

# Combinatorial Optimization

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# 1 Organization

## 1.1 Prerequisites

- Basic knowledge of graph algorithms
- Linear Programming (LP Duality)
- Programming skills in C++

## 1.2 Exam

- Qualification requires 50% of the points in theoretical & programming exercises
- Oral exam

## 1.3 Books

- "Combinatorial Optimization", Korte & Vygen
- "Understanding & Using Linear Programming", B. Gärtner, J. Matousek
- Skript (theorems & definitions)
- Further on website

# 2 Matchings

## 2.1 Introduction

**Definition 1.**

1. A *matching*  $M$  in a graph  $G = (V, E)$  is a set of pairwise disjoint edges, i.e. they don't have a common endpoint.  
 $\nu(G) := \max.$  cardinality of a matching in  $G$
2. An *edge cover*  $C$  of a graph  $G = (V, E)$  is a subset of  $E$  s.t.  $C = \bigcup_{e \in C} e$ .  
 $\zeta(G) := \min.$  cardinality of an edge cover in  $G$
3. A matching is called *perfect* (or *1-factor*) if it is an edge cover
4.  $v \in V$  with  $v \in e \in M$  is called *M-covered*
5.  $v \in V$  is called *M-exposed* if it is not *M-covered*

**Definition 2.**

1. A *stable set* (independent set)  $S$  is a set of pairwise non-adjacent vertices.

$$\alpha(G) := \text{max. cardinality of a stable set}$$

2. A *vertex cover*  $C$  is a subset of  $V$  s.t.  $E = \bigcup_{\{x,y\} \in E, x \in C} \{x,y\}$

$$\tau(G) := \text{min. cardinality of a vertex cover}$$

**Lemma 3.**

1.  $\alpha(G) + \tau(G) = |V|$
2.  $\nu(G) + \zeta(G) = |V|$  if  $G$  has no isolated vertices
3.  $\zeta(G) = \alpha(G)$  if  $G$  is bipartite and has no isolated vertices

**Problem.** Cardinalty Matching Problem

Input: Graph  $G = (V, E)$

Task: Find a maximum cardinality matching

**Problem.** Maximum Weight Matching Problem (MWMP)

Input: Graph  $G, c : E \rightarrow \mathbb{R}$

Task: Find a matching  $M$  maximizing  $c(M)$

**Problem.** Minimum Weight Perfect Matching (MWMPMP)

Input: Graph  $G, c : E \rightarrow \mathbb{R}$

Task: Find a perfect matching of minimum weight or decide that no perfect matching exists in  $G$

**Lemma 4.** *The MWMP is equivalent to the MWMPMP (i.e. there exists a transformation with linear complexity)*

*Proof.* Given a MWMPMP instance  $(G, c)$ , define  $c' := K - c$  ( $K := 1 + \sum_{e \in E} |c(e)|$ ).

$\Rightarrow$  Any maximum weight matching is a maximum cardinality matching

Given a MVMP instance  $(G, c)$ , define  $G'$  as 2 copies of  $G$  where the 2 copies of a vertex are joined by an edge.

$\Rightarrow G'$  has a perfect matching. Define:

$$c'(e) := \begin{cases} -c(e) & \text{if } e \text{ is in the first copy} \\ 0 & \text{else} \end{cases}$$

A minimum weight perfect matching in  $G'$  gives us a maximum weight matching in  $G$ .  $\square$

**Definition 5.** Let  $G = (V, E)$  be a graph and  $M \subseteq E$  a matching in  $G$ . A path  $P$  is *M-alternating* if its edges are alternatingly in and not in  $M$ . If both end points of this path are *M-exposed*,  $P$  is an *M-augmenting* path.

**Lemma 6.** *Given a matching  $M$  in  $G$  and an inclusion-wise maximal  $M$ -alternating path  $P$ ,*

$$M \Delta P := M \setminus P \cup P \setminus M$$

*is a matching. If  $P$  is  $M$ -augmenting, then  $|M \Delta P| = |M| + 1$ .*

**Theorem 7** (Petersen 1891, Berge 1957). *Augmenting Path Theorem Given a graph  $G = (V, E)$  and a matching  $M$  in  $G$ :*

$$|M| = \nu(G) \Leftrightarrow \nexists M\text{-augmenting path } P \text{ in } G$$

*Proof.*

" $\Rightarrow$ ": Clear

" $\Leftarrow$ ": Assume there exists a matching  $M'$  with  $|M'| > |M|$ . Let  $G' := (V, M \Delta M')$ .

$$\Rightarrow |\delta_{G'}(v)| \leq 2 \quad \forall v \in V$$

$\Rightarrow G'$  is the union of disjoint circuits and paths

$\Rightarrow$  all circuits are even and have the same number of edges from  $M$  and  $M'$

$\Rightarrow \exists$  a path  $P$  in  $G'$  starting and ending with an edge in  $M'$

$\Rightarrow P$  is an alternating path

□

## 2.2 Bipartite Matching

**Theorem 8** (König 1931). *If  $G$  is bipartite, then  $\nu(G) = \tau(G)$*

*Proof.* Add vertices  $s$  and  $t$  edges between them to all vertices of the respective partition. Direct all edges from  $s$  to  $t$ . Then  $\nu(G)$  is maximum number of disjoint  $s$ - $t$ -paths. Menger  $\Rightarrow$  This is equal to the minimum number of vertices that disconnect  $t$  from  $s$ . □

**Theorem 9** (Hall 1935). *Let  $G = (A \dot{\cup} B, E)$  be a bipartite graph. Then:*

$$G \text{ has a matching covering } A \Leftrightarrow |\Gamma(X)| \geq |X| \quad \forall X \subseteq A$$

**Corollary 10.** *Marriage Theorem*

$$|\Gamma(X)| \geq |X| \quad \forall X \subseteq A \text{ and } |A| = |B| \Leftrightarrow G \text{ has a perfect matching}$$

**Definition 11.** The MWPM for bipartite graphs is called *Assignment Problem*.

**Theorem 12.** *The Assignment Problem can be solved in time  $O(nm + n^2 \log m)$ .*

*Proof.* Use the Successive Shortest Paths algorithm in an auxiliary graph. □

## 2.3 The Tutte Matrix & Randomized Matching

**Definition 13.** Let  $G$  be a simple, undirected graph. Let  $G'$  be an orientation of  $G$  and  $(X_e)_{e \in E(G)}$ . The *Tutte matrix* is defined as

$$T_G(X) := (T_{vw}^*)_{v,w \in V(G)}$$

where

$$t_{vw}^* := \begin{cases} X_{\{v,w\}} & \text{if } (v,w) \in E(G) \\ -X_{\{v,w\}} & \text{if } -(v,w) \in E(G) \\ 0 & \text{else} \end{cases}$$

*Remark 14.*  $T_G(X)$  is skew-symmetric (i.e.  $T_G(X) = -(T_G(X))^t$ ).  $\text{rank}(T_G(X))$  is independent of the orientation of  $G$ .  $\det(T_G(X))$  is a polynomial in  $X$ .

**Theorem 15** (Tutte). *A simple graph  $G$  has a perfect matching  $\Leftrightarrow \det(T_G(X)) \neq 0$*

*Proof.* Let  $V(G) = \{v_1, \dots, v_n\}$  and  $S_n$  be the permutation group.

$$\det T_G(X) = \sum_{\pi \in S_n} \text{sgn} \pi \cdot \prod_{i=1}^n t_{v_i, v_{\pi(i)}}^*$$

Let  $S'_n := \{\pi \in S_n \mid \prod_{i=1}^n t_{v_i, v_{\pi(i)}}^* \neq 0\}$ . Each  $\pi \in S_n$  corresponds to a digraph  $H_\pi := (V(G), \{(v_i, v_{\pi(i)}) \mid i \in [n]^1\})$ . We have  $|\delta^+(v)| = 1 = |\delta^-(v)| \forall v \in V(H_\pi) \Rightarrow H_\pi$  is the union of disjoint circuits. If  $\pi \in S'_n$ , then  $H_\pi \subset \overset{\Leftrightarrow}{G'}$ .

If there exists  $\pi \in S'_n$  s.t.  $H_\pi$  is a collection of even circuits, then this immediately yields a perfect matching in  $G$  (using every second edge of each circuit).

Otherwise,  $\forall \pi \in S'_n$ ,  $H_\pi$  contains an odd circuit. Let  $r(\pi) \in S'_n$  arise from  $\pi$  by reversing edges on the unique odd circuit containing a vertex with minimum index  $\Rightarrow r(r(\pi)) = \pi$  and  $\text{sgn}(\pi) = \text{sgn}(r(\pi))$ . The second part is true since for reversing an odd cycle, we need an even number of swaps. Let  $v_{i_1}, \dots, v_{i_{2k+1}}$  be the "first" odd circuit. Then  $r(\pi)$  is attained by  $2k$  swaps: For  $j = 1, \dots, k$  swap  $(\pi(i_{2j-1}), \pi(i_{2k}))$  and  $(\pi(i_{2j}), \pi(i_{2k+1}))$ .

$\prod_{i=1}^n t_{v_i, v_{\pi(i)}}^* = -\prod_{i=1}^n t_{v_i, v_{r(\pi(i))}}^*$  since there is an odd number of sign changes to  $t^*$ .  $\Rightarrow \det(T_G(X)) = 0$ . We have shown that if  $G$  has no perfect matching, then  $\det T_G(X) = 0$ .

Assume that  $G$  has a perfect matching  $M$ . Define  $\pi$  as  $\pi(i) = j, \pi(j) = i$  where  $\{i, j\} \in M$ .  $\Rightarrow \prod_{i=1}^n t_{v_i, v_{\pi(i)}}^* = \prod_{e \in M} -X_e^2$  cannot be canceled out. In particular,  $\det T_G(X) \neq 0$ .  $\square$

*Remark 16.* Picking  $X' \in [0, 1]^{E(G)}$  at random, we almost surely have (since the zero set of a non-zero polynomial is a set of measure zero):

$$\det T_G(X') \neq 0 \Leftrightarrow G \text{ has a perfect matching}$$

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<sup>1</sup>This is an abbreviation for  $\{1, \dots, n\}$ .

**Theorem 17** (Lovász 1979). *Let  $G$  be a simple graph and  $X \in [0, 1]^{E(G)}$  chosen randomly. Then almost surely  $\text{rank}(T_G(X)) = 2\nu(G)$ .*

## 2.4 Tutte's Matching Theorem

Let  $X \subseteq V(G)$ .  $G - X$  consists of even and odd (in terms of the number of vertices) connected components. We define  $q_G(X)$  to be the number of odd components in  $G - X$ .

**Definition 18.** A graph  $G$  satisfies the *Tutte Condition* if  $q_G(X) \leq |X|$  for all  $X \subseteq V(G)$ .  $\emptyset \neq X \subseteq V(G)$  is called *barrier* if  $q_G(X) = |X|$ .

**Lemma 19.** *For any graph  $G$  and any  $X \subseteq V(G)$ :*

$$q_G(X) - |X| \equiv |V(G)| \pmod{2}$$

**Definition 20.** A graph  $G$  is *factor-critical* if  $G - v$  has a perfect matching for all  $v \in V(G)$ . A matching is called *near-perfect* if it covers  $|V(G)| - 1$  vertices.

**Lemma 21.** *If  $G$  is factor-critical, then it is connected.*

**Theorem 22** (Tutte 1947). *A graph  $G$  has a perfect matching  $\Leftrightarrow$  Tutte Condition holds (i.e.  $q_G(X) \leq |X| \ \forall X \subseteq V(G)$ )*

*Proof.*

" $\Rightarrow$ ": Clear

" $\Leftarrow$ ": We proceed by induction on  $|V(G)|$ . The case  $|V(G)| = 2$  is clear.

Generally, if the Tutte Condition holds, then  $|V(G)|$  must be even (pick  $X = \emptyset$ ). Proposition 19  $\Rightarrow$   $q_G(X) - |X|$  is even. Every  $x \in V(G)$  induces a barrier  $\{x\}$ . Let  $X$  be a maximum barrier. Then  $G - X$  doesn't have any even components (since otherwise a single vertex of such a component could be added to  $X$ ).

**Claim:** Each odd component is factor-critical.

Let  $C$  be an odd component in  $G - X$ ,  $v \in V(C)$ . Assume that  $C - v$  does not have a perfect matching. Induction Hypothesis  $\Rightarrow$   $C - v$  violates Tutte Condition.

$$\Rightarrow \exists Y \subseteq V(C - v) : q_{C-v}(Y) > |Y|$$

$$\stackrel{19}{\Rightarrow} q_{C-v}(Y) \geq |Y| + 2$$

Observe  $X \cap \{v\} = Y \cap \{v\} = X \cap Y = \emptyset$ :

$$\begin{aligned} q_G(X \cup Y \cup \{v\}) &= q_G(X) - 1 + q_C(Y \cup \{v\}) \\ &= |X| - 1 + q_{C-v}(Y) \\ &\geq |X| - 1 + |Y| + 2 \\ &= |X \cup Y| + 1 \\ &= |X \cup Y \cup \{v\}| \end{aligned}$$

$\Rightarrow X \cup Y \cup \{v\}$  is a barrier

$\Rightarrow$  Claim

Let  $G'$  arise from  $G$  by contracting each odd component into a single vertex. We have  $V(G') = X \dot{\cup} Z$  and  $G'$  is bipartite. We have to show that  $G'$  has a perfect matching. If not, then  $\exists A \subseteq Z : |\Gamma_{G'}(A)| < |A|$   
 $\Rightarrow q_G(\Gamma_{G'}(A)) \geq |A| > |\Gamma_{G'}(A)|$  which contradicts the Tutte Condition.

□

**Theorem 23** (Berge 1958).

$$|V(G)| = 2\nu(G) + \max_{X \subseteq V(G)} (q_G(X) - |X|)$$