Combinatorial Optimization

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Contents

0 Organization

- Prerequisites
  - Basic knowledge of graph algorithms
  - Linear Programming (LP Duality)
  - Programming skills in C++
- Exam
  - Qualification requires 50% of the points in theoretical & programming exercises
  - Oral exam
- Books
  - "Combinatorial Optimization", Korte & Vygen
  - "Understanding & Using Linear Programming", B. Gärtner, J. Matouset
  - Skript (theorems & definitions)
  - Further book recommendations are on the website

1 Matchings

1.1 Introduction

Definition 1.1.

1. A *matching* M in a graph $G = (V, E)$ is a set of pairwise disjointed edges, i.e. they don't have a common endpoint.

   $\nu(G) :=$ max. cardinality of a matching in G

2. An *edge cover* C of a graph $G = (V, E)$ is a subset of E s.t. $C = \bigcup_{e \in C} e$.

   $\zeta(G) :=$ min. cardinality of an edge cover in G

3. A matching is called *perfect* (or *1-factor*) if it is an edge cover

4. $v \in V$ with $v \in e \in M$ is called *M-covered*

5. $v \in V$ is called *M-exposed* if it is not M-covered

Definition 1.2.

1. A *stable set* (independent set) S is a set of pairwise non-adjacent vertices.

   $\alpha(G) :=$ max. cardinality of a stable set

2. A *vertex cover* C is a subset of V s.t. $E = \bigcup_{\{x,y\}\in E,\ x\in G}\{x,y\}$
   $\tau(G) \coloneqq$ min. cardinality of a vertex cover

Lemma 1.3.

1. *$\alpha(G) + \tau(G) = |V|$*
2. *$\nu(G) + \zeta(G) = |V|$ if G has no isolated vertices*
3. *$\zeta(G) = \alpha(G)$ if G is bipartite and has no isolated vertices*

Problem. Cardinalty Matching Problem
Input: Graph $G = (V, E)$
Task: Find a maximum cardinality matching

Problem. Maximum Weight Matching Problem (MWMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a matching M maximizing $c(M)$

Problem. Minimum Weight Perfect Matching (MWPMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a perfect matching of minimum weight or decide that no perfect matching exists in G

Lemma 1.4. *The MWMP is equivalent to the MWPMP (i.e. there exists a transformation with linear complexity)*

Proof. Given a MWPMP instance (G, c), define $c' \coloneqq K - c$ ($K \coloneqq 1 + \sum_{e\in E} |c(e)|$).
$\Rightarrow$ Any maximum weight matching is a maximum cardinality matching

Given a MVMP instance (G, c), define G' as 2 copies of G where the 2 copies of a vertex are joined by an edge.
$\Rightarrow$ G' has a perfect matching. Define:

$$c'(e) \coloneqq \begin{cases} -c(e) & \text{if } e \text{ is in the first copy} \\ 0 & \text{else} \end{cases}$$

A minimum weight perfect matching in G' gives us a maximum weight matching in G. $\square$

Definition 1.5. Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching in G. A path P is *M-alternating* if its edges are alternatingly in and not in M. If both end points of this path are M-exposed, P is an *M-augmenting* path.

Lemma 1.6. *Given a matching M in G and an inclusion-wise maximal M-alternating path P,*

$$M\Delta P \coloneqq M \setminus P \cup P \setminus M$$

is a matching. If P is M-augmenting, then $|M\Delta P| = |M| + 1$.

Theorem 1.7 (Petersen 1891, Berge 1957)**.** *Augmenting Path Theorem Given a graph $G = (V, E)$ and a matching M in G:*

$$|M| = \nu(G) \Leftrightarrow \nexists M\text{-augmenting path } P \text{ in } G$$

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": Assume there exists a matching M' with $|M'| > |M|$. Let $G' := (V, M \Delta M')$.
$\Rightarrow |\delta_{G'}(v)| \leq 2 \ \forall v \in V$
$\Rightarrow G'$ is the union of disjoint circuits and paths
$\Rightarrow$ all circuits are even and have the same number of edges from M and M'
$\Rightarrow \exists$ a path P in G' starting and ending with an edge in M'
$\Rightarrow P$ is an alternating path

□

1.2 Bipartite Matching

Theorem 1.8 (König 1931)**.** *If G is bipartite, then $\nu(G) = \tau(G)$*

Proof. Add vertices s and t edges between them to all vertices of the respective partition. Direct all edges from s to t. Then $\nu(G)$ is maximum number of disjoint s-t-paths. Menger $\Rightarrow$ This is equal to the minimum number of vertices that disconnect t from s. □

Theorem 1.9 (Hall 1935)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph. Then:*

$$G \text{ has a matching covering } A \Leftrightarrow |\Gamma(X)| \geq |X| \quad \forall X \subseteq A$$

Corollary 1.10. *Marriage Theorem*

$$|\Gamma(X)| \geq |X| \ \ \forall X \subseteq A \text{ and } |A| = |B| \Leftrightarrow G \text{ has a perfect matching}$$

Definition 1.11. The MWPMP for bipartite graphs is called *Assignment Problem.*

Theorem 1.12. *The Assignment Problem can be solved in time $O(nm + n^2 \log m)$.*

Proof. Use the Successive Shortest Paths algorithm in an auxiliary graph. □

1.3 The Tutte Matrix & Randomized Matching

Definition 1.13. Let G be a simple, undirected graph. Let G' be an orientation of G and $(X_e)_{e\in E(G)}$. The *Tutte matrix* is defined as

$$T_G(X) \coloneqq (T^*_{vw})_{v,w\in V(G)}$$

where

$$t^*_{vw} \coloneqq \begin{cases} X_{\{v,w\}} & \text{if } (v,w)\in E(G) \\ -X_{\{v,w\}} & \text{if } -(v,w)\in E(G) \\ 0 & \text{else} \end{cases}$$

Remark 1.14. $T_G(X)$ is shew-symmetric (i.e. $T_G(X) = -(T_G(X))^t$). $\operatorname{rank}(T_G(X))$ is independent of the orientation of G. $\det(T_G(X))$ is a polyomial in X.

Theorem 1.15 (Tutte)**.** *A simple graph G has a perfect matching* $\Leftrightarrow \det(T_G(X)) \neq 0$

Proof. Let $V(G) = \{v_1,\dots,v_n\}$ and S_n be the permutation group.

$$\det T_G(X) = \sum_{\pi\in S_n} \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} t^*_{v_i,v_{\pi(i)}}$$

Let $S'_n \coloneqq \{\pi\in S_n \mid \prod_{i=1}^n t^*_{v_i,v_{\pi_i}} \neq 0\}$. Each $\pi\in S_n$ corresponds to a digraph $H_\pi \coloneqq (V(G), \{(v_i, v_{\pi(i)}) \mid i\in [n]$[1]$\})$. We have $|\delta^+(v) = 1 = |\delta^-(v)||\ \ \forall v\in V(H_\pi) \Rightarrow H_\pi$ is the union of disjoint circuits. If $\pi\in S'_n$, then $H_\pi \subset \overleftrightarrow{G'}$.

If there exists $\pi\in S'_n$ s.t. H_π is a collection of even circuits, then this immediately yields a perfect matching in G (using every second edge of each circuit).

Otherwise, $\forall\pi\in S'_n$, H_π contains an odd circuit. Let $r(\pi)\in S'_n$ arise from π by reversing edges on the unique odd circuit containing a vertex with minimum index $\Rightarrow r(r(\pi)) = \pi$ and $\operatorname{sgn}(\pi) = \operatorname{sgn}(r(\pi))$. The second part is true since for reversing an odd cycle, we need an even number of swaps. Let $v_{i_1},\dots,v_{i_{2k+1}}$ be the "first" odd circuit. Then $r(\pi)$ is attained by $2k$ swaps: For $j = 1,\dots,k$ swap $(\pi(i_{2j-1}), \pi(i_{2k}))$ and $(\pi(i_{2j}), \pi(i_{2k+1}))$.

$\prod_{i=1}^n t^*_{v_iv_{\pi(i)}} = -\prod_{i=1}^n t^*_{v_iv_{r(\pi(i))}}$ since there is an odd number of sign changes to t^*. $\Rightarrow \det(T_G(X)) = 0$. We have shown that if G has no perfect matching, then $\det T_G(X) = 0$.

Assume that G has a perfect matching M. Define π as $\pi(i) = j, \pi(j) = i$ where $\{i,j\}\in M$. $\Rightarrow \prod_{i=1}^n t^*_{v_iv_{\pi(i)}} = \prod_{e\in M} -X_e^2$ cannot be canceled out. In particular, $\det T_G(X) \neq 0$. □

Remark 1.16. Picking $X' \in [0,1]^{E(G)}$ at random, we almost surely have (since the zero set of a non-zero polynomial is a set of measure zero):

$$\det T_G(X') \neq 0 \Leftrightarrow G \text{ has a perfect matching}$$

[1] This is an abbreviation for $\{1,\dots,n\}$.

Theorem 1.17 (Lovász 1979)**.** *Let G be a simple graph and $X \in [0,1]^{E(G)}$ chosen randomly. Then almost surely* $\operatorname{rank}(T_G(X)) = 2\nu(G)$.

1.4 Tutte's Matching Theorem

Let $X \subseteq V(G)$. $G - X$ consists of even and odd (in terms of the number of vertices) connected components. We define $q_G(X)$ to be the number of odd components in $G - X$.

Definition 1.18. A graph G satisfies the *Tutte Condition* if $q_G(X) \leq |X|$ for all $X \subseteq V(G)$. $\emptyset \neq X \subseteq V(G)$ is called *barrier* if $q_G(X) = |X|$.

Lemma 1.19. *For any graph G and any $X \subseteq V(G)$:*

$$q_G(X) - |X| \equiv |V(G)| \mod 2$$

Definition 1.20. A graph G is *factor-critical* if $G-v$ has a perfect matching for all $v \in V(G)$. A matching is called *near-perfect* if it covers $|V(G)| - 1$ vertices.

Lemma 1.21. *If G is factor-critical, then it is connected.*

Theorem 1.22 (Tutte 1947)**.** *A graph G has a perfect matching $\Leftrightarrow$ Tutte Condition holds (i.e. $q_G(X) \leq |X| \;\; \forall X \subseteq V(G)$)*

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": We proceed by induction on $|V(G)|$. The case $|V(G)| = 2$ is clear.

Generally, if the Tutte Condition holds, then $|V(G)|$ must be even (pick $X = \emptyset$). Proposition 1.19 $\Rightarrow$ $q_G(X) - |X|$ is even. Every $x \in V(G)$ induces a barrier $\{x\}$. Let X be a maximum barrier. Then $G - X$ doesn't have any even components (since otherwise a single vertex of such a component could be added to X).

Claim: Each odd component is factor-critical.
Let C be an odd component in $G - X$, $v \in V(C)$. Assume that $C - v$ does not have a perfect matching. Induction Hypothesis $\Rightarrow$ $C - v$ violates Tutte Condition.
$\Rightarrow \exists Y \subseteq V(C - v) : q_{C-v}(Y) > |Y|$
$\overset{1.19}{\Rightarrow} q_{C-v}(Y) \geq |Y| + 2$
Observe $X \cap \{v\} = Y \cap \{v\} = X \cap Y = \emptyset$:

$$\begin{aligned} q_G(X \cup Y \cup \{v\}) &= q_G(X) - 1 + q_C(Y \cup \{v\}) \\ &= |X| - 1 + q_{C-v}(Y) \\ &\geq |X| - 1 + |Y| + 2 \\ &= |X \cup Y| + 1 \\ &= |X \cup Y \cup \{v\}| \end{aligned}$$

$\Rightarrow X \cup Y \cup \{v\}$ is a barrier
$\Rightarrow$ Claim

Let G' arise from G by contracting each odd component into a single vertex. We have $V(G') = X \dot{\cup} Z$ and G' is bipartite. We have to show that G' has a perfect matching. If not, then $\exists A \subseteq Z : |\Gamma_{G'}(A)| < |A|$ $\Rightarrow q_G(\Gamma_{G'}(A)) \geq |A| > |\Gamma_{G'}(A)|$ which contradicts the Tutte Condition.

$\square$

Theorem 1.23 (Berge 1958)**.**

$$|V(G)| = 2\nu(G) + \max_{X \subseteq V(G)} (q_G(X) - |X|)$$