

# Combinatorial Optimization

Dozent: Stephan Held

January 24, 2023

## Contents

<b>0</b>	<b>Organization</b>	<b>3</b>
<b>1</b>	<b>Matchings</b>	<b>3</b>
1.1	Introduction . . . . .	3
1.2	Bipartite Matching . . . . .	5
1.3	The Tutte Matrix & Randomized Matching . . . . .	6
1.4	Tutte's Matching Theorem . . . . .	7
1.5	Ear Decompositions of Factor-Critical Graphs . . . . .	9
1.6	Edmond's Matching Algorithm . . . . .	12
1.6.1	Growing forest - $O(n^3)$ . . . . .	14
1.7	Gallai-Edmonds Decomposition . . . . .	19
1.8	Minimum Weight Perfect Matching . . . . .	20
1.8.1	The Matching Polytope . . . . .	23
<b>2</b>	<b><math>T</math>-Joins and <math>b</math>-Matchings</b>	<b>28</b>
2.2	$T$ -Join Applications . . . . .	29
2.2.1	TSP Approximation . . . . .	29
2.2.2	Shortest Paths in Undirected Graphs . . . . .	30
2.2.3	Chinese Postman Problem . . . . .	30
2.3	$T$ -Joins and $T$ -Cuts . . . . .	31
2.3.1	$T$ -join Polytope . . . . .	34
2.4	Excursus: Gomory-Hu Trees . . . . .	34
2.5	Finding Minimum-Capacity $T$ -Cuts . . . . .	36
2.6	$b$ -Matchings . . . . .	37
2.7	Padberg-Rao Theorem . . . . .	40
<b>3</b>	<b>The TSP Polytope</b>	<b>42</b>
3.1	The Spanning Tree Polytope . . . . .	42
3.2	The Held-Karp Polytope . . . . .	45
3.3	Further Inequalities for the TSP . . . . .	46

<b>4</b>	<b>Matroids &amp; Generalizations</b>	<b>47</b>
4.1.1	Matroid Constructions . . . . .	51
4.2	Matroid Intersection . . . . .	51
4.4	Polymatroids . . . . .	51
4.4.1	Applications of Matroid Intersection . . . . .	57
4.5	Submodular Function Maximization . . . . .	57
4.5.2	Randomized USM . . . . .	59
4.6	Submodular Function Minimization . . . . .	61
<b>5</b>	<b>Splitting-Off Lemma and Connectivity</b>	<b>66</b>
5.1	Splitting-Off Lemma . . . . .	66
5.2	Construction of $2k$ -edge-connected graphs . . . . .	67

## 0 Organization

- Prerequisites
  - Basic knowledge of graph algorithms
  - Linear Programming (LP Duality)
  - Programming skills in C++
- Exam
  - Qualification requires 50% of the points in theoretical & programming exercises
  - Oral exam
- Books
  - "Combinatorial Optimization", Korte & Vygen
  - "Understanding & Using Linear Programming", B. Gärtner, J. Matousek
  - Skript (theorems & definitions)
  - Further book recommendations are on the website

## 1 Matchings

### 1.1 Introduction

**Definition 1.1.**

1. A *matching*  $M$  in a graph  $G = (V, E)$  is a set of pairwise disjoint edges, i.e. they don't have a common endpoint.  
 $\nu(G) := \max.$  cardinality of a matching in  $G$
2. An *edge cover*  $C$  of a graph  $G = (V, E)$  is a subset of  $E$  s.t.  $V = \bigcup_{e \in C} e$ .  
 $\zeta(G) := \min.$  cardinality of an edge cover in  $G$
3. A matching is called *perfect* (or *1-factor*) if it is an edge cover
4.  $v \in V$  with  $v \in e \in M$  is called  *$M$ -covered*
5.  $v \in V$  is called  *$M$ -exposed* if it is not  $M$ -covered

**Definition 1.2.**

1. A *stable set* (independent set)  $S$  is a set of pairwise non-adjacent vertices.

$\alpha(G) := \max.$  cardinality of a stable set

2. A *vertex cover*  $C$  is a subset of  $V$  s.t.  $E = \bigcup_{\{x,y\} \in E, x \in C} \{x, y\}$

$\tau(G) := \min.$  cardinality of a vertex cover

**Lemma 1.3.**

1.  $\alpha(G) + \tau(G) = |V|$

2.  $\nu(G) + \zeta(G) = |V|$  if  $G$  has no isolated vertices

3.  $\zeta(G) = \alpha(G)$  if  $G$  is bipartite and has no isolated vertices

**Problem.** Cardinality Matching Problem

Input: Graph  $G = (V, E)$

Task: Find a maximum cardinality matching

**Problem.** Maximum Weight Matching Problem (MWMP)

Input: Graph  $G, c : E \rightarrow \mathbb{R}$

Task: Find a matching  $M$  maximizing  $c(M)$

**Problem.** Minimum Weight Perfect Matching (MWMPMP)

Input: Graph  $G, c : E \rightarrow \mathbb{R}$

Task: Find a perfect matching of minimum weight or decide that no perfect matching exists in  $G$

**Lemma 1.4.** *The MWMP is equivalent to the MWMPMP (i.e. there exists a transformation with linear complexity)*

*Proof.* Given a MWMPMP instance  $(G, c)$ , define  $c' := K - c$  ( $K := 1 + \sum_{e \in E} |c(e)|$ ).

$\Rightarrow$  Any maximum weight matching is a maximum cardinality matching

Given a MWMP instance  $(G, c)$ , define  $G'$  as 2 copies of  $G$  where the 2 copies of a vertex are joined by an edge.

$\Rightarrow G'$  has a perfect matching. Define:

$$c'(e) := \begin{cases} -c(e) & \text{if } e \text{ is in the first copy} \\ 0 & \text{else} \end{cases}$$

A minimum weight perfect matching in  $G'$  gives us a maximum weight matching in  $G$ .  $\square$

**Definition 1.5.** Let  $G = (V, E)$  be a graph and  $M \subseteq E$  a matching in  $G$ . A path  $P$  is *M-alternating* if its edges are alternatingly in and not in  $M$ . If both end points of this path are *M-exposed*,  $P$  is an *M-augmenting* path.



Figure 1: Example of the construction in Theorem 1.8

**Lemma 1.6.** *Given a matching  $M$  in  $G$  and an inclusion-wise maximal  $M$ -alternating path  $P$ ,*

$$M \Delta P := M \setminus P \cup P \setminus M$$

*is a matching. If  $P$  is  $M$ -augmenting, then  $|M \Delta P| = |M| + 1$ .*

**Theorem 1.7** (Petersen 1891, Berge 1957). *Augmenting Path Theorem*  
*Given a graph  $G = (V, E)$  and a matching  $M$  in  $G$ :*

$$|M| = \nu(G) \Leftrightarrow \nexists M\text{-augmenting path } P \text{ in } G$$

*Proof.*

" $\Rightarrow$ ": Clear

" $\Leftarrow$ ": Assume there exists a matching  $M'$  with  $|M'| > |M|$ . Let  $G' := (V, M \Delta M')$ .

$$\Rightarrow |\delta_{G'}(v)| \leq 2 \quad \forall v \in V$$

$\Rightarrow G'$  is the union of disjoint circuits and paths

$\Rightarrow$  all circuits are even and have the same number of edges from  $M$  and  $M'$

$\Rightarrow \exists$  a path  $P$  in  $G'$  starting and ending with an edge in  $M'$

$\Rightarrow P$  is an alternating path

□

## 1.2 Bipartite Matching

**Theorem 1.8** (König 1931). *If  $G$  is bipartite, then  $\nu(G) = \tau(G)$*

*Proof.* Add vertices  $s$  and  $t$  edges between them to all vertices of the respective partition. Direct all edges from  $s$  to  $t$ . Then  $\nu(G)$  is maximum number of disjoint  $s$ - $t$ -paths. Menger  $\Rightarrow$  This is equal to the minimum number of vertices that disconnect  $t$  from  $s$ . □

**Theorem 1.9** (Hall 1935). *Let  $G = (A \dot{\cup} B, E)$  be a bipartite graph. Then:*

$$G \text{ has a matching covering } A \Leftrightarrow |\Gamma(X)| \geq |X| \quad \forall X \subseteq A$$

**Corollary 1.10.** *Marriage Theorem*

$$|\Gamma(X)| \geq |X| \quad \forall X \subseteq A \text{ and } |A| = |B| \Leftrightarrow G \text{ has a perfect matching}$$

**Definition 1.12.** The MWPMP for bipartite graphs is called *Assignment Problem*.

**Theorem 1.13.** *The Assignment Problem can be solved in time  $O(nm + n^2 \log m)$ .*

*Proof.* Use the Successive Shortest Paths algorithm in an auxiliary graph.  $\square$

### 1.3 The Tutte Matrix & Randomized Matching

**Definition 1.14.** Let  $G$  be a simple, undirected graph. Let  $G'$  be an orientation of  $G$  and  $(X_e)_{e \in E(G)}$ . The *Tutte matrix* is defined as

$$T_G(X) := (t_{vw}^*)_{v,w \in V(G)}$$

where

$$t_{vw}^* := \begin{cases} X_{\{v,w\}} & \text{if } (v,w) \in E(G) \\ -X_{\{v,w\}} & \text{if } (w,v) \in E(G) \\ 0 & \text{else} \end{cases}$$

*Remark 1.15.*  $T_G(X)$  is skew-symmetric (i.e.  $T_G(X) = -(T_G(X))^t$ ).  $\text{rank}(T_G(X))$  is independent of the orientation of  $G$ .  $\det(T_G(X))$  is a polynomial in  $X$ .

**Theorem 1.16** (Tutte). *A simple graph  $G$  has a perfect matching  $\Leftrightarrow \det(T_G(X)) \neq 0$*

*Proof.* Let  $V(G) = \{v_1, \dots, v_n\}$  and  $S_n$  be the permutation group.

$$\det T_G(X) = \sum_{\pi \in S_n} \text{sgn} \pi \cdot \prod_{i=1}^n t_{v_i, v_{\pi(i)}}^*$$

Let  $S'_n := \{\pi \in S_n \mid \prod_{i=1}^n t_{v_i, v_{\pi(i)}}^* \neq 0\}$ . Each  $\pi \in S_n$  corresponds to a digraph  $H_\pi := (V(G), \{(v_i, v_{\pi(i)}) \mid i \in [n]^1\})$ . We have  $|\delta^+(v)| = 1 = |\delta^-(v)| \quad \forall v \in V(H_\pi) \Rightarrow H_\pi$  is the union of disjoint circuits. If  $\pi \in S'_n$ , then  $H_\pi \subset \overset{\Leftrightarrow}{G'}$ .

If there exists  $\pi \in S'_n$  s.t.  $H_\pi$  is a collection of even circuits, then this immediately yields a perfect matching in  $G$  (using every second edge of each circuit).

---

<sup>1</sup>This is an abbreviation for  $\{1, \dots, n\}$ .

Otherwise,  $\forall \pi \in S'_n$ ,  $H_\pi$  contains an odd circuit. Let  $r(\pi) \in S'_n$  arise from  $\pi$  by reversing edges on the unique odd circuit containing a vertex with minimum index  $\Rightarrow r(r(\pi)) = \pi$  and  $\text{sgn}(\pi) = \text{sgn}(r(\pi))$ . The second part is true since for reversing an odd cycle, we need an even number of swaps. Let  $v_{i_1}, \dots, v_{i_{2k+1}}$  be the "first" odd circuit. Then  $r(\pi)$  is attained by  $2k$  swaps: For  $j = 1, \dots, k$  swap  $(\pi(i_{2j-1}), \pi(i_{2k}))$  and  $(\pi(i_{2j}), \pi(i_{2k+1}))$ .

$\prod_{i=1}^n t_{v_i v_{\pi(i)}}^* = - \prod_{i=1}^n t_{v_i v_{r(\pi(i))}}^*$  since there is an odd number of sign changes to  $t^*$ .  $\Rightarrow \det(T_G(X)) = 0$ . We have shown that if  $G$  has no perfect matching, then  $\det T_G(X) = 0$ .

Assume that  $G$  has a perfect matching  $M$ . Define  $\pi$  as  $\pi(i) = j, \pi(j) = i$  where  $\{i, j\} \in M$ .  $\Rightarrow \prod_{i=1}^n t_{v_i v_{\pi(i)}}^* = \prod_{e \in M} -X_e^2$  cannot be canceled out. In particular,  $\det T_G(X) \neq 0$ .  $\square$

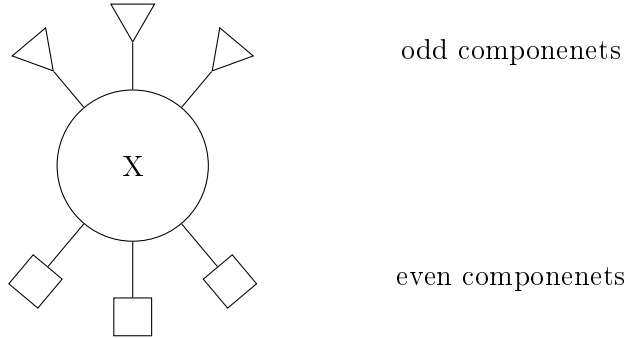
*Remark 1.17.* Picking  $X' \in [0, 1]^{E(G)}$  at random, we almost surely have (since the zero set of a non-zero polynomial is a set of measure zero):

$$\det T_G(X') \neq 0 \Leftrightarrow G \text{ has a perfect matching}$$

**Theorem 1.18** (Lovász 1979). *Let  $G$  be a simple graph and  $X \in [0, 1]^{E(G)}$  chosen randomly. Then almost surely  $\text{rank}(T_G(X)) = 2\nu(G)$ .*

#### 1.4 Tutte's Matching Theorem

Let  $X \subseteq V(G)$ .  $G - X$  consists of even and odd (in terms of the number of vertices) connected components. We define  $q_G(X)$  to be the number of odd components in  $G - X$ .



**Definition 1.19.** A graph  $G$  satisfies the *Tutte Condition* if  $q_G(X) \leq |X|$  for all  $X \subseteq V(G)$ .  $\emptyset \neq X \subseteq V(G)$  is called *barrier* if  $q_G(X) = |X|$ .

**Proposition 1.20.** *For any graph  $G$  and any  $X \subseteq V(G)$ :*

$$q_G(X) - |X| \equiv |V(G)| \pmod{2}$$

**Definition 1.21.** A graph  $G$  is *factor-critical* if  $G - v$  has a perfect matching for all  $v \in V(G)$ . A matching is called *near-perfect* if it covers  $|V(G)| - 1$  vertices.

**Proposition 1.22.** *If  $G$  is factor-critical, then it is connected.*

**Theorem 1.23** (Tutte 1947). *A graph  $G$  has a perfect matching  $\Leftrightarrow$  Tutte Condition holds (i.e.  $q_G(X) \leq |X| \ \forall X \subseteq V(G)$ )*

*Proof.*

" $\Rightarrow$ ": Clear

" $\Leftarrow$ ": We proceed by induction on  $|V(G)|$ . The case  $|V(G)| = 2$  is clear.

Generally, if the Tutte Condition holds, then  $|V(G)|$  must be even (pick  $X = \emptyset$ ). Proposition 1.20  $\Rightarrow q_G(X) - |X|$  is even. Every  $x \in V(G)$  induces a barrier  $\{x\}$ . Let  $X$  be a maximum barrier. Then  $G - X$  doesn't have any even components (since otherwise a single vertex of such a component could be added to  $X$ ).

**Claim:** Each odd component is factor-critical.

Let  $C$  be an odd component in  $G - X$ ,  $v \in V(C)$ . Assume that  $C - v$  does not have a perfect matching. Induction Hypothesis  $\Rightarrow C - v$  violates Tutte Condition.

$\Rightarrow \exists Y \subseteq V(C - v) : q_{C-v}(Y) > |Y|$

$\stackrel{1.20}{\Rightarrow} q_{C-v}(Y) \geq |Y| + 2$

Observe  $X \cap \{v\} = Y \cap \{v\} = X \cap Y = \emptyset$ :

$$\begin{aligned} q_G(X \cup Y \cup \{v\}) &= q_G(X) - 1 + q_C(Y \cup \{v\}) \\ &= |X| - 1 + q_{C-v}(Y) \\ &\geq |X| - 1 + |Y| + 2 \\ &= |X \cup Y| + 1 \\ &= |X \cup Y \cup \{v\}| \end{aligned}$$

$\Rightarrow X \cup Y \cup \{v\}$  is a barrier

$\Rightarrow$  Claim

Let  $G'$  arise from  $G$  by contracting each odd component into a single vertex. We have  $V(G') = X \dot{\cup} Z$  and  $G'$  is bipartite. We have to show that  $G'$  has a perfect matching. If not, then  $\exists A \subseteq Z : |\Gamma_{G'}(A)| < |A|$   
 $\Rightarrow q_G(\Gamma_{G'}(A)) \geq |A| > |\Gamma_{G'}(A)|$  which contradicts the Tutte Condition.

□



**Theorem 1.24** (Berge 1958).

$$|V(G)| = 2\nu(G) + \max_{X \subseteq V(G)} (q_G(X) - |X|)$$

*Proof.* For  $X \subseteq V(G)$ , any matching has at least  $q_G(X) - |X|$  uncovered vertices, so " $\geq$ " holds.

For the other inequality, add  $k := \max_{X \subseteq V(G)} (q_G(X) - |X|)$  new vertices and connect them to all existing vertices, yielding a new graph  $H$ .

We claim that  $H$  has a perfect matching. This then implies:

$$2\nu(G) + k \geq 2\nu(H) - k = |V(H)| - k = |V(G)|$$

Assume that  $H$  does not have a perfect matching. Then by Tutte's Theorem, there exists  $Y \subseteq V(H)$  with  $q_H(Y) > |Y|$ . By 1.20,  $k \equiv |V(G)| \pmod{2}$ , therefore  $|V(H)|$  is even, so  $Y \neq \emptyset$ .  $Y$  must contain all new vertices, otherwise  $H - Y$  would be connected<sup>2</sup> and  $q_H(Y) \leq 1 \leq |Y|$ .

$$\Rightarrow q_G(Y \cap |V(G)|) = q_H(Y) > |Y| = |Y \cap V(G)| + k$$

which is a contradiction to the choice of  $k$ . □

## 1.5 Ear Decompositions of Factor-Critical Graphs

**Definition 1.25.** Let  $G$  be a graph. An *ear decomposition* of  $G$  is a sequence  $r, P_1, \dots, P_k$  with  $G = (r, \emptyset) + P_1 + \dots + P_k$  such that each  $P_i$  is either a path with exactly the endpoints located in  $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$  or a circuit where exactly one of the vertices belongs to  $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$ .

$P_1, \dots, P_k$  are called *ears*. If  $|V(P_1)| \geq 3$  and  $P_2, \dots, P_k$  are paths we call it a *proper ear decomposition*.

**Theorem 1.27** (Whitney 1932). *Let  $G$  be an undirected graph. Then:*

$$G \text{ 2-connected} \Leftrightarrow G \text{ has a proper ear decomposition}$$

**Definition 1.28.** An ear decomposition is *odd* if every ear has odd length (in terms of the number of edges).

**Theorem 1.29.** *Let  $G$  be an undirected graph. Then*

$$G \text{ factor-critical} \Leftrightarrow G \text{ has an odd ear decomposition}$$

*The first vertex  $r$  of the ear decomposition can be chosen arbitrarily.*

*Proof.*

---

<sup>2</sup>Note that  $Y$  cannot contain all old vertices, since otherwise  $q_H(Y) < |Y|$ .

" $\Leftarrow$ ": Let  $G$  be a graph with an odd ear decomposition  $r, P_1, \dots, P_k$ .  $P_1$  is an odd circuit, so it is factor-critical. We use induction on the number of ears. Let  $P$  be the last ear and  $G'$  be  $G$  before adding  $P$ . By the induction hypothesis,  $G'$  is factor-critical. Given  $v \in V(G)$ , we have to show that  $G - v$  has a perfect matching.

Case 1:  $v \in V(G')$ . Then  $G' - v$  has a perfect matching. Adding every second edge of  $P$  (excluding the endpoints) to it, yields a perfect matching of  $G - v$ .

Case 2:  $v \in V(G) \setminus V(G')$ . Let  $x, y$  be the endpoints of  $P$ . Without loss of generality let  $P_{[v,x]}$  be even. There exists a perfect matching in  $G' - x$ . Together with every second edge of  $P_{[v,y]}$  and  $P_{[v,x]}$  this is a perfect matching in  $G - v$ .

" $\Rightarrow$ ": Let  $r \in V(G)$  be any vertex. Let  $M$  be a perfect matching in  $G - r$ . Suppose we have an odd ear decomposition for  $G' \subseteq G$  with  $r \in V(G')$  and  $M \cap E(G')$  is a near-perfect matching in  $G'$  (i.e. all vertices in  $G'$  except for  $r$  are matched with other vertices in  $G'$ ).

If  $G' \neq G$ , there is an edge  $\{x, y\} \in E(G) \setminus E(G')$  with  $x \in V(G')$  (by Proposition 1.22). If  $y \in V(G')$ , then  $\{x, y\}$  can be chosen as the next ear. Otherwise, we construct an  $M$ -alternating odd ear, starting with  $\{x, y\}$ : Let  $N$  be a matching in  $G - y$ .  $M \Delta N$  contains a  $y$ - $r$ -path  $P$ . Let  $w$  be the first vertex in  $P \cap V(G')$ .  $w$  is  $M$ -exposed in  $P_{[y,w]}$ ,  $y$  is  $N$ -exposed in  $P_{[y,w]}$ . Therefore  $P_{[y,w]}$  is even and together with  $\{x, y\}$  it forms an  $M$ -alternating odd ear.

Inductively, this argument yields an odd ear decomposition. □

**Definition 1.30.** Let  $G$  be factor-critical and  $M$  a near-perfect matching. An  $M$ -alternating ear decomposition is an odd ear decomposition such that each ear is an  $M$ -alternating path or circuit  $C$  with:

$$|E(C) \cap M| = |E(C) \setminus M| - 1$$

**Corollary 1.31.** *For any factor-critical graph  $G$  and any near-perfect matching  $M$  in  $G$ , there exists in  $M$ -alternating ear decomposition of  $G$ .*

**Definition 1.32.** Let  $G$  be factor-critical,  $M$  a near-perfect matching and  $r, P_1, \dots, P_k$  an  $M$ -alternating ear decomposition of  $G$ .  $\mu, \varphi : V(G) \rightarrow V(G)$  are associated with the ear decomposition if:

- $\{x, y\} \in M \Rightarrow \mu(x) = y$
- $\{x, y\} \in E(P_i) \setminus M$  and  $x \notin \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$

$$\Rightarrow \varphi(x) = y$$

$$\bullet \mu(r) = \varphi(r) = r$$

**Proposition 1.33.** *Let  $G$  be a factor-critical graph and  $\mu, \varphi$  functions associated with an  $M$ -alternating ear decomposition. Then this ear decomposition is unique up to the order of the ears. The Ear-Decomposition-Algorithm (algorithm 1) correctly determines an explicit list of the ears in linear time.*

*Proof.* Step 3 determines ears uniquely. The algorithm clearly runs in linear time.  $\square$

---

**Algorithm 1:** Ear Decomposition Algorithm

---

**Input:** Factor-critical graph  $G$ , functions  $\mu, \varphi$  associated with an  $M$ -alternating ear decomposition

**Output:** An  $M$ -alternating ear decomposition  $r, P_1, \dots, P_k$

```

1  $X := \{r\}$  where  $r$  is the vertex with  $\mu(r) = r$ 
2  $k := 0$ ,  $S :=$  empty stack
3 while  $X \neq V(G)$  do
4   if  $S$  is non-empty then
5      $\mid$  Let  $v \in V(G) \setminus X$  be an endpoint of the topmost element of the
        $\mid$  stack
6   else
7      $\mid$  Choose  $v \in V(G) \setminus X$  arbitrarily
8    $x := v$ ,  $y := \mu(v)$ ,  $P := (\{x, y\}, \{\{x, y\}\})$ 
9   while  $\varphi(\varphi(x)) = x$  do
10     $\mid P := P + \{x, \varphi(x)\} + \{\varphi(x), \mu(\varphi(x))\}$ 
11     $\mid x := \mu(\varphi(x))$ 
12  while  $\varphi(\varphi(y)) = y$  do
13     $\mid P := P + \{y, \varphi(y)\} + \{\varphi(y), \mu(\varphi(y))\}$ 
14     $\mid y := \mu(\varphi(y))$ 
15   $P := P + \{x, \varphi(x)\} + \{y, \varphi(y)\}$ 
16   $P$  is the ear containing  $y$  as an inner vertex. Put  $P$  on  $S$ .
17  while Both endpoints of the topmost element  $P$  of the stack  $S$  are in
     $\mid X$  do
18     $\mid$  Delete  $P$  from  $S$ 
19     $\mid k := k + 1$ ,  $P_k := P$ ,  $X := X \cup V(P)$ 
20 forall  $\{y, z\} \in E(G) \setminus (E(P_1) \cup \dots \cup E(P_k))$  do
21    $\mid k := k + 1$ ,  $P_k := (\{y, z\}, \{\{y, z\}\})$ 
22 return  $r, P_1, \dots, P_k$ 

```

---

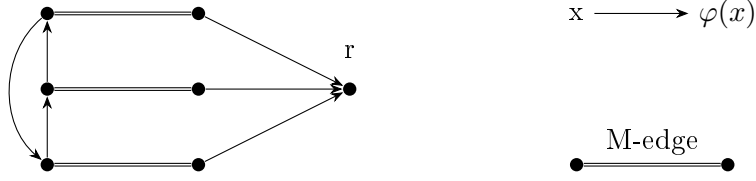


Figure 2: Counter example for the reverse implication of lemma 1.34

**Lemma 1.34.** *Let  $G$  be factor-critical and  $\mu, \varphi$  associated with an  $M$ -alternating ear decomposition. Then the maximal path given by the initial sequence*

$$x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \dots \quad (1)$$

*defines an  $M$ -alternating  $x$ - $r$ -path of even length.*

*Proof.* We proceed by induction on the number of ears. Let  $x \in V(G) \setminus \{r\}$  and  $P_i$  be the ear containing  $x$ . A subsequence of (1) is a subpath  $Q$  of  $P_i$  from  $x$  to  $y \in \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$ .  $Q$  starts with a matching edge and ends with a non-matching edge, so it has even length. If  $y = r$ , we are done, otherwise the statement follows from the induction hypothesis.  $\square$

## 1.6 Edmond's Matching Algorithm

**Definition 1.35.** Let  $G$  be a graph,  $M$  a matching in  $G$ . A *blossom* in  $G$  with respect to  $M$  is a factor-critical subgraph  $B$  of  $G$  such that  $|M \cap E(B)| = \frac{|V(B)|-1}{2}$ . The vertex  $r \in V(B)$  that is exposed by  $M \cap E(B)$  is called the *base* of  $B$ .

**Definition 1.36.** Let  $G$  be a graph,  $M$  a matching in  $G$ ,  $B$  a blossom and  $Q$  a  $M$ -alternating  $v$ - $r$ -path of even length from  $v \in V(G)$  that is  $M$ -exposed to the base  $r$  of  $B$ . Additionally, let  $E(Q) \cap E(B) = \emptyset$ .  $B + Q$  is called an  *$M$ -flower*.

**Lemma 1.37.** *Let  $G$  be a graph,  $M$  a matching in  $G$ . Suppose there is a  $M$ -flower  $B + Q$ . Let  $G', M'$  result from  $G$  and  $M$  by contracting  $V(B)$  into a single vertex. Then:*

$$M \text{ maximum matching in } G \Leftrightarrow M' \text{ maximum matching in } G'$$

*Proof.*

" $\Leftarrow$ ": Assume that  $M$  is not maximum in  $G$ .  $N := M \Delta E(Q)$  is a matching with  $|N| = |M|$ .

$\Rightarrow \exists N$ -augmenting path  $P$  in  $G$ . At least one endpoint  $x$  of  $P$  is not in  $V(B)$  (since  $B$  contains only one  $N$ -exposed vertex). If  $P$  and  $B$  are disjoint, let  $y$  be the other endpoint of  $P$ . Otherwise, let  $y$  be the

first vertex on  $P$  in  $B$ .  $P' := P_{[x,y]}$  is an  $N'$ -augmenting path in  $G'$ , so  $|N'| = |M'| < \mu(G')$ .

" $\Rightarrow$ ": Assume that  $M'$  is not maximum in  $G'$ , so there exists a matching  $N'$  in  $G'$  with  $|N'| > |M'|$ . Let  $N_0$  arise from  $N'$  in  $G$ , then  $N_0$  contains  $\leq 1$  vertex from  $V(B)$ . Since  $B$  is factor-critical,  $N_0$  can be extended by  $k := \frac{|V(G)|-1}{2}$  edges to a matching  $N$  in  $G$ . We have

$$|N| = |N_0| + k = |N'| + k > |M'| + k = |M|$$

so  $M$  is not maximum. □

**Lemma 1.39.** *Let  $G$  be a graph,  $M$  a matching in  $G$ .  $X \subseteq V(G)$  is the set of exposed vertices. We can find a shortest  $M$ -alternating  $X$ - $X$ -walk of positive length in  $O(|E(G)|)$  time.*

*Proof.* Define  $D := (V(G), A)$  where:

$$A := \{(u, v) \mid \exists x \in V(G) : \{u, x\} \in E(G), \{x, v\} \in M\}$$

A shortest  $X - \Gamma_G(X)$ -path in  $D$  corresponds to a shortest  $X$ - $X$ -walk in  $G$ . □

**Theorem 1.40.** *Let  $P = v_0, \dots, v_t$  be a shortest  $M$ -alternating  $X$ - $X$ -walk in  $G$ . Then either*

- $P$  is an  $M$ -augmenting path or
- $v_0, \dots, v_j$  is an  $M$ -flower for some  $j \leq t$ .

*Proof.* If  $P$  is not a path, choose  $i < j$  such that  $v_i = v_j$  and  $j$  minimal. Then  $v_0, \dots, v_{j-1}$  are distinct vertices. If  $j - i$  is even, deleting  $v_{i-1}, \dots, v_j$  from  $P$  yields a shorter walk, so  $j - i$  is odd.

Case 1:  $j$  is even. Then  $i$  is odd and therefore  $v_{i+1} = v_{j-1}$  must be the matching mate of  $V_i = v_j$  which contradicts the minimality of  $j$ .

Case 2:  $j$  is odd. Then  $i$  is even, so  $v_0, \dots, v_i$  is an  $M$ -alternating path of even length and  $v_i, \dots, v_j$  is an  $M$ -alternating odd circuit, i.e. a blossom. □

**Theorem 1.41.** *Given a graph  $G$ , a maximum cardinality matching can be found in time  $O(n^2m)$  where  $n := |V(G)|, m := |E(G)|$*

---

**Algorithm 2:** Edmond's Augmenting Path Search

---

**Input:** Graph  $G$ , matching  $M$ **Output:** An  $M$ -augmenting path (if one exists)

```
1  $X :=$  set of exposed vertices
2 if  $\exists M$ -alternating  $X$ - $X$ -walk of positive length then
3    $P = v_0, \dots, v_t :=$  a shortest such walk
4   if  $P$  is a path then
5     return  $P$ 
6   else
7     Choose  $j$  as in Theorem 1.40
8      $v_0, \dots, v_j$  is an  $M$ -flower with blossom  $B$ 
9     Recurse on  $G/B$ 
10    Augment an  $M/B$ -augmenting path in  $G/B$  to an
11     $M$ -augmenting path  $P'$  in  $G$ 
12    return  $P'$ 
12 else
13    $\nexists M$ -augmenting path
```

---

*Proof.* Start with  $M = \emptyset$  and iteratively find  $M$ -augmenting path  $P$ , set  $M := M \Delta E(P)$ . If no such path exists, then  $M$  is maximum.  $P$  can be found in time  $O(mn)^3$ . Since  $M$  is maximum after at most  $\frac{n}{2}$  augmentation, we have total running time  $O(n^2m)$ .  $\square$

### 1.6.1 Growing forest - $O(n^3)$

**Definition 1.42.** Let  $G$  be a graph,  $M$  a matching in  $G$ . An *alternating forest* with respect to  $M$  in  $G$  is a forest  $F$  in  $G$  where:

- $V(F)$  contains all  $M$ -exposed vertices, each tree of  $F$  contains exactly one exposed vertex, its *root*.
- We call  $v \in V(G)$  an outer (inner) vertex if it has even (odd) distance from the root of its component.
- $\forall v \in V(F)$  the unique path from  $v$  to the root of its component is  $M$ -alternating.
- $v \in V(G) \setminus V(F)$  is called *out-of-forest*.

Clearly, inner vertices always have degree 2 (we always assume that there are no matching edges that can immediately be added to  $F$ ).

---

<sup>3</sup>Here,  $m$  is the time required for finding a walk and the recursion depth is bounded by  $n$ .

**Proposition 1.43.** *In any alternating forest, the number of outer vertices that are not the root equals the number of inner vertices.*

*Proof.* For all outer vertices, there exists exactly one inner vertex on its path to the root.  $\square$

**Lemma 1.44.** *Given a graph  $G$ , a matching  $M$ , an alternating forest  $F$  with respect to  $M$  in  $G$ . Then, either  $M$  is a maximum matching or  $\exists$  outer vertex  $x \in V(F)$ , an edge  $\{x, y\} \notin E(F)$  such that one of the following holds:*

- *Grow:*  $y \notin V(F)$  and therefore  $\{y, z\} \in M$  with  $z \notin V(F)$ . In this case,  $y, z$  and  $\{x, y\}, \{y, z\}$  can be added to  $F$ .
- *Augment:*  $y$  is an outer vertex in a different connected component in  $F$ . In this case,  $M$  can be augmented along  $P(x) \cup \{x, y\} \cup P(y)$  where  $P(z)$  denotes the unique path from  $z \in V(F)$  to the root of its connected component.
- *Shrink:*  $y$  is an outer vertex in the same component as  $x$ . Let  $r$  be the first vertex on  $P(x)$  that is also on  $P(y)$ . Then  $|\delta_F(r)| \geq 3$ , so  $y$  is an outer vertex and  $|E(F_{[x,r]})|, |E(F_{[y,r]})|$  are even. Together with  $\{x, y\}$  these paths form a blossom with  $\geq 3$  vertices.

*Proof.* We show that if none of these cases apply,  $M$  is maximum. If none of the cases apply, then every outer vertex only has inner vertices as neighbors. Let  $X$  be the set of inner vertices,  $s := |X|$  and  $t$  be the number of outer vertices. All outer vertices are isolated in  $G - X$ , so  $q_G(X) - |X| = t - s$ . By Berge's formula (1.24),  $t - s$  vertices are exposed by any matching, so  $M$  is maximum.  $\square$

**Definition 1.45.** Let  $G$  be a graph,  $M$  a matching in  $G$ . A subgraph  $F$  of  $G$  is a *general blossom forest* with respect to  $M$  if there exists a partition  $V(F) = V_1 \dot{\cup} \dots \dot{\cup} V_k$  such that  $F_i = F[V_i]$  is a maximal factor-critical subgraph of  $F$  with  $|M \cap E(F_i)| = \frac{|V_i| - 1}{2}$  ( $i \in [k]$ ) and after contracting each  $V_i$ , we obtain an  $M$ -alternating forest  $F'$ .  $F_i$  is called an outer (inner) blossom if  $V_i$  is an outer (inner) vertex in  $F'$ .

A *special blossom forest* is a general blossom forest where each inner blossom is a single vertex.

Store a special blossom forest with 3 functions  $\mu, \varphi, \rho : V(G) \rightarrow V(G)$ :

$$\begin{aligned}\mu(x) &:= \begin{cases} x & \text{if } x \text{ is exposed in } M \\ y & \text{if } \{x, y\} \in M \end{cases} \\ \varphi(x) &:= \begin{cases} x & \text{if } x \text{ is the base of an outer blossom or } x \text{ is out-of-forest} \\ y & \text{if } x \text{ is an inner vertex and } \{x, y\} \in E(F) \setminus M \\ y & \text{if } x \text{ is an outer vertex (i.e. in an outer blossom)} \\ & \text{and } \mu, \varphi \text{ are associated with an } M\text{-alternating} \\ & \text{ear decomposition of } x\text{'s blossom, } \{x, y\} \in \\ & E(F) \setminus M \end{cases} \\ \rho(x) &:= \begin{cases} x & \text{if } x \text{ is an inner vertex or out-of-forest} \\ y & \text{if } x \text{ is an outer vertex and } y \text{ is the base of the} \\ & \text{outer blossom containing } x \text{ (} y = x \text{ is possible).} \end{cases}\end{aligned}$$

**Proposition 1.46.** *Let  $F$  be a special blossom forest with respect to  $M$  and  $\mu, \varphi, \rho$  as above. Then:*

1. *For all outer vertices  $x$ ,  $P(x) :=$  maximal path given by subsequence of  $x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \dots$  is an  $M$ -alternating path from  $x$  to  $q$  where  $q$  is the root of the component containing  $x$ .*
2. *A vertex  $x$  is*
  - *an outer vertex  $\Leftrightarrow \mu(x) = x \vee \varphi(\mu(x)) \neq \mu(x)$*
  - *an inner vertex  $\Leftrightarrow \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x$*
  - *out-of-forest  $\Leftrightarrow \mu(x) \neq x \wedge \varphi(x) = x \wedge \varphi(\mu(x)) = \mu(x)$*

*Proof.*

1. By definition of  $\mu, \varphi$  and lemma 1.34 some initial subsequence of  $P(x)$  ends at the base  $r$  of the blossom containing  $x$ . If  $r = q$ , we are done. Otherwise  $\mu(r), \varphi(\mu(r))$  are next elements in a sequence leading to outer vertex  $\varphi(\mu(r))$ . This can be iterated.
2. Since the conditions are mutually exclusive, it suffices to show one implication for all the statements.
  - If  $x$  is outer, it is a root ( $\mu(x) = x$ ) or  $P(x)$  is a path of length at least 2, so  $\varphi(\mu(x)) \neq \mu(x)$ .
  - If  $x$  is inner, then  $\mu(x)$  is the base of an outer blossom. Therefore  $\varphi(\mu(x)) = \mu(x)$ .  $P(\mu(x))$  is a path of length at least 2, so  $\varphi(x) \neq x$ .



- If  $x$  is out-of-forest, then  $x$  is covered by  $M$  so  $\mu(x) \neq x$ . By definition of  $\varphi$ ,  $\varphi(x) = x$ .  $\mu(x)$  is out-of-forest as well, so  $\varphi(\mu(x)) = \mu(x)$ .

□

**Lemma 1.47.** *Following invariants hold:*

- a)  $\{\{x, \mu(x)\} \mid x \in V(G), \mu(x) \neq x\}$  is a matching
- b)  $\{\{x, \mu(x)\} \mid \underbrace{x \in V(G), \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x}_{\text{inner vertices}}\} \cup \{\{x, \varphi(x)\} \mid x \in V(G), \varphi(x) \neq x\}$  forms the edge set of a special blossom forest.
- c)  $\mu, \varphi, \rho$  satisfy the conditions in definition 1.45 (special blossom forest).

*Proof.* a) holds as  $\mu$  only changes in *Augment*. b) is correct after initialization and after the reset in the *Augment* step. It is preserved by *Grow* steps.

In a *Shrink* step,  $r$  (the first vertex that the paths from  $x, y$  to the root share) is a root or has  $|\delta(r)| = 3$  (i.e. it is the base of a blossom), so it is an outer vertex. We define a blossom  $B := \{v \in V(G) \mid \varphi(v) \in V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})\}$ . Consider  $\{u, v\} \in F$  with  $u \in B, v \notin B$ . If  $\{u, v\} \in M$ , we have  $u = r, v = \mu(r)$  (since  $F[B]$  contains a near-perfect matching).  $u$  was an outer vertex before shrinking and  $F[B]$  being factor-critical follows from c) and the characterization by ear-decompositions.

For c), it's clear that  $\mu$  always represents a matching.  $\varphi(x) = x$  if  $x$  is not an outer vertex. Therefore,  $\mu + \varphi$  represent an  $M$ -alternating ear decomposition of  $B$ . During *Shrink*,  $\varphi(v)$  is not changed if  $\varphi(v) = r$ . Therefore, the odd ear decomposition for  $B' :=$  blossom containing  $r$ , is the correct starting point. The next ear is  $P(x)_{[x,x']} \cup P(y)_{[y,y']} + \{x, y\}$ , where  $x'$  ( $y'$ ) is the first vertex in  $B'$  on  $P(x)_{[x,r]}$  ( $P(y)_{[y,r]}$ ).

For each ear  $Q$  of a former blossom  $B'' \subseteq B$ ,  $Q \setminus (E(P(x)) \cup E(P(y)))$  form a new ear (since it is created by removing an even path).  $\varphi, \mu$  represent this ear-decomposition. □

**Theorem 1.48.** *Edmond's cardinality matching algorithm correctly determines a maximum matching in  $O(n^3)$  time, where  $n := |V(G)|$ .*

*Proof.* By lemma 1.47 and proposition 1.46, the algorithm maintains a special blossom forest. Let  $M, F$  be the final matching and forest.  $x$  an outer

---

**Algorithm 3:** Edmond's Cardinality Matching Algorithm

---

**Input:** A graph  $G$

**Output:** A maximum matching  $M$  (defined by  $\{x, \mu(x)\}$ )

```
1  $\mu(v) := v, \varphi(v) := v, \rho(v) := v, scanned(v) := \text{false}$  for all  $v \in V(G)$ 
  // Outer Vertex Scan:
2 while  $\exists$  outer vertex  $x$  with  $scanned(x) = \text{false}$  do
3   Let  $y$  be a neighbor of  $x$  such that  $y$  is either out-of-forest or  $y$  is
     outer with  $\rho(y) \neq \rho(x)$ 
4   if such a  $y$  does not exist then
5      $scanned(x) := \text{true}$ , continue
  // Grow:
6   if  $y$  is out-of-forest then
7      $\varphi(y) := x$ , continue
  // Augment:
8   else if  $P(x)$  and  $P(y)$  are vertex-disjoint then
9      $\mu(\varphi(v)) = v, \mu(v) = \varphi(v)$  for all  $v \in V(P(x) \cup P(y))$  with odd
       distance from  $x$  or  $y$  on  $P(x)$  or  $P(y)$ , respectively
10     $\mu(x) := y, \mu(y) := x$ 
11     $\varphi(v) := v, \rho(v) := v, scanned(v) := \text{false}$  for all  $v \in V(G)$ 
  // Shrink:
12  else
13    Let  $r$  be the first vertex on  $V(P(x)) \cap V(P(y))$  with  $\rho(r) = r$ 
14    forall  $v \in V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})$  with odd distance from  $x$  or
        $y$  on  $P(x)_{[x,r]}$  or  $P(y)_{[y,r]}$ , respectively and  $\rho(\varphi(v)) \neq r$  do
15       $\varphi(\varphi(v)) := v$ 
16      if  $\rho(x) \neq r$  then
17         $\varphi(x) := y$ 
18      if  $\rho(y) \neq r$  then
19         $\varphi(y) := x$ 
20      forall  $v \in V(G)$  with  $\rho(v) \in V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})$  do
21         $\rho(v) := r$ 
22 return  $\mu$ 
```

---

vertex implies that  $\forall y \in \Gamma(x) : y$  is inner and  $\varphi(y) = \varphi(x)$ . Define:

$X :=$  set of inner vertices

$B :=$  set of bases of (outer) blossoms

Then every unmatched vertex is in  $B$ . Matched vertices in  $B$  have matching mates in  $X$  and  $|B| = |X| + |V(G)| - 2|M|$ . (Outer) blossoms are odd connected components in  $G - X$ , so by Berge's theorem (1.24), at least  $|B| - |X|$  vertices remain uncovered by any matching, so  $M$  is maximum.

We now consider the running time: The status (outer, inner, out-of-forest) for a given vertex can be checked in constant time (proposition 1.46). Therefore, *Grow*, *Augment* and *Shrink* can be implemented in  $O(n)$  time. There are at most  $n$  calls to *Grow* and *Shrink* per augment and at most  $\frac{n}{2}$  *Augments*. This implies the running time  $O(n^3)$ .  $\square$

*Remark 1.49.* The time for *Shrink* can be reduced to  $O(\log n)$  using a binary tree, leading to a running time of  $O(nm \log n)$  in total. Tarjan (1974), Gabow & Tarjan (1983) proved a running time of  $O(nm\alpha(m, n))$  (where  $\alpha$  is the inverse Ackermann function) or  $O(nm)$ .

*Remark 1.50.* It's not necessary to reset everything after augmenting. It suffices to reset the 2 trees that were changed by the augmentation. Gabow & Tarjan (1983) showed that it's possible to augment all paths of the same length in  $O(m)$  time. There are  $2\sqrt{\nu(G)} + 2$  different path lengths, so in total this results in a running time of  $O(\sqrt{nm})$ .

*Remark 1.51* (Skew-symmetric flows). Goldberg & Karzanov (2003) (and Fremuth-Pagen & Jungnickel (2003)) used *Generalized Max-Flow* to achieve a running time of  $O(\sqrt{nm} \frac{\log \frac{m}{n}}{\log n})$ .

## 1.7 Gallai-Edmonds Decomposition

**Proposition 1.52.** *Let  $G$  be a graph,  $X \subseteq V(G)$  with  $|V(G)| - 2\nu(G) = q_G(X) - |X|$ . Then any maximum matching of  $G$*

- *contains a perfect matching in the even components of  $G - X$ .*
- *contains a near-perfect matching in odd components of  $G - X$ .*
- *matches all  $x \in X$  to distinct odd components.*

*Proof.* Follows directly from Berge's theorem (1.24).  $\square$

**Theorem 1.53.** *Let  $G$  be a graph and:*

$$Y := \{v \in V(G) \mid \exists \text{ maximum matching that exposes } v\}$$

*Define  $X := \Gamma(Y)$  and  $W := V(G) \setminus (X \cup Y)$ . Then:*

1.  $X$  attains  $\max_{X' \subseteq V(G)} q_G(X') - |X'|$ .
2.  $G[Y]$  is the union of factor-critical subgraphs and  $G[W]$  is the union of even connected components.
3. Any maximum matching in  $G$ 
  - contains a perfect matching in  $G[W]$ .
  - contains a near-perfect matching in each component of  $G[Y]$ .
  - matches all  $x \in X$  to distinct connected components

$Y, X, W$  is called Gallai-Edmonds decomposition of  $G$ .

*Proof.* Consider the matching  $M$  and special blossom forest  $F$  at the end of the algorithm. Let  $X'$  ( $Y'$ ) be the set of inner (outer) vertices and  $W'$  the set of out-of-forest vertices.  $X', Y', W'$  satisfy 1., 2. and 3. by the proof of theorem 1.48.

Proposition 1.52 implies that any maximum matching covers all vertices in  $V(G) \setminus Y'$ , so  $Y \subseteq Y'$ . For the other inclusion, let  $v \in Y'$ . Then  $M \Delta P(v)$  is a maximum matching exposing  $v$ , so  $v \in Y$  and  $Y' = Y$ . By definition,  $X = X'$  and  $W = W'$ .  $\square$

**Corollary 1.54.** *A graph  $G$  has a perfect matching  $\Leftrightarrow \forall U \subseteq V(G)$ ,  $G - U$  has at most  $|U|$  factor-critical components.*

## 1.8 Minimum Weight Perfect Matching

We use the following Integer Programming formulation:

$$\begin{aligned}
& \min \sum_{e \in E(G)} c_e x_e \\
& \text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 1 \quad v \in V(G) \\
& \quad \quad x_e \in \{0, 1\}
\end{aligned}$$

and the corresponding relaxation where we only require  $x_e \geq 0$ . The dual problem of this is:

$$\begin{aligned}
& \max \sum_{v \in V(G)} z_v \\
& \text{s.t.} \quad z_v + z_w \leq c_e \quad \{v, w\} \in E(G)
\end{aligned}$$

**Proposition 1.55** (Hungarian Method). *Let  $G$  be a graph,  $c \in \mathbb{R}^{E(G)}$  and  $z \in \mathbb{R}^{V(G)}$  with  $z_v + z_w \leq c_e$  for all  $e = \{v, w\} \in E(G)$ . Define:*

$$G_z := (V(G), \{e = \{v, w\} \mid z_v + z_w = c_e\})$$

Let  $M$  be a matching in  $G_z$ ,  $F$  a maximal alternating forest in  $G_z$  with respect to  $M$ . Let  $X/Y$  be the set of inner/outer vertices. Then:

1. If  $M$  is a perfect matching in  $G_z$ , then it is a minimum-weight perfect matching in  $G$ .
2. If  $\Gamma_G(y) \subseteq X$  for all  $y \in Y$ , then  $M$  is a maximum matching.
3. If neither 1. nor 2. hold, define:

$$\epsilon := \min \left\{ \min_{e=\{v,w\} \in E(G[Y])} \frac{c_e - z_v - z_w}{2}, \min_{e \in \delta(Y) \cap \delta(V(F))} c_e - z_v - z_w \right\}$$

Set  $z'_v := z_v - \epsilon$  for all  $v \in X$ ,  $z'_v := z_v + \epsilon$  for all  $v \in Y$  and  $z'_v := z_v$  for all  $v \in V(G) \setminus (X \cup Y)$ . Then  $z'$  is a feasible dual solution and  $M \cup E(F) \subseteq E(G_{z'})$ . Additionally,  $\Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$  for some  $y \in Y$ .

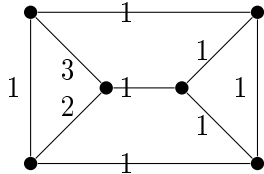
*Proof.* 1. Let  $M'$  be a minimum-weight perfect matching.

$$\begin{aligned} \sum_{e \in M'} c_e &= \sum_{v \in V(G)} z_v + \sum_{e=\{v,w\} \in M'} (c_e - z_v - z_w) \\ &\geq \sum_{v \in V(G)} z_v \\ &= \sum_{v \in V(G)} z_v + \sum_{e=\{v,w\} \in M} (c_e - z_v - z_w) \\ &= \sum_{e \in M} c_e \end{aligned}$$

2. Each outer vertex is an odd blossom (singleton) of  $G - x$ . By Berge (1.24), at least  $|Y| - |X|$  vertices remain uncovered.
3.  $z'$  stays feasible by the choice of  $\epsilon$ . Edges in  $E(F), M$  remain tight. By 1. and 2.,  $\exists y \in Y : \Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$ .

□

*Remark 1.56.* For bipartite graphs, the adjacency matrix is totally unimodular, so the LP has integral vertices.



We define  $\mathcal{A} := \{X \subseteq V(G) \text{ odd}\}$  and add the blossom inequalities

$$\sum_{e \in \delta(X)} x_e \geq 1 \quad \forall X \in \mathcal{A}$$

to the LP relaxation. The new dual problem is then:

$$\begin{aligned} \max \quad & \sum_{A \in \mathcal{A}} z_A \\ \text{s.t.} \quad & \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A \leq c_e \\ & z_A \geq 0 \quad (A \in \mathcal{A}, |A| \geq 3) \end{aligned}$$

Edmond's Algorithm starts with an empty matching  $x = 0$  and dual feasible solution:

$$z_A := \begin{cases} \frac{1}{2} \min\{c(e) \mid e \in \delta(A)\} & |A| = 1 \\ 0 & \text{else} \end{cases}$$

We always ensure that  $z$  is dual feasible and that  $(x, z)$  satisfy complementary slackness:

$$\begin{aligned} x_e > 0 &\Rightarrow \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A = c_e \\ z_A > 0, |A| > 1 &\Rightarrow \sum_{e \in \delta(A)} x_e = 1 \end{aligned}$$

**Definition 1.57.**  $c_z(e) := c(e) - \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A$  is the *reduced cost* of  $e$ .

**Theorem 1.58.** *There are at most  $\frac{7}{2}|V(G)|^2$  of the repeat-until loop in algorithm 4.*

*Proof.*  $\mathcal{B}$  is laminar at any time, i.e. for  $X, Y \in \mathcal{B}$  we have  $(X \subseteq Y) \vee (Y \subseteq X) \vee (X \cap Y = \emptyset)$ . Therefore  $|\mathcal{B}| \leq 2|V(G)|$ .

**Observation.** *Any  $U$  added to  $\mathcal{B}$  during Shrink will not be "unpacked" before the next Augment.*

*Proof.* After *Shrink*, there exists an even length  $M$ -augmenting  $R$ - $U$ -path. It remains in  $G_z$  until the next *Augment* or until  $U$  is included in another blossom  $U' \supseteq U$  which is not resolved before an *Augment* (inductively).  $\square$

Between 2 augments:

- $\# \text{ Unpacks} \leq |\mathcal{B}|$  at beginning of the sequence

- $\# \text{ Shrinks} \leq |\mathcal{B}|$  at the end of the sequence

Therefore, there are at most  $4|V(G)|$  *Unpack* and *Shrink* operations between 2 augments. For each dual change without *Unpack*, we have:  $z_B > 0 \quad \forall B \in \mathcal{B}$ , so  $\epsilon$  is not determined by  $z_B$ . Therefore  $\exists e = \{X, Y\}$  with  $X \notin \mathcal{X}, Y \in \mathcal{Y}$  where  $c_z(e)$  becomes 0.

Case 1:  $X \notin \mathcal{Y}$ . Then  $|V(G_z) \setminus (\mathcal{X} \cup \mathcal{Y})|$  decreases.

Case 2:  $X \in \mathcal{Y}$ . Then  $\exists X$ - $Y$   $M$ -alternating walk in the next iteration.

In particular, such a dual change can occur at most  $|V(G)|$  times between 2 augmentations.

In total, there are at most  $\frac{1}{2}|V(G)|$  *Augment* steps. Therefore, there are  $\frac{1}{2}|V(G)|^2(4 + |V(G)| + 2|V(G)|)$   $\square$

---

**Algorithm 4:** Minimum-Weight Perfect Matching

---

**Input:** Graph  $G$  with edge weights  $c : E(G) \rightarrow \mathbb{R}$

**Output:** A minimum-weight perfect matching  $M$  in  $(G, c)$

---

**Corollary 1.59.** *A minimum-weight perfect matching can be computed in  $O(n^2m)$  time where  $n := |V(G)|$  and  $m = |E(G)|$ .*

*Proof.* Theorem 1.58 times  $O(m)$ .  $\square$

*Remark 1.60.* To achieve  $O(n^3)$  running time, one can modify the algorithm:

1. Use a General Blossom Forest to avoid recomputing the  $R$ - $R$ -walks from scratch. We then have mappings  $\mu_v, \varphi_v^i, \rho_v^i$  for  $1 \leq i \leq k_v$  where  $k_v$  is the number of blossoms that contain  $v$ .
2. Store all vertices in a heap (ordered by their criticality for dual-feasibility) to speed up the computation of  $\epsilon$ .

Gabow (1990) showed a running time of  $O(n(m + n \log n))$ . Gabow & Tarjan (1991) showed a running time of  $O(m \log(nW) \sqrt{n \alpha(m, n) \log n})$  where  $W := \max_{e \in E(G)} |c(e)|$ .

### 1.8.1 The Matching Polytope

**Theorem 1.61.** *Let  $G$  be a graph. The set of vectors  $x \in \mathbb{R}^{E(G)}$  satisfying*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(v)) &= 1 & v &\in V(G) \\ x(\delta(A)) &\geq 1 & A &\subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

is the convex hull of all perfect matchings in  $G$ . It is called the perfect matching polytope.

*Proof.* For any objective function  $c : E(G) \rightarrow \mathbb{R}$ , the minimum-weight perfect matching algorithm produces an integral primal and a dual solution that satisfy complementary slackness. In particular, all vertices of the polytope are integral.  $\square$

**Theorem 1.62.** Let  $G$  be a graph. The set of vectors  $x \in \mathbb{R}^{E(G)}$  satisfying

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(v)) &\leq 1 & v &\in V(G) \\ x(E(G[A])) &\leq \frac{|A| - 1}{2} & A &\subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

is the convex hull of all matchings in  $G$ . It is called the matching polytope.

*Proof.* Any matching solution  $x$  satisfies these conditions. Let  $x$  be any solution that satisfies the conditions. We have to show that  $x$  is a convex combination of matching solutions. Define  $H$  by:

$$\begin{aligned} V(H) &:= \{(v, i) \mid v \in V(G), i \in \{1, 2\}\} \\ E(H) &:= \{ \{(v, i), (w, i)\} \mid \{v, w\} \in E(G), i \in \{1, 2\} \} \\ &\quad \cup \{ \{(v, 1), (v, 2)\} \mid v \in V(G) \} \end{aligned}$$

We set  $y_{\{(v,i),(w,i)\}} := x_{\{v,w\}}$  for all  $\{v, w\} \in E(G), i \in \{1, 2\}$  and  $y_{\{(v,1),(v,2)\}} := 1 - x(\delta(v))$  for all  $v \in V(G)$ . Then  $y \geq 0$  and  $y(\delta_H(x)) = 1$  for all  $x \in V(H)$ .

**Claim.**  $y$  satisfies the inequalities of the perfect matching polytope (in particular the blossom inequalities).

If this is true, by 1.62  $y$  is a convex combination of perfect matchings.  $H[\{(v, 1) \mid v \in V(G)\}]$  is isomorphic to  $G$ , so  $x$  is a convex combination of matchings in  $G$ .

We now prove the claim: Let  $X \subseteq V(H)$  with  $|X|$  odd. We have to show that  $y(\delta_H(X)) \geq 1$ . Define:

$$\begin{aligned} A &:= \{v \in V(G) \mid (v, 1) \in X, (v, 2) \notin X\} \\ B &:= \{v \in V(G) \mid (v, 1) \in X, (v, 2) \in X\} \\ C &:= \{v \in V(G) \mid (v, 1) \notin X, (v, 2) \in X\} \end{aligned}$$

Define  $A_i := A \cap (V(G) \times \{i\})$  and  $B_i := B \cap (V(G) \times \{i\})$ .  $|B_1 \cup B_2|$  is even, so (since  $|X|$  is odd)  $|A|$  or  $|C|$  is odd. Without loss of generality, let



$|A|$  be odd.

$$\begin{aligned}
\sum_{e \in \delta_H(X)} y_e &\geq \sum_{v \in A_1} \underbrace{\sum_{e \in \delta_H(v)} y_e}_{=1} - 2 \cdot \sum_{e \in E(H[A_1])} y_e - \sum_{e \in \delta(A_1) \cap \delta(B_1)} y_e \\
&+ \sum_{e \in \delta(A_2) \cap \delta(B_2)} y_e \\
&= |A_1| - 2 \cdot \sum_{e \in E(G[A])} x_e \\
&\geq |A_1| - (|A| - 1) \\
&= 1
\end{aligned}$$

□

**Theorem 1.63.** *The matching polyhedron is TDI (Totally Dual Integral), i.e. for all  $c \in \mathbb{Z}^{E(G)}$  for which the dual program of  $(\max c^t x \text{ s.t. } \dots)$  has a finite optimum solution, it has an integral optimum solution.*

*Proof.* The dual is

$$\begin{aligned}
\min \quad & \sum_{v \in V(G)} y_v + \sum_{A \in \mathcal{A}, |A| > 1} \frac{|A| - 1}{2} z_A \\
\text{s.t.} \quad & \sum_{v \in e} y_v + \sum_{A \in \mathcal{A}, |A| > 1, e \in E(G[A])} z_A \geq c(e) \quad e \in E(G) \\
& y, z \geq 0
\end{aligned}$$

Let  $(G, c)$  be a counterexample such that  $|V(G)| + |E(G)| + \sum_{e \in E(G)} |c(e)|$  is minimum. Then:

- $c(e) \geq 1$  for all  $e \in E(G)$ , since otherwise  $e$  could be deleted.
- $G$  has no isolated vertices.

**Claim.** *In an optimum solution  $(y, z)$ ,  $y = 0$ .*

*Proof.* If  $y_v > 0$ , then  $x(\delta(v)) = 1$  for all optimum solutions  $x$ . Decreasing  $c(e)$  by 1 for all  $e \in \delta(v)$  yields a smaller feasible instance  $(G, c')$  where the weight of  $x$  is decreased by 1 and  $x$  remains optimum. By assumption,  $(G, c')$  is not a counterexample, so there exists an integral optimum solution  $(y', z')$ . Increasing  $y'_v$  by one yields some optimum in  $(G, c)$  which has optimum integral solution  $(y' + \mathbb{1}_v, z')$ . □

Let  $(y = 0, z)$  be a dual optimum solution such that

$$\sum_{A \in \mathcal{A}, |A| > 1} |A|^2 z_A$$

is maximum.

**Claim.**  $\mathcal{F} := \{A : z_A > 0\}$  is laminar.

If not, there exist  $X, Y \in \mathcal{F} : X \cap Y, X \setminus Y, Y \setminus X \neq \emptyset$ . We proceed by "uncrossing". Let  $\epsilon := \min\{z_X, z_Y\} > 0$ .

Case 1:  $|X \cap Y|$  is odd. Then  $|X \cup Y|$  is odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_Y &:= z_Y - \epsilon \\ z'_{X \cap Y} &:= z_{X \cap Y} + \epsilon & (\text{unless } |X \cap Y| = 1) \\ z'_{X \cup Y} &:= z_{X \cup Y} + \epsilon \\ z'_A &:= z_A & \text{elsewhere} \end{aligned}$$

Then  $(y, z')$  is a dual optimum solution.

Case 2:  $|X \cap Y|$  is even. Then  $|X \setminus Y|$  and  $|Y \setminus X|$  are odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_Y &:= z_Y - \epsilon \\ z'_{X \setminus Y} &:= z_{X \setminus Y} + \epsilon & \text{unless } |X \setminus Y| = 1 \\ z'_{Y \setminus X} &:= z_{Y \setminus X} + \epsilon & \text{unless } |Y \setminus X| = 1 \\ z'_A &:= z_A & \text{elsewhere} \\ y'_v &:= \epsilon & \forall v \in X \cap Y \\ y'_v &:= 0 & \forall v \notin X \cap Y \end{aligned}$$

Then  $(y', z')$  is feasible. The objective value is:

$$\begin{aligned} & \sum_{v \in V(G)} y'_v + \sum_{A \in \mathcal{A}, |A| > 1} z'_A \frac{|A| - 1}{2} \\ &= \epsilon \cdot |X \cap Y| + \sum_{A \in \mathcal{A}, |A| > 1} \frac{|A| - 1}{2} \\ &+ \epsilon \left( \frac{|X \setminus Y| - 1}{2} + \frac{|Y \setminus X| - 1}{2} - \frac{|X| - 1}{2} - \frac{|Y| - 1}{2} \right) \\ &= \text{objective}(y, z) \end{aligned}$$

Therefore  $(y', z')$  is an optimum solution with  $y' \neq 0$ , which is a contradiction to the previous claim.

We can conclude that  $\mathcal{F}$  is laminar.

Let  $A \in \mathcal{F}$  with  $z_A \notin \mathbb{Z}$  and  $|A|$  is maximal. Define  $\epsilon := z_A - \lfloor z_A \rfloor > 0$ . Let  $A_1, \dots, A_k$  be the inclusion-wise maximal proper subsets of  $A$  in  $\mathcal{F}$ . Since  $\mathcal{F}$  is laminar,  $A_i \cap A_j = \emptyset$  for  $i \neq j$ . Define:

$$\begin{aligned} z'_A &:= z_A - \epsilon \\ z'_{A_i} &:= z_A + \epsilon & 1 \leq i \leq k \\ z'_D &:= z_D & \text{elsewhere} \end{aligned}$$

Then  $(y, z')$  is dual feasible with objective value:

$$\sum_{B \in \mathcal{A}, |B| > 1} \frac{|B| - 1}{2} z'_B < \sum_{B \in \mathcal{A}, |B| > 1} \frac{|B| - 1}{2} z_B$$

This contradicts the optimality of  $(y, z)$ , so there exists no counter example.  $\square$

**Theorem 1.64.** *Let  $G$  be a graph.*

$$\begin{aligned} P &:= \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) \leq 1 \quad \forall v \in V(G)\} \\ Q &:= \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) = 1 \quad \forall v \in V(G)\} \end{aligned}$$

*are called the fractional matching polytope and the fractional perfect matching polytope. If  $G$  is bipartite, then  $P$  and  $Q$  are integral.*

*Proof.* The adjacency matrices of bipartite graphs are totally unimodular.  $\square$

**Theorem 1.65.** *Let  $G$  be a graph. The vertices of the fractional perfect matching polytope satisfy*

$$x_e = \begin{cases} \frac{1}{2} & \text{if } e \in E(C_1) \cup \dots \cup E(C_k) \\ 1 & \text{if } e \in M \\ 0 & \text{else} \end{cases}$$

*where  $C_1, \dots, C_k$  are vertex-disjoint odd circuits and  $M$  is a perfect matching in  $G - (V(C_1) \cup \dots \cup V(C_k))$ .*

*Proof.* Exercise 6.3  $\square$

## 2 $T$ -Joins and $b$ -Matchings

**Definition 2.1.** Let  $G$  be a graph,  $T \subseteq V(G)$ . A subset  $J \subseteq E(G)$  is called  $T$ -join if  $T$  is the set of odd-degree vertices in  $(V(G), J)$ .

**Proposition 2.2.** Let  $G$  be a graph,  $T, T' \subseteq V(G)$ ,  $J$  a  $T$ -join and  $J'$  a  $T'$ -join. Then  $J \Delta J'$  is a  $T \Delta T'$ -join.

*Proof.* For  $v \in V(G)$ :

$$\begin{aligned} |\delta_{J \Delta J'}(v)| &\equiv |\delta_J(v)| + |\delta_{J'}(v)| \\ &\equiv |\{v\} \cap T| + |\{v\} \cap T'| \\ &\equiv |\{v\} \cap (T \Delta T')| \pmod{2} \end{aligned}$$

□

**Proposition 2.3.** Let  $G$  be a graph,  $T \subseteq V(G)$ .

$\exists$   $T$ -join in  $G \Leftrightarrow |V(C) \cap T|$  even for each connected component  $C$

*Proof.*

" $\Rightarrow$ ": Let  $J$  be a  $T$ -join. For each connected component  $C$ :

$$\sum_{v \in V(C)} |J \cap \delta(v)| = 2 |J \cap E(C)|$$

Therefore  $|J \cap \delta(v)|$  is odd for an even number of vertices and  $|V(C) \cap T|$  is even.

" $\Leftarrow$ ": Partition  $T$  into pairs  $\{v_1, w_1\}, \dots, \{v_k, w_k\}$  such that  $v_i$  and  $w_i$  are in the same component for all  $i$ . Let  $P_i$  be a  $v_i$ - $w_i$ -path in  $G$ . Define  $J := E(P_1) \Delta E(P_2) \Delta \dots \Delta E(P_k)$ . By proposition 2.2, this is a  $T$ -join.

□

**Theorem 2.4.** Let  $G$  be a graph,  $c : E(G) \rightarrow \mathbb{R}$  and  $T \subseteq V(G)$ . In strongly polynomial time (e.g.  $O(n^2m)$ ) we can determine if a  $T$ -join exists and if so, compute a minimum-weight  $T$ -join.

*Proof.* In  $O(m)$  ( $m := |E(G)|$ ), we can check if a  $T$ -join exists. If so:

1. Eliminate negative weights.

$$\begin{aligned} N &:= \{e \in E(G) \mid c(e) < 0\} \\ U &:= \{v \in V(G) \mid |\delta_N(v)| \text{ odd}\} \\ T' &:= T \Delta U \\ c'(e) &:= |c(e)| \qquad e \in E(G) \end{aligned}$$

**Claim.** *If  $J'$  is a minimum  $T'$ -join with respect to  $c'$ , then  $J' \Delta N$  is a minimum  $T$ -join with respect to  $c$ .*

Let  $\tilde{J}$  be a  $T$ -join. Then  $\tilde{J} \Delta N$  is a  $T'$ -join, so  $c'(\tilde{J}) \leq c'(\tilde{J} \Delta N)$  and

$$c(J) = c'(J') + c(N) \leq c'(\tilde{J} \Delta N) + c(N) = c(\tilde{J})$$

which proves the claim.

2. We can now assume that  $c \geq 0$ . A minimum-weight  $T$ -join does not have cycles of positive weight. We can eliminate cycles of weight 0 without changing the cost. We can then restrict ourselves to searching for collections of  $T$ - $T$ -paths.

Let  $K_T$  be the metric closure of  $T$  with respect to  $G$ . It can be computed in  $O(n \cdot (m + n \log n))$  by using Dijkstra for all vertices. Find a minimum-weight perfect matching  $M$  in  $K_T$ . Each  $e = \{s, t\} \in M$  induces a path  $P_{s,t}$ . Then the symmetric difference  $\Delta_{\{s,t\} \in M} E(P_{s,t})$  is a minimum-weight  $T$ -join in  $G$ .

□

**Corollary 2.6.** *A maximum-weight  $T$ -join can be computed as fast as a minimum-weight  $T$ -join.*

*Proof.* Set  $c' := -c$ .

□

**Corollary 2.7.** *Let  $G$  be a graph,  $c : E(G) \rightarrow \mathbb{R}$ . We can find a cycle of negative length in  $G$  in  $O(n^2 m)$  time.*

*Proof.* Apply theorem 2.4 to  $T = \emptyset$ . If  $c(J) < 0$ ,  $(V(G), J)$  contains a cycle  $C$ . If  $c(C) = 0$ , we can eliminate it and recurse, otherwise return  $C$ . □

## 2.2 $T$ -Join Applications

### 2.2.1 TSP Approximation

Let  $(K_n, c)$  with  $c$  metric be an instance of the TSP. Consider the *Double tree algorithm*:

1. Compute a minimum spanning tree  $T$ .
2.  $T' := T + T$  (doubling all edges). Then  $T'$  is Eulerian.
3. Walk along  $T'$  and add vertices to the TSP tour in the order of their first appearance, yielding a tour  $T^*$ . Since  $c$  is metric, we have  $c(T^*) \leq c(T') \leq 2c(T)$ . Since the cost of  $T$  is a lower bound for the cost of a tour, we have  $c(T^*) \leq 2\text{OPT}$  (where  $\text{OPT}$  is the cost of a shortest TSP tour).

---

**Algorithm 5:** Christofides Algorithm (1976)

---

**Input:** Complete metric graph  $(K_n, c)$

**Output:** A TSP-tour  $T$

- 1 Find MST  $T_{\text{MST}}$  in  $(K_n, c)$
  - 2  $W := \{v \in V(K_n) \mid |\delta_{T_{\text{MST}}}(v)| \text{ odd}\}$
  - 3  $J :=$  minimum-weight  $W$ -Join in  $(K_n, c)$
  - 4 Add cities to  $T$  in the order of first appearance in a Eulerian walk of  $T_{\text{MST}} + J$ .
  - 5 **return**  $T$
- 

**Theorem 2.8.** *Algorithm 5 is a  $\frac{3}{2}$ -approximation algorithm for the metric TSP, i.e. for the computed tour  $T$  we have:*

$$c(T) \leq \frac{3}{2} \text{OPT}$$

*Proof.* We have  $c(T_{\text{MST}}) \leq \text{OPT}$  and  $\text{OPT}(W) \leq \text{OPT}(V(K_n))$  (since  $c$  is metric). Any tour through the vertices in  $W$  can be decomposed into 2 matchings. Therefore,  $c(J) \leq \frac{1}{2} \text{OPT}(W) \leq \frac{1}{2} \text{OPT}$ . It follows that  $c(T) \leq (1 + \frac{1}{2}) \text{OPT}$ .  $\square$

### 2.2.2 Shortest Paths in Undirected Graphs

The naive reduction to digraphs requires non-negative weights.

**Corollary 2.9.** *Given an undirected graph  $G$ ,  $c : E(G) \rightarrow \mathbb{R}$  such that each circuit has length at least 0. Then for  $s, t \in V(G)$ , a shortest  $s$ - $t$ -path can be found in  $O(n^2m)$  time, where  $n := |V(G)|$ ,  $m := |E(G)|$ .*

*Proof.* Choose  $T := \{s, t\}$ . Apply theorem 2.4 to get a minimum-weight  $T$ -join  $J$ .  $J$  can be partitioned into circuits of length 0 and an  $s$ - $t$ -path of length  $c(J)$ .  $\square$

### 2.2.3 Chinese Postman Problem

**Definition 2.10.** A walk  $C = \{v_0, e_1, v_1, \dots, e_t, v_t\}$  is called a Chinese postman tour if  $v_0 = v_t$  and each edge in  $E(G)$  is visited at least once. The Chinese Postman Problem is the problem of finding a shortest Chinese postman tour in  $G$  with respect to  $c : E(G) \rightarrow \mathbb{R}_{\geq 0}$ .

**Corollary 2.11.** *The Chinese postman problem can be solved in  $O(n^2m)$  time, where  $n := |V(G)|$ ,  $m := |E(G)|$ .*

*Proof.* Set  $T := \{v \in V(G) \mid |\delta(v)| \text{ odd}\}$  and let  $J$  be a minimum-weight  $T$ -join. Compute a Eulerian tour  $C$  in  $G + J$ . Let  $C'$  be a shortest Chinese

postman tour. Let  $J' :=$  set of edges occuring in  $C'$  an even number of times (at least twice). Then  $J'$  is a  $T$ -join, so  $c(J') \geq c(J)$  and:

$$c(C') \geq c(E(G)) + c(J') \geq c(E(G)) + c(J) = c(C)$$

□

### 2.3 $T$ -Joins and $T$ -Cuts

**Definition 2.12.** Let  $G$  be a graph and  $T \subseteq V(G)$ . A  $T$ -cut is a cut  $C = \delta(X)$  with  $X \subseteq V(G)$  and  $|X \cap T|$  odd.

**Proposition 2.13.** Let  $G$  be a graph,  $T \subseteq V(G)$ ,  $|T|$  even. Then:

1. For any  $T$ -join  $J$  and any  $T$ -cut  $C$ :  $J \cap C \neq \emptyset$ .
2. The inclusion-wise minimal  $T$ -cuts ( $T$ -joins) are exactly the inclusion-wise minimal edge sets intersecting all  $T$ -joins (all  $T$ -cuts).

*Proof.* For 1., let  $C = \delta(X)$  with  $|X \cap T|$  odd be a  $T$ -cut. Then the edges in  $J \cap C$  either belong to a path passing through  $X$  or have an endpoint in  $T$ . Therefore  $|J \cap C|$  is odd, in particular the set is non-empty.

For 2., we prove in an exercise that each edge set intersecting all  $T$ -joins ( $T$ -cuts) contains a  $T$ -cut ( $T$ -join). Therefore minimal such sets are  $T$ -cuts ( $T$ -joins). Remark: The minimum cardinality of a  $T$ -join is at least as large as the maximum number of edge-disjoint  $T$ -cuts<sup>4</sup>. □

**Theorem 2.14** (Seymour (1981)). Let  $G$  be bipartite,  $T \subseteq V(G)$  such that there exists a  $T$ -join. Then:

$$\min. \text{ cardinality of a } T\text{-join} = \max. \text{ number of edge-disjoint } T\text{-cuts}$$

The maximum is attained by a crossfree family  $\mathcal{C}$  of cuts, i.e.

$$\forall X, Y \in \mathcal{C} : X \subseteq Y \vee Y \subseteq X \vee X \cap Y = \emptyset \vee X \cup Y = V(G)$$

*Proof.* If  $T = \emptyset$ , the statement is clear. Let  $T \neq \emptyset$ . We proceed by induction on  $|V(G)| + |T|$ . Let  $J$  be a minimum-cardinality  $T$ -join. Set:

$$c(e) := \begin{cases} -1 & e \in J \\ 1 & e \in E(G) \setminus J \end{cases}$$

**Claim.** Every circuit  $C$  has  $c(C) \geq 0$ .

---

<sup>4</sup>In general, the two numbers are not equal: Consider  $K_4$  and  $T = V(K_4)$ . A minimum  $T$ -join consists of 2 edges but there are no 2 edge-disjoint  $T$ -cuts.

$$\begin{aligned}
c(C) &= c(C \setminus J) + c(C \cap J) + |J \setminus C| - |J \cap C| \\
&= \left| \underbrace{C \Delta J}_{T\text{-join}} \right| - |J| \geq 0
\end{aligned}$$

Let  $P$  be a minimum length walk in  $(G, c)$  traversing no edge more than once such that  $|E(P)|$  is minimum. Then  $P$  is a path. Let  $t$  be the last vertex in  $P$  and  $f$  the edge entering  $t$ . Then  $f \in J$ , otherwise  $c(f) = 1$  and deleting  $f$  would yield a shorter path. Furthermore,  $|\delta_J(t)| = 1$ , otherwise we could add the other edge from  $J \cap \delta(t)$  to shorten  $c(P)$ .

**Claim.** *Each circuit  $C$  that contains  $t$  but not  $f$  has  $c(C) > 0$ .*

Case 1:  $t$  is the only vertex in  $V(C) \cap V(P)$ . Let  $e \ni t$  be an edge on  $C$  incident to  $t$ . Then  $c(e) = 1$  (since  $\delta_J(t) = \{f\}$ ) and  $P' := P + C - e$  yields a shorter walk if  $c(C) \leq 0$ .

Case 2:  $V(C) \cap V(P)$  contains another vertex  $x$ . Let  $u$  be the last vertex on  $P$  before  $t$  that is also on  $C$ . Define  $P' := P_{[u, t]}$ .  $C$  can be split into 2  $u$ - $t$ -paths  $C', C''$ . By minimality of  $P$ ,  $c(P') < 0$ .  $P' + C', P' + C''$  are circuits (by choice of  $u$ ). By the first claim,  $c(C'), c(C'') > 0$ , so also  $c(C) > 0$ .

*Shrink:*  $\{t\} \cup \Gamma(t)$  to a new vertex  $v_0$ . This yields a bipartite graph  $G'$ . If  $|T \cap (\{t\} \cup \Gamma(t))|$  is odd, set  $T' := T \setminus (\{t\} \cup \Gamma(t)) \cup \{v_0\}$ . Otherwise,  $T' := T \setminus (\{t\} \cup \Gamma(t))$ . Define  $J := J \setminus \{f\}$ .

**Claim.**  *$J'$  is a minimum cardinality  $T'$ -join in  $G'$ .*

If not, there exists a  $T'$ -join  $J''$  with  $|J''| < |J'|$ .  $J'' \Delta J'$  is an  $\emptyset$ -Join. Therefore, there exists a circuit  $C'$  where  $|C' \setminus J'| < |C' \setminus J''| = |C' \cap J'|$  (since  $G$  is bipartite). If  $C'$  results from a circuit  $C$  in  $G$  not containing  $T$ , then  $|C \setminus J| < |C \cap J|$ . This is a contradiction to the minimality of  $J$ .

Therefore  $C'$  results from a circuit containing  $T$ .

Case 1:  $C$  traverses  $f$ . Then

$$\begin{aligned}
|C' \setminus J'| - |C' \cap J'| &= |C \setminus J| - |C \cap J| \\
&> 0
\end{aligned}$$

which is a contradiction.



Case 2: By the second claim,  $c(C) > 0$ , so since  $G$  is bipartite  $c(C) \geq 2$  and  $|C \setminus J| \geq |C \cap J| + 2$ . Therefore

$$\begin{aligned} |C' \setminus J'| &= |C \setminus J| - 2 \\ &\geq |C \cap J| \\ &= |C' \cap J'| \end{aligned}$$

which is a contradiction to the assumption.

By the induction hypothesis on  $G'$ ,  $G'$  has cross-free  $T'$ -cuts  $D_1, \dots, D_{|J'|}$ . Together with  $\delta(t)$ , we get  $|J'| + 1 = |J|$   $T$ -cuts. Since  $\Gamma(t)$  was contracted in  $G'$ , they are cross-free.  $\square$

**Corollary 2.15.** *Let  $G$  be a graph,  $c : E(G) \rightarrow \mathbb{Z}_{\geq 0}$ ,  $T \subseteq V(G)$  such that a  $T$ -join exists. The minimum cost of a  $T$ -join equals half the maximum number of  $T$ -cuts covering each edge  $e$  at most  $2 \cdot c(e)$  times. This maximum is attained by a cross-free family of  $T$ -cuts.*

*Proof.* Let  $E_0 := \{e \in E(G) \mid c(e) = 0\}$ . Contract the connected components in  $(V(G), E_0)$  and replace each  $e \in E(G)$  by a path of length  $2 \cdot c(e) > 0$ . The resulting graph  $G'$  is bipartite. Let

$$T' := \{v \in V(G') \mid v \text{ corresponds to a connected component } X \text{ in } G \text{ with } |X \cap T| \text{ odd}\}$$

Let  $k$  be the minimum cost of a  $T$ -join in  $G$ .

**Claim.** *The minimum cardinality of a  $T'$ -join in  $G'$  is  $2k$ .*

" $\leq$ ": Every  $T$ -join  $J$  in  $J$  corresponds to a  $T'$ -join  $J'$  in  $G'$  with  $|J'| \leq 2c(J)$ .

" $\geq$ ": Let  $J'$  be a  $T'$ -join in  $G'$ .  $J'$  corresponds to an edge set  $J \subseteq E(G)$ . Let  $\bar{T} := T \Delta \{v \in V(G) \mid |\delta(v) \cap J| \text{ odd}\}$ . For each connected component  $X$  in  $(V(G), E_0)$ :

$$|\delta(X) \cap J| \equiv |X \cap T| \pmod{2}$$

Therefore  $|X \cap \bar{T}|$  is even, so by proposition 2.3, there exists a  $\bar{T}$ -join  $\bar{J}$  in  $(V(G), E_0)$ . Then  $J \cup \bar{J}$  is a  $T$ -join of weight  $c(J) = \frac{|J'|}{2}$ .

By theorem 2.14, there exist  $2k$  pairwise disjoint  $T'$ -cuts in  $G'$ . In  $G$  this yields  $2k$   $T$ -cuts such that every edge  $e$  is covered by at most  $2 \cdot c(e)$  cuts and they can be created cross-free.  $\square$

### 2.3.1 $T$ -join Polytope

We define the  $T$ -join polytope:

$$\begin{aligned} P_{T\text{-join}} &:= \text{conv}\{x \in \mathbb{R}^{E(G)} \mid x \text{ incidence vector of a } T\text{-join}\} \\ P_{T\text{-join}}^\uparrow &:= P_{T\text{-join}} + \mathbb{R}_{\geq 0}^{E(G)} \end{aligned}$$

**Corollary 2.16.**  $P_{T\text{-join}}^\uparrow$  is determined by

$$\begin{aligned} x_e &\geq 0 & e \in E(G) \\ x(\delta(X)) &\geq 1 & \forall T\text{-cuts } \delta(X) \end{aligned}$$

*Proof.* " $\subseteq$ " is clear. Assume that the other inclusion does not hold. Then there exists  $w : E(G) \rightarrow \mathbb{Q}$  such that the minimum weight of a  $T$ -join  $\alpha > \min w^t x$  where  $x$  satisfies the stated inequalities. Without loss of generality,  $w \in \mathbb{Z}_{\geq 0}^{E(G)}$ , both cones are identical ( $\mathbb{R}_{\geq 0}^{E(G)}$ ). By corollary 2.15, there exist  $T$ -cuts  $C_1, \dots, C_{2\alpha}$  such that each edge  $e$  is covered at most  $2w(e)$  times.

$$y_C := \frac{1}{2} \text{number of times } C \text{ occurs in } C_1, \dots, C_{2\alpha}$$

Then  $y$  is a feasible solution to the dual:

$$\begin{aligned} &\max_{C \text{ } T\text{-cut}} y_C \\ \text{s.t. } &\sum_{C \text{ } T\text{-cut}, e \in C} y_e \leq w(e) & e \in E(G) \\ &y \geq 0 \end{aligned}$$

$\sum_C y_C = \alpha$  is a lower bound for the minimization problem which is a contradiction to the assumed inequality.  $\square$

## 2.4 Excursus: Gomory-Hu Trees

Let  $G$  be a graph,  $u : E(G) \rightarrow \mathbb{R}_{\geq 0}$ . Find  $\emptyset \subsetneq X \subsetneq V(G)$  minimizing  $u(\delta(X))$ . One approach:  $\binom{|V(G)|}{2}$   $s$ - $t$ -cut computations (this can clearly be reduced to  $|V(G)| - 1$  by fixing  $s$ ).

**Definition 2.17.** For  $s, t \in V(G)$ , denote by  $\lambda_{st}$  the minimum capacity of an  $s$ - $t$ -cut (or *local edge connectivity* of  $s, t$ ).

**Lemma 2.18.** For all  $u, v, w \in V(G)$ :

$$\lambda_{uw} \geq \min\{\lambda_{uv}, \lambda_{vw}\}$$

*Proof.* Let  $\delta(A)$  be a  $u$ - $w$ -cut. If  $v \in A$ , then  $\delta(A)$  is a  $v$ - $w$ -cut, so  $u(\delta(A)) \geq \lambda_{vw}$ . Otherwise,  $\delta(A)$  is a  $u$ - $v$ -cut, so  $u(\delta(A)) \geq \lambda_{uv}$ .  $\square$

**Definition 2.19.** Let  $G$  be a graph,  $u : E(G) \rightarrow \mathbb{R}_{\geq 0}$ . A tree  $T$  is a Gomory-Hu tree for  $(G, u)$  if  $V(T) = V(G)$  and

$$\lambda_{st} = \min_{e \in E(T_{[s,t]})} u(\delta_G(C_e)) \quad \forall s, t \in V(G)$$

where  $C_e$  and  $V(G) \setminus C_e$  are the connected components of  $T - e$ <sup>5</sup>.

**Lemma 2.20.** Given  $(G, u)$  and a tree  $T$  with  $V(T) = V(G)$ :

$T$  Gomory-Hu tree  $\Leftrightarrow \forall e = \{s, t\} \in E(T)$  is a minimum capacity  $s$ - $t$ -cut

*Proof.* " $\Rightarrow$ " follows directly from the definition. For the other direction, let  $s, t \in V(G)$  and  $e = \{u, v\} \in \arg \min_{e \in E(T_{s,t})} \lambda_{uv}$ . Without loss of generality,  $s \in C_e$ ,  $t \in V(G) \setminus C_e$ , so  $\delta(C_e)$  is an  $s$ - $t$ -cut. Therefore:  $\lambda_{st} \leq u(\delta(C_e)) = \lambda_e$  (with  $\lambda_e := \lambda_{uv}$ ). By lemma 2.20 and induction,  $\lambda_{st} \geq \min\{\lambda_{v'w'} \mid \{v', w'\} \in E(T_{[s,t]})\} = \lambda_{uv}$ . Therefore  $\lambda_{st} = \lambda_{uv}$ .  $\square$

Idea: Choose  $r, s \in V(G)$  and compute a minimum capacity  $r$ - $s$ -cut  $\delta(R)$ . Without loss of generality  $r \in R$ . Construct a graph  $G_R$  by shrinking  $S := V(G) \setminus R$  into a single vertex. Find a minimum capacity  $p$ - $q$ -cut (where  $p, q \in R$  are chosen arbitrarily) in  $G_R$ . This partitions  $R$  into 2 parts. Continue this process until  $V(G)$  is partitioned into singletons.

**Lemma 2.21.** Let  $(G, u)$  as above,  $s, t \in V(G)$ ,  $\delta(A)$  a minimum capacity  $s$ - $t$ -cut in  $G$  and  $s', t' \in V(G) \setminus A$ . Let  $(G', u')$  arise from  $(G, u)$  by contracting  $A$  into a single vertex. Then for any minimum capacity  $s'$ - $t'$ -cut  $\delta_{G'}(K \cup \{A\})$  in  $(G', u')$ ,  $\delta_G(K \cup A)$  is a minimum capacity  $s'$ - $t'$ -cut in  $(G, u)$ .

*Proof.* Without loss of generality,  $s \in A$ . We show:  $\exists$  min. capacity  $s'$ - $t'$ -cut  $\delta(A')$  in  $(G, u)$  such that  $A \subseteq A'$ . Let  $\delta(C)$  be any  $s'$ - $t'$ -cut in  $(G, u)$ . Without loss of generality,  $s \in C$ .  $u(\delta(\cdot))$  is a submodular function, i.e.  $u(\delta(A)) + u(\delta(B)) \geq u(\delta(A \cap B)) + u(\delta(A \cup B))$ <sup>6</sup>.

$\delta(A \cap C)$  is an  $s$ - $t$ -cut, so  $u(\delta(A \cap C)) \geq \lambda_{st} = u(\delta(A))$ . Therefore,  $u(\delta(A \cup C)) \leq u(\delta(C)) = \lambda_{s't'}$ . Since  $s' \in A \cup C$ ,  $A \cup C$  is a minimum capacity  $s'$ - $t'$ -cut.  $\square$

In general, we now choose a component  $X$  with  $|X| \geq 2$ . Contract connected components in  $T - \{X\}$ , yielding a graph  $(G', u')$ . Choose  $s, t \in X$ , minimum  $s$ - $t$ -cut  $\delta(A')$  in  $(G', u')$ .  $X = (X \cap A') \dot{\cup} (X \cap (V(G') \setminus A'))$ .

**Lemma 2.22.** At the end of *MinCut*:

1.  $A \dot{\cup} B = V(G)$

<sup>5</sup> $\delta(C_e)$  is called *fundamental cut* induced by  $e$

<sup>6</sup>This holds with equality, if we add  $2u(E(A, B))$  to the right side

2.  $E(A, B)$  is a minimum  $s$ - $t$ -cut in  $(G, u)$

*Proof.* Elements of  $V(T)$  are non-empty subsets of  $V(G)$  and  $V(T)$  form a partition of  $V(G)$ . Therefore  $A \dot{\cup} B$  is a partition of  $V(G)$ . 2. follows from successive application of lemma 2.21 to each connected component of  $T - X$ .  $\square$

**Lemma 2.23.** *At any time before FinishTree:  $w(e) = u(\delta_G(\bigcup_{Z \in C_e} Z))$  for all  $e \in E(T)$ . Moreover,  $\forall e = \{P, Q\} \in E(T)$  there exist  $p \in P, q \in Q$ :  $w(e) = \lambda_{pq}$ .*

*Proof.* At the start,  $E(T) = \emptyset$ . We show that both properties are always satisfied. Let  $X, s, t, A', B', A, B$  as determined by ChooseComponents, Contract and MinCut. Edges in  $E(T) \setminus \delta(X)$  are not affected. For new edges both conditions are true after ModifyTree.

Consider an edge  $e \in \{Y, X\}$  that is replaced by  $e'$  in ModifyTree. Without loss of generality  $Y \subseteq A$ , so  $e' = \{X \cap A, Y\}$ . We show that both statements hold for  $e'$ .  $w(e) = w(e') = u(\delta_G(\bigcup_{Z \in C_e} Z)) = u(\delta_G(\bigcup_{Z \in C_{e'}} Z'))$  so 1. holds. Assume  $p \in X, q \in Y$ :  $\lambda_{pq} = w(e)$ . If  $p \in X \cap A$ , we are done.

If  $p \in X \cap B$ , we claim:  $\lambda_{sq} = \lambda_{pq}$ . This then implies  $w(e') = w(e) = \lambda_{pq} = \lambda_{sq}$ . By lemma 2.20,  $\lambda_{sq} \geq \min\{\lambda_{st}, \lambda_{tp}, \lambda_{pq}\}$ . By lemma 2.22,  $E(A, B)$  is a minimum  $s$ - $t$ -cut. By lemma 2.21 and since  $s, q \in A$ ,  $\lambda_{sq}$  does not change when contracting  $B$ . Adding  $\{t, p\}$  with sufficiently high capacity does not change  $\lambda_{sq}$ . Therefore  $\lambda_{sq} \geq \min\{\lambda_{st}, \lambda_{pq}\} = \lambda_{pq}$  because  $E(A, B)$  is also a  $p$ - $q$ -cut.  $w(e)$  is the capacity of a cut separating  $s, q$ , so  $\lambda_{sq} \leq w(e) = \lambda_{pq}$ .  $\square$

**Theorem 2.24** (Min Cut, Gomory & Hu (1961)). *Every undirected graph  $G$  with edge capacities  $c : E(G) \rightarrow \mathbb{R}_{\geq 0}$  has a Gomory-Hu-tree. It can be computed using  $n - 1$  Min- $s$ - $t$ -cut computations, e.g. in  $O(n^3 \sqrt{m})$  time (using the Push-Relabel algorithm for computing the minimum cuts) where  $n := |V(G)|$  and  $m := |E(G)|$ .*

*Proof.* Algorithm-Hu-Algorithm computes a Gomory-Hu-tree (lemma 2.23). It uses  $n - 1$  iterations in each of which we need  $O(n^2 \sqrt{m})$  for Push-Relabel. Everything else can be handled in  $O(\min\{n^3, n^2 m\})$  time.  $\square$

## 2.5 Finding Minimum-Capacity $T$ -Cuts

**Theorem 2.25** (Padberg & Rao (1987)). *Given a graph  $G, u : E(G) \rightarrow \mathbb{R}_{\geq 0}$ , a Gomory-Hu-tree  $H$  for  $(G, u)$ ,  $T \subseteq V(G)$  ( $|T| \geq 2$  even), a minimum capacity  $T$ -cut can be found among the fundamental cuts of  $H$ . A minimum capacity  $T$ -cut can be computed in  $O(n^3 \sqrt{m})$  time.*

*Proof.* Let  $\delta_G(X)$  be a minimum capacity  $T$ -cut in  $G$ . Let  $J$  be the set of edges in  $E(H)$  for where  $|C_e \cap T|$  is odd (where  $C_e$  is a connected component of  $H - e$ ). For all  $x \in V(G)$ :

$$\begin{aligned} |\delta_J(x)| &\equiv \sum_{e \in \delta_H(x)} |C_e \cap T| \\ &\stackrel{T \text{ even}}{\equiv} |\{x\} \cap T| \pmod{2} \end{aligned}$$

Therefore  $J$  is a  $T$ -join in  $H$ . Since  $T$ -cuts and  $T$ -joins intersect, there is  $f \in J \cap \delta_H(X)$ .

$$\begin{aligned} u(\delta_G(X)) &\geq \min\{u(\delta_G(Y)) \mid |Y \cap f| = 1\} \\ &= u(\delta_G(C_f)) \end{aligned}$$

We conclude that  $\delta_G(C_f)$  is a minimum-capacity  $T$ -cut.  $\square$

## 2.6 $b$ -Matchings

**Definition 2.26.** Let  $G$  be a graph,  $u : E(G) \rightarrow \mathbb{N}_0 \cup \{\infty\}$  and  $b : V(G) \rightarrow \mathbb{N}_0$ . A  $b$ -matching is a function  $f : E(G) \rightarrow \mathbb{N}_0$  such that  $f(e) \leq u(e)$  and  $f(\delta(v)) \leq b(v)$  for all  $e \in E(G)$  and  $v \in V(G)$ .

- If  $u \equiv 1$ , the instance is called *simple*.
- If  $b \equiv 1$ , this is equivalent to a matching.
- If  $f(\delta(v)) = b(v)$  for all  $v \in V(G)$ , it is called *perfect*.
- Simple perfect  $b$ -matchings are called  *$b$ -factors*.

*Example.* A TSP-tour is a 2-factor. Therefore valid inequalities for 2-factors are valid for TSP.

**Theorem 2.27** (Edmonds (1965)). *Let  $G$  be a graph,  $b : V(G) \rightarrow \mathbb{N}$ . The  $b$ -matching polytope of  $(G, \infty)$  is the set of vectors  $x \in \mathbb{R}_{\geq 0}^{E(G)}$  satisfying:*

$$\begin{aligned} x_e &\geq 0 & e \in E(G) \\ x(\delta(v)) &\leq b(v) & v \in V(G) \\ \sum_{e \in E(G[X])} x_e &\leq \lfloor \frac{1}{2} \sum_{v \in X} b(v) \rfloor & X \subseteq V(G) \end{aligned}$$

*Proof.* Clearly, any  $b$ -matching satisfies these inequalities. Let  $x$  satisfy the inequalities. Without loss of generality  $b \geq 1$ . Define  $H$  by splitting each

$v \in V(G)$  into  $b(v)$  copies, i.e.:

$$\begin{aligned} X_v &:= \{(v, i) \mid i \in [b(v)]\} & v \in V(G) \\ V(H) &:= \bigcup_{v \in V(G)} X_v \\ E(H) &:= \{\{v', w'\} \mid \{v, w\} \in E(G), v' \in X_v, w' \in X_w\} \\ y_{e'} &:= \frac{1}{b(v) \cdot b(w)} x_{\{v, w\}} & e' = \{v', w'\} \in E(H), v' \in X_v, w' \in X_w \end{aligned}$$

**Claim.**  $y$  is a convex combination of matchings in  $H$ . Contracting all  $X_v$  ( $v \in V(G)$ ) yields a convex combination of  $b$ -matchings for  $x$ .

We show that  $Y$  is contained in the matching polytope, i.e.:

$$\begin{aligned} y_e &\geq 0 & e \in E(H) \\ \sum_{e \in E(H[A])} y_e &\leq \frac{|A| - 1}{2} & A \subseteq V(H), |A| \text{ odd} \end{aligned}$$

If  $\forall v \in V(H)$ :  $X_v \subseteq A$  or  $X_v \cap A = \emptyset$ , this follows directly from the given inequalities. Otherwise, let  $a, b \in X_v$  such that  $a \in A, b \notin A$ .

$$\begin{aligned} 2 \sum_{e \in E(H[A])} y_e &= \sum_{c \in A \setminus \{a\}} \sum_{e \in E(\{c\}, A \setminus \{c\})} y_e + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e \\ &\leq \sum_{c \in A \setminus \{a\}} \sum_{e \in \delta_H(c) \setminus \{\{c, b\}\}} y_e + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e \\ &= \sum_{c \in A \setminus \{a\}} \sum_{e \in \delta_H(c)} y_e - \underbrace{\sum_{e \in E(\{b\}, A \setminus \{a\})} y_e + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e}_{=0} \\ &\leq |A| - 1 \end{aligned}$$

□

**Theorem 2.28** (Edmonds & Johnson (1970)). *Let  $G$  be a graph,  $u : E(G) \rightarrow \mathbb{N} \cup \{\infty\}$ ,  $b : V(G) \rightarrow \mathbb{N}$ . The  $b$ -matching polytope is given by:*

$$\begin{aligned} x &\geq 0 \\ x &\leq u \\ x(\delta(v)) &\leq b(v) & v \in V(G) \\ \sum_{e \in E(G[X])} x_e + \sum_{e \in F} x_e &\leq \underbrace{\left\lfloor \frac{1}{2} \left( \sum_{v \in X} b(v) + \sum_{e \in F} u(e) \right) \right\rfloor}_{\text{Gomory-Chvátal-Cut}} & X \subseteq V(G), F \subseteq \delta(X) \end{aligned}$$

*Proof.*

" $\subseteq$ ": Let  $x$  be an incidence vector of  $b$ -matchings. Then  $x \leq u$  and  $x(\delta(v)) \leq b(v)$  for all  $v \in V(G)$ .

$$\begin{aligned} \sum_{e \in E(G[X])} x_e + \sum_{e \in F} x_e &= \frac{1}{2} \left( \sum_{v \in X} \sum_{e \in \delta(v)} x_e + \sum_{e \in F} x_e - \sum_{e \in \delta(X) \setminus F} x_e \right) \\ &\leq \frac{1}{2} \left( \sum_{v \in X} b(v) + \sum_{e \in F} u(e) \right) \end{aligned}$$

Since the left hand side is integral, the right hand side can be rounded down.

" $\supseteq$ ": Let  $x$  satisfy all the inequalities. We have to show that  $x$  is a convex combinations of  $b$ -matchings. Let  $H$  arise from  $G$  by subdividing each edge  $e = \{v, w\}$  with  $u(e) \neq \infty$  by 2 new vertices  $(e, v), (e, w)$  and a path  $v-(e, v)-(e, w)-w$ , where  $b((e, v)) = u(e) = b((e, w))$ . Set  $y_{\{v, (e, v)\}} := x_e =: y_{\{(e, w), w\}}$  and  $y_{\{(e, v), (e, w)\}} := u(e) - x_e$ . If  $u(e) = \infty$ ,  $y_e := x_e$ .

**Claim.**  $y$  is in the  $b$ -matching polytope of  $(H, \infty)$ . This then implies that  $x$  is contained in the capacitated  $b$ -matching polytope of  $(G, u)$ .

$y(\delta_H(v)) \leq b(v)$  clearly holds for all  $v \in V(H)$ . Assume that there exists  $A \subseteq V(H)$  with:

$$y(E(H[A])) > \lfloor \frac{1}{2}b(A) \rfloor$$

Let  $B := A \cap V(G)$ . For  $\{v, w\} \in E(G[B])$ , we may assume that  $(e, v), (e, w) \in A$ . If  $(e, v) \in A$ , we may assume  $v \in A$ :

Case 1: If  $(e, w) \in A$ , we can remove  $(e, v)$  and  $(e, w)$ .

Case 2: If  $(e, w) \notin A$ , we can remove  $(e, v)$ .

There are 3 remaining cases. Define:

$$F := \{e = \{v, w\} \in E(G) \mid |A \cap \{(e, v), (e, w)\}| = 1\}$$

Then

$$\begin{aligned} x(E(G[B])) + x(F) &= y(E(H[A])) - \sum_{\substack{e \in E(G[B]) \\ u(e) < \infty}} u(e) \\ &> \lfloor \frac{1}{2}b(A) \rfloor - \sum_{\substack{e \in E(G[B]) \\ u(e) < \infty}} u(e) \\ &= \lfloor \frac{1}{2}(b(B) + \sum_{e \in F} u(e)) \rfloor \end{aligned}$$

which is a contradiction to the feasibility of  $x$ . Therefore,  $y$  satisfies the inequalities w.r.t.  $(H, \infty)$ . Let  $e \in P := b$ -matching polytope for  $(H, \infty)$ , then  $y \in \{z \in P \mid \sum_{e \in \delta(v)} z_e = b(v) \forall v \in V(H) \setminus V(G)\}$ . Therefore,  $y$  is the convex combination of  $b$ -matchings  $f_1, \dots, f_m$  in  $(H, \infty)$  with  $f_i(\delta(v)) = b(v)$  for all  $v \in V(H) \setminus V(G)$ . We get:

$$f_i(\{v, (e, v)\}) = f_i(\{w, (e, w)\}) \leq u(e) \quad \forall e = \{v, w\} \in E(G)$$

Set:

$$f'_i(e) := \begin{cases} f_i(v, (e, v)) & e = \{v, w\} \in E(G), u(e) < \infty \\ f_i(e) & e = \{v, w\} \in E(G), u(e) = \infty \end{cases}$$

Then  $x$  is a convex combination of  $f'_1, \dots, f'_m$  (of  $b$ -matchings).

□

## 2.7 Padberg-Rao Theorem

**Lemma 2.30.** *Let  $G$  be a graph,  $|E(G)| \geq 1$ ,  $T \subseteq V(G)$  with  $|T|$  even,  $c, c' : E(G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ . There exists a  $O(n^2m)$  time algorithm that finds a vertex set  $X \subseteq V(G)$  and  $F \subseteq \delta(X)$  such that  $|X \cap T| + |F|$  is odd and*

$$c(\delta(X) \setminus F) + c'(F)$$

*is minimum.*

*Proof.* Without loss of generality,  $G$  is connected: Otherwise, add edges  $e$  with  $c(e) = 0$  and  $c'(e) = \infty$ . Let

$$\begin{aligned} d(e) &:= \min\{c(e), c'(e)\} & e \in E(G) \\ E' &:= \{e \in E(G) \mid c'(e) < c(e)\} \\ V' &:= \{v \in V(G) \mid |\delta_{E'}(v)| \text{ odd}\} \\ T' &:= T \Delta V' \end{aligned}$$

Since  $E'$  is a  $V'$ -join, for  $X \subseteq V(G)$ :

$$|X \cap T| + |\delta(X) \cap E'| \equiv |X \cap T| + |X \cap T'| \equiv |X \cap T'| \pmod{2}$$

Compute a Gomory-Hu-Tree  $H$  for  $(G, d)$ . For  $f \in E(H)$ , let  $\delta(C_f)$  be the fundamental cut of  $f$  (i.e.  $C_f$  is a connected component in  $H - f$ ). Let  $g_f \in \arg \min_{e \in \delta_G(C_f)} |c(e) - c'(e)|$ . Let:

$$F_f := \begin{cases} \delta_G(C_f) \cap E' & \text{if } |C_f \cap T'| \text{ is odd} \\ \delta_G(C_f) \cap E' \Delta \{g_f\} & \text{else} \end{cases}$$

Finally, choose  $f \in E(H)$  minimizing  $c(\delta(C_f) \setminus F_f) + c'(F_f)$  and output  $C_f, F_f$ . The running time is dominated by the computation of  $H$ .

It remains to show correctness: Let  $X^*, F^*$  be an optimum solution.



Case 1:  $|X^* \cap T'|$  is odd.  $J' := \{f \in E(H) \mid |C_f \cap T'| \text{ odd}\}$  is a  $T'$ -join in  $H$ . Therefore,  $J'$  intersects the  $T'$ -cut  $\delta_H(X^*)$ . Let  $f \in \delta_H(X^*)$  with  $|C_f \cap T'|$  odd. Then  $d(\delta_G(C_f)) \leq d(\delta_G(X^*)) \leq \text{obj}(X^*)$ , since  $H$  is a Gomory-Hu-tree. By construction,  $F_f = \delta_G(C_f) \cap E'$  and:

$$c(\delta_G(C_f) \setminus F_f) + c'(F_f) \leq d(\delta_G(X^*)) \leq \text{obj}(X^*)$$

Case 2:  $|X^* \cap T'|$  is even. Let  $g^* \in \arg \min_{e \in \delta(X^*)} |c(e) - c'(e)|$ .  $H + g^*$  has a unique circuit that contains some  $f \in \delta_H(X^*)$ . Then

$$\begin{aligned} c(\delta_G(X^*) \setminus F^*) + c'(F^*) &= d(\delta(X^*)) + |c(g^*) - c'(g^*)| \\ &\geq d(\delta_G(C_f)) + |c(g^*) - c'(g^*)| \\ &\stackrel{g^* \in \delta_G(C_f)}{\geq} c(\delta_G(C_f) \setminus F_f) + c'(F_f) \end{aligned}$$

□

**Theorem 2.31** (Padberg & Rao (1987)). *Let  $G$  be a graph,  $u : E(G) \rightarrow \mathbb{N} \cup \{\infty\}$  and  $b : V(G) \rightarrow \mathbb{N}$ . Then the separation problem for the  $b$ -matching polytope can be solved in  $O(n^2m)$  time.*

*Proof.*  $0 \leq X \leq u$  and  $x(\delta(v)) \leq b(v)$  for all  $v \in V(G)$  can be checked in linear time. It remains to check:

$$x(E(G[X])) + x(F) \leq \lfloor \frac{1}{2} (b(X) + u(F)) \rfloor \quad X \subseteq V(G), F \subseteq \delta(X)$$

If  $b(X) + u(F)$  is even (i.e. no rounding is done), this is implied by the other inequalities. Otherwise, the inequality is violated iff:

$$b(X) - 2x(E(G[X])) + u(F) - 2x(F) < 1$$

Extend  $G$  to  $H$  by adding a new vertex  $z$  and edges  $\{z, v\}$  for every  $v \in V(G)$ . Set:

$$\begin{aligned} b(z) &:= b(V(G)) \\ T &:= \{v \in V(H) \mid b(v) \text{ odd}\} \\ E' &:= \{e \in E(G) \mid u(e) < \infty \text{ and odd}\} \\ c(e) &:= \begin{cases} x_e & e \in E' \\ \min\{x_e, u(e) - x_e\} & e \in E(G) \setminus E' \\ b(v) - x(\delta(v)) & e = \{z, v\} \in E(H) \end{cases} \\ c'(e) &:= \begin{cases} u(e) - x_e & e \in E' \\ \infty & e \in E(H) \setminus E' \end{cases} \end{aligned}$$

For  $X \subseteq V(G)$ , let  $D_X := \{e \in \delta_G(X) \setminus E' \mid u(e) \leq 2x_e\}$ . Then  $\forall X \subseteq V(G)$ ,  $F \subseteq \delta_G(X) \cap E'$ ,

$$|X \cap T| + |F| \equiv b(X) + u(F \cup D_X) \pmod{2}$$

and:

$$\begin{aligned} c(\delta_H(X) \setminus F) + c'(F) &= b(X) - \sum_{v \in X} x(\delta_G(v)) + \sum_{e \in (\delta_G(X) \cap E') \setminus F} x_e \\ &+ \sum_{e \in \delta_G(X) \setminus E'} \min\{x_e, u(e) - x_e\} + \sum_{e \in F} u(e) - x_e \\ &= b(X) - 2x(E(G[X])) + \sum_{e \in F \cup D_X} u(e) - 2x_e \end{aligned}$$

Apply lemma 2.30 to  $H, T, c, c'$ : If there exists  $X \subseteq V(H)$ ,  $F \subseteq \delta_H(X)$  with  $c(\delta(X) \setminus F) + c'(F) < 1$ , then  $F \subseteq E'$  and without loss of generality  $z \notin X$  (otherwise use the complement). We get

$$b(X) - 2x(E(G[X])) + \sum_{e \in F \cup D_X} u(e) - 2x_e < 1$$

Setting  $F' := F \cup D_X$  yields a violating of the corresponding inequality.

For the other direction, note that if the inequality holds for  $X \subseteq V(G)$  and  $F \subseteq \delta(X)$ , then without loss of generality,  $D_X \subseteq F \subseteq E' \cup D_X$  (since adding edges in  $D_X \setminus F$  increases the violation). Then:

$$c(\delta_H(X) \setminus (F \setminus D_X)) + c'(F \setminus D_X) < 1$$

Therefore, the  $b$ -matching polytope can be separated in polynomial time.  $\square$

**Corollary 2.32.** *The Maximum-Weight  $b$ -Matching Problem can be solved in polynomial time.*

*Proof.* Use the Ellipsoid method together with theorem 2.31.  $\square$

### 3 The TSP Polytope

#### 3.1 The Spanning Tree Polytope

**Theorem 3.1** (Edmonds (1967)). *Let  $G$  be a connected graph,  $n := |V(G)|$ . Then*

$$P_{ST} := \{x \in [0, 1]^{E(G)} \mid x(E(G)) = n-1, \forall \emptyset \neq X \subsetneq V(G) : \sum_{e \in E(G[X])} x_e \leq |X|-1\}$$

*is the convex hull of incidence vectors of spanning trees. It is called the spanning tree polytope.*

*Proof.* Let  $T$  be a spanning tree with incidence vector  $x$ . Then  $x \in P_{ST}$  and as  $x \in \{0, 1\}^{E(G)}$ ,  $x$  is a vertex.

For the other direction, let  $x \in P_{ST} \cap \mathbb{Z}^{E(G)}$ . Then  $x$  cannot contain cycles, so it is a forest. Since  $x(E(G)) = n - 1$ , it is a spanning tree.

**Claim.**  $P_{ST}$  is integral.

Let  $c : E(G) \rightarrow \mathbb{R}$  and  $T$  be a minimum spanning tree produced by Kruskals algorithm. Let  $E(T) := \{f_1, \dots, f_{n-1}\}$  in order of addition, i.e.  $c(f_1) \leq c(f_2) \leq \dots \leq c(f_{n-1})$ . Let  $X_k \subseteq V(G)$  be the connected component in  $(V(G), \{f_1, \dots, f_k\})$  containing  $f_k$ . Let  $x^*$  be the incidence vector of  $T$ .

**Claim.**  $x^*$  is an optimum solution to

$$\begin{aligned} & \min c^t x \\ & \text{s.t.} \quad 1^t x = n - 1 \\ & x(E(G[X])) \leq |X| - 1 \quad \forall \emptyset \subsetneq X \subseteq V(G) \end{aligned}$$

The dual problem is:

$$\begin{aligned} & \max - \sum_{\emptyset \subsetneq X \subseteq V(G)} (|X| - 1) z_X \\ & \text{s.t.} \quad - \sum_{e \subseteq X \subseteq V(G)} z_X \leq c(e) \quad e \in E(G) \\ & \quad \quad \quad z_X \geq 0 \quad \emptyset \subsetneq X \subsetneq V(G) \end{aligned}$$

Construct a dual solution  $z^*$ : For  $k \in \{1, \dots, n-2\}$ , set  $z_{X_k}^* := c(f_l) - c(f_k) \geq 0$  where  $l$  is the minimum index larger than  $k$  with  $X_k \cap f_l \neq \emptyset$ . Define  $z_{V(G)}^* = -c(f_{n-1})$  and  $z_A^* := 0$  for all other  $A \subseteq V(G)$ .

For  $e = \{v, w\} \in E(G)$ :

$$- \sum_{e \subseteq X \subseteq V(G)} z_X = c(f_i) \leq c(e)$$

where  $i$  is the smallest index such that  $e \subseteq X_i$ . Therefore,  $z^*$  is dual feasible. For tree edges, we have equality, so for  $x_e > 0$  the dual constraint is tight. Let  $\emptyset \subsetneq X \subseteq V(G)$  with  $z_X^* > 0$ . Then  $T[X]$  is connected, so the primal constraint is tight. Complementary slackness implies that  $x^*, z^*$  are optimum primal/dual solutions.

*Remark.* If  $c \in \mathbb{Z}^{E(G)}$ , then  $z^*$  is an integral optimum dual solution, so the system is TDI.

□

**Theorem 3.2** (Fulkerson (1974)). *Let  $G$  be a digraph,  $c : E(G) \rightarrow \mathbb{Z}_{\geq 0}$ ,  $r \in V(G)$  such that  $G$  contains an  $r$ -arborescence. Then the minimum weight of an  $r$ -arborescence spanning  $V(G)$  equals the maximum number of  $r$ -cuts  $C_1, \dots, C_t$  (where repetitions are allowed) such that no edge  $e$  is contained in more than  $c(e)$  of the cuts.*

*Proof.* Consider the  $(r\text{-cuts}) \times (\text{edges})$  matrix  $A$ , where

$$A_{Ce} = \begin{cases} 1 & e \in C \\ 0 & \text{otherwise} \end{cases}$$

Consider the LP and its dual:

$$\begin{aligned} \min \{ & c^t x \mid x \in \mathbb{R}^{E(G)}, Ax \geq 1, x \geq 0 \} \\ \max \{ & 1^t y \mid y \in \mathbb{R}^{r\text{-cuts}}, A^t y \leq c, y \geq 0 \} \end{aligned}$$

**Claim.** *The system is TDI.*

*Proof.* Let  $y$  be an optimum dual solution maximizing

$$\sum_{\emptyset \subsetneq X \subseteq V(G) \setminus \{r\}} y_{\delta^-(X)} |X|^2$$

**Claim.**  $\mathcal{F} := \{X \subseteq V(G) \mid y_{\delta^-(X)} > 0\}$  is laminar.

Suppose that there are  $X, Y \in \mathcal{F}$  with  $X \cap Y, X \setminus Y, Y \setminus X \neq \emptyset$ . Let:

$$\begin{aligned} \epsilon &:= \min\{y_{\delta^-(X)}, y_{\delta^-(Y)}\} \\ y'_{\delta^-(X)} &:= y_{\delta^-(X)} - \epsilon \\ y'_{\delta^-(Y)} &:= y_{\delta^-(Y)} - \epsilon \\ y'_{\delta^-(X \cap Y)} &:= y_{\delta^-(X \cap Y)} + \epsilon \\ y'_{\delta^-(X \cup Y)} &:= y_{\delta^-(X \cup Y)} + \epsilon \\ y' &:= y \quad \text{everywhere else} \end{aligned}$$

Then  $y'$  is a dual optimum solution which contradicts the maximality of  $y$ .

By Ghoulia-Houri, if the set of rows can be partitioned  $\mathcal{R} = \mathcal{R}_1 \dot{\cup} \mathcal{R}_2$  such that for all columns  $j$ :

$$\sum_{r \in \mathcal{R}_1} a_{rj} - \sum_{r \in \mathcal{R}_2} a_{rj} \in \{-1, 0, 1\}$$

then  $A$  is totally unimodular. Let  $\mathcal{R}_1, \mathcal{R}_2$  be a partition of the laminar family  $\mathcal{F}$  alternating between each level. Let  $A' \subseteq A$  consist of rows with positive support (i.e. rows in  $\mathcal{F}$ ). Then by this argument,  $A'$  is totally unimodular. In particular, for  $c \in \mathbb{Z}_{\geq 0}$ , we find an integral optimum dual solution.  $\square$

Since the system is TDI, there exists an integral optimum primal solution  $x$ .  $\square$

**Corollary 3.3.** *Let  $G$  be a digraph,  $c : E(G) \rightarrow \mathbb{R}_{\geq 0}$  and  $r \in V(G)$  such that a spanning  $r$ -arborescence exists. Then*

$$\min\{c^t x \mid x \geq 0, x(\delta^+(X)) \geq 1 \ \forall r \in X \subsetneq V(G)\}$$

*has an integral solution which is the incidence vector of a minimum-weight spanning  $r$ -arborescence plus (possibly) edges of weight 0.*

### 3.2 The Held-Karp Polytope

**Proposition 3.4.** *Let  $n \in \mathbb{Z}_{\geq 3}$ . The incidence vectors  $x$  of TSP tours in  $K_n$  are described by:*

$$\begin{aligned} x(\delta(v)) &= 2 & v &\in V(G) \\ x(\delta(X)) &\geq 2 & \emptyset \neq X &\subsetneq V(G) \\ x &\in \{0, 1\}^{E(K_n)} \end{aligned}$$

*Proof.* Integrality and the first inequality imply that  $x$  is the incidence vector of a collection of cycles. By the second inequality (which is called the *subtour elimination constraint*), there is exactly one cycle.  $\square$

Relaxing the integrality (i.e. only requiring  $x \in [0, 1]$ ) yields the *subtour polytope* (or Held-Karp-polytope).

**Proposition 3.5.** *Let  $n \in \mathbb{Z}_{\geq 2}$ ,  $x \in [0, 1]^{E(G)}$  with  $x(\delta(v)) = 2$  for all  $v \in V(K_n)$ . Then the following are equivalent:*

1.  $x(\delta(X)) \geq 2$  for all  $\emptyset \neq X \subsetneq V(G)$  (i.e. 3.4).
2.  $x(E(K_n[X])) \leq |X| - 1$  for all  $\emptyset \neq X \subsetneq V(G)$ .
3.  $x(E(K_n[X])) \leq |X| - 1$  for all  $\emptyset \neq X \subseteq V(K_n) \setminus \{r\}$ .

*Proof.*

$$\begin{aligned}
2 &\leq x(\delta(V(G) \setminus X)) \\
&= x(\delta(X)) \\
&= \sum_{v \in X} x(\delta(v)) - 2x(E(K_n[X])) \\
&= 2|X| - 2x(E(K_n[X]))
\end{aligned}$$

□

**Theorem 3.6** (Wolsey (1980)). *Let  $(K_n, c)$  with  $c$  metric and*

$$P_{HK} = \{x \in \mathbb{R}_{\geq 0}^{E(K_n)} \mid x(\delta(v)) = 2 \forall v \in V(K_n), x(\delta(X)) \geq 2 \forall \emptyset \neq X \subsetneq V(K_n)\}$$

*be the Held-Karp polytope. Then:*

$$\min\{c^t x \mid x \in P_{HK} \cap \mathbb{Z}^{E(K_n)}\} \leq \frac{3}{2} \min\{c^t x \mid x \in P_{HK}\}$$

*Proof.* Let  $x^* \in \arg \min\{c^x \mid x \in P_{HK}\}$ ,  $Y$  be a minimum spanning tree in  $(K_n, c)$  and  $J$  a minimum-weight odd( $Y$ )-join.  $\frac{n-1}{n}x^* \in P_{ST}$  and  $\frac{1}{2}x^* \in P_{\text{odd}(Y)\text{-join}}$ . We get:

$$\begin{aligned}
c(Y) + c(J) &\leq \frac{n-1}{n}c^t x^* + \frac{1}{2}c^t x^* \\
&< \frac{3}{2}c^t x^*
\end{aligned}$$

□

**Conjecture 3.7.** *If for  $(K_n, c)$ ,  $c$  is metric, then:*

$$\min\{c^t x \mid x \in P_{HK} \cap \mathbb{Z}^{E(G)}\} \leq \frac{4}{3} \min\{c^x \mid x \in P_{HK}\}$$

### 3.3 Further Inequalities for the TSP

Consider the *2-matching inequalities*:

$$x(E(G[H])) + x(F) \leq |H| + \lfloor \frac{|F|}{2} \rfloor \quad \forall H \subseteq V(G), F \subseteq \delta(H), |F| \text{ odd}$$

**Theorem 3.8.** *Let  $H, T_1, \dots, T_k \subseteq V(G)$  such that:*

1.  $|H \cap T_i| \geq 1$  for  $i \in [k]$
2.  $|T_i \setminus H| \geq 1$  for  $i \in [k]$

3.  $T_i \cap T_j = \emptyset$  for  $i \neq j$

4.  $k$  is odd

Then

$$x(E(G[H])) + \sum_{i=1}^k x(E(G[T_i])) \leq |H| + \sum_{i=1}^k (|T_i| - 1) - \frac{k+1}{2}$$

is a valid inequality for the TSP polytope. They're called comb inequalities.  $H$  is called handle,  $T_i$  are called teeth and  $(H, T_1, \dots, T_k)$  is a comb.

*Proof.* Let  $(H, T_1, \dots, T_k)$  be a comb. Generate the inequality as a Gomory-Chvátal-cut: Multiply the following inequalities by  $\frac{1}{2}$ , add them together and round:

- $x(\delta(v)) = 2$  for  $v \in H$
- $-x_e \leq 0$  for  $e \in \delta(H) \setminus \bigcup_{i=1}^k E(G[T_i])$
- $x(\delta(X)) \geq 2$  for  $X = T_i, H \cap T_i, T_i \setminus H$  ( $i \in [k]$ )

□

The complexity of comb separation is an open question.

**Theorem 3.9** (Fiorini et al. (1985)). *There is no polyhedron with polynomially many facets, whose projection is the TSP polytope.*

*Proof.* Omitted.

□

**Definition 3.10.** Let  $P \subseteq \mathbb{R}^n$  be a polyhedron. A polyhedron  $Q \subseteq \mathbb{R}^m$  is an *extension* of  $P$  if there exists a projective map  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$  with  $\pi(Q) = P$ . The *extension complexity* of a polyhedron  $P$  is the minimum number of facets of an extension  $Q$  of  $P$ .

Rothvoss (2013) proved that the matching polytope has an exponential extension complexity.

## 4 Matroids & Generalizations

**Definition 4.1.** A set system  $(E, \mathcal{F})$  (where  $\mathcal{F} \subseteq 2^E$ ) is an independent system if:

- i)  $\emptyset \in \mathcal{F}$
- ii)  $X \in \mathcal{F} \Rightarrow \forall Y \subseteq X : Y \in \mathcal{F}$ 
  - Elements in  $\mathcal{F}$  are called *independent*.

- Inclusion-wise maximal sets  $A \in \mathcal{F}$  are called *bases*. Its cardinality is called  $\text{rank}(A)$ .
  - Inclusion-wise minimal sets  $A \in \mathcal{F}$  are *circuits*.
- iii)  $\forall X, Y \in \mathcal{F}$  with  $|X| < |Y|$ :  $\exists y \in Y \setminus X$  such that  $X \cup \{y\} \in \mathcal{F}$ .

This is equivalent to:

- iii)'  $\forall X, Y \in \mathcal{F}$  with  $|X| + 1 = |Y|$ :  $\exists y \in Y$  such that  $X \cup \{y\} \in \mathcal{F}$ .
- iii)"  $\forall X \subseteq E$  and  $A, A' \subseteq X$  maximal with  $A, A' \in \mathcal{F}$ :  $\text{rank}(A) = \text{rank}(A')$ .

If  $\mathcal{M} = (E, \mathcal{F})$  is a matroid, then  $r(\mathcal{M}) = r(E)$ . The rank function is defined by:

$$r : 2^E \rightarrow \mathbb{N}$$

$$r(A) := \max_{B \subseteq A, B \in \mathcal{F}} |B|$$

---

**Algorithm 6:** Greedy Algorithm for independent systems

---

**Input:** Independent system  $(E, \mathcal{F})$ ,  $c : E \rightarrow \mathbb{R}$

**Output:**  $X \in \mathcal{F}$  with the objective of maximizing  $c(X)$

```

1  $X \leftarrow \emptyset$ 
2 while  $\exists x \in X$  with  $c(x) > 0$  and  $X \cup \{x\} \in \mathcal{F}$  do
3   | Choose  $x \in \arg \max_{x \notin X, X \cup \{x\} \in \mathcal{F}} c(x)$ 
4   |  $X \leftarrow X \cup \{x\}$ 
5 return  $X$ 

```

---

**Theorem 4.2.**  $(E, \mathcal{F})$  is a matroid  $\Leftrightarrow$  algorithm 6 finds an optimum solution for every cost function  $c$ .

*Example 4.3.*

- *Cycle matroid:*  $E$  is the edge set of an undirected graph,  $\mathcal{F}$  is the set of forests. Then  $(E, \mathcal{F})$  is a matroid. Matroids that can be represented this way are called *graphic matroids*.
- $A \in \mathbb{R}^{m \times n}$ ,  $E = [n]$  and  $\mathcal{F}$  is the set of linearly independent subsets of  $E$ . This is called a *vector matroid*.
- *Uniform matroid:*  $E$  is a finite set,  $k \in \mathbb{Z}_{\geq 0}$  and  $\mathcal{F} := \{X \subseteq E \mid |X| \leq k\}$ .
- *Matching matroid:*  $G$  is an undirected graph,  $E := V(G)$  and  $\mathcal{F} := \{F \subseteq E \mid \exists \text{ matching in } G \text{ covering } F\}$ .
- *Gammoids:*  $G$  is a graph (directed or undirected),  $E, U \subseteq V(G)$ .  $X \in \mathcal{F}$  if there exist  $|X|$  vertex-disjoint  $U$ - $X$ -paths.



- *Transversal matroid*:  $G$  is a bipartite graph with  $V(G) = E \dot{\cup} U$  and  $(E, U)$  is a gammoid.  $\mathcal{F}$  is the set of subsets of  $E$  that are covered by some matching.

*Example 4.4.* Independent systems that are not matroids:

- Matchings
- Stable sets and cliques
- Subsets of TSP tours or Steiner trees
- Feasible solutions of knapsack problems

**Theorem 4.5** (Edmonds (1970)). *Let  $(E, \mathcal{F})$  be a matroid and  $r : 2^E \rightarrow \mathbb{N}$  its rank function. Then the matroid polytope of  $(E, \mathcal{F})$  (i.e. the convex hull of incidence vectors of independent sets) can be described by:*

$$\{x \in \mathbb{R}^E \mid x \geq 0, \sum_{e \in A} x_e \leq r(A) \ \forall A \subseteq E\}$$

*Proof.* The polytope contains all incidence vectors of independent sets. We have to show that the vertices of the polytope are integral, or equivalently:

$$\max\{c^t x \mid x \geq 0, \sum_{e \in A} x_e \leq r(A) \ \forall A \subseteq E\}$$

attains an integral optimum for all  $c \in \mathbb{R}^E$ . Let  $x^0$  be the incidence vector of the set  $J$  found by the greedy algorithm (algorithm 6).

**Claim.**  $x^0$  is an optimum solution in the polytope.

The dual problem is

$$\begin{aligned} \min \quad & \sum_{A \subseteq E} r(A) y_A \\ \sum_{A \subseteq E, e \in A} y_A & \geq c(e) & e \in E \\ y & \geq 0 \end{aligned}$$

Our goal is to find a dual solution in complementary slackness with  $x^0$ , so  $x_e > 0 \Rightarrow \sum_{A \subseteq E, e \in A} y_A = c(e)$  and  $y_A > 0 \Rightarrow x(A) = r(A)$ .

Consider the Dual Greedy Algorithm:

1. Order  $E$  as  $\{e_1, \dots, e_n\}$  with:

$$c(e_1) \geq \dots \geq c(e_m) \geq 0 \geq c(e_{m+1}) \geq \dots \geq c(e_n)$$

2.  $T_i := \{e_1, \dots, e_i\}$  for  $1 \leq i \leq m$ ,  $T_0 := \emptyset$  and

$$y_A^0 := \begin{cases} c(e_i) - c(e_{i+1}) & A = T_i \text{ for } i \in \{1, \dots, m-1\} \\ c(e_m) & A = T_m \\ 0 & \text{else} \end{cases}$$

$y \geq 0$  and for  $j > m$ ,  $c(e_j) \leq 0$  so the inequality is satisfied. If  $j \leq m$ , then:

$$\sum_{A \subseteq E, e_j \in A} y_A = \sum_{i=j}^m y_{T_i}^0 = c(e_j)$$

Therefore,  $y$  is dual feasible. If  $x_e^0 > 0$ , the corresponding dual constraint is tight. Let  $y_A^0 > 0$ , so  $A = T_i$  for some  $i$ . We have to show that  $x^0(A) = r(A)$ , i.e.  $J \cap T_i$  is a basis of  $T_i$ . If not, there exists  $e_k \in T_i \setminus J$  with  $(J \cap T_i) \cup \{e_k\} \in \mathcal{F}$  and  $c(e_k) > c(e_j)$ . Since the algorithm didn't add  $e_k$ , this is a contradiction.  $\square$

**Corollary 4.6.** *Let  $\mathcal{M} = (E, \mathcal{F})$  be a matroid,  $c \in \mathbb{R}^E$  and  $J \in \mathcal{F}$ . Then  $J$  is a maximum-weight independent set if and only if:*

- a)  $\forall e \in J : c(e) \geq 0$
- b)  $\forall e \notin J, J \cup \{e\} \in \mathcal{F} : c(e) \leq 0$
- c)  $\forall e \notin J, f \in J, (J \cup \{e\}) \setminus \{f\} \in \mathcal{F} : c(e) \leq c(f)$

*Proof.*

" $\Rightarrow$ ": Clear

" $\Leftarrow$ ": Take a dual solution  $y^0$  from the dual greedy algorithm. By a),  $\sum_{e \in A} y_A = c(e)$  for all  $e \in J$ . If there exists  $A \subseteq E$  with  $y_A > 0$  and  $x(A) < r(A)$ , then  $\exists i$  with  $c(e_i) > c(e_{i+1})$  and  $J \cap T_i$  is not a basis of  $T_i = A$ . Therefore, there exists  $e \in T_i \setminus J$  with  $(J \cap T_i) \cup \{e\} \in \mathcal{F}$ . If  $\{e\} \cup J \in \mathcal{F}$ , this would contradict b). Otherwise, extend  $(J \cap T_i) \cup \{e\}$  to a basis  $J'$  of  $J \cup \{e\}$ . Then  $|J'| = |J|$ , so  $J' = (J \cup \{e\}) \setminus \{f\}$  for some  $f \in T_i$ , which is a contradiction to c).  $\square$

**Theorem 4.7.** *Let  $G$  be an undirected graph. The forest polytope of  $G$  is given by:*

$$\{x \in \mathbb{R}^{E(G)} \mid x(E(G[T])) \leq |T| - 1 \ \forall \emptyset \neq T \subseteq V(G)\}$$

*Proof.* Apply theorem 4.5 to the cycle matroid.  $\square$

#### 4.1.1 Matroid Constructions

**Proposition 4.8** (Disjoint Union). *Given matroids  $\mathcal{M}_1 = (E_1, \mathcal{F}_1)$  and  $\mathcal{M}_2 = (E_2, \mathcal{F}_2)$  with  $E_1 \cap E_2 = \emptyset$ ,  $\mathcal{M} := \mathcal{M}_1 \oplus \mathcal{M}_2 := (E, \mathcal{F})$  where  $E = E_1 \dot{\cup} E_2$  and  $\mathcal{F} = \{J_1 \cup J_2 \mid J_1 \in \mathcal{F}_1, J_2 \in \mathcal{F}_2\}$  is a matroid with rank function*

$$r(A) = r(A \cap E_1) + r(A \cap E_2)$$

where  $r_i$  is the rank function of  $\mathcal{M}_i$ .

**Proposition 4.9** (Partition Matroid). *Let  $E = E_1 \dot{\cup} \dots \dot{\cup} E_k$  and  $\mathcal{F} := \{J \subseteq E(G) \mid |J \cap E_i| \leq 1 \forall i \in [k]\}$ . Then  $(E, \mathcal{F})$  is a matroid with rank function:*

$$r(A) = |\{i \in [k] \mid E_i \cap A \neq \emptyset\}|$$

**Proposition 4.10** (Restriction Matroid). *Let  $\mathcal{M} = (E, \mathcal{F})$  be a matroid and  $B \subseteq E$ . Then  $\mathcal{M}' := \mathcal{M} \setminus B := (E \setminus B, \{J \subseteq E \setminus B \mid J \in \mathcal{F}\})$  is a matroid.*

**Proposition 4.11** (Contraction Matroid). *Let  $\mathcal{M} = (E, \mathcal{F})$  be a matroid and  $B \subseteq E$ . Choose an arbitrary basis  $J$  of  $B$  (i.e.  $J \in \mathcal{F}$  and  $r(J) = r(B)$ ). Then  $\mathcal{M}' := \mathcal{M}/B := (E \setminus B, \{J' \subseteq E \setminus B \mid J' \cup J \in \mathcal{F}\})$  is a matroid.  $\mathcal{M}$  is independent of the chosen basis  $J$ . Its rank function is*

$$r'(A) = r(A \cup B) - r(B)$$

**Corollary 4.12.** *Let  $\mathcal{M} = (E, \mathcal{F})$  be a matroid and  $B \subseteq E$ . Then  $\mathcal{M}' := (\mathcal{M} \setminus B) \oplus (\mathcal{M}/(E \setminus B))$  is a matroid on  $E$ . The bases of  $\mathcal{M}'$  are those bases of  $\mathcal{M}$  that intersect  $B$  in a basis of  $B$ .*

**Proposition 4.13** (Matroid Minors). *Let  $\mathcal{M} = (E, \mathcal{F})$  be a matroid and  $\emptyset = T_0 \subseteq T_1 \subseteq \dots \subseteq T_{l+1} = \mathcal{F}$ . The bases of  $T_l$  in  $\mathcal{M}$  that intersect  $T_i$  ( $1 \leq i \leq l$ ) are the bases of  $T_l$  in the matroid  $\mathcal{N} := \mathcal{N}_0 \oplus \dots \oplus \mathcal{N}_l$  where for each  $i$ ,  $\mathcal{N}_i := (\mathcal{M}/T_i) \setminus (E \setminus T_{i+1})$ .  $\mathcal{N}$  is called a minor of  $\mathcal{M}$ .*

#### 4.2 Matroid Intersection

Finding  $\arg \max\{|J| \mid J \in \mathcal{F}_1 \cap \mathcal{F}_2\}$  for matroids  $(E, \mathcal{F}_1)$  and  $(E, \mathcal{F}_2)$  can be done similarly to bipartite matching in  $O(|E|^2)$ . Weighted matroid intersection (of 2 matroids) can also be done in polynomial time.

Computing  $\max\{|J| \mid J \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3\}$  is NP-hard.

#### 4.4 Polymatroids

For the rank function  $r$  of a matroid,  $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$  for all  $X, Y \in E$ , so the rank function is *submodular*.

**Definition 4.34.** A *polymatroid* is the polytope

$$P(f\{x \in \mathbb{R}^{E(G)} \mid x \geq 0, x(A) \leq f(A) \forall A \subseteq E\})$$

where  $E$  is a finite set and  $f : 2^E \rightarrow \mathbb{R}_{\geq 0}$  is submodular.

**Proposition 4.35.** For any polymatroid  $P(f)$ ,  $f$  can be chosen such that  $f(\emptyset) = 0$  and  $f$  is monotone, i.e.  $A \subseteq B$  implies  $f(A) \leq f(B)$ .

**Proposition 4.36.** Let  $E = \{e_1, \dots, e_n\}$ ,  $f : 2^E \rightarrow \mathbb{R}_{\geq 0}$  submodular with  $f(\emptyset) \geq 0$ ,  $B : E \rightarrow \mathbb{R}$  with  $b(e_1) \leq f(e_1)$  and  $b(e_i) \leq f(\{e_1, \dots, e_i\}) - f(\{e_1, \dots, e_{i-1}\})$  for  $i \in \{2, \dots, n\}$ . Then  $\sum_{a \in A} b(a) \leq f(A)$  for all  $A \subseteq E$ .

*Proof.* Induction on  $i = \max\{j \mid e_j \in A\}$ . For  $A = \emptyset$ , the statement is trivial. For  $i \geq 1$ :

$$\begin{aligned} b(A) &= b(A \setminus \{e_i\}) + b(e_i) \\ &\leq f(A \setminus \{e_i\}) + b(e_i) \\ &\leq f(A \setminus \{e_i\}) + f(\{e_1, \dots, e_i\}) - f(\{e_1, \dots, e_{i-1}\}) \\ &\leq f(A) \end{aligned}$$

□

---

**Algorithm 7:** Polymatroid Greedy Algorithm

---

**Input:** Finite set  $E$  and  $f : 2^E \rightarrow \mathbb{R}_{\geq 0}$  submodular and monotone  
(given by an oracle) and  $c : E \rightarrow \mathbb{R}$

**Output:**  $x \in P(f)$  maximizing  $c^t x$

```

1 Sort  $E = \{e_1, \dots, e_n\}$  such that:
   $c(e_1) \geq \dots \geq c(e_k) > 0 \geq c(e_{k+1}) \geq \dots \geq c(e_n)$ 
2 if  $k \geq 1$  then
3    $x_{e_1} \leftarrow f(\{e_1\})$ 
4   for  $i = 2, \dots, k$  do
5      $x_{e_i} \leftarrow f(\{e_1, \dots, e_i\}) - f(\{e_1, \dots, e_{i-1}\})$ 
6   for  $i = k + 1, \dots, n$  do
7      $x_{e_i} \leftarrow 0$ 

```

---

**Theorem 4.37.** The Polymatroid Greedy algorithm correctly finds  $x \in P(f)$  maximizing  $c^t x$ . If  $f$  is integral, then  $x$  is also integral.

*Proof.* Let  $x$  be the output of algorithm 7. If  $f$  is integral,  $x$  is integral by construction. Assume that there exists  $y \in \mathbb{R}_{\geq 0}^E$  with  $c^t y > c^t x$ . For

$i \in [k-1]$ , define  $d_j := c(e_j) - c(e_{j+1})$  and  $d_k := c(e_k)$ .

$$\begin{aligned} \sum_{j=1}^k d_j \sum_{i=1}^j x_i &= c^t x \\ &< c^t y \\ &= \sum_{j=1}^k d_j \sum_{i=1}^j y_i \end{aligned}$$

Therefore, there exists  $j \in [k]$  such that

$$\sum_{i=1}^j y_i > \sum_{i=1}^j x_i = f(\{e_1, \dots, e_j\})$$

so  $y$  is not contained in the polymatroid.  $\square$

**Theorem 4.38.** *Let  $E$  be finite and  $f, g : 2^E \rightarrow \mathbb{R}_{\geq 0}$  submodular. Then*

$$\begin{array}{ll} x(A) \leq f(A) & A \subseteq E \\ x(A) \leq g(A) & A \subseteq E \end{array}$$

*is TDI.*

*Proof.* Consider the primal-dual pair:

$$\begin{array}{ll} \max c^t x & \min_{A \subseteq E} f(A)y_A + g(A)z_A \\ x(A) \leq f(A) \quad A \subseteq E & \sum_{e \in A \subseteq E} (y_A + z_A) \geq c(e) \\ x(A) \leq g(A) \quad A \subseteq E & y, z \geq 0 \end{array}$$

**Claim.** *Let  $Ax \leq b, x \geq 0$  be a linear program. If for any  $c \in \mathbb{Z}^n$  where the dual has a solution, it has an optimum solution  $y_i^*$  such that the rows of  $A$  where  $y_i^* > 0$  (plus possibly basic 0-entries) forms a TU matrix. Then  $Ax \leq b, x \geq 0$  is TDI.*

*Proof.* Let  $c, y^*$  be as above. We have for the dual:

$$\min\{y^t b \mid yA \geq c, y \geq 0\} = \min\{y^t b' \mid yA' \geq c, y \geq 0\}$$

" $\leq$ " is clear. Since the restriction of  $y^*$  is feasible for the right hand side, the other inequality also holds. Since  $A'$  is TU, the right hand system is TDI, so  $y^*$  can be chosen integrally if  $c$  is integral.  $\square$

Let  $c : E \rightarrow \mathbb{Z}_{\geq 0}$  and  $y, z$  be an optimum dual solution such that

$$\sum_{A \subseteq E} (y_A + z_A) \cdot |A| \cdot |E \setminus A|$$

is minimum.

**Claim.**  $\mathcal{F} := \{A \subseteq E \mid y_A > 0\}$  is a chain.

Otherwise, there are  $A, B \in \mathcal{F}$  with  $A \cap B \neq A, B \cap A \neq B$ . Let

$$\begin{aligned} \epsilon &:= \min\{y_A, y_B\} \\ y'_A &:= y_A - \epsilon \\ y'_B &:= y_B - \epsilon \\ y'_{A \cup B} &:= y_{A \cup B} + \epsilon \\ y'_{A \cap B} &:= y_{A \cap B} + \epsilon \\ y_S &:= y_S \quad \text{elsewhere} \end{aligned}$$

$y', z$  is feasible and optimal by submodularity but the term above gets smaller, which is a contradiction. Similarly,  $\mathcal{F}' := \{A \subseteq E \mid z_A > 0\}$  is a chain.

Let  $M, M'$  be the matrices with column set  $E$  and row set  $\mathcal{F}, \mathcal{F}'$ . Then  $\begin{pmatrix} M \\ M' \end{pmatrix}$  is TU:  $A_1 \geq \dots \geq A_p \in \mathcal{F}$  and  $B_1 \geq \dots \geq B_q \in \mathcal{F}'$ . Define

$$\begin{aligned} \mathcal{R}_i &:= \{A_i \mid i \text{ odd}\} \cup \{B_i \mid i \text{ even}\} \\ \mathcal{R}_2 &:= \{A_i \mid i \text{ even}\} \cup \{B_i \mid i \text{ odd}\} \end{aligned}$$

These sets satisfy Ghoulia-Houri, so the system is TDI.  $\square$

**Corollary 4.39.** *Let  $(E, \mathcal{F}_1), (E, \mathcal{F}_2)$  be two matroids. Then the convex hull of incidence vectors  $x \in \mathcal{F}_1 \cap \mathcal{F}_2$  is the polytope*

$$\{x \in \mathbb{R}_{\geq 0}^E \mid x(A) \leq \min\{r_1(A), r_2(A)\} \ \forall A \subseteq E\}$$

where  $r_1, r_2$  are the rank functions of the matroids.

*Proof.* By theorem 4.38, the inequality system is TDI, so since  $r_1, r_2$  are integral, the polytope is integral. Integral vectors in the polytope correspond exactly to incidence vectors of sets in  $\mathcal{F}_1 \cap \mathcal{F}_2$ .  $\square$

**Corollary 4.40.** *Let  $f, g : 2^E \rightarrow \mathbb{R}_{\geq 0}$  be submodular, monotone with  $f(\emptyset) = g(\emptyset) = 0$ . Then:*

$$\underbrace{\max\{\mathbb{1}^t x \mid x \in P(f) \cap P(g)\}}_{(**)} = \min_{A \subseteq E} f(A) + g(E \setminus A)$$

*Proof.* The dual of  $(**)$  is:

$$\min\left\{\sum_{A \subseteq E} (f(A)y_A + g(A)z_A) \mid y, z \geq 0, \sum_{E \supseteq A \ni e} y_A + z_A \geq 1 \ \forall e \in E\right\}$$

" $\geq$ ": By theorem 4.38, the dual has an integral optimum solution  $y, z$ . Let:

$$B := \bigcup_{\substack{A \subseteq E \\ y_A \geq 1}} A \qquad C := \bigcup_{\substack{A \subseteq E \\ z_A \geq 1}} A$$

Since  $y, z$  are integral, the dual constraint implies  $E = B \cup C$ , so  $E \setminus B \subseteq C$ . Therefore:

$$\begin{aligned} \sum_{A \subseteq E} (f(A)y_A + g(A)z_A) &\geq f(B) + g(C) \\ &\geq f(B) + g(E \setminus B) \end{aligned}$$

" $\leq$ ": For  $A \subseteq E$ , we construct the feasible dual solution  $y_A := 1$  and  $z_{E \setminus A} := 0$ , everything else 0 which has cost  $f(A) + g(E \setminus A)$ . By LP-duality, any primal solution attains at most this value.

□

- $f : 2^E \rightarrow \mathbb{R}$  supermodular:

$$f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y) \quad \forall X, Y \subseteq E$$

- $f : 2^E \rightarrow \mathbb{R}$  modular:

$$f(X) + f(Y) = f(X \cup Y) + f(X \cap Y) \quad \forall X, Y \subseteq E$$

- $f(A)$  submodular implies  $f(E \setminus A)$  submodular.

**Corollary 4.41** (Frank's Discrete Sandwich Theorem (1982)). *Let  $E$  be a finite set,  $f : 2^E \rightarrow \mathbb{R}$  supermodular,  $g : 2^E \rightarrow \mathbb{R}$  submodular with  $f(A) \leq g(A)$  for all  $A \subseteq E$ . Then there exists a modular function  $h : 2^E \rightarrow \mathbb{R}$  with  $f(A) \leq h(A) \leq g(A)$  for all  $A \subseteq E$ . If  $f, g$  are integral,  $h$  can be chosen integral.*

*Proof.*

- Without loss of generality,  $f(\emptyset) = g(\emptyset)$  and  $f(E) = g(E)$ .
- Let  $M := 2 \cdot \max\{|f(A)| + |g(A)| \mid A \subseteq E\}$  and:

$$\begin{aligned} f'(A) &:= g(E) - f(E \setminus A) + M \cdot |A| \\ g'(A) &:= g(A) - f(\emptyset) + M \cdot |A| \end{aligned}$$

$f', g'$  are submodular, nonnegative, monotone and  $f'(\emptyset) = 0 = g'(\emptyset)$ .

- By corollary 4.40:

$$\begin{aligned}
& \max\{\mathbb{1}^t x \mid x \in P(f') \cap P(g')\} \\
&= \min_{A \subseteq E} (f'(A) + g'(E \setminus A)) \\
&= \min_{A \subseteq E} (g(E) - f(E \setminus A) + M \cdot |A|) + (g(E \setminus A) - f(\emptyset) + M \cdot |E \setminus A|) \\
&\geq g(E) + M \cdot |E| - f(\emptyset)
\end{aligned}$$

- Let  $x \in P(f') \cap P(g')$  such that  $\mathbb{1}^t x = g(E) - f(\emptyset) + M \cdot |E|$ . If  $f, g$  are integral, we can choose it such that  $x \in \mathbb{Z}^E$ .
- Define:

$$\begin{aligned}
h'(A) &:= \sum_{e \in A} x_e & A \subseteq E \\
h(A) &:= h'(A) + f(\emptyset) - M \cdot |A| & A \subseteq E
\end{aligned}$$

Then  $h$  is modular and for  $A \subseteq E$ :

$$\begin{aligned}
h(A) &\leq g'(A) + f(\emptyset) - M \cdot |A| \\
&= g(A) \\
h(A) &= \mathbb{1}^t x - h'(E \setminus A) + f(\emptyset) - M \cdot |A| \\
&\geq g(E) + M \cdot |E \setminus A| - f'(E \setminus A) \\
&= f(A)
\end{aligned}$$

□

**Definition 4.42.** Let  $f : 2^E \rightarrow \mathbb{R}$  be a function. For  $x \in \mathbb{R}_{\geq 0}^E$ , there exist unique  $k \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_k > 0$  and sets  $\emptyset \subsetneq T_1 \subsetneq \dots \subsetneq T_k \subseteq E$  such that  $x = \sum_{i=1}^k \lambda_i \chi^{T_i}$  where  $\chi^{T_i}$  is the incidence vector of  $T_i$ . The Lovász extension of  $f$  is defined as:

$$\begin{aligned}
f' : \mathbb{R}_{\geq 0}^E &\rightarrow \mathbb{R} \\
x &\mapsto \sum_{i=1}^k \lambda_i f(T_i)
\end{aligned}$$

**Lemma 4.43.** Let  $f : 2^E \rightarrow \mathbb{R}$  be submodular and  $f'$  its Lovász extension. Then:

$$f'(x) = \max\{x^t y \mid y \in P(f)\}$$

*Proof.* Exercise

□

**Theorem 4.44.**

$$f \text{ submodular} \Leftrightarrow f' \text{ convex}$$



#### 4.4.1 Applications of Matroid Intersection

**Orientations:** Let  $G$  be an undirected graph and  $k : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ . Does there exist an orientation  $\vec{G}$  of  $G$  such that  $|\delta_{\vec{G}}^-(v)| \leq k(v)$  for all  $v \in V(G)$ ?

Let  $D := (V(G), \{(v, w), (w, v) \mid \{v, w\} \in E(G)\})$ . We define:

- $(A, \mathcal{F}_1)$  as the partition matroid on  $\bigsqcup_{\{v, w\} \in E(G)} \{(v, w), (w, v)\}$
- $(A, \mathcal{F}_2)$  as the (generalized) partition matroid on  $\bigsqcup_{v \in V(G)} \delta_D^-(v)$  allowing  $\leq k$  elements from  $\delta_D^-(v)$  for all  $v \in V(G)$ .

Then such an orientation  $\vec{G}$  exists  $\Leftrightarrow$  there exists  $F \in \mathcal{F}_1 \cap \mathcal{F}_2$  with  $|F| = |E|$ .

**Theorem.**  $G$  has an orientation  $\vec{G}$  such that  $|\delta_{\vec{G}}^-(v)| \leq k(v)$  for all  $v \in V(G)$  if and only if:

$$\forall P \subseteq V(G) : |E(G[P])| \leq \sum_{v \in P} k(v)$$

**Two disjoint spanning trees:** For a matroid  $\mathcal{M} = (E, \mathcal{F})$  we define  $\mathcal{M}^* := (E, \mathcal{F}^*)$  where:

$$\mathcal{F}^* := \{A \subseteq E \mid E \setminus A \text{ contains a basis of } \mathcal{F}\}$$

$\mathcal{M}^*$  is a matroid with rank function  $r_{\mathcal{M}^*}(X) = |X| + r_{\mathcal{M}}(E \setminus X) - |E|$ .

**Proposition.** Let  $G$  be a graph and  $\mathcal{M} = (E, \mathcal{F})$  its graphic matroid. Then:

$$G \text{ has 2 disjoint spanning trees} \Leftrightarrow \max_{I \in \mathcal{F} \cap \mathcal{F}^*} |I| = |V(G)| - 1$$

#### 4.5 Submodular Function Maximization

Recall:  $f : 2^E \rightarrow \mathbb{R}$  is called submodular if  $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$  for all  $A, B \subseteq E$ . Equivalently,  $f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$  for all  $X \subseteq Y \subseteq E$  and  $x \in E \setminus Y$ .

**Problem** (USM: "unconstrained submodular function maximization"). Given a submodular function  $f : 2^E \rightarrow \mathbb{R}$ , find  $S \subseteq E$  maximizing  $f(S)$ .

*Example.* For a given graph  $G$ , define  $f(X) := |\delta(X)|$ . Maximizing  $f(X)$  corresponds to the maximum cut problem (which is NP-hard).

---

**Algorithm 8:** Deterministic Double Greedy

---

**Input:** Finite set  $E$ , submodular function  $f : 2^E \rightarrow \mathbb{R}^+$

**Output:**  $S \subseteq E$

```
1  $X_0 \leftarrow \emptyset, Y_0 \leftarrow E$ 
2 for  $i = 1, \dots, n$  do
3    $a_i \leftarrow f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})$ 
4    $b_i \leftarrow f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})$ 
5   if  $a_i \geq b_i$  then
6      $X_i \leftarrow X_{i-1} \cup \{e_i\}, Y_i \leftarrow Y_{i-1}$ 
7   else
8      $X_i \leftarrow X_{i-1}, Y_i \leftarrow Y_{i-1} \setminus \{e_i\}$ 
9 return  $S \leftarrow X_n$ 
```

---

**Lemma 4.45.** *For every  $1 \leq i \leq n$ ,  $a_i + b_i \geq 0$ .*

*Proof.* By the equivalent characterization of submodularity and since  $X_i \subseteq Y_i$  for all  $i$ :

$$\begin{aligned} a_i + b_i &= f(X_{i-1} \cup \{e_i\}) - f(X_{i-1}) + f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) \\ &= (f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})) - (f(Y_{i-1}) - f(Y_{i-1} \setminus \{e_i\})) \\ &\geq 0 \end{aligned}$$

□

Let  $\text{OPT}$  be the optimum solution and  $\text{OPT}_i := (\text{OPT} \cup X_i) \cap Y_i$ , so  $\text{OPT}_i$  coincides with  $X_i$  and  $Y_i$  on the first  $i$  elements and with  $\text{OPT}$  on the rest. In particular,  $\text{OPT}_0 = \text{OPT}$  and  $\text{OPT}_n = X_n$ .

**Lemma 4.46.** *For every  $1 \leq i \leq n$ , we have:*

$$f(\text{OPT}_{i-1}) - f(\text{OPT}_i) \leq (f(X_i) - f(X_{i-1})) + (f(Y_i) - f(Y_{i-1}))$$

*Proof.* Without loss of generality assume that  $a_i \geq b_i$ , so the second summand is 0. Then  $\text{OPT}_i = \text{OPT}_{i-1} \cup \{e_i\}$ . We need to show:

$$f(\text{OPT}_{i-1}) - f(\text{OPT}_i) \leq f(X_i) - f(X_{i-1}) = a_i$$

Case 1:  $e_i \in \text{OPT}_{i-1}$ . Then the left side is 0 and so by lemma 4.45,  $a_i \geq 0$ .

Case 2:  $e_i \notin \text{OPT}_{i-1}$ . Then

$$\text{OPT}_{i-1} = (\text{OPT} \cup X_{i-1}) \cap Y_{i-1} \subseteq Y_{i-1} \setminus \{e_i\}$$

so by submodularity:

$$\begin{aligned} f(\text{OPT}_{i-1}) - f(\text{OPT}_{i-1} \cup \{e_i\}) &\leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) \\ &= b_i \leq a_i \end{aligned}$$

□

**Theorem 4.47** (Buchbinder et al.). *Algorithm 8 returns a  $\frac{1}{3}$ -approximation for USM.*

*Proof.* By lemma 4.46:

$$\sum_{i=1}^n (f(\text{OPT}_{i-1}) - f(\text{OPT}_i)) \leq \sum_{i=1}^n (f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}))$$

Since both sides are telescopic sums:

$$\begin{aligned} f(\text{OPT}_0) - f(\text{OPT}_n) &\leq f(X_n) - f(X_0) + f(Y_n) - f(Y_0) \\ &\leq f(\underbrace{X_n}_{=\text{OPT}_n}) + f(\underbrace{Y_n}_{=\text{OPT}_n}) \end{aligned}$$

In total,  $f(\text{OPT}_0) \leq 3f(\text{OPT}_n)$ . □

*Remark 4.48.* If  $f$  is arbitrary, we can simply add a constant to it to make it non-negative. The analysis is tight.

#### 4.5.2 Randomized USM

---

**Algorithm 9:** Randomized Double Greedy

---

**Input:** Finite set  $E$ , submodular function  $f : 2^E \rightarrow \mathbb{R}_+$

**Output:**  $S \subseteq E$

```

1  $X_0 \leftarrow \emptyset, Y_0 \leftarrow E$ 
2 for  $i = 1, \dots, n$  do
3    $a_i \leftarrow f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})$ 
4    $b_i \leftarrow f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})$ 
5    $p \leftarrow \begin{cases} 1 & b_i \leq 0 \\ 0 & a_i \leq 0 \\ \frac{a_i}{a_i + b_i} & \text{else} \end{cases}$ 
6   with probability  $p$  do
7      $X_i \leftarrow X_{i-1} \cup \{e_i\}, Y_i \leftarrow Y_{i-1}$ 
8   else
9      $X_i \leftarrow X_{i-1}, Y_i \leftarrow Y_{i-1} \setminus \{e_i\}$ 
10 return  $X_n$ 
```

---

**Lemma 4.49.** *For  $i \in \{1, \dots, n\}$ :*

$$\mathbb{E} \left[ \underbrace{f(\text{OPT}_{i-1}) - f(\text{OPT}_i)}_{\text{I}} \right] \leq \frac{1}{2} \mathbb{E} \left[ f(X_i) - f(X_{i-1}) + \underbrace{f(Y_i) - f(Y_{i-1})}_{\text{II}} \right]$$

*Proof.* We can consider each  $X_{i-1}$  separately, so we condition on some event of the form  $X_{i-1} = S_{i-1}$  where  $S_{i-1} \subseteq \{e_1, \dots, e_{i-1}\}$  is fixed and the probability that  $X_{i-1} = S_{i-1}$  is non-zero.

Case 1:  $b_i \leq 0$ . Then  $p = 1$  and  $Y_i = Y_{i-1} = S_{i-1} \cup \{e_i, \dots, e_n\}$  and  $X_i = S_{i-1} \cup \{e_i\}$ .

**Claim.**

$$f(\text{OPT}_{i-1}) - f(\text{OPT}_{i-1} \cup \{e_i\}) \leq \frac{1}{2}f(X_i) - f(X_{i-1}) = \frac{a_i}{2}$$

- If  $e_i \in \text{OPT}$ ,  $0 \leq \frac{a_i}{2}$ .
- If  $e_i \notin \text{OPT}$ , then by submodularity:

$$f(\text{OPT}_{i-1}) - f(\text{OPT}_{i-1} \cup \{e_i\}) \leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) = b_i \leq 0 \leq \frac{a_i}{2}$$

The statement then follows directly from the claim.

Case 2:  $a_i \leq 0$ . This is analogous to case 1.

Case 3:  $a_i, b_i > 0$ .

$$\begin{aligned} & \mathbb{E}[f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1})] \\ &= p \cdot (f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})) + (1-p) \cdot (f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})) \\ &= \frac{a_i^2 + b_i^2}{a_i + b_i} \end{aligned}$$

We have found a value for the right side of the inequality. Now, we upper-bound the left side.

$$\begin{aligned} & \mathbb{E}[f(\text{OPT}_{i-1}) - f(\text{OPT}_i)] \\ &= p(f(\text{OPT}_{i-1}) - f(\text{OPT}_{i-1} \cup \{e_i\})) \\ &+ (1-p) \underbrace{(f(\text{OPT}_{i-1}) - f(\text{OPT}_{i-1} \setminus \{e_i\}))}_{\text{III}} \\ &\stackrel{(*)}{\leq} \frac{a_i b_i}{a_i + b_i} \end{aligned}$$

To see (\*):

Case 3.1: If  $e_i \notin \text{OPT}_{i-1}$ , then III is 0 and as  $\text{OPT}_{i-1} = (\text{OPT} \cup X_{i-1}) \cap Y_{i-1}$  by submodularity:

$$f(\text{OPT}_{i-1}) - f(\text{OPT}_{i-1} \cup \{e_i\}) \leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) = b_i$$

Case 3.2: If  $e_i \in \text{OPT}_{i-1}$ , then the first term of the LHS is 0. By submodularity:

$$f(\text{OPT}_{i-1}) - f(\text{OPT}_{i-1} \setminus \{e_i\}) \leq f(X_{i-1} \cup \{e_i\}) - f(X_{i-1}) = a_i$$

Now  $\frac{a_i b_i}{a_i + b_i} \leq \frac{1}{2} \frac{a_i^2 + b_i^2}{a_i + b_i}$  by the binomial formula.

□

**Theorem 4.50.** *Algorithm 9 returns a solution  $S$  with*

$$\mathbb{E}[f(S)] \geq \frac{f(\text{OPT})}{2}$$

*Proof.* Summing up lemma 4.49 for all  $i \in \{1, \dots, n\}$  and collapsing the telescopic sums yields:

$$\begin{aligned} \mathbb{E}[f(\text{OPT}_0) - f(\text{OPT}_n)] &\leq \frac{1}{2} \mathbb{E}[f(X_n) - f(X_0) + f(Y_n) - f(Y_0)] \\ &\leq \frac{\mathbb{E}[f(X_n) + f(Y_n)]}{2} \end{aligned}$$

In total,  $\mathbb{E}[f(\text{OPT}_n)] \geq \frac{f(\text{OPT})}{2}$ .

□

*Remark.* There is no  $\frac{1}{2} + \epsilon$ -approximation for  $\epsilon > 0$  that only uses a polynomial number of oracle calls.

## 4.6 Submodular Function Minimization

**Problem** (Submodular Function Minimization). Given a finite set  $U$  and a submodular function  $f : 2^U \rightarrow \mathbb{R}$  with  $f(\emptyset) = 0$ , find a set  $S \subseteq U$  with  $f(S)$  minimum.

**Definition 4.53.** Let  $U$  be finite and  $f : 2^U \rightarrow \mathbb{R}$  submodular. Then the *base polyhedron* is defined as:

$$B(f) := \{x \in \mathbb{R}^U \mid x(A) \leq f(A) \ \forall A \subseteq U\}, \ x(U) = f(U)$$

*Example.* Let  $U = \{1, 2\}$  and  $f(\{1\}) = 2$ ,  $f(\{2\}) = -2$ ,  $f(\{1, 2\}) = -1$ .

**Theorem 4.54.** *The vertices of the base polyhedron are given by the vectors  $b^<$  for all total orders  $<$  of  $U$  where:*

$$b^<(u) := f(\{v \in U \mid v \leq u\}) - f(\{v \in U \mid v < u\})$$

*Proof.* Exercise

□

**Theorem 4.55.** *Let  $f : 2^U \rightarrow \mathbb{R}$  be submodular,  $f(\emptyset) = 0$ . Then*

$$\min_{S \subseteq U} f(S) = \max\{x^-(U) \mid x \in B(f)\}$$

where  $x^-(U) = \sum_{u \in U} x^-(u) = \sum_{u \in U} \min\{0, x(u)\}$ .

*Proof.* Exercise □

*Idea:* Maintain  $x \in B(f)$  and represent it by a convex combination of the vertices. By Carathéodory,  $|U|$  vertices are enough.

---

**Algorithm 10:** Schrijver's Algorithm

---

**Input:** Finite set  $U = \{1, \dots, n\}$ , submodular function  $f : 2^U \rightarrow \mathbb{R}$   
with  $f(\emptyset) = 0$

**Output:**  $X \subseteq U$  with  $f(X)$  minimum

```

1  $k \leftarrow 1, <_1 \leftarrow$  any total order on  $U, x \leftarrow b^{<_1}$ 
2 Build Graph:
3    $D \leftarrow (U, A)$  where  $A = \{(u, v) \mid u <_i v \text{ for some } 1 \leq i \leq k\}$ 
4    $P \leftarrow \{u \in U \mid x(u) > 0\}$ 
5    $N \leftarrow \{u \in U \mid x(u) < 0\}$ 
6    $X \leftarrow$  set of vertices not reachable from  $P$  in  $D$ 
7   if  $N \subseteq X$  then
8     return  $X$ 
9 Find Augmentation:
10  Let  $d(v)$  denote the distance from  $P$  to  $v$  in  $D$ 
11  Choose  $t \in N \setminus X$  with  $(d(t), t)$  lexicographically maximum
12  Choose  $s$  maximal with  $(s, t) \in A$  and  $d(s) = d(t) - 1$ 
13  Let  $i \in \{1, \dots, k\}$  such that  $\alpha = |\{v \in U \mid s <_i v \leq_i t\}|$  is maximum.
    [Let  $\beta$  be the number of indices attaining  $\alpha$ ]
14 Change Solution:
15  Compute  $0 \leq \epsilon \leq -x(t)$  and write  $x' = x + \epsilon(\chi^t - \chi^s)$  as an explicit
    convex combination of  $\leq n$  vectors from  $b^{<_1}, \dots, b^{<_k}$  and
     $b^{<_i^{s,u}} \forall s <_i u \leq_i t$  (where  $b^{<_i^{s,u}}$  arises from  $b^{<_i}$  by placing  $u$  directly
    before  $s$ ) such that  $b^{<_i}$  does not occur if  $x'(t) < 0$ 
16   $x \leftarrow x'$ , rename the vectors in the convex combination of  $x$  as
     $b^{<_1}, \dots, b^{<_{k'}}, k \leftarrow k'$ 
17 go to Build Graph

```

---

*Example.* In the example from above, let  $<_1$  be  $1 <_1 2$ ,  $k = 1$  and  $x = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ .

**Theorem 4.56.** *Algorithm 10 returns an optimum solution if it terminates.*

*Proof.* If the algorithm terminates,  $D$  does not contain a  $P$ - $N$ -path. Since  $N \subseteq X \subseteq U \setminus P$ ,  $\sum_{u \in X} x(u) \leq \sum_{w \in W} x(w)$  for all  $W \subseteq U$ . No edge enters  $X$ , so  $X = \emptyset$  or for all  $j \in \{1, \dots, k\}$  there exists  $v \in X$  with  $X = \{u \in U \mid u \leq_j v\}$ . Therefore  $\sum_{u \in X} b^{<_j}(u) = f(X)$  for all  $j$  (by definition of  $b_u^{<}$ ). By theorem 4.36 (and again the definition of  $b_u^{<}$ ),  $\sum_{u \in W} b^{<_j}(u) \leq f(W) \forall W \subseteq U, j \in \{1, \dots, k\}$ . We get (where  $\lambda_j$  are the factors in the

convex combination):

$$\begin{aligned}
f(W) &\geq \sum_{j=1}^k \lambda_j \sum_{u \in W} b^{<_j}(u) \\
&= \sum_{u \in W} \sum_{i=1}^k \lambda_j b^{<_j}(u) \\
&= \sum_{u \in W} x(u) \\
&\geq \sum_{u \in X} x(u) \\
&= \sum_{u \in X} \sum_{j=1}^k \lambda_j b^{<_j}(u) \\
&= \sum_{j=1}^k \lambda_j \sum_{u \in X} b^{<_j}(u) \\
&= f(X)
\end{aligned}$$

□

**Theorem 4.57.** *Each iteration can be performed in  $O(n^3 + \gamma n^2)$  time where  $\gamma$  is the time required for an oracle call.*

*Proof.* BuildGraph and FindAugmentation can both be implemented in  $O(n^3)$ . We need to show that ChangeSolution can be done in  $O(n^3 + \gamma n^2)$ . Let  $x = \lambda_1 b^{<_1} + \dots + \lambda_k b^{<_k}$  and  $s <_i t$ .

**Claim.** *For some  $\delta > 0$ ,  $\delta(\chi^t - \chi^s)$  can be written as a convex combination of the vectors  $b^{<_i^{s,u}} - b^{<_i}$  for  $s <_i u \leq_i t$  in  $O(\gamma n^2)$  time.*

How do  $b^{<_i^{s,u}}$  and  $b^{<_i}$  compare?

- Let  $s <_i v \leq_i t$ . Then by definition  $b^{<_i^{s,v}}(u) = b^{<_i}(u)$  for  $u <_i s$  or  $u >_i v$ .
- For  $s \leq_i u <_i v$ :  $b^{<_i^{s,v}}(u) = f(\{w \in U \mid w \leq_i^{s,v} u\}) - f(\{w \in U \mid w <_i^{s,v} u\}) \leq f(\{w \in U \mid w \leq_i u\}) - f(\{w \in U \mid w <_i u\}) = b^{<_i}(u)$  by submodularity.
- For  $u = v$  we have by submodularity:

$$\begin{aligned}
b^{<_i^{s,v}}(u) &= f(\{w \in U \mid w \leq_i^{s,v} u\}) - f(\{w \in U \mid w <_i^{s,v} u\}) \\
&\geq f(\{w \in U \mid w \leq_i u\}) - f(\{w \in U \mid w <_i u\}) \\
&= b^{<_i}(u)
\end{aligned}$$

*Proof of claim:*

- If  $\exists s <_i v <_i t$  such that  $b^{<_{i^{s,v}}}(v) = b^{<_i}(v)$  choose  $\delta = 0$  and  $\lambda_v = 1$ .
- Otherwise for all  $s <_i v \leq_i t$  we have  $b^{<_{i^{s,v}}}(v) > b^{<_i}(v)$ . Look at the matrix  $M = (b^{<_{i^{s,v}}} - b^{<_i})_{vu}$  with rows  $s <_i v \leq_i t$  and columns for  $u \in U$ . Then

$$\chi^t - \chi^s = \sum_{s <_i v \leq_i t} \kappa_v (b^{<_{i^{s,v}}} - b^{<_i})$$

is a non-negative combination for

$$\kappa_v = \frac{\chi_v^t - \sum_{v <_i w \leq_i t} \kappa_w (b^{<_{i^{s,w}}}(v) - b^{<_i}(v))}{b^{<_{i^{s,v}}}(v) - b^{<_i}(v)}$$

- By scaling, we get a convex combination.

Set  $\epsilon := \min\{\lambda_i \delta, -x(t)\}$ .

- If  $\epsilon = \lambda_i \delta$  then:

$$x' = \sum_{j=1}^k \lambda_j b^{<_i} + \lambda_i \sum_{s <_i v \leq_i t} \kappa_v (b^{<_{i^{s,v}}} - b^{<_i})$$

$b^{<_i}$  cancels out.

- Otherwise,  $x'(t) = 0$ .

We can then use Gaussian elimination to get  $\leq n$  vectors in  $O(n^3)$ .  $\square$

**Theorem 4.58.** *The number of iterations is bounded by  $O(n^5)$ .*

*Proof.*

**Claim.**  *$d(w)$  never decreases for  $w \in U$ .*

If  $(v, w)$  was added after a new vertex  $b^{<_{i^{s,u}}}$  was added to the convex combination in ChangeSolution, then  $s \leq_i w <_i v \leq t$  in that iteration. In particular  $d(w) \leq d(s) + 1 = d(t) \leq d(v) + 1$ , so adding the edge  $(v, w)$  does not decrease  $d(w)$ . Additionally, ChangeSolution does not add any elements to  $P$  which proves the claim.

We call a sequence of iterations with the same  $s$  and  $t$  a *block*. Each block has  $O(n^2)$  iterations as the pair  $(\alpha, \beta)$  decreases lexicographically.

**Claim.** *The number of blocks is bounded by  $O(n^3)$ .*

We consider different reasons for ending a block:



- a)  $d(v)$  increases for some  $v \in U$ , in which case  $v$  may become the new  $t$  or  $s$ .
- b)  $t$  is removed from  $N$ .
- c)  $(s, t)$  is removed from  $A$ .

We now bound the number of blocks of each type:

- The number of blocks of type a) is bounded by  $O(n^2)$  since  $d(w)$  never decreases.
- We claim that for all  $t^* \in U$  there are at most  $O(n^2)$  iterations with  $t = t^*$  and  $x'(t) = 0$ : Between such iterations some  $d(v)$  ( $v \in U$ ) must change. We have just shown that this only happens  $O(n^2)$  times. Since there are  $n$  choices for  $t^*$ , there are  $O(n^3)$  blocks of type b).
- We claim that there are  $O(n^3)$  types of type c). It suffices to show that  $d(t)$  changes between 2 blocks with the pair  $(s, t)$ . For  $s, t \in U$ , call  $s$  *t-boring* if one of the following holds:

- $(s, t) \notin A$  or
- $d(t) \leq d(s)$

Let  $s^*, t^* \in U$  and consider the time after a block  $s = s^*, t = t^*$  is ending because  $(s^*, t^*)$  is removed from  $A$  until a subsequent increase of  $d(t^*)$ .

We prove that each  $v \in \{s^*, \dots, n\}$  is  $t^*$ -boring during this period. At the beginning, each  $v \in \{s^* + 1, \dots, n\}$  is  $t^*$ -boring by the maximal choice of  $s^*$ .  $s^*$  is  $t^*$ -boring because the arc  $(s^*, t^*)$  was removed. As  $d(t^*)$  remains constant and  $d$  never decreases, we only need to check the introduction of new arcs.

Suppose for  $v \in \{s^*, \dots, n\}$ ,  $(v, t^*)$  is added in an iteration with pair  $(s, t)$ . Then  $s \leq_i t^* <_i v \leq_i t$ , so  $d(t^*) \leq d(s) + 1 = d(t) \leq d(v) + 1$ .

Case 1:  $s > v$ . Then  $d(t^*) \leq d(s)$ , either because  $s = t^*$  or  $s$  was  $t^*$ -boring and  $(s, t^*) \in A$ .

Case 2:  $s < v$ . Then  $d(t) \leq d(v)$ , either because  $v = t$  or by choice of  $s$  and since  $(v, t) \in A$ .

In either case, we have one strict inequality, so  $d(t^*) \leq d(v)$  and  $v$  remains  $t^*$ -boring as claimed.

$d(t)$  can increase  $O(n)$  times and there are  $O(n^2)$  pairs  $(s, t)$ .

In total, the total number of iterations is:

$$O(n^5) = \underbrace{O(n^2)}_{\text{iterations per block}} \cdot \underbrace{O(n^3)}_{\text{number of blocks}}$$

□

**Theorem 4.59.** *The submodular function minimization problem can be solved in time  $O(n^8 + n^7\gamma)$ , where  $\gamma$  is the time required for a call to the function oracle.*

**Corollary 4.60.** *Linear functions over the intersection of 2 polymatroids can be optimized in polynomial time.*

*Remark.*

- The fastest known algorithm has a running time of  $O(n^6 + n^5\gamma)$  (Orlin, 2009 and Sidford, Wong, Lee, 2015).
- There is also a weakly polynomial algorithm  $O((n^5 + n^4\gamma)(\log M))$  where  $M = \max_X f(X)$ .

*Remark.*  $[0, 1]^n$  can be partitioned into  $n!$   $n$ -simplices (induced by the  $n!$  orders on  $\{1, \dots, n\}$ ). For each simplex, there exists a unique linear interpolation/extension of a function on the corners of the simplex to its interior. This corresponds to the definition of the Lovász extension.

In particular, a function is submodular  $\Leftrightarrow$  the combination of the linear interpolations is convex.

## 5 Splitting-Off Lemma and Connectivity

### 5.1 Splitting-Off Lemma

**Lemma 5.1** (Lovász). *Let  $G$  be a (multi-)graph with  $V(G) = V \dot{\cup} \{s\}$  with  $|\delta(s)|$  even and  $k \geq 2$  such that:*

$$|\delta(U)| \geq k \quad \forall \emptyset \neq U \subsetneq V \quad (2)$$

*Then  $\forall \{s, t\} \in E : \exists u \in \Gamma(s)$  such that*

$$G' := G - \{s, t\} - \{s, u\} + \{u, t\}$$

*satisfies (2).*

*Remark.* If  $t = u$ , then  $G'$  contains a loop which does not change the connectivity when it gets deleted.

*Proof.* If  $|\Gamma(s)| = 1$ , then the statement is clear since for all  $U \subsetneq V$  with  $t \in U$ :

$$|\delta_G(U)| = \underbrace{|\delta_{G-s}(U)|}_{=|\delta_G(V \setminus U)| \geq k} + |E[U, \{s\}]|$$

Therefore removing edges incident to  $s$  maintains the connectivity. Assume now that  $|\Gamma(s)| > 1$ . Fix  $t \in \Gamma(s)$ .

**Claim.** *We can find  $u \in \Gamma(s) \setminus \{t\}$  such that  $G'$  satisfies (2).*

If not, then for all  $u \in \Gamma(s)$  there exists  $U \subsetneq V$  such that  $|\delta_{G'}(U)| < k$ . Then  $t, u \in U$ , else  $|\delta_{G'}(U)| = |\delta_G(U)|$ . Also,  $|\delta_G(U)| \leq k + 1$ . Let:

$$\mathcal{C} := \{U \subsetneq V \mid t \in U, |\delta_G(U)| \leq k + 1\}$$

This covers  $\Gamma(s)$ . Then  $\forall U \in \mathcal{C}$

$$1 \geq \underbrace{|\delta_G(U)|}_{\leq k+1} - \underbrace{|\delta_G(U \cup \{s\})|}_{\geq k} = |E(\{s\}, U)| - |E(\{s\}, V \setminus U)|$$

so  $|E(\{s\}, U)| \leq |E(\{s\}, V \setminus U)| + 1$ . Since  $|\delta(s)|$  is even, there cannot be equality, so:

$$|E(\{s\}, U)| \leq |E(\{s\}, V \setminus U)|$$

Now  $\{s, t\} \in E(\{s\}, U)$  for all  $U \in \mathcal{C}$ . In particular, we need  $> 2$  sets from  $\mathcal{C}$  to cover  $\delta(s)$ . Take  $U_1, U_2, U_3 \in \mathcal{C}$  such that  $U_1 \setminus (U_2 \cup U_3)$ ,  $U_2 \setminus (U_1 \cup U_3)$ ,  $U_3 \setminus (U_1 \cup U_2)$  are nonempty. Then

$$\begin{aligned} & |\delta(U_1)| + |\delta(U_2)| + |\delta(U_3)| \\ & \geq |\delta(U_1 \cap U_2 \cap U_3)| + |\delta(U_1 \setminus (U_2 \cup U_3))| \\ & \quad + |\delta(U_2 \setminus (U_1 \cup U_3))| + |\delta(U_3 \setminus (U_1 \cup U_2))| \end{aligned}$$

This "3-way submodularity" can be proved by considering how often each edge is counted on both sides. Actually, the left side is larger by 2 since the edge  $\{s, t\}$  is counted three times here but only once on the right side.

Each term on the left side is at most  $k + 1$ . Each term on the right is at least  $k$ . In total  $3(k + 1) \geq 4k + 2$ , so  $k \leq 1$  in contradiction to the assumption.  $\square$

## 5.2 Construction of $2k$ -edge-connected graphs

**Lemma 5.2.** *Every minimal  $k$ -edge-connected (multi-)graph has a vertex of degree  $k$ .*

*Proof.* Let  $G$  be such a graph. Then every cut has at least  $k$  edges and every edge is part of a cut with (at most)  $k$  edges. Let  $X \subsetneq V(G)$  be minimum set such that  $|\delta(X)| = k$ . If  $|X| = 1$ , we are done. Otherwise, by minimality  $G[X]$  is connected. Let  $e \in E(G[X])$ , then  $\exists T \subsetneq V(G)$  with  $e \in \delta(T)$  and  $|\delta(T)| = k$ .

Case 1:  $T \cup X = V(G)$ . Then  $|\delta(X \setminus T)| = |\delta(T)| = k$  in contradiction to the minimality of  $X$ .

Case 2:  $T \cup X \neq V(G)$ . Then  $|\delta(X \cap T)| = k$  by submodularity of  $|\delta(\cdot)|$ :

$$|\delta(X)| + |\delta(T)| \geq |\delta(X \cap T)| + |\delta(X \cup T)|$$

This again contradicts the minimality of  $X$ .

□

**Theorem 5.3.** *Let  $M_{2k}$  be a multigraph with 2 vertices joined by  $2k$  edges. Any  $2k$ -edge-connected graph with at least 2 vertices can be built from  $M_{2k}$  by iteratively applying:*

1. *Adding edges (possibly loops)*
2. *Pinching  $k$  edges: Take  $k$  edges  $(\{v_i, w_i\})_{i=1}^k$ , add a new vertex  $s$  and replace each edge  $\{v_i, w_i\}$  by  $\{s, v_i\}$  and  $\{s, w_i\}$  for  $1 \leq i \leq k$ .*

*Proof.* Start with any  $2k$ -edge-connected graph  $G$ . Then do 1. and 2. in reverse, i.e.:

1. Delete a maximal set of edges (while maintaining  $2k$ -edge-connectivity)
2. By lemma 5.2, there is a vertex  $s$  with  $|\delta(s)| = 2k$ . Split off  $s$ , as in lemma 5.1.

At the end,  $M_{2k}$  remains since both operations maintain  $2k$ -edge-connectivity.

□

**Theorem 5.4** (Nash-Williams). *An undirected graph  $G$  is  $2k$ -edge-connected if and only if there is an orientation  $\vec{G}$  of  $G$  that is  $k$ -edge-connected.*

*Proof.* If  $\vec{G}$  is  $k$ -edge-connected, then each cut contains  $k$  outgoing and  $k$  incoming edges, so  $G$  is  $2k$ -edge-connected.

For the other implication, let  $G$  be  $2k$ -edge-connected. Take  $M_{2k}$  and orient  $k$  edges in each direction. Apply theorem 5.3 and preserve the orientation. This preserves  $k$ -edge-connectivity in the oriented graph. □

*Remark 5.5.* Nash-Williams actually proved that each graph  $G$  has an orientation  $\vec{G}$  for which  $\lambda(x, y, \vec{G}) \geq \lfloor \frac{\lambda(x, y, G)}{2} \rfloor \forall x, y \in V(G)$  where  $\lambda(x, y, H)$  denotes the local edge-connectivity, so the number of edge-disjoint  $x$ - $y$ -paths in  $H$ .