

# Combinatorial Optimization

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## 0 Organization

- Prerequisites
  - Basic knowledge of graph algorithms
  - Linear Programming (LP Duality)
  - Programming skills in C++
- Exam
  - Qualification requires 50% of the points in theoretical & programming exercises
  - Oral exam
- Books
  - "Combinatorial Optimization", Korte & Vygen
  - "Understanding & Using Linear Programming", B. Gärtner, J. Matousek
  - Skript (theorems & definitions)
  - Further book recommendations are on the website

## 1 Matchings

### 1.1 Introduction

#### Definition 1.1.

1. A *matching*  $M$  in a graph  $G = (V, E)$  is a set of pairwise disjoint edges, i.e. they don't have a common endpoint.  
 $\nu(G) := \max.$  cardinality of a matching in  $G$
2. An *edge cover*  $C$  of a graph  $G = (V, E)$  is a subset of  $E$  s.t.  $V = \bigcup_{e \in C} e$ .  
 $\zeta(G) := \min.$  cardinality of an edge cover in  $G$
3. A matching is called *perfect* (or *1-factor*) if it is an edge cover
4.  $v \in V$  with  $v \in e \in M$  is called  *$M$ -covered*
5.  $v \in V$  is called  *$M$ -exposed* if it is not  *$M$ -covered*

#### Definition 1.2.

1. A *stable set* (independent set)  $S$  is a set of pairwise non-adjacent vertices.  
 $\alpha(G) := \max.$  cardinality of a stable set

2. A *vertex cover*  $C$  is a subset of  $V$  s.t.  $E = \bigcup_{\{x,y\} \in E, x \in C} \{x, y\}$   
 $\tau(G) := \min.$  cardinality of a vertex cover

**Lemma 1.3.**

1.  $\alpha(G) + \tau(G) = |V|$
2.  $\nu(G) + \zeta(G) = |V|$  if  $G$  has no isolated vertices
3.  $\zeta(G) = \alpha(G)$  if  $G$  is bipartite and has no isolated vertices

**Problem.** Cardinalty Matching Problem

Input: Graph  $G = (V, E)$

Task: Find a maximum cardinality matching

**Problem.** Maximum Weight Matching Problem (MWMP)

Input: Graph  $G, c : E \rightarrow \mathbb{R}$

Task: Find a matching  $M$  maximizing  $c(M)$

**Problem.** Minimum Weight Perfect Matching (MWPM)

Input: Graph  $G, c : E \rightarrow \mathbb{R}$

Task: Find a perfect matching of minimum weight or decide that no perfect matching exists in  $G$

**Lemma 1.4.** *The MWMP is equivalent to the MWPM (i.e. there exists a transformation with linear complexity)*

*Proof.* Given a MWPM instance  $(G, c)$ , define  $c' := K - c$  ( $K := 1 + \sum_{e \in E} |c(e)|$ ).

$\Rightarrow$  Any maximum weight matching is a maximum cardinality matching

Given a MVMP instance  $(G, c)$ , define  $G'$  as 2 copies of  $G$  where the 2 copies of a vertex are joined by an edge.

$\Rightarrow G'$  has a perfect matching. Define:

$$c'(e) := \begin{cases} -c(e) & \text{if } e \text{ is in the first copy} \\ 0 & \text{else} \end{cases}$$

A minimum weight perfect matching in  $G'$  gives us a maximum weight matching in  $G$ .  $\square$

**Definition 1.5.** Let  $G = (V, E)$  be a graph and  $M \subseteq E$  a matching in  $G$ . A path  $P$  is *M-alternating* if its edges are alternatingly in and not in  $M$ . If both end points of this path are *M-exposed*,  $P$  is an *M-augmenting* path.

**Lemma 1.6.** *Given a matching  $M$  in  $G$  and an inclusion-wise maximal M-alternating path  $P$ ,*

$$M \Delta P := M \setminus P \cup P \setminus M$$

*is a matching. If  $P$  is M-augmenting, then  $|M \Delta P| = |M| + 1$ .*



Figure 1: Example of the construction in Theorem 1.8

**Theorem 1.7** (Petersen 1891, Berge 1957). *Augmenting Path Theorem*  
 Given a graph  $G = (V, E)$  and a matching  $M$  in  $G$ :

$$|M| = \nu(G) \Leftrightarrow \nexists M\text{-augmenting path } P \text{ in } G$$

*Proof.*

" $\Rightarrow$ ": Clear

" $\Leftarrow$ ": Assume there exists a matching  $M'$  with  $|M'| > |M|$ . Let  $G' := (V, M \Delta M')$ .

$$\Rightarrow |\delta_{G'}(v)| \leq 2 \quad \forall v \in V$$

$\Rightarrow G'$  is the union of disjoint circuits and paths

$\Rightarrow$  all circuits are even and have the same number of edges from  $M$  and  $M'$

$\Rightarrow \exists$  a path  $P$  in  $G'$  starting and ending with an edge in  $M'$

$\Rightarrow P$  is an alternating path

□

## 1.2 Bipartite Matching

**Theorem 1.8** (König 1931). *If  $G$  is bipartite, then  $\nu(G) = \tau(G)$*

*Proof.* Add vertices  $s$  and  $t$  edges between them to all vertices of the respective partition. Direct all edges from  $s$  to  $t$ . Then  $\nu(G)$  is maximum number of disjoint  $s$ - $t$ -paths. Menger  $\Rightarrow$  This is equal to the minimum number of vertices that disconnect  $t$  from  $s$ . □

**Theorem 1.9** (Hall 1935). *Let  $G = (A \dot{\cup} B, E)$  be a bipartite graph. Then:*

$$G \text{ has a matching covering } A \Leftrightarrow |\Gamma(X)| \geq |X| \quad \forall X \subseteq A$$

**Corollary 1.10.** *Marriage Theorem*

$$|\Gamma(X)| \geq |X| \quad \forall X \subseteq A \text{ and } |A| = |B| \Leftrightarrow G \text{ has a perfect matching}$$

**Definition 1.12.** The MWPMP for bipartite graphs is called *Assignment Problem*.

**Theorem 1.13.** *The Assignment Problem can be solved in time  $O(nm + n^2 \log m)$ .*

*Proof.* Use the Successive Shortest Paths algorithm in an auxiliary graph.  $\square$

### 1.3 The Tutte Matrix & Randomized Matching

**Definition 1.14.** Let  $G$  be a simple, undirected graph. Let  $G'$  be an orientation of  $G$  and  $(X_e)_{e \in E(G)}$ . The *Tutte matrix* is defined as

$$T_G(X) := (T_{vw}^*)_{v,w \in V(G)}$$

where

$$t_{vw}^* := \begin{cases} X_{\{v,w\}} & \text{if } (v,w) \in E(G) \\ -X_{\{v,w\}} & \text{if } -(v,w) \in E(G) \\ 0 & \text{else} \end{cases}$$

*Remark 1.15.*  $T_G(X)$  is skew-symmetric (i.e.  $T_G(X) = -(T_G(X))^t$ ).  $\text{rank}(T_G(X))$  is independent of the orientation of  $G$ .  $\det(T_G(X))$  is a polynomial in  $X$ .

**Theorem 1.16** (Tutte). *A simple graph  $G$  has a perfect matching  $\Leftrightarrow \det(T_G(X)) \neq 0$*

*Proof.* Let  $V(G) = \{v_1, \dots, v_n\}$  and  $S_n$  be the permutation group.

$$\det T_G(X) = \sum_{\pi \in S_n} \text{sgn} \pi \cdot \prod_{i=1}^n t_{v_i, v_{\pi(i)}}^*$$

Let  $S'_n := \{\pi \in S_n \mid \prod_{i=1}^n t_{v_i, v_{\pi(i)}}^* \neq 0\}$ . Each  $\pi \in S_n$  corresponds to a digraph  $H_\pi := (V(G), \{(v_i, v_{\pi(i)}) \mid i \in [n]\})$ . We have  $|\delta^+(v)| = 1 = |\delta^-(v)| \ \forall v \in V(H_\pi) \Rightarrow H_\pi$  is the union of disjoint circuits. If  $\pi \in S'_n$ , then  $H_\pi \subset G'$ .

If there exists  $\pi \in S'_n$  s.t.  $H_\pi$  is a collection of even circuits, then this immediately yields a perfect matching in  $G$  (using every second edge of each circuit).

Otherwise,  $\forall \pi \in S'_n$ ,  $H_\pi$  contains an odd circuit. Let  $r(\pi) \in S'_n$  arise from  $\pi$  by reversing edges on the unique odd circuit containing a vertex with minimum index  $\Rightarrow r(r(\pi)) = \pi$  and  $\text{sgn}(\pi) = \text{sgn}(r(\pi))$ . The second part is true since for reversing an odd cycle, we need an even number of swaps. Let  $v_{i_1}, \dots, v_{i_{2k+1}}$  be the "first" odd circuit. Then  $r(\pi)$  is attained by  $2k$  swaps: For  $j = 1, \dots, k$  swap  $(\pi(i_{2j-1}), \pi(i_{2k}))$  and  $(\pi(i_{2j}), \pi(i_{2k+1}))$ .

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<sup>1</sup>This is an abbreviation for  $\{1, \dots, n\}$ .

$\prod_{i=1}^n t_{v_i v_{\pi(i)}}^* = -\prod_{i=1}^n t_{v_i v_{r(\pi(i))}}^*$  since there is an odd number of sign changes to  $t^*$ .  $\Rightarrow \det(T_G(X)) = 0$ . We have shown that if  $G$  has no perfect matching, then  $\det T_G(X) = 0$ .

Assume that  $G$  has a perfect matching  $M$ . Define  $\pi$  as  $\pi(i) = j, \pi(j) = i$  where  $\{i, j\} \in M$ .  $\Rightarrow \prod_{i=1}^n t_{v_i v_{\pi(i)}}^* = \prod_{e \in M} -X_e^2$  cannot be canceled out. In particular,  $\det T_G(X) \neq 0$ .  $\square$

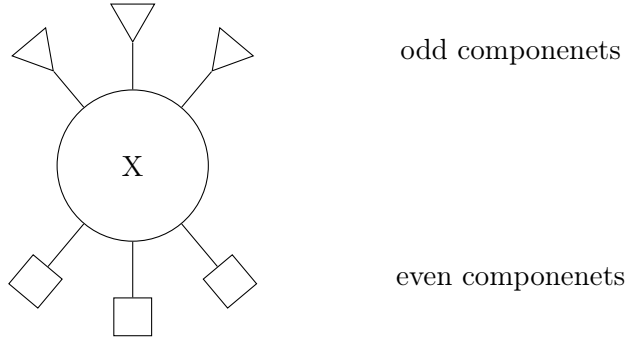
*Remark 1.17.* Picking  $X' \in [0, 1]^{E(G)}$  at random, we almost surely have (since the zero set of a non-zero polynomial is a set of measure zero):

$$\det T_G(X') \neq 0 \Leftrightarrow G \text{ has a perfect matching}$$

**Theorem 1.18** (Lovász 1979). *Let  $G$  be a simple graph and  $X \in [0, 1]^{E(G)}$  chosen randomly. Then almost surely  $\text{rank}(T_G(X)) = 2\nu(G)$ .*

#### 1.4 Tutte's Matching Theorem

Let  $X \subseteq V(G)$ .  $G - X$  consists of even and odd (in terms of the number of vertices) connected components. We define  $q_G(X)$  to be the number of odd components in  $G - X$ .



**Definition 1.19.** A graph  $G$  satisfies the *Tutte Condition* if  $q_G(X) \leq |X|$  for all  $X \subseteq V(G)$ .  $\emptyset \neq X \subseteq V(G)$  is called *barrier* if  $q_G(X) = |X|$ .

**Proposition 1.20.** *For any graph  $G$  and any  $X \subseteq V(G)$ :*

$$q_G(X) - |X| \equiv |V(G)| \pmod{2}$$

**Definition 1.21.** A graph  $G$  is *factor-critical* if  $G - v$  has a perfect matching for all  $v \in V(G)$ . A matching is called *near-perfect* if it covers  $|V(G)| - 1$  vertices.

**Proposition 1.22.** *If  $G$  is factor-critical, then it is connected.*

**Theorem 1.23** (Tutte 1947). *A graph  $G$  has a perfect matching  $\Leftrightarrow$  Tutte Condition holds (i.e.  $q_G(X) \leq |X| \forall X \subseteq V(G)$ )*

*Proof.*

" $\Rightarrow$ ": Clear

" $\Leftarrow$ ": We proceed by induction on  $|V(G)|$ . The case  $|V(G)| = 2$  is clear.

Generally, if the Tutte Condition holds, then  $|V(G)|$  must be even (pick  $X = \emptyset$ ). Proposition 1.20  $\Rightarrow q_G(X) - |X|$  is even. Every  $x \in V(G)$  induces a barrier  $\{x\}$ . Let  $X$  be a maximum barrier. Then  $G - X$  doesn't have any even components (since otherwise a single vertex of such a component could be added to  $X$ ).

**Claim:** Each odd component is factor-critical.

Let  $C$  be an odd component in  $G - X$ ,  $v \in V(C)$ . Assume that  $C - v$  does not have a perfect matching. Induction Hypothesis  $\Rightarrow C - v$  violates Tutte Condition.

$\Rightarrow \exists Y \subseteq V(C - v) : q_{C-v}(Y) > |Y|$

$\stackrel{1.20}{\Rightarrow} q_{C-v}(Y) \geq |Y| + 2$

Observe  $X \cap \{v\} = Y \cap \{v\} = X \cap Y = \emptyset$ :

$$\begin{aligned} q_G(X \cup Y \cup \{v\}) &= q_G(X) - 1 + q_C(Y \cup \{v\}) \\ &= |X| - 1 + q_{C-v}(Y) \\ &\geq |X| - 1 + |Y| + 2 \\ &= |X \cup Y| + 1 \\ &= |X \cup Y \cup \{v\}| \end{aligned}$$

$\Rightarrow X \cup Y \cup \{v\}$  is a barrier

$\Rightarrow$  Claim

Let  $G'$  arise from  $G$  by contracting each odd component into a single vertex. We have  $V(G') = X \dot{\cup} Z$  and  $G'$  is bipartite. We have to show that  $G'$  has a perfect matching. If not, then  $\exists A \subseteq Z : |\Gamma_{G'}(A)| < |A|$   
 $\Rightarrow q_G(\Gamma_{G'}(A)) \geq |A| > |\Gamma_{G'}(A)|$  which contradicts the Tutte Condition.

□

**Theorem 1.24** (Berge 1958).

$$|V(G)| = 2\nu(G) + \max_{X \subseteq V(G)} (q_G(X) - |X|)$$

*Proof.* For  $X \subseteq V(G)$ , any matching has at least  $q_G(X) - |X|$  uncovered vertices, so " $\geq$ " holds.

For the other inequality, add  $k := \max_{X \subseteq V(G)} (q_G(X) - |X|)$  new vertices and connect them to all existing vertices, yielding a new graph  $H$ .

We claim that  $H$  has a perfect matching. This then implies:

$$2\nu(G) + k \geq 2\nu(H) - k = |V(H)| - k = |V(G)|$$

Assume that  $H$  does not have a perfect matching. Then by Tutte's Theorem, there exists  $Y \subseteq V(H)$  with  $q_H(Y) > |Y|$ . By 1.20,  $k \equiv |V(G)| \pmod{2}$ , therefore  $|V(H)|$  is even, so  $Y \neq \emptyset$ .  $Y$  must contain all new vertices, otherwise  $H - Y$  would be connected<sup>2</sup> and  $q_H(Y) \leq 1 \leq |Y|$ .

$$\Rightarrow q_G(Y \cap |V(G)|) = q_H(Y) > |Y| = |Y \cap V(G)| + k$$

which is a contradiction to the choice of  $k$ .  $\square$

## 1.5 Ear Decompositions of Factor-Critical Graphs

**Definition 1.25.** Let  $G$  be a graph. An *ear decomposition* of  $G$  is a sequence  $r, P_1, \dots, P_k$  with  $G = (r, \emptyset) + P_1 + \dots + P_k$  such that each  $P_i$  is either a path with exactly the endpoints located in  $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$  or a circuit where exactly one of the vertices belongs to  $\{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$ .

$P_1, \dots, P_k$  are called *ears*. If  $|V(P_1)| \geq 3$  and  $P_2, \dots, P_k$  are paths we call it a *proper ear decomposition*.

**Theorem 1.27** (Whitney 1932). *Let  $G$  be an undirected graph. Then:*

$$G \text{ 2-connected} \Leftrightarrow G \text{ has a proper ear decomposition}$$

**Definition 1.28.** An ear decomposition is *odd* if every ear has odd length (in terms of the number of edges).

**Theorem 1.29.** *Let  $G$  be an undirected graph. Then*

$$G \text{ factor-critical} \Leftrightarrow G \text{ has an odd ear decomposition}$$

*The first vertex  $r$  of the ear decomposition can be chosen arbitrarily.*

*Proof.*

" $\Leftarrow$ ": Let  $G$  be a graph with an odd ear decomposition  $r, P_1, \dots, P_k$ .  $P_1$  is an odd circuit, so it is factor-critical. We use induction on the number of ears. Let  $P$  be the last ear and  $G'$  be  $G$  before adding  $P$ . By the induction hypothesis,  $G'$  is factor-critical. Given  $v \in V(G)$ , we have to show that  $G - v$  has a perfect matching.

Case 1:  $v \in V(G')$ . Then  $G' - v$  has a perfect matching. Adding every second edge of  $P$  (excluding the endpoints) to it, yields a perfect matching of  $G - v$ .

Case 2:  $v \in V(G) \setminus V(G')$ . Let  $x, y$  be the endpoints of  $P$ . Without loss of generality let  $P_{[v,x]}$  be even. There exists a perfect matching in  $G' - x$ . Together with every second edge of  $P_{[v,y]}$  and  $P_{[v,x]}$  this is a perfect matching in  $G - v$ .

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<sup>2</sup>Note that  $Y$  cannot contain all old vertices, since otherwise  $q_H(Y) < |Y|$ .



" $\Rightarrow$ ": Let  $r \in V(G)$  be any vertex. Let  $M$  be a perfect matching in  $G - r$ . Suppose we have an odd ear decomposition for  $G' \subseteq G$  with  $r \in V(G')$  and  $M \cap E(G')$  is a near-perfect matching in  $G'$  (i.e. all vertices in  $G'$  except for  $r$  are matched with other vertices in  $G'$ ).

If  $G' \neq G$ , there is an edge  $\{x, y\} \in E(G) \setminus E(G')$  with  $x \in V(G')$  (by Proposition 1.22). If  $y \in V(G')$ , then  $\{x, y\}$  can be chosen as the next ear. Otherwise, construct an  $M$ -alternating odd ear, starting with  $\{x, y\}$ . Let  $N$  be a matching in  $G - y$ .  $M \Delta N$  contains a  $y$ - $r$ -path  $P$ . Let  $w$  be the first vertex in  $P \cap V(G')$ .  $w$  is  $M$ -exposed in  $P_{[y, w]}$ ,  $y$  is  $N$ -exposed in  $P_{[y, w]}$ . Therefore  $P_{[y, w]}$  is even and together with  $\{x, y\}$  it forms an  $M$ -alternating odd ear.

Inductively, this argument yields an odd ear decomposition. □

**Definition 1.30.** Let  $G$  be factor-critical and  $M$  a near-perfect matching. An  $M$ -alternating ear decomposition is an odd ear decomposition such that each ear is an  $M$ -alternating path or circuit  $C$  with:

$$|E(C) \cap M| = |E(C) \setminus M| - 1$$

**Corollary 1.31.** For any factor-critical graph  $G$  and any near-perfect matching  $M$  in  $G$ , there exists in  $M$ -alternating ear decomposition of  $G$ .

**Definition 1.32.** Let  $G$  be factor-critical,  $M$  a near-perfect matching and  $r, P_1, \dots, P_k$  an  $M$ -alternating ear decomposition of  $G$ .  $\mu, \varphi : V(G) \rightarrow V(G)$  are associated with the ear decomposition if:

- $\{x, y\} \in M \Rightarrow \mu(x) = y$
- $\{x, y\} \in E(P_i) \setminus M$  and  $x \notin \{r\} \cup \bigcup_{j \in [i]} V(P_j) \Rightarrow \varphi(x) = y$
- $\mu(r) = \varphi(r) = r$

**Proposition 1.33.** Let  $G$  be a factor-critical graph and  $\mu, \varphi$  functions associated with an  $M$ -alternating ear decomposition. Then this ear decomposition is unique up to the order of the ears. The Ear-Decomposition-Algorithm correctly determines an explicit list of the ears in linear time.

*Proof.* Step 3 determines ears uniquely. The algorithm clearly runs in linear time. □

**Lemma 1.34.** Let  $G$  be factor-critical and  $\mu, \varphi$  associated with an  $M$ -alternating ear decomposition. Then the maximal path given by the initial sequence

$$x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \dots \tag{1}$$

defines an  $M$ -alternating  $x$ - $r$ -path of even length.

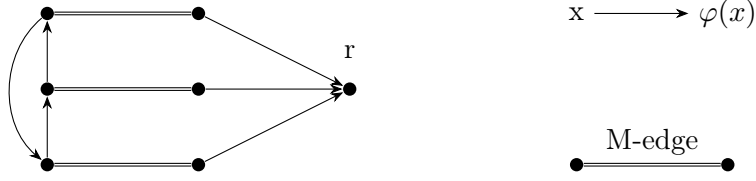


Figure 2: Counter example for the reverse proposition.

*Proof.* We proceed by induction on the number of ears. Let  $x \in V(G) \setminus \{r\}$  and  $P_i$  be the ear containing  $x$ . A subsequence of (1) is a subpath  $Q$  of  $P_i$  from  $x$  to  $y \in \{r\} \cup \bigcup_{j \in [i-1]} V(P_j)$ .  $Q$  starts with a matching edge and ends with a non-matching edge, so it has even length. If  $y = r$ , we are done, otherwise the statement follows from the induction hypothesis.  $\square$

## 1.6 Edmond's Matching Algorithm

**Definition 1.35.** Let  $G$  be a graph,  $M$  a matching in  $G$ . A *blossom* in  $G$  with respect to  $M$  is a factor-critical subgraph of  $B$  of  $G$  such that  $|M \cap E(B)| = \frac{|V(B)|-1}{2}$ . The vertex  $r \in V(B)$  that is exposed by  $M$  is called the *base* of  $B$ .

**Definition 1.36.** Let  $G$  be a graph,  $M$  a matching in  $G$ ,  $B$  a blossom and  $Q$  a  $M$ -alternating  $v$ - $r$ -path of even length from  $v \in V(G)$  that is  $M$ -exposed to the base  $r$  of  $B$ . Additionally, let  $E(Q) \cap E(B) = \emptyset$ .  $B + Q$  is called a  $M$ -flower.

**Lemma 1.37.** Let  $G$  be a graph,  $M$  a matching in  $G$ . Suppose there is a  $M$ -flower  $B + Q$ . Let  $G', M'$  result from  $G$  and  $M$  by contracting  $V(B)$  into a single vertex. Then:

$$M \text{ maximum matching in } G \Leftrightarrow M \text{ maximum matching in } G'$$

*Proof.*

" $\Leftarrow$ ": Assume that  $M$  is not maximum in  $G$ .  $N := M \Delta E(Q)$  is a matching with  $|N| = |M|$ .

$\Rightarrow \exists N$ -augmenting path  $P$  in  $G$ . At least one endpoint  $x$  of  $P$  is in  $V(B)$ . If  $P$  and  $B$  are disjoint, let  $y$  be the other endpoint of  $P$ . Otherwise, let  $y$  be the first vertex on  $P$  in  $B$ .  $P' := P_{[x,y]}$  is an  $N'$ -augmenting path in  $G'$ , so  $|N'| = |M'| < \mu(G')$ .

" $\Rightarrow$ ": Assume that  $M'$  is not maximum in  $G'$ , so there exists a matching  $N'$  in  $G'$  with  $|N'| > |M'|$ . Let  $N_0$  arise from  $N'$  in  $G$ , then  $N_0$  contains  $\leq 1$  vertex from  $V(B)$ . Since  $B$  is factor-critical,  $N_0$  can be extended by  $k := \frac{|V(G)|-1}{2}$  edges to a matching  $N$  in  $G$ . We have

$$|N| = |N_0| + k = |N'| + k > |M'| + k = |M|$$

so  $M$  is not maximum.

□

**Lemma 1.39.** *Let  $G$  be a graph,  $M$  a matching in  $G$ .  $X \subseteq V(G)$  is the set of exposed vertices. We can find a shortest  $M$ -alternating  $X$ - $X$ -walk of positive length in  $O(|E(G)|)$  time.*

*Proof.* Define  $D := (V(G), A)$  where:

$$A := \{(u, v) \mid \exists x \in V(G) : \{u, x\} \in E(G), \{x, v\} \in M\}$$

A shortest  $X - \Gamma_G(X)$ -path in  $D$  corresponds to a shortest  $X$ - $X$ -walk in  $G$ . □

**Theorem 1.40.** *Let  $P = v_0, \dots, v_t$  be a shortest  $M$ -alternating  $X$ - $X$ -walk in  $G$ . Then either*

- $P$  is an  $M$ -augmenting path or
- $v_0, \dots, v_j$  is an  $M$ -flower for some  $j \leq t$ .

*Proof.* If  $P$  is not a path, choose  $i < j$  such that  $v_i = v_j$  and  $j$  minimal. Then  $v_0, \dots, v_{j-1}$  are distinct vertices. If  $j - i$  is even, deleting  $v_{i-1}, \dots, v_j$  from  $P$  yields a shorter walk, so  $j - i$  is odd.

Case 1:  $j$  is even. Then  $i$  is odd and therefore  $v_{i+1} = v_{j-1}$  must be the matching mate of  $V_i = v_j$  which contradicts the minimality of  $j$ .

Case 2:  $j$  is odd. Then  $i$  is even, so  $v_0, \dots, v_i$  is an  $M$ -alternating path of even length and  $v_i, \dots, v_j$  is an  $M$ -alternating odd circuit, i.e. a blossom. □

**Theorem 1.41.** *Given a graph  $G$ , a maximum cardinality matching can be found in time  $O(n^2m)$  where  $n := |V(G)|, m := |E(G)|$*

*Proof.* Start with  $M = \emptyset$  and iteratively find  $M$ -augmenting path  $P$ , set  $M := M \Delta E(P)$ . If no such path exists, then  $M$  is maximum.  $P$  can be found in time  $O(mn)^3$ . Since  $M$  is maximum after at most  $\frac{n}{2}$  augmentation, we have total running time  $O(n^2m)$ . □

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<sup>3</sup>Here,  $m$  is the time required for finding a walk and the recursion depth is bounded by  $n$ .

---

**Algorithm 1:** Edmond's Augmenting Path Search

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**Input:** Graph  $G$ , matching  $M$

**Output:** An  $M$ -augmenting path (if one exists)

```
1  $X :=$  set of exposed vertices
2 if  $\exists M$ -alternating  $X$ - $X$ -walk of positive length then
3    $P = v_0, \dots, v_t :=$  a shortest such walk
4   if  $P$  is a path then
5     return  $P$ 
6   else
7     Choose  $j$  as in Theorem 1.40
8      $v_0, \dots, v_j$  is an  $M$ -flower with blossom  $B$ 
9     Recurse on  $G/B$ 
10    Augment an  $M/B$ -augmenting path in  $G/B$  to an
11     $M$ -augmenting path  $P'$  in  $G$ 
12    return  $P'$ 
12 else
13    $\nexists M$ -augmenting path
```

---

### 1.6.1 Growing forest - $O(n^3)$

**Definition 1.42.** Let  $G$  be a graph,  $M$  a matching in  $G$ . An *alternating forest* with respect to  $M$  in  $G$  is a forest  $F$  in  $G$  where:

- $V(F)$  contains all  $M$ -exposed vertices, each tree of  $F$  contains exactly one exposed vertex, its *root*.
- We call  $v \in V(G)$  an outer (inner) vertex if it has even (odd) distance from the root of its component.
- $\forall v \in V(F)$  the unique path from  $v$  to the root of its component is  $M$ -alternating.
- $v \in V(G) \setminus V(F)$  is called *out-of-forest*.

Clearly, inner vertices always have degree 2 (we always assume that there are no matching edges that can immediately be added to  $F$ ).

**Proposition 1.43.** *In any alternating forest, the number of outer vertices that are not the root equals the number of inner vertices.*

*Proof.* For all outer vertices, there exists exactly one inner vertex on its path to the root.  $\square$

**Lemma 1.44.** *Given a graph  $G$ , a matching  $M$ , an alternating forest  $F$  with respect to  $M$  in  $G$ . Then, either  $M$  is a maximum matching or  $\exists$  outer vertex  $x \in V(F)$ , an edge  $\{x, y\} \notin E(F)$  such that one of the following holds:*

- *Grow*:  $y \notin V(F)$  and therefore  $\{y, z\} \in M$  with  $z \notin V(F)$ . In this case,  $y, z$  and  $\{x, y\}, \{y, z\}$  can be added to  $F$ .
- *Augment*:  $y$  is an outer vertex in a different connected component in  $F$ . In this case,  $M$  can be augmented along  $P(x) \cup \{x, y\} \cup P(y)$  where  $P(z)$  denotes the unique path from  $z \in V(F)$  to the root of its connected component.
- *Shrink*:  $y$  is an outer vertex in the same component as  $x$ . Let  $r$  be the first vertex on  $P(x)$  that is also on  $P(y)$ . Then  $|\delta_F(r)| \geq 3$ , so  $y$  is an outer vertex and  $|E(F_{[x,r]})|, |E(F_{[y,r]})|$  are even. Together with  $\{x, y\}$  these paths form a blossom with  $\geq 3$  vertices.

*Proof.* We show that if none of these cases apply,  $M$  is maximum. Let  $X$  be the set of inner vertices,  $s := |X|$  and  $t$  be the number of outer vertices. All outer vertices are isolated in  $G - X$ , so  $G - X$  and  $q_G(X) - |X| = t - s$ . By Berge's formula (1.24),  $t - s$  vertices are exposed by any matching, so  $M$  is maximum.  $\square$

**Definition 1.45.** Let  $G$  be a graph,  $M$  a matching in  $G$ . A subgraph  $F$  of  $G$  is a *general blossom forest* with respect to  $M$  if there exists a partition  $V(F) = V_1 \dot{\cup} \dots \dot{\cup} V_k$  such that  $F_i = F[V_i]$  is a maximal factor-critical subgraph of  $F$  with  $|M \cap E(F_i)| = \frac{|V_i| - 1}{2}$  ( $i \in [k]$ ) and after contracting each  $V_i$ , we obtain an  $M$ -alternating forest  $F'$ .  $F_i$  is called an *outer (inner) blossom* if  $V_i$  is an outer (inner) vertex in  $F'$ .

A *special blossom forest* is a general blossom forest where each inner blossom is a single vertex.

Store a special blossom forest with 3 functions  $\mu, \varphi, \rho : V(G) \rightarrow V(G)$ :

$$\begin{aligned} \mu(x) &:= \begin{cases} x & \text{if } x \text{ is exposed in } M \\ y & \text{if } \{x, y\} \in M \end{cases} \\ \varphi(x) &:= \begin{cases} x & \text{if } x \text{ is the base of an outer blossom or } x \text{ is out-of-forest} \\ y & \text{if } x \text{ is an inner vertex and } \{x, y\} \in E(F) \setminus M \\ y & \text{if } x \text{ is an outer vertex (i.e. in an outer blossom)} \\ & \text{and } \mu, \varphi \text{ are associated with an } M\text{-alternating} \\ & \text{ear decomposition of } x\text{'s blossom, } \{x, y\} \in \\ & E(F) \setminus M \end{cases} \\ \rho(x) &:= \begin{cases} x & \text{if } x \text{ is an inner vertex or out-of-forest} \\ y & \text{if } x \text{ is an outer vertex and } y \text{ is the base of the} \\ & \text{outer blossom containing } x \text{ (} y = x \text{ is possible).} \end{cases} \end{aligned}$$

**Proposition 1.46.** Let  $F$  be a special blossom forest with respect to  $M$  and  $\mu, \varphi, \rho$  as above. Then:

1. For all outer vertices  $x$ ,  $P(x) :=$  maximal path given by subsequence of  $x, \mu(x), \varphi(\mu(x)), \mu(\varphi(\mu(x))), \dots$  is an  $M$ -alternating path from  $x$  to  $q$  where  $q$  is the root of the component containing  $x$ .
2. A vertex  $x$  is
  - an outer vertex  $\Leftrightarrow \mu(x) = x \vee \varphi(\mu(x)) \neq \mu(x)$
  - an inner vertex  $\Leftrightarrow \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x$
  - out-of-forest  $\Leftrightarrow \mu(x) \neq x \wedge \varphi(x) = x \wedge \varphi(\mu(x)) = \mu(x)$

*Proof.*

1. By definition of  $\mu, \varphi$  and lemma 1.33 some initial subsequence of  $P(x)$  ends at the base  $r$  of the blossom containing  $x$ . If  $r = q$ , we are done. Otherwise  $\mu(r), \varphi(\mu(r))$  are next elements in a sequence leading to outer vertex  $\varphi(\mu(r))$ . This can be iterated.
2. Since the conditions are mutually exclusive, it suffices to show one implication for all the statements.
  - If  $x$  is outer, it is a root ( $\mu(x) = x$ ) or  $P(x)$  is a path of length at least 2, so  $\varphi(\mu(x)) \neq \mu(x)$ .
  - If  $x$  is inner, then  $\mu(x)$  is the base of an outer blossom. Therefore  $\varphi(\mu(x)) = \mu(x)$ .  $P(\mu(x))$  is a path of length at least 2, so  $\varphi(x) \neq x$ .
  - If  $x$  is out-of-forest, then  $x$  is covered by  $M$  so  $\mu(x) \neq x$ . By definition of  $\varphi$ ,  $\varphi(x) = x$ .  $\mu(x)$  is out-of-forest as well, so  $\varphi(\mu(x)) = \mu(x)$ .

□

**Lemma 1.47.** *Following invariants hold:*

- a)  $\{\{x, \mu(x)\} \mid x \in V(G), \mu(x) \neq x\}$  is a matching
- b)  $\{\{x, \mu(x)\} \mid \underbrace{x \in V(G), \varphi(\mu(x)) = \mu(x) \wedge \varphi(x) \neq x}_{\text{inner vertices}}\} \cup \{\{x, \varphi(x)\} \mid x \in V(G), \varphi(x) \neq x\}$  forms the edge set of a special blossom forest.
- c)  $\mu, \varphi, \rho$  satisfy the conditions in definition 1.45 (special blossom forest).

*Proof.* a) holds as  $\mu$  only changes in *Augment*. b) is correct after initialization and after the reset in the *Augment* step. It is preserved by *Grow* steps.

In a *Shrink* step,  $r$  (the first vertex that the paths from  $x, y$  to the root share) is a root or has  $|\delta(r)| = 3$  (i.e. it is the base of a blossom), so it is an outer vertex. We define a blossom  $B := \{v \in V(G) \mid \varphi(v) \in$

$V(P(x)_{[x,r]}) \cup V(P(y)_{[y,r]})\}$ . Consider  $\{u, v\} \in F$  with  $u \in B, v \notin B$ . If  $\{u, v\} \in M$ , we have  $u = r, v = \mu(r)$  (since  $F[B]$  contains a near-perfect matching).  $u$  was an outer vertex before shrinking and  $F[B]$  being factor-critical follows from c) and the characterization by ear-decompositions.

For c), it's clear that  $\mu$  always represents a matching.  $\varphi(x) = x$  if  $x$  is not an outer vertex. Therefore,  $\mu + \varphi$  represent an  $M$ -alternating ear decomposition of  $B$ . During *Shrink*,  $\varphi(v)$  is not changed if  $\varphi(v) = r$ . Therefore, the odd ear decomposition for  $B' := \text{blossom containing } r$ , is the correct starting point. The next ear is  $P(x)_{[x,x']} \cup P(y)_{[y,y']} + \{x, y\}$ , where  $x'$  ( $y'$ ) is the first vertex in  $B'$  on  $P(x)_{[x,r]}$  ( $P(y)_{[y,r]}$ ).

For each ear  $Q$  of a former blossom  $B'' \subseteq B$ ,  $Q \setminus (E(P(x)) \cup E(P(y)))$  form a new ear (since it is created by removing an even path).  $\varphi, \mu$  represent this ear-decomposition.  $\square$

**Theorem 1.48.** *Edmond's cardinality matching algorithm correctly determines a maximum matching in  $O(n^3)$  time, where  $n := |V(G)|$ .*

*Proof.* By lemma 1.47 and proposition 1.46, the algorithm maintains a special blossom forest. Let  $M, F$  be the final matching and forest.  $x$  an outer vertex implies that  $\forall y \in \Gamma(x) : y$  is inner and  $\varphi(y) = \varphi(x)$ . Define:

$B := \text{set of inner vertices}$

$B := \text{set of bases of (outer) blossoms}$

Then every unmatched vertex is in  $B$ . Matched vertices in  $B$  have matching mates in  $X$  and  $|B| = |X| + |V(G)| - 2|M|$ . (Outer) blossoms are odd connected components in  $G - X$ , so by Berge's theorem (1.24), at least  $|B| - |X|$  vertices remain uncovered by any matching, so  $M$  is maximum.

We now consider the running time: The status (outer, inner, out-of-forest) for a given vertex can be checked in constant time (proposition 1.46). Therefore, *Grow*, *Augment* and *Shrink* can be implemented in  $O(n)$  time. There are at most  $n$  calls to *Grow* and *Shrink* per augment and at most  $\frac{n}{2}$  *Augments*. This implies the running time  $O(n^3)$ .  $\square$

*Remark 1.49.* The time for *Shrink* can be reduced to  $O(\log n)$  using a binary tree, leading to a running time of  $O(nm \log n)$  in total. Tarjan (1974), Gabow & Tarjan (1983) proved a running time of  $O(nm\alpha(m, n))$  (where  $\alpha$  is the inverse Ackermann function) or  $O(nm)$ .

*Remark 1.50.* It's not necessary to reset everything after augmenting. It suffices to reset the 2 trees that were changed by the augmentation. Gabow & Tarjan (1983) showed that it's possible to augment all paths of the same length in  $O(m)$  time. There are  $2\sqrt{\nu(G)} + 2$  different path lengths, so in total this results in a running time of  $O(\sqrt{nm})$ .

*Remark 1.51* (Skew-symmetric flows). Goldberg & Karzanov (2003) (and Fremuth-Pagen & Jungnickel (2003)) used *Generalized Max-Flow* to achieve a running time of  $O(\sqrt{nm} \frac{\log \frac{m}{n}}{\log n})$ .

## 1.7 Gallai-Edmonds Decomposition

**Proposition 1.52.** *Let  $G$  be a graph,  $X \subseteq V(G)$  with  $|V(G)| - 2\nu(G) = q_G(X) - |X|$ . Then any maximum matching of  $G$*

- *contains a perfect matching in the even components of  $G - X$ .*
- *contains a near-perfect matching in odd components of  $G - X$ .*
- *matches all  $x \in X$  to distinct odd components.*

*Proof.* Follows directly from Berge's theorem (1.24). □

**Theorem 1.53.** *Let  $G$  be a graph and:*

$$Y := \{v \in V(G) \mid \exists \text{ maximum matching that exposes } v\}$$

*Define  $X := \Gamma(Y)$  and  $W := V(G) \setminus (X \cup Y)$ . Then:*

1.  *$X$  attains  $\max_{X' \subseteq V(G)} q_G(X') - |X'|$ .*
2.  *$G[Y]$  is the union of factor-critical subgraphs and  $G[W]$  is the union of even connected components.*
3. *Any maximum matching in  $G$* 
  - *contains a perfect matching in  $G[W]$ .*
  - *contains a near-perfect matching in each component of  $G[Y]$ .*
  - *matches all  $x \in X$  to distinct connected components*

$Y, X, W$  is called Gallai-Edmonds decomposition of  $G$ .

*Proof.* Consider the matching  $M$  and special blossom forest  $F$  at the end of the algorithm. Let  $X'$  ( $Y'$ ) be the set of inner (outer) vertices and  $W'$  the set of out-of-forest vertices.

**Claim.**  $X', Y', W'$  satisfy 1., 2. and 3.

(Proof of theorem 1.48).

Proposition 1.52 implies that any maximum matching covers all vertices in  $V(G) \setminus Y'$ , so  $Y \subseteq Y'$ . For the other inclusion, let  $v \in Y'$ . Then  $M \Delta P(v)$  is a maximum matching exposing  $v$ , so  $v \in Y$  and  $Y' = Y$ . By definition,  $X = X'$  and  $W = W'$ . □

**Corollary 1.54.** *A graph  $G$  has a perfect matching  $\Leftrightarrow \forall U \subseteq V(G)$ ,  $G - U$  has at most  $|U|$  factor-critical components.*



## 1.8 Minimum Weight Perfect Matching

We use the following Integer Programming formulation:

$$\begin{aligned} \min \quad & \sum_{e \in E(G)} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e = 1 \quad v \in V(G) \\ & x_e \in \{0, 1\} \end{aligned}$$

and the corresponding relaxation where we only require  $x_e \geq 0$ . The dual problem of this is:

$$\begin{aligned} \max \quad & \sum_{v \in V(G)} z_v \\ \text{s.t.} \quad & z_v + z_w \leq c_e \quad \{v, w\} \in E(G) \end{aligned}$$

**Proposition 1.55** (Hungarian Method). *Let  $G$  be a graph,  $c \in \mathbb{R}^{E(G)}$  and  $z \in \mathbb{R}^{V(G)}$  with  $z_v + z_w \leq c_e$  for all  $e = \{v, w\} \in E(G)$ . Define:*

$$G_z := (V(G), \{e = \{v, w\} \mid z_v + z_w = c_e\})$$

*Let  $M$  be a matching in  $G_z$ ,  $F$  a maximal alternating forest in  $G_z$  with respect to  $M$ . Let  $X/Y$  be the set of inner/outer vertices. Then:*

1. *If  $M$  is a perfect matching in  $G_z$ , then it is a minimum-weight perfect matching in  $G$ .*
2. *If  $\Gamma_G(y) \subseteq X$  for all  $y \in Y$ , then  $M$  is a maximum matching.*
3. *If neither 1. nor 2. hold, define:*

$$\epsilon := \min \left\{ \min_{e=\{v,w\} \in E(G[Y])} \frac{c_e - z_v - z_w}{2}, \min_{e \in \delta(Y) \cap \delta(V(F))} c_e - z_v - z_w \right\}$$

*Set  $z'_v := z_v - \epsilon$  for all  $v \in X$ ,  $z'_v := z_v + \epsilon$  for all  $v \in Y$  and  $z'_v := z_v$  for all  $v \in V(G) \setminus (X \cup Y)$ . Then  $z'$  is a feasible dual solution and  $M \cup E(F) \subseteq E(G_{z'})$ . Additionally,  $\Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$  for some  $y \in Y$ .*

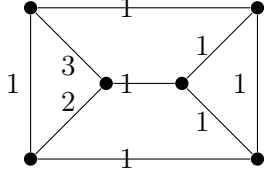
*Proof.* 1. Let  $M'$  be a minimum-weight perfect matching.

$$\begin{aligned} \sum_{e \in M'} c_e &= \sum_{v \in V(G)} z_v + \sum_{e=\{v,w\} \in M'} (c_e - z_v - z_w) \\ &\geq \sum_{v \in V(G)} z_v \\ &= \sum_{v \in V(G)} z_v + \sum_{e=\{v,w\} \in M} (c_e - z_v - z_w) \\ &= \sum_{e \in M} c_e \end{aligned}$$

2. Each outer vertex is an odd blossom (singleton) of  $G - x$ . By Berge (1.24), at least  $|Y| - |X|$  vertices remain uncovered.
3.  $z'$  stays feasible by the choice of  $\epsilon$ . Edges in  $E(F), M$  remain tight. By 1. and 2.,  $\exists y \in Y : \Gamma_{G_{z'}}(y) \setminus X \neq \emptyset$ .

□

*Remark 1.56.* For bipartite graphs, the adjacency matrix is totally unimodular, so the LP has integral vertices.



We define  $\mathcal{A} := \{X \subseteq V(G) \text{ odd}\}$  and add the blossom inequalities

$$\sum_{e \in \delta(X)} x_e \geq 1 \quad \forall X \in \mathcal{A}$$

to the LP relaxation. The new dual problem is then:

$$\begin{aligned} & \max \sum_{A \in \mathcal{A}} z_A \\ & \text{s.t.} \quad \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A \leq c_e \\ & \quad z_A \geq 0 \quad (A \in \mathcal{A}, |A| \geq 3) \end{aligned}$$

Edmond's Algorithm starts with an empty matching  $x = 0$  and dual feasible solution:

$$z_A := \begin{cases} \frac{1}{2} \min\{c(e) \mid e \in \delta(A)\} & |A| = 1 \\ 0 & \text{else} \end{cases}$$

We always ensure that  $z$  is dual feasible and that  $(x, z)$  satisfy complementary slackness:

$$\begin{aligned} x_e > 0 & \Rightarrow \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A = c_e \\ z_A > 0, |A| > 1 & \Rightarrow \sum_{e \in \delta(A)} x_e = 1 \end{aligned}$$

**Definition 1.57.**  $c_z(e) := c(e) - \sum_{A \in \mathcal{A}, e \in \delta(A)} z_A$  is the *reduced cost* of  $e$ .

**Theorem 1.58.** *There are at most  $\frac{7}{2}|V(G)|^2$  of the repeat-until loop in algorithm 4.*

*Proof.*  $\mathcal{B}$  is laminar at any time, i.e. for  $X, Y \in \mathcal{B}$  we have  $(X \subseteq Y) \vee (Y \subseteq X) \vee (X \cap Y = \emptyset)$ . Therefore  $|\mathcal{B}| \leq 2|V(G)|$ .

**Observation.** *Any  $U$  added to  $\mathcal{B}$  during Shrink will not be "unpacked" before the next Augment.*

*Proof.* After *Shrink*, there exists an even length  $M$ -augmenting  $R$ - $U$ -path. It remains in  $G_z$  until the next *Augment* or until  $U$  is included in another blossom  $U' \supseteq U$  which is not resolved before an *Augment* (inductively).  $\square$

Between 2 augments:

- $\# \text{ Unpacks} \leq |\mathcal{B}|$  at beginning of the sequence
- $\# \text{ Shrinks} \leq |\mathcal{B}|$  at the end of the sequence

Therefore, there are at most  $4|V(G)|$  *Unpack* and *Shrink* operations between 2 augments. For each dual change without *Unpack*, we have:  $z_B > 0 \quad \forall B \in \mathcal{B}$ , so  $\epsilon$  is not determined by  $z_B$ . Therefore  $\exists e = \{X, Y\}$  with  $X \notin \mathcal{X}, Y \in \mathcal{Y}$  where  $c_z(e)$  becomes 0.

Case 1:  $X \notin \mathcal{Y}$ . Then  $|V(G_z) \setminus (\mathcal{X} \cup \mathcal{Y})|$  decreases.

Case 2:  $X \in \mathcal{Y}$ . Then  $\exists X$ - $Y$   $M$ -alternating walk in the next iteration.

In particular, such a dual change can occur at most  $|V(G)|$  times between 2 augmentations.

In total, there are at most  $\frac{1}{2}|V(G)|$  *Augment* steps. Therefore, there are  $\frac{1}{2}|V(G)|^2 (4 + |V(G)| + 2|V(G)|)$   $\square$

---

**Algorithm 2:** Minimum-Weight Perfect Matching

---

**Input:** Graph  $G$  with edge weights  $c : E(G) \rightarrow \mathbb{R}$

**Output:** A minimum-weight perfect matching  $M$  in  $(G, c)$

---

**Corollary 1.59.** *A minimum-weight perfect matching can be computed in  $O(n^2m)$  time where  $n := |V(G)|$  and  $m = |E(G)|$ .*

*Proof.* Theorem 1.58 times  $O(m)$ .  $\square$

*Remark 1.60.* To achieve  $O(n^3)$  running time, one can modify the algorithm:

1. Use a General Blossom Forest to avoid recomputing the  $R$ - $R$ -walks from scratch. We then have mappings  $\mu_v, \varphi_v^i, \rho_v^i$  for  $1 \leq i \leq k_v$  where  $k_v$  is the number of blossoms that contain  $v$ .

2. Store all vertices in a heap (ordered by their criticality for dual-feasibility) to speed up the computation of  $\epsilon$ .

Gabow (1990) showed a running time of  $O(n(m+n \log n))$ . Gabow & Tarjan (1991) showed a running time of  $O(m \log(nW) \sqrt{n\alpha(m, n) \log n})$  where  $W := \max_{e \in E(G)} |c(e)|$ .

**Theorem 1.61.** *Let  $G$  be a graph. The set of vectors  $x \in \mathbb{R}^{E(G)}$  satisfying*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(v)) &= 1 & v &\in V(G) \\ x(\delta(A)) &\geq 1 & A &\subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

*is the convex hull of all perfect matchings in  $G$ . It is called the perfect matching polytope.*

*Proof.* For any objective function  $c : E(G) \rightarrow \mathbb{R}$ , the minimum-weight perfect matching algorithm produces an integral primal and a dual solution that satisfy complementary slackness. In particular, all vertices of the polytope are integral.  $\square$

**Theorem 1.62.** *Let  $G$  be a graph. The set of vectors  $x \in \mathbb{R}^{E(G)}$  satisfying*

$$\begin{aligned} x_e &\geq 0 & e &\in E(G) \\ x(\delta(v)) &\leq 1 & v &\in V(G) \\ x(E(G[A])) &\leq \frac{|A| - 1}{2} & A &\subseteq V(G) \text{ with } |A| \text{ odd} \end{aligned}$$

*is the convex hull of all matchings in  $G$ . It is called the matching polytope.*

*Proof.* Any matching solution  $x$  satisfies these conditions. Let  $x$  be any solution that satisfies the conditions. We have to show that  $x$  is a convex combination of matching solutions. Define  $H$  by:

$$\begin{aligned} V(H) &:= \{(v, i) \mid v \in V(G), i \in \{1, 2\}\} \\ E(H) &:= \{ \{(v, i), (w, i)\} \mid \{v, w\} \in E(G), i \in \{1, 2\} \} \\ &\quad \cup \{ \{(v, 1), (v, 2)\} \mid v \in V(G) \} \end{aligned}$$

We set  $y_{\{(v, i), (w, i)\}} := x_{\{v, w\}}$  for all  $\{v, w\} \in E(G), i \in \{1, 2\}$  and  $y_{\{(v, 1), (v, 2)\}} := 1 - x(\delta(v))$  for all  $v \in V(G)$ . Then  $y \geq 0$  and  $y(\delta_H(x)) = 1$  for all  $x \in V(H)$ .

**Claim.**  *$y$  satisfies the inequalities of the perfect matching polytope (in particular the blossom inequalities).*

If this is true, by 1.62  $y$  is a convex combination of perfect matchings.  $H[\{(v, 1) \mid v \in V(G)\}]$  is isomorphic to  $G$ , so  $x$  is a convex combination of matchings in  $G$ .

We now prove the claim: Let  $X \subseteq V(G)$  with  $|X|$  odd. We have to show that  $y(\delta_H(X)) \geq 1$ . Define:

$$\begin{aligned} A &:= \{v \in V(G) \mid (v, 1) \in X, (v, 2) \notin X\} \\ B &:= \{v \in V(G) \mid (v, 1) \in X, (v, 2) \in X\} \\ C &:= \{v \in V(G) \mid (v, 1) \notin X, (v, 2) \in X\} \end{aligned}$$

Define  $A_i := A \cap (V(G) \times \{i\})$  and  $B_i := B \cap (V(G) \times \{i\})$ .  $|B_1 \cup B_2|$  is even, so (since  $|X|$  is odd)  $|A|$  or  $|C|$  is odd. Without loss of generality, let  $|A|$  be odd.

$$\begin{aligned} \sum_{e \in \delta_H(X)} y_e &\geq \sum_{v \in A_1} \underbrace{\sum_{e \in \delta_H(v)} y_e}_{=1} - 2 \cdot \sum_{e \in E(H[A_1])} y_e - \sum_{e \in \delta(A_1) \cap \delta(B_1)} y_e \\ &\quad + \sum_{e \in \delta(A_2) \cap \delta(B_2)} y_e \\ &= |A_1| - 2 \cdot \sum_{e \in E(G[A])} x_e \\ &\geq |A_1| - (|A| - 1) \\ &= 1 \end{aligned}$$

□

**Theorem 1.63.** *The matching polyhedron is TDI (Totally Dual Integral), i.e. for all  $c \in \mathbb{Z}^{E(G)}$  for which the dual program of  $(\max c^t x \text{ s.t. } \dots)$  has a finite optimum solution, it has an integral optimum solution.*

*Proof.* The dual is

$$\begin{aligned} \min \quad & \sum_{v \in V(G)} y_v + \sum_{A \in \mathcal{A}, |A| > 1} \frac{|A| - 1}{2} z_A \\ \text{s.t.} \quad & \sum_{v \in e} y_v + \sum_{A \in \mathcal{A}, |A| > 1, e \in E(G[A])} z_A \geq c(e) \quad e \in E(G) \\ & y, z \geq 0 \end{aligned}$$

Let  $(G, c)$  be a counterexample such that  $|V(G)| + |E(G)| + \sum_{e \in E(G)} |c(e)|$  is minimum. Then:

- $c(e) \geq 1$  for all  $e \in E(G)$ , since otherwise  $e$  could be deleted.
- $G$  has no isolated vertices.

**Claim.** *In an optimum solution  $(y, z)$ ,  $y = 0$ .*

*Proof.* If  $y_v > 0$ , then  $x(\delta(v)) = 1$  for all optimum solutions  $x$ . Decreasing  $c(e)$  by 1 for all  $e \in \delta(v)$  yields a smaller feasible instance  $(G, c')$  where the weight of  $x$  is decreased by 1 and  $x$  remains optimum. By assumption,  $(G, c')$  is not a counterexample, so there exists an integral optimum solution  $(y', z')$ . Increasing  $y'_v$  by one yields some optimum in  $(G, c)$  which has optimum integral solution  $(y' + \mathbb{1}_v, z')$ .  $\square$

Let  $(y = 0, z)$  be a dual optimum solution such that

$$\sum_{A \in \mathcal{A}, |A| > 1} |A|^2 z_A$$

is maximum.

**Claim.**  $\mathcal{F} := \{A : z_A > 0\}$  is laminar.

If not, there exist  $X, Y \in \mathcal{F} : X \cap Y, X \setminus Y, Y \setminus X \neq \emptyset$ . We proceed by "uncrossing". Let  $\epsilon := \{z_X, z_Y\} > 0$ .

Case 1:  $|X \cap Y|$  is odd. Then  $|X \cup Y|$  is odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_Y &:= z_Y - \epsilon \\ z'_{X \cap Y} &:= z_{X \cap Y} + \epsilon & (\text{unless } |X \cap Y| = 1) \\ z'_{X \cup Y} &:= z_{X \cup Y} + \epsilon \\ z'_A &:= z_A & \text{elsewhere} \end{aligned}$$

Then  $(y, z')$  is a dual optimum solution.

Case 2:  $|X \cap Y|$  is even. Then  $|X \setminus Y|$  and  $|Y \setminus X|$  are odd. Define:

$$\begin{aligned} z'_X &:= z_X - \epsilon \\ z'_Y &:= z_Y - \epsilon \\ z'_{X \setminus Y} &:= z_{X \setminus Y} + \epsilon & \text{unless } |X \setminus Y| = 1 \\ z'_{Y \setminus X} &:= z_{Y \setminus X} + \epsilon & \text{unless } |Y \setminus X| = 1 \\ z'_A &:= z_A & \text{elsewhere} \\ y'_v &:= \epsilon & \forall v \in X \cap Y \\ y'_v &:= 0 & \forall v \notin X \cap Y \end{aligned}$$

Then  $(y', z')$  is feasible. The objective value is:

$$\begin{aligned}
& \sum_{v \in V(G)} y'_v + \sum_{A \in \mathcal{A}, |A| > 1} z'_A \frac{|A| - 1}{2} \\
&= \epsilon \cdot |X \cap Y| + \sum_{A \in \mathcal{A}, |A| > 1} \frac{|A| - 1}{2} \\
&+ \epsilon \left( \frac{|X \setminus Y| - 1}{2} + \frac{|Y \setminus X| - 1}{2} - \frac{|X| - 1}{2} - \frac{|Y| - 1}{2} \right) \\
&= \text{objective}(y, z)
\end{aligned}$$

Therefore  $(y', z')$  is an optimum solution with  $y' \neq 0$ , which is a contradiction to the previous claim.

We can conclude that  $\mathcal{F}$  is laminar.

Let  $A \in \mathcal{F}$  with  $z_A \notin \mathbb{Z}$  and  $|A|$  is maximal. Define  $\epsilon := z_A - \lfloor z_A \rfloor > 0$ . Let  $A_1, \dots, A_k$  be the inclusion-wise maximal proper subsets of  $A$  in  $\mathcal{F}$ . Since  $\mathcal{F}$  is laminar,  $A_i \cap A_j = \emptyset$  for  $i \neq j$ . Define:

$$\begin{aligned}
z'_A &:= z_A - \epsilon \\
z'_{A_i} &:= z_A + \epsilon & 1 \leq i \leq k \\
z'_D &:= z_D & \text{elsewhere}
\end{aligned}$$

Then  $(y, z')$  is dual feasible with objective value:

$$\sum_{B \in \mathcal{A}, |B| > 1} \frac{|B| - 1}{2} z'_B < \sum_{B \in \mathcal{A}, |B| > 1} \frac{|B| - 1}{2} z_B$$

This contradicts the optimality of  $(y, z)$ , so there exists no counter example.  $\square$

**Theorem 1.64.** *Let  $G$  be a graph.*

$$P := \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) \leq 1 \quad \forall v \in V(G)\}$$

*is the functional matching polytope.*

$$Q := \{x \in \mathbb{R}_{\geq 0}^{E(G)} \mid x(\delta(v)) = 1 \quad \forall v \in V(G)\}$$

*If  $G$  is bipartite, then  $P$  and  $Q$  are integral.*

*Proof.* The adjacency matrices of bipartite graphs are totally unimodular.  $\square$

**Theorem 1.65.** *Let  $G$  be a graph. The vertices of the fractional perfect matching polytope satisfy*

$$x_e = \begin{cases} \frac{1}{2} & \text{if } e \in E(C_1) \cup \dots \cup E(C_k) \\ 1 & \text{if } e \in M \\ 0 & \text{else} \end{cases}$$

where  $C_1, \dots, C_k$  are vertex-disjoint odd circuits and  $M$  is a perfect matching in  $G - (V(C_1) \cup \dots \cup V(C_k))$ .

*Proof.* Exercise 6.3 □

## 2 $T$ -Joins and $b$ -Matchings

**Definition 2.1.** Let  $G$  be a graph,  $T \subseteq V(G)$ . A subset  $J \subseteq E(G)$  is called  $T$ -join if  $T$  is the set of odd-degree vertices in  $(V(G), J)$ .

**Proposition 2.2.** *Let  $G$  be a graph,  $T, T' \subseteq V(G)$ ,  $J$  a  $T$ -join and  $J'$  a  $T'$ -join. Then  $J \Delta J'$  is a  $T \Delta T'$ -join.*

*Proof.* For  $v \in V(G)$ :

$$\begin{aligned} |\delta_{J \cap J'}(v)| &\equiv |\delta_J(v)| + |\delta_{J'}(v)| \\ &\equiv |\{v\} \cap T| + |\{v\} \cap T'| \\ &\equiv |\{v\} \cap (T \Delta T')| \pmod{2} \end{aligned}$$

□

**Proposition 2.3.** *Let  $G$  be a graph,  $T \subseteq V(G)$ .*

$$\exists \text{ } T\text{-join in } G \Leftrightarrow |V(C) \cap T| \text{ for each connected component } C$$

*Proof.*

" $\Rightarrow$ ": Let  $J$  be a  $T$ -join. For each connected component  $C$ :

$$\sum_{v \in V(C)} |J \cap \delta(v)| = 2 |J \cap E(C)|$$

Therefore  $|J \cap \delta(v)|$  is odd for an even number of vertices and  $|V(C) \cap T|$  is even.

" $\Leftarrow$ ": Partition  $T$  into pairs  $\{v_1, w_1\}, \dots, \{v_k, w_k\}$  such that  $v_i$  and  $w_i$  are in the same component for all  $i$ . Let  $P_i$  be a  $v_i$ - $w_i$ -path in  $G$ . Define  $J := E(P_1) \Delta E(P_2) \Delta \dots \Delta E(P_k)$ . By proposition 2.2, this is a  $T$ -join.

□



**Theorem 2.4.** *Let  $G$  be a graph,  $c : E(G) \rightarrow \mathbb{R}$  and  $T \subseteq V(G)$ . In strongly polynomial time (e.g.  $O(n^2m)$ ) we can determine if a  $T$ -join exists and if so, compute a minimum-weight  $T$ -join.*

*Proof.* In  $O(m)$  ( $m := |E(G)|$ ), we can check if a  $T$ -join exists. If so:

1. Eliminate negative weights.

$$\begin{aligned} N &:= \{e \in E(G) \mid c(e) < 0\} \\ U &:= \{v \in V(G) \mid |\delta_N(v)| \text{ odd}\} \\ T' &:= T \Delta U \\ c'(e) &:= |c(e)| \quad e \in E(G) \end{aligned}$$

**Claim.** *If  $J'$  is a minimum  $T'$ -join with respect to  $c'$ , then  $J' \Delta N$  is a minimum  $T$ -join with respect to  $c$ .*

Let  $\tilde{J}$  be a  $T$ -join. Then  $\tilde{J} \Delta N$  is a  $T'$ -join, so  $c'(\tilde{J}) \leq c'(\tilde{J} \Delta N)$  and

$$c(J) = c'(J') + c(N) \leq c'(\tilde{J} \Delta N) + c(N) = c(\tilde{J})$$

which proves the claim.

2. We can now assume that  $c \geq 0$ . A minimum-weight  $T$ -join does not have cycles of positive weight. We can eliminate cycles of weight 0 without changing the cost. We can then restrict ourselves to searching for collections of  $T$ - $T$ -paths.

Let  $K_T$  be the metric closure of  $T$  with respect to  $G$ . It can be computed in  $O(n \cdot (m + n \log n))$  by using Dijkstra for all vertices. Find a minimum-weight perfect matching  $M$  in  $K_T$ . Each  $e = \{s, t\} \in M$  induces a path  $P_{s,t}$ . Then the symmetric difference  $\Delta_{\{s,t\} \in M} E(P_{s,t})$  is a minimum-weight  $T$ -join in  $G$ .

□

**Corollary 2.6.** *A maximum-weight  $T$ -join can be computed as fast as a minimum-weight  $T$ -join.*

*Proof.* Set  $c' := -c$ .

□

**Corollary 2.7.** *Let  $G$  be a graph,  $c : E(G) \rightarrow \mathbb{R}$ . We can find a cycle of negative length in  $G$  in  $O(n^2m)$  time.*

*Proof.* Apply theorem 2.4 to  $T = \emptyset$ . If  $c(J) < 0$ ,  $(V(G), J)$  contains a cycle  $C$ . If  $c(C) = 0$ , we can eliminate it and recurse, otherwise return  $C$ .

□