □

Theorem 1.9 (Hall 1935)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph. Then:*

$$G \text{ has a matching covering } A \Leftrightarrow |\Gamma(X)| \geq |X| \quad \forall X \subseteq A$$

Corollary 1.10. *Marriage Theorem*

$$|\Gamma(X)| \geq |X| \;\; \forall X \subseteq A \text{ and } |A| = |B| \Leftrightarrow G \text{ has a perfect matching}$$

Definition 1.12. The MWPMP for bipartite graphs is called *Assignment Problem.*

Theorem 1.13. *The Assignment Problem can be solved in time $O(nm + n^2 \log m)$.*

Proof. Use the Successive Shortest Paths algorithm in an auxiliary graph. □

1.3 The Tutte Matrix & Randomized Matching

Definition 1.14. Let G be a simple, undirected graph. Let G' be an orientation of G and $(X_e)_{e \in E(G)}$. The *Tutte matrix* is defined as

$$T_G(X) \coloneqq (t^*_{vw})_{v,w \in V(G)}$$

where

$$t^*_{vw} \coloneqq \begin{cases} X_{\{v,w\}} & \text{if } (v,w) \in E(G') \\ -X_{\{v,w\}} & \text{if } (w,v) \in E(G') \\ 0 & \text{else} \end{cases}$$

Remark 1.15. $T_G(X)$ is shew-symmetric (i.e. $T_G(X) = -(T_G(X))^t$). $\operatorname{rank}(T_G(X))$ is independent of the orientation of G. $\det(T_G(X))$ is a polyomial in X.

Theorem 1.16 (Tutte). *A simple graph G has a perfect matching* $\Leftrightarrow \det(T_G(X)) \neq 0$

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and S_n be the permutation group.

$$\det T_G(X) = \sum_{\pi \in S_n} \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} t^*_{v_i, v_{\pi(i)}}$$

Let $S'_n \coloneqq \{\pi \in S_n \mid \prod_{i=1}^{n} t^*_{v_i, v_{\pi_i}} \neq 0\}$. Each $\pi \in S_n$ corresponds to a digraph $H_\pi \coloneqq (V(G), \{(v_i, v_{\pi(i)}) \mid i \in [n][1]\})$. We have $|\delta^+(v) = 1 = |\delta^-(v)|| \;\; \forall v \in V(H_\pi) \Rightarrow H_\pi$ is the union of disjoint circuits. If $\pi \in S'_n$, then $H_\pi \overset{\Leftrightarrow}{\subset G'}$.

If there exists $\pi \in S'_n$ s.t. H_π is a collection of even circuits, then this immediately yields a perfect matching in G (using every second edge of each circuit).

[1] This is an abbreviation for $\{1, \ldots, n\}$.

Otherwise, $\forall \pi \in S'_n$, H_π contains an odd circuit. Let $r(\pi) \in S'_n$ arise from π by reversing edges on the unique odd circuit containing a vertex with minimum index $\Rightarrow r(r(\pi)) = \pi$ and $\operatorname{sgn}(\pi) = \operatorname{sgn}(r(\pi))$. The second part is true since for reversing an odd cycle, we need an even number of swaps. Let $v_{i_1}, \dots, v_{i_{2k+1}}$ be the "first" odd circuit. Then $r(\pi)$ is attained by $2k$ swaps: For $j = 1, \dots, k$ swap $(\pi(i_{2j-1}), \pi(i_{2k}))$ and $(\pi(i_{2j}), \pi(i_{2k+1}))$.

$\prod_{i=1}^n t^*_{v_i v_{\pi(i)}} = -\prod_{i=1}^n t^*_{v_i v_{r(\pi(i))}}$ since there is an odd number of sign changes to t^*. $\Rightarrow \det(T_G(X)) = 0$. We have shown that if G has no perfect matching, then $\det T_G(X) = 0$.

Assume that G has a perfect matching M. Define π as $\pi(i) = j, \pi(j) = i$ where $\{i,j\} \in M$. $\Rightarrow \prod_{i=1}^n t^*_{v_i v_{\pi(i)}} = \prod_{e \in M} -X_e^2$ cannot be canceled out. In particular, $\det T_G(X) \neq 0$. $\square$

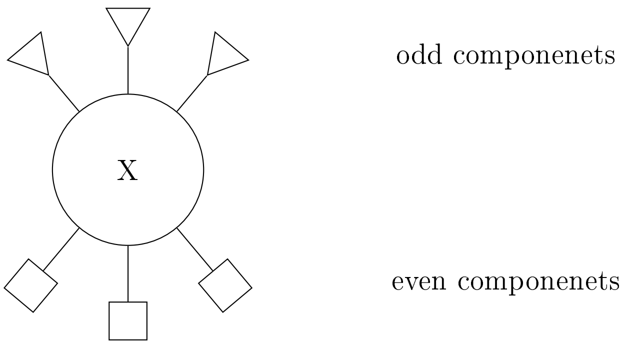
Remark 1.17. Picking $X' \in [0,1]^{E(G)}$ at random, we almost surely have (since the zero set of a non-zero polynomial is a set of measure zero):

$$\det T_G(X') \neq 0 \Leftrightarrow G \text{ has a perfect matching}$$

Theorem 1.18 (Lovász 1979). *Let G be a simple graph and $X \in [0,1]^{E(G)}$ chosen randomly. Then almost surely* $\operatorname{rank}(T_G(X)) = 2\nu(G)$.

1.4 Tutte's Matching Theorem

Let $X \subseteq V(G)$. $G - X$ consists of even and odd (in terms of the number of vertices) connected components. We define $q_G(X)$ to be the number of odd components in $G - X$.



Definition 1.19. A graph G satisfies the *Tutte Condition* if $q_G(X) \leq |X|$ for all $X \subseteq V(G)$. $\emptyset \neq X \subseteq V(G)$ is called *barrier* if $q_G(X) = |X|$.

Proposition 1.20. *For any graph G and any $X \subseteq V(G)$:*

$$q_G(X) - |X| \equiv |V(G)| \mod 2$$

Definition 1.21. A graph G is *factor-critical* if $G-v$ has a perfect matching for all $v \in V(G)$. A matching is called *near-perfect* if it covers $|V(G)|-1$ vertices.

Proposition 1.22. *If G is factor-critical, then it is connected.*

Theorem 1.23 (Tutte 1947)**.** *A graph G has a perfect matching $\Leftrightarrow$ Tutte Condition holds (i.e. $q_G(X) \leq |X| \ \forall X \subseteq V(G)$)*

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": We proceed by induction on $|V(G)|$. The case $|V(G)| = 2$ is clear.

Generally, if the Tutte Condition holds, then $|V(G)|$ must be even (pick $X = \emptyset$). Proposition 1.20 $\Rightarrow q_G(X) - |X|$ is even. Every $x \in V(G)$ induces a barrier $\{x\}$. Let X be a maximum barrier. Then $G - X$ doesn't have any even components (since otherwise a single vertex of such a component could be added to X).

Claim: Each odd component is factor-critical.
Let C be an odd component in $G - X$, $v \in V(C)$. Assume that $C - v$ does not have a perfect matching. Induction Hypothesis $\Rightarrow C - v$ violates Tutte Condition.
$\Rightarrow \exists Y \subseteq V(C - v) : q_{C-v}(Y) > |Y|$
$\overset{1.20}{\Rightarrow} q_{C-v}(Y) \geq |Y| + 2$
Observe $X \cap \{v\} = Y \cap \{v\} = X \cap Y = \emptyset$:

$$\begin{aligned} q_G(X \cup Y \cup \{v\}) &= q_G(X) - 1 + q_C(Y \cup \{v\}) \\ &= |X| - 1 + q_{C-v}(Y) \\ &\geq |X| - 1 + |Y| + 2 \\ &= |X \cup Y| + 1 \\ &= |X \cup Y \cup \{v\}| \end{aligned}$$

$\Rightarrow X \cup Y \cup \{v\}$ is a barrier
$\Rightarrow$ Claim

Let G' arise from G by contracting each odd component into a single vertex. We have $V(G') = X \dot\cup Z$ and G' is bipartite. We have to show that G' has a perfect matching. If not, then $\exists A \subseteq Z : |\Gamma_{G'}(A)| < |A|$ $\Rightarrow q_G(\Gamma_{G'}(A)) \geq |A| > |\Gamma_{G'}(A)|$ which contradicts the Tutte Condition.

$\square$

Theorem 1.24 (Berge 1958)**.**

$$|V(G)| = 2\nu(G) + \max_{X\subseteq V(G)} (q_G(X) - |X|)$$

Proof. For $X \subseteq V(G)$, any matching has at least $q_G(X) - |X|$ uncovered vertices, so "$\geq$" holds.

For the other inequality, add $k := \max_{X\subseteq V(G)}(q_G(X) - |X|)$ new vertices and connect them to all existing vertices, yielding a new graph H.

We claim that H has a perfect matching. This then implies:

$$2\nu(G) + k \geq 2\nu(H) - k = |V(H)| - k = |V(G)|$$

Assume that H does not have a perfect matching. Then by Tutte's Theorem, there exists $Y \subseteq V(H)$ with $q_H(Y) > |Y|$. By 1.20, $k \equiv |V(G)| \mod 2$, therefore $|V(H)|$ is even, so $Y \neq \emptyset$. Y must contain all new vertices, otherwise $H - Y$ would be connected[2] and $q_H(Y) \leq 1 \leq |Y|$.

$$\Rightarrow q_G(Y \cap |V(G)|) = q_H(Y) > |Y| = |Y \cap V(G)| + k$$

which is a contradiction to the choice of k. □

1.5 Ear Decompositions of Factor-Critical Graphs

Definition 1.25. Let G be a graph. An *ear decomposition* of G is a sequence $r, P_1, \ldots, P_k$ with $G = (r, \emptyset) + P_1 + \ldots + P_k$ such that each P_i is either a path with exactly the endpoints located in $\{r\} \cup \bigcup_{j\in[i-1]} V(P_j)$ or a circuit where exactly one of the vertices belongs to $\{r\} \cup \bigcup_{j\in[i-1]} V(P_j)$.

$P_1, \ldots, P_k$ are called *ears*. If $|V(P_1)| \geq 3$ and $P_2, \ldots, P_k$ are paths we call it a *proper* ear decomposition.

Theorem 1.27 (Whitney 1932)**.** *Let G be an undirected graph. Then:*

$$G \text{ 2-connected} \Leftrightarrow G \text{ has a proper ear decomposition}$$

Definition 1.28. An ear decomposition is *odd* if every ear has odd length (in terms of the number of edges).

Theorem 1.29. *Let G be an undirected graph. Then*

$$G \text{ factor-critical} \Leftrightarrow G \text{ has an odd ear decomposition}$$

The first vertex r of the ear decomposition can be chosen arbitrarily.

Proof.

[2] Note that Y cannot contain all old vertices, since otherwise $q_H(Y) < |Y|$.

"$\Leftarrow$": Let G be a graph with an odd ear decomposition $r, P_1, \ldots, P_k$. P_1 is an odd circuit, so it is factor-critical. We use induction on the number of ears. Let P be the last ear and G' be G before adding P. By the induction hypothesis, G' is factor-critical. Given $v \in V(G)$, we have to show that $G - v$ has a perfect matching.

Case 1: $v \in V(G')$. Then $G' - v$ has a perfect matching. Adding every second edge of P (excluding the endpoints) to it, yields a perfect matching of $G - v$.

Case 2: $v \in V(G) \setminus V(G')$. Let x, y be the endpoints of P. Without loss of generality let $P_{[v,x]}$ be even. There exists a perfect matching in $G' - x$. Together with every second edge of $P_{[v,y]}$ and $P_{[v,x]}$ this is a perfect matching in $G - v$.

"$\Rightarrow$": Let $r \in V(G)$ be any vertex. Let M be a perfect matching in $G - r$. Suppose we have an odd ear decomposition for $G' \subseteq G$ with $r \in V(G')$ and $M \cap E(G')$ is a near-perfect matching in G' (i.e. all vertices in G' except for r are matched with other vertices in G').

If $G' \neq G$, there is an edge $\{x, y\} \in E(G) \setminus E(G')$ with $x \in V(G')$ (by Proposition 1.22). If $y \in V(G')$, then $\{x, y\}$ can be chosen as the next ear. Otherwise, we construct an M-alternating odd ear, starting with $\{x, y\}$: Let N be a matching in $G - y$. $M \Delta N$ contains a y-r-path P. Let w be the first vertex in $P \cap V(G')$. w is M-exposed in $P_{[y,w]}$, y is N-exposed in $P_{[y,w]}$. Therefore $P_{[y,w]}$ is even and together with $\{x, y\}$ it forms an M-alternating odd ear.

Inductively, this argument yields an odd ear decomposition.

$\square$

Definition 1.30. Let G be factor-critical and M a near-perfect matching. An M-alternating ear decomposition is an odd ear decomposition such that each ear is an M-alternating path or circuit C with:

$$|E(C) \cap M| = |E(C) \setminus M| - 1$$

Corollary 1.31. *For any factor-critical graph G and any near-perfect matching M in G, there exists in M-alternating ear decomposition of G.*

Definition 1.32. Let G be factor-critical, M a near-perfect matching and $r, P_1, \ldots, P_k$ an M-alternating ear decomposition of G. $\mu, \varphi : V(G) \to V(G)$ are *associated with the ear decomposition* if:

- $\{x, y\} \in M \Rightarrow \mu(x) = y$
- $\{x, y\} \in E(P_i) \setminus M$ and $x \notin \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$

$\Rightarrow \varphi(x) = y$

- $\mu(r) = \varphi(r) = r$

Proposition 1.33. *Let G be a factor-critical graph and μ, φ functions associated with an M-alternating ear decomposition. Then this ear decomposition is unique up to the order of the ears. The Ear-Decomposition-Algorithm (algorithm 1) correctly determines an explicit list of the ears in linear time.*

Proof. Step 3 determines ears uniquely. The algorithm clearly runs in linear time. $\square$

Algorithm 1: Ear Decomposition Algorithm

Input: Factor-critical graph G, functions μ, φ associated with an M-alternating ear decomposition

Output: An M-alternating ear decomposition $r, P_1, \dots, P_k$

1 $X := \{r\}$ where r is the vertex with $\mu(r) = r$
2 $k := 0$, $S :=$ empty stack
3 **while** $X \neq V(G)$ **do**
4   **if** *S is non-empty* **then**
5   Let $v \in V(G) \setminus X$ be an endpoint of the topmost element of the stack
6   **else**
7   Choose $v \in V(G) \setminus X$ arbitrarily
8     $x := v,\ y := \mu(v),\ P := (\{x, y\}, \{\{x, y\}\})$
9   **while** $\varphi(\varphi(x)) = x$ **do**
10         $P := P + \{x, \varphi(x)\} + \{\varphi(x), \mu(\varphi(x))\}$
11         $x := \mu(\varphi(x))$
12   **while** $\varphi(\varphi(y)) = y$ **do**
13         $P := P + \{y, \varphi(y)\} + \{\varphi(y), \mu(\varphi(y))\}$
14         $y := \mu(\varphi(y))$
15     $P := P + \{x, \varphi(x)\} + \{y, \varphi(y)\}$
16   P is the ear containing y as an inner vertex. Put P on S.
17   **while** *Both endpoints of the topmost element P of the stack S are in X* **do**
18   Delete P from S
19         $k := k + 1,\ P_k := P,\ X := X \cup V(P)$
20 **forall** $\{y, z\} \in E(G) \setminus (E(P_1) \cup \ldots \cup E(P_k))$ **do**
21     $k := k + 1,\ P_k := (\{y, z\}, \{\{y, z\}\})$
22 **return** $r, P_1, \dots, P_k$

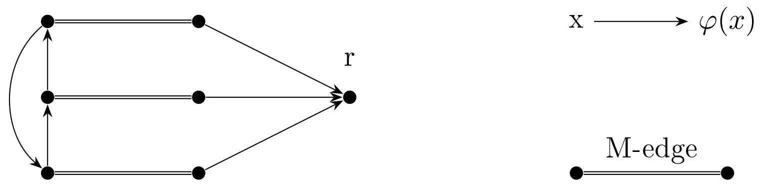


Figure 2: Counter example for the reverse implication of lemma 1.34

Lemma 1.34. *Let G be factor-critical and μ, φ associated with an M-alternating ear decomposition. Then the maximal path given by the initial sequence*

$$x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \ldots \tag{1}$$

defines an M-alternating x-r-path of even length.

Proof. We proceed by induction on the number of ears. Let $x \in V(G) \setminus \{r\}$ and P_i be the ear containing x. A subsequence of (1) is a subpath Q of P_i from x to $y \in \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$. Q starts with a matching edge and ends with a non-matching edge, so it has even length. If $y = r$, we are done, otherwise the statement follows from the induction hypothesis. $\square$

1.6 Edmond's Matching Algorithm

Definition 1.35. Let G be a graph, M a matching in G. A *blossom* in G with respect to M is a factor-critical subgraph B of G such that $|M \cap E(B)| = \frac{|V(B)|-1}{2}$. The vertex $r \in V(B)$ that is exposed by $M \cap E(B)$ is called the *base* of B.

Definition 1.36. Let G be a graph, M a matching in G, B a blossom and Q a M-alternating v-r-path of even length from $v \in V(G)$ that is M-exposed to the base r of B. Additionally, let $E(Q) \cap E(B) = \emptyset$. $B + Q$ is called an *M-flower*.

Lemma 1.37. *Let G be a graph, M a matching in G. Suppose there is a M-flower $B + Q$. Let G', M' result from G and M by contracting $V(B)$ into a single vertex. Then:*

$$M \text{ maximum matching in } G \Leftrightarrow M' \text{ maximum matching in } G'$$

Proof.

"$\Leftarrow$": Assume that M is not maximum in G. $N := M \Delta E(Q)$ is a matching with $|N| = |M|$.
$\Rightarrow$ $\exists N$-augmenting path P in G. At least one endpoint x of P is not in $V(B)$ (since B contains only one N-exposed vertex). If P and B are disjoint, let y be the other endpoint of P. Otherwise, let y be the

first vertex on P in B. $P' := P_{[x,y]}$ is an N'-augmenting path in G', so $|N'| = |M'| < \mu(G')$.

"$\Rightarrow$": Assume that M' is not maximum in G', so there exists a matching N' in G' with $|N'| > |M'|$. Let N_0 arise from N' in G, then N_0 contains ≤ 1 vertex from $V(B)$. Since B is factor-critical, N_0 can be extended by $k := \frac{|V(G)|-1}{2}$ edges to a matching N in G. We have

$$|N| = |N_0| + k = \left|N'\right| + k > \left|M'\right| + k = |M|$$

so M is not maximum.

□

Lemma 1.39. *Let G be a graph, M a matching in G. $X \subseteq V(G)$ is the set of exposed vertices. We can find a shortest M-alternating X-X-walk of positive length in $O(|E(G)|)$ time.*

Proof. Define $D := (V(G), A)$ where:

$$A := \{(u,v) \mid \exists x \in V(G) : \{u,x\} \in E(G), \{x,v\} \in M\}$$

A shortest $X - \Gamma_G(X)$-path in D corresponds to a shortest X-X-walk in G. □

Theorem 1.40. *Let $P = v_0, \dots, v_t$ be a shortest M-alternating X-X-walk in G. Then either*

- *P is an M-augmenting path* or
- *$v_0, \dots, v_j$ is an M-flower for some $j \leq t$.*

Proof. If P is not a path, choose $i < j$ such that $v_i = v_j$ and j minimal. Then $v_0, \dots, v_{j-1}$ are distinct vertices. If $j - i$ is even, deleting $v_{i-1}, \dots, v_j$ from P yields a shorter walk, so $j - i$ is odd.

Case 1: j is even. Then i is odd and therefore $v_{i+1} = v_{j-1}$ must be the matching mate of $V_i = v_j$ which contradicts the minimality of j.

Case 2: j is odd. Then i is even, so $v_0, \dots, v_i$ is an M-alternating path of even length and $v_i, \dots, v_j$ is an M-alternating odd circuit, i.e. a blossom.

□

Theorem 1.41. *Given a graph G, a maximum cardinality matching can be found in time $O(n^2 m)$ where $n := |V(G)|, m := |E(G)|$*

Algorithm 2: Edmond's Augmenting Path Search

Input: Graph G, matching M

Output: An M-augmenting path (if one exists)

$X \coloneqq$ set of exposed vertices
if $\exists M$*-alternating* X*-*X*-walk of positive length* **then**
    $P = v_0, \ldots, v_t \coloneqq$ a shortest such walk
    if P *is a path* **then**
        return P
    else
        Choose j as in Theorem 1.40
        $v_0, \ldots, v_j$ is an M-flower with blossom B
        Recurse on G/B
        Augment an M/B-augmenting path in G/B to an M-augmenting path P' in G
        return P'
else
    $\nexists M$-augmenting path

Proof. Start with $M = \emptyset$ and iteratively find M-augmenting path P, set $M \coloneqq M \Delta E(P)$. If no such path exists, then M is maximum. P can be found in time $O(mn)$[3]. Since M is maximum after at most $\frac{n}{2}$ augmentation, we have total running time $O(n^2 m)$. $\square$

1.6.1 Growing forest - $O(n^3)$

Definition 1.42. Let G be a graph, M a matching in G. An *alternating forest* with respect to M in G is a forest F in G where:

- $V(F)$ contains all M-exposed vertices, each tree of F contains exactly one exposed vertex, its *root*.
- We call $v \in V(G)$ an outer (inner) vertex if it has even (odd) distance from the root of its component.
- $\forall v \in V(F)$ the unique path from v to the root of its component is M-alternating.
- $v \in V(G) \setminus V(F)$ is called *out-of-forest*.

Clearly, inner vertices always have degree 2 (we always assume that there are no matching edges that can immediately be added to F).

[3]Here, m is the time required for finding a walk and the recursion depth is bounded by n.

Proposition 1.43. *In any alternating forest, the number of outer vertices that are not the root equals the number of inner vertices.*

Proof. For all outer vertices, there exists exactly one inner vertex on its path to the root. $\square$

Lemma 1.44. *Given a graph G, a matching M, an alternating forest F with respect to M in G. Then, either M is a maximum matching or $\exists$ outer vertex $x \in V(F)$, an edge $\{x, y\} \notin E(F)$ such that one of the following holds:*

- *Grow: $y \notin V(F)$ and therefore $\{y, z\} \in M$ with $z \notin V(F)$. In this case, y, z and $\{x, y\}, \{y, z\}$ can be added to F.*
- *Augment: y is an outer vertex in a different connected component in F. In this case, M can be augmented along $P(x) \cup \{x, y\} \cup P(y)$ where $P(z)$ denotes the unique path from $z \in V(F)$ to the root of its connected component.*
- *Shrink: y is an outer vertex in the same component as x. Let r be the first vertex on $P(x)$ that is also on $P(y)$. Then $|\delta_F(r)| \geq 3$, so r is an outer vertex and $\left|E(F_{[x,r]})\right|, \left|E(F_{[y,r]})\right|$ are even. Together with $\{x, y\}$ these paths form a blossom with ≥ 3 vertices.*

Proof. We show that if none of these cases apply, M is maximum. If none of the cases apply, then every outer vertex only has inner vertices as neighbors. Let X be the set of inner vertices, $s := |X|$ and t be the number of outer vertices. All outer vertices are isolated in $G - X$, so $q_G(X) - |X| = t - s$. By Berge's formula (1.24), $t - s$ vertices are exposed by any matching, so M is maximum. $\square$

Definition 1.45. Let G be a graph, M a matching in G. A subgraph F of G is a *general blossom forest* with respect to M if there exists a partition $V(F) = V_1 \dot\cup \ldots \dot\cup V_k$ such that $F_i = F[V_i]$ is a maximal factor-critical subgraph of F with $|M \cap E(F_i)| = \frac{|V_i|-1}{2}$ $(i \in [k])$ and after contracting each V_i, we obtain an M-alternating forest F'. F_i is called an outer (inner) blossom if V_i is an outer (inner) vertex in F'.

A *special blossom forest* is a general blossom forest where each inner blossom is a single vertex.

Store a special blossom forest with 3 functions $\mu, \varphi, \rho : V(G) \to V(G)$:

$$\mu(x) := \begin{cases} x & \text{if } x \text{ is exposed in } M \\ y & \text{if } \{x, y\} \in M \end{cases}$$

$$\varphi(x) := \begin{cases} x & \text{if } x \text{ is the base of an outer blossom or } x \text{ is out-of-forest} \\ y & \text{if } x \text{ is an inner vertex and } \{x, y\} \in E(F) \setminus M \\ y & \text{if } x \text{ is an outer vertex (i.e. in an outer blossom) and } \mu, \varphi \text{ are associated with an } M\text{-alternating ear decomposition of } x\text{'s blossom, } \{x, y\} \in E(F) \setminus M \end{cases}$$

$$\rho(x) := \begin{cases} x & \text{if } x \text{ is an inner vertex or out-of-forest} \\ y & \text{if } x \text{ is an outer vertex and } y \text{ is the base of the outer blossom containing } x \text{ } (y = x \text{ is possible}). \end{cases}$$

Proposition 1.46. *Let F be a special blossom forest with respect to M and μ, φ, ρ as above. Then:*

1. *For all outer vertices x, $P(x) :=$ maximal path given by subsequence of $x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \ldots$ is an M-alternating path from x to q where q is the root of the component containing x.*
2. *A vertex x is*
   - *an outer vertex* $\Leftrightarrow \mu(x) = x \vee \varphi(\mu(x)) \neq \mu(x)$
   - *an inner vertex* $\Leftrightarrow \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x$
   - *out-of-forest* $\Leftrightarrow \mu(x) \neq x \wedge \varphi(x) = x \wedge \varphi(\mu(x)) = \mu(x)$

Proof.

1. By definition of μ, φ and lemma 1.34 some initial subsequence of $P(x)$ ends at the base r of the blossom containing x. If $r = q$, we are done. Otherwise $\mu(r), \varphi(\mu(r))$ are next elements in a sequence leading to outer vertex $\varphi(\mu(r))$. This can be iterated.
2. Since the conditions are mutually exclusive, it suffices to show one implication for all the statements.
   - If x is outer, it is a root ($\mu(x) = x$) or $P(x)$ is a path of length at least 2, so $\varphi(\mu(x)) \neq \mu(x)$.
   - If x is inner, then $\mu(x)$ is the base of an outer blossom. Therefore $\varphi(\mu(x)) = \mu(x)$. $P(\mu(x))$ is a path of length at least 2, so $\varphi(x) \neq x$.

- If x is out-of-forest, then x is covered by M so $\mu(x) \neq x$. By definition of φ, $\varphi(x) = x$. $\mu(x)$ is out-of-forest as well, so $\varphi(\mu(x)) = \mu(x)$.

$\square$

Lemma 1.47. *Following invariants hold:*

a) $\{\{x, \mu(x)\} \mid x \in V(G), \mu(x) \neq x\}$ *is a matching*

b) $\{\{x, \mu(x)\} \mid \underbrace{x \in V(G), \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x}_{\textit{inner vertices}}\} \cup \{\{x, \varphi(x)\} \mid x \in V(G), \varphi(x) \neq x\}$ *forms the edge set of a special blossom forest.*

c) μ, φ, ρ *satisfy the conditions in definition 1.45 (special blossom forest).*

Proof. a) holds as μ only changes in *Augment*. b) is correct after initialization and after the reset in the *Augment* step. It is preserved by *Grow* steps.

In a *Shrink* step, r (the first vertex that the paths from x, y to the root share) is a root or has $|\delta(r)| = 3$ (i.e. it is the base of a blossom), so it is an outer vertex. We define a blossom $B := \{v \in V(G) \mid \varphi(v) \in V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})\}$. Consider $\{u, v\} \in F$ with $u \in B, v \notin B$. If $\{u, v\} \in M$, we have $u = r, v = \mu(r)$ (since $F[B]$ contains a near-perfect matching). u was an outer vertex before shrinking and $F[B]$ being factor-critical follows from c) and the characterization by ear-decompositions.

For c), it's clear that μ always represents a matching. $\varphi(x) = x$ if x is not an outer vertex. Therefore, $\mu + \varphi$ represent an M-alternating ear decomposition of B. During *Shrink*, $\varphi(v)$ is not changed if $\varphi(v) = r$. Therefore, the odd ear decomposition for $B' :=$ blossom containing r, is the correct starting point. The next ear is $P(x)_{[x,x']} \cup P(y)_{[y,y']} + \{x, y\}$, where x' (y') is the first vertex in B' on $P(x)_{[x,r]}$ ($P(y)_{[y,r]}$).

For each ear Q of a former blossom $B'' \subseteq B$, $Q \setminus (E(P(x)) \cup E(P(y)))$ form a new ear (since it is created by removing an even path). φ, μ represent this ear-decomposition. $\square$

Theorem 1.48. *Edmond's cardinality matching algorithm correctly determines a maximum matching in $O(n^3)$ time, where $n := |V(G)|$.*

Proof. By lemma 1.47 and proposition 1.46, the algorithm maintains a special blossom forest. Let M, F be the final matching and forest. x an outer

Algorithm 3: Edmond's Cardinality Matching Algorithm

Input: A graph G

Output: A maximum matching M (defined by $\{x, \mu(x)\}$)

$\mu(v) := v,\ \varphi(v) := v,\ \rho(v) := v,\ \mathit{scanned}(v) :=$ false for all $v \in V(G)$

`// Outer Vertex Scan:`

while $\exists$ *outer vertex* x *with* $\mathit{scanned}(x) =$ false **do**

  Let y be a neighbor of x such that y is either out-of-forest or y is outer with $\rho(y) \neq \rho(x)$

  if *such a* y *does not exist* **then**

    $\mathit{scanned}(x)$ =true, **continue**

  `// Grow:`

  if y *is out-of-forest* **then**

    $\varphi(y) := x$, **continue**

  `// Augment:`

  else if $P(x)$ *and* $P(y)$ *are vertex-disjoint* **then**

    $\mu(\varphi(v)) = v,\ \mu(v) = \varphi(v)$ for all $v \in V(P(x) \cup P(y))$ with odd distance from x or y on $P(x)$ or $P(y)$, respectively

    $\mu(x) := y,\ \mu(y) := x$

    $\varphi(v) := v, \rho(v) := v, \mathit{scanned}(v) :=$ false for all $v \in V(G)$

  `// Shrink:`

  else

    Let r be the first vertex on $V(P(x)) \cap V(P(y))$ with $\rho(r) = r$

    forall $v \in V(P(x)_{[x,r]}) \cup V(P(y)_{y,r})$ *with odd distance from* x *or* y *on* $P(x)_{[x,r]}$ *or* $P(y)_{[y,r]}$, *respectively and* $\rho(\varphi(v)) \neq r$ **do**

      $\varphi(\varphi(v)) := v$

    if $\rho(x) \neq r$ **then**

      $\varphi(x) := y$

    if $\rho(y) \neq r$ **then**

      $\varphi(y) := x$

    forall $v \in V(G)$ *with* $\rho(v) \in V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})$ **do**

      $\rho(v) := r$

return μ

vertex implies that $\forall y \in \Gamma(x) : y$ is inner and $\varphi(y) = \varphi(x)$. Define:

$$X \coloneqq \text{set of inner vertices}$$
$$B \coloneqq \text{set of bases of (outer) blossoms}$$

Then every unmatched vertex is in B. Matched vertices in B have matching mates in X and $|B| = |X| + |V(G)| - 2|M|$. (Outer) blossoms are odd connected components in $G - X$, so by Berge's theorem (1.24), at least $|B| - |X|$ vertices remain uncovered by any matching, so M is maximum.

We now consider the running time: The status (outer, inner, out-of-forest) for a given vertex can be checked in constant time (proposition 1.46). Therefore, *Grow*, *Augment* and *Shrink* can be implemented in $O(n)$ time. There are at most n calls to *Grow* and *Shrink* per augment and at most $\frac{n}{2}$ *Augments*. This implies the running time $O(n^3)$. □

Remark 1.49. The time for *Shrink* can be reduced to $O(\log n)$ using a binary tree, leading to a running time of $O(nm \log n)$ in total. Tarjan (1974), Gabow & Tarjan (1983) proved a running time of $O(nm\alpha(m,n))$ (where α is the inverse Ackermann function) or $O(nm)$.

Remark 1.50. It's not necessary to reset everything after augmenting. It suffices to reset the 2 trees that were changed by the augmentation. Gabow & Tarjan (1983) showed that it's possible to augment all paths of the same length in $O(m)$ time. There are $2\sqrt{\nu(G)} + 2$ different path lengths, so in total this results in a running time of $O(\sqrt{n}m)$.

Remark 1.51 (Skew-symmetric flows). Goldberg & Karzanov (2003) (and Fremuth-Pagen & Jungnickel (2003)) used *Generalized Max-Flow* to achieve a running time of $O(\sqrt{n}m\frac{\log \frac{m}{n}}{\log n})$.

1.7 Gallai-Edmonds Decomposition

Proposition 1.52. *Let G be a graph, $X \subseteq V(G)$ with $|V(G)| - 2\nu(G) = q_G(X) - |X|$. Then any maximum matching of G*

- *contains a perfect matching in the even components of $G - X$.*
- *contains a near-perfect matching in odd components of $G - X$.*
- *matches all $x \in X$ to distinct odd components.*

Proof. Follows directly from Berge's theorem (1.24). □

Theorem 1.53. *Let G be a graph and:*

$$Y \coloneqq \{v \in V(G) \mid \exists \text{ maximum matching that exposes } v\}$$

Define $X \coloneqq \Gamma(Y)$ and $W \coloneqq V(G) \setminus (X \cup Y)$. Then:

1. *X attains* $\max_{X' \subseteq V(G)} q_G(X') - |X'|$.
2. *$G[Y]$ is the union of factor-critical subgraphs and $G[W]$ is the union of even connected components.*
3. *Any maximum matching in G*
   - *contains a perfect matching in $G[W]$.*
   - *contains a near-perfect matching in each component of $G[Y]$.*
   - *matches all $x \in X$ to distinct connected components*

Y, X, W is called Gallai-Edmonds decomposition *of G.*

Proof. Consider the matching M and special blossom forest F at the end of the algorithm. Let X' (Y') be the set of inner (outer) vertices and W' the set of out-of-forest vertices. X', Y', W' satisfy 1., 2. and 3. by the proof of theorem 1.48.

Proposition 1.52 implies that any maximum matching covers all vertices in $V(G) \setminus Y'$, so $Y \subseteq Y'$. For the other inclusion, let $v \in Y'$. Then $M \Delta P(v)$ is a maximum matching exposing v, so $v \in Y$ and $Y' = Y$. By definition, $X = X'$ and $W = W'$. $\square$

Corollary 1.54. *A graph G has a perfect matching $\Leftrightarrow \forall U \subseteq V(G)$, $G - U$ has at most $|U|$ factor-critical components.*

1.8 Minimum Weight Perfect Matching

We use the following Integer Programming formulation:

$$\begin{aligned} \min \sum_{e \in E(G)} c_e x_e & \\ \text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 1 & \qquad v \in V(G) \\ x_e \in \{0,1\} & \end{aligned}$$

and the corresponding relaxation where we only require $x_e \geq 0$. The dual problem of this is:

$$\begin{aligned} \max \sum_{v \in V(G)} z_v & \\ \text{s.t.} \quad z_v + z_w \leq c_e & \qquad \{v, w\} \in E(G) \end{aligned}$$

Proposition 1.55 (Hungarian Method)**.** *Let G be a graph, $c \in \mathbb{R}^{E(G)}$ and $z \in \mathbb{R}^{V(G)}$ with $z_v + z_w \leq c_e$ for all $e = \{v, w\} \in E(G)$. Define:*

$$G_z := (V(G), \{e = \{v, w\} \mid z_v + z_w = c_e\})$$

Let M be a matching in G_z, F a maximal alternating forest in G_z with respect to M. Let X/Y be the set of inner/outer vertices. Then:

1. *If M is a perfect matching in G_z, then it is a minimum-weight perfect matching in G.*

2. *If $\Gamma_G(y) \subseteq X$ for all $y \in Y$, then M is a maximum matching.*

3. *If neither 1. nor 2. hold, define:*

$$\epsilon := \min\{\min_{e=\{v,w\}\in E(G[Y])} \frac{c_e - z_v - z_w}{2}, \min_{e\in\delta(Y)\cap\delta(V(F))} c_e - z_v - z_w\}$$

   Set $z'_v := z_v - \epsilon$ for all $v \in X$, $z'_v := z_v + \epsilon$ for all $v \in Y$ and $z'_v := z_v$ for all $v \in V(G) \setminus (X \cup Y)$. Then z' is a feasible dual solution and $M \cup E(F) \subseteq E(G_{z'})$. Additionally, $\Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$ for some $y \in Y$.

Proof.

1. Let M' be a minimum-weight perfect matching.

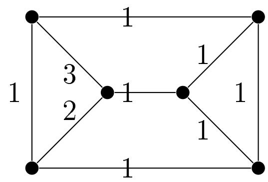
$$\begin{aligned}\sum_{e\in M'} c_e &= \sum_{v\in V(G)} z_v + \sum_{e=\{v,w\}\in M'} (c_e - z_v - z_w)\\ &\geq \sum_{v\in V(G)} z_v\\ &= \sum_{v\in V(G)} z_v + \sum_{e=\{v,w\}\in M} (c_e - z_v - z_w)\\ &= \sum_{e\in M} c_e\end{aligned}$$

2. Each outer vertex is an odd blossom (singleton) of $G - x$. By Berge (1.24), at least $|Y| - |X|$ vertices remain uncovered.

3. z' stays feasible by the choice of ϵ. Edges in $E(F), M$ remain tight. By 1. and 2., $\exists y \in Y :\ \Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$.

$\square$

Remark 1.56. For bipartite graphs, the adjacency matrix is totally unimodular, so the LP has integral vertices.



We define $\mathcal{A} := \{X \subseteq V(G) \text{ odd}\}$ and add the blossom inequalities

$$\sum_{e \in \delta(X)} x_e \geq 1 \qquad \forall X \in \mathcal{A}$$

to the LP relaxation. The new dual problem is then:

$$\begin{aligned} & \max \sum_{A \in \mathcal{A}} z_A \\ \text{s.t.} \quad & \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A \leq c_e \\ & z_A \geq 0 \qquad (A \in \mathcal{A}, |A| \geq 3) \end{aligned}$$

Edmond's Algorithm starts with an empty matching $x = 0$ and dual feasible solution:

$$z_A := \begin{cases} \frac{1}{2} \min\{c(e) \mid e \in \delta(A)\} & |A| = 1 \\ 0 & \text{else} \end{cases}$$

We always ensure that z is dual feasible and that (x, z) satisfy complementary slackness:

$$\begin{aligned} x_e > 0 \Rightarrow \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A = c_e \\ z_A > 0, |A| > 1 \Rightarrow \sum_{e \in \delta(A)} x_e = 1 \end{aligned}$$

Definition 1.57. $c_z(e) := c(e) - \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A$ is the *reduced cost* of e.

Theorem 1.58. *There are at most $\frac{7}{2} |V(G)|^2$ of the repeat-until loop in algorithm 4.*

Proof. $\mathcal{B}$ is laminar at any time, i.e. for $X, Y \in \mathcal{B}$ we have $(X \subseteq Y) \vee (Y \subseteq X) \vee (X \cap Y = \emptyset)$. Therefore $|\mathcal{B}| \leq 2 |V(G)|$.

Observation. *Any U added to $\mathcal{B}$ during* Shrink *will not be "unpacked" before the next* Augment*.*

Proof. After *Shrink*, there exists an even length M-augmenting R-U-path. It remains in G_z until the next *Augment* or until U is included in another blossom $U' \supseteq U$ which is not resolved before an *Augment* (inductively). ☐

Between 2 augments:

- # *Unpacks* $\leq |\mathcal{B}|$ at beginning of the sequence

- # *Shrinks* $\leq |\mathcal{B}|$ at the end of the sequence

Therefore, there are at most $4\,|V(G)|$ *Unpack* and *Shrink* operations between 2 augments. For each dual change without *Unpack*, we have: $z_B > 0 \quad \forall B \in \mathcal{B}$, so ϵ is not determined by z_B. Therefore $\exists e = \{X, Y\}$ with $X \notin \mathcal{X}, Y \in \mathcal{Y}$ where $c_z(e)$ becomes 0.

Case 1: $X \notin \mathcal{Y}$. Then $|V(G_z) \setminus (\mathcal{X} \cup \mathcal{Y})|$ decreases.

Case 2: $X \in \mathcal{Y}$. Then $\exists$ X-Y M-alternating walk in the next iteration.

In particular, such a dual change can occur at most $|V(G)|$ times between 2 augmentations.

In total, there are at most $\frac{1}{2}\,|V(G)|$ *Augment* steps. Therefore, there are $\frac{1}{2}\,|V(G)|^2\,(4 + |V(G)| + 2\,|V(G)|)$ □

Algorithm 4: Minimum-Weight Perfect Matching

Input: Graph G with edge weights $c : E(G) \to \mathbb{R}$
Output: A minimum-weight perfect matching M in (G, c)

Corollary 1.59. *A minimum-weight perfect matching can be computed in $O(n^2 m)$ time where $n := |V(G)|$ and $m = |E(G)|$.*

Proof. Theorem 1.58 times $O(m)$. □

Remark 1.60. To achieve $O(n^3)$ running time, one can modify the algorithm:

1. Use a General Blossom Forest to avoid recomputing the R-R-walks from scratch. We then have mappings $\mu_v, \varphi_v^i, \rho_v^i$ for $1 \leq i \leq k_v$ where k_v is the number of blossoms that contain v.
2. Store all vertices in a heap (ordered by their criticality for dual-feasibility) to speed up the computation of ϵ.

Gabow (1990) showed a running time of $O(n(m+n\log n))$. Gabow & Tarjan (1991) showed a running time of $O(m \log(nW)\sqrt{n\alpha(m,n)\log n})$ where $W := \max_{e\in E(G)} |c(e)|$.

1.8.1 The Matching Polytope

Theorem 1.61. *Let G be a graph. The set of vectors $x \in \mathbb{R}^{E(G)}$ satisfying*

$$\begin{aligned} x_e &\geq 0 & & e \in E(G) \\ x(\delta(v)) &= 1 & & v \in V(G) \\ x(\delta(A)) &\geq 1 & & A \subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

is the convex hull of all perfect matchings in G. It is called the perfect matching polytope.

Proof. For any objective function $c : E(G) \to \mathbb{R}$, the minimum-weight perfect matching algorithm produces an integral primal and a dual solution that satisfy complementary slackness. In particular, all vertices of the polytope are integral. $\square$

Theorem 1.62. *Let G be a graph. The set of vectors $x \in \mathbb{R}^{E(G)}$ satisfying*

$$\begin{aligned} x_e &\geq 0 && e \in E(G) \\ x(\delta(v)) &\leq 1 && v \in V(G) \\ x(E(G[A])) &\leq \frac{|A|-1}{2} && A \subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

is the convex hull of all matchings in G. It is called the matching polytope.

Proof. Any matching solution x satisfies these conditions. Let x be any solution that satisfies the conditions. We have to show that x is a convex combination of matching solutions. Define H by:

$$\begin{aligned} V(H) :=& \{(v,i) \mid v \in V(G), i \in \{1,2\}\} \\ E(H) :=& \{\{(v,i),(w,i)\} \mid \{v,w\} \in E(G), i \in \{1,2\}\} \\ & \cup \{\{(v,1),(v,2)\} \mid v \in V(G)\} \end{aligned}$$

We set $y_{\{(v,i),(w,i)\}} := x_{\{v,w\}}$ for all $\{v,w\} \in E(G), i \in \{1,2\}$ and $y_{\{(v,1),(v,2)\}} := 1 - x(\delta(v))$ for all $v \in V(G)$. Then $y \geq 0$ and $y(\delta_H(x)) = 1$ for all $x \in V(H)$.

Claim. *y satisfies the inequalities of the perfect matching polytope (in particular the blossom inequalities).*

If this is true, by 1.62 y is a convex combination of perfect matchings. $H[\{(v,1) \mid v \in V(G)\}]$ is isomorphic to G, so x is a convex combination of matchings in G.

We now prove the claim: Let $X \subseteq V(H)$ with $|X|$ odd. We have to show that $y(\delta_H(X)) \geq 1$. Define:

$$\begin{aligned} A :=& \{v \in V(G) \mid (v,1) \in X, (v,2) \notin X\} \\ B :=& \{v \in V(G) \mid (v,1) \in X, (v,2) \in X\} \\ C :=& \{v \in V(G) \mid (v,1) \notin X, (v,2) \in X\} \end{aligned}$$

Define $A_i := A \cap (V(G) \times \{i\})$ and $B_i := B \cap (V(G) \times \{i\})$. $|B_1 \cup B_2|$ is even, so (since $|X|$ is odd) $|A|$ or $|C|$ is odd. Without loss of generality, let

$|A|$ be odd.

$$\begin{aligned}\sum_{e\in\delta_H(X)} y_e &\geq \sum_{v\in A_1} \underbrace{\sum_{e\in\delta_H(v)} y_e}_{=1} - 2\cdot \sum_{e\in E(H[A_1])} y_e - \sum_{e\in\delta(A_1)\cap\delta(B_1)} y_e \\ &+ \sum_{e\in\delta(A_2)\cap\delta(B_2)} y_e \\ &= |A_1| - 2\cdot \sum_{e\in E(G[A])} x_e \\ &\geq |A_1| - (|A|-1) \\ &= 1\end{aligned}$$

$\square$

Theorem 1.63. *The matching polyhedron is TDI (Totally Dual Integral), i.e. for all $c \in \mathbb{Z}^{E(G)}$ for which the dual program of ($\max c^t x s.t. \dots$) has a finite optimum solution, it has an integral optimum solution.*

Proof. The dual is

$$\begin{aligned}\min \sum_{v\in V(G)} y_v &+ \sum_{A\in\mathcal{A},\ |A|>1} \frac{|A|-1}{2} z_A \\ s.t. \sum_{v\in e} y_v &+ \sum_{A\in\mathcal{A},\ |A|>1, e\in E(G[A])} z_A \geq c(e) \qquad e\in E(G) \\ & \qquad y,z\geq 0\end{aligned}$$

Let (G,c) be a counterexample such that $|V(G)| + |E(G)| + \sum_{e\in E(G)} |c(e)|$ is minimum. Then:

- $c(e) \geq 1$ for all $e \in E(G)$, since otherwise e could be deleted.
- G has no isolated vertices.

Claim. *In an optimum solution (y,z), $y = 0$.*

Proof. If $y_v > 0$, then $x(\delta(v)) = 1$ for all optimum solutions x. Decreasing $c(e)$ by 1 for all $e \in \delta(v)$ yields a smaller feasible instance (G,c') where the weight of x is decreased by 1 and x remains optimum. By assumption, (G,c') is not a counterexample, so there exists an integral optimum solution (y',z'). Increasing y'_v by one yields some optimum in (G,c) which has optimum integral solution $(y' + \mathbb{1}_v, z')$. $\square$

Let $(y = 0, z)$ be a dual optimum solution such that

$$\sum_{A \in \mathcal{A}, |A|>1} |A|^2 z_A$$

is maximum.

Claim. *$\mathcal{F} := \{A : z_A > 0\}$ is laminar.*

If not, there exist $X, Y \in \mathcal{F}: \ X \cap Y, X \setminus Y, Y \setminus X \neq \emptyset$. We proceed by "uncrossing". Let $\epsilon := \min\{z_X, z_Y\} > 0$.

Case 1: $|X \cap Y|$ is odd. Then $|X \cup Y|$ is odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_y &:= z_y - \epsilon \\ z'_{X \cap Y} &:= z_{X \cap Y} + \epsilon && \text{(unless } |X \cap Y| = 1) \\ z'_{X \cup Y} &:= z_{X \cup Y} + \epsilon \\ z'_A &:= z_A && \text{elsewhere} \end{aligned}$$

Then (y, z') is a dual optimum solution.

Case 2: $|X \cap Y|$ is even. Then $|X \setminus Y|$ and $|Y \setminus X|$ are odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_Y &:= z_Y - \epsilon \\ z'_{X \setminus Y} &:= z_{X \setminus Y} + \epsilon && \text{unless } |X \setminus Y| = 1 \\ z'_{Y \setminus X} &:= z_{Y \setminus X} + \epsilon && \text{unless } |Y \setminus X| = 1 \\ z'_A &:= z_A && \text{elsewhere} \\ y'_v &:= \epsilon && \forall v \in X \cap Y \\ y'_v &:= 0 && \forall v \notin X \cap Y \end{aligned}$$

Then (y', z') is feasible. The objective value is:

$$\begin{aligned} &\sum_{v \in V(G)} y'_v + \sum_{A \in \mathcal{A},\ |A|>1} z'_A \frac{|A|-1}{2} \\ =&\epsilon \cdot |X \cap Y| + \sum_{A \in \mathcal{A},\ |A|>1} \frac{|A|-1}{2} \\ +&\epsilon \left(\frac{|X \setminus Y| - 1}{2} + \frac{|Y \setminus X| - 1}{2} - \frac{|X| - 1}{2} - \frac{|Y| - 1}{2} \right) \\ =&\text{objective}(y, z) \end{aligned}$$

Therefore (y', z') is an optimum solution with $y' \neq 0$, which is a contradiction to the previous claim.

We can conclude that $\mathcal{F}$ is laminar.

Let $A \in \mathcal{F}$ with $z_A \notin \mathbb{Z}$ and $|A|$ is maximal. Define $\epsilon := z_A - \lfloor z_A \rfloor > 0$. Let $A_1, \dots, A_k$ be the inclusion-wise maximal proper subsets of A in $\mathcal{F}$. Since $\mathcal{F}$ is laminar, $A_i \cap A_j = \emptyset$ for $i \neq j$. Define:

$$\begin{aligned} z'_A &:= z_A - \epsilon \\ z'_{A_i} &:= z_A + \epsilon && 1 \leq i \leq k \\ z'_D &:= z_D && \text{elsewhere} \end{aligned}$$

Then (y, z') is dual feasible with objective value:

$$\sum_{B \in \mathcal{A},\ |B|>1} \frac{|B|-1}{2} z'_B < \sum_{B \in \mathcal{A},\ |B|>1} \frac{|B|-1}{2} z_B$$

This contradicts the optimality of (y, z), so there exists no counter example. □

Theorem 1.64. *Let G be a graph.*

$$\begin{aligned} P &:= \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) \leq 1 \quad \forall v \in V(G)\} \\ Q &:= \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) = 1 \quad \forall v \in V(G)\} \end{aligned}$$

are called the fractional matching polytope *and the* fractional perfect matching polytope. *If G is bipartite, then P and Q are integral.*

Proof. The adjacency matrices of bipartite graphs are totally unimodular. □

Theorem 1.65. *Let G be a graph. The vertices of the fractional perfect matching polytope satisfy*

$$x_e = \begin{cases} \frac{1}{2} & \text{if } e \in E(C_1) \cup \ldots \cup E(C_k) \\ 1 & \text{if } e \in M \\ 0 & \text{else} \end{cases}$$

where $C_1, \dots, C_k$ are vertex-disjoint odd circuits and M is a perfect matching in $G - (V(C_1) \cup \ldots \cup V(C_k))$.

Proof. Exercise 6.3 □

2 T-Joins and b-Matchings

Definition 2.1. Let G be a graph, $T \subseteq V(G)$. A subset $J \subseteq E(G)$ is called *T-join* if T is the set of odd-degree vertices in $(V(G), J)$.

Proposition 2.2. *Let G be a graph, $T, T' \subseteq V(G)$, J a T-join and J' a T'-join. Then $J \Delta J'$ is a $T \Delta T'$-join.*

Proof. For $v \in V(G)$:

$$\begin{aligned} |\delta_{J\Delta J'}(v)| &\equiv |\delta_J(v)| + |\delta_{J'}(v)| \\ &\equiv |\{v\} \cap T| + \left|\{v\} \cap T'\right| \\ &\equiv \left|\{v\} \cap (T \Delta T')\right| \mod 2 \end{aligned}$$

□

Proposition 2.3. *Let G be a graph, $T \subseteq V(G)$.*

$$\exists\ T\text{-join in } G \Leftrightarrow |V(C) \cap T| \text{ even for each connected component } C$$

Proof.

"$\Rightarrow$": Let J be a T-join. For each connected component C:

$$\sum_{v \in V(C)} |J \cap \delta(v)| = 2\,|J \cap E(C)|$$

Therefore $|J \cap \delta(v)|$ is odd for an even number of vertices and $|V(C) \cap T|$ is even.

"$\Leftarrow$": Partition T into pairs $\{v_1, w_1\}, \ldots, \{v_k, w_k\}$ such that v_i and w_i are in the same component for all i. Let P_i be a v_i-w_i-path in G. Define $J := E(P_1)\Delta E(P_2)\Delta \ldots \Delta E(P_k)$. By proposition 2.2, this is a T-join.

□

Theorem 2.4. *Let G be a graph, $c : E(G) \to \mathbb{R}$ and $T \subseteq V(G)$. In strongly polynomial time (e.g. $O(n^2 m)$) we can determine if a T-join exists and if so, compute a minimum-weight T-join.*

Proof. In $O(m)$ ($m := |E(G)|$), we can check if a T-join exists. If so:

1. Eliminate negative weights.

$$\begin{aligned} N &:= \{e \in E(G) \mid c(e) < 0\} \\ U &:= \{v \in V(G) \mid |\delta_N(v)| \text{ odd}\} \\ T' &:= T \Delta U \\ c'(e) &:= |c(e)| \qquad e \in E(G) \end{aligned}$$

Claim. *If J' is a minimum T'-join with respect to c', then $J' \Delta N$ is a minimum T-join with respect to c.*

Let $\tilde{J}$ be a T-join. Then $\tilde{J} \Delta N$ is a T'-join, so $c'(\tilde{J}) \leq c'(\tilde{J} \Delta N)$ and

$$c(J) = c'(J') + c(N) \leq c'(\tilde{J} \Delta N) + c(N) = c(\tilde{J})$$

which proves the claim.

2. We can now assume that $c \geq 0$. A minimum-weight T-join does not have cycles of positive weight. We can eliminate cycles of weight 0 without changing the cost. We can then restrict ourselves to searching for collections of T-T-paths.

   Let K_T be the metric closure of T with respect to G. It can be computed in $O(n \cdot (m + n \log n))$ by using Dijkstra for all vertices. Find a minimum-weight perfect matching M in K_T. Each $e = \{s, t\} \in M$ induces a path $P_{s,t}$. Then the symmetric difference $\Delta_{\{s,t\} \in M} E(P_{s,t})$ is a minimum-weight T-join in G.

□

Corollary 2.6. *A maximum-weight T-join can be computed as fast as a minimum-weight T-join.*

Proof. Set $c' := -c$. □

Corollary 2.7. *Let G be a graph, $c : E(G) \to \mathbb{R}$. We can find a cycle of negative length in G in $O(n^2 m)$ time.*

Proof. Apply theorem 2.4 to $T = \emptyset$. If $c(J) < 0$, $(V(G), J)$ contains a cycle C. If $c(C) = 0$, we can eliminate it and recurse, otherwise return C. □

2.2 T-Join Applications

2.2.1 TSP Approximation

Let (K_n, c) with c metric be an instance of the TSP. Consider the *Double tree algorithm*:

1. Compute a minimum spanning tree T.
2. $T' := T + T$ (doubling all edges). Then T' is Eulerian.
3. Walk along T' and add vertices to the TSP tour in the order of their first appearance, yielding a tour T^*. Since c is metric, we have $c(T^*) \leq c(T') \leq 2c(T)$. Since the cost of T is a lower bound for the cost of a tour, we have $c(T^*) \leq 2\text{OPT}$ (where OPT is the cost of a shortest TSP tour).

Algorithm 5: Christofides Algorithm (1976)

Input: Complete metric graph (K_n, c)

Output: A TSP-tour T

Find MST T_{MST} in (K_n, c)

$W := \{v \in V(K_n) \mid |\delta_{T_{\text{MST}}}(v)| \text{ odd}\}$

$J :=$ minimum-weight W-Join in (K_n, c)

Add cities to T in the order of first appearance in a Eulerian walk of $T_{\text{MST}} + J$.

return T

Theorem 2.8. *Algorithm 5 is a $\frac{3}{2}$-approximation algorithm for the metric TSP, i.e. for the computed tour T we have:*

$$c(T) \leq \frac{3}{2}\text{OPT}$$

Proof. We have $c(T_{\text{MST}}) \leq \text{OPT}$ and $\text{OPT}(W) \leq \text{OPT}(V(K_n))$ (since c is metric). Any tour through the vertices in W can be decomposed into 2 matchings. Therefore, $c(J) \leq \frac{1}{2}\text{OPT}(W) \leq \frac{1}{2}\text{OPT}$. It follows that $c(T) \leq (1 + \frac{1}{2})\text{OPT}$. □

2.2.2 Shortest Paths in Undirected Graphs

The naive reduction to digraphs requires non-negative weights.

Corollary 2.9. *Given an undirected graph G, $c : E(G) \to \mathbb{R}$ such that each ciruit has length at least 0. Then for $s, t \in V(G)$, a shortest s-t-path can be found in $O(n^2 m)$ time, where $n := |V(G)|$, $m := |E(G)|$.*

Proof. Choose $T := \{s, t\}$. Apply theorem 2.4 to get a minimum-weight T-join J. J can be partitioned into circuits of length 0 and an s-t-path of length $c(J)$. □

2.2.3 Chinese Postman Problem

Definition 2.10. A walk $C = \{v_0, e_1, v_1, \ldots, e_t, v_t\}$ is called a Chinese postman tour if $v_0 = v_t$ and each edge in $E(G)$ is visited at least once. The Chinese Postman Problem is the problem of finding a shortest Chinese postman tour in G with respect to $c : E(G) \to \mathbb{R}_{\geq 0}$.

Corollary 2.11. *The Chinese postman problem can be solved in $O(n^2 m)$ time, where $n := |V(G)|$, $m := |E(G)|$.*

Proof. Set $T := \{v \in V(G) \; |\delta(v)| \text{ odd}\}$ and let J be a minimum-weight T-join. Compute a Eulerian tour C in $G + J$. Let C' be a shortest Chinese postman tour. Let $J' :=$ set of edges occuring in C' an even number of times (at least twice). Then J' is a T-join, so $c(J') \geq c(J)$ and:

$$c(C') \geq c(E(G)) + c(J') \geq c(E(G)) + c(J) = c(C)$$

□

2.3 T-Joins and T-Cuts

Definition 2.12. Let G be a graph and $T \subseteq V(G)$. A T-cut is a cut $C = \delta(X)$ with $X \subseteq V(G)$ and $|X \cap T|$ odd.

Proposition 2.13. *Let G be a graph, $T \subseteq V(G)$, $|T|$ even. Then:*

1. *For any T-join J and any T-cut C: $J \cap C \neq \emptyset$.*
2. *The inclusion-wise minimal T-cuts (T-joins) are exactly the inclusion-wise minimal edge sets intersecting all T-joins (all T-cuts).*

Proof. For 1., let $C = \delta(X)$ with $|X \cap T|$ odd be a T-cut. Then the edges in $J \cap C$ either belong to a path passing through X or have an endpoint in T. Therefore $|J \cap C|$ is odd, in particular the set is non-empty.

For 2., we prove in an exercise that each edge set intersecting all T-joins (T-cuts) contains a T-cut (T-join). Therefore minimal such sets are T-cuts (T-joins). Remark: The minimum cardinality of a T-join is at least as large as the maximum number of edge-disjoint T-cuts[4]. □

Theorem 2.14 (Seymour (1981))**.** *Let G be bipartite, $T \subseteq V(G)$ such that there exists a T-join. Then:*

$$\textit{min. cardinality of a } T\textit{-join} = \textit{max. number of edge-disjoint } T\textit{-cuts}$$

The maximum is attained by a crossfree family $\mathcal{C}$ of cuts, i.e.

$$\forall X, Y \in \mathcal{C} : X \subseteq Y \vee Y \subseteq X \vee X \cap Y = \emptyset \vee X \cup Y = V(G)$$

Proof. If $T = \emptyset$, the statement is clear. Let $T \neq \emptyset$. We proceed by induction on $|V(G)| + |T|$. Let J be a minimum-cardinality T-join. Set:

$$c(e) := \begin{cases} -1 & e \in J \\ 1 & e \in E(G) \setminus J \end{cases}$$

Claim. *Every circuit C has $c(C) \geq 0$.*

[4] In general, the two numbers are not equal: Consider K_4 and $T = V(K_4)$. A minimum T-join consists of 2 edges but there are no 2 edge-disjoint T-cuts.

$$\begin{aligned} c(C) &= \underbrace{c(C \setminus J)}_{=|C\setminus J|} + \underbrace{c(C \cap J)}_{=-|C\cap J|} + |J \setminus C| - |J \setminus C| \\ &= \left|\underbrace{C \Delta J}_{T\text{-join}}\right| - |J| \geq 0 \end{aligned}$$

Let P be a minimum length walk in (G, c) traversing no edge more than once such that $|E(P)|$ is minimum. Then P is a path. Let t be the last vertex in P and f the edge entering t. Then $f \in J$, otherwise $c(f) = 1$ and deleting f would yield a shorter path. Furthermore, $|\delta_J(t)| = 1$, otherwise we could add the other edge from $J \cap \delta(t)$ to shorten $c(P)$.

Claim. *Each circuit C that contains t but not f has $c(C) > 0$.*

Case 1: t is the only vertex in $V(C) \cap V(P)$. Let $e \ni t$ be an edge on C incident to t. Then $c(e) = 1$ (since $\delta_J(t) = \{f\}$) and $P' := P + C - e$ yields a shorter walk if $c(C) \leq 0$.

Case 2: $V(C) \cap V(P)$ contains another vertex x. Let u be the last vertex on P before t that is also on C. Define $P' := P_{[u,t]}$. C can be split into 2 u-t-paths C', C''. By minimality of P, $c(P') < 0$. $P' + C', P' + C''$ are circuits (by choice of u). By the first claim, $c(C'), c(C'') > 0$, so also $c(C) > 0$.

Shrink: $\{t\} \cup \Gamma(t)$ to a new vertex v_0. This yields a bipartite graph G'. If $|T \cap (\{t\} \cup \Gamma(t))|$ is odd, set $T' := T \setminus (\{t\} \cup \Gamma(t)) \cup \{v_0\}$. Otherwise, $T' := T \setminus (\{t\} \cup \Gamma(t))$. Define $J' := J \setminus \{f\}$.

Claim. *J' is a minimum cardinality T'-join in G'.*

If not, there exists a T'-join J'' with $|J''| < |J'|$. $J'' \Delta J'$ is an $\emptyset$-Join. Therefore, there exists a circuit C' where $|C' \setminus J'| < |C' \setminus J''| = |C' \cap J'|$ (since G is bipartite). If C' results from a circuit C in G not containing t, then $|C \setminus J| < |C \cap J|$. This is a contradiction to the minimality of J.

Therefore C' results from a circuit containing t.

Case 1: C traverses f. Then

$$\begin{aligned} \left|C' \setminus J'\right| - \left|C' \cap J'\right| &= |C \setminus J| - |C \cap J| \\ &> 0 \end{aligned}$$

which is a contradiction.

Case 2: By the second claim, $c(C) > 0$, so since G is bipartite $c(C) \geq 2$ and $|C \setminus J| \geq |C \cap J| + 2$. Therefore

$$\begin{aligned} |C' \setminus J'| &= |C \setminus J| - 2 \\ &\geq |C \cap J| \\ &= |C' \cap J'| \end{aligned}$$

which is a contradiction to the assumption.

By the induction hypothesis on G', G' has cross-free T'-cuts $D_1, \dots, D_{|J'|}$. Together with $\delta(t)$, we get $|J'| + 1 = |J|$ T-cuts. Since $\Gamma(t)$ was contracted in G', they are cross-free. □

Corollary 2.15. *Let G be a graph, $c : E(G) \to \mathbb{Z}_{\geq 0}$, $T \subseteq V(G)$ such that a T-join exists. The minimum cost of a T-join equals half the maximum number of T-cuts covering each edge e at most $2 \cdot c(e)$ times. This maximum is attained by a cross-free family of T-cuts.*

Proof. Let $E_0 := \{e \in E(G) \mid c(e) = 0\}$. Contract the connected components in $(V(G), E_0)$ and replace each $e \in E(G)$ by a path of length $2 \cdot c(e) > 0$. The resulting graph G' is bipartite. Let

$T' := \{v \in V(G') \mid v$ corresponds to a connected component X in G with $|X \cap T|$ odd$\}$

Let k be the minimum cost of a T-join in G.

Claim. *The minimum cardinality of a T'-join in G' is $2k$.*

"$\leq$": Every T-join J in J corresponds to a T'-join J' in G' with $|J'| \leq 2c(J)$.

"$\geq$": Let J' be a T'-join in G'. J' corresponds to an edge set $J \subseteq E(G)$. Let $\overline{T} := T \Delta \{v \in V(G) \mid |\delta(v) \cap J| \text{ odd}\}$. For each connected component X in $(V(G), E_0)$:

$$|\delta(X) \cap J| \equiv |X \cap T| \mod 2$$

Therefore $|X \cap \overline{T}|$ is even, so by proposition 2.3, there exists a $\overline{T}$-join $\overline{J}$ in $(V(G), E_0)$. Then $J \cup \overline{J}$ is a T-join of weight $c(J) = \frac{|J'|}{2}$.

By theorem 2.14, there exist $2k$ pairwise disjoint T'-cuts in G'. In G this yields $2k$ T-cuts such that every edge e is covered by at most $2 \cdot c(e)$ cuts and they can be created cross-free. □

2.3.1 T-join Polytope

We define the T-join polytope:

$$P_{T\text{-join}} := \text{conv}\{x \in \mathbb{R}^{E(G)} \mid x \text{ incidence vector of a } T\text{-join}\}$$
$$P^{\uparrow}_{T\text{-join}} := P_{T\text{-join}} + \mathbb{R}^{E(G)}_{\geq 0}$$

Corollary 2.16. *$P^{\uparrow}_{T\text{-join}}$ is determined by*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(X)) &\geq 1 & \forall &T\text{-cuts } \delta(X) \end{aligned}$$

Proof. "$\subseteq$" is clear. Assume that the other inclusion does not hold. Then there exists $w : E(G) \to \mathbb{Q}$ such that the minimum weight of a T-join $\alpha > \min w^t x$ where x satisfies the stated inequalities. Without loss of generality, $w \in \mathbb{Z}^{E(G)}_{\geq 0}$, both cones are identical ($\mathbb{R}^{E(G)}_{\geq 0}$). By corollary 2.15, there exist T-cuts $C_1, \ldots, C_{2\alpha}$ such that each edge e is covered at most $2w(e)$ times.

$$y_C := \frac{1}{2}\text{number of times } C \text{ occurs in } C_1, \ldots, C_{2\alpha}$$

Then y is a feasible solution to the dual:

$$\begin{aligned} &\max_{C\ T\text{-cut}} y_C \\ \text{s.t.} &\sum_{C\ T\text{-cut},\ e \in C} y_e \leq w(e) \qquad e \in E(G) \\ &y \geq 0 \end{aligned}$$

$\sum_C y_C = \alpha$ is a lower bound for the minimization problem which is a contradiction to the assumed inequality. $\square$

2.4 Excursus: Gomory-Hu Trees

Let G be a graph, $u : E(G) \to \mathbb{R}_{\geq 0}$. Find $\emptyset \subsetneq X \subsetneq V(G)$ minimizing $u(\delta(X))$. One approach: $\binom{|V(G)|}{2}$ s-t-cut computations (this can clearly be reduced to $|V(G)| - 1$ by fixing s).

Definition 2.17. For $s, t \in V(G)$, denote by λ_{st} the minimum capacity of an s-t-cut (or *local edge connectivity* of s, t).

Lemma 2.18. *For all $u, v, w \in V(G)$:*

$$\lambda_{uw} \geq \min\{\lambda_{uv}, \lambda_{vw}\}$$

Proof. Let $\delta(A)$ be a u-w-cut. If $v \in A$, then $\delta(A)$ is a v-w-cut, so $u(\delta(A)) \geq \lambda_{vw}$. Otherwise, $\delta(A)$ is a u-v-cut, so $u(\delta(A)) \geq \lambda_{uv}$. $\square$

Definition 2.19. Let G be a graph, $u : E(G) \to \mathbb{R}_{\geq 0}$. A tree T is a Gomory-Hu tree for (G, u) if $V(T) = V(G)$ and

$$\lambda_{st} = \min_{e \in E(T_{[s,t]})} u(\delta_G(C_e)) \qquad \forall s, t \in V(G)$$

where C_e and $V(G) \setminus C_e$ are the connected components of $T - e$[5].

Lemma 2.20. *Given (G, u) and a tree T with $V(T) = V(G)$:*

$$T \text{ Gomory-Hu tree} \Leftrightarrow \forall e = \{s, t\} \in E(T) \text{ is a minimum capacity } s\text{-}t\text{-cut}$$

Proof. "$\Rightarrow$" follows directly from the definition. For the other direction, let $s, t \in V(G)$ and $e = \{u, v\} \in \arg\min_{e \in E(T_{s,t})} \lambda_{uv}$. Without loss of generality, $s \in C_e$, $t \in V(G) \setminus C_e$, so $\delta(C_e)$ is an s-t-cut. Therefore: $\lambda_{st} \leq u(\delta(C_e)) = \lambda_e$ (with $\lambda_e := \lambda_{uv}$). By lemma 2.20 and induction, $\lambda_{st} \geq \min\{\lambda_{v'w'} \mid \{v', w'\} \in E(T_{[s,t]})\} = \lambda_{uv}$. Therefore $\lambda_{st} = \lambda_{uv}$. $\square$

Idea: Choose $r, s \in V(G)$ and compute a minimum capacity r-s-cut $\delta(R)$. Without loss of generality $r \in R$. Construct a graph G_R by shrinking $S := V(G) \setminus R$ into a single vertex. Find a minimum capacity p-q-cut (where $p, q \in R$ are chosen arbitrarily) in G_R. This partitions R into 2 parts. Continue this process until $V(G)$ is partitioned into singletons.

Lemma 2.21. *Let (G, u) as above, $s, t \in V(G)$, $\delta(A)$ a minimum capacity s-t-cut in G and $s', t' \in V(G) \setminus A$. Let (G', u') arise from (G, u) by contracting A into a single vertex. Then for any minimum capacity s'-t'-cut $\delta_{G'}(K \cup \{A\})$ in (G', u'), $\delta_G(K \cup A)$ is a minimum capacity s'-t'-cut in (G, u).*

Proof. Without loss of generality, $s \in A$. We show: $\exists$ min. capacity s'-t'-cut $\delta(A')$ in (G, u) such that $A \subseteq A'$. Let $\delta(C)$ be any s'-t'-cut in (G, u). Without loss of generality, $s \in C$. $u(\delta(\cdot))$ is a submodular function, i.e. $u(\delta(A)) + u(\delta(C)) \geq u(\delta(A \cap C)) + u(\delta(A \cup C))$ [6].

$\delta(A \cap C)$ is an s-t-cut, so $u(\delta(A \cap C)) \geq \lambda_{st} = u(\delta(A))$. Therefore, $u(\delta(A \cup C)) \leq u(\delta(C)) = \lambda_{s't'}$. Since $s' \in A \cup C$, $A \cup C$ is a minimum capacity s'-t'-cut. $\square$

In general, we now choose a component X wih $|X| \geq 2$. Contract connected components in $T - \{X\}$, yielding a graph (G', u'). Choose $s, t \in X$, minimum s-t-cut $\delta(A')$ in (G', u'). $X = (X \cap A') \dot{\cup} (X \cap (V(G') \setminus A'))$.

Lemma 2.22. *At the end of MinCut:*

1. $A \dot{\cup} B = V(G)$

[5] $\delta(C_e)$ is called *fundamental cut* induced by e

[6] This holds with equality, if we add $2u(E(A, B))$ to the right side

2. $E(A, B)$ is a minimum s-t-cut in (G, u)

Proof. Elements of $V(T)$ are non-empty subsets of $V(G)$ and $V(T)$ form a partition of $V(G)$. Therefore $A \dot{\cup} B$ is a partition of $V(G)$. 2. follows from successive application of lemma 2.21 to each connected component of $T - X$. $\square$

Lemma 2.23. *At any time before FinishTree: $w(e) = u(\delta_G(\bigcup_{Z \in C_e} Z))$ for all $e \in E(T)$. Moreover, $\forall e = \{P, Q\} \in E(T)$ there exist $p \in P, q \in Q$: $w(e) = \lambda_{pq}$.*

Proof. At the start, $E(T) = \emptyset$. We show that both properties are always satisfied. Let X, s, t, A', B', A, B as determined by ChooseComponents, Contract and MinCut. Edges in $E(T) \setminus \delta(X)$ are not affected. For new edges both conditions are true after ModifyTree.

Consider an edge $e \in \{X, Y\}$ that is replaced by e' in ModifyTree. Without loss of generality $Y \subseteq A$, so $e' = \{X \cap A, Y\}$. We show that both statements hold for e'. $w(e) = w(e') = u(\delta_G(\bigcup_{Z \in C_e} Z)) = u(\delta_G(\bigcup_{Z \in C_{e'}} Z'))$ so 1. holds. Assume $p \in X, q \in Y$: $\lambda_{pq} = w(e)$. If $p \in X \cap A$, we are done.

If $p \in X \cap B$, we claim: $\lambda_{sq} = \lambda_{pq}$. This then implies $w(e') = w(e) = \lambda_{pq} = \lambda_{sq}$. By lemma 2.20, $\lambda_{sq} \geq \min\{\lambda_{st}, \lambda_{tp}, \lambda_{pq}\}$. By lemma 2.22, $E(A, B)$ is a minimum s-t-cut. By lemma 2.21 and since $s, q \in A$, λ_{sq} does not change when contracting B. Adding $\{t, p\}$ with sufficiently high capacity does not change λ_{sq}. Therefore $\lambda_{sq} \geq \min\{\lambda_{st}, \lambda_{pq}\} = \lambda_{pq}$ because $E(A, B)$ is also a p-q-cut. $w(e)$ is the capacity of a cut separating s, q, so $\lambda_{sq} \leq w(e) = \lambda_{pq}$. $\square$

Theorem 2.24 (Min Cut, Gomory & Hu (1961))**.** *Every undirected graph G with edge capacities $e : E(G) \to \mathbb{R}_{\geq 0}$ has a Gomory-Hu-tree. It can be computed using $n - 1$ Min-s-t-cut computations, e.g. in $O(n^3\sqrt{m})$ time (using the Push-Relabel algorithm for computing the minimum cuts) where $n := |V(G)|$ and $m := |E(G)|$.*

Proof. Gomory-Hu-Algorithm computes a Gomory-Hu-tree (lemma 2.23). It uses $n - 1$ iterations in each of which we need $O(n^2\sqrt{m})$ for Push-Relabel. Everything else can be handled in $O(\min\{n^3, n^2 m\})$ time. $\square$

2.5 Finding Minimum-Capacity T-Cuts

Theorem 2.25 (Padberg & Rao (1987))**.** *Given a graph $G, u : E(G) \to \mathbb{R}_{\geq 0}$, a Gomory-Hu-tree H for (G, u), $T \subseteq V(G)$ ($|T| \geq 2$ even), a minimum capacity T-cut can be found among the fundamental cuts of H. A minimum capacity T-cut can be computed in $O(n^3\sqrt{m})$ time.*

Proof. Let $\delta_G(X)$ be a minimum capacity T-cut in G. Let J be the set of edges in $E(H)$ for where $|C_e \cap T|$ is odd (where C_e is a connected component of $H - e$). For all $x \in V(G)$:

$$\begin{aligned} |\delta_J(x)| &\equiv \sum_{e \in \delta_H(x)} |C_e \cap T| \\ &\overset{T \text{ even}}{\equiv} |\{x\} \cap T| \qquad \mod 2 \end{aligned}$$

Therefore J is a T-join in H. Since T-cuts and T-joins intersect, there is $f \in J \cap \delta_H(X)$.

$$\begin{aligned} u(\delta_G(X)) &\geq \min\{u(\delta_G(Y)) \mid |Y \cap f| = 1\} \\ &= u(\delta_G(C_f)) \end{aligned}$$

We conclude that $\delta_G(C_f)$ is a minimum-capacity T-cut. □

2.6 b-Matchings

Definition 2.26. Let G be a graph, $u : E(G) \to \mathbb{N}_0 \cup \{\infty\}$ and $b : V(G) \to \mathbb{N}_0$. A *b-matching* is a function $f : E(G) \to \mathbb{N}_0$ such that $f(e) \leq u(e)$ and $f(\delta(v)) \leq b(v)$ for all $e \in E(G)$ and $v \in V(G)$.

- If $u \equiv 1$, the instance is called *simple*.
- If $b \equiv 1$, this is equivalent to a matching.
- If $f(\delta(v)) = b(v)$ for all $v \in V(G)$, it is called *perfect*.
- Simple perfect b-matchings are called *b-factors*.

Example. A TSP-tour is a 2-factor. Therefore valid inequalities for 2-factors are valid for TSP.

Theorem 2.27 (Edmonds (1965))**.** *Let G be a graph, $b : V(G) \to \mathbb{N}$. The b-matching polytope of (G, ∞) is the set of vectors $x \in \mathbb{R}_{\geq 0}^{E(G)}$ satisfying:*

$$\begin{aligned} x_e &\geq 0 && e \in E(G) \\ x(\delta(v)) &\leq b(v) && v \in V(G) \\ \sum_{e \in E(G[X])} x_e &\leq \lfloor \frac{1}{2} \sum_{v \in X} b(v) \rfloor && X \subseteq V(G) \end{aligned}$$

Proof. Clearly, any b-matching satisfies these inequalities. Let x satisfy the inequalities. Without loss of generality $b \geq 1$. Define H by splitting each

$v \in V(G)$ into $b(v)$ copies, i.e.:

$$\begin{aligned}
X_v &:= \{(v,i) \mid i \in [b(v)]\} \qquad v \in V(G) \\
V(H) &:= \bigcup_{v \in V(G)} X_v \\
E(H) &:= \{\{v', w'\} \mid \{v,w\} \in E(G), v' \in X_v, w' \in X_w\} \\
y_{e'} &:= \frac{1}{b(v) \cdot b(w)} x_{\{v,w\}} \qquad e' = \{v', w'\} \in E(H), v' \in X_v, w' \in X_w
\end{aligned}$$

Claim. *y is a convex combination of matchings in H. Contracting all X_v ($v \in V(G)$) yields a convex combination of b-matchings for x.*

We show that y is contained in the matching polytope, i.e.:

$$\begin{aligned}
y_e &\geq 0 && e \in E(H) \\
\sum_{e \in E(H[A])} y_e &\leq \frac{|A|-1}{2} && A \subseteq V(H),\ |A| \text{ odd}
\end{aligned}$$

If $\forall v \in V(H)$: $X_v \subseteq A$ or $X_v \cap A = \emptyset$, this follows directly from the given inequalities. Otherwise, let $a, b \in X_v$ such that $a \in A, b \notin A$.

$$\begin{aligned}
2 \sum_{e \in E(H[A])} y_e &= \sum_{c \in A \setminus \{a\}} \sum_{e \in E(\{c\}, A \setminus \{c\})} y_e + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e \\
&\leq \sum_{c \in A \setminus \{a\}} \sum_{e \in \delta_H(c) \setminus \{\{c,b\}\}} + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e \\
&= \sum_{c \in A \setminus \{a\}} \sum_{e \in \delta_H(c)} y_e \underbrace{- \sum_{e \in E(\{b\}, A \setminus \{a\})} y_e + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e}_{=0} \\
&\leq |A| - 1
\end{aligned}$$

□

Theorem 2.28 (Edmonds & Johnson (1970))**.** *Let G be a graph, $u : E(G) \to \mathbb{N} \cup \{\infty\}$, $b : V(G) \to \mathbb{N}$. The b-matching polytope is given by:*

$$\begin{aligned}
x &\geq 0 \\
x &\leq u \\
x(\delta(v)) &\leq b(v) && v \in V(G) \\
\sum_{e \in E(G[X])} x_e + \sum_{e \in F} x_e &\leq \underbrace{\left\lfloor \frac{1}{2} \left(\sum_{v \in X} b(v) + \sum_{e \in F} u(e) \right) \right\rfloor}_{\textit{Gomory-Chvátal-Cut}} && X \subseteq V(G), F \subseteq \delta(X)
\end{aligned}$$

Proof.

"$\subseteq$": Let x be an incidence vector of b-matchings. Then $x \leq u$ and $x(\delta(v)) \leq b(v)$ for all $v \in V(G)$.

$$\sum_{e \in E(G[X])} x_e + \sum_{e \in F} x_e = \frac{1}{2}\left(\sum_{v \in X} \sum_{e \in \delta(x)} x_e + \sum_{e \in F} x_e - \sum_{e \in \delta(X) \setminus F} x_e\right)$$
$$\leq \frac{1}{2}\left(\sum_{v \in X} b(v) + \sum_{e \in F} u(e)\right)$$

Since the left hand side is integral, the right hand side can be rounded down.

"$\supseteq$": Let x satisfy all the inequalities. We have to show that x is a convex combinations of b-matchings. Let H arise from G by subdividing each edge $e = \{v, w\}$ with $u(e) \neq \infty$ by 2 new vertices $(e, v), (e, w)$ and a path v-(e, v)-(e, w)-w, where $b((e, v)) = u(e) = b((e, w))$. Set $y_{\{v,(e,v)\}} \coloneqq x_e \eqqcolon y_{\{(e,w),w\}}$ and $y_{\{(e,v),(e,w)\}} \coloneqq u(e) - x_e$. If $u(e) = \infty$, $y_e \coloneqq x_e$.

Claim. *y is in the b-matching polytope of (H, ∞). This then implies that x is contained in the capacitated b-matching polytope of (G, u).*

$y(\delta_H(v)) \leq b(v)$ clearly holds for all $v \in V(H)$. Assume that there exists $A \subseteq V(H)$ with:

$$y(E(H[A])) > \lfloor \frac{1}{2} b(A) \rfloor$$

Let $B \coloneqq A \cap V(G)$. For $\{v, w\} \in E(G[B])$, we may assume that $(e, v), (e, w) \in A$. If $(e, v) \in A$, we may assume $v \in A$:

Case 1: If $(e, w) \in A$, we can remove (e, v) and (e, w).

Case 2: If $(e, w) \notin A$, we can remove (e, v).

There are 3 remaining cases. Define:

$$F \coloneqq \{e = \{v, w\} \in E(G) \mid |A \cap \{(e, v), (e, w)\}| = 1\}$$

Then

$$x(E(G[B])) + x(F) = y(E(H[A])) - \sum_{\substack{e \in E(G[B]) \\ u(e) < \infty}} u(e)$$
$$> \lfloor \frac{1}{2} b(A) \rfloor - \sum_{\substack{e \in E(G[B]) \\ u(e) < \infty}} u(e)$$
$$= \lfloor \frac{1}{2}(b(B) + \sum_{e \in F} u(e)) \rfloor$$

which is a contradiction to the feasibility of x. Therefore, y satisfies the inequalities w.r.t. (H, ∞). Let $e \in P :=$ b-matching polytope for (H, ∞), then $y \in \{z \in P \mid \sum_{e \in \delta(v)} z_e = b(v) \forall v \in V(H) \setminus V(G)\}$. Therefore, y is the convex combination of b-matchings $f_1, \ldots, f_m$ in (H, ∞) with $f_i(\delta(v)) = b(v)$ for all $v \in V(H) \setminus V(G)$. We get:

$$f_i(\{v, (e, v)\}) = f_i(\{w, (e, w)\}) \leq u(e) \qquad \forall e = \{v, w\} \in E(G)$$

Set:

$$f_i'(e) := \begin{cases} f_i(v, (e, v)) & e = \{v, w\} \in E(G),\ u(e) < \infty \\ f_i(e) & e = \{v, w\} \in E(G),\ u(e) = \infty \end{cases}$$

Then x is a convex combination of $f_1', \ldots, f_m'$ (of b-matchings).

□

2.7 Padberg-Rao Theorem

Lemma 2.30. *Let G be a graph, $|E(G)| \geq 1$, $T \subseteq V(G)$ with $|T|$ even, $c, c' : E(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. There exists a $O(n^2 m)$ time algorithm that finds a vertex set $X \subseteq V(G)$ and $F \subseteq \delta(X)$ such that $|X \cap T| + |F|$ is odd and*

$$c(\delta(X) \setminus F) + c'(F)$$

is minimum.

Proof. Without loss of generality, G is connected: Otherwise, add edges e with $c(e) = 0$ and $c'(e) = \infty$. Let

$$\begin{aligned} d(e) &:= \min\{c(e), c'(e)\} & e \in E(G) \\ E' &:= \{e \in E(G) \mid c'(e) < c(e)\} \\ V' &:= \{v \in V(G) \mid |\delta_{E'}(v)| \text{ odd}\} \\ T' &:= T \Delta V' \end{aligned}$$

Since E' is a V'-join, for $X \subseteq V(G)$:

$$|X \cap T| + \left|\delta(X) \cap E'\right| \equiv |X \cap T| + \left|X \cap T'\right| \equiv \left|X \cap T'\right| \mod 2$$

Compute a Gomory-Hu-Tree H for (G, d). For $f \in E(H)$, let $\delta(C_f)$ be the fundamental cut of f (i.e. C_f is a connected component in $H - f$). Let $g_f \in \arg\min_{e \in \delta_G(C_f)} |c(e) - c'(e)|$. Let:

$$F_f := \begin{cases} \delta_G(C_f) \cap E' & \text{if } |C_f \cap T'| \text{ is odd} \\ \delta_G(C_f) \cap E' \Delta \{g_f\} & \text{else} \end{cases}$$

Finally, choose $f \in E(H)$ minimizing $c(\delta(C_f) \setminus F_f) + c'(F_f)$ and output C_f, F_f. The running time is dominated by the computation of H.

It remains to show correctness: Let X^*, F^* be an optimum solution.

Case 1: $|X^* \cap T'|$ is odd. $J' := \{f \in E(H) \mid |C_f \cap T'| \text{ odd}\}$ is a T'-join in H. Therefore, J' intersects the T'-cut $\delta_H(X^*)$. Let $f \in \delta_H(X^*)$ with $|C_f \cap T'|$ odd. Then $d(\delta_G(C_f)) \leq d(\delta_G(X^*)) \leq \text{obj}(X^*)$, since H is a Gomory-Hu-tree. By construction, $F_f = \delta_G(C_f) \cap E'$ and:

$$c(\delta_G(C_f) \setminus F_f) + c'(F_f) \leq d(\delta_G(X^*)) \leq \text{obj}(X^*)$$

Case 2: $|X^* \cap T'|$ is even. Let $g^* \in \arg\min_{e \in \delta(X^*)} |c(e) - c'(e)|$. $H + g^*$ has a unique circuit that contains some $f \in \delta_H(X^*)$. Then

$$\begin{aligned} c(\delta_G(X^*) \setminus F^*) + c'(F^*) &= d(\delta(X^*)) + \left|c(g^*) - c'(g^*)\right| \\ &\geq d(\delta_G(C_f)) + \left|c(g^*) - c'(g^*)\right| \\ &\overset{g^* \in \delta_G(C_f)}{\geq} c(\delta(C_f) \setminus F_f) + c'(F_f) \end{aligned}$$

□

Theorem 2.31 (Padberg & Rao (1987)). *Let G be a graph, $u : E(G) \to \mathbb{N} \cup \{\infty\}$ and $b : V(G) \to \mathbb{N}$. Then the separation problem for the b-matching polytope can be solved in $O(n^2 m)$ time.*

Proof. $0 \leq X \leq u$ and $x(\delta(v)) \leq b(v)$ for all $v \in V(G)$ can be checked in linear time. It remains to check:

$$x(E(G[X])) + x(F) \leq \lfloor \frac{1}{2} (b(X) + u(F)) \rfloor \qquad X \subseteq V(G), F \subseteq \delta(X)$$

If $b(X) + u(F)$ is even (i.e. no rounding is done), this is implied by the other inequarities. Otherwise, the inequality is violated iff:

$$b(X) - 2x(E(G[X])) + u(F) - 2x(F) < 1$$

Extend G to H by adding a new vertex z and edges $\{z, v\}$ for every $v \in V(G)$. Set:

$$\begin{aligned} b(z) &:= b(V(G)) \\ T &:= \{v \in V(H) \mid b(v) \text{ odd}\} \\ E' &:= \{e \in E(G) \mid u(e) < \infty \text{ and odd}\} \\ c(e) &:= \begin{cases} x_e & e \in E' \\ \min\{x_e, u(e) - x_e\} & e \in E(G) \setminus E' \\ b(v) - x(\delta(v)) & e = \{z, v\} \in E(H) \end{cases} \\ c'(e) &:= \begin{cases} u(e) - x_e & e \in E' \\ \infty & e \in E(H) \setminus E' \end{cases} \end{aligned}$$

For $X \subseteq V(G)$, let $D_X := \{e \in \delta_G(X) \setminus E' \mid u(e) \leq 2x_e\}$. Then $\forall X \subseteq V(G)$, $F \subseteq \delta_G(X) \cap E'$,

$$|X \cap T| + |F| \equiv b(X) + u(F \cup D_X) \mod 2$$

and:

$$\begin{aligned} c(\delta_H(X) \setminus F) + c'(F) &= b(X) - \sum_{v \in X} x(\delta_G(v)) + \sum_{e \in (\delta_G(X) \cap E') \setminus F} x_e \\ &\quad + \sum_{e \in \delta_G(X) \setminus E'} \min\{x_e, u(e) - x_e\} + \sum_{e \in F} u(e) - x_e \\ &= b(X) - 2x(E(G[X])) + \sum_{e \in F \cup D_X} u(e) - 2x_e \end{aligned}$$

Apply lemma 2.30 to H, T, c, c': If there exists $X \subseteq V(H)$, $F \subseteq \delta_H(X)$ with $c(\delta(X) \setminus F) + c'(F) < 1$, then $F \subseteq E'$ and without loss of generality $z \notin X$ (otherwise use the complement). We get

$$b(X) - 2x(E(G[X])) + \sum_{e \in F \cup D_X} u(e) - 2x_e < 1$$

Setting $F' := F \cup D_X$ yields a violating of the corresponding inequality.

For the other direction, note that if the inequality holds for $X \subseteq V(G)$ and $F \subseteq \delta(X)$, then without loss of generality, $D_X \subseteq F \subseteq E' \cup D_X$ (since adding edges in $D_X \setminus F$ increases the violation). Then:

$$c(\delta_H(X) \setminus (F \setminus D_X)) + c'(F \setminus D_X) < 1$$

Therefore, the b-matching polytope can be separated in polynomial time. $\square$

Corollary 2.32. *The Maximum-Weight b-Matching Problem can be solved in polynomial time.*

Proof. Use the Ellipsoid method together with theorem 2.31. $\square$

3 The TSP Polytope

3.1 The Spanning Tree Polytope

Theorem 3.1 (Edmonds (1967))**.** *Let G be a connected graph, $n := |V(G)|$. Then*

$$P_{ST} := \{x \in [0,1]^{E(G)} \mid x(E(G)) = n-1, \forall \emptyset \neq X \subsetneq V(G) : \sum_{e \in E(G[X])} x_e \leq |X|-1\}$$

is the convex hull of incidence vectors of spanning trees. It is called the spanning tree polytope.

Proof. Let T be a spanning tree with incidence vector x. Then $x \in P_{ST}$ and as $x \in \{0,1\}^{E(G)}$, x is a vertex.

For the other direction, let $x \in P_{ST} \cap \mathbb{Z}^{E(G)}$. Then x cannot contain cycles, so it is a forest. Since $x(E(G)) = n-1$, it is a spanning tree.

Claim. *P_{ST} is integral.*

Let $c : E(G) \to \mathbb{R}$ and T be a minimum spanning tree produced by Kruskals algorithm. Let $E(T) \coloneqq \{f_1, \dots, f_{n-1}\}$ in order of addition, i.e. $c(f_1) \le c(f_2) \le \dots \le c(f_{n-1})$. Let $X_k \subseteq V(G)$ be the connected component in $(V(G), \{f_1, \dots, f_k\})$ containing f_k. Let x^* be the incidence vector of T.

Claim. *x^* is an optimum solution to*

$$\begin{aligned} &\min c^t x \\ s.t. \quad & 1^t x = n-1 \\ & x(E(G[X])) \le |X| - 1 && \forall \emptyset \subsetneq X \subseteq V(G) \\ & x \ge 0 \end{aligned}$$

The dual problem is:

$$\begin{aligned} \max \quad & -\sum_{\emptyset \subsetneq X \subseteq V(G)} (|X|-1) z_X \\ s.t. \quad & -\sum_{e \subseteq X \subseteq V(G)} z_X \le c(e) && e \in E(G) \\ & z_X \ge 0 && \emptyset \subsetneq X \subsetneq V(G) \end{aligned}$$

Construct a dual solution z^*: For $k \in \{1, \dots, n-2\}$, set $z^*_{X_k} \coloneqq c(f_l) - c(f_k) \ge 0$ where l is the minimum index larger than k with $X_k \cap f_l \ne \emptyset$. Define $z^*_{V(G)} = -c(f_{n-1})$ and $z^*_A \coloneqq 0$ for all other $A \subseteq V(G)$.

For $e = \{v, w\} \in E(G)$:

$$-\sum_{e \subseteq X \subseteq V(G)} z_X = c(f_i) \le c(e)$$

where i is the smallest index such that $e \subseteq X_i$. Therefore, z^* is dual feasible. For tree edges, we have equality, so for $x_e > 0$ the dual constraint is tight. Let $\emptyset \subsetneq X \subseteq V(G)$ with $z^*_X > 0$. Then $T[X]$ is connected, so the primal constraint is tight. Complementary slackness implies that x^*, z^* are optimum primal/dual solutions.

Remark. If $c \in \mathbb{Z}^{E(G)}$, then z^* is an integral optimum dual solution, so the system is TDI.

$\square$

Theorem 3.2 (Fulkerson (1974)). *Let G be a digraph, $c : E(G) \to \mathbb{Z}_{\geq 0}$, $r \in V(G)$ such that G contains an r-arborescence. Then the minimum weight of an r-arborescence spanning $V(G)$ equals the maximum number of r-cuts $C_1, \ldots, C_t$ (where repetitions are allowed) such that no edge e is contained in more than $c(e)$ of the cuts.*

Proof. Consider the (r-cuts)$\times$(edges) matrix A, where

$$A_{Ce} = \begin{cases} 1 & e \in C \\ 0 & \text{otherwise} \end{cases}$$

Consider the LP and its dual:

$$\min\{c^t x \mid x \in \mathbb{R}^{E(G)},\ Ax \geq 1, x \geq 0\}$$
$$\max\{1^t y \mid y \in \mathbb{R}^{r\text{-cuts}},\ A^t y \leq c,\ y \geq 0\}$$

Claim. *The system is TDI.*

Proof. Let y be an optimum dual solution maximizing

$$\sum_{\emptyset \subsetneq X \subseteq V(G) \setminus \{r\}} y_{\delta^-(X)} \, |X|^2$$

Claim. $\mathcal{F} := \{X \subseteq V(G) \mid y_{\delta^-(X)} > 0\}$ *is laminar.*

Suppose that there are $X, Y \in \mathcal{F}$ with $X \cap Y,\ X \setminus Y,\ Y \setminus X \neq \emptyset$. Let:

$$\begin{aligned} \epsilon &:= \min\{y_{\delta^-(X)}, y_{\delta^-(Y)}\} \\ y'_{\delta^-(X)} &:= y_{\delta^-(X)} - \epsilon \\ y'_{\delta^-(Y)} &:= y_{\delta^-(Y)} - \epsilon \\ y'_{\delta^-(X \cap Y)} &:= y_{\delta^-(X \cap Y)} + \epsilon \\ y'_{\delta^-(X \cup Y)} &:= y_{\delta^-(X \cup Y)} + \epsilon \\ y' &:= y \qquad\qquad\qquad\qquad \text{everywhere else} \end{aligned}$$

Then y' is a dual optimum solution which contradicts the maximality of y.

By Ghoulia-Houri, if the set of rows can be partitioned $\mathcal{R} = \mathcal{R}_1 \dot\cup \mathcal{R}_2$ such that for all columns j:

$$\sum_{r \in \mathcal{R}_1} a_{rj} - \sum_{r \in \mathcal{R}_r} a_{rj} \in \{-1, 0, 1\}$$

then A is totally unimodular. Let $\mathcal{R}_1, \mathcal{R}_2$ be a partition of the laminar family $\mathcal{F}$ alternating between each level. Let $A' \subseteq A$ consist of rows with positive support (i.e. rows in $\mathcal{F}$). Then by this argument, A' is totally unimodular. In particular, for $c \in \mathbb{Z}_{\geq 0}$, we find an integral optimum dual solution. □

Since the system is TDI, there exists an integral optimum primal solution x. □

Corollary 3.3. *Let G be a digraph, $c : E(G) \to \mathbb{R}_{\geq 0}$ and $r \in V(G)$ such that a spanning r-arborescence exists. Then*

$$\min\{c^t x \mid x \geq 0,\ x(\delta^+(X)) \geq 1\ \forall r \in X \subsetneq V(G)\}$$

has an integral solution which is the incidence vector of a minimum-weight spanning r-arborescence plus (possibly) edges of weight 0.

3.2 The Held-Karp Polytope

Proposition 3.4. *Let $n \in \mathbb{Z}_{\geq 3}$. The incidence vectors x of TSP tours in K_n are described by:*

$$\begin{aligned} x(\delta(v)) &= 2 & v &\in V(G) \\ x(\delta(X)) &\geq 2 & \emptyset \neq X &\subsetneq V(G) \\ x &\in \{0,1\}^{E(K_n)} \end{aligned}$$

Proof. Integrality and the first inequality imply that x is the incidence vector of a collection of cycles. By the second inequality (which is called the *subtour elimination constraint*), there is exactly one cycle. □

Relaxing the integrality (i.e. only requiring $x \in [0,1]$) yields the *subtour polytope* (or Held-Karp-polytope).

Proposition 3.5. *Let $n \in \mathbb{Z}_{\geq 2}$, $x \in [0,1]^{E(G)}$ with $x(\delta(v)) = 2$ for all $v \in V(K_n)$. Then the following are equivalent:*

1. *$x(\delta(X)) \geq 2$ for all $\emptyset \neq X \subsetneq V(G)$ (i.e. 3.4).*
2. *$x(E(K_n[X])) \leq |X| - 1$ for all $\emptyset \neq X \subsetneq V(G)$.*
3. *$x(E(K_n[X])) \leq |X| - 1$ for all $\emptyset \neq X \subseteq V(K_n) \setminus \{r\}$.*

Proof.

$$\begin{aligned}2 &\leq x(\delta(V(G) \setminus X)) \\ &= x(\delta(X)) \\ &= \sum_{v \in X} x(\delta(v)) - 2x(E(K_n[X])) \\ &= 2\,|X| - 2x(E(K_n[X]))\end{aligned}$$

□

Theorem 3.6 (Wolsey (1980)). *Let (K_n, c) with c metric and*

$$P_{HK} = \{x \in \mathbb{R}_{\geq 0}^{E(K_n)} \mid x(\delta(v)) = 2\ \forall v \in V(K_n),\ x(\delta(X)) \geq 2\ \forall \emptyset \neq X \subsetneq V(K_n)\}$$

be the Held-Karp polytope. Then:

$$\min\{c^t x \mid x \in P_{HK} \cap \mathbb{Z}^{E(K_n)}\} \leq \frac{3}{2} \min\{c^t x \mid x \in P_{HK}\}$$

Proof. Let $x^* \in \arg\min\{c^x \mid x \in P_{HK}\}$, Y be a minimum spanning tree in (K_n, c) and J a minimum-weight odd(Y)-join. $\frac{n-1}{n} x^* \in P_{ST}$ and $\frac{1}{2} x^* \in P_{\text{odd}(Y)\text{-join}}$. We get:

$$\begin{aligned}c(Y) + c(J) &\leq \frac{n-1}{n} c^t x^* + \frac{1}{2} c^t x^* \\ &< \frac{3}{2} c^t x^*\end{aligned}$$

□

Conjecture 3.7. *If for (K_n, c), c is metric, then:*

$$\min\{c^t x \mid x \in P_{HK} \cap \mathbb{Z}^{E(G)}\} \leq \frac{4}{3} \min\{c^x \mid x \in P_{HK}\}$$

3.3 Further Inequalities for the TSP

Consider the *2-matching inequalities*:

$$x(E(G[H])) + x(F) \leq |H| + \lfloor \frac{|F|}{2} \rfloor \qquad \forall H \subseteq V(G),\ F \subseteq \delta(H),\ |F| \text{ odd}$$

Theorem 3.8. *Let $H, T_1, \ldots, T_k \subseteq V(G)$ such that:*

1. *$|H \cap T_i| \geq 1$ for $i \in [k]$*
2. *$|T_i \setminus H| \geq 1$ for $i \in [k]$*

3. $T_i \cap T_j = \emptyset$ *for* $i \neq j$

4. k *is odd*

Then

$$x(E(G[H])) + \sum_{i=1}^{k} x(E(G[T_i])) \leq |H| + \sum_{i=1}^{k}(|T_i| - 1) - \frac{k+1}{2}$$

is a valid inequality for the TSP polytope. They're called comb inequalities. *H is called* handle, *T_i are called* teeth *and* $(H, T_1, \ldots, T_k)$ *is a comb.*

Proof. Let $(H, T_1, \ldots, T_k)$ be a comb. Generate the inequality as a Gomory-Chvátal-cut: Multiply the following inequalities by $\frac{1}{2}$, add them together and round:

- $x(\delta(v)) = 2$ for $v \in H$
- $-x_e \leq 0$ for $e \in \delta(H) \setminus \bigcup_{i=1}^{k} E(G[T_i])$
- $x(\delta(X)) \geq 2$ for $X = T_i, H \cap T_i, T_i \setminus H$ $(i \in [k])$

□

The complexity of comb separation is an open question.

Theorem 3.9 (Fiorini et al. (1985))**.** *There is no polyhedron with polynomially many facets, whose projection is the TSP polytope.*

Proof. Omitted. □

Definition 3.10. Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron $Q \subseteq \mathbb{R}^m$ is an *extension* of P if there exists a projective map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ with $\pi(Q) = P$. The *extension complexity* of a polyhedron P is the minimum number of facets of an extension Q of P.

Rothvoss (2013) proved that the matching polytope has an exponential extension complexity.

4 Matroids & Generalizations

Definition 4.1. A set system $(E, \mathcal{F})$ (where $\mathcal{F} \subseteq 2^E$) is an independent system if:

i) $\emptyset \in \mathcal{F}$

ii) $X \in \mathcal{F} \Rightarrow \forall Y \subseteq X : Y \in \mathcal{F}$

- Elements in $\mathcal{F}$ are called *independent*.

- Inclusion-wise maximal sets $A \in \mathcal{F}$ are called *bases*. Its cardinality is called $\text{rank}(A)$.
- Inclusion-wise minimal sets $A \in \mathcal{F}$ are *circuits*.

An indepndent system $(E, \mathcal{F})$ is a matroid if the following axiom holds:

iii) $\forall X, Y \in \mathcal{F}$ with $|X| < |Y|$: $\exists y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{F}$. This is equivalent to:

iii)' $\forall X, Y \in \mathcal{F}$ with $|X| + 1 = |Y|$: $\exists y \in Y$ such that $X \cup \{y\} \in \mathcal{F}$.

iii)" $\forall X \subseteq E$ and $A, A' \subseteq X$ maximal with $A, A' \in \mathcal{F}$: $\text{rank}(A) = \text{rank}(A')$.

If $\mathcal{M} = (E, \mathcal{F})$ is a matroid, then $r(\mathcal{M}) = r(E)$. The rank function is defined by:

$$\begin{aligned} r &: 2^E \to \mathbb{N} \\ r(A) &:= \max_{B \subseteq A, B \in \mathcal{F}} |B| \end{aligned}$$

Algorithm 6: Greedy Algorithm for independent systems

Input: Independent system $(E, \mathcal{F})$, $c : E \to \mathbb{R}$
Output: $X \in \mathcal{F}$ with the objective of maximizing $c(X)$

```
X ← ∅
while ∃x ∈ X with c(x) > 0 and X ∪ {x} ∈ F do
    Choose x ∈ arg max_{x∉X, X∪{x}∈F} c(x)
    X ← X ∪ {x}
return X
```

Theorem 4.2. *$(E, \mathcal{F})$ is a matroid $\Leftrightarrow$ algorithm 6 finds an optimum solution for every cost function c.*

Example 4.3.

- *Cycle matroid*: E is the edge set of an undirected graph, $\mathcal{F}$ is the set of forests. Then $(E, \mathcal{F})$ is a matroid. Matroids that can be represented this way are called *graphic matroids*.
- $A \in \mathbb{R}^{m \times n}$, $E = [n]$ and $\mathcal{F}$ is the set of linearly independent subsets of E. This is called a *vector matroid*.
- *Uniform matroid*: E is a finite set, $k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{F} := \{X \subseteq E \mid |X| \leq k\}$.
- *Matching matroid*: G is an undirected graph, $E := V(G)$ and $\mathcal{F} := \{F \subseteq E \mid \exists \text{ matching in } G \text{ covering } F\}$.

- *Gammoids*: G is a graph (directed or undirected), $E, U \subseteq V(G)$. $X \in \mathcal{F}$ if there exist $|X|$ vertex-disjoint U-X-paths.
- *Transversal matroid*: G is a bipartite graph with $V(G) = E \dot{\cup} U$ and (E, U) is a gammoid. $\mathcal{F}$ is the set of subsets of E that are covered by some matching.

Example 4.4. Independent systems that are not matroids:

- Matchings
- Stable sets and cliques
- Subsets of TSP tours or Steiner trees
- Feasible solutions of knapsack problems

Theorem 4.5 (Edmonds (1970)). *Let $(E, \mathcal{F})$ be a matroid and $r : 2^E \to \mathbb{N}$ its rank function. Then the* matroid polytope *of $(E, \mathcal{F})$ (i.e. the convex hull of incidence vectors of independent sets) can be described by:*

$$\{x \in \mathbb{R}^E \mid x \geq 0, \sum_{e \in A} x_e \leq r(A) \ \forall A \subseteq E\}$$

Proof. The polytope contains all incidence vectors of independent sets. We have to show that the vertices of the polytope are integral, or equivalently:

$$\max\{c^t x \mid x \geq 0, \sum_{e \in A} x_e \leq r(A) \ \forall A \subseteq E\}$$

attains an integral optimum for all $c \in \mathbb{R}^E$. Let x^0 be the incidence vector of the set J found by the greedy algorithm (algorithm 6).

Claim. *x^0 is an optimum solution in the polytope.*

The dual problem is

$$\begin{aligned} \min \sum_{A \subseteq E} r(A) y_A & \\ \sum_{A \subseteq E,\ e \in A} y_A \geq c(e) & \qquad\qquad e \in E \\ y \geq 0 & \end{aligned}$$

Our goal is to find a dual solution in complementary slackness with x^0, so $x_e > 0 \Rightarrow \sum_{A \subseteq E,\ e \in A} y_A = c(e)$ and $y_A > 0 \Rightarrow x(A) = r(A)$.

Consider the Dual Greedy Algorithm:

1. Order E as $\{e_1, \ldots, e_n\}$ with:
$$c(e_1) \geq \ldots \geq c(e_m) \geq 0 \geq c(e_{m+1}) \geq \ldots \geq c(e_n)$$

2. $T_i \coloneqq \{e_1, \ldots, e_i\}$ for $1 \leq i \leq m$, $T_0 \coloneqq \emptyset$ and
$$y_A^0 \coloneqq \begin{cases} c(e_i) - c(e_{i+1}) & A = T_i \text{ for } i \in \{1, \ldots, m-1\} \\ c(e_m) & A = T_m \\ 0 & \text{else} \end{cases}$$

$y \geq 0$ and for $j > m$, $c(e_j) \leq 0$ so the inequality is satisfied. If $j \leq m$, then:
$$\sum_{A \subseteq E,\ e_j \in A} y_A = \sum_{i=j}^{m} y_{T_i}^0 = c(e_j)$$

Therefore, y is dual feasible. If $x_e^0 > 0$, the corresponding dual constraint is tight. Let $y_A^0 > 0$, so $A = T_i$ for some i. We have to show that $x^0(A) = r(A)$, i.e. $J \cap T_i$ is a basis of T_i. If not, there exists $e_k \in T_i \setminus J$ with $(J \cap T_i) \cup \{e_k\} \in \mathcal{F}$ and $c(e_k) > c(e_j)$. Since the algorithm didn't add e_k, this is a contradiction. □

Corollary 4.6. *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid, $c \in \mathbb{R}^E$ and $J \in \mathcal{F}$. Then J is a maximum-weight independent set if and only if:*

a) $\forall e \in J :\ c(e) \geq 0$

b) $\forall e \notin J,\ J \cup \{e\} \in \mathcal{F} :\ c(e) \leq 0$

c) $\forall e \notin J, f \in J, (J \cup \{e\}) \setminus \{f\} \in \mathcal{F} :\ c(e) \leq c(f)$

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": Take a dual solution y^0 from the dual greedy algorithm.By a), $\sum_{e \in A} y_A = c(e)$ for all $e \in J$. If there exists $A \subseteq E$ with $y_A > 0$ and $x(A) < r(A)$, then $\exists i$ with $c(e_i) > c(e_{i+1})$ and $J \cap T_i$ is not a basis of $T_i = A$. Therefore, there exists $e \in T_i \setminus J$ with $(J \cap T_i) \cup \{e\} \in \mathcal{F}$. If $\{e\} \cup J \in \mathcal{F}$, this would contradict b). Otherwise, extend $(J \cap T_i) \cup \{e\}$ to a basis J' of $J \cup \{e\}$. Then $|J'| = |J|$, so $J' = (J \cup \{e\}) \setminus \{f\}$ for some $f \in T_i$, which is a contradiction to c).

□

Theorem 4.7. *Let G be an undirected graph. The forest polytope of G is given by:*
$$\{x \in \mathbb{R}^{E(G)} \mid x(E(G[T])) \leq |T| - 1 \ \forall \emptyset \neq T \subseteq V(G)\}$$

Proof. Apply theorem 4.5 to the cycle matroid. $\square$

4.1.1 Matroid Constructions

Proposition 4.8 (Disjoint Union). *Given matroids $\mathcal{M}_1 = (E_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{F}_2)$ with $E_1 \cap E_2 = \emptyset$, $\mathcal{M} := \mathcal{M}_1 \oplus \mathcal{M}_2 := (E, \mathcal{F})$ where $E = E_1 \dot\cup E_2$ and $\mathcal{F} = \{J_1 \cup J_2 \mid J_1 \in \mathcal{F}_1,\ J_2 \in \mathcal{F}_2\}$ is a matroid with rank function*

$$r(A) = r(A \cap E_1) + r(A \cap E_2)$$

where r_i is the rank function of $\mathcal{M}_i$.

Proposition 4.9 (Partition Matroid). *Let $E = E_1 \dot\cup \ldots \dot\cup E_k$ and $\mathcal{F} := \{J \subseteq E(G) \mid |J \cap E_i| \leq 1 \forall i \in [k]\}$. Then $(E, \mathcal{F})$ is a matroid with rank function:*

$$r(A) = |\{i \in [k] \mid E_i \cap A \neq \emptyset\}|$$

Proposition 4.10 (Restriction Matroid). *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $B \subseteq E$. Then $\mathcal{M}' := \mathcal{M} \setminus B := (E \setminus B, \{J \subseteq E \setminus B \mid J \in \mathcal{F}\})$ is a matroid.*

Proposition 4.11 (Contraction Matroid). *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $B \subseteq E$. Choose an arbitrary basis J of B (i.e. $J \in \mathcal{F}$ and $r(J) = r(B)$). Then $M' := \mathcal{M}/B := (E \setminus B, \{J' \subseteq E \setminus B \mid J' \cup J \in \mathcal{F}\})$ is a matroid. $\mathcal{M}$ is independent of the chosen basis J. Its rank function is*

$$r'(A) = r(A \cup B) - r(B)$$

Corollary 4.12. *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $B \subseteq E$. Then $\mathcal{M}' := (\mathcal{M} \setminus B) \oplus (\mathcal{M}/(E \setminus B))$ is a matroid on E. The bases of $\mathcal{M}'$ are those bases of $\mathcal{M}$ that intersect B in a basis of B.*

Proposition 4.13 (Matroid Minors). *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $\emptyset = T_0 \subseteq T_1 \subseteq \ldots \subseteq T_{l+1} = \mathcal{F}$. The bases of T_l in $\mathcal{M}$ that intersect T_i ($1 \leq i \leq l$) are the bases of T_l in the matroid $\mathcal{N} := \mathcal{N}_0 \oplus \ldots \oplus \mathcal{N}_l$ where for each i, $\mathcal{N}_i := (\mathcal{M}/T_i) \setminus (E \setminus T_{i+1})$. $\mathcal{N}$ is called a minor of $\mathcal{M}$.*

4.2 Matroid Intersection

Finding $\arg\max\{|J| \mid J \in \mathcal{F}_1 \cap \mathcal{F}_2\}$ for matroids $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$ can be done similarly to bipartite matching in $O(|E|^2)$. Weighted matroid intersection (of 2 matroids) can also be done in polynomial time.

Computing $\max\{|J| \mid J \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3\}$ is NP-hard.

4.4 Polymatroids

For the rank function r of a matroid, $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ for all $X, Y \in E$, so the rank function is *submodular.*

Definition 4.34. A *polymatroid* is the polytope

$$P(f) \coloneqq \{x \in \mathbb{R}^{E(G)} \mid x \geq 0,\ x(A) \leq f(A)\ \forall A \subseteq E\}$$

where E is a finite set and $f : 2^e \to \mathbb{R}_{\geq 0}$ is submodular.

Proposition 4.35. *For any polymatroid $P(f)$, f can be chosen such that $f(\emptyset) = 0$ and f is monotone, i.e. $A \subseteq B$ implies $f(A) \leq f(B)$.*

Proposition 4.36. *Let $E = \{e_1, \ldots, e_n\}$, $f : 2^E \to \mathbb{R}_{\geq 0}$ submodular with $f(\emptyset) \geq 0$, $b : E \to \mathbb{R}$ with $b(e_1) \leq f(e_1)$ and $b(e_i) \leq f(\{e_1, \ldots, e_i\}) - f(\{e_1, \ldots, e_{i-1}\})$ for $i \in \{2, \ldots, n\}$. Then $\sum_{a \in A} b(a) \leq f(A)$ for all $A \subseteq E$.*

Proof. Induction on $i = \max\{j \mid e_j \in A\}$. For $A = \emptyset$, the statement is trivial. For $i \geq 1$:

$$\begin{aligned} b(A) &= b(A \setminus \{e_i\}) + b(e_i) \\ &\leq f(A \setminus \{e_i\}) + b(e_i) \\ &\leq f(A \setminus \{e_i\}) + f(\{e_1, \ldots, e_i\}) - f(\{e_1, \ldots, e_{i-1}\}) \\ &\leq f(A) \end{aligned}$$

□

Algorithm 7: Polymatroid Greedy Algorithm

Input: Finite set E and $f : 2^E \to \mathbb{R}_{\geq 0}$ submodular and monotone (given by an oracle) and $c : E \to \mathbb{R}$

Output: $x \in P(f)$ maximizing $c^t x$

Sort $E = \{e_1, \ldots, e_n\}$ such that:
$c(e_1) \geq \ldots \geq c(e_k) > 0 \geq c(e_{k+1}) \geq \ldots \geq c(e_n)$

if $k \geq 1$ **then**
- $x_{e_1} \leftarrow f(\{e_1\})$
- **for** $i = 2, \ldots, k$ **do**
  - $x_{e_i} \leftarrow f(\{e_1, \ldots, e_i\}) - f(\{e_1, \ldots, e_{i-1}\})$
- **for** $i = k+1, \ldots, n$ **do**
  - $x_{e_i} \leftarrow 0$

Theorem 4.37. *The Polymatroid Greedy algorithm correctly finds $x \in P(f)$ maximizing $c^t x$. If f is integral, then x is also integral.*

Proof. Let x be the output of algorithm 7. If f is integral, x is integral by construction. Assume that there exists $y \in \mathbb{R}^E_{\geq 0}$ with $c^t y > c^t x$. For $i \in [k-1]$, define $d_j \coloneqq c(e_j) - c(e_{j+1})$ and $d_k \coloneqq c(e_k)$.

$$
\begin{aligned}
\sum_{j=1}^{k} d_j \sum_{i=1}^{j} x_i &= c^t x \\
&< c^t y \\
&= \sum_{j=1}^{k} d_j \sum_{i=1}^{j} y_i
\end{aligned}
$$

Therefore, there exists $j \in [k]$ such that

$$\sum_{i=1}^{j} y_i > \sum_{i=1}^{j} x_i = f(\{e_1, \ldots, e_j\})$$

so y is not contained in the polymatroid. □

Theorem 4.38. *Let E be finite and $f, g : 2^E \to \mathbb{R}_{\geq 0}$ submodular. Then*

$$
\begin{aligned}
x(A) &\leq f(A) & A &\subseteq E \\
x(A) &\leq g(A) & A &\subseteq E \\
x &\geq 0
\end{aligned}
$$

is TDI.

Proof. Consider the primal-dual pair:

$$
\begin{array}{lll|lll}
\max c^t x & & & & \min \sum_{A \subseteq E} f(A) y_A + g(A) z_A & \\
x(A) \leq f(A) & A \subseteq E & & & \sum_{e \in A \subseteq E} (y_A + z_A) \geq c(e) & e \in E \\
x(A) \leq g(A) & A \subseteq E & & & y, z \geq 0 & \\
x \geq 0 & & & & &
\end{array}
$$

Claim. *Let $Ax \leq b, x \geq 0$ be a linear program. If for any $c \in \mathbb{Z}^n$ where the dual is feasible and bounded, it has an optimum solution y_i^* such that the rows of A where $y_i^* > 0$ (plus possibly basic 0-entries) forms a TU matrix A'. Then $Ax \leq b, x \geq 0$ is TDI.*

Proof. Let c, y^* be as above. We claim:

$$\min\{y^t b \mid A^t y \geq c,\ y \geq 0\} = \min\{y^t b' \mid (A')^t y \geq c,\ y \geq 0\}$$

"$\leq$" is clear. Since the restriction of y^* is feasible for the right hand side, the other inequality also holds. Since A' is TU, the right hand system is TDI, so y^* can be chosen integrally if c is integral. $\square$

Let $c : E \to \mathbb{Z}_{\geq 0}$ and y, z be an optimum dual solution such that

$$\sum_{A \subseteq E} (y_A + z_A) \cdot |A| \cdot |E \setminus A|$$

is minimum.

Claim. *$\mathcal{F} := \{A \subseteq E \mid y_A > 0\}$ is a chain.*

Otherwise, there are $A, B \in \mathcal{F}$ with $A \cap B \neq A, B \cap A \neq B$. Let

$$\begin{aligned}
\epsilon &:= \min\{y_A, y_B\} \\
y'_A &:= y_A - \epsilon \\
y'_B &:= y_B - \epsilon \\
y'_{A\cup B} &:= y_{A\cup B} + \epsilon \\
y'_{A\cap B} &:= y_{A\cap B} + \epsilon \\
y_S &:= y_S \qquad \text{elsewhere}
\end{aligned}$$

y', z is feasible and optimal by submodularity but the term above gets smaller, which is a contradiction. Similarly, $\mathcal{F}' := \{A \subseteq E \mid z_A > 0\}$ is a chain.

Let M, M' be the matrices with column set E and row set $\mathcal{F}, \mathcal{F}'$. Then $\begin{pmatrix} M \\ M' \end{pmatrix}$ is TU: $A_1 \geq \ldots \geq A_p \in \mathcal{F}$ and $B_1 \geq \ldots \geq B_q \in \mathcal{F}'$. Define

$$\begin{aligned}
\mathcal{R}_1 &:= \{A_i \mid i \text{ odd}\} \cup \{B_i \mid i \text{ even}\} \\
\mathcal{R}_2 &:= \{A_i \mid i \text{ even}\} \cup \{B_i \mid i \text{ odd}\}
\end{aligned}$$

These sets satisfy Ghoulia-Houri, so the system is TDI. $\square$

Corollary 4.39. *Let $(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$ be two matroids. Then the convex hull of incidence vectors $x \in \mathcal{F}_1 \cap \mathcal{F}_2$ is the polytope*

$$\{x \in \mathbb{R}^E_{\geq 0} \mid x(A) \leq \min\{r_1(A), r_2(a)\} \ \forall A \subseteq E\}$$

where r_1, r_2 are the rank functions of the matroids.

Proof. By theorem 4.38, the inequality system is TDI, so since r_1, r_2 are integral, the polytope is integral. Integral vectors in the polytope correspond exactly to incidence vectors of sets in $\mathcal{F}_1 \cap \mathcal{F}_2$. $\square$

Corollary 4.40. *Let $f, g : 2^E \to \mathbb{R}_{\geq 0}$ be submodular, monotone with $f(\emptyset) = g(\emptyset) = 0$. Then:*

$$\underbrace{\max\{\mathbb{1}^t x \mid x \in P(f) \cap P(g)\}}_{(**)} = \min_{A \subseteq E} f(A) + g(E \setminus A)$$

Proof. The dual of $(**)$ is:

$$\min\{\sum_{A \subseteq E} (f(A)y_A + g(A)z_A) \mid y, z \geq 0, \sum_{E \supseteq A \ni e} y_A + z_A \geq 1 \; \forall e \in E\}$$

"$\geq$": By theorem 4.38, the dual has an integral optimum solution y, z. Let:

$$B \coloneqq \bigcup_{\substack{A \subseteq E \\ y_A \geq 1}} A \qquad\qquad C \coloneqq \bigcup_{\substack{A \subseteq E \\ z_A \geq 1}} A$$

Since y, z are integral, the dual constraint implies $E = B \cup C$, so $E \setminus B \subseteq C$. Therefore:

$$\begin{aligned} \sum_{A \subseteq E} (f(A)y_A + g(A)z_A) &\geq f(B) + g(C) \\ &\geq f(B) + g(E \setminus B) \end{aligned}$$

"$\leq$": For $A \subseteq E$, we construct the feasible dual solution $y_A \coloneqq 1$ and $z_{E \setminus A} \coloneqq 1$, everything else 0 which has cost $f(A) + g(E \setminus A)$. By LP-duality, any primal solution attains at most this value.

□

- $f : 2^E \to \mathbb{R}$ supermodular:

$$f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y) \qquad \forall X, Y \subseteq E$$

- $f : 2^E \to \mathbb{R}$ modular:

$$f(X) + f(Y) = f(X \cup Y) + f(X \cap Y) \qquad \forall X, Y \subseteq E$$

- $f(A)$ submodular implies $f(E \setminus A)$ submodular.

Corollary 4.41 (Frank's Discrete Sandwich Theorem (1982))**.** *Let E be a finite sit, $f : 2^E \to \mathbb{R}$ supermodular, $g : 2^E \to \mathbb{R}$ submodular with $f(A) \leq g(A)$ for all $A \subseteq E$. Then there exists a modular function $h : 2^E \to \mathbb{R}$ with $f(A) \leq h(A) \leq g(A)$ for all $A \subseteq E$. If f, g are integral, h can be chosen integral.*

Proof.

- Without loss of generality, $f(\emptyset) = g(\emptyset)$ and $f(E) = g(E)$.
- Let $M \coloneqq 2 \cdot \max\{|f(A)| + |g(A)| \mid A \subseteq E\}$ and:

$$\begin{aligned} f'(A) &\coloneqq g(E) - f(E \setminus A) + M \cdot |A| \\ g'(A) &\coloneqq g(A) - f(\emptyset) + M \cdot |A| \end{aligned}$$

  f', g' are submodular, nonnegative, monotone and $f'(\emptyset) = 0 = g'(\emptyset)$.
- By corollary 4.40:

$$\begin{aligned} &\max\{\mathbb{1}^t x \mid x \in P(f') \cap P(g')\} \\ =&\min_{A \subseteq E}(f'(A) + g'(E \setminus A)) \\ =&\min_{A \subseteq E}(g(E) - f(E \setminus A) + M \cdot |A|) + (g(E \setminus A) - f(\emptyset) + M \cdot |E \setminus A|) \\ \geq& g(E) + M \cdot |E| - f(\emptyset) \end{aligned}$$

- Let $x \in P(f') \cap P(g')$ such that $\mathbb{1}^t x = g(E) - f(\emptyset) + M \cdot |E|$. If f, g are integral, we can choose it such that $x \in \mathbb{Z}^E$.
- Define:

$$\begin{aligned} h'(A) &\coloneqq \sum_{e \in A} x_e && A \subseteq E \\ h(A) &\coloneqq h'(A) + f(\emptyset) - M \cdot |A| && A \subseteq E \end{aligned}$$

  Then h is modular and for $A \subseteq E$:

$$\begin{aligned} h(A) &\leq g'(A) + f(\emptyset) - M \cdot |A| \\ &= g(A) \\ h(A) &= 1^t x - h'(E \setminus A) + f(\emptyset) - M \cdot |A| \\ &\geq g(E) + M \cdot |E \setminus A| - f'(E \setminus A) \\ &= f(A) \end{aligned}$$

$\square$

Definition 4.42. Let $f : 2^E \to \mathbb{R}$ be a function. For $x \in \mathbb{R}^E_{\geq 0}$, there exist unique $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k > 0$ and sets $\emptyset \subsetneq T_1 \subsetneq \dots \subsetneq T_k \subseteq E$ such that $x = \sum_{i=1}^k \lambda_i \chi^{T_i}$ where χ^{T_i} is the incidence vector of T_i. The Lovász extension of f is defined as:

$$\begin{aligned} f' : \mathbb{R}^E_{\geq 0} &\to \mathbb{R} \\ x &\mapsto \sum_{i=1}^k \lambda_i f(T_i) \end{aligned}$$

Lemma 4.43. *Let $f : 2^E \to \mathbb{R}$ be submodular and f' its Lovász extension. Then:*

$$f'(x) = \max\{x^t y \mid y \in P(f)\}$$

Proof. Exercise $\square$

Theorem 4.44.

$$f \text{ submodular} \Leftrightarrow f' \text{ convex}$$

4.4.1 Applications of Matroid Intersection

Orientations: Let G be an undirected graph and $k : V(G) \to \mathbb{Z}_{\geq 0}$. Does there exist an orientation $\vec{G}$ of G such that $\left|\delta^-_{\vec{G}}(v)\right| \leq k(v)$ for all $v \in V(G)$?

Let $D := (V(G), \{(v,w),(w,v) \mid \{v,w\} \in E(G)\})$. We define:

- $(A, \mathcal{F}_1)$ as the partition matroid on $\bigsqcup_{\{v,w\} \in E(G)} \{(v,w),(w,v)\}$
- $(A, \mathcal{F}_2)$ as the (generalized) partition matroid on $\bigsqcup_{v \in V(G)} \delta^-_D(v)$ allowing $\leq k$ elements from $\delta^-_D(v)$ for all $v \in V(G)$.

Then such an orientation $\vec{G}$ exists $\Leftrightarrow$ there exists $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ with $|F| = |E|$.

Theorem. *G has an orientation $\vec{G}$ such that $\left|\delta^1_{\vec{G}}(v)\right| \leq k(v)$ for all $v \in V(G)$ if and only if:*

$$\forall P \subseteq V(G): \ |E(G[P])| \leq \sum_{v \in P} k(v)$$

Two disjoint spanning trees: For a matroid $\mathcal{M} = (E, \mathcal{F})$ we define $\mathcal{M}^* := (E, \mathcal{F}^*)$ where:

$$\mathcal{F}^* := \{A \subseteq E \mid E \setminus A \text{ contains a basis of } \mathcal{F}\}$$

$\mathcal{M}^*$ is a matroid with rank function $r_{\mathcal{M}^*}(X) = |X| + r_{\mathcal{M}}(E \setminus X) - |E|$.

Proposition. *Let G be a graph and $\mathcal{M} = (E, \mathcal{F})$ its graphic matroid. Then:*

$$G \text{ has 2 disjoint spanning trees} \Leftrightarrow \max_{I \in \mathcal{F} \cap \mathcal{F}^*} |I| = |V(G)| - 1$$

4.5 Submodular Function Maximization

Recall: $f : 2^E \to \mathbb{R}$ is called submodular if $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for all $A, B \subseteq E$. Equivalently, $f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$ for all $X \subseteq Y \subseteq E$ and $x \in E \setminus Y$.

Problem (USM: "unconstrainted submodular function maximization")**.** Given a submodular function $f : 2^E \to \mathbb{R}$, find $S \subseteq E$ maximizing $f(S)$.

Example. For a given graph G, define $f(X) \coloneqq |\delta(X)|$. Maximizing $f(X)$ corresponds to the maximum cut problem (which is NP-hard).

Algorithm 8: Deterministic Double Greedy

Input: Finite set E, submodular function $f : 2^E \to \mathbb{R}^+$
Output: $S \subseteq E$
$X_0 \leftarrow \emptyset,\ Y_0 \leftarrow E$
for $i = 1, \ldots, n$ **do**
  $a_i \leftarrow f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})$
  $b_i \leftarrow f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})$
  if $a_i \geq b_i$ **then**
    $X_i \leftarrow X_{i-1} \cup \{e_i\},\ Y_i \leftarrow Y_{i-1}$
  else
    $X_i \leftarrow X_{i-1},\ Y_i \leftarrow Y_{i-1} \setminus \{e_i\}$
return $S \leftarrow X_n$

Lemma 4.45. *For every $1 \leq i \leq n$, $a_i + b_i \geq 0$.*

Proof. By the equivalent characterization of submodularity and since $X_i \subseteq Y_i$ for all i:

$$\begin{aligned} a_i + b_i &= f(X_{i-1} \cup \{e_i\}) - f(X_{i-1}) + f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) \\ &= (f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})) - (f(Y_{i-1}) - f(Y_{i-1} \setminus \{e_i\})) \\ &\geq 0 \end{aligned}$$

□

Let OPT be the optimum solution and $\mathrm{OPT}_i \coloneqq (\mathrm{OPT} \cup X_i) \cap Y_i$, so OPT_i coincides with X_i and Y_i on the first i elements and with OPT on the rest. In particular, $\mathrm{OPT}_0 = \mathrm{OPT}$ and $\mathrm{OPT}_n = X_n$.

Lemma 4.46. *For every $1 \leq i \leq n$, we have:*

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i) \leq (f(X_i) - f(X_{i-1})) + (f(Y_i) - f(Y_{i-1}))$$

Proof. Without loss of generality assume that $a_i \geq b_i$, so the second summand is 0. Then $\mathrm{OPT}_i = \mathrm{OPT}_{i-1} \cup \{e_i\}$. We need to show:

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i) \leq f(X_i) - f(X_{i-1}) = a_i$$

Case 1: $e_i \in \mathrm{OPT}_{i-1}$. Then the left side is 0 and so by lemma 4.45, $a_i \geq 0$.

Case 2: $e_i \notin \mathrm{OPT}_{i-1}$. Then

$$\mathrm{OPT}_{i-1} = (\mathrm{OPT} \cup X_{i-1}) \cap Y_{i-1} \subseteq Y_{i-1} \setminus \{e_i\}$$

so by submodularity:

$$\begin{aligned} f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) &\leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) \\ &= b_i \leq a_i \end{aligned}$$

$\square$

Theorem 4.47 (Buchbinder et al.)**.** *Algorithm 8 returns a $\frac{1}{3}$-approximation for USM.*

Proof. By lemma 4.46:

$$\sum_{i=1}^{n}(f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i)) \leq \sum_{i=1}^{n}(f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}))$$

Since both sides are telescopic sums:

$$\begin{aligned} f(\mathrm{OPT}_0) - f(\mathrm{OPT}_n) &\leq f(X_n) - f(X_0) + f(Y_n) - f(Y_0) \\ &\leq f(\underbrace{X_n}_{=\mathrm{OPT}_n}) + f(\underbrace{Y_n}_{=\mathrm{OPT}_n}) \end{aligned}$$

In total, $f(\mathrm{OPT}_0) \leq 3f(\mathrm{OPT}_n)$. $\square$

Remark 4.48. If f is arbitrary, we can simply add a constant to it to make it non-negative. The analysis is tight.

4.5.2 Randomized USM

Lemma 4.49. *For $i \in \{1, \ldots, n\}$:*

$$\mathbb{E}\left[\underbrace{f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i)}_{\mathrm{I}}\right] \leq \frac{1}{2}\mathbb{E}\left[f(X_i) - f(X_{i-1}) + \underbrace{f(Y_i) - f(Y_{i-1})}_{\mathrm{II}}\right]$$

Proof. We can consider each X_{i-1} separately, so we condition on some event of the form $X_{i-1} = S_{i-1}$ where $S_{i-1} \subseteq \{e_1, \ldots, e_{i-1}\}$ is fixed and the probability that $X_{i-1} = S_{i-1}$ is non-zero.

Case 1: $b_i \leq 0$. Then $p = 1$ and $Y_i = Y_{i-1} = S_{i-1} \cup \{e_i, \ldots, e_n\}$ and $X_i = S_{i-1} \cup \{e_i\}$.

Claim.

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) \leq \frac{1}{2}f(X_i) - f(X_{i-1}) = \frac{a_i}{2}$$

- If $e_i \in \mathrm{OPT}$, $0 \leq \frac{a_i}{2}$.

Algorithm 9: Randomized Double Greedy

Input: Finite set E, submodular function $f : 2^E \to \mathbb{R}_+$

Output: $S \subseteq E$

$X_0 \leftarrow \emptyset,\ Y_0 \leftarrow E$

for $i = 1, \ldots, n$ **do**

    $a_i \leftarrow f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})$

    $b_i \leftarrow f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})$

    $p \leftarrow \begin{cases} 1 & b_i \leq 0 \\ 0 & a_i \leq 0 \\ \frac{a_i}{a_i+b_i} & \text{else} \end{cases}$

    with probability p **do**

        $X_i \leftarrow X_{i-1} \cup \{e_i\},\ Y_i \leftarrow Y_{i-1}$

    else

        $X_i \leftarrow X_{i-1},\ Y_i \leftarrow Y_{i-1} \setminus \{e_i\}$

return X_n

- If $e_i \notin \mathrm{OPT}$, then by submodularity:

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) \leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) = b_i \leq 0 \leq \frac{a_i}{2}$$

The statement then follows directly from the claim.

Case 2: $a_i \leq 0$. This is analogous to case 1.

Case 3: $a_i, b_i > 0$.

$$\begin{aligned} &\mathbb{E}\left[f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1})\right] \\ =&p \cdot (f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})) + (1-p) \cdot (f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})) \\ =&\frac{a_i^2 + b_i^2}{a_i + b_i} \end{aligned}$$

We have found a value for the right side of the inequality. Now, we upper-bound the left side.

$$\begin{aligned} &\mathbb{E}\left[f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i)\right] \\ =&p\left(f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\})\right) \\ +&(1-p)\underbrace{\left(f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \setminus \{e_i\})\right)}_{\text{III}} \\ \overset{(*)}{\leq}&\frac{a_i b_i}{a_i + b_i} \end{aligned}$$

To see $(*)$:

Case 3.1: If $e_i \notin \mathrm{OPT}_{i-1}$, then III is 0 and as $\mathrm{OPT}_{i-1} = (\mathrm{OPT} \cup X_{i-1}) \cap Y_{i-1}$ by submodularity:

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) \leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) = b_i$$

Case 3.2: If $e_i \in \mathrm{OPT}_{i-1}$, then the first term of the LHS is 0. By submodularity:

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \setminus \{e_i\}) \leq f(X_{i-1} \cup \{e_i\}) - f(X_{i-1}) = a_i$$

Now $\frac{a_i b_i}{a_i + b_i} \leq \frac{1}{2} \frac{a_i^2 + b_i^2}{a_i + b_i}$ by the binomial formula.

□

Theorem 4.50. *Algorithm 9 returns a solution S with*

$$\mathbb{E}[f(S)] \geq \frac{f(\mathrm{OPT})}{2}$$

Proof. Summing up lemma 4.49 for all $i \in \{1, \ldots, n\}$ and collapsing the telescopic sums yields:

$$\begin{aligned} \mathbb{E}[f(\mathrm{OPT}_0) - f(\mathrm{OPT}_n)] &\leq \frac{1}{2} \mathbb{E}[f(X_n) - f(X_0) + f(Y_n) - f(Y_0)] \\ &\leq \frac{\mathbb{E}[f(X_n) + f(Y_n)]}{2} \end{aligned}$$

In total, $\mathbb{E}[f(\mathrm{OPT}_n)] \geq \frac{f(\mathrm{OPT})}{2}$. □

Remark. There is no $\frac{1}{2} + \epsilon$-approximation for $\epsilon > 0$ that only uses a polynomial number of oracle calls.

4.6 Submodular Function Minimization

Problem (Submodular Function Minimization)**.** Given a finite set U and a submodular function $f : 2^U \to \mathbb{R}$ with $f(\emptyset) = 0$, find a set $S \subseteq U$ with $f(S)$ minimum.

Definition 4.53. Let U be finite and $f : 2^U \to \mathbb{R}$ submodular. Then the *base polyhedron* is defined as:

$$B(f) \coloneqq \{x \in \mathbb{R}^U \mid x(A) \leq f(A) \ \forall A \subseteq U,\ x(U) = f(U)\}$$

Example. Let $U = \{1, 2\}$ and $f(\{1\}) = 2,\ f(\{2\}) = -2,\ f(\{1, 2\}) = -1$.

Theorem 4.54. *The vertices of the base polyhedron are given by the vectors $b^<$ for all total orders $<$ of U where:*

$$b^<(u) := f(\{v \in U \mid v \leq u\}) - f(\{v \in U \mid v < u\})$$

Proof. Exercise □

Theorem 4.55. *Let $f : 2^U \to \mathbb{R}$ be submodular, $f(\emptyset) = 0$. Then*

$$\min_{S \subseteq U} f(S) = \max\{x^-(U) \mid x \in B(f)\}$$

where $x^-(U) = \sum_{u \in U} x^-(u) = \sum_{u \in U} \min\{0, x(u)\}$.

Proof. Exercise □

Idea: Maintain $x \in B(f)$ and represent it by a convex combination of the vertices. By Carathéodory, $|U|$ vertices are enough.

Algorithm 10: Schrijver's Algorithm

Input: Finite set $U = \{1, \dots, n\}$, submodular function $f : 2^U \to \mathbb{R}$ with $f(\emptyset) = 0$

Output: $X \subseteq U$ with $f(X)$ minimum

$k \leftarrow 1$, $<_1 \leftarrow$ any total order on U, $x \leftarrow b^{<_1}$

Build Graph:

- $D \leftarrow (U, A)$ where $A = \{(u, v) \mid u <_i v$ for some $1 \leq i \leq k\}$
- $P \leftarrow \{u \in U \mid x(u) > 0\}$
- $N \leftarrow \{u \in U \mid x(u) < 0\}$
- $X \leftarrow$ set of vertices not reachable from P in D
- **if** $N \subseteq X$ **then**
  - **return** X

Find Augmentation:

- Let $d(v)$ denote the distance from P to v in D
- Choose $t \in N \setminus X$ with $(d(t), t)$ lexicographically maximum
- Choose s maximal with $(s, t) \in A$ and $d(s) = d(t) - 1$
- Let $i \in \{1, \dots, k\}$ such that $\alpha = |\{v \in U \mid s <_i v \leq_i t\}|$ is maximum. [Let β be the number of indices attaining α]

Change Solution:

- Compute $0 \leq \epsilon \leq -x(t)$ and write $x' = x + \epsilon(\chi^t - \chi^s)$ as an explicit convex combination of $\leq n$ vectors from $b^{<_1}, \dots, b^{<_k}$ and $b^{<_i^{s,u}}$ $\forall s <_i u \leq_i t$ (where $<_i^{s,u}$ arises from $<_i$ by placing u directly before s) such that $b^{<_i}$ does not occur if $x'(t) < 0$
- $x \leftarrow x'$, rename the vectors in the convex combination of x as $b^{<_1}, \dots, b^{<_{k'}}$, $k \leftarrow k'$

go to Build Graph

Example. In the example from above, let $<_1$ be $1 <_1 2$, $k = 1$ and $x = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Theorem 4.56. *Algorithm 10 returns an optimum solution if it terminates.*

Proof. If the algorithm terminates, D does not contain a P-N-path. Since $N \subseteq X \subseteq U \setminus P$, $\sum_{u \in X} x(u) \leq \sum_{w \in W} x(w)$ for all $W \subseteq U$. No edge enters X, so $X = \emptyset$ or for all $j \in \{1, \ldots, k\}$ there exists $v \in X$ with $X = \{u \in U \mid u \leq_j v\}$. Therefore $\sum_{u \in X} b^{<_j}(u) = f(X)$ for all j (by definition of $b_u^{<}$). By theorem 4.36 (and again the definition of $b_u^{<}$), $\sum_{u \in W} b^{<_j}(u) \leq f(W)\ \forall W \subseteq U, j \in \{1, \ldots, k\}$. We get (where λ_j are the factors in the convex combination):

$$
\begin{aligned}
f(W) &\geq \sum_{j=1}^{k} \lambda_j \sum_{u \in W} b^{<_j}(u) \\
&= \sum_{u \in W} \sum_{i=1}^{k} \lambda_j b^{<_j}(u) \\
&= \sum_{u \in W} x(u) \\
&\geq \sum_{u \in X} x(u) \\
&= \sum_{u \in X} \sum_{j=1}^{k} \lambda_j b^{<_j}(u) \\
&= \sum_{j=1}^{k} \lambda_j \sum_{u \in X} b^{<_j}(u) \\
&= f(X)
\end{aligned}
$$

□

Theorem 4.57. *Each iteration can be performed in $O(n^3 + \gamma n^2)$ time where γ is the time required for an oracle call.*

Proof. BuildGraph and FindAugmentation can both be implemented in $O(n^3)$. We need to show that ChangeSolution can be done in $O(n^3 + \gamma n^2)$. Let $x = \lambda_1 b^{<_1} + \ldots + \lambda_k b^{<_k}$ and $s <_i t$.

Claim. *For some $\delta > 0$, $\delta(\chi^t - \chi^s)$ can be written as a convex combination of the vectors $b^{<_i^{s,u}} - b^{<_i}$ for $s <_i u \leq_i t$ in $O(\gamma n^2)$ time.*

How do $b^{<_i^{s,u}}$ and $b^{<_i}$ compare?

- Let $s <_i v \leq_i t$. Then by definition $b^{<_i^{s,v}}(u) = b^{<_i}(u)$ for $u <_i s$ or $u >_i v$.
- For $s \leq_i u <_i v$: $b^{<_i^{s,v}}(u) = f(\{w \in U \mid w \leq_i^{s,v} u\}) - f(\{w \in U \mid w <_i^{s,v} u\}) \leq f(\{w \in U \mid w \leq_i u\}) - f(\{w \in U \mid w <_i u\}) = b^{<_i}(u)$ by submodularity.
- For $u = v$ we have by submodularity:

$$\begin{aligned} b^{<_i^{s,v}}(u) &= f(\{w \in U \mid w \leq_i^{s,v} u\}) - f(\{w \in U \mid w <_i^{s,v} u\}) \\ &\geq f(\{w \in U \mid w \leq_i u\}) - f(\{w \in U \mid w <_i u\}) \\ &= b^{<_i}(u) \end{aligned}$$

Proof of claim:

- If $\exists s <_i v <_i t$ such that $b^{<_i^{s,v}}(v) = b^{<_i}(v)$ choose $\delta = 0$ and $\lambda_v = 1$.
- Otherwise for all $s <_i v \leq_i t$ we have $b^{<_i^{s,v}}(v) > b^{<_i}(v)$. Look at the matrix $M = (b^{<_i^{s,v}} - b^{<_i})_{vu}$ with rows $s <_i v \leq_i t$ and columns for $u \in U$. Then

$$\chi^t - \chi^s = \sum_{s <_i v \leq_i t} \kappa_v (b^{<_i^{s,v}} - b^{<_i})$$

is a non-negative combination for

$$\kappa_v = \frac{\chi_v^t - \sum_{v <_i w \leq_i t} \kappa_w (b^{<_i^{s,w}}(v) - b^{<_i}(v))}{b^{<_i^{s,v}}(v) - b^{<_i}(v)}$$

- By scaling, we get a convex combination.

Set $\epsilon := \min\{\lambda_i \delta, -x(t)\}$.

- If $\epsilon = \lambda_i \delta$ then:

$$x' = \sum_{j=1}^{k} \lambda_j b^{<_i} + \lambda_i \sum_{s <_i v \leq_i t} \kappa_v (b^{<_i^{s,v}} - b^{<_i})$$

$b^{<_i}$ cancels out.

- Otherwise, $x'(t) = 0$.

We can then use Gaussian elimination to get $\leq n$ vectors in $O(n^3)$. $\square$

Theorem 4.58. *The number of iterations is bounded by $O(n^5)$.*

Proof.

Claim. *$d(w)$ never decreases for $w \in U$.*

If (v, w) was added after a new vertex $b^{<_i^{s,u}}$ was added to the convex combination in ChangeSolution, then $s \leq_i w <_i v \leq t$ in that iteration. In particular $d(w) \leq d(s) + 1 = d(t) \leq d(v) + 1$, so adding the edge (v, w) does not decrease $d(w)$. Additionally, ChangeSolution does not add any elements to P which proves the claim.

We call a sequence of iterations with the same s and t a *block*. Each block has $O(n^2)$ iterations as the pair (α, β) decreases lexicographically.

Claim. *The number of blocks is bounded by $O(n^3)$.*

We consider different reasons for ending a block:

a) $d(v)$ increases for some $v \in U$, in which case v may become the new t or s.

b) t is removed from N.

c) (s, t) is removed from A.

We now bound the number of blocks of each type:

- The number of blocks of type a) is bounded by $O(n^2)$ since $d(w)$ never decreases.
- We claim that for all $t^* \in U$ there are at most $O(n^2)$ iterations with $t = t^*$ and $x'(t) = 0$: Between such iterations some $d(v)$ ($v \in U$) must change. We have just shown that this only happens $O(n^2)$ times. Since there are n choices for t^*, there are $O(n^3)$ blocks of type b).
- We claim that there are $O(n^3)$ types of type c). It suffices to show that $d(t)$ changes between 2 blocks with the pair (s, t). For $s, t \in U$, call s *t-boring* if one of the following holds:

  - $(s, t) \notin A$ *or*
  - $d(t) \leq d(s)$

  Let $s^*, t^* \in U$ and consider the time after a block $s = s^*, t = t^*$ is ending because (s^*, t^*) is removed from A until a subsequent increase of $d(t^*)$.

  We prove that each $v \in \{s^*, \ldots, n\}$ is t^*-boring during this period. At the beginning, each $v \in \{s^* + 1, \ldots, n\}$ is t^*-boring by the maximal choice of s^*. s^* is t^*-boring because the arc (s^*, t^*) was removed. As $d(t^*)$ remains constant and d never decreases, we only need to check the introduction of new arcs.

Suppose for $v \in \{s^*, \dots, n\}$, (v, t^*) is added in an iteration with pair (s, t). Then $s \leq_i t^* <_i v \leq_i t$, so $d(t^*) \leq d(s) + 1 = d(t) \leq d(v) + 1$.

Case 1: $s > v$. Then $d(t^*) \leq d(s)$, either because $s = t^*$ or s was t^*-boring and $(s, t^*) \in A$.

Case 2: $s < v$. Then $d(t) \leq d(v)$, either because $v = t$ or by choice of s and since $(v, t) \in A$.

In either case, we have one strict inequality, so $d(t^*) \leq d(v)$ and v remains t^*-boring as claimed.

$d(t)$ can increase $O(n)$ times and there are $O(n^2)$ pairs (s, t).

In total, the total number of iterations is:

$$O(n^5) = \underbrace{O(n^2)}_{\text{iterations per block}} \cdot \underbrace{O(n^3)}_{number of blocks}$$

□

Theorem 4.59. *The submodular function minimization problem can be solved in time $O(n^8 + n^7\gamma)$, where γ is the time required for a call to the function oracle.*

Corollary 4.60. *Linear functions over the intersection of 2 polymatroids can be optimized in polynomial time.*

Remark.

- The fastest known algorithm has a running time of $O(n^6 + n^5\gamma)$ (Orlin, 2009 and Sidford, Wong, Lee, 2015).
- There is also a weakly polynomial algorithm $O((n^5 + n^4\gamma)(\log M))$ where $M = \max_X f(X)$.

Remark. $[0,1]^n$ can be partitioned into $n!$ n-simplices (induced by the $n!$ orders on $\{1, \dots, n\}$). For each simplex, there exists a unique linear interpolation/extension of a function on the corners of the simplex to its interior. This corresponds to the definition of the Lovász extension.

In particular, a function is submodular $\Leftrightarrow$ the combination of the linear interpolations is convex.

5 Splitting-Off Lemma and Connectivity

5.1 Splitting-Off Lemma

Lemma 5.1 (Lovász). *Let G be a (multi-)graph with $V(G) = V \dot\cup \{s\}$ with $|\delta(s)|$ even and $k \geq 2$ such that:*

$$|\delta(U)| \geq k \qquad \forall \emptyset \neq U \subsetneq V \tag{2}$$

Then $\forall \{s,t\} \in E: \ \exists u \in \Gamma(s)$ such that

$$G' \coloneqq G - \{s,t\} - \{s,u\} + \{u,t\}$$

satisfies (2).

Remark. If $t = u$, then G' contains a loop which does not change the connectivity when it gets deleted.

Proof. If $|\Gamma(s)| = 1$, then the statement is clear since for all $U \subsetneq V$ with $t \in U$:

$$|\delta_G(U)| = \underbrace{|\delta_{G-s}(U)|}_{=|\delta_G(V \setminus U)| \geq k} \ |E[U, \{s\}]|$$

Therefore removing edges incident to s maintins the connectivity. Assume now that $|\Gamma(s)| > 1$. Fix $t \in \Gamma(s)$.

Claim. *We can find $u \in \Gamma(s) \setminus \{t\}$ such that G' satisfies (2).*

If not, then for all $u \in \Gamma(s)$ there exists $U \subsetneq V$ such that $|\delta_{G'}(U)| < k$. Then $t, u \in U$, else $|\delta_{G'}(U)| = |\delta_G(U)|$. Also, $|\delta_G(U)| \leq k+1$. Let:

$$\mathcal{C} \coloneqq \{U \subsetneq V \mid t \in U, \ |\delta_G(U)| \leq k+1\}$$

This covers $\Gamma(s)$. Then $\forall U \in \mathcal{C}$

$$1 \geq \underbrace{|\delta_G(U)|}_{\leq k+1} - \underbrace{|\delta_G(U \cup \{s\})|}_{\geq k} = |E(\{s\}, U)| - |E(\{s\}, V \setminus U)|$$

so $|E(\{s\}, U)| \leq |E(\{s\}, V \setminus U)| + 1$. Since $|\delta(s)|$ is even, there cannot be equality, so:

$$|E(\{s\}, U)| \leq |E(\{s\}, V \setminus U)|$$

Now $\{s,t\} \in E(\{s\}, U)$ for all $U \in \mathcal{C}$. In particular, we need > 2 sets from $\mathcal{C}$ to cover $\delta(s)$. Take $U_1, U_2, U_3 \in \mathcal{C}$ such that $U_1 \setminus (U_2 \cup U_3)$, $U_2 \setminus (U_1 \cup U_3)$, $U_3 \setminus (U_1 \cup U_2)$ are nonempty. Then

$$\begin{aligned}
&|\delta(U_1)| + |\delta(U_2)| + |\delta(U_3)| \\
&\geq |\delta(U_1 \cap U_2 \cap U_3)| + |\delta(U_1 \setminus (U_2 \cup U_3))| \\
&+ |\delta(U_2 \setminus (U_1 \cup U_3))| + |\delta(U_3 \setminus (U_1 \cup U_2))|
\end{aligned}$$

This "3-way submodularity" can be proved by considering how often each edge is counted on both sides. Actually, the left side is larger by 2 since the edge $\{s, t\}$ is counted three times here but only once on the right side.

Each term on the left side is at most $k+1$. Each term on the right is at least k. In total $3(k+1) \geq 4k+2$, so $k \leq 1$ in contradiction to the assumption. $\square$

5.2 Construction of $2k$-edge-connected graphs

Lemma 5.2. *Every minimal k-edge-connected (multi-)graph has a vertex of degree k.*

Proof. Let G be such a graph. Then every cut has at least k edges and every edge is part of a cut with (at most) k edges. Let $X \subsetneq V(G)$ be minimum set such that $|\delta(X)| = k$. If $|X| = 1$, we are done. Otherwise, by minimality $G[X]$ is connected. Let $e \in E(G[X])$, then $\exists T \subsetneq V(G)$ with $e \in \delta(T)$ and $|\delta(T)| = k$.

Case 1: $T \cup X = V(G)$. Then $|\delta(X \setminus T)| = |\delta(T)| = k$ in contradiction to the minimality of X.

Case 2: $T \cup X \neq V(G)$. Then $|\delta(X \cap T)| = k$ by submodularity of $|\delta(\cdot)|$:

$$|\delta(X)| + |\delta(T)| \geq |\delta(X \cap T)| + |\delta(X \cup T)|$$

This again contradicts the minimality of X.

$\square$

Theorem 5.3. *Let M_{2k} be a multigraph with 2 vertices joined by $2k$ edges. Any $2k$-edge-connected graph with at least 2 vertices can be built from M_{2k} by iteratively applying:*

1. *Adding edges (possibly loops)*
2. *Pinching k edges: Take k edges $(\{v_i, w_i\})_{i=1}^k$, add a new vertex s and replace each edge $\{v_i, w_i\}$ by $\{s, v_i\}$ and $\{s, w_i\}$ for $1 \leq i \leq k$.*

Proof. Start with any $2k$-edge-connected graph G. Then do 1. and 2. in reverse, i.e.:

1. Delete a maximal set of edges (while maintaining $2k$-edge-connectivity)
2. By lemma 5.2, there is a vertex s with $|\delta(s)| = 2k$. Split off s, as in lemma 5.1.

At the end, M_{2k} remains since both operations maintain $2k$-edge-connectivity. $\square$

Theorem 5.4 (Nash-Williams). *An undirected graph G is $2k$-edge-connected if and only if there is an orientation $\vec{G}$ of G that is k-edge-connected.*

Proof. If $\vec{G}$ is k-edge-connected, then each cut contains k outgoing and k incoming edges, so G is $2k$-edge-connected.

For the other implication, let G be $2k$-edge-connected. Take M_{2k} and orient k edges in each direction. Apply theorem 5.3 and preserve the orientation. This preserves k-edge-connectivity in the oriented graph. $\square$

Remark 5.5. Nash-Williams actually proved that each graph G has an orientation $\vec{G}$ for which $\lambda(x, y, \vec{G}) \geq \lfloor \frac{\lambda(x,y,G)}{2} \rfloor$ $\forall x, y \in V(G)$ where $\lambda(x, y, H)$ denotes the local edge-connectivity, so the number of edge-disjoint x-y-paths in H.

Remark (Lovász Extension). For the exercises regarding the Lovász extension, we need $f(\emptyset) = 0$ and monotonicity for the polymatroid definition that requires $x \geq 0$. Allowing for negative vectors in the polymatroid, the polymatroid greedy algorithm still works without monotonicity. Therefore also the Lovász identity works without assuming monotonicity.

5.3 Connectivity Augmentation

Problem (Connectivity Augmentation). Given a graph G and $k \geq 1$, find a minimum multiset F choses from $\{\{v, w\} \mid v, w \in V(G),\ v \neq w\}$ such that $G + F$ is a k-edge-connected graph.

Lemma 5.6. *Given a graph G and a degree requirement $x : V(G) \to \mathbb{N}$, there exists a multiset F chosen from $\{\{v, w\} \mid v, w \in V(G),\ v \neq w\}$ such that $G + F$ is k-edge-connected and*

$$|\delta_F(v)| + 2l(v)^7 = x(v) \qquad \textit{for some } l : V(G) \to \mathbb{N}$$

if and only if:

1. *$x(V(G))$ is even*
2. *$|\delta_G(U)| + x(U) \geq k$ for all $\emptyset \neq U \subsetneq V(G)$*

Proof.

"$\Rightarrow$": The sum $\sum_{v \in V(G)} |\delta_F(v)|$ is even independent of F which implies 1. Since $G + F$ is k-edge-connected, we also get 2.

"$\Leftarrow$": Add a new vertex s and $x(v)$ edges between v and s for all $v \in V(G)$, resulting in a new graph G'. Then $|\delta(s)|$ is even since $x(V(G))$ is even. Let $\emptyset \neq U \subsetneq V(G)$. Then $|\delta_{G'}(U)| = |\delta_G(U)| + x(U) \geq k$ by 2.

[7] one can think of this as loops that one is allowed to drop

Therefore G' is k-edge-connected. Now apply splitting-off to s. This preserves k-edge-connectivity. Choose $l(v)$ to be the number of loops created at $v \in V(G)$ by splitting off. Set F to be all new edges except for the loops.

□

Theorem 5.7 (Watanabe, Nakamura)**.** *A graph G can be augmented to a k-edge-connected graph by adding γ edges if and only if any collection $\mathcal{U}$ of disjoint proper subsets of $V(G)$ satisfies:*

$$\sum_{U \in \mathcal{U}} (k - |\delta_G(U)|) \leq 2\gamma$$

Proof.

"$\Rightarrow$": Each summand on the left measures how many edges are missing in $\delta(U)$ for k-edge-connectivity. Each edge that is added can be part of at most 2 such cuts since the elements of $\mathcal{U}$ are disjoint.

"$\Leftarrow$": We want to apply lemma 5.6 so we need to find a suitable degree constraint. Introduce degree constraints starting with $x(v) = k$ for all $v \in V(G)$. Decrease x arbitrarily while preserving:

$$x(U) \geq \max\{0, k - |\delta_G(u)|\} \qquad \forall \emptyset \neq U \subsetneq V(G)$$

Now for all $v \in V(G)$ with $x(v) > 0$ there exists a set $v \in U \subsetneq V(G)$ with $x(U) = k - |\delta_G(U)|$. Now:

- By definition of x, it satisfies condition 2 of lemma 5.6.
- If we show that $x(V(G)) \leq 2\gamma$ then $\sum_{v \in V(G)} |\delta_F(v)| \leq 2\gamma$, so $|F| \leq \gamma$.
- If $x(V(G))$ is odd then $x(V(G)) < 2\gamma$, so we can increase x on any vertex $v \in V(G)$ in order to restore 2.

Let $\mathcal{U} := \{U \subsetneq V(G) \mid x(U) = k - |\delta_G(U)|\,,\ U \text{ maximal}\}$ be the set of maximal tight subsets.

Case 1: There are $S, T \in \mathcal{U}$ with $S \cup T = V(G)$ and $S \neq T$. Then $(V(G) \setminus S) \cap (V(G) \setminus T) = \emptyset$, so

$$\begin{aligned} x(V(G)) &\leq x(S) + x(T) \\ &= (k - |\delta_G(V(G) \setminus S)|) + (k - |\delta_G(V(G) \setminus T)|) \\ &\leq 2\gamma \end{aligned}$$

Case 2: $S \cup T \neq V(G)$ for all $S, T \in \mathcal{U}$. Let $S, T \in \mathcal{U}$ with $S \cap T \neq \emptyset$. Then:

$$\begin{aligned}
x(S) + x(T) &= (k - |\delta_G(S)|) + (k - |\delta_G(T)|) \\
&\overset{\text{submodularity}}{\leq} (k - |\delta_G(S \cap T)|) + (k - |\delta_G(S \cup T)|) \\
&\leq x(S \cap T) + x(S \cup T) \\
&= x(S) + x(T)
\end{aligned}$$

We have equality everywhere, so $S \cap T$ and $S \cup T$ are tight. This contradicts the maximality of S or T.

Case 3: $S \cup T \neq V(G)$ and $S \cap T = \emptyset$ for all $S, T \in \mathcal{U}$. Since it contains all $v \in V(G)$, $\mathcal{U}$ is then a partition of $V(G)$ and:

$$\begin{aligned}
x(V(G)) &= \sum_{U \in \mathcal{U}} x(U) \\
&= \sum_{U \in \mathcal{U}} (k - |\delta_G(U)|) \\
&\leq 2\gamma
\end{aligned}$$

In all cases, $x(V(G)) \leq 2\gamma$. If $x(V(G))$ is odd, we increase $x(v)$ for some $v \in V(G)$ by 1. This maintains condition 2 (since 2γ is even) and restores condition 1 of lemma 5.6.

$\square$

Remark. The proof provides a polynomial time algorithm where splitting off and the x-calculation can be done by min-cut computations.

6 Survivable Network Design

Problem (SND)**.** Given an undirected graph G with weights $c : E(G) \to \mathbb{R}_{\geq 0}$ and connectivity requirements $r_{xy} \in \mathbb{Z}_{\geq 0}$ for each unordered pair $x, y \in V(G)$, find a minimum weight subgraph H such that for each x, y there are at least r_{xy} edge-disjoint paths from x to y in H.

Example 6.1 (Steiner Tree Problem). Given a graph G, $T \subseteq V(G)$ and $c : E(G) \to \mathbb{R}$ find a minimum weight edge set $F \subseteq E(G)$ such that all $t \in T$ are in the same connected component of $(V(G), F)$.

This can be reduced to SND by setting: $r_{xy} := \begin{cases} 1 & x, y \in T \\ 0 & \text{else} \end{cases}$

Example 6.2. For $r_{xy} = k$ for all $x, y \in V(G)$ we look for k-edge-connected subgraphs.

We now formulate an ILP description. Let $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ be defined by $f(\emptyset) = f(V(G)) = 0$ and $f(S) \coloneqq \max_{x \in S,\ y \in V(G) \setminus S} r_{x,y}$ for $\emptyset \neq S \subsetneq V(G)$. The SND problem can now be formulated as (SNDIP):

$$
\begin{aligned}
\min \sum_{e \in E(G)} c(e) x_e & \\
\text{s.t.} \sum_{e \in \delta(S)} x_e \geq f(S) & \qquad \forall S \subsetneq V(G) \\
x_e \in \{0,1\} &
\end{aligned}
$$

We also consider the relaxation SNDLP where we only require $x_e \in [0,1]$.

Definition 6.3. A function $f : 2^U \to \mathbb{R}_{\geq 0}$ is *proper* if it satisfies the following 3 constraints:

i) $f(S) = f(U \setminus S) \qquad \forall S \subseteq U$

ii) $f(A \cup B) \leq \max\{f(A), f(B)\} \qquad \forall A, B \subseteq U,\ A \cap B = \emptyset$

iii) $f(\emptyset) = 0$

$\Rightarrow$ the function of SND is proper.

Definition 6.4. A function $f : 2^U \to \mathbb{Z}_{\geq 0}$ is called *weakly supermodular* if for $A, B \subseteq U$:

i) $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$ *or*

ii) $f(A) + f(B) \leq f(A \setminus B) + f(B \setminus A)$

Proposition 6.5. *A proper function is weakly supermodular.*

Proof. As f is proper, we have:

(1) $f(A) \leq \max\{f(A \cap B), f(A \setminus B)\}$

(2) $f(B) \leq \max\{F(A \cap B), f(B \setminus A)\}$

(3) $f(A) = f(U \setminus A) \leq \max\{f(B \setminus A), f(U \setminus (B \cup A))\} = \max\{f(B \setminus A), f(B \cup A)\}$

(4) $f(B) \leq \max\{f(A \setminus B), f(A \cup B)\}$

Depending on which of $f(B \setminus A),\ f(A \setminus B),\ f(A \cup B),\ f(A \cap B)$ is minimum, we add those inequalities containing it. In either case, it follows that f is weakly supermodular. $\square$

Goal: Solve the separation problem for proper functions by Gomory-Hu trees. The crucial part is having a bound on $f(S)$.

Lemma 6.6. *Let G be an undirected graph, $u : E(G) \to \mathbb{R}_{\geq 0}$, $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ proper. Let H be a Gomory-Hu tree for (G, u). Then for $\emptyset \neq S \subsetneq V(G)$ we have:*

i) $\sum_{e' \in \delta_G(S)} u(e') \geq \max_{e \in \delta_H(S)} \sum_{e' \in \delta_G(C_e)} u(e')$

ii) $f(S) \leq \max_{e \in \delta_G(S)} f(C_e)$

Proof. i) follows directly from the Gomory-Hu tree property. For ii), let $X_1, \ldots, X_k$ be the connected components of $H - S$. Then for each $i \in [k]$ (where we choose C_e such that $C_e \cap X_i = \emptyset$):

$$V(H) \setminus X_i = \dot{\bigcup}_{e \in \delta_H(X_i)} C_e$$

Now

$$\begin{aligned} f(X_i) &= f(V(H) \setminus X_i) \\ &= f(\dot{\bigcup}_{e \in \delta_H(X_i)} C_e) \\ &\overset{f \text{ proper}}{\leq} \max_{e \in \delta_H(X_i)} f(C_e) \end{aligned}$$

so:

$$\begin{aligned} f(S) &= f(V(H) \setminus S) \\ &= f(\dot{\bigcup}_{i \in [k]} X_i) \\ &\overset{f \text{ proper}}{\leq} \max_{i \in [k]} f(X_i) \\ &\leq \max_{e \in \delta_H(S)} f(C_e) \end{aligned}$$

$\square$

Theorem 6.7. *Let G be an undirected graph, $x \in \mathbb{R}_{\geq 0}^{E(G)}$ and $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ proper. Then we can find in $O(n^4 + n\theta)$ a set $S \subseteq V(G)$ with*

$$\sum_{e \in \delta_G(S)} x_e < f(S)$$

or decide that no such set exists.

Proof. Compute a Gomory-Hu tree H for (G, x). For each $\emptyset \neq S \subsetneq V(G)$, there is $e \in \delta_H(S)$ with $f(S) \leq f(C_e)$ by part ii) of lemma 6.6. By part i) of the lemma, $f(S) - x(\delta_G(S)) \leq f(C_e) - x(\delta_G(C_e))$. Since $\emptyset \neq C_e \subsetneq V(G)$:

$$\max_{\emptyset \neq S \subsetneq V(G)} f(S) - x(\delta_G(S)) = \max_{e \in E(H)} f(C_e) - x(\delta_G(C_e))$$

In particular, it suffices to check the inequality for fundamental cuts of the Gomory-Hu tree. □

Remark. With theorem 6.7, we can check whether there exists an integral feasible solution (and compute it).

6.1 Jain's Iterative LP Rounding

Idea: We can now solve the survivable network design LP. We round up edges with $x_e \geq \frac{1}{2}$. Fix them to 1 and compute the LP solution.

Part a: The rounding gives a 2-approximation

Part b: There is always an $e \in E(G)$ with $x_e \geq \frac{1}{2}$ in the LP solution.

6.1.1 Iterative Rounding

Let x^* be an optimum solution of (SNDLP) and $E_{\geq \frac{1}{2}}$ the set of edges with $x_e \geq \frac{1}{2}$. Consider $G_{\text{res}} \coloneqq G - E_{\geq \frac{1}{2}}$ and adjust (SNDLP):

$$\begin{aligned} \min \sum_{e \in E(G_{\text{res}})} & c(e) x_e \\ \text{s.t.} \sum_{e \in \delta_{G_{\text{res}}}(S)} & x_e \geq f(S) - \left| E_{\geq \frac{1}{2}} \cap \delta_G(S) \right| \qquad S \subseteq V(G) \end{aligned}$$

Remark. This is equivalent to fixing $x_e = 1$ for all $e \in E_{\geq \frac{1}{2}}$. In particular, we can still separate the inequalities using theorem 6.7.

Theorem 6.8. *Let z^* and z^*_{res} be the optimum values for (SNDLP) and its restriction to G_{res}. Let E_{res} be an integral solution of the restriction with $c(E_{\text{res}}) \leq 2z^*_{\text{res}}$. Then $E_{\text{res}} \cup E_{\geq \frac{1}{2}}$ is an integral solution to (SNDLP) with $c(E_{\text{res}} \cup E_{\geq \frac{1}{2}}) \leq 2z^*$.*

Proof. $E_{\text{res}} \cup E_{\geq \frac{1}{2}}$ is clearly a feasible solution to (SNDLP). x^* is an optimum solution to (SNDLP) (with value z^*). Its restriction to G_{res} is feasible, so:

$$\begin{aligned}
z^*_{\text{res}} &\le z^* - \sum_{e \in E_{\ge \frac{1}{2}}} c(e) x^*_e \\
\Leftrightarrow 2z^* &\ge 2z^*_{\text{res}} + \sum_{e \in E_{\ge \frac{1}{2}}} 2c(e) x^*_e \\
&\ge 2z^*_{\text{res}} + \sum_{e \in E_{\ge \frac{1}{2}}} c(e) \\
&\ge \sum_{e \in E_{\text{res}}} c(e) + \sum_{e \in E_{\ge \frac{1}{2}}} c(e) \\
&= c(E_{\text{res}} \cup E_{\ge \frac{1}{2}})
\end{aligned}$$

□

Algorithm 11: Jain's Algorithm

Input: Graph G, weights $c : E(G) \to \mathbb{R}_{\ge 0}$ and $f : 2^{V(G)} \to \mathbb{Z}_{\ge 0}$ proper
Output: An integral solution to (SNDLP)

```
E_sol ← ∅, f' ← f, G' ← G
repeat
    Find an optimum basis solution x* of (SNDLP) for (G', f')
    Add all edges e with x_e ≥ 1/2 to E_sol
    G' ← G − E_sol, f'(S) ← f(S) − |E_sol ∩ δ_G(S)| for S ⊆ V(G)
until x* = 0
return E_sol
```

Theorem 6.9. *Let x be a basic feasible solution to (SNDLP). Then there exists an edge $e \in E(G)$ with $x_e \ge \frac{1}{2}$.*

6.1.2 Uncrossing

Goal: Find a large family of laminar sets, each with a significant f-value and tight constraint.

We can assume that there is no edge e with $x_e = 0$. If there exists $x_e \ge \frac{1}{2}$, we are done, so assume $x_e \in (0, \frac{1}{2})$. Call $A \subsetneq V(G)$ *tight* if $x(\delta(A)) = f(A)$. Let $\mathcal{A}(A)$ be the row in the constraint matrix induced by A.

Lemma 6.10. *For 2 tight sets A, B one of the following holds:*

1. *$A \setminus B$ and $B \setminus A$ are tight and $\mathcal{A}(A) + \mathcal{A}(B) = \mathcal{A}(A \setminus B) + \mathcal{A}(B \setminus A)$*
2. *$A \cap B$ and $A \cup B$ are tight and $\mathcal{A}(A) + \mathcal{A}(B) = \mathcal{A}(A \cap B) + \mathcal{A}(A \cup B)$*

Proof. Let $S_1 := A \setminus B,\ S_2 := A \cap B,\ S_3 := B \setminus A,\ S_4 := V(G) \setminus (A \cup B)$. By tightness, we have:

$$\begin{aligned} f(A) &= x(E(S_1,S_3)) + x(E(S_1,S_4)) + x(E(S_2,S_3)) + x(E(S_2,S_4)) \\ f(B) &= x(E(S_1,S_2)) + x(E(S_1,S_3)) + x(E(S_4,S_2)) + x(E(S_4,S_3)) \end{aligned}$$

By feasibility:

$$\begin{aligned} f(A \setminus B) = f(S_1) &\le x(E(S_1,S_2)) + x(E(S_1,S_3)) + x(E(S_1,S_4)) \\ f(B \setminus A) = f(S_3) &\le x(E(S_1,S_3)) + x(E(S_2,S_3)) + x(E(S_4,S_3)) \end{aligned}$$

As f is weakly supermodular, we have:

$$\begin{aligned} f(A) + f(B) &\le f(A \setminus B) + f(B \setminus A) \qquad \textit{or} \\ f(A) + f(B) &\le f(A \cap B) + f(A \cup B) \end{aligned}$$

We only consider the first case (the second case is similar). By adding the above inequalities and comparing the terms, we see $2x(E(S_2,S_4)) \le 0$. Since $x > 0$ (by assumption), $E(S_2,S_4) = \emptyset$. In particular,

$$\mathcal{A}(A) + \mathcal{A}(B) = \mathcal{A}(A \setminus B) + \mathcal{A}(B \setminus A)$$

□

Let $\mathcal{T}$ be the family of tight sets. For a family $\mathcal{F} \subseteq \mathcal{T}$, define $\operatorname{span}(\mathcal{F}) := \operatorname{span}(\{\mathcal{A}(S) \mid S \in \mathcal{F}\})$.

Lemma 6.11. *For any maximal laminar family $\mathcal{L} \subseteq \mathcal{T}$ of tight sets,* $\operatorname{span}(\mathcal{L}) = \operatorname{span}(\mathcal{T})$.

Using the lemma, we can take any basis $\mathcal{B} \subseteq \mathcal{L}$. Since we assumed that $x_e \in (0,1)$, we get $\dim \operatorname{span}(T) = |E(G)|$.

Proof of lemma. "$\subseteq$" is clear. If the other inclusion doesn't hold, there exists $S \in \mathcal{T}$ with $\mathcal{A}(S) \notin \operatorname{span}(\mathcal{L})$. Choose S such that it crosses a minimum number of sets in $\mathcal{L}$ (since $\mathcal{L}$ is maximal, S crosses some set in $\mathcal{L}$). Let $L \in \mathcal{L}$ cross S. By lemma 6.10, we have either:

1. $S \setminus L$ and $L \setminus S$ are tight and $\mathcal{A}(S) + \mathcal{A}(L) = \mathcal{A}(S \setminus L) + \mathcal{A}(L \setminus S)$ or
2. $S \cup L$ and $S \cap L$ are tight and $\mathcal{A}(S) + \mathcal{A}(L) = \mathcal{A}(S \cup L) + \mathcal{A}(S \cap L)$.

Case 1: 1. holds, so $S \setminus L$ and $L \setminus S$ are tight.

Case 1.1 $\mathcal{A}(S \setminus L) \notin \operatorname{span}(\mathcal{L})$.

Claim. *If $L' \in \mathcal{L}$ crosses $S \setminus L$, then L' also crosses S. In particular (since $S \setminus L$) doesn't cross L we get a contradiction to the minimality of S.*

We get $(S \setminus L) \cap L' \neq \emptyset$, so $S \cap L' \neq \emptyset$ and $L' \setminus L \neq \emptyset$. Additionally $(S \setminus L) \setminus L' \neq \emptyset$ and $(S \setminus L') \neq \emptyset$. Since $L, L' \in \mathcal{L}$, we have $L \subseteq L'$ or $L \cap L' = \emptyset$.

Claim. *In both cases $L' \setminus S \neq \emptyset$.*

If $L \subseteq L'$, $\emptyset \neq L \setminus S \subseteq L' \setminus S$. If $L \cap L' = \emptyset$, then $\emptyset \neq L' \setminus (S \setminus L) = L' \setminus S$.

□

Lemma 6.12. *Given a basic optimum LP solution $0 < x < 1$, there exists a laminar family $\mathcal{B}$ with:*

1. $|\mathcal{B}| = |E(G)|$
2. $\{\mathcal{A}(B) \mid B \in \mathcal{B}\}$ *are linearly independent.*
3. $\dim \operatorname{span}(\mathcal{B}) = |E(G)|$
4. $f(B) \geq 1$ *for all* $B \in \mathcal{B}$

Proof. Since x is a basic solution, $\dim(\operatorname{span}(\mathcal{T})) = |E(G)|$. Choose $\mathcal{B}$ by lemma 6.11 such that 1. 2. and 3. are satisfied. $f(B) < 0$ is impossible for a (tight) set $B \in \mathcal{B}$. If $f(B) = 0$, then $x(\delta(B)) = 0$, so $\mathcal{A}(B) = 0$ which is also impossible since $\mathcal{B}$ is a basis. □

We want to show that there exists $x_e \geq \frac{1}{2}$. The idea is to assign a token to each set in $\mathcal{B}$ for each edge in its cut.

If not, $0 < x < \frac{1}{2}$. Let F be the branching representing $\mathcal{B}$ (i.e. $V(F) = \mathcal{B}$). Define the half complement $y_e \coloneqq \frac{1}{2} - x_e \in (0, \frac{1}{2})$ and

$$\begin{aligned}
\operatorname{coreq}(S) &\coloneqq y(\delta(S)) \\
&= \frac{1}{2} |\delta_G(S)| - x(\delta_G(S)) \\
&\overset{\text{tightness}}{=} \frac{1}{2} |\delta_G(S)| - f(S)
\end{aligned}$$

Proposition 6.13. *For $S \in \mathcal{T}$,* $\operatorname{coreq}(S) \in \mathbb{Z} + \frac{1}{2}$. *Additionally,* $\operatorname{coreq}(S) \notin \mathbb{N}$ *if and only if $|\delta_G(S)|$ is odd.*

Lemma 6.14 (Vazirani)**.** *Suppose $S \in V(F)$ with α children, all of which have a corequirement of $\frac{1}{2}$ and S has β tokens such that $\alpha + \beta = 3$. Then* $\operatorname{coreq}(S) = \frac{1}{2}$.

Proof. Since each child C has a corequirement of $\frac{1}{2}$, $|\delta(C)|$ is odd. Since $\alpha + \beta = 3$, we can show by case enumeration that $|\delta(S)|$ is odd. We get $\mathrm{coreq}(S) \notin \mathbb{N}$. It suffices to show $\mathrm{coreq}(S) < \frac{3}{2}$:

$$\begin{aligned}\mathrm{coreq}(S) &= y(\delta(S)) \\ &\leq \sum_{C \text{ child of } S} \mathrm{coreq}(S) + \sum_{\substack{e=\{x,y\}\in\delta(S) \\ \text{token was donated to} \\ x\in S}}\end{aligned}$$

If $\beta \geq 1$, we are done. If $\beta = 0$, then $\alpha = 3$, so there must be an edge between 2 children and $y(\delta(S)) < \sum_C \mathrm{coreq}(C)$. □

Lemma 6.15. *If $S \in V(F)$ has one child C, then S must own at least 2 tokens.*

Proof. S owns at least 1 token, otherwise $\mathcal{A}(S) = \mathcal{A}(C)$. If S owns 1 token, then $\mathbb{N} \ni |f(S) - f(C)| = |x(\delta(S)) - x(\delta(C))| \in (0, \frac{1}{2})$ which is a contradiction. □

Lemma 6.16. *If $S \in V(F)$ has 2 children C_1, C_2 with* $\mathrm{coreq}(C_1) = \frac{1}{2}$ *then S must own a token.*

Proof. If not, $\mathcal{A}(S), \mathcal{A}(C_1), \mathcal{A}(C_2)$ are linearly independent, so $\delta(C_1) \subseteq \delta(S)$ and $\delta(C_1) \subseteq \delta(C_2)$ are impossible. Let:

$$\begin{aligned} a &\coloneqq y(\delta(S) \cap \delta(C_2)) > 0 \\ b &\coloneqq y(\delta(C_1) \cap \delta(C_2)) > 0 \end{aligned}$$

Then $a + b = \mathrm{coreq}(C_1) = \frac{1}{2}$. Then $|\delta(C_1)|$ is odd, so $|\delta(S)| \equiv |\delta(C_2)| \mod 2$.

$\mathrm{coreq}(S) - \mathrm{coreq}(C_2) = a - b$. $-\frac{1}{2} < a - b < \frac{1}{2}$, so $\mathrm{coreq}(S) = \mathrm{coreq}(C_2)$ which is a contradiction. □

Lemma 6.17. *Let $0 < x < \frac{1}{2}$ be a basic optimum solution to (SNDLP). Consider a subarborescence rooted at $R \in V(F)$. Then we can redistribute tokens such that R gets at least 3 tokens and each proper descendant of R gets 2 tokens. If* $\mathrm{coreq}(R) \neq \frac{1}{2}$, *$R$ gets ≥ 4 tokens.*

Proof. Proceed by induction on the height of the subarborescence. For leaves, this holds. Let $\mathrm{surplus}(S) \coloneqq \#\text{assigned tokens} - 2$. Let R not be a leaf:

Case 1: R has 4 children. Then we can simply move up tokens from its children.

Case 2: R has 3 children. If one has a surplus of ≥ 2, we are done. Otherwise, they all have a surplus of 1, so a corequirement of $\frac{1}{2}$, so $\text{coreq}(R) = \frac{1}{2}$.

Case 3: R has 2 children. If both have a surplus of at least 2, we are done. Otherwise, one child C_1 has a surplus of 1, so a corequirement of $\frac{1}{2}$. Then by the previous lemma, R owns a token. If both children of a surplus of 1, then we again get $\text{coreq}(R) = \frac{1}{2}$.

Case 4: R has 1 child. Then S owns 2 tokens. This works the same way as the previous case (but using a different lemma).

□

This lemma then proves Theorem 6.9.