A Mode-sum Representation of the Singular Green Function and its Derivatives via the Hadamard Parametrix

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Declaration of Authorship

I hereby declare that I am the sole author of this project and that all work presented is my own, with the exception of references made to the work of other authors. This work has not been submitted as an exercise for a degree to any other university or college. I give Technological University Dublin authority to lend this project report to other institutes or individuals for research purposes.

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Abstract

So-called ultraviolet divergences create significant difficulties in the study of quantum fields in curved spacetimes, as quantities become infinitely large when two points of a manifold are brought closer together in their 'coincidence limit'. In this thesis, we follow closely the work of Taylor and Breen in their mode-sum representation of the singular Green propagator - a term that captures all of these problematic infinities, by assuming the form of the Hadamard parametrix. A litany of methods are applied to divergent terms to regularize them appropriately, beginning with an expansion in "extended coordinates", calling regularly upon Legendre functions for their physical applications and reducing divergences at the coincidence limit to a mathematically useful sum of modes, capable then of being subtracted mode-by-mode to permit the calculation of expectation values. We then extend upon the previous work and calculate some of the derivatives required to eventually calculate the renormalised stress-energy tensor.

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Chapter 1

Introduction

In 1905 Albert Einstein published the Special Theory of Relativity, changing our perception of time and space. Rather than absolute, separate entities that retain their characteristic scales and properties in all frames of reference, in fact they are together integrated into a 4-dimensional manifold called *spacetime* that properly accounts for the paradoxes he discovered in his insightful '*gedankenexperiments*', or thought experiments. Our ontological reality was thus transformed and now depended on the observers frame of reference. Einstein based his observations and work on two postulates:

1. The laws of physics are invariant in all inertial frames.

2. The speed of light in a vacuum is a constant for all observers.

Over the next 10 years, Einstein built upon the Special Theory and worked intensively on the General Theory of Relativity (GR). In 1915 he published the field equations that described with remarkable insight how the gravitational field was actually a manifestation of spacetime. The interaction between matter and spacetime curvature is today often simplified surprisingly aptly as "*spacetime tells matter how to move; matter tells space-time how to curve*". In their tensor form, the equations are written generally as

$$G_{ab} - \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \tag{1.1}$$

where G_{ab} is known as the Einstein tensor, Λ is the cosmological constant and T_{ab} is the Stress-Energy tensor - an object that is of much significance in general relativity and quantum field theory. It encapsulates, in tensor form, the density and flux of energy and momentum in spacetime. We will describe T_{ab} in more detail in section 1.7 below. G is Newton's gravitational constant, with a value of 6.67408×10^{-11} m^3 kg^{-1} s^{-2}.

1.1 Spacetime Curvature

The Minkowski spacetime is a Euclidean 3-dimensional space integrated with a fourth dimension of time, known as a flat spacetime. Flat spacetime is the backdrop of the framework within which special relativity is constructed and valid. Special relativity is in fact a special case of the more general framework of general relativity. A point in spacetime is generally called an *event*, with coordinates of (ct, x, y, z) in Minkowski space. The Greek superscript indices that are conventional run over the number of dimensions, starting at 0, and denoted

$$x^{\mu}: \quad \begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}$$

for 4-dimensional Minkowski space. In this work, we will instead use Latin superscripts a, b, c... in keeping with the cited literature. We also set $c = G = 1$, making our equations that bit more convenient. This is equivalent to working in units where 1 second equals 3 x 10^8 m. The line element ds in flat spacetime is given by

$$ds^2 = \eta_{ab} dx^a dx^b$$

where $\eta_{\mu\nu}$ is the Minkowski *metric* - a 4x4 matrix unique to flat spacetimes given by

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The generalised metric g_{ab} gives the line element as

$$ds^2 = g_{ab} dx^a dx^b$$

and curvature depends on the metric. The generalised derivative of the coordinate space is

$$\nabla_a X^b = \partial_a X^b + \Gamma^b_{ac} X^c \qquad , \qquad \nabla_a X_b = \partial_a X_b - \Gamma^c_{ab} X_c$$

where X^b is a vector field (for a scalar field $\phi(x)$, $\nabla_a \phi(x) = \partial_a \phi(x)$) and Γ^b_{ac} is the Christoffel symbol - a non-tensor object that encodes a *connection* between the vectors of tangent spaces for nearby points. It is constructed from the metric and is given by

$$\Gamma^b_{ac} = \tfrac{1}{2} g^{bd}(\partial_a g_{cd} + \partial_c g_{da} - \partial_d g_{ac}).$$

The d'Alembertian operator $\Box$ is the product of two covariant derivative operators, and is defined for general curvature as

$$\Box = g^{ab} \nabla_a \nabla_b.$$

or equivalently, and sometimes more usefully, when working with a scalar field as

$$\Box\phi(x) = \tfrac{1}{\sqrt{g}} \partial_a (\sqrt{g} \partial^a (\phi(x)).$$

This work will extend methods by Taylor and Breen (2017) that considered a spherically symmetric black hole spacetime given by the line element

$$ds^2 = g_{ab} dx^a dx^b = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{1.2}$$

where the metric and its inverse are given respectively by

$$g_{ab} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \quad , \quad g^{ab} = \begin{pmatrix} -\frac{1}{f(r)} & 0 & 0 & 0 \\ 0 & f(r) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}.$$

Curvature is primarily described in terms of Riemannian geometry, where we have the Riemann tensor obtained from the connection as

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}.$$

The Riemann tensor is a property of the intrinsic geometry of a spacetime and is required to calculate the Einstein tensor G_{ab}, a truly fundamental object in general relativity, given by

$$G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab}.$$

Here, R is known as the *Ricci scalar*

$$R = R^a{}_a = g^{ab} R_{ab}$$

and R_{ab} is the symmetric *Ricci tensor*, which is a contraction of the Riemann tensor, given by

$$R_{ab} = R^c{}_{acd}.$$

Finally, a (very) brief outline on general relativity would not be complete without discussing *geodesics*. The definition of a geodesic is a curve along which the tangent vector is parallel transported [1], where the tangent vector to a path parameterised by λ is given as $dx^a(\lambda)/d\lambda$. The condition of parallel transport is given by

$$\frac{d^2 x^a(\lambda)}{d\lambda^2} + \Gamma^a{}_{bc} \frac{dx^b(\lambda)}{d\lambda} \frac{dx^c(\lambda)}{d\lambda} = 0\,.$$

This equation helps us in general relativity redefine the concept of a 'straight line', which of course should depend on curvature, and hence on the connection and metric. A geodesic is very useful in that it describes the path followed by unaccelerated test particles in a spacetime. We say imaginary, or test, particles, as we require that they have negligible mass - paying due attention to the fact that mass affects curvature. Light follows geodesic paths, paths that can be demonstrably curved due to the presence of sufficient mass. The first empirical evidence supporting this prediction of relativity was in 1919 when British astronomers Eddington and Dyson measured the deflection of starlight by the sun during a solar eclipse. The empirical evidence since has only supported the predictions of general relativity, notably including the LIGO (Laser Interferometer Gravitational-Wave Observatory) experiment's first detection of gravitational waves in 2016.

Einstein's equation for general relativity encapsulates all of the information above into a surprisingly simple expression:

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} \;=\; 8\pi G T_{ab}. \tag{1.3}$$

The curvature of spacetime is thus a fundamental concept in cosmology, as it presents tangible clues to the beginnings, evolution and end-state of the universe. It is the best description we currently have about gravity, but its assimilation into the world of quantum field theory presents many real, conceptual and mathematical problems.

We will now describe (briefly) the key concepts of quantum field theory (QFT), focussing on semiclassical QFT, limiting the scope of the discussion to relevance to the work ahead.

1.2 Quantum Field Theory in Curved Spacetimes

The fundamental premise of quantum field theory is that all particles and waves within the universe are excitations of a quantum field that permeate all space and time. Three of the four fundamental forces (electromagnetism, strong nuclear and weak nuclear) have been successfully quantised and accounted for in QFT. The Standard Model of particle physics integrates these three forces into useful models, using the techniques and concepts of QFT, to describe particle interactions and account for all known elementary particles to date. While there are gaps in the theory, it is self-consistent and has produced the models that best explain phenomena witnessed in a wide-ranging set of observations, from stellar physics to particle colliders in CERN or FermiLab.

However a complete quantum theory-one that permits the quantisation of gravity-has so far not been possible. 'Quantum Field Theory in Curved Spacetimes' (QFTCS) essentially takes the quantum theory, already defined on a flat, Minkowski spacetime η_{ab}, and generalises it to arbitrarily curved spacetimes g_{ab}. This permits the development of an effective field theory [1] that merges the basic principles of classical general relativity with those of QFT, applicable within appropriate constraints as defined. The metric g_{ab} is treated classically with the explicit goal of uncovering useful and interesting physics, without having to first solve the problem of how to quantise gravity. It is likely that general relativity is not a complete description of gravity, as it does not merge well with a world that is fundamentally quantum in nature, despite its success at the correct scales-larger than the Planck length, but smaller than the typical scale of quantum electrodynamics (i.e. between 10^{-33} cm and 10^{-11} cm). As we move forward with QFTCS, we will park the discussion regarding the fundamental nature of gravity and rest assured that modern evidence, empirical and theoretical, in cosmology certainly indicates that its study is a fruitful endeavour.

Two notable modern discoveries should help reinforce the motivation for studying quantum fields in curved spacetimes. One is the recent discovery that the expansion of the universe is accelerating. The discovery earned Perlmutter, Schmidt and Reiss the 2011 Nobel Prize in physics, but more importantly it is readily interpreted as a universe with a non-zero vacuum energy (discussed below). Heretofore, it had been expected that all of the matter in the universe, as was expected from general relativity, should decelerate the expansion owing to gravity. It is not currently known what the value of the vacuum energy is, but it is thus clear that the study of matter (specifically, the quantum fields that they are excitations of) in curved spacetimes is a good candidate for a potential explanation of the mystery. The second notable discovery was that of S.W. Hawking when it was shown, using QFTCS, that black holes can emit particles formed by quantum field fluctuations close to its event horizon. This ultimately creates a situation in which a black hole loses a small quantity of mass with each emission, and hence 'evaporates' over time. While this finding leads to the unresolved 'black hole information paradox' [2], it is one that was not possible

[1] An effective field theory is one that can describe physical observations and make predictions only within certain assumptions, conditions or boundaries.

[2] There have been laudable recent efforts to resolve this paradox, and whether or not it remains a paradox is the matter of some debate.

in a fully classical field theory [2].

1.3 Definitions

QFT quantises classical fields (as one might expect) where operator valued functions of space and time are the basic degrees of freedom. Therefore, there is essentially an infinite number of degrees of freedom-an undesired infinity in QFT that must be dealt with.

The vacuum state, or quantum vacuum state, is one of an infinite number of possible quantum states. It is the one with the lowest possible energy at any point in spacetime. A quantum state itself is a mathematical representation of the distribution of probabilities for the possible outcomes of an experiment or observation. We define the vacuum state $|0\rangle$ (in a flat spacetime) as

$$a_{\mathbf{p}} |0\rangle = 0$$

where $a_{\mathbf{p}}$ is an annihilation operator with three-momentum $\mathbf{p}$. The vacuum state is normalised by $\langle 0|0\rangle = 1$. If a creation operator acts on the vacuum state we get

$$a^\dagger_{\mathbf{p}} |0\rangle = |\mathbf{p}\rangle$$

which is interpreted as the momentum eigenstate of a single particle. Creation and annihilation operators have the commutation relations

$$[a_{\mathbf{p}_i}, a_{\mathbf{p}_j}] = 0 \tag{1.4}$$

$$[a^\dagger_{\mathbf{p}_i}, a^\dagger_{\mathbf{p_j}}] = 0 \tag{1.5}$$

$$[a_{\mathbf{p}_i}, a^\dagger_{\mathbf{p}_j}] = \delta_{ij} \tag{1.6}$$

for all values of i and j. Single particle states satisfy

$$\langle \mathbf{p}|\mathbf{q}\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{1.7}$$

where $\delta^{(3)}(\mathbf{p}-\mathbf{q})$ is the Dirac delta function in three dimensional momentum space. This function is ubiquitous in QFT and defined as

$$\delta(p) = \begin{cases} \infty & \text{if p = 0} \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

which therefore requires normalisation, as introducing infinity is something we generally seek to avoid. The Dirac Delta function is thus normalised as

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx \equiv f(a).$$

Fourier transforms give us a convenient representation in momentum space as

$$\delta^{(3)}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\, e^{i\,\mathbf{p}\cdot\mathbf{x}}.$$

In quantum mechanics, the *expectation value* is the probabilistic expected outcome of a measurement of some quantum state. For example, given a quantum state ψ, the expectation value of observable A being in the state ψ is given as

$$\langle\psi|\, A\, |\psi\rangle = \langle A\rangle\,.$$

The vacuum expectation value (VEV) of a scalar field, ϕ, is then given as

$$\langle 0|\, \phi\, |0\rangle = \langle\phi\rangle\,.$$

However, if we try to calculate VEV of a state of annihilation and creation operators using Eq. 1.7 we obtain

$$\langle 0|\, a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}\, |0\rangle = \langle\mathbf{p}|\mathbf{p}\rangle = (2\pi)^3\delta^3(0) = \infty$$

which clearly has no physical meaning. In QFT, these objects are operator valued distributions, rather than functions. The way we interpret the same exercise for a scalar field, say, where we expect that $\langle 0|\,\phi(\mathbf{x})\,|0\rangle = 0$ but that

$$\langle 0|\,\phi(\mathbf{x})\phi(\mathbf{x})\,|0\rangle \;=\; \langle \mathbf{x}|\mathbf{x}\rangle \;=\; \delta^3(0) = \infty\,,$$

is by recognising that the infinity is attributed to *density fluctuations* at that point in space. Using another Fourier transform, we can localise a wavepacket in space using its momentum representation by

$$|\phi(\mathbf{x})\rangle \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-i\,\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p})\,|\mathbf{p}\rangle\,.$$

1.4 Quantum Fields

As this work will concentrate on the scalar quantum field, only a brief explanation of some of the fields in QFT is required, and only for context and further discussion later. We have previously mentioned that all particles and waves are excitations of quantum fields, but let us qualify that briefly.

Why do we need quantum fields in the first place? Despite its enormous success at predicting the outcomes of experiments and observations, the formulation of quantum mechanics assumes that the observer has access to all possible states and measurements of a system. However, special relativity expressly forbids this. Information about a system can only be transmitted at a finite speed, immediately reducing the knowledge available to the observer. Operator-valued fields, rather than locally defined observables, that obey the rules of special relativity are the currency of QFT and how we speak about quantum fields. An arbitrary operator-valued field O in such a scheme must not permit influence faster than the speed of light. This is expressed at the commutation relation for two different spacetimes $\mathbf{x}$ and $\mathbf{y}$ that are space-like separated as

$$[O(\mathbf{x}), O(\mathbf{y})] \;=\; 0\,.$$

QFT has been used successfully to describe the particles, interactions and dynamics of each of the quantum fields within the Standard Model. The only fundamental scalar quantum field that has been actually observed is the Higgs field [3]. The sheer usefulness of the scalar field in an effective field theory encourages us to pursue work on it in curved spacetimes and as a precursor to fields of higher-spin.

1.5 Quantisation of the Scalar Field in Curved Spacetime

The canonical prescription for quantising a field commences with identifying the classical Lagrangian density. For the scalar field $\phi(t, \mathbf{x})$ in curved spacetime, this is given as

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\right)$$

where ξ is a dimensionless constant known as the coupling constant. In the literature, there are two usual choices for ξ: $\xi = 0$ for minimal coupling and $\xi = \frac{1}{6}$ for conformal coupling. We also impose commutation relations on the scalar field as

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{x}^{'})] = 0 \tag{1.9}$$

$$[\pi(t, \mathbf{x}), \pi(t, \mathbf{x}^{'})] = 0 \tag{1.10}$$

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}^{'}))] = \frac{i}{\sqrt{-g}}\delta^{d-1}(\mathbf{x} - \mathbf{x}^{'}) \tag{1.11}$$

where d is the number of dimensions and π is the conjugate momentum, defined as usual by

$$\pi = \frac{\partial\mathcal{L}}{\partial(\nabla_0\phi)}\,.$$

These are known as equal-time commutation relations. Having defined the field, we calculate the equations of motion using the Lagrangian density and the principle of least action. Solving

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\right) = 0$$

we find the equation of motion to be

$$(\Box - m^2 - \xi R)\phi = 0.$$

This equation is known as the **Klein-Gordon** equation. The scalar field operator can be expanded out in terms of its modes, using creation and annihilation operators, as

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} (a_{\mathbf{p}} e^{-i\,\mathbf{p}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{p}} e^{i\,\mathbf{p}\cdot\mathbf{x}}) \tag{1.12}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. Once normal ordering[3] is applied, three important operators of the quantised field are the Hamiltonian, four-momentum and particle number operators given respectively by

$$H = \frac{1}{2} \int \frac{d^3\mathbf{p}}{2\pi^3 2E_{\mathbf{p}}} E_{\mathbf{p}} a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} \tag{1.13}$$

$$P^a = \frac{1}{2} \int \frac{d^3\mathbf{p}}{2\pi^3 2E_{\mathbf{p}}} \mathbf{p}^a a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} \tag{1.14}$$

$$N = \frac{1}{2} \int \frac{d^3\mathbf{p}}{2\pi^3 2E_{\mathbf{p}}} a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} \tag{1.15}$$

1.6 Renormalisation

Despite all of its successes, QFT also introduces a number of significant difficulties. We will limit discussion of these again to their relevance to the work ahead, and in relatively brief terms. Having already spoken of effective field theories, we acknowledge that some of these difficulties can be shelved once appropriate constraints or techniques are applied. As always however, we must remind ourselves that conclusions drawn must be qualified with respect to the initial caveats. We will focus here on one of the most immediate difficulties that presents itself in the discipline and the technique used to handle it - renormalisation.

The source of the difficulty has to do with the vacuum state in QFT. To briefly explain the issue, consider the general Hamiltonian expressed as the Hamiltonian density $\mathcal{H}$ integrated over all space as

$$H = \int d^{n-1}x\mathcal{H}$$

[3] When creation and annihilation operators are misplaced in an equation, we encounter meaningless infinities. To circumvent this mathematical irregularity, normal ordering is applied. Essentially, creation operators are placed on the left, annihilation operators on the right. There is more to it, but this is the essential concept and produces a consistent framework.

where n is related to the number operator $\hat{n}$ - an operator producing a number of excitations above the ground state of a field, where

$$\hat{n}_{\mathbf{p}} = \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}}$$

and for an eigenstate $|n\rangle$ we have

$$\hat{n}_{\mathbf{p}} |n\rangle = n_{\mathbf{p}} |n\rangle$$

giving the actual number n of excitations with momentum $\mathbf{p}$. These can be interpreted as particles in QFT, but in momentum space. The Hamiltonian for a scalar field is

$$H = \int d^{n-1} \left[\hat{n}_{\mathbf{p}} + \frac{1}{2}\delta^{(n-1)}(0) \right] E_{\mathbf{p}}$$

The factor $\delta^{n-1}(0)$ is a source of infinity. Even when measured in the vacuum state, the Hamiltonian is infinite. The total energy is an integral of an infinite energy over an infinite momentum space. Renormalisation is the process of subtracting an infinite constant from the Hamiltonian density to obtain a sensible vacuum state. It is a technique that addresses the relationship between a quantum theory and a classical.

Once renormalised, it is possible then to define a vacuum state. For flat spacetime, it is possible to define a vacuum state, as described earlier. For generally curved spacetimes however, there is no preferred way to decompose the field into fixed modes of frequency. In other words, it is very difficult (if not impossible) to decide on a basis from which observers in any inertial frame can agree on the same number of particles. Hence it is also much more difficult to construct an appropriate renormalisation procedure. Another complication is that in GR, energy is a source of curvature, so subtracting energy then alters the curvature.

Notwithstanding the fundamental difficulties involved, many techniques have been developed to define a renormalised stress-energy tensor, usually for a given set of constraints - such as for a given spacetime, or for a particular symmetry. The underlying aim of renormalisation is to reduce a generally curved spacetime to a flat, Minkowskian spacetime on sufficiently small scales, subtracting away the divergences that appear as a result of curvature.

1.7 The Semiclassical Einstein Equations

The semiclassical Einstein equations are a useful tool in investigation of quantum fields on curved spacetimes. The stress-energy tensor in classical field theory is given as

$$T^{ab} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_a \phi)}\partial^b\phi - g^{ab}\mathcal{L} = \frac{2}{\sqrt{g}}\frac{\partial S}{\partial g_{ab}}, \tag{1.16}$$

where $S = S(\phi, g_{ab})$. Further, since this action S is defined as

$$S = -\frac{\sqrt{-g}}{2}\int d^n x\{g^{ab}\partial_a\phi\partial_b\phi + [m^2 + \xi R]\phi^2\}$$

then [4]

$$\begin{aligned} T^{ab} = (1-2\xi)\nabla^a\phi\nabla^b\phi + (2\xi - \tfrac{1}{2})g^{ab}\nabla_c\phi\nabla^c\phi - 2\xi\phi\nabla^a\nabla^b\phi \\ + 2\xi g^{ab}\phi\Box\phi + \xi(R^{ab} - \tfrac{1}{2}Rg^{ab})\phi^2 - \frac{m^2}{2}g^{ab}\phi^2. \end{aligned} \tag{1.17}$$

T^{ab} is a tensor object that contains information about matter, radiation and forces (non-gravitational), equating them through GR to the resultant gravitational field at a point in spacetime. In the semiclassical approach to QFTCS, T_{ab} is replaced with its expectation value. As we have seen in section 1.6, there does not exist a unique value of the quantum vacuum state for arbitrary curvature, so this expectation value must be with respect to some field or state, such that

$$\langle T_{ab}\rangle = \langle\psi|T_{ab}|\psi\rangle$$

where $|\psi\rangle$ is some normalised quantum state. This expectation value of the stress-energy tensor plays a crucial role in semiclassical gravity. The semiclassical version of the Einstein field equations is then

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} \ = \ 8\pi G\,\langle T_{ab}\rangle\,.$$

However, $\langle T_{ab}\rangle$ is formally an infinite value, and renormalisation is required to make sense of it. Solving the equation above gives the next order approximation to quantum gravity. Suppose quantum gravity has a solution of g_{ab}, then the semiclassical approximation $\hat{g}_{ab}$ is given by

$$\hat{g}_{ab} = g_{ab} + \hbar \hat{g}_{,ab} + \mathcal{O}(\hbar^2)$$

where $\hbar = \frac{h}{2\pi}$, and h is Planck's constant. One method of renormalisation is the 'point-splitting' approach, used in conjunction with the Hadamard representation of the Green function - all of which will be discussed in the next chapter.

1.8 Summary

We have briefly introduced the prerequisite concepts of both general relativity and QFT, the scope being limited to its relevance to the work following this chapter. We have discussed topics of particular importance to the work ahead, including the metric tensor, spacetime curvature, the Dirac delta function, expectation values of quantum fields, renormalisation and the stress-energy tensor. Having armed ourselves with the basic toolkit for the study of quantum fields in curved spacetimes, let us now proceed to a more in-depth discussion in Chapter 2 wherein we will examine the singular propagator and the Hadamard parametrix.

Chapter 2

The Singular Propagator

This thesis is based upon, and will extend slightly, upon the work of Brown and Ottewill [5] and Taylor and Breen [6] in particular. In order for a theory to be renormalised - to define certain physical characteristics such as the vacuum state, or the scattering matrix - it must first be regularised. The cited work deals with exactly this, a particular method of regularising the stress-energy tensor by subtracting away the divergences. Let us spend some time in this chapter understanding this work.

2.1 The Euclidean Green Function

The literature considers a quantum scalar field on a static, spherically symmetric black hole spacetime of the form

$$ds^2 = -f(r)dt^2 + fr^2/f(r) + r^2 d\Omega_{d-2}^2$$

where Ω_{d-2} is a point on the $(d-2)$-sphere. By performing what is known as a Wick rotation, transforming the time coordinate to imaginary, proper time coordinates as $t \to i\tau$, we obtain a Euclidean metric

$$ds^2 = f(r)d\tau^2 + fr^2/f(r) + r^2 d\Omega_{d-2}^2$$

rather than the former Minkowskian metric above. Much of QFT is based upon Euclidean mathematics, having been used extensively by some of the pioneering theoreticians such as Schrödinger, Feynman and Schwinger [7] for the particular properties it introduces. It has been suggested that the latter likely "viewed the Euclidean formalism as empirically equivalent to, but better mathematically behaved, than the Minkowski space theory" [8]. A Green's, or Green, function - depending on where you read it - in QFT is a function that propagates a particle (or a field excitation, to be exact). In its simplest terms, it has the general property that for a wave function ϕ initially at position y and time t_y,

$$\phi(x, t_x) = \int G(x, t_x, y, t_y)\phi(y, t_y)dy$$

the Green function G propagates the particle from state $\phi(y, t_y)$ to $\phi(x, t_x)$. Note that $\phi(x, t_x) = \langle x|\phi(t)\rangle$. There is usually one very important constraint enforced for a Green function-the particle must not propagate back in time[1]. This can be stated in several ways, one of which is to apply a Heaviside step function θ, where only positive time generates the positive component of the Green function, say G^+ as

$$G^+(x, t_x, y, t_y) = \theta(t_x - t_y)\left\langle x(t_x)\middle|y(t_y)\right\rangle$$

where $t_x > t_y$. If $t_y > t_x$ then the Green function becomes zero and the particle does not propagate. This is known as the time-retarded Green function. The time-advanced Green function G^- works the other way, for $t_y > t_x$ and is similarly given by

$$G^-(x, t_x, y, t_y) = \theta(t_y - t_x)\left\langle y(t_y)\middle|x(t_x)\right\rangle.$$

Thus the Green function is generally decomposed as

$$G(x, t_x, y, t_y) = G^+(x, t_x, y, t_y) + G^-(x, t_x, y, t_y).$$

Another important definition of the function is that for a linear operator $\hat{L}$

$$\hat{L}G(x, t) = \delta(x - t).$$

This of course is a very rudimentary explanation of the Green function, however it is useful in developing a physical understanding. Green functions are dependent on

[1]To be clear, QFT absolutely permits particles to travel back in time. However, such particles are actually referred to as virtual particles, and are an essential concept in the theory.

the physical system being studied. Above, we spoke of the space and time domain, but more common perhaps is the momentum and time domain. An example is a function that propagates a particle with momentum state $|\mathbf{p}\rangle$ for a positive interval of time Δt :

$$G(\mathbf{p}, \Delta t) = \theta(\Delta t) e^{-iE_{\mathbf{p}}\Delta t}$$

Another is the very useful and famous Feynman propagator, defined as [9]

$$G(x,y) = \theta(t_x - t_y) \langle \Omega | \phi(x)\phi^\dagger(y) | \Omega \rangle + \theta(t_y - t_x) \langle \Omega | \phi^\dagger(y)\phi(x) | \Omega \rangle$$

where Ω is the ground state of the system. This is interpreted as two-step event. The first term is a particle being created at (t_y, y) and being annihilated at (t_x, x), travelling forwards in time. The second term is a virtual particle being created at (t_x, x) and being annihilated at (t_y, y), travelling backwards in time. Although this is still in its simplest form, the concept of using Green function propagators for real and virtual particles is a cornerstone of QFT, and demonstrates how important Green functions are to the discipline. In fact Richard Feynman, having simplified some of the complex mathematics involved in calculating the propagation of particles to simple diagrammatic constructs, famously publicised these concepts on his family van.

FIGURE 2.1: Feynman propagators, in their diagram form, on Richard Feynman's van.

The mathematical expression for a free Feynman propagator (one without interactions) makes use of a time ordering operator[2] T to help keep track of particles, anti-particles and virtual particles. It orders expressions so that the earliest is on the left,

[2]Actually T is not strictly an operator, but an ordering symbol.

and latest on the right, so that a Green function is then

$$\begin{aligned}G(x,y) &= \langle\Omega|\, T\phi(x)\phi^{\dagger}(y)\,|\Omega\rangle \\ &= \theta(t_x - t_y)\,\langle\Omega|\,\phi(x)\phi^{\dagger}(y)\,|\Omega\rangle + \theta(t_y - t_x)\,\langle\Omega|\,\phi^{\dagger}(y)\phi(x)\,|\Omega\rangle\,.\end{aligned}$$

The full general, non-interacting expression in flat spacetime is then given as

$$G(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon}$$

where $\epsilon > 0$. This term essentially forces the Green function to remain regular. The latter term is the Fourier component of the Feynman propagator [9], and is the proagator of a particle with momentum p, ie

$$\Delta(p) = \frac{i}{p^2 - m^2 + i\epsilon}.$$

It is this term that proves so useful in particle interaction study in QFT, particularly as it directly permits intuitive study of particle interactions and creation or annihilation, especially via Feynman diagrams.

Having clarified the purpose of Green functions, it is also important to emphasise the fast that they contain all of the physical information of the system [7], such as energy and scattering (interaction) properties. The Euclidean Green function of concern for this work has the following mode-sum representation [6] for $d = 4$:

$$G(x,x^{'}) = \frac{\kappa}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} \frac{(2l+1)}{\Omega_2} P_l(\cos\gamma) g_{nl}(r,r^{'}) \tag{2.1}$$

where $\Omega_2 = 2\pi^{3/2}/\Gamma(3/2) = 4\pi$ and $P_l(x)$ is the Legendre polynomial - a special case of the general Gegenbauer polynomial for $d = 4$ (see the next section). With respect to the time interval $\Delta\tau = \tau - \tau^{'}$, we have enforced a period of $2\pi/\kappa$ via the definition

$$\delta(\tau - \tau^{'}) = \frac{\kappa}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}.$$

γ is the geodesic distance on the 2-sphere. $g_{nl}(r, r^{'})$ is known as the radial Green function, satisfying

$$\left[\frac{d}{dr}\left(r^2 f(r)\frac{d}{dr}\right) - r^2\left(\frac{n^2\kappa^2}{f(r)} + m^2 + \xi R(r)\right) - l(l+1)\right] g_{nl}(r, r^{'}) = -\delta(r - r^{'}) \quad (2.2)$$

as described in Appendix A. κ is the surface gravity of a black hole - an important aspect and context of Breen's work [4]. The period of $\frac{2\pi}{\kappa}$ is significant in this work as the context remains specific to black-hole spacetimes, where Hawking and Gibbons [10] showed that the temperature radiated away from a black-hole via Hawking radiation is given as $T = \frac{\kappa_0}{2\pi}$, where κ_0 is the black hole's surface gravity at the horizon. Breen concluded that "any quantum state which possesses the same periodicity, and therefore temperature, as an horizon of the space-time would be expected to be regular on that horizon" [4]. Hence, building a period of $\frac{2\pi}{\kappa}$ into our mode-sum should ensure regularisation is possible everywhere in the exterior of the black hole.

2.2 Legendre Polynomials

Rather than a thorough description of Legendre polynomials, in this section we will explain them in appropriate depth so that their function in this work and the cited literature becomes clear. As mentioned above, this work is confined to $d = 4$, when Legendre polynomials are employed. Much of the cited literature, such as [4] and [6], deals with the general cases, for any d, wherein Legendre polynomials may be generalised to Gegenbauer polynomials. As they are outside of the scope of this work, we will not discuss Gegenbauer polynomials any further.

Legendre polynomials are mathematical objects that are ubiquitous in physics and engineering applications. They appear frequently in work relating to quantum physics, fluid and wave mechanics, thermodynamics and electromagnetism, for example. Owing to the fact that they are defined in an orthogonal basis, they prove an extremely efficient way of quantifying more complex functions. There are many ways to define Legendre polynomials, one of which stems from Legendre's differential equation:

$$\frac{d}{dx}\left((1 - x^2)\frac{d}{dx}P_l(x)\right) + l(l+1)P_l(x) = 0.$$

When $-1 \leq x \leq 1$, or equivalently and as per this work - when $x = \cos\theta$ (where $\theta \in \mathbb{R}$) - then the Legendre polynomials of degree l (also referred to as *modes*) are defined as

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

for $l \in \mathbb{W}$. These modes are, in physical terms, the spatial components comprising a waveform, related to angular separation. The *mode-sum*, is the summation of all modes to return the waveform - similar to Fourier decomposition to a sum of sin waves. Another common definition is via a generating function, given as

$$\sum_{l=0}^{\infty} P_l(x) a^l = \frac{1}{\sqrt{a^2 - 2ax + 1}} \qquad , \qquad |a| < 1.$$

Legendre polynomials satisfy the recursion relations

$$(2l+1)xP_l(x) - (l+1)P_{l+1} - lP_{l-1} = 0$$

and

$$(x^2 - 1)\frac{dP_l(x)}{dx} = l(xP_l - P_{l-1}) = \frac{l(l+1)}{2l+1}(P_{l+1} - P_{l-1}).$$

Importantly for many applications, including QFT, these polynomials are orthogonal. The orthogonality of Legendre polynomials is defined over the interval $[-1, 1]$ as

$$\int_{-1}^{1} P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'}$$

so that

$$\int_{-1}^{1} P_l(x) P_{l'}(x) dx = 0 \qquad , \qquad l \neq l^{'}.$$

A Legendre polynomial of degree l has $l - 1$ inflection points, where $\frac{dP_l(x)}{dx} = 0$. The polynomial is even if l is even, and odd otherwise. Fig. 2.2 below graphs the first few.

Similar to how any continuous function $f(x)$ can be represented as a Fourier series, so too it can be represented as a sum of Legendre polynomials. This is referred to as *completeness*. Specifically, if we let a function $f_n(x)$ be the sum of Legendre

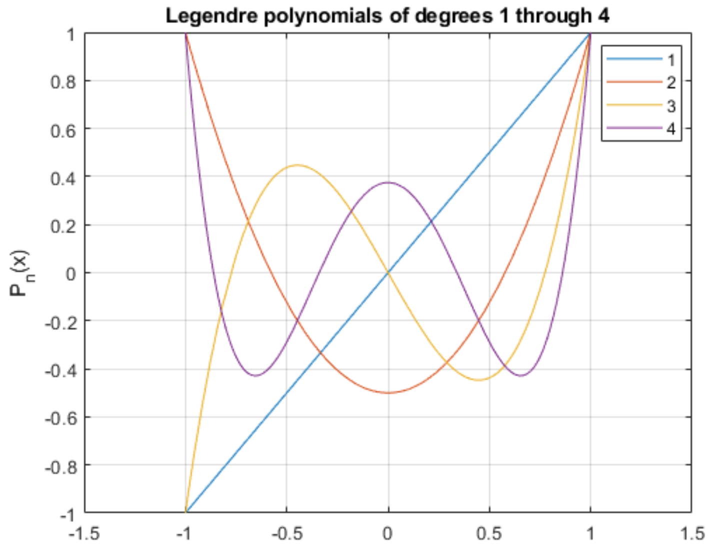

FIGURE 2.2: Legendre Polynomials plotted for $l = 1, 2, 3, 4$ over the defined range $-1 \leq x \leq 1$ [11] .

polynomials by defining

$$f_n(x) = \sum_{l=0}^{n} c_l P_l(x),$$

and further define the constants c_l as

$$c_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) dx,$$

then $f_n(x) \to f(x)$ as $l \to \infty$. In the next chapter, we will call upon the Legendre completeness relation

$$\int_{-1}^{1} P_l(x) P_{l'}(x) dx = \frac{\Gamma(n+1)}{(l+\frac{1}{2})l!} \delta_{ll'} \tag{2.3}$$

to assist us with regularization. Looking at Eq. 2.1, the summation of Legendre Polynomials to ∞ is a factor in the equation permitting us to define a Green function exactly using completeness.

Associated Legendre polynomials are another mathematical object ubiquitous in physics, derived directly from Legendre polynomials. They are often described in the context of 'spherical harmonics' - basis functions defined on the surface of a sphere, permitting the study of eigenfunctions in dynamic and spherically symmetric phenomena. In keeping with the literature, we denote them as $Q_l^m(x)$ for

$-l \leq m \leq l$ and are derived directly from Legendre polynomials as

$$Q_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l.$$

They satisfy the differential equation

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + l(l+1) - \frac{m^2}{1-x^2}\right]Q_l^m(x) = 0.$$

and their orthogonality is defined generally as

$$\int_{-1}^{1} Q_l^m(x)Q_{l'}^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}.$$

We will find them useful as they permit us to change the coordinate system of a Legendre polynomial when integrated. For the general function $f(x,z,j)$ we will call upon an identity of the type

$$\int_{-1}^{1} f(x,z)P_l(x)dx = g(z,l)Q_l^m(z)$$

in the regularization of the Hadamard parametrix. Describing Legendre polynomials in terms of its 'spherical harmonics' will thus be a powerful tool in the work ahead. The Legendre polynomials we will encounter throughout this work are of the form $P_l(\cos\gamma)$ where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ where θ and ϕ are the angular coordinates of a sphere.

2.3 Point-splitting

Point-splitting is a mathematical technique that isolates divergences and singular behaviour in quantum fields, permitting their removal and hence meaningful physical quantities can be calculated. Wald [12] showed that this technique - for any spacetime evolution - gives meaningful and finite renormalized values of the expectation value of T_{ab}. Herein, we essentially treat it as a two-point operator-valued distribution $T_{ab}(x,x')$. This function and the Green function representation in Eq. 2.1 above, depends on the terms (x,x') rather than say (x,y). This is not mere notation, but a unique concept known as *point-splitting*, or alternatively as *point-separation*. The points x and x' are two spacetime points where $x \neq x'$, but are brought close together, meaning that $\sigma(x, x') \rightarrow 0$, where $\sigma(x, x')$ is half the squared distance along the shortest geodesic connecting x and x' [13]. The geodesic distance is denoted $s(x - x')$ (a scalar quantity) and σ is then defined simply as

$$\sigma = \frac{1}{2}s^2.$$

Further, σ satisfies

$$2\sigma = \nabla_a \sigma \nabla^a \sigma,$$

and $\sigma < 0$ if x and x' are timelike related, $\sigma > 0$ if they are spacelike related and $\sigma = 0$ if the geodesic is a null geodesic. Importantly, we assume that x' is "in a *normal neighborhood* of x, meaning that geodesics emanating from x do not intersect" [14] - that is, there is only one geodesic that satisfies the above equation.

The **coincidence limits** (the limit as the two points are brought together such that the geodesic distance becomes 0) as $x \rightarrow x'$ are defined as [13]:

$$\sigma(x, x') \rightarrow 0 \quad , \quad \nabla_a \sigma(x, x') \rightarrow 0 \quad , \quad \nabla_a \nabla_b \sigma(x, x') \rightarrow g_{ab} \quad , \quad \nabla_a \nabla_{b'} \sigma(x, x') \rightarrow -g_{ab}$$

The biscalar form of the Van Vleck-Morette determinant Δ, is defined [5] as a function of x and x' by

$$\Delta = -\frac{1}{\sqrt{g(x)}} \, det(\nabla_a \nabla_{b'} \sigma(x, x')) \, \frac{1}{\sqrt{g(x')}}$$

where $\Delta \rightarrow 1$ in the coincidence limit. The classical expression for the stress-energy tensor of a scalar field is given by

$$\begin{aligned} T^{ab} = {} & (1 - 2\xi)\phi^{;a}\phi^{;b} + (2\xi - \tfrac{1}{2})g^{ab}\phi_{;c}\phi^{;c} - 2\xi\phi\phi^{;ab} \\ & + 2\xi g^{ab}\phi\Box\phi + \xi(R^{ab} - \tfrac{1}{2}Rg^{ab})\phi^2 - \tfrac{m^2}{2}g^{ab}\phi^2 \end{aligned} \tag{2.4}$$

where we adopt the notation $\phi^{;a} = \nabla^a \phi$ in keeping with the literature. T^{ab} is a divergent quantity If we consider the massless ($m = 0$) and minimally coupled ($\xi =$

0) scalar field, the above becomes

$$T^{ab} = \phi^{;a}\phi^{;b} - \frac{1}{2}g^{ab}\phi_{;c}\phi^{;c}.$$

Using the point-splitting approach, we can also now define the expectation value (with respect to a Hadamard quantum state $|\psi\rangle$) of the above stress-energy tensor [15] as

$$\begin{aligned} \langle\psi|\, T_{ab}\,|\psi\rangle &= \frac{1}{2}\lim_{x\to x'}\left\{\left[\partial_a\partial_{b'} - \frac{1}{2}g_{ab}\partial_c\partial^{c'}\right]G(x,x')\right\} \\ &= \lim_{x\to x'}\tau^{ab}\left[-iG(x,x')\right] \end{aligned} \tag{2.5}$$

where $G(x,x')$ is the point separated Green function, and the *operator* τ^{ab} is related to Eq. 1.17 in the coincidence limit [4] as

$$\begin{aligned} \tau^{ab} = (1-2\xi)g_{b'}^{\ b}\nabla^a\nabla^{b'} + (2\xi - \tfrac{1}{2})g^{ab}g_{c'}^{\ c}\nabla^{c'}\nabla_c \\ - 2\xi\nabla^a\nabla^b + 2\xi g^{ab}\nabla_c\nabla^c + \xi(R^{ab} - \tfrac{1}{2}Rg^{ab}) - \tfrac{m^2}{2}g^{ab}. \end{aligned} \tag{2.6}$$

where $g_{b'}^{\ b}$ is the bivector of parallel transport. It is defined by

$$\sigma^{;c}g_{ab';c} = 0$$

with the boundary condition

$$\lim_{x\to x'} g_{ab'} = g_{ab}.$$

A bivector of parallel transport "parallel transports a vector at x' to a vector at x" [4]. Parallel transporting a vector in a spacetime with general curvature is the concept of moving a vector in the tangent space along a path while keeping it constant. It is inherently dependent on the path between x and x', and hence is dependent on the connection. For the Euclidean Green function G_E, the expectation value of the stress-tensor is given by [4]

$$\langle T_{ab}\rangle = \mathcal{R}\left[\lim_{x\to x'}\tau^{ab}G_E(x,x')\right] \tag{2.7}$$

where $\mathcal{R}$ indicates the real part only. This will be implied throughout, so the $\mathcal{R}$ notation can be dropped. Due to the behaviour of the Green function (see the next section on ultraviolet divergence) in Eq. 2.5, the expectation value $\langle\psi| T_{ab} |\psi\rangle$ is divergent, and must be renormalised to make it physically useful. The geometric parts of the Hadamard state are the cause of the divergences.

2.3.1 Ultraviolet Divergences

At this point, a discussion of *ultraviolet divergences* is appropriate, before we look at their treatment in the renormalisation process. The term ultraviolet in this context is a throwback to one of the initial motivating problems that ultimately led to the development quantum mechanics, emotively referred to as 'the Ultraviolet Catastrophe'. The problem at the time (circa late 19^{th} Century) was that classical physics predicted that black bodies should radiate more energy per photon as wavelength decreased - to the extent that at the ultraviolet end of the spectrum, the quantities of energy emitted approached infinity; they diverged. Max Planck's quantisation of electromagnetic radiation ultimately solved the problem, and so began quantum physics.

The ultraviolet divergences that we must deal with, are not related to black bodies, but to the necessarily small distances we have defined in the point-splitting procedure above. A similar divergence in resulting calculations is experienced when we calculate the expectation values of products of field operators [15]. In the coincidence limit, we will have to deal with some terms via renormalisation precisely due to this behaviour. Eq. 2.6 above, for example, is divergent in this limit [4].

2.4 The Hadamard Parametrix

In order for a four-dimensional spacetime to have a unique singularity structure, the Green function is required to be a *Hadamard state* - that is, when x and $x^{'}$ are close:

$$G(x,x^{'}) = \frac{U(x,x^{'})}{\sigma + i\epsilon} + V(x,x^{'}) \ln{(\sigma + i\epsilon)} + W(x,x^{'}) \tag{2.8}$$

where the functions U, V and W are all regular at $x = x^{'}$, and $U(x,x) = 1$ [1], and the term $i\epsilon$ again helps maintain regularity. In fact, the terms U and V are "geometrical quantities independent of the quantum state, and only W carries information about the state" [15]. Since $U(x,x^{'}) \to 1$ in the coincidence limit, it is thus independent of the quantum state, holding no information about it. We choose the Hadamard parametrix as it has been shown [12] that the Hadamard geometrical singularity structure is preserved throughout spacetime. Hence it is a good candidate for renormalization. We first introduce the two-point function $G_s(x,x^{'}) = G(x,x^{'}) - W(x,x^{'})$ using the Hadamard form of the Euclidean Green function as per Eq. 2.8 for a singular propagator $G_s(x,x^{'})$.

2.4.1 The Singular Propagator

As per the regularisation scheme we wish to follow in [5, 6], then for even dimensions the Green function for the singular propagator G_s is defined as

$$G_s(x,x^{'}) = \frac{1}{8\pi^2}\left(\frac{U(x,x^{'})}{\sigma(x,x^{'})} + V(x,x^{'})\log\left(\frac{2\sigma(x,x^{'})}{\ell^2}\right)\right) \tag{2.9}$$

where ℓ is a length scale ensuring that the log term is dimensionless. Both U and V are uniquely determined and purely geometrical objects, depending only on the geometry of the geodesic. W, however, is not uniquely determined - it depends on both the quantum state and the spacetime, not just on the geometry [16]. We remove W to arrive at the expression above, so that we focus our interest entirely on the divergences as a result of geometry (the metric and its derivaties) - the Ultraviolet divergences mentioned above. The singular propagator satisfies

$$(\Box - m^2 - \xi R)G_s(x,x^{'}) = \delta(x,x^{'}) + S(x,x^{'})$$

where $S(x,x^{'})$ is an arbitrary smooth biscalar, since this guarantees that the singularities within cancel with the singularities of the original Green function $G(x,x^{'})$ [6]. Upon subtracting W it is clear that $G_s(x,x^{'})$ depends only on geometry, and not on the quantum state, thereby capturing the divergences. By permitting this inhomogenous wave equation to admit the Hadamard solutions, and by making frequent use of the identities $\sigma_a\sigma^a = 2\sigma$ and $\Delta^{1/2}(\Box\sigma - 4) + 2\sigma^a\Delta^{1/2}_{;b} = 0$, we can derive the wave equation for V (see Appendix B.1) as

$$(\Box - m^2 - \xi R)V = 0 \tag{2.10}$$

and for W as

$$(\Box - m^2 - \xi R)W = -2V - (\Box - m^2 - \xi R)\Delta^{\frac{1}{2}} - 2\sigma^a(V_a - V\Delta^{\frac{1}{2}}_{;a}\Delta^{-\frac{1}{2}}). \tag{2.11}$$

The wave equation applied to $U(x, x^{'})$ gives

$$\sigma(\Box - m^2 - \xi R)U = (d-2)U^{;a}\sigma_{;a} - (d-2)U\sigma^{;a}\Delta^{\frac{1}{2}}_{;a}\Delta^{-\frac{1}{2}} \tag{2.12}$$

as in Appendix B.1. Crucially, as in Taylor and Breen's work [6], we can expand U and V by assuming the Hadamard ansatz for a series solution as

$$U(x, x^{'}) = \sum_{p=0}^{\frac{d}{2}-2} U_p(x, x^{'})\sigma^p \quad , \quad V(x, x^{'}) = \sum_{p=0}^{\infty} V_p(x, x^{'})\sigma^p. \tag{2.13}$$

Since we are restricting this work to $d = 4$, then $U = U_0$.

2.4.2 Renormalisation

The geometric part of the Hadamard state, in the coincidence limit, causes divergence as described. Specifically, the terms $1/\sigma$ and $\ln\sigma$ are problematic in the limit. The divergent behaviour of Eq. 2.5 can be dealt with however, provided we can find the appropriate divergences to subtract, leaving us with a regular expression - the renormalised expectation value of the stress energy tensor operator (with respect to the Hadamard quantum state, that is). This process is known as the Hadamard regularization prescription in the literature [17] and can be summarised as follows.

We wish to subtract from the divergent Green's function a term which has the same geometrical singularity structure as $x \to x^{'}$. In Eq. 2.5 the term $G(x, x^{'})$ can be naturally split into the geometric and singular terms (those containing σ and causing divergence in the limit), denoted by say $G_s(x, x^{'})$ (the 'singular propagator' [6]) and the quantum state term containing W. The regularized Green function $G_{reg}(x, x^{'})$

can then be defined as

$$G_{reg}(x,x^{'}) = G(x,x^{'}) - G_s(x,x^{'})$$

where [4]

$$G_s(x,x^{'}) = \frac{\Delta^{\frac{1}{2}}}{\sigma} + V\log(\lambda\sigma)$$

for a constant λ. With a regularized Green function, it is then possible to calculate a renormalised expectation value of the stress energy tensor, $\langle T_{ab}\rangle_{ren}$, one which can now act as a physically meaningful source in Einstein's field equations.

2.4.3 Recursion Relations

By applying the Hadamard ansatz of Eq. 2.13 to the wave equations of Eq. 2.12 and Eq. 2.10 respectively, we can determine the recursion relations for each of U_p and V_p (Appendix B.1) as

$$(\Box - m^2 - \xi R)U_p = 2p\sigma^{;a}\nabla_a U_{p+1} + 2p(p+1)U_{p+1} - 2pU_{p+1}\Delta^{-\frac{1}{2}}\sigma^{;a}\Delta^{\frac{1}{2}}_{;a} \tag{2.14}$$

and

$$(\Box - m^2 - \xi R)V_p = 2(p+1)V_{p+1}\sigma^{;a}\Delta^{\frac{1}{2}}_{;a}\Delta^{-\frac{1}{2}} - 2(p+1)\nabla_a V_{p+1}\sigma^{;a} - (p+1)(2p+4)V_{p+1}. \tag{2.15}$$

The boundary conditions for U and V are derived in Appendix B.2 and are shown to be

$$U_0 = \Delta^{\frac{1}{2}} \tag{2.16}$$

and

$$(\Box - m^2 - \xi R)U_0 - 2\nabla^a V_0\sigma_{;a} - 2\nabla_a U_0\Delta^{-\frac{1}{2}} + 2V_0 = 0. \tag{2.17}$$

These will be useful once we introduce the expansion in the next chapter.

2.5 Computation

Although the next chapter deals with the coordinate expansion that is the main character of this work, it is worth discussing at this point why the above recursion relations were calculated. We will decide upon an appropriate expansion for U_p and V_p via the Hadamard ansatz, however we can see from the previous section that in order to satisfy the wave equation the recursion relations insist that the quantities $\Box U_p$ and $\Box V_p$ be known. For $d = 4$ we will need to use Mathematica to calculate $\Box U_0$, $\Box V_0$ and $\Box V_1$ in order to equate the ansatz with the new coordinate expansion and determine values coefficient by coefficient. This can be done by hand of course, however even for $d = 4$ the number of terms in the expansion is quite large.

2.6 Summary

We have summarised the interpretation of a Green function, and discussed its function as a propagator of particles in QFT. With a better understanding of the roles of the Euclidean and radial Green functions, it was possible then to examine the role of the Hadamard state - a specific description of a Green function required in order for a four-dimensional spacetime to have a unique singularity structure. We discussed its individual terms, and examined some of its properties as a result of the wave equation. We also took the time to look at Legendre polynomials and point-splitting, since the rest of this work will feature both heavily. Let us now proceed to the unique and judiciously-chose coordinate expansion of Taylor and Breen [6] and demonstrate the methodology and justification step-by-step, before attempting the regularisation of the mode-sum in Chapter 4.

Chapter 3

Extended Coordinate Expansion

In the work of Taylor and Breen [6], a unique expansion of the *Synge's world function* $\sigma(x, x^{'})$ is chosen in the form

$$\sigma = \sum_{ijk} \sigma_{ijk}(r) w^i \Delta r^j s^k$$

where

$$w^2 = \frac{2}{\kappa^2}(1 - \cos \kappa \Delta \tau) \qquad , \qquad s^2 = f(r) w^2 + 2r^2(1 - \cos \gamma).$$

Since σ, to lowest order, is $\sigma(x, x^{'}) = \frac{1}{2} g_{ab} \Delta x^a \Delta x^b + \mathcal{O}(\Delta x^3)$, we may formally treat w and s as $\mathcal{O}(\epsilon) \sim \mathcal{O}(\Delta x)$ quantities - noting that we will actually have no terms of $\mathcal{O}(\epsilon)$. This particular coordinate expansion in $(w, \Delta r, s)$ is the unique work of Breen and Taylor, referred to as *extended coordinates* in the literature, and the choice of definition is made clear in [6] and in this chapter. From this choice of coordinate expansion,

$$w^i = \left(\frac{2}{\kappa^2}\right)^{i/2} (1 - \cos \kappa \Delta \tau)^{i/2} \qquad \text{and} \qquad s^k = \left(f(r) w^2 + 2r^2(1 - \cos \gamma)\right)^{k/2}.$$

We now have a coordinate expansion polynomial in $\cos \kappa \Delta \tau$ and $\cos \gamma$ - an important point for the regularization scheme ahead [1]. We can use the defining equation $\sigma^a \sigma_a = 2\sigma$ to determine the coefficients $\sigma_{ijk}(r)$, hence

$$\frac{1}{f(r)} \frac{\partial \sigma}{\partial \tau} + f(r) \frac{\partial \sigma}{\partial r} + \frac{4}{r^2} \frac{\partial \sigma}{\partial \gamma} = 2 \sum_{ijk} \sigma_{ijk}(r) w^i \Delta r^j s^k \tag{3.1}$$

[1] It is important to point out that although w and s are actually of half integer powers, we will end up with only even powers of both in the expansion - and so the polynomial assertion holds.

In the new coordinate scheme, we have that (as calculated in Appendix C.1):

$$\sigma_r^{'} = \sigma_r + \left(\frac{f^{'}(r)w^2}{2s} + \frac{s}{r} - \frac{f(r)w^2}{sr} \right) \sigma_s - \sigma_{\Delta r};$$

$$\sigma_\tau^{'} = \left(1 - \frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}} \left(-\frac{w}{s} f(r)\sigma_s - \sigma_w \right);$$

and

$$\sigma_\gamma^{'} = r\left(1 - \frac{f(r)w^2}{s^2} - \frac{s^2}{4r^2} - \frac{f(r)^2w^4}{4r^2s^2} + \frac{f(r)w^2}{2r^2}\right)^{\frac{1}{2}} \sigma_s$$

where we have adopted the notation

$$\sigma_s = \frac{\partial \sigma}{\partial s} = \sum_{ijk} \sigma_{ijk}(r) k w^i \Delta r^j s^{k-1},$$

$$\sigma_w = \frac{\partial \sigma}{\partial w} = \sum_{ijk} \sigma_{ijk}(r) i w^{i-1} \Delta r^j s^k,$$

$$\sigma_{\Delta r} = \frac{\partial \sigma}{\partial \Delta r} = \sum_{ijk} \sigma_{ijk}(r) j w^i \Delta r^{j-1} s^k$$

and

$$\sigma_r = \frac{\partial \sigma}{\partial r} = \sum_{ijk} \sigma_{ijk}^{'}(r) w^i \Delta r^j s^k.$$

We then obtain Eq. 3.1 in extended coordinates as

$$\begin{aligned}
\sigma = \frac{1}{2f(r)} & \left(1 - \frac{w^2\kappa^2}{4}\right)\left(-\frac{w}{s} f(r)\sigma_s - \sigma_w \right)^2 \\
& + f(r)\left[\sigma_r + \left(\frac{f^{'}(r)w^2}{2s} + \frac{s}{r} - \frac{f(r)w^2}{sr} \right)\sigma_s - \sigma_{\Delta r} \right]^2 \\
& + \left(1 - \frac{f(r)w^2}{s^2} - \frac{s^2}{4r^2} - \frac{f(r)^2w^4}{4r^2s^2} + \frac{f(r)w^2}{2r^2}\right)\sigma_s^2 .
\end{aligned} \tag{3.2}$$

3.1 'Bookkeeping'

A 'bookkeeping mechanism' is introduced at this point by Taylor and Breen [6] via a dimensionless parameter ϵ, which will do nothing more than keep track of the order of each term in $(w, \Delta r, s)$. Ultimately, ϵ will be set $= 1$, however its use in determining the order of the extended coordinates is employed in Mathematica, where the calculations are performed for this work.

Let us demonstrate the bookkeeping terms' use and ensure that the correct order is maintained. We can keep track of ϵ using the following method [2]. First, we let

$$\sigma = \sigma_{ijk}(r)\epsilon^{i+j+k}w^i\Delta r^k s^k$$

and note that the order of ϵ is equal to the sum of the indices on the w, Δr and s terms. Then differentiating, say by s, we obtain

$$\frac{\partial\sigma}{\partial s} = k\sigma_{ijk}\epsilon^{i+j+k}w^i\Delta r^k s^{k-1}.$$

So ϵ is of one order higher than the coordinate indices summed. We can rectify this by dividing by ϵ, or by multiplying by s (or alternatively by either Δr or w). That is, for example:

$$\frac{1}{\epsilon}\frac{\partial\sigma}{\partial s} = k\sigma_{ijk}\epsilon^{i+j+k-1}w^i\Delta r^k s^{k-1} \qquad \text{or} \qquad s\frac{\partial\sigma}{\partial s} = k\sigma_{ijk}\epsilon^{i+j+k}w^i\Delta r^k s^k$$

where either operation restores the correct order to the respective equation. Next, we have that

$$\sigma_s^2 = \left(\frac{\partial\sigma}{\partial s}\right)^2 = k^2\sigma_{ijk}^2\epsilon^{2i+2j+2k}w^{2i}\Delta r^{2j}s^{2k-2}.$$

So the order of ϵ is wrong again, but can be fixed using the correct multiplier/divisor. Of more relevance, if we look at the term

$$\left(1 - \frac{f(r)w^2}{s^2} - \frac{s^2}{4r^2} - \frac{f(r)^2w^4}{4r^2s^2} + \frac{f(r)w^2}{2r^2}\right)\sigma_s^2$$

from the change of coordinates in the preceding section, we then insert ϵ accordingly, obtaining

$$\left(1 - \frac{f(r)w^2}{s^2} - \epsilon^2\frac{s^2}{4r^2} - \epsilon^2\frac{f(r)^2w^4}{4r^2s^2} + \epsilon^2\frac{f(r)w^2}{2r^2}\right)\sigma_s^2\,.$$

[2] We drop the summation term over ijk at this point, treating it as implied.

To verify this bookkeeping method works, let us check term by term that the order of ϵ is consistent with the summed orders of the extended coordinates:

$$\begin{aligned}
\sigma_s^2 &= k^2\sigma_{ijk}^2\epsilon^{2i+2j+2k}w^{2i}\Delta r^{2j}s^{2k-2} \\
\frac{f(r)w^2}{s^2}\sigma_s^2 &= f(r)k^2\sigma_{ijk}^2\epsilon^{2i+2j+2k}w^{2i+2}\Delta r^{2j}s^{2k-4} \\
\epsilon^2\frac{s^2}{4r^2}\sigma_s^2 &= \frac{k^2}{4r^2}\sigma_{ijk}^2\epsilon^{2i+2j+2k+2}w^{2i}\Delta r^{2j}s^{2k} \\
\epsilon^2\frac{f(r)^2w^4}{4r^2s^2}\sigma_s^2 &= \frac{k^2f(r)^2}{4r^2}\sigma_{ijk}^2\epsilon^{2i+2j+2k+2}w^{2i+4}\Delta r^{2j}s^{2k-4} \\
\epsilon^2\frac{f(r)w^2}{2r^2}\sigma_s^2 &= \frac{k^2f(r)}{2r^2}\sigma_{ijk}^2\epsilon^{2i+2j+2k+2}w^{2i+2}\Delta r^{2j}s^{2k-2}
\end{aligned}$$

Each expression above evidently contains an excess factor of ϵ^2. This can be accounted for via the Mathematica expression $\sigma_s = D[\sigma, s]/\epsilon$, which is $\frac{1}{\epsilon}\frac{\partial\sigma}{\partial s}$. It is equivalent instead to writing

$$\left(\frac{1}{\epsilon^2} - \frac{f(r)w^2}{\epsilon^2s^2} - \frac{s^2}{4r^2} - \frac{f(r)^2w^4}{4r^2s^2} + \frac{f(r)w^2}{2r^2}\right)\sigma_s^2$$

and then just letting $\sigma_s = D[\sigma, s]$. So we can rewrite Eq. 3.2, inserting ϵ appropriately, as

$$\begin{aligned}
2\sigma = &\frac{1}{f(r)}\left(1 - \frac{\epsilon^2w^2\kappa^2}{4}\right)\left(-\frac{w}{\epsilon s}f(r)\sigma_s - \frac{\sigma_w}{\epsilon}\right)^2 \\
&+ f(r)\left[\sigma_r + \left(\frac{f'(r)w^2}{2s} + \frac{s}{r} - \frac{f(r)w^2}{sr}\right)\sigma_s - \frac{\sigma_{\Delta r}}{\epsilon}\right]^2 \\
&+ \left(1 - \frac{f(r)w^2}{s^2} - \epsilon^2\frac{s^2}{4r^2} - \epsilon^2\frac{f(r)^2w^4}{4r^2s^2} + \epsilon^2\frac{f(r)w^2}{2r^2}\right)\frac{\sigma_s^2}{\epsilon^2}
\end{aligned} \tag{3.3}$$

as calculated in Appendix C.1. The reason we take the trouble to bookkeep is that we must determine the coefficients $\sigma_{ijk}(r)$ of the extended coordinate expansion $\sigma = \sum_{ijk}\sigma_{ijk}(r)w^i\Delta r^js^k$, and the order of the extended coordinates in the ensuing expansion. To leading order, we have $\sigma = \frac{1}{2}\epsilon^2(s^2 + \frac{\Delta r^2}{f(r)}) + \mathcal{O}(\epsilon^3)$, as in Eq. 3.4 below, and where σ obeys the boundary condition $\nabla_a\nabla_b\sigma = g_{ab}$. As this is the smallest magnitude of powers in the expansion, we infer without difficulty that

$$\sigma_{000} = \sigma_{100} = \sigma_{010} = \sigma_{001} = 0$$

and that σ_{002} contains the term $\frac{1}{2}$, while σ_{020} contains the term $\frac{1}{2f(r)}$. These values can be hard-coded into Mathematica prior to calculating the series expansion. We find that the odd values of i and j in the expansion are zero. This becomes apparent in the code, as coefficients with odd i or j are computed $= 0$. To see why, consider the expanding σ "in terms of an expansion in powers of the coordinate separation" [4] $\Delta x^\alpha = (x^\alpha - x'^\alpha)$, where $x^\alpha = (t, r, \gamma(\theta, \phi))$, as

$$\sigma = \frac{1}{2} g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta + A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma + B_{\alpha\beta\gamma\delta} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma \Delta x^\delta + \dots \tag{3.4}$$

where $A_{\alpha\beta\gamma}$, $B_{\alpha\beta\gamma\delta}$ ... are tensor functions of $g_{\alpha\beta}$ of increasing rank. With the defining equation $2\sigma = \sigma_a \sigma^a$ in mind, let us calculate σ_a (up to $\mathcal{O}(\Delta x^3)$ is sufficient to make this point):

$$\begin{aligned}
\sigma_a &= \partial_a \big(\frac{1}{2} g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta\big) + \partial_a \big(A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma\big) + \dots \\
&= \partial_\alpha \big(\frac{1}{2} g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta\big) + \partial_\beta \big(\frac{1}{2} g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta\big) \\
&\quad + \partial_\alpha \big(A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma\big) + \partial_\beta \big(A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma\big) + \partial_\gamma \big(A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma\big) + \dots \\
&= \frac{1}{2} g_{\alpha\beta,\alpha} \Delta x^\alpha \Delta x^\beta + \frac{1}{2} g_{\alpha\beta} \Delta x^\beta + \frac{1}{2} g_{\alpha\beta,\beta} \Delta x^\alpha \Delta x^\beta + \frac{1}{2} g_{\alpha\beta} \Delta x^\alpha \\
&\quad + A_{\alpha\beta\gamma,\alpha} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma + A_{\alpha\beta\gamma} \Delta x^\beta \Delta x^\gamma + A_{\alpha\beta\gamma,\beta} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma + A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\gamma \\
&\quad + A_{\alpha\beta\gamma,\gamma} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma + A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\beta + \dots \\
&= g_{\alpha a} \Delta x^a + \big(g_{\alpha\beta,a} + 3A_{a\alpha\beta}\big) \Delta x^\alpha \Delta x^\beta + \big(A_{\alpha\beta\gamma,a} + 4B_{a\alpha\beta\gamma}\big) \Delta x^\alpha \Delta x^\beta \Delta x^\gamma + \dots
\end{aligned}$$

where $B_{a\alpha\beta\gamma}$ is a factor introduced when differentiating the original $\mathcal{O}(\Delta x^4)$ term. It is clear then that the lowest order term in the expansion of $\sigma_a \sigma^a$ will be $\mathcal{O}(\Delta x^2)$ and it can be shown [4] that the coefficient of the $\mathcal{O}(\Delta x^3)$ term, $A_{\alpha\beta\gamma}$, is given by

$$A_{\alpha\beta\gamma} = -\frac{1}{4} g_{(\alpha\beta,\gamma)}.$$

Since the metric $g_{\alpha\beta}$ is known to be diagonal, and that the only coordinate γ can be is $\gamma = r$ due to the symmetry of the spacetime, we then have the $\mathcal{O}(\Delta x^3)$ term

$$A_{\alpha\alpha r} \mathcal{O}(\Delta x^3) = -\frac{1}{4} g_{(\alpha\alpha,r)} \Delta x^{(\alpha} \Delta x^\alpha \Delta x^{r)}$$

where the brackets conventionally denote symmetry of the indices. We find then that we only obtain odd powers of radial separation in the expansion of σ. Further, since $w \sim \Delta\tau$, $s \sim \Delta\tau$ and $s \sim \Delta\gamma$, all odd powers of w and s in the expansion have expansion coefficients $= 0$, since these coefficients are differentiated with respect to r only. The most important point to note is that, if we were to examine each coefficient for higher order terms in the expansion, they all depend exclusively on the metric tensor, and thus contain the same symmetries.

Moving into Mathematica, the Maclaurin series expansion of, for example, $\sigma_r'^2$ (we remove the prime at this point so as not to cause confusion with derivatives) is expressed as `Series[`σ_r^2`,{`ϵ`,0,`$\sigma_{n_{max}}$`}]`. This is a series expansion about $\epsilon = 0$ up to order $\sigma_{n_{max}}$ defined by

$$\begin{aligned}\sigma_r^2(\epsilon) &= \sum_{n=0}^{\sigma_{n_{max}}} \sigma_r^{2(n)}(0)\frac{(\epsilon-0)^n}{n!} \\ &= \sigma_r^2(0) + \frac{\partial}{\partial\epsilon}\sigma_r^2(0)\cdot\epsilon + \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\sigma_r^2(0)\cdot\epsilon^2 + \cdots + \frac{1}{\sigma_{n_{max}}!}\frac{\partial^{\sigma_{n_{max}}}}{\partial\epsilon^{\sigma_{n_{max}}}}\sigma_r^2(0)\cdot\epsilon^{\sigma_{n_{max}}}.\end{aligned}$$

We write the expansions for σ_γ and σ_τ using the same method, up to the same order. Combining these expansions with Eq. 3.3, we are almost in a position to equate coefficients. The final step before analysing in Mathematica is to subtract the summation

$$2\sigma = \sum_{ijk}\sigma_{ijk}(r)\epsilon^{i+j+k}w^i\Delta r^j s^k \tag{3.5}$$

as written explicitly in Eq. 3.3, with $i+j+k \geq 2$ (as we have discussed), and $i+j+k \leq \sigma_{n_{max}}$. Due to our treatment of w and s as $\mathcal{O}(\Delta x) \sim \mathcal{O}(\epsilon)$ terms, the information we are interested in lies in the terms between $\mathcal{O}(\epsilon^3)$ and $\mathcal{O}(\epsilon^{\sigma_{n_{max}}})$. The subtraction of 2σ permits this. n_{max} is the highest order of expansion required, and chosen as $n_{max} = d + 2(m-1) = 6$ in [6]. The term m is explained in section 3.4. Thus, with the criteria outlined above, Mathematica returns coefficients of 0 for $\mathcal{O}(\epsilon^{\leq 2})$. For $\mathcal{O}(\epsilon^3)$, the coefficients calculated are

$$4\sigma_{030}(r) + \frac{f'(r)}{f(r)^2} \quad , \quad \frac{1}{2}\left(-\frac{4}{r} + 8\sigma_{012}(r)\right) \quad \text{and} \quad \frac{1}{2}\left(\frac{4f(r)}{r} + 8\sigma_{210}(r) - 2f'(r)\right)$$

3.2 Solving for Coefficients

In [6], analogous expansions to 3.5 are assumed for $U_p(x,x^{'})$ and $V_p(x,x^{'})$ in Eq. 2.13 of the form

$$U_p(x,x^{'}) = \sum_{ijk} u^{(p)}_{ijk}(r)\epsilon^{i+j+k}w^i\Delta r^j s^k \qquad , \qquad V_p(x,x^{'}) = \sum_{ijk} v^{(p)}_{ijk}(r)\epsilon^{i+j+k}w^i\Delta r^j s^k$$

with the goal of determining the coefficients $u^{(p)}_{ijk}(r)$, $u^{(p)}_{ijk}(r)$ and $\sigma_{ijk}(r)$, for values of p as determined by the recursion relations found earlier. This is a task extremely well suited for Mathematica making use of the equations derived in Appendix C.2. We begin by taking the defining equation for σ,

$$\Box\sigma = 4 - 2\Delta^{-\frac{1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}$$

and, using $U_0 = \Delta^{\frac{1}{2}}$, rearrange as

$$\sigma^a\nabla_a U_0 + \frac{1}{2}(\Box\sigma - 4)U_0 = 0.$$

To solve this for coefficients of U_0, we first need an expression for $\Box\sigma$ in our new coordinate system, which we derive in Appendix C.2 as

$$\begin{aligned}\Box\sigma = &\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 f\sigma_r\right) - \frac{1}{\epsilon}\frac{\partial}{\partial\Delta r}\left(r^2 f\sigma_r\right) + \left(\frac{s}{r} + \frac{w^2}{2sr}(rf^{'} - 2f)\right)\frac{\partial}{\partial s}\left(r^2 f\sigma_r\right)\right] \\ &+ \frac{1}{f}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial^2\sigma}{\partial w^2} + \left(\frac{1}{\epsilon^2} + \frac{fw^2}{2r^2} - \frac{s^2}{4r^2} - \frac{fw^4\kappa^2}{4s^2} - \frac{f^2w^4}{4s^2r^2}\right)\frac{\partial^2\sigma}{\partial s^2} \\ &+ \frac{2w}{s}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial\sigma}{\partial w\partial s} - \frac{w\kappa^2}{4f}\sigma_w \\ &+ \left(\frac{2}{s\epsilon^2} - \frac{3s}{4r^2} - \frac{w^2\kappa^2}{2s} + \frac{fw^2}{2r^2 s} + \frac{f\kappa^2 w^4}{4s^3} + \frac{f^2w^4}{4s^3r^2}\right)\sigma_s.\end{aligned}$$

We then require an expression for the term $\sigma^a\nabla_a U_0$, derived in Appendix C.2 again as

$$\sigma^a \nabla_a U_0 = \frac{1}{f}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\sigma_w \frac{\partial U_0}{\partial w} + \frac{w}{s}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\left(\sigma_s \frac{\partial U_0}{\partial w} + \sigma_w \frac{\partial U_0}{\partial s}\right)$$
$$+ \left[\frac{1}{\epsilon^2} - \frac{s^2}{4r^2} + \frac{4w^2}{2r^2} - \frac{f w^4 \kappa^2}{4s^2}\left(1 + \frac{f}{\kappa^2 r^2}\right)\right]\sigma_s \frac{\partial U_0}{\partial s}$$
$$+ f\sigma_r \left[\left(\frac{f' w^2}{2s} + \frac{s}{r} - \frac{f w^2}{sr}\right)\frac{\partial U_0}{\partial s} + \frac{\partial U_0}{\partial r} - \frac{1}{\epsilon}\frac{\partial U_0}{\partial \Delta r}\right].$$

We then calculate the coefficients $u_{ijk}^{(0)}$ and turn our attention to Eq. 2.17:

$$(\Box - m^2 - \xi R)U_0 - 2\nabla^a V_0 \sigma_{;a} - 2\nabla_a U_0 \Delta^{-\frac{1}{2}} + 2V_0 = 0.$$

We recognise that, in order to solve this equation, we may rewrite as per Appendix C.3

$$(\Box - m^2 - \xi R)U_0 + 2\sigma^a \nabla_a V_0 - V_0(\Box\sigma - 4) + 2V_0 = 0. \tag{3.6}$$

In order to find the coefficients $v_{ijk}^{(0)}$ we therefore need to calculate the term $\Box U_0$ as in Appendix C.3:

$$\Box U_0 = \frac{1}{r^2}\left[\frac{\partial}{\partial r} - \frac{1}{\epsilon}\frac{\partial}{\partial \Delta r} + \left(\frac{s}{r} + \frac{w^2}{2sr}(rf' - 2f)\right)\frac{\partial}{\partial s}\right]\left[r^2 f\left(\left(\frac{f' w^2}{2s} + \frac{s}{r} - \frac{f w^2}{sr}\right)\frac{\partial U_0}{\partial s} + \frac{\partial U_0}{\partial r} - \frac{1}{\epsilon}\frac{\partial U_0}{\partial \Delta r}\right)\right]$$
$$+ \frac{1}{f}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial^2 U_0}{\partial w^2} + \left(\frac{1}{\epsilon^2} + \frac{f w^2}{2r^2} - \frac{s^2}{4r^2} - \frac{f w^4 \kappa^2}{4s^2} - \frac{f^2 w^4}{4s^2 r^2}\right)\frac{\partial^2 U_0}{\partial s^2}$$
$$+ \frac{2w}{s}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial U_0}{\partial w \partial s} - \frac{w\kappa^2}{4f}\frac{\partial U_0}{\partial w}$$
$$+ \left(\frac{2}{s\epsilon^2} - \frac{3s}{4r^2} - \frac{w^2\kappa^2}{2s} + \frac{f w^2}{2r^2 s} + \frac{f\kappa^2 w^4}{4s^3} + \frac{f^2 w^4}{4s^3 r^2}\right)\frac{\partial U_0}{\partial s}.$$

Finally, to solve Eq. 3.6 we also need $\sigma^a \nabla_a V_0$, derived in Appendix C.4 as

$$\sigma^a \nabla_a V_0 = \frac{1}{f}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\sigma_w \frac{\partial V_0}{\partial w} + \frac{w}{s}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\left(\sigma_s \frac{\partial V_0}{\partial w} + \sigma_w \frac{\partial V_0}{\partial s}\right)$$
$$+ \left[\frac{1}{\epsilon^2} - \frac{s^2}{4r^2} + \frac{4w^2}{2r^2} - \frac{f w^4 \kappa^2}{4s^2}\left(1 + \frac{f}{\kappa^2 r^2}\right)\right]\sigma_s \frac{\partial V_0}{\partial s}$$
$$+ f\sigma_r \left[\left(\frac{f' w^2}{2s} + \frac{s}{r} - \frac{f w^2}{sr}\right)\frac{\partial V_0}{\partial s} + \frac{\partial V_0}{\partial r} - \frac{1}{\epsilon}\frac{\partial V_0}{\partial \Delta r}\right].$$

All that remains to calculate now for our scheme, as determined in the recursion relations, is the coefficients of V_1. We use Eq. 2.15, which simplifies to

$$4V_1 + 2\sigma^a \nabla_a V_1 + V_1(\Box\sigma - 4) + \Box V_0 - (m^2 + \xi R)V_0 = 0$$

to calculate the coefficients $v^{(1)}_{ijk}$. With the required coefficients calculated in Mathematica, they may be grouped in terms of their coordinates and for given orders of ϵ, in order to proceed to the next section where we rename these groupings as *direct* or *tail* Hadamard coefficients.

3.3 Expansion of the Hadamard Parametrix

We now consider expanding the Hadamard parametrix in terms of the new coordinates, by setting $\Delta r = 0$. Via Mathematica or similar, it can be shown [6] that for $m = 2$ and $d = 4$,

$$\begin{aligned}\frac{U}{\sigma} + V\log\left(\frac{2\sigma}{\ell^2}\right) &= \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}^{(+)}_{ij}(r)\epsilon^{2i-2}\frac{w^{2i+2j}}{s^{2+2j}} + \sum_{i=1}^{2}\sum_{j=1}^{i}\mathcal{D}^{(-)}_{ij}(r)\epsilon^{2i-2}\frac{w^{2i-2j}}{s^{2-2j}} \\ &\quad + \log\left(\frac{2\sigma}{\ell^2}\right)\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}^{(l)}_{ij}(r)\epsilon^{2i}s^{2i-2j}w^{2j} + \sum_{i=1}^{1}\sum_{j=0}^{i}\mathcal{T}^{(p)}_{ij}(r)\epsilon^{2i}s^{2i-2j}w^{2j} \\ &\quad + \sum_{i=1}^{1}\sum_{j=0}^{1-i}\mathcal{T}^{(r)}_{ij}(r)\epsilon^{2i}s^{-2j-2}w^{2i+2j+2} + \mathcal{O}(\epsilon^4\log\epsilon),\end{aligned} \tag{3.7}$$

where $\mathcal{D}^{\pm}_{ij} = \mathcal{D}^{\pm}_{ij}(r)$ are the coefficients from the *direct* part U/σ and $\mathcal{T}_{ij} = \mathcal{T}_{ij}(r)$ are from the *tail* part $V\log(2\sigma/\ell^2)$. m is the *truncation order* term, and is explained further in section 3.4 below, but it has to do with the convergence of the mode-sum. In the code, we focus on the explicit direct and tail Hadamard terms - the $\mathcal{O}(\epsilon^4\log\epsilon)$ term can be dropped for our expressions, once we recognise that it is used in general to identify those terms that tend to zero as fast, or faster, than $\mathcal{O}(\epsilon^4\log\epsilon)$. The superscripts of $\mathcal{T}$ (l, p and r) refer to coefficients containing a *logarithm*, those that are *polynomial* and those that are *rational* in s^2 and w^2 respectively. Considering the expansion of the direct term first, we may explicitly write that

$$\begin{aligned}\frac{U_0}{\sigma} = \mathcal{D}^{+}_{00}\frac{1}{s^2\epsilon^2} + \mathcal{D}^{+}_{10}\frac{w^2}{s^2} + \mathcal{D}^{+}_{11}\frac{w^2}{s^4} + \mathcal{D}^{+}_{20}\epsilon^2\frac{w^4}{s^2} + \mathcal{D}^{+}_{21}\epsilon^2\frac{w^6}{s^4} + \mathcal{D}^{+}_{22}\epsilon^2\frac{w^8}{s^6} \\ + \mathcal{D}^{-}_{11} + \mathcal{D}^{-}_{21}\epsilon^2 w^2 + \mathcal{D}^{-}_{22}\epsilon^2 s^2\end{aligned} \tag{3.8}$$

for which each coefficient can be identified from the code. The distinction between $\mathcal{D}^{+}_{ij}$ and $\mathcal{D}^{-}_{ij}$ arises from the more general definition of the direct coefficients,

$$\frac{U}{\sigma} = \sum_{i=-2}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}(r)\epsilon^{2i-2}\frac{w^{2i+2}}{s^{2+2j}}$$

however we choose to subsume the factor of -1 for $-2 \leq i \leq -1$ into the corresponding coefficients of $\mathcal{D}_{ij}$ and then refer to them as $\mathcal{D}^{-}_{ij}$. Thus, by keeping the summation over i positive, we can let

$$\sum_{i=-2}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}(r)\epsilon^{2i-2}\frac{w^{2i+2}}{s^{2+2j}} = \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}^{(+)}_{ij}(r)\epsilon^{2i-2}\frac{w^{2i+2j}}{s^{2+2j}} + \sum_{i=1}^{2}\sum_{j=1}^{i}\mathcal{D}^{(-)}_{ij}(r)\epsilon^{2i-2}\frac{w^{2i-2j}}{s^{2-2j}}.$$

Referring back to Eq. 3.8 we notice that the $\mathcal{D}^{-}_{ij}$ coefficients for $i \geq 1$ are associated with even powers of either s or w only. This holds in the general case for any d, and it is very clear for $d = 4$. The advantage of having selected this particular coordinate expansion becomes apparent at this point, as we realise now that even powers of either s or w are polynomials in $\cos\gamma$ or $\cos\kappa\Delta\tau$ respectively - of the first order in this case. The entire point of this method, and renormalisation in general, is to identify and subtract the terms that are divergent. Terms polynomial in s^2 and w^2 in this case are polynomial in $\cos\gamma$ or $\cos\kappa\Delta\tau$ and when expressed as Legendre polynomials, do not contribute to the divergence we seek to subtract. At the coincidence limit of our point-separation technique, they do not contribute. Hence they are not useful to the method herein and we remove all $\mathcal{D}^{-}_{ij}$ terms except for the zeroth order polynomial, or the term $\mathcal{D}^{-}_{11}$. Eq. 3.7 thus becomes

$$\frac{U}{\sigma} + V\log\left(\frac{2\sigma}{\ell^2}\right) = \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\epsilon^{2i-2}\frac{w^{2i+2j}}{s^{2+2j}} + \mathcal{D}_{11}^{(-)}(r)$$
$$+\log\left(\frac{2\sigma}{\ell^2}\right)\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\epsilon^{2i}s^{2i-2j}w^{2j} + \sum_{i=1}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(p)}(r)\epsilon^{2i}s^{2i-2j}w^{2j}$$
$$+\sum_{i=1}^{1}\sum_{j=0}^{1-i}\mathcal{T}_{ij}^{(r)}(r)\epsilon^{2i}s^{-2j-2}w^{2i+2j+2}.$$

The tail part of the expansion is written explicitly as

$$V\log\left(\frac{2\sigma}{\ell^2}\right) = \log\left(\frac{\epsilon^2 s^2}{\ell^2}\right)\left(\mathcal{T}_{00}^{(l)} + \mathcal{T}_{10}^{(l)}\epsilon^2 s^2 + \mathcal{T}_{11}^{(l)}\epsilon^2 w^2\right)$$
$$+\mathcal{T}_{10}^{(p)}\epsilon^2 s^2 + \mathcal{T}_{11}^{(p)}\epsilon^2 w^2 + \mathcal{T}_{10}^{(r)}\epsilon^2\frac{w^4}{s^2}.$$

An immediate consequence of the polynomial argument made for the direct part is that the same consequence follows for the two $\mathcal{T}_{ij}^{p}$ terms above. They will similarly fail to contribute divergences at the coincidence limit since they can be expressed finitely, and so do not contain information of value to this method. We may remove them without losing any of the divergence. Eq. 3.7 then further simplifies to

$$\frac{U}{\sigma} + V\log\left(\frac{2\sigma}{\ell^2}\right) = \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\epsilon^{2i-2}\frac{w^{2i+2j}}{s^{2+2j}} + \mathcal{D}_{11}^{(-)}(r)$$
$$+\log\left(\frac{\epsilon^2 s^2}{\ell^2}\right)\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(1)}(r)\epsilon^{2i}s^{2i-2j}w^{2j}$$
$$+\sum_{i=1}^{1}\sum_{j=0}^{1-i}\mathcal{T}_{ij}^{(r)}(r)\epsilon^{2i}s^{-2j-2}w^{2i+2j+2}.$$

Having established that even powers of s and w are polynomials in cos, the $\mathcal{T}_{ij}^{r}$ terms are, by definition, rational[3]. "Unlike the polynomial terms, these are not ordinary integrable functions near coincidence" [6], hence we retain them. These terms, although not divergent, will assist us in speeding up the convergence of the mode-sum. We notice that we have only one of them, $\mathcal{T}_{10}$, hence Eq. 3.7 finally reduces

[3] A rational function is any function which can be written as the ratio of two polynomial functions, where the polynomial in the denominator is not equal to zero. Note that the terms containing $\mathcal{D}_{ij}^{+}$ for $i \geq 1$ are also rational.

to

$$\frac{U}{\sigma} + V \log\left(\frac{2\sigma}{\ell^2}\right) = \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}_{ij}^{(+)}(r)\epsilon^{2i-2}\frac{w^{2i+2j}}{s^{2+2j}} + \mathcal{D}_{11}^{(-)}(r)$$
$$+ \log\left(\frac{\epsilon^2 s^2}{\ell^2}\right)\sum_{i=0}^{l}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(1)}(r)\epsilon^{2i}s^{2i-2j}w^{2j} \tag{3.9}$$
$$+ \mathcal{T}_{10}^{(r)}(r)\epsilon^2\frac{w^4}{s^2}$$

or explicitly:

$$\frac{U}{\sigma} + V\log\left(\frac{2\sigma}{\ell^2}\right) = \mathcal{D}_{00}^{+}\frac{1}{s^2\epsilon^2} + \mathcal{D}_{10}^{+}\frac{w^2}{s^2} + \mathcal{D}_{11}^{+}\frac{w^4}{s^4} + \mathcal{D}_{20}^{+}\epsilon^2\frac{w^4}{s^2} + \mathcal{D}_{21}^{+}\epsilon^2\frac{w^6}{s^4} + \mathcal{D}_{22}^{+}\epsilon^2\frac{w^8}{s^6}$$
$$+ \mathcal{D}_{11}^{-} + \log\left(\frac{\epsilon^2 s^2}{\ell^2}\right)\left(\mathcal{T}_{00}^{(l)} + \mathcal{T}_{10}^{(l)}\epsilon^2 s^2 + \mathcal{T}_{11}^{(l)}\epsilon^2 w^2\right) + \mathcal{T}_{10}^{(r)}\epsilon^2\frac{w^4}{s^2}.$$

Using ϵ to isolate the desired orders of w and s in Mathematica, the values of the coefficients are isolated and given in the tables below (we let $f(r) = f$ and $f' = \frac{\partial f}{\partial r}$ for simplicity):

TABLE 3.1: $\mathcal{D}_{ij}^{(+)}(r)$ coefficients for $d = 4$ in terms of general $f(r)$

$\mathcal{D}_{ij}^{(+)}(r)$ coefficients for $d=4$			
	$j=0$	$j=1$	$j=2$
$i=0$	2		
$i=1$	$-\frac{f}{12r^2}(r^2 f'' + 2f - 2rf' - 2)$	$\frac{f}{24r^2}(r^2(f'^2 - 4\kappa^2) - 4f\left(rf'+1\right) + 4f^2)$	
$i=2$	$\frac{f}{2880r^4}(-5r^2\left(4\kappa^2 - f'^2\right) \times \left(r^2 f'' - 2rf' - 2\right) - 8f^2(3r^3 f^{(3)} - 7r^2 f'' + 19rf' + 10) + f(9r^4 f''^2 - 20r^2 f'' + 86r^2 f'^2 + 4rf' \times (3r^3 f^{(3)} - 14r^2 f'' + 20) - 40\kappa^2 r^2 + 4) + 76f^3)$	$-\frac{f}{2880r^4}(r^4(-20\kappa^2 f'^2 + f'^4 + 64\kappa^4) + r^2 f \times \left(-20\kappa^2\left(r^2 f'' - 6\right) + 120\kappa^2 rf' - 30rf'^3 + f'^2\left(11r^2 f'' - 30\right)\right) + 4f^3(11r^2 f'' - 52rf' - 40) - 2f^2(10r^2 f'' - 67r^2 f'^2 + f'(22r^3 f'' - 80r) + 60\kappa^2 r^2 - 28) + 104f^4)$	$\frac{f^2}{1152r^4}(r^2(f'^2 - 4\kappa^2) - 4f(rf'+1) + 4f^2)^2$

TABLE 3.2: $\mathcal{D}_{ij}^{(-)}(r)$ coefficients for $d=4$ in terms of general $f(r)$

$\mathcal{D}_{ij}^{(-)}(r)$ coefficients for $d=4$			
	$j=0$	$j=1$	$j=2$
$i=0$			
$i=1$		$-\dfrac{f'}{6r}$	
$i=2$		0	0

TABLE 3.3: $\mathcal{T}_{ij}^{(l)}(r)$ coefficients for $d=4$ in terms of general $f(r)$

$\mathcal{T}_{ij}^{(l)}(r)$ coefficients for $d=4$		
	$j=0$	$j=1$
$i=0$	$\frac{1}{12r^2}\Big(-(6\xi-1)r\left(rf''+4f'\right) + (2-12\xi)f+6m^2r^2+12\xi-2\Big)$	
$i=1$	$\frac{1}{480r^4}\Big(-60m^2\xi r^4f''+10m^2r^4f'' + 30\xi^2r^4f''^2-10\xi r^4f''^2+r^4f''^2 - 120\xi^2r^2f''+20\xi r^2f'' + 4\left(120\xi^2-25\xi+1\right)r^2f'^2 - 2rf'\Big((1-5\xi)r^3f^{(3)} + 2\left(-60\xi^2+5\xi+1\right)r^2f'' + 10m^2(12\xi-1)r^2 + 40\xi(6\xi-1) + 2f\Big(5\xi r^4f^{(4)}-r^4f^{(4)}+40\xi r^3f^{(3)} - 7r^3f^{(3)}+4\left(15\xi^2+10\xi-2\right)r^2f'' + 2\left(120\xi^2-40\xi+3\right)rf' - 60m^2\xi r^2-120\xi^2+20\xi + 4\left(30\xi^2-10\xi+1\right)f^2 + 30m^4r^4+120m^2\xi r^2+120\xi^2-4\Big)$	$\frac{f}{480r^4}(-2\left(r^2f''+2\right)\big((1-5\xi)r^2f'' + 5m^2r^2+10\xi-2+2(1-10\xi)r^2f'^2 + rf'\Big((10\xi-1)r^3f^{(3)} + 4(20\xi-3)r^2f''+20(6\xi-1) + rf\left((16-80\xi)f'\right. + r\Big(r^2f^{(4)}+(6-20\xi)rf^{(3)} + (12-80\xi)f''+20m^2 + 8(5\xi-1)f^2)$

TABLE 3.4: $\mathcal{T}_{ij}^{(r)}(r)$ coefficients for $d=4$ in terms of general $f(r)$

$\mathcal{T}_{ij}^{(r)}(r)$ coefficients for $d=4$		
	$j=0$	$j=1$
$i=0$		
$i=1$	$\frac{f}{576r^4}(r^2\left(f'^2-4\kappa^2\right)$ $-4f\left(rf'+1\right)+4f^2)$ $\times(6\xi r^2 f''-r^2 f''+4(6\xi-1)rf'$ $+2(6\xi-1)f-6m^2r^2-12\xi+2)$	

3.4 Convergence

Say we chose $m=0$ as the truncation order instead of $m=2$ (for $d=4$), our expansion of the Hadamard Parametrix would instead simply be

$$\frac{U}{\sigma}+V\log\left(\frac{2\sigma}{\ell^2}\right)=\mathcal{D}_{00}^{+}\frac{1}{s^2\epsilon^2}+\mathcal{O}(\log\epsilon)$$

which does not in fact capture all the singular terms, since no tail terms feature in the expansion. However, for $m=1$, we obtain

$$\frac{U}{\sigma}+V\log\left(\frac{2\sigma}{\ell^2}\right)=\mathcal{D}_{00}^{+}\frac{1}{s^2\epsilon^2}+\mathcal{D}_{10}^{+}\frac{w^2}{s^2\epsilon^2}+\mathcal{D}_{11}^{+}\frac{w^4}{s^4\epsilon^2}+\log\left(\frac{\epsilon^2s^2}{\ell^2}\right)\mathcal{T}_{00}^{(l)}+\mathcal{O}(\epsilon\log\epsilon)$$

we do capture all singular terms. However *convergence* of the "mode-sum expression for the regularized Green function" [6] is less efficient than for higher order terms, eg $m=2$ or $m=3$. By taking $m=2$ as we have done in this work, we then have coordinates expanded as $\frac{1}{s^2}, \frac{w^2}{s^2}, \frac{w^4}{s^4}, \frac{w^4}{s^2}, \frac{w^6}{s^4}, \frac{w^8}{s^6}, \log(\epsilon^2s^2/\ell^2), \log(\epsilon^2s^2/\ell^2)s^2$ and $\log(\epsilon^2s^2/\ell^2)w^2$ and the resulting mode-sum converges faster, as described succinctly by Taylor and Breen. This choice of m is also the least for which all of the direct and tail Hadamard terms are included in the mode-sum - a requirement for the calculation of the renormalised stress-energy tensor. While polynomial terms in w^2 and s^2 vanish in the coincidence limit, the presence of terms such as w^4/s^2 in the expansion aid convergence, so we retain them. We may also observe at this point, aided by this justification, that owing to our choice of new coordinates w and s, we have not confined our point-splitting to any particular direction. Since w is a function of temporal seperation and s is a function of w and γ (geodesic distance on the 2-sphere), we have a natural mechanism of point-splitting which does not create a difficulty when performing mode-sum calculations. This is unlike previous work where committing to a particular direction of point-splitting prevented mode-by-mode regularisation. Hence, we have gained a natural advantage through the choice of coordinates introduced by Taylor and Breen. The speed of convergence of the mode-sum can thus be controlled by the ability to subtract the resulting mode-sum expression as a function of both summation over n and over l independently, thanks to our definitions of w and s.

3.5 Summary

Having defined our new choice of coordinates, dependent only on $\Delta\tau$, r and γ, we expanded Synge's world function σ and introduced a bookkeeping term ϵ to maintain strict computational oversight on the order of the expansion in terms of coordinates $(w, s, \Delta r, r)$. Combined with the symmetry of our chosen spherically-symmetric black hole spacetime, we discovered that only orders of ϵ^2 and higher featured in the expansion, and only even powers of our coordinates w and s survived. Using then our recurrence relations and the defining equation for σ, we solved for the coefficients of the expansions of Hadamard terms $U(x, x^{'})$ and $V(x, x^{'})$ and thus expanded the Hadamard parametrix. Isolating and calculating the direct and tail terms within this expansion was conducted in Mathematica. In the next chapter, we will investigate how to regularise our coordinate-expanded Hadamard terms, by making extensive use of the properties and various identities of Legendre functions.

Chapter 4

Mode-Sum Representation of the Hadamard Parametrix

Having now an expression for the Hadamard parametrix in terms of direct and tail Hadamard coefficients and contributing terms in (w, s) at the coincidence limit - Eq. 3.9 - the goal now is to express alternatively as a function of multipole moments using Fourier frequency modes. This method is particularly beneficial when applying the point-splitting method as it permits splitting in any direction owing to the spherical harmonics intrinsic to the choice of Legendre polynomials and Fourier multipole modes n. In this way, the Hadamard parametrix can be expressed as the sum of mode contributions, so that they may be subtracted mode-by-mode, removing the divergence and thus permitting the regularisation of quantities such as the vacuum polarisation or stress-energy tensor. As previously discussed, for a mode n and Legendre polynomial degree l the Euclidean Green function is represented as

$$G(x, x^{'}) = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) g_{nl}(r, r^{'}) \tag{4.1}$$

where $g_{nl}(r, r^{'})$ is the radial Green function. Changing coordinate system from Euclidean to spherical permits a mode-sum decomposition in terms of Legendre polynomials, or 'spherical harmonics'. Summing to infinity over n and l returns a complete description of the Hadamard parametrix due to the *completeness* property of Legendre polynomials. The task then is to take the divergent terms at coincidence and equate them to mode-sum descriptions (as with the Euclidean Green function above), and multiply by appropriate *regularisation parameters* to convert to closed-form expressions that may later be subtracted. Due to the existence of the log term in the tail, the parameters for the direct and the tail need to be approached separately.

4.1 Regularisation Parameters for the Direct Part

We first "assume a Fourier frequency and multipole decomposition" [6] of the form

$$\frac{w^{2i\pm 2j}}{s^{2\pm 2j}} = \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma) \overset{[d]}{\Psi}{}^{(\pm)}_{nl}(i,j\mid r)$$

where $\overset{[d]}{\Psi}{}^{(\pm)}_{nl}(i,j \mid r)$ is the regularization parameter, so that we can subtract from Eq. 4.1 mode-by-mode. We wish to invert the equation above and determine the parameters $\overset{[d]}{\Psi}{}^{(\pm)}_{nl}(i,j \mid r)$. As in [6], we first multiply both sides by $e^{-in'\kappa\Delta\tau}P_{l'}(x)$ where we let $x = \cos\gamma$ to obtain

$$\frac{w^{2i\pm 2j}}{s^{2\pm 2j}} e^{-in'\kappa\Delta\tau}P_{l'}(x) = \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau - in'\Delta\tau} \sum_{l=0}^{\infty}(2l+1)P_l(x)P_{l'}(x) \overset{[d]}{\Psi}{}^{(\pm)}_{nl}(i,j\mid r).$$

We then integrate, giving

$$\int_{-1}^{1}\frac{w^{2i\pm 2j}}{s^{2\pm 2j}} P_{l'}(x)dx \int_0^{2\pi/\kappa} e^{-in'\kappa\Delta\tau}d\Delta\tau = \sum_{n=-\infty}^{\infty}\int_0^{2\pi/\kappa} e^{in\kappa\Delta\tau - in'\Delta\tau}d\Delta\tau \sum_{l=0}^{\infty}(2l+1)$$
$$\times \int_{-1}^{1} P_l(x)P_{l'}(x)dx \overset{[d]}{\Psi}{}^{(\pm)}_{nl}(i,j\mid r).$$

Using the completeness relations

$$\int_0^{2\pi/\kappa} e^{in\Delta\tau - in'\Delta\tau}d\Delta\tau = \frac{2\pi}{\kappa}\delta_{nn'}$$

and

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{\Gamma(n+1)}{(l+\frac{1}{2})l!}\delta_{ll'}$$

we obtain, when $n = n'$ and $l = l'$:

$$\int_{-1}^{1} \frac{w^{2i\pm 2j}}{s^{2\pm 2j}} P_l(x)dx \int_0^{2\pi/\kappa} e^{-in\kappa\Delta\tau} d\Delta\tau = \frac{2\pi}{\kappa} \frac{(2l+1)\Gamma(n+1)}{(l+\frac{1}{2})l!} \overset{[d]}{\Psi}{}^{(\pm)}_{nl} (i,j \mid r)$$
$$= \frac{4\pi}{\kappa} \frac{\Gamma(n+1)}{l!} \overset{[d]}{\Psi}{}^{(\pm)}_{nl} (i,j \mid r).$$

Thus

$$\overset{[d]}{\Psi}{}^{(\pm)}_{nl} (i,j \mid r) = \frac{\kappa}{4\pi} \frac{l!}{\Gamma(n+1)} \int_{-1}^{1} \frac{w^{2i\pm 2j}}{s^{2\pm 2j}} P_l(x)dx \int_0^{2\pi/\kappa} e^{-in\kappa\Delta\tau} d\Delta\tau.$$

Setting $z = 1 + \frac{f^2}{\kappa^2 r^2}(1 - \cos\kappa t)$, we employ the identity [18]

$$\int_{-1}^{1} \frac{P_l(x)}{(z-x)^{1\pm j}} dx = \frac{2(-1)^j \Gamma(l+1)(z^2-1)^{\mp j/2}}{l!\Gamma(1\pm j)} Q_l^{\pm j}(z),$$

and that

$$w^{2i\pm 2j} = \frac{2^{i\pm j}}{\kappa^{2i\pm 2j}} (1 - \cos\kappa\Delta\tau)^{i\pm j}$$

to obtain

$$\begin{aligned}
\overset{[d]}{\Psi}{}^{(\pm)}_{nl} (i,j \mid r) &= \frac{\kappa}{4\pi} \frac{l!}{\Gamma(n+1)} \int_{-1}^{1} \frac{P_l(x)}{(z-x)^{1\pm j}} \frac{w^{2i\pm 2j}(z-x)^{i\pm j}}{s^{2\pm 2j}} dx \int_0^{2\pi/\kappa} e^{-in\kappa\Delta\tau} d\Delta\tau \\
&= \frac{\kappa}{2\pi} \frac{(-1)^j \Gamma(l+1)}{\Gamma(n+1)\Gamma(1\pm j)} Q_l^{\pm j}(z) \frac{2^{i\pm j}(z-x)^{i\pm j}}{\kappa^{2i\pm 2j} s^{2\pm 2j}} \int_0^{2\pi/\kappa} (1-\cos\kappa\Delta\tau)^{i\pm j} e^{-in\kappa\Delta\tau} (z^2-1)^{\mp j/2} d\Delta\tau \\
&= \frac{\kappa}{2\pi} \frac{(-1)^j}{\Gamma(1\pm j)} Q_l^{\pm j}(z) \frac{2^{i\pm j}(z-x)^{i\pm j}}{\kappa^{2i\pm 2j} s^{2\pm 2j}} \int_0^{2\pi/\kappa} (1-\cos\kappa\Delta\tau)^{i\pm j} e^{-in\kappa\Delta\tau} (z^2-1)^{\mp j/2} d\Delta\tau
\end{aligned} \tag{4.2}$$

Since $s^2 = fw^2 + (1-\cos\gamma) = fw^2 + (1-x)$, then $z - x = s^2/2r^2$ and

$$(z-x)^{1\pm j} = \frac{s^{2\pm 2j}}{2^{1\pm j} r^{2\pm 2j}}. \tag{4.3}$$

We therefore find that

$$\overset{[d]}{\Psi}{}^{(\pm)}_{nl} (i,j \mid r) = \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i\pm 2j} r^{2\pm 2j} \Gamma(1\pm j)} \int_0^{2\pi/\kappa} (1-\cos\kappa t)^{i\pm j} e^{-in\kappa t} (z^2-1)^{\mp j/2} Q_l^{\pm j}(z) dt. \tag{4.4}$$

We now look at $\overset{[d]}{\Psi}{}^{(+)}_{nl} (i,j \mid r)$ and $\overset{[d]}{\Psi}{}^{(-)}_{nl} (i,j \mid r)$ separately.

4.1.1 The Time Integral

In order to make the regularization procedures in the following sections more tractable, we wish to find a more useful expression for

$$\int_0^{2\pi/\kappa} (1-\cos\kappa t)^i e^{-in\kappa t} \cos p\kappa t dt.$$

For this task, we turn to binomial expansions and exponential representation of the trigonometric functions, as in [6]. First, we expand $(1-\cos i\kappa t)^i$ as

$$\begin{aligned}(1-\cos i\kappa t)^i &= \sum_{m=0}^{i} \binom{i}{m} 1^{i-m}(-\cos\kappa t)^m \\ &= \sum_{m=0}^{i} \binom{i}{m} (-1)^m (\cos\kappa t)^m \\ &= \sum_{m=0}^{i} \binom{i}{m} (-1)^m (\frac{1}{2}(e^{i\kappa t}+e^{-i\kappa t}))^m \\ &= \sum_{m=0}^{i} \binom{i}{m} \frac{(-1)^m}{2^m} (e^{i\kappa t}+e^{-i\kappa t})^m.\end{aligned}$$

Further expanding, we obtain

$$\begin{aligned}(1-\cos\kappa t)^i &= \sum_{m=0}^{i} \binom{i}{m} \frac{(-1)^m}{2^m} \sum_{s=0}^{m} \binom{m}{s} (e^{i\kappa t})^{m-s}(e^{-i\kappa t})^s \\ &= \sum_{m=0}^{i} \binom{i}{m} \frac{(-1)^m}{2^m} \sum_{s=0}^{m} \binom{m}{s} e^{im\kappa t} e^{-2is\kappa t} \\ &= \sum_{m=0}^{i} \sum_{s=0}^{m} \binom{i}{m} \binom{m}{s} \frac{(-1)^m}{2^m} e^{im\kappa t} e^{-2is\kappa t}\end{aligned}$$

Since,

$$\cos p\kappa t = \frac{1}{2}(e^{ip\kappa t}+e^{-ip\kappa t})$$

the time integral thus becomes

$$\int_0^{2\pi/\kappa} (1-\cos\kappa t)^i e^{-in\kappa t} \cos p\kappa t dt = \frac{1}{2} \sum_{m=0}^{i} \sum_{s=0}^{m} \binom{i}{m} \binom{m}{s} \frac{(-1)^m}{2^m} \int_0^{2\pi/\kappa} e^{im\kappa t} e^{-2is\kappa t} e^{-in\kappa t} (e^{ip\kappa t}+e^{-ip\kappa t}) dt.$$

Simplifying this expression, we find that

$$\binom{i}{m}\binom{m}{s} = \frac{i!}{m!(i-m)!} \frac{m!}{s!(m-s)!} = \frac{i!}{s!(i-m)!(m-s)!}$$

hence

$$\begin{aligned}\int_0^{2\pi/\kappa}(1-\cos\kappa t)^i e^{-in\kappa t}\cos p\kappa t dt &= \frac{1}{2}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m}\\ &\quad\times\int_0^{2\pi/\kappa}e^{im\kappa t}e^{-2is\kappa t}e^{-in\kappa t}(e^{ip\kappa t}+e^{-ip\kappa t})dt\\ &= \frac{1}{2}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m}\\ &\quad\times\int_0^{2\pi/\kappa}\left(e^{i(p-(2s+n-m))\kappa t}+e^{i(p-(m-2s-n))\kappa t}\right)dt\\ &= \frac{\pi}{\kappa}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m}\left(\delta_{p,2s+n-m}+\delta_{p,m-2s-n}\right).\end{aligned}$$

We have in the process also found that

$$\begin{aligned}\int_0^{2\pi/\kappa}(1-\cos\kappa t)^i e^{-in\kappa t}dt &= \sum_{m=0}^{i}\sum_{s=0}^{m}\binom{i}{m}\binom{m}{s}\frac{(-1)^m}{2^m}\int_0^{2\pi/\kappa}e^{im\kappa t}e^{-2is\kappa t}e^{-in\kappa t}dt\\ &= \frac{2\pi}{\kappa}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m}\delta_{m,2s+n}\\ &= \frac{2\pi}{\kappa}\sum_{s=0}^{i}\frac{i!}{s!(i-2s-n)!(s+n)!}\frac{(-1)^n}{2^{2s+n}}\end{aligned}$$

4.1.2 $\overset{[d]}{\Psi}{}^{(+)}_{nl}$

By defining

$$\eta \equiv \sqrt{1+\frac{f}{\kappa^2 r^2}}$$

and using the fact that

$$(z^2-1)^{-j/2}Q_l^j(z) = \frac{(-1)^j}{2^j(1-\cos\kappa t)^j}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j Q_l^j(z)$$

we can substitue into Eq 4.4 to obtain

$$\overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r) = \frac{\kappa}{2\pi}\frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j\int_0^{2\pi/\kappa}(1-\cos\kappa t)^i e^{-in\kappa t}Q_l(z)dt.$$

Now we may rewrite the associated Legendre function $Q_l(z)$ without a time dependency by using the theorem [19]

$$Q_l(z) = P_l(\eta)Q_l(\eta) + 2\sum_{p=1}^{\infty}(-1)^p P_l^{-p}(\eta)Q_l^p(\eta)\cos p\kappa t \tag{4.5}$$

and the time integral is resolved using the identity derived in section 4.1.1. Thus we obtain

$$\begin{aligned}
{}^{[d]}\Psi_{nl}^{(+)}(i,j\mid r) &= \frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j P_l(\eta)Q_l(\eta)\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m}\delta_{m,2s+n} \\
&\quad + \frac{\kappa}{\pi}\frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j \sum_{p=1}^{\infty}(-1)^p P_l^{-p}(\eta)Q_l^p(\eta) \\
&\quad \times\left(\frac{\pi}{\kappa}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m}\left(\delta_{p,2s+n-m}+\delta_{p,m-2s-n}\right)\right).
\end{aligned}$$

The only p terms that survive are $p = 2s+n-m$ and $p = m-2s-m$, as the δ terms collapse at these points, we drop the summations and we arrive at the equation

$$\begin{aligned}
{}^{[d]}\Psi_{nl}^{(+)}(i,j\mid r) &= \frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m} \\
&\quad\times\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j\Big[P_l(\eta)Q_l(\eta)\delta_{m,2s+n} + (-1)^{(2s+n-m)}P_l^{(m-2s-n)}(\eta)Q_l^{(2s+n-m)}(\eta) \\
&\quad + (-1)^{(m-2s-n)}P_l^{(2s+n-m)}(\eta)Q_l^{(m-2s-n)}(\eta)\Big].
\end{aligned} \tag{4.6}$$

This expression is easily implemented then in Mathematica and for given input i and j, the appropriate regularisation expression is calculated. This alternative expression was compared, for many values, with the equivalent expression of [6] and results were in agreement.

4.1.3 ${}^{[d]}\Psi_{nl}^{(-)}$

From Eq. 4.4, we now want an explicit and simplified expression for ${}^{[d]}\Psi_{nl}^{(-)}$. We begin with

$$ {}^{[d]}\Psi_{nl}^{(-)}(i,j\mid r) = \frac{\kappa}{2\pi}\frac{2^{i-1}(-1)^j}{\kappa^{2i-2j}r^{2-2j}\Gamma(1-j)}\int_0^{2\pi/\kappa}(1-\cos\kappa t)^{i-j}e^{-in\kappa t}(z^2-1)^{j/2}Q_l^{-j}(z)dt.$$

As per [6], we will employ the identity [19]

$$(z^2-1)^{j/2}Q_l^{-j} = \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4l+1}{(l-k+\frac{1}{2})_{j+1}} Q_{l+j-2k}(z)$$

to obtain

$$\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) = \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i-2j} r^{2-2j}\Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4l+1}{(l-k+\frac{1}{2})_{j+1}} \\ \times \int_0^{2\pi/\kappa} (1-\cos\kappa t)^{i-j} e^{-in\kappa t} Q_{l+j-2k}(z) dt.$$

We now repeat the same procedure as in the previous section and apply the theorem in Eq. 4.5:

$$\begin{aligned}
\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) &= \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i-2j} r^{2-2j}\Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}} \\
&\quad \times \int_0^{2\pi/\kappa} (1-\cos\kappa t)^{i-j} e^{-in\kappa t} \\
&\qquad \times \left(P_{l+j-2k}(\eta) Q_{l+j-2k}(\eta) + 2\sum_{p=1}^{\infty} (-1)^p P^{-p}_{l+j-2k}(\eta) Q^{p}_{l+j-2k}(\eta) \cos p\kappa t \right) dt \\
&= \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i-2j} r^{2-2j}\Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}} \\
&\quad \times \Bigg(P_{l+j-2k}(\eta) Q_{l+j-2k}(\eta) \int_0^{2\pi/\kappa} (1-\cos\kappa t)^{i-j} e^{-in\kappa t} dt \\
&\qquad + 2\sum_{p=1}^{\infty} (-1)^p P^{-p}_{l+j-2k}(\eta) Q^{p}_{l+j-2k}(\eta) \int_0^{2\pi/\kappa} (1-\cos\kappa t)^{i-j} e^{-in\kappa t} \cos p\kappa t dt \Bigg) \\
&= \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i-2j} r^{2-2j}\Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}} \\
&\quad \times \Bigg(P_{l+j-2k}(\eta) Q_{l+j-2k}(\eta) \sum_{m=0}^{i-j} \sum_{s=0}^{m} \frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m} \int_0^{2\pi/\kappa} e^{im\kappa t} e^{-2is\kappa t} e^{-in\kappa t} dt \\
&\quad + 2\sum_{p=1}^{\infty} (-1)^p P^{-p}_{l+j-2k}(\eta) Q^{p}_{l+j-2k}(\eta) \\
&\qquad \times \frac{1}{2} \sum_{m=0}^{i-j} \sum_{s=0}^{m} \frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m} \int_0^{2\pi/\kappa} e^{im\kappa t} e^{-2is\kappa t} e^{-in\kappa t} (e^{ip\kappa t} + e^{-ip\kappa t}) dt \Bigg),
\end{aligned}$$

simplifying to

$$\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) = \frac{\kappa}{2\pi}\frac{2^{i-1}(-1)^j}{\kappa^{2i-2j}r^{2-2j}\Gamma(1-j)}\sum_{k=0}^{j}\frac{(-1)^k}{2^{j+1}}\binom{j}{k}\frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}}\sum_{m=0}^{i-j}\sum_{s=0}^{m}\frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m}$$
$$\times\Big[P_{l+j-2k}(\eta)Q_{l+j-2k}(\eta)\delta_{m,2s+n}+\sum_{p=1}^{\infty}(-1)^pP^{-p}_{l+j-2k}(\eta)Q^{p}_{l+j-2k}(\eta)\delta_{p,2s+n-m}$$
$$+\sum_{p=1}^{\infty}(-1)^pP^{-p}_{l+j-2k}(\eta)Q^{p}_{l+j-2k}(\eta)\delta_{p,m-2s-n}\Big].$$

As before, the only surviving p terms are $p=2s+n-m$ and $p=m-2s-n$, thus we are left with

$$\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) = \frac{\kappa}{2\pi}\frac{2^{i-1}(-1)^j}{\kappa^{2i-2j}r^{2-2j}\Gamma(1-j)}\sum_{k=0}^{j}\frac{(-1)^k}{2^{j+1}}\binom{j}{k}\frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}}\sum_{m=0}^{i-j}\sum_{s=0}^{m}\frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m}$$
$$\times\Big[P_{l+j-2k}(\eta)Q_{l+j-2k}(\eta)\delta_{m,2s+n}+(-1)^{(2s+n-m)}P^{(m-n-2s)}_{l+j-2k}(\eta)Q^{(2s+n-m)}_{l+j-2k}(\eta)$$
$$+(-1)^{(m-2s-n)}P^{(2s+n-m)}_{l+j-2k}(\eta)Q^{(m-2s-n)}_{l+j-2k}(\eta)\Big]. \tag{4.7}$$

This expression was again programmed into Mathematica and for a range of values of i and j, results were in agreement with the equivalent expression in [6]. Having now derived the regularisation parameters $\overset{[d]}{\Psi}{}^{(\pm)}_{nl}$ for the direct part, the next task is to do likewise for the tail.

4.2 Tail Regularisation Parameters $\overset{[d]}{\chi}_{nl}$

We begin with assuming the expansion of the form

$$s^{2i-2j}w^{2j}\log\left(\frac{s^2}{l^2}\right)=\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l\cos\gamma\,\overset{[d]}{\chi}_{nl}(i,j|r)$$

where $\overset{[d]}{\chi}_{nl}(i,j|r)$ are the regularisation parameters. Using completeness relations (show) we can represent these tail regularisation parameters [6] as

$$\overset{[d]}{\chi}_{nl}(i,j|r)=\frac{\kappa}{4\pi}\int_0^{2\pi/\kappa}\int_{-1}^{1}w^{2j}e^{-in\kappa\Delta\tau}\log\left(\frac{s^2}{l^2}\right)s^{2i-2j}P_l(x)dxd\Delta\tau. \tag{4.8}$$

Mirroring the approach of Taylor and Breen, we will integrate by parts to make the integral more tractable. In particular, we seek a more tractable expression for the general integral of the form

$$\int_{-1}^{1} \log{(z-x)}(z-k)^k P_l(x) dx.$$

We can then apply this general result to Eq. 4.8 having made the appropriate variable substitutions.

4.3 Integration by Parts

Eq. (42) of [6] reduces in our scheme to

$$\int_{-1}^{1} \log{(z-x)}(z-k)^k P_l(x) dx$$

and by substituting in the standard definition of a Legendre function

$$P_l(x) = \frac{(-1)^l}{2^l l!} \frac{d^l}{dx^l} \left[(1-x^2)^l \right],$$

we then wish to find a useful expression for

$$\frac{(-1)^l}{2^l l!} \int_{-1}^{1} \log{(z-x)}(z-x)^k \frac{d^l}{dx^l} \left[(1-x^2)^l \right] dx. \tag{4.9}$$

To do so, we integrate by parts a number of times. Let us use n to keep track of the number of integrations we perform. For the first iteration ($n = 1$), using the standard notation $\int udv = uv - \int vdu$, we let

$$u = \log{(z-x)}(z-k)^k \qquad \text{and} \qquad dv = \frac{d^l}{dx^l} \left[(1-x^2)^l \right] dx.$$

Then

$$du = -(z-x)^{k-1} \left[1 + k \log{(z-k)^k} \right] dx$$

and

$$v = \int_{-1}^{1} dv = \int_{-1}^{1} \frac{d^l}{dx^l} \left[(1-x^2)^l \right] dx = \frac{d^{l-1}}{dx^{l-1}} \left[(1-x^2)^l \right].$$

A key feature of this integration, making the expressions we will work with much simpler, is revealed when we look closer at the uv term. For any order of integration

n we know, for every order $l-n$, that $l-n<l$. Thus when calculating the term $\frac{d^{l-n}}{dx^{l-n}}\left[(1-x^2)^l\right]$ in the uv part, we are always left with a term $f(x)(1-x^2)$ where $f(x)$ is some arbitrary function. Now, when we evaluate this over the interval $[-1,1]$, we find that, true for each order of integration:

$$\left[uv\right]_{-1}^{1} = \left[f(x)(1-x^2)^l\right]_{-1}^{1} = f(1)(1-1)^l - f(-1)(1-1)^l = 0.$$

Hence, we discard the uv term for each integration, permitting us to rewrite Eq. 4.9 (ignoring for now the constant term in front) as

$$\int_{-1}^{1}(z-x)^{k-1}\left[1+k\log{(z-x)^k}\right]\frac{d^{l-1}}{dx^{l-1}}\left[(1-x^2)^l\right]dx.$$

Let us proceed with $n=2$, $n=3$ and $n=4$ to determine the pattern and thus generalise.

- $n=2$:

  First, we let $u=(z-x)^{k-1}\left[1+k\log{(z-k)^k}\right]$ and $dv=\frac{d^{l-1}}{dx^{l-1}}\left[(1-x^2)^l\right]dx$. Then

  $$du = -(z-x)^{k-2}\left[(k-1)+k+k(k-1)\log{(z-x)}\right]dx$$

  and since $v=\int\frac{d^{l-1}}{dx^{l-1}}\left[(1-x^2)^l\right]dx=\frac{d^{l-2}}{dx^{l-2}}\left[(1-x^2)^l\right]$, then the expression for $n=2$ becomes

  $$\int_{-1}^{1}(z-x)^{k-2}\left[(k-1)+k+k(k-1)\log{(z-x)}\right]\frac{d^{l-2}}{dx^{l-2}}\left[(1-x^2)^l\right]dx$$

- $n=3$:

  Using similar choices for u and dv, we obtain and so the whole expression for $n=3$ becomes

  $$\int_{-1}^{1}(z-x)^{k-3}\left[(k-1)(k-2)+k(k-2)+k(k-1)\right.$$
  $$\left.+k(k-1)(k-2)\log{(z-x)}\right]\frac{d^{l-3}}{dx^{l-3}}\left[(1-x^2)^l\right]dx$$

- $n = 4$

  Repeating, we obtain

$$\int_{-1}^{1} (z-x)^{k-4} \Big[(k-1)(k-2)(k-3) + k(k-2)(k-3) + k(k-1)(k-3) \\ + k(k-1)(k-2) + k(k-1)(k-2)(k-3) \log(z-x) \Big] \\ \times \frac{d^{l-4}}{dx^{l-4}} \left[(1-x^2)^l \right] dx.$$

Continuing this procedure then, we find that for n integration by parts:

$$\int_{-1}^{1} (z-x)^{k-n} \Big[\sum_{i=0}^{n-1} \frac{(k)_n}{k-i} + (k)_n \log(z-x) \Big] \frac{d^{l-n}}{dx^{l-n}} \left[(1-x^2)^l \right] dx \tag{4.10}$$

where $(k)_n$ is the *Pochhammer* symbol, defined as $(k)_n = k(k-1)(k-2)...(k-(n-1)) = \frac{\Gamma(k+n)}{\Gamma(k)}$. We may now consider integrating n times by letting $n = k$, obtaining

$$\int_{-1}^{1} \Big[\sum_{i=0}^{k-1} \frac{k!}{k-i} + k! \log(z-x) \Big] \frac{d^{l-k}}{dx^{l-k}} \left[(1-x^2)^l \right] dx$$

using the fact that $(k)_k = k!$. Integrating by parts again (using m this time to count):

- $m = 1$:

  Letting $u = \sum_{i=0}^{k-1} \frac{k!}{k-i} + k! \log(z-x)$, and $dv = \frac{d^{l-k}}{dx^{l-k}} \left[(1-x^2)^l \right] dx$ we obtain

$$du = -\frac{k!}{(z-x)} dx$$

  and

$$v = \frac{d^{l-k-1}}{dx^{l-k-1}} \left[(1-x^2)^l \right].$$

  Thus the integral becomes

$$\int_{-1}^{1} \frac{k!}{(z-x)} \frac{d^{l-k-1}}{dx^{l-k-1}} \left[(1-x^2)^l \right] dx.$$

- $m = 2$:

  We obtain:

$$du = \frac{k!}{(z-x)^2} dx \qquad , \qquad v = \frac{d^{l-k-2}}{dx^{l-k-2}} \left[(1-x^2)^l \right]$$

and the integral now becomes

$$-\int_{-1}^{1}\frac{k!}{(z-x)^2}\frac{d^{l-k-2}}{dx^{l-k-2}}\left[(1-x^2)^l\right]dx.$$

- $m=3$:

  We obtain:

$$du=-\frac{2k!}{(z-x)^3}dx \quad , \quad v=\frac{d^{l-k-3}}{dx^{l-k-3}}\left[(1-x^2)^l\right]$$

  and the integral becomes

$$\int_{-1}^{1}\frac{2k!}{(z-x)^3}\frac{d^{l-k-3}}{dx^{l-k-3}}\left[(1-x^2)^l\right]dx.$$

Continuing this process m times produces the integral

$$\int_{-1}^{1}\frac{(-1)^{m+1}(m-1)!k!}{(z-x)^m}\frac{d^{l-k-m}}{dx^{l-k-m}}\left[(1-x^2)^l\right]dx$$

and thus by performing $l-k>0$ integrations, we obtain (after reintroducing the constant in front of the integral $\frac{(-1)^l}{2^l l!}$):

$$\frac{(-1)^l}{2^l l!}\int_{-1}^{1}(-1)^{l-k+1}(l-k+1)!k!(z-x)^{k-l}(1-x^2)^l dx$$

or, since $(-1)^{2l}=1$:

$$\frac{1}{2^l l!}\int_{-1}^{1}(-1)^{k+1}(l-k+1)!k!(z-x)^{k-l}(1-x^2)^l dx \quad , \quad l>k. \tag{4.11}$$

For the case where $l\leq k$, we instead let $n=l$ in Eq. 4.10 and obtain

$$\frac{1}{2^l l!}\int_{-1}^{1}(-1)^l(z-x)^{k-l}\left[\sum_{i=0}^{l-1}\frac{(k)_l}{k-i}+(k)_l\log(z-x)\right](1-x^2)^l dx \quad , \quad l\leq k.$$

We can now summarise as per Eq. 45 in [6] as

$$\int_{-1}^{1}\log(z-x)(z-x)^k P_l(x)dx=\frac{1}{2^l l!}\int_{-1}^{1}B_{lk}(z,x)(1-x^2)dx, \tag{4.12}$$

where

$$B_{lk} = \begin{cases} (-1)^{k+1}(l-k+1)!k!(z-x)^{k-l}, & l > k \\ (-1)^l (z-x)^{k-l}(k)_l \Big[\sum_{i=0}^{l-1} \frac{1}{k-i} + \log{(z-x)} \Big], & l \leq k. \end{cases}$$

Examining the summation term in more detail, we have explicitly as

$$\sum_{i=0}^{l-1} \frac{1}{k-i} = \frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \cdots + \frac{1}{k-l+1}.$$

Turning to harmonic numbers H_n, defined as

$$H_n \equiv \sum_{k=1}^{n} \frac{1}{k}$$

we may then write that

$$\sum_{i=0}^{l-1} \frac{1}{k-i} = H_k - H_{k-l}.$$

Secondly, given that the function $\psi(n) = H_{n-1} - \gamma$, where γ is the *Euler–Mascheroni constant*, we can state that

$$\sum_{i=0}^{l-1} \frac{1}{k-i} = \psi(k+1) - \psi(k-l+1).$$

Thus we may then write B_{lk} as in [6]

$$B_{lk} = \begin{cases} (-1)^{k+1}(l-k+1)!k!(z-x)^{k-l}, & l > k \\ (-1)^l (z-x)^{k-l}(k-l+1)_l \Big[\log{(z-x)} + \psi(k+1) - \psi(k+1-l) \Big], & l \leq k. \end{cases} \tag{4.13}$$

$\psi(k)$ is known as the digamma function, and may also be defined as

$$\psi(k) = \frac{\Gamma'(k)}{\Gamma(k)} = \frac{d}{dk} \log{(\Gamma(k))}.$$

4.3.1 $l > k$.

For the first of these expressions for B_{lk}, we may express Eq. 4.11 in terms of the Legendre function of the second kind as

$$\int_{-1}^{1} \log (z-x)(z-x)^k P_l(x)dx = \frac{2k!}{l!}(1)_l(z^2-1)^{\frac{1}{2}(k+1)}Q_l^{-k-1}(z) \qquad , \qquad l > k.$$

Noting that we were originally interested in solving the integral

$$\int_{-1}^{1} \log(s^2/l^2)s^{2i-2j}P_l(x)dx$$

we must first then let $k = i - j$ and, since we earlier defined $s^2 = fw^2 + 2r^2(1-\cos\gamma)$ with $w^2 = \frac{2}{\kappa^2}(1-\cos\kappa\Delta\tau)$ and $z = 1 + \frac{f}{\kappa^2 r^2}(1-\cos\kappa\tau)$, we then have that

$$s^2 = 2r^2(z-x)$$

where $x = \cos\gamma$. Hence $(s^2)^{i-j} = (2r^2)^{i-j}(z-x)^{i-j}$. The log term vanishes, as we have seen in Eq. 4.13. Substituting, we then obtain

$$\begin{aligned}
\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{\kappa}{4\pi}\frac{l!}{\Gamma(l+1)}\int_0^{2\pi/\kappa}\int_{-1}^{1} w^{2j}e^{-in\kappa\Delta\tau}\log\left(\frac{s^2}{l^2}\right)s^{2i-2j}P_l(x)dxd\Delta\tau \\
&= \frac{\kappa}{4\pi}\frac{l!}{\Gamma(l+1)}\int_0^{2\pi/\kappa} w^{2j}e^{-in\kappa\Delta\tau}(2r^2)^{i-j}\frac{2(i-j)!}{l!}(1)_l(z^2-1)^{\frac{1}{2}(i-j+1)}Q_l^{j-i-1}(z)d\Delta\tau \\
&= \frac{\kappa}{2\pi}(2r^2)^{i-j}(i-j)!\int_0^{2\pi/\kappa} w^{2j}e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(i-j+1)}Q_l^{j-i-1}(z)d\Delta\tau
\end{aligned}$$

where we used the fact that $(1)_l = \frac{\Gamma(l+1)}{\Gamma(1)}$. As in [6], we now make use of two identities. The first involves using [19]

$$(z^2-1)^{j/2}Q_l^{-j}(z) = \sum_{k=0}^{j}\frac{(-1)^k}{2^{j+1}}\binom{j}{k}\frac{(2l+2j-4k+1)}{(l-k+\frac{1}{2})_{j+1}}Q_{l+j-2k}(z)$$

to obtain

$$
\begin{aligned}
\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{\kappa}{2\pi}(2r^2)^{i-j}(i-j)!\int_0^{2\pi/\kappa} w^{2j}e^{-in\kappa\Delta\tau} \\
&\quad\times\sum_{k=0}^{i-j+1}\frac{(-1)^k}{2^{i-j+2}}\binom{i-j+1}{k}\frac{(2l+2i-2j-4k+3)}{(l-k+\frac{1}{2})_{i-j+2}}Q_{l+i-j-2k+1}(z)d\Delta\tau \\
&= \frac{\kappa}{2\pi}(2r^2)^{i-j}(i-j)!w^{2j}\sum_{k=0}^{i-j+1}\frac{(-1)^k}{2^{i-j+2}}\binom{i-j+1}{k}\frac{(2l+2i-2j-4k+3)}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\quad\times\int_0^{2\pi/\kappa} e^{-in\kappa\Delta\tau}Q_{l+i-j-2k+1}(z)d\Delta\tau.
\end{aligned}
$$

The second identity we make use of is the addition theorem, given explicitly here as

$$
\begin{aligned}
Q_{l+i-j-2k+1}(z) &= P_{l+i-j-2k+1}(\eta)Q_{l+i-j-2k+1}(\eta) \\
&\quad + 2\sum_{p=1}^{\infty}(-1)^p P^{-p}_{l+i-j-2k+1}(\eta)Q^{p}_{l+i-j-2k+1}(\eta)\cos p\kappa t.
\end{aligned}
$$

We also recall that w is defined as $w^2 = \frac{2}{\kappa^2}(1-\cos\kappa\Delta\tau)$. Thus the regularisation parameter is rewritten as

$$
\begin{aligned}
\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{\kappa}{2\pi}(2r^2)^{i-j}(i-j)!\frac{2^j}{\kappa^{2j}}\sum_{k=0}^{i-j+1}\frac{(-1)^k}{2^{i-j+2}}\binom{i-j+1}{k}\frac{(2l+2i-2j-4k+3)}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\quad\times\int_0^{2\pi/\kappa}(1-\cos\kappa\Delta\tau)^j e^{-in\kappa\Delta\tau}\Big[P_{l+i-j-2k+1}(\eta)Q_{l+i-j-2k+1}(\eta) \\
&\qquad + 2\sum_{p=1}^{\infty}(-1)^p P^{-p}_{l+i-j-2k+1}(\eta)Q^{p}_{l+i-j-2k+1}(\eta)\cos p\kappa\tau\Big]d\Delta\tau \\
&= \frac{\kappa}{\pi}\frac{2^{j-3}r^{2i-2j}(i-j)!}{\kappa^{2j}}\sum_{k=0}^{i-j+1}(-1)^k\binom{i-j+1}{k}\frac{(2l+2i-2j-4k+3)}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\quad\times\Big[P_{l+i-j-2k+1}(\eta)Q_{l+i-j-2k+1}(\eta)\int_0^{2\pi/\kappa}(1-\cos\kappa\Delta\tau)^j e^{-in\kappa\Delta\tau}d\Delta\tau \\
&\qquad + 2\sum_{p=1}^{\infty}(-1)^p P^{-p}_{l+i-j-2k+1}(\eta)Q^{p}_{l+i-j-2k+1}(\eta)\int_0^{2\pi/\kappa}(1-\cos\kappa\Delta\tau)^j e^{-in\kappa\Delta\tau}\cos p\kappa\tau d\Delta\tau\Big] \\
&= \frac{2^{j-3}r^{2i-2j}(i-j)!}{\pi\kappa^{2j-1}}\sum_{k=0}^{i-j+1}(-1)^k\binom{i-j+1}{k}\frac{(2l+2i-2j-4k+3)}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\quad\times\Big[P_{l+i-j-2k+1}(\eta)Q_{l+i-j-2k+1}(\eta)\frac{2\pi}{\kappa}\sum_{m=0}^{j}\sum_{s=0}^{m}\frac{j!}{s!(j-m)!(m-s)!}\frac{(-1)^m}{2^m}\delta_{m,2s+n} \\
&\qquad + 2\sum_{p=1}^{\infty}(-1)^p P^{-p}_{l+i-j-2k+1}(\eta)Q^{p}_{l+i-j-2k+1}(\eta) \\
&\qquad\quad\times\frac{\pi}{\kappa}\sum_{m=0}^{j}\sum_{s=0}^{m}\frac{j!}{s!(j-m)!(m-s)!}\frac{(-1)^m}{2^m}\left(\delta_{p,2s+n-m}+\delta_{p,m-2s-n}\right)\Big].
\end{aligned}
$$

Simplifying, we are left with the resulting in the regularisation parameters

$$\begin{aligned}\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{2^{j-2}r^{2i-2j}(i-j)!}{\kappa^{2j}}\sum_{k=0}^{i-j+1}(-1)^k\binom{i-j+1}{k}\frac{(2l+2i-2j-4k+3)}{(l-k+\frac{1}{2})_{i-j+2}}\\ &\times\sum_{m=0}^{j}\sum_{s=0}^{m}\frac{j!}{s!(j-m)!(m-s)!}\frac{(-1)^m}{2^m}\\ &\times\Big[P_{l+i-j-2k+1}(\eta)Q_{l+i-j-2k+1}(\eta)\delta_{m,2s+n}\\ &+\sum_{p=1}^{\infty}(-1)^p P^{-p}_{l+i-j-2k+1}(\eta)Q^{p}_{l+i-j-2k+1}(\eta)\left(\delta_{p,2s+n-m}+\delta_{p,m-2s-n}\right)\Big].\end{aligned} \tag{4.14}$$

Despite the fact that the summation over p is an infinite one, there is only one surviving p term thanks to the Kronecker deltas. Thus,

$$\begin{aligned}\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{2^{j-1}r^{2i-2j}(i-j)!}{\kappa^{2j}}\sum_{k=0}^{i-j+1}(-1)^k\binom{i-j+1}{k}\frac{(l+i-j-2k+\frac{3}{2})}{(l-k+\frac{1}{2})_{i-j+2}}\\ &\times\sum_{m=0}^{j}\sum_{s=0}^{m}\frac{j!}{s!(j-m)!(m-s)!}\frac{(-1)^m}{2^m}\\ &\times\Big[P_{l+i-j-2k+1}(\eta)Q_{l+i-j-2k+1}(\eta)\delta_{m,2s+n}\\ &+P^{(m-n-2s)}_{l+i-j-2k+1}(\eta)Q^{(2s+n-m)}_{l+i-j-2k+1}(\eta)\\ &+(-1)^n P^{(2s+n-m)}_{l+i-j-2k+1}(\eta)Q^{(m-2s-n)}_{l+i-j-2k+1}(\eta)\Big]\end{aligned} \tag{4.15}$$

or, by removing the $\delta_{m,2s+n}$ term, and the relevant summation:

$$\begin{aligned}\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{2^{j-1}r^{2i-2j}(i-j)!}{\kappa^{2j}}\sum_{k=0}^{i-j+1}(-1)^k\binom{i-j+1}{k}\frac{(l+i-j-2k+\frac{3}{2})}{(l-k+\frac{1}{2})_{i-j+2}}\\ &\times\Bigg[\sum_{m=0}^{j}\frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!}\frac{(-1)^m}{2^m}P_{l+i-j-2k+1}(\eta)Q_{l+i-j-2k+1}(\eta)\\ &+\sum_{m=0}^{j}\sum_{s=0}^{m}\frac{j!}{s!(j-m)!(m-s)!}\frac{(-1)^m}{2^m}\Big(P^{(m-n-2s)}_{l+i-j-2k+1}(\eta)Q^{(2s+n-m)}_{l+i-j-2k+1}(\eta)\\ &+(-1)^n P^{(2s+n-m)}_{l+i-j-2k+1}(\eta)Q^{(m-2s-n)}_{l+i-j-2k+1}(\eta)\Big)\Bigg].\end{aligned}$$

4.3.2 $l \leq k$.

Given the argument made in [6] on the difficulties for the case where $l \leq k$, here we show how the regularisation parameters can be expressed as a derivative with respect to the exponent. First, we begin with the double integral representation of the tail regularisation parameter as before:

$$\overset{[d]}{\chi}_{nl}(i,j|r) = \frac{\kappa}{4\pi}\frac{l!}{\Gamma(l+1)}\int_0^{2\pi/\kappa}\int_{-1}^{1} w^{2j}e^{-in\kappa\Delta\tau}\log\left(\frac{s^2}{l^2}\right)s^{2i-2j}P_l(x)dxd\Delta\tau.$$

Making the substitutions as before, we obtain

$$\overset{[d]}{\chi}_{nl}(i,j|r) = \frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j(2r)^{i-j}\int_0^{2\pi/\kappa}\int_{-1}^{1}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\log\left(\frac{s^2}{\ell^2}\right)(z-x)^{i-j}P_l(x)dxd\Delta\tau.$$

With $s^2 = 2r^2(z-x)$, we then have

$$\begin{aligned}
\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j(2r)^{i-j}\int_0^{2\pi/\kappa}\int_{-1}^{1}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\log\left(\frac{2r^2}{\ell^2}(z-x)\right)(z-x)^{i-j}P_l(x)dxd\Delta\tau \\
&= \frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j(2r)^{i-j}\int_0^{2\pi/\kappa}\int_{-1}^{1}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
&\qquad\times\left[\log\left(\frac{2r^2}{\ell^2}\right)+\log(z-x)\right](z-x)^{i-j}P_l(x)dxd\Delta\tau \\
&= \frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j(2r)^{i-j}\int_0^{2\pi/\kappa}\int_{-1}^{1}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\log\left(\frac{2r^2}{\ell^2}\right)(z-x)^{i-j}P_l(x)dxd\Delta\tau \\
&\quad+\frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j(2r)^{i-j}\int_0^{2\pi/\kappa}\int_{-1}^{1}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\log(z-x)(z-x)^{i-j}P_l(x)dxd\Delta\tau
\end{aligned}$$

and by recalling the relevant expression in Eq. 4.13, we can then write

$$\begin{aligned}
\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j\frac{(2r)^{i-j}}{2^l l!}\int_0^{2\pi/\kappa}\int_{-1}^{1}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
&\qquad\times\left[(-1)^l(k-l+1)_l(z-x)^{k-l}\log\left(\frac{2r^2}{\ell^2}\right)\right]dxd\Delta\tau \\
&\quad+\frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j\frac{(2r)^{i-j}}{2^l l!}\int_0^{2\pi/\kappa}\int_{-1}^{1}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
&\qquad\times\left[(-1)^l(k-l+1)_l(z-x)^{k-l}\left\{\log(z-x)+\psi(k+1)+\psi(k+1-l)\right\}\right]dxd\Delta\tau \\
&= \frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j\frac{(2r)^{i-j}(-1)^l}{2^l l!}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
&\quad\times\int_{-1}^{1}(z-x)^{k-l}(k-l+1)_l\left\{\log(z-x)+\psi(k+1)+\psi(k+1-l)+\log\left(\frac{2r^2}{\ell^2}\right)\right\}dxd\Delta\tau.
\end{aligned} \tag{4.16}$$

Breen and Taylor [6] found that this expression can be expressed in terms of a derivative with respect to an exponent λ:

$$\overset{[d]}{\chi}_{nl}(i,j|r) = \frac{\kappa}{4\pi}\left(\frac{2}{\kappa^2}\right)^j \frac{(2r)^{i-j}(-1)^l}{2^l l!}\Big[\frac{d}{d\lambda}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\ \times \int_{-1}^{1}(1-x^2)^l(\lambda+1-l)_l(z-x)^{\lambda-l}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} dx d\Delta\tau\Big]_{\lambda=i-j}. \tag{4.17}$$

Working backwards from the equation expressed in this way to get Eq. 4.16 above is an informative exercise, so we shall do it here. Beginning with the above equation, we are then essentially interested in finding the following derivative:

$$\frac{d}{d\lambda}\left[(\lambda+1-l)_l(z-x)^{\lambda-l}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\right].$$

Taking each term individually,

$$\begin{aligned}
\frac{d}{d\lambda}(\lambda+1-l)_l &= \frac{d}{d\lambda}\left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-l)}\right) \\
&= \frac{\Gamma(\lambda+1-l)\Gamma'(\lambda+1)-\Gamma(\lambda+1)\Gamma'(\lambda+1-l)}{(\Gamma(\lambda+1-l))^2} \\
&= \frac{\Gamma(\lambda+1-l)\psi(\lambda+1)\Gamma(\lambda+1)-\Gamma(\lambda+1)\psi(\lambda+1-l)\Gamma(\lambda+1-l)}{(\Gamma(\lambda+1-l))^2} \\
&= \frac{\psi(\lambda+1)\Gamma(\lambda+1)-\Gamma(\lambda+1)\psi(\lambda+1-l)}{\Gamma(\lambda+1-l)} \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-l)}\left[\psi(\lambda+1)-\psi(\lambda+1-l)\right] \\
&= (\lambda+1-l)_l\left[\psi(\lambda+1)-\psi(\lambda+1-l)\right], \\
\frac{d}{d\lambda}(z-x)^{\lambda-l} &= \log(z-x)(z-x)^{\lambda}, \qquad \text{and} \\
\frac{d}{d\lambda}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} &= \log\left(\frac{2r^2}{\ell^2}\right)\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}.
\end{aligned}$$

Combining these terms via the product rule, we obtain

$$\begin{aligned}
\frac{d}{d\lambda}\left[(\lambda+1-l)_l(z-x)^{\lambda-l}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\right] &= (\lambda+1-l)_l(z-x)^{\lambda-l}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} \\
&\quad\times\left[\log(z-x)+\log\left(\frac{2r^2}{\ell^2}\right)+\psi(\lambda+1)-\psi(\lambda+1-l)\right].
\end{aligned}$$

However we note that this expression isn't quite right, owing to the $2r^2/\ell^2$ term, which doesn't feature in Eq 4.16. We need to evaluate the expression at $\lambda = k = i-j$.

Therefore

$$\frac{d}{d\lambda}\left[(\lambda+1-l)_l(z-x)^{\lambda-l}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\right]_{\lambda=i-j} = (k+1-l)_l(z-x)^{k-l}$$
$$\times\left[\log(z-x)+\log\left(\frac{2r^2}{\ell^2}\right)+\psi(k+1)-\psi(k+1-l)\right]$$

and it becomes clear then that Eq. 4.17 is equivalent to Eq. 4.16. From [20] (p. 182), we know that the associated Legendre's function of the second kind can be defined as

$$Q_n^m(z) = \frac{(-1)^m(n+m)!}{2^{n+1}n!}(z^2-1)^{m/2}\int_{-1}^{1}\frac{(1-x^2)^n}{(z-x)^{n+m+1}}dx \qquad , \qquad z \notin [-1,1]$$

or rewritten more familiarly as

$$\int_{-1}^{1}(1-x^2)^n(z-x)^{-n-m-1}dx = \frac{2^{n+1}n!}{(-1)^m(n+m)!}(z^2-1)^{-m/2}Q_n^m(z).$$

Referring back to Eq. 4.17 then, the integral over x we are concerned with is

$$\int_{-1}^{1}(1-x^2)^l(z-x)^{\lambda-l}dx,$$

hence if we replace n with l, and m with $-(\lambda+1)$, we obtain

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^l(z-x)^{\lambda-l}dx &= \frac{2^{l+1}l!}{(-1)^{-\lambda+1}(l-\lambda-1)!}(z^2-1)^{1/2(\lambda+1)}Q_l^{-\lambda-1}(z)\\ &= \frac{2^{l+1}\Gamma(l+1)}{(-1)^{-\lambda+1}\Gamma(l-\lambda)}(z^2-1)^{1/2(\lambda+1)}Q_l^{-\lambda-1}(z).\end{aligned}$$

Realising that the function $\Gamma(l-\lambda)$ is only defined for integers $l-\lambda \geq 1$, and we do have a small number of modes l for which $\Gamma(l-\lambda)$ is thus not defined, we look to "Olver's definition of the associated Legendre function of the second kind" $\boldsymbol{Q}(z)$ rather than the above $Q(z)$ - as per the work in [6] - otherwise the $Q(z)$ will not be defined either. Using the fact that $Q(z)$ is related to $\boldsymbol{Q}(z)$ by [20] (p. 178)

$$\boldsymbol{Q}_l^{-\lambda-1}(z) = \frac{e^{(\lambda+1)\pi i}Q_l^{-\lambda-1}(z)}{\Gamma(l-\lambda)}$$

we can then express the integral without the problematic function $\Gamma(l-\lambda)$ as

$$\int_{-1}^{1}(1-x^2)^l(z-x)^{\lambda-l}dx = \frac{2^{l+1}\Gamma(l+1)}{(-1)^{1-\lambda}e^{(\lambda+1)\pi i}}(z^2-1)^{1/2(\lambda+1)}\boldsymbol{Q}_l^{-\lambda-1}(z).$$

Taking a closer look at the denominator, we can express more simply as

$$(-1)^{1-\lambda}e^{(\lambda+1)\pi i} = (-1)^{\lambda}e^{i\lambda\pi} = (-1)^{\lambda}(\cos\lambda\pi + i\sin\lambda\pi) = 1 \quad , \quad \lambda\in\mathbb{Z}.$$

Since $\lambda = i-j$ our integral can be expressed as a function of Olver's function as

$$\int_{-1}^{1}(1-x^2)^l(z-x)^{\lambda-l}dx = 2^{l+1}\Gamma(l+1)(z^2-1)^{1/2(\lambda+1)}\boldsymbol{Q}_l^{-\lambda-1}(z).$$

Olver's Legendre function of the second kind is defined by

$$\boldsymbol{Q}_l^{-\lambda-1}(z) = \frac{\sqrt{\pi}}{2^{l+1}z^{l-\lambda}}(z^2-1)^{\frac{1}{2}(-\lambda-1)}\mathbf{F}\Big(\frac{1}{2}(-\lambda-1)+\frac{l}{2}+\frac{1}{2},\frac{1}{2}(-\lambda-1)+\frac{l}{2}+1;l+\frac{3}{2};\frac{1}{z^2}\Big)$$

where $\mathbf{F}(a,b;c;z)$ is the regularised hypergeometric function defined as

$$\mathbf{F}\Big(\frac{1}{2}(-\lambda-1)+\frac{l}{2}+\frac{1}{2},\frac{1}{2}(-\lambda-1)+\frac{l}{2}+1;l+\frac{3}{2};\frac{1}{z^2}\Big) = \sum_{k=0}^{\infty}\frac{(\frac{1}{2}(l-\lambda))_k(\frac{1}{2}(l-\lambda+1))_k}{z^{2k}\Gamma(l+\frac{3}{2}+k)k!}.$$

Inserting $\boldsymbol{Q}_l^{-\lambda-1}(z)$ then into Eq. 4.17, we therefore obtain the regularisation parameter

$$\begin{aligned}\overset{[d]}{\chi}_{nl}(i,j|r) &= \frac{\kappa}{4\pi}\Big(\frac{2}{\kappa^2}\Big)^j\frac{(2r)^{i-j}(-1)^l}{2^l l!}\Big[\frac{d}{d\lambda}(\lambda+1-l)_l\Big(\frac{2r^2}{\ell^2}\Big)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\\ &\quad\times(z^2-1)^{\frac{1}{2}(\lambda+1)}2^{l+1}\Gamma(l+1)\boldsymbol{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j}\\ &= \frac{\kappa}{2\pi}\Big(\frac{2}{\kappa^2}\Big)^j(2r)^{i-j}(-1)^l\Big[\frac{d}{d\lambda}(\lambda+1-l)_l\Big(\frac{2r^2}{\ell^2}\Big)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\\ &\quad\times(z^2-1)^{\frac{1}{2}(\lambda+1)}\boldsymbol{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j}.\end{aligned} \tag{4.18}$$

4.4 The Hadamard Parametrix

In Chapter 2 we discussed the Green function of the singular propagator $G_s(x,x')$ and its useful properties in the context of renormalisation - chiefly that it contains all of the geometric divergences common to those problematic in the coincidence limit. Having applied the Hadamard parametrix to it, we defined it as

$$G_s(x,x^{'}) = \frac{1}{8\pi^2}\left(\frac{U(x,x^{'})}{\sigma(x,x^{'})} + V(x,x^{'})\log\left(\frac{2\sigma(x,x^{'})}{\ell^2}\right)\right).$$

After the work in Chapters 3 and 4 we are now in a position to redefine it in terms of the mode-sum prescription of Breen and Taylor [6] and in their selected, judiciously chosen coordinate system $(w, s, \Delta r)$ with $\Delta r \to 0$ as

$$\begin{aligned}
G_s(x,x^{'}) &= \frac{1}{8\pi^2}\left\{\frac{U(x,x^{'})}{\sigma(x,x^{'})} + V(x,x^{'})log\left(\frac{2\sigma(x,x^{'})}{\ell^2}\right)\right\} \\
&= \frac{1}{8\pi^2}\Bigg\{\sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\frac{w^{2i+2j}}{s^{2+2j}} + \mathcal{D}_{11}^{(-)}(r) \\
&\quad + \log(s^2/\ell^2)\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)s^{2i-2j}w^{2j} + \mathcal{T}_{10}^{(r)}(r)\frac{w^4}{s^2}\Bigg\} \\
&= \frac{1}{8\pi^2}\Bigg\{\sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(x)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) \\
&\quad + \mathcal{D}_{11}^{(-)}(r) \\
&\qquad + \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(x)\overset{[d]}{\chi}{}_{nl}(i,j|r) \\
&\qquad\quad + \mathcal{T}_{10}^{(r)}(r)\sum_{l=0}^{\infty}(2l+1)P_l(x)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg\} \\
&= \frac{1}{8\pi^2}\sum_{l=0}^{\infty}(2l+1)P_l(x)\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\tau}\Bigg\{\sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) \\
&\qquad + \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r) + \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}{}_{nl}(i,j|r)\Bigg\} + \frac{1}{8\pi^2}\mathcal{D}_{11}^{(-)}(r).
\end{aligned}$$

Replacing our variable x with its actual value of $\cos\gamma$, we are now ready to replicate the main result of Breen and Taylor [6]: a regularised, mode-sum prescription in extended coordinates $(w, \Delta r, s)$ for the singular Green's function propagator. For the case of $d = 4$, we thus have

$$G_s(x, x') = \frac{1}{8\pi^2} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \Bigg\{ \sum_{i=0}^{2} \sum_{j=0}^{i} \mathcal{D}_{ij}^{(+)}(r) \overset{[d]}{\Psi}{}_{nl}^{(+)} (i, j \mid r) + \mathcal{T}_{10}^{(r)}(r) \overset{[d]}{\Psi}{}_{nl}^{(-)} (2, 0 \mid r) + \sum_{i=0}^{1} \sum_{j=0}^{i} \mathcal{T}_{ij}^{(l)}(r) \overset{[d]}{\chi}{}_{nl} (i, j|r) \Bigg\} + \frac{1}{8\pi^2} \mathcal{D}_{11}^{(-)}(r). \tag{4.19}$$

4.5 Summary

In this chapter we took a mode-sum representation of the Green function, an infinite sum of $e^{in\kappa\Delta\tau}$ and Legendre functions $P_l(\cos\gamma)$ for modes n and l, and calculated regularisation terms for each of the terms from Chapter 3 in (w, s). To do so, we calculated a tractable equivalent of the time integral, and used many properties of Legendre and associated Legendre functions, of both the first and second kind. We calculated four regularisation terms, and were then able to write a complete description of the singular Green propagator using only these terms and the direct and tail coefficients - thus achieving our mode-sum expression. In the next chapter, we will subtract this mode sum from the Euclidean Green mode-sum representation, and by taking the coincidence limit we will then renormalise our Green function. We will extend upon previous work by taking the derivatives of this renormalised Green function in the context of calculating the renormalised stress-energy tensor.

Chapter 5

The Renormalised Stress-Energy Tensor

In this chapter we will consider some of the calculations required for the computation of the renormalised stress-energy tensor: the derivatives of the renormalised Green function. To do so, we must revert to the original mode-sum description of the singular propagator as we are not specifically now working towards the renormalised vaccuum expectation value - a quantity that requires the coincidence limit to be taken. Hence the polynomial terms that were disregarded in Chapter 3 are reintroduced. From this point on, we work with the mode-sum description of the singular Green function

$$
\begin{aligned}
G_s(x,x') = \frac{1}{8\pi^2} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) \Bigg\{ & \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}_{ij}^{(+)}(r) \overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j \mid r) \\
& + \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}_{ij}^{(-)}(r) \overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j \mid r) + \sum_{i=0}^{1}\sum_{j=0}^{i} \mathcal{T}_{ij}^{(l)}(r) \overset{[d]}{\chi}{}_{nl}(i,j|r) + \sum_{j=0}^{1} \mathcal{T}_{1j}^{(p)}(r) \overset{[d]}{\chi}{}_{nl}(1,j|r) \\
& + \mathcal{T}_{10}^{(r)}(r) \overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0 \mid r) \Bigg\} + \mathcal{O}(\epsilon^4 \log\epsilon),
\end{aligned}
$$

having reintroduced the terms that did not contribute in the coincidence limit. This function will henceforth be used to normalise the Green function, permitting us then to find its derivatives.

5.1 Schwarzschild Spacetime

We decide at this point to introduce a specific metric function $f(r)$ given by

$$f(r) = 1 - \frac{2M}{r}$$

where M is the mass of a black hole of radius r. This is a well known function applicable in spherically symmetric spacetimes. We do so for two reasons. First, the derivatives we will encounter in this section may be made less complicated by calculating explicitly, and secondly because we maintain consistency with the primary works upon which this work is based on. The same methodology may be applied to other appropriate values of $f(r)$. Let us first write the direct and tail Hadamard coefficients for this choice of $f(r)$.

TABLE 5.1: $\mathcal{D}_{ij}^{(+)}(r)$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\mathcal{D}_{ij}^{(+)}(r)$ coefficients for $d = 4$			
	$j = 0$	$j = 1$	$j = 2$
$i = 0$	2		
$i = 1$	$\frac{M(r-2M)}{r^4}$	$-\frac{(2M-r)}{6r^5} \times \left(9M^2 - 4Mr - \kappa^2 r^4\right)$	
$i = 2$	$\frac{M(2M-r)}{12r^8} \times (27M^2 - 19Mr - \kappa^2 r^4 + 3r^2)$	$\frac{(2M-r)}{180r^9} \times (675M^4 - 705M^3 r + M^2\left(220r^2 - 75\kappa^2 r^4\right) + Mr^3\left(35\kappa^2 r^2 - 18\right) + 4\kappa^4 r^8)$	$\frac{(r-2M)^2}{72r^{10}} \times (-9M^2 + 4Mr + \kappa^2 r^4)^2$

TABLE 5.2: $\mathcal{D}_{ij}^{(-)}(r)$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\mathcal{D}_{ij}^{(-)}(r)$ coefficients for $d = 4$			
	$j = 0$	$j = 1$	$j = 2$
$i = 0$			
$i = 1$		$-\dfrac{M}{3r^3}$	
$i = 2$		$\dfrac{M(11M - 4r)(2M - r)}{20r^7}$	$\dfrac{M(2M - r)}{20r^6}$

TABLE 5.3: $\mathcal{T}_{ij}^{(l)}(r)$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\mathcal{T}_{ij}^{(l)}(r)$ coefficients for $d = 4$		
	$j = 0$	$j = 1$
$i = 0$	$\dfrac{M^2}{2}$	
$i = 1$	$\dfrac{M^4}{16}$	0

TABLE 5.4: $\mathcal{T}_{ij}^{(r)}(r)$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\mathcal{T}_{ij}^{(r)}(r)$ coefficients for $d = 4$		
	$j = 0$	$j = 1$
$i = 0$		
$i = 1$	$\dfrac{M^2(2M - r)}{24r^5}\left(9M^2 - 4Mr - \kappa^2 r^4\right)$	

TABLE 5.5: $\mathcal{T}_{ij}^{(p)}(r)$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\mathcal{T}_{ij}^{(p)}(r)$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$		
	$j = 0$	$j = 1$
$i = 0$		
$i = 1$	$\dfrac{M^3}{12r^3}$	$\dfrac{M^3(2M - r)}{4r^4}$

5.2 The Renormalised Green Function G_{ren}

We could now compute the regularised vacuum polarization quantity

$$\langle\phi^2\rangle_{reg} = \lim_{x\to x'} \{G(x,x') - G_s(x,x')\}$$

however since this approach is explored in [6], we can afford to investigate a different quantity - the renormalised stress-energy tensor. First, we define the standard mode-sum representation of the Green function as in [6] as

$$G_E(x,x') = \frac{\kappa}{8\pi^2}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)N_{nl}p_{nl}(r_<)q_{nl}(r_>).$$

The crux of the problem is that taking the limit of $G_E(x,x')$ as $x \to x'$ diverges, so we captured the divergences in $G_s(x,x')$. We obtained a mode-sum representation in Chapter 4 for this singular propagator in Eq. 4.19, thus we can then write the regularised vacuum polarization, where $\Delta r = 0$ and $\Delta\tau = 0$ in the coincidence limit, as

$$\begin{aligned}\langle\phi^2\rangle_{reg} &= \lim_{x\to x'}\left[G_E(x,x') - G_s(x,x')\right]\\ &= \frac{1}{8\pi^2}\Bigg[\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\Bigg\{N_{nl}p_{nl}(r)q_{nl}(r)\\ &\quad -\sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) - \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\\ &\quad -\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r)\Bigg\} - \mathcal{D}_{11}^{(-)}(r)\Bigg].\end{aligned} \tag{5.1}$$

We will now define the renormalised Green function as $G_{ren} = G_E - G_s$ and consider the stress-energy tensor T_{ab}. The diagonal components of the point-split renormalised expectation value of the energy-stress tensor is given in [4] as

$$\begin{aligned}\langle T^{\nu}{}_{\nu}\rangle_{ren} &= (1-2\xi)g^{\nu\nu'}G_{ren\,;\nu\nu'} + (2\xi - \tfrac{1}{2})g^{\alpha\alpha'}G_{ren\,;\alpha\alpha'} - 2\xi g^{\nu\nu}G_{ren\,;\nu\nu}\\ &\quad + 2\xi g^{\alpha\alpha}G_{ren\,;\alpha\alpha} + \xi(R^{\nu}{}_{\nu} - \tfrac{1}{2}R)G_{ren} - \frac{m^2}{2}G_{ren} + \frac{2v_1}{8\pi^2}\end{aligned} \tag{5.2}$$

where ν is not summed over, and we adopt the notation

$$G_{ren}(x,x') = \lim_{x\to x'} G(x,x')$$

so that, for example,

$$g_{\mu\nu'}G_{ren\,;\nu\nu'} = \lim_{x\to x'}\Big(g_{\mu\nu'}G_{E\,;\nu\nu'} - g_{\mu\nu'}G_{s\,;\nu\nu'}\Big).$$

We will take these derivatives in terms of $x = (t, r, \theta, \phi)$. In doing so, it will be convenient for us to have the metric connections, or Christoffel symbols, at our disposal.

5.3 Christoffel Symbols

The Christoffel symbol is defined by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right)$$

and given our choice of coordinates, we will need to calculate

$$\begin{array}{ccccccc} \Gamma^t_{tr}, & \Gamma^t_{rt}, & \Gamma^r_{rr}, & \Gamma^r_{\theta\theta}, & \Gamma^r_{\phi\phi}, & \Gamma^r_{tt}, & \Gamma^\theta_{r\theta} \\ \Gamma^\theta_{\theta r}, & \Gamma^\theta_{\phi\phi}, & \Gamma^\phi_{r\phi}, & \Gamma^\phi_{\phi r}, & \Gamma^\phi_{\theta\phi}, & \Gamma^\phi_{\phi\theta}. & \end{array}$$

for the metric [1]

$$ds^2 = f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$

as described previously in Chapter 1. The remainder of the possible Christoffel symbols not listed above turn out to be zero, proof of which is left to the reader as an exercise. Let us calculate some of these now for demonstration purposes.

[1]Note that we should technically be using τ instead of t as we are dealing with the Euclidean metric.

- Γ^r_{rr}

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{1}{2}g^{rd}\left(\partial_r g_{dr} + \partial_r g_{dr} - \partial_d g_{rr}\right) \\
&= \frac{1}{2}g^{rr}\left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) \\
&= \frac{1}{2}f(r)\left(\partial_r \frac{1}{f(r)} + \partial_r \frac{1}{f(r)} - \partial_r \frac{1}{f(r)}\right) \\
&= \frac{f(r)}{2}\left(-\frac{f^{'}(r)}{\left(f(r)\right)^2}\right) \\
&= -\frac{f^{'}(r)}{2f(r)}.
\end{aligned}$$

- $\Gamma^\phi_{r\phi}$

$$\begin{aligned}
\Gamma^\phi_{r\phi} &= \frac{1}{2}g^{\phi d}\left(\partial_r g_{d\phi} + \partial_\phi g_{d\phi} - \partial_d g_{r\phi}\right) \\
&= \frac{1}{2}g^{\phi\phi}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{r\phi}\right) \\
&= \frac{1}{2r^2\sin^2\theta}\left(\partial_r r^2\sin^2\theta\right) \\
&= \frac{2r\sin^2\theta}{2r^2\sin^2\theta} \\
&= \frac{1}{r}.
\end{aligned}$$

- Γ^t_{tr}

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{1}{2}g^{td}\left(\partial_t g_{dr} + \partial_r g_{dt} - \partial_d g_{tr}\right) \\
&= \frac{1}{2}g^{tt}\left(\partial_t g_{tr} + \partial_r g_{tt} - \partial_t g_{tr}\right) \\
&= \frac{1}{2}\frac{1}{f(r)}\left(\partial_r f(r)\right) \\
&= \frac{f^{'}(r)}{2f(r)}.
\end{aligned}$$

...and so on for the non-vanishing connections. We summarise them here:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{f^{'}(r)}{2f(r)} \quad , \quad \Gamma^r_{rr} = -\frac{f^{'}(r)}{2f(r)} \quad , \quad \Gamma^r_{\theta\theta} = -rf(r) \quad , \quad \Gamma^r_{\phi\phi} = -rf(r)\sin^2\theta$$
$$\Gamma^r_{tt} = \frac{f(r)f^{'}(r)}{2} \quad , \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \quad , \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \quad , \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}$$
$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.$$

In other work, for example [4], it is necessary to consider the bivectors of parallel transport in the computation of the renormalised quantities, since most of the work on the Green function is prior to renormalisation. Expressions in such work necessarily involved the bivectors owing to the point-seperation method prior to the coincidence limit being taken. However the advantage of having calculated the singular propagator found in [6] and explicitly followed in this work, $G_s(x,x^{'})$, is that we thus managed to find a convergent mode sum that describes the renormalised quantity. Owing to this, the bivectors of parallel transport that are required in the cited work converge on the regular spacetime metric.

5.4 Derivatives of G_{ren}

We now proceed with taking the covariant derivatives[2] of G at x. We define G as

$$
\begin{aligned}
G &= G_E(x,x^{'}) - G_s(x,x^{'}) \\
&= \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Bigg\{N_{nl}p_{nl}(r)q_{nl}(r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) \\
&\quad - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) - \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
&\quad - \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg\} + \mathcal{O}(\epsilon^4\log\epsilon).
\end{aligned}
$$

and G_{ren} as G in the coincidence limit, where $\Delta\tau \to 0$ and $\gamma \to 0$, given then by

$$
\begin{aligned}
G_{ren} &= G_E(x,x^{'}) - G_s(x,x^{'}) \\
&= \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\Bigg\{N_{nl}p_{nl}(r)q_{nl}(r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) \\
&\quad - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) - \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
&\quad - \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg\}.
\end{aligned}
$$

[2]We are deciding that G is a scalar at x, and a vector at $x^{'}$. We could of course do this the other way around, once we accounted for this correctly when taking the derivatives $G_{;\mu\nu'}$

The first point we notice is that the metric tensor is diagonal, so we are only required to calculate covariant derivatives of the form $G_{;\mu\mu}$. The second point to note, as mentioned at the start of the chapter is that we have reintroduced those polynomial tail terms that were previously disregarded, as they did not contribute in the coincidence limit. We now take these one at a time, calculating $g^{\mu\mu}G_{;\mu\mu}$ in the process (we are not applying Einstein's summation convention to this particular argument - we simply wish to convey that we are working with the diagonal only), where the square brackets denote the partial coincidence limit ($r_> \to r_<, \theta \to \theta^{'}, \phi \to \phi^{'}$) - we will need these terms later. We will use the fact that since $\cos\gamma = \cos\theta\cos\theta^{'} + \sin\theta\sin\theta^{'}\cos(\phi - \phi^{'})$, then $P_l(\cos\gamma) \to 1$ in the coincidence limit [3].

Since G is in terms of τ after our earlier Wick rotation, we are careful to calculate $\partial/\partial\tau$ instead of $\partial/\partial t$ - we simply let

$$\frac{\partial}{\partial t} = i\frac{\partial}{\partial \tau}$$

and thus

$$\partial_t G = i\frac{\partial}{\partial\tau}G = iG_{,\tau}$$

However, the application of the second covariant derivative at x requires us to now use the metric connection via the general rule

$$\nabla_c G_a = \partial_c G_a - \Gamma^b_{ac}G_b.$$

We will now take the derivatives with respect to the coordinates.

- $g^{tt^{'}}G_{ren\,;tt^{'}}$

[3] We choose to apply the partial coincidence limit to the temporal dimension since we earlier set $\Delta r = 0$.

First,

$$\begin{aligned}
G_{;tt'} &= i\frac{\partial}{\partial\tau}i\frac{\partial}{\partial\tau}G = -\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'}G \\
&= -\frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty} n^2\kappa^2 e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Bigg\{N_{nl}p_{nl}(r)q_{nl}(r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) \\
&\quad - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) - \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
&\quad - \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg\} + \mathcal{O}(\epsilon^2\log\epsilon).
\end{aligned}$$

With $g^{tt'} = -\sqrt{f(r)f(r')}$ we obtain

$$\begin{aligned}
g^{tt'}G_{;tt'} = \frac{\sqrt{f(r)f(r')}}{8\pi^2}&\sum_{n=-\infty}^{\infty} n^2\kappa^2 e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Bigg\{N_{nl}p_{nl}(r)q_{nl}(r) \\
&- \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) \\
&- \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
&- \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg\} + \mathcal{O}(\epsilon^2\log\epsilon).
\end{aligned}$$

and therefore

$$\begin{aligned}
g^{tt'}G_{ren\,;tt'} = \frac{\sqrt{f(r)f(r')}}{8\pi^2}&\sum_{n=-\infty}^{\infty} n^2\kappa^2\sum_{l=0}^{\infty}(2l+1)\Bigg\{N_{nl}p_{nl}(r)q_{nl}(r) \\
&- \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) \\
&- \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
&- \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg\}.
\end{aligned} \tag{5.3}$$

- $g^{tt}G_{ren\,;tt}$

Since

$$g^{tt}G_{;tt} = -\frac{1}{f}G_{;tt}$$

and since Γ^r_{tt} is the only non-zero connection, we obtain:

$$g^{tt}G_{;tt} = -\frac{1}{f}\left(-\frac{\partial^2}{\partial\tau^2}G - \Gamma^r_{tt}\frac{\partial}{\partial r}G \right).$$

Taking these terms separately,

$$\begin{aligned}
\frac{\partial^2}{\partial\tau^2}G = \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty} n^2\kappa^2 e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Bigg\{ & N_{nl}p_{nl}(r)q_{nl}(r) \\
& -\sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}^{(+)}_{ij}(r)\overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}^{(-)}_{ij}(r)\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) \\
& -\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}^{(l)}_{ij}(r)\overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}^{(p)}_{1j}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
& -\mathcal{T}^{(r)}_{10}(r)\overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0\mid r)\Bigg\} + \mathcal{O}(\epsilon^2\log\epsilon),
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
\frac{\partial}{\partial r}G = \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Bigg\{ & N_{nl}\frac{\partial p_{nl}(r)q_{nl}(r)}{\partial r} \\
& -\sum_{i=0}^{2}\sum_{j=0}^{i}\left[\frac{\partial\mathcal{D}^{(+)}_{ij}(r)}{\partial r}\overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r) + \mathcal{D}^{(+)}_{ij}(r)\frac{\partial\overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r)}{\partial r}\right] \\
& -\sum_{i=0}^{2}\sum_{j=0}^{i}\left[\frac{\partial\mathcal{D}^{(-)}_{ij}(r)}{\partial r}\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) + \mathcal{D}^{(-)}_{ij}(r)\frac{\partial\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r)}{\partial r}\right] \\
& -\sum_{i=0}^{1}\sum_{j=0}^{i}\left[\frac{\partial\mathcal{T}^{(l)}_{ij}(r)}{\partial r}\overset{[d]}{\chi}_{nl}(i,j|r) + \mathcal{T}^{(l)}_{ij}(r)\frac{\partial\overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r}\right] \\
& -\sum_{j=0}^{1}\left[\frac{\mathcal{T}^{(p)}_{1j}(r)}{\partial r}\overset{[d]}{\chi}_{nl}(1,j|r) + \mathcal{T}^{(p)}_{1j}(r)\frac{\partial\overset{[d]}{\chi}_{nl}(1,j|r)}{\partial r}\right] \\
& -\left[\frac{\partial\mathcal{T}^{(r)}_{10}(r)}{\partial r}\overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0\mid r) + \mathcal{T}^{(r)}_{10}(r)\frac{\partial\overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0\mid r)}{\partial r}\right]\Bigg\} \\
& +\mathcal{O}(\epsilon^3\log\epsilon).
\end{aligned} \tag{5.5}$$

Each of the derivatives of the coefficients, for $f(r) = 1 - \frac{2M}{r}$ are given in the tables below. We first note that all $\frac{\partial\mathcal{T}^{(l)}_{ij}(r)}{\partial r} = 0$. The rest are calculated in the tables below:

TABLE 5.6: $\frac{\partial \mathcal{D}^{(+)}_{ij}(r)}{\partial r}$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\frac{\partial \mathcal{D}^{(+)}_{ij}(r)}{\partial r}$ coefficients for $d = 4$			
	$j = 0$	$j = 1$	$j = 2$
$i = 0$	0		
$i = 1$	$\frac{M(8M-3r)}{r^5}$	$\frac{M}{3r^6}(45M^2 - 34Mr - \kappa^2 r^4 + 6r^2)$	
$i = 2$	$\frac{M}{12r^9}(-432M^3 + 455M^2 r + 8\kappa^2 M r^4 - 150Mr^2 - 3\kappa^2 r^5 + 15r^3)$	$-\frac{M}{180r^{10}}(12150M^4 - 16680M^3 r + M^2 \times (8015r^2 - 750\kappa^2 r^4) + 4Mr^3(145\kappa^2 r^2 - 384) + r^4\left(8\kappa^4 r^4 - 105\kappa^2 r^2 + 90\right)$	$-\frac{M(2M-r)}{18r^{11}}(9M^2 - 4Mr - \kappa^2 r^4) \times (45M^2 - 34Mr - \kappa^2 r^4 + 6r^2)$

TABLE 5.7: $\frac{\partial \mathcal{D}^{(-)}_{ij}(r)}{\partial r}$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\frac{\partial \mathcal{D}^{(-)}_{ij}(r)}{\partial r}$ coefficients for $d = 4$			
	$j = 0$	$j = 1$	$j = 2$
$i = 0$			
$i = 1$		$\frac{M}{r^4}$	
$i = 2$		$-\frac{M}{10r^8}(77M^2 - 57Mr + 10r^2)$	$\frac{M(5r-12M)}{20r^7}$

TABLE 5.8: $\frac{\partial \mathcal{T}^{(r)}_{ij}(r)}{\partial r}$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\frac{\partial \mathcal{T}^{(r)}_{ij}(r)}{\partial r}$ coefficients for $d = 4$		
	$j = 0$	$j = 1$
$i = 0$		
$i = 1$	$\frac{M^3}{12r^6}(-45M^2 + 34Mr + \kappa^2 r^4 - 6r^2)$	

TABLE 5.9: $\frac{\partial \mathcal{T}_{ij}^{(p)}(r)}{\partial r}$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$

$\frac{\partial \mathcal{T}_{ij}^{(p)}(r)}{\partial r}$ coefficients for $d = 4$ with $f(r) = 1 - \frac{2M}{r}$		
	$j = 0$	$j = 1$
$i = 0$		
$i = 1$	$-\dfrac{M^3}{4r^4}$	$\dfrac{M^3(3r-8M)}{4r^5}$

In Appendix D we calculate the derivatives of the mode-sum expressions, and list them here.

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j \mid r)}{\partial r} = {} & \frac{2^{i-j}}{\kappa^{2i+2j} r^{2+2j} \Gamma(1+j)} \sum_{m=0}^{i} \sum_{s=0}^{m} \frac{i!}{s!(i-m)!(m-s)!} \frac{(-1)^m}{2^m} \\
& \times \left(\frac{1}{\eta} \frac{\partial}{\partial \eta} \right)^j \Big[P_l(\eta) Q_l(\eta) \delta_{m,2s+n} + (-1)^{(2s+n-m)} P_l^{(m-2s-n)}(\eta) Q_l^{(2s+n-m)}(\eta) \\
& \qquad + (-1)^{(m-2s-n)} P_l^{(2s+n-m)}(\eta) Q_l^{(m-2s-n)}(\eta) \Big] \\
& \times \Bigg\{ -\frac{(j+1)}{r} - \frac{j(rf' - f^2)}{\kappa^2 r \eta^2} + \frac{rf' - f}{2fl\eta} \Bigg[\Bigg\{ \Big(\eta P_l(\eta) - P_{l-1}(\eta) \Big) Q_l(\eta) \\
& \qquad + P_l(\eta) \Big(\eta Q_l(\eta) - Q_{l-1}(\eta) \Big) \Bigg\} \delta_{m,2s+n} \\
& + (-1)^{(2s+n-m)} \Big\{ 2l\eta P_l^{m-2s-n}(\eta) Q_l^{2s+n-m}(\eta) \\
& \qquad - (l+2s-m+n) P_l^{m-2s-n}(\eta) Q_{l-1}^{2s-m+n} - (l-2s-n+m) P_{l-1}^{m-2s-n}(\eta) Q_l^{2s-m+n} \Big\} \\
& + (-1)^{(m-2s-n)} \Big\{ 2l\eta P_l^{2s-m+n}(\eta) Q_l^{m-n-2s}(\eta) \\
& \qquad - (l-2s+m-n) P_l^{2s-m+n}(\eta) Q_{l-1}^{m-2s-n} \\
& \qquad - (l+2s+n-m) P_{l-1}^{2s-m+n}(\eta) Q_l^{m-2s-n} \Big\} \Bigg] \Bigg\}.
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r)}{\partial r} &= \frac{\kappa}{2\pi}\frac{2^{i-1}(-1)^j}{\kappa^{2i-2j}r^{2-2j}\Gamma(1-j)}\sum_{k=0}^{j}\frac{(-1)^k}{2^{j+1}}\binom{j}{k}\frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}}\sum_{m=0}^{i-j}\sum_{s=0}^{m}\frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m}\\
&\times\Bigg\{\frac{2j-2}{r}\Bigg[P_{l+j-2k}(\eta)Q_{l+j-2k}(\eta)\delta_{m,2s+n}+(-1)^{(2s+n-m)}P^{(m-n-2s)}_{l+j-2k}(\eta)Q^{(2s+n-m)}_{l+j-2k}(\eta)\\
&\qquad+(-1)^{(m-2s-n)}P^{(2s+n-m)}_{l+j-2k}(\eta)Q^{(m-2s-n)}_{l+j-2k}(\eta)\Bigg]\\
&+\frac{rf'-f}{f(l+j-2k)\eta}\Bigg[\Bigg\{\Big(\eta P_{l+j-2k}(\eta)-P_{l+j-2k-1}(\eta)\Big)Q_{l+j-2k}(\eta)\\
&\qquad+P_{l+j-2k}(\eta)\Big(\eta Q_{l+j-2k}(\eta)-Q_{l+j-2k-1}(\eta)\Big)\Bigg\}\delta_{m,2s+n}\\
&+(-1)^{(2s+n-m)}\Big\{2(l+j-2k)\eta P^{m-2s-n}_{l+j-2k}(\eta)Q^{2s+n-m}_{l+j-2k}(\eta)\\
&\quad-(l+j-2k+2s-m+n)P^{m-2s-n}_{l+j-2k}(\eta)Q^{2s-m+n}_{l+j-2k-1}\\
&\qquad-(l+j-2k-2s-n+m)P^{m-2s-n}_{l+j-2k-1}(\eta)Q^{2s-m+n}_{l+j-2k}\Big\}\\
&+(-1)^{(m-2s-n)}\Big\{2(l+j-2k)\eta P^{2s+n-m}_{l+j-2k}(\eta)Q^{m-2s-n}_{l+j-2k}(\eta)\\
&\quad-(l+j-2k-2s+m-n)P^{2s+n-m}_{l+j-2k}(\eta)Q^{m-2s-n}_{l+j-2k-1}\\
&\qquad-(l+j-2k+2s+n-m)P^{2s+n-m}_{l+j-2k-1}(\eta)Q^{m-2s-n}_{l+j-2k}\Big\}\Bigg]\Bigg\}.
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} = {} & \frac{2^{j-1}r^{2i-2j}(i-j)!}{\kappa^{2j}} \sum_{k=0}^{i-j+1} (-1)^k \binom{i-j+1}{k} \frac{(l+i-j-2k+\frac{3}{2})}{(l-k+\frac{1}{2})_{i-j+2}} \\
& \times \Bigg\{ \frac{2i-2j}{r} \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} P_{l+i-j-2k+1}(\eta) Q_{l+i-j-2k+1}(\eta) \\
& \qquad + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \Big(P^{(m-n-2s)}_{l+i-j-2k+1}(\eta) Q^{(2s+n-m)}_{l+i-j-2k+1}(\eta) \\
& \qquad\quad + (-1)^n P^{(2s+n-m)}_{l+i-j-2k+1}(\eta) Q^{(m-2s-n)}_{l+i-j-2k+1}(\eta) \Big) \Bigg] \\
& + \frac{rf' - f}{f(l+i-j-2k+1)\eta} \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} \\
& \qquad \times \Big\{ \Big(\eta P_{l+i-j-2k+1}(\eta) - P_{l+i-j-2k}(\eta) \Big) Q_{l+i-j-2k+1}(\eta) \\
& \qquad + P_{l+i-j-2k+1}(\eta) \Big(\eta Q_{l+i-j-2k+1}(\eta) - Q_{l+i-j-2k}(\eta) \Big) \Big\} \\
& + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \\
& \times \Big\{ 2(1+i-j-2k+l)\eta P^{m-n-2s}_{l+i-j-2k+1}(\eta) Q^{2s+n-m}_{l+i-j-2k+1} \\
& \qquad - (2s+n-m+l+i-j-2k+1) \\
& \qquad\quad \times [P^{m-n-2s}_{l+i-j-2k+1}(\eta) Q^{2s-m+n}_{i-j-2k+l} + P^{m-n-2s}_{i-j-2k+l}(\eta) Q^{2s-m+n}_{l+i-j-2k+1}(\eta)] \\
& \qquad + (-1)^n \Big[2(1+i-j-2k+l)\eta P^{2s-m+n}_{l+i-j-2k+1}(\eta) Q^{m-n-2s}_{l+i-j-2k+1} \\
& \qquad - (m-2s-n+l+i-j-2k+1) \\
& \qquad\quad \times [P^{2s-m+n}_{l+i-j-2k+1}(\eta) Q^{m-2s-n}_{l+i-j-2k} + P^{2s-m+n}_{l+i-j-2k}(\eta) Q^{m-2s-n}_{l+i-j-2k+1}(\eta)] \Big] \Big\} \Bigg] \Bigg\}, \\
& \qquad\quad l > i-j.
\end{aligned} \tag{5.8}
$$

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} &= \frac{\kappa}{2\pi}\left(\frac{2}{\kappa^2}\right)^j 2^{i-j}(i-j)(r)^{i-j-1}(-1)^l \Big[\frac{d}{d\lambda}(\lambda+1-l)_l\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
&\qquad \times (z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j} \\
&\quad + \frac{\kappa}{2\pi}\left(\frac{2}{\kappa^2}\right)^j (2r)^{i-j}(-1)^l\Bigg[\frac{2(\lambda-i+j)\ell^2}{r}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} \\
&\qquad \times \frac{d}{d\lambda}\Big[(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j} \\
&\qquad + \frac{2}{r}\Big\{(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big\}\Bigg] \\
&= \frac{\kappa}{2\pi r}\left(\frac{2}{\kappa^2}\right)^j (2r)^{i-j}(-1)^l\Bigg[(i-j)\Big[\frac{d}{d\lambda}(\lambda+1-l)_l\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
&\quad \times (z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j} \\
&\qquad + 2(\lambda-i+j)\ell^2\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} \\
&\qquad \times \frac{d}{d\lambda}\Big[(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j} \\
&\quad + 2\Big\{(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big\}\Bigg], \\
&\qquad\qquad l \le i-j.
\end{aligned}
\tag{5.9}
$$

We combine Eq. 5.4 and Eq. 5.5 and simplify to obtain

$$
\begin{aligned}
g^{tt}G_{;tt} = \frac{1}{8\pi^2} & \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) \\
& \times \Bigg\{ \frac{n^2\kappa^2}{f} N_{nl} p_{nl}(r) q_{nl}(r) + \frac{f'}{2} N_{nl} \frac{\partial p_{nl}(r) q_{nl}(r)}{\partial r} \\
& - \sum_{i=0}^{2}\sum_{j=0}^{i} \left[\frac{n^2\kappa^2}{f} \mathcal{D}_{ij}^{(+)}(r) \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r) + \frac{f'}{2}\left(\frac{\partial \mathcal{D}_{ij}^{(+)}(r)}{\partial r} \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r) + \mathcal{D}_{ij}^{(+)}(r) \frac{\partial \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r)}{\partial r} \right) \right] \\
& - \sum_{i=0}^{2}\sum_{j=0}^{i} \left[\frac{n^2\kappa^2}{f} \mathcal{D}_{ij}^{(-)}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r) + \frac{f'}{2}\left(\frac{\partial \mathcal{D}_{ij}^{(-)}(r)}{\partial r} \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r) + \mathcal{D}_{ij}^{(-)}(r) \frac{\partial \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r)}{\partial r} \right) \right] \\
& - \sum_{i=0}^{1}\sum_{j=0}^{i} \left[\frac{n^2\kappa^2}{f} \mathcal{T}_{ij}^{(l)}(r) \overset{[d]}{\chi}_{nl}(i,j|r) + \frac{f'}{2} \mathcal{T}_{ij}^{(l)}(r) \frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} \right] \\
& - \sum_{j=0}^{1} \left[\frac{n^2\kappa^2}{f} \mathcal{T}_{1j}^{(p)}(r) \overset{[d]}{\chi}_{nl}(1,j|r) + \frac{f'}{2}\left(\frac{\partial \mathcal{T}_{1j}^{(p)}(r)}{\partial r} \overset{[d]}{\chi}_{nl}(1,j|r) + \mathcal{T}_{1j}^{(p)}(r) \frac{\partial \overset{[d]}{\chi}_{nl}(1,j|r)}{\partial r} \right) \right] \\
& - \frac{n^2\kappa^2}{f} \mathcal{T}_{10}^{(r)}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r) - \frac{f'}{2}\left(\frac{\partial \mathcal{T}_{10}^{(r)}(r)}{\partial r} \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r) + \mathcal{T}_{10}^{(r)}(r) \frac{\partial \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r)}{\partial r} \right) \Bigg\} \\
& \qquad + \mathcal{O}(\epsilon^2 \log\epsilon)
\end{aligned}
$$

and in the coincidence limit

$$
\begin{aligned}
g^{tt}G_{ren\,;tt} = \frac{1}{8\pi^2} & \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (2l+1) \\
& \times \Bigg\{ \frac{n^2\kappa^2}{f} N_{nl} p_{nl}(r) q_{nl}(r) + \frac{f'}{2} N_{nl} \frac{\partial p_{nl}(r) q_{nl}(r)}{\partial r} \\
& - \sum_{i=0}^{2}\sum_{j=0}^{i} \left[\frac{n^2\kappa^2}{f} \mathcal{D}_{ij}^{(+)}(r) \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r) + \frac{f'}{2}\left(\frac{\partial \mathcal{D}_{ij}^{(+)}(r)}{\partial r} \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r) + \mathcal{D}_{ij}^{(+)}(r) \frac{\partial \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r)}{\partial r} \right) \right] \\
& - \sum_{i=0}^{2}\sum_{j=0}^{i} \left[\frac{n^2\kappa^2}{f} \mathcal{D}_{ij}^{(-)}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r) + \frac{f'}{2}\left(\frac{\partial \mathcal{D}_{ij}^{(-)}(r)}{\partial r} \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r) + \mathcal{D}_{ij}^{(-)}(r) \frac{\partial \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r)}{\partial r} \right) \right] \\
& - \sum_{i=0}^{1}\sum_{j=0}^{i} \left[\frac{n^2\kappa^2}{f} \mathcal{T}_{ij}^{(l)}(r) \overset{[d]}{\chi}_{nl}(i,j|r) + \frac{f'}{2} \mathcal{T}_{ij}^{(l)}(r) \frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} \right] \\
& - \sum_{j=0}^{1} \left[\frac{n^2\kappa^2}{f} \mathcal{T}_{1j}^{(p)}(r) \overset{[d]}{\chi}_{nl}(1,j|r) + \frac{f'}{2}\left(\frac{\partial \mathcal{T}_{1j}^{(p)}(r)}{\partial r} \overset{[d]}{\chi}_{nl}(1,j|r) + \mathcal{T}_{1j}^{(p)}(r) \frac{\partial \overset{[d]}{\chi}_{nl}(1,j|r)}{\partial r} \right) \right] \\
& - \frac{n^2\kappa^2}{f} \mathcal{T}_{10}^{(r)}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r) - \frac{f'}{2}\left(\frac{\partial \mathcal{T}_{10}^{(r)}(r)}{\partial r} \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r) + \mathcal{T}_{10}^{(r)}(r) \frac{\partial \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r)}{\partial r} \right) \Bigg\},
\end{aligned}
$$

simplifying to

$$
\begin{aligned}
g^{tt}G_{ren\,;tt} &= \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1) \\
&\times\left(\frac{n^2\kappa^2}{f}+\frac{f'}{2}\frac{\partial}{\partial r}\right)\Bigg(N_{nl}p_{nl}(r)q_{nl}(r)-\sum_{i=0}^{2}\sum_{j=0}^{i}\left[\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r)+\mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r)\right] \\
&\quad-\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}{}_{nl}(i,j|r)-\sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}{}_{nl}(1,j|r)-\mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg) \\
&=\left(\frac{n^2\kappa^2}{f}+\frac{f'}{2}\frac{\partial}{\partial r}\right)G_{ren}
\end{aligned}
\tag{5.10}
$$

where the coefficients and derivatives are listed above.

- $g^{\theta\theta'}G_{ren\,;\theta\theta'}$

$$g^{\theta\theta'}G_{ren\,;\theta\theta'}=g^{\theta\theta'}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta'}G_{ren}.$$

Hence we first need to consider $\frac{\partial}{\partial\theta'}G$. The only term in Eq. 5.4 that depends on θ' is $P_l(\cos\gamma)$, since $\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$. Via the chain rule we calculate that

$$
\begin{aligned}
\frac{\partial P_l(\cos\gamma)}{\partial\theta'} &= \frac{\partial P_l(\cos\gamma)}{\partial\cos\gamma}\frac{\partial\cos\gamma}{\partial\theta'} \\
&= \frac{\cos\gamma P_l(\cos\gamma)-P_{l-1}(\cos\gamma)}{\cos^2\gamma-1}\Big(\sin\theta\cos\theta'\cos(\phi-\phi')-\cos\theta\sin\theta'\Big).
\end{aligned}
$$

We next require

$$\frac{\partial}{\partial\theta}\frac{\partial P_l(\cos\gamma)}{\partial\theta'}=\frac{\partial}{\partial\theta}\left[\frac{\cos\gamma P_l(\cos\gamma)-P_{l-1}(\cos\gamma)}{\cos^2\gamma-1}\Big(\sin\theta\cos\theta'\cos(\phi-\phi')-\cos\theta\sin\theta'\Big)\right].$$

Although this expression is relatively easily calculated using Mathematica or similar, the expression is too large to print considering that we are only interested in the partial coincidence limit, where $\theta\to\theta'$ and $\phi\to\phi'$:

$$\left[\frac{\partial^2 P_l(\cos\gamma)}{\partial\theta\theta'}\right]_{\theta=\theta',\phi=\phi'}=\frac{1}{2}l(l+1).$$

And since $g^{\theta\theta'} = \frac{1}{r^2}$ [4], then

$$
\begin{aligned}
g^{\theta\theta'} G_{;\theta\theta'} = \frac{1}{8r^2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} \frac{l(l+1)(2l+1)}{2} \Bigg\{ & N_{nl} p_{nl}(r) q_{nl}(r) - \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}^{(+)}_{ij}(r) \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r) \\
& - \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}^{(-)}_{ij}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r) - \sum_{i=0}^{1}\sum_{j=0}^{i} \mathcal{T}^{(l)}_{ij}(r) \overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1} \mathcal{T}^{(p)}_{1j}(r) \overset{[d]}{\chi}_{nl}(1,j|r) \\
& - \mathcal{T}^{(r)}_{10}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r) \Bigg\} + \mathcal{O}(\epsilon^2 \log \epsilon).
\end{aligned}
$$

Thus

$$
\begin{aligned}
g^{\theta\theta'} G_{ren\,;\theta\theta'} &= \frac{1}{8r^2\pi^2} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \frac{l(l+1)(2l+1)}{2} \Bigg\{ N_{nl} p_{nl}(r) q_{nl}(r) - \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}^{(+)}_{ij}(r) \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r) \\
&\quad - \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}^{(-)}_{ij}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r) - \sum_{i=0}^{1}\sum_{j=0}^{i} \mathcal{T}^{(l)}_{ij}(r) \overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1} \mathcal{T}^{(p)}_{1j}(r) \overset{[d]}{\chi}_{nl}(1,j|r) \\
&\quad - \mathcal{T}^{(r)}_{10}(r) \overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0 \mid r) \Bigg\} \\
&= \frac{l(l+1)}{2r^2} G_{ren}.
\end{aligned} \tag{5.11}
$$

Due to the symmetries of the spacetime, we may conclude that

$$
g^{\phi\phi'} G_{ren\,;\phi\phi'} = g^{\theta\theta'} G_{ren\,;\theta\theta'}. \tag{5.12}
$$

- $g^{\theta\theta} G_{ren\,;\theta\theta}$

$$
g^{\theta\theta} G_{;\theta\theta} = g^{\theta\theta} \left(\frac{\partial^2}{\partial\theta^2} G - \Gamma^r_{\theta\theta} \frac{\partial}{\partial r} G \right).
$$

Using the fact that, in the partial coincidence limit [4],

$$
\left[\frac{\partial^2 P_l(\cos\gamma)}{\partial\theta^2} \right]_{\theta=\theta',\phi=\phi'} = -\frac{1}{2} l(l+1)
$$

and with $g^{\theta\theta} = \frac{1}{r^2}$ and $\Gamma^r_{\theta\theta} = -rf$, we obtain

$$
\begin{aligned}
g^{\theta\theta}G_{;\theta\theta} = \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Bigg\{ &-\frac{l(l+1)}{2r^2}N_{nl}p_{nl}(r)q_{nl}(r) + \frac{f}{r}P_l(\cos\gamma)N_{nl}\frac{\partial p_{nl}(r)q_{nl}(}{\partial r} \\
&+\sum_{i=0}^{2}\sum_{j=0}^{i}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{D}^{(+)}_{ij}(r)\overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r) \\
&\quad + \frac{f}{r}P_l(\cos\gamma)\Bigg(\frac{\partial\mathcal{D}^{(+)}_{ij}(r)}{\partial r}\overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r) + \mathcal{D}^{(+)}_{ij}(r)\frac{\partial\overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r)}{\partial r}\Bigg)\Bigg] \\
&+\sum_{i=0}^{2}\sum_{j=0}^{i}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{D}^{(-)}_{ij}(r)\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) \\
&\quad + \frac{f}{r}P_l(\cos\gamma)\Bigg(\frac{\partial\mathcal{D}^{(-)}_{ij}(r)}{\partial r}\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r) + \mathcal{D}^{(-)}_{ij}(r)\frac{\partial\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j\mid r)}{\partial r}\Bigg)\Bigg] \\
&+\sum_{i=0}^{2}\sum_{j=0}^{i}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{T}^{(l)}_{ij}(r)\overset{[d]}{\chi}_{nl}(i,j|r) + \frac{f}{r}P_l(\cos\gamma)\mathcal{T}^{(l)}_{ij}(r)\frac{\partial\overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r}\Bigg] \\
&+\sum_{j=0}^{1}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{T}^{(p)}_{1j}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
&\quad + \frac{f}{r}P_l(\cos\gamma)\Bigg(\frac{\partial\mathcal{T}^{(p)}_{1j}(r)}{\partial r}\overset{[d]}{\chi}_{nl}(1,j|r) + \mathcal{T}^{(p)}_{1j}(r)\frac{\partial\overset{[d]}{\chi}_{nl}(1,j|r)}{\partial r}\Bigg)\Bigg] \\
&+\frac{l(l+1)}{2r^2}\mathcal{T}^{(r)}_{10}(r)\overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0\mid r) \\
&\quad + \frac{f}{r}P_l(\cos\gamma)\Bigg(\frac{\partial\mathcal{T}^{(r)}_{10}(r)}{\partial r}\overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0\mid r) + \mathcal{T}^{(r)}_{10}(r)\frac{\partial\overset{[d]}{\Psi}{}^{(-)}_{nl}(2,0\mid r)}{\partial r}\Bigg)\Bigg\} \\
&\qquad + \mathcal{O}(\epsilon^2\log\epsilon).
\end{aligned}
$$

In the coincidence limit,

$$
\begin{aligned}
g^{\theta\theta}G_{ren\,;\theta\theta} = \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\Bigg\{ & -\frac{l(l+1)}{2r^2}N_{nl}p_{nl}(r)q_{nl}(r) + \frac{f}{r}N_{nl}\frac{\partial p_{nl}(r)q_{nl}(r)}{\partial r} \\
& + \sum_{i=0}^{2}\sum_{j=0}^{i}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) \\
& \qquad + \frac{f}{r}\Bigg(\frac{\partial\mathcal{D}_{ij}^{(+)}(r)}{\partial r}\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) + \mathcal{D}_{ij}^{(+)}(r)\frac{\partial\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r)}{\partial r}\Bigg)\Bigg] \\
& + \sum_{i=0}^{2}\sum_{j=0}^{i}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) \\
& \qquad + \frac{f}{r}\Bigg(\frac{\partial\mathcal{D}_{ij}^{(-)}(r)}{\partial r}\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) + \mathcal{D}_{ij}^{(-)}(r)\frac{\partial\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r)}{\partial r}\Bigg)\Bigg] \\
& + \sum_{i=0}^{1}\sum_{j=0}^{i}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r) + \frac{f}{r}\mathcal{T}_{ij}^{(l)}(r)\frac{\partial\overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r}\Bigg] \\
& + \sum_{j=0}^{1}\Bigg[\frac{l(l+1)}{2r^2}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}_{nl}(1,j|r) \\
& \qquad + \frac{f}{r}\Bigg(\frac{\partial\mathcal{T}_{1j}^{(p)}(r)}{\partial r}\overset{[d]}{\chi}_{nl}(1,j|r) + \mathcal{T}_{1j}^{(p)}(r)\frac{\partial\overset{[d]}{\chi}_{nl}(1,j|r)}{\partial r}\Bigg)\Bigg] \\
& + \frac{l(l+1)}{2r^2}\mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r) \\
& \qquad + \frac{f}{r}\Bigg(\frac{\partial\mathcal{T}_{10}^{(r)}(r)}{\partial r}\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r) + \mathcal{T}_{10}^{(r)}(r)\frac{\partial\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)}{\partial r}\Bigg)\Bigg\},
\end{aligned}
$$

simplifying to

$$
\begin{aligned}
g^{\theta\theta}G_{ren\,;\theta\theta} &= -\frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\left(\frac{l(l+1)}{2r^2} - \frac{f}{r}\frac{\partial}{\partial r}\right)\Bigg(N_{nl}p_{nl}(r)q_{nl}(r) \\
&\quad - \sum_{i=0}^{2}\sum_{j=0}^{i}\left[\mathcal{D}_{ij}^{(+)}(r)\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) + \mathcal{D}_{ij}^{(-)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r)\right] \\
&\quad - \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\overset{[d]}{\chi}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\overset{[d]}{\chi}_{nl}(1,j|r) - \mathcal{T}_{10}^{(r)}(r)\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg) \\
&= \left(\frac{f}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{2r^2}\right)G_{ren}.
\end{aligned}
\tag{5.13}
$$

Again, due to the symmetries of the spacetime, we conclude that

$$g^{\phi\phi} G_{ren\,;\phi\phi} = g^{\theta\theta} G_{ren\,;\theta\theta}. \tag{5.14}$$

- $g^{rr} G_{ren\,;rr}$

  We may now make use of the fact that the Green function G_{ren} satisfies the wave equation[4]

  $$(\Box - m^2 - \xi R) G_{ren} = 0$$

  and calculate [4]

  $$g^{rr} G_{ren\,;rr} = -g^{tt} G_{ren;tt} - 2 g^{\theta\theta} G_{ren;\theta\theta} + (m^2 + \xi R) G_{ren}$$

  since we have already computed the required terms, and recalling that $g^{\phi\phi} G_{ren\,;\phi\phi} = g^{\theta\theta} G_{ren\,;\theta\theta}$ by symmetry. Thus

[4]The alternative is to take double derivatives of the mode-sum with respect to r, which would be quite lengthy.

$$
\begin{aligned}
g^{rr}G_{ren\,;rr} = {} & \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1) \\
& \times\Bigg\{\Big[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)\Big]N_{nl}\,p_{nl}(r)q_{nl}(r)-\Big[\frac{f'}{2}+\frac{2f}{r}\Big]N_{nl}\frac{\partial p_{nl}(r)q_{nl}(r)}{\partial r} \\
& -\sum_{i=0}^{2}\sum_{j=0}^{i}\Bigg[\Big[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)\Big]\mathcal{D}^{(+)}_{ij}(r)\,\overset{[d]}{\Psi}{}^{(+)}_{nl}\,(i,j\mid r) \\
& \qquad +\Big[\frac{f'}{2}-\frac{2f}{r}\Big]\Bigg(\frac{\partial\mathcal{D}^{(+)}_{ij}(r)}{\partial r}\,\overset{[d]}{\Psi}{}^{(+)}_{nl}\,(i,j\mid r)+\mathcal{D}^{(+)}_{ij}(r)\frac{\partial\,\overset{[d]}{\Psi}{}^{(+)}_{nl}\,(i,j\mid r)}{\partial r}\Bigg)\Bigg] \\
& -\sum_{i=0}^{2}\sum_{j=0}^{i}\Bigg[\Big[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)\Big]\mathcal{D}^{(-)}_{ij}(r)\,\overset{[d]}{\Psi}{}^{(-)}_{nl}\,(i,j\mid r) \\
& \qquad +\Big[\frac{f'}{2}-\frac{2f}{r}\Big]\Bigg(\frac{\partial\mathcal{D}^{(-)}_{ij}(r)}{\partial r}\,\overset{[d]}{\Psi}{}^{(-)}_{nl}\,(i,j\mid r)+\mathcal{D}^{(-)}_{ij}(r)\frac{\partial\,\overset{[d]}{\Psi}{}^{(-)}_{nl}\,(i,j\mid r)}{\partial r}\Bigg)\Bigg] \\
& -\sum_{i=0}^{1}\sum_{j=0}^{i}\Bigg[\Big[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)\Big]\mathcal{T}^{(l)}_{ij}(r)\,\overset{[d]}{\chi}_{nl}\,(i,j|r) \\
& \qquad +\Big[\frac{f'}{2}-\frac{2f}{r}\Big]\mathcal{T}^{(l)}_{ij}(r)\frac{\partial\,\overset{[d]}{\chi}_{nl}\,(i,j|r)}{\partial r}\Bigg] \\
& -\sum_{j=0}^{1}\Bigg[\Big[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)\Big]\mathcal{T}^{(p)}_{1j}(r)\,\overset{[d]}{\chi}_{nl}\,(1,j|r) \\
& \qquad +\Big[\frac{f'}{2}-\frac{2f}{r}\Big]\Bigg(\frac{\partial\mathcal{T}^{(p)}_{1j}(r)}{\partial r}\,\overset{[d]}{\chi}_{nl}\,(1,j|r)+\mathcal{T}^{(p)}_{1j}(r)\frac{\partial\,\overset{[d]}{\chi}_{nl}\,(1,j|r)}{\partial r}\Bigg)\Bigg] \\
& -\Big[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)\Big]\mathcal{T}^{(r)}_{10}(r)\,\overset{[d]}{\Psi}{}^{(-)}_{nl}\,(2,0\mid r) \\
& \qquad -\Big[\frac{f'}{2}+\frac{2f}{r}\Big]\Bigg(\frac{\partial\mathcal{T}^{(r)}_{10}(r)}{\partial r}\,\overset{[d]}{\Psi}{}^{(-)}_{nl}\,(2,0\mid r)+\mathcal{T}^{(r)}_{10}(r)\frac{\partial\,\overset{[d]}{\Psi}{}^{(-)}_{nl}\,(2,0\mid r)}{\partial r}\Bigg)\Bigg\}
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
g^{rr}G_{ren\,;rr} &= \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1) \\
&\quad\times\Bigg\{\left[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)-\left(\frac{f^{'}}{2}-\frac{2f}{r}\right)\frac{\partial}{\partial r}\right]\Bigg(N_{nl}p_{nl}(r)q_{nl}(r) \\
&\qquad -\sum_{i=0}^{2}\sum_{j=0}^{i}\left[\mathcal{D}_{ij}^{(+)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r)-\mathcal{D}_{ij}^{(-)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r)\right] \\
&\qquad -\sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\,\overset{[d]}{\chi}{}_{nl}(i,j|r)-\sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\,\overset{[d]}{\chi}{}_{nl}(1,j|r)-\mathcal{T}_{10}^{(r)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg) \\
&\qquad +\frac{4f}{r}\frac{\partial}{\partial r}\left(N_{nl}p_{nl}(r)q_{nl}(r)+\mathcal{T}_{10}^{(r)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\right)\Bigg\} \\
&= \left[\frac{l(l+1)}{r^2}-\frac{n^2\kappa^2}{f}+(m+\xi R)-\left(\frac{f^{'}}{2}-\frac{2f}{r}\right)\frac{\partial}{\partial r}\right]G_{ren} \\
&\qquad +\frac{4f}{r}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\frac{\partial}{\partial r}\left(N_{nl}p_{nl}(r)q_{nl}(r)+\mathcal{T}_{10}^{(r)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\right).
\end{aligned}
\tag{5.15}
$$

5.5 $g^{rr'}G_{ren\,;rr'}$

The attentive reader will notice that the derivative $g^{rr'}G_{ren\,;rr'}$ is missing from this work. This is not an oversight, but stems from section 3.3 where we set $\Delta r = 0$ at the beginning of the expansion. Thus, this derivative cannot be calculated, as we have essentially removed the point $r^{'}$ from our prescription. This was a simplification step, permitting us to consider the partial coincidence limit throughout the remainder of the work. If we had not done so, we would have expanded in terms of $(w, s, \Delta r)$, rather than our (w, s) terms, and increased the complexity of the coefficients.

Thus in order to calculate $g^{rr'}G_{ren\,;rr'}$ we could revert to the initial expansion and re-examine the resulting expressions in terms of coefficients, convergence and the resulting mode-sum. This is outside of the scope of this work, but would be a logical step for further work.

5.6 Summary

Armed with a mode-sum description of the singular Green propagator from Chapter 4, we subtracted this from the Euclidean Green function, took the coincidence limit and calculated our renormalised Green function. With a view to the terms required in the eventual calculation of the renormalised stress-energy tensor, we chose $f(r) = 1 - \frac{2M}{r}$ for our spherically symmetric black hole spacetime, and calculated the derivatives of the direct and tail coefficients with respect to r. We also calculated the derivatives of each of the four regularisation terms from Chapter 4, since they are required for the next step - calculating the derivatives of the renormalised Green function. We presented a succinct description of each mode-sum derivative, other than $g^{rr'} G_{ren\,;rr'}$.

Chapter 6

Conclusion

In the first two chapters of this thesis, we introduced the basic concepts behind two of the most successful physical theories of the twentieth century - general relativity and quantum field theory. We then addressed one of technical problems however that prevents the successful unification of these theories - the so called ultraviolet divergences that generate significant difficulties, and have so far been an obstacle to substantial progress in the field. The study of quantum field theory in curved spacetimes (QFTCS) is an active area of study, within which much of the literature and inspiration for this work is grounded. One potentially fruitful technique is to approximate a full theory of quantum gravity via semiclassical gravity, in which we work towards the expectation value of the stress-energy tensor, or of the vacuum polarization, or other physical quantities.

In Chapters 3 and 4, we worked through the novel approach of Taylor and Breen in devising a new expansion of the world function σ using 'extended coordinates' w and s, and by keeping careful track of the order, we then calculated the coefficients of the various terms in (w, s). We assumed a Hadamard parametrix as a complete description of the singular propagator - an assumption well supported by the cited literature - and we carefully considered the contribution of each term in the expansion to the coincidence limit, in terms of convergence of the resulting mode-sum. The resulting mode-sum as a function of the modes n and l, related to the quantum number and angular momentum respectively. We used a technique known as point-splitting, in which we seperate two spacetime coordinates, only to bring them together in the coincidence limit as we subtracted mode-by-mode from the singular propagator to arrive at a finite Green propagator - the renormalised Green function.

Chapters 3 and 4 work towards the same goal as that of [6] - calculating the expectation value of the vacuum polarisation. This involves taking the conincidence limit, and hence disregarding certain terms due to lack of contribution. However in Chapter 5, we reintroduced these terms and targeted the derivatives of the renormalised Green function as our quantities of interest. In this novel work, we calculated five distinct derivatives of the Green function in terms of our mode-sum prescription (seven by symmetry) with respect to the coordinates of the spherically symmetric black hole spacetime of interest. These derivatives are expressed in terms of the derivatives with respect to r only of the direct and tail coefficients, and the regularisation parameters calculated in Chapter 4, due to the symmetry of the spacetime, as we discussed in section 3.1.

There are some exciting opportunities for further work extending upon the work of Taylor and Breen, including reverting to the step where we set $\Delta r = 0$, and reworking the expansion and subsequent mode-sum so that the derivative $G_{;rr'}$ is possible. With each of the derivatives calculated, it is then a logical step to calculate the renormalised expectation value for the stress-energy tensor, where we would first require other quantities such as the Riemann tensor $R^{a}{}_{bcd}$ and Ricci scalar R. Armed with the renormalised stress-energy tensor, we may use it to solve the Einstein field equations, hence finding a first order approximation to quantum gravity.

As discussed in the earlier chapters, it is not expected that the methods of semiclassical gravity would ultimately provide the fundamental connection between quantum field theory and gravity, however they are evidently useful in certain physical systems - such as the black hole horizon featured in this work - and may shed some light on some interesting physics or other, insightful mathematical techniques that could help propel the field forward into new discoveries.

Appendix A

The Radial Green Function

To show that the radial Green function g_{nl} satisfies Eq. 2.2, we begin with

$$(\Box - m^2 - \xi R)\frac{\kappa}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\gamma)g_{nl}(r,r^{'})$$
$$= \frac{\kappa}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\gamma)\frac{\delta(r-r^{'})}{r^2}.$$

We begin with calculating the covariant derivative term:

$$\Box\left[\frac{\kappa}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\gamma)g_{nl}(r,r^{'})\right]$$
$$= -\frac{\kappa}{2f\pi}\sum_{n=-\infty}^{\infty} n^2\kappa^2 e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\gamma)g_{nl}(r,r^{'})$$
$$+\frac{f\kappa}{\pi r}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\gamma)g^{'}_{nl}(r,r^{'})$$
$$+\frac{\kappa f}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\gamma)g^{''}_{nl}(r,r^{'})$$
$$+\frac{\kappa}{2\pi r^2}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}l(l+1)P_l(\cos\gamma)g_{nl}(r,r^{'}).$$

Thus after equating the summations on either side, we are left with

$$-\frac{n^2\kappa^2}{f}g_{nl}(r,r^{'}) + \frac{2f}{r}g^{'}_{nl}(r,r^{'}) + \frac{l(l+1)}{r^2}g_{nl}(r,r^{'}) - \xi R g_{nl}(r,r^{'}) - m^2 g_{nl}(r,r^{'}) = \frac{\delta(r,r^{'})}{r^2}$$

which simplifies to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 f\frac{d}{dr}g_{nl}(r,r^{'})\right) - \left(\frac{n^2\kappa^2}{f} + \frac{l(l+1)}{r^2} + m^2 + \xi R\right)g_{nl}(r,r^{'}) = \frac{\delta(r-r^{'})}{r^2}$$

Appendix B

$U(x,x^{'})$ and $V(x,x^{'})$

B.1 Solving the Wave Equation for U, V and W

The inhomogenous wave equation

$$(\Box - \xi R - m^2)G(x,x^{'}) = -\delta(x,x^{'})$$

admits the Hadamard solutions [5]

$$G(x,x^{'}) = \frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma} + V\ln(\sigma) + W\right].$$

Applying this solution to the wave equation we can write

$$\Box(\tfrac{\Delta^{1/2}}{\sigma}) + \Box(V\ln\sigma) + \Box W = (m^2+\xi R)(\tfrac{\Delta^{1/2}}{\sigma} + V\ln\sigma + W). \tag{B.1}$$

We solve term by term on the LHS, then combine and equate with the RHS. The first term gives

$$\begin{aligned}
\Box(\tfrac{\Delta^{1/2}}{\sigma}) &= g^{ab}\nabla_a\nabla_b(\tfrac{\Delta^{1/2}}{\sigma}) \\
&= g^{ab}\nabla_a\left[\frac{\nabla_b\Delta^{1/2}}{\sigma} - \frac{\Delta^{1/2}\nabla_b\sigma}{\sigma^2}\right] \\
&= g^{ab}\left[\frac{\sigma\nabla_a\nabla_b\Delta^{1/2} - \nabla_b\Delta^{1/2}\nabla_a\sigma}{\sigma^2} - \left(\frac{\sigma^2(\nabla_a\Delta^{1/2}\nabla_b\sigma + \Delta^{1/2}\nabla_a\nabla_b\sigma) - \Delta^{1/2}\nabla_b\sigma\nabla_a\sigma^2}{\sigma^4}\right)\right] \\
&= g^{ab}\left[\frac{\nabla_a\nabla_b\Delta^{1/2}}{\sigma} - \frac{\nabla_b\Delta^{1/2}\nabla_a\sigma}{\sigma^2} - \frac{\nabla_a\Delta^{1/2}\nabla_b\sigma + \Delta^{1/2}\nabla_a\nabla_b\sigma}{\sigma^2} + \frac{\Delta^{1/2}\nabla_b\sigma\nabla_a\sigma^2}{\sigma^4}\right] \\
&= \frac{\Box\Delta^{1/2}}{\sigma} - \frac{g^{ab}\nabla_b\Delta^{1/2}\sigma_a}{\sigma^2} - \frac{g^{ab}\nabla_a\Delta^{1/2}\sigma_b}{\sigma^2} - \frac{\Delta^{1/2}\Box\sigma}{\sigma^2} + \frac{g^{ab}\Delta^{1/2}\sigma_b(2\sigma\sigma_a)}{\sigma^4}.
\end{aligned}$$

Using that $\sigma_a\sigma^a = 2\sigma$ and $\Delta^{1/2}(\Box\sigma - 4) + 2\sigma^a\nabla_b\Delta^{1/2} = 0$ we then have

$$
\begin{aligned}
\Box(\tfrac{\Delta^{1/2}}{\sigma}) &= \frac{\Box\Delta^{1/2}}{\sigma} - \frac{g^{ab}\nabla_b\Delta^{1/2}\sigma_a}{\sigma^2} - \frac{g^{ab}\nabla_a\Delta^{1/2}\sigma_b}{\sigma^2} - \frac{\Delta^{1/2}\Box\sigma}{\sigma^2} + \frac{4\Delta^{1/2}}{\sigma^2} \\
&= \frac{\Box\Delta^{1/2}}{\sigma} - \frac{\sigma^b\nabla_b\Delta^{1/2}}{\sigma^2} - \frac{g^{ab}\sigma_b\delta^b_a\nabla_b\Delta^{1/2}}{\sigma^2} - \frac{1}{\sigma^2}(\Delta^{1/2}\Box\sigma - 4\Delta^{1/2}) \\
&= \frac{\Box\Delta^{1/2}}{\sigma} - \frac{\sigma^b\nabla_b\Delta^{1/2}}{\sigma^2} - \frac{\sigma^b\nabla_b\Delta^{1/2}}{\sigma^2} + \frac{2\sigma^a\nabla_b\Delta^{1/2}}{\sigma^2} \\
&= \frac{\Box\Delta^{1/2}}{\sigma}.
\end{aligned}
$$

The next term on the LHS of Eq. B.1 gives

$$
\begin{aligned}
\Box(V\ln\sigma) &= g^{ab}\nabla_a\nabla_b(V\ln\sigma) \\
&= g^{ab}\nabla_a[(\nabla_b V)\ln\sigma + V(\nabla_b\ln\sigma)] \\
&= g^{ab}\nabla_a[(\nabla_b V)\ln\sigma + V\frac{\sigma_b}{\sigma}] \\
&= g^{ab}\left[\nabla_a[(\nabla_b V)\ln\sigma] + \nabla_a(V\frac{\sigma_b}{\sigma})\right] \\
&= g^{ab}\left[(\nabla_a\nabla_b V)\ln\sigma + (\nabla_b V)(\nabla_a\ln\sigma) + (\nabla_a V)\frac{\sigma_b}{\sigma} + V(\nabla_a\frac{\sigma_b}{\sigma})\right] \\
&= g^{ab}\left[(\nabla_a\nabla_b V)\ln\sigma + \nabla_b V\frac{\sigma_a}{\sigma} + \nabla_a V\frac{\sigma_b}{\sigma} + V\left(\frac{\sigma\nabla_a\sigma_b - \sigma_b\sigma_a}{\sigma^2}\right)\right] \\
&= (\Box V)\ln\sigma + g^{ab}\nabla_b V\frac{\sigma_a}{\sigma} + g^{ab}\nabla_a V\frac{\sigma_b}{\sigma} + g^{ab}V\frac{\nabla_a\sigma_b}{\sigma} - g^{ab}\frac{V\sigma_a\sigma_b}{\sigma^2} \\
&= (\Box V)\ln\sigma + g^{ab}V_b\frac{\sigma_a}{\sigma} + g^{ab}V_a\frac{\sigma_b}{\sigma} + V\frac{\nabla_a\sigma^a}{\sigma} - V\frac{2\sigma}{\sigma^2} \\
&= (\Box V)\ln\sigma + \frac{V_b\sigma^b}{\sigma} + \frac{V_a\sigma^a}{\sigma} + \frac{V\Box\sigma}{\sigma} - \frac{V2\sigma}{\sigma^2} \\
&= (\Box V)\ln\sigma + \frac{2V_a\sigma^a}{\sigma} + \frac{V(4 - 2\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}})}{\sigma} - \frac{2V}{\sigma} \\
&= (\Box V)\ln\sigma + \frac{2V}{\sigma} + \frac{2\sigma^a(V_a - V\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}))}{\sigma}
\end{aligned}
$$

With these expressions, we can now revisit Eq. B.1 and equate respective terms. The LHS can now be rewritten as

$$
\Box(\tfrac{\Delta^{1/2}}{\sigma}) + \Box(V\ln\sigma) + (\Box V)\ln\sigma + \frac{2V}{\sigma} + \frac{2\sigma^a(V_a - V\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}))}{\sigma} + \Box W
$$

and by equating this with the RHS we obtain:

$$\frac{\Box\Delta^{1/2}}{\sigma} + (\Box V)\ln\sigma + \frac{2V}{\sigma} + \frac{2\sigma^a(V_a - V\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}))}{\sigma} + \Box W = (m^2+\xi R)\frac{\Delta^{1/2}}{\sigma}$$
$$+(m^2+\xi R)V\ln\sigma$$
$$+(m^2+\xi R)W.$$

Looking at the $\ln\sigma$ term first, we can see that

$$(\Box V)\ln\sigma = (m^2+\xi R)V\ln\sigma$$

and thus

$$(\Box - m^2 - \xi R)V = 0. \tag{B.2}$$

Using this fact simplifies matters further when rearranging, and we now see that

$$(\Box - m^2 - \xi R)W = -(\Box - m^2 - \xi R)\frac{\Delta^{1/2}}{\sigma} - \frac{2V}{\sigma} - \frac{2\sigma^a(V_a - V\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}})}{\sigma}$$

or more conveniently

$$\sigma(\Box - m^2 - \xi R)W = -2V - (\Box - m^2 - \xi R)\Delta^{1/2} - 2\sigma^a(V_a - V\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}). \tag{B.3}$$

The inhomogenous wave equation [6]

$$(\Box - \xi R - m^2)G_s(x,x^{'}) = \delta(x,x^{'}) + S(x,x^{'})$$

for the singular two-point function $G_s(x,x^{'})$ admits, in even dimensions, the following Hadamard parametrix

$$G(x,x^{'}) = \frac{\Gamma(\frac{d}{2}-1)}{2(2\pi)^{d/2}}\left\{\frac{U(x,x^{'})}{\sigma(x,x^{'})^{\frac{d}{2}-1}} + V(x,x^{'})log(2\sigma(x,x^{'})/l^2)\right\}.$$

Applying this solution to the wave equation we can write

$$\Box\left\{\frac{\Gamma(\frac{d}{2}-1)}{2(2\pi)^{d/2}}\frac{U}{\sigma^{\frac{d}{2}-1}}\right\} + \Box\left\{\frac{\Gamma(\frac{d}{2}-1)}{2(2\pi)^{d/2}}Vlog\left(\frac{2\sigma}{l^2}\right)\right\} = (m^2+\xi R)\frac{\Gamma(\frac{d}{2}-1)}{2(2\pi)^{d/2}}\left\{\frac{U}{\sigma^{\frac{d}{2}-1}} + Vlog\left(\frac{2\sigma}{l^2}\right)\right\} + S$$

or more conveniently as

$$\Box\left\{\frac{U}{\sigma^{\frac{d}{2}-1}}\right\}+\Box\left\{Vlog\left(\frac{2\sigma}{l^2}\right)\right\}=(m^2+\xi R)\left\{\frac{U}{\sigma^{\frac{d}{2}-1}}+Vlog\left(\frac{2\sigma}{l^2}\right)\right\}+S\frac{2(2\pi)^{d/2}}{\Gamma(\frac{d}{2}-1)}. \tag{B.4}$$

Similar to the workings for Eq. B.1 above, we take each term on the LHS of this equation and calculate, then equate the terms on each side.

$$\begin{aligned}
\Box\left(\frac{U}{\sigma^{\frac{d}{2}-1}}\right) &= g^{ab}\nabla_a\nabla_b\left(\frac{U}{\sigma^{\frac{d}{2}-1}}\right)\\
&= g^{ab}\nabla_a\left(\frac{\sigma^{\frac{d}{2}-1}U_b-U(\frac{d}{2}-1)(\sigma^{\frac{d}{2}-2})\sigma_b}{\sigma^{d-2}}\right)\\
&= g^{ab}\nabla_a\left(\frac{U_b}{\sigma^{\frac{d}{2}-1}}-\frac{U(\frac{d}{2}-1)\sigma_b}{\sigma^{\frac{d}{2}}}\right)\\
&= g^{ab}\left(\frac{\sigma^{\frac{d}{2}-1}\nabla_aU_b-U_b(\frac{d}{2}-1)\sigma^{\frac{d}{2}-2}\sigma_a}{\sigma^{d-2}}\right)-\\
&\qquad g^{ab}\left(\frac{\sigma^{\frac{d}{2}}(U_a(\frac{d}{2}-1)\sigma_b+U(\frac{d}{2}-1)\nabla_a\sigma_b)-U(\frac{d}{2}-1)\sigma_b(\frac{d}{2})\sigma^{\frac{d}{2}-1}\sigma_a}{\sigma^{d}}\right)\\
&= g^{ab}\frac{\nabla_aU_b}{\sigma^{\frac{d}{2}-1}}-g^{ab}\frac{U_b(\frac{d}{2}-1)\sigma_a}{\sigma^{\frac{d}{2}}}-g^{ab}\frac{U_a(\frac{d}{2}-1)\sigma_b}{\sigma^{\frac{d}{2}}}-\\
&\qquad g^{ab}\frac{U(\frac{d}{2}-1)\nabla_a\sigma_b}{\sigma^{\frac{d}{2}}}+g^{ab}\frac{U(\frac{d}{2}-1)\sigma_b(\frac{d}{2})\sigma_a}{\sigma^{\frac{d}{2}+1}}\\
&= \frac{\Box U}{\sigma^{\frac{d}{2}-1}}-g^{ab}\frac{\frac{d}{2}-1}{\sigma^{\frac{d}{2}}}(U_b\sigma_a+U_a\sigma_b)-\frac{U(\frac{d}{2}-1)g^{ab}\nabla_a\nabla_b\sigma}{\sigma^{\frac{d}{2}}}+\frac{U(\frac{d}{2}-1)\sigma^a\sigma_a(\frac{d}{2})}{\sigma^{\frac{d}{2}+1}}\\
&= \frac{\Box U}{\sigma^{\frac{d}{2}-1}}-\frac{\frac{d}{2}-1}{\sigma^{\frac{d}{2}}}(U^a\sigma_a+U_a\sigma^a)-\frac{U(\frac{d}{2}-1)\Box\sigma}{\sigma^{\frac{d}{2}}}+\frac{Ud(\frac{d}{2}-1)\sigma}{\sigma^{\frac{d}{2}+1}}\\
&= \frac{\Box U}{\sigma^{\frac{d}{2}-1}}-\frac{(d-2)(U^a\sigma_a)}{\sigma^{\frac{d}{2}}}-\frac{U(\frac{d}{2}-1)(d-2\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}})}{\sigma^{\frac{d}{2}}}+\frac{Ud(\frac{d}{2}-1)}{\sigma^{\frac{d}{2}}}\\
&= \frac{\Box U}{\sigma^{\frac{d}{2}-1}}-\frac{(d-2)(U^a\sigma_a)}{\sigma^{\frac{d}{2}}}+\frac{(d-2)U\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}}{\sigma^{\frac{d}{2}}}.
\end{aligned}$$

The next term on the LHS of Eq. B.4 is calculated as

$$\begin{aligned}
\Box(Vlog(\tfrac{2\sigma}{l^2})) &= g^{ab}\nabla_a\nabla_b(Vlog(\tfrac{2\sigma}{l^2}))\\
&= g^{ab}\nabla_a(V_b log(\tfrac{2\sigma}{l^2}) + V\tfrac{\sigma_b}{\sigma})\\
&= g^{ab}\left((\nabla_a\nabla_b V)log(\tfrac{2\sigma}{l^2}) + V_b\tfrac{\sigma_a}{\sigma} + V_a\tfrac{\sigma_b}{\sigma} + V\frac{\sigma\nabla_a\nabla_b\sigma - \sigma_b\sigma_a}{\sigma^2}\right)\\
&= (\Box V)log(\tfrac{2\sigma}{l^2}) + \frac{g^{ab}}{\sigma}(V_b\sigma_a + V_a\sigma_b) + \frac{V\Box\sigma}{\sigma} - V\frac{\sigma^b\sigma_b}{\sigma^2}\\
&= (\Box V)log(\tfrac{2\sigma}{l^2}) + \frac{V^a\sigma_a + V_a\sigma^a}{\sigma} + \frac{V(d - 2\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}})}{\sigma} - \frac{2V}{\sigma}\\
&= (\Box V)log(\tfrac{2\sigma}{l^2}) + \frac{2V^a\sigma_a}{\sigma} - \frac{2V\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}}{\sigma} + \frac{(d-2)V}{\sigma}.
\end{aligned}$$

Thus Eq. B.4 becomes

$$\begin{aligned}
&\frac{\Box U}{\sigma^{\frac{d}{2}-1}} - \frac{(d-2)(U^a\sigma_a)}{\sigma^{\frac{d}{2}}} + \frac{(d-2)U\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}}{\sigma^{\frac{d}{2}}}\\
&\quad + (\Box V)log(\tfrac{2\sigma}{l^2}) + \frac{2V^a\sigma_a}{\sigma} - \frac{2V\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}}{\sigma} + \frac{(d-2)V}{\sigma}\\
&\qquad = (m^2+\xi R)\left\{\frac{U}{\sigma^{\frac{d}{2}-1}} + Vlog\left(\frac{2\sigma}{l^2}\right)\right\} + S\frac{2(2\pi)^{d/2}}{\Gamma(\frac{d}{2}-1)}.
\end{aligned}$$

First, examining the $log(\frac{2\sigma}{l^2})$ terms, we see that

$$(\Box V)log(\tfrac{2\sigma}{l^2}) = (m^2+\xi R)Vlog(\tfrac{2\sigma}{l^2})$$

and thus

$$(\Box - m^2 - \xi R)V = 0.$$

Multiplying by $\sigma^{\frac{d}{2}}$ and applying the above condition for V we then get

$$\begin{aligned}
&\sigma\Box U - (d-2)U^a\sigma_a + (d-2)U\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}\\
&\quad + 2\sigma^{\frac{d}{2}-1}V^a\sigma_a - 2\sigma^{\frac{d}{2}-1}V\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}} + (d-2)\sigma^{\frac{d}{2}-1}V\\
&\qquad = (m^2+\xi R)\sigma U + S\frac{2(2\pi)^{d/2}\sigma^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)}
\end{aligned}$$

which can be arranged to show

$$\sigma(\Box - m^2 - \xi R)U = S\frac{2(2\pi)^{d/2}\sigma^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)} + (d-2)U^a\sigma_a - (d-2)U\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}} - 2\sigma^{\frac{d}{2}-1}V^a\sigma_a + 2\sigma^{\frac{d}{2}-1}V\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}} - (d-2)\sigma^{\frac{d}{2}-1}V. \tag{B.5}$$

Since the term $S(x,x^{'})$ is chosen to be an arbitrary smooth biscalar, we will use it as a term that is dependent on $V(x,x^{'})$ only. Hence, with respect to $U(x,x^{'})$ only, we have

$$\sigma(\Box - m^2 - \xi R)U = (d-2)U^a\sigma_a - (d-2)U\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{\frac{-1}{2}}. \tag{B.6}$$

B.2 Recurrence Relations for U and V

By using Eq. B.6, and by assuming the Hadamard ansatz

$$U(x,x^{'}) = \sum_{p=0}^{\frac{d}{2}-2} U_p(x,x^{'})\sigma^p, \tag{B.7}$$

we will find the recursion relations that each coefficient U_p satisfies, as in [6]. We begin by applying the ansatz to the wave equation.

$$\begin{aligned}\sigma(\Box - m^2 - \xi R)U &= \sigma(\Box - m^2 - \xi R)\sum_{p=0}^{\frac{d}{2}-2} U_p\sigma^p \\ &= \sigma\Box\sum_{p=0}^{\frac{d}{2}-2} U_p\sigma^p - \sigma(m^2 + \xi R)\sum_{p=0}^{\frac{d}{2}-2} U_p\sigma^p\end{aligned} \tag{B.8}$$

We first need to examine the term $\Box U_p\sigma^p$ for any p.

$$\begin{aligned}
\Box U_p\sigma^p &= g^{ab}\nabla_a\nabla_b(U_p\sigma^p) \\
&= g^{ab}\nabla_a[(\nabla_b U_p)\sigma^p + U_p(\nabla_b\sigma^p)] \\
&= g^{ab}\nabla_a[(\nabla_b U_p)\sigma^p + U_p(p\sigma^{p-1}\sigma_b)] \\
&= g^{ab}[(\nabla_a\nabla_b U_p)\sigma^p + (\nabla_b U_p)p\sigma^{p-1}\sigma_a + \nabla_a U_p(p\sigma^{p-1}\sigma_b) \\
&\qquad + U_p(p(p-1)\sigma^{p-2}\sigma_a\sigma_b) + U_p(p\sigma^{p-1}\nabla_a\nabla_b\sigma)] \\
&= \sigma^p\Box U_p + g^{ab}(\nabla_b U_p)p\sigma^{p-1}\sigma_a + g^{ab}(\nabla_a U_p)p\sigma^{p-1}\sigma_b \\
&\qquad + U_p(p(p-1)\sigma^{p-2}\sigma^b\sigma_b) + U_p(p\sigma^{p-1}\Box\sigma) \\
&= \sigma^p\Box U_p + p\sigma^{p-1}(\nabla^a U_p\sigma_a + \nabla_a U_p\sigma^a) \\
&\qquad + 2p(p-1)U_p\sigma^{p-2}\sigma + pU_p\sigma^{p-1}\Box\sigma
\end{aligned}$$

We can simplify this equation using that $\nabla^a U_p\sigma_a = \nabla_a U_p\sigma^a$ and that $\Box\sigma = d - 2\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}}$. We then have

$$\begin{aligned}
\Box U_p\sigma^p &= \sigma^p\Box U_p + 2p\sigma^{p-1}\nabla_a U_p\sigma^a + 2p(p-1)U_p\sigma^{p-1} + pU_p\sigma^{p-1}(d - 2\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}}) \\
&= \sigma^p\Box U_p + 2p\sigma^{p-1}\nabla_a U_p\sigma^a + p(2(p-1)+d)U_p\sigma^{p-1} - 2pU_p\sigma^{p-1}\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}}
\end{aligned}$$

Summing over p, we can then write

$$\begin{aligned}
\sigma\Box\sum_{p=0}^{\frac{d}{2}-2} U_p\sigma^p &= \sigma\sum_{p=0}^{\frac{d}{2}-2}\Big\{\sigma^p\Box U_p + 2p\sigma^{p-1}\nabla_a U_p\sigma^a \\
&\qquad + p(2(p-1)+d)U_p\sigma^{p-1} - 2pU_p\sigma^{p-1}\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}}\Big\} \\
&= \sum_{p=0}^{\frac{d}{2}-2}\Big\{\sigma^{p+1}\Box U_p + 2p\sigma^p\nabla_a U_p\sigma^a \\
&\qquad + p(2(p-1)+d)U_p\sigma^p - 2pU_p\sigma^p\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}}\Big\}.
\end{aligned}$$

Eq. B.8 then becomes

$$\begin{aligned}
\sigma(\Box - m^2 - \xi R)U = \sum_{p=0}^{\frac{d}{2}-2}\Big\{&\sigma^{p+1}(\Box - m^2 - \xi R)U_p + 2p\sigma^p\nabla_a U_p\sigma^a \\
&+ p(2(p-1)+d)U_p\sigma^p - 2pU_p\sigma^p\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}}\Big\}
\end{aligned} \tag{B.9}$$

The next step is to look at the RHS of Eq. B.6 and then equate with the above. Applying the Hadamard ansatz, we get

$$(d-2)\sigma^a\nabla_a U-(d-2)U\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}=(d-2)\sigma^a\nabla_a\sum_{p=0}^{\frac{d}{2}-2}U_p\sigma^p-(d-2)\sum_{p=0}^{\frac{d}{2}-2}U_p\sigma^p\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}$$

Focusing on the RHS of this equation, we find the equation for any p, then we can simply sum over all p.

$$\begin{aligned}
(d-2)\sigma^a\nabla_a(U_p\sigma^p)-&(d-2)U_p\sigma^p\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}\\
&=(d-2)\sigma^a\left[(\nabla_aU_p)\sigma^p+U_pp\sigma^{p-1}\sigma_a\right]-(d-2)U_p\sigma^p\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}\\
&=(d-2)\sigma^a(\nabla_aU_p)\sigma^p+(d-2)pU_p\sigma^{p-1}\sigma^a\sigma_a-(d-2)U_p\sigma^p\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}\\
&=(d-2)\sigma^a(\nabla_aU_p)\sigma^p+2(d-2)pU_p\sigma^p-(d-2)U_p\sigma^p\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}
\end{aligned}$$

And so, summing over all p, we can equate this result that in Eq. B.8, giving

$$\begin{aligned}
\sum_{p=0}^{\frac{d}{2}-2}&\left\{\sigma^{p+1}(\Box-m^2-\xi R)U_p+2p\sigma^p\nabla_aU_p\sigma^a+p(2(p-1)+d)U_p\sigma^p-2pU_p\sigma^p\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}}\right\}\\
&=\sum_{p=0}^{\frac{d}{2}-2}\left\{(d-2)\sigma^a(\nabla_aU_p)\sigma^p+2(d-2)pU_p\sigma^p\right.\\
&\qquad\left.-(d-2)U_p\sigma^p\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}\right\}.
\end{aligned}$$

Rearrange to get

$$\begin{aligned}
\sum_{p=0}^{\frac{d}{2}-2}\sigma^{p+1}(\Box-m^2-\xi R)U_p=\sum_{p=0}^{\frac{d}{2}-2}&\left\{(d-2-2p)\sigma^a\nabla_aU_p\sigma^p\right.\\
&+[2p(d-2)-p(2(p-1)+d)]U_p\sigma^p\\
&\left.-(d-2-2p)U_p\sigma^p\Delta^{\frac{-1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}\right\}.
\end{aligned}$$

We notice however that for $p=0$, we end up with terms of $U_0\mathcal{O}(\sigma)$ on the LHS equated to terms of U_0 only on the RHS. To rectify and hence produce recurrence relations for U_p we step from $p \to p+1$ on the RHS. We can the remove the summation and tidy up the terms to get

$$\begin{aligned}\sigma^{p+1}(\Box - m^2 - \xi R)U_p = U_0 + (d-2p-4)\sigma^a \nabla_a U_{p+1}\sigma^{p+1} \\ + (p+1)(d-2p-4)U_{p+1}\sigma^{p+1} \\ - (d-2p-4)U_{p+1}\sigma^{p+1}\Delta^{\frac{-1}{2}}\sigma^a \nabla_a \Delta^{\frac{1}{2}}.\end{aligned}$$

We obtain the recurrence relation for U_p by dividing across by σ^{p+1}.

$$\begin{aligned}(\Box - m^2 - \xi R)U_p = (d-2p-4)\sigma^a \nabla_a U_{p+1} \\ + (p+1)(d-2p-4)U_{p+1} \\ - (d-2p-4)U_{p+1}\Delta^{\frac{-1}{2}}\sigma^a \nabla_a \Delta^{\frac{1}{2}}.\end{aligned} \tag{B.10}$$

We notice from this equation however, that for the final term of U_p, where $p = \frac{d}{2} - 2$, that $(\Box - m^2 - \xi R)U_{\frac{d}{2}-2} = 0$. We will have to look at the recurrence relations for V_p to identify a suitable boundary for the maximum p. To find the recurrence relations in V_p, the calculations are the same, until we arrive at the equivalent of Eq. B.9 for $V(x,x')$:

$$\begin{aligned}(\Box - m^2 - \xi R)V = \sum_{p=0}^{\infty} \Big\{ \sigma^p(\Box - m^2 - \xi R)V_p + 2p\sigma^{p-1}\nabla_a V_p \sigma^a \\ + p(2(p-1)+d)V_p\sigma^{p-1} - 2pV_p\sigma^{p-1}\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}} \Big\}.\end{aligned}$$

Since $(\Box - m^2 - \xi R)V = 0$, then

$$\sum_{p=0}^{\infty} \sigma^p(\Box - m^2 - \xi R)V_p = \sum_{p=0}^{\infty} \Big\{ 2pV_p\sigma^{p-1}\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}} - 2p\sigma^{p-1}\nabla_a V_p\sigma^a - p(2(p-1)+d)V_p\sigma^{p-1} \Big\}.$$

As for U_p above, we can see here that when $p = 0$, there is a mismatch with the σ terms, so we need to shift $p \to p+1$ on the RHS and then drop the summation terms:

$$\begin{aligned}\sigma^p(\Box - m^2 - \xi R)V_p = V_o + 2(p+1)V_{p+1}\sigma^p\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}} \\ - 2(p+1)\sigma^p\nabla_a V_{p+1}\sigma^a - (p+1)(2p+d)V_{p+1}\sigma^p\end{aligned}$$

Then dividing across by σ^p gives the recursion relation

$$(\Box - m^2 - \xi R)V_p = 2(p+1)V_{p+1}\sigma^a(\Delta^{\frac{1}{2}})_a\Delta^{\frac{-1}{2}} - 2(p+1)\nabla_a V_{p+1}\sigma^a - (p+1)(2p+d)V_{p+1}. \tag{B.11}$$

To derive the boundary conditions (BC), we use Eq. (A.5) and make use of the ansatz in Eq. B.7. Combining, and using S in place of its full expression, we get

$$\begin{aligned}\sum_{p=0}^{\frac{d}{2}-2}\sigma^{p+1}(\Box - m^2 - \xi R)U_p = S + \sum_{p=0}^{\frac{d}{2}-2}\Bigg[&(d-2-2p)\sigma^a\nabla_a U_p\sigma^p \\ &+\Big(2p(d-2) - p(2(p-1)+d)\Big)U_p\sigma^p \\ &-(d-2-2p)U_p\sigma^p\Delta^{-\frac{1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Bigg] \\ +\sum_{p=0}^{\infty}\Bigg[&-2\sigma^{p+\frac{d}{2}-1}\nabla^a V_p\sigma_a - 4p\sigma^{p+\frac{d}{2}-1}V_p \\ &+2\sigma^{p+\frac{d}{2}-1}V_p\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{-\frac{1}{2}} - (d-2)\sigma^{\Delta^{-\frac{1}{2}}}V_p\Bigg]\end{aligned}$$

Since we must equate terms of the correct order of σ, we note that for any given p the LHS is one order higher than the RHS. For all p on the LHS from $p=0$ to $p=\frac{d}{2}-1$ we must equate to terms of $p+1$ on the RHS. However, noting that the σ^p term for $p=\frac{d}{2}-2$ on the LHS then has a equivalent out of range on the RHS, we must equate it to $p=0$ on the RHS. As such:

$$\begin{aligned}\sigma^{\frac{d}{2}-1}(\Box - m^2 - \xi R)U_{\frac{d}{2}-2} = S + (d-2)\sigma^a\nabla_a U_0 - (d-2)U_0\Delta^{-\frac{1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}} \\ -2\sigma^{\frac{d}{2}-1}\nabla^a V_0\sigma_a + 2\sigma^{\frac{d}{2}-1}\nabla_a\Delta^{\frac{1}{2}}\Delta^{-\frac{1}{2}} - (d-2)\sigma^{\frac{d}{2}-1}V_0.\end{aligned}$$

As $U_{\frac{d}{2}-2}$ should not depend on U_0, we insist that the term $(d-2)\sigma^a\nabla_a U_0 - (d-2)U_0\Delta^{-\frac{1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}} = 0$. Thus

$$\begin{aligned}(d-2)\sigma^a\nabla_a U_0 - (d-2)U_0\Delta^{-\frac{1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}} &= 0 \\ \sigma^a\nabla_a U_0 &= U_0\Delta^{-\frac{1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}.\end{aligned}$$

This can only hold true if $U_0 = \Delta^{\frac{1}{2}}$, which is the first BC. With the U_0 terms removed, we then have

$$\sigma^{\frac{d}{2}-1}\big(\Box - m^2 - \xi R\big)U_{\frac{d}{2}-2} = 2\sigma^{\frac{d}{2}-1}\nabla^a V_0 \sigma_a + 2\sigma^{\frac{d}{2}-1}\nabla_a \Delta^{\frac{1}{2}}\Delta^{-\frac{1}{2}} - (d-2)\sigma^{\frac{d}{2}-1}V_0. \tag{B.12}$$

Dividing across by $\sigma^{\frac{d}{2}-1}$ and rearranging, we get the next BC:

$$\big(\Box - m^2 - \xi R\big)U_{\frac{d}{2}-2} - 2\nabla^a V_0 \sigma_a - 2\nabla_a \Delta^{\frac{1}{2}}\Delta^{-\frac{1}{2}} + (d-2)V_0 = 0, \tag{B.13}$$

which will be implemented in Mathematica code, and solved using the expansions of Chapter 3.

Appendix C

Expansion of Hadamard Parametrix

C.1 Changing Coordinates

We first require a change of coordinates, from the $(\Delta\tau, \Delta r, r, \gamma)$ system to the $(s, w, r, \Delta r)$ system. To do this, we use that $\sigma_\phi = \frac{\partial x^a}{\partial \phi}\frac{\partial \sigma}{\partial x^a} = \frac{\partial x^a}{\partial \phi}\sigma_a$. Thus

$$
\begin{aligned}
\sigma_r^{'} &= \frac{\partial r}{\partial r}\frac{\partial \sigma}{\partial r} + \frac{\partial s}{\partial r}\frac{\partial \sigma}{\partial s} + \frac{\partial r}{\partial r}\frac{\partial \sigma}{\partial \Delta r} + \frac{\partial w}{\partial r}\frac{\partial \sigma}{\partial w} \\
&= \frac{\partial r}{\partial r}\sigma_r + \frac{\partial s}{\partial r}\sigma_s + \frac{\partial r}{\partial \Delta r}\sigma_{\Delta r} + \frac{\partial w}{\partial r}\sigma_w \\
&= \sigma_r + \frac{\partial}{\partial r}\big(f(r)w^2 + 2r^2(1-\cos\gamma)\big)^{\frac{1}{2}}\sigma_s + \frac{\partial}{\partial r}(r^{'} - r)\sigma_{\Delta r} + \frac{\partial w}{\partial r}\sigma_w \\
&= \sigma_r + \frac{1}{2S}(f^{'}(r)w^2 + 4r(1-\cos\gamma))\sigma_s - \sigma_{\Delta r} \\
&= \sigma_r + \left(\frac{f^{'}(r)w^2}{2s} + \frac{2r}{s} - \frac{2r\cos\gamma}{s}\right)\sigma_s - \sigma_{\Delta r}.
\end{aligned}
$$

To remove the γ dependency, we use that $\cos\gamma = 1 - \frac{s^2 - f(r)w^2}{2r^2}$ hence

$$
\begin{aligned}
\sigma_r^{'} &= \sigma_r + \left(\frac{f^{'}(r)w^2}{2s} + \frac{2r}{s} - \frac{2r}{s}\left(1 - \frac{s^2 - f(r)w^2}{2r^2}\right)\right)\sigma_s - \sigma_{\Delta r} \\
&= \sigma_r + \left(\frac{f^{'}(r)w^2}{2s} + \frac{s}{r} - \frac{f(r)w^2}{sr}\right)\sigma_s - \sigma_{\Delta r}.
\end{aligned}
\tag{C.1}
$$

Next we calculate $\sigma^{'}_{\tau}$.

$$
\begin{aligned}
\sigma^{'}_{\tau} &= \frac{\partial r}{\partial \tau}\frac{\partial \sigma}{\partial r} + \frac{\partial s}{\partial \tau}\frac{\partial \sigma}{\partial s} + \frac{\partial r}{\partial \tau}\frac{\partial \sigma}{\partial \Delta r} + \frac{\partial w}{\partial \tau}\frac{\partial \sigma}{\partial w} \\
&= \frac{\partial r}{\partial \tau}\sigma_r + \frac{\partial s}{\partial \tau}\sigma_s + \frac{\partial r}{\partial \tau}\sigma_{\Delta r} + \frac{\partial w}{\partial \tau}\sigma_w \\
&= \frac{1}{2s}\left(f(r)2w\frac{\partial w}{\partial \tau}\right)\sigma_s + \frac{\partial w}{\partial \tau}\sigma_w \\
&= \frac{1}{2s}\left(f(r)2w\frac{\partial}{\partial \tau}\left(\frac{2}{\kappa^2}(1-\cos\kappa\Delta\tau)\right)^{\frac{1}{2}}\right)\sigma_s + \frac{\partial}{\partial \tau}\left(\frac{2}{\kappa^2}(1-\cos\kappa\Delta\tau)\right)^{\frac{1}{2}}\Bigg)\sigma_w \\
&= \frac{1}{2s}\left(f(r)2w\frac{1}{2w}\frac{2}{\kappa^2}(-\kappa\sin\kappa\Delta\tau)\right)\sigma_s + \frac{1}{2w}\frac{2}{\kappa^2}(-\kappa\sin\kappa\Delta\tau)\sigma_w \\
&= -\frac{f(r)\sin\kappa\Delta\tau}{s\kappa}\sigma_s - \frac{\sin\kappa\Delta\tau}{w\kappa}\sigma_w.
\end{aligned}
$$

However we wish to remove the dependency on $\Delta\tau$ to complete the coordinate transformation. We do so by using that $\cos\kappa\Delta\tau = 1 - \frac{w^2\kappa^2}{2}$ and $\sin\kappa\Delta\tau^2 + \cos\kappa\Delta\tau^2 = 1$. Hence, $\sin\kappa\Delta\tau = w\kappa(1-\frac{w^2\kappa^2}{4})^{\frac{1}{2}}$ and

$$
\begin{aligned}
\sigma^{'}_{\tau} &= -\frac{f(r)}{s\kappa}w\kappa(1-\tfrac{w^2\kappa^2}{4})^{\frac{1}{2}}\sigma_s - \frac{1}{w\kappa}(w\kappa(1-\tfrac{w^2\kappa^2}{4})^{\frac{1}{2}})\sigma_w \\
&= \left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left(-\frac{w}{s}f(r)\sigma_s - \sigma_w\right).
\end{aligned} \tag{C.2}
$$

Finally, $\sigma^{'}_{\gamma}$ is given by

$$
\begin{aligned}
\sigma^{'}_{\gamma} &= \frac{\partial r}{\partial \gamma}\frac{\partial \sigma}{\partial r} + \frac{\partial s}{\partial \gamma}\frac{\partial \sigma}{\partial s} + \frac{\partial r}{\partial \gamma}\frac{\partial \sigma}{\partial \Delta r} + \frac{\partial w}{\partial \gamma}\frac{\partial \sigma}{\partial w} \\
&= \frac{\partial r}{\partial \gamma}\sigma_r + \frac{\partial s}{\partial \gamma}\sigma_s + \frac{\partial r}{\partial \gamma}\sigma_{\Delta r} + \frac{\partial w}{\partial \gamma}\sigma_w \\
&= \frac{\partial}{\partial \gamma}\left(f(r)w^2 + 2r^2(1-\cos\gamma)\right)^{\frac{1}{2}}\sigma_s \\
&= \frac{1}{2s}(2r^2\sin\gamma)\sigma_s \\
&= \frac{r^2\sin\gamma}{s}\sigma_s\,,
\end{aligned}
$$

where we again must complete the change of coordinates. Similar to above, we find that $\sin\gamma = \left(2(\frac{s^2-f(r)w^2}{2r^2}) - (\frac{s^2-f(r)w^2}{2r^2})^2\right)^{\frac{1}{2}}$. Thus

$$\begin{aligned}\sigma_\gamma^{'} &= \frac{r^2}{s}\left(2(\frac{s^2-f(r)w^2}{2r^2}) - (\frac{s^2-f(r)w^2}{2r^2})^2\right)^{\frac{1}{2}}\sigma_s \\ &= \frac{r^2}{s}\left[\frac{s^2}{r^2}\left(1-\frac{f(r)w^2}{s^2}-\frac{s^2}{4r^2}-\frac{f(r)^2w^4}{4r^2s^2}+\frac{f(r)w^2}{2r^2}\right)\right]^{\frac{1}{2}}\sigma_s \\ &= r\left(1-\frac{f(r)w^2}{s^2}-\frac{s^2}{4r^2}-\frac{f(r)^2w^4}{4r^2s^2}+\frac{f(r)w^2}{2r^2}\right)^{\frac{1}{2}}\sigma_s\,.\end{aligned} \tag{C.3}$$

Thus we can now write the line element in the new coordinates as

$$2\sigma = \frac{1}{f(r)}\sigma_\tau^2 + f(r)\sigma_r^2 + \frac{1}{r^2}\sigma_\gamma^2.$$

Expanding, we get

$$\begin{aligned}2\sigma = \frac{1}{f(r)}&\left(1-\frac{w^2\kappa^2}{4}\right)\left(-\frac{w}{s}f(r)\sigma_s - \sigma_w\right)^2 \\ &+ f(r)\left[\sigma_r + \left(\frac{f^{'}(r)w^2}{2s}+\frac{s}{r}-\frac{f(r)w^2}{sr}\right)\sigma_s - \sigma_{\Delta r}\right]^2 \\ &+ \left(1-\frac{f(r)w^2}{s^2}-\frac{s^2}{4r^2}-\frac{f(r)^2w^4}{4r^2s^2}+\frac{f(r)w^2}{2r^2}\right)\sigma_s^2\,.\end{aligned} \tag{C.4}$$

C.2 Expansions of $U_0(x,x^{'})$ and $V_0(x,x^{'})$ in $(s,w,\Delta r,r)$

As we consider $d=4$ only, we need only consider $U_0 = \Delta^{\frac{1}{2}}$ as shown above. In order to expand U_0, we will rely on the defining equation

$$\Box\sigma = 4 - 2\sigma^a\nabla_a\Delta^{\frac{1}{2}}\Delta^{-\frac{1}{2}}.$$

Having written σ_r, σ_τ and σ_γ in terms of $(s,w,\Delta r,r)$ we can now calculate $\Box\sigma$ in this basis too. First,

$$\begin{aligned}\Box\sigma &= \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\partial^a\sigma) \\ &= \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b\sigma)\end{aligned}$$

Where the inverse of the spherically-symmetric is given by

$$g_{ab} = \begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

where $\gamma = \gamma(\theta,\phi)$, we obtain a determinant of $g = r^2\sin\theta$. Thus, the d'Alembert operator then gives

$$\begin{aligned}\Box\sigma &= \frac{1}{r^2}\partial_a(r^2 g^{ab}\partial_b\sigma)\\ &= \frac{1}{r^2}\partial_r(r^2 f(r)\partial_r\sigma) + \frac{1}{r^2}\partial_\tau(r^2\frac{1}{f(r)}\partial_\tau\sigma) + \frac{1}{r^2}\frac{1}{\sin\gamma}\partial_\gamma(\sin\gamma\partial_\gamma\sigma)\\ &= \frac{1}{r^2}\partial_r(r^2 f(r)\sigma_r) + \partial_\tau(\tfrac{1}{f(r)}\sigma_\tau) + \frac{1}{r^2}\frac{1}{\sin\gamma}\partial_\gamma(\sin\gamma\sigma_\gamma).\end{aligned} \tag{C.5}$$

We drop the r dependency in $f(r)$ at this point and treat it as implicit, and recognise that the partial derivatives ∂_r, ∂_τ and ∂_γ must be transformed into the $(s, w, \Delta r, r)$ coordinate system too. For example,

$$\partial_\tau = \frac{\partial w}{\partial\tau}\frac{\partial}{\partial w} + \frac{\partial s}{\partial\tau}\frac{\partial}{\partial s} + \frac{\partial r}{\partial\tau}\frac{\partial}{\partial r} + \frac{\partial\Delta r}{\partial\tau}\frac{\partial}{\partial\Delta r}$$

Calculating term by term in Eq. C.5,

$$\begin{aligned}\frac{1}{r^2}\partial_r(r^2 f\sigma_r) &= \frac{1}{r^2}\left(\frac{\partial w}{\partial r}\frac{\partial}{\partial w} + \frac{\partial s}{\partial r}\frac{\partial}{\partial s} + \frac{\partial r}{\partial r}\frac{\partial}{\partial r} + \frac{\partial\Delta r}{\partial r}\frac{\partial}{\partial\Delta r}\right)\left(r^2 f\sigma_r\right)\\ &= \frac{1}{r^2}\left(\frac{\partial s}{\partial r}\frac{\partial}{\partial s} + \frac{\partial}{\partial r} - \frac{\partial}{\partial\Delta r}\right)\left(r^2 f\sigma_r\right)\\ &= \frac{1}{r^2}\frac{\partial s}{\partial r}\frac{\partial}{\partial s}\left(r^2 f\sigma_r\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\sigma_r\right) - \frac{1}{r^2}\frac{\partial}{\partial\Delta r}\left(r^2 f\sigma_r\right)\\ &= \frac{1}{r^2}\left[\left(\frac{f' w^2}{2s} + \frac{s}{r} - \frac{f w^2}{sr}\right)\frac{\partial}{\partial s}\left(r^2 f\sigma_r\right) + \frac{\partial}{\partial r}\left(r^2 f\sigma_r\right)\right.\\ &\qquad\left. - \frac{\partial}{\partial\Delta r}\left(r^2 f\sigma_r\right)\right]\end{aligned}$$

$$
\begin{aligned}
\partial_\tau(\tfrac{1}{f}\sigma_\tau) &= \left(\frac{\partial w}{\partial \tau}\frac{\partial}{\partial w} + \frac{\partial s}{\partial \tau}\frac{\partial}{\partial s} + \frac{\partial r}{\partial r}\frac{\partial}{\partial r} + \frac{\partial \Delta r}{\partial r}\frac{\partial}{\partial \Delta r}\right)\left(\frac{1}{f}\sigma_\tau\right) \\
&= \left(\frac{\partial w}{\partial \tau}\frac{\partial}{\partial w} + \frac{\partial s}{\partial \tau}\frac{\partial}{\partial s}\right)\left(\frac{1}{f}\sigma_\tau\right) \\
&= \frac{\partial w}{\partial \tau}\frac{\partial}{\partial w}\left(\frac{1}{f}\sigma_\tau\right) + \frac{\partial s}{\partial \tau}\frac{\partial}{\partial s}\left(\frac{1}{f}\sigma_\tau\right) \\
&= -\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\frac{\partial}{\partial w}\left[\frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left(-\frac{wf}{s}\sigma_s - \sigma_w\right)\right] \\
&\qquad - \frac{wf}{s}\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\frac{\partial}{\partial s}\left[\frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left(-\frac{wf}{s}\sigma_s - \sigma_w\right)\right] \\
&= -\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left[\frac{1}{2f}\left(1-\frac{w^2\kappa^2}{4}\right)^{-\frac{1}{2}}\left(-\frac{2w\kappa^2}{4}\right)\left(-\frac{wf}{s}\sigma_s - \sigma_s\right)\right. \\
&\qquad \left. + \frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left(-\frac{f}{s}\sigma_s - \frac{wf}{s}\frac{\partial^2\sigma}{\partial w\partial s} - \frac{\partial^2\sigma}{\partial w^2}\right)\right] \\
&\quad - \frac{wf}{s}\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left[\frac{wf}{s^2} - \frac{wf}{s}\frac{\partial^2\sigma}{\partial s^2} - \frac{\partial^2\sigma}{\partial w\partial s}\right] \\
&= -\frac{1}{2f}\left(-\frac{2w\kappa^2}{4}\right)\left(-\frac{wf}{s}\sigma_s - \sigma_w\right) - \frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)\left(-\frac{f}{s}\sigma_s - \frac{wf}{s}\frac{\partial^2\sigma}{\partial w\partial s} - \frac{\partial^2\sigma}{\partial w^2}\right) \\
&\qquad - \frac{w}{s}\left(1-\frac{w^2\kappa^2}{4}\right)\left[\frac{wf}{s^2} - \frac{wf}{s}\frac{\partial^2\sigma}{\partial s^2} - \frac{\partial^2\sigma}{\partial w\partial s}\right] \\
&= \Big[\frac{-w^2\kappa^2}{4s} + \frac{1}{s}\left(1-\frac{w^2\kappa^2}{4}\right) - \frac{w^2 f}{s^3}\left(1-\frac{w^2\kappa^2}{4}\right)\Big]\sigma_s - \frac{w\kappa^2}{4f}\sigma_w \\
&\qquad + \frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)\frac{\partial^2\sigma}{\partial w^2} + \frac{w^2 f}{s^2}\left(1-\frac{w^2\kappa^2}{4}\right)\frac{\partial^2\sigma}{\partial s^2} + \frac{2w}{s}\left(1-\frac{w^2\kappa^2}{4}\right)\frac{\partial^2\sigma}{\partial w\partial s}
\end{aligned}
$$

Finally,

$$\begin{aligned}
\frac{1}{r^2}\frac{1}{\sin\gamma}\partial_\gamma(\sin\gamma\partial_\gamma\sigma) &= \frac{1}{r^2}\frac{1}{\sin\gamma}\frac{\partial s}{\partial\gamma}\frac{\partial}{\partial s}\Bigg(\sin\gamma\sigma_\gamma\Bigg)\\
&= \frac{1}{r^2\sin\gamma}\frac{r^2\sin\gamma}{s}\frac{\partial}{\partial s}\Bigg(\sin\gamma\sigma_\gamma\Bigg)\\
&= \frac{1}{s}\Big[\frac{\partial}{\partial s}(\sin\gamma)\sigma_\gamma + \sin\gamma\frac{\partial}{\partial s}(\sigma_\gamma)\Big]\\
&= \frac{1}{s}\Big[\frac{1}{2}\Bigg(2\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)-\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)^2\Bigg)^{-\frac{1}{2}}\Bigg(\frac{2s}{r^2}-2\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)\frac{s}{r^2}\Bigg)\sigma_\gamma\\
&\qquad +\sin\gamma\frac{\partial}{\partial s}\sigma_\gamma\Big]\\
&= \frac{1}{2s}\frac{1}{\sin\gamma}\Bigg(\frac{2s}{r^2}-2\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)\frac{s}{r^2}\Bigg)\frac{r^2\sin\gamma}{s}\sigma_s+\frac{\sin\gamma}{s}\frac{\partial}{\partial s}\Bigg(\frac{r^2\sin\gamma}{s}\sigma_s\Bigg)\\
&= \frac{r^2}{s^2}\Bigg(\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)\frac{s}{r^2}\Bigg)\sigma_s\\
&\qquad +\frac{\sin\gamma}{s}\Big[\frac{\partial}{\partial s}\Bigg(\frac{r^2}{s}\Bigg)\sin\gamma\sigma_s+\frac{r^2}{s}\frac{\partial}{\partial s}\Bigg(\sin\gamma\Bigg)\sigma_s+\frac{r^2}{s}\sin\gamma\frac{\partial^2\sigma}{\partial s^2}\Big]\\
&= \Bigg(\frac{1}{s}-\frac{1}{s}\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)\Bigg)\sigma_s\\
&\qquad +\frac{\sin\gamma}{s}\Big[-\frac{r^2}{s}\sin\gamma\sigma_s+\frac{r^2}{s}\Bigg(\frac{1}{2}\frac{1}{\sin\gamma}\Bigg(\frac{2s}{r^2}-2\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)\frac{s}{r^2}\Bigg)\Bigg)\sigma_s\\
&\qquad\qquad +\frac{r^2}{s}\sin\gamma\frac{\partial^2\sigma}{\partial s^2}\Big]\\
&= \Bigg(\frac{1}{s}-\frac{s}{2r^2}+\frac{fw^2}{2sr^2}\Bigg)\sigma_s\\
&\qquad -\frac{r^2}{s^3}\sin^2\gamma\sigma_s+\frac{\sin\gamma}{s}\Bigg(\frac{r^2}{s}\frac{1}{2}\frac{1}{\sin\gamma}\Bigg(\frac{2s}{r^2}-2(\frac{s^2-fw^2}{2r^2})\frac{s}{r^2}\Bigg)\Bigg)\sigma_s\\
&\qquad\qquad +\frac{\sin\gamma}{s}\frac{r^2}{s}\sin\gamma\frac{\partial^2\sigma}{\partial s^2}\\
&= \Bigg(\frac{1}{s}-\frac{s}{2r^2}+\frac{fw^2}{2sr^2}\Bigg)\sigma_s\\
&\qquad -\Big[\frac{r^2}{s^3}\Bigg(\frac{s^2}{r^2}-\frac{fw^2}{r^2}\Bigg)-\frac{r^2}{s^3}\Bigg(\frac{s^4}{4r^2}+\frac{f^2w^4}{4r^4}-\frac{s^2fw^2}{2r^4}\Bigg)\Big]\sigma_s\\
&\qquad\quad +\Bigg(\frac{1}{s}-\frac{1}{s}\Bigg(\frac{s^2-fw^2}{2r^2}\Bigg)\Bigg)\sigma_s\\
&\qquad\qquad +\frac{r^2}{s^2}\Big[\frac{s^2}{r^2}\Bigg(1-\frac{fw^2}{s^2}-\frac{s^2}{4r^2}-\frac{f^2w^4}{4r^2s^2}+\frac{fw^2}{2r^2}\Bigg)\Big]\frac{\partial^2\sigma}{\partial s^2}\\
&= \Bigg(\frac{1}{s}-\frac{3s}{4r^2}+\frac{fw^2}{2sr^2}+\frac{f^2w^4}{4s^3r^2}+\frac{fw^2}{s^3}\Bigg)\sigma_s\\
&\quad +\Bigg(1-\frac{fw^2}{s}-\frac{s^2}{4r^2}-\frac{f^2w^4}{4r^2s^2}+\frac{fw^2}{2r^2}\Bigg)\frac{\partial^2\sigma}{\partial s^2}
\end{aligned}$$

Inserting these three equations into Eq. C.5, we obtain

$$
\begin{aligned}
\Box\sigma = \frac{1}{r^2}\Big[&\frac{\partial}{\partial r}\left(r^2 f\sigma_r\right) - \frac{\partial}{\partial \Delta r}\left(r^2 f\sigma_r\right) + \left(\frac{s}{r} + \frac{w^2}{2sr}(rf' - 2f)\right)\frac{\partial}{\partial s}\left(r^2 f\sigma_r\right)\Big] \\
&+ \frac{1}{f}\left(1 - \frac{w^2\kappa^2}{4}\right)\frac{\partial^2\sigma}{\partial w^2} + \left(1 + \frac{fw^2}{2r^2} - \frac{s^2}{4r^2} - \frac{fw^4\kappa^2}{4s^2} - \frac{f^2w^4}{4s^2r^2}\right)\frac{\partial^2\sigma}{\partial s^2} \\
&+ \frac{2w}{s}\left(1 - \frac{w^2\kappa^2}{4}\right)\frac{\partial\sigma}{\partial w\partial s} - \frac{w\kappa^2}{4f}\sigma_w \\
&+ \left(\frac{2}{s} - \frac{3s}{4r^2} - \frac{w^2\kappa^2}{2s} + \frac{fw^2}{2r^2 s} + \frac{f\kappa^2 w^4}{4s^3} + \frac{f^2w^4}{4s^3r^2}\right)\sigma_s.
\end{aligned}
$$

Recalling that to keep track of coefficients in the expansion, we must insert the book-keeping ϵ into the appropriate terms as before. We then obtain

$$
\begin{aligned}
\Box\sigma = \frac{1}{r^2}\Big[&\frac{\partial}{\partial r}\left(r^2 f\sigma_r\right) - \frac{1}{\epsilon}\frac{\partial}{\partial \Delta r}\left(r^2 f\sigma_r\right) + \left(\frac{s}{r} + \frac{w^2}{2sr}(rf' - 2f)\right)\frac{\partial}{\partial s}\left(r^2 f\sigma_r\right)\Big] \\
&+ \frac{1}{f}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial^2\sigma}{\partial w^2} + \left(\frac{1}{\epsilon^2} + \frac{fw^2}{2r^2} - \frac{s^2}{4r^2} - \frac{fw^4\kappa^2}{4s^2} - \frac{f^2w^4}{4s^2r^2}\right)\frac{\partial^2\sigma}{\partial s^2} \\
&+ \frac{2w}{s}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial\sigma}{\partial w\partial s} - \frac{w\kappa^2}{4f}\sigma_w \\
&+ \left(\frac{2}{s\epsilon^2} - \frac{3s}{4r^2} - \frac{w^2\kappa^2}{2s} + \frac{fw^2}{2r^2 s} + \frac{f\kappa^2 w^4}{4s^3} + \frac{f^2w^4}{4s^3r^2}\right)\sigma_s.
\end{aligned}
\tag{C.6}
$$

As we have seen, for $d = 4$ the defining equation for σ is

$$\Box\sigma = 4 - 2\Delta^{-\frac{1}{2}}\sigma^a\nabla_a\Delta^{\frac{1}{2}}.$$

Rearranging, and letting $\Delta^{\frac{1}{2}} = U_0$, we can write

$$\sigma^a\nabla_a U_0 + \frac{1}{2}(\Box\sigma - 4)U_0 = 0. \tag{C.7}$$

To make use of this equation and find the coefficients in the expansion of U_0, we now require $\sigma^a\nabla_a\Delta^{\frac{1}{2}}$ in terms of $(s, w, \Delta r, r)$. First, we have that

$$
\begin{aligned}
\sigma^a\nabla_a U_0 &= g^{ab}\sigma_b\nabla_a U_0 \\
&= g^{\tau\tau}\sigma_\tau\nabla_\tau U_0 + g^{rr}\sigma_r\nabla_r U_0 + g^{\gamma\gamma}\sigma_\gamma\nabla_\gamma U_0 \\
&= g^{\tau\tau}\sigma_\tau\partial_\tau U_0 + g^{rr}\sigma_r\partial_r U_0 + g^{\gamma\gamma}\sigma_\gamma\partial_\gamma U_0
\end{aligned}
\tag{C.8}
$$

Similar to above, we require the derivatives of U_0 to be in terms of coordinates $(s, w, \Delta r, r)$. Thus,

$$\begin{aligned}
\partial_r U_0 &= \frac{\partial w}{\partial r}\frac{\partial U_0}{\partial w} + \frac{\partial s}{\partial r}\frac{\partial U_0}{\partial s} + \frac{\partial r}{\partial r}\frac{\partial U_0}{\partial r} + \frac{\partial \Delta r}{\partial r}\frac{\partial U_0}{\partial \Delta r} \\
&= \left(\frac{f' w^2}{2s} + \frac{s}{r} - \frac{f w^2}{sr}\right)\frac{\partial U_0}{\partial s} + \frac{\partial U_0}{\partial r} - \frac{\partial U_0}{\partial \Delta r} \\
\partial_\tau U_0 &= \frac{\partial w}{\partial \tau}\frac{\partial U_0}{\partial w} + \frac{\partial s}{\partial \tau}\frac{\partial U_0}{\partial s} + \frac{\partial r}{\partial \tau}\frac{\partial U_0}{\partial r} + \frac{\partial \Delta r}{\partial \tau}\frac{\partial U_0}{\partial \Delta r} \\
&= -\left(1 - \frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\frac{\partial U_0}{\partial w} - \frac{wf}{s}\left(1 - \frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\frac{\partial U_0}{\partial s} \\
&= -\left(1 - \frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left(\frac{\partial U_0}{\partial w} + \frac{wf}{s}\frac{\partial U_0}{\partial s}\right) \\
\partial_\gamma U_0 &= \frac{\partial w}{\partial \gamma}\frac{\partial U_0}{\partial \gamma} + \frac{\partial s}{\partial \gamma}\frac{\partial U_0}{\partial s} + \frac{\partial r}{\partial r}\frac{\partial U_0}{\partial \gamma} + \frac{\partial \Delta r}{\partial \gamma}\frac{\partial U_0}{\partial \Delta r} \\
&= r\left(1 - \frac{f w^2}{s^2} - \frac{s^2}{4r^2} - \frac{f^2 w^4}{4r^2 s^2} + \frac{f w^2}{2r^2}\right)^{\frac{1}{2}}\frac{\partial U_0}{\partial s}
\end{aligned}$$

Then Eq. C.8 becomes

$$
\begin{aligned}
\sigma^a \nabla_a U_0 = &-\frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left(-\frac{wf}{s}\sigma_s-\sigma_w\right)\left(1-\frac{w^2\kappa^2}{4}\right)^{\frac{1}{2}}\left(\frac{\partial U_0}{\partial w}+\frac{wf}{s}\frac{\partial U_0}{\partial s}\right) \\
&+f\left(\sigma_r+\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\sigma_s-\sigma_{\Delta r}\right)\left[\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\frac{\partial U_0}{\partial s}+\frac{\partial U_0}{\partial r}-\frac{\partial U_0}{\partial \Delta r}\right] \\
&+\frac{1}{r^2}\left[r\left(1-\frac{fw^2}{s^2}-\frac{s^2}{4r^2}-\frac{f^2w^4}{4r^2s^2}+\frac{fw^2}{2r^2}\right)^{\frac{1}{2}}\sigma_s\right]\left[r\left(1-\frac{fw^2}{s^2}-\frac{s^2}{4r^2}-\frac{f^2w^4}{4r^2s^2}+\frac{fw^2}{2r^2}\right)^{\frac{1}{2}}\frac{\partial U_0}{\partial s}\right] \\
= &-\frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)\left(-\frac{wf}{s}\sigma_s-\sigma_w\right)\left(\frac{\partial U_0}{\partial w}+\frac{wf}{s}\frac{\partial U_0}{\partial s}\right) \\
&+f\sigma_r\left[\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\frac{\partial U_0}{\partial s}+\frac{\partial U_0}{\partial r}-\frac{\partial U_0}{\partial \Delta r}\right] \\
&\quad+f\sigma_s\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\left[\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\frac{\partial U_0}{\partial s}+\frac{\partial U_0}{\partial r}-\frac{\partial U_0}{\partial \Delta r}\right] \\
&\qquad-f\sigma_{\Delta r}\left[\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\frac{\partial U_0}{\partial s}+\frac{\partial U_0}{\partial r}-\frac{\partial U_0}{\partial \Delta r}\right] \\
&+\left(1-\frac{fw^2}{s^2}-\frac{s^2}{4r^2}-\frac{f^2w^4}{4r^2s^2}+\frac{fw^2}{2r^2}\right)\sigma_s\frac{\partial U_0}{\partial s} \\
= &\ \frac{1}{f}\left(1-\frac{w^2\kappa^2}{4}\right)\sigma_w\frac{\partial U_0}{\partial w}+\frac{w}{s}\left(1-\frac{w^2\kappa^2}{4}\right)\left(\sigma_s\frac{\partial U_0}{\partial w}+\sigma_w\frac{\partial U_0}{\partial s}\right) \\
&\qquad+\left[1-\frac{s^2}{4r^2}+\frac{4w^2}{2r^2}-\frac{fw^4\kappa^2}{4s^2}\left(1+\frac{f}{\kappa^2r^2}\right)\right]\sigma_s\frac{\partial U_0}{\partial s} \\
&+\left[f\sigma_r+f\sigma_s\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)-f\sigma_{\Delta r}\right]\left[\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\frac{\partial U_0}{\partial s}+\frac{\partial U_0}{\partial r}-\frac{\partial U_0}{\partial \Delta r}\right]
\end{aligned}
$$

Remembering that in the new coordinate system, we obtained

$$
\sigma_r' = \sigma_r+\sigma_s\left(\frac{f' w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)-\sigma_{\Delta r}
$$

so by removing the prime (for consistency) and using this calculated term in the notebook, and by inserting the bookkeeping term ϵ as before, we then have:

$$
\begin{aligned}
\sigma^a \nabla_a U_0 = &\ \frac{1}{f}\left(\frac{1}{\epsilon^2}-\frac{w^2\kappa^2}{4}\right)\sigma_w\frac{\partial U_0}{\partial w}+\frac{w}{s}\left(\frac{1}{\epsilon^2}-\frac{w^2\kappa^2}{4}\right)\left(\sigma_s\frac{\partial U_0}{\partial w}+\sigma_w\frac{\partial U_0}{\partial s}\right) \\
&\qquad+\left[\frac{1}{\epsilon^2}-\frac{s^2}{4r^2}+\frac{4w^2}{2r^2}-\frac{fw^4\kappa^2}{4s^2}\left(1+\frac{f}{\kappa^2r^2}\right)\right]\sigma_s\frac{\partial U_0}{\partial s} \\
&+f\sigma_r\left[\left(\frac{f'w^2}{2s}+\frac{s}{r}-\frac{fw^2}{sr}\right)\frac{\partial U_0}{\partial s}+\frac{\partial U_0}{\partial r}-\frac{1}{\epsilon}\frac{\partial U_0}{\partial \Delta r}\right]
\end{aligned}
\tag{C.9}
$$

At this point we return to Mathematica to solve Eq. C.9 for the coefficients of U_0 now in terms of $(s, w, \Delta r, r)$, having expanded the terms $\Box\sigma$ and $\sigma^a \nabla_a U_0$ appropriately.

C.3 Calculating $\Box U_0$ in $(s, w, \Delta r, r)$

Eq. B.13 for $d = 4$ and $U_0 = \Delta^{\frac{1}{2}}$ is

$$(\Box - m^2 - \xi R) U_0 + 2\sigma^a \nabla_a V_0 - 2\frac{V_0}{U_0}\sigma^a \nabla_a U_0 + 2V_0 = 0$$

which is rewritten using the defining equation $\Delta^{\frac{1}{2}}(\Box\sigma - 4) + 2\sigma^a \nabla_a \Delta^{\frac{1}{2}} = 0$ as

$$(\Box - m^2 - \xi R) U_0 + 2\sigma^a \nabla_a V_0 - V_0(\Box\sigma - 4) + 2V_0 = 0. \tag{C.10}$$

Hence we now need an expression for $\Box U_0$ that we can implement in Mathematica, before turning our attention then to V_0 and $\sigma_a \nabla^a V_0$. Using the derived expression for $\partial_r U_0$ above we have

$$r^2 f \partial_r U_0 = r^2 f \left(\left(\frac{f' w^2}{2s} + \frac{s}{r} - \frac{f w^2}{sr} \right) \frac{\partial U_0}{\partial s} + \frac{\partial U_0}{\partial r} - \frac{\partial U_0}{\partial \Delta r} \right).$$

Hence,

$$\begin{aligned}
\Box U_0 = {} & \frac{1}{r^2}\left[\frac{\partial}{\partial r} - \frac{1}{\epsilon}\frac{\partial}{\partial \Delta r} + \left(\frac{s}{r} + \frac{w^2}{2sr}(rf' - 2f)\right)\frac{\partial}{\partial s}\right]\left[r^2 f\left(\left(\frac{f' w^2}{2s} + \frac{s}{r} - \frac{f w^2}{sr}\right)\frac{\partial U_0}{\partial s} + \frac{\partial U_0}{\partial r} - \frac{1}{\epsilon}\frac{\partial U_0}{\partial \Delta r}\right)\right] \\
& + \frac{1}{f}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial^2 U_0}{\partial w^2} + \left(\frac{1}{\epsilon^2} + \frac{f w^2}{2r^2} - \frac{s^2}{4r^2} - \frac{f w^4 \kappa^2}{4s^2} - \frac{f^2 w^4}{4s^2 r^2}\right)\frac{\partial^2 U_0}{\partial s^2} \\
& + \frac{2w}{s}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\frac{\partial U_0}{\partial w \partial s} - \frac{w\kappa^2}{4f}\frac{\partial U_0}{\partial w} \\
& + \left(\frac{2}{s\epsilon^2} - \frac{3s}{4r^2} - \frac{w^2\kappa^2}{2s} + \frac{f w^2}{2r^2 s} + \frac{f\kappa^2 w^4}{4s^3} + \frac{f^2 w^4}{4s^3 r^2}\right)\frac{\partial U_0}{\partial s}.
\end{aligned} \tag{C.11}$$

C.4 Calculating V_0 and V_1 in $(s, w, \Delta r, r)$

The equation we are interested in implementing in Mathematica to solve for coefficients for V_0 is Eq. C.10. Hence we now require an expression for the term $\sigma^a \nabla_a V_0$.

Luckily, since U_0 and V_0 are expanded in the chosen coordinate system as

$$U_p(x,x') = \sum_{ijk} u^{(p)}_{ijk}(r)\epsilon^{i+j+k} w^i \Delta r^j s^k$$
$$V_p(x,x') = \sum_{ijk} v^{(p)}_{ijk}(r)\epsilon^{i+j+k} w^i \Delta r^j s^k$$

the expression for $\sigma^a\nabla_a V_0$ is essentially identical to $\sigma^a\nabla_a U_0$ from earlier. Thus we have

$$\begin{aligned}\sigma^a\nabla_a V_0 = \frac{1}{f}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\sigma_w\frac{\partial V_0}{\partial w} + \frac{w}{s}\left(\frac{1}{\epsilon^2} - \frac{w^2\kappa^2}{4}\right)\left(\sigma_s\frac{\partial V_0}{\partial w} + \sigma_w\frac{\partial V_0}{\partial s}\right) \\ + \left[\frac{1}{\epsilon^2} - \frac{s^2}{4r^2} + \frac{4w^2}{2r^2} - \frac{fw^4\kappa^2}{4s^2}\left(1 + \frac{f}{\kappa^2 r^2}\right)\right]\sigma_s\frac{\partial V_0}{\partial s} \\ + f\sigma_r\left[\left(\frac{f'w^2}{2s} + \frac{s}{r} - \frac{fw^2}{sr}\right)\frac{\partial V_0}{\partial s} + \frac{\partial V_0}{\partial r} - \frac{1}{\epsilon}\frac{\partial V_0}{\partial \Delta r}\right].\end{aligned} \tag{C.12}$$

With expanded equation for V_0 we then need V_1. We use the recursion relation in Eq. B.12 for $d = 4$ and $p = 0$ to obtain

$$4V_1 + 2\sigma^a\nabla_a V_1 - 2\frac{V_1}{U_0}\sigma^a\nabla_a U_0 + \Box V_0 - (m^2 + \xi R)V_0 = 0$$

which computationally simplifies to

$$4V_1 + 2\sigma^a\nabla_a V_1 + V_1(\Box\sigma - 4) + \Box V_0 - (m^2 + \xi R)V_0 = 0.$$

We note that for the same reasons as above, $\Box V_0$ is identical to $\Box U_0$ with the appropriate change $U_0 \to V_0$ in Eq. C.11. Similarly, $\sigma^a\nabla_a V_1$ is equivalent to Eq. C.12 with the substitution $V_0 \to V_1$.

Appendix D

Derivatives of the Green function

We now differentiate each mode-sum expression from Chapter 4 with respect to r.

$$\frac{\partial \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r)}{\partial r} = \frac{\partial}{\partial r}\left\{ \frac{2^{i-j-1}}{\kappa^{2i+2j} r^{2+2j} \Gamma(1+j)} \sum_{m=0}^{i} \sum_{s=0}^{m} \frac{i!}{s!(i-m)!(m-s)!} \frac{(-1)^m}{2^m} \right.$$
$$\times \left(\frac{1}{\eta}\frac{\partial}{\partial \eta}\right)^j \Big[P_l(\eta) Q_l(\eta) \delta_{m,2s+n} + (-1)^{(2s+n-m)} P_l^{(m-2s-n)}(\eta) Q_l^{(2s+n-m)}(\eta)$$
$$\left. + (-1)^{(m-2s-n)} P_l^{(2s+n-m)}(\eta) Q_l^{(m-2s-n)}(\eta) \Big] \right\}$$

We recall that we defined

$$\eta = \sqrt{1 + \frac{f^2(r)}{\kappa^2 r^2}},$$

and so

$$\frac{\partial \eta}{\partial r} = \frac{r f f' - f^2}{\kappa^2 r^3 \eta}$$

will be useful when using the chain rule. We will also use that

$$\frac{\partial}{\partial r}\left(\frac{1}{\eta}\right)^j = \frac{-j(r f f' - f^2)}{\kappa^2 r^3 \eta^{j+2}}.$$

Now we can write that

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r)}{\partial r} = {} & \frac{-2^{i-j}(j+1)}{\kappa^{2i+2j}r^{3+2j}\Gamma(1+j)}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m} \\
& \times \left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j \Big[P_l(\eta)Q_l(\eta)\delta_{m,2s+n} + (-1)^{(2s+n-m)}P_l^{(m-2s-n)}(\eta)Q_l^{(2s+n-m)}(\eta) \\
& \quad + (-1)^{(m-2s-n)}P_l^{(2s+n-m)}(\eta)Q_l^{(m-2s-n)}(\eta)\Big] \\
& + \frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m} \\
& \times \frac{-j(rff' - f^2)}{\kappa^2 r^3\eta^{j+2}}\left(\frac{\partial}{\partial\eta}\right)^j \Big[P_l(\eta)Q_l(\eta)\delta_{m,2s+n} + (-1)^{(2s+n-m)}P_l^{(m-2s-n)}(\eta)Q_l^{(2s+n-m)}(\eta) \\
& \quad + (-1)^{(m-2s-n)}P_l^{(2s+n-m)}(\eta)Q_l^{(m-2s-n)}(\eta)\Big] \\
& + \frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m} \\
& \times \left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j \Big[\left(\frac{\partial P_l(\eta)}{\partial r}Q_l(\eta) + P_l(\eta)\frac{\partial Q_l(\eta)}{\partial r}\right)\delta_{m,2s+n} \\
& + (-1)^{(2s+n-m)}\left(\frac{\partial P_l^{(m-2s-n)}(\eta)}{\partial r}Q_l^{(2s+n-m)}(\eta) + P_l^{(m-2s-n)}(\eta)\frac{\partial Q_l^{(2s+n-m)}(\eta)}{\partial r}\right) \\
& + (-1)^{(m-2s-n)}\left(\frac{\partial P_l^{(2s+n-m)}(\eta)}{\partial r}Q_l^{(m-2s-n)}(\eta) + P_l^{(2s+n-m)}(\eta)\frac{\partial Q_l^{(m-2s-n)}(\eta)}{\partial r}\right)\Big].
\end{aligned}
\tag{D.1}
$$

The next task is to calculate the derivatives of the Legendre functions using the chain rule. For $P_l(\eta)$, this is given by

$$
\begin{aligned}
\frac{\partial P_l(\eta)}{\partial r} &= \frac{\partial P_l(\eta)}{\partial \eta}\frac{\partial \eta}{\partial r} \\
&= \frac{l}{\eta^2-1}\Big(\eta P_l(\eta) - P_{l-1}(\eta)\Big)\frac{rff' - f^2}{\kappa^2 r^3\eta} \\
&= \Big(\eta P_l(\eta) - P_{l-1}(\eta)\Big)\frac{rf' - f}{fl\eta}.
\end{aligned}
$$

For the Legendre function of the second kind, we similarly find that

$$
\frac{\partial Q_l(\eta)}{\partial r} = \Big(\eta Q_l(\eta) - Q_{l-1}(\eta)\Big)\frac{rf' - f}{fl\eta}.
$$

For the associated Legendre functions, we obtain

$$\begin{aligned}
\frac{\partial P_l^x(\eta)}{\partial r} &= \frac{\partial P_l^x(\eta)}{\partial \eta}\frac{\partial \eta}{\partial r} \\
&= \frac{1}{\eta^2-1}\Big(\eta l P_l^x(\eta) - (x+l)P_{l-1}^x(\eta)\Big)\frac{rff^{'}-f^2}{\kappa^2 r^3 \eta} \\
&= \Big(\eta l P_l^x(\eta) - (x+l)P_{l-1}^x(\eta)\Big)\frac{rf^{'}-f}{fl\eta}
\end{aligned}$$

and

$$\frac{\partial Q_l^x(\eta)}{\partial r} = \Big(\eta l Q_l^x(\eta) - (x+l)Q_{l-1}^x(\eta)\Big)\frac{rf^{'}-f}{fl\eta}.$$

Thus Eq. D.1 becomes

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j\mid r)}{\partial r} &= \frac{2^{i-j}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)} \sum_{m=0}^{i}\sum_{s=0}^{m} \frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m} \\
&\quad \times \left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j \Big[P_l(\eta)Q_l(\eta)\delta_{m,2s+n} + (-1)^{(2s+n-m)}P_l^{(m-2s-n)}(\eta)Q_l^{(2s+n-m)}(\eta) \\
&\qquad + (-1)^{(m-2s-n)}P_l^{(2s+n-m)}(\eta)Q_l^{(m-2s-n)}(\eta)\Big]\Big[-\frac{(j+1)}{r} - \frac{j(rf' - f^2)}{\kappa^2 r\eta^2}\Big] \\
&\quad + \frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)} \sum_{m=0}^{i}\sum_{s=0}^{m} \frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m} \\
&\quad \times \left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j \frac{rf'-f}{fl\eta}\Bigg[\Bigg\{\Big(\eta P_l(\eta) - P_{l-1}(\eta)\Big)Q_l(\eta) \\
&\qquad + P_l(\eta)\Big(\eta Q_l(\eta) - Q_{l-1}(\eta)\Big)\Bigg\}\delta_{m,2s+n} \\
&\quad + (-1)^{(2s+n-m)}\Bigg\{\Big(\eta l P_l^{m-2s-n}(\eta) - (m-2s-n+l)P_{l-1}^{m-2s-n}(\eta)\Big)Q_l^{2s+n-m}(\eta) \\
&\qquad + P_l^{m-2s-n}(\eta)\Big(\eta l Q_l^{2s+n-m}(\eta) - (2s+n-m+l)Q_{l-1}^{2s+n-m}(\eta)\Big)\Bigg\} \\
&\quad + (-1)^{(m-2s-n)}\Bigg\{\Big(\eta l P_l^{2s+n-m}(\eta) - (2s+n-m+l)P_{l-1}^{2s+n-m}(\eta)\Big)Q_l^{m-2s-n}(\eta) \\
&\qquad + P_l^{2s+n-m}(\eta)\Big(\eta l Q_l^{m-2s-n}(\eta) - (m-2s-n+l)Q_{l-1}^{m-2s-n}(\eta)\Bigg\}\Big)\Bigg] \\
&= \frac{2^{i-j}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)} \sum_{m=0}^{i}\sum_{s=0}^{m} \frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m} \\
&\quad \times \left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j \Big[P_l(\eta)Q_l(\eta)\delta_{m,2s+n} + (-1)^{(2s+n-m)}P_l^{(m-2s-n)}(\eta)Q_l^{(2s+n-m)}(\eta) \\
&\qquad + (-1)^{(m-2s-n)}P_l^{(2s+n-m)}(\eta)Q_l^{(m-2s-n)}(\eta)\Big] \\
&\quad \times \Bigg\{-\frac{(j+1)}{r} - \frac{j(rf'-f^2)}{\kappa^2 r\eta^2} + \frac{rf'-f}{2fl\eta}\Bigg[\Bigg\{\Big(\eta P_l(\eta) - P_{l-1}(\eta)\Big)Q_l(\eta) \\
&\qquad + P_l(\eta)\Big(\eta Q_l(\eta) - Q_{l-1}(\eta)\Big)\Bigg\}\delta_{m,2s+n} \\
&\qquad + (-1)^{(2s+n-m)}\Bigg\{\Big(\eta l P_l^{m-2s-n}(\eta) - (m-2s-n+l)P_{l-1}^{m-2s-n}(\eta)\Big)Q_l^{2s+n-m}(\eta) \\
&\qquad + P_l^{m-2s-n}(\eta)\Big(\eta l Q_l^{2s+n-m}(\eta) - (2s+n-m+l)Q_{l-1}^{2s+n-m}(\eta)\Big)\Bigg\} \\
&\qquad + (-1)^{(m-2s-n)}\Bigg\{\Big(\eta l P_l^{2s+n-m}(\eta) - (2s+n-m+l)P_{l-1}^{2s+n-m}(\eta)\Big)Q_l^{m-2s-n}(\eta) \\
&\qquad + P_l^{2s+n-m}(\eta)\Big(\eta l Q_l^{m-2s-n}(\eta) - (m-2s-n+l)Q_{l-1}^{m-2s-n}(\eta)\Big)\Bigg\}\Bigg]\Bigg\}
\end{aligned}
$$

which simplifies to:

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\Psi}{}^{(+)}_{nl}(i,j \mid r)}{\partial r} = {} & \frac{2^{i-j}}{\kappa^{2i+2j} r^{2+2j} \Gamma(1+j)} \sum_{m=0}^{i} \sum_{s=0}^{m} \frac{i!}{s!(i-m)!(m-s)!} \frac{(-1)^m}{2^m} \\
& \times \left(\frac{1}{\eta} \frac{\partial}{\partial \eta} \right)^j \Big[P_l(\eta) Q_l(\eta) \delta_{m,2s+n} + (-1)^{(2s+n-m)} P_l^{(m-2s-n)}(\eta) Q_l^{(2s+n-m)}(\eta) \\
& \qquad + (-1)^{(m-2s-n)} P_l^{(2s+n-m)}(\eta) Q_l^{(m-2s-n)}(\eta) \Big] \\
\times \Bigg\{ & - \frac{(j+1)}{r} - \frac{j(rf' - f^2)}{\kappa^2 r \eta^2} + \frac{rf' - f}{2fl\eta} \Bigg[\Bigg\{ \Big(\eta P_l(\eta) - P_{l-1}(\eta) \Big) Q_l(\eta) \\
& \qquad + P_l(\eta) \Big(\eta Q_l(\eta) - Q_{l-1}(\eta) \Big) \Bigg\} \delta_{m,2s+n} \\
& + (-1)^{(2s+n-m)} \Big\{ 2l\eta P_l^{m-2s-n}(\eta) Q_l^{2s+n-m}(\eta) \\
& \quad - (l+2s-m+n) P_l^{m-2s-n}(\eta) Q_{l-1}^{2s-m+n} - (l-2s-n+m) P_{l-1}^{m-2s-n}(\eta) Q_l^{2s-m+n} \Big\} \\
& + (-1)^{(m-2s-n)} \Big\{ 2l\eta P_l^{2s-m+n}(\eta) Q_l^{m-n-2s}(\eta) \\
& \quad - (l-2s+m-n) P_l^{2s-m+n}(\eta) Q_{l-1}^{m-2s-n} \\
& \qquad - (l+2s+n-m) P_{l-1}^{2s-m+n}(\eta) Q_l^{m-2s-n} \Big\} \Bigg] \Bigg\}.
\end{aligned}
$$

Next, turning to the derivative of $\overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r)$, we obtain

$$\begin{aligned}
\frac{\partial \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r)}{\partial r} &= \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j(2j-2)}{\kappa^{2i-2j}r^{3-2j}\Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}} \sum_{m=0}^{i-j} \sum_{s=0}^{m} \frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m} \\
&\quad \times \Bigg[P_{l+j-2k}(\eta) Q_{l+j-2k}(\eta) \delta_{m,2s+n} + (-1)^{(2s+n-m)} P^{(m-n-2s)}_{l+j-2k}(\eta) Q^{(2s+n-m)}_{l+j-2k}(\eta) \\
&\qquad + (-1)^{(m-2s-n)} P^{(2s+n-m)}_{l+j-2k}(\eta) Q^{(m-2s-n)}_{l+j-2k}(\eta) \Bigg] \\
&+ \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i-2j}r^{2-2j}\Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}} \sum_{m=0}^{i-j} \sum_{s=0}^{m} \frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m} \\
&\quad \times \frac{rf' - f}{f(l+j-2k)\eta} \Bigg[\Bigg\{ \Big(\eta P_{l+j-2k}(\eta) - P_{l+j-2k-1}(\eta) \Big) Q_{l+j-2k}(\eta) \\
&\qquad P_{l+j-2k}(\eta) \Big(\eta Q_{l+j-2k}(\eta) - Q_{l+j-2k-1}(\eta) \Big) \Bigg\} \delta_{m,2s+n} \\
&+ (-1)^{(2s+n-m)} \Bigg\{ \Big(\eta(l+j-2k) P^{m-n-2s}_{l+j-2k}(\eta) \\
&\qquad - (m-n-2s+l+j-2k) P^{m-n-2s}_{l+j-2k-1}(\eta) \Big) Q^{(2s+n-m)}_{l+j-2k}(\eta) \\
&\quad + P^{(m-n-2s)}_{l+j-2k}(\eta) \Big(\eta(l+j-2k) Q^{2s+n-m}_{l+j-2k}(\eta) - (2s+n-m+l+j-2k) Q^{2s+n-m}_{l+j-2k-1}(\eta) \Big) \Bigg\} \\
&+ (-1)^{(m-2s-n)} \Bigg\{ \Big(\eta(l+j-2k) P^{2s+n-m}_{l+j-2k}(\eta) \\
&\qquad - (2s+n-m+l+j-2k) P^{2s+n-m}_{l+j-2k-1}(\eta) \Big) Q^{m-2s-n}_{l+j-2k}(\eta) \\
&\quad + P^{2s+n-m}_{l+j-2k}(\eta) \Big(\eta(l+j-2k) Q^{m-2s-n}_{l+j-2k}(\eta) - (m-2s-n+l+j-2k) Q^{m-2s-n}_{l+j-2k-1}(\eta) \Big) \Bigg\} \Bigg]
\end{aligned}$$

which simplifies to:

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\Psi}{}^{(-)}_{nl}(i,j \mid r)}{\partial r} &= \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i-2j} r^{2-2j} \Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}} \sum_{m=0}^{i-j} \sum_{s=0}^{m} \frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m} \\
&\quad \times \Bigg\{ \frac{2j-2}{r} \Bigg[P_{l+j-2k}(\eta) Q_{l+j-2k}(\eta) \delta_{m,2s+n} + (-1)^{(2s+n-m)} P^{(m-n-2s)}_{l+j-2k}(\eta) Q^{(2s+n-m)}_{l+j-2k}(\eta) \\
&\qquad + (-1)^{(m-2s-n)} P^{(2s+n-m)}_{l+j-2k}(\eta) Q^{(m-2s-n)}_{l+j-2k}(\eta) \Bigg] \\
&\quad + \frac{rf' - f}{f(l+j-2k)\eta} \Bigg[\Bigg\{ \Big(\eta P_{l+j-2k}(\eta) - P_{l+j-2k-1}(\eta) \Big) Q_{l+j-2k}(\eta) \\
&\qquad P_{l+j-2k}(\eta) \Big(\eta Q_{l+j-2k}(\eta) - Q_{l+j-2k-1}(\eta) \Big) \Bigg\} \delta_{m,2s+n} \\
&\quad + (-1)^{(2s+n-m)} \Big\{ 2(l+j-2k)\eta P^{m-2s-n}_{l+j-2k}(\eta) Q^{2s+n-m}_{l+j-2k}(\eta) \\
&\qquad - (l+j-2k+2s-m+n) P^{m-2s-n}_{l+j-2k}(\eta) Q^{2s-m+n}_{l+j-2k-1} \\
&\qquad - (l+j-2k-2s-n+m) P^{m-2s-n}_{l+j-2k-1}(\eta) Q^{2s-m+n}_{l+j-2k} \Big\} \\
&\quad + (-1)^{(m-2s-n)} \Big\{ 2(l+j-2k)\eta P^{2s+n-m}_{l+j-2k}(\eta) Q^{m-2s-n}_{l+j-2k}(\eta) \\
&\qquad - (l+j-2k-2s+m-n) P^{2s+n-m}_{l+j-2k}(\eta) Q^{m-2s-n}_{l+j-2k-1} \\
&\qquad - (l+j-2k+2s+n-m) P^{2s+n-m}_{l+j-2k-1}(\eta) Q^{m-2s-n}_{l+j-2k} \Big\} \Bigg] \Bigg\}.
\end{aligned}
$$

We now turn to the tail regularisation parameters and calculate $\frac{\partial}{\partial r}$, for the case $l > i-j$ first:

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} &= \frac{2^{j-1}(2i-2j)r^{2i-2j-1}(i-j)!}{\kappa^{2j}} \sum_{k=0}^{i-j+1} (-1)^k \binom{i-j+1}{k} \frac{(l+i-j-2k+\frac{3}{2})}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\quad \times \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} P_{l+i-j-2k+1}(\eta) Q_{l+i-j-2k+1}(\eta) \\
&\qquad + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \Big(P^{(m-n-2s)}_{l+i-j-2k+1}(\eta) Q^{(2s+n-m)}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad + (-1)^n P^{(2s+n-m)}_{l+i-j-2k+1}(\eta) Q^{(m-2s-n)}_{l+i-j-2k+1}(\eta) \Big) \Bigg] \\
&\quad + \frac{2^{j-1} r^{2i-2j}(i-j)!}{\kappa^{2j}} \sum_{k=0}^{i-j+1} (-1)^k \binom{i-j+1}{k} \frac{(l+i-j-2k+\frac{3}{2})}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\quad \times \frac{rf' - f}{f(l+i-j-2k+1)\eta} \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} \\
&\qquad \times \Big\{ \Big(\eta P_{l+i-j-2k+1}(\eta) - P_{l+i-j-2k}(\eta) \Big) Q_{l+i-j-2k+1}(\eta) \\
&\qquad + P_{l+i-j-2k+1}(\eta) \Big(\eta Q_{l+i-j-2k+1}(\eta) - Q_{l+i-j-2k}(\eta) \Big) \Big\} \\
&\quad + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \\
&\quad \times \Big\{ \Big(\eta(l+i-j-2k+1) P^{m-n-2s}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad - (m-n-2s+l+i-j-2k+1) P^{m-n-2s}_{l+i-j-2k}(\eta) \Big) Q^{2s+n-m}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad + P^{m-n-2s}_{l+i-j-2k+1}(\eta) \Big(\eta(l+i-j-2k+1) Q^{2s+n-m}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad\qquad - (2s+n-m+l+i-j-2k+1) Q^{2s+n-m}_{l+i-j-2k}(\eta) \Big) \\
&\qquad + (-1)^n \Big[\Big(\eta(l+i-j-2k+1) P^{2s+n-m}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad - (2s+n-m+l+i-j-2k+1) P^{2s+n-m}_{l+i-j-2k}(\eta) \Big) Q^{m-2s-n}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad + P^{2s+n-m}_{l+i-j-2k+1}(\eta) \Big(\eta(l+i-j-2k+1) Q^{m-2s-n}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad\qquad - (m-2s-n+l+i-j-2k+1) Q^{m-2s-n}_{l+i-j-2k}(\eta) \Big) \Big] \Big\} \Bigg]
\end{aligned}
$$

simplifying to

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} &= \frac{2^{j-1} r^{2i-2j} (i-j)!}{\kappa^{2j}} \sum_{k=0}^{i-j+1} (-1)^k \binom{i-j+1}{k} \frac{(l+i-j-2k+\frac{3}{2})}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\times \Bigg\{ \frac{2i-2j}{r} \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} P_{l+i-j-2k+1}(\eta) Q_{l+i-j-2k+1}(\eta) \\
&\qquad + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \Big(P^{(m-n-2s)}_{l+i-j-2k+1}(\eta) Q^{(2s+n-m)}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad + (-1)^n P^{(2s+n-m)}_{l+i-j-2k+1}(\eta) Q^{(m-2s-n)}_{l+i-j-2k+1}(\eta) \Big) \Bigg] \\
&\quad + \frac{rf' - f}{f(l+i-j-2k+1)\eta} \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} \\
&\qquad \times \Bigg\{ \Big(\eta P_{l+i-j-2k+1}(\eta) - P_{l+i-j-2k}(\eta) \Big) Q_{l+i-j-2k+1}(\eta) \\
&\qquad + P_{l+i-j-2k+1}(\eta) \Big(\eta Q_{l+i-j-2k+1}(\eta) - Q_{l+i-j-2k}(\eta) \Big) \Bigg\} \\
&\quad + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \\
&\quad \times \Bigg\{ \Big(\eta(l+i-j-2k+1) P^{m-n-2s}_{l+i-j-2k+1}(\eta) \\
&\qquad - (m-n-2s+l+i-j-2k+1) P^{m-n-2s}_{l+i-j-2k}(\eta) \Big) Q^{2s+n-m}_{l+i-j-2k+1}(\eta) \\
&\qquad + P^{m-n-2s}_{l+i-j-2k+1}(\eta) \Big(\eta(l+i-j-2k+1) Q^{2s+n-m}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad - (2s+n-m+l+i-j-2k+1) Q^{2s+n-m}_{l+i-j-2k}(\eta) \Big) \\
&\quad + (-1)^n \Bigg[\Big(\eta(l+i-j-2k+1) P^{2s+n-m}_{l+i-j-2k+1}(\eta) \\
&\qquad - (2s+n-m+l+i-j-2k+1) P^{2s+n-m}_{l+i-j-2k}(\eta) \Big) Q^{m-2s-n}_{l+i-j-2k+1}(\eta) \\
&\qquad + P^{2s+n-m}_{l+i-j-2k+1}(\eta) \Big(\eta(l+i-j-2k+1) Q^{m-2s-n}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad - (m-2s-n+l+i-j-2k+1) Q^{m-2s-n}_{l+i-j-2k}(\eta) \Big) \Bigg] \Bigg\} \Bigg] \Bigg\}
\end{aligned}
$$

and further (via Mathematica) to:

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} &= \frac{2^{j-1} r^{2i-2j}(i-j)!}{\kappa^{2j}} \sum_{k=0}^{i-j+1} (-1)^k \binom{i-j+1}{k} \frac{(l+i-j-2k+\frac{3}{2})}{(l-k+\frac{1}{2})_{i-j+2}} \\
&\times \Bigg\{ \frac{2i-2j}{r} \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} P_{l+i-j-2k+1}(\eta) Q_{l+i-j-2k+1}(\eta) \\
&\qquad + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \Big(P^{(m-n-2s)}_{l+i-j-2k+1}(\eta) Q^{(2s+n-m)}_{l+i-j-2k+1}(\eta) \\
&\qquad\qquad + (-1)^n P^{(2s+n-m)}_{l+i-j-2k+1}(\eta) Q^{(m-2s-n)}_{l+i-j-2k+1}(\eta) \Big) \Bigg] \\
&\quad + \frac{rf' - f}{f(l+i-j-2k+1)\eta} \Bigg[\sum_{m=0}^{j} \frac{j!}{(\frac{1}{2}(m-n))!(j-m)!(\frac{1}{2}(m+n))!} \frac{(-1)^m}{2^m} \\
&\qquad \times \Bigg\{ \Big(\eta P_{l+i-j-2k+1}(\eta) - P_{l+i-j-2k}(\eta) \Big) Q_{l+i-j-2k+1}(\eta) \\
&\qquad + P_{l+i-j-2k+1}(\eta) \Big(\eta Q_{l+i-j-2k+1}(\eta) - Q_{l+i-j-2k}(\eta) \Big) \Bigg\} \\
&\quad + \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \\
&\quad \times \Bigg\{ 2(1+i-j-2k+l)\eta P^{m-n-2s}_{l+i-j-2k+1}(\eta) Q^{2s+n-m}_{l+i-j-2k+1} \\
&\qquad - (2s+n-m+l+i-j-2k+1) \\
&\qquad\qquad \times \left[P^{m-n-2s}_{l+i-j-2k+1}(\eta) Q^{2s-m+n}_{i-j-2k+l} + P^{m-n-2s}_{i-j-2k+l}(\eta) Q^{2s-m+n}_{l+i-j-2k+1}(\eta) \right] \\
&\qquad + (-1)^n \Bigg[2(1+i-j-2k+l)\eta P^{2s-m+n}_{l+i-j-2k+1}(\eta) Q^{m-n-2s}_{l+i-j-2k+1} \\
&\qquad - (m-2s-n+l+i-j-2k+1) \\
&\qquad\qquad \times \left[P^{2s-m+n}_{l+i-j-2k+1}(\eta) Q^{m-2s-n}_{l+i-j-2k} + P^{2s-m+n}_{l+i-j-2k}(\eta) Q^{m-2s-n}_{l+i-j-2k+1}(\eta) \right] \Bigg] \Bigg\} \Bigg] \Bigg\}
\end{aligned}
$$

.

Finally then, the tail regularisation parameter for the case $l \leq i-j$:

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} = & \frac{\kappa}{2\pi}\left(\frac{2}{\kappa^2}\right)^j 2^{i-j}(i-j)(r)^{i-j-1}(-1)^l \Big[\frac{d}{d\lambda}(\lambda+1-l)_l\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
& \times (z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j} \\
& + \frac{\kappa}{2\pi}\left(\frac{2}{\kappa^2}\right)^j (2r)^{i-j}(-1)^l \frac{\partial}{\partial r}\Big[\frac{d}{d\lambda}(\lambda+1-l)_l\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau} \\
& \times (z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j}.
\end{aligned}
\tag{D.2}
$$

We thus need to consider the term

$$
\frac{\partial}{\partial r}\Big[\frac{d}{d\lambda}(\lambda+1-l)_l\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j}.
$$

By carefully examining the product rule, we find that this statement is equivalent to

$$
\begin{aligned}
& \frac{\partial}{\partial r}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\frac{d}{d\lambda}\Big[(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j} \\
& + \frac{\partial}{\partial r}\frac{d}{d\lambda}\Big[\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\Big]_{\lambda=i-j}\left\{(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right\}.
\end{aligned}
$$

Since

$$
\frac{\partial}{\partial r}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} = \frac{2(\lambda-i+j)\ell^2}{r}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}
$$

and

$$
\frac{\partial}{\partial r}\frac{d}{d\lambda}\Big[\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\Big]_{\lambda=i-j} = \frac{2}{r}
$$

we may then rewrite as

$$
\begin{aligned}
& \frac{2(\lambda-i+j)\ell^2}{r}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\frac{d}{d\lambda}\Big[(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\Big]_{\lambda=i-j} \\
& + \frac{2}{r}\left\{(\lambda+1-l)_l\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right\}.
\end{aligned}
$$

Substituting this expression into Eq. D.2, we obtain

$$
\begin{aligned}
\frac{\partial \overset{[d]}{\chi}_{nl}(i,j|r)}{\partial r} &= \frac{\kappa}{2\pi}\left(\frac{2}{\kappa^2}\right)^j 2^{i-j}(i-j)(r)^{i-j-1}(-1)^l \left[\frac{d}{d\lambda}(\lambda+1-l)_l \left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} \int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\right.\\
&\qquad \left.\times\, (z^2-1)^{\frac{1}{2}(\lambda+1)} \mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right]_{\lambda=i-j}\\
&\quad + \frac{\kappa}{2\pi}\left(\frac{2}{\kappa^2}\right)^j (2r)^{i-j}(-1)^l \left[\frac{2(\lambda-i+j)\ell^2}{r}\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\right.\\
&\qquad \times \frac{d}{d\lambda}\left[(\lambda+1-l)_l \int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right]_{\lambda=i-j}\\
&\qquad \left.+\frac{2}{r}\left\{(\lambda+1-l)_l \int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right\}\right]\\
&= \frac{\kappa}{2\pi r}\left(\frac{2}{\kappa^2}\right)^j (2r)^{i-j}(-1)^l \left[(i-j)\left[\frac{d}{d\lambda}(\lambda+1-l)_l\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}\right.\right.\\
&\quad \left.\times\,(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right]_{\lambda=i-j}\\
&\qquad + 2(\lambda-i+j)\ell^2\left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j}\\
&\qquad \times \frac{d}{d\lambda}\left[(\lambda+1-l)_l \int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right]_{\lambda=i-j}\\
&\quad \left.+2\left\{(\lambda+1-l)_l \int_0^{2\pi/\kappa}(1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}(z^2-1)^{\frac{1}{2}(\lambda+1)}\mathcal{Q}_l^{-\lambda-1}(z)d\Delta\tau\right\}\right].
\end{aligned}
$$

Bibliography

[1] Sean M. Carroll. *Spacetime and Geometry*. Pearson, 2016. ISBN: 9789332571655.

[2] S. W. Hawking. “Particle Creation by Black Holes”. In: *Communications in Mathematical Physics* 43.3 (1975), pp. 199–220. URL: https://www.brainmaster.com/software/pubs/physics/Hawking%20Particle%20Creation.pdf.

[3] Peter W. Higgs. “Broken symmetries, massless particles and gauge fields”. In: *Phys. Lett.* 12 (1964), pp. 132–133. DOI: 10.1016/0031-9163(64)91136-9.

[4] Cormac Breen. “Quantum Field Theory in Black Hole Space-Times with Multiple Horizons”. PhD thesis. University College Dublin, 2011.

[5] M. R. Brown and A. C. Ottewill. “Photon propagators and the definition and approximation of renormalized stress tensors in curved space-time”. In: *Physical Review D* 34.6 (Sept. 1986), pp. 1776–1787.

[6] Peter Taylor and Cormac Breen. “A Mode-Sum Prescription for Vacuum Polarization in Even Dimensions”. In: (Aug. 2017).

[7] Julian Schwinger. “On the Euclidean Structure of Relativistic Field Theory”. In: *Proceedings of the National Academy of Sciences of the United States of America* 44.9 (1958), pp. 956–965. ISSN: 00278424. URL: http://www.jstor.org/stable/90042.

[8] Michael E. Miller. “The origins of Schwingers Euclidean Greens functions”. In: *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 50 (2015), pp. 5–12. ISSN: 1355-2198. URL: http://www.jstor.org/stable/90042.

[9] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. CRC Press, 1995. ISBN: 9780201503975.

[10] G. W. Gibbons and S. W. Hawking. “Cosmological event horizons, thermodynamics, and particle creation”. In: *Phys. Rev. D* 15 (10 1977), pp. 2738–2751. DOI: `10.1103/PhysRevD.15.2738`. URL: `https://link.aps.org/doi/10.1103/PhysRevD.15.2738`.

[11] *legendreP*. `https://www.mathworks.com/help/symbolic/legendrep.html`. Accessed: 2021-02-21.

[12] Stephen A. Fulling, Mark Sweeny, and Robert M. Wald. “Singularity structure of the two-point function quantum field theory in curved spacetime”. In: *Communications in Mathematical Physics* 63.3 (1978), pp. 257 –264. DOI: `cmp/1103904566`. URL: `https://doi.org/`.

[13] Bryce S. DeWitt. “Quantum field theory in curved spacetime”. In: *Physics Reports* 19.6 (1975), pp. 295 –357. ISSN: 0370-1573. DOI: `https://doi.org/10.1016/0370-1573(75)90051-4`. URL: `http://www.sciencedirect.com/science/article/pii/0370157375900514`.

[14] Leonard Parker and David Toms. *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009. DOI: `10.1017/CBO9780511813924`.

[15] Larry H. Ford. *Quantum Field Theory in Curved Spacetime*. 1997.

[16] Yves Décanini and Antoine Folacci. “Hadamard renormalization of the stress-energy tensor for a quantized scalar field in a general spacetime of arbitrary dimension”. In: *Phys. Rev. D* 78 (4 2008), p. 044025. DOI: `10.1103/PhysRevD.78.044025`. URL: `https://link.aps.org/doi/10.1103/PhysRevD.78.044025`.

[17] Robert M. Wald. *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*. Chicago Lectures in Physics. Chicago, IL: University of Chicago Press, Oct. 1995. ISBN: 978-0-226-87027-4.

[18] H. S. Cohl. “On a generalization of the generating function for Gegenbauer polynomials”. In: *Integral Transforms and Special Functions* 24 (10 2013), p. 807. DOI: `10.1080/10652469.2012.761613`. URL: `https://doi.org/10.1080/10652469.2012.761613`.

[19] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Seventh. Elsevier/Academic Press, Amsterdam, 2007. ISBN: 978-0-12-373637-6; 0-12-373637-4.

[20] F. W. J. Olver. *Asymptotics and Special Functions*. New York: Academic University Press, 1974. ISBN: 978-0-12-525850-0.