Combinatorial Optimization

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Contents

0 Organization

- Prerequisites
  - Basic knowledge of graph algorithms
  - Linear Programming (LP Duality)
  - Programming skills in C++
- Exam
  - Qualification requires 50% of the points in theoretical & programming exercises
  - Oral exam
- Books
  - "Combinatorial Optimization", Korte & Vygen
  - "Understanding & Using Linear Programming", B. Gärtner, J. Matouset
  - Skript (theorems & definitions)
  - Further book recommendations are on the website

1 Matchings

1.1 Introduction

Definition 1.1.

1. A *matching* M in a graph $G = (V, E)$ is a set of pairwise disjointed edges, i.e. they don't have a common endpoint.

   $\nu(G) :=$ max. cardinality of a matching in G

2. An *edge cover* C of a graph $G = (V, E)$ is a subset of E s.t. $V = \bigcup_{e \in C} e$.

   $\zeta(G) :=$ min. cardinality of an edge cover in G

3. A matching is called *perfect* (or *1-factor*) if it is an edge cover
4. $v \in V$ with $v \in e \in M$ is called *M-covered*
5. $v \in V$ is called *M-exposed* if it is not M-covered

Definition 1.2.

1. A *stable set* (independent set) S is a set of pairwise non-adjacent vertices.

$\alpha(G) :=$ max. cardinality of a stable set

2. A *vertex cover* C is a subset of V s.t. $E = \bigcup_{\{x,y\}\in E,\ x\in G} \{x, y\}$

   $\tau(G) :=$ min. cardinality of a vertex cover

Lemma 1.3.

1. $\alpha(G) + \tau(G) = |V|$

2. $\nu(G) + \zeta(G) = |V|$ *if* G *has no isolated vertices*

3. $\zeta(G) = \alpha(G)$ *if* G *is bipartite and has no isolated vertices*

Problem. Cardinality Matching Problem
Input: Graph $G = (V, E)$
Task: Find a maximum cardinality matching

Problem. Maximum Weight Matching Problem (MWMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a matching M maximizing $c(M)$

Problem. Minimum Weight Perfect Matching (MWPMP)
Input: Graph G, $c : E \to \mathbb{R}$
Task: Find a perfect matching of minimum weight or decide that no perfect matching exists in G

Lemma 1.4. *The MWMP is equivalent to the MWPMP (i.e. there exists a transformation with linear complexity)*

Proof. Given a MWPMP instance (G, c), define $c' := K - c$ ($K := 1 + \sum_{e\in E} |c(e)|$).
$\Rightarrow$ Any maximum weight matching is a maximum cardinality matching

Given a MWMP instance (G, c), define G' as 2 copies of G where the 2 copies of a vertex are joined by an edge.
$\Rightarrow$ G' has a perfect matching. Define:

$$c'(e) := \begin{cases} -c(e) & \text{if } e \text{ is in the first copy} \\ 0 & \text{else} \end{cases}$$

A minimum weight perfect matching in G' gives us a maximum weight matching in G. □

Definition 1.5. Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching in G. A path P is M*-alternating* if its edges are alternatingly in and not in M. If both end points of this path are M-exposed, P is an M*-augmenting* path.

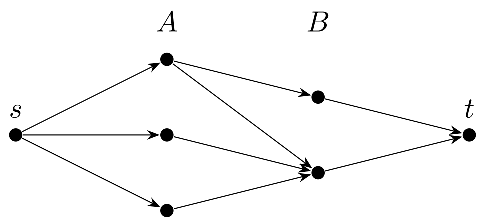


Figure 1: Example of the construction in Theorem 1.8

Lemma 1.6. *Given a matching M in G and an inclusion-wise maximal M-alternating path P,*

$$M \Delta P := M \setminus P \cup P \setminus M$$

is a matching. If P is M-augmenting, then $|M \Delta P| = |M| + 1$.

Theorem 1.7 (Petersen 1891, Berge 1957)**.** *Augmenting Path Theorem Given a graph $G = (V, E)$ and a matching M in G:*

$$|M| = \nu(G) \Leftrightarrow \nexists M\text{-augmenting path } P \text{ in } G$$

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": Assume there exists a matching M' with $|M'| > |M|$. Let $G' := (V, M \Delta M')$.
$\Rightarrow |\delta_{G'}(v)| \leq 2 \ \forall v \in V$
$\Rightarrow G'$ is the union of disjoint circuits and paths
$\Rightarrow$ all circuits are even and have the same number of edges from M and M'
$\Rightarrow \exists$ a path P in G' starting and ending with an edge in M'
$\Rightarrow P$ is an alternating path

□

1.2 Bipartite Matching

Theorem 1.8 (König 1931)**.** *If G is bipartite, then $\nu(G) = \tau(G)$*

Proof. Add vertices s and t edges between them to all vertices of the respective partition. Direct all edges from s to t. Then $\nu(G)$ is maximum number of disjoint s-t-paths. Menger $\Rightarrow$ This is equal to the minimum number of vertices that disconnect t from s. □

Theorem 1.9 (Hall 1935)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph. Then:*

$$G \text{ has a matching covering } A \Leftrightarrow |\Gamma(X)| \geq |X| \quad \forall X \subseteq A$$

Corollary 1.10. *Marriage Theorem*

$$|\Gamma(X)| \geq |X| \;\; \forall X \subseteq A \textit{ and } |A| = |B| \Leftrightarrow G \textit{ has a perfect matching}$$

Definition 1.12. The MWPMP for bipartite graphs is called *Assignment Problem.*

Theorem 1.13. *The Assignment Problem can be solved in time $O(nm + n^2 \log m)$.*

Proof. Use the Successive Shortest Paths algorithm in an auxiliary graph. □

1.3 The Tutte Matrix & Randomized Matching

Definition 1.14. Let G be a simple, undirected graph. Let G' be an orientation of G and $(X_e)_{e\in E(G)}$. The *Tutte matrix* is defined as

$$T_G(X) \coloneqq (t^*_{vw})_{v,w\in V(G)}$$

where

$$t^*_{vw} \coloneqq \begin{cases} X_{\{v,w\}} & \text{if } (v,w) \in E(G') \\ -X_{\{v,w\}} & \text{if } (w,v) \in E(G') \\ 0 & \text{else} \end{cases}$$

Remark 1.15. $T_G(X)$ is shew-symmetric (i.e. $T_G(X) = -(T_G(X))^t$). $\operatorname{rank}(T_G(X))$ is independent of the orientation of G. $\det(T_G(X))$ is a polyomial in X.

Theorem 1.16 (Tutte)**.** *A simple graph G has a perfect matching* $\Leftrightarrow \det(T_G(X)) \neq 0$

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ and S_n be the permutation group.

$$\det T_G(X) = \sum_{\pi\in S_n} \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} t^*_{v_i, v_{\pi(i)}}$$

Let $S'_n \coloneqq \{\pi \in S_n \mid \prod_{i=1}^n t^*_{v_i,v_{\pi_i}} \neq 0\}$. Each $\pi \in S_n$ corresponds to a digraph $H_\pi \coloneqq (V(G), \{(v_i, v_{\pi(i)}) \mid i \in [n][1]\})$. We have $|\delta^+(v) = 1 = |\delta^-(v)|| \;\; \forall v \in V(H_\pi) \Rightarrow H_\pi$ is the union of disjoint circuits. If $\pi \in S'_n$, then $H_\pi \subset \overset{\Leftrightarrow}{G'}$.

If there exists $\pi \in S'_n$ s.t. H_π is a collection of even circuits, then this immediately yields a perfect matching in G (using every second edge of each circuit).

[1]This is an abbreviation for $\{1, \dots, n\}$.

Otherwise, $\forall \pi \in S'_n$, H_π contains an odd circuit. Let $r(\pi) \in S'_n$ arise from π by reversing edges on the unique odd circuit containing a vertex with minimum index $\Rightarrow r(r(\pi)) = \pi$ and $\mathrm{sgn}(\pi) = \mathrm{sgn}(r(\pi))$. The second part is true since for reversing an odd cycle, we need an even number of swaps. Let $v_{i_1}, \ldots, v_{i_{2k+1}}$ be the "first" odd circuit. Then $r(\pi)$ is attained by $2k$ swaps: For $j = 1, \ldots, k$ swap $(\pi(i_{2j-1}), \pi(i_{2k}))$ and $(\pi(i_{2j}), \pi(i_{2k+1}))$.

$\prod_{i=1}^n t^*_{v_i v_{\pi(i)}} = -\prod_{i=1}^n t^*_{v_i v_{r(\pi(i))}}$ since there is an odd number of sign changes to t^*. $\Rightarrow \det(T_G(X)) = 0$. We have shown that if G has no perfect matching, then $\det T_G(X) = 0$.

Assume that G has a perfect matching M. Define π as $\pi(i) = j, \pi(j) = i$ where $\{i, j\} \in M$. $\Rightarrow \prod_{i=1}^n t^*_{v_i v_{\pi(i)}} = \prod_{e \in M} -X_e^2$ cannot be canceled out. In particular, $\det T_G(X) \neq 0$. □

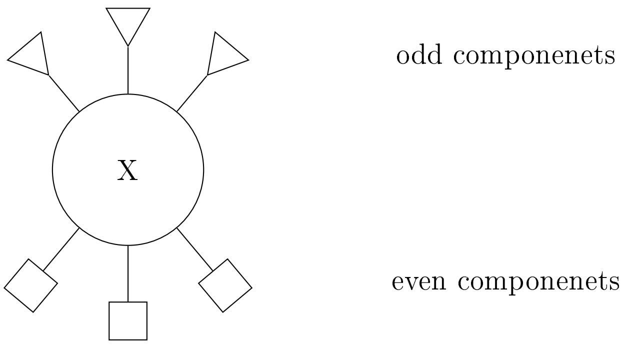
Remark 1.17. Picking $X' \in [0,1]^{E(G)}$ at random, we almost surely have (since the zero set of a non-zero polynomial is a set of measure zero):

$$\det T_G(X') \neq 0 \Leftrightarrow G \text{ has a perfect matching}$$

Theorem 1.18 (Lovász 1979). *Let G be a simple graph and $X \in [0,1]^{E(G)}$ chosen randomly. Then almost surely* $\mathrm{rank}(T_G(X)) = 2\nu(G)$.

1.4 Tutte's Matching Theorem

Let $X \subseteq V(G)$. $G - X$ consists of even and odd (in terms of the number of vertices) connected components. We define $q_G(X)$ to be the number of odd components in $G - X$.



Definition 1.19. A graph G satisfies the *Tutte Condition* if $q_G(X) \leq |X|$ for all $X \subseteq V(G)$. $\emptyset \neq X \subseteq V(G)$ is called *barrier* if $q_G(X) = |X|$.

Proposition 1.20. *For any graph G and any $X \subseteq V(G)$:*

$$q_G(X) - |X| \equiv |V(G)| \mod 2$$

Definition 1.21. A graph G is *factor-critical* if $G-v$ has a perfect matching for all $v \in V(G)$. A matching is called *near-perfect* if it covers $|V(G)|-1$ vertices.

Proposition 1.22. *If G is factor-critical, then it is connected.*

Theorem 1.23 (Tutte 1947)**.** *A graph G has a perfect matching $\Leftrightarrow$ Tutte Condition holds (i.e. $q_G(X) \leq |X| \ \forall X \subseteq V(G)$)*

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": We proceed by induction on $|V(G)|$. The case $|V(G)| = 2$ is clear.

Generally, if the Tutte Condition holds, then $|V(G)|$ must be even (pick $X = \emptyset$). Proposition 1.20 $\Rightarrow$ $q_G(X) - |X|$ is even. Every $x \in V(G)$ induces a barrier $\{x\}$. Let X be a maximum barrier. Then $G - X$ doesn't have any even components (since otherwise a single vertex of such a component could be added to X).

Claim: Each odd component is factor-critical.
Let C be an odd component in $G - X$, $v \in V(C)$. Assume that $C - v$ does not have a perfect matching. Induction Hypothesis $\Rightarrow$ $C - v$ violates Tutte Condition.
$\Rightarrow \exists Y \subseteq V(C - v) : q_{C-v}(Y) > |Y|$
$\overset{1.20}{\Rightarrow} q_{C-v}(Y) \geq |Y| + 2$
Observe $X \cap \{v\} = Y \cap \{v\} = X \cap Y = \emptyset$:

$$\begin{aligned} q_G(X \cup Y \cup \{v\}) &= q_G(X) - 1 + q_C(Y \cup \{v\}) \\ &= |X| - 1 + q_{C-v}(Y) \\ &\geq |X| - 1 + |Y| + 2 \\ &= |X \cup Y| + 1 \\ &= |X \cup Y \cup \{v\}| \end{aligned}$$

$\Rightarrow X \cup Y \cup \{v\}$ is a barrier
$\Rightarrow$ Claim

Let G' arise from G by contracting each odd component into a single vertex. We have $V(G') = X \dot\cup Z$ and G' is bipartite. We have to show that G' has a perfect matching. If not, then $\exists A \subseteq Z : |\Gamma_{G'}(A)| < |A|$ $\Rightarrow q_G(\Gamma_{G'}(A)) \geq |A| > |\Gamma_{G'}(A)|$ which contradicts the Tutte Condition.

□

Theorem 1.24 (Berge 1958)**.**

$$|V(G)| = 2\nu(G) + \max_{X \subseteq V(G)} (q_G(X) - |X|)$$

Proof. For $X \subseteq V(G)$, any matching has at least $q_G(X) - |X|$ uncovered vertices, so "$\geq$" holds.

For the other inequality, add $k := \max_{X \subseteq V(G)}(q_G(X) - |X|)$ new vertices and connect them to all existing vertices, yielding a new graph H.

We claim that H has a perfect matching. This then implies:

$$2\nu(G) + k \geq 2\nu(H) - k = |V(H)| - k = |V(G)|$$

Assume that H does not have a perfect matching. Then by Tutte's Theorem, there exists $Y \subseteq V(H)$ with $q_H(Y) > |Y|$. By 1.20, $k \equiv |V(G)| \mod 2$, therefore $|V(H)|$ is even, so $Y \neq \emptyset$. Y must contain all new vertices, otherwise $H - Y$ would be connected[2] and $q_H(Y) \leq 1 \leq |Y|$.

$$\Rightarrow q_G(Y \cap |V(G)|) = q_H(Y) > |Y| = |Y \cap V(G)| + k$$

which is a contradiction to the choice of k. $\square$

1.5 Ear Decompositions of Factor-Critical Graphs

Definition 1.25. Let G be a graph. An *ear decomposition* of G is a sequence $r, P_1, \dots, P_k$ with $G = (r, \emptyset) + P_1 + \dots + P_k$ such that each P_i is either a path with exactly the endpoints located in $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$ or a circuit where exactly one of the vertices belongs to $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$.

$P_1, \dots, P_k$ are called *ears*. If $|V(P_1)| \geq 3$ and $P_2, \dots, P_k$ are paths we call it a *proper* ear decomposition.

Theorem 1.27 (Whitney 1932)**.** *Let G be an undirected graph. Then:*

$$G \text{ 2-connected} \Leftrightarrow G \text{ has a proper ear decomposition}$$

Definition 1.28. An ear decomposition is *odd* if every ear has odd length (in terms of the number of edges).

Theorem 1.29. *Let G be an undirected graph. Then*

$$G \text{ factor-critical} \Leftrightarrow G \text{ has an odd ear decomposition}$$

The first vertex r of the ear decomposition can be chosen arbitrarily.

Proof.

[2] Note that Y cannot contain all old vertices, since otherwise $q_H(Y) < |Y|$.

"$\Leftarrow$": Let G be a graph with an odd ear decomposition $r, P_1, \dots, P_k$. P_1 is an odd circuit, so it is factor-critical. We use induction on the number of ears. Let P be the last ear and G' be G before adding P. By the induction hypothesis, G' is factor-critical. Given $v \in V(G)$, we have to show that $G - v$ has a perfect matching.

Case 1: $v \in V(G')$. Then $G' - v$ has a perfect matching. Adding every second edge of P (excluding the endpoints) to it, yields a perfect matching of $G - v$.

Case 2: $v \in V(G) \setminus V(G')$. Let x, y be the endpoints of P. Without loss of generality let $P_{[v,x]}$ be even. There exists a perfect matching in $G' - x$. Together with every second edge of $P_{[v,y]}$ and $P_{[v,x]}$ this is a perfect matching in $G - v$.

"$\Rightarrow$": Let $r \in V(G)$ be any vertex. Let M be a perfect matching in $G - r$. Suppose we have an odd ear decomposition for $G' \subseteq G$ with $r \in V(G')$ and $M \cap E(G')$ is a near-perfect matching in G' (i.e. all vertices in G' except for r are matched with other vertices in G').

If $G' \neq G$, there is an edge $\{x, y\} \in E(G) \setminus E(G')$ with $x \in V(G')$ (by Proposition 1.22). If $y \in V(G')$, then $\{x, y\}$ can be chosen as the next ear. Otherwise, we construct an M-alternating odd ear, starting with $\{x, y\}$: Let N be a matching in $G - y$. $M \Delta N$ contains a y-r-path P. Let w be the first vertex in $P \cap V(G')$. w is M-exposed in $P_{[y,w]}$, y is N-exposed in $P_{[y,w]}$. Therefore $P_{[y,w]}$ is even and together with $\{x, y\}$ it forms an M-alternating odd ear.

Inductively, this argument yields an odd ear decomposition.

$\square$

Definition 1.30. Let G be factor-critical and M a near-perfect matching. An M-alternating ear decomposition is an odd ear decomposition such that each ear is an M-alternating path or circuit C with:

$$|E(C) \cap M| = |E(C) \setminus M| - 1$$

Corollary 1.31. *For any factor-critical graph G and any near-perfect matching M in G, there exists in M-alternating ear decomposition of G.*

Definition 1.32. Let G be factor-critical, M a near-perfect matching and $r, P_1, \dots, P_k$ an M-alternating ear decomposition of G. $\mu, \varphi : V(G) \to V(G)$ are *associated with the ear decomposition* if:

- $\{x, y\} \in M \Rightarrow \mu(x) = y$
- $\{x, y\} \in E(P_i) \setminus M$ and $x \notin \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$

$\Rightarrow \varphi(x) = y$

- $\mu(r) = \varphi(r) = r$

Proposition 1.33. *Let G be a factor-critical graph and μ, φ functions associated with an M-alternating ear decomposition. Then this ear decomposition is unique up to the order of the ears. The Ear-Decomposition-Algorithm (algorithm 1) correctly determines an explicit list of the ears in linear time.*

Proof. Step 3 determines ears uniquely. The algorithm clearly runs in linear time. □

Algorithm 1: Ear Decomposition Algorithm

Input: Factor-critical graph G, functions μ, φ associated with an M-alternating ear decomposition

Output: An M-alternating ear decomposition $r, P_1, \ldots, P_k$

1 $X := \{r\}$ where r is the vertex with $\mu(r) = r$
2 $k := 0$, $S :=$ empty stack
3 **while** $X \neq V(G)$ **do**
4   **if** *S is non-empty* **then**
5   Let $v \in V(G) \setminus X$ be an endpoint of the topmost element of the stack
6   **else**
7   Choose $v \in V(G) \setminus X$ arbitrarily
8     $x := v,\ y := \mu(v),\ P := (\{x, y\}, \{\{x, y\}\})$
9   **while** $\varphi(\varphi(x)) = x$ **do**
10         $P := P + \{x, \varphi(x)\} + \{\varphi(x), \mu(\varphi(x))\}$
11         $x := \mu(\varphi(x))$
12   **while** $\varphi(\varphi(y)) = y$ **do**
13         $P := P + \{y, \varphi(y)\} + \{\varphi(y), \mu(\varphi(y))\}$
14         $y := \mu(\varphi(y))$
15     $P := P + \{x, \varphi(x)\} + \{y, \varphi(y)\}$
16   P is the ear containing y as an inner vertex. Put P on S.
17   **while** *Both endpoints of the topmost element P of the stack S are in X* **do**
18   Delete P from S
19         $k := k + 1,\ P_k := P,\ X := X \cup V(P)$
20 **forall** $\{y, z\} \in E(G) \setminus (E(P_1) \cup \ldots \cup E(P_k))$ **do**
21     $k := k + 1,\ P_k := (\{y, z\}, \{\{y, z\}\})$
22 **return** $r, P_1, \ldots, P_k$

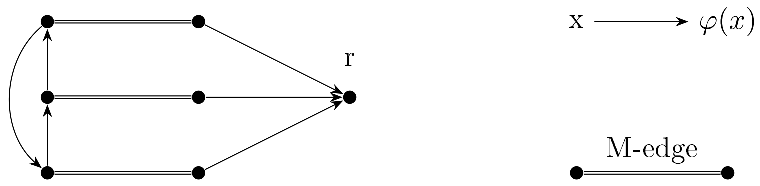


Figure 2: Counter example for the reverse implication of lemma 1.34

Lemma 1.34. *Let G be factor-critical and μ, φ associated with an M-alternating ear decomposition. Then the maximal path given by the initial sequence*

$$x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \ldots \tag{1}$$

defines an M-alternating x-r-path of even length.

Proof. We proceed by induction on the number of ears. Let $x \in V(G) \setminus \{r\}$ and P_i be the ear containing x. A subsequence of (1) is a subpath Q of P_i from x to $y \in \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$. Q starts with a matching edge and ends with a non-matching edge, so it has even length. If $y = r$, we are done, otherwise the statement follows from the induction hypothesis. $\square$

1.6 Edmond's Matching Algorithm

Definition 1.35. Let G be a graph, M a matching in G. A *blossom* in G with respect to M is a factor-critical subgraph B of G such that $|M \cap E(B)| = \frac{|V(B)|-1}{2}$. The vertex $r \in V(B)$ that is exposed by $M \cap E(B)$ is called the *base* of B.

Definition 1.36. Let G be a graph, M a matching in G, B a blossom and Q a M-alternating v-r-path of even length from $v \in V(G)$ that is M-exposed to the base r of B. Additionally, let $E(Q) \cap E(B) = \emptyset$. $B + Q$ is called an *M-flower*.

Lemma 1.37. *Let G be a graph, M a matching in G. Suppose there is a M-flower $B + Q$. Let G', M' result from G and M by contracting $V(B)$ into a single vertex. Then:*

$$M \text{ maximum matching in } G \Leftrightarrow M' \text{ maximum matching in } G'$$

Proof.

"$\Leftarrow$": Assume that M is not maximum in G. $N := M \Delta E(Q)$ is a matching with $|N| = |M|$.
$\Rightarrow$ $\exists N$-augmenting path P in G. At least one endpoint x of P is not in $V(B)$ (since B contains only one N-exposed vertex). If P and B are disjoint, let y be the other endpoint of P. Otherwise, let y be the

first vertex on P in B. $P' := P_{[x,y]}$ is an N'-augmenting path in G', so $|N'| = |M'| < \mu(G')$.

"$\Rightarrow$": Assume that M' is not maximum in G', so there exists a matching N' in G' with $|N'| > |M'|$. Let N_0 arise from N' in G, then N_0 contains ≤ 1 vertex from $V(B)$. Since B is factor-critical, N_0 can be extended by $k := \frac{|V(G)|-1}{2}$ edges to a matching N in G. We have

$$|N| = |N_0| + k = \left|N'\right| + k > \left|M'\right| + k = |M|$$

so M is not maximum.

$\square$

Lemma 1.39. *Let G be a graph, M a matching in G. $X \subseteq V(G)$ is the set of exposed vertices. We can find a shortest M-alternating X-X-walk of positive length in $O(|E(G)|)$ time.*

Proof. Define $D := (V(G), A)$ where:

$$A := \{(u,v) \mid \exists x \in V(G) : \{u,x\} \in E(G), \{x,v\} \in M\}$$

A shortest $X - \Gamma_G(X)$-path in D corresponds to a shortest X-X-walk in G. $\square$

Theorem 1.40. *Let $P = v_0, \dots, v_t$ be a shortest M-alternating X-X-walk in G. Then either*

- *P is an M-augmenting path* or
- *$v_0, \dots, v_j$ is an M-flower for some $j \leq t$.*

Proof. If P is not a path, choose $i < j$ such that $v_i = v_j$ and j minimal. Then $v_0, \dots, v_{j-1}$ are distinct vertices. If $j - i$ is even, deleting $v_{i-1}, \dots, v_j$ from P yields a shorter walk, so $j - i$ is odd.

Case 1: j is even. Then i is odd and therefore $v_{i+1} = v_{j-1}$ must be the matching mate of $V_i = v_j$ which contradicts the minimality of j.

Case 2: j is odd. Then i is even, so $v_0, \dots, v_i$ is an M-alternating path of even length and $v_i, \dots, v_j$ is an M-alternating odd circuit, i.e. a blossom.

$\square$

Theorem 1.41. *Given a graph G, a maximum cardinality matching can be found in time $O(n^2 m)$ where $n := |V(G)|, m := |E(G)|$*

Algorithm 2: Edmond's Augmenting Path Search

Input: Graph G, matching M

Output: An M-augmenting path (if one exists)

$X :=$ set of exposed vertices
if $\exists M$*-alternating* X*-*X*-walk of positive length* **then**
  $P = v_0, \ldots, v_t :=$ a shortest such walk
  if P *is a path* **then**
    return P
  else
    Choose j as in Theorem 1.40
    $v_0, \ldots, v_j$ is an M-flower with blossom B
    Recurse on G/B
    Augment an M/B-augmenting path in G/B to an M-augmenting path P' in G
    return P'
else
  $\nexists M$-augmenting path

Proof. Start with $M = \emptyset$ and iteratively find M-augmenting path P, set $M := M \Delta E(P)$. If no such path exists, then M is maximum. P can be found in time $O(mn)$[3]. Since M is maximum after at most $\frac{n}{2}$ augmentation, we have total running time $O(n^2 m)$. $\square$

1.6.1 Growing forest - $O(n^3)$

Definition 1.42. Let G be a graph, M a matching in G. An *alternating forest* with respect to M in G is a forest F in G where:

- $V(F)$ contains all M-exposed vertices, each tree of F contains exactly one exposed vertex, its *root*.
- We call $v \in V(G)$ an outer (inner) vertex if it has even (odd) distance from the root of its component.
- $\forall v \in V(F)$ the unique path from v to the root of its component is M-alternating.
- $v \in V(G) \setminus V(F)$ is called *out-of-forest*.

Clearly, inner vertices always have degree 2 (we always assume that there are no matching edges that can immediately be added to F).

[3] Here, m is the time required for finding a walk and the recursion depth is bounded by n.

Proposition 1.43. *In any alternating forest, the number of outer vertices that are not the root equals the number of inner vertices.*

Proof. For all outer vertices, there exists exactly one inner vertex on its path to the root. □

Lemma 1.44. *Given a graph G, a matching M, an alternating forest F with respect to M in G. Then, either M is a maximum matching or $\exists$ outer vertex $x \in V(F)$, an edge $\{x, y\} \notin E(F)$ such that one of the following holds:*

- *Grow: $y \notin V(F)$ and therefore $\{y, z\} \in M$ with $z \notin V(F)$. In this case, y, z and $\{x, y\}, \{y, z\}$ can be added to F.*
- *Augment: y is an outer vertex in a different connected component in F. In this case, M can be augmented along $P(x) \cup \{x, y\} \cup P(y)$ where $P(z)$ denotes the unique path from $z \in V(F)$ to the root of its connected component.*
- *Shrink: y is an outer vertex in the same component as x. Let r be the first vertex on $P(x)$ that is also on $P(y)$. Then $|\delta_F(r)| \geq 3$, so y is an outer vertex and $\left|E(F_{[x,r]})\right|, \left|E(F_{[y,r]})\right|$ are even. Together with $\{x, y\}$ these paths form a blossom with ≥ 3 vertices.*

Proof. We show that if none of these cases apply, M is maximum. If none of the cases apply, then every outer vertex only has inner vertices as neighbors. Let X be the set of inner vertices, $s := |X|$ and t be the number of outer vertices. All outer vertices are isolated in $G - X$, so $q_G(X) - |X| = t - s$. By Berge's formula (1.24), $t - s$ vertices are exposed by any matching, so M is maximum. □

Definition 1.45. Let G be a graph, M a matching in G. A subgraph F of G is a *general blossom forest* with respect to M if there exists a partition $V(F) = V_1 \dot{\cup} \ldots \dot{\cup} V_k$ such that $F_i = F[V_i]$ is a maximal factor-critical subgraph of F with $|M \cap E(F_i)| = \frac{|V_i|-1}{2}$ ($i \in [k]$) and after contracting each V_i, we obtain an M-alternating forest F'. F_i is called an outer (inner) blossom if V_i is an outer (inner) vertex in F'.

A *special blossom forest* is a general blossom forest where each inner blossom is a single vertex.

Store a special blossom forest with 3 functions $\mu, \varphi, \rho : V(G) \to V(G)$:

$$\mu(x) \coloneqq \begin{cases} x & \text{if } x \text{ is exposed in } M \\ y & \text{if } \{x,y\} \in M \end{cases}$$

$$\varphi(x) \coloneqq \begin{cases} x & \text{if } x \text{ is the base of an outer blossom or } x \text{ is out-of-forest} \\ y & \text{if } x \text{ is an inner vertex and } \{x,y\} \in E(F) \setminus M \\ y & \text{if } x \text{ is an outer vertex (i.e. in an outer blossom) and } \mu, \varphi \text{ are associated with an } M\text{-alternating ear decomposition of } x\text{'s blossom, } \{x,y\} \in E(F) \setminus M \end{cases}$$

$$\rho(x) \coloneqq \begin{cases} x & \text{if } x \text{ is an inner vertex or out-of-forest} \\ y & \text{if } x \text{ is an outer vertex and } y \text{ is the base of the outer blossom containing } x \ (y = x \text{ is possible).} \end{cases}$$

Proposition 1.46. *Let F be a special blossom forest with respect to M and μ, φ, ρ as above. Then:*

1. *For all outer vertices x, $P(x) \coloneqq$ maximal path given by subsequence of $x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \ldots$ is an M-alternating path from x to q where q is the root of the component containing x.*
2. *A vertex x is*
   - *an outer vertex* $\Leftrightarrow \mu(x) = x \vee \varphi(\mu(x)) \neq \mu(x)$
   - *an inner vertex* $\Leftrightarrow \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x$
   - *out-of-forest* $\Leftrightarrow \mu(x) \neq x \wedge \varphi(x) = x \wedge \varphi(\mu(x)) = \mu(x)$

Proof.

1. By definition of μ, φ and lemma 1.34 some initial subsequence of $P(x)$ ends at the base r of the blossom containing x. If $r = q$, we are done. Otherwise $\mu(r), \varphi(\mu(r))$ are next elements in a sequence leading to outer vertex $\varphi(\mu(r))$. This can be iterated.
2. Since the conditions are mutually exclusive, it suffices to show one implication for all the statements.
   - If x is outer, it is a root ($\mu(x) = x$) or $P(x)$ is a path of length at least 2, so $\varphi(\mu(x)) \neq \mu(x)$.
   - If x is inner, then $\mu(x)$ is the base of an outer blossom. Therefore $\varphi(\mu(x)) = \mu(x)$. $P(\mu(x))$ is a path of length at least 2, so $\varphi(x) \neq x$.

- If x is out-of-forest, then x is covered by M so $\mu(x) \neq x$. By definition of φ, $\varphi(x) = x$. $\mu(x)$ is out-of-forest as well, so $\varphi(\mu(x)) = \mu(x)$.

$\square$

Lemma 1.47. *Following invariants hold:*

a) $\{\{x, \mu(x)\} \mid x \in V(G), \mu(x) \neq x\}$ *is a matching*

b) $\{\{x, \mu(x)\} \mid \underbrace{x \in V(G), \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x}_{\text{inner vertices}}\} \cup \{\{x, \varphi(x)\} \mid x \in V(G), \varphi(x) \neq x\}$ *forms the edge set of a special blossom forest.*

c) μ, φ, ρ *satisfy the conditions in definition 1.45 (special blossom forest).*

Proof. a) holds as μ only changes in *Augment*. b) is correct after initialization and after the reset in the *Augment* step. It is preserved by *Grow* steps.

In a *Shrink* step, r (the first vertex that the paths from x, y to the root share) is a root or has $|\delta(r)| = 3$ (i.e. it is the base of a blossom), so it is an outer vertex. We define a blossom $B := \{v \in V(G) \mid \varphi(v) \in V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})\}$. Consider $\{u, v\} \in F$ with $u \in B, v \notin B$. If $\{u, v\} \in M$, we have $u = r, v = \mu(r)$ (since $F[B]$ contains a near-perfect matching). u was an outer vertex before shrinking and $F[B]$ being factor-critical follows from c) and the characterization by ear-decompositions.

For c), it's clear that μ always represents a matching. $\varphi(x) = x$ if x is not an outer vertex. Therefore, $\mu + \varphi$ represent an M-alternating ear decomposition of B. During *Shrink*, $\varphi(v)$ is not changed if $\varphi(v) = r$. Therefore, the odd ear decomposition for $B' :=$ blossom containing r, is the correct starting point. The next ear is $P(x)_{[x,x']} \cup P(y)_{[y,y']} + \{x, y\}$, where x' (y') is the first vertex in B' on $P(x)_{[x,r]}$ ($P(y)_{[y,r]}$).

For each ear Q of a former blossom $B'' \subseteq B$, $Q \setminus (E(P(x)) \cup E(P(y)))$ form a new ear (since it is created by removing an even path). φ, μ represent this ear-decomposition. $\square$

Theorem 1.48. *Edmond's cardinality matching algorithm correctly determines a maximum matching in* $O(n^3)$ *time, where* $n := |V(G)|$.

Proof. By lemma 1.47 and proposition 1.46, the algorithm maintains a special blossom forest. Let M, F be the final matching and forest. x an outer

Algorithm 3: Edmond's Cardinality Matching Algorithm

Input: A graph G

Output: A maximum matching M (defined by $\{x, \mu(x)\}$)

$\mu(v) := v,\ \varphi(v) := v,\ \rho(v) := v,\ \mathit{scanned}(v) :=$ false for all $v \in V(G)$

`// Outer Vertex Scan:`

while $\exists$ *outer vertex* x *with* $\mathit{scanned}(x) =$ false **do**

- Let y be a neighbor of x such that y is either out-of-forest or y is outer with $\rho(y) \neq \rho(x)$
- **if** *such a* y *does not exist* **then**
  - $\mathit{scanned}(x)$ =true, **continue**
- `// Grow:`
- **if** y *is out-of-forest* **then**
  - $\varphi(y) := x$, **continue**
- `// Augment:`
- **else if** $P(x)$ *and* $P(y)$ *are vertex-disjoint* **then**
  - $\mu(\varphi(v)) = v,\ \mu(v) = \varphi(v)$ for all $v \in V(P(x) \cup P(y))$ with odd distance from x or y on $P(x)$ or $P(y)$, respectively
  - $\mu(x) := y,\ \mu(y) := x$
  - $\varphi(v) := v, \rho(v) := v, \mathit{scanned}(v) :=$ false for all $v \in V(G)$
- `// Shrink:`
- **else**
  - Let r be the first vertex on $V(P(x)) \cap V(P(y))$ with $\rho(r) = r$
  - **forall** $v \in V(P(x)_{[x,r]}) \cup V(P(y)_{y,r})$ *with odd distance from* x *or* y *on* $P(x)_{[x,r]}$ *or* $P(y)_{[y,r]}$, *respectively and* $\rho(\varphi(v)) \neq r$ **do**
    - $\varphi(\varphi(v)) := v$
  - **if** $\rho(x) \neq r$ **then**
    - $\varphi(x) := y$
  - **if** $\rho(y) \neq r$ **then**
    - $\varphi(y) := x$
  - **forall** $v \in V(G)$ *with* $\rho(v) \in V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})$ **do**
    - $\rho(v) := r$

return μ

vertex implies that $\forall y \in \Gamma(x) : y$ is inner and $\varphi(y) = \varphi(x)$. Define:

$$X := \text{set of inner vertices}$$
$$B := \text{set of bases of (outer) blossoms}$$

Then every unmatched vertex is in B. Matched vertices in B have matching mates in X and $|B| = |X| + |V(G)| - 2|M|$. (Outer) blossoms are odd connected components in $G - X$, so by Berge's theorem (1.24), at least $|B| - |X|$ vertices remain uncovered by any matching, so M is maximum.

We now consider the running time: The status (outer, inner, out-of-forest) for a given vertex can be checked in constant time (proposition 1.46). Therefore, *Grow*, *Augment* and *Shrink* can be implemented in $O(n)$ time. There are at most n calls to *Grow* and *Shrink* per augment and at most $\frac{n}{2}$ *Augments*. This implies the running time $O(n^3)$. □

Remark 1.49. The time for *Shrink* can be reduced to $O(\log n)$ using a binary tree, leading to a running time of $O(nm \log n)$ in total. Tarjan (1974), Gabow & Tarjan (1983) proved a running time of $O(nm\alpha(m, n))$ (where α is the inverse Ackermann function) or $O(nm)$.

Remark 1.50. It's not necessary to reset everything after augmenting. It suffices to reset the 2 trees that were changed by the augmentation. Gabow & Tarjan (1983) showed that it's possible to augment all paths of the same length in $O(m)$ time. There are $2\sqrt{\nu(G)} + 2$ different path lengths, so in total this results in a running time of $O(\sqrt{n}m)$.

Remark 1.51 (Skew-symmetric flows). Goldberg & Karzanov (2003) (and Fremuth-Pagen & Jungnickel (2003)) used *Generalized Max-Flow* to achieve a running time of $O(\sqrt{n}m\frac{\log \frac{m}{n}}{\log n})$.

1.7 Gallai-Edmonds Decomposition

Proposition 1.52. *Let G be a graph, $X \subseteq V(G)$ with $|V(G)| - 2\nu(G) = q_G(X) - |X|$. Then any maximum matching of G*

- *contains a perfect matching in the even components of $G - X$.*
- *contains a near-perfect matching in odd components of $G - X$.*
- *matches all $x \in X$ to distinct odd components.*

Proof. Follows directly from Berge's theorem (1.24). □

Theorem 1.53. *Let G be a graph and:*

$$Y := \{v \in V(G) \mid \exists \text{ maximum matching that exposes } v\}$$

Define $X := \Gamma(Y)$ and $W := V(G) \setminus (X \cup Y)$. Then:

1. *X attains* $\max_{X' \subseteq V(G)} q_G(X') - |X'|$.
2. *$G[Y]$ is the union of factor-critical subgraphs and $G[W]$ is the union of even connected components.*
3. *Any maximum matching in G*
   - *contains a perfect matching in $G[W]$.*
   - *contains a near-perfect matching in each component of $G[Y]$.*
   - *matches all $x \in X$ to distinct connected components*

Y, X, W is called Gallai-Edmonds decomposition *of G.*

Proof. Consider the matching M and special blossom forest F at the end of the algorithm. Let X' (Y') be the set of inner (outer) vertices and W' the set of out-of-forest vertices. X', Y', W' satisfy 1., 2. and 3. by the proof of theorem 1.48.

Proposition 1.52 implies that any maximum matching covers all vertices in $V(G) \setminus Y'$, so $Y \subseteq Y'$. For the other inclusion, let $v \in Y'$. Then $M \Delta P(v)$ is a maximum matching exposing v, so $v \in Y$ and $Y' = Y$. By definition, $X = X'$ and $W = W'$. $\square$

Corollary 1.54. *A graph G has a perfect matching $\Leftrightarrow \forall U \subseteq V(G)$, $G - U$ has at most $|U|$ factor-critical components.*

1.8 Minimum Weight Perfect Matching

We use the following Integer Programming formulation:

$$\begin{aligned} \min \quad & \sum_{e \in E(G)} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e = 1 \qquad v \in V(G) \\ & x_e \in \{0, 1\} \end{aligned}$$

and the corresponding relaxation where we only require $x_e \geq 0$. The dual problem of this is:

$$\begin{aligned} \max \quad & \sum_{v \in V(G)} z_v \\ \text{s.t.} \quad & z_v + z_w \leq c_e \qquad \{v, w\} \in E(G) \end{aligned}$$

Proposition 1.55 (Hungarian Method)**.** *Let G be a graph, $c \in \mathbb{R}^{E(G)}$ and $z \in \mathbb{R}^{V(G)}$ with $z_v + z_w \leq c_e$ for all $e = \{v, w\} \in E(G)$. Define:*

$$G_z := (V(G), \{e = \{v, w\} \mid z_v + z_w = c_e\})$$

Let M be a matching in G_z, F a maximal alternating forest in G_z with respect to M. Let X/Y be the set of inner/outer vertices. Then:

1. *If M is a perfect matching in G_z, then it is a minimum-weight perfect matching in G.*
2. *If $\Gamma_G(y) \subseteq X$ for all $y \in Y$, then M is a maximum matching.*
3. *If neither 1. nor 2. hold, define:*

$$\epsilon := \min\{\min_{e=\{v,w\}\in E(G[Y])} \frac{c_e - z_v - z_w}{2}, \min_{e\in\delta(Y)\cap\delta(V(F))} c_e - z_v - z_w\}$$

   Set $z'_v := z_v - \epsilon$ for all $v \in X$, $z'_v := z_v + \epsilon$ for all $v \in Y$ and $z'_v := z_v$ for all $v \in V(G) \setminus (X \cup Y)$. Then z' is a feasible dual solution and $M \cup E(F) \subseteq E(G_{z'})$. Additionally, $\Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$ for some $y \in Y$.

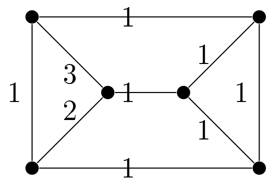
Proof. 1. Let M' be a minimum-weight perfect matching.

$$\begin{aligned}\sum_{e\in M'} c_e &= \sum_{v\in V(G)} z_v + \sum_{e=\{v,w\}\in M'} (c_e - z_v - z_w) \\ &\geq \sum_{v\in V(G)} z_v \\ &= \sum_{v\in V(G)} z_v + \sum_{e=\{v,w\}\in M} (c_e - z_v - z_w) \\ &= \sum_{e\in M} c_e\end{aligned}$$

2. Each outer vertex is an odd blossom (singleton) of $G - x$. By Berge (1.24), at least $|Y| - |X|$ vertices remain uncovered.
3. z' stays feasible by the choice of ϵ. Edges in $E(F), M$ remain tight. By 1. and 2., $\exists y \in Y : \Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$.

□

Remark 1.56. For bipartite graphs, the adjacency matrix is totally unimodular, so the LP has integral vertices.



We define $\mathcal{A} := \{X \subseteq V(G) \text{ odd}\}$ and add the blossom inequalities

$$\sum_{e \in \delta(X)} x_e \geq 1 \qquad \forall X \in \mathcal{A}$$

to the LP relaxation. The new dual problem is then:

$$\begin{aligned} &\max \sum_{A \in \mathcal{A}} z_A \\ \text{s.t.} \quad &\sum_{A \in \mathcal{A}, e \in \delta(A)} z_A \leq c_e \\ &z_A \geq 0 \qquad (A \in \mathcal{A}, |A| \geq 3) \end{aligned}$$

Edmond's Algorithm starts with an empty matching $x = 0$ and dual feasible solution:

$$z_A := \begin{cases} \frac{1}{2} \min\{c(e) \mid e \in \delta(A)\} & |A| = 1 \\ 0 & \text{else} \end{cases}$$

We always ensure that z is dual feasible and that (x, z) satisfy complementary slackness:

$$\begin{aligned} x_e > 0 \Rightarrow \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A = c_e \\ z_A > 0, |A| > 1 \Rightarrow \sum_{e \in \delta(A)} x_e = 1 \end{aligned}$$

Definition 1.57. $c_z(e) := c(e) - \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A$ is the *reduced cost* of e.

Theorem 1.58. *There are at most* $\frac{7}{2} |V(G)|^2$ *of the repeat-until loop in algorithm 4.*

Proof. $\mathcal{B}$ is laminar at any time, i.e. for $X, Y \in \mathcal{B}$ we have $(X \subseteq Y) \vee (Y \subseteq X) \vee (X \cap Y = \emptyset)$. Therefore $|\mathcal{B}| \leq 2 |V(G)|$.

Observation. *Any U added to $\mathcal{B}$ during* Shrink *will not be "unpacked" before the next* Augment.

Proof. After *Shrink*, there exists an even length M-augmenting R-U-path. It remains in G_z until the next *Augment* or until U is included in another blossom $U' \supseteq U$ which is not resolved before an *Augment* (inductively). □

Between 2 augments:

- \# *Unpacks* $\leq |\mathcal{B}|$ at beginning of the sequence

- $\#$ *Shrinks* $\leq |\mathcal{B}|$ at the end of the sequence

Therefore, there are at most $4\,|V(G)|$ *Unpack* and *Shrink* operations between 2 augments. For each dual change without *Unpack*, we have: $z_B > 0 \quad \forall B \in \mathcal{B}$, so ϵ is not determined by z_B. Therefore $\exists e = \{X, Y\}$ with $X \notin \mathcal{X}, Y \in \mathcal{Y}$ where $c_z(e)$ becomes 0.

Case 1: $X \notin \mathcal{Y}$. Then $|V(G_z) \setminus (\mathcal{X} \cup \mathcal{Y})|$ decreases.

Case 2: $X \in \mathcal{Y}$. Then $\exists$ X-Y M-alternating walk in the next iteration.

In particular, such a dual change can occur at most $|V(G)|$ times between 2 augmentations.

In total, there are at most $\frac{1}{2}\,|V(G)|$ *Augment* steps. Therefore, there are $\frac{1}{2}\,|V(G)|^2\,(4 + |V(G)| + 2\,|V(G)|)$ $\square$

Algorithm 4: Minimum-Weight Perfect Matching

Input: Graph G with edge weights $c : E(G) \to \mathbb{R}$
Output: A minimum-weight perfect matching M in (G, c)

Corollary 1.59. *A minimum-weight perfect matching can be computed in $O(n^2 m)$ time where $n := |V(G)|$ and $m = |E(G)|$.*

Proof. Theorem 1.58 times $O(m)$. $\square$

Remark 1.60. To achieve $O(n^3)$ running time, one can modify the algorithm:

1. Use a General Blossom Forest to avoid recomputing the R-R-walks from scratch. We then have mappings $\mu_v, \varphi_v^i, \rho_v^i$ for $1 \leq i \leq k_v$ where k_v is the number of blossoms that contain v.
2. Store all vertices in a heap (ordered by their criticality for dual-feasibility) to speed up the computation of ϵ.

Gabow (1990) showed a running time of $O(n(m + n \log n))$. Gabow & Tarjan (1991) showed a running time of $O(m \log(nW)\sqrt{n\alpha(m,n) \log n})$ where $W := \max_{e \in E(G)} |c(e)|$.

1.8.1 The Matching Polytope

Theorem 1.61. *Let G be a graph. The set of vectors $x \in \mathbb{R}^{E(G)}$ satisfying*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(v)) &= 1 & v &\in V(G) \\ x(\delta(A)) &\geq 1 & A &\subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

is the convex hull of all perfect matchings in G. It is called the perfect matching polytope.

Proof. For any objective function $c : E(G) \to \mathbb{R}$, the minimum-weight perfect matching algorithm produces an integral primal and a dual solution that satisfy complementary slackness. In particular, all vertices of the polytope are integral. $\square$

Theorem 1.62. *Let G be a graph. The set of vectors $x \in \mathbb{R}^{E(G)}$ satisfying*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(v)) &\leq 1 & v &\in V(G) \\ x(E(G[A])) &\leq \frac{|A|-1}{2} & A &\subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

is the convex hull of all matchings in G. It is called the matching polytope.

Proof. Any matching solution x satisfies these conditions. Let x be any solution that satisfies the conditions. We have to show that x is a convex combination of matching solutions. Define H by:

$$\begin{aligned} V(H) &:= \{(v,i) \mid v \in V(G), i \in \{1,2\}\} \\ E(H) &:= \{\{(v,i),(w,i)\} \mid \{v,w\} \in E(G), i \in \{1,2\}\} \\ &\cup \{\{(v,1),(v,2)\} \mid v \in V(G)\} \end{aligned}$$

We set $y_{\{(v,i),(w,i)\}} := x_{\{v,w\}}$ for all $\{v,w\} \in E(G), i \in \{1,2\}$ and $y_{\{(v,1),(v,2)\}} := 1 - x(\delta(v))$ for all $v \in V(G)$. Then $y \geq 0$ and $y(\delta_H(x)) = 1$ for all $x \in V(H)$.

Claim. *y satisfies the inequalities of the perfect matching polytope (in particular the blossom inequalities).*

If this is true, by 1.62 y is a convex combination of perfect matchings. $H[\{(v,1) \mid v \in V(G)\}]$ is isomorphic to G, so x is a convex combination of matchings in G.

We now prove the claim: Let $X \subseteq V(H)$ with $|X|$ odd. We have to show that $y(\delta_H(X)) \geq 1$. Define:

$$\begin{aligned} A &:= \{v \in V(G) \mid (v,1) \in X, (v,2) \notin X\} \\ B &:= \{v \in V(G) \mid (v,1) \in X, (v,2) \in X\} \\ C &:= \{v \in V(G) \mid (v,1) \notin X, (v,2) \in X\} \end{aligned}$$

Define $A_i := A \cap (V(G) \times \{i\})$ and $B_i := B \cap (V(G) \times \{i\})$. $|B_1 \cup B_2|$ is even, so (since $|X|$ is odd) $|A|$ or $|C|$ is odd. Without loss of generality, let

$|A|$ be odd.

$$\begin{aligned}\sum_{e\in\delta_H(X)} y_e &\geq \sum_{v\in A_1} \underbrace{\sum_{e\in\delta_H(v)} y_e}_{=1} - 2\cdot \sum_{e\in E(H[A_1])} y_e - \sum_{e\in\delta(A_1)\cap\delta(B_1)} y_e \\ &+ \sum_{e\in\delta(A_2)\cap\delta(B_2)} y_e \\ &= |A_1| - 2\cdot \sum_{e\in E(G[A])} x_e \\ &\geq |A_1| - (|A|-1) \\ &= 1\end{aligned}$$

$\square$

Theorem 1.63. *The matching polyhedron is TDI (Totally Dual Integral), i.e. for all $c \in \mathbb{Z}^{E(G)}$ for which the dual program of ($\max c^t x s.t. \dots$) has a finite optimum solution, it has an integral optimum solution.*

Proof. The dual is

$$\begin{aligned}\min \sum_{v\in V(G)} y_v + \sum_{A\in\mathcal{A},\ |A|>1} \frac{|A|-1}{2} z_A & \\ s.t. \sum_{v\in e} y_v + \sum_{A\in\mathcal{A},\ |A|>1, e\in E(G[A])} z_A \geq c(e) & \qquad e\in E(G) \\ y, z \geq 0 &\end{aligned}$$

Let (G,c) be a counterexample such that $|V(G)| + |E(G)| + \sum_{e\in E(G)} |c(e)|$ is minimum. Then:

- $c(e) \geq 1$ for all $e \in E(G)$, since otherwise e could be deleted.
- G has no isolated vertices.

Claim. *In an optimum solution (y,z), $y = 0$.*

Proof. If $y_v > 0$, then $x(\delta(v)) = 1$ for all optimum solutions x. Decreasing $c(e)$ by 1 for all $e \in \delta(v)$ yields a smaller feasible instance (G, c') where the weight of x is decreased by 1 and x remains optimum. By assumption, (G,c') is not a counterexample, so there exists an integral optimum solution (y', z'). Increasing y'_v by one yields some optimum in (G,c) which has optimum integral solution $(y' + \mathbb{1}_v, z')$. $\square$

Let $(y=0,z)$ be a dual optimum solution such that

$$\sum_{A\in\mathcal{A},|A|>1} |A|^2\, z_A$$

is maximum.

Claim. $\mathcal{F} := \{A : z_A > 0\}$ *is laminar.*

If not, there exist $X, Y \in \mathcal{F}:\ \ X\cap Y, X\setminus Y, Y\setminus X \neq \emptyset$. We proceed by "uncrossing". Let $\epsilon := \min\{z_X, z_Y\} > 0$.

Case 1: $|X\cap Y|$ is odd. Then $|X\cup Y|$ is odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_y &:= z_y - \epsilon \\ z'_{X\cap Y} &:= z_{X\cap Y} + \epsilon && \text{(unless } |X\cap Y| = 1\text{)} \\ z'_{X\cup Y} &:= z_{X\cup Y} + \epsilon \\ z'_A &:= z_A && \text{elsewhere} \end{aligned}$$

Then (y, z') is a dual optimum solution.

Case 2: $|X\cap Y|$ is even. Then $|X\setminus Y|$ and $|Y\setminus X|$ are odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_Y &:= z_Y - \epsilon \\ z'_{X\setminus Y} &:= z_{X\setminus Y} + \epsilon && \text{unless } |X\setminus Y| = 1 \\ z'_{Y\setminus X} &:= z_{Y\setminus X} + \epsilon && \text{unless } |Y\setminus X| = 1 \\ z'_A &:= z_A && \text{elsewhere} \\ y'_v &:= \epsilon && \forall v\in X\cap Y \\ y'_v &:= 0 && \forall v\notin X\cap Y \end{aligned}$$

Then (y', z') is feasible. The objective value is:

$$\begin{aligned} &\sum_{v\in V(G)} y'_v + \sum_{A\in\mathcal{A},\ |A|>1} z'_A \frac{|A|-1}{2} \\ =&\epsilon\cdot|X\cap Y| + \sum_{A\in\mathcal{A},\ |A|>1} \frac{|A|-1}{2} \\ +&\epsilon\left(\frac{|X\setminus Y|-1}{2} + \frac{|Y\setminus X|-1}{2} - \frac{|X|-1}{2} - \frac{|Y|-1}{2}\right) \\ =&\text{objective}(y,z) \end{aligned}$$

Therefore (y', z') is an optimum solution with $y'\neq 0$, which is a contradiction to the previous claim.

We can conclude that $\mathcal{F}$ is laminar.

Let $A \in \mathcal{F}$ with $z_A \notin \mathbb{Z}$ and $|A|$ is maximal. Define $\epsilon := z_A - \lfloor z_A \rfloor > 0$. Let $A_1, \ldots, A_k$ be the inclusion-wise maximal proper subsets of A in $\mathcal{F}$. Since $\mathcal{F}$ is laminar, $A_i \cap A_j = \emptyset$ for $i \neq j$. Define:

$$\begin{aligned} z'_A &:= z_A - \epsilon && \\ z'_{A_i} &:= z_A + \epsilon && 1 \le i \le k \\ z'_D &:= z_D && \text{elsewhere} \end{aligned}$$

Then (y, z') is dual feasible with objective value:

$$\sum_{B \in \mathcal{A},\ |B|>1} \frac{|B|-1}{2} z'_B < \sum_{B \in \mathcal{A},\ |B|>1} \frac{|B|-1}{2} z_B$$

This contradicts the optimality of (y, z), so there exists no counter example. □

Theorem 1.64. *Let G be a graph.*

$$\begin{aligned} P &:= \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) \leq 1 \quad \forall v \in V(G)\} \\ Q &:= \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) = 1 \quad \forall v \in V(G)\} \end{aligned}$$

are called the fractional matching polytope *and the* fractional perfect matching polytope. *If G is bipartite, then P and Q are integral.*

Proof. The adjacency matrices of bipartite graphs are totally unimodular. □

Theorem 1.65. *Let G be a graph. The vertices of the fractional perfect matching polytope satisfy*

$$x_e = \begin{cases} \frac{1}{2} & \textit{if } e \in E(C_1) \cup \ldots \cup E(C_k) \\ 1 & \textit{if } e \in M \\ 0 & \textit{else} \end{cases}$$

where $C_1, \ldots, C_k$ are vertex-disjoint odd circuits and M is a perfect matching in $G - (V(C_1) \cup \ldots \cup V(C_k))$.

Proof. Exercise 6.3 □

2 T-Joins and b-Matchings

Definition 2.1. Let G be a graph, $T \subseteq V(G)$. A subset $J \subseteq E(G)$ is called *T-join* if T is the set of odd-degree vertices in $(V(G), J)$.

Proposition 2.2. *Let G be a graph, $T, T' \subseteq V(G)$, J a T-join and J' a T'-join. Then $J \Delta J'$ is a $T \Delta T'$-join.*

Proof. For $v \in V(G)$:

$$\begin{aligned} |\delta_{J\Delta J'}(v)| &\equiv |\delta_J(v)| + |\delta_{J'}(v)| \\ &\equiv |\{v\} \cap T| + \left|\{v\} \cap T'\right| \\ &\equiv \left|\{v\} \cap (T \Delta T')\right| \mod 2 \end{aligned}$$

□

Proposition 2.3. *Let G be a graph, $T \subseteq V(G)$.*

$$\exists\ T\text{-join in } G \Leftrightarrow |V(C) \cap T| \ \text{ even for each connected component } C$$

Proof.

"$\Rightarrow$": Let J be a T-join. For each connected component C:

$$\sum_{v \in V(C)} |J \cap \delta(v)| = 2\,|J \cap E(C)|$$

Therefore $|J \cap \delta(v)|$ is odd for an even number of vertices and $|V(C) \cap T|$ is even.

"$\Leftarrow$": Partition T into pairs $\{v_1, w_1\}, \ldots, \{v_k, w_k\}$ such that v_i and w_i are in the same component for all i. Let P_i be a v_i-w_i-path in G. Define $J := E(P_1)\Delta E(P_2)\Delta \ldots \Delta E(P_k)$. By proposition 2.2, this is a T-join.

□

Theorem 2.4. *Let G be a graph, $c : E(G) \to \mathbb{R}$ and $T \subseteq V(G)$. In strongly polynomial time (e.g. $O(n^2 m)$) we can determine if a T-join exists and if so, compute a minimum-weight T-join.*

Proof. In $O(m)$ ($m := |E(G)|$), we can check if a T-join exists. If so:

1. Eliminate negative weights.

$$\begin{aligned} N &:= \{e \in E(G) \mid c(e) < 0\} \\ U &:= \{v \in V(G) \mid |\delta_N(v)| \text{ odd}\} \\ T' &:= T \Delta U \\ c'(e) &:= |c(e)| && e \in E(G) \end{aligned}$$

Claim. *If J' is a minimum T'-join with respect to c', then $J' \Delta N$ is a minimum T-join with respect to c.*

Let $\tilde{J}$ be a T-join. Then $\tilde{J} \Delta N$ is a T'-join, so $c'(\tilde{J}) \leq c'(\tilde{J} \Delta N)$ and

$$c(J) = c'(J') + c(N) \leq c'(\tilde{J} \Delta N) + c(N) = c(\tilde{J})$$

which proves the claim.

2. We can now assume that $c \geq 0$. A minimum-weight T-join does not have cycles of positive weight. We can eliminate cycles of weight 0 without changing the cost. We can then restrict ourselves to searching for collections of T-T-paths.

   Let K_T be the metric closure of T with respect to G. It can be computed in $O(n \cdot (m + n \log n))$ by using Dijkstra for all vertices. Find a minimum-weight perfect matching M in K_T. Each $e = \{s, t\} \in M$ induces a path $P_{s,t}$. Then the symmetric difference $\Delta_{\{s,t\} \in M} E(P_{s,t})$ is a minimum-weight T-join in G.

□

Corollary 2.6. *A maximum-weight T-join can be computed as fast as a minimum-weight T-join.*

Proof. Set $c' := -c$. □

Corollary 2.7. *Let G be a graph, $c : E(G) \to \mathbb{R}$. We can find a cycle of negative length in G in $O(n^2 m)$ time.*

Proof. Apply theorem 2.4 to $T = \emptyset$. If $c(J) < 0$, $(V(G), J)$ contains a cycle C. If $c(C) = 0$, we can eliminate it and recurse, otherwise return C. □

2.2 T-Join Applications

2.2.1 TSP Approximation

Let (K_n, c) with c metric be an instance of the TSP. Consider the *Double tree algorithm*:

1. Compute a minimum spanning tree T.
2. $T' := T + T$ (doubling all edges). Then T' is Eulerian.
3. Walk along T' and add vertices to the TSP tour in the order of their first appearance, yielding a tour T^*. Since c is metric, we have $c(T^*) \leq c(T') \leq 2c(T)$. Since the cost of T is a lower bound for the cost of a tour, we have $c(T^*) \leq 2\text{OPT}$ (where OPT is the cost of a shortest TSP tour).

Algorithm 5: Christofides Algorithm (1976)

Input: Complete metric graph (K_n, c)
Output: A TSP-tour T
Find MST T_{MST} in (K_n, c)
$W \coloneqq \{v \in V(K_n) \mid |\delta_{T_{\text{MST}}}(v)| \text{ odd}\}$
$J \coloneqq$ minimum-weight W-Join in (K_n, c)
Add cities to T in the order of first appearance in a Eulerian walk of $T_{\text{MST}} + J$.
return T

Theorem 2.8. *Algorithm 5 is a $\frac{3}{2}$-approximation algorithm for the metric TSP, i.e. for the computed tour T we have:*

$$c(T) \leq \frac{3}{2}\text{OPT}$$

Proof. We have $c(T_{\text{MST}}) \leq \text{OPT}$ and $\text{OPT}(W) \leq \text{OPT}(V(K_n))$ (since c is metric). Any tour through the vertices in W can be decomposed into 2 matchings. Therefore, $c(J) \leq \frac{1}{2}\text{OPT}(W) \leq \frac{1}{2}\text{OPT}$. It follows that $c(T) \leq (1 + \frac{1}{2})\text{OPT}$. □

2.2.2 Shortest Paths in Undirected Graphs

The naive reduction to digraphs requires non-negative weights.

Corollary 2.9. *Given an undirected graph G, $c : E(G) \to \mathbb{R}$ such that each ciruit has length at least 0. Then for $s, t \in V(G)$, a shortest s-t-path can be found in $O(n^2 m)$ time, where $n \coloneqq |V(G)|$, $m \coloneqq |E(G)|$.*

Proof. Choose $T \coloneqq \{s, t\}$. Apply theorem 2.4 to get a minimum-weight T-join J. J can be partitioned into circuits of length 0 and an s-t-path of length $c(J)$. □

2.2.3 Chinese Postman Problem

Definition 2.10. A walk $C = \{v_0, e_1, v_1, \dots, e_t, v_t\}$ is called a Chinese postman tour if $v_0 = v_t$ and each edge in $E(G)$ is visited at least once. The Chinese Postman Problem is the problem of finding a shortest Chinese postman tour in G with respect to $c : E(G) \to \mathbb{R}_{\geq 0}$.

Corollary 2.11. *The Chinese postman problem can be solved in $O(n^2 m)$ time, where $n \coloneqq |V(G)|$, $m \coloneqq |E(G)|$.*

Proof. Set $T \coloneqq \{v \in V(G) \; |\delta(v)| \text{ odd}\}$ and let J be a minimum-weight T-join. Compute a Eulerian tour C in $G + J$. Let C' be a shortest Chinese postman tour. Let $J' :=$ set of edges occuring in C' an even number of times (at least twice). Then J' is a T-join, so $c(J') \geq c(J)$ and:

$$c(C') \geq c(E(G)) + c(J') \geq c(E(G)) + c(J) = c(C)$$

□

2.3 T-Joins and T-Cuts

Definition 2.12. Let G be a graph and $T \subseteq V(G)$. A T-cut is a cut $C = \delta(X)$ with $X \subseteq V(G)$ and $|X \cap T|$ odd.

Proposition 2.13. *Let G be a graph, $T \subseteq V(G)$, $|T|$ even. Then:*

1. *For any T-join J and any T-cut C: $J \cap C \neq \emptyset$.*
2. *The inclusion-wise minimal T-cuts (T-joins) are exactly the inclusion-wise minimal edge sets intersecting all T-joins (all T-cuts).*

Proof. For 1., let $C = \delta(X)$ with $|X \cap T|$ odd be a T-cut. Then the edges in $J \cap C$ either belong to a path passing through X or have an endpoint in T. Therefore $|J \cap C|$ is odd, in particular the set is non-empty.

For 2., we prove in an exercise that each edge set intersecting all T-joins (T-cuts) contains a T-cut (T-join). Therefore minimal such sets are T-cuts (T-joins). Remark: The minimum cardinality of a T-join is at least as large as the maximum number of edge-disjoint T-cuts[4]. □

Theorem 2.14 (Seymour (1981))**.** *Let G be bipartite, $T \subseteq V(G)$ such that there exists a T-join. Then:*

$$\textit{min. cardinality of a } T\textit{-join} = \textit{max. number of edge-disjoint } T\textit{-cuts}$$

The maximum is attained by a crossfree family $\mathcal{C}$ of cuts, i.e.

$$\forall X, Y \in \mathcal{C} : X \subseteq Y \vee Y \subseteq X \vee X \cap Y = \emptyset \vee X \cup Y = V(G)$$

Proof. If $T = \emptyset$, the statement is clear. Let $T \neq \emptyset$. We proceed by induction on $|V(G)| + |T|$. Let J be a minimum-cardinality T-join. Set:

$$c(e) := \begin{cases} -1 & e \in J \\ 1 & e \in E(G) \setminus J \end{cases}$$

Claim. *Every circuit C has $c(C) \geq 0$.*

[4] In general, the two numbers are not equal: Consider K_4 and $T = V(K_4)$. A minimum T-join consists of 2 edges but there are no 2 edge-disjoint T-cuts.

$$
\begin{aligned}
c(C) &= \underbrace{c(C \setminus J)}_{=|C\setminus J|} + \underbrace{c(C \cap J)}_{=-|C\cap J|} + |J \setminus C| - |J \setminus C| \\
&= \left|\underbrace{C \Delta J}_{T\text{-join}}\right| - |J| \geq 0
\end{aligned}
$$

Let P be a minimum length walk in (G, c) traversing no edge more than once such that $|E(P)|$ is minimum. Then P is a path. Let t be the last vertex in P and f the edge entering t. Then $f \in J$, otherwise $c(f) = 1$ and deleting f would yield a shorter path. Furthermore, $|\delta_J(t)| = 1$, otherwise we could add the other edge from $J \cap \delta(t)$ to shorten $c(P)$.

Claim. *Each circuit C that contains t but not f has $c(C) > 0$.*

Case 1: t is the only vertex in $V(C) \cap V(P)$. Let $e \ni t$ be an edge on C incident to t. Then $c(e) = 1$ (since $\delta_J(t) = \{f\}$) and $P' := P + C - e$ yields a shorter walk if $c(C) \leq 0$.

Case 2: $V(C) \cap V(P)$ contains another vertex x. Let u be the last vertex on P before t that is also on C. Define $P' := P_{[u,t]}$. C can be split into 2 u-t-paths C', C''. By minimality of P, $c(P') < 0$. $P' + C', P' + C''$ are circuits (by choice of u). By the first claim, $c(C'), c(C'') > 0$, so also $c(C) > 0$.

Shrink: $\{t\} \cup \Gamma(t)$ to a new vertex v_0. This yields a bipartite graph G'. If $|T \cap (\{t\} \cup \Gamma(t))|$ is odd, set $T' := T \setminus (\{t\} \cup \Gamma(t)) \cup \{v_0\}$. Otherwise, $T' := T \setminus (\{t\} \cup \Gamma(t))$. Define $J' := J \setminus \{f\}$.

Claim. *J' is a minimum cardinality T'-join in G'.*

If not, there exists a T'-join J'' with $|J''| < |J'|$. $J'' \Delta J'$ is an $\emptyset$-Join. Therefore, there exists a circuit C' where $|C' \setminus J'| < |C' \setminus J''| = |C' \cap J'|$ (since G is bipartite). If C' results from a circuit C in G not containing T, then $|C \setminus J| < |C \cap J|$. This is a contradiction to the minimality of J.

Therefore C' results from a circuit containing T.

Case 1: C traverses f. Then

$$
\begin{aligned}
|C' \setminus J'| - |C' \cap J'| &= |C \setminus J| - |C \cap J| \\
&> 0
\end{aligned}
$$

which is a contradiction.

Case 2: By the second claim, $c(C) > 0$, so since G is bipartite $c(C) \geq 2$ and $|C \setminus J| \geq |C \cap J| + 2$. Therefore

$$\begin{aligned} |C' \setminus J'| &= |C \setminus J| - 2 \\ &\geq |C \cap J| \\ &= |C' \cap J'| \end{aligned}$$

which is a contradiction to the assumption.

By the induction hypothesis on G', G' has cross-free T'-cuts $D_1, \ldots, D_{|J'|}$. Together with $\delta(t)$, we get $|J'| + 1 = |J|$ T-cuts. Since $\Gamma(t)$ was contracted in G', they are cross-free. □

Corollary 2.15. *Let G be a graph, $c : E(G) \to \mathbb{Z}_{\geq 0}$, $T \subseteq V(G)$ such that a T-join exists. The minimum cost of a T-join equals half the maximum number of T-cuts covering each edge e at most $2 \cdot c(e)$ times. This maximum is attained by a cross-free family of T-cuts.*

Proof. Let $E_0 := \{e \in E(G) \mid c(e) = 0\}$. Contract the connected components in $(V(G), E_0)$ and replace each $e \in E(G)$ by a path of length $2 \cdot c(e) > 0$. The resulting graph G' is bipartite. Let

$T' := \{v \in V(G') \mid v$ corresponds to a connected component X in G with $|X \cap T|$ odd$\}$

Let k be the minimum cost of a T-join in G.

Claim. *The minimum cardinality of a T'-join in G' is $2k$.*

"$\leq$": Every T-join J in J corresponds to a T'-join J' in G' with $|J'| \leq 2c(J)$.

"$\geq$": Let J' be a T'-join in G'. J' corresponds to an edge set $J \subseteq E(G)$. Let $\overline{T} := T \Delta \{v \in V(G) \mid |\delta(v) \cap J| \text{ odd}\}$. For each connected component X in $(V(G), E_0)$:

$$|\delta(X) \cap J| \equiv |X \cap T| \mod 2$$

Therefore $|X \cap \overline{T}|$ is even, so by proposition 2.3, there exists a $\overline{T}$-join $\overline{J}$ in $(V(G), E_0)$. Then $J \cup \overline{J}$ is a T-join of weight $c(J) = \frac{|J'|}{2}$.

By theorem 2.14, there exist $2k$ pairwise disjoint T'-cuts in G'. In G this yields $2k$ T-cuts such that every edge e is covered by at most $2 \cdot c(e)$ cuts and they can be created cross-free. □

2.3.1 T-join Polytope

We define the T-join polytope:

$$P_{T\text{-join}} := \operatorname{conv}\{x \in \mathbb{R}^{E(G)} \mid x \text{ incidence vector of a } T\text{-join}\}$$
$$P^{\uparrow}_{T\text{-join}} := P_{T\text{-join}} + \mathbb{R}^{E(G)}_{\geq 0}$$

Corollary 2.16. *$P^{\uparrow}_{T\text{-join}}$ is determined by*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(X)) &\geq 1 & &\forall T\text{-cuts } \delta(X) \end{aligned}$$

Proof. "$\subseteq$" is clear. Assume that the other inclusion does not hold. Then there exists $w : E(G) \to \mathbb{Q}$ such that the minimum weight of a T-join $\alpha > \min w^t x$ where x satisfies the stated inequalities. Without loss of generality, $w \in \mathbb{Z}^{E(G)}_{\geq 0}$, both cones are identical ($\mathbb{R}^{E(G)}_{\geq 0}$). By corollary 2.15, there exist T-cuts $C_1, \ldots, C_{2\alpha}$ such that each edge e is covered at most $2w(e)$ times.

$$y_C := \frac{1}{2}\text{number of times } C \text{ occurs in } C_1, \ldots, C_{2\alpha}$$

Then y is a feasible solution to the dual:

$$\begin{aligned} &\max_{C\ T\text{-cut}} y_C \\ \text{s.t.} &\sum_{C\ T\text{-cut},\ e \in C} y_e \leq w(e) & e \in E(G) \\ &y \geq 0 \end{aligned}$$

$\sum_C y_C = \alpha$ is a lower bound for the minimization problem which is a contradiction to the assumed inequality. ☐

2.4 Excursus: Gomory-Hu Trees

Let G be a graph, $u : E(G) \to \mathbb{R}_{\geq 0}$. Find $\emptyset \subsetneq X \subsetneq V(G)$ minimizing $u(\delta(X))$. One approach: $\binom{|V(G)|}{2}$ s-t-cut computations (this can clearly be reduced to $|V(G)| - 1$ by fixing s).

Definition 2.17. For $s, t \in V(G)$, denote by λ_{st} the minimum capacity of an s-t-cut (or *local edge connectivity* of s, t).

Lemma 2.18. *For all $u, v, w \in V(G)$:*

$$\lambda_{uw} \geq \min\{\lambda_{uv}, \lambda_{vw}\}$$

Proof. Let $\delta(A)$ be a u-w-cut. If $v \in A$, then $\delta(A)$ is a v-w-cut, so $u(\delta(A)) \geq \lambda_{vw}$. Otherwise, $\delta(A)$ is a u-v-cut, so $u(\delta(A)) \geq \lambda_{uv}$. ☐

Definition 2.19. Let G be a graph, $u : E(G) \to \mathbb{R}_{\geq 0}$. A tree T is a Gomory-Hu tree for (G, u) if $V(T) = V(G)$ and

$$\lambda_{st} = \min_{e \in E(T_{[s,t]})} u(\delta_G(C_e)) \qquad \forall s, t \in V(G)$$

where C_e and $V(G) \setminus C_e$ are the connected components of $T - e$[5].

Lemma 2.20. *Given (G, u) and a tree T with $V(T) = V(G)$:*

$$T \text{ Gomory-Hu tree} \Leftrightarrow \forall e = \{s, t\} \in E(T) \text{ is a minimum capacity } s\text{-}t\text{-cut}$$

Proof. "$\Rightarrow$" follows directly from the definition. For the other direction, let $s, t \in V(G)$ and $e = \{u, v\} \in \arg\min_{e \in E(T_{s,t})} \lambda_{uv}$. Without loss of generality, $s \in C_e$, $t \in V(G) \setminus C_e$, so $\delta(C_e)$ is an s-t-cut. Therefore: $\lambda_{st} \leq u(\delta(C_e)) = \lambda_e$ (with $\lambda_e := \lambda_{uv}$). By lemma 2.20 and induction, $\lambda_{st} \geq \min\{\lambda_{v'w'} \mid \{v', w'\} \in E(T_{[s,t]})\} = \lambda_{uv}$. Therefore $\lambda_{st} = \lambda_{uv}$. □

Idea: Choose $r, s \in V(G)$ and compute a minimum capacity r-s-cut $\delta(R)$. Without loss of generality $r \in R$. Construct a graph G_R by shrinking $S := V(G) \setminus R$ into a single vertex. Find a minimum capacity p-q-cut (where $p, q \in R$ are chosen arbitrarily) in G_R. This partitions R into 2 parts. Continue this process until $V(G)$ is partitioned into singletons.

Lemma 2.21. *Let (G, u) as above, $s, t \in V(G)$, $\delta(A)$ a minimum capacity s-t-cut in G and $s', t' \in V(G) \setminus A$. Let (G', u') arise from (G, u) by contracting A into a single vertex. Then for any minimum capacity s'-t'-cut $\delta_{G'}(K \cup \{A\})$ in (G', u'), $\delta_G(K \cup A)$ is a minimum capacity s'-t'-cut in (G, u).*

Proof. Without loss of generality, $s \in A$. We show: $\exists$ min. capacity s'-t'-cut $\delta(A')$ in (G, u) such that $A \subseteq A'$. Let $\delta(C)$ be any s'-t'-cut in (G, u). Without loss of generality, $s \in C$. $u(\delta(\cdot))$ is a submodular function, i.e. $u(\delta(A)) + u(\delta(B)) \geq u(\delta(A \cap B)) + u(\delta(A \cup B))$ [6].

$\delta(A \cap C)$ is an s-t-cut, so $u(\delta(A \cap C)) \geq \lambda_{st} = u(\delta(A))$. Therefore, $u(\delta(A \cup C)) \leq u(\delta(C)) = \lambda_{s't'}$. Since $s' \in A \cup C$, $A \cup C$ is a minimum capacity s'-t'-cut. □

In general, we now choose a component X wih $|X| \geq 2$. Contract connected components in $T - \{X\}$, yielding a graph (G', u'). Choose $s, t \in X$, minimum s-t-cut $\delta(A')$ in (G', u'). $X = (X \cap A') \dot{\cup} (X \cap (V(G') \setminus A'))$.

Lemma 2.22. *At the end of MinCut:*

1. $A \dot{\cup} B = V(G)$

[5] $\delta(C_e)$ is called *fundamental cut* induced by e

[6] This holds with equality, if we add $2u(E(A, B))$ to the right side

2. $E(A,B)$ is a minimum s-t-cut in (G,u)

Proof. Elements of $V(T)$ are non-empty subsets of $V(G)$ and $V(T)$ form a partition of $V(G)$. Therefore $A \dot\cup B$ is a partition of $V(G)$. 2. follows from successive application of lemma 2.21 to each connected component of $T - X$. $\square$

Lemma 2.23. *At any time before FinishTree: $w(e) = u(\delta_G(\bigcup_{Z \in C_e} Z))$ for all $e \in E(T)$. Moreover, $\forall e = \{P,Q\} \in E(T)$ there exist $p \in P, q \in Q$: $w(e) = \lambda_{pq}$.*

Proof. At the start, $E(T) = \emptyset$. We show that both properties are always satisfied. Let X, s, t, A', B', A, B as determined by ChooseComponents, Contract and MinCut. Edges in $E(T) \setminus \delta(X)$ are not affected. For new edges both conditions are true after ModifyTree.

Consider an edge $e \in \{Y, X\}$ that is replaced by e' in ModifyTree. Without loss of generality $Y \subseteq A$, so $e' = \{X \cap A, Y\}$. We show that both statements hold for e'. $w(e) = w(e') = u(\delta_G(\bigcup_{Z \in C_e} Z)) = u(\delta_G(\bigcup_{Z \in C_{e'}} Z'))$ so 1. holds. Assume $p \in X, q \in Y$: $\lambda_{pq} = w(e)$. If $p \in X \cap A$, we are done.

If $p \in X \cap B$, we claim: $\lambda_{sq} = \lambda_{pq}$. This then implies $w(e') = w(e) = \lambda_{pq} = \lambda_{sq}$. By lemma 2.20, $\lambda_{sq} \geq \min\{\lambda_{st}, \lambda_{tp}, \lambda_{pq}\}$. By lemma 2.22, $E(A,B)$ is a minimum s-t-cut. By lemma 2.21 and since $s, q \in A$, λ_{sq} does not change when contracting B. Adding $\{t, p\}$ with sufficiently high capacity does not change λ_{sq}. Therefore $\lambda_{sq} \geq \min\{\lambda_{st}, \lambda_{pq}\} = \lambda_{pq}$ because $E(A,B)$ is also a p-q-cut. $w(e)$ is the capacity of a cut separating s, q, so $\lambda_{sq} \leq w(e) = \lambda_{pq}$. $\square$

Theorem 2.24 (Min Cut, Gomory & Hu (1961))**.** *Every undirected graph G with edge capacities $e : E(G) \to \mathbb{R}_{\geq 0}$ has a Gomory-Hu-tree. It can be computed using $n-1$ Min-s-t-cut computations, e.g. in $O(n^3\sqrt{m})$ time (using the Push-Relabel algorithm for computing the minimum cuts) where $n := |V(G)|$ and $m := |E(G)|$.*

Proof. Gomory-Hu-Algorithm computes a Gomory-Hu-tree (lemma 2.23). It uses $n-1$ iterations in each of which we need $O(n^2\sqrt{m})$ for Push-Relabel. Everything else can be handled in $O(\min\{n^3, n^2 m\})$ time. $\square$

2.5 Finding Minimum-Capacity T-Cuts

Theorem 2.25 (Padberg & Rao (1987))**.** *Given a graph $G, u : E(G) \to \mathbb{R}_{\geq 0}$, a Gomory-Hu-tree H for (G,u), $T \subseteq V(G)$ ($|T| \geq 2$ even), a minimum capacity T-cut can be found among the fundamental cuts of H. A minimum capacity T-cut can be computed in $O(n^3\sqrt{m})$ time.*

Proof. Let $\delta_G(X)$ be a minimum capacity T-cut in G. Let J be the set of edges in $E(H)$ for where $|C_e \cap T|$ is odd (where C_e is a connected component of $H - e$). For all $x \in V(G)$:

$$\begin{aligned} |\delta_J(x)| &\equiv \sum_{e \in \delta_H(x)} |C_e \cap T| \\ &\overset{T \text{ even}}{\equiv} |\{x\} \cap T| \qquad \mod 2 \end{aligned}$$

Therefore J is a T-join in H. Since T-cuts and T-joins intersect, there is $f \in J \cap \delta_H(X)$.

$$\begin{aligned} u(\delta_G(X)) &\geq \min\{u(\delta_G(Y)) \mid |Y \cap f| = 1\} \\ &= u(\delta_G(C_f)) \end{aligned}$$

We conclude that $\delta_G(C_f)$ is a minimum-capacity T-cut. $\square$

2.6 b-Matchings

Definition 2.26. Let G be a graph, $u : E(G) \to \mathbb{N}_0 \cup \{\infty\}$ and $b : V(G) \to \mathbb{N}_0$. A *$b$-matching* is a function $f : E(G) \to \mathbb{N}_0$ such that $f(e) \leq u(e)$ and $f(\delta(v)) \leq b(v)$ for all $e \in E(G)$ and $v \in V(G)$.

- If $u \equiv 1$, the instance is called *simple*.
- If $b \equiv 1$, this is equivalent to a matching.
- If $f(\delta(v)) = b(v)$ for all $v \in V(G)$, it is called *perfect*.
- Simple perfect b-matchings are called *b-factors*.

Example. A TSP-tour is a 2-factor. Therefore valid inequalities for 2-factors are valid for TSP.

Theorem 2.27 (Edmonds (1965))**.** *Let G be a graph, $b : V(G) \to \mathbb{N}$. The b-matching polytope of (G, ∞) is the set of vectors $x \in \mathbb{R}_{\geq 0}^{E(G)}$ satisfying:*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(v)) &\leq b(v) & v &\in V(G) \\ \sum_{e \in E(G[X])} x_e &\leq \Big\lfloor \frac{1}{2} \sum_{v \in X} b(v) \Big\rfloor & X &\subseteq V(G) \end{aligned}$$

Proof. Clearly, any b-matching satisfies these inequalities. Let x satisfy the inequalities. Without loss of generality $b \geq 1$. Define H by splitting each

$v \in V(G)$ into $b(v)$ copies, i.e.:

$$\begin{aligned}
X_v &\coloneqq \{(v,i) \mid i \in [b(v)]\} \qquad v \in V(G) \\
V(H) &\coloneqq \bigcup_{v \in V(G)} X_v \\
E(H) &\coloneqq \{\{v', w'\} \mid \{v,w\} \in E(G), v' \in X_v, w' \in X_w\} \\
y_{e'} &\coloneqq \frac{1}{b(v) \cdot b(w)} x_{\{v,w\}} \qquad e' = \{v', w'\} \in E(H), v' \in X_v, w' \in X_w
\end{aligned}$$

Claim. *y is a convex combination of matchings in H. Contracting all X_v ($v \in V(G)$) yields a convex combination of b-matchings for x.*

We show that y is contained in the matching polytope, i.e.:

$$\begin{aligned}
y_e &\geq 0 && e \in E(H) \\
\sum_{e \in E(H[A])} y_e &\leq \frac{|A|-1}{2} && A \subseteq V(H),\ |A| \text{ odd}
\end{aligned}$$

If $\forall v \in V(H)$: $X_v \subseteq A$ or $X_v \cap A = \emptyset$, this follows directly from the given inequalities. Otherwise, let $a, b \in X_v$ such that $a \in A, b \notin A$.

$$\begin{aligned}
2 \sum_{e \in E(H[A])} y_e &= \sum_{c \in A \setminus \{a\}} \sum_{e \in E(\{c\}, A \setminus \{c\})} y_e + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e \\
&\leq \sum_{c \in A \setminus \{a\}} \sum_{e \in \delta_H(c) \setminus \{\{c,b\}\}} + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e \\
&= \sum_{c \in A \setminus \{a\}} \sum_{e \in \delta_H(c)} y_e \underbrace{- \sum_{e \in E(\{b\}, A \setminus \{a\})} y_e + \sum_{e \in E(\{a\}, A \setminus \{a\})} y_e}_{=0} \\
&\leq |A| - 1
\end{aligned}$$

□

Theorem 2.28 (Edmonds & Johnson (1970))**.** *Let G be a graph, $u : E(G) \to \mathbb{N} \cup \{\infty\}$, $b : V(G) \to \mathbb{N}$. The b-matching polytope is given by:*

$$\begin{aligned}
x &\geq 0 \\
x &\leq u \\
x(\delta(v)) &\leq b(v) && v \in V(G) \\
\sum_{e \in E(G[X])} x_e + \sum_{e \in F} x_e &\leq \underbrace{\lfloor \frac{1}{2} \left(\sum_{v \in X} b(v) + \sum_{e \in F} u(e) \right) \rfloor}_{\text{Gomory-Chvátal-Cut}} && X \subseteq V(G), F \subseteq \delta(X)
\end{aligned}$$

Proof.

"$\subseteq$": Let x be an incidence vector of b-matchings. Then $x \leq u$ and $x(\delta(v)) \leq b(v)$ for all $v \in V(G)$.

$$\sum_{e \in E(G[X])} x_e + \sum_{e \in F} x_e = \frac{1}{2}\left(\sum_{v \in X}\sum_{e \in \delta(x)} x_e + \sum_{e \in F} x_e - \sum_{e \in \delta(X)\setminus F} x_e\right)$$
$$\leq \frac{1}{2}\left(\sum_{v \in X} b(v) + \sum_{e \in F} u(e)\right)$$

Since the left hand side is integral, the right hand side can be rounded down.

"$\supseteq$": Let x satisfy all the inequalities. We have to show that x is a convex combinations of b-matchings. Let H arise from G by subdividing each edge $e = \{v, w\}$ with $u(e) \neq \infty$ by 2 new vertices $(e, v), (e, w)$ and a path v-(e, v)-(e, w)-w, where $b((e, v)) = u(e) = b((e, w))$. Set $y_{\{v,(e,v)\}} := x_e =: y_{\{(e,w),w\}}$ and $y_{\{(e,v),(e,w)\}} := u(e) - x_e$. If $u(e) = \infty$, $y_e := x_e$.

Claim. *y is in the b-matching polytope of (H, ∞). This then implies that x is contained in the capacitated b-matching polytope of (G, u).*

$y(\delta_H(v)) \leq b(v)$ clearly holds for all $v \in V(H)$. Assume that there exists $A \subseteq V(H)$ with:

$$y(E(H[A])) > \lfloor \frac{1}{2} b(A) \rfloor$$

Let $B := A \cap V(G)$. For $\{v, w\} \in E(G[B])$, we may assume that $(e, v), (e, w) \in A$. If $(e, v) \in A$, we may assume $v \in A$:

Case 1: If $(e, w) \in A$, we can remove (e, v) and (e, w).

Case 2: If $(e, w) \notin A$, we can remove (e, v).

There are 3 remaining cases. Define:

$$F := \{e = \{v, w\} \in E(G) \mid |A \cap \{(e, v), (e, w)\}| = 1\}$$

Then

$$x(E(G[B])) + x(F) = y(E(H[A])) - \sum_{\substack{e \in E(G[B]) \\ u(e) < \infty}} u(e)$$
$$> \lfloor \frac{1}{2} b(A) \rfloor - \sum_{\substack{e \in E(G[B]) \\ u(e) < \infty}} u(e)$$
$$= \lfloor \frac{1}{2}(b(B) + \sum_{e \in F} u(e)) \rfloor$$

which is a contradiction to the feasibility of x. Therefore, y satisfies the inequalities w.r.t. (H, ∞). Let $e \in P :=$ b-matching polytope for (H, ∞), then $y \in \{z \in P \mid \sum_{e \in \delta(v)} z_e = b(v) \forall v \in V(H) \setminus V(G)\}$. Therefore, y is the convex combination of b-matchings $f_1, \dots, f_m$ in (H, ∞) with $f_i(\delta(v)) = b(v)$ for all $v \in V(H) \setminus V(G)$. We get:

$$f_i(\{v, (e, v)\}) = f_i(\{w, (e, w)\}) \leq u(e) \qquad \forall e = \{v, w\} \in E(G)$$

Set:

$$f_i'(e) := \begin{cases} f_i(v, (e, v)) & e = \{v, w\} \in E(G),\ u(e) < \infty \\ f_i(e) & e = \{v, w\} \in E(G),\ u(e) = \infty \end{cases}$$

Then x is a convex combination of $f_1', \dots, f_m'$ (of b-matchings).

□

2.7 Padberg-Rao Theorem

Lemma 2.30. *Let G be a graph, $|E(G)| \geq 1$, $T \subseteq V(G)$ with $|T|$ even, $c, c' : E(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. There exists a $O(n^2 m)$ time algorithm that finds a vertex set $X \subseteq V(G)$ and $F \subseteq \delta(X)$ such that $|X \cap T| + |F|$ is odd and*

$$c(\delta(X) \setminus F) + c'(F)$$

is minimum.

Proof. Without loss of generality, G is connected: Otherwise, add edges e with $c(e) = 0$ and $c'(e) = \infty$. Let

$$\begin{aligned} d(e) &:= \min\{c(e), c'(e)\} & e \in E(G) \\ E' &:= \{e \in E(G) \mid c'(e) < c(e)\} \\ V' &:= \{v \in V(G) \mid |\delta_{E'}(v)| \text{ odd}\} \\ T' &:= T \Delta V' \end{aligned}$$

Since E' is a V'-join, for $X \subseteq V(G)$:

$$|X \cap T| + \left|\delta(X) \cap E'\right| \equiv |X \cap T| + \left|X \cap T'\right| \equiv \left|X \cap T'\right| \mod 2$$

Compute a Gomory-Hu-Tree H for (G, d). For $f \in E(H)$, let $\delta(C_f)$ be the fundamental cut of f (i.e. C_f is a connected component in $H - f$). Let $g_f \in \arg\min_{e \in \delta_G(C_f)} |c(e) - c'(e)|$. Let:

$$F_f := \begin{cases} \delta_G(C_f) \cap E' & \text{if } |C_f \cap T'| \text{ is odd} \\ \delta_G(C_f) \cap E' \Delta \{g_f\} & \text{else} \end{cases}$$

Finally, choose $f \in E(H)$ minimizing $c(\delta(C_f) \setminus F_f) + c'(F_f)$ and output C_f, F_f. The running time is dominated by the computation of H.

It remains to show correctness: Let X^*, F^* be an optimum solution.

Case 1: $|X^* \cap T'|$ is odd. $J' := \{f \in E(H) \mid |C_f \cap T'| \text{ odd}\}$ is a T'-join in H. Therefore, J' intersects the T'-cut $\delta_H(X^*)$. Let $f \in \delta_H(X^*)$ with $|C_f \cap T'|$ odd. Then $d(\delta_G(C_f)) \leq d(\delta_G(X^*)) \leq \text{obj}(X^*)$, since H is a Gomory-Hu-tree. By construction, $F_f = \delta_G(C_f) \cap E'$ and:

$$c(\delta_G(C_f) \setminus F_f) + c'(F_f) \leq d(\delta_G(X^*)) \leq \text{obj}(X^*)$$

Case 2: $|X^* \cap T'|$ is even. Let $g^* \in \arg\min_{e \in \delta(X^*)} |c(e) - c'(e)|$. $H + g^*$ has a unique circuit that contains some $f \in \delta_H(X^*)$. Then

$$\begin{aligned} c(\delta_G(X^*) \setminus F^*) + c'(F^*) &= d(\delta(X^*)) + \big|c(g^*) - c'(g^*)\big| \\ &\geq d(\delta_G(C_f)) + \big|c(g^*) - c'(g^*)\big| \\ &\overset{g^* \in \delta_G(C_f)}{\geq} c(\delta(C_f) \setminus F_f) + c'(F_f) \end{aligned}$$

$\square$

Theorem 2.31 (Padberg & Rao (1987)). *Let G be a graph, $u : E(G) \to \mathbb{N} \cup \{\infty\}$ and $b : V(G) \to \mathbb{N}$. Then the separation problem for the b-matching polytope can be solved in $O(n^2 m)$ time.*

Proof. $0 \leq X \leq u$ and $x(\delta(v)) \leq b(v)$ for all $v \in V(G)$ can be checked in linear time. It remains to check:

$$x(E(G[X])) + x(F) \leq \lfloor \frac{1}{2}(b(X) + u(F)) \rfloor \qquad X \subseteq V(G), F \subseteq \delta(X)$$

If $b(X) + u(F)$ is even (i.e. no rounding is done), this is implied by the other inequarities. Otherwise, the inequality is violated iff:

$$b(X) - 2x(E(G[X])) + u(F) - 2x(F) < 1$$

Extend G to H by adding a new vertex z and edges $\{z, v\}$ for every $v \in V(G)$. Set:

$$\begin{aligned} b(z) &:= b(V(G)) \\ T &:= \{v \in V(H) \mid b(v) \text{ odd}\} \\ E' &:= \{e \in E(G) \mid u(e) < \infty \text{ and odd}\} \\ c(e) &:= \begin{cases} x_e & e \in E' \\ \min\{x_e, u(e) - x_e\} & e \in E(G) \setminus E' \\ b(v) - x(\delta(v)) & e = \{z, v\} \in E(H) \end{cases} \\ c'(e) &:= \begin{cases} u(e) - x_e & e \in E' \\ \infty & e \in E(H) \setminus E' \end{cases} \end{aligned}$$

For $X \subseteq V(G)$, let $D_X := \{e \in \delta_G(X) \setminus E' \mid u(e) \leq 2x_e\}$. Then $\forall X \subseteq V(G),\ F \subseteq \delta_G(X) \cap E'$,

$$|X \cap T| + |F| \equiv b(X) + u(F \cup D_X) \mod 2$$

and:

$$\begin{aligned} c(\delta_H(X) \setminus F) + c'(F) = b(X) - \sum_{v \in X} x(\delta_G(v)) + \sum_{e \in (\delta_G(X) \cap E') \setminus F} x_e \\ + \sum_{e \in \delta_G(X) \setminus E'} \min\{x_e, u(e) - x_e\} + \sum_{e \in F} u(e) - x_e \\ = b(X) - 2x(E(G[X])) + \sum_{e \in F \cup D_X} u(e) - 2x_e \end{aligned}$$

Apply lemma 2.30 to H, T, c, c': If there exists $X \subseteq V(H),\ F \subseteq \delta_H(X)$ with $c(\delta(X) \setminus F) + c'(F) < 1$, then $F \subseteq E'$ and without loss of generality $z \notin X$ (otherwise use the complement). We get

$$b(X) - 2x(E(G[X])) + \sum_{e \in F \cup D_X} u(e) - 2x_e < 1$$

Setting $F' := F \cup D_X$ yields a violating of the corresponding inequality.

For the other direction, note that if the inequality holds for $X \subseteq V(G)$ and $F \subseteq \delta(X)$, then without loss of generality, $D_X \subseteq F \subseteq E' \cup D_X$ (since adding edges in $D_X \setminus F$ increases the violation). Then:

$$c(\delta_H(X) \setminus (F \setminus D_X)) + c'(F \setminus D_X) < 1$$

Therefore, the b-matching polytope can be separated in polynomial time. □

Corollary 2.32. *The Maximum-Weight b-Matching Problem can be solved in polynomial time.*

Proof. Use the Ellipsoid method together with theorem 2.31. □

3 The TSP Polytope

3.1 The Spanning Tree Polytope

Theorem 3.1 (Edmonds (1967))**.** *Let G be a connected graph, $n := |V(G)|$. Then*

$$P_{ST} := \{x \in [0,1]^{E(G)} \mid x(E(G)) = n-1, \forall \emptyset \neq X \subsetneq V(G) : \sum_{e \in E(G[X])} x_e \leq |X|-1\}$$

is the convex hull of incidence vectors of spanning trees. It is called the spanning tree polytope.

Proof. Let T be a spanning tree with incidence vector x. Then $x \in P_{ST}$ and as $x \in \{0,1\}^{E(G)}$, x is a vertex.

For the other direction, let $x \in P_{ST} \cap \mathbb{Z}^{E(G)}$. Then x cannot contain cycles, so it is a forest. Since $x(E(G)) = n-1$, it is a spanning tree.

Claim. *P_{ST} is integral.*

Let $c : E(G) \to \mathbb{R}$ and T be a minimum spanning tree produced by Kruskals algorithm. Let $E(T) \coloneqq \{f_1, \dots, f_{n-1}\}$ in order of addition, i.e. $c(f_1) \leq c(f_2) \leq \dots \leq c(f_{n-1})$. Let $X_k \subseteq V(G)$ be the connected component in $(V(G), \{f_1, \dots, f_k\})$ containing f_k. Let x^* be the incidence vector of T.

Claim. *x^* is an optimum solution to*

$$\begin{aligned} &\min c^t x \\ s.t.\quad &1^t x = n-1 \\ &x(E(G[X])) \leq |X| - 1 \qquad\qquad \forall \emptyset \subsetneq X \subseteq V(G) \end{aligned}$$

The dual problem is:

$$\begin{aligned} \max\ &- \sum_{\emptyset \subsetneq X \subseteq V(G)} (|X|-1) z_A \\ s.t.\quad &- \sum_{e \subseteq X \subseteq V(G)} z_X \leq c(e) \qquad\qquad e \in E(G) \\ &z_X \geq 0 \qquad\qquad \emptyset \subsetneq X \subsetneq V(G) \end{aligned}$$

Construct a dual solution z^*: For $k \in \{1, \dots, n-2\}$, set $z^*_{X_k} \coloneqq c(f_l) - c(f_k) \geq 0$ where l is the minimum index larger than k with $X_k \cap f_l \neq \emptyset$. Define $z^*_{V(G)} = -c(f_{n-1})$ and $z^*_A \coloneqq 0$ for all other $A \subseteq V(G)$.

For $e = \{v, w\} \in E(G)$:

$$- \sum_{e \subseteq X \subseteq V(G)} z_X = c(f_i) \leq c(e)$$

where i is the smallest index such that $e \subseteq X_i$. Therefore, z^* is dual feasible. For tree edges, we have equality, so for $x_e > 0$ the dual constraint is tight. Let $\emptyset \subsetneq X \subseteq V(G)$ with $z^*_X > 0$. Then $T[X]$ is connected, so the primal constraint is tight. Complementary slackness implies that x^*, z^* are optimum primal/dual solutions.

Remark. If $c \in \mathbb{Z}^{E(G)}$, then z^* is an integral optimum dual solution, so the system is TDI.

$\square$

Theorem 3.2 (Fulkerson (1974)). *Let G be a digraph, $c : E(G) \to \mathbb{Z}_{\geq 0}$, $r \in V(G)$ such that G contains an r-arborescence. Then the minimum weight of an r-arborescence spanning $V(G)$ equals the maximum number of r-cuts $C_1, \dots, C_t$ (where repetitions are allowed) such that no edge e is contained in more than $c(e)$ of the cuts.*

Proof. Consider the (r-cuts)$\times$(edges) matrix A, where

$$A_{Ce} = \begin{cases} 1 & e \in C \\ 0 & \text{otherwise} \end{cases}$$

Consider the LP and its dual:

$$\begin{aligned} &\min\{c^t x \mid x \in \mathbb{R}^{E(G)},\ Ax \geq 1, x \geq 0\} \\ &\max\{1^t y \mid y \in \mathbb{R}^{r\text{-cuts}},\ A^t y \leq c,\ y \geq 0\} \end{aligned}$$

Claim. *The system is TDI.*

Proof. Let y be an optimum dual solution maximizing

$$\sum_{\emptyset \subsetneq X \subseteq V(G) \setminus \{r\}} y_{\delta^-(X)} \, |X|^2$$

Claim. *$\mathcal{F} := \{X \subseteq V(G) \mid y_{\delta^-(X)} > 0\}$ is laminar.*

Suppose that there are $X, Y \in \mathcal{F}$ with $X \cap Y,\ X \setminus Y,\ Y \setminus X \neq \emptyset$. Let:

$$\begin{aligned} \epsilon &:= \min\{y_{\delta^-(X)}, y_{\delta^-(Y)}\} \\ y'_{\delta^-(X)} &:= y_{\delta^-(X)} - \epsilon \\ y'_{\delta^-(Y)} &:= y_{\delta^-(Y)} - \epsilon \\ y'_{\delta^-(X \cap Y)} &:= y_{\delta^-(X \cap Y)} + \epsilon \\ y'_{\delta^-(X \cup Y)} &:= y_{\delta^-(X \cup Y)} + \epsilon \\ y' &:= y \qquad \qquad \qquad \qquad \text{everywhere else} \end{aligned}$$

Then y' is a dual optimum solution which contradicts the maximality of y.

By Ghoulia-Houri, if the set of rows can be partitioned $\mathcal{R} = \mathcal{R}_1 \dot\cup \mathcal{R}_2$ such that for all columns j:

$$\sum_{r \in \mathcal{R}_1} a_{rj} - \sum_{r \in \mathcal{R}_r} a_{rj} \in \{-1, 0, 1\}$$

then A is totally unimodular. Let $\mathcal{R}_1, \mathcal{R}_2$ be a partition of the laminar family $\mathcal{F}$ alternating between each level. Let $A' \subseteq A$ consist of rows with positive support (i.e. rows in $\mathcal{F}$). Then by this argument, A' is totally unimodular. In particular, for $c \in \mathbb{Z}_{\geq 0}$, we find an integral optimum dual solution. $\square$

Since the system is TDI, there exists an integral optimum primal solution x. $\square$

Corollary 3.3. *Let G be a digraph, $c : E(G) \to \mathbb{R}_{\geq 0}$ and $r \in V(G)$ such that a spanning r-arborescence exists. Then*

$$\min\{c^t x \mid x \geq 0,\ x(\delta^+(X)) \geq 1\ \forall r \in X \subsetneq V(G)\}$$

has an integral solution which is the incidence vector of a minimum-weight spanning r-arborescence plus (possibly) edges of weight 0.

3.2 The Held-Karp Polytope

Proposition 3.4. *Let $n \in \mathbb{Z}_{\geq 3}$. The incidence vectors x of TSP tours in K_n are described by:*

$$\begin{aligned} x(\delta(v)) &= 2 && v \in V(G) \\ x(\delta(X)) &\geq 2 && \emptyset \neq X \subsetneq V(G) \\ x &\in \{0,1\}^{E(K_n)} \end{aligned}$$

Proof. Integrality and the first inequality imply that x is the incidence vector of a collection of cycles. By the second inequality (which is called the *subtour elimination constraint*), there is exactly one cycle. $\square$

Relaxing the integrality (i.e. only requiring $x \in [0,1]$) yields the *subtour polytope* (or Held-Karp-polytope).

Proposition 3.5. *Let $n \in \mathbb{Z}_{\geq 2}$, $x \in [0,1]^{E(G)}$ with $x(\delta(v)) = 2$ for all $v \in V(K_n)$. Then the following are equivalent:*

1. *$x(\delta(X)) \geq 2$ for all $\emptyset \neq X \subsetneq V(G)$ (i.e. 3.4).*
2. *$x(E(K_n[X])) \leq |X| - 1$ for all $\emptyset \neq X \subsetneq V(G)$.*
3. *$x(E(K_n[X])) \leq |X| - 1$ for all $\emptyset \neq X \subseteq V(K_n) \setminus \{r\}$.*

Proof.

$$\begin{aligned}2 &\leq x(\delta(V(G) \setminus X)) \\ &= x(\delta(X)) \\ &= \sum_{v \in X} x(\delta(v)) - 2x(E(K_n[X])) \\ &= 2\,|X| - 2x(E(K_n[X]))\end{aligned}$$

□

Theorem 3.6 (Wolsey (1980)). *Let* (K_n, c) *with* c *metric and*

$$P_{HK} = \{x \in \mathbb{R}^{E(K_n)}_{\geq 0} \mid x(\delta(v)) = 2 \;\forall v \in V(K_n),\; x(\delta(X)) \geq 2 \;\forall \emptyset \neq X \subsetneq V(K_n)\}$$

be the Held-Karp polytope. Then:

$$\min\{c^t x \mid x \in P_{HK} \cap \mathbb{Z}^{E(K_n)}\} \leq \frac{3}{2} \min\{c^t x \mid x \in P_{HK}\}$$

Proof. Let $x^* \in \arg\min\{c^x \mid x \in P_{HK}\}$, Y be a minimum spanning tree in (K_n, c) and J a minimum-weight odd(Y)-join. $\frac{n-1}{n} x^* \in P_{ST}$ and $\frac{1}{2} x^* \in P_{\text{odd}(Y)\text{-join}}$. We get:

$$\begin{aligned}c(Y) + c(J) &\leq \frac{n-1}{n} c^t x^* + \frac{1}{2} c^t x^* \\ &< \frac{3}{2} c^t x^*\end{aligned}$$

□

Conjecture 3.7. *If for* (K_n, c), *c is metric, then:*

$$\min\{c^t x \mid x \in P_{HK} \cap \mathbb{Z}^{E(G)}\} \leq \frac{4}{3} \min\{c^x \mid x \in P_{HK}\}$$

3.3 Further Inequalities for the TSP

Consider the *2-matching inequalities*:

$$x(E(G[H])) + x(F) \leq |H| + \lfloor \frac{|F|}{2} \rfloor \qquad \forall H \subseteq V(G),\; F \subseteq \delta(H),\; |F| \text{ odd}$$

Theorem 3.8. *Let* $H, T_1, \ldots, T_k \subseteq V(G)$ *such that:*

1. $|H \cap T_i| \geq 1$ *for* $i \in [k]$
2. $|T_i \setminus H| \geq 1$ *for* $i \in [k]$

3. $T_i \cap T_j = \emptyset$ *for* $i \neq j$

4. k *is odd*

Then

$$x(E(G[H])) + \sum_{i=1}^{k} x(E(G[T_i])) \leq |H| + \sum_{i=1}^{k}(|T_i| - 1) - \frac{k+1}{2}$$

is a valid inequality for the TSP polytope. They're called comb inequalities. H *is called* handle, T_i *are called* teeth *and* $(H, T_1, \ldots, T_k)$ *is a comb.*

Proof. Let $(H, T_1, \ldots, T_k)$ be a comb. Generate the inequality as a Gomory-Chvátal-cut: Multiply the following inequalities by $\frac{1}{2}$, add them together and round:

- $x(\delta(v)) = 2$ for $v \in H$
- $-x_e \leq 0$ for $e \in \delta(H) \setminus \bigcup_{i=1}^{k} E(G[T_i])$
- $x(\delta(X)) \geq 2$ for $X = T_i, H \cap T_i, T_i \setminus H$ $(i \in [k])$

□

The complexity of comb separation is an open question.

Theorem 3.9 (Fiorini et al. (1985))**.** *There is no polyhedron with polynomially many facets, whose projection is the TSP polytope.*

Proof. Omitted. □

Definition 3.10. Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron $Q \subseteq \mathbb{R}^m$ is an *extension* of P if there exists a projective map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ with $\pi(Q) = P$. The *extension complexity* of a polyhedron P is the minimum number of facets of an extension Q of P.

Rothvoss (2013) proved that the matching polytope has an exponential extension complexity.

4 Matroids & Generalizations

Definition 4.1. A set system $(E, \mathcal{F})$ (where $\mathcal{F} \subseteq 2^E$) is an independent system if:

i) $\emptyset \in \mathcal{F}$

ii) $X \in \mathcal{F} \Rightarrow \forall Y \subseteq X : Y \in \mathcal{F}$

- Elements in $\mathcal{F}$ are called *independent*.

- Inclusion-wise maximal sets $A \in \mathcal{F}$ are called *bases*. Its cardinality is called $\operatorname{rank}(A)$.
- Inclusion-wise minimal sets $A \in \mathcal{F}$ are *circuits*.

An indepndent system $(E, \mathcal{F})$ is a matroid if the following axiom holds:

iii) $\forall X, Y \in \mathcal{F}$ with $|X| < |Y|$: $\exists y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{F}$. This is equivalent to:

iii)' $\forall X, Y \in \mathcal{F}$ with $|X| + 1 = |Y|$: $\exists y \in Y$ such that $X \cup \{y\} \in \mathcal{F}$.

iii)" $\forall X \subseteq E$ and $A, A' \subseteq X$ maximal with $A, A' \in \mathcal{F}$: $\operatorname{rank}(A) = \operatorname{rank}(A')$.

If $\mathcal{M} = (E, \mathcal{F})$ is a matroid, then $r(\mathcal{M}) = r(E)$. The rank function is defined by:

$$r : 2^E \to \mathbb{N}$$
$$r(A) := \max_{B \subseteq A, B \in \mathcal{F}} |B|$$

Algorithm 6: Greedy Algorithm for independent systems

Input: Independent system $(E, \mathcal{F})$, $c : E \to \mathbb{R}$
Output: $X \in \mathcal{F}$ with the objective of maximizing $c(X)$
$X \leftarrow \emptyset$
while $\exists x \in X$ *with* $c(x) > 0$ *and* $X \cup \{x\} \in \mathcal{F}$ **do**
  Choose $x \in \arg\max_{x \notin X,\ X \cup \{x\} \in \mathcal{F}} c(x)$
  $X \leftarrow X \cup \{x\}$
return X

Theorem 4.2. *$(E, \mathcal{F})$ is a matroid $\Leftrightarrow$ algorithm 6 finds an optimum solution for every cost function c.*

Example 4.3.

- *Cycle matroid*: E is the edge set of an undirected graph, $\mathcal{F}$ is the set of forests. Then $(E, \mathcal{F})$ is a matroid. Matroids that can be represented this way are called *graphic matroids*.
- $A \in \mathbb{R}^{m \times n}$, $E = [n]$ and $\mathcal{F}$ is the set of linearly independent subsets of E. This is called a *vector matroid*.
- *Uniform matroid*: E is a finite set, $k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{F} := \{X \subseteq E \mid |X| \leq k\}$.
- *Matching matroid*: G is an undirected graph, $E := V(G)$ and $\mathcal{F} := \{F \subseteq E \mid \exists \text{ matching in } G \text{ covering } F\}$.

- *Gammoids*: G is a graph (directed or undirected), $E, U \subseteq V(G)$. $X \in \mathcal{F}$ if there exist $|X|$ vertex-disjoint U-X-paths.
- *Transversal matroid*: G is a bipartite graph with $V(G) = E \dot{\cup} U$ and (E, U) is a gammoid. $\mathcal{F}$ is the set of subsets of E that are covered by some matching.

Example 4.4. Independent systems that are not matroids:

- Matchings
- Stable sets and cliques
- Subsets of TSP tours or Steiner trees
- Feasible solutions of knapsack problems

Theorem 4.5 (Edmonds (1970)). *Let $(E, \mathcal{F})$ be a matroid and $r : 2^E \to \mathbb{N}$ its rank function. Then the* matroid polytope *of $(E, \mathcal{F})$ (i.e. the convex hull of incidence vectors of independent sets) can be described by:*

$$\{x \in \mathbb{R}^E \mid x \geq 0, \sum_{e \in A} x_e \leq r(A) \ \forall A \subseteq E\}$$

Proof. The polytope contains all incidence vectors of independent sets. We have to show that the vertices of the polytope are integral, or equivalently:

$$\max\{c^t x \mid x \geq 0, \sum_{e \in A} x_e \leq r(A) \ \forall A \subseteq E\}$$

attains an integral optimum for all $c \in \mathbb{R}^E$. Let x^0 be the incidence vector of the set J found by the greedy algorithm (algorithm 6).

Claim. *x^0 is an optimum solution in the polytope.*

The dual problem is

$$\begin{aligned} &\min \sum_{A \subseteq E} r(A) y_A \\ \sum_{A \subseteq E,\ e \in A} &y_A \geq c(e) \qquad\qquad e \in E \\ &y \geq 0 \end{aligned}$$

Our goal is to find a dual solution in complementary slackness with x^0, so $x_e > 0 \Rightarrow \sum_{A \subseteq E,\ e \in A} y_A = c(e)$ and $y_A > 0 \Rightarrow x(A) = r(A)$.

Consider the Dual Greedy Algorithm:

1. Order E as $\{e_1, \ldots, e_n\}$ with:
$$c(e_1) \geq \ldots \geq c(e_m) \geq 0 \geq c(e_{m+1}) \geq \ldots \geq c(e_n)$$

2. $T_i \coloneqq \{e_1, \ldots, e_i\}$ for $1 \leq i \leq m$, $T_0 \coloneqq \emptyset$ and
$$y_A^0 \coloneqq \begin{cases} c(e_i) - c(e_{i+1}) & A = T_i \text{ for } i \in \{1, \ldots, m-1\} \\ c(e_m) & A = T_m \\ 0 & \text{else} \end{cases}$$

$y \geq 0$ and for $j > m$, $c(e_j) \leq 0$ so the inequality is satisfied. If $j \leq m$, then:
$$\sum_{A \subseteq E,\ e_j \in A} y_A = \sum_{i=j}^{m} y_{T_i}^0 = c(e_j)$$

Therefore, y is dual feasible. If $x_e^0 > 0$, the corresponding dual constraint is tight. Let $y_A^0 > 0$, so $A = T_i$ for some i. We have to show that $x^0(A) = r(A)$, i.e. $J \cap T_i$ is a basis of T_i. If not, there exists $e_k \in T_i \setminus J$ with $(J \cap T_i) \cup \{e_k\} \in \mathcal{F}$ and $c(e_k) > c(e_j)$. Since the algorithm didn't add e_k, this is a contradiction. $\square$

Corollary 4.6. *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid, $c \in \mathbb{R}^E$ and $J \in \mathcal{F}$. Then J is a maximum-weight independent set if and only if:*

a) $\forall e \in J: \ c(e) \geq 0$

b) $\forall e \notin J, \ J \cup \{e\} \in \mathcal{F}: \ c(e) \leq 0$

c) $\forall e \notin J, f \in J, (J \cup \{e\}) \setminus \{f\} \in \mathcal{F}: \ c(e) \leq c(f)$

Proof.

"$\Rightarrow$": Clear

"$\Leftarrow$": Take a dual solution y^0 from the dual greedy algorithm.By a), $\sum_{e \in A} y_A = c(e)$ for all $e \in J$. If there exists $A \subseteq E$ with $y_A > 0$ and $x(A) < r(A)$, then $\exists i$ with $c(e_i) > c(e_{i+1})$ and $J \cap T_i$ is not a basis of $T_i = A$. Therefore, there exists $e \in T_i \setminus J$ with $(J \cap T_i) \cup \{e\} \in \mathcal{F}$. If $\{e\} \cup J \in \mathcal{F}$, this would contradict b). Otherwise, extend $(J \cap T_i) \cup \{e\}$ to a basis J' of $J \cup \{e\}$. Then $|J'| = |J|$, so $J' = (J \cup \{e\}) \setminus \{f\}$ for some $f \in T_i$, which is a contradiction to c).

$\square$

Theorem 4.7. *Let G be an undirected graph. The forest polytope of G is given by:*
$$\{x \in \mathbb{R}^{E(G)} \mid x(E(G[T])) \leq |T| - 1 \ \forall \emptyset \neq T \subseteq V(G)\}$$

Proof. Apply theorem 4.5 to the cycle matroid. □

4.1.1 Matroid Constructions

Proposition 4.8 (Disjoint Union). *Given matroids $\mathcal{M}_1 = (E_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{F}_2)$ with $E_1 \cap E_2 = \emptyset$, $\mathcal{M} := \mathcal{M}_1 \oplus \mathcal{M}_2 := (E, \mathcal{F})$ where $E = E_1 \dot{\cup} E_2$ and $\mathcal{F} = \{J_1 \cup J_2 \mid J_1 \in \mathcal{F}_1,\ J_2 \in \mathcal{F}_2\}$ is a matroid with rank function*

$$r(A) = r(A \cap E_1) + r(A \cap E_2)$$

where r_i is the rank function of $\mathcal{M}_i$.

Proposition 4.9 (Partition Matroid). *Let $E = E_1 \dot{\cup} \dots \dot{\cup} E_k$ and $\mathcal{F} := \{J \subseteq E(G) \mid |J \cap E_i| \leq 1 \forall i \in [k]\}$. Then $(E, \mathcal{F})$ is a matroid with rank function:*

$$r(A) = |\{i \in [k] \mid E_i \cap A \neq \emptyset\}|$$

Proposition 4.10 (Restriction Matroid). *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $B \subseteq E$. Then $\mathcal{M}' := \mathcal{M} \setminus B := (E \setminus B, \{J \subseteq E \setminus B \mid J \in \mathcal{F}\})$ is a matroid.*

Proposition 4.11 (Contraction Matroid). *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $B \subseteq E$. Choose an arbitrary basis J of B (i.e. $J \in \mathcal{F}$ and $r(J) = r(B)$). Then $M' := \mathcal{M}/B := (E \setminus B, \{J' \subseteq E \setminus B \mid J' \cup J \in \mathcal{F}\})$ is a matroid. $\mathcal{M}$ is independent of the chosen basis J. Its rank function is*

$$r'(A) = r(A \cup B) - r(B)$$

Corollary 4.12. *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $B \subseteq E$. Then $\mathcal{M}' := (\mathcal{M} \setminus B) \oplus (\mathcal{M}/(E \setminus B))$ is a matroid on E. The bases of $\mathcal{M}'$ are those bases of $\mathcal{M}$ that intersect B in a basis of B.*

Proposition 4.13 (Matroid Minors). *Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid and $\emptyset = T_0 \subseteq T_1 \subseteq \dots \subseteq T_{l+1} = \mathcal{F}$. The bases of T_l in $\mathcal{M}$ that intersect T_i $(1 \leq i \leq l)$ are the bases of T_l in the matroid $\mathcal{N} := \mathcal{N}_0 \oplus \dots \oplus \mathcal{N}_l$ where for each i, $\mathcal{N}_i := (\mathcal{M}/T_i) \setminus (E \setminus T_{i+1})$. $\mathcal{N}$ is called a minor of $\mathcal{M}$.*

4.2 Matroid Intersection

Finding $\arg\max\{|J| \mid J \in \mathcal{F}_1 \cap \mathcal{F}_2\}$ for matroids $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$ can be done similarly to bipartite matching in $O(|E|^2)$. Weighted matroid intersection (of 2 matroids) can also be done in polynomial time.

Computing $\max\{|J| \mid J \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3\}$ is NP-hard.

4.4 Polymatroids

For the rank function r of a matroid, $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ for all $X, Y \in E$, so the rank function is *submodular.*

Definition 4.34. A *polymatroid* is the polytope

$$P(f\{x \in \mathbb{R}^{E(G)} \mid x \geq 0,\ x(A) \leq f(A)\ \forall A \subseteq E\}$$

where E is a finite set and $f : 2^e \to \mathbb{R}_{\geq 0}$ is submodular.

Proposition 4.35. *For any polymatroid $P(f)$, f can be chosen such that $f(\emptyset) = 0$ and f is monotone, i.e. $A \subseteq B$ implies $f(A) \leq f(B)$.*

Proposition 4.36. *Let $E = \{e_1, \ldots, e_n\}$, $f : 2^E \to \mathbb{R}_{\geq 0}$ submodular with $f(\emptyset) \geq 0$, $b : E \to \mathbb{R}$ with $b(e_1) \leq f(e_1)$ and $b(e_i) \leq f(\{e_1, \ldots, e_i\}) - f(\{e_1, \ldots, e_{i-1}\})$ for $i \in \{2, \ldots, n\}$. Then $\sum_{a \in A} b(a) \leq f(A)$ for all $A \subseteq E$.*

Proof. Induction on $i = \max\{j \mid e_j \in A\}$. For $A = \emptyset$, the statement is trivial. For $i \geq 1$:

$$\begin{aligned} b(A) &= b(A \setminus \{e_i\}) + b(e_i) \\ &\leq f(A \setminus \{e_i\}) + b(e_i) \\ &\leq f(A \setminus \{e_i\}) + f(\{e_1, \ldots, e_i\}) - f(\{e_1, \ldots, e_{i-1}\}) \\ &\leq f(A) \end{aligned}$$

$\square$

Algorithm 7: Polymatroid Greedy Algorithm

Input: Finite set E and $f : 2^E \to \mathbb{R}_{\geq 0}$ submodular and monotone (given by an oracle) and $c : E \to \mathbb{R}$

Output: $x \in P(f)$ maximizing $c^t x$

```
Sort E = {e_1, ..., e_n} such that:
  c(e_1) ≥ ... ≥ c(e_k) > 0 ≥ c(e_{k+1}) ≥ ... ≥ c(e_n)
if k ≥ 1 then
    x_{e_1} ← f({e_1})
    for i = 2, ..., k do
        x_{e_i} ← f({e_1, ..., e_i}) − f({e_1, ..., e_{i−1}})
    for i = k + 1, ..., n do
        x_{e_i} ← 0
```

Theorem 4.37. *The Polymatroid Greedy algorithm correctly finds $x \in P(f)$ maximizing $c^t x$. If f is integral, then x is also integral.*

Proof. Let x be the output of algorithm 7. If f is integral, x is integral by construction. Assume that there exists $y \in \mathbb{R}^E_{\geq 0}$ with $c^t y > c^t x$. For $i \in [k-1]$, define $d_j := c(e_j) - c(e_{j+1})$ and $d_k := c(e_k)$.

$$\begin{aligned}\sum_{j=1}^{k} d_j \sum_{i=1}^{j} x_i &= c^t x \\ &< c^t y \\ &= \sum_{j=1}^{k} d_j \sum_{i=1}^{j} y_i\end{aligned}$$

Therefore, there exists $j \in [k]$ such that

$$\sum_{i=1}^{j} y_i > \sum_{i=1}^{j} x_i = f(\{e_1, \ldots, e_j\})$$

so y is not contained in the polymatroid. $\square$

Theorem 4.38. *Let E be finite and $f, g : 2^E \to \mathbb{R}_{\geq 0}$ submodular. Then*

$$\begin{aligned} x(A) &\leq f(A) & A &\subseteq E \\ x(A) &\leq g(A) & A &\subseteq E \\ x &\geq 0 \end{aligned}$$

is TDI.

Proof. Consider the primal-dual pair:

$$\begin{array}{llcll} \max c^t x & & & \min \sum_{A \subseteq E} f(A) y_A + g(A) z_A & \\ x(A) \leq f(A) & A \subseteq E & & \sum_{e \in A \subseteq E} (y_A + z_A) \geq c(e) & e \in E \\ x(A) \leq g(A) & A \subseteq E & & y, z \geq 0 & \\ x \geq 0 & & & & \end{array}$$

Claim. *Let $Ax \leq b, x \geq 0$ be a linear program. If for any $c \in \mathbb{Z}^n$ where the dual is feasible and bounded, it has an optimum solution y_i^* such that the rows of A where $y_i^* > 0$ (plus possibly basic 0-entries) forms a TU matrix A'. Then $Ax \leq b, x \geq 0$ is TDI.*

Proof. Let c, y^* be as above. We claim:

$$\min\{y^t b \mid A^t y \geq c,\ y \geq 0\} = \min\{y^t b' \mid (A')^t y \geq c,\ y \geq 0\}$$

"$\leq$" is clear. Since the restriction of y^* is feasible for the right hand side, the other inequality also holds. Since A' is TU, the right hand system is TDI, so y^* can be chosen integrally if c is integral. $\square$

Let $c : E \to \mathbb{Z}_{\geq 0}$ and y, z be an optimum dual solution such that

$$\sum_{A \subseteq E} (y_A + z_A) \cdot |A| \cdot |E \setminus A|$$

is minimum.

Claim. *$\mathcal{F} := \{A \subseteq E \mid y_A > 0\}$ is a chain.*

Otherwise, there are $A, B \in \mathcal{F}$ with $A \cap B \neq A, B \cap A \neq B$. Let

$$\begin{aligned} \epsilon &:= \min\{y_A, y_B\} \\ y'_A &:= y_A - \epsilon \\ y'_B &:= y_B - \epsilon \\ y'_{A \cup B} &:= y_{A \cup B} + \epsilon \\ y'_{A \cap B} &:= y_{A \cap B} + \epsilon \\ y_S &:= y_S \qquad \text{elsewhere} \end{aligned}$$

y', z is feasible and optimal by submodularity but the term above gets smaller, which is a contradiction. Similarly, $\mathcal{F}' := \{A \subseteq E \mid z_A > 0\}$ is a chain.

Let M, M' be the matrices with column set E and row set $\mathcal{F}, \mathcal{F}'$. Then $\begin{pmatrix} M \\ M' \end{pmatrix}$ is TU: $A_1 \geq \ldots \geq A_p \in \mathcal{F}$ and $B_1 \geq \ldots \geq B_q \in \mathcal{F}'$. Define

$$\begin{aligned} \mathcal{R}_1 &:= \{A_i \mid i \text{ odd}\} \cup \{B_i \mid i \text{ even}\} \\ \mathcal{R}_2 &:= \{A_i \mid i \text{ even}\} \cup \{B_i \mid i \text{ odd}\} \end{aligned}$$

These sets satisfy Ghoulia-Houri, so the system is TDI. $\square$

Corollary 4.39. *Let $(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$ be two matroids. Then the convex hull of incidence vectors $x \in \mathcal{F}_1 \cap \mathcal{F}_2$ is the polytope*

$$\{x \in \mathbb{R}^E_{\geq 0} \mid x(A) \leq \min\{r_1(A), r_2(a)\} \ \forall A \subseteq E\}$$

where r_1, r_2 are the rank functions of the matroids.

Proof. By theorem 4.38, the inequality system is TDI, so since r_1, r_2 are integral, the polytope is integral. Integral vectors in the polytope correspond exactly to incidence vectors of sets in $\mathcal{F}_1 \cap \mathcal{F}_2$. $\square$

Corollary 4.40. *Let $f, g : 2^E \to \mathbb{R}_{\geq 0}$ be submodular, monotone with $f(\emptyset) = g(\emptyset) = 0$. Then:*

$$\underbrace{\max\{\mathbb{1}^t x \mid x \in P(f) \cap P(g)\}}_{(**)} = \min_{A \subseteq E} f(A) + g(E \setminus A)$$

Proof. The dual of $(**)$ is:

$$\min\{\sum_{A \subseteq E} (f(A)y_A + g(A)z_A) \mid y, z \geq 0, \sum_{E \supseteq A \ni e} y_A + z_A \geq 1 \ \forall e \in E\}$$

"$\geq$": By theorem 4.38, the dual has an integral optimum solution y, z. Let:

$$B \coloneqq \bigcup_{\substack{A \subseteq E \\ y_A \geq 1}} A \qquad\qquad C \coloneqq \bigcup_{\substack{A \subseteq E \\ z_A \geq 1}} A$$

Since y, z are integral, the dual constraint implies $E = B \cup C$, so $E \setminus B \subseteq C$. Therefore:

$$\begin{aligned} \sum_{A \subseteq E} (f(A)y_A + g(A)z_A) &\geq f(B) + g(C) \\ &\geq f(B) + g(E \setminus B) \end{aligned}$$

"$\leq$": For $A \subseteq E$, we construct the feasible dual solution $y_A \coloneqq 1$ and $z_{E \setminus A} \coloneqq 1$, everything else 0 which has cost $f(A) + g(E \setminus A)$. By LP-duality, any primal solution attains at most this value.

$\square$

- $f : 2^E \to \mathbb{R}$ supermodular:

$$f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y) \qquad \forall X, Y \subseteq E$$

- $f : 2^E \to \mathbb{R}$ modular:

$$f(X) + f(Y) = f(X \cup Y) + f(X \cap Y) \qquad \forall X, Y \subseteq E$$

- $f(A)$ submodular implies $f(E \setminus A)$ submodular.

Corollary 4.41 (Frank's Discrete Sandwich Theorem (1982))**.** *Let E be a finite sit, $f : 2^E \to \mathbb{R}$ supermodular, $g : 2^E \to \mathbb{R}$ submodular with $f(A) \leq g(A)$ for all $A \subseteq E$. Then there exists a modular function $h : 2^E \to \mathbb{R}$ with $f(A) \leq h(A) \leq g(A)$ for all $A \subseteq E$. If f, g are integral, h can be chosen integral.*

Proof.

- Without loss of generality, $f(\emptyset) = g(\emptyset)$ and $f(E) = g(E)$.
- Let $M \coloneqq 2 \cdot \max\{|f(A)| + |g(A)| \mid A \subseteq E\}$ and:

$$
\begin{aligned}
f'(A) &\coloneqq g(E) - f(E \setminus A) + M \cdot |A| \\
g'(A) &\coloneqq g(A) - f(\emptyset) + M \cdot |A|
\end{aligned}
$$

  f', g' are submodular, nonnegative, monotone and $f'(\emptyset) = 0 = g'(\emptyset)$.
- By corollary 4.40:

$$
\begin{aligned}
&\max\{\mathbb{1}^t x \mid x \in P(f') \cap P(g')\} \\
=&\min_{A \subseteq E}(f'(A) + g'(E \setminus A)) \\
=&\min_{A \subseteq E}(g(E) - f(E \setminus A) + M \cdot |A|) + (g(E \setminus A) - f(\emptyset) + M \cdot |E \setminus A|) \\
\geq& g(E) + M \cdot |E| - f(\emptyset)
\end{aligned}
$$

- Let $x \in P(f') \cap P(g')$ such that $\mathbb{1}^t x = g(E) - f(\emptyset) + M \cdot |E|$. If f, g are integral, we can choose it such that $x \in \mathbb{Z}^E$.
- Define:

$$
\begin{aligned}
h'(A) &\coloneqq \sum_{e \in A} x_e & A \subseteq E \\
h(A) &\coloneqq h'(A) + f(\emptyset) - M \cdot |A| & A \subseteq E
\end{aligned}
$$

  Then h is modular and for $A \subseteq E$:

$$
\begin{aligned}
h(A) &\leq g'(A) + f(\emptyset) - M \cdot |A| \\
&= g(A) \\
h(A) &= 1^t x - h'(E \setminus A) + f(\emptyset) - M \cdot |A| \\
&\geq g(E) + M \cdot |E \setminus A| - f'(E \setminus A) \\
&= f(A)
\end{aligned}
$$

□

Definition 4.42. Let $f : 2^E \to \mathbb{R}$ be a function. For $x \in \mathbb{R}^E_{\geq 0}$, there exist unique $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k > 0$ and sets $\emptyset \subsetneq T_1 \subsetneq \ldots \subsetneq T_k \subseteq E$ such that $x = \sum_{i=1}^k \lambda_i \chi^{T_i}$ where χ^{T_i} is the incidence vector of T_i. The Lovász extension of f is defined as:

$$
\begin{aligned}
f' : \mathbb{R}^E_{\geq 0} &\to \mathbb{R} \\
x &\mapsto \sum_{i=1}^k \lambda_i f(T_i)
\end{aligned}
$$

Lemma 4.43. *Let $f : 2^E \to \mathbb{R}$ be submodular and f' its Lovász extension. Then:*
$$f'(x) = \max\{x^t y \mid y \in P(f)\}$$

Proof. Exercise $\square$

Theorem 4.44.
$$f \text{ submodular} \Leftrightarrow f' \text{ convex}$$

4.4.1 Applications of Matroid Intersection

Orientations: Let G be an undirected graph and $k : V(G) \to \mathbb{Z}_{\geq 0}$. Does there exist an orientation $\vec{G}$ of G such that $\left|\delta^-_{\vec{G}}(v)\right| \leq k(v)$ for all $v \in V(G)$?

Let $D := (V(G), \{(v,w),(w,v) \mid \{v,w\} \in E(G)\})$. We define:

- $(A, \mathcal{F}_1)$ as the partition matroid on $\bigsqcup_{\{v,w\}\in E(G)}\{(v,w),(w,v)\}$
- $(A, \mathcal{F}_2)$ as the (generalized) partition matroid on $\bigsqcup_{v\in V(G)} \delta^-_D(v)$ allowing $\leq k$ elements from $\delta^-_D(v)$ for all $v \in V(G)$.

Then such an orientation $\vec{G}$ exists $\Leftrightarrow$ there exists $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ with $|F| = |E|$.

Theorem. *G has an orientation $\vec{G}$ such that $\left|\delta^1_{\vec{G}}(v)\right| \leq k(v)$ for all $v \in V(G)$ if and only if:*
$$\forall P \subseteq V(G) : \ |E(G[P])| \leq \sum_{v\in P} k(v)$$

Two disjoint spanning trees: For a matroid $\mathcal{M} = (E, \mathcal{F})$ we define $\mathcal{M}^* := (E, \mathcal{F}^*)$ where:
$$\mathcal{F}^* := \{A \subseteq E \mid E \setminus A \text{ contains a basis of } \mathcal{F}\}$$

$\mathcal{M}^*$ is a matroid with rank function $r_{\mathcal{M}^*}(X) = |X| + r_{\mathcal{M}}(E \setminus X) - |E|$.

Proposition. *Let G be a graph and $\mathcal{M} = (E, \mathcal{F})$ its graphic matroid. Then:*
$$G \text{ has 2 disjoint spanning trees} \Leftrightarrow \max_{I\in\mathcal{F}\cap\mathcal{F}^*} |I| = |V(G)| - 1$$

4.5 Submodular Function Maximization

Recall: $f : 2^E \to \mathbb{R}$ is called submodular if $f(A)+f(B) \geq f(A\cap B)+f(A\cup B)$ for all $A, B \subseteq E$. Equivalently, $f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$ for all $X \subseteq Y \subseteq E$ and $x \in E \setminus Y$.

Problem (USM: "unconstrainted submodular function maximization")**.** Given a submodular function $f : 2^E \to \mathbb{R}$, find $S \subseteq E$ maximizing $f(S)$.

Example. For a given graph G, define $f(X) \coloneqq |\delta(X)|$. Maximizing $f(X)$ corresponds to the maximum cut problem (which is NP-hard).

Algorithm 8: Deterministic Double Greedy

Input: Finite set E, submodular function $f : 2^E \to \mathbb{R}^+$
Output: $S \subseteq E$

$X_0 \leftarrow \emptyset,\ Y_0 \leftarrow E$
for $i = 1, \dots, n$ **do**
  $a_i \leftarrow f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})$
  $b_i \leftarrow f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})$
  if $a_i \geq b_i$ **then**
    $X_i \leftarrow X_{i-1} \cup \{e_i\},\ Y_i \leftarrow Y_{i-1}$
  else
    $X_i \leftarrow X_{i-1},\ Y_i \leftarrow Y_{i-1} \setminus \{e_i\}$
return $S \leftarrow X_n$

Lemma 4.45. *For every $1 \leq i \leq n$, $a_i + b_i \geq 0$.*

Proof. By the equivalent characterization of submodularity and since $X_i \subseteq Y_i$ for all i:

$$\begin{aligned} a_i + b_i &= f(X_{i-1} \cup \{e_i\}) - f(X_{i-1}) + f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) \\ &= (f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})) - (f(Y_{i-1}) - f(Y_{i-1} \setminus \{e_i\})) \\ &\geq 0 \end{aligned}$$

□

Let OPT be the optimum solution and $\mathrm{OPT}_i \coloneqq (\mathrm{OPT} \cup X_i) \cap Y_i$, so OPT_i coincides with X_i and Y_i on the first i elements and with OPT on the rest. In particular, $\mathrm{OPT}_0 = \mathrm{OPT}$ and $\mathrm{OPT}_n = X_n$.

Lemma 4.46. *For every $1 \leq i \leq n$, we have:*

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i) \leq (f(X_i) - f(X_{i-1})) + (f(Y_i) - f(Y_{i-1}))$$

Proof. Without loss of generality assume that $a_i \geq b_i$, so the second summand is 0. Then $\mathrm{OPT}_i = \mathrm{OPT}_{i-1} \cup \{e_i\}$. We need to show:

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i) \leq f(X_i) - f(X_{i-1}) = a_i$$

Case 1: $e_i \in \mathrm{OPT}_{i-1}$. Then the left side is 0 and so by lemma 4.45, $a_i \geq 0$.

Case 2: $e_i \notin \mathrm{OPT}_{i-1}$. Then

$$\mathrm{OPT}_{i-1} = (\mathrm{OPT} \cup X_{i-1}) \cap Y_{i-1} \subseteq Y_{i-1} \setminus \{e_i\}$$

so by submodularity:

$$\begin{aligned} f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) &\le f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) \\ &= b_i \le a_i \end{aligned}$$

□

Theorem 4.47 (Buchbinder et al.)**.** *Algorithm 8 returns a $\frac{1}{3}$-approximation for USM.*

Proof. By lemma 4.46:

$$\sum_{i=1}^{n}(f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i)) \le \sum_{i=1}^{n}(f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}))$$

Since both sides are telescopic sums:

$$\begin{aligned} f(\mathrm{OPT}_0) - f(\mathrm{OPT}_n) &\le f(X_n) - f(X_0) + f(Y_n) - f(Y_0) \\ &\le f(\underbrace{X_n}_{=\mathrm{OPT}_n}) + f(\underbrace{Y_n}_{=\mathrm{OPT}_n}) \end{aligned}$$

In total, $f(\mathrm{OPT}_0) \le 3f(\mathrm{OPT}_n)$. □

Remark 4.48. If f is arbitrary, we can simply add a constant to it to make it non-negative. The analysis is tight.

4.5.2 Randomized USM

Lemma 4.49. *For $i \in \{1, \dots, n\}$:*

$$\mathbb{E}\left[\underbrace{f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i)}_{\mathrm{I}}\right] \le \frac{1}{2}\mathbb{E}\left[f(X_i) - f(X_{i-1}) + \underbrace{f(Y_i) - f(Y_{i-1})}_{\mathrm{II}}\right]$$

Proof. We can consider each X_{i-1} separately, so we condition on some event of the form $X_{i-1} = S_{i-1}$ where $S_{i-1} \subseteq \{e_1, \dots, e_{i-1}\}$ is fixed and the probability that $X_{i-1} = S_{i-1}$ is non-zero.

Case 1: $b_i \le 0$. Then $p = 1$ and $Y_i = Y_{i-1} = S_{i-1} \cup \{e_i, \dots, e_n\}$ and $X_i = S_{i-1} \cup \{e_i\}$.

Claim.

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) \le \frac{1}{2}f(X_i) - f(X_{i-1}) = \frac{a_i}{2}$$

- If $e_i \in \mathrm{OPT}$, $0 \le \frac{a_i}{2}$.

Algorithm 9: Randomized Double Greedy

Input: Finite set E, submodular function $f : 2^E \to \mathbb{R}_+$
Output: $S \subseteq E$
$X_0 \leftarrow \emptyset,\ Y_0 \leftarrow E$
for $i = 1, \ldots, n$ **do**
  $a_i \leftarrow f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})$
  $b_i \leftarrow f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})$
  $p \leftarrow \begin{cases} 1 & b_i \leq 0 \\ 0 & a_i \leq 0 \\ \frac{a_i}{a_i+b_i} & \text{else} \end{cases}$
  with probability p **do**
    $X_i \leftarrow X_{i-1} \cup \{e_i\},\ Y_i \leftarrow Y_{i-1}$
  else
    $X_i \leftarrow X_{i-1},\ Y_i \leftarrow Y_{i-1} \setminus \{e_i\}$
return X_n

- If $e_i \notin \mathrm{OPT}$, then by submodularity:

  $$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) \leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) = b_i \leq 0 \leq \frac{a_i}{2}$$

The statement then follows directly from the claim.

Case 2: $a_i \leq 0$. This is analogous to case 1.

Case 3: $a_i, b_i > 0$.

$$\begin{aligned} & \mathbb{E}\left[f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1})\right] \\ =& p \cdot \left(f(X_{i-1} \cup \{e_i\}) - f(X_{i-1})\right) + (1-p) \cdot \left(f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1})\right) \\ =& \frac{a_i^2 + b_i^2}{a_i + b_i} \end{aligned}$$

We have found a value for the right side of the inequality. Now, we upper-bound the left side.

$$\begin{aligned} & \mathbb{E}\left[f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_i)\right] \\ =& p\left(f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\})\right) \\ &+ (1-p)\underbrace{\left(f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \setminus \{e_i\})\right)}_{\mathrm{III}} \\ \overset{(*)}{\leq}& \frac{a_i b_i}{a_i + b_i} \end{aligned}$$

To see $(*)$:

Case 3.1: If $e_i \notin \mathrm{OPT}_{i-1}$, then III is 0 and as $\mathrm{OPT}_{i-1} = (\mathrm{OPT} \cup X_{i-1}) \cap Y_{i-1}$ by submodularity:

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \cup \{e_i\}) \leq f(Y_{i-1} \setminus \{e_i\}) - f(Y_{i-1}) = b_i$$

Case 3.2: If $e_i \in \mathrm{OPT}_{i-1}$, then the first term of the LHS is 0. By submodularity:

$$f(\mathrm{OPT}_{i-1}) - f(\mathrm{OPT}_{i-1} \setminus \{e_i\}) \leq f(X_{i-1} \cup \{e_i\}) - f(X_{i-1}) = a_i$$

Now $\frac{a_i b_i}{a_i + b_i} \leq \frac{1}{2} \frac{a_i^2 + b_i^2}{a_i + b_i}$ by the binomial formula.

□

Theorem 4.50. *Algorithm 9 returns a solution S with*

$$\mathbb{E}[f(S)] \geq \frac{f(\mathrm{OPT})}{2}$$

Proof. Summing up lemma 4.49 for all $i \in \{1, \ldots, n\}$ and collapsing the telescopic sums yields:

$$\begin{aligned}\mathbb{E}[f(\mathrm{OPT}_0) - f(\mathrm{OPT}_n)] &\leq \frac{1}{2}\mathbb{E}[f(X_n) - f(X_0) + f(Y_n) - f(Y_0)] \\ &\leq \frac{\mathbb{E}[f(X_n) + f(Y_n)]}{2}\end{aligned}$$

In total, $\mathbb{E}[f(\mathrm{OPT}_n)] \geq \frac{f(\mathrm{OPT})}{2}$. □

Remark. There is no $\frac{1}{2} + \epsilon$-approximation for $\epsilon > 0$ that only uses a polynomial number of oracle calls.

4.6 Submodular Function Minimization

Problem (Submodular Function Minimization)**.** Given a finite set U and a submodular function $f : 2^U \to \mathbb{R}$ with $f(\emptyset) = 0$, find a set $S \subseteq U$ with $f(S)$ minimum.

Definition 4.53. Let U be finite and $f : 2^U \to \mathbb{R}$ submodular. Then the *base polyhedron* is defined as:

$$B(f) \coloneqq \{x \in \mathbb{R}^U \mid x(A) \leq f(A)\ \forall A \subseteq U,\ x(U) = f(U)\}$$

Example. Let $U = \{1, 2\}$ and $f(\{1\}) = 2,\ f(\{2\}) = -2,\ f(\{1, 2\}) = -1$.

Theorem 4.54. *The vertices of the base polyhedron are given by the vectors* $b^<$ *for all total orders* $<$ *of* U *where:*

$$b^<(u) \coloneqq f(\{v \in U \mid v \le u\}) - f(\{v \in U \mid v < u\})$$

Proof. Exercise $\square$

Theorem 4.55. *Let* $f : 2^U \to \mathbb{R}$ *be submodular,* $f(\emptyset) = 0$. *Then*

$$\min_{S \subseteq U} f(S) = \max\{x^-(U) \mid x \in B(f)\}$$

where $x^-(U) = \sum_{u \in U} x^-(u) = \sum_{u \in U} \min\{0, x(u)\}$.

Proof. Exercise $\square$

Idea: Maintain $x \in B(f)$ and represent it by a convex combination of the vertices. By Carathéodory, $|U|$ vertices are enough.

Algorithm 10: Schrijver's Algorithm

Input: Finite set $U = \{1, \dots, n\}$, submodular function $f : 2^U \to \mathbb{R}$ with $f(\emptyset) = 0$

Output: $X \subseteq U$ with $f(X)$ minimum

$k \leftarrow 1$, $<_1 \leftarrow$ any total order on U, $x \leftarrow b^{<_1}$

Build Graph:

- $D \leftarrow (U, A)$ where $A = \{(u, v) \mid u <_i v \text{ for some } 1 \le i \le k\}$
- $P \leftarrow \{u \in U \mid x(u) > 0\}$
- $N \leftarrow \{u \in U \mid x(u) < 0\}$
- $X \leftarrow$ set of vertices not reachable from P in D
- **if** $N \subseteq X$ **then**
  - **return** X

Find Augmentation:

- Let $d(v)$ denote the distance from P to v in D
- Choose $t \in N \setminus X$ with $(d(t), t)$ lexicographically maximum
- Choose s maximal with $(s, t) \in A$ and $d(s) = d(t) - 1$
- Let $i \in \{1, \dots, k\}$ such that $\alpha = |\{v \in U \mid s <_i v \le_i t\}|$ is maximum. [Let β be the number of indices attaining α]

Change Solution:

- Compute $0 \le \epsilon \le -x(t)$ and write $x' = x + \epsilon(\chi^t - \chi^s)$ as an explicit convex combination of $\le n$ vectors from $b^{<_1}, \dots, b^{<_k}$ and $b^{<_i^{s,u}}$ $\forall s <_i u \le_i t$ (where $<_i^{s,u}$ arises from $<_i$ by placing u directly before s) such that $b^{<_i}$ does not occur if $x'(t) < 0$
- $x \leftarrow x'$, rename the vectors in the convex combination of x as $b^{<_1}, \dots, b^{<_{k'}}$, $k \leftarrow k'$

go to Build Graph

Example. In the example from above, let $<_1$ be $1 <_1 2$, $k = 1$ and $x = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Theorem 4.56. *Algorithm 10 returns an optimum solution if it terminates.*

Proof. If the algorithm terminates, D does not contain a P-N-path. Since $N \subseteq X \subseteq U \setminus P$, $\sum_{u \in X} x(u) \leq \sum_{w \in W} x(w)$ for all $W \subseteq U$. No edge enters X, so $X = \emptyset$ or for all $j \in \{1, \ldots, k\}$ there exists $v \in X$ with $X = \{u \in U \mid u \leq_j v\}$. Therefore $\sum_{u \in X} b^{<_j}(u) = f(X)$ for all j (by definition of $b_u^{<}$). By theorem 4.36 (and again the definition of $b_u^{<}$), $\sum_{u \in W} b^{<_j}(u) \leq f(W)\ \forall W \subseteq U, j \in \{1, \ldots, k\}$. We get (where λ_j are the factors in the convex combination):

$$
\begin{aligned}
f(W) &\geq \sum_{j=1}^{k} \lambda_j \sum_{u \in W} b^{<_j}(u) \\
&= \sum_{u \in W} \sum_{i=1}^{k} \lambda_j b^{<_j}(u) \\
&= \sum_{u \in W} x(u) \\
&\geq \sum_{u \in X} x(u) \\
&= \sum_{u \in X} \sum_{j=1}^{k} \lambda_j b^{<_j}(u) \\
&= \sum_{j=1}^{k} \lambda_j \sum_{u \in X} b^{<_j}(u) \\
&= f(X)
\end{aligned}
$$

□

Theorem 4.57. *Each iteration can be performed in $O(n^3 + \gamma n^2)$ time where γ is the time required for an oracle call.*

Proof. BuildGraph and FindAugmentation can both be implemented in $O(n^3)$. We need to show that ChangeSolution can be done in $O(n^3 + \gamma n^2)$. Let $x = \lambda_1 b^{<_1} + \ldots + \lambda_k b^{<_k}$ and $s <_i t$.

Claim. *For some $\delta > 0$, $\delta(\chi^t - \chi^s)$ can be written as a convex combination of the vectors $b^{<_i^{s,u}} - b^{<_i}$ for $s <_i u \leq_i t$ in $O(\gamma n^2)$ time.*

How do $b^{<_i^{s,u}}$ and $b^{<_i}$ compare?

- Let $s <_i v \leq_i t$. Then by definition $b^{<_i^{s,v}}(u) = b^{<_i}(u)$ for $u <_i s$ or $u >_i v$.
- For $s \leq_i u <_i v$: $b^{<_i^{s,v}}(u) = f(\{w \in U \mid w \leq_i^{s,v} u\}) - f(\{w \in U \mid w <_i^{s,v} u\}) \leq f(\{w \in U \mid w \leq_i u\}) - f(\{w \in U \mid w <_i u\}) = b^{<_i}(u)$ by submodularity.
- For $u = v$ we have by submodularity:

$$\begin{aligned} b^{<_i^{s,v}}(u) &= f(\{w \in U \mid w \leq_i^{s,v} u\}) - f(\{w \in U \mid w <_i^{s,v} u\}) \\ &\geq f(\{w \in U \mid w \leq_i u\}) - f(\{w \in U \mid w <_i u\}) \\ &= b^{<_i}(u) \end{aligned}$$

Proof of claim:

- If $\exists s <_i v <_i t$ such that $b^{<_i^{s,v}}(v) = b^{<_i}(v)$ choose $\delta = 0$ and $\lambda_v = 1$.
- Otherwise for all $s <_i v \leq_i t$ we have $b^{<_i^{s,v}}(v) > b^{<_i}(v)$. Look at the matrix $M = (b^{<_i^{s,v}} - b^{<_i})_{vu}$ with rows $s <_i v \leq_i t$ and columns for $u \in U$. Then

$$\chi^t - \chi^s = \sum_{s <_i v \leq_i t} \kappa_v (b^{<_i^{s,v}} - b^{<_i})$$

is a non-negative combination for

$$\kappa_v = \frac{\chi^t_v - \sum_{v <_i w \leq_i t} \kappa_w (b^{<_i^{s,w}}(v) - b^{<_i}(v))}{b^{<_i^{s,v}}(v) - b^{<_i}(v)}$$

- By scaling, we get a convex combination.

Set $\epsilon := \min\{\lambda_i \delta, -x(t)\}$.

- If $\epsilon = \lambda_i \delta$ then:

$$x' = \sum_{j=1}^{k} \lambda_j b^{<_i} + \lambda_i \sum_{s <_i v \leq_i t} \kappa_v (b^{<_i^{s,v}} - b^{<_i})$$

$b^{<_i}$ cancels out.

- Otherwise, $x'(t) = 0$.

We can then use Gaussian elimination to get $\leq n$ vectors in $O(n^3)$. $\square$

Theorem 4.58. *The number of iterations is bounded by $O(n^5)$.*

Proof.

Claim. *$d(w)$ never decreases for $w \in U$.*

If (v, w) was added after a new vertex $b^{<_i^{s,u}}$ was added to the convex combination in ChangeSolution, then $s \leq_i w <_i v \leq t$ in that iteration. In particular $d(w) \leq d(s) + 1 = d(t) \leq d(v) + 1$, so adding the edge (v, w) does not decrease $d(w)$. Additionally, ChangeSolution does not add any elements to P which proves the claim.

We call a sequence of iterations with the same s and t a *block*. Each block has $O(n^2)$ iterations as the pair (α, β) decreases lexicographically.

Claim. *The number of blocks is bounded by $O(n^3)$.*

We consider different reasons for ending a block:

a) $d(v)$ increases for some $v \in U$, in which case v may become the new t or s.

b) t is removed from N.

c) (s, t) is removed from A.

We now bound the number of blocks of each type:

- The number of blocks of type a) is bounded by $O(n^2)$ since $d(w)$ never decreases.
- We claim that for all $t^* \in U$ there are at most $O(n^2)$ iterations with $t = t^*$ and $x'(t) = 0$: Between such iterations some $d(v)$ $(v \in U)$ must change. We have just shown that this only happens $O(n^2)$ times. Since there are n choices for t^*, there are $O(n^3)$ blocks of type b).
- We claim that there are $O(n^3)$ types of type c). It suffices to show that $d(t)$ changes between 2 blocks with the pair (s, t). For $s, t \in U$, call s *t-boring* if one of the following holds:
  - *$(s, t) \notin A$ or*
  - *$d(t) \leq d(s)$*

  Let $s^*, t^* \in U$ and consider the time after a block $s = s^*, t = t^*$ is ending because (s^*, t^*) is removed from A until a subsequent increase of $d(t^*)$.

  We prove that each $v \in \{s^*, \ldots, n\}$ is t^*-boring during this period. At the beginning, each $v \in \{s^* + 1, \ldots, n\}$ is t^*-boring by the maximal choice of s^*. s^* is t^*-boring because the arc (s^*, t^*) was removed. As $d(t^*)$ remains constant and d never decreases, we only need to check the introduction of new arcs.

Suppose for $v \in \{s^*, \dots, n\}$, (v, t^*) is added in an iteration with pair (s, t). Then $s \leq_i t^* <_i v \leq_i t$, so $d(t^*) \leq d(s) + 1 = d(t) \leq d(v) + 1$.

Case 1: $s > v$. Then $d(t^*) \leq d(s)$, either because $s = t^*$ or s was t^*-boring and $(s, t^*) \in A$.

Case 2: $s < v$. Then $d(t) \leq d(v)$, either because $v = t$ or by choice of s and since $(v, t) \in A$.

In either case, we have one strict inequality, so $d(t^*) \leq d(v)$ and v remains t^*-boring as claimed.

$d(t)$ can increase $O(n)$ times and there are $O(n^2)$ pairs (s, t).

In total, the total number of iterations is:

$$O(n^5) = \underbrace{O(n^2)}_{\text{iterations per block}} \cdot \underbrace{O(n^3)}_{number of blocks}$$

□

Theorem 4.59. *The submodular function minimization problem can be solved in time $O(n^8 + n^7\gamma)$, where γ is the time required for a call to the function oracle.*

Corollary 4.60. *Linear functions over the intersection of 2 polymatroids can be optimized in polynomial time.*

Remark.

- The fastest known algorithm has a running time of $O(n^6 + n^5\gamma)$ (Orlin, 2009 and Sidford, Wong, Lee, 2015).
- There is also a weakly polynomial algorithm $O((n^5 + n^4\gamma)(\log M))$ where $M = \max_X f(X)$.

Remark. $[0, 1]^n$ can be partitioned into $n!$ n-simplices (induced by the $n!$ orders on $\{1, \dots, n\}$). For each simplex, there exists a unique linear interpolation/extension of a function on the corners of the simplex to its interior. This corresponds to the definition of the Lovász extension.

In particular, a function is submodular $\Leftrightarrow$ the combination of the linear interpolations is convex.

5 Splitting-Off Lemma and Connectivity

5.1 Splitting-Off Lemma

Lemma 5.1 (Lovász). *Let G be a (multi-)graph with $V(G) = V\dot{\cup}\{s\}$ with $|\delta(s)|$ even and $k \geq 2$ such that:*

$$|\delta(U)| \geq k \qquad \forall \emptyset \neq U \subsetneq V \tag{2}$$

Then $\forall \{s,t\} \in E:\ \exists u \in \Gamma(s)$ such that

$$G' \coloneqq G - \{s,t\} - \{s,u\} + \{u,t\}$$

satisfies (2).

Remark. If $t = u$, then G' contains a loop which does not change the connectivity when it gets deleted.

Proof. If $|\Gamma(s)| = 1$, then the statement is clear since for all $U \subsetneq V$ with $t \in U$:

$$|\delta_G(U)| = \underbrace{|\delta_{G-s}(U)|}_{=|\delta_G(V\setminus U)|\geq k} \ |E[U,\{s\}]|$$

Therefore removing edges incident to s maintins the connectivity. Assume now that $|\Gamma(s)| > 1$. Fix $t \in \Gamma(s)$.

Claim. *We can find $u \in \Gamma(s) \setminus \{t\}$ such that G' satisfies (2).*

If not, then for all $u \in \Gamma(s)$ there exists $U \subsetneq V$ such that $|\delta_{G'}(U)| < k$. Then $t, u \in U$, else $|\delta_{G'}(U)| = |\delta_G(U)|$. Also, $|\delta_G(U)| \leq k+1$. Let:

$$\mathcal{C} \coloneqq \{U \subsetneq V \mid t \in U,\ |\delta_G(U)| \leq k+1\}$$

This covers $\Gamma(s)$. Then $\forall U \in \mathcal{C}$

$$1 \geq \underbrace{|\delta_G(U)|}_{\leq k+1} - \underbrace{|\delta_G(U \cup \{s\})|}_{\geq k} = |E(\{s\},U)| - |E(\{s\}, V \setminus U)|$$

so $|E(\{s\},U)| \leq |E(\{s\},V\setminus U)| + 1$. Since $|\delta(s)|$ is even, there cannot be equality, so:

$$|E(\{s\},U)| \leq |E(\{s\},V\setminus U)|$$

Now $\{s,t\} \in E(\{s\},U)$ for all $U \in \mathcal{C}$. In particular, we need > 2 sets from $\mathcal{C}$ to cover $\delta(s)$. Take $U_1, U_2, U_3 \in \mathcal{C}$ such that $U_1 \setminus (U_2 \cup U_3)$, $U_2 \setminus (U_1 \cup U_3)$, $U_3 \setminus (U_1 \cup U_2)$ are nonempty. Then

$$\begin{aligned}
&|\delta(U_1)| + |\delta(U_2)| + |\delta(U_3)| \\
&\geq |\delta(U_1 \cap U_2 \cap U_3)| + |\delta(U_1 \setminus (U_2 \cup U_3))| \\
&+ |\delta(U_2 \setminus (U_1 \cup U_3))| + |\delta(U_3 \setminus (U_1 \cup U_2))|
\end{aligned}$$

This "3-way submodularity" can be proved by considering how often each edge is counted on both sides. Actually, the left side is larger by 2 since the edge $\{s, t\}$ is counted three times here but only once on the right side.

Each term on the left side is at most $k+1$. Each term on the right is at least k. In total $3(k+1) \geq 4k+2$, so $k \leq 1$ in contradiction to the assumption. □

5.2 Construction of $2k$-edge-connected graphs

Lemma 5.2. *Every minimal k-edge-connected (multi-)graph has a vertex of degree k.*

Proof. Let G be such a graph. Then every cut has at least k edges and every edge is part of a cut with (at most) k edges. Let $X \subsetneq V(G)$ be minimum set such that $|\delta(X)| = k$. If $|X| = 1$, we are done. Otherwise, by minimality $G[X]$ is connected. Let $e \in E(G[X])$, then $\exists T \subsetneq V(G)$ with $e \in \delta(T)$ and $|\delta(T)| = k$.

Case 1: $T \cup X = V(G)$. Then $|\delta(X \setminus T)| = |\delta(T)| = k$ in contradiction to the minimality of X.

Case 2: $T \cup X \neq V(G)$. Then $|\delta(X \cap T)| = k$ by submodularity of $|\delta(\cdot)|$:

$$|\delta(X)| + |\delta(T)| \geq |\delta(X \cap T)| + |\delta(X \cup T)|$$

This again contradicts the minimality of X.

□

Theorem 5.3. *Let M_{2k} be a multigraph with 2 vertices joined by $2k$ edges. Any $2k$-edge-connected graph with at least 2 vertices can be built from M_{2k} by iteratively applying:*

1. *Adding edges (possibly loops)*
2. *Pinching k edges: Take k edges $(\{v_i, w_i\})_{i=1}^{k}$, add a new vertex s and replace each edge $\{v_i, w_i\}$ by $\{s, v_i\}$ and $\{s, w_i\}$ for $1 \leq i \leq k$.*

Proof. Start with any $2k$-edge-connected graph G. Then do 1. and 2. in reverse, i.e.:

1. Delete a maximal set of edges (while maintaining $2k$-edge-connectivity)
2. By lemma 5.2, there is a vertex s with $|\delta(s)| = 2k$. Split off s, as in lemma 5.1.

At the end, M_{2k} remains since both operations maintain $2k$-edge-connectivity.
□

Theorem 5.4 (Nash-Williams)**.** *An undirected graph G is $2k$-edge-connected if and only if there is an orientation $\vec{G}$ of G that is k-edge-connected.*

Proof. If $\vec{G}$ is k-edge-connected, then each cut contains k outgoing and k incoming edges, so G is $2k$-edge-connected.

For the other implication, let G be $2k$-edge-connected. Take M_{2k} and orient k edges in each direction. Apply theorem 5.3 and preserve the orientation. This preserves k-edge-connectivity in the oriented graph. □

Remark 5.5. Nash-Williams actually proved that each graph G has an orientation $\vec{G}$ for which $\lambda(x, y, \vec{G}) \geq \lfloor \frac{\lambda(x,y,G)}{2} \rfloor \ \forall x, y \in V(G)$ where $\lambda(x, y, H)$ denotes the local edge-connectivity, so the number of edge-disjoint x-y-paths in H.

Remark (Lovász Extension). For the exercises regarding the Lovász extension, we need $f(\emptyset) = 0$ and monotonicity for the polymatroid definition that requires $x \geq 0$. Allowing for negative vectors in the polymatroid, the polymatroid greedy algorithm still works without monotonicity. Therefore also the Lovász identity works without assuming monotonicity.

5.3 Connectivity Augmentation

Problem (Connectivity Augmentation)**.** Given a graph G and $k \geq 1$, find a minimum multiset F choses from $\{\{v, w\} \mid v, w \in V(G),\ v \neq w\}$ such that $G + F$ is a k-edge-connected graph.

Lemma 5.6. *Given a graph G and a degree requirement $x : V(G) \to \mathbb{N}$, there exists a multiset F chosen from $\{\{v, w\} \mid v, w \in V(G),\ v \neq w\}$ such that $G + F$ is k-edge-connected and*

$$|\delta_F(v)| + 2l(v)^{7} = x(v) \qquad \textit{for some } l : V(G) \to \mathbb{N}$$

if and only if:

1. *$x(V(G))$ is even*
2. *$|\delta_G(U)| + x(U) \geq k$ for all $\emptyset \neq U \subsetneq V(G)$*

Proof.

"⇒": The sum $\sum_{v \in V(G)} |\delta_F(v)|$ is even independent of F which implies 1. Since $G + F$ is k-edge-connected, we also get 2.

"⇐": Add a new vertex s and $x(v)$ edges between v and s for all $v \in V(G)$, resulting in a new graph G'. Then $|\delta(s)|$ is even since $x(V(G))$ is even. Let $\emptyset \neq U \subsetneq V(G)$. Then $|\delta_{G'}(U)| = |\delta_G(U)| + x(U) \geq k$ by 2.

[7] one can think of this as loops that one is allowed to drop

Therefore G' is k-edge-connected. Now apply splitting-off to s. This preserves k-edge-connectivity. Choose $l(v)$ to be the number of loops created at $v \in V(G)$ by splitting off. Set F to be all new edges except for the loops.

$\square$

Theorem 5.7 (Watanabe, Nakamura). *A graph G can be augmented to a k-edge-connected graph by adding γ edges if and only if any collection $\mathcal{U}$ of disjoint proper subsets of $V(G)$ satisfies:*

$$\sum_{U \in \mathcal{U}} (k - |\delta_G(U)|) \leq 2\gamma$$

Proof.

"$\Rightarrow$": Each summand on the left measures how many edges are missing in $\delta(U)$ for k-edge-connectivity. Each edge that is added can be part of at most 2 such cuts since the elements of $\mathcal{U}$ are disjoint.

"$\Leftarrow$": We want to apply lemma 5.6 so we need to find a suitable degree constraint. Introduce degree constraints starting with $x(v) = k$ for all $v \in V(G)$. Decrease x arbitrarily while preserving:

$$x(U) \geq \max\{0, k - |\delta_G(u)|\} \qquad \forall \emptyset \neq U \subsetneq V(G)$$

Now for all $v \in V(G)$ with $x(v) > 0$ there exists a set $v \in U \subsetneq V(G)$ with $x(U) = k - |\delta_G(U)|$. Now:

- By definition of x, it satisfies condition 2 of lemma 5.6.
- If we show that $x(V(G)) \leq 2\gamma$ then $\sum_{v \in V(G)} |\delta_F(v)| \leq 2\gamma$, so $|F| \leq \gamma$.
- If $x(V(G))$ is odd then $x(V(G)) < 2\gamma$, so we can increase x on any vertex $v \in V(G)$ in order to restore 2.

Let $\mathcal{U} := \{U \subsetneq V(G) \mid x(U) = k - |\delta_G(U)|\,,\ U \text{ maximal}\}$ be the set of maximal tight subsets.

Case 1: There are $S, T \in \mathcal{U}$ with $S \cup T = V(G)$ and $S \neq T$. Then $(V(G) \setminus S) \cap (V(G) \setminus T) = \emptyset$, so

$$\begin{aligned} x(V(G)) &\leq x(S) + x(T) \\ &= (k - |\delta_G(V(G) \setminus S)|) + (k - |\delta_G(V(G) \setminus T)|) \\ &\leq 2\gamma \end{aligned}$$

Case 2: $S \cup T \neq V(G)$ for all $S, T \in \mathcal{U}$. Let $S, T \in \mathcal{U}$ with $S \cap T \neq \emptyset$. Then:

$$\begin{aligned} x(S) + x(T) &= (k - |\delta_G(S)|) + (k - |\delta_G(T)|) \\ &\overset{\text{submodularity}}{\leq} (k - |\delta_G(S \cap T)|) + (k - |\delta_G(S \cup T)|) \\ &\leq x(S \cap T) + x(S \cup T) \\ &= x(S) + x(T) \end{aligned}$$

We have equality everywhere, so $S \cap T$ and $S \cup T$ are tight. This contradicts the maximality of S or T.

Case 3: $S \cup T \neq V(G)$ and $S \cap T = \emptyset$ for all $S, T \in \mathcal{U}$. Since it contains all $v \in V(G)$, $\mathcal{U}$ is then a partition of $V(G)$ and:

$$\begin{aligned} x(V(G)) &= \sum_{U \in \mathcal{U}} x(U) \\ &= \sum_{U \in \mathcal{U}} (k - |\delta_G(U)|) \\ &\leq 2\gamma \end{aligned}$$

In all cases, $x(V(G)) \leq 2\gamma$. If $x(V(G))$ is odd, we increase $x(v)$ for some $v \in V(G)$ by 1. This maintains condition 2 (since 2γ is even) and restores condition 1 of lemma 5.6.

□

Remark. The proof provides a polynomial time algorithm where splitting off and the x-calculation can be done by min-cut computations.

6 Survivable Network Design

Problem (SND)**.** Given an undirected graph G with weights $c : E(G) \to \mathbb{R}_{\geq 0}$ and connectivity requirements $r_{xy} \in \mathbb{Z}_{\geq 0}$ for each unordered pair $x, y \in V(G)$, find a minimum weight subgraph H such that for each x, y there are at least r_{xy} edge-disjoint paths from x to y in H.

Example 6.1 (Steiner Tree Problem). Given a graph G, $T \subseteq V(G)$ and $c : E(G) \to \mathbb{R}$ find a minimum weight edge set $F \subseteq E(G)$ such that all $t \in T$ are in the same connected component of $(V(G), F)$.

This can be reduced to SND by setting: $r_{xy} := \begin{cases} 1 & x, y \in T \\ 0 & \text{else} \end{cases}$

Example 6.2. For $r_{xy} = k$ for all $x, y \in V(G)$ we look for k-edge-connected subgraphs.

We now formulate an ILP description. Let $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ be defined by $f(\emptyset) = f(V(G)) = 0$ and $f(S) \coloneqq \max_{x \in S,\ y \in V(G) \setminus S} r_{x,y}$ for $\emptyset \neq S \subsetneq V(G)$. The SND problem can now be formulated as (SNDIP):

$$\begin{aligned} \min \sum_{e \in E(G)} & c(e) x_e \\ \text{s.t.} \sum_{e \in \delta(S)} & x_e \geq f(S) \qquad \forall S \subsetneq V(G) \\ & x_e \in \{0, 1\} \end{aligned}$$

We also consider the relaxation SNDLP where we only require $x_e \in [0, 1]$.

Definition 6.3. A function $f : 2^U \to \mathbb{R}_{\geq 0}$ is *proper* if it satisfies the following 3 constraints:

i) $f(S) = f(U \setminus S) \qquad \forall S \subseteq U$

ii) $f(A \cup B) \leq \max\{f(A), f(B)\} \qquad \forall A, B \subseteq U,\ A \cap B = \emptyset$

iii) $f(\emptyset) = 0$

$\Rightarrow$ the function of SND is proper.

Definition 6.4. A function $f : 2^U \to \mathbb{Z}_{\geq 0}$ is called *weakly supermodular* if for $A, B \subseteq U$:

i) $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$ *or*

ii) $f(A) + f(B) \leq f(A \setminus B) + f(B \setminus A)$

Proposition 6.5. *A proper function is weakly supermodular.*

Proof. As f is proper, we have:

(1) $f(A) \leq \max\{f(A \cap B), f(A \setminus B)\}$

(2) $f(B) \leq \max\{F(A \cap B), f(B \setminus A)\}$

(3) $f(A) = f(U \setminus A) \leq \max\{f(B \setminus A), f(U \setminus (B \cup A))\} = \max\{f(B \setminus A), f(B \cup A)\}$

(4) $f(B) \leq \max\{f(A \setminus B), f(A \cup B)\}$

Depending on which of $f(B \setminus A),\ f(A \setminus B),\ f(A \cup B),\ f(A \cap B)$ is minimum, we add those inequalities containing it. In either case, it follows that f is weakly supermodular. $\square$

Goal: Solve the separation problem for proper functions by Gomory-Hu trees. The crucial part is having a bound on $f(S)$.

Lemma 6.6. *Let G be an undirected graph, $u : E(G) \to \mathbb{R}_{\geq 0}$, $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ proper. Let H be a Gomory-Hu tree for (G, u). Then for $\emptyset \neq S \subsetneq V(G)$ we have:*

i) $\sum_{e' \in \delta_G(S)} u(e') \geq \max_{e \in \delta_H(S)} \sum_{e' \in \delta_G(C_e)} u(e')$

ii) $f(S) \leq \max_{e \in \delta_G(S)} f(C_e)$

Proof. i) follows directly from the Gomory-Hu tree property. For ii), let $X_1, \ldots, X_k$ be the connected components of $H - S$. Then for each $i \in [k]$ (where we choose C_e such that $C_e \cap X_i = \emptyset$):

$$V(H) \setminus X_i = \dot{\bigcup}_{e \in \delta_H(X_i)} C_e$$

Now

$$\begin{aligned} f(X_i) &= f(V(H) \setminus X_i) \\ &= f(\dot{\bigcup}_{e \in \delta_H(X_i)} C_e) \\ &\overset{f \text{ proper}}{\leq} \max_{e \in \delta_H(X_i)} f(C_e) \end{aligned}$$

so:

$$\begin{aligned} f(S) &= f(V(H) \setminus S) \\ &= f(\dot{\bigcup}_{i \in [k]} X_i) \\ &\overset{f \text{ proper}}{\leq} \max_{i \in [k]} f(X_i) \\ &\leq \max_{e \in \delta_H(S)} f(C_e) \end{aligned}$$

□

Theorem 6.7. *Let G be an undirected graph, $x \in \mathbb{R}_{\geq 0}^{E(G)}$ and $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ proper. Then we can find in $O(n^4 + n\theta)$ a set $S \subseteq V(G)$ with*

$$\sum_{e \in \delta_G(S)} x_e < f(S)$$

or decide that no such set exists.

Proof. Compute a Gomory-Hu tree H for (G, x). For each $\emptyset \neq S \subsetneq V(G)$, there is $e \in \delta_H(S)$ with $f(S) \leq f(C_e)$ by part ii) of lemma 6.6. By part i) of the lemma, $f(S) - x(\delta_G(S)) \leq f(C_e) - x(\delta_G(C_e))$. Since $\emptyset \neq C_e \subsetneq V(G)$:

$$\max_{\emptyset \neq S \subsetneq V(G)} f(S) - x(\delta_G(S)) = \max_{e \in E(H)} f(C_e) - x(\delta_G(C_e))$$

In particular, it suffices to check the inequality for fundamental cuts of the Gomory-Hu tree. □

Remark. With theorem 6.7, we can check whether there exists an integral feasible solution (and compute it).

6.1 Jain's Iterative LP Rounding

Idea: We can now solve the survivable network design LP. We round up edges with $x_e \geq \frac{1}{2}$. Fix them to 1 and compute the LP solution.

Part a: The rounding gives a 2-approximation

Part b: There is always an $e \in E(G)$ with $x_e \geq \frac{1}{2}$ in the LP solution.

6.1.1 Iterative Rounding

Let x^* be an optimum solution of (SNDLP) and $E_{\geq \frac{1}{2}}$ the set of edges with $x_e \geq \frac{1}{2}$. Consider $G_{\text{res}} \coloneqq G - E_{\geq \frac{1}{2}}$ and adjust (SNDLP):

$$\begin{aligned} \min \sum_{e \in E(G_{\text{res}})} & c(e) x_e \\ \text{s.t.} \sum_{e \in \delta_{G_{\text{res}}}(S)} & x_e \geq f(S) - \left| E_{\geq \frac{1}{2}} \cap \delta_G(S) \right| \qquad\qquad S \subseteq V(G) \end{aligned}$$

Remark. This is equivalent to fixing $x_e = 1$ for all $e \in E_{\geq \frac{1}{2}}$. In particular, we can still separate the inequalities using theorem 6.7.

Theorem 6.8. *Let z^* and z^*_{res} be the optimum values for (SNDLP) and its restriction to G_{res}. Let E_{res} be an integral solution of the restriction with $c(E_{\text{res}}) \leq 2z^*_{\text{res}}$. Then $E_{\text{res}} \cup E_{\geq \frac{1}{2}}$ is an integral solution to (SNDLP) with $c(E_{\text{res}} \cup E_{\geq \frac{1}{2}}) \leq 2z^*$.*

Proof. x^* is an optimum solution to (SNDLP). Its restriction to G_{res} is feasible. Additionally, $E_{\text{res}} \cup E_{\geq \frac{1}{2}}$ is a feasible solution to (SNDLP) so:

$$\begin{aligned}
z^*_{\text{res}} &\leq z^* - \sum_{e \in E_{\geq \frac{1}{2}}} c(e) x^*_e \\
\Leftrightarrow 2z^* &\geq 2z^*_{\text{res}} + \sum_{e \in E_{\geq \frac{1}{2}}} 2c(e) x^*_e \\
&\geq 2z^*_{\text{res}} + \sum_{e \in E_{\geq \frac{1}{2}}} c(e) \\
&\geq \sum_{e \in E_{\text{res}}} c(e) + \sum_{e \in E_{\geq \frac{1}{2}}} c(e) \\
&= c(E_{\text{res}} \cup E_{\geq \frac{1}{2}})
\end{aligned}$$

□

Algorithm 11: Jain's Algorithm

Input: Graph G, weights $c : E(G) \to \mathbb{R}_{\geq 0}$ and $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ proper
Output: An integral solution to (SNDLP)

$E_{\text{sol}} \leftarrow \emptyset,\ f' \leftarrow f,\ G' \leftarrow G$
repeat
  Find an optimum basis solution x^* of (SNDLP) for (G', f')
  Add all edges e with $x_e \geq \frac{1}{2}$ to E_{sol}
  $G' \leftarrow G - E_{\text{sol}},\ f'(S) \leftarrow f(S) - |E_{\text{sol}} \cap \delta_G(S)|$ for $S \subseteq V(G)$
until $x^* = 0$
return E_{sol}

Theorem 6.9. *Let x be a basic feasible solution to (SNDLP). Then there exists an edge $e \in E(G)$ with $x_e \geq \frac{1}{2}$.*

6.1.2 Uncrossing

Goal: Find a large family of laminar sets, each with a significant f-value and tight constraint.

We can assume that there is no edge e with $x_e = 0$. If there exists $x_e \geq \frac{1}{2}$, we are done, so assume $x_e \in (0, \frac{1}{2})$. Call $A \subsetneq V(G)$ *tight* if $x(\delta(A)) = f(A)$. Let $\mathcal{A}(A)$ be the row in the constraint matrix induced by A.

Lemma 6.10. *For 2 tight sets A, B one of the following holds:*

1. *$A \setminus B$ and $B \setminus A$ are tight and $\mathcal{A}(A) + \mathcal{A}(B) = \mathcal{A}(A \setminus B) + \mathcal{A}(B \setminus A)$*
2. *$A \cap B$ and $A \cup B$ are tight and $\mathcal{A}(A) + \mathcal{A}(B) = \mathcal{A}(A \cap B) + \mathcal{A}(A \cup B)$*

Proof. Let $S_1 := A \setminus B,\ S_2 := A \cap B,\ S_3 := B \setminus A,\ S_4 := V(G) \setminus (A \cup B)$. By tightness, we have:

$$
\begin{aligned}
f(A) &= x(E(S_1,S_3)) + x(E(S_1,S_4)) + x(E(S_2,S_3)) + x(E(S_2,S_4)) \\
f(B) &= x(E(S_1,S_2)) + x(E(S_1,S_3)) + x(E(S_4,S_2)) + x(E(S_4,S_3))
\end{aligned}
$$

By feasibility:

$$
\begin{aligned}
f(A \setminus B) = f(S_1) &\le x(E(S_1,S_2)) + x(E(S_1,S_3)) + x(E(S_1,S_4)) \\
f(B \setminus A) = f(S_3) &\le x(E(S_1,S_3)) + x(E(S_2,S_3)) + x(E(S_4,S_3))
\end{aligned}
$$

As f is weakly supermodular, we have:

$$
\begin{aligned}
f(A) + f(B) &\le f(A \setminus B) + f(B \setminus A) \qquad \textit{or} \\
f(A) + f(B) &\le f(A \cap B) + f(A \cup B)
\end{aligned}
$$

We only consider the first case (the second case is similar). By adding the above inequalities and comparing the terms, we see $2x(E(S_2,S_4)) \le 0$. Since $x > 0$ (by assumption), $E(S_2,S_4) = \emptyset$. In particular,

$$\mathcal{A}(A) + \mathcal{A}(B) = \mathcal{A}(A \setminus B) + \mathcal{A}(B \setminus A)$$

□

Let $\mathcal{T}$ be the family of tight sets. For a family $\mathcal{F} \subseteq \mathcal{T}$, define $\operatorname{span}(\mathcal{F}) := \operatorname{span}(\{\mathcal{A}(S) \mid S \in \mathcal{F}\})$.

Lemma 6.11. *For any maximal laminar family $\mathcal{L} \subseteq \mathcal{T}$ of tight sets,* $\operatorname{span}(\mathcal{L}) = \operatorname{span}(\mathcal{T})$.

Using the lemma, we can take any basis $\mathcal{B} \subseteq \mathcal{L}$. Since we assumed that $x_e \in (0,1)$, we get $\dim \operatorname{span}(T) = |E(G)|$.

Proof of lemma. "$\subseteq$" is clear. If the other inclusion doesn't hold, there exists $S \in \mathcal{T}$ with $\mathcal{A}(S) \notin \operatorname{span}(\mathcal{L})$. Choose S such that it crosses a minimum number of sets in $\mathcal{L}$ (since $\mathcal{L}$ is maximal, S crosses some set in $\mathcal{L}$). Let $L \in \mathcal{L}$ cross S. By lemma 6.10, we have either:

1. $S \setminus L$ and $L \setminus S$ are tight and $\mathcal{A}(S) + \mathcal{A}(L) = \mathcal{A}(S \setminus L) + \mathcal{A}(L \setminus S)$ or
2. $S \cup L$ and $S \cap L$ are tight and $\mathcal{A}(S) + \mathcal{A}(L) = \mathcal{A}(S \cup L) + \mathcal{A}(S \cap L)$.

Case 1: 1. holds, so $S \setminus L$ and $L \setminus S$ are tight.

Case 1.1 $\mathcal{A}(S \setminus L) \notin \operatorname{span}(\mathcal{L})$.

Claim. *If $L' \in \mathcal{L}$ crosses $S \setminus L$, then L' also crosses S. In particular (since $S \setminus L$) doesn't cross L we get a contradiction to the minimality of S.*

We get $(S \setminus L) \cap L' \neq \emptyset$, so $S \cap L' \neq \emptyset$ and $L' \setminus L \neq \emptyset$. Additionally $(S \setminus L) \setminus L' \neq \emptyset$ and $(S \setminus L') \neq \emptyset$. Since $L, L' \in \mathcal{L}$, we have $L \subseteq L'$ or $L \cap L' = \emptyset$.

Claim. *In both cases $L' \setminus S \neq \emptyset$.*

If $L \subseteq L'$, $\emptyset \neq L \setminus S \subseteq L' \setminus S$. If $L \cap L' = \emptyset$, then $\emptyset \neq L' \setminus (S \setminus L) = L' \setminus S$.

□

Lemma 6.12. *Given a basic optimum LP solution $0 < x < 1$, there exists a laminar family $\mathcal{B}$ with:*

1. $|\mathcal{B}| = |E(G)|$
2. $\{\mathcal{A}(B) \mid B \in \mathcal{B}\}$ *are linearly independent.*
3. $\dim \operatorname{span}(\mathcal{B}) = |E(G)|$
4. $f(B) \geq 1$ *for all* $B \in \mathcal{B}$

Proof. Since x is a basic solution, $\dim(\operatorname{span}(\mathcal{T})) = |E(G)|$. Choose $\mathcal{B}$ by lemma 6.11 such that 1. 2. and 3. are satisfied. $f(B) < 0$ is impossible for a (tight) set $B \in \mathcal{B}$. If $f(B) = 0$, then $x(\delta(B)) = 0$, so $\mathcal{A}(B) = 0$ which is also impossible since $\mathcal{B}$ is a basis. □

We want to show that there exists $x_e \geq \frac{1}{2}$. The idea is to assign a token to each set in $\mathcal{B}$ for each edge in its cut.

If not, $0 < x < \frac{1}{2}$. Let F be the branching representing $\mathcal{B}$ (i.e. $V(F) = \mathcal{B}$). Define the half complement $y_e \coloneqq \frac{1}{2} - x_e \in (0, \frac{1}{2})$ and

$$\begin{aligned}
\operatorname{coreq}(S) &\coloneqq y(\delta(S)) \\
&= \frac{1}{2} |\delta_G(S)| - x(\delta_G(S)) \\
&\overset{\text{tightness}}{=} \frac{1}{2} |\delta_G(S)| - f(S)
\end{aligned}$$

Proposition 6.13. *For $S \in \mathcal{T}$, $\operatorname{coreq}(S) \in \mathbb{Z} + \frac{1}{2}$. Additionally, $\operatorname{coreq}(S) \notin \mathbb{N}$ if and only if $|\delta_G(S)|$ is odd.*

Lemma 6.14 (Vazirani)**.** *Suppose $S \in V(F)$ with α children, all of which have a corequirement of $\frac{1}{2}$ and S has β tokens such that $\alpha + \beta = 3$. Then* $\operatorname{coreq}(S) = \frac{1}{2}$.

Proof. Since each child C has a corequirement of $\frac{1}{2}$, $|\delta(C)|$ is odd. Since $\alpha+\beta=3$, we can show by case enumeration that $|\delta(S)|$ is odd. We get $\mathrm{coreq}(S)\notin\mathbb{N}$. It suffices to show $\mathrm{coreq}(S)<\frac{3}{2}$:

$$\begin{aligned}\mathrm{coreq}(S) &= y(\delta(S))\\ &\leq \sum_{C \text{ child of } S} \mathrm{coreq}(S) + \sum_{\substack{e=\{x,y\}\in\delta(S)\\ \text{token was donated to}\\ x\in S}}\end{aligned}$$

If $\beta\geq 1$, we are done. If $\beta=0$, then $\alpha=3$, so there must be an edge between 2 children and $y(\delta(S))<\sum_C \mathrm{coreq}(C)$. □

Lemma 6.15. *If $S\in V(F)$ has one child C, then S must own at least 2 tokens.*

Proof. S owns at least 1 token, otherwise $\mathcal{A}(S)=\mathcal{A}(C)$. If S owns 1 token, then $\mathbb{N}\ni|f(S)-f(C)|=|x(\delta(S))-x(\delta(C))|\in(0,\frac{1}{2})$ which is a contradiction. □

Lemma 6.16. *If $S\in V(F)$ has 2 children C_1,C_2 with* $\mathrm{coreq}(C_1)=\frac{1}{2}$ *then S must own a token.*

Proof. If not, $\mathcal{A}(S),\mathcal{A}(C_1),\mathcal{A}(C_2)$ are linearly independent, so $\delta(C_1)\subseteq\delta(S)$ and $\delta(C_1)\subseteq\delta(C_2)$ are impossible. Let:

$$\begin{aligned}a &:= y(\delta(S)\cap\delta(C_2))>0\\ b &:= y(\delta(C_1)\cap\delta(C_2))>0\end{aligned}$$

Then $a+b=\mathrm{coreq}(C_1)=\frac{1}{2}$. Then $|\delta(C_1)|$ is odd, so $|\delta(S)|\equiv|\delta(C_2)| \mod 2$.

$\mathrm{coreq}(S)-\mathrm{coreq}(C_2)=a-b$. $-\frac{1}{2}<a-b<\frac{1}{2}$, so $\mathrm{coreq}(S)=\mathrm{coreq}(C_2)$ which is a contradiction. □

Lemma 6.17. *Let $0<x<\frac{1}{2}$ be a basic optimum solution to (SNDLP). Consider a subarborescence rooted at $R\in V(F)$. Then we can redistribute tokens such that R gets at least 3 tokens and each proper descendant of R gets 2 tokens. If* $\mathrm{coreq}(R)\neq\frac{1}{2}$, *$R$ gets ≥ 4 tokens.*

Proof. Proceed by induction on the height of the subarborescence. For leaves, this holds. Let $\mathrm{surplus}(S):=\#\text{assigned tokens}-2$. Let R not be a leaf:

Case 1: R has 4 children. Then we can simply move up tokens from its children.

Case 2: R has 3 children. If one has a surplus of ≥ 2, we are done. Otherwise, they all have a surplus of 1, so a corequirement of $\frac{1}{2}$, so $\operatorname{coreq}(R) = \frac{1}{2}$.

Case 3: R has 2 children. If both have a surplus of at least 2, we are done. Otherwise, one child C_1 has a surplus of 1, so a corequirement of $\frac{1}{2}$. Then by the previous lemma, R owns a token. If both children of a surplus of 1, then we again get $\operatorname{coreq}(R) = \frac{1}{2}$.

Case 4: R has 1 child. Then S owns 2 tokens. This works the same way as the previous case (but using a different lemma).

$\square$

This lemma then proves Theorem 6.9.