A Mode-sum Representation of the Singular Green Function and its Derivatives via the Hadamard Parametrix

MSc Dissertation Presentation

Eoin Scanlon

School of Mathematical Sciences
Technological University Dublin

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Presentation Overview

1. Quantum Field Theory in Curved Spacetimes
2. Expansion in Extended Coordinates
3. Mode-Sum Representation of the Hadamard Parametrix
4. Derivatives of the Renormalised Green Function
5. Conclusion and Further Work

Spherically Symmetric Spacetime and Einstein Field Equations

The spacetime background chosen in [Taylor and Breen, 2017] has the line element

$$ds^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

for general $f(r)$.

Wick rotation $t \to -i\tau$ results in *Euclidean* metric.

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}.$$

Spacetime Curvature

All of the information about the curvature of a spacetime is contained within the Ricci curvature tensor R_{ab} and the scalar curvature R, both dependent on the metric g_{ab}.

"Spacetime tells matter how to move; matter tells space-time how to curve".

The Semiclassical Einstein Field Equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} \; = \; 8\pi G \langle T_{ab} \rangle$$

where $\langle T_{ab} \rangle = \langle \psi | T_{ab} | \psi \rangle$ is a **divergent quantity**. Using *point-splitting* we can define the expectation value (with respect to a Hadamard quantum state $|\psi\rangle$) of T_{ab} as

$$\langle \psi | \, T_{ab} \, | \psi \rangle = \lim_{x \to x'} \tau_{ab} \big[-iG(x, x') \big]$$

where

$$\tau^{ab} = (1 - 2\xi) g_{b'}{}^{b} \nabla^{a} \nabla^{b'} + (2\xi - \tfrac{1}{2}) g^{ab} g_{c'}{}^{c} \nabla^{c'} \nabla_{c} - 2\xi \nabla^{a} \nabla^{b}$$
$$+ 2\xi g^{ab} \nabla_{c} \nabla^{c} + \xi (R^{ab} - \tfrac{1}{2} R g^{ab}) - \tfrac{m^2}{2} g^{ab}.$$

Green Function and its Mode-Sum Representation

In its simplest terms, the Green function G propagates the particle from state $\phi(y, t_y)$ to $\phi(x, t_x)$, and is defined as

$$\phi(x, t_x) = \int G(x, t_x, y, t_y)\phi(y, t_y)dy$$

In this work, we assign the Green function the mode-sum form

$$G(x, x^{'}) = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)g_{nl}(r, r^{'})$$

where $g_{nl}(r, r^{'})$ is known as the radial Green function, satisfying

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 f\frac{d}{dr}g_{nl}(r, r^{'})\right) - \left(\frac{n^2\kappa^2}{f} + \frac{l(l+1)}{r^2} + m^2 + \xi R\right)g_{nl}(r, r^{'}) = \frac{\delta(r - r^{'})}{r^2}$$

The Hadamard Parametrix and Renormalisation

For $d = 4$, the Green function for the singular propagator G_s is defined as

$$G_s(x, x') = \frac{1}{8\pi^2}\left(\frac{U(x, x')}{\sigma(x, x')} + V(x, x')\log\left(\frac{2\sigma(x, x')}{\ell^2}\right)\right)$$

which is a **Hadamard state**. The terms $\sigma(x, x')$ and $\log(\sigma(x, x'))$ are problematic in the **coincidence limit**.

By expressing $G_s(x, x')$ as a mode-sum, we may regularise the Green function by

$$G_{reg}(x, x') = G(x, x') - G_s(x, x')$$

subtracting mode-by-mode in the code.

Recursion Relations and Boundaries

We further assume the Hadamard ansatz for U and V as

$$U(x,x^{'}) = U_0(x,x^{'}) \qquad , \qquad V(x,x^{'}) = \sum_{p=0}^{1} V_p(x,x^{'})\sigma^p.$$

Applying the wave equation we derive the recursion relation

$$(\Box - m^2 - \xi R)V_p = 2(p+1)V_{p+1}\sigma^{\,;a}\Delta^{\frac{1}{2}}_{\;;a}\Delta^{-\frac{1}{2}} - 2(p+1)\nabla_a V_{p+1}\sigma^{\,;a} - (p+1)(2p+4)V_{p+1}$$

and boundary conditions

$$U_0 = \Delta^{\frac{1}{2}}$$

and

$$(\Box - m^2 - \xi R)U_0 - 2\nabla^a V_0 \sigma_{\,;a} - 2\nabla_a U_0 \Delta^{-\frac{1}{2}} + 2V_0 = 0.$$

Extended Coordinates

A unique expansion of $\sigma(x, x^{'})$ is chosen in the form

$$\sigma = \sum_{ijk} \sigma_{ijk}(r) w^i \Delta r^j s^k$$

where

$$w^2 = \frac{2}{\kappa^2}(1 - \cos \kappa \Delta \tau) \qquad , \qquad s^2 = f(r) w^2 + 2r^2(1 - \cos \gamma).$$

Using the defining equation $\sigma^a \sigma_a = 2\sigma$ to determine the coefficients $\sigma_{ijk}(r)$, we obtain σ in extended coordinates to the correct order using ϵ as a *bookkeeping* mechanism as

$$\sigma = \sum_{ijk} \sigma_{ijk}(r) \epsilon^{i+j+k} w^i \Delta r^k s^k.$$

Expansion in $(w, s, \Delta r)$

Then, by assuming similar expansions for $U(x, x^{'})$ and $V(x, x^{'})$ as

$$U_0(x, x^{'}) = \sum_{ijk} u^{(0)}_{ijk}(r)\epsilon^{i+j+k} w^i \Delta r^j s^k \qquad , \qquad V_p(x, x^{'}) = \sum_{ijk} v^{(p)}_{ijk}(r)\epsilon^{i+j+k} w^i \Delta r^j s^k$$

and by calculating the quantities $\Box\sigma$, $\sigma^a \nabla_a U_0$, $\Box U_0$ and $\sigma^a \nabla_a V_0$ in Mathematica, we can solve for the coefficients u^0_{ijk}, v^0_{ijk} and v^1_{ijk}.

Direct and Tail Parts of the Hadamard Expansion

Setting $\Delta r = 0$ in Mathematica, we look at each of the terms in the expansion in extended coordinates. The **direct** parts of the expansion are those from the U/σ term, and the **tail** parts are from the $V \log(2\sigma/\ell^2)$ term.

The expansion is written then as

$$\begin{aligned}
\frac{U}{\sigma} + V \log\left(\frac{2\sigma}{\ell^2}\right) &= \sum_{i=0}^{2}\sum_{j=0}^{i} \mathcal{D}_{ij}^{(+)}(r)\epsilon^{2i-2}\frac{w^{2i+2j}}{s^{2+2j}} + \mathcal{D}_{11}^{(-)}(r) \\
&\quad + \log\left(\frac{2\sigma}{\ell^2}\right)\sum_{i=0}^{1}\sum_{j=0}^{i} \mathcal{T}_{ij}^{(l)}(r)\epsilon^{2i}s^{2i-2j}w^{2j} + \sum_{i=1}^{1}\sum_{j=0}^{i} \mathcal{T}_{ij}^{(p)}(r)\epsilon^{2i}s^{2i-2j}w^{2j} \\
&\quad + \sum_{i=1}^{1}\sum_{j=0}^{1-i} \mathcal{T}_{ij}^{(r)}(r)\epsilon^{2i}s^{-2j-2}w^{2i+2j+2}
\end{aligned}$$

and we calculate each of the direct and tail coefficients.

Regularisation of the Direct Terms

We assume a Fourier frequency and multipole decomposition of the form

$$\frac{w^{2i\pm2j}}{s^{2\pm2j}} = \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Psi_{nl}^{(\pm)}(i,j \mid r).$$

By inverting, integrating and using specific completeness relations as identified in the literature, we obtain

$$\Psi_{nl}^{(\pm)}(i,j \mid r) = \frac{\kappa}{2\pi}\frac{2^{i-1}(-1)^j}{\kappa^{2i\pm2j}r^{2\pm2j}\Gamma(1\pm j)}\int_0^{2\pi/\kappa}(1-\cos\kappa t)^{i\pm j}e^{-in\kappa t}(z^2-1)^{\mp j/2}Q_l^{\pm j}(z)dt.$$

Regularisation of the Direct Terms - $\Psi_{nl}^{[d]^{(+)}}$

By choosing a suitable value of $\eta(r)$, we can rewrite $Q_l(z)$ without a time dependency by using the theorem [Gradshteyn, 2007]

$$Q_l(z) = P_l(\eta)Q_l(\eta) + 2\sum_{p=1}^{\infty}(-1)^p P_l^{-p}(\eta)Q_l^p(\eta)\cos p\kappa t,$$

and calculate

$$\Psi_{nl}^{(+)}(i,j \mid r) = \frac{2^{i-j-1}}{\kappa^{2i+2j}r^{2+2j}\Gamma(1+j)}\sum_{m=0}^{i}\sum_{s=0}^{m}\frac{i!}{s!(i-m)!(m-s)!}\frac{(-1)^m}{2^m}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j \times\Big[P_l(\eta)Q_l(\eta)\delta_{m,2s+n} + (-1)^{(2s+n-m)}P_l^{(m-2s-n)}(\eta)Q_l^{(2s+n-m)}(\eta) + (-1)^{(m-2s-n)}P_l^{(2s+n-m)}(\eta)Q_l^{(m-2s-n)}(\eta)\Big].$$

Regularisation of the Direct Terms - $\Psi_{nl}^{[d]^{(-)}}$

Employing the identity [Gradshteyn, 2007]

$$(z^2-1)^{j/2} Q_l^{-j} = \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4l+1}{(l-k+\frac{1}{2})_{j+1}} Q_{l+j-2k}(z)$$

and performing similar calculations, we obtain

$$\Psi_{nl}^{(-)}(i,j \mid r) = \frac{\kappa}{2\pi} \frac{2^{i-1}(-1)^j}{\kappa^{2i-2j} r^{2-2j} \Gamma(1-j)} \sum_{k=0}^{j} \frac{(-1)^k}{2^{j+1}} \binom{j}{k} \frac{2l+2j-4k+1}{(l-k+\frac{1}{2})_{j+1}}$$
$$\times \sum_{m=0}^{i-j} \sum_{s=0}^{m} \frac{(i-j)!(-1)^m}{s!(i-j-m)!(m-s)!2^m} \Big[P_{l+j-2k}(\eta) Q_{l+j-2k}(\eta) \delta_{m,2s+n} + (-1)^{(2s+n-m)}$$
$$\times P_{l+j-2k}^{(m-n-2s)}(\eta) Q_{l+j-2k}^{(2s+n-m)}(\eta) + (-1)^{(m-2s-n)} P_{l+j-2k}^{(2s+n-m)}(\eta) Q_{l+j-2k}^{(m-2s-n)}(\eta) \Big].$$

Regularisation of the Tail Terms

We assume an expansion of the form

$$s^{2i-2j} w^{2j} \log\left(\frac{s^2}{l^2}\right) = \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} (2l+1) P_l \cos\gamma \chi_{nl}(i,j|r)$$

and using completeness relations we can represent the tail regularisation parameters [Taylor and Breen, 2017] as

$$\chi_{nl}(i,j|r) = \frac{\kappa}{4\pi} \int_0^{2\pi/\kappa} \int_{-1}^{1} w^{2j} e^{-in\kappa\Delta\tau} \log\left(\frac{s^2}{l^2}\right) s^{2i-2j} P_l(x) dx d\Delta\tau.$$

Regularisation of the Tail Terms - $l > k$

We may then make use of some unique identities and find that (where $\alpha = l + i - j - 2k + 1$):

$$\chi_{nl}(i,j|r) = \frac{2^{j-1} r^{2i-2j}(i-j)!}{\kappa^{2j}} \sum_{k=0}^{i-j+1} (-1)^k \binom{i-j+1}{k} \frac{(\alpha + \frac{1}{2})}{(l-k+\frac{1}{2})_{i-j+2}}$$
$$\times \sum_{m=0}^{j} \sum_{s=0}^{m} \frac{j!}{s!(j-m)!(m-s)!} \frac{(-1)^m}{2^m} \Big[P_\alpha(\eta) Q_\alpha(\eta) \delta_{m,2s+n}$$
$$+ P_\alpha^{(m-n-2s)}(\eta) Q_\alpha^{(2s+n-m)}(\eta) + (-1)^n P_\alpha^{(2s+n-m)}(\eta) Q_\alpha^{(m-2s-n)}(\eta) \Big].$$

Regularisation of the Tail Terms - $l \leq k$

With "Olver's definition of the associated Legendre function of the second kind" $\boldsymbol{\mathcal{Q}}(z)$ defined as

$$\boldsymbol{\mathcal{Q}}_l^{-\lambda-1}(z) = \frac{e^{(\lambda+1)\pi i}\, Q_l^{-\lambda-1}(z)}{\Gamma(l-\lambda)}$$

rather than the standard $Q(z)$, we then obtain for the case where $l \leq k$

$$\chi_{nl}(i,j|r) = \frac{\kappa}{2\pi}\left(\frac{2}{\kappa^2}\right)^j (2r)^{i-j}(-1)^l \Big[\frac{d}{d\lambda}(\lambda+1-l)_l \left(\frac{2r^2}{\ell^2}\right)^{\lambda-i+j} \int_0^{2\pi/\kappa} (1-\cos\kappa\tau)^j e^{-in\kappa\Delta\tau}$$
$$\times\, (z^2-1)^{\frac{1}{2}(\lambda+1)} \boldsymbol{\mathcal{Q}}_l^{-\lambda-1}(z)\, d\Delta\tau\Big]_{\lambda=i-j}.$$

for $l \leq k$.

Mode-Sum Prescription of Singular Green Function

The mode-sum description of the singular Green function is then given by

$$
\begin{aligned}
G_s(x, x^{'}) = \frac{1}{8\pi^2} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) \Bigg\{ \sum_{i=0}^{2} \sum_{j=0}^{i} \mathcal{D}_{ij}^{(+)}(r) \Psi_{nl}^{(+)}(i, j \mid r) \\
+ \sum_{i=0}^{2} \sum_{j=0}^{i} \mathcal{D}_{ij}^{(-)}(r) \Psi_{nl}^{(-)}(i, j \mid r) + \sum_{i=0}^{1} \sum_{j=0}^{i} \mathcal{T}_{ij}^{(l)}(r) \chi_{nl}(i, j|r) \\
+ \sum_{j=0}^{1} \mathcal{T}_{1j}^{(p)}(r) \chi_{nl}(1, j|r) + \mathcal{T}_{10}^{(r)}(r) \Psi_{nl}^{(-)}(2, 0 \mid r) \Bigg\} + \mathcal{O}(\epsilon^4 \log \epsilon).
\end{aligned}
$$

The Renormalised Green Function . . . (1)

We may now define the renormalised Green function as

$$G_{ren}(x, x') = \lim_{x \to x'} \left(G_E(x, x') - G_s(x, x')\right)$$

and the diagonal components of the point-split renormalised expectation value of the energy-stress tensor are given by as

$$\begin{aligned}\langle T^{\nu}{}_{\nu}\rangle_{ren} = (1-2\xi)g^{\nu\nu'}G_{ren\,;\nu\nu'} + (2\xi - \tfrac{1}{2})g^{\alpha\alpha'}G_{ren\,;\alpha\alpha'} - 2\xi g^{\nu\nu}G_{ren\,;\nu\nu} \\ + 2\xi g^{\alpha\alpha}G_{ren\,;\alpha\alpha} + \xi(R^{\nu}{}_{\nu} - \tfrac{1}{2}R)G_{ren} - \frac{m^2}{2}G_{ren} + \frac{2v_1}{8\pi^2}\end{aligned}$$

noting that this is not a summation over ν. v_1 is a function of R_{abcd}, R, g_{ab}, ξ, and m.

The Renormalised Green Function . . . (2)

In the coincidence limit during the calculation of the renormalised Green function, $\Delta\tau \to 0$ and $\gamma \to 0$, thus $e^{in\kappa\Delta\tau} \to 1$ and $P_l(\cos\gamma) \to 1$, and we obtain

$$G_{ren} = \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\Bigg\{N_{nl}p_{nl}(r)q_{nl}(r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(+)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j\mid r) - \sum_{i=0}^{2}\sum_{j=0}^{i}\mathcal{D}_{ij}^{(-)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j\mid r) - \sum_{i=0}^{1}\sum_{j=0}^{i}\mathcal{T}_{ij}^{(l)}(r)\,\overset{[d]}{\chi}{}_{nl}(i,j|r) - \sum_{j=0}^{1}\mathcal{T}_{1j}^{(p)}(r)\,\overset{[d]}{\chi}{}_{nl}(1,j|r) - \mathcal{T}_{10}^{(r)}(r)\,\overset{[d]}{\Psi}{}_{nl}^{(-)}(2,0\mid r)\Bigg\}.$$

Derivatives of the Renormalised Green Function . . . (1)

The following derivatives are then calculated in the dissertation, which is an extension of the previous work and a necessary step towards the renormalised stress-energy tensor:

$$g^{tt'} G_{ren\,;tt'}$$
$$g^{tt} G_{ren\,;tt}$$
$$g^{rr} G_{ren\,;rr}$$
$$g^{\phi\phi'} G_{ren\,;\phi\phi'}$$
$$g^{\phi\phi} G_{ren\,;\phi\phi}$$
$$g^{\theta\theta} G_{ren\,;\theta\theta}$$
$$g^{\theta\theta'} G_{ren\,;\theta\theta'}$$

Note that we cannot calculate $g^{rr'} G_{ren\,;rr'}$, as earlier in the expansion we set $\Delta r = 0$. We select $f(r) = 1 - \frac{2M}{r}$ and calculate these derivatives explicitly.

Derivatives of the Renormalised Green Function . . . (2)

We also calculate the following derivatives of the direct and tail coefficient terms, explicitly for our choice of $f(r)$:

$$\frac{\partial \mathcal{D}_{ij}^{(+)}(r)}{\partial r}$$

$$\frac{\partial \mathcal{D}_{ij}^{(-)}(r)}{\partial r}$$

$$\frac{\partial \mathcal{T}_{ij}^{(r)}(r)}{\partial r}$$

$$\frac{\partial \mathcal{T}_{ij}^{(l)}(r)}{\partial r} = 0$$

$$\frac{\partial \mathcal{T}_{ij}^{(p)}(r)}{\partial r}$$

and the derivatives of the regularisation terms:

$$\frac{\partial \overset{[d]}{\Psi}{}_{nl}^{(+)}(i,j|r)}{\partial r}$$

$$\frac{\partial \overset{[d]}{\Psi}{}_{nl}^{(-)}(i,j|r)}{\partial r}$$

$$\frac{\partial \overset{[d]}{\chi}{}_{nl}(i,j|r)}{\partial r}, \; l > k$$

$$\frac{\partial \overset{[d]}{\chi}{}_{nl}(i,j|r)}{\partial r}, \; l \leq k$$

Conclusion

- We sought to approximate a full theory of quantum gravity via semiclassical gravity.

The Hadamard Parametrix for the Singular Propagator

$$G_s = \frac{1}{8\pi^2}\left(\frac{U}{\sigma} + V \log\left(\frac{2\sigma}{\ell^2}\right)\right)$$

- We worked through Taylor and Breen's novel expansion of σ using *extended coordinates* w and s, with $\Delta r = 0$.
- We calculated a mode sum over modes n and l for G_s, classifying the terms as either **direct** (from $\frac{U}{\sigma}$) or **tail** (from $V \log\left(\frac{2\sigma}{\ell^2}\right)$), keeping track of the order of ϵ and of its form as a function of (w, s).
- By assuming Fourier frequency and and multipole decompositions, we were able to derive **regularisation** terms for both direct and tail components of the expansion.
- Our mode-sum expansion was then a product of direct and tail coefficients and regularisation terms, which could then be subtracted from the Euclidean mode-sum.
- Worked towards a calculation of $\langle T_{\mu\nu}\rangle_{ren}$, by calculating some of the derivatives $g^{\mu\nu} G_{ren\,;\mu\nu}$ and $g^{\mu\nu'} G_{ren\,;\mu\nu'}$.

Further Work

This work was a logical extension of the work of Taylor and Breen, working towards a renormalised stress-energy tensor rather than the renormalised vacuum polarisation value in [Taylor and Breen, 2017]. Some of the next steps required for a calculation of the stress-energy tensor would include

- Calculating $g^{rr'} G_{ren\,;rr'}$.
- Calculating the Riemann tensor $R^{a}{}_{bcd}$ and Ricci scalar R.
- Having calculated a renormalised stress-energy tensor, the next step would be to solve the semi-classical Einstein field equations, hence finding a first order approximation to quantum gravity.

Thank you for your time.

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