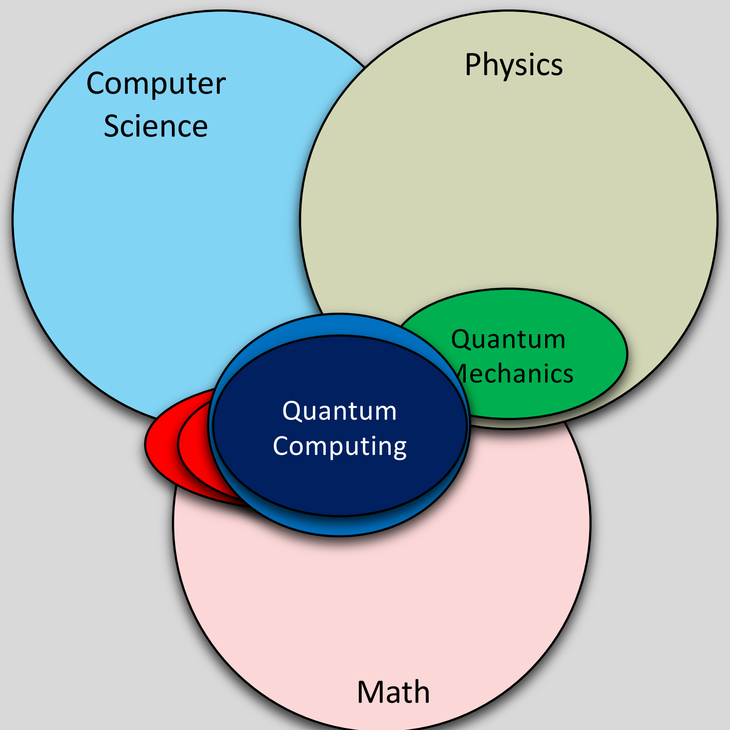
Quantum Computing

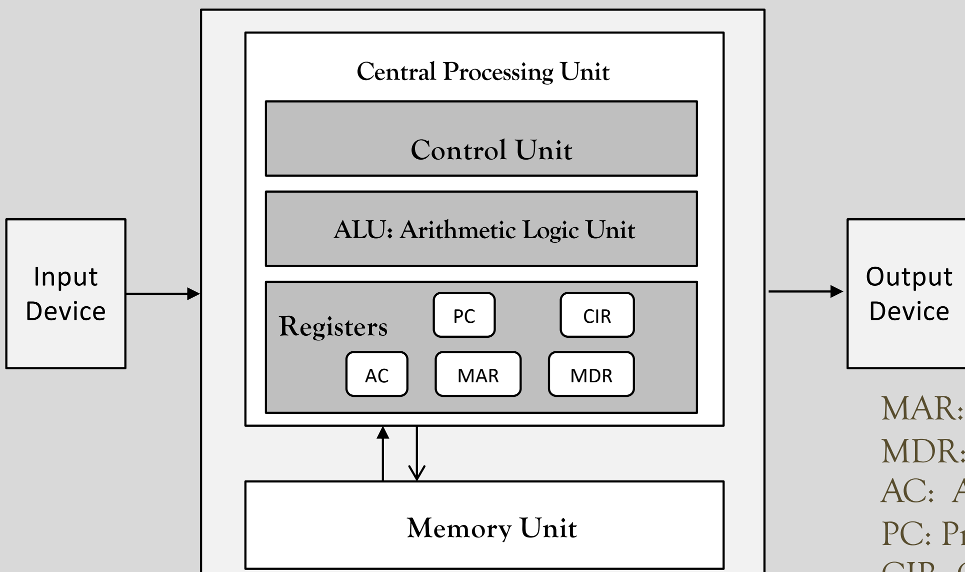
A first approximation from computer science and engineering

Quantum Information Science



Classical Computing
(application-centric approach – Von Neumann architecture)

- An automated calculator,
  - To estimate ballistic trajectories,

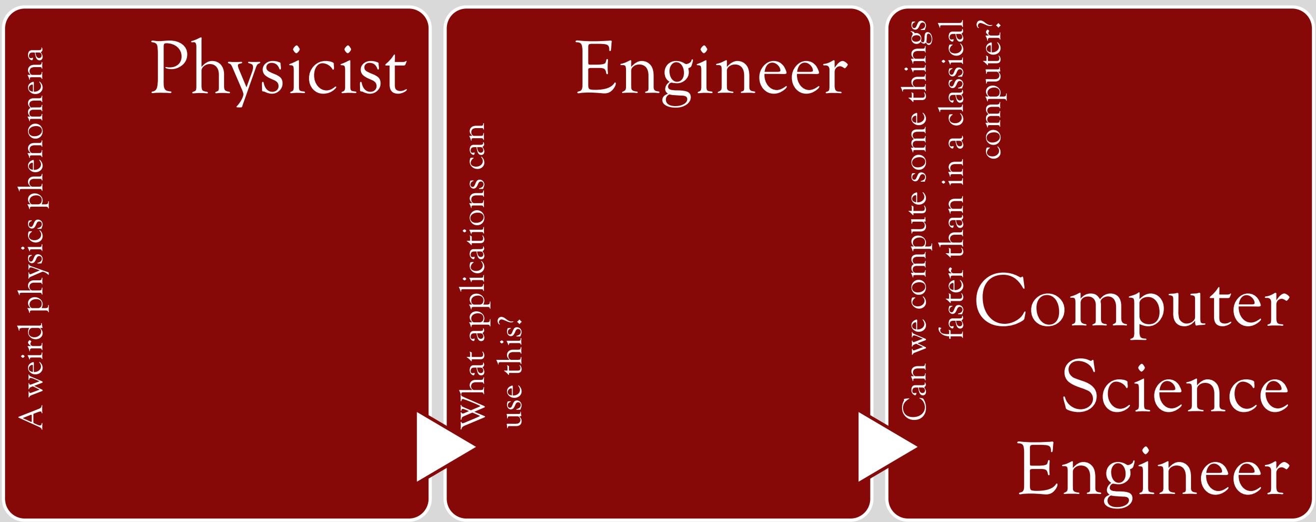


MAR: Memory Address Register
MDR: Memory Data Register
AC: Acumulator
PC: Program Counter
CIR: Current Instruction Register

Classical Computing (application-centric approach)

- Improvements:
  - Faster hardware,
  - Optimized hardware,
  - Better software,
  - Optimized Software for optimized hardware,
  - Storage.
- Wide application spectrum:
  - Accounting, AI, time-waste, science

Quantum Computing
(technology-centric approach)

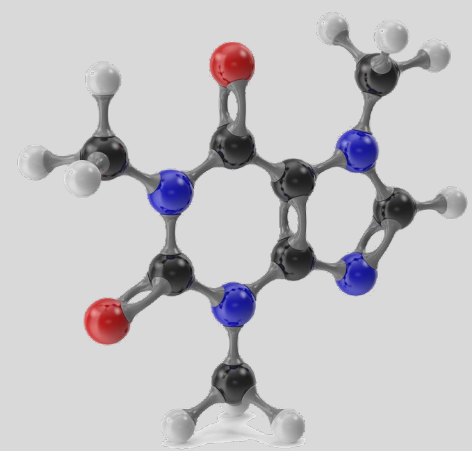


Limits on Classical Computing

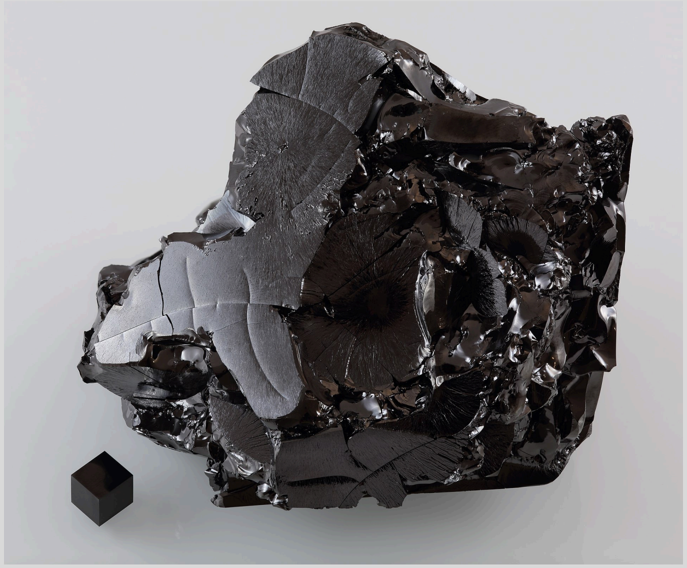
- Yes, classical computers are faster than humans,
- Many complex calculations are –however- intractable with classical machines,
  - Thanks to that, we have cryptography,
  - But there are some natural problems interesting to us:
    - Chemical simulations for improved materials, medicines, vaccines, etc.
      - Search space is exponential, even small molecules, can't be solved with classical computers

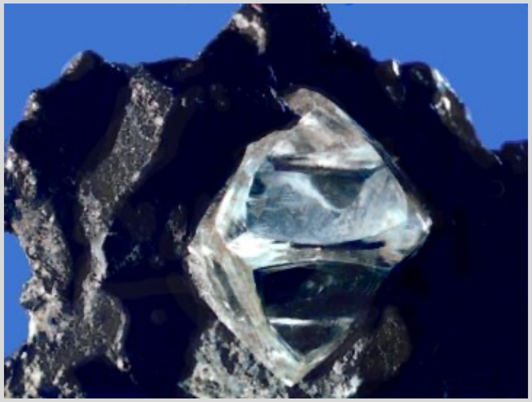
Limits on Classical Computing (.)

- Caffeine ($C_8H_{10}N_4O_2$):
  - *24* atoms – *10^{48}* bits to describe all possible energy arrangements,
  - *160* qubits should be enough:
    - *2^{160}=1.46x10^{48}*,
- And caffeine is a pretty simple molecule.

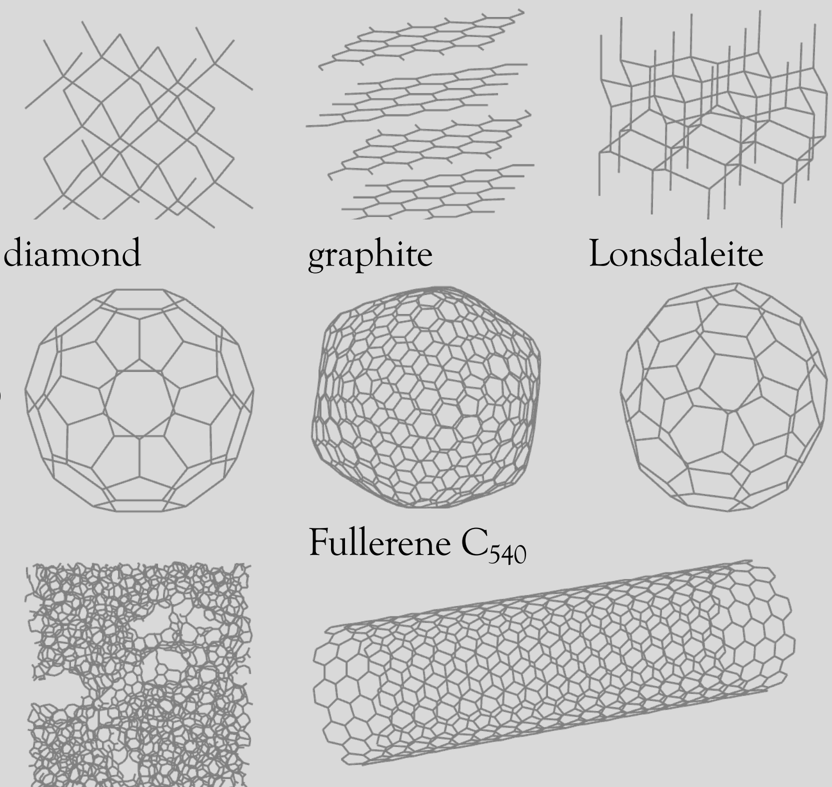


Science of Materials (carbon)

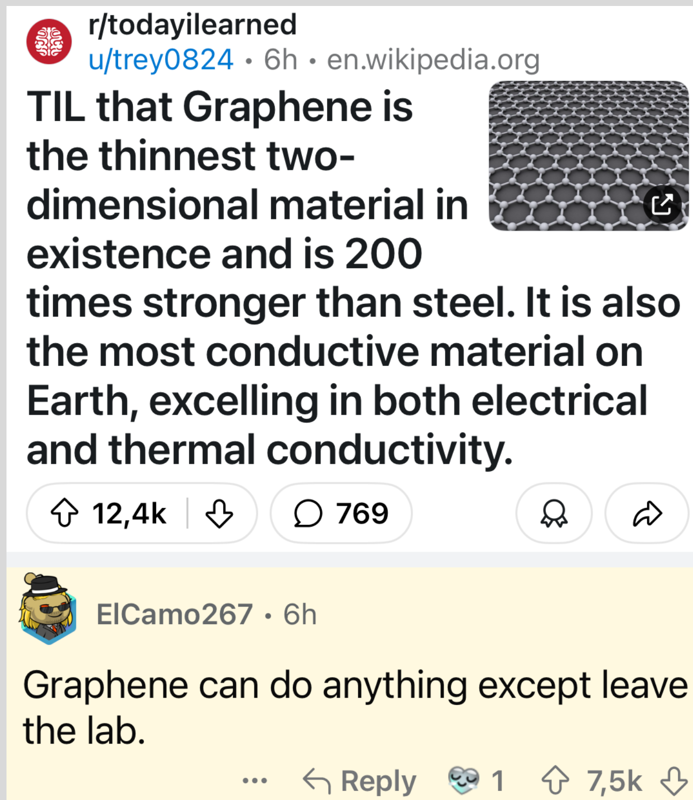




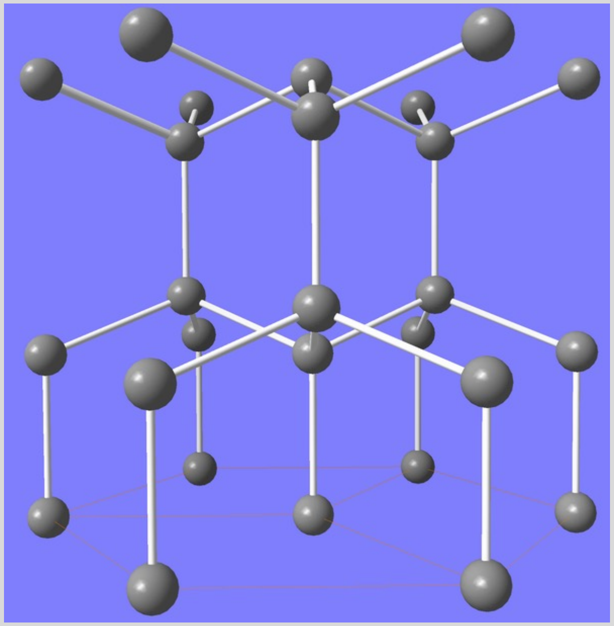
Science of Materials (carbon allotropes)



Science of Materials (carbon allotropes)(.)

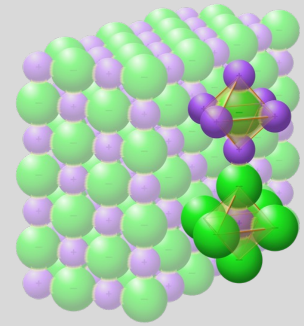


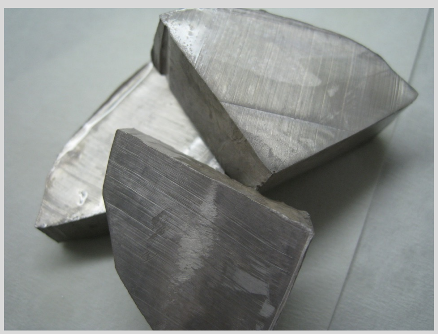
Science of Materials (carbon allotropes)(Lonsdaleite)



- Hexagonal lattice,
  - Diamond is a cubic lattice,
- Discovered in 1967,
- Harder than diamond (58% harder),

Science of Materials
(Natrium + Chloride = Salt)





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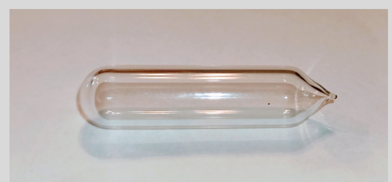

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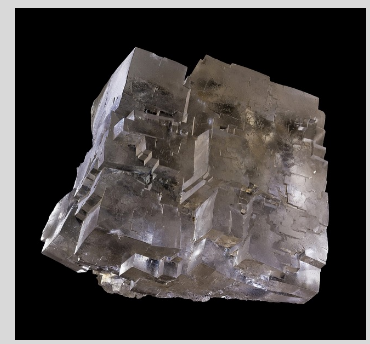
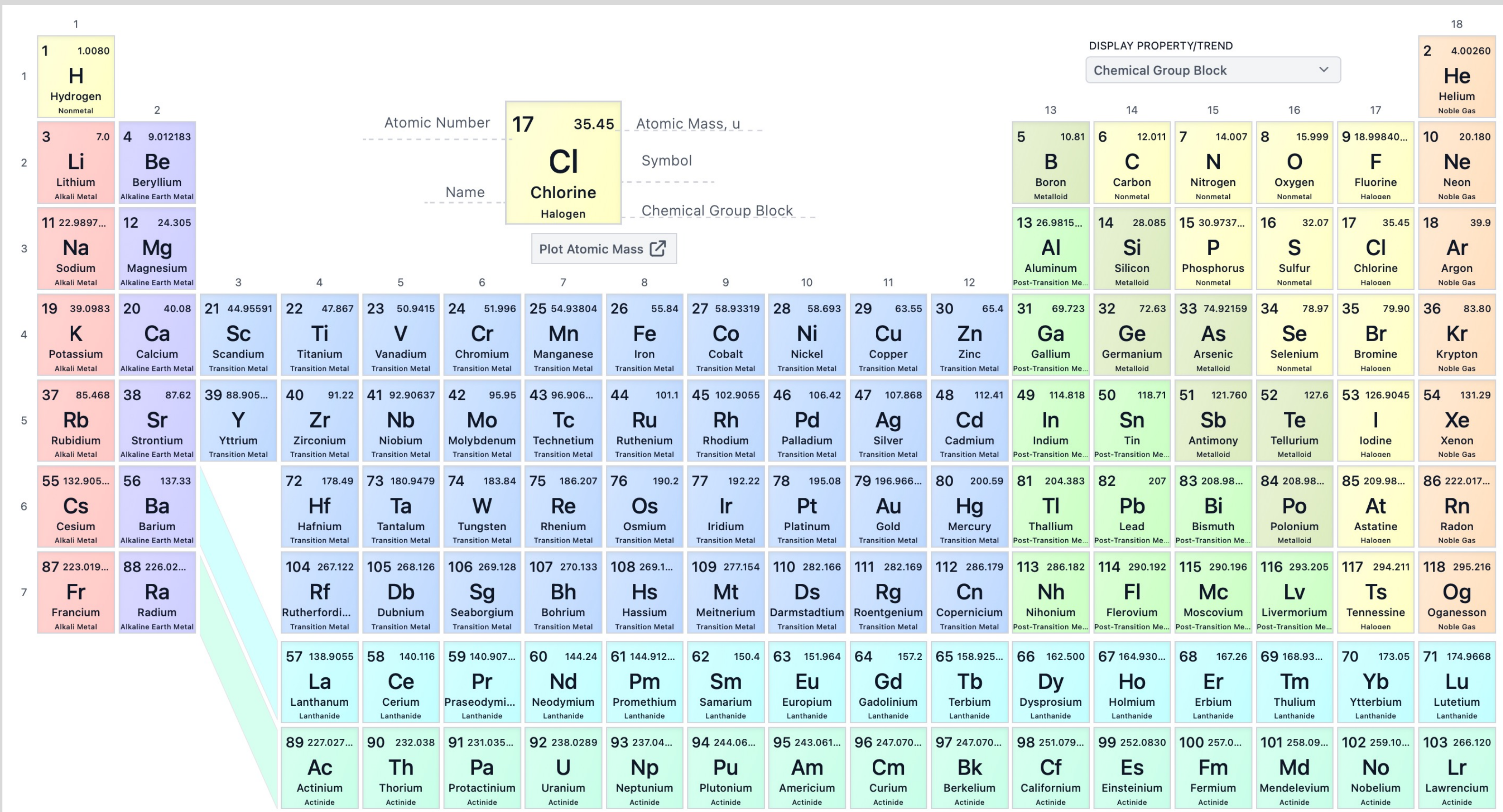


Reactive, metal, soft, poisonous, good conductor

Reactive, corrosive, toxic

Harmless, biologically essential.

Science of Materials



Science of Materials

electrek

BYD SODIUM-ION BATTERY

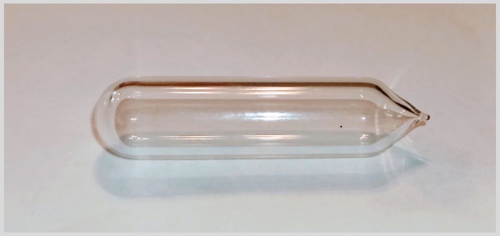
BYD breaks ground on its first sodium-ion EV battery plant

Peter Johnson | Jan 5 2024 - 11:41 am PT | 60 Comments

The world's largest EV maker, BYD, broke ground on its first sodium-ion battery plant this week. BYD is investing $1.4 billion (RMB 10 billion) with 30 GWh planned annual capacity.

You likely heard that BYD just topped Tesla in overall EV volume to become the largest electric car maker globally. However, BYD is also a top global battery manufacturer.

Science of Materials (Hydrogen + Oxygen = Water)



Explosive gas

+

Promotes violent combustion

=

Liquid... puts out fires

Energy in batteries

AI already uses as much energy as a small country. It's only the beginning.

The energy needed to support data storage is expected to double by 2026. You can do something to stop it.

By Brian Calvert | Mar 28, 2024, 8:00am EDT

SHARE

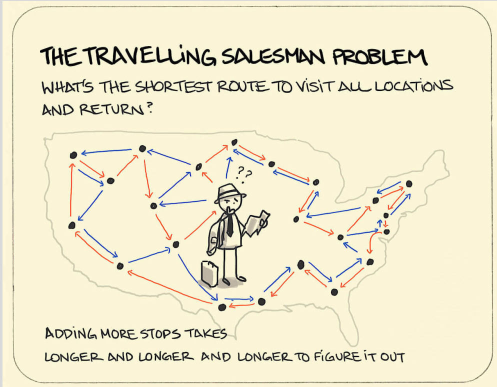
Paige Vickers/Vox; Getty Images

***Brian Calvert** is an environmental journalist currently based in Pacifica, California.*

The travelling salesman problem (TSP)

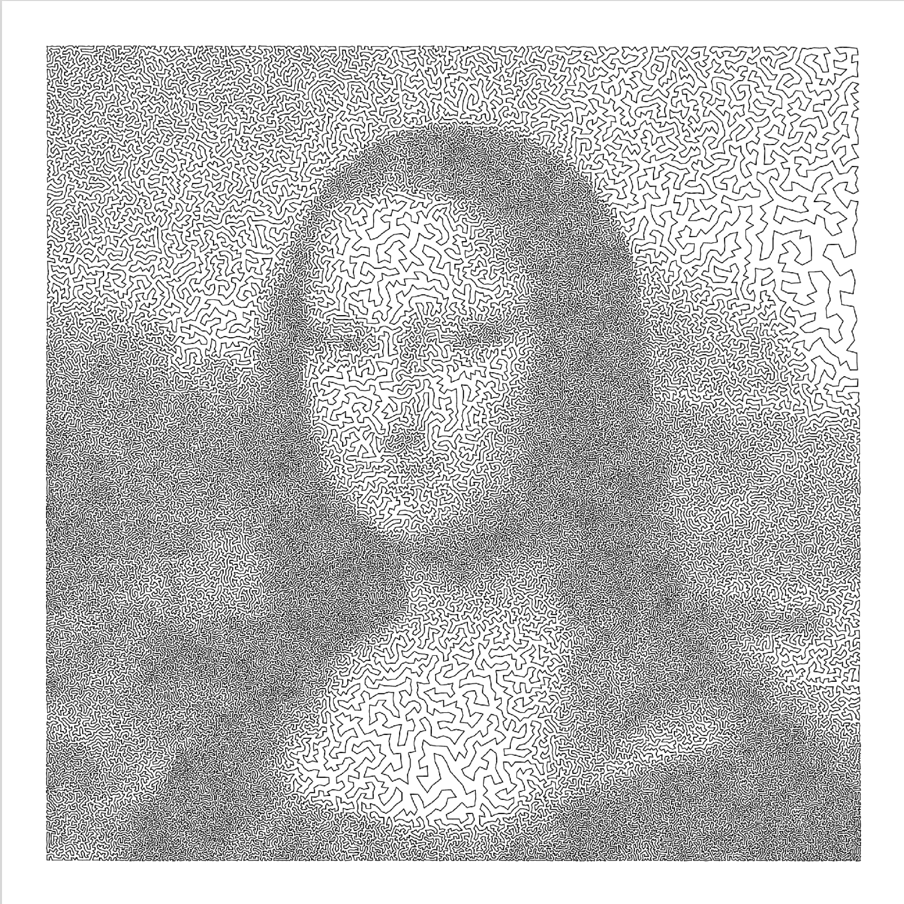
- Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?
- Suppose you want to visit coffee-shop, bank and drugstore:
  - Coffee-shop, drugstore, bank,
  - Coffee-shop, bank, drugstore,
  - Bank, coffee-shop, drugstore,
  - Bank, drugstore, coffee-shop,
  - Drugstore, bank, coffee-shop,
  - Drugstore, coffee-shop, bank.
- With 20 stops, you will have *20! = 2430'000,000'000,000'000* possible solutions.
- UPS drivers never turn-left.

3 x 2 = 6 = 3!



https://theconversation.com/why-ups-drivers-dont-turn-left-and-you-probably-shouldnt-either-71432

TSP Art



Development of Quantum Computers

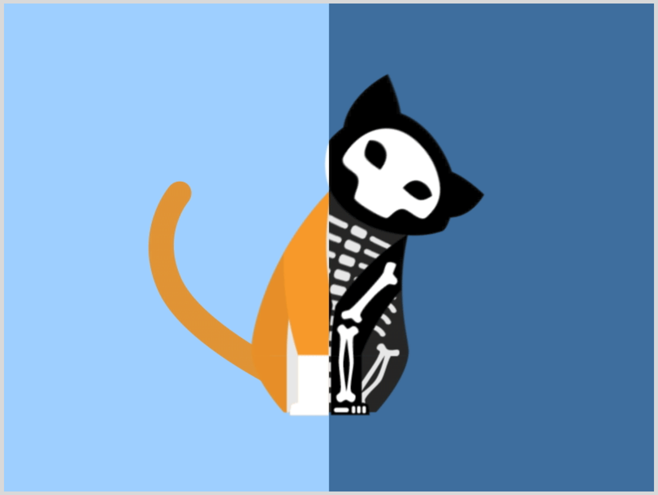
- Were not designed to mimic how humans perform tasks,
- They use unique features that can be harnessed for some computations that are lengthy when done like humans,
- A solution in search of a problem,
- We don’t care about “Copenhagen interpretation” or “many worlds interpretation”,
  - Our interpretation is plain and simple:
    - “shut-up and calculate”.

A physics algorithm for chemistry and biology

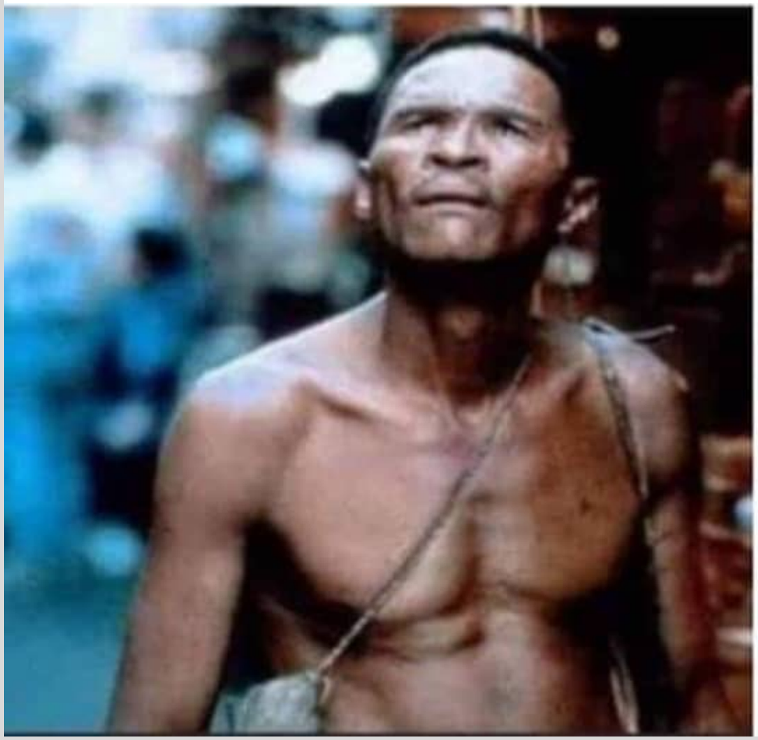
1. Start with the wave function Ψ(x), which describes an electron located at the point x.

2. Insert the wave into the Schrödinger equation HΨ(x) = i(*h*/2π) ∂_tΨ(x). (H, which is known as the Hamiltonian, corresponds to the energy of the system.)
3. Each solution of this equation is labeled an index n, so in general, Ψ(x) is a sum or superposition of all these multiple states.
4. When a measurement is made, the wave function "collapses," leaving only one state $\Psi(x)_n$, i.e., all the other waves are set to zero. The probability of finding the electron in this state is given by the absolute value of $\Psi(x)_n$.

$$\imath\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle = \hat{H}\left|\Psi\right\rangle$$



Estudiante de cualquier ingeniería en clase de QC



Estudiante de informática en clase de QC

Summary

- The differences are at the most basic level,
- We don't yet have languages that hide those basic details,
- To understand how quantum computers perform computation, we need to start with the basic quantum operations,
- This is very different from how people learn to program classical computers today.

Preliminaries

Math, physics, philosophy, computability and complexity... may be you can stay at the main road

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$$
$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$$
$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$
$$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}$$

$$\sin(-\alpha) = -\sin(\alpha)$$
$$\cos(-\alpha) = \cos(\alpha)$$

$$\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$
$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$
$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$
$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Some trigonometry

Useful to have in hand...

The path to the most beautiful equation

Let the function $f(\theta) = (\cos\theta + \imath\sin\theta)\, e^{-\imath\theta}$, with $\quad \theta \in \mathbb{R}$.

$$\frac{df(\theta)}{d\theta} = e^{-\imath\theta}(-\sin\theta + \imath\cos\theta) - \imath e^{-\imath\theta}(\cos\theta + \imath\sin\theta) = e^{-\imath\theta}\left(-\sin\theta + \imath\cos\theta - \imath\cos\theta - \imath^2\sin\theta\right) = 0,$$

So $f(\theta)$ is a constant. Let's calculate $f(0)$

$$f(0) = (1 + \imath 0)\, e^0 = 1.$$

So $f(\theta)$ as a constant is always 1, so $e^{\imath\theta} = \cos\theta + \imath\sin\theta$.

With $\theta = \pi, \quad e^{\imath\pi} = -1, \quad e^{\imath\pi} + 1 = 0.$

$$e^{\imath\pi} + 1 = 0$$

$$e^{\imath\theta} = \cos\theta + \imath\sin\theta$$

- You can also prove it using Taylor series:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$e^{\imath\theta} = 1 + \frac{\imath\theta}{1!} + \frac{(\imath\theta)^2}{2!} + \frac{(\imath\theta)^3}{3!} + \frac{(\imath\theta)^4}{4!} + \frac{(\imath\theta)^5}{5!} + \frac{(\imath\theta)^6}{6!} + \frac{(\imath\theta)^7}{7!} + \ldots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + \imath\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right) = \cos\theta + \imath\sin\theta$$

- In home you can extend this property to square matrices **A**:

$$e^{\imath\mathbf{A}} = \sum_{n=0}^{\infty}\frac{(\imath\mathbf{A})^n}{n!}, \qquad \sin(\imath\mathbf{A}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\imath\mathbf{A})^{2n+1}, \qquad \cos(\imath\mathbf{A}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\imath\mathbf{A})^{2n} \qquad\qquad e^{\imath\mathbf{A}} = \cos\mathbf{A} + \imath\sin\mathbf{A}$$

- Also: $e^{\imath\pi\mathbf{I}} + \mathbf{I} = \mathbf{0}$
  - Hint: take the expansion with **A=πI** to get:

$$e^{\imath\pi\mathbf{I}} = \frac{\imath^0\pi^0\mathbf{I}^0}{0!} + \frac{\imath\pi\mathbf{I}}{1!} + \frac{(\imath\pi\mathbf{I})^2}{2!} + \frac{(\imath\pi\mathbf{I})^3}{3!} + \frac{(\imath\pi\mathbf{I})^4}{4!} + \frac{(\imath\pi\mathbf{I})^5}{5!} + \frac{(\imath\pi\mathbf{I})^6}{6!} + \frac{(\imath\pi\mathbf{I})^7}{7!} + \ldots = \mathbf{I}\left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \ldots\right) + \imath\mathbf{I}\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots\right) = \mathbf{I}\cancelto{-1}{\cos\pi} + \imath\mathbf{I}\cancelto{0}{\sin\pi} = -\mathbf{I}.$$

And keep following...

- Remember Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

- Suppose a matrix B such that $\mathbf{A} = \theta \mathbf{B}$ and $\begin{cases} \mathbf{B}^1 = \mathbf{B}, \\ \mathbf{B}^2 = \mathbf{I}, \\ \mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \mathbf{I}\mathbf{B} = \mathbf{B} \\ \mathbf{B}^4 = \mathbf{B}^3\mathbf{B} = \mathbf{B}\mathbf{B} = \mathbf{I} \end{cases}$

$$e^{\imath\theta\mathbf{B}} = \frac{(\imath\theta\mathbf{B})^0}{0!} + \frac{(\imath\theta\mathbf{B})^1}{1!} + \frac{(\imath\theta\mathbf{B})^2}{2!} + \frac{(\imath\theta\mathbf{B})^3}{3!} + \frac{(\imath\theta\mathbf{B})^4}{4!} + \frac{(\imath\theta\mathbf{B})^5}{5!} + \frac{(\imath\theta\mathbf{B})^6}{6!} + \frac{(\imath\theta\mathbf{B})^7}{7!} + \frac{(\imath\theta\mathbf{B})^8}{8!} + \frac{(\imath\theta\mathbf{B})^9}{9!} + \frac{(\imath\theta\mathbf{B})^{10}}{10!} + \frac{(\imath\theta\mathbf{B})^{11}}{11!} + \ldots$$

$$= \mathbf{I} + \frac{\imath\theta\mathbf{B}}{1!} + \frac{\imath^2\theta^2\mathbf{I}}{2!} + \frac{\imath^3\theta^3\mathbf{B}}{3!} + \frac{\imath^4\theta^4\mathbf{I}}{4!} + \frac{\imath^5\theta^5\mathbf{B}}{5!} + \frac{\imath^6\theta^6\mathbf{I}}{6!} + \frac{\imath^7\theta^7\mathbf{B}}{7!} + \frac{\imath^8\theta^8\mathbf{I}}{8!} + \frac{\imath^9\theta^9\mathbf{B}}{9!} + \frac{\imath^{10}\theta^{10}\mathbf{I}}{10!} + \frac{\imath^{11}\theta^{11}\mathbf{B}}{11!} + \ldots$$

$$e^{\imath\theta\mathbf{B}} = \mathbf{I}\underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \ldots\right)}_{\cos\theta} + \imath\mathbf{B}\underbrace{\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!}\ldots\right)}_{\sin\theta}$$

$$= \mathbf{I}\cos\theta + \imath\mathbf{B}\sin\theta$$

$$e^{\pm\imath\theta\mathbf{B}} = \mathbf{I}\cos\theta \pm \imath\mathbf{B}\sin\theta.$$

And keep following more...

- Remember Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

- Suppose the matrix

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{J}^0 = \mathbf{I}$$

$$\mathbf{J}^1 = \mathbf{J}$$

$$\mathbf{J}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$$

$$\mathbf{J}^3 = \mathbf{J}^2\mathbf{J} = -\mathbf{I}\mathbf{J} = -\mathbf{J}$$

$$\begin{cases} \mathbf{J}^4 = \mathbf{J}^3\mathbf{J} = -\mathbf{J}\mathbf{J} = -\mathbf{J}^2 = \mathbf{I} \\ \mathbf{J}^5 = \mathbf{J}^4\mathbf{J} = \mathbf{I}\mathbf{J} = \mathbf{J} \\ \mathbf{J}^6 = \mathbf{J}^5\mathbf{J} = \mathbf{J}\mathbf{J} = -\mathbf{I} \\ \mathbf{J}^7 = \mathbf{J}^6\mathbf{J} = -\mathbf{I}\mathbf{J} = -\mathbf{J} \end{cases}$$

$$\begin{aligned} e^{\pi\mathbf{J}} &= \frac{\pi^0\mathbf{J}^0}{0!} + \frac{\pi^1\mathbf{J}^1}{1!} + \frac{\pi^2\mathbf{J}^2}{2!} + \frac{\pi^3\mathbf{J}^3}{3!} + \frac{\pi^4\mathbf{J}^4}{4!} + \frac{\pi^5\mathbf{J}^5}{5!} + \frac{\pi^6\mathbf{J}^6}{6!} + \frac{\pi^7\mathbf{J}^7}{7!} + \frac{\pi^8\mathbf{J}^8}{8!} + \frac{\pi^9\mathbf{J}^9}{9!} + \frac{\pi^{10}\mathbf{J}^{10}}{10!} + \frac{\pi^{11}\mathbf{J}^{11}}{11!} + \dots \\ &= \mathbf{I} + \pi\mathbf{J} - \frac{\pi^2\mathbf{I}}{2!} - \frac{\pi^3\mathbf{J}}{3!} + \frac{\pi^4\mathbf{I}}{4!} + \frac{\pi^5\mathbf{J}}{5!} - \frac{\pi^6\mathbf{I}}{6!} - \frac{\pi^7\mathbf{J}}{7!} + \frac{\pi^8\mathbf{I}}{8!} + \frac{\pi^9\mathbf{J}}{9!} - \frac{\pi^{10}\mathbf{I}}{10!} - \frac{\pi^{11}\mathbf{J}}{11!} + \dots \end{aligned}$$

$$= \mathbf{I}\underbrace{\left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \frac{\pi^{10}}{10!} + \dots\right)}_{\cos\pi\, =\, -1} + \mathbf{J}\underbrace{\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \frac{\pi^{11}}{11!} + \dots\right)}_{\sin\pi\, =\, 0} = -\mathbf{I}$$

$$e^{\pi\mathbf{J}} + \mathbf{I} = \mathbf{0} \qquad\qquad e^{\imath\pi} + 1 = 0$$

The matrix $\mathbf{J}$ acts as $\imath = \sqrt{-1}$ for matrices

And keep following more and more...

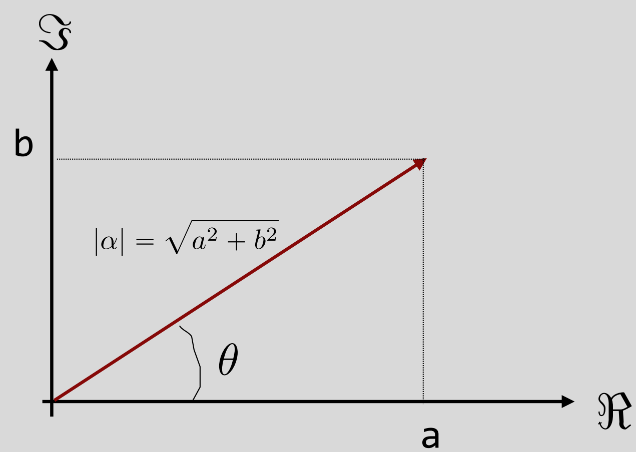
- Moivre's Theorem: $(\cos\theta + \imath\sin\theta)^n = \cos(n\theta) + \imath\sin(n\theta)$

- Proof: We know that $e^{\imath\theta} = \cos\theta + \imath\sin\theta,$ so

$$e^{n\imath\theta} = (\cos\theta + \imath\sin\theta)^n = \cos(n\theta) + \imath\sin(n\theta)$$

Complex Numbers

- A pair of real numbers, part imaginary, part real



$$\begin{aligned} \alpha &= a + \imath b \\ \imath^2 &= -1 \\ \Re(\alpha) &= a \\ \Im(\alpha) &= b \\ |\alpha| &= \sqrt{a^2+b^2} \end{aligned}$$

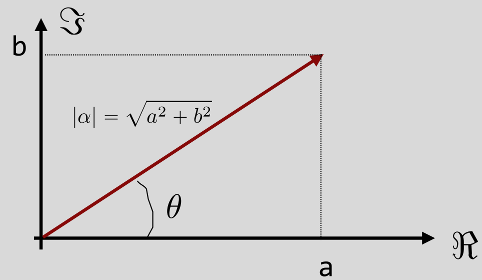
$$\alpha = a + \imath b \equiv \sqrt{a^2+b^2}\;\; e^{\imath\theta}$$

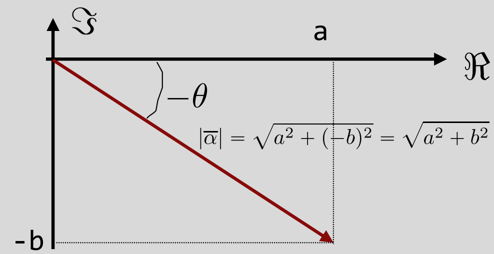
$$\sqrt{a^2+b^2}\;\; e^{\imath\theta} = \sqrt{a^2+b^2}\,(\cos\theta + \imath\sin\theta) = \cancel{\sqrt{a^2+b^2}}\left(\frac{a}{\cancel{\sqrt{a^2+b^2}}} + \imath\frac{b}{\cancel{\sqrt{a^2+b^2}}}\right) = a + \imath b$$

So, both notations are equivalent $\quad a + \imath b \equiv \sqrt{a^2+b^2}\;\; e^{\imath\theta}, \quad \text{with} \quad \theta = \arctan\left(\frac{b}{a}\right)$

Complex Numbers
(complex conjugate)

- If $\alpha = a + \imath b, \quad \overline{\alpha} = a - \imath b$





$$\alpha = a + \imath b \equiv \sqrt{a^2 + b^2}\ \ e^{\imath\theta}, \quad \text{with} \quad \theta = \arctan\left(\frac{b}{a}\right)$$

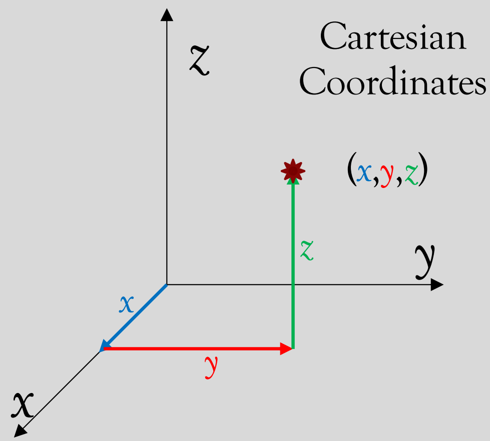
$$\overline{\alpha} = a - \imath b \equiv \sqrt{a^2 + b^2}\ \ e^{-\imath\theta}, \quad \text{with} \quad \theta = \arctan\left(\frac{-b}{a}\right) = -\arctan\left(\frac{b}{a}\right) = -\theta$$

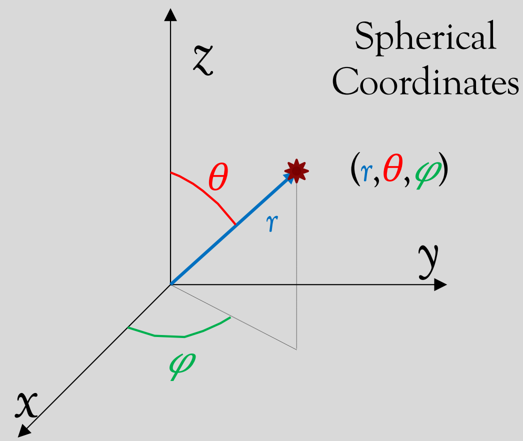
$$\alpha\ \overline{\alpha} = (a + \imath b)\,(a - \imath b) = a^2 - \imath^2 b^2 = a^2 + b^2$$

$$\alpha\ \overline{\alpha} = \left(\sqrt{a^2 + b^2}\ e^{\imath\theta}\right)\left(\sqrt{a^2 + b^2}\ e^{-\imath\theta}\right) = \left(a^2 + b^2\right) e^{\imath\theta - \imath\theta} = a^2 + b^2$$

Remember $\sin(-\theta) = -\sin(\theta), \quad \cos(-\theta) = \cos(\theta), \quad \tan(-\theta) = -\tan(\theta)$

3D: Cartesian and Spherical Coordinates





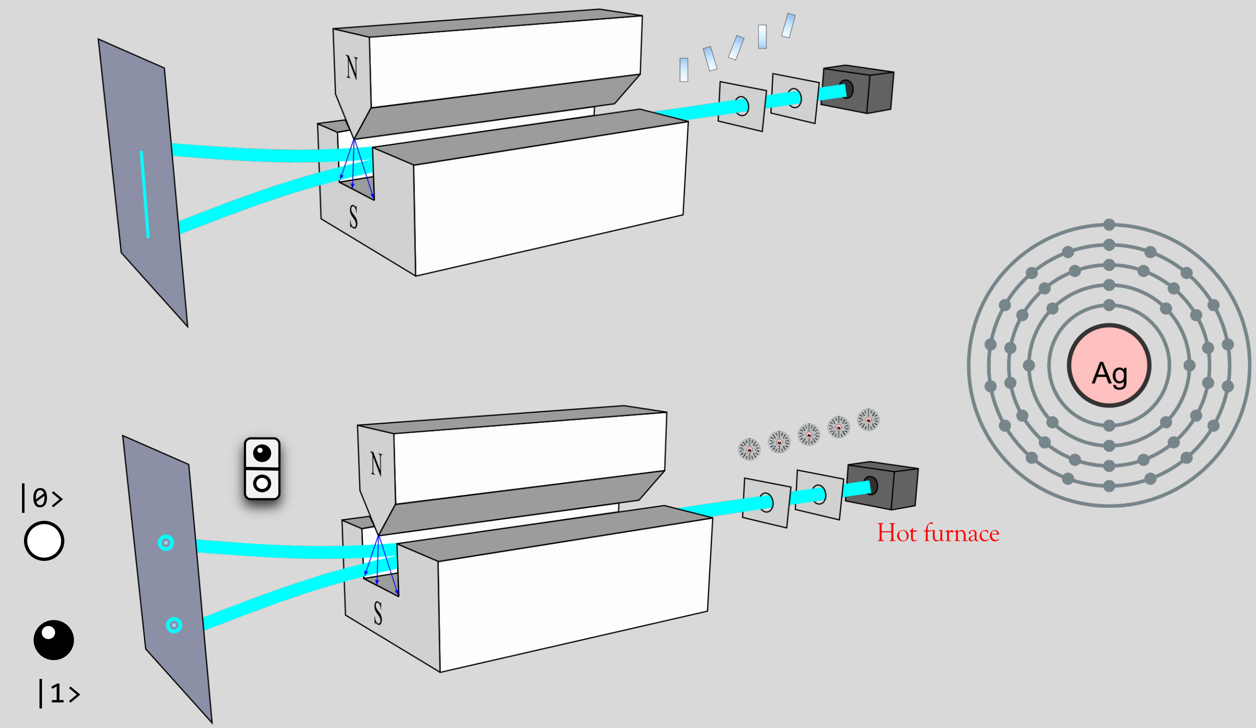
Cartesian to Spherical:

$$r^2 = x^2 + y^2 + z^2, \qquad \tan\phi = \frac{y}{x} \qquad \tan\theta = \frac{\sqrt{x^2+y^2}}{z}.$$

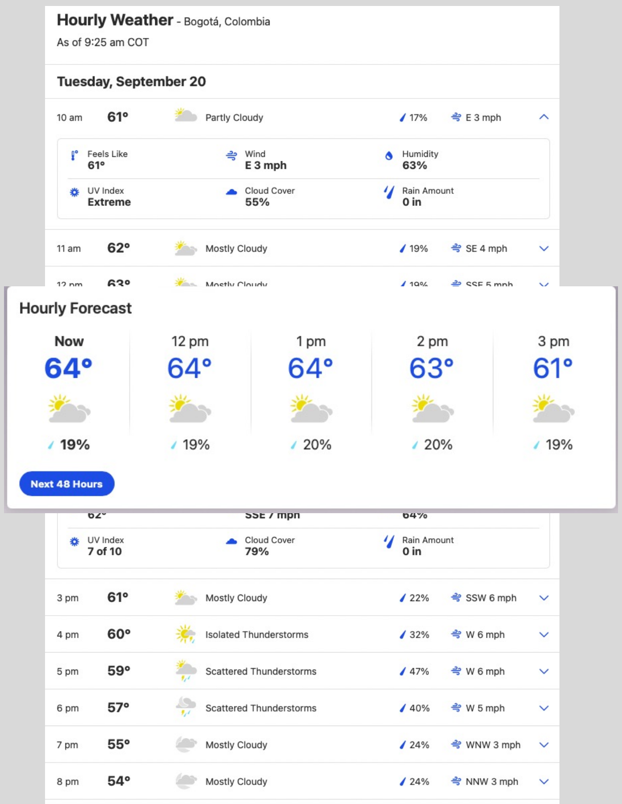
Spherical to Cartesian:

$$x = r\sin\theta\cos\theta, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

The Stern-Gerlach experiment



The Weather for Bogotá



- Will it rain at 2 pm? Sunny day?
  - Of 100,000 days with similar conditions to today, in average:
    - 20,000 days rain,
    - 80,000 days not rain,
- Probability is not about predicting,
  - Unless p=0, p=*1*,
- Question: if we choose a million bits at random and the first 999,999 of them happen to come out to be zero, what is the pro bability that the final digit is zero?

Probability in quantum computing

- Each qubit in superposition has a probability of being measured *0* or *1*,
  - It's not entirely `|0>` and not entirely `|1>`,
- To perform a computation we will need several qubits,
- Qubits starts with independent probabilities, but then they become multi-qubit combinations.

Random Thoughts
(or thoughts about the word *random*)

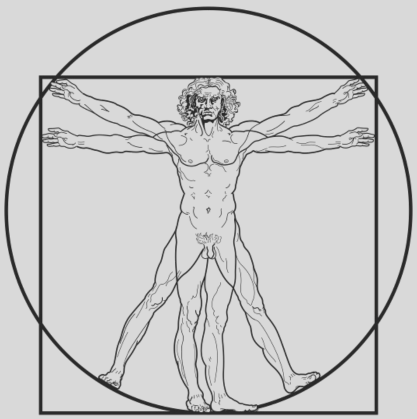
- A random comment (*un comentario al azar*),
  - Indicates surprise... I wasn't expecting the comment,
    - Of course, the comment's author is not surprised at all,
- Draw a card at random (*seleccione una carta al azar*),
  - Equal probability (low) for every card...
- Quantum measurements have random outcomes:
  - The outcome has a known probability,
  - The outcome of a single measurement is not guaranteed:
    - It's indeterminate or nondeterministic,
    - But the state of the system is changed by measurement!

Summary

- Probability is not:
  - A prediction about the outcome of an individual action,
  - Confirmed or refuted through a single experiment,
- Probability is:
  - A prediction of the frequency of an outcome of many, many actions,
  - Confirmed or refuted through many, many experiments,
- For an independent event, you can multiply to get the probability of the combined event.

What is measurement?

- Measurement consists of a **question**, a **device** and a **method**,
  - **Question**: What is this man height?
  - **Device**: metric tape,
  - **Method**: take one end of the tape, put on the floor and extend the device until reach the head,



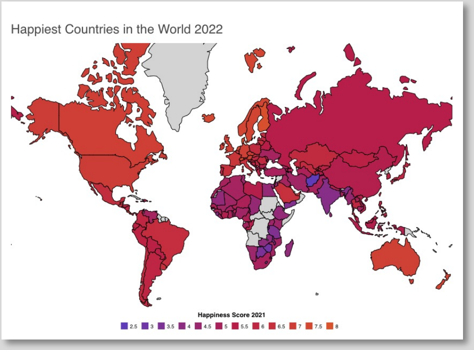
What is measurement? (.)

- Measurement consists of a **question**, a **device** and a **method**,
  - **Question**: How long can you hold your breath?
  - **Device:** stopwatch,
  - **Method**: start the stopwatch and stop when you breath,

Try a double measurement.
Some measurements affects the item being measured!

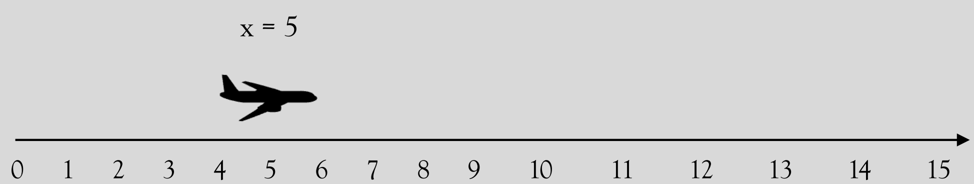
What is measurement? (..)

- Measurement consists of a **question**, a **device** and a **method**,
  - **Question**: What are the happiest countries in the world?
  - **Device:** polls,
  - **Method**: gross domestic product per capita, social support, healthy life expectancy, freedom to make your own life choices, generosity of the general population, and perceptions of internal and external corruption levels,



Some measurements give only partial information!

A slightly different way of measurement



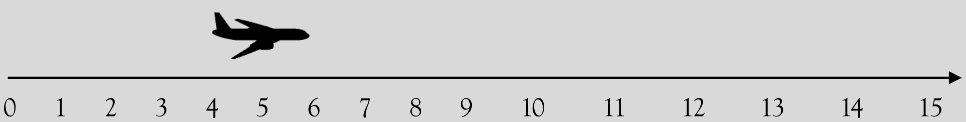
$$|\psi\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

statevector:

Probability of plane being at different positions

Position, speed, colors, orbits, spins, …

A plane on the Bermuda triangle



$$|\psi\rangle = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

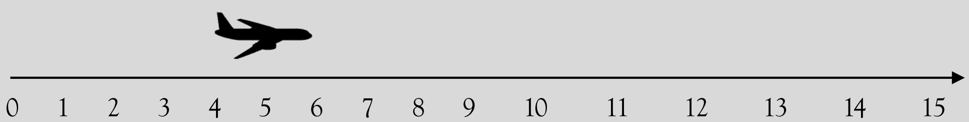
statevector:

Probability of plane being at different positions

$$16\left(\frac{1}{4}\right)^2 = 16\frac{1}{16} = 1$$

Position, speed, colors, orbits, spins, ...

A plane on the Bermuda triangle (after some information)



$$|\psi\rangle = \frac{1}{8}\begin{bmatrix}1\\1\\1\\1\\1\\1\\5\\1\\5\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}$$

statevector:

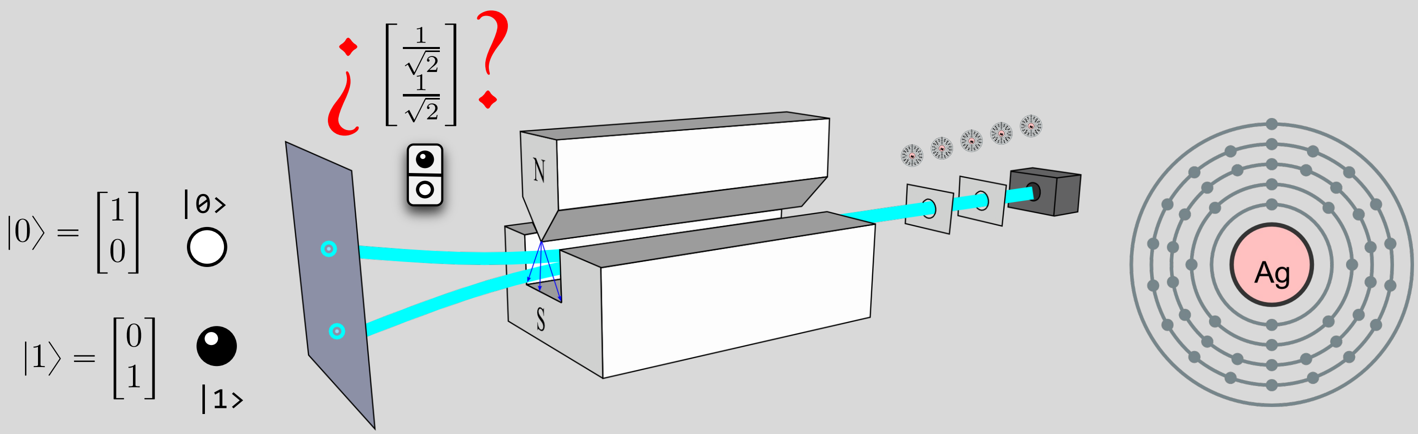
Probability of plane being at different positions

$$2\left(\frac{5}{8}\right)^2 + 14\left(\frac{1}{8}\right)^2 = 2\frac{25}{64} + 14\frac{1}{64} = \frac{50}{64} + \frac{14}{64} = \frac{64}{64} = 1$$

Position, speed, colors, orbits, spins, ...

The Stern-Gerlach experiment and the information

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

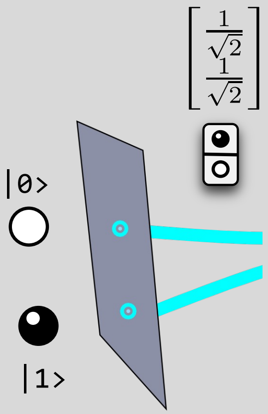


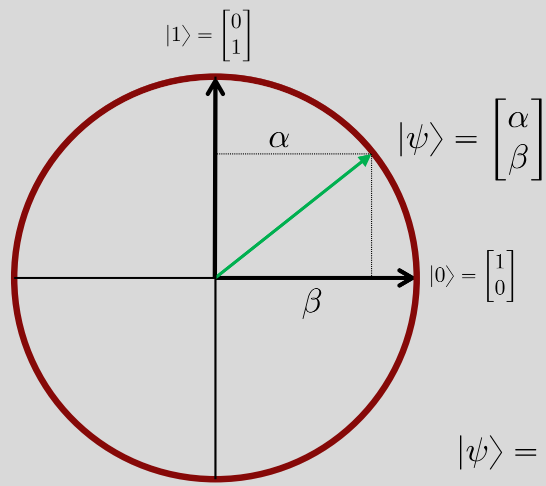
Measurement is taking out information from the system

Why $\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$? Well, because $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$





$$|\psi\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix} = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

: circumference (not circle) of possible states

Computational complexity

- Suppose you have a deck of playing cards, and you want to have it ordered,
- How many steps you will need to order the entire deck of n cards?
  - n^2 ?
  - $n \log n$?
  - $n!$?
- Does it matter?
  - Throw in more RAM and more processors and the problem will go away...

Computational complexity (bogo-sort)

```
while not isInOrder(deck):
    shuffle(desk)
```

- How many steps?
  - Suppose a deck with 2 cards,
    - `1, 2`
    - `2, 1`
  - Suppose now a deck with 3 cards,
    - `1, 2, 3`
    - `1, 3, 2`
    - `2, 1, 3`
    - `2, 3, 1`
    - `3, 1, 2`
    - `3, 2, 1`
- For a deck of n cards you will need (worst case) $n!$ steps,

Computational complexity (selection-sort)

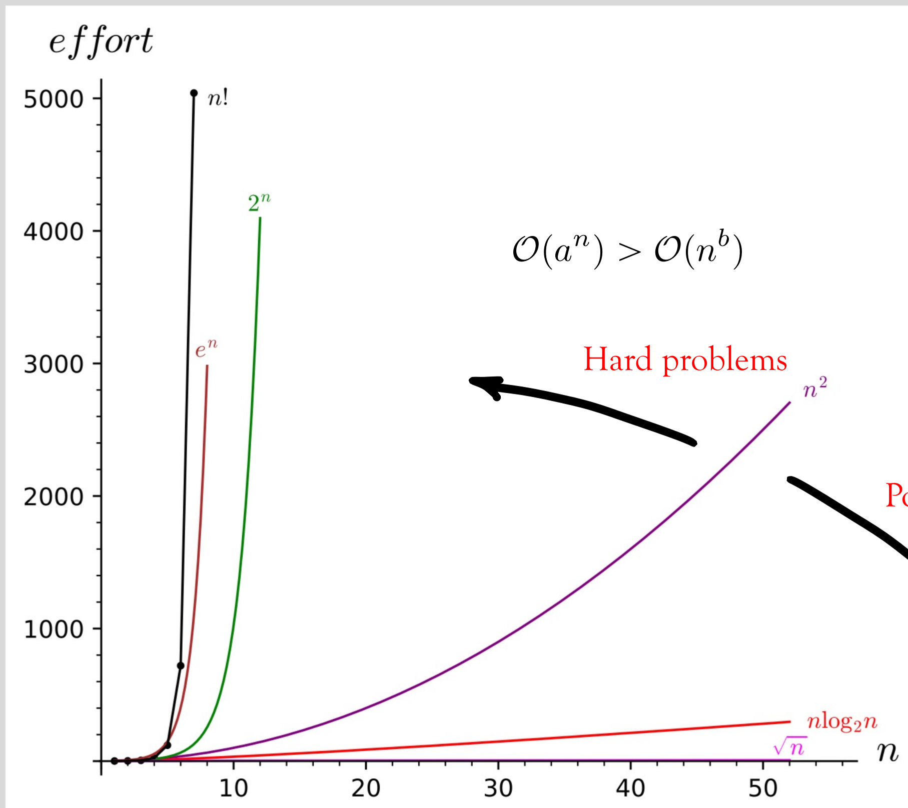
- How many steps?
  - To select the first card you will need $n\text{-}1$ checks,
    - To select the second card $n\text{-}2$ checks,
- For a deck of n cards you will need (worst case) n^2 steps,

```
Select smallest card in the deck,
Sort the remaining deck
```

Computational complexity (all kind of sorts)

Name	Best	Average	Worst	Memory	Stable	Method	Other notes
Quicksort	$n \log n$	$n \log n$	n^2	$\log n$	No	Partitioning	Quicksort is usually done in-place with $O(\log n)$ stack space.[5][6]
Merge sort	$n \log n$	$n \log n$	$n \log n$	n	Yes	Merging	Highly parallelizable (up to $O(\log n)$ using the Three Hungarians' Algorithm).[7]
In-place merge sort	—	—	$n \log^2 n$	1	Yes	Merging	Can be implemented as a stable sort based on stable in-place merging.[8]
Introsort	$n \log n$	$n \log n$	$n \log n$	$\log n$	No	Partitioning & Selection	Used in several STL implementations.
Heapsort	$n \log n$	$n \log n$	$n \log n$	1	No	Selection	
Insertion sort	n	n^2	n^2	1	Yes	Insertion	$O(n + d)$, in the worst case over sequences that have *d* inversions.
Block sort	n	$n \log n$	$n \log n$	1	Yes	Insertion & Merging	Combine a block-based $O(n)$ in-place merge algorithm[9] with a bottom-up merge sort.
Timsort	n	$n \log n$	$n \log n$	n	Yes	Insertion & Merging	Makes *n-1* comparisons when the data is already sorted or reverse sorted.
Selection sort	n^2	n^2	n^2	1	No	Selection	Stable with $O(n)$ extra space, when using linked lists, or when made as a variant of Insertion Sort instead of swapping the two items.[10]
Cubesort	n	$n \log n$	$n \log n$	n	Yes	Insertion	Makes *n-1* comparisons when the data is already sorted or reverse sorted.
Shellsort	$n \log n$	$n^{4/3}$	$n^{3/2}$	1	No	Insertion	Small code size.
Bubble sort	n	n^2	n^2	1	Yes	Exchanging	Tiny code size.
Exchange sort	n^2	n^2	n^2	1	No	Exchanging	Tiny code size.
Tree sort	$n \log n$	$n \log n$	$n \log n$ (balanced)	n	Yes	Insertion	When using a self-balancing binary search tree.
Cycle sort	n^2	n^2	n^2	1	No	Selection	In-place with theoretically optimal number of writes.
Library sort	$n \log n$	$n \log n$	n^2	n	No	Insertion	Similar to a gapped insertion sort. It requires randomly permuting the input to warrant with-high-probability time bounds, which makes it not stable.
Patience sorting	n	$n \log n$	$n \log n$	n	No	Insertion & Selection	Finds all the longest increasing subsequences in $O(n \log n)$.
Smoothsort	n	$n \log n$	$n \log n$	1	No	Selection	An adaptive variant of heapsort based upon the Leonardo sequence rather than a traditional binary heap.
Strand sort	n	n^2	n^2	n	Yes	Selection	
Tournament sort	$n \log n$	$n \log n$	$n \log n$	n[11]	No	Selection	Variation of Heapsort.
Cocktail shaker sort	n	n^2	n^2	1	Yes	Exchanging	A variant of Bubblesort which deals well with small values at end of list
Comb sort	$n \log n$	n^2	n^2	1	No	Exchanging	Faster than bubble sort on average.
Gnome sort	n	n^2	n^2	1	Yes	Exchanging	Tiny code size.
Odd–even sort	n	n^2	n^2	1	Yes	Exchanging	Can be run on parallel processors easily.

Computational complexity



Hard problems

Polynomial is easy

aspect_ratio = 0.01

Computational Complexity

- Looking for an element in an unordered list is:

$$\mathcal{O}(n)$$

- Looking for an element in an ordered list with random access is:

$$\mathcal{O}(\log_2 n)$$

Computational Complexity (cryptography applications)

- Multiply two n-digits numbers is $\mathcal{O}(n^2)$
  - Given a number, finding out the factors is: $\mathcal{O}(e^{n^{\frac{1}{3}}})$
    - Hard to calculate, easy to verify

- Given b calculate $a = g^b \mod p$ requires $\log_2 b$
  - Given a calculate b from a is: $\mathcal{O}(2^{\frac{1}{2}\log(n-1)})$
    - Hard to calculate, easy to verify

Depends on the computational model: digital computer

Search Problems

(v.g. RSA, hard to solve, easy to verify)

```
In [270]: reset()

          import time

          print ('Time is %f secs.  Generating p' % time.time())
          p = random_prime(2^1024)
          print ('Done. Time is %f secs.  Now generating q' % time.time())
          q = random_prime(2^1024)
          print ('Done. Time is %f secs.  Now multiplying p and q' % time.time())
          N = p*q
          print ('Done. Time is %f secs.  p*q = %d * %d = %d' % (time.time(), p, q, N))
```

2.05 s
3.39 s
0.45 ms

```
Time is 1644926448.192391 secs.  Generating p
Done. Time is 1644926450.244496 secs.  Now generating q
Done. Time is 1644926453.630617 secs.  Now multiplying p and q
Done. Time is 1644926453.631071 secs.  p*q = 6467095961165892555506790608221411256486181981153644068064582053
1382507680311159812348901607050070031334353275037656193407344410807115829967118016518029973548055519345258882
4742899267305081438101370875219769213977247579113627285217156736862086093546964430472016433352031676268473021
19952473074972662130887253 * 1162671416340562924660425940685846112893289877739811092343126603116663442013837
3199274330809246170805833371951543192575805777898948819557332491005531746708705742712730916208440172861520928
0183282680979625969899628212906486222645136668119953703652816703891608785977703744895489756510708558519853528
48545381549 = 75191076207790824230092235855081350920718467661014015917644814568947693544862215911798315609100
9613062899314676789187570883998694649864285749578897925162508877197679222702201851642928084092293320902405305
9283165608213228122626591235946952110104888395421349243785592020391832889786146958859030524213263290480129630
3159094700064875817903130162788487837386954773660798248162302024264460684118191441309957599984963916422062087
9657162412249626907011123681347516630516044443566617961983823955315512474682263274530783307202742844637036320
9879141177643737253902997528189177266405401325611286014486421188190116043474785494897
```

0.45 ms 10^6 pasos

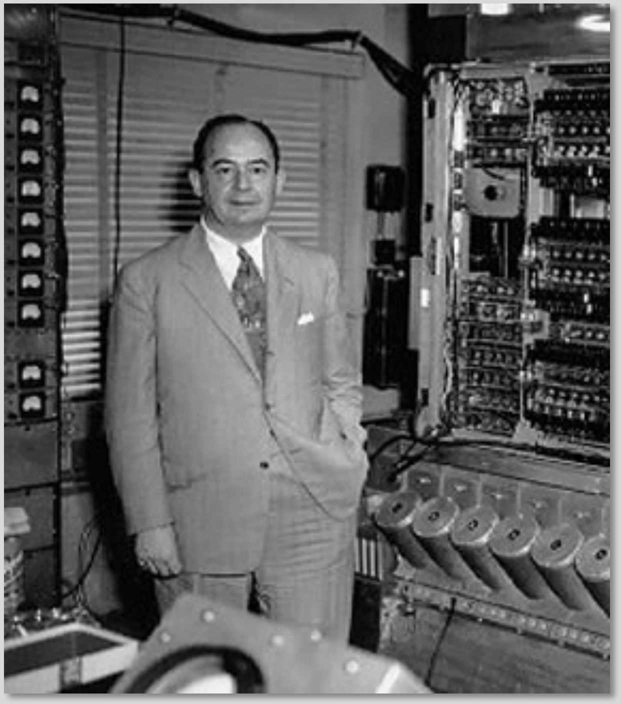
300×10^{12} años $1{,}5 \times 10^{35}$ pasos

What is to compute?

https://physicstoday.scitation.org/do/10.1063/pt.5.9084/full/

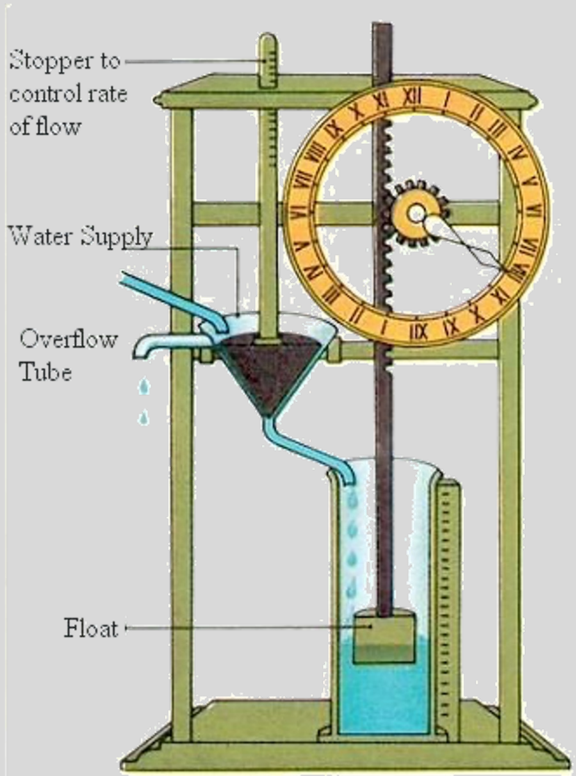
Why *ordinateur*?

- François Girard, 1995, Head of IBM France, trying to enter the new French Market with *«la nouvelle machine électronique destinée au traitement de l'information»*,
    - Tried *compter* (to calculate), very similar to *calculatrice*,
        - Bad marketing feeling,
- He asked to his humanities professor, Jacques Perret:
    - He recommended latin for "to arrange": *ordinare*,
        - *Ordinare* → *Ordiratrice électronique* → *l'ordinateur*,
            - It's a trademark.

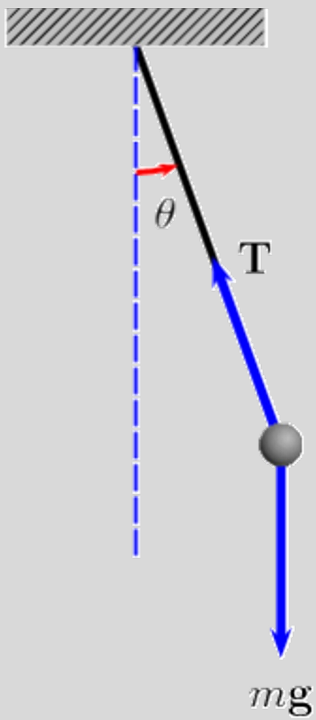


Analog Computers
(a sundial)

Analog Computers
(a clepsydra)



Analog Computers
(a pendulum)

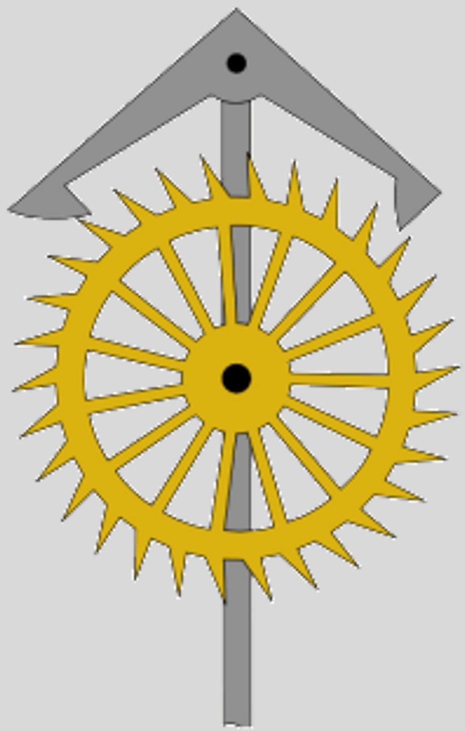


- For $\theta_0 \ll 1$ rad (57°3)

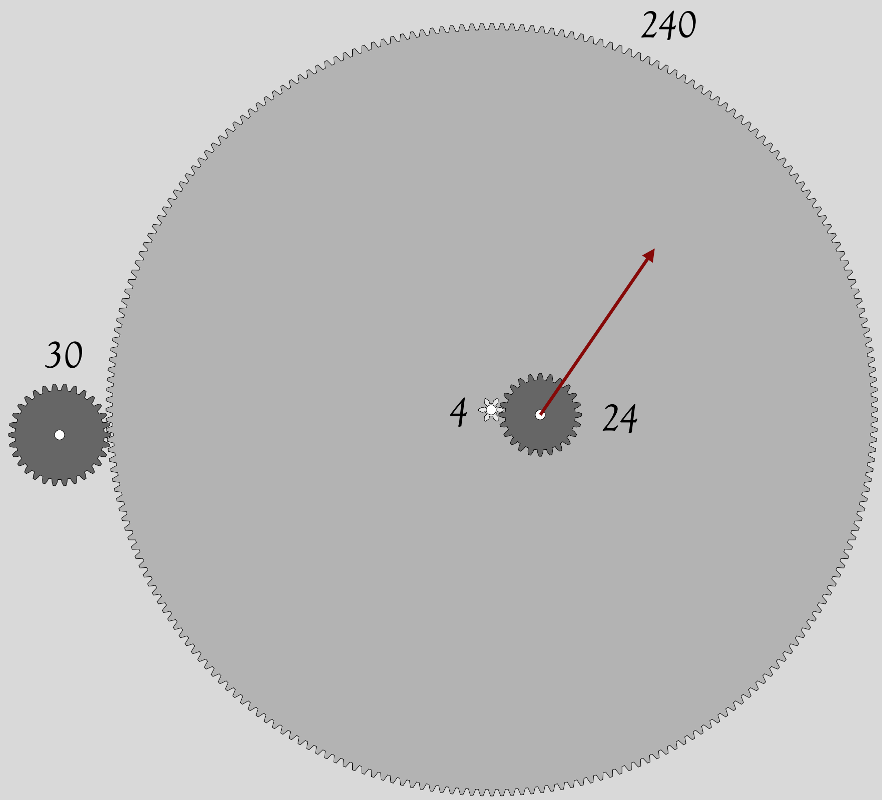
$$T \approx 2\pi\sqrt{\frac{L}{g}}$$

- Depends only on g (9.8 m/s^2), and L (pendulum length),
- Let's L = *1 meter*, T = *2 sec*

Analog Computers
(anchor escapement)

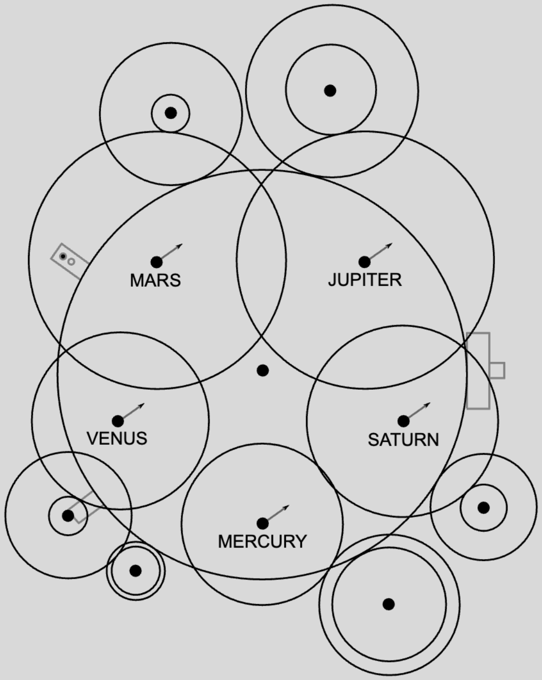


Analog Computers
(anchor escapement and gears)



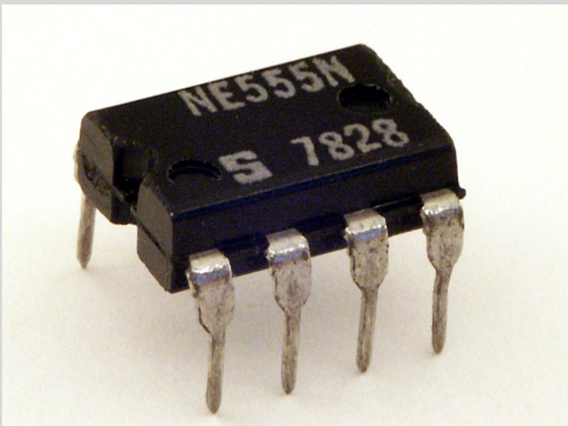
Analog Computers
(orrery)

Analog Computers
(Antikythera)



Computational Complexity
(multi-axis tourbillon)

Digital Clocks
(quartz precision)



- Just count pulses,
  - With a stable 555 you will get really good precision and accuracy,
    - And really cheap,
  - Sorry Swiss industry and its tourbillons,
- Actually is pretty easy
  - See a couple of slides from here the NAND gate and adders based on NAND gates.

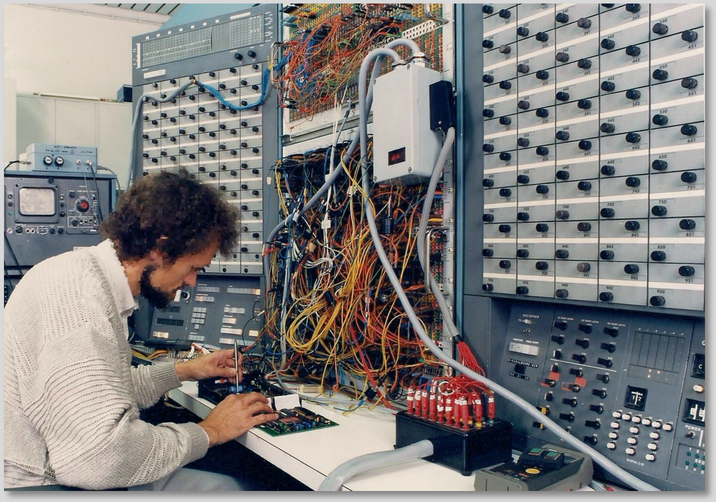
Analog Computers
(equinoccial building - Santa Marta de Tera)

Analog Computers
(Maya piramid - Kukulkán en Chichén Itzá)

Computational Complexity
(analog computers)

Computational Complexity
(analog computers)(.)

Akat-1 (1959)



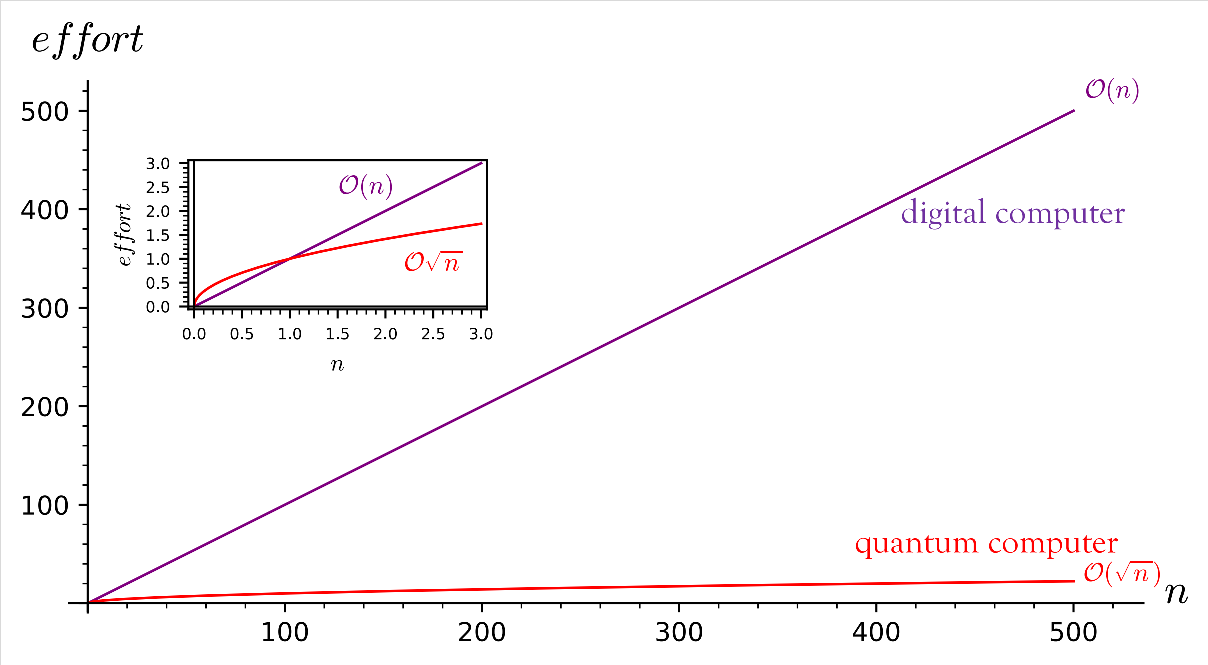
EAI 8800 (1986)

Computational Complexity (analog computers)(..)

- Problems:
  - No general use, only specific purpose:
    - Sea navigation, jet propulsion, calculator, spring simulations (AKAT-1), dynamic system simulation, …
  - Low precision, hard to control errors,
    - we know how to deal with errors in digital computers and get arbitrary precision,
- But the simulation of physical processes can be really fast in analog computers,
- Quantum computers are both analog and digital:
  - While in **superposition** are analog, when the qubit **collapses** turns on digital,
  - But we need quantum error correction.

Computational Complexity
(Grover's algorithm)

Looking for an element in an unordered list is:

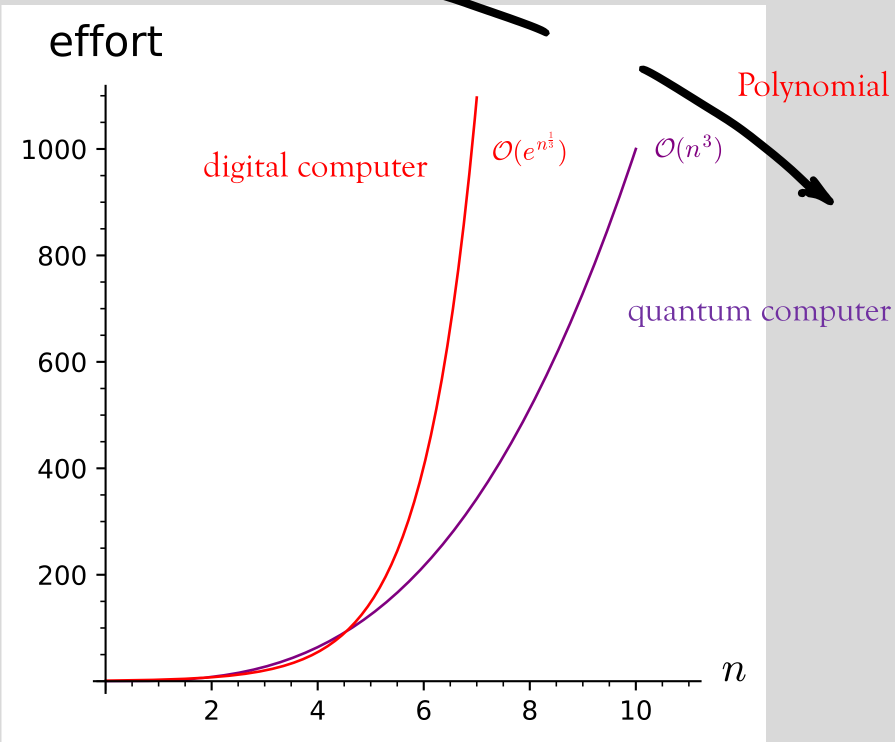


Computational Complexity
(Shor's algorithm)

Factorize a number:

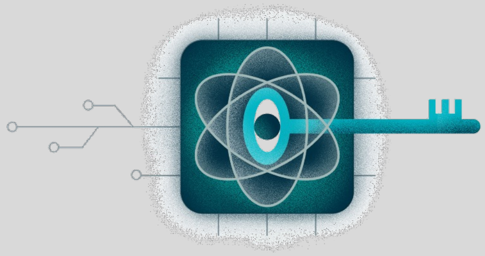
Hard problems

Polynomial is easy



Shor's Algorithm

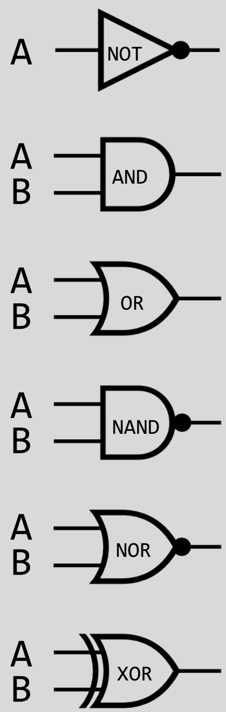
- A big infosec headache:
  - Using a quantum computer can break in polynomial time (easy) the RSA algorithm,
    - The discrete logarithm (Diffie-Hellman key exchange),
    - The Elliptic Curve Diffie-Hellman exchange,
- The Race
  - 2001, IBM factored 15 into 3x5, using 7 photonic qubits:
    - https://cryptome.org/shor-nature.pdf
  - 2012 factored 21 into 3x7 also using solid-state qubits:
    - https://arxiv.org/abs/1111.4147
  - 2019 factored 35 into 5x7 using IBM Q System One.
    - https://arxiv.org/abs/1903.00768
- What to do?
  - Security in depth,
  - Post-quantum cryptography.



Logic and Quantum Gates

Logical gates (Boolean), Quantum gates and computability

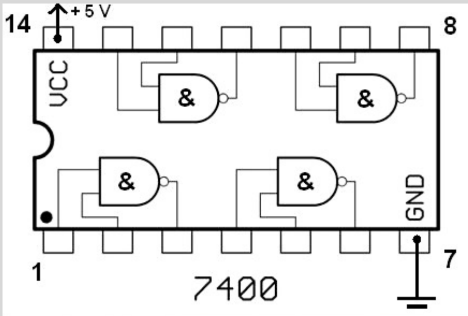
Logic Gates

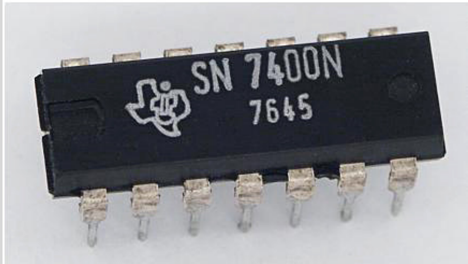


INPUT		OUTPUT					
A	B	NOT A	AND	OR	NAND	NOR	XOR
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0					
1	1	0	1	1	0	0	0

Logic Gates

- A **functionally complete** set of Boolean operators is one that can express all possible truth tables,
  - A set of gates which is functionally complete is called a **universal set** of gates,
  - Every possible logic gate can be implemented as a network of gates,
- Examples:
  - The set of logical connectives {¬,∧} (Not, And) is functionally complete,
  - The singleton {NOR} is functionally complete (https://proofwiki.org/wiki/Functionally_Complete_Logical_Connectives/NOR),
  - The singleton {NAND} is functionally complete (https://proofwiki.org/wiki/Functionally_Complete_Logical_Connectives/NAND),
    - `¬A ≡ A NAND A`
    - `A ∧ B ≡ ¬(A NAND B) ≡ (A NAND B) NAND (A NAND B)`
    - `A ∨ B ≡ (A NAND A) NAND (B NAND B)`

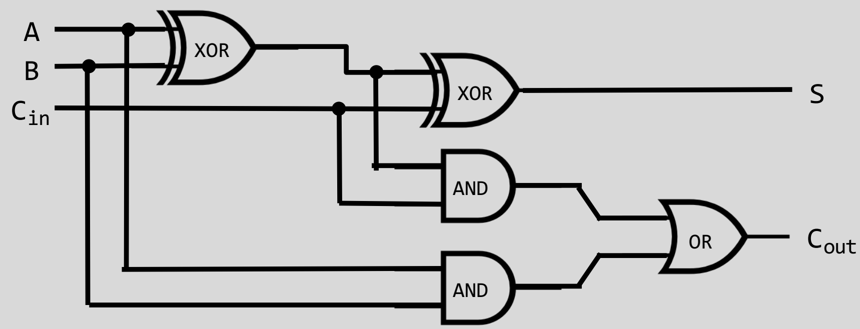


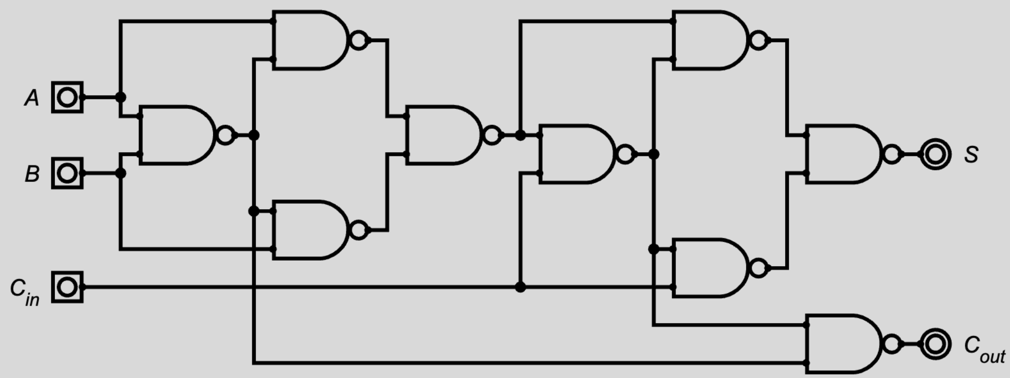


https://www.amazon.com/Types-74HCxx-Assortment-High-Speed-Si-Gate/dp/B08BR1PPQM/ref=sr_1_2?crid=3OTQXRQ1DS80B&keywords=7400+ttl&qid=1664285373&qu=eyJxc2MiOiIxLjQ0IiwicXNhIjoiMS41OSIsInFzcCI6IjEuMDAifQ%3D%3D&sprefix=7400+ttl%2Caps%2C129&sr=8-2

Full Adders using Logic Gates

INPUTS			OUTPUTS	
A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1





A full 4-bit CPU using 7400

ARITH-MATIC

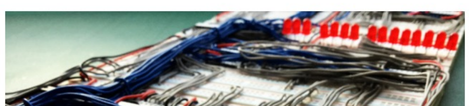
Building the Cambridge-1: A 4-bit homebrew 7400 based CPU

With a handful of 7400 series ICs and an appreciation of the fetch-decode-execute cycle, it's possible to build a simple digital computer in less than 30 days.

I have, for quite a while, had some ideas swirling around in my head to build a 4-bit 7400 series based CPU. And like all good ideas they seemed to be caught up in a never-ending purgatory - never seeing the light of day because 'life' gets in the way. However, last month I finally bit the bullet and decided to actually build this imagined machine and I had the perfect excuse - the Retro Computer Festival was just around the corner.

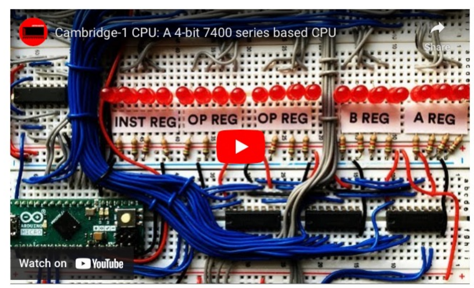
For the uninitiated, the Retro Computer Festival is now a yearly event hosted at the Centre for Computing History in Cambridge, England. It is not only a great exhibition of fantastic retro machines, but an opportunity for a like minded community of computing enthusiasts and makers to get together and share super cool stuff.

In this spirit, I decided to finally build my imagined 4-bit machine and set myself a challenge: what was the simplest computer I could build in 30 days? This question became the genesis of the Cambridge-1 CPU.



comparator to generate the flags for jump instructions. And, before I forget, there's a (non-7400 series) SRAM chip to store the software!

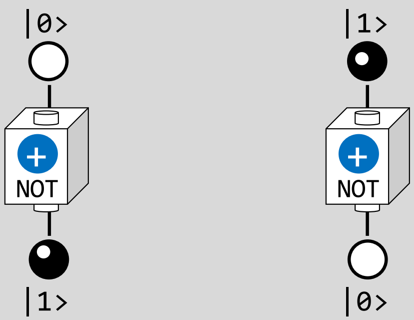
The machine operates from a simple Instruction Set - comprising of 16 instructions (defined by a 4-bit word) and the basic functions you'd expect: binary arithmetic (adding and subtracting), memory manipulation, jump instructions, IO functions etc.



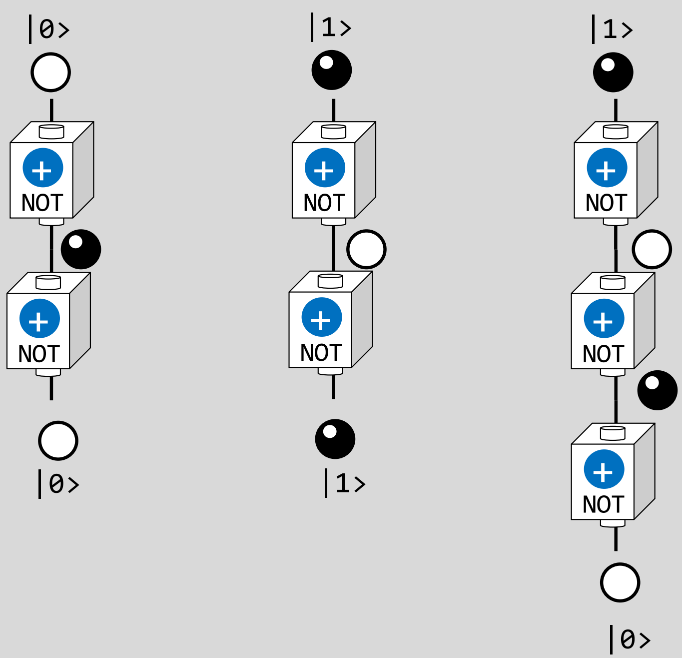
Manuals

https://arith-matic.com/notebook/4bit-7400-homebrew-computer-cpu

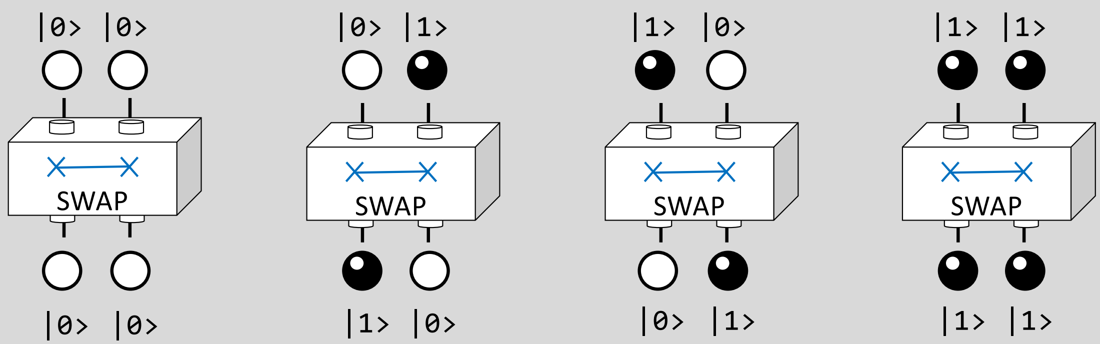
The operation NOT



The operation NOT (examples)

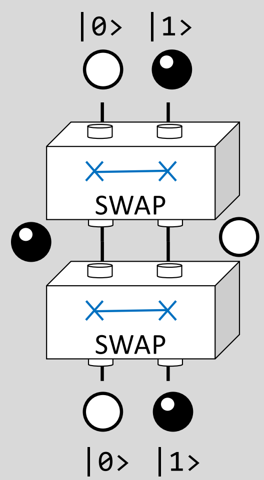


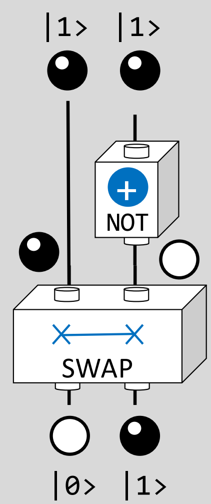
The operation `SWAP`



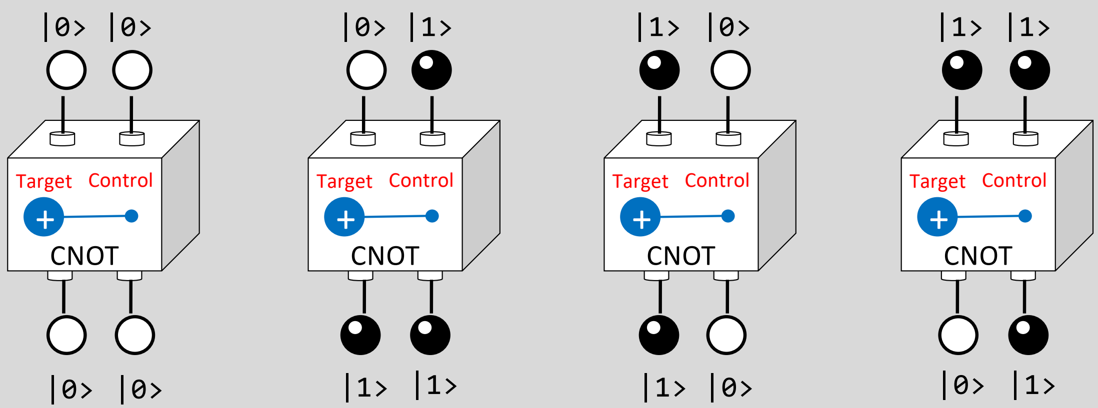
The balls swap colors relative to one another. They do not swap locations.

The operation SWAP (exercise)



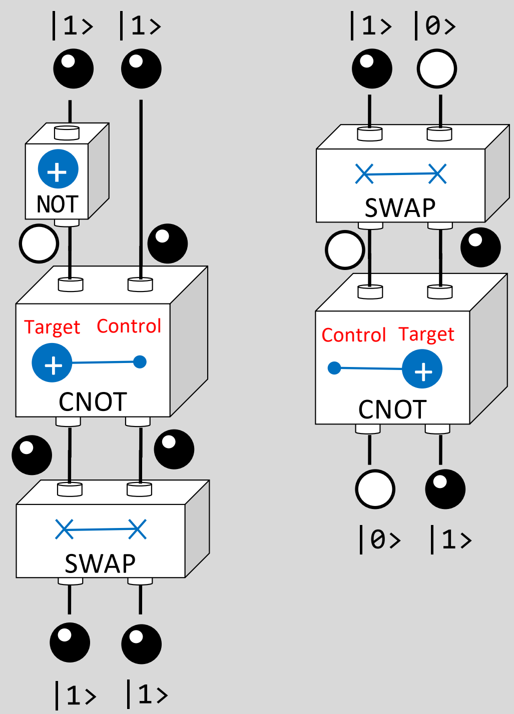


The operation **CNOT** (Controlled - **NOT**)

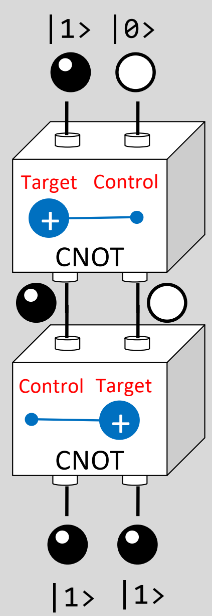


Control toggles the NOT operation

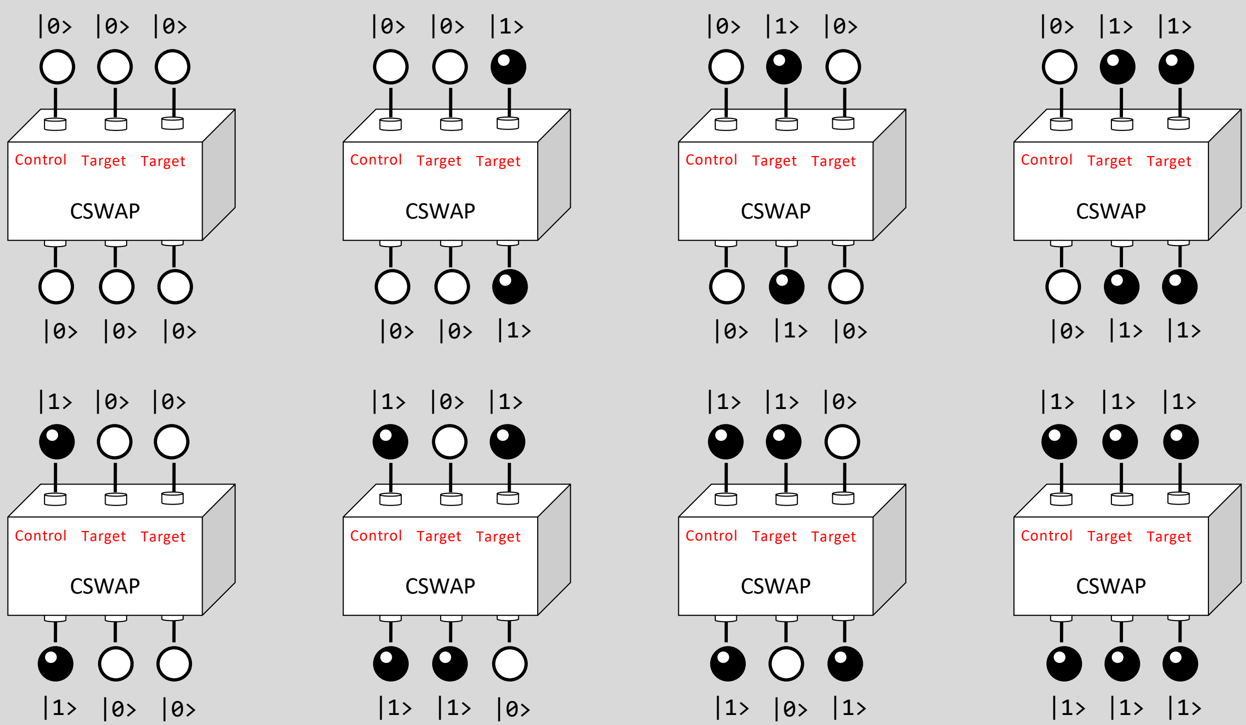
The operation CNOT (Controlled-NOT)(exercise)



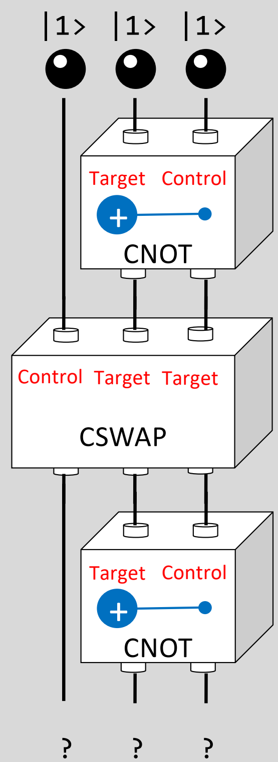
The operation **CNOT** (Controlled-**NOT**)(exercise)(.)



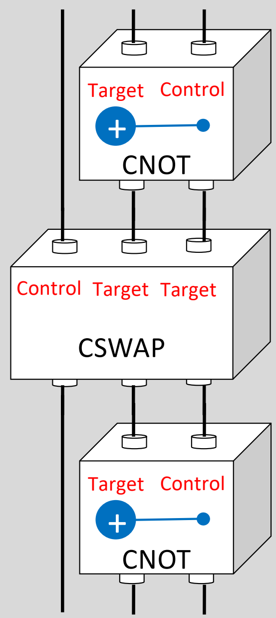
The operation CSWAP (Controlled-SWAP aka. Fredkin gate)



The operation CSWAP (Controlled-SWAP)(exercise)



The operation CSWAP (Building a CCNOT)



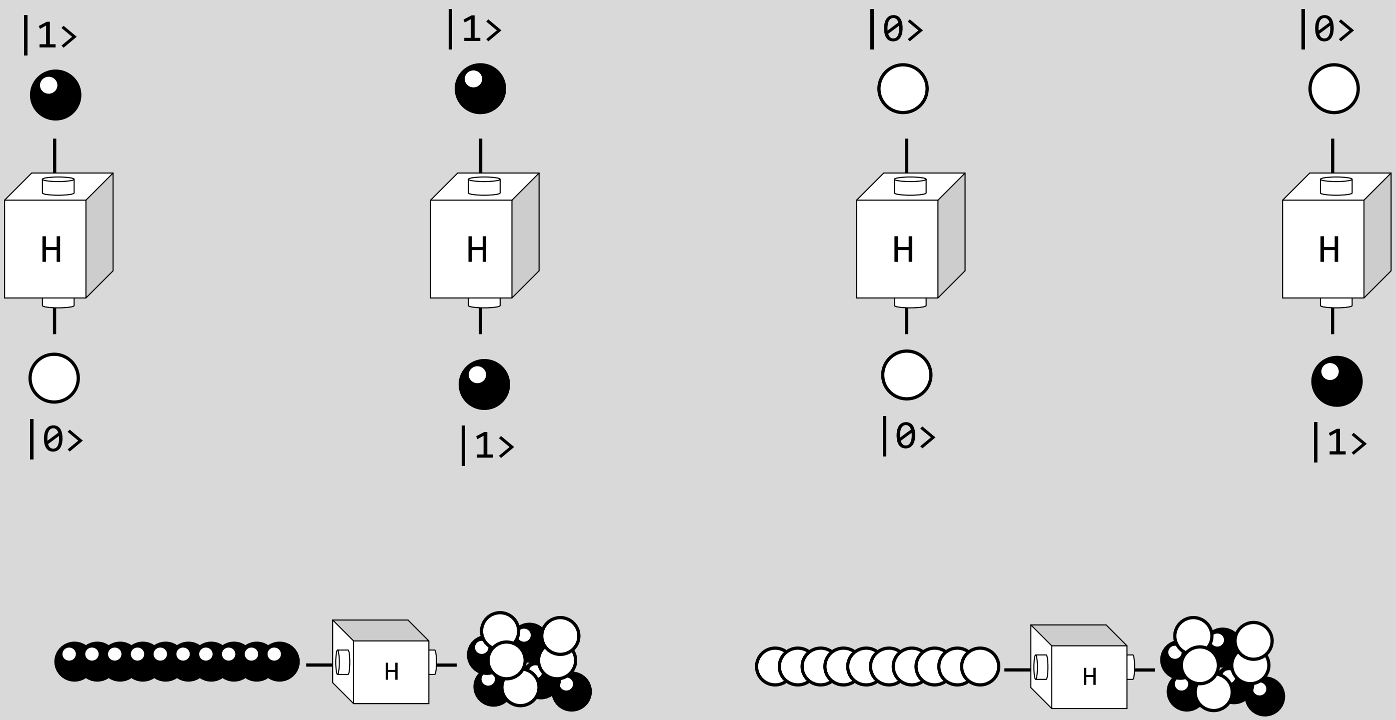
Input	Output
○ ○ ○	○ ○ ○
○ ○ ●	○ ○ ●
○ ● ○	○ ● ○
○ ● ●	○ ● ●
● ○ ○	● ○ ○
● ○ ●	● ○ ●
● ● ○	● ● ●
● ● ●	● ● ○

First two balls never change color.

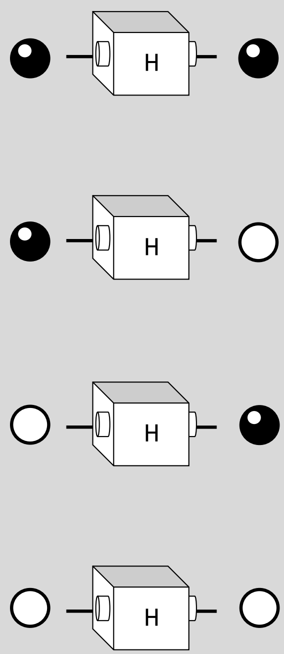
When both of the first two balls are black, a `NOT` is applied to the third ball, otherwise the third ball is unaffected.

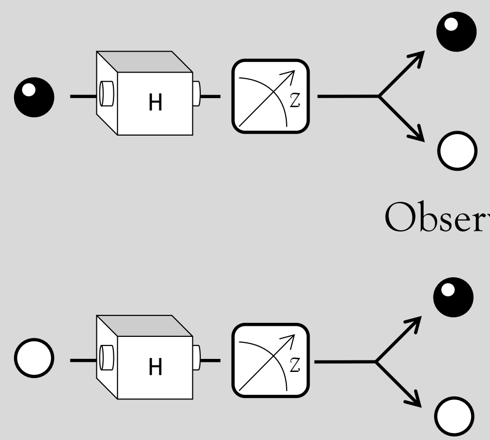
`CCNOT`: Controlled-controlled - `NOT`

The H Gate



The H Gate (.)

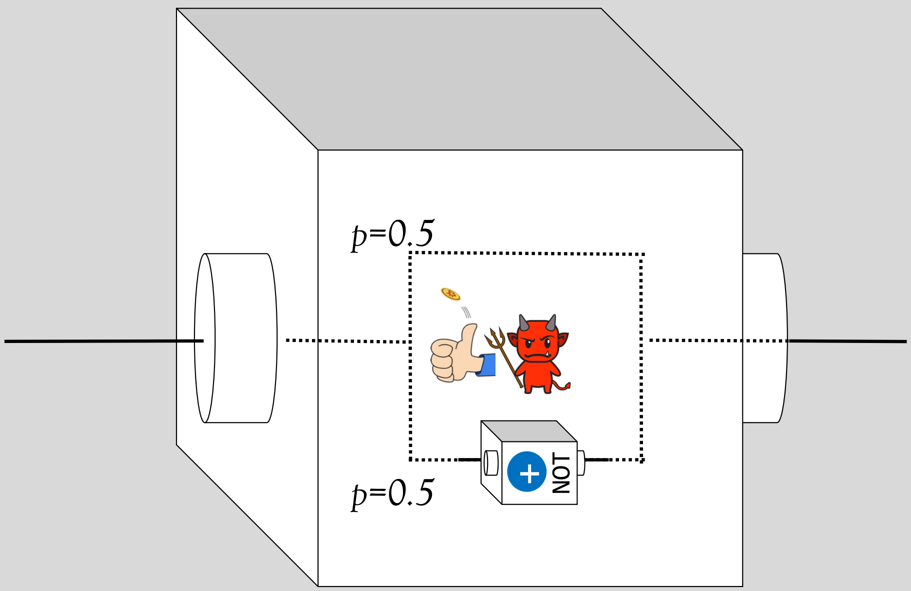


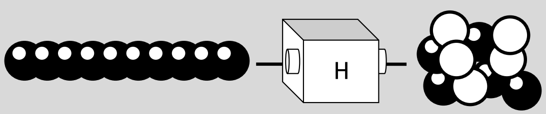


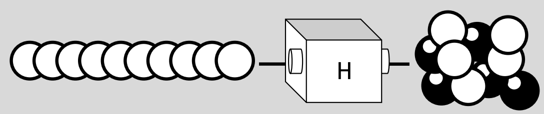
Observed outcome is random

However, is predictable (p=0.5)

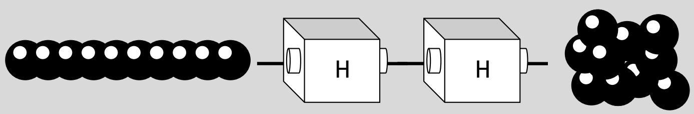
The `H` Gate
(random nature)



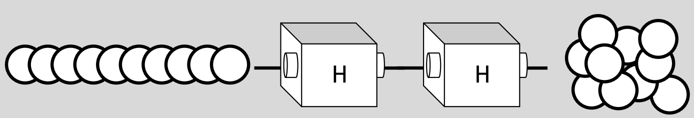




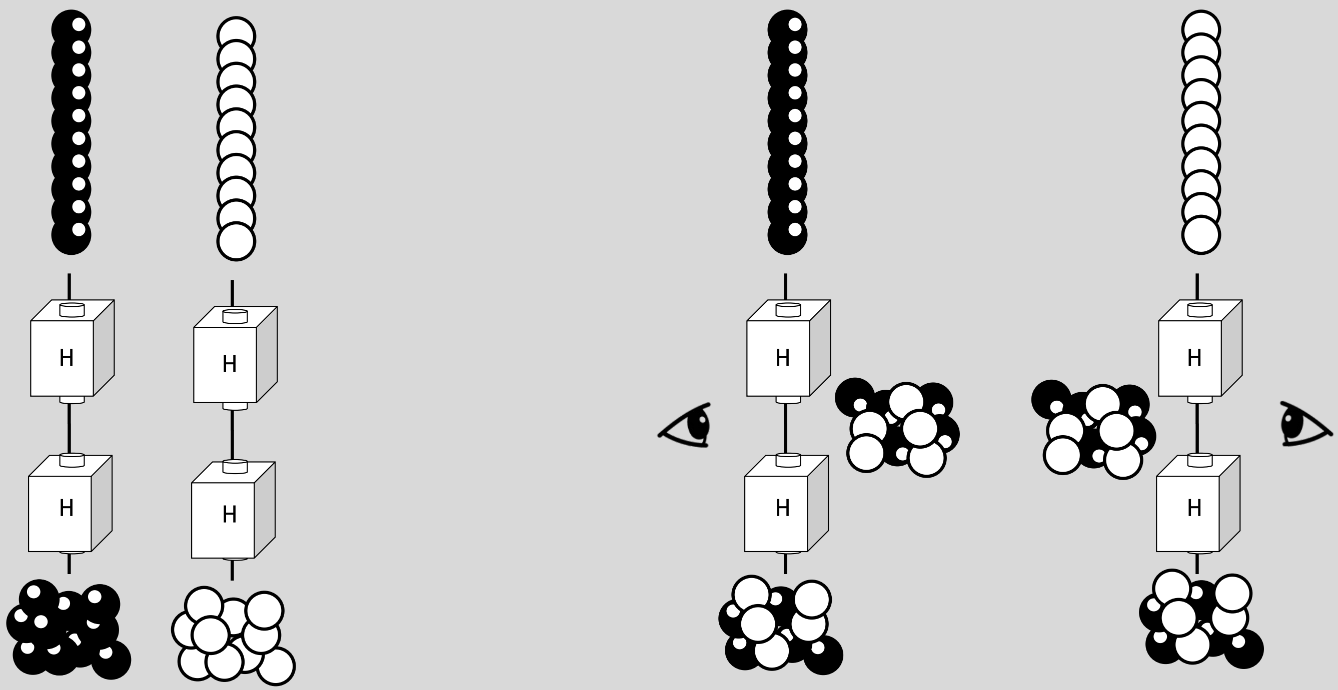
The H Gate (quantum weirdness)(double H)



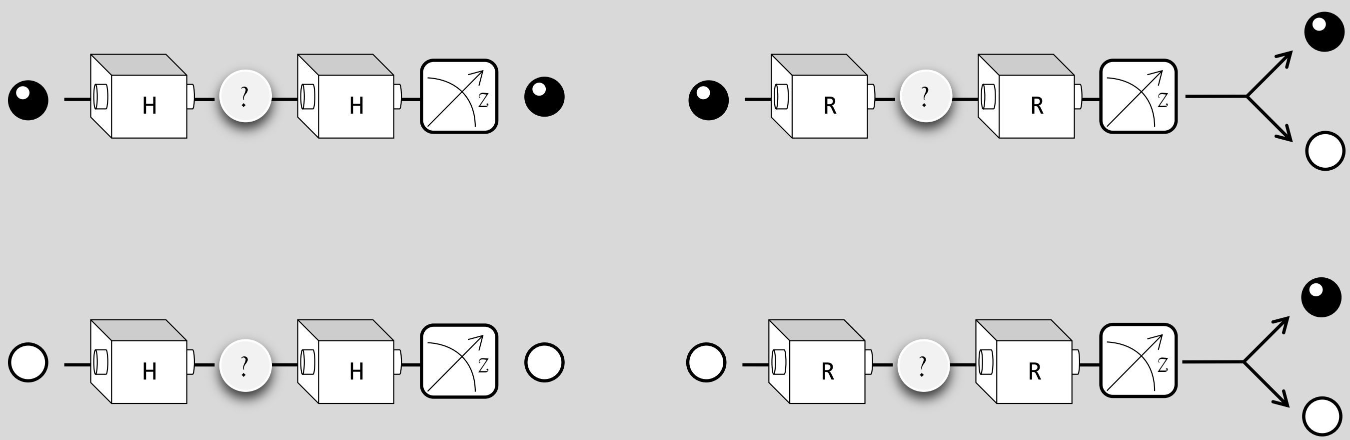
Not really random
(random with state / memory)



The `H` Gate
(quantum weirdness)(.)



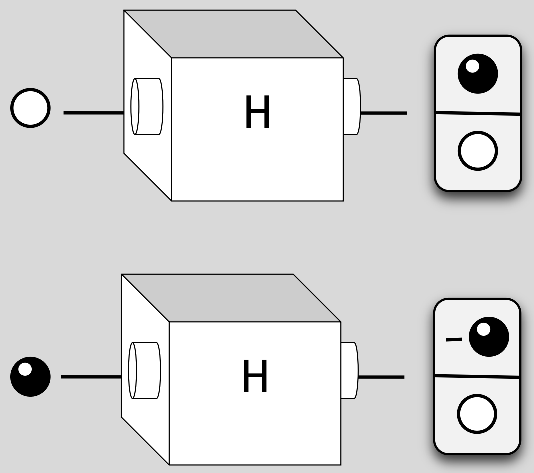
The `H` Gate
(appears random, but it is not)



Two instances always brings it back to the initial color.

After the `H` gate but before measurement, the balls are not `b/w`, but something more complex

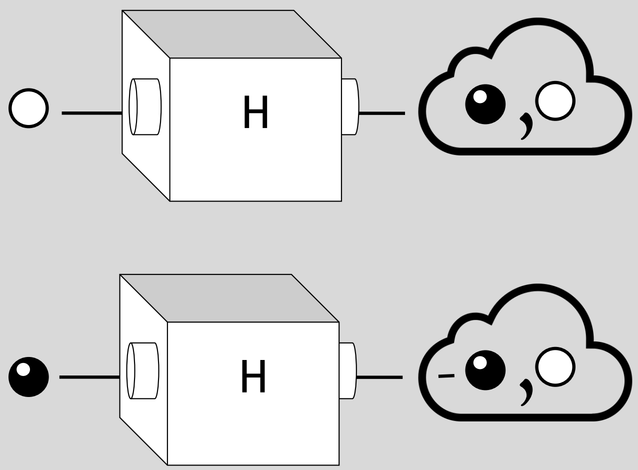
The H Gate
(superposition state)



It *'remembers'* that used to be black

The H Gate

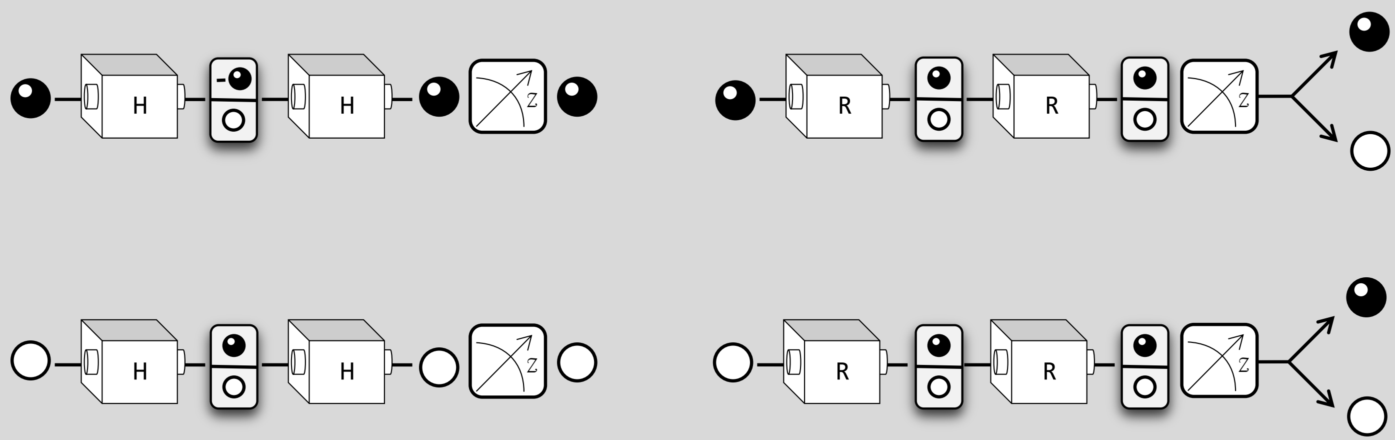
(superposition state)(alternative description)



'the mist'

It *'remembers'* that used to be black

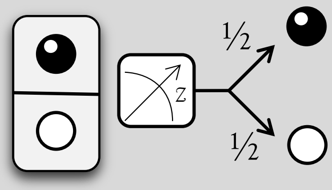
The `H` Gate
(superposition state)(.)

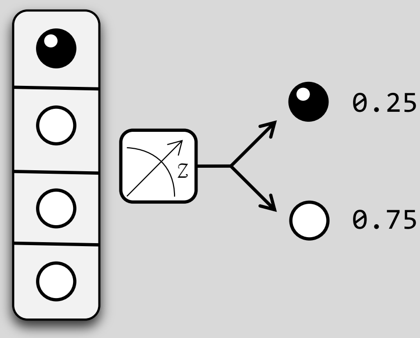


Two instances always brings it back to the initial color.

After the `H` gate but before measurement, the balls are not `b/w`, but something more complex.

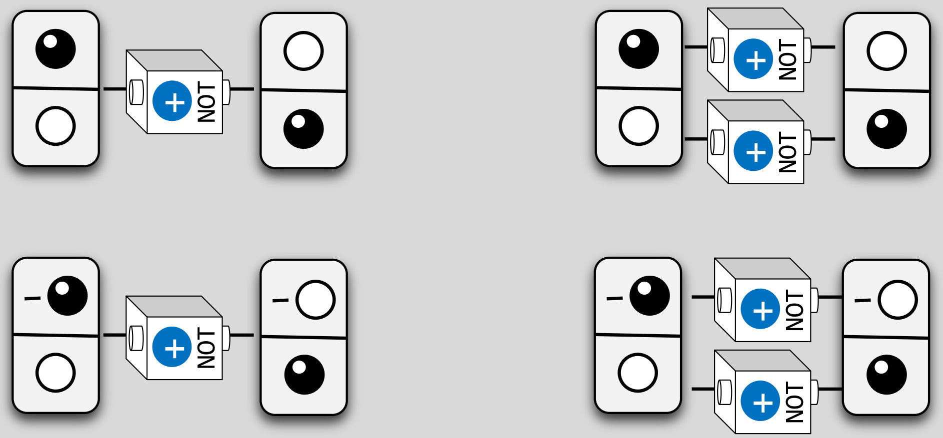
The H Gate
(superposition state)(beyond ½ ½)





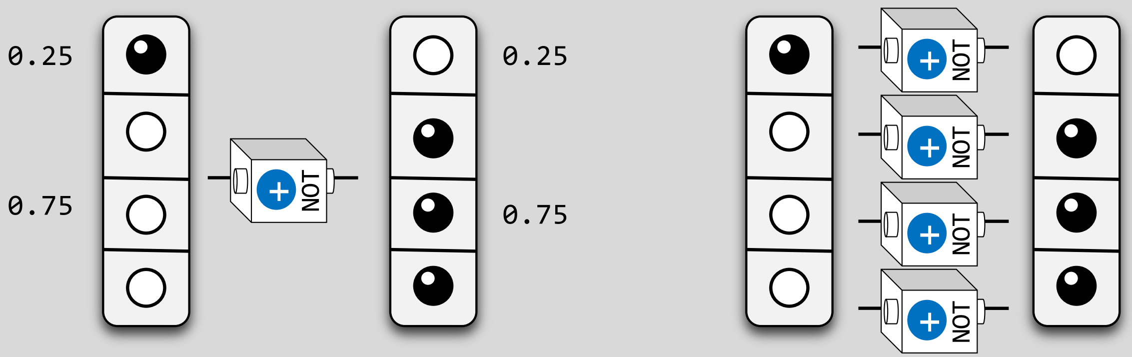
Each state has the same probability

The `H` Gate
(superposition as input)

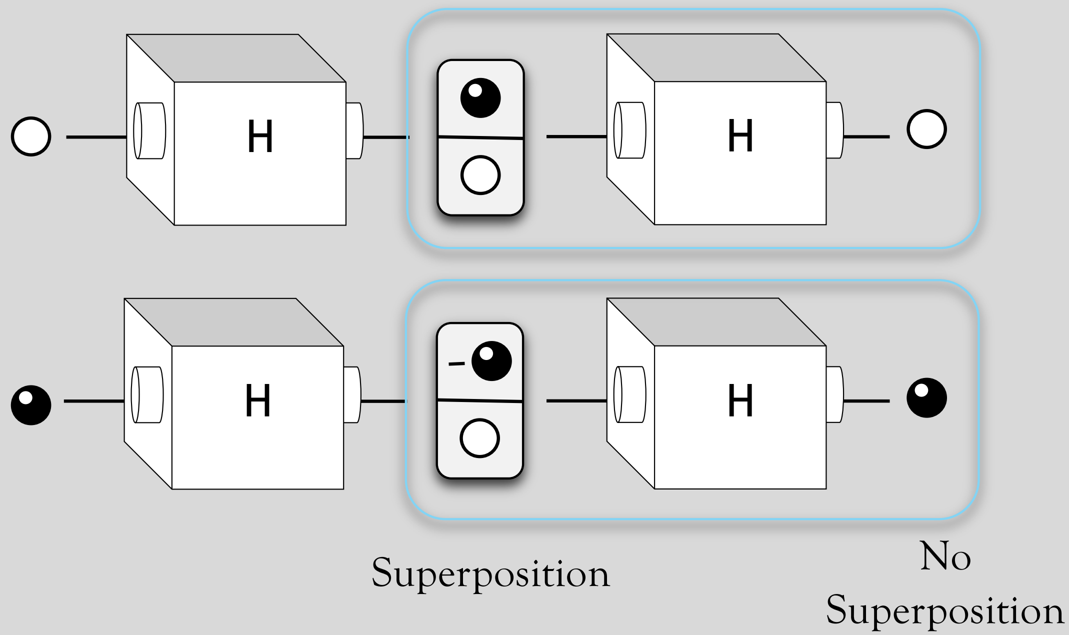


Apply a `NOT` gate to each state

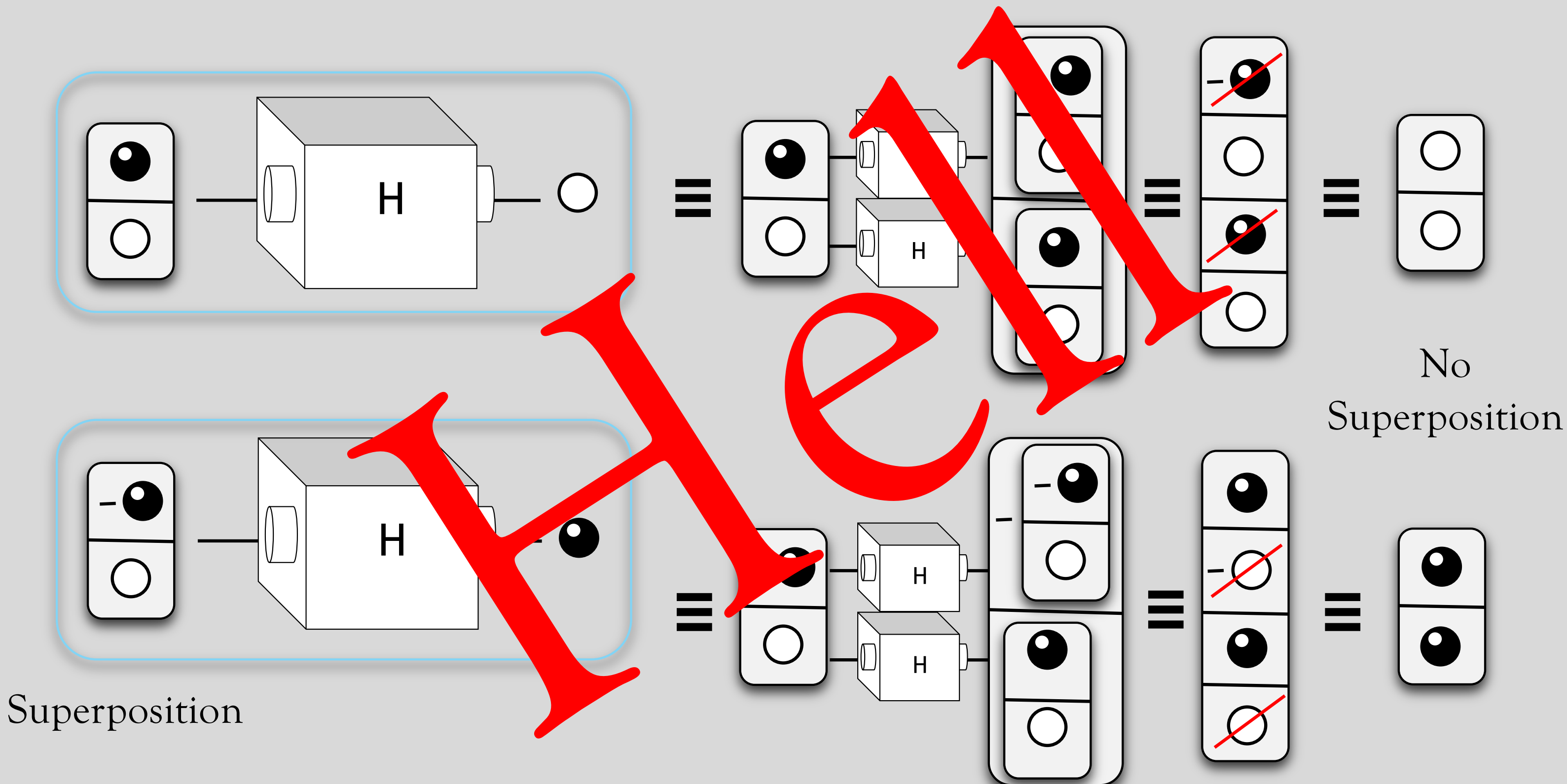
The `H` Gate
(superposition as input)(.)



The H Gate
(the odd behavior)



The H Gate
(the odd behavior)



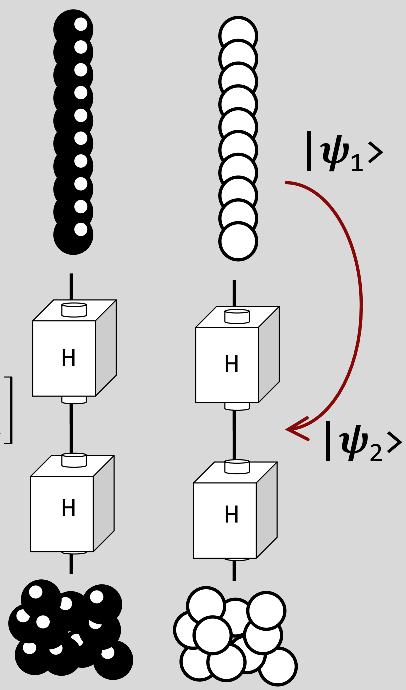
The H Gate (random nature)

Matrices as linear transformations for kets

$$\mathbf{H}\left|0\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = \left|0\right\rangle$$

$$\mathbf{H}\left|1\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} = \left|1\right\rangle$$

$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$



Transformation of a ket into another!

$$\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$$

$$\mathbf{H}\left|\psi\right\rangle = \mathbf{H}\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\left(\alpha\begin{bmatrix}1\\ 0\end{bmatrix} + \beta\begin{bmatrix}0\\ 1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha + \beta\\ \alpha - \beta\end{bmatrix}$$

Summary

- The `H` gate puts a qubit in superposition,
  - It's not entirely `|0>` and not entirely `|1>`,
- An `H` gate applied to a pure state `|0>` or `|1>` results in *50/50* chance of measuring *0* or *1*,
- Two `H` gates in sequence reverse each other, resulting in the original input,
- Thus, there is more to state that just the probability of measuring *0* or *1*, there is also **phase**,
- Our calculation with the phase value accurately models / predicts this reversing behavior.

Summary
(.)

- Because qubits can exist in superposition, the outcomes of quantum operations can be probabilistic,

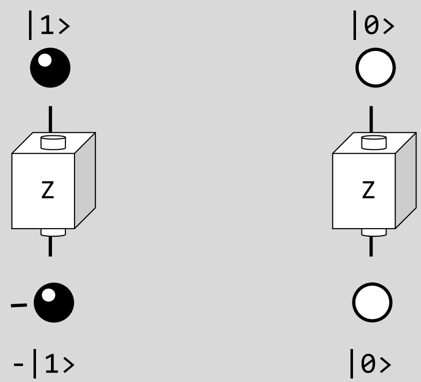
- Measurement "*collapses*" the state of a qubit in superposition,
  - Just like measuring the strength of a vase by using a hammer destroys the vase, measuring the state of a quantum bit can also modify its value,
    - And the qubit turns into a classical bit.

- It's way more comfortable to think in matrices as linear transformations than in b/w balls.

Detour to linear transformations

Matrices as linear transformations for vectors (kets)… eigen-vectors and eigen-values and its geometrical interpretation

The Z Gate
(phase flip gate)



What is the matrix?

$$\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{Z}\ |0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

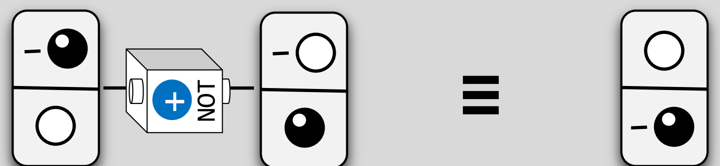
$$\mathbf{Z}\ |1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{Z}\ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{Z} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{Z}\ \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

The `NOT` Gate (superposition as input)



What is the matrix?

$$\mathbf{NOT} = \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{X}\ |0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$\mathbf{X}\ |1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

Eigen-val, eigen-vec

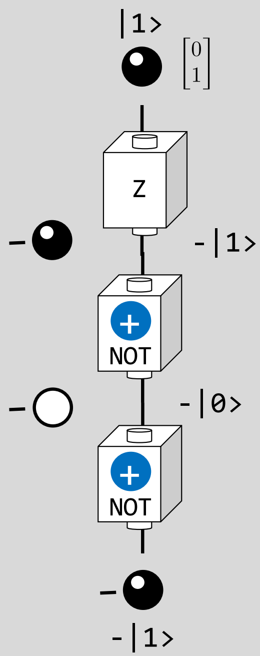
$$\mathbf{X}\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{X}\ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

Negative phase is an attribute of the **qubit**... by convention phase sign is placed with the black once the computation is done

$$\mathbf{X}\ \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

The Z Gate
(phase flip gate)(.)

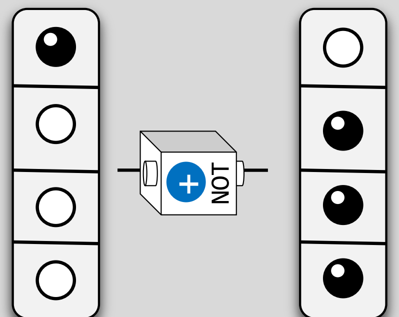


With no superposition, phase symbol
is necessary on the white symbol

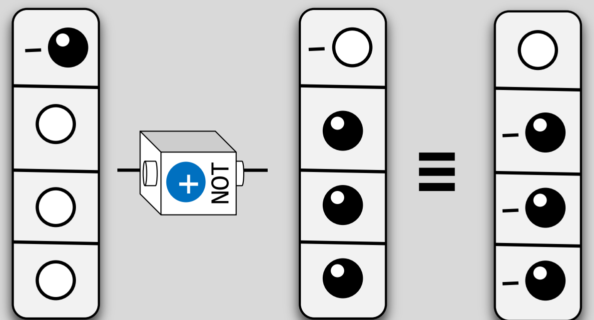
$$\mathbf{XXZ}\ |1\rangle = \cancelto{\mathbf{I}}{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

Phase and Complex Superposition

$$\mathbf{X}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$$



$$\mathbf{X}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{bmatrix}$$



Phase is associated with the entire **qubit**, not the **zero** or the **one**. However, by convention, we place the negative sign with the **one** term. And if there is no one term, it'll go on the **zero**.

Summary

- Phase is associated with the entire qubit, not `|0>` or `|1>`,
- By convention, we place the negative sign with the `|1>` term,
  - If there is no `|1>` term, it goes on the `|0>`,
- For visual representation, if there is more than one black ball in a single qubit's superposition state, we will put the negative sign on **all** of the black balls.

Rabbit or Duck

•

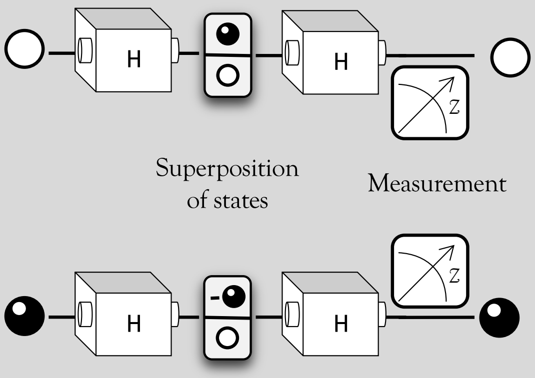
Two different values for the image until decided upon by an observer!

https://en.wikipedia.org/wiki/Rabbit%E2%80%93duck_illusion

Revisiting the H gate

What is the matrix?

$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



No intermedia measurement,
predictable outcome

$$\mathbf{H}\ |0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{H}\ |1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

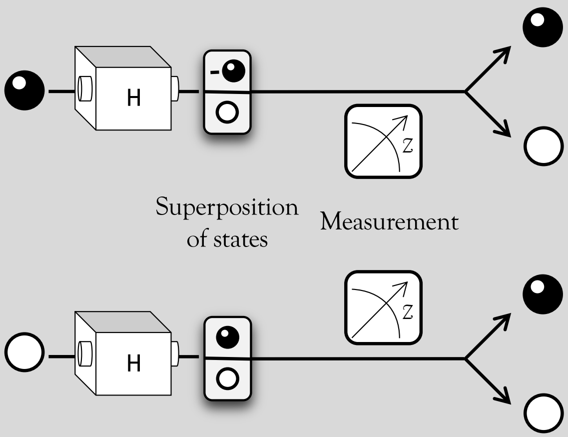
$$\mathbf{H}\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$\mathbf{H}\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

What if we measure in the middle?

$$\mathbf{H}\ \mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \mathbf{I}$$

Revisiting the H gate (.)



$$\mathbf{H}\ |1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

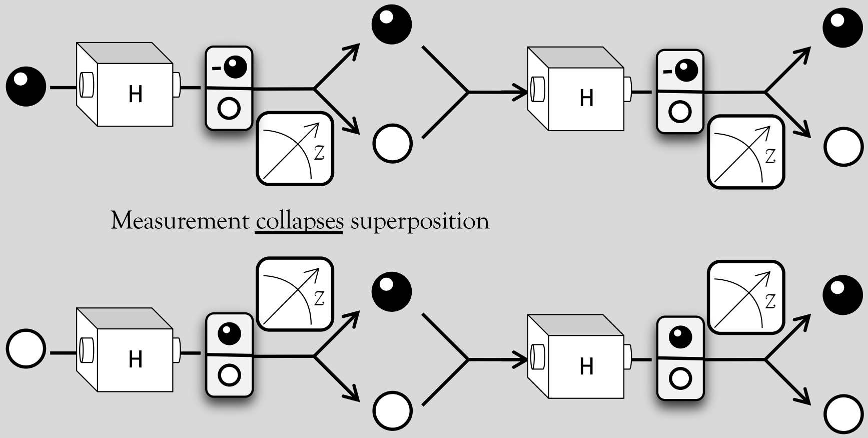
Measurement does not reveal full state.

Measurement reveals neither phase nor probabilities involved.

$$\mathbf{H}\ |0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

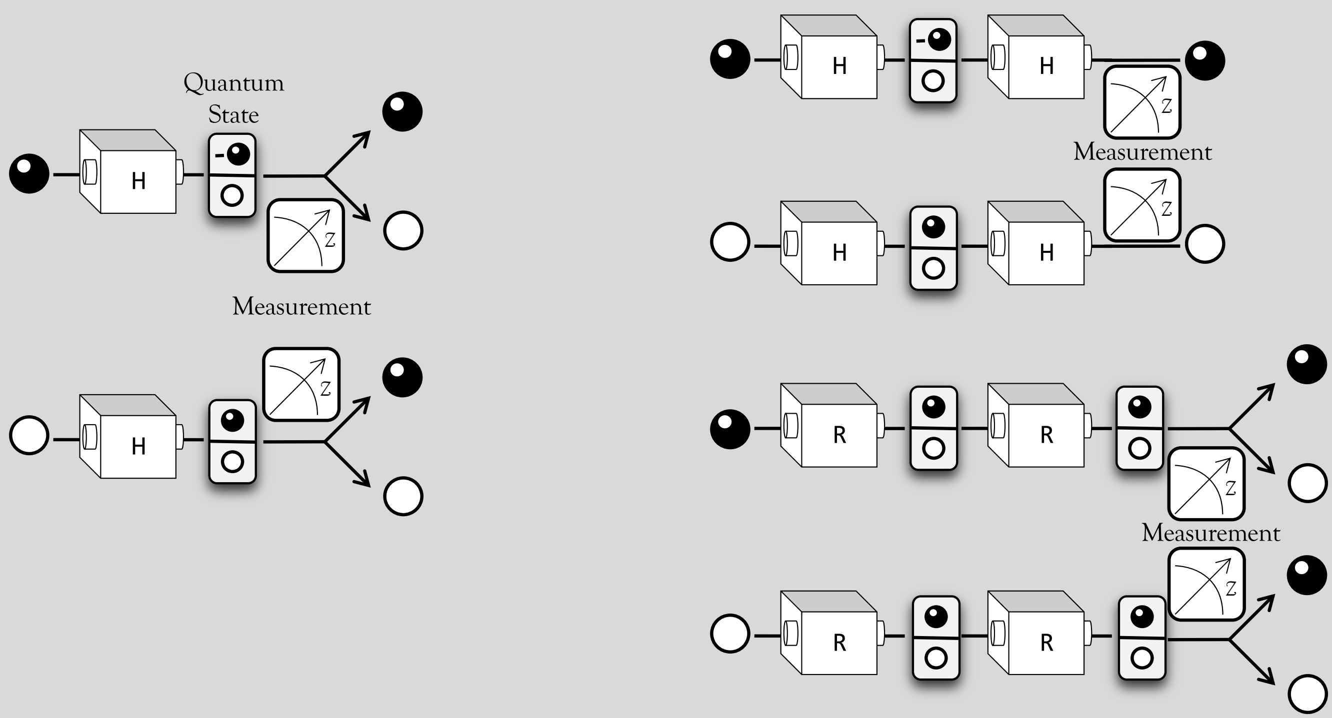
What if put an H-gate later?

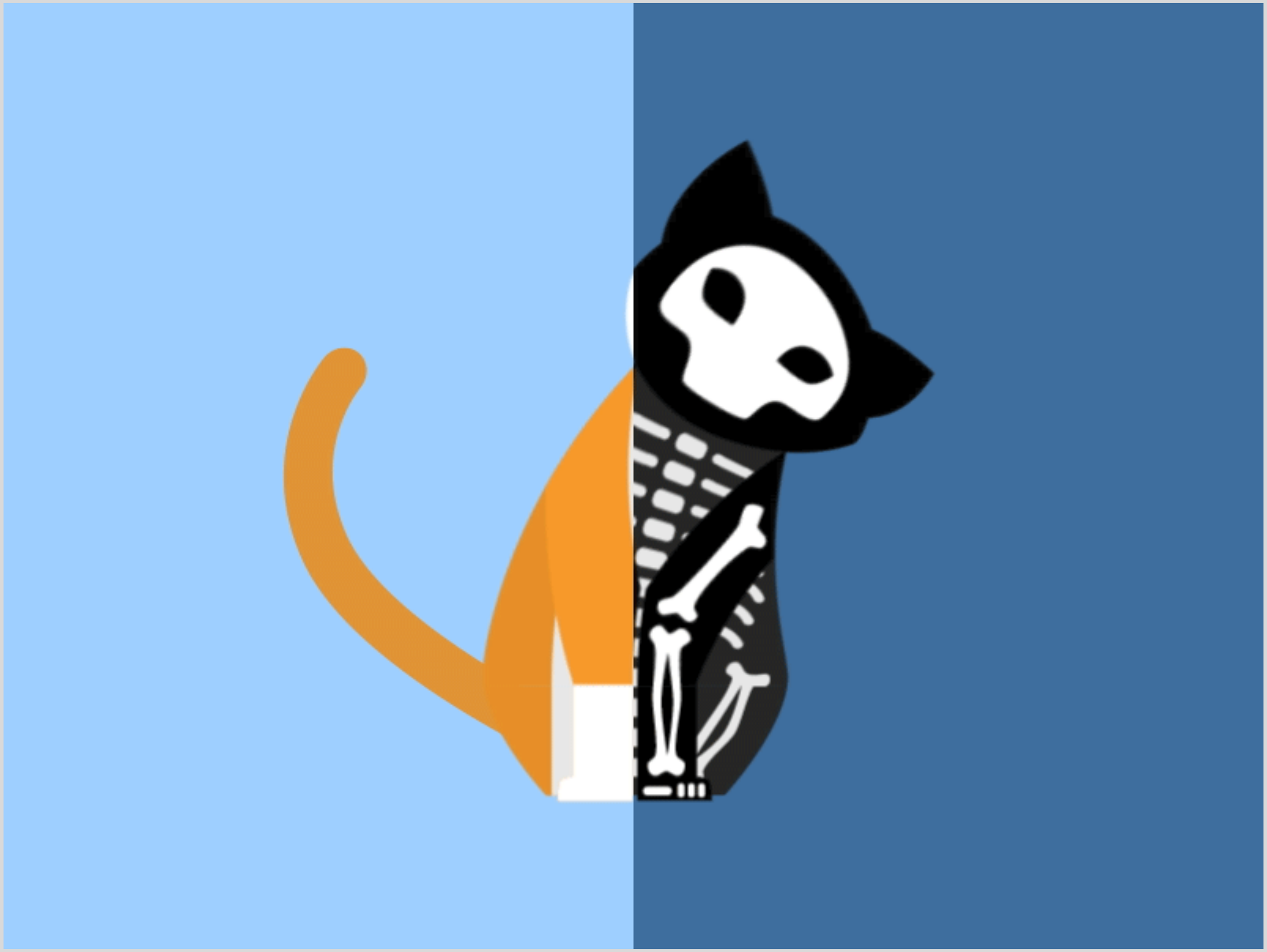
Revisiting the `H` gate (..)



Intermedia measurement,
unpredictable outcome

Revisiting the `H` gate (...)





Summary

- Superposition:
  - A single object **can be** multiple things at once,
  - State is **suspended** as a **combination** of multiple values,
- Quantum superposition:
  - A qubit is a superposition of two values: `|0> or |1>`,
  - Part of quantum state is the **probability** of measuring `0` or *1*,
  - The **probability** that a measurement detects one or the other can be **manipulated** through quantum operations,
  - **Measurement** cannot detect the entire state, only an individual `0` or *1*,
  - The act of **measurement collapses the superposition**, making the qubit become only the measured value `0` or *1*,
    - **We can't measure for phase and white and black at the same time.**

Summary (.)

- "By July of 1947 a prototype ten-stage adder was found to "function reliably for periods of several days," performing a complete carry through all ten stages in 0.6 microseconds. In August a ten-stage shift register was put through a life test for the entire month, and in February of 1948 a prototype accumulator was run, "at a-rate of about 100,000 additions per second," through five billion operations without a mistake."
- Logs entries:
  - "False start machine or human?" reads the first entry for a blast wave calculation run in February 1953. "Found Trouble in code—I hope!"
  - "Code error, machine not guilty," by Barricelli on March 4, 1953.
  - "What's the use? GOOD NIGHT", is recorded at 11:00 p.m. on May 7, 1953.
  - "Damnit—I can be just as stubborn as this thing" on June 14, 1953. "I'll never know why you have to load these codes twice sometimes to make them go, but they go usually the second time."
  - "I have now duplicated BOTH RESULTS how will I know which is right assuming one result is correct?" July 10, 1953. "This now is the 3rd different output", "I know when I'm licked."
  - 1953: "if only this machine would be just a little consistent".
  - "THE HELL WITH IT," is the final entry for June 17, 1956 .
  - "M/C OK. All troubles were code troubles" March 6, 1958.
- On one very hot day in May there was trouble with the IBM card equipment, and the machine log records: "IBM machine putting a tar-like substance on cards." The next log entry explains: "Tar is tar from roof."

See… "Turing's cathedral" – George Dyson

Bits y Qubits

Computer's anatomy and Bra-ket - Vector notation for Qubits

A classical computer

- Basic info unit: bit (binary digit) holds a 1 or 0 (at any given time)
  - A byte is made by 8 bits,
    - A nibble is made by 4 bits,
- A classical variable stores in n bits, one of 2^n possible values,
  - Programming languages hides effectively a lot of the associated complexity,
    - We know that very well in infosec:
      - And sometimes is really bad!

A quantum computer

- Basic info unit: qubit (quantum bit)
  - Holds `|0>`, `|1>` or `|0>` and `|1>` (some probability of measuring *0* or *1*)
  - Phase: positive (+) or negative (-),
- A quantum variable stores in n qubits, up to 2^n possible values, with a distinct probability of measuring each individual value,
  - In some way the computer evaluates in each step every one of the 2^n possible values,
    - Works on all possible inputs at the same time,
    - Variables are in "**quantum state**", a superposition of values,
    - Answers are always probabilistic,
      - When we read a quantum register it is forced into only *1* state,
        - We need a clever method that will allow a quantum register to evolve into a superposition of only acceptable solutions,
  - Programming languages still doesn't hide the associated complexity.

Quantum State: Bra-ket Notation

Expresses **probability of measuring** each of the possible states:

$$a\,|\text{👻}\rangle + b\,|\text{🎃}\rangle$$

Probability amplitude

ket

$|a|^2$: Probability of measuring 👻

$|b|^2$: Probability of measuring 🎃

Constrained by the equation: $|a|^2 + |b|^2 = 1, \quad a, b \in \mathbb{C}$

$$|a_r + \imath a_i| = \sqrt{a_r^2 + a_i^2}$$

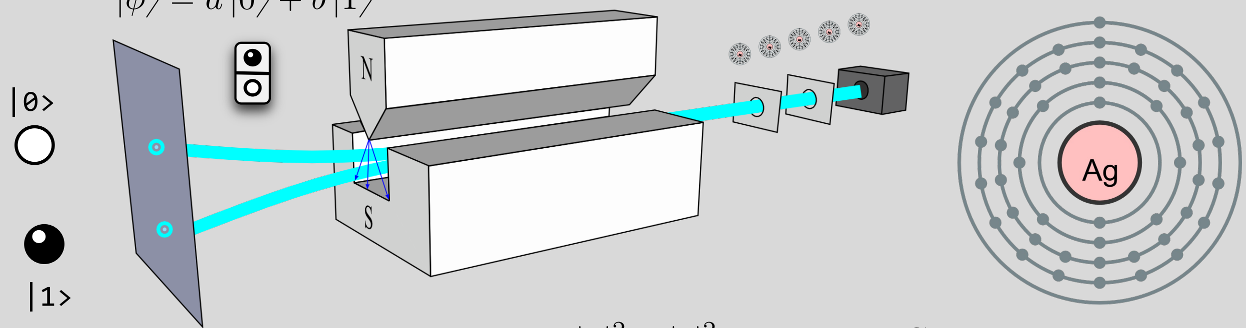
Bra-ket notation also indicates phase (+/-)

The Stern-Gerlach experiment and the information

$$|\phi\rangle = a\,|0\rangle + b\,|1\rangle$$

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$



$$|a|^2 + |b|^2 = 1, \quad a, b \in \mathbb{C}$$

$|a|^2$: Probability of measuring "0"

$|b|^2$: Probability of measuring "1"

$$|a_r + \imath a_i| = \sqrt{a_r^2 + a_i^2}$$

Ludwig Wittgenstein – Tractatus Logico-Philosophicus:

Theories and concepts we build are ladders or nets we use to reach the truth, but we must throw them away upon getting there.

And what about the balls?

|0>

Probability of measuring: 0: *1.0* 1: *0.0*

Phase: Positive (+)

Quantum State: $|\psi\rangle = 1\,|0\rangle + 0\,|1\rangle$

|1>

Probability of measuring: 0: 0.0 1: 1.0

Phase: Positive (+)

Quantum State: $|\psi\rangle = 0\,|0\rangle + 1\,|1\rangle$

Probability of measuring: 0: *0.5* 1: *0.5*

Phase: Positive (+)

Quantum State: $|\psi\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$

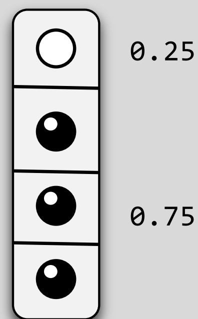
-

Probability of measuring: 0: *0.5* 1: *0.5*

Phase: Negative (-)

Quantum State: $|\psi\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle$

And what about the balls?
(.)



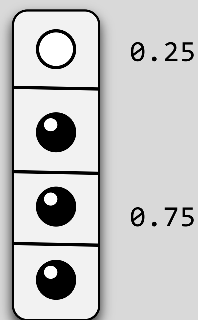
$$|\psi\rangle = \frac{1}{\sqrt{4}}|0\rangle + \frac{\sqrt{3}}{\sqrt{4}}|1\rangle$$

$$\left|\frac{1}{\sqrt{4}}\right|^2 + \left|\frac{\sqrt{3}}{\sqrt{4}}\right|^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$p(\text{measuring } 0) = \left|\frac{1}{\sqrt{4}}\right|^2 = \frac{1}{4}$$

$$p(\text{measuring } 1) = \left|\frac{\sqrt{3}}{\sqrt{4}}\right|^2 = \frac{3}{4}$$

And what about the balls?
(beyond the balls)



$$|\psi\rangle = \frac{\imath}{\sqrt{4}}\,|0\rangle - \imath\frac{\sqrt{3}}{\sqrt{4}}\,|1\rangle\,,$$

$$\left|0 + \frac{\imath}{\sqrt{4}}\right|^2 + \left|0 - \imath\frac{\sqrt{3}}{\sqrt{4}}\right|^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$p(\text{measuring } 0) = \left|0 + \frac{\imath}{\sqrt{4}}\right|^2 = \frac{1}{4}$$

$$p(\text{measuring } 1) = \left|0 - \imath\frac{\sqrt{3}}{\sqrt{4}}\right|^2 = \frac{3}{4}$$

Bra-ket algebra

- Conventions improves readability:
  - Component for `|0>` goes always first,
- Quantum notation simplifies in the same way as algebraic expressions:

$$|\psi\rangle = 1\,|0\rangle + 0\,|1\rangle = |0\rangle$$

- Less popular: factor out equivalent constants

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle = |0\rangle + |1\rangle$$

Qubit Bra-ket – Vector Notation

Expresses probability of measuring each possible state, and indicates phase (+/-)

$$|\psi\rangle = \underbrace{a}_{\text{Probability amplitude}} |0\rangle \underbrace{\pm}_{\text{Phase}} b\,|1\rangle$$

$|a|^2$: Prob. of measuring 0

$|b|^2$: Prob. of measuring 1

$$|a|^2 + |b|^2 = 1$$

Ket

$$|\psi\rangle = \begin{bmatrix} a \\ \pm b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\,|0\rangle \pm b\,|1\rangle$$

Bra-ket Notation

$$|a|^2 + |b|^2 = 1, \quad a, b \in \mathbb{C}$$

Vector Notation

$$|a_r + \imath a_i| = \sqrt{a_r^2 + a_i^2}$$

Some examples

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

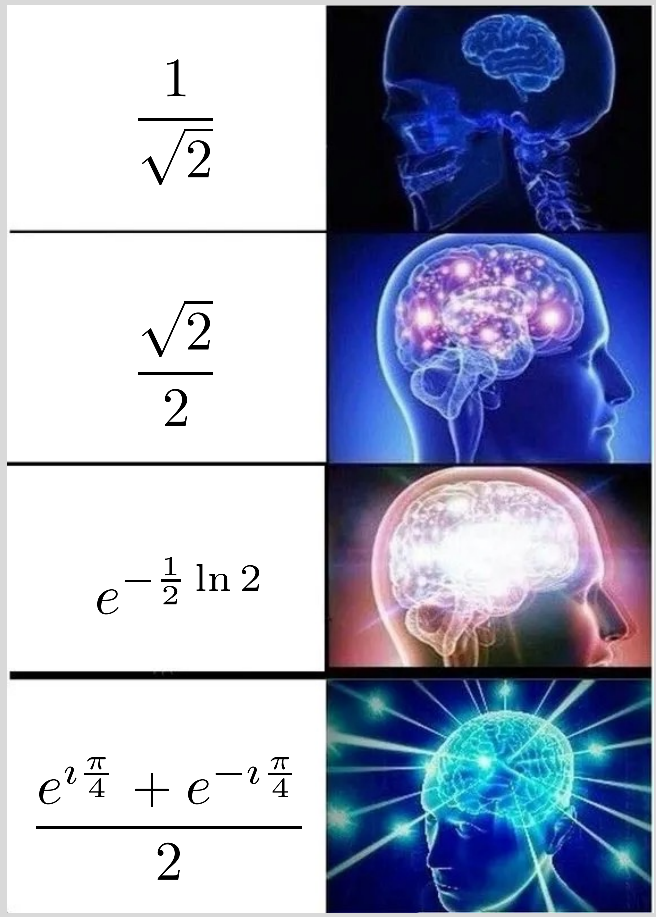
$$|\psi\rangle = \frac{1}{2}|1\rangle + \frac{\sqrt{3}}{2}|0\rangle \qquad \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$|\psi\rangle = \frac{\sqrt{2}}{2}|0\rangle + \imath\frac{\sqrt{2}}{2}|1\rangle \qquad \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix}$$

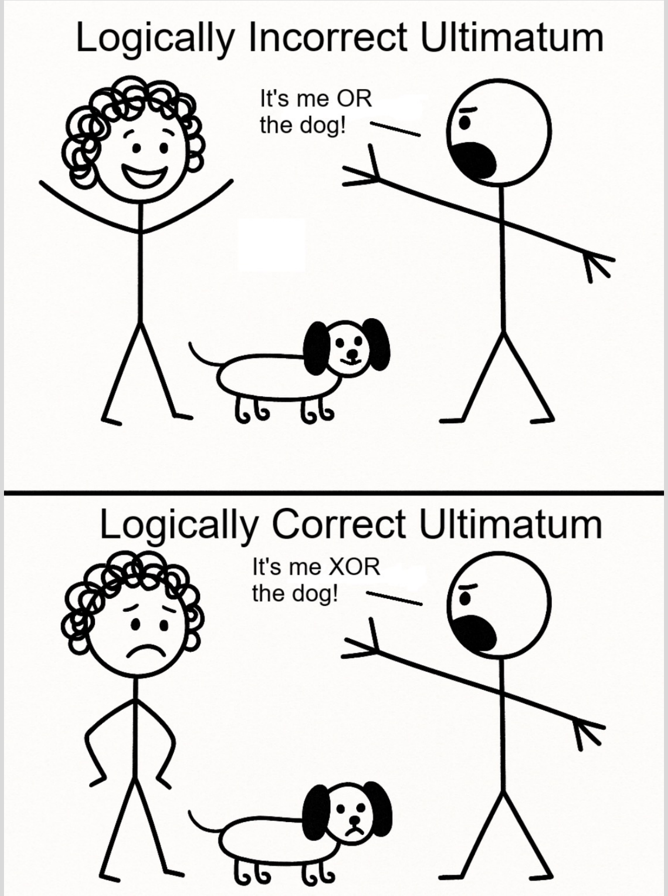
$$|\psi\rangle = \frac{\sqrt{2}}{2}|0\rangle - \imath\frac{\sqrt{2}}{2}|1\rangle \qquad \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$|\psi\rangle = \imath|1\rangle \qquad \begin{bmatrix} 0 \\ \imath \end{bmatrix}$$

A meme just for fun



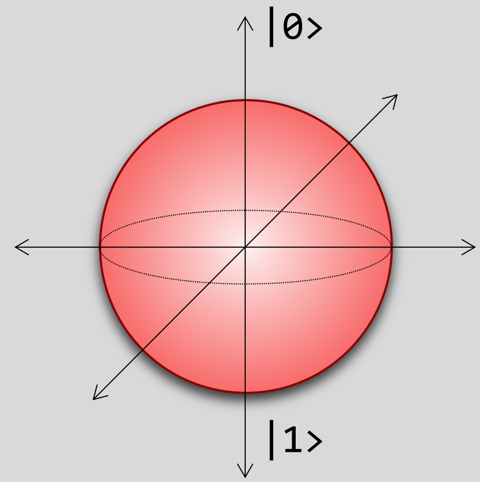
Another meme just for fun



Bits and Qubits

- Classical information bits have value of only *0* or *1*,
- Quantum state can hold a state of superposition:
  - **Before** measurement: $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
  - **After** measurement: $|0\rangle$ or $|1\rangle$
- How would a single bit or qubit be represented visually?

Visualizing bits and qubits



Classical bit – only possible states are *0* and *1*. Represented as endpoint of a line (nothing between).

Qubit – measurement allows state measurement of only $|0\rangle \longmapsto 0$ or $|1\rangle \longmapsto 1$, but superposition and phase allows an infinite set of states on the **surface** of a unit sphere. The unit sphere reflects that for the qubit

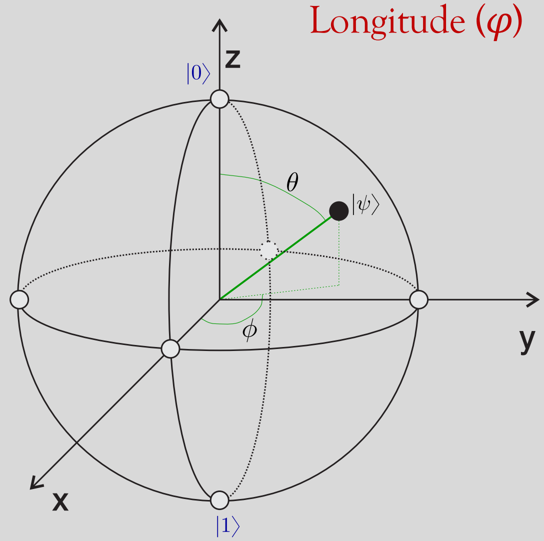
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha, \beta \in \mathbb{C}$$

$$|\alpha_r + \imath\alpha_i| = \sqrt{\alpha_r^2 + \alpha_i^2}$$

Quantum State: Bloch sphere Notation

A sphere, with diameter *2* units, with antipodal points corresponding to a pair of mutually orthogonal state vectors `|0>` and `|1>`

Latitude (θ)
Longitude (φ)



$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + (\cos\phi + \imath\sin\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi, \quad \theta, \phi \in \mathbb{R}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

$$\theta = 0, \phi = 0, \quad |\psi\rangle = \cancelto{1}{\cos(0)}|0\rangle + \cancelto{0}{e^{\imath\phi}\sin(0)}|1\rangle = |0\rangle,$$

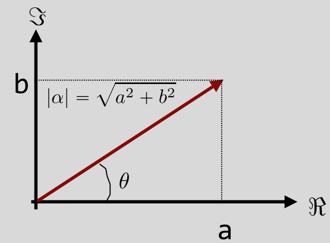
$$\theta = \pi, \phi = 0, \quad |\psi\rangle = \cancelto{0}{\cos\left(\frac{\pi}{2}\right)}|0\rangle + \cancelto{1}{e^{\imath\phi}\sin\left(\frac{\pi}{2}\right)}|1\rangle = |1\rangle.$$

That's why $\theta/2$

`|0⟩` and `|1⟩` are perpendicular states geometrically at 90°. On the Bloch sphere should be at 180°.

Quantum State: Bloch sphere Notation (explanation)

Two complex numbers, each one:



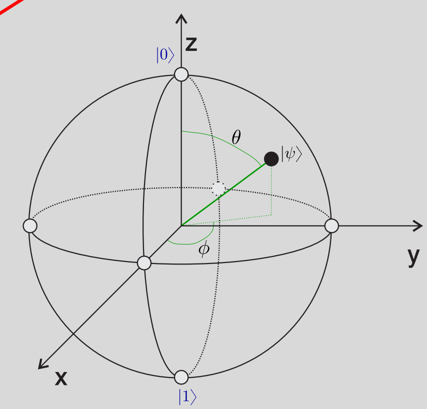
Relative phase (observable)

Global phase (not observable)

$$a + \imath b = \Upsilon e^{\imath\theta}, \quad \Upsilon = \sqrt{a^2+b^2}$$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle, \quad \alpha,\beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1$$

$$|\psi\rangle = \Upsilon_0 e^{\imath\theta_0}\,|0\rangle + \Upsilon_1 e^{\imath\theta_1}\,|1\rangle = e^{\imath\theta_0}\left(\Upsilon_0\,|0\rangle + \Upsilon_1 e^{\imath(\theta_1-\theta_0)}\,|1\rangle\right).$$



Notice that $\left|e^{\imath\theta_0}\right|^2 = |\cos\theta_0 + \imath\sin\theta_0|^2 = (\cos\theta_0 + \imath\sin\theta_0)(\cos\theta_0 - \imath\sin\theta_0) = \cos^2\theta_0 + \sin^2\theta_0 = 1.$

$$\text{So} \quad |\psi\rangle = \left(\Upsilon_0\,|0\rangle + \Upsilon_1 e^{\imath(\theta_1-\theta_0)}\,|1\rangle\right) = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$\text{So} \quad \Upsilon_0 = \cos\frac{\theta}{2}, \quad \Upsilon_1 = \sin\frac{\theta}{2}, \quad \phi = \theta_1 - \theta_0, \quad \Upsilon_0^2 + \Upsilon_1^2 = 1.$$

A qubit's representation based on two **real** angles.

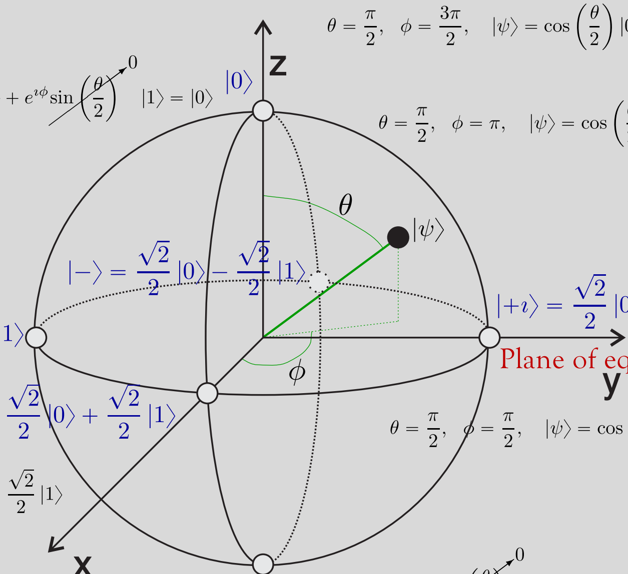
Quantum State: Bloch sphere Notation (.)

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + (\cos\phi + \imath\sin\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi$$

$$\theta = \frac{\pi}{2}, \quad \phi = \frac{3\pi}{2}, \quad |\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \frac{\sqrt{2}}{2}|0\rangle + \left(\cancelto{0}{\cos\frac{3\pi}{2}} + \imath\cancelto{-1}{\sin\frac{3\pi}{2}}\right)\frac{\sqrt{2}}{2}|1\rangle = \frac{\sqrt{2}}{2}|0\rangle - \imath\frac{\sqrt{2}}{2}|1\rangle$$

$$\theta = 0, \quad |\psi\rangle = \cancelto{1}{\cos\left(\frac{\theta}{2}\right)}|0\rangle + e^{\imath\phi}\cancelto{0}{\sin\left(\frac{\theta}{2}\right)}|1\rangle = |0\rangle$$

$$\theta = \frac{\pi}{2}, \quad \phi = \pi, \quad |\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \frac{\sqrt{2}}{2}|0\rangle + \left(\cancelto{-1}{\cos\pi} + \imath\cancelto{0}{\sin\pi}\right)\frac{\sqrt{2}}{2}|1\rangle = \frac{\sqrt{2}}{2}|0\rangle - \frac{\sqrt{2}}{2}|1\rangle$$



$$|-\rangle = \frac{\sqrt{2}}{2}|0\rangle - \frac{\sqrt{2}}{2}|1\rangle$$

$$|+\imath\rangle = \frac{\sqrt{2}}{2}|0\rangle + \imath\frac{\sqrt{2}}{2}|1\rangle$$

$$|-\imath\rangle = \frac{\sqrt{2}}{2}|0\rangle - \imath\frac{\sqrt{2}}{2}|1\rangle$$

Plane of equal superpositions |0> |1>. Only changes the phase (φ).

$$|+\rangle = \frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$$

$$\theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{2}, \quad |\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \frac{\sqrt{2}}{2}|0\rangle + \left(\cancelto{0}{\cos\frac{\pi}{2}} + \imath\cancelto{1}{\sin\frac{\pi}{2}}\right)\frac{\sqrt{2}}{2}|1\rangle = \frac{\sqrt{2}}{2}|0\rangle + \imath\frac{\sqrt{2}}{2}|1\rangle$$

$$\theta = \frac{\pi}{2}, \quad \phi = 0, \quad |\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$$

$$\theta = \pi, \quad |\psi\rangle = \cancelto{0}{\cos\left(\frac{\theta}{2}\right)}|0\rangle + e^{\imath\phi}\cancelto{1}{\sin\left(\frac{\theta}{2}\right)}|1\rangle = |1\rangle$$

$$\theta, \phi \in \mathbb{R}$$

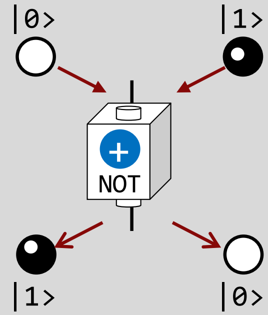
A Gate as a Matrix and a matrix as a linear transformation

$$|\psi\rangle = 1\,|0\rangle + 0\,|1\rangle$$

$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi\rangle = 0\,|0\rangle + 1\,|1\rangle$$

$$|\psi\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$|\psi\rangle = 0\,|0\rangle + 1\,|1\rangle$$

$$|\psi\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

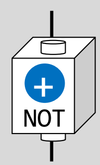
$$|\psi\rangle = 1\,|0\rangle + 0\,|1\rangle$$

$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

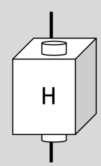
A Gate as a Matrix and a matrix as a linear transformation (.)

Let's represent a Quantum Gate as a linear transformation

$$\underbrace{\begin{bmatrix} - & - \\ - & - \end{bmatrix}}_{\text{Quantum Gate}} \underbrace{\begin{bmatrix} - \\ - \end{bmatrix}}_{\text{Initial Qubit}} = \underbrace{\begin{bmatrix} - \\ - \end{bmatrix}}_{\text{Result Qubit}}$$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



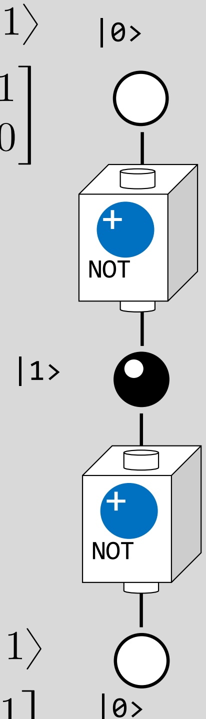
$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A Matrix is a physical process

A vector is a mathematical gadget to describe the state of a quantum system

A Gate as a Matrix and a matrix as a linear transformation (`NOT` gate)

$$|\psi\rangle = 1\,|0\rangle + 0\,|1\rangle \qquad \begin{bmatrix}1\\0\end{bmatrix}$$



$$|\psi\rangle = 0\,|0\rangle + 1\,|1\rangle \qquad \begin{bmatrix}0\\1\end{bmatrix}$$

$$|\psi\rangle = 1\,|0\rangle + 0\,|1\rangle \qquad \begin{bmatrix}1\\0\end{bmatrix}$$

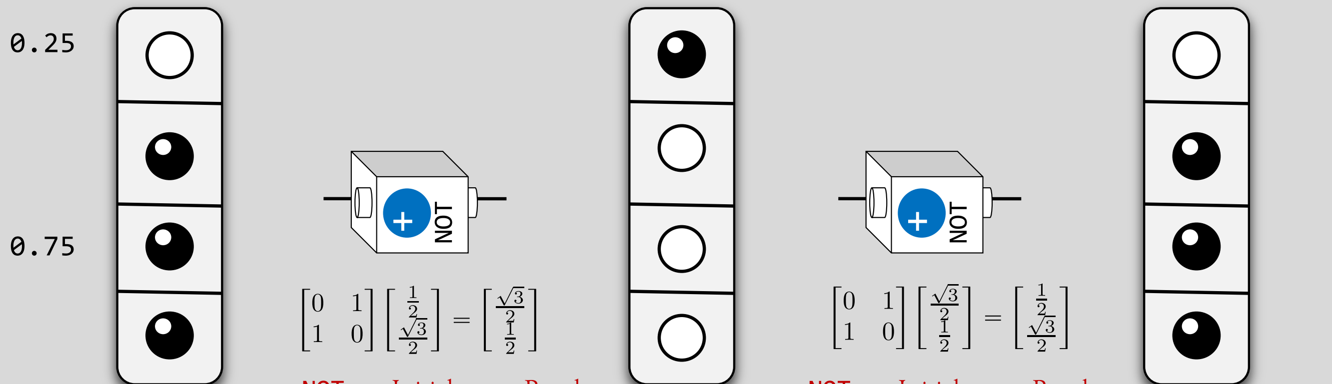
$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

`NOT` Gate — Initial Qubit — Result Qubit

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

A Gate as a Matrix and a matrix as a linear transformation (`NOT` gate)

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad |\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \qquad |\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$



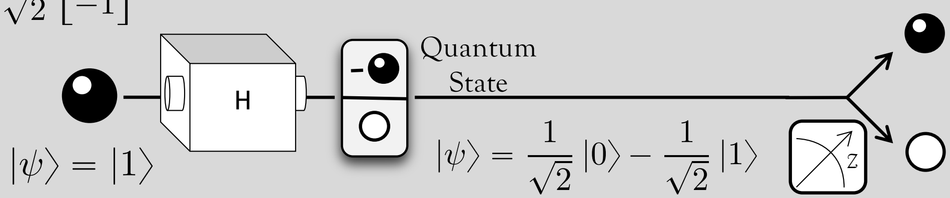
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

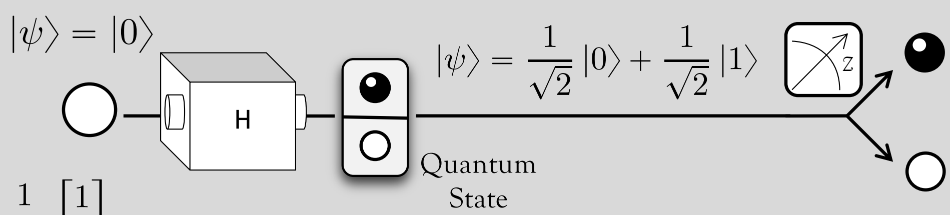
`NOT` Gate — Initial Qubit — Result Qubit — `NOT` Gate — Initial Qubit — Result Qubit

$$\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \qquad \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

A Gate as a Matrix and a matrix as a linear transformation (H gate)

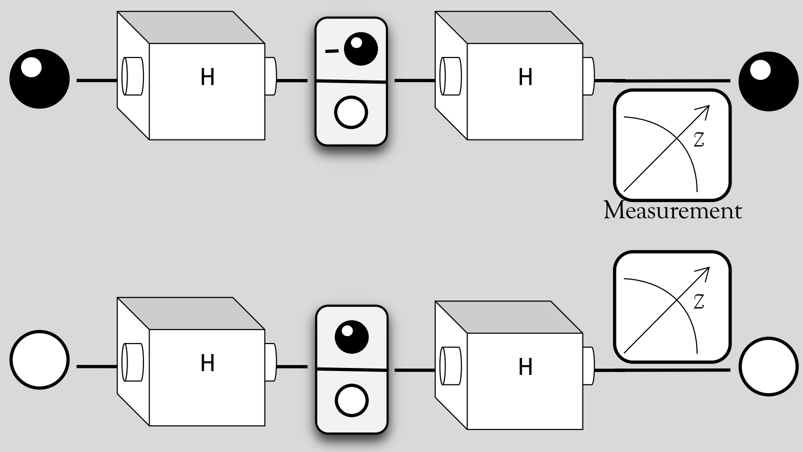
$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$





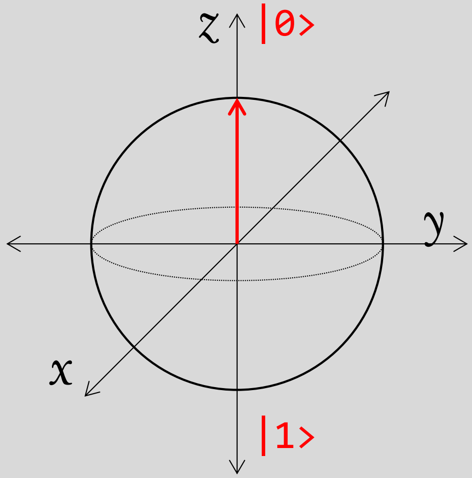
$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

A Gate as a Matrix and a matrix as a linear transformation (H gate)

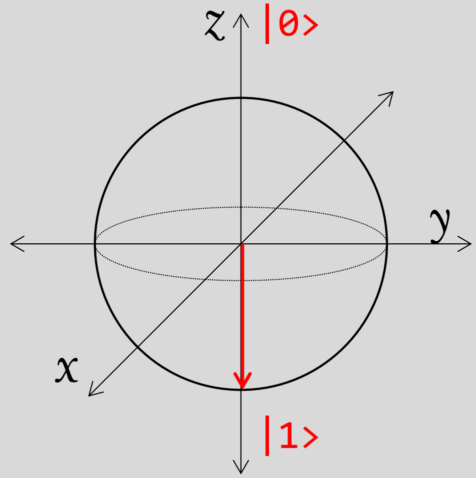


$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \mathbb{I}_{2\times 2}$$

Qubits on the Bloch Sphere

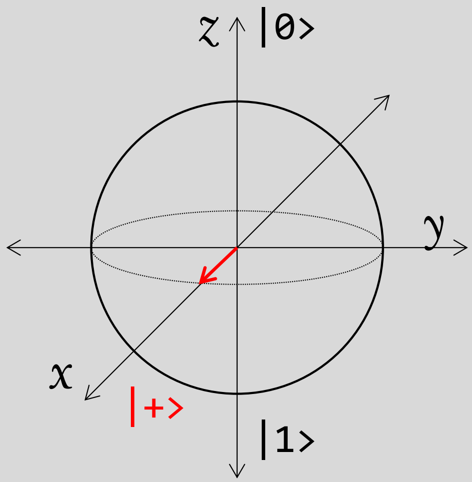


$$1\,|0\rangle + 0\,|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

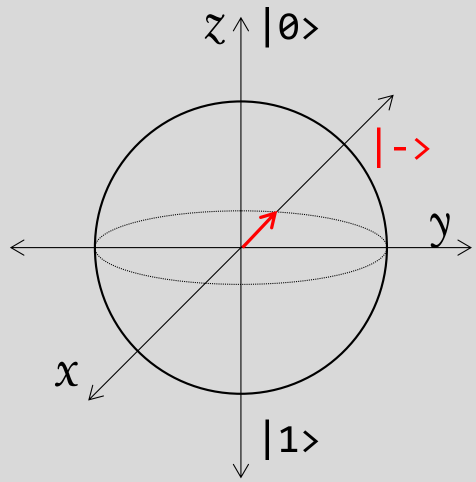


$$0\,|0\rangle + 1\,|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Qubits on the Bloch Sphere (.)



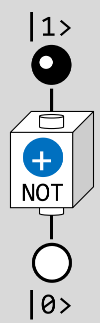
$$\frac{1\,|0\rangle + 1\,|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$



$$\frac{1\,|0\rangle - 1\,|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

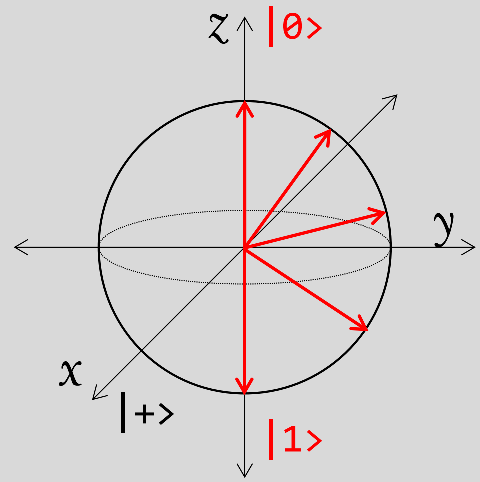
The equator of the Bloch sphere shows infinite qubits with *50/50* possibilities of measuring `|0>` or `|1>`

Behavior of the `NOT` (X) gate



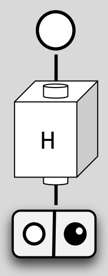
$$0\,|0\rangle + 1\,|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = 1\,|0\rangle + 0\,|1\rangle$$



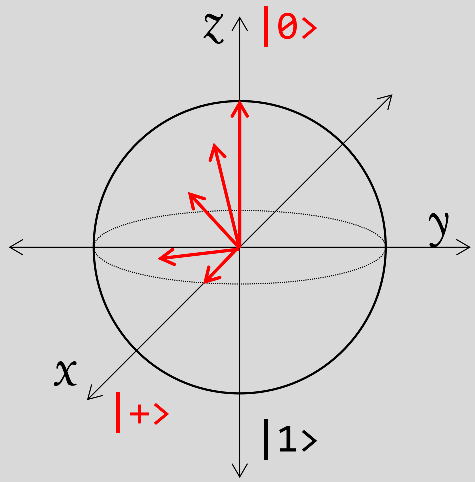
The `NOT` implements a rotation of π around the *x*-axis!

Behavior of the H gate



$$1\,|0\rangle + 0\,|1\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1\,|0\rangle + 1\,|1\rangle}{\sqrt{2}}$$



The H implements a rotation of $\pi/2$ around the y-axis!

The Bloch Sphere: a deeper look

- Qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ falls on sphere's surface
  - Remember, $|\alpha|^2 + |\beta|^2 = 1, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha_r + \imath\alpha_i| = \sqrt{\alpha_r^2 + \alpha_i^2}$
- Single qubit quantum operations implements rotations around x, y, and z axis to transform the state of $|\psi\rangle$
- We can describe a qubit as:

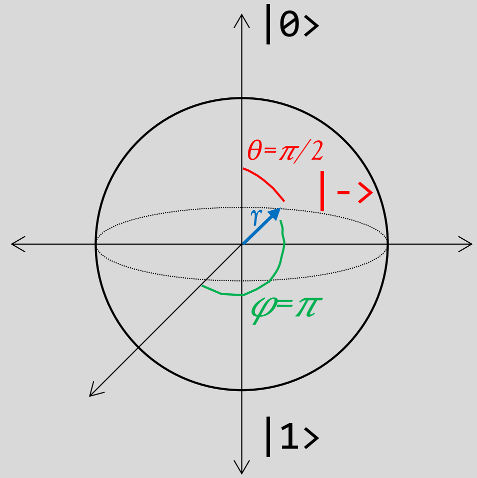
$$\alpha\,|0\rangle + \beta\,|1\rangle \equiv \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \equiv \underbrace{\cos\left(\frac{\theta}{2}\right)}_{\text{Probability amplitudes}} |0\rangle + \underbrace{e^{\imath\phi}}_{\text{Phase}} \underbrace{\sin\left(\frac{\theta}{2}\right)}\,|1\rangle \equiv \cos\left(\frac{\theta}{2}\right)|0\rangle + (\cos\phi + \imath\sin\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle, \qquad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi$$

Example: check location of $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

So, $\quad \theta = \pi/2, \quad \phi = \pi$

(Notice that $\;e^{\imath\pi} = \cos\pi + \imath\sin\pi = -1$)



Exercises

- Given the ket: $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$
  - Identify its position on the Bloch sphere,
  - Apply the "`Z`" gate... Identify the new position,
  - Apply the "`Y`" gate... Identify the new position,
  - Apply the "`X`" gate... Identify the new position,
  - Apply the "`H`" gate... Identify the new position,
  - Remember $-\imath\,\mathbf{X}\,\mathbf{Y}\,\mathbf{Z} = \mathbf{I}$
- Is the "`NOT`" gate an authentic NOT gate?
  - Yes, for pure states, because $\mathbf{X}|0\rangle = |1\rangle, \mathbf{X}|1\rangle = |0\rangle,$
    - So, the "`NOT`" gate moves a ket to its antipode in the Bloch sphere?

Exercises

- Intuition is a bitch! $$\frac{1}{2}\begin{bmatrix} 1+\imath \\ 1-\imath \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} \quad ?$$

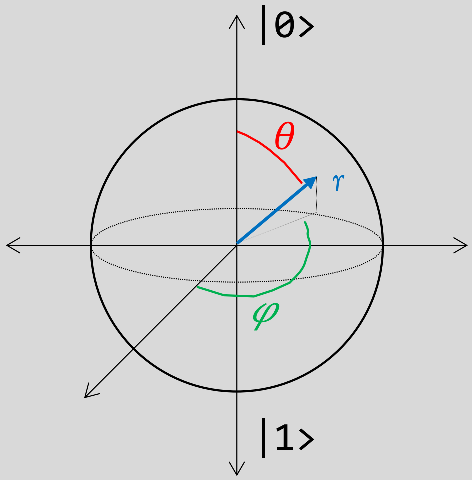
$$\frac{1}{2}\begin{bmatrix} 1+\imath \\ 1-\imath \end{bmatrix} = \frac{1}{2}\frac{1-\imath}{1-\imath}\begin{bmatrix} 1+\imath \\ 1-\imath \end{bmatrix} = \frac{1}{2}\frac{1}{1-\imath}\begin{bmatrix} (1-\imath)\,(1+\imath) \\ (1-\imath)\,(1-\imath) \end{bmatrix} = \frac{1}{2}\frac{1}{1-\imath}\begin{bmatrix} 1-\imath^2 \\ \cancel{1}-2\imath+\cancel{\imath^2} \end{bmatrix}$$

$$= \frac{1}{\cancel{2}}\frac{1}{1-\imath}\begin{bmatrix} \cancel{2} \\ -\cancel{2}\imath \end{bmatrix} = \frac{1}{1-\imath}\begin{bmatrix} 1 \\ -\imath \end{bmatrix} = \frac{1}{1-\imath}\frac{2}{\sqrt{2}}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ -\imath \end{bmatrix} = \frac{1}{1-\imath}\frac{2}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |-\imath\rangle$$

Not observable

Summary

- Angle θ influences the probability of observing |0> or |1>,
  - As increases θ, more likely to observe a |1>,
  - Actually for $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle,\quad p\left(|0\rangle\right) = \cos^2\left(\frac{\theta}{2}\right),\quad p\left(|1\rangle\right) = \sin^2\left(\frac{\theta}{2}\right).$



- Angle φ influences phase between |0> or |1>,
  - Phase is never measured, but is important for quantum algorithms,
  - The phase difference between complex component |0> and |1> is the **relative phase**: $|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$

- **Global phase** common to |0> and |1> has no observable impact on quantum state and is typically discarded:

$$-|\psi\rangle = -\left(\alpha|0\rangle + \beta|1\rangle\right) = -\alpha|0\rangle - \beta|1\rangle \equiv \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$$
$$e^{\imath\gamma}|\psi\rangle = e^{\imath\gamma}\left(\alpha|0\rangle + \beta|1\rangle\right) \equiv \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$$

More on this later…

Bra-ket algebra (Measurement)

- Any 'Ket' `|a>` has a corresponding 'Bra' `<a|`

$$|\psi\rangle = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_n \end{bmatrix}, \qquad \langle\psi| = \left[\ \overline{\psi_0}, \overline{\psi_1}, \ldots \overline{\psi_n}\ \right], \qquad \overline{\psi_i} \text{ is the complex conjugate of } \psi_i,$$

column vector

row vector

$$\overline{a + \imath b} = a - \imath b$$

- Note that:

$$\langle a|\,|b\rangle = \langle a\,|\,b\rangle = \overline{a_0}b_0 + \overline{a_1}b_1 + \ldots + \overline{a_n}b_n$$ escalar

$$\langle 0\,|\,0\rangle = [\ \overline{1}, \overline{0}\ ] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\times 1 \ + \ 0\times 0 = 1$$ normalized magnitude is *1*

$$\langle 0\,|\,1\rangle = [\ \overline{1}, \overline{0}\ ] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\times 0 \ + \ 0\times 1 = 0$$ orthogonality

$$\langle 1\,|\,0\rangle = [\ \overline{0}, \overline{1}\ ] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0\times 1 \ + \ 1\times 0 = 0$$ orthogonality

$$\langle 1\,|\,1\rangle = [\ \overline{0}, \overline{1}\ ] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\times 0 \ + \ 1\times 1 = 1$$ normalized magnitude is *1*

Bra-ket algebra (Measurement)(.)

- Measuring is applying bra-ket… Given a qubit described by |ψ> , the probability that the qubit is measured as |x> is:

$$\left|\langle x \mid \psi\rangle^2\right| = \left|(\overline{x_0}\psi_0 + \overline{x_1}\psi_1 + \ldots + \overline{x_n}\psi_n)^2\right|$$

- Let a qubit $|q_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{\imath}{\sqrt{2}}|1\rangle$

  - Probability of q_0 measure 0 is $\left|\langle 0 \mid q_0\rangle^2\right| = \left|\left(\left[\,\overline{1},\overline{0}\,\right]\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{\imath}{\sqrt{2}}\end{bmatrix}\right)^2\right| = \left|\left(\frac{1}{\sqrt{2}}\right)^2\right| = \frac{1}{2}$

  - Probability of q_0 measure 1 is $\left|\langle 1 \mid q_0\rangle^2\right| = \left|\left(\left[\,\overline{0},\overline{1}\,\right]\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{\imath}{\sqrt{2}}\end{bmatrix}\right)^2\right| = \left|\left(\frac{\imath}{\sqrt{2}}\right)^2\right| = \left|\frac{-1}{2}\right| = \frac{1}{2}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = (\alpha_r + \imath\alpha_i)|0\rangle + (\beta_r + \imath\beta_i)|1\rangle$$

$$\left|\langle\psi \mid \psi\rangle^2\right| = \left|\overline{\psi_0}\psi_0 + \overline{\psi_1}\psi_1 + \ldots + \overline{\psi_n}\psi_n\right|^2 = \left|\left(\left[\,\overline{\alpha},\overline{\beta}\,\right]\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right)^2\right| = \left|\left(\left[\,\alpha_r - \imath\alpha_i, \beta_r - \imath\beta_i\,\right]\begin{bmatrix}\alpha_r + \imath\alpha_i\\ \beta_r + \imath\beta_i\end{bmatrix}\right)^2\right| = \left|(\alpha_r^2 + \alpha_i^2 + \beta_r^2 + \beta_i^2)^2\right| = \left|\left(|\alpha|^2 + |\beta|^2\right)^2\right| = \left|1^2\right| = 1$$

Normalization condition

Bra-ket algebra (Measurement)(..)

- Let a qubit $|q_1\rangle = \imath\,|1\rangle$
  - Probability of q_1 measure 0 is $\left|\langle 0\,|\,q_1\rangle^2\right| = \left|\left(\left[\,\bar{1},\bar{0}\,\right]\begin{bmatrix}0\\ \imath\end{bmatrix}\right)^2\right| = \left|(0)^2\right| = 0$
  - Probability of q_1 measure 1 is $\left|\langle 1\,|\,q_1\rangle^2\right| = \left|\left(\left[\,\bar{0},\bar{1}\,\right]\begin{bmatrix}0\\ \imath\end{bmatrix}\right)^2\right| = \left|(\imath)^2\right| = |-1| = 1$
- Let a qubit $|q_2\rangle = |1\rangle$
  - Probability of q_2 measure 0 is $\left|\langle 0\,|\,q_2\rangle^2\right| = \left|\left(\left[\,\bar{1},\bar{0}\,\right]\begin{bmatrix}0\\ 1\end{bmatrix}\right)^2\right| = \left|(0)^2\right| = 0$
  - Probability of q_2 measure 1 is $\left|\langle 1\,|\,q_2\rangle^2\right| = \left|\left(\left[\,\bar{0},\bar{1}\,\right]\begin{bmatrix}0\\ 1\end{bmatrix}\right)^2\right| = \left|(1)^2\right| = 1$

measurement is the only way to extract information from a qubit, so these two states are equivalent in all ways that are physically relevant

Bra-ket algebra (Measurement)(...)

- Let a qubit $|q_2\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \sqrt{2}\,|1\rangle\right)$
  - Probability of q_2 measure 0 is $\left|\left\langle 0 \middle| \frac{1}{\sqrt{3}}\left(|0\rangle + \sqrt{2}\,|1\rangle\right)\right\rangle^2\right| = \left|\left(\frac{1}{\sqrt{3}}\cancelto{1}{\langle 0|0\rangle} + \sqrt{\frac{2}{3}}\cancelto{0}{\langle 0|1\rangle}\right)^2\right| = \frac{1}{3}$
  - Probability of q_2 measure 1 is $\frac{2}{3}$
- Let's measure qubit $|q_3\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$ in the basis {|+>,|->}:
  - Probability of q_3 measure **+** is

$$\left|\left\langle + \middle| \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right\rangle^2\right| = \left|\left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right)^2\right| = \left|\frac{1}{2}\left(\cancelto{1}{\langle 0|0\rangle} - \cancelto{0}{\langle 0|1\rangle} + \cancelto{0}{\langle 1|0\rangle} - \cancelto{1}{\langle 1|1\rangle}\right)^2\right| = 0$$

Of course, we already know that $|q_3\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$ and |+> and |-> are orthogonal

  - Probability of q_3 measure **-** is 1

Bra-ket algebra (Measurement)(weirdness)

- Suppose a qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle,\ \alpha, \beta \in \mathbb{C}$ is measured and got:

$$\left|\langle 0\,|\,\psi\rangle^2\right| = \left|\left(\left[\,\overline{1}, \overline{0}\,\right]\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right)^2\right| = 1$$

$$\left|\langle 1\,|\,\psi\rangle^2\right| = \left|\left(\left[\,\overline{0}, \overline{1}\,\right]\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right)^2\right| = 0$$

- If we measure again, there is a *100%* chance of finding the qubit in the state `|0>`
  - We already know, the act of measurement changes the state of the qubit: α=1, β=0
    - **collapses** the quantum state into a classical bit,
      - Extract **information** from the qubit and destroy it.
    - and we lost all the advantages of quantum computing.

Math detour
(you can stay safely at the main road)

- A vector space $\mathbb{V}$ over a field $\mathbb{F}$ is a set of vectors where two conditions hold:
  - If $|a\rangle, |b\rangle \in \mathbb{V}, \quad |a\rangle + |b\rangle \in \mathbb{V},$
  - For any $n \in \mathbb{F}, |a\rangle \in \mathbb{V}, \quad n\,|a\rangle \in \mathbb{V}.$
- A set of vectors S spans a subspace of the vector space $\mathbb{V}$ ($\mathbb{V}_S \subseteq \mathbb{V}$), if we can write any vector in $\mathbb{V}_S$ as a linear combination of vectors in S:
  - For any $|v_s\rangle \in \mathbb{V}_S, \quad |v_s\rangle = f_1\,|v_1\rangle + f_2\,|v_2\rangle + \ldots + f_n\,|v_n\rangle = \sum_i f_i\,|v_i\rangle, \quad f_i \in \mathbb{F}, \quad |v_i\rangle \in S$

General idea: shrink-down vector spaces and express them in terms of only a few vectors

Math detour
(vector spaces and basis)

- A basis is a linearly independent spanning set.
- The basis of a vector space is the minimal possible set that spans the entire space:
  - The size of the basis set is the dimension of the vector space,
- In quantum computing we have basis:
  - `|0>,|1>`
  - `|+i>,|-i>`
  - `|+>,|->`

Math detour
(Hilbert spaces and inner products)

- A Hilbert space is a vector space with an inner product. Let's define for two vectors `|a>,|b>`

$$\langle a|\,|b\rangle = \langle a\,|\,b\rangle = [a_1^*, a_2^*, ..., a_n^*]\begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix} = a_1^* b_1 + a_2^* b_2 + ... + a_n^* b_n$$

- We can use a Hilbert space to represent a quantum system with an additional condition:

$$\langle \phi|\,|\phi\rangle = \langle \phi\,|\,\phi\rangle = 1$$

The probability of the quantum system being measured in the state that it is in must be *1.*

The sum of the probabilities of finding the quantum system in any particular state must equal *1.*

The surface of the Bloch sphere, along the inner product between qubit state vectors is a valid Hilbert space.

Math detour
(scalar product)

- Let a qubit $|\psi\rangle = a\,|\text{👻}\rangle + b\,|\text{🎃}\rangle$

  Probability amplitude

  ket

  $|a|^2$: Prob. measuring 👻

  $|b|^2$: Prob. of measuring 🎃

  Constrained by the equation: $|a|^2 + |b|^2 = 1, \quad a, b \in \mathbb{C}$ $\qquad |a_r + \imath a_i| = \sqrt{a_r^2 + a_i^2}$ $\qquad |a|^2 + |b|^2 = 1 = a_r^2 + a_i^2 + b_r^2 + b_i^2$

- Let's multiply the qubit for a complex number $\Upsilon e^{\imath\theta} = \Upsilon\left(\cos\theta + \imath\sin\theta\right)$

$$\Upsilon e^{\imath\theta}\,|\psi\rangle = \Upsilon e^{\imath\theta}\left(a\,|\text{👻}\rangle + b\,|\text{🎃}\rangle\right) = \Upsilon e^{\imath\theta} a\,|\text{👻}\rangle + \Upsilon e^{\imath\theta} b\,|\text{🎃}\rangle$$

- Calculate now its magnitude

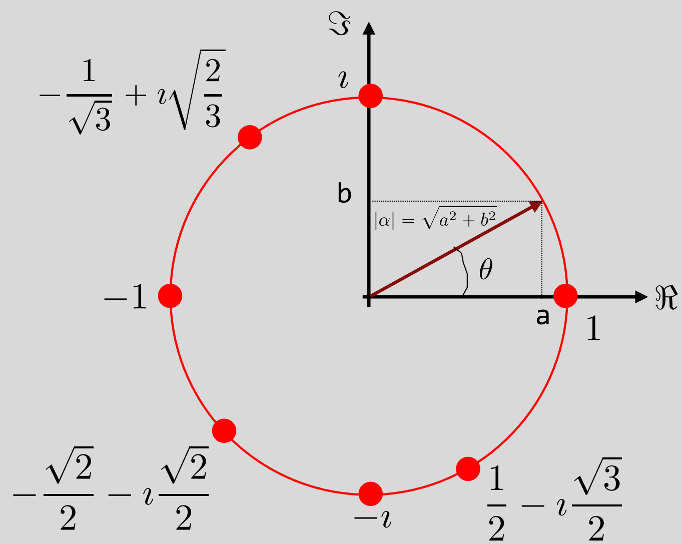
$$\left|\Upsilon e^{\imath\theta} a\right|^2 + \left|\Upsilon e^{\imath\theta} b\right|^2 = \left|\Upsilon e^{\imath\theta}(a_r + \imath a_i)\right|^2 + \left|\Upsilon e^{\imath\theta}(b_r + \imath b_i)\right|^2 = \left|\Upsilon e^{\imath\theta} a_r + \imath\Upsilon e^{\imath\theta} a_i\right|^2 + \left|\Upsilon e^{\imath\theta} b_r + \imath\Upsilon e^{\imath\theta} b_i\right|^2$$

$$= \Upsilon^2 e^{2\imath\theta} a_r^2 + \Upsilon^2 e^{2\imath\theta} a_i^2 + \Upsilon^2 e^{2\imath\theta} b_r^2 + \Upsilon^2 e^{2\imath\theta} b_i^2 = \Upsilon^2 e^{2\imath\theta}\cancelto{1}{\left(a_r^2 + a_i^2 + b_r^2 + b_i^2\right)} = \Upsilon^2 e^{2\imath\theta} = \Upsilon^2\left(e^{\imath\theta}\right)^2 = \Upsilon^2\left(\cos\theta + \imath\sin\theta\right)^2$$

$$= \Upsilon^2\left(\cos\theta + \imath\sin\theta\right)\left(\cos\theta - \imath\sin\theta\right) = \Upsilon^2\cancelto{1}{\left(\cos^2\theta + \sin^2\theta\right)} = \Upsilon^2$$

Math detour
(scalar product)(.)

- Multiply a qubit $|\psi\rangle = a\,|👻\rangle + b\,|🎃\rangle$ for a scalar (complex number) $\Upsilon e^{\imath\theta}$ have for observable effect a change in magnitude $\left|\Upsilon e^{\imath\theta}\,|\psi\rangle\right|^2 = \left|\Upsilon e^{\imath\theta}\,(a\,|👻\rangle + b\,|🎃\rangle)\right|^2 = \Upsilon^2$
- A lot of complex numbers have for magnitude *1* $(\Upsilon^2 = 1)$:



$$\alpha = a + \imath b$$
$$\imath^2 = -1$$
$$\Re(\alpha) = a$$
$$\Im(\alpha) = b$$
$$|\alpha| = \sqrt{a^2 + b^2}$$

Math detour (Unitary matrices)

- Unitary matrices preserve the inner product,
- Proof:
  - Suppose $\langle \phi \,|\, \phi \rangle = 1$ and $\mathbf{U}$ an unitary matrix ($\mathbf{U}^{\dagger} = \mathbf{U}^{-1}$),
  - We want to show that if $|\phi'\rangle = \mathbf{U}\,|\phi\rangle$, then $\langle \phi' \,|\, \phi' \rangle = 1$
  - Remember from the 'primer' that $(\mathbf{AB})^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger}$
  - So, $\langle \phi' \,|\, \phi' \rangle = \left\langle (\mathbf{U}\,|\phi\rangle)^{\dagger} \;\middle|\; (\mathbf{U}\,|\phi\rangle) \right\rangle = \left\langle \left(|\phi\rangle^{\dagger}\,|\mathbf{U}^{\dagger} \right) \;\middle|\; (\mathbf{U}\,|\phi\rangle) \right\rangle$

$$= \left(\langle \phi |\, \mathbf{U}^{\dagger} \right) \;\; \mathbf{U}\,|\phi\rangle = \langle \phi |\, \left(\mathbf{U}^{\dagger}\mathbf{U} \right) |\phi\rangle = \langle \phi |\, \mathbf{I}\,|\phi\rangle = \langle \phi \,|\, \phi \rangle = 1 \blacksquare$$

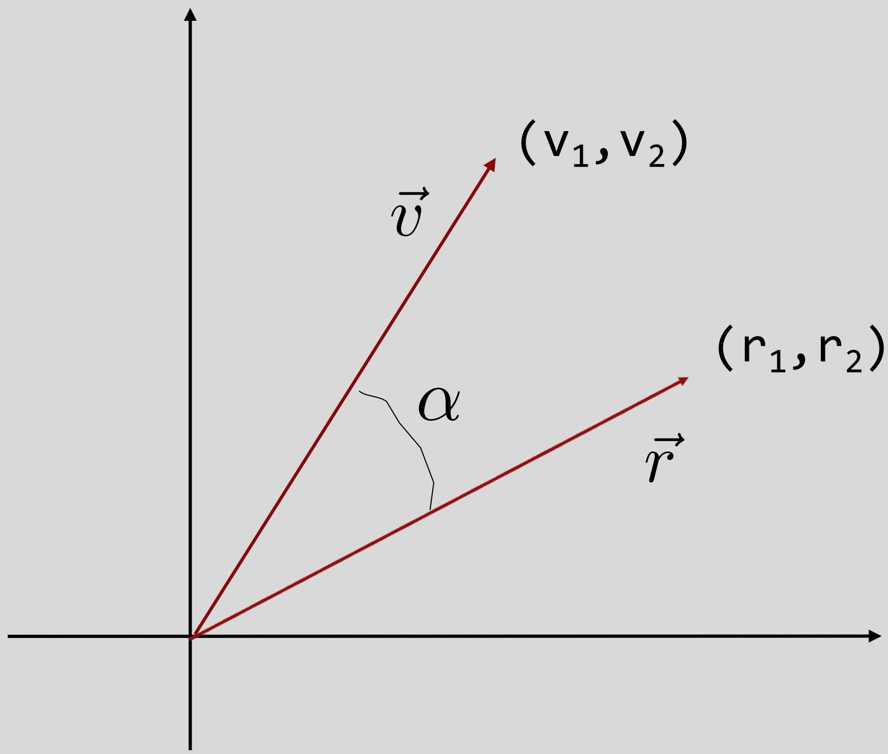
No matter how you transform a vector under a sequence of unitary matrices, the normalization condition still holds true.

Unitary evolution sends quantum states to other valid quantum states.

For a single qubit in a Hilbert space represented by the Bloch sphere, unitary transformations correspond to rotations of state vectors to different points in the sphere surface, not changing the length of the state vector in any way.

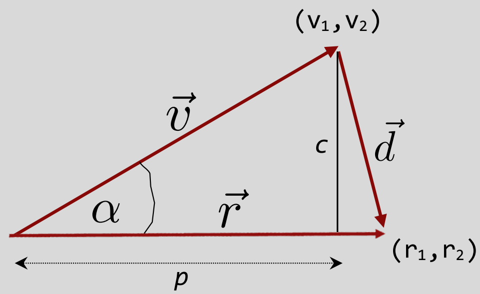
Math detour

(how to measure similarity between vectors)



Math detour

(how to measure vectors)(cosine similarity)



From conditions: $\vec{v} + \vec{d} = \vec{r}, \quad |\vec{v}| = 1, \quad |\vec{r}| = 1$

$$\cos\alpha = \frac{p}{|\vec{v}|} = p = \vec{v}\cdot\vec{r}$$

From the left triangle: $|\vec{v}|^2 = p^2 + c^2, \quad 1 = p^2 + c^2, \quad c^2 = 1 - p^2$ [1]

From the right triangle: $(|\vec{r}| - p)^2 + c^2 = |\vec{d}|^2, \quad |\vec{d}|^2 = (1-p)^2 + c^2 = 1 - 2p + p^2 + c^2$

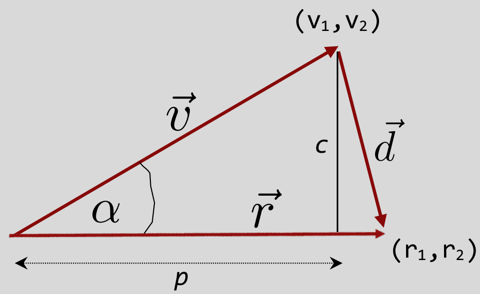
$$|\vec{d}|^2 = 1 - 2p + p^2 + 1 - p^2 = 2 - 2p \quad [2]$$

From conditions: $\vec{d} = \vec{r} - \vec{v}, \quad d_x = r_x - v_x, \quad d_y = r_y - v_y.$

$$|\vec{d}|^2 = d_x^2 + d_y^2 = (r_x - v_x)^2 + (r_y - v_y)^2 = r_x^2 - 2r_xv_x + v_x^2 + r_y^2 - 2r_yv_y + v_y^2 = (r_x^2 + r_y^2) + (v_x^2 + v_y^2) - 2(r_xv_x + r_yv_y)$$

$$|\vec{d}|^2 = 1 + 1 - 2(r_xv_x + r_yv_y). \quad \text{Using [2]}: \quad 2 - 2\vec{r}\cdot\vec{v} = 2 - 2p, \quad \text{so} \quad p = \vec{r}\cdot\vec{v}$$

Math detour
(how to measure non-unit vectors)



From conditions: $\vec{v} + \vec{d} = \vec{r}$,

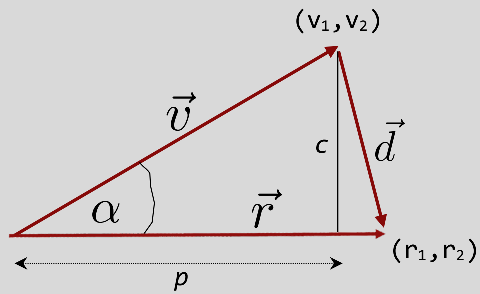
$$\cos\alpha = \frac{p}{|\vec{v}|} = p = \vec{v}\cdot\vec{r}$$

More general: $|\vec{v}|, |\vec{r}| \neq 1. \quad \vec{v} = |\vec{v}|\ \hat{v}, \quad \vec{r} = |\vec{r}|\ \hat{r}, \quad \text{where} \quad |\hat{v}| = |\hat{r}| = 1.$

$$\cos\alpha = \hat{v}\cdot\hat{r} = \frac{\vec{v}\cdot\vec{r}}{|\vec{v}||\vec{r}|}$$

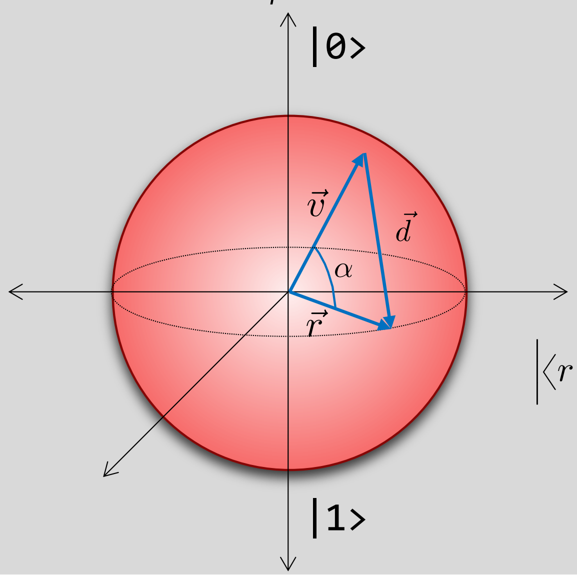
Math detour

(how to measure vectors)(Bloch sphere)



From conditions: $\vec{v} + \vec{d} = \vec{r}$,

$$\cos\alpha = \frac{p}{|\vec{v}|} = p = \vec{v} \cdot \vec{r}$$



See that:

$$-1 \leqslant \cos\alpha \leqslant 1, \quad 0 \leqslant \cos^2\alpha \leqslant 1.$$

Given a qubit described by |v> , the probability that the qubit is measured as |r> is:

$$\left|\langle r \,|\, v\rangle^2\right| = \left|(\langle \overline{r_0}, \overline{r_1}| \;\;|\;\; |v_0, v_1\rangle)^2\right| = \left|(\overline{r_0}v_0 + \overline{r_1}v_1)^2\right| = \cos^2\alpha$$

Quantum Entanglement

Difficult to grasp, very useful however

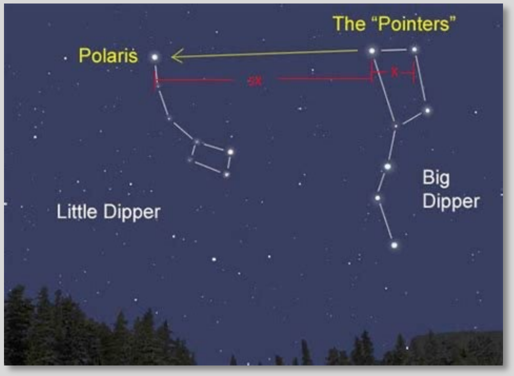
How to navigate without GPS?
(magnetic compass)

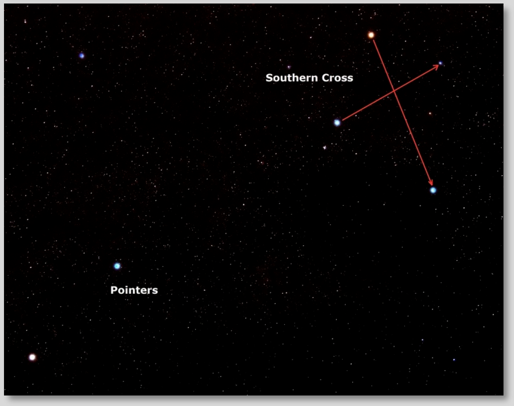
Some metal objects rotates and orient to the North

How to navigate without GPS?
(icelandic spar - calcita)

Sun's light refraction allows to detect the sun in the mist

How to navigate without GPS?
(stars)





Polaris is close to the North.

The Southern cross is close to the South.

How to start a revolution?
(Boltzmann)

'entropy' as a measure of disorder (1875).

Shannon (1948) Information Theory.

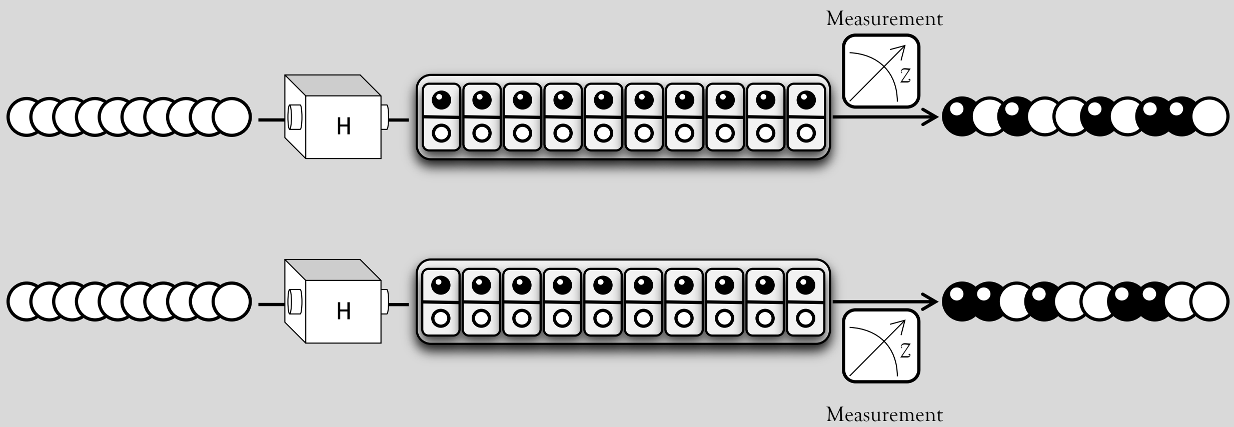
Nevertheless, the entire industrial revolution was built using Boltzmann work.

Quantum Entanglement

- Observations:
  - There exist quantum operations that, when applied to two qubits, make their outcomes dependent,
- Uses:
  - Put multi-qubit numbers in superposition,
  - Transport quantum state over long distances,
- Explanation:
  - No idea,
  - We can model / predict the math with precision, but we don't know the reason.

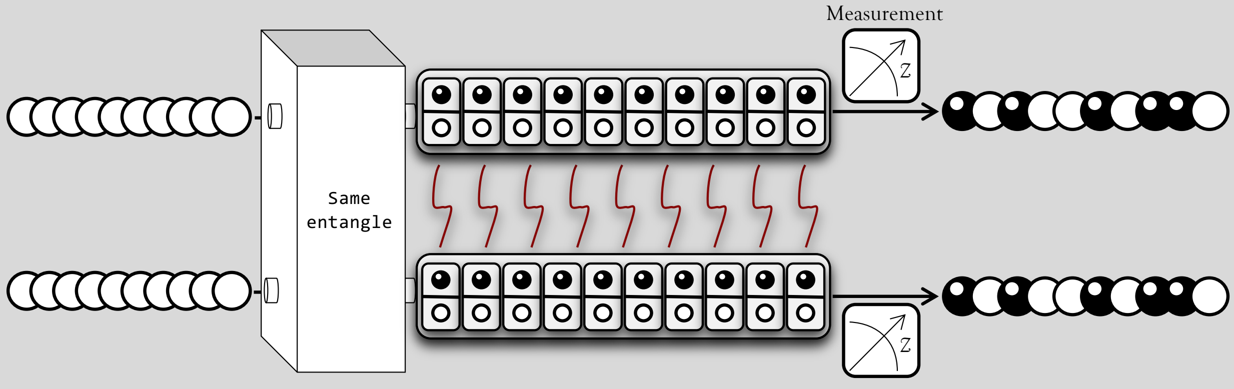
Two `H` gates
(no entanglement)

- Upon leaving the H gate, each individual qubit has the same (quantum) state,
- Upon measurement:
  - Each individual qubit has a *0.5* probability of being measured white or black,
  - Each pair of qubits has a *0.25* probability of combination (`WW`, `WB`, `BW`, `BB`).



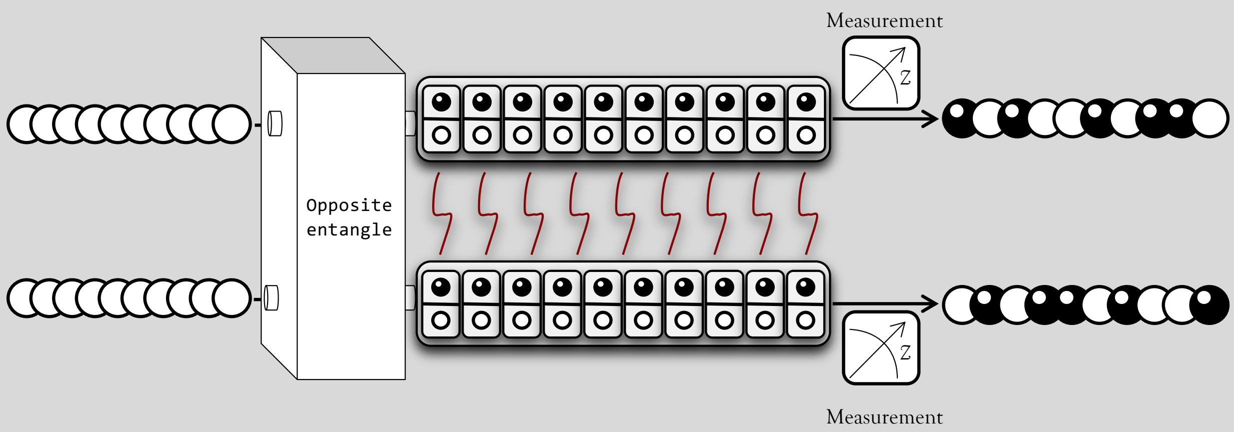
Two `H` gates (same entanglement)

- Upon measurement:
    - Each individual qubit has a *0.5* probability of being measured white or black,
    - Each pair of qubits has a *0.5* probability of combination (`WW`, `BB`),
- The first measurement determines the result of its pair measurement!



Two `H` gates (opposite entanglement)

- Upon measurement:
  - Each individual qubit has a *0.5* probability of being measured white or black,
  - Each pair of qubits has a *0.5* probability of combination (`WB`, `BW`),
- The first measurement determines the result of its pair measurement!



QC to determine a global property

- Given a function *f(x)* you want to find an *x* such that:
  - *f(x)* is minimum,
  - Period of *f(x)* if *f(x)* is periodic,
- Naïve solution: compute *f(x)* for many values in order to get sufficient information about the global property,
- Q solution: create superposition states so the function can be applied to many possible inputs simultaneously,
  - But if we measure, we will get a single result,
- We can induce a quantum interference effect, which will reveal the global property we want.

Why is entanglement important?

- Imagine a variable to store a *4*-bits number

Qubits

Variable (*4* bits number)

Variable holds a superposition of four numbers

0.25 each

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

They are working cooperatively… if one qubit measures *1* all the others qubit will measure *0*

0 All other combinations

With no entanglement, probability would be $0.75^4 = 0.316$

Entanglement is present when the probability of outcomes does not follow the independent probability calculations. Qubits are not independent.

Summary

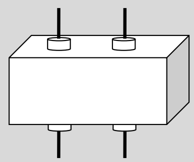
- Entanglement allows the outcome of one qubit to depend on the outcome of a different qubit,
- Entanglement is required to make multiple qubits work together to create superpositions of particular values,
- Entanglement and Superposition provides the power to QC:
  - The limitations of measurement curtail that power,
    - Measurement provides only limited information,
    - Measurement collapses the superposition.

Multiple Qubits

Notations and Calculation

Suppose a gate with two (independent) qubits

$|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle \quad |\psi_2\rangle = a_2|0\rangle + b_2|1\rangle$



TWO INDEPENDENT QUBITS

(not entangled)

Qubit₁: $|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle$

Qubit₂: $|\psi_2\rangle = a_2|0\rangle + b_2|1\rangle$

Qubits 1 and 2: $|\psi_{1,2}\rangle = a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle$

Tensor product

|0> |0>

$$\begin{aligned}|\psi_1\rangle &= 1|0\rangle + 0|1\rangle\\ |\psi_2\rangle &= 1|0\rangle + 0|1\rangle\\ |\psi_{1,2}\rangle &= 1|00\rangle + 0|01\rangle + 0|10\rangle + 0|11\rangle\end{aligned} \quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

|0> |1>

$$\begin{aligned}|\psi_1\rangle &= 1|0\rangle + 0|1\rangle\\ |\psi_2\rangle &= 0|0\rangle + 1|1\rangle\\ |\psi_{1,2}\rangle &= 0|00\rangle + 1|01\rangle + 0|10\rangle + 0|11\rangle\end{aligned} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

|1> |0>

$$\begin{aligned}|\psi_1\rangle &= 0|0\rangle + 1|1\rangle\\ |\psi_2\rangle &= 1|0\rangle + 0|1\rangle\\ |\psi_{1,2}\rangle &= 0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle\end{aligned} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

|1> |1>

$$\begin{aligned}|\psi_1\rangle &= 0|0\rangle + 1|1\rangle\\ |\psi_2\rangle &= 0|0\rangle + 1|1\rangle\\ |\psi_{1,2}\rangle &= 0|00\rangle + 0|01\rangle + 0|10\rangle + 1|11\rangle\end{aligned} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Suppose a gate with two (independent) qubits (beyond balls)

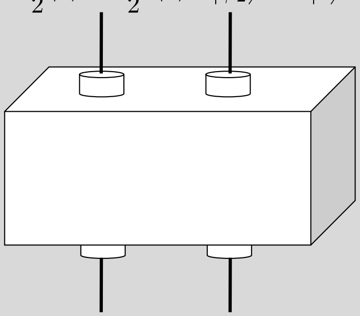
TWO INDEPENDENT QUBITS
(not entangled)

Qubit1:
$|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle$

Qubit2:
$|\psi_2\rangle = a_2|0\rangle + b_2|1\rangle$

Qubits 1 and 2:
$|\psi_{1,2}\rangle = a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle$

$|\psi_1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \quad |\psi_2\rangle = 0|0\rangle + 1|1\rangle$

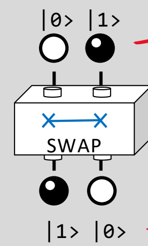


$$|\psi_{1,2}\rangle = \cancelto{0}{\left(\frac{1}{2}\times 0\right)}|00\rangle + \left(\frac{1}{2}\times 1\right)|01\rangle + \cancelto{0}{\left(\frac{\sqrt{3}}{2}\times 0\right)}|10\rangle + \left(\frac{\sqrt{3}}{2}\times 1\right)|11\rangle = 0|00\rangle + \frac{1}{2}|01\rangle + 0|10\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

A SWAP gate with |0> |1>

- SWAP gate matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$|\psi_1\rangle = 1\,|0\rangle + 0\,|1\rangle$ $\quad |\psi_2\rangle = 0\,|0\rangle + 1\,|1\rangle$



$$|\psi_{1,2}\rangle = 0\,|00\rangle + 1\,|01\rangle + 0\,|10\rangle + 0\,|11\rangle$$

- Result is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

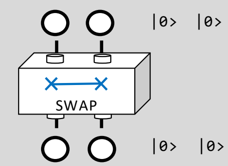
$$|\psi_{1,2}\rangle = 0\,|00\rangle + 0\,|01\rangle + 1\,|10\rangle + 0\,|11\rangle$$

A `SWAP` gate for balls

- `SWAP` gate matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

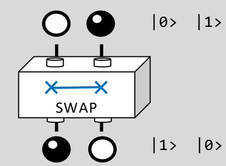
$$|\psi_1\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_2\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_{1,2}\rangle = (1\times 1)\,|00\rangle + \cancelto{0}{(1\times 0)}\,|01\rangle + \cancelto{0}{(0\times 1)}\,|10\rangle + \cancelto{0}{(0\times 0)}\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



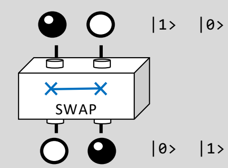
$$|\psi_1\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_2\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{1,2}\rangle = \cancelto{0}{(1\times 0)}\,|00\rangle + (1\times 1)\,|01\rangle + \cancelto{0}{(0\times 0)}\,|10\rangle + \cancelto{0}{(0\times 1)}\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



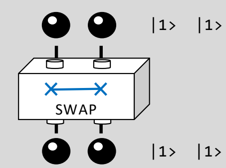
$$|\psi_1\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_2\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_{1,2}\rangle = \cancelto{0}{(0\times 1)}\,|00\rangle + \cancelto{0}{(0\times 0)}\,|01\rangle + (1\times 1)\,|10\rangle + \cancelto{0}{(1\times 0)}\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

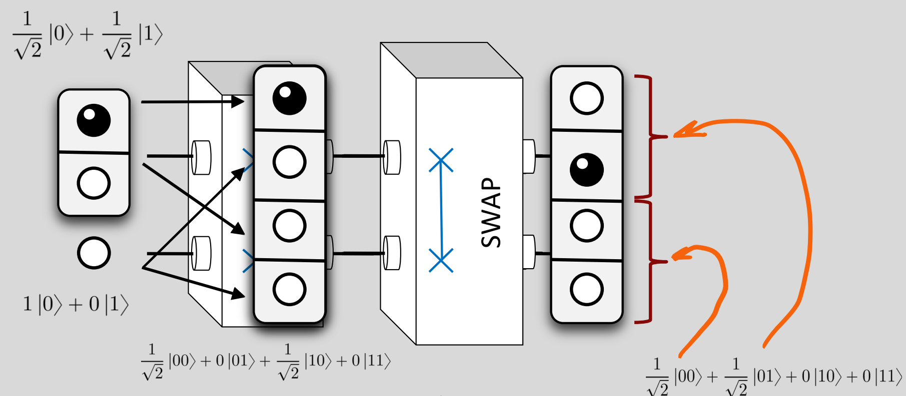


$$|\psi_1\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_2\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{1,2}\rangle = \cancelto{0}{(0\times 0)}\,|00\rangle + \cancelto{0}{(0\times 1)}\,|01\rangle + \cancelto{0}{(1\times 0)}\,|10\rangle + (1\times 1)\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



A `SWAP` gate with superposition input

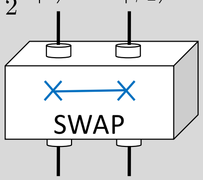


$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

A SWAP gate beyond balls

SWAP gate matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$|\psi_1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ $\quad |\psi_2\rangle = 0\,|0\rangle + 1\,|1\rangle$



$$|\psi_{1,2}\rangle = \cancelto{0}{\left(\frac{1}{2}\times 0\right)}|00\rangle + \left(\frac{1}{2}\times 1\right)|01\rangle + \cancelto{0}{\left(\frac{\sqrt{3}}{2}\times 0\right)}|10\rangle + \left(\frac{\sqrt{3}}{2}\times 1\right)|11\rangle = 0\,|00\rangle + \frac{1}{2}|01\rangle + 0\,|10\rangle + \frac{\sqrt{3}}{2}|11\rangle = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$|\psi_1\rangle = 0\,|0\rangle + 1\,|1\rangle$ $\quad |\psi_2\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

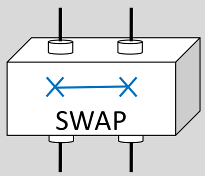
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$|\psi_{1,2}\rangle = \cancelto{0}{\left(0\times\frac{1}{2}\right)}|00\rangle + \cancelto{0}{\left(0\times\frac{\sqrt{3}}{2}\right)}|01\rangle + \left(1\times\frac{1}{2}\right)|10\rangle + \left(1\times\frac{\sqrt{3}}{2}\right)|11\rangle = 0\,|00\rangle + 0\,|01\rangle + \frac{1}{2}|10\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

A `SWAP` gate beyond balls (generic way)

`SWAP` gate matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$|\psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$ $\quad |\psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$



$|\psi_1\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$ $\quad |\psi_2\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$

$$|\psi_{1,2}\rangle = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle \equiv \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 \\ \beta_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\beta_2 \end{bmatrix}$$

Can you see the swap?

$$|\psi_{1,2}\rangle = \alpha_1\alpha_2 |00\rangle + \beta_1\alpha_2 |01\rangle + \alpha_1\beta_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

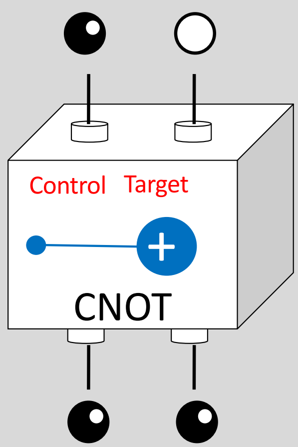
A CNOT gate (control |1> target |0>)

- CNOT gate matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Control bit always goes left

$|\psi_{control}\rangle = 0\,|0\rangle + 1\,|1\rangle$ $\quad |\psi_{target}\rangle = 1\,|0\rangle + 0\,|1\rangle$

$$|\psi_{control,target}\rangle = 0\,|00\rangle + 0\,|01\rangle + 1\,|10\rangle + 0\,|11\rangle \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



- Result is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$0\,|00\rangle + 0\,|01\rangle + 0\,|10\rangle + 1\,|11\rangle$

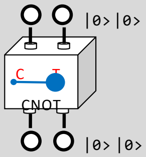
A CNOT gate for balls

CNOT gate matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Control bit always goes left

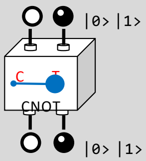
$$|\psi_{control}\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_{target}\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_{control,target}\rangle = (1\times 1)\,|00\rangle + \cancel{(1\times 0)}^{0}\,|01\rangle + \cancel{(0\times 1)}^{0}\,|10\rangle + \cancel{(0\times 0)}^{0}\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



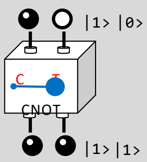
$$|\psi_{control}\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_{target}\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{control,target}\rangle = \cancel{(1\times 0)}^{0}\,|00\rangle + (1\times 1)\,|01\rangle + \cancel{(0\times 0)}^{0}\,|10\rangle + \cancel{(0\times 1)}^{0}\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



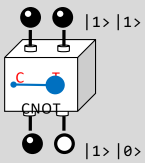
$$|\psi_{control}\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{target}\rangle = 1\,|0\rangle + 0\,|1\rangle$$
$$|\psi_{control,target}\rangle = \cancel{(0\times 1)}^{0}\,|00\rangle + \cancel{(0\times 0)}^{0}\,|01\rangle + (1\times 1)\,|10\rangle + \cancel{(1\times 0)}^{0}\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

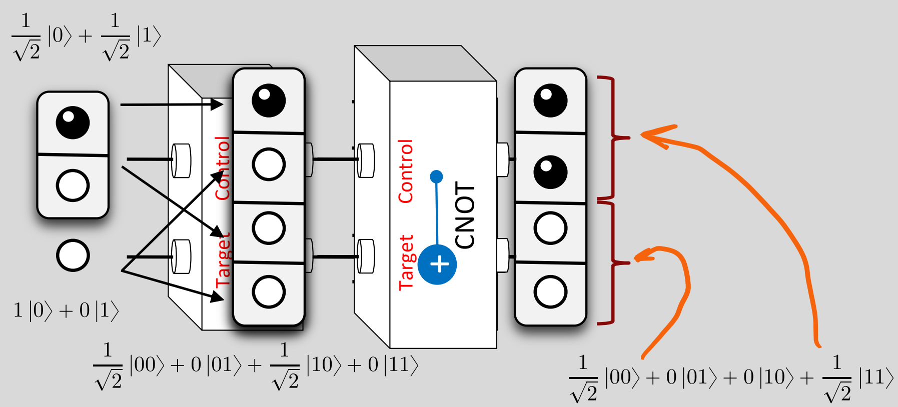


$$|\psi_{control}\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{target}\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{control,target}\rangle = \cancel{(0\times 0)}^{0}\,|00\rangle + \cancel{(0\times 1)}^{0}\,|01\rangle + \cancel{(1\times 0)}^{0}\,|10\rangle + (1\times 1)\,|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



A `CNOT` with superposition input



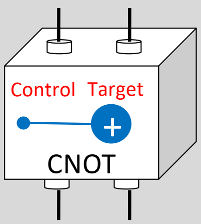
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A `CNOT` gate beyond balls

`CNOT` gate matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Control bit always goes left

$$|\psi_{control}\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad |\psi_{target}\rangle = 0\,|0\rangle + 1\,|1\rangle$$



$$|\psi_{control,target}\rangle = \cancelto{0}{\left(\frac{1}{2}\times 0\right)}|00\rangle + \left(\frac{1}{2}\times 1\right)|01\rangle + \cancelto{0}{\left(\frac{\sqrt{3}}{2}\times 0\right)}|10\rangle + \left(\frac{\sqrt{3}}{2}\times 1\right)|11\rangle \qquad \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

$$0\,|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{2}|10\rangle + 0\,|11\rangle$$

A CCNOT gate (control |1>,|1> target |1>)

- `CCNOT` gate matrix is:

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$

Control bit always goes left

Input	Output
○○○	○○○
○○●	○○●
○●○	○●○
○●●	○●●
●○○	●○○
●○●	●○●
●●○	●●●
●●●	●●○

$$|\psi_{control-1}\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{control-2}\rangle = 0\,|0\rangle + 1\,|1\rangle$$
$$|\psi_{target}\rangle = 0\,|0\rangle + 1\,|1\rangle$$

$$|\psi_{control-1,control-2,target}\rangle = 0\,|000\rangle + 0\,|001\rangle + 0\,|010\rangle + 0\,|011\rangle + 0\,|100\rangle + 0\,|101\rangle + 0\,|110\rangle + 1\,|111\rangle$$

$$\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix}$$

- Result is:

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{bmatrix} \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix}$$

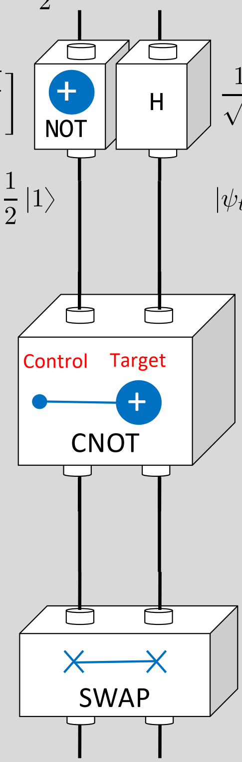
$$0\,|000\rangle + 0\,|001\rangle + 0\,|010\rangle + 0\,|011\rangle + 0\,|100\rangle + 0\,|101\rangle + 1\,|110\rangle + 0\,|111\rangle$$

Quantum computing

(an exercise)

$|\psi_1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad |\psi_2\rangle = 0\,|0\rangle + 1\,|1\rangle$

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$



$|\psi_{control}\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \qquad |\psi_{target}\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$$|\psi_{control,target}\rangle = \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)|00\rangle - \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)|01\rangle + \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right)|10\rangle - \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right)|11\rangle$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\\ \frac{1}{2\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\\ \frac{1}{2\sqrt{2}}\end{bmatrix}$$

$$\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle - \frac{1}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\\ \frac{1}{2\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\\ \frac{1}{2\sqrt{2}}\end{bmatrix}$$

$$\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle - \frac{1}{2\sqrt{2}}|01\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$$

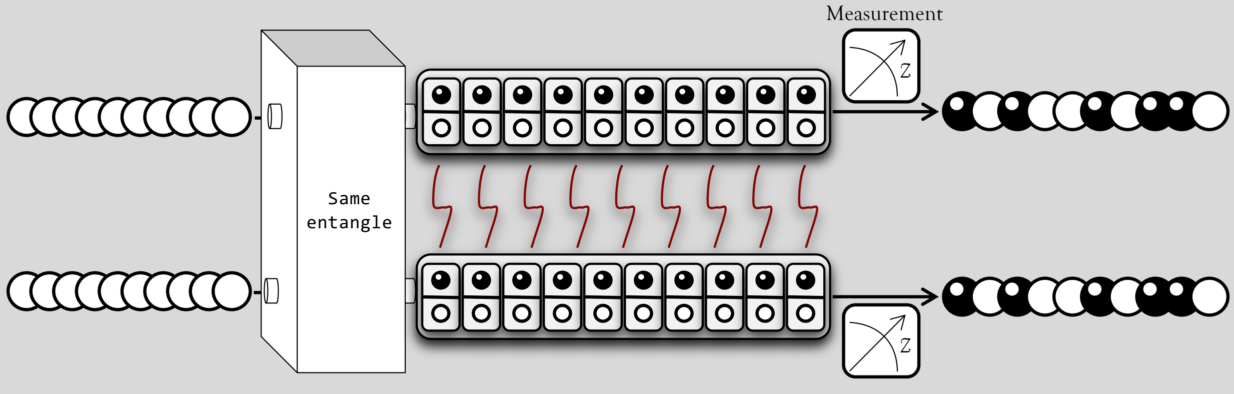
$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \qquad |\psi_2\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$

Entanglement Implementation

A circuit that entangles two Qubits

Remember
(same entanglement)

- Upon measurement:
  - Each individual qubit has a 0.5 probability of being measured white or black,
  - Each pair of qubits has a 0.5 probability of combination (`WW`, `BB`),
- The first measurement determines the result of its pair measurement!

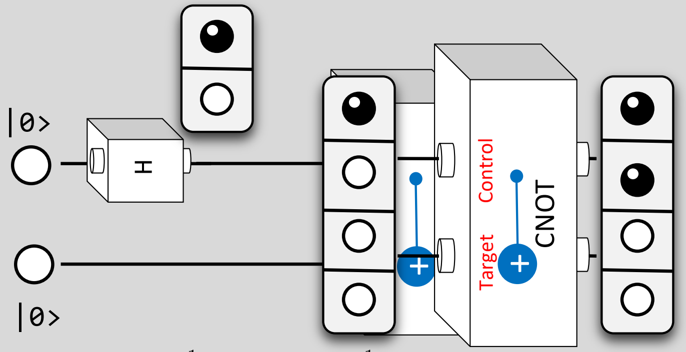


What about this quantum circuit?

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$1|0\rangle + 0|1\rangle$

$1|0\rangle + 0|1\rangle$



0.5 - 0.5 probability: **both black** or **both white**

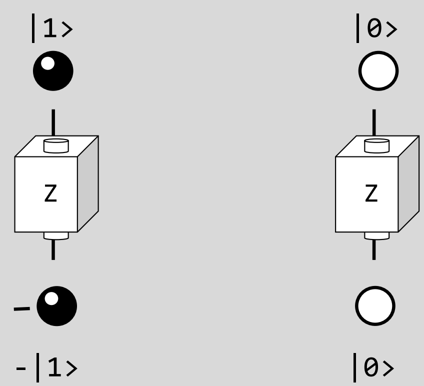
$$\frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + 0|11\rangle$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + 0|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

0.5 both white — *0.5* both black

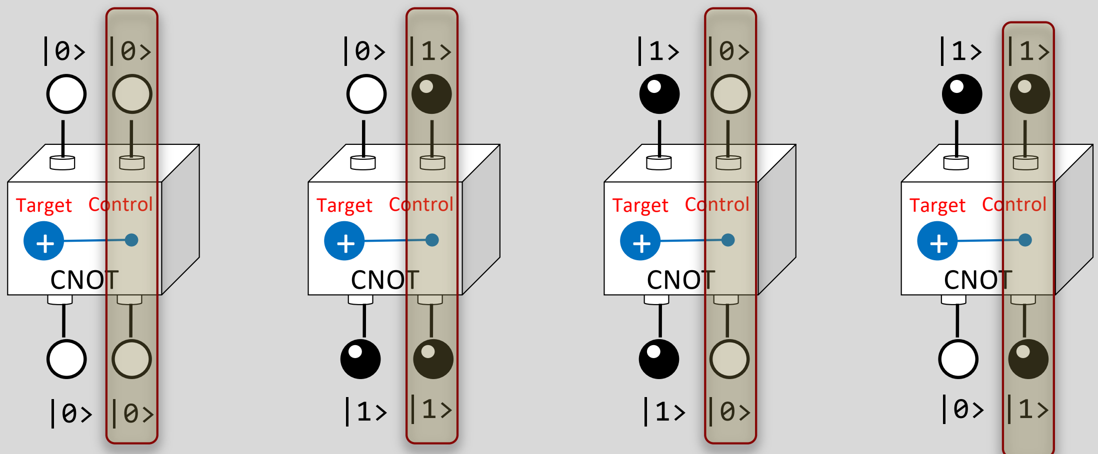
0 b/w or w/b

Do you remember the Z Gate?
(phase flip gate)



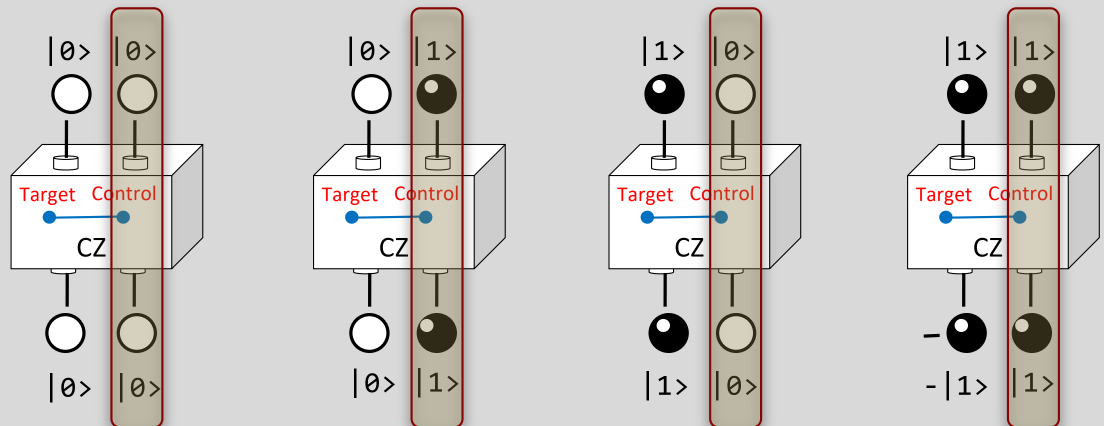
Do you remember the `CNOT` Gate?

(controlled `NOT`)



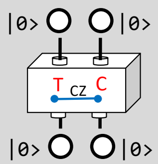
Control toggles the `NOT` operation
Control never changes !!!

The gate CZ
(controlled Z –control phase changes)

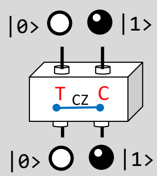


Control toggles the Z operation
Control never changes !!!

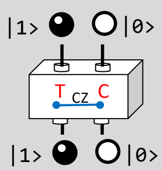
The gate CZ (controlled Z)(the math)



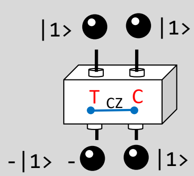
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$



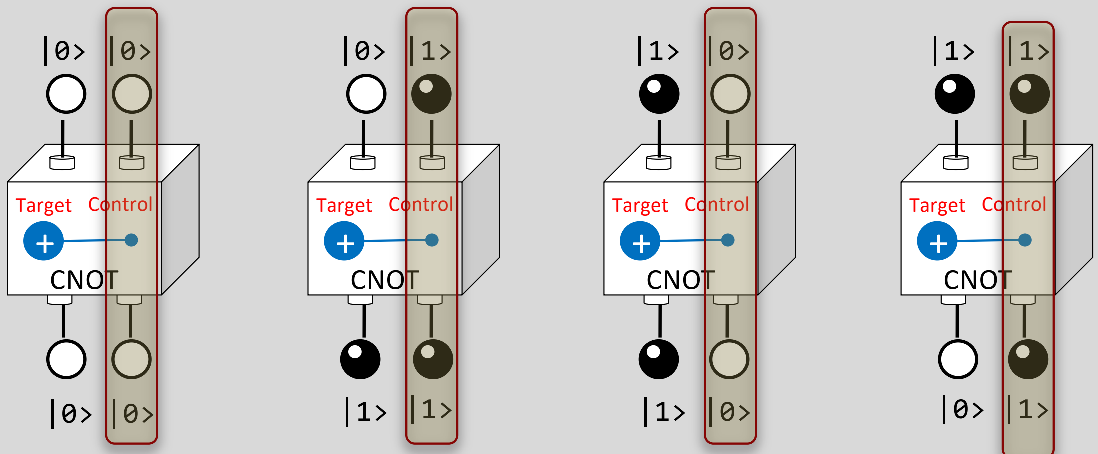
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -|11\rangle$$

The entire **|11>** pair toggles its phase, not any individual qubits

Do you remember the `CNOT` Gate? (controlled `NOT`)



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

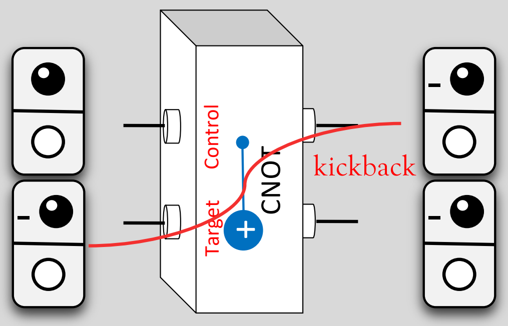
Control toggles the `NOT` operation
Control never changes !!! or does it?

As a matter of fact, **`Target`** can affect **`Control`**

CNOT with superposition

$$|\psi_{control}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi_{target}\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



$$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

It changed **Control**' s phase!

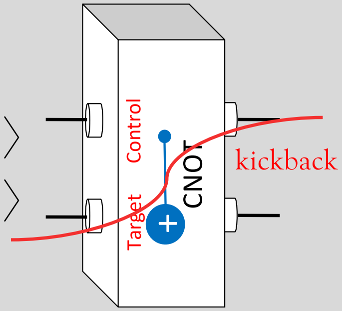
Target remains unchanged.

$$|\psi_{control,target}\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \equiv \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

CNOT with superposition (generic way)

$$|\psi_{control}\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$$
$$|\psi_{target}\rangle = \alpha_2 |0\rangle - \beta_2 |1\rangle$$



$$\alpha_1 |0\rangle - \beta_1 |1\rangle$$
$$\alpha_2 |0\rangle - \beta_2 |1\rangle$$

It changed **Control**' s phase!

Target remains unchanged.

$$|\psi_{control,target}\rangle = \alpha_1\alpha_2 |00\rangle - \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle - \beta_1\beta_2 |11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1\alpha_2 \\ -\alpha_1\beta_2 \\ \beta_1\alpha_2 \\ -\beta_1\beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 \\ -\alpha_1\beta_2 \\ -\beta_1\beta_2 \\ \beta_1\alpha_2 \end{bmatrix} \equiv^? \begin{bmatrix} \alpha_1\alpha_2 \\ -\alpha_1\beta_2 \\ -\beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}$$

True if $\beta_2=\alpha_2$ $\quad |\psi_{target}\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$

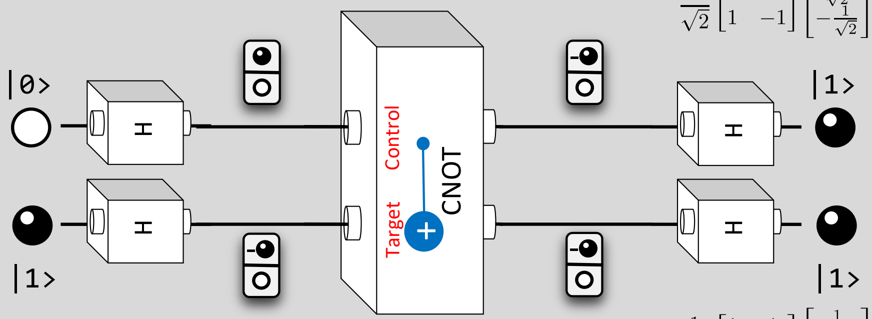
H + CNOT + superposition

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} \equiv 0\left|0\right\rangle + 1\left|1\right\rangle$$

$$\left|\psi_1\right\rangle = 1\left|0\right\rangle + 0\left|1\right\rangle$$

$$\left|\psi_2\right\rangle = 0\left|0\right\rangle + 1\left|1\right\rangle$$



$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} \equiv 0\left|0\right\rangle + 1\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$\left|\psi_1\right\rangle \otimes \left|\psi_2\right\rangle = \frac{1}{2}\left|00\right\rangle - \frac{1}{2}\left|01\right\rangle + \frac{1}{2}\left|10\right\rangle - \frac{1}{2}\left|11\right\rangle$$

$$\frac{1}{2}\left|00\right\rangle - \frac{1}{2}\left|01\right\rangle - \frac{1}{2}\left|10\right\rangle + \frac{1}{2}\left|11\right\rangle = \left(\frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle\right)$$

The **`CNOT`** inside changes **`Control`**'s phase. The **`H`** gates turned a phase flip into a bit flip

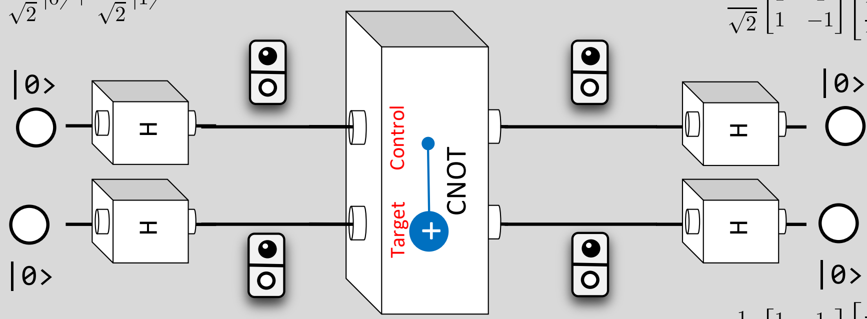
H + CNOT + superposition
(please check my math)

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \equiv 1\left|0\right\rangle + 0\left|1\right\rangle$$

$$\left|\psi_1\right\rangle = 1\left|0\right\rangle + 0\left|1\right\rangle$$

$$\left|\psi_2\right\rangle = 1\left|0\right\rangle + 0\left|1\right\rangle$$



$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \equiv 1\left|0\right\rangle + 0\left|1\right\rangle$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$\frac{1}{2}\left|00\right\rangle + \frac{1}{2}\left|01\right\rangle + \frac{1}{2}\left|10\right\rangle + \frac{1}{2}\left|11\right\rangle = \left(\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\right)$$

$$\left|\psi_1\right\rangle \otimes \left|\psi_2\right\rangle = \frac{1}{2}\left|00\right\rangle + \frac{1}{2}\left|01\right\rangle + \frac{1}{2}\left|10\right\rangle + \frac{1}{2}\left|11\right\rangle$$

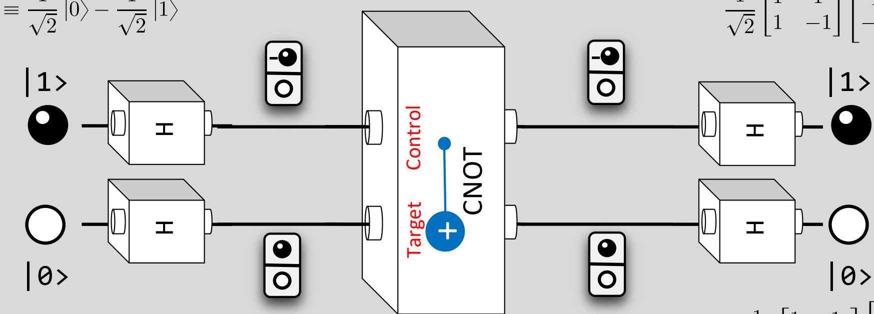
H + CNOT + superposition (please check my math)

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \equiv 0\left|0\right\rangle + 1\left|1\right\rangle$$

$$\left|\psi_1\right\rangle = 0\left|0\right\rangle + 1\left|1\right\rangle$$

$$\left|\psi_2\right\rangle = 1\left|0\right\rangle + 0\left|1\right\rangle$$



$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \equiv 1\left|0\right\rangle + 0\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

$$\frac{1}{2}\left|00\right\rangle + \frac{1}{2}\left|01\right\rangle - \frac{1}{2}\left|10\right\rangle - \frac{1}{2}\left|11\right\rangle = \left(\frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\right)$$

$$\left|\psi_1\right\rangle \otimes \left|\psi_2\right\rangle = \frac{1}{2}\left|00\right\rangle + \frac{1}{2}\left|01\right\rangle - \frac{1}{2}\left|10\right\rangle - \frac{1}{2}\left|11\right\rangle$$

H + CNOT + superposition
(please check my math)

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} \equiv 1\left|0\right\rangle + 0\left|1\right\rangle$$

$$\left|\psi_1\right\rangle = 0\left|0\right\rangle + 1\left|1\right\rangle$$

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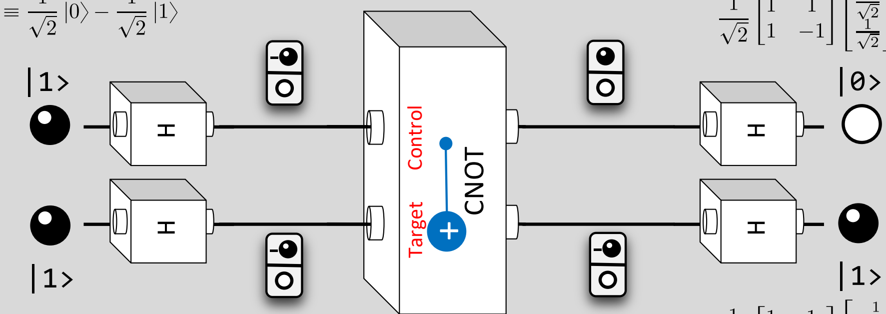


$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} \equiv 0\left|0\right\rangle + 1\left|1\right\rangle$$

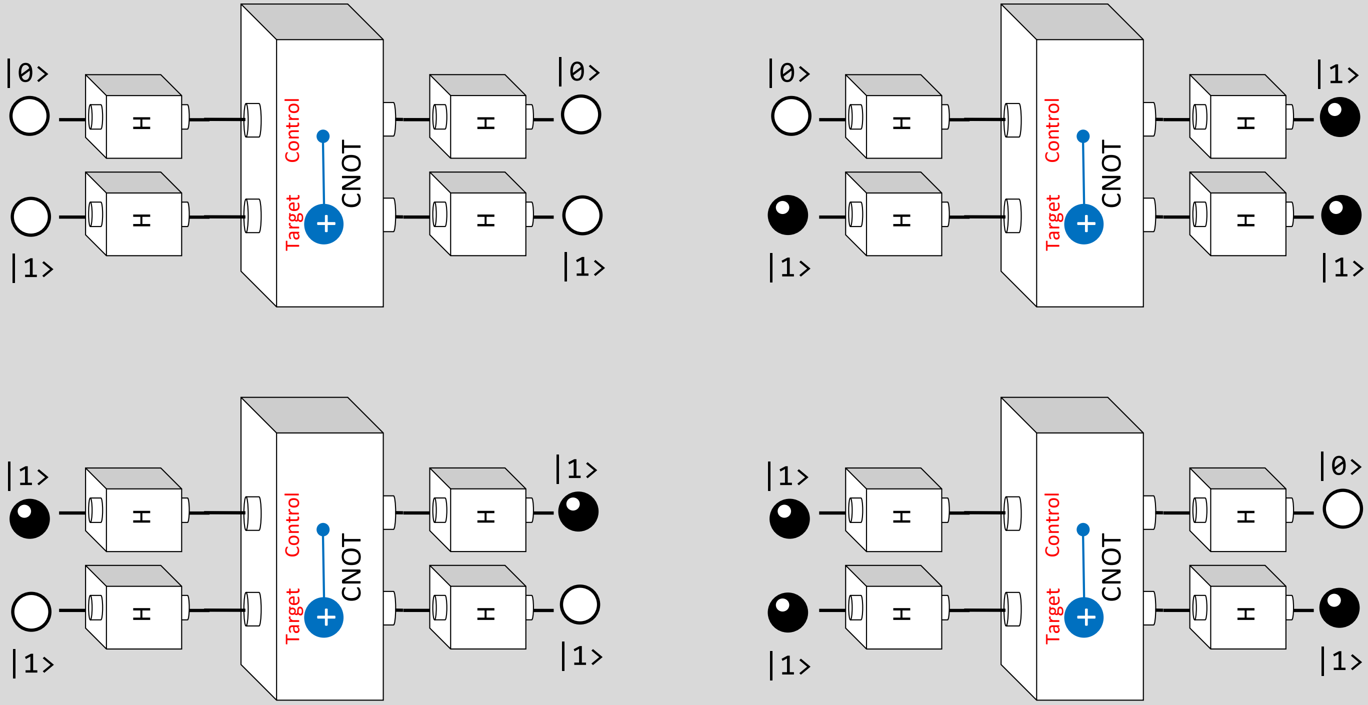
$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{bmatrix}$$

$$\frac{1}{2}\left|00\right\rangle - \frac{1}{2}\left|01\right\rangle + \frac{1}{2}\left|10\right\rangle - \frac{1}{2}\left|11\right\rangle = \left(\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle\right)$$

$$\left|\psi_1\right\rangle \otimes \left|\psi_2\right\rangle = \frac{1}{2}\left|00\right\rangle - \frac{1}{2}\left|01\right\rangle - \frac{1}{2}\left|10\right\rangle + \frac{1}{2}\left|11\right\rangle$$

H + CNOT + superposition (please check my math)

The **CNOT** inside changes **Control**'s phase.
The **H** gates turned a phase flip into a bit flip



How to *factorize* a tensor product?

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \overset{?}{=} \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}, \quad \text{so} \quad xw = abcd = yz$$

$$\begin{bmatrix} 1 \\ z/x \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ yz/x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$xw = yz$$

More general, let $uv = x$, so $a = u, b = uz/x, c = v, d = vy/x = y/u$, with $x \neq 0$

$$\begin{bmatrix} u \\ uz/x \end{bmatrix} \otimes \begin{bmatrix} v \\ y/u \end{bmatrix} = \begin{bmatrix} uv \\ y \\ uzv/x \\ yz/x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Most common Quantum Gates

	Operator	Gate(s)	Matrix
NOT	**Pauli-X (X)**	**X**	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
	Pauli-Y (Y)	**Y**	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Change phase	**Pauli-Z (Z)**	**Z**	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
	Hadamard (H)	**H**	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
	Phase (S, P)	**S**	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
	$\pi/8$ (T)	**T**	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
	Controlled Not (CNOT, CX)		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
	Controlled Z (CZ)	**Z**	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
	SWAP		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
	Toffoli (CCNOT, CCX, TOFF)		$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Pauli Gates

$$\sigma_0 = \mathbf{I}, \quad \sigma_1 = \mathbf{X}, \quad \sigma_2 = \mathbf{Y}, \quad \sigma_3 = \mathbf{Z},$$

$$\mathbf{X}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{Y}^2 = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} = \begin{bmatrix} -\imath^2 & 0 \\ 0 & -\imath^2 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{Z}^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{XY} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} = \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix} = \imath \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \imath\mathbf{Z} = -\mathbf{YX}$$

$$\mathbf{YZ} = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix} = \imath \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \imath\mathbf{X} = -\mathbf{ZY}$$

$$\mathbf{ZX} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \imath \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} = \imath\mathbf{Y} = -\mathbf{XZ}$$

$$\mathbf{XYZ} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \imath & 0 \\ 0 & \imath \end{bmatrix} = \imath\mathbf{I}$$

The X gate

$$\mathbf{X}\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \underset{\text{physics}}{\sim} |1\rangle$$

$$\mathbf{X}\,|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \underset{\text{physics}}{\sim} |0\rangle$$

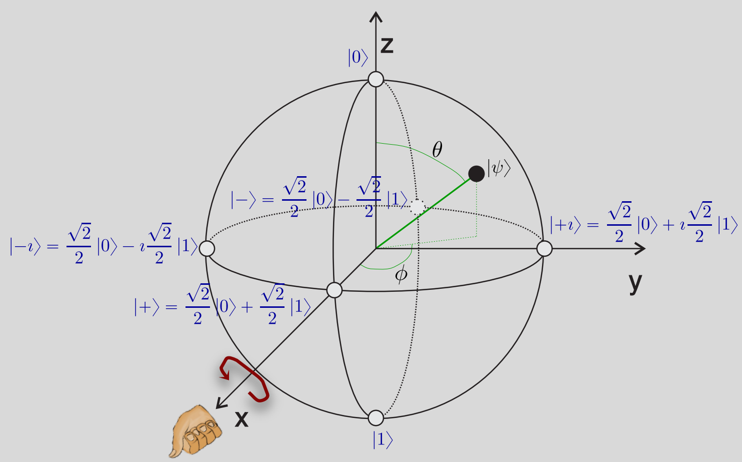
$$\mathbf{X}\,|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = |+\rangle \underset{\text{physics}}{\sim} |+\rangle$$

$$\mathbf{X}\,|-\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = -\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = -\,|-\rangle \underset{\text{physics}}{\sim} |-\rangle$$

$$\mathbf{X}\,|+\imath\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \imath\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \imath\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \imath\,|-\imath\rangle \underset{\text{physics}}{\sim} |-\imath\rangle$$

$$\mathbf{X}\,|-\imath\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\imath\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = -\imath\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = -\imath\,|+\imath\rangle \underset{\text{physics}}{\sim} |+\imath\rangle$$

The X gate rotates the qubit around the X-axis



The Y gate

$$\mathbf{Y}\left|0\right\rangle = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \imath \end{bmatrix} = \imath\left|1\right\rangle \underset{\text{physics}}{\sim} \left|1\right\rangle$$

$$\mathbf{Y}\left|1\right\rangle = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\imath \\ 0 \end{bmatrix} = -\imath\left|0\right\rangle \underset{\text{physics}}{\sim} \left|0\right\rangle$$

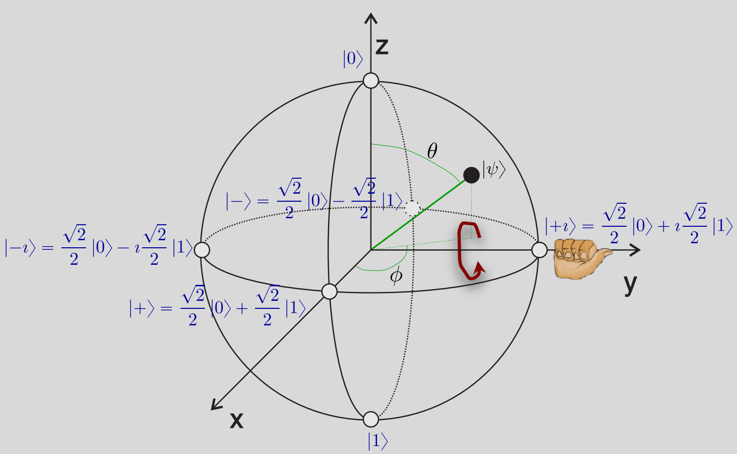
$$\mathbf{Y}\left|+\right\rangle = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\imath\frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = -\imath\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = -\imath\left|-\right\rangle \underset{\text{physics}}{\sim} \left|-\right\rangle$$

$$\mathbf{Y}\left|-\right\rangle = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \imath\frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \imath\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \imath\left|+\right\rangle \underset{\text{physics}}{\sim} \left|+\right\rangle$$

$$\mathbf{Y}\left|+\imath\right\rangle = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \left|+\imath\right\rangle \underset{\text{physics}}{\sim} \left|+\imath\right\rangle$$

$$\mathbf{Y}\left|-\imath\right\rangle = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = -\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = -\left|-\imath\right\rangle \underset{\text{physics}}{\sim} \left|-\imath\right\rangle$$

The Y gate rotates the qubit around the Y-axis



The Z gate

$$\mathbf{Z}\,|0\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle \underset{\text{physics}}{\sim} |0\rangle$$

$$\mathbf{Z}\,|1\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} = -\begin{bmatrix}0\\1\end{bmatrix} = -\,|1\rangle \underset{\text{physics}}{\sim} |1\rangle$$

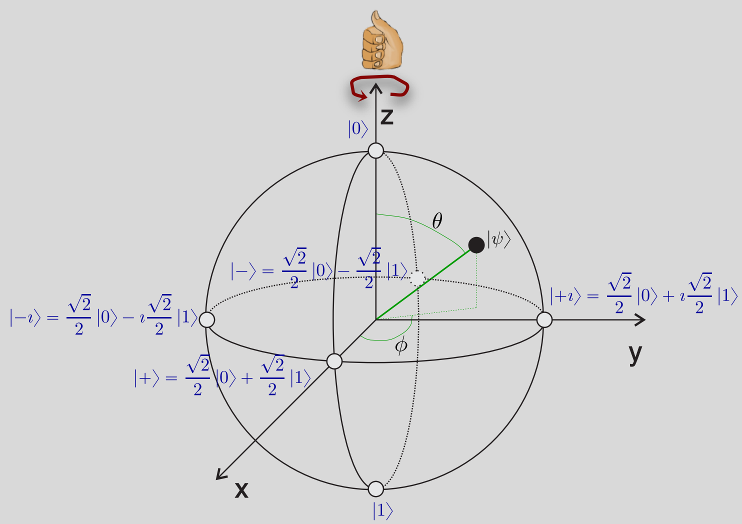
$$\mathbf{Z}\,|+\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{bmatrix} = |-\rangle \underset{\text{physics}}{\sim} |-\rangle$$

$$\mathbf{Z}\,|-\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix} = |+\rangle \underset{\text{physics}}{\sim} |+\rangle$$

$$\mathbf{Z}\,|+\imath\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \imath\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\imath\frac{\sqrt{2}}{2}\end{bmatrix} = |-\imath\rangle \underset{\text{physics}}{\sim} |-\imath\rangle$$

$$\mathbf{Z}\,|-\imath\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\imath\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ \imath\frac{\sqrt{2}}{2}\end{bmatrix} = |+\imath\rangle \underset{\text{physics}}{\sim} |+\imath\rangle$$

The Z gate rotates the qubit around the Z-axis



The H gate

$$\mathbf{H}\,|0\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = |+\rangle \underset{\text{physics}}{\sim} |+\rangle$$

$$\mathbf{H}\,|1\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = |-\rangle \underset{\text{physics}}{\sim} |-\rangle$$

$$\mathbf{H}\,|+\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \underset{\text{physics}}{\sim} |0\rangle$$

$$\mathbf{H}\,|-\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \underset{\text{physics}}{\sim} |1\rangle$$

$$\mathbf{H}\,|+\imath\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\imath \\ 1-\imath \end{bmatrix} = \frac{1}{2}\frac{1}{1-\imath}\begin{bmatrix} 1-\imath^2 \\ \cancel{1} - 2\imath + \cancel{\imath^2} \end{bmatrix}$$

$$= \frac{1}{1-\imath}\begin{bmatrix} 1 \\ -\imath \end{bmatrix} = \frac{1}{1-\imath}\frac{2}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |-\imath\rangle$$

$$\mathbf{H}\,|-\imath\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-\imath \\ 1+\imath \end{bmatrix} = \frac{1}{2}\frac{1}{1+\imath}\begin{bmatrix} 1-\imath^2 \\ \cancel{1} + 2\imath + \cancel{\imath^2} \end{bmatrix}$$

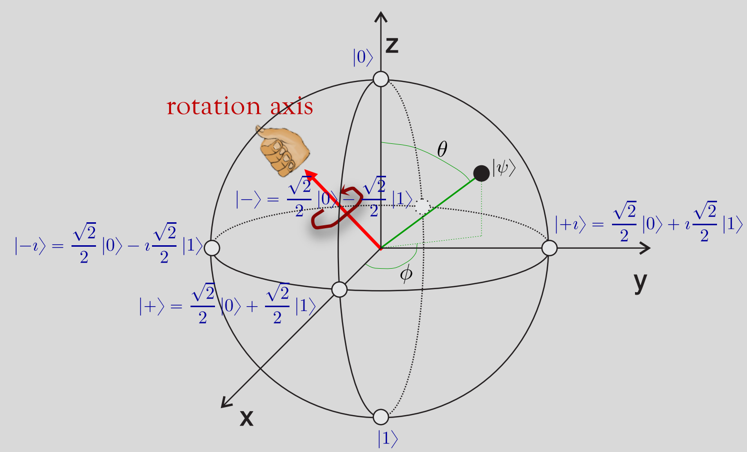
$$= \frac{1}{1+\imath}\begin{bmatrix} 1 \\ \imath \end{bmatrix} = \frac{1}{1+\imath}\frac{2}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |+\imath\rangle$$

Eigen-vectors: $\begin{bmatrix} \cos\left(\frac{3\pi}{8}\right) \\ -\sin\left(\frac{3\pi}{8}\right) \end{bmatrix}, \quad \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) \end{bmatrix}$

$$\cos\left(\frac{3\pi}{8}\right)|0\rangle - \sin\left(\frac{3\pi}{8}\right)|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad \theta = \frac{3\pi}{4}, \phi = \pi.$$

$$\cos\left(\frac{\pi}{8}\right)|0\rangle - \sin\left(\frac{\pi}{8}\right)|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad \theta = \frac{\pi}{4}, \phi = 0.$$

The H gate rotates the qubit π around the axis on the diagonal Z+X



Eigen-decomposition

- Any square matrix **A** that commutes with its **Hermitian conjugate** (i.e. $\mathbf{A}\ \mathbf{A}^\dagger = \mathbf{A}^\dagger\ \mathbf{A}$) is a **normal** matrix and can be decomposed as:

$$\mathbf{A} = \sum_i \lambda_i \vec{v_i}\ \vec{v_i}^\dagger$$

- As **X, Y, Z** satisfies this criterion, so we have:

$$\mathbf{X} = |+\rangle\langle+| \quad - \quad |-\rangle\langle-|$$

$$\mathbf{Y} = |+\imath\rangle\langle+\imath| \quad - \quad |-\imath\rangle\langle-\imath|$$

$$\mathbf{Z} = |0\rangle\langle0| \quad - \quad |1\rangle\langle1|$$

Remember:

$\overline{z},\ z^*$: complex conjugate of the complex number z

$\overline{(1+\imath)} = (1+\imath)^* = 1-\imath$

$\overline{\mathbf{A}},\ \mathbf{A}^*$: complex conjugate of the matrix $\mathbf{A}$

$\mathbf{A}^T$: transpose of the matrix $\mathbf{A}$

$\mathbf{A}^\dagger$: Hermitian conjugate or transposed conjugate of the matrix $\mathbf{A}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$$

Eigen-decomposition (summary)

- If two matrices $\mathbf{A}$ and $\mathbf{B}$ have the same eigen-vectors and eigenvalues, they are the same matrix,
- Similarly, if two operations can be represented by the same eigen-states and eigen-phases, the operations are indistinguishable from each other,
- If after applying an operation $\mathbf{U}$ the state of a qubit is only modified by a global phase, we say that the state is an eigenstate of $\mathbf{U}$.

The stuff in qiskit

```
import math
import numpy as np
#import qiskit.quantum_info as qi
import qiskit
from qiskit import QuantumCircuit, QuantumRegister #, execute, Aer, IBMQ
from qiskit.visualization import plot_bloch_multivector, plot_bloch_vector
from qiskit.quantum_info import Statevector


qr = qiskit.QuantumRegister(4)
qc = qiskit.QuantumCircuit(qr)
qc.initialize([math.sqrt(3)/2, 1/2], 0)
qc.initialize([math.sqrt(3)/2, 1/2], 1)
qc.initialize([math.sqrt(3)/2, 1/2], 2)
qc.initialize([math.sqrt(3)/2, 1/2], 3)
qc.x(qr[1])
qc.y(qr[2])
qc.z(qr[3])

# qc = qc.reverse_bits()

print(qc.draw())
print(qiskit.quantum_info.Statevector.from_instruction(qc))

state = Statevector(qc)
plot_bloch_multivector(state)
```

Rotating phase gates (gates S and T)

- Similar to gate Z, both S and T rotate qubit phase (φ value):

$$\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{\imath\pi}{4}} \end{bmatrix}$$

- Using gate S with the basis

$$\mathbf{S}\,|{+\imath}\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = |-\rangle \qquad \mathbf{S}\,|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = |{+\imath}\rangle \qquad \mathbf{S}\,|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$\mathbf{S}\,|{-\imath}\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = |+\rangle \qquad \mathbf{S}\,|-\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = |{-\imath}\rangle \qquad \mathbf{S}\,|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \imath \end{bmatrix} = \imath\,|1\rangle$$

S gate at the Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + (\cos\phi + \imath\sin\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi$$

$$\mathbf{S}\,|+\imath\rangle = |-\rangle$$

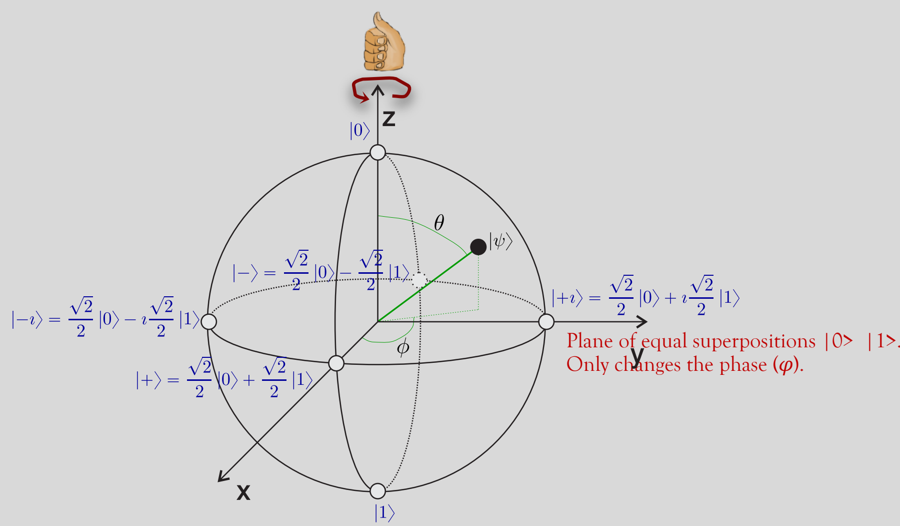
$$\mathbf{S}\,|-\imath\rangle = |+\rangle$$

$$\mathbf{S}\,|+\rangle = |+\imath\rangle$$

$$\mathbf{S}\,|-\rangle = |-\imath\rangle$$

$$\mathbf{S}\,|0\rangle = |0\rangle$$

$$\mathbf{S}\,|1\rangle = \imath\,|1\rangle, \quad \phi = \frac{\pi}{2}, \quad \theta = \pi$$



S gate increases φ in $\pi/2$

Relations between phase gates S, T, Z

- $S^2 = Z$: $$\mathbf{S}\,\mathbf{S} = \begin{bmatrix}1 & 0\\ 0 & \imath\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \imath\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \mathbf{Z}$$

- $T^2 = S$: $$\mathbf{T}\,\mathbf{T} = \begin{bmatrix}1 & 0\\ 0 & e^{\frac{\imath\pi}{4}}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{\frac{\imath\pi}{4}}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & e^{\frac{\imath\pi}{2}}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & \cancelto{0}{\cos\left(\frac{\pi}{2}\right)} + \imath\cancelto{1}{\sin\left(\frac{\pi}{2}\right)}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & \imath\end{bmatrix} = \mathbf{S}$$

- So equivalently: $$\sqrt{\mathbf{Z}} = \mathbf{S}, \quad \sqrt{\mathbf{S}} = \mathbf{T}, \quad \mathbf{T}^4 = \mathbf{Z}$$

S gate increases φ in $\pi/2$, so T gate is also called $\pi/8$

Gates' Square Root

- Remember that

$$e^{\pm\imath\theta} = \cos\theta \pm \imath\sin\theta, \qquad e^{\pm\imath\theta\mathbf{B}} = \mathbf{I}\cos\theta \pm \imath\mathbf{B}\sin\theta.$$

For involutory matrices `B` such that `B²=I`

For example, `X²=I`, `Y²=I`, `Z²=I`, `H²=I`,

Let's start with $e^{-\imath\theta\mathbf{B}} = \mathbf{I}\cos\theta - \imath\mathbf{B}\sin\theta$

Let's make $\theta = \pi/2$: $e^{-\imath\frac{\pi}{2}\mathbf{B}} = \mathbf{I}\cancelto{0}{\cos\frac{\pi}{2}} - \imath\,\mathbf{B}\cancelto{1}{\sin\frac{\pi}{2}} = -\imath\,\mathbf{B}$

Multiply both sides by $\imath$, we got: $\imath e^{-\imath\frac{\pi}{2}\mathbf{B}} = \mathbf{B}$

Remember that $e^{\imath\theta} = \cos\theta + \imath\sin\theta$, with $\theta = \pi/2$, we got: $e^{\imath\frac{\pi}{2}} = \cancelto{0}{\cos\frac{\pi}{2}} + \imath\cancelto{1}{\sin\frac{\pi}{2}} = \imath$. So $\imath = e^{\imath\frac{\pi}{2}}$

Replacing the $\imath$ just calculated we got: $\mathbf{B} = e^{\imath\frac{\pi}{2}}e^{-\imath\frac{\pi}{2}\mathbf{B}}$

So $\sqrt{\mathbf{B}} = \sqrt{e^{\imath\frac{\pi}{2}}e^{-\imath\frac{\pi}{2}\mathbf{B}}} = e^{\imath\frac{\pi}{4}}e^{-\imath\frac{\pi}{4}\mathbf{B}}$

Remember that $\mathbf{B}^2 = \mathbf{I}$, so $e^{-\imath\frac{\pi}{4}\mathbf{B}} = \mathbf{I}\cos\frac{\pi}{4} - \imath\mathbf{B}\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(\mathbf{I} - \imath\mathbf{B})$, and $e^{\imath\frac{\pi}{4}} = \cos\frac{\pi}{4} + \imath\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+\imath)$

So $\sqrt{\mathbf{B}} = \frac{1}{\sqrt{2}}(1+\imath)\frac{1}{\sqrt{2}}(\mathbf{I} - \imath\mathbf{B})$.

$\sqrt{\mathbf{X}} = \sqrt{\mathbf{NOT}}, \quad \sqrt{\mathbf{Y}}, \quad \sqrt{\mathbf{Z}}, \quad \sqrt{\mathbf{H}}$

- For any involutory matrix `A (B²=I)`: $\sqrt{\mathbf{A}} = \frac{1+\imath}{2}(\mathbf{I} - \imath\mathbf{A})$

Easier proof?

$$\left(\sqrt{\mathbf{A}}\right)^2 = \mathbf{A} = \left(\frac{1+\imath}{2}\right)^2 (\mathbf{I} - \imath\mathbf{A})^2 = \frac{\cancel{1} + 2\imath - \cancel{1}}{4}\left(\mathbf{I}^2 - 2\imath\mathbf{A} + \imath^2\mathbf{A}^2\right) = \frac{\imath}{2}\left(\cancel{\mathbf{I}} - 2\imath\mathbf{A} - \cancel{\mathbf{I}}\right) = \frac{-\imath^2}{\cancel{2}}\cancel{2}\mathbf{A} = \mathbf{A}$$

- As `X`, `Y`, `Z`, `H` are involutory, so:

$$\sqrt{\mathbf{X}} = \frac{1+\imath}{2}(\mathbf{I} - \imath\mathbf{X}) = \frac{1+\imath}{2}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & \imath\\\imath & 0\end{bmatrix}\right) = \frac{1+\imath}{2}\left(\begin{bmatrix}1 & -\imath\\-\imath & 1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1+\imath & 1-\imath\\1-\imath & 1+\imath\end{bmatrix}$$

$$\sqrt{\mathbf{Y}} = \frac{1+\imath}{2}(\mathbf{I} - \imath\mathbf{Y}) = \frac{1+\imath}{2}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\right) = \frac{1+\imath}{2}\left(\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1+\imath & -1-\imath\\1+\imath & 1+\imath\end{bmatrix}$$

$$\sqrt{\mathbf{Z}} = \frac{1+\imath}{2}(\mathbf{I} - \imath\mathbf{Z}) = \frac{1+\imath}{2}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}\imath & 0\\0 & -\imath\end{bmatrix}\right) = \frac{1+\imath}{2}\begin{bmatrix}1-\imath & 0\\0 & 1+\imath\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1-\imath^2 & 0\\0 & \cancel{1} + 2\imath + \cancel{\imath^2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\imath\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & \imath\end{bmatrix}$$

$$\sqrt{\mathbf{H}} = \frac{1+\imath}{2}(\mathbf{I} - \imath\mathbf{H}) = \frac{1+\imath}{2}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \imath\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\right) = \frac{1+\imath}{2}\begin{bmatrix}1-\imath\frac{\sqrt{2}}{2} & -\imath\frac{\sqrt{2}}{2}\\-\imath\frac{\sqrt{2}}{2} & 1+\imath\frac{\sqrt{2}}{2}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}(1+\imath)\left(2-\imath\sqrt{2}\right) & (1+\imath)\left(-\imath\sqrt{2}\right)\\(1+\imath)\left(-\imath\sqrt{2}\right) & (1+\imath)\left(2+\imath\sqrt{2}\right)\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}2-\imath\sqrt{2}+2\imath-\imath^2\sqrt{2} & -\imath\sqrt{2}-\imath^2\sqrt{2}\\-\imath\sqrt{2}-\imath^2\sqrt{2} & 2+\imath\sqrt{2}+2\imath+\imath^2\sqrt{2}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}\sqrt{2}(1-\imath)+2+2\imath & \sqrt{2}(1-\imath)\\\sqrt{2}(1-\imath) & \sqrt{2}(\imath-1)+2+2\imath\end{bmatrix}$$

An alternative $\sqrt{\mathbf{NOT}}$

- The gate $\sqrt{\mathbf{NOT}} = \frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix}$
- It's equivalent to $\frac{1}{\sqrt{2\imath}}\begin{bmatrix} \imath & 1 \\ 1 & \imath \end{bmatrix}$

$$\frac{1}{\sqrt{2\imath}}\begin{bmatrix} \imath & 1 \\ 1 & \imath \end{bmatrix}\frac{1}{\sqrt{2\imath}}\begin{bmatrix} \imath & 1 \\ 1 & \imath \end{bmatrix} = \frac{1}{2\imath}\begin{bmatrix} 0 & 2\imath \\ 2\imath & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathbf{NOT} \quad \blacksquare$$

Remember that $e^{\pm\imath\theta} = \cos\theta \pm \imath\sin\theta.$

So, $e^{\imath\pi/4} = \cos(\pi/4) + \imath\sin(\pi/4) = \frac{1}{\sqrt{2}}(1+\imath)$, and, $e^{-\imath\pi/4} = \cos(-\pi/4) + \imath\sin(-\pi/4) = \frac{1}{\sqrt{2}}(1-\imath)$, and, $e^{\imath\pi/2} = \cos(\pi/2) + \imath\sin(\pi/2) = \imath.$

Hence $\frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{2}e^{\imath\pi/4} & \sqrt{2}e^{-\imath\pi/4} \\ \sqrt{2}e^{-\imath\pi/4} & \sqrt{2}e^{\imath\pi/4} \end{bmatrix} = \frac{\sqrt{2}}{2}\frac{\sqrt{\imath}}{\sqrt{\imath}}\begin{bmatrix} e^{\imath\pi/4} & e^{-\imath\pi/4} \\ e^{-\imath\pi/4} & e^{\imath\pi/4} \end{bmatrix} = \frac{1}{\sqrt{2\imath}}e^{\imath\pi/4}\begin{bmatrix} e^{\imath\pi/4} & e^{-\imath\pi/4} \\ e^{-\imath\pi/4} & e^{\imath\pi/4} \end{bmatrix} = \frac{1}{\sqrt{2\imath}}\begin{bmatrix} e^{\imath\pi/2} & 1 \\ 1 & e^{\imath\pi/2} \end{bmatrix} = \frac{1}{\sqrt{2\imath}}\begin{bmatrix} \imath & 1 \\ 1 & \imath \end{bmatrix} \quad \blacksquare$

Interpretation:
X is a π rotation around X-axis… √X is a half rotation (√X√X = X)… you can rotate clockwise ($\pi/2$) or counterclockwise ($3\pi/2$)… Two equivalent expressions.

$\sqrt{\mathbf{X}} = \sqrt{\mathbf{NOT}}$

$$\sqrt{\mathbf{X}}\,|0\rangle = \frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\imath \\ 1-\imath \end{bmatrix} = \frac{1}{2}\frac{1}{1-\imath}\begin{bmatrix} 1-\imath^2 \\ \cancel{1}-2\imath+\cancel{\imath^2} \end{bmatrix}$$

$$= \frac{1}{1-\imath}\begin{bmatrix} 1 \\ -\imath \end{bmatrix} = \frac{1}{1-\imath}\frac{2}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |-\imath\rangle$$

$$\sqrt{\mathbf{X}}\,|1\rangle = \frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-\imath \\ 1+\imath \end{bmatrix} = \frac{1}{2}\frac{1}{1+\imath}\begin{bmatrix} 1-\imath^2 \\ \cancel{1}+2\imath+\cancel{\imath^2} \end{bmatrix}$$

$$= \frac{1}{1+\imath}\frac{2}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |+\imath\rangle$$

$$\sqrt{\mathbf{X}}\,|+\rangle = \frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} 1+\cancel{\imath}+1-\cancel{\imath} \\ 1-\cancel{\imath}+1+\cancel{\imath} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = |+\rangle \underset{\text{physics}}{\sim} |+\rangle$$

$$\sqrt{\mathbf{X}}\,|-\rangle = \frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \cancel{1}+\imath-\cancel{1}+\imath \\ \cancel{1}-\imath-\cancel{1}-\imath \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle \underset{\text{physics}}{\sim} |-\rangle$$

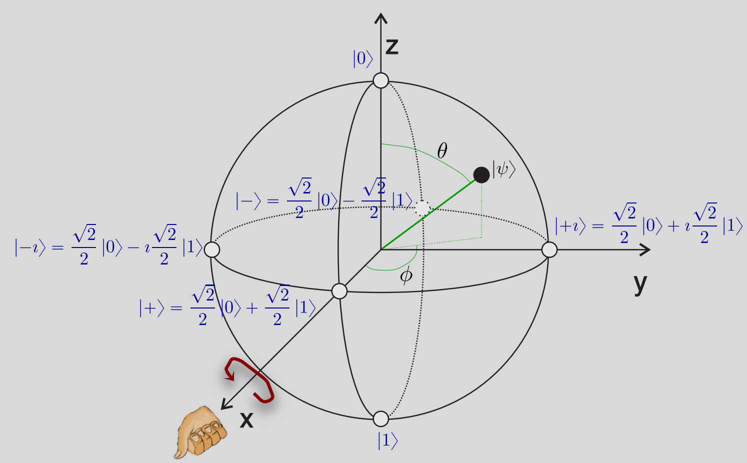
$$\sqrt{\mathbf{X}}\,|+\imath\rangle = \frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \cancel{1}+\imath+\imath-\cancel{\imath^2} \\ \cancel{1}-\cancel{\imath}+\cancel{\imath}+\cancel{\imath^2} \end{bmatrix}$$

$$= \imath\frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \underset{\text{physics}}{\sim} |0\rangle$$

$$\sqrt{\mathbf{X}}\,|-\imath\rangle = \frac{1}{2}\begin{bmatrix} 1+\imath & 1-\imath \\ 1-\imath & 1+\imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \cancel{1}+\cancel{\imath}-\cancel{\imath}+\cancel{\imath^2} \\ 1-\imath-\imath-\imath^2 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}(1-\imath)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \underset{\text{physics}}{\sim} |1\rangle$$

The $\sqrt{\text{X}}$ gate rotates $\pi/2$ the qubit around the X-axis



$\sqrt{\mathbf{Y}}$

$$\sqrt{\mathbf{Y}}\,|0\rangle = \frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \imath+1 \\ \imath+1 \end{bmatrix}$$

$$= \frac{1}{2}(\imath+1)\frac{2}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |+\rangle$$

$$\sqrt{\mathbf{Y}}\,|1\rangle = \frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\imath-1 \\ +\imath+1 \end{bmatrix}$$

$$= \frac{1}{2}(-\imath-1)\frac{2}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |-\rangle$$

$$\sqrt{\mathbf{Y}}\,|+\rangle = \frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \cancel{\imath}+\cancel{1}-\cancel{\imath}-\cancel{1} \\ \imath+1+\imath+1 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 \\ \imath+1 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}(\imath+1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \underset{\text{physics}}{\sim} |1\rangle$$

$$\sqrt{\mathbf{Y}}\,|-\rangle = \frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \imath+1+\imath+1 \\ \cancel{\imath}+\cancel{1}-\cancel{\imath}-\cancel{1} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} \imath+1 \\ 0 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}(\imath+1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \underset{\text{physics}}{\sim} |0\rangle$$

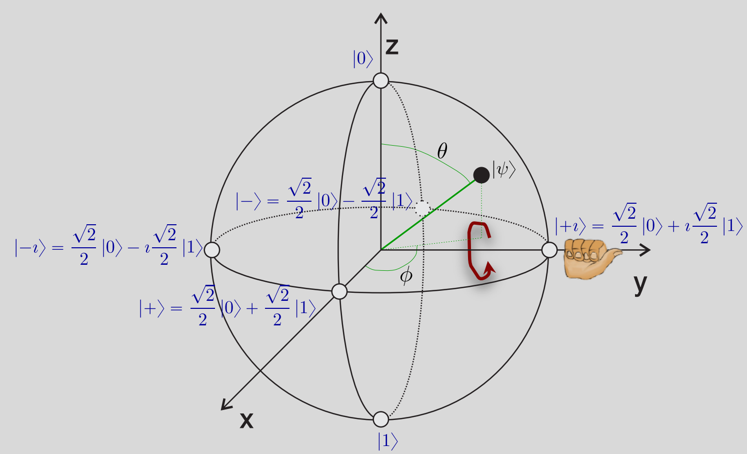
$$\sqrt{\mathbf{Y}}\,|+\imath\rangle = \frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \cancel{\imath}+1-\imath^2-\cancel{\imath} \\ \imath+\cancel{1}+\cancel{\imath^2}+\imath \end{bmatrix}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} 2 \\ 2\imath \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = |+\imath\rangle \underset{\text{physics}}{\sim} |+\imath\rangle$$

$$\sqrt{\mathbf{Y}}\,|-\imath\rangle = \frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \imath+\cancel{1}+\cancel{\imath^2}+\imath \\ \cancel{\imath}+1-\imath^2-\cancel{\imath} \end{bmatrix}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} 2\imath \\ 2 \end{bmatrix} = -\frac{\imath}{\imath}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = |-\imath\rangle \underset{\text{physics}}{\sim} |-\imath\rangle$$

The $\sqrt{\text{Y}}$ gate rotates $\pi/2$ the qubit around the Y-axis



$\sqrt{\mathbf{Z}}$

$$\sqrt{\mathbf{Z}}\,|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \underset{\text{physics}}{\sim} |0\rangle$$

$$\sqrt{\mathbf{Z}}\,|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \imath \end{bmatrix} = \imath\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \imath\,|1\rangle \underset{\text{physics}}{\sim} |1\rangle$$

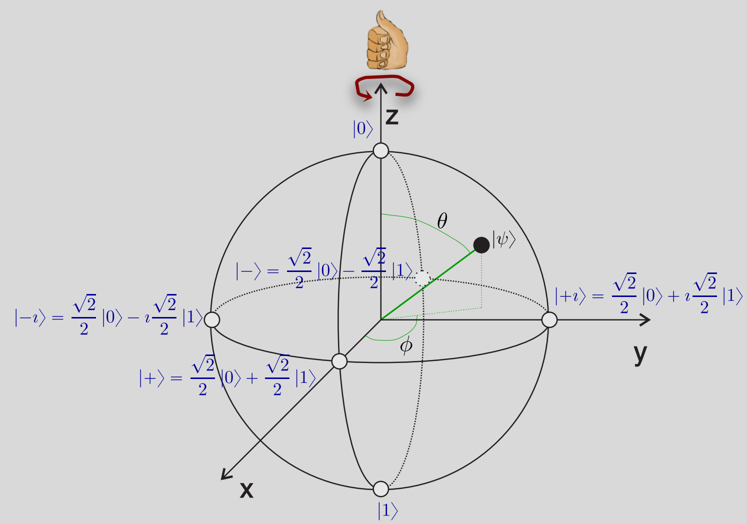
$$\sqrt{\mathbf{Z}}\,|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = |+\imath\rangle \underset{\text{physics}}{\sim} |+\imath\rangle$$

$$\sqrt{\mathbf{Z}}\,|-\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = |-\imath\rangle \underset{\text{physics}}{\sim} |-\imath\rangle$$

$$\sqrt{\mathbf{Z}}\,|+\imath\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = |-\rangle \underset{\text{physics}}{\sim} |-\rangle$$

$$\sqrt{\mathbf{Z}}\,|-\imath\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = |+\rangle \underset{\text{physics}}{\sim} |+\rangle$$

The $\sqrt{Z}$ gate rotates $\pi/2$ the qubit around the Z-axis



$\sqrt{\mathbf{H}}$

$$\sqrt{\mathbf{H}}\,|0\rangle = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2}+2\imath+2 & -(\imath-1)\sqrt{2} \\ -(\imath-1)\sqrt{2} & (\imath-1)\sqrt{2}+2\imath+2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2}+2\imath+2 \\ -(\imath-1)\sqrt{2} \end{bmatrix}$$

$$\sqrt{\mathbf{H}}\,|1\rangle = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2}+2\imath+2 & -(\imath-1)\sqrt{2} \\ -(\imath-1)\sqrt{2} & (\imath-1)\sqrt{2}+2\imath+2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2} \\ (\imath-1)\sqrt{2}+2\imath+2 \end{bmatrix}$$

$$\sqrt{\mathbf{H}}\,|+\rangle = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2}+2\imath+2 & -(\imath-1)\sqrt{2} \\ -(\imath-1)\sqrt{2} & (\imath-1)\sqrt{2}+2\imath+2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -\sqrt{2}\left((\imath-1)\sqrt{2}-2\imath-2\right)-2\imath+2 \\ -\sqrt{2}\left(-(i-1)\sqrt{2}-2\imath-2\right)-2\imath+2 \end{bmatrix}$$

$$\sqrt{\mathbf{H}}\,|-\rangle = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2}+2\imath+2 & -(\imath-1)\sqrt{2} \\ -(\imath-1)\sqrt{2} & (\imath-1)\sqrt{2}+2\imath+2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -\sqrt{2}\left((\imath-1)\sqrt{2}-2\imath-2\right)+2\imath-2 \\ \sqrt{2}\left(-(\imath-1)\sqrt{2}-2\imath-2\right)-2\imath+2 \end{bmatrix}$$

$$\sqrt{\mathbf{H}}\,|+\imath\rangle = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2}+2\imath+2 & -(\imath-1)\sqrt{2} \\ -(\imath-1)\sqrt{2} & (\imath-1)\sqrt{2}+2\imath+2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -\sqrt{2}\left((\imath-1)\sqrt{2}-2\imath-2\right)+2\imath+2 \\ -\sqrt{2}\left(-(\imath-1)\sqrt{2}-2\imath-2\right)-2\imath+2 \end{bmatrix}$$

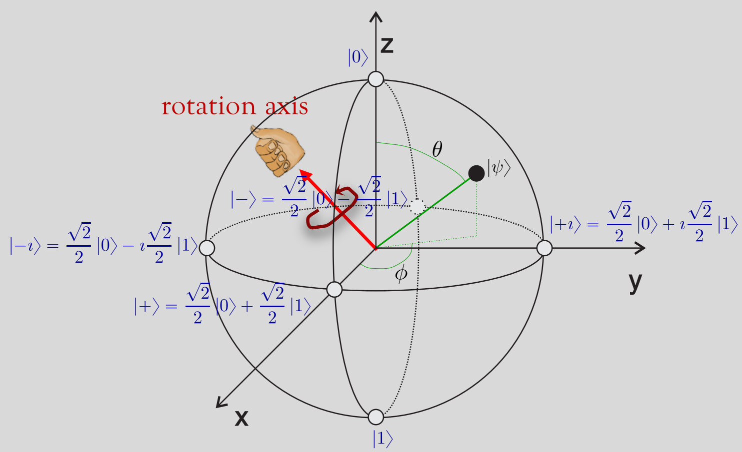
$$\sqrt{\mathbf{H}}\,|-\imath\rangle = \frac{1}{4}\begin{bmatrix} -(\imath-1)\sqrt{2}+2\imath+2 & -(\imath-1)\sqrt{2} \\ -(\imath-1)\sqrt{2} & (\imath-1)\sqrt{2}+2\imath+2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -\sqrt{2}\left((i-1)\sqrt{2}-2\imath-2\right)-2\imath-2 \\ \sqrt{2}\left(-(i-1)\sqrt{2}-2\imath-2\right)-2\imath+2 \end{bmatrix}$$

Eigen-vectors: $\begin{bmatrix} \cos\left(\frac{3\pi}{8}\right) \\ -\sin\left(\frac{3\pi}{8}\right) \end{bmatrix}, \quad \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) \end{bmatrix}$

$$\cos\left(\frac{3\pi}{8}\right)|0\rangle - \sin\left(\frac{3\pi}{8}\right)|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad \theta = \frac{3\pi}{4}, \phi = \pi.$$

$$\cos\left(\frac{\pi}{8}\right)|0\rangle - \sin\left(\frac{\pi}{8}\right)|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad \theta = \frac{\pi}{4}, \phi = 0.$$

The $\sqrt{\text{H}}$ gate rotates the qubit $\pi/2$ around the axis on the diagonal Z+X



Fractional phase shift gate

$$\mathbf{R}_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi\imath}{2^k}} \end{bmatrix}.$$

$$\mathbf{R}_0 = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi\imath} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \cancelto{1}{\cos(2\pi)} + \imath\cancelto{0}{\sin(2\pi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi\imath}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\pi\imath} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{Z}$$

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi\imath}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{\pi\imath}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} = \mathbf{S}$$

$$\mathbf{R}_3 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi\imath}{8}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{\pi\imath}{4}} \end{bmatrix} = \mathbf{T}$$

Eigenvectors, Eigenvalues for Fractional phase shift gate

$$\mathbf{R}_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi\imath}{2^k}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi\imath}{2^k}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi\imath}{2^k}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{\frac{2\pi\imath}{2^k}} \end{bmatrix} = e^{\frac{2\pi\imath}{2^k}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$k = 0$	$k = 1$	$k = 2$	$k = 3$	$\dots$
$\lambda_1 = +1$	$\lambda_1 = +1$	$\lambda_1 = +1$	$\lambda_1 = +1$	$\dots$
$\lambda_2 = +1$	$\lambda_2 = -1$	$\lambda_2 = +\imath$	$\lambda_2 = \frac{\sqrt{2}}{2}(1+\imath)$	$\dots$
I	**Z**	**S**	**T**	$\dots$

Generalized Rotation Operations

Rotation of arbitrary angle γ around x-axis:

$$\mathbf{R}_x(\gamma) = e^{-\imath\frac{\gamma}{2}\mathbf{X}} = \cos\left(\frac{\gamma}{2}\right)\mathbf{I} - \imath\sin\left(\frac{\gamma}{2}\right)\mathbf{X} = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & 0 \\ 0 & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix} - \imath\sin\left(\frac{\gamma}{2}\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\gamma}{2} & -\imath\sin\frac{\gamma}{2} \\ -\imath\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{bmatrix}$$

Rotation of arbitrary angle γ around y-axis:

$$\mathbf{R}_y(\gamma) = e^{-\imath\frac{\gamma}{2}\mathbf{Y}} = \cos\left(\frac{\gamma}{2}\right)\mathbf{I} - \imath\sin\left(\frac{\gamma}{2}\right)\mathbf{Y} = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & 0 \\ 0 & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix} - \imath\sin\left(\frac{\gamma}{2}\right)\begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{bmatrix}$$

Rotation of arbitrary angle γ around z-axis:

$$\mathbf{R}_z(\gamma) = e^{-\imath\frac{\gamma}{2}\mathbf{Z}} = e^{\wedge}\left(-\imath\frac{\gamma}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = e^{\wedge}\begin{bmatrix} -\imath\frac{\gamma}{2} & 0 \\ 0 & \imath\frac{\gamma}{2} \end{bmatrix} = \begin{bmatrix} e^{-\imath\frac{\gamma}{2}} & 0 \\ 0 & e^{\imath\frac{\gamma}{2}} \end{bmatrix}$$

Rotation of arbitrary angle γ around 'h'-axis:

$$\mathbf{R}_H(\gamma) = e^{-\imath\frac{\gamma}{2}\mathbf{H}} = \cos\left(\frac{\gamma}{2}\right)\mathbf{I} - \imath\sin\left(\frac{\gamma}{2}\right)\mathbf{H} = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & 0 \\ 0 & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix} - \imath\sin\left(\frac{\gamma}{2}\right)\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\gamma}{2} - \imath\sin\left(\frac{\gamma}{2}\right)\frac{\sqrt{2}}{2} & -\imath\sin\left(\frac{\gamma}{2}\right)\frac{\sqrt{2}}{2} \\ -\imath\sin\left(\frac{\gamma}{2}\right)\frac{\sqrt{2}}{2} & \cos\frac{\gamma}{2} + \imath\sin\left(\frac{\gamma}{2}\right)\frac{\sqrt{2}}{2} \end{bmatrix}$$

Global phase has no observable impact.

Sanity Check:

$$\mathbf{R}_x(\pi) = \begin{bmatrix} \cancelto{0}{\cos\frac{\pi}{2}} & -\imath\cancelto{1}{\sin\frac{\pi}{2}} \\ -\imath\cancelto{1}{\sin\frac{\pi}{2}} & \cancelto{0}{\cos\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 0 & -\imath \\ -\imath & 0 \end{bmatrix} = -\imath\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underset{\text{physics}}{\sim} \mathbf{NOT}$$

$$\mathbf{R}_y(\pi) = \begin{bmatrix} \cancelto{0}{\cos\frac{\pi}{2}} & -\cancelto{1}{\sin\frac{\pi}{2}} \\ \cancelto{1}{\sin\frac{\pi}{2}} & \cancelto{0}{\cos\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\imath\begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} \underset{\text{physics}}{\sim} \mathbf{Y}$$

$$\mathbf{R}_z(\pi) = \begin{bmatrix} -e^{\imath\frac{\pi}{2}} & 0 \\ 0 & e^{\imath\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} \cancelto{0}{\cos\frac{\pi}{2}} - \imath\cancelto{1}{\sin\frac{\pi}{2}} & 0 \\ 0 & \cancelto{0}{\cos\frac{\pi}{2}} + \imath\cancelto{1}{\sin\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} -\imath & 0 \\ 0 & \imath \end{bmatrix} = -\imath\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underset{\text{physics}}{\sim} \mathbf{Z}$$

$$\mathbf{R}_H(\pi) = \begin{bmatrix} \cancelto{0}{\cos\frac{\pi}{2}} - \imath\cancelto{1}{\sin\frac{\pi}{2}}\frac{\sqrt{2}}{2} & -\imath\cancelto{1}{\sin\frac{\pi}{2}}\frac{\sqrt{2}}{2} \\ -\imath\cancelto{1}{\sin\frac{\pi}{2}}\frac{\sqrt{2}}{2} & \cancelto{0}{\cos\frac{\pi}{2}} + \imath\cancelto{1}{\sin\frac{\pi}{2}}\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} -\imath & -\imath \\ -\imath & \imath \end{bmatrix} = -\imath\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underset{\text{physics}}{\sim} \mathbf{H}$$

Eigenvectors, Eigenvalues for generalized rotation gates

$$\mathbf{R_X}(\gamma)\,|+\rangle = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -\imath\sin\left(\frac{\gamma}{2}\right) \\ -\imath\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) - \imath\sin\left(\frac{\gamma}{2}\right)\right) \\ \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) - \imath\sin\left(\frac{\gamma}{2}\right)\right) \end{bmatrix} = e^{-\imath\frac{\gamma}{2}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |+\rangle\,.$$

eigenvalues eigenvectors

$$\mathbf{R_X}(\gamma)\,|-\rangle = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -\imath\sin\left(\frac{\gamma}{2}\right) \\ -\imath\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) + \imath\sin\left(\frac{\gamma}{2}\right)\right) \\ -\frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) + \imath\sin\left(\frac{\gamma}{2}\right)\right) \end{bmatrix} = e^{\imath\frac{\gamma}{2}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |-\rangle\,.$$

eigenvalues eigenvectors

$$\mathbf{R_Y}(\gamma)\,|+\imath\rangle = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) - \imath\sin\left(\frac{\gamma}{2}\right)\right) \\ \imath\frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) - \imath\sin\left(\frac{\gamma}{2}\right)\right) \end{bmatrix} = e^{-\imath\frac{\gamma}{2}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |+\imath\rangle\,.$$

$$\mathbf{R_Y}(\gamma)\,|-\imath\rangle = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) + \imath\sin\left(\frac{\gamma}{2}\right)\right) \\ -\imath\frac{\sqrt{2}}{2}\left(\cos\left(\frac{\gamma}{2}\right) + \imath\sin\left(\frac{\gamma}{2}\right)\right) \end{bmatrix} = e^{\imath\frac{\gamma}{2}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} \underset{\text{physics}}{\sim} |-\imath\rangle\,.$$

eigenvalues eigenvectors

$$\mathbf{R_Z}(\gamma)\,|0\rangle = \begin{bmatrix} e^{-\imath\frac{\gamma}{2}} & 0 \\ 0 & e^{\imath\frac{\gamma}{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-\imath\frac{\gamma}{2}} \\ 0 \end{bmatrix} = e^{-\imath\frac{\gamma}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \underset{\text{physics}}{\sim} |0\rangle\,.$$

$$\mathbf{R_Z}(\gamma)\,|1\rangle = \begin{bmatrix} e^{-\imath\frac{\gamma}{2}} & 0 \\ 0 & e^{\imath\frac{\gamma}{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{\imath\frac{\gamma}{2}} \end{bmatrix} = e^{\imath\frac{\gamma}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underset{\text{physics}}{\sim} |1\rangle\,.$$

Eigenvectors, Eigenvalues for H gate

- Verify that: $$\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{2}+1\\ 1\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix}\sqrt{2}+2\\ \sqrt{2}\end{bmatrix} = (+1)\begin{bmatrix}\sqrt{2}+1\\ 1\end{bmatrix}.$$

$$\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}-\sqrt{2}+1\\ 1\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix}-\sqrt{2}+2\\ -\sqrt{2}\end{bmatrix} = (-1)\begin{bmatrix}-\sqrt{2}+1\\ 1\end{bmatrix}.$$

- But, better to normalize these vectors. So, the corresponding eigenvectors – eigenvalues are:

$$(+1)\begin{bmatrix}\sqrt{2}+1\\ 1\end{bmatrix} \xRightarrow[|\vec{v}|=1]{\text{Normalize}} (+1)\frac{1}{\sqrt{\left(\sqrt{2}+1^2\right)^2+1}}\begin{bmatrix}\sqrt{2}+1\\ 1\end{bmatrix} = (+1)\begin{bmatrix}\frac{\sqrt{2}+1}{\sqrt{4+2\sqrt{2}}}\\ \frac{1}{\sqrt{4+2\sqrt{2}}}\end{bmatrix}.$$

eigenvalues eigenvectors

$$(-1)\begin{bmatrix}-\sqrt{2}+1\\ 1\end{bmatrix} \xRightarrow[|\vec{v}|=1]{\text{Normalize}} (-1)\frac{1}{\sqrt{\left(-\sqrt{2}+1^2\right)^2+1}}\begin{bmatrix}-\sqrt{2}+1\\ 1\end{bmatrix} = (-1)\begin{bmatrix}\frac{-\sqrt{2}+1}{\sqrt{4-2\sqrt{2}}}\\ \frac{1}{\sqrt{4-2\sqrt{2}}}\end{bmatrix}.$$

Eigenvectors, Eigenvalues for S gate

- Verify that:

$$\begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \imath \end{bmatrix} = (+\imath) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- Vectors are already normalized.

Eigenvectors, Eigenvalues for T gate

$$\left(\cos\left(\frac{\pi}{4}\right)+\imath\sin\left(\frac{\pi}{4}\right)\right)\begin{bmatrix}0\\1\end{bmatrix} = \left(\frac{\sqrt{2}}{2}+\imath\frac{\sqrt{2}}{2}\right)\begin{bmatrix}0\\1\end{bmatrix}$$

- Verify that:

$$\begin{bmatrix}1 & 0\\0 & e^{\frac{\imath\pi}{4}}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\e^{\frac{\imath\pi}{4}}\end{bmatrix} = (+e^{\frac{\imath\pi}{4}})\begin{bmatrix}0\\1\end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix}1 & 0\\0 & e^{\frac{\imath\pi}{4}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = (+1)\begin{bmatrix}1\\0\end{bmatrix}.$$

- Vectors are already normalized.

Eigenvectors, Eigenvalues for `SQRT_X` gate

- Verify that:

$$\frac{1}{2}\begin{bmatrix} \imath+1 & -\imath+1 \\ -\imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = (+1)\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

eigenvalues eigenvectors

$$\frac{1}{2}\begin{bmatrix} \imath+1 & -\imath+1 \\ -\imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} 2\imath \\ -2\imath \end{bmatrix} = (+\imath)\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

- Vectors are already normalized.

Eigenvectors, Eigenvalues for `SQRT_Y` gate

- Verify that:

$$\frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \cancel{\imath}+1-\imath^2-\cancel{\imath} \\ \imath+\cancel{1}+\cancel{\imath^2}+\imath \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} 2 \\ 2\imath \end{bmatrix} = (+1)\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix}.$$

eigenvalues eigenvectors

$$\frac{1}{2}\begin{bmatrix} \imath+1 & -\imath-1 \\ \imath+1 & \imath+1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} \imath+\cancel{1}+\cancel{\imath^2}+\imath \\ \cancel{\imath}+1-\imath^2-\cancel{\imath} \end{bmatrix} = \frac{1}{2}\frac{\sqrt{2}}{2}\begin{bmatrix} 2\imath \\ 2 \end{bmatrix} = \frac{\sqrt{2}}{2}\imath\begin{bmatrix} 1 \\ \frac{1}{\imath} \end{bmatrix} = \frac{\sqrt{2}}{2}\imath\begin{bmatrix} 1 \\ \frac{1}{\imath}\frac{\imath^2}{\imath^2} \end{bmatrix} = \frac{\sqrt{2}}{2}\imath\begin{bmatrix} 1 \\ -\imath \end{bmatrix} = (+\imath)\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix}.$$

- Vectors are already normalized.

Eigenvectors, Eigenvalues for `SQRT_Z` gate

- Verify that:

$$\begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \imath \end{bmatrix} = (+\imath) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- Vectors are already normalized.

Eigenvectors, Eigenvalues for `SQRT_H` gate

- Verify that:

$$\begin{bmatrix} \frac{\sqrt{2}(1-\imath)+2+2\imath}{4} & \frac{\sqrt{2}(1-\imath)}{4} \\ \frac{\sqrt{2}(1-\imath)}{4} & \frac{\sqrt{2}(-1+\imath)+2+2\imath}{4} \end{bmatrix} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \sqrt{2}\,(1-\imath)+2+2\imath-\sqrt{2}\left(\sqrt{2}\,(1-\imath)+2+2\imath\right)+\sqrt{2}\,(1-\imath) \\ \sqrt{2}\,(1-\imath)-\sqrt{2}\sqrt{2}\,(1-\imath)+\sqrt{2}\,(-1+\imath)+2+2\imath \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \cancel{\sqrt{2}}-\imath\sqrt{2}+\cancel{2}+2\imath-\cancel{2}+2\imath-\cancel{2\sqrt{2}}-2\imath\sqrt{2}+\cancel{\sqrt{2}}-\imath\sqrt{2} \\ \cancel{\sqrt{2}}-\cancel{\imath\sqrt{2}}-\cancel{2}+2\imath-\cancel{\sqrt{2}}+\cancel{\imath\sqrt{2}}+\cancel{2}+2\imath \end{bmatrix} =$$

$$\frac{1}{\cancel{4}} \begin{bmatrix} \cancel{4}\imath-\cancel{4}\imath\sqrt{2} \\ \cancel{4}\imath \end{bmatrix} = \imath \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} = (+\imath) \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} \frac{\sqrt{2}(1-\imath)+2+2\imath}{4} & \frac{\sqrt{2}(1-\imath)}{4} \\ \frac{\sqrt{2}(1-\imath)}{4} & \frac{\sqrt{2}(-1+\imath)+2+2\imath}{4} \end{bmatrix} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \sqrt{2}\,(1-\imath)+2+2\imath+\sqrt{2}\left(\sqrt{2}\,(1-\imath)+2+2\imath\right)+\sqrt{2}\,(1-\imath) \\ \sqrt{2}\,(1-\imath)+\sqrt{2}\sqrt{2}\,(1-\imath)+\sqrt{2}\,(-1+\imath)+2+2\imath \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \sqrt{2}-\cancel{\imath\sqrt{2}}+2+\cancel{2\imath}+2-\cancel{2\imath}+2\sqrt{2}+2\cancel{\imath\sqrt{2}}+\sqrt{2}-\cancel{\imath\sqrt{2}} \\ \cancel{\sqrt{2}}-\cancel{\imath\sqrt{2}}+2-\cancel{2\imath}-\cancel{\sqrt{2}}+\cancel{\imath\sqrt{2}}+2+\cancel{2\imath} \end{bmatrix} =$$

$$\frac{1}{\cancel{4}} \begin{bmatrix} \cancel{4}+\cancel{4}\sqrt{2} \\ \cancel{4} \end{bmatrix} = (+1) \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}.$$

- But, better to normalize these vectors. So, the corresponding eigenvectors – eigenvalues are:

$$(+\imath) \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} \xrightarrow[|\vec{v}|=1]{\text{Normalize}} (+\imath) \frac{1}{\sqrt{\left(1-\sqrt{2}\right)^2+1^2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} = (+\imath) \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}.$$

eigenvalues eigenvectors

$$(+1) \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \xrightarrow[|\vec{v}|=1]{\text{Normalize}} (+1) \frac{1}{\sqrt{\left(1+\sqrt{2}\right)^2+1^2}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} = (+1) \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}.$$

Eigenvectors, Eigenvalues for `SWAP` gate

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = (+1)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = (+1)\begin{bmatrix}0\\1\\1\\0\end{bmatrix} \xRightarrow[|\vec{v}|=1]{\text{Normalize}} (+1)\begin{bmatrix}0\\\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\0\end{bmatrix}.$$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix} = (-1)\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \xRightarrow[|\vec{v}|=1]{\text{Normalize}} (-1)\begin{bmatrix}0\\\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\0\end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = (+1)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Eigenvectors, Eigenvalues for `CNOT` gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = (+1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \xRightarrow[|\vec{v}|=1]{\text{Normalize}} (+1) \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \xRightarrow[|\vec{v}|=1]{\text{Normalize}} (-1) \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Eigenvectors, Eigenvalues for **CY** gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\imath \\ 0 & 0 & \imath & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\imath \\ 0 & 0 & \imath & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\imath \\ 0 & 0 & \imath & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \imath \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\imath^2 \\ \imath \end{bmatrix} = (+1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ \imath \end{bmatrix} \xrightarrow[|\vec{v}|=1]{\text{Normalize}} (+1) \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{2} \\ \imath\frac{\sqrt{2}}{2} \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\imath \\ 0 & 0 & \imath & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\imath \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \imath^2 \\ \imath \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ \imath \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\imath \end{bmatrix} \xrightarrow[|\vec{v}|=1]{\text{Normalize}} (-1) \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{2} \\ -\imath\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Eigenvectors, Eigenvalues for CZ gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Eigenvectors, Eigenvalues for **CH** gate

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\0&0&\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=(+1)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\0&0&\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=(+1)\begin{bmatrix}0\\1\\0\\0\end{bmatrix}.$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\0&0&\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}0\\0\\1\\\sqrt{2}-1\end{bmatrix}=\begin{bmatrix}0\\0\\\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(\sqrt{2}-1\right)\\\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\left(\sqrt{2}-1\right)\end{bmatrix}=\begin{bmatrix}0\\0\\\cancel{\frac{\sqrt{2}}{2}}+\frac{2}{2}-\cancel{\frac{\sqrt{2}}{2}}\\\frac{\sqrt{2}}{2}-\frac{1}{1}+\frac{\sqrt{2}}{2}\end{bmatrix}=(+1)\begin{bmatrix}0\\0\\1\\\sqrt{2}-1\end{bmatrix}$$

$$\xrightarrow[|\vec{v}|=1]{\text{Normalize}}(+1)\frac{1}{\sqrt{\left(\sqrt{2}-1\right)^2+1^2}}\begin{bmatrix}0\\0\\1\\\sqrt{2}-1\end{bmatrix}=(+1)\frac{1}{\sqrt{2-2\sqrt{2}+1+1}}\begin{bmatrix}0\\0\\1\\\sqrt{2}-1\end{bmatrix}=(+1)\begin{bmatrix}0\\0\\\frac{1}{4-2\sqrt{2}}\\\frac{\sqrt{2}-1}{4-2\sqrt{2}}\end{bmatrix}.$$

eigenvalues eigenvectors

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\0&0&\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}0\\0\\1\\-\sqrt{2}-1\end{bmatrix}=\begin{bmatrix}0\\0\\\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(-\sqrt{2}-1\right)\\\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\left(-\sqrt{2}-1\right)\end{bmatrix}=\begin{bmatrix}0\\0\\\cancel{\frac{\sqrt{2}}{2}}-1-\cancel{\frac{\sqrt{2}}{2}}\\\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}\end{bmatrix}=\begin{bmatrix}0\\0\\-1\\\sqrt{2}+1\end{bmatrix}=(-1)\begin{bmatrix}0\\0\\1\\-\sqrt{2}-1\end{bmatrix}$$

$$\xrightarrow[|\vec{v}|=1]{\text{Normalize}}(-1)\frac{1}{\sqrt{\left(-\sqrt{2}-1\right)^2+1^2}}\begin{bmatrix}0\\0\\1\\-\sqrt{2}-1\end{bmatrix}=(-1)\frac{1}{\sqrt{2+2\sqrt{2}+1+1}}\begin{bmatrix}0\\0\\1\\-\sqrt{2}-1\end{bmatrix}=(-1)\begin{bmatrix}0\\0\\\frac{1}{4+2\sqrt{2}}\\\frac{-\sqrt{2}-1}{4+2\sqrt{2}}\end{bmatrix}.$$

Eigenvectors, Eigenvalues for `CCNOT` gate

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
:
\left\{
\begin{array}{ll}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\
\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\
\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}^T & \text{with } \textit{eigenvalue } -1
\end{array}
\right\}.
$$

Eigenvectors, Eigenvalues for CCY gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\imath \\ 0 & 0 & 0 & 0 & 0 & 0 & \imath & 0 \end{bmatrix} : \left\{ \begin{array}{ll} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & \imath \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\imath \end{bmatrix}^T & \text{with } \textit{eigenvalue } -1 \end{array} \right\}.$$

Eigenvectors, Eigenvalues for CCZ gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} : \left\{ \begin{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T & \text{with } \textit{eigenvalue } -1 \end{matrix} \right\}.$$

Eigenvectors, Eigenvalues for CCH gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} : \left\{ \begin{array}{ll} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & \sqrt{2}-1 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\sqrt{2}-1 \end{bmatrix}^T & \text{with } \textit{eigenvalue } -1 \end{array} \right\}.$$

Eigenvectors, Eigenvalues for `CSWAP` gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} : \left\{ \begin{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T & \text{with } \textit{eigenvalue } 1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}^T & \text{with } \textit{eigenvalue } -1 \end{matrix} \right\}.$$

Eigenstates

- An eigenstate for a qubit is a state that is left alone by a quantum operation,
  - At most a global phase is applied.

Pauli gates and qiskit

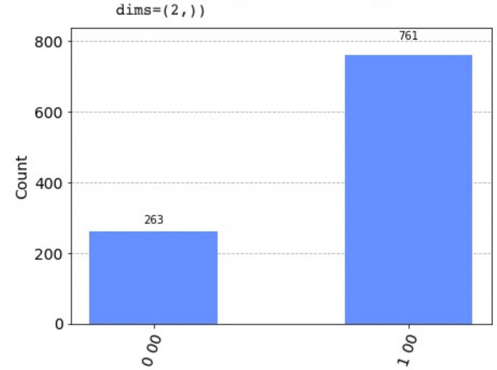
```
# Playing with the Pauli's gates X,Y,Z

from qiskit import QuantumCircuit, assemble, Aer
from qiskit.visualization import plot_histogram, plot_bloch_vector
from math import sqrt, pi

qreg = qiskit.QuantumRegister(1)
circuit = qiskit.QuantumCircuit(qreg,creg)
initial_state = [sqrt(3)/2, 1/2]      # podría ser [1/sqrt(2), 1j/sqrt(2)]
circuit.initialize(initial_state, 0)   # Apply initialisation operation to the 0th qubit
circuit.x(qreg[0]) # Apply an X gate to the first qubit. Change for y, z
circuit.save_statevector()   # Tell simulator to save statevector
qobj = assemble(circuit)      # Create a Qobj from the circuit for the simulator to run
result = sim.run(qobj).result() # Do the simulation and return the result
out_state = result.get_statevector()
print(out_state) # Display the output state vector
circuit.measure_all()
circuit.draw()

qobj = assemble(circuit)
result = sim.run(qobj).result()
counts = result.get_counts()
plot_histogram(counts)
```

```
Statevector([0.5      +0.j, 0.8660254+0.j],
            dims=(2,))
```



Full Adder using Quantum Gates
(Feynman Adder - 1984)

Quantum Mechanical Computers

By Richard P. Feynman

Introduction

This work is a part of an effort to analyze the physical limitations of computers due to the laws of physics. For example, Bennett[1] has made a careful study of the free energy dissipation that must accompany computation. He found it to be virtually zero. He suggested to me the question of the limitations due to quantum mechanics and the uncertainty principle. I have found that, aside from the obvious limitation to size if the working parts are to be made of atoms, there is no fundamental limit from these sources either.

We are here considering ideal machines; the effects of small imperfections will be considered later. This study is one of principle; our aim is to exhibit some Hamiltonian for a system which could serve as a computer. We are not concerned with whether we have the most efficient system, nor how we could best implement it.

Since the laws of quantum physics are reversible in time, we shall have to consider computing engines which obey such reversible laws. This problem already occurred to Bennett[1], and to Fredkin and Toffoli[2], and a great deal of thought has been given to it. Since it may not be familiar to you here, I shall review this, and in doing so, take the opportunity to review, very briefly, the conclusions of Bennett[2], for we shall confirm them all when we analyze our quantum system.

It is a result of computer science that a universal computer can be made by a suitably complex network of interconnected primitive elements. Following the usual classical analysis we can imagine the interconnections to be ideal wires carrying one of two standard voltages representing the local 1 and 0. We can take the primitive elements to be just two, NOT and AND (actually just the one element NAND = NOT AND suffices, for if one input is set at 1 the output is the NOT of the other input). They are symbolized in Fig. 1, with the logical values resulting on the outgoing wires, resulting from different combinations of input wires.

From a logical point of view, we must consider the wires in detail, for in other systems, and our quantum system in particular, we may not have wires as such. We see we really have two more logical primitives, FAN OUT when two wires are connected to one, and EXCHANGE, when wires are crossed. In the usual computer the NOT and NAND primitives are implemented by transistors, possibly as in Fig. 2.

What is the minimum free energy that must be expended to operate an ideal computer made of such primitives? Since, for example, when the AND operates the output line, c' is being determined to be one of two values no matter what it was before the entropy change is $ln(2)$ units. This represents a heat generation of $kT\,ln(2)$ at temperature T. For many years it was thought that this represented an absolute minimum to the quantity of heat per primitive step that had to be dissipated in making a calculation.

The question is academic at this time. In actual machines we are quite concerned with the heat dissipation question, but the transistor system used actually dissipates about $10^{10}kT$! As Bennett[3] has pointed out, this arises because to change a wire's voltage we dump it to ground through a resistance; and to build it up again we feed charge, again through a resistance, to the wire. It could be greatly reduced if energy could be stored in an inductance, or other reactive element.

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However, it is apparently very difficult to make inductive elements on silicon wafers with present techniques. Even Nature, in her DNA copying machine, dissipates about 100 kT per bit copied. Being, at present, so very far from this $kT\,ln(2)$ figure, it seems ridiculous to argue that even this is too high and the minimum is really essentially zero. But, we are going to be even more ridiculous later and consider bits written on one atom instead of the present 10^{11} atoms. Such nonsense is very entertaining to professors like me. I hope you will find it interesting and entertaining also.

What Bennett pointed out was that this former limit was wrong because it is not necessary to use irreversible primitives. Calculations can be done with reversible machines containing only reversible primitives. If this is done the minimum free energy required is independent of the complexity or number of logical steps in the calculation. If anything, it is kT per bit of the output answer.

But even this, which might be considered the free energy needed to clear the computer for further use, might also be considered as part of what you are going to do with the answer—the information in the result if you transmit it to another point. This is a limit only achieved ideally if you compute with a reversible computer at infinitesimal speed.

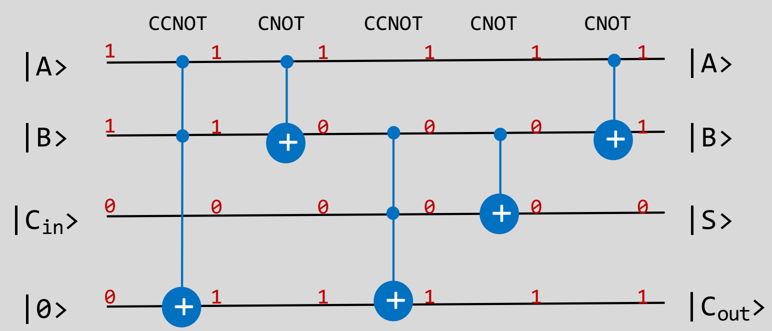
Computation with a reversible machine

We will now describe three reversible primitives that could be used to make a universal machine (Toffoli[4]). The first is the NOT which evidently loses no information, and is reversible, being reversed by acting again with NOT. Because the conventional symbol is not symmetrical we shall use an X on the wire instead (see Fig. 3a).

Next is what we shall call the CONTROLLED NOT (see Fig. 3b). There are two entering lines, a and b and two exiting lines, a' and b'. The a' is always the same as a, which is the control line. If the control is activated $a = 1$ then the out b' is the NOT of b. Otherwise b is unchanged, $b' = b$. The table of values

Full Adder using Quantum Gates (Feynman Adder)

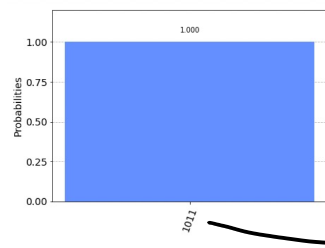
INPUTS			OUTPUTS	
A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



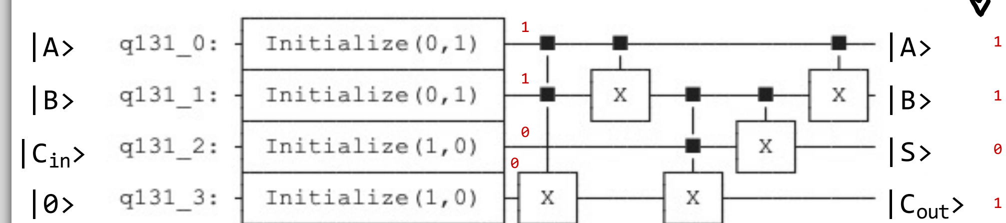
Full Adder using Quantum Gates (Feynman Adder)(qiskit)

INPUTS			OUTPUTS	
A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

```python
# el adder de Feynman

from qiskit import QuantumCircuit, assemble, Aer
from qiskit.visualization import plot_histogram, plot_bloch_vector
from math import sqrt, pi

qreg = qiskit.QuantumRegister(4)
circuit = qiskit.QuantumCircuit(qreg)
circuit.initialize([0, 1], 0)   # Apply initialisation operation to the 0th qubit, podría ser [1/sqrt(2), 1j/sqrt(2)], [sqrt(3)/2, 1/2], ...
circuit.initialize([0, 1], 1)   # Apply initialisation operation to the 1th qubit, podría ser [1/sqrt(2), 1j/sqrt(2)], [sqrt(3)/2, 1/2], ...
circuit.initialize([1, 0], 2)   # Apply initialisation operation to the 2th qubit, podría ser [1/sqrt(2), 1j/sqrt(2)], [sqrt(3)/2, 1/2], ...
circuit.initialize([1, 0], 3)   # Apply initialisation operation to the 3th qubit, podría ser [1/sqrt(2), 1j/sqrt(2)], [sqrt(3)/2, 1/2], ...

circuit.ccx(qreg[0], qreg[1], qreg[3]) # Apply a CCNOT gate where the control are the first and second qubit and the target is the third.
circuit.cx(qreg[0], qreg[1])           # Apply a CNOT gate where the control is the first qubit and the target is the second.
circuit.ccx(qreg[1], qreg[2], qreg[3]) # Apply a CCNOT gate where the control are the first and second qubit and the target is the third.
circuit.cx(qreg[1], qreg[2])           # Apply a CNOT gate where the control is the first qubit and the target is the second.
circuit.cx(qreg[0], qreg[1])           # Apply a CNOT gate where the control is the first qubit and the target is the second.
circuit.draw()

circuit.save_statevector()      # Tell simulator to save statevector
qobj = assemble(circuit)        # Create a Qobj from the circuit for the simulator to run
result = sim.run(qobj).result()  # Do the simulation and return the result
counts = result.get_counts()
plot_histogram(counts)
```



Qiskit: top qubit is less significant. Bottom qubit is the most significant.

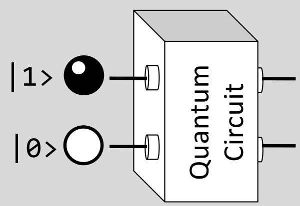


IBM Quantum Tools

- IBM Quantum Experience: https://www.ibm.com/quantum-computing
  - *'Drag and drop'* gates,
  - You can simulate your code or even *'take a number'* and run it in a real quantum computer,
- Qiskit: https://qiskit.org/
  - Traditional code (python), almost self-explanatory,
  - Better to run as cells Jupyter,
    - Using Colaboratory (https://colab.research.google.com/notebooks/intro.ipynb),
    - Or running your own Qiskit installation (https://qiskit.org/documentation/getting_started.html).

Qubit ordering in Qiskit

Textbook convention



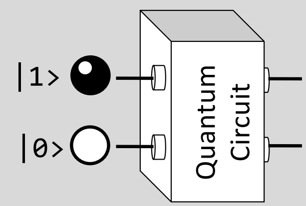
|10>

$$0\,|00\rangle + 0\,|01\rangle + 1\,|10\rangle + 0\,|11\rangle$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Top qubit is left in ket. Bottom qubit is right in ket.

Qiskit convention



|01>

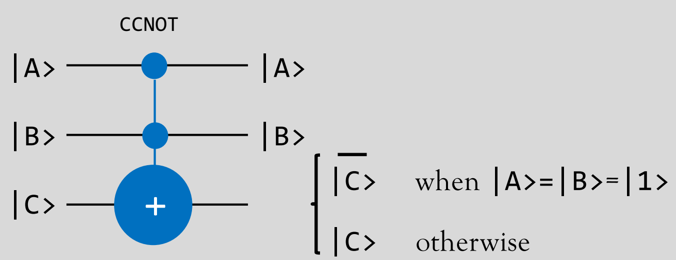
$$0\,|00\rangle + 1\,|01\rangle + 0\,|10\rangle + 0\,|11\rangle$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Top qubit is right in ket. Bottom qubit is left in ket.
Top qubit is less significant. Bottom qubit is the most significant.

Universal Quantum Gates

- The Hadamard gate, the Toffoli (`CCNOT`) gate and the `T` gate are universal:
  - Any operation possible on a quantum computer can be reduced to sequences of these gates,
  - Any Boolean function can be build using only Toffoli gates,



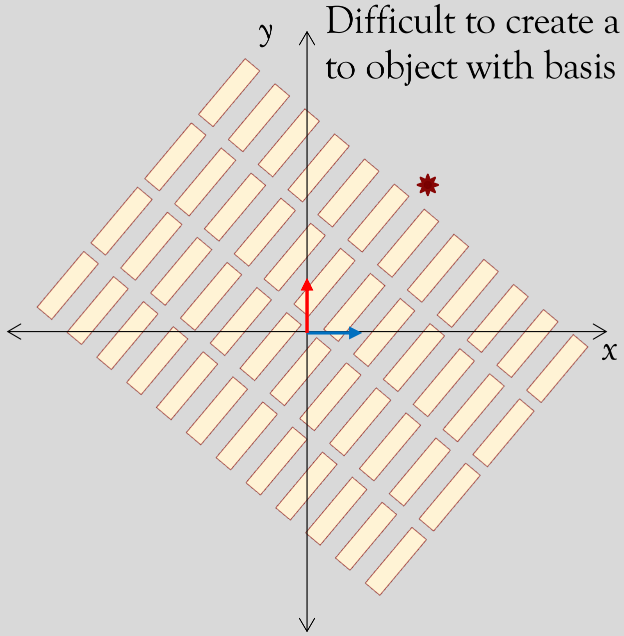
So `NAND(a,b) = CCNOT(a,b,1)`

Change of basis

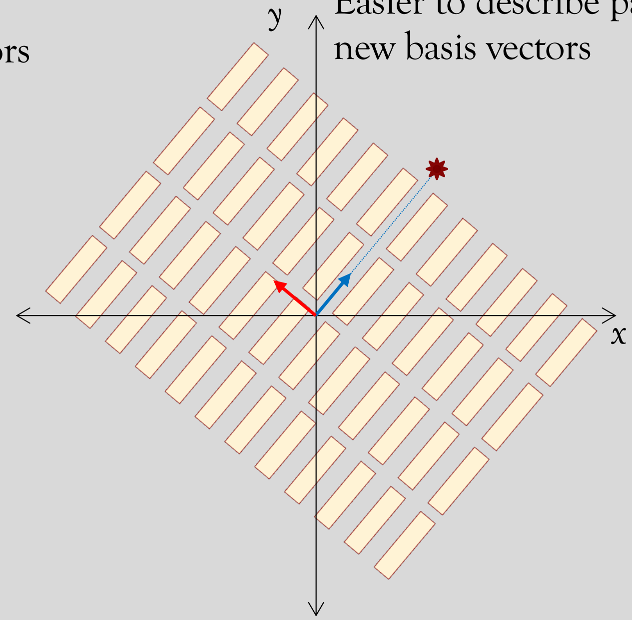
Sometimes a convenient and useful simplification

A reference framework to locate an object

Standard basis:
Difficult to create a path
to object with basis vectors

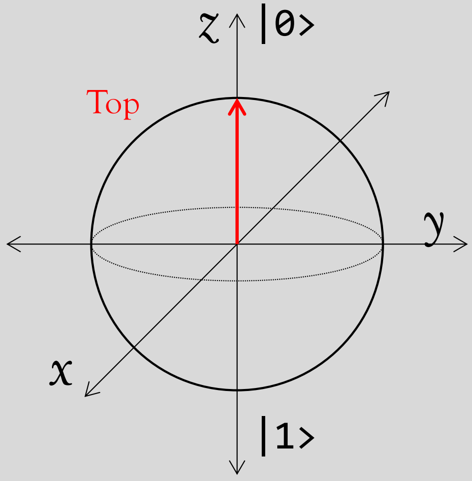


New basis:
Easier to describe path with
new basis vectors

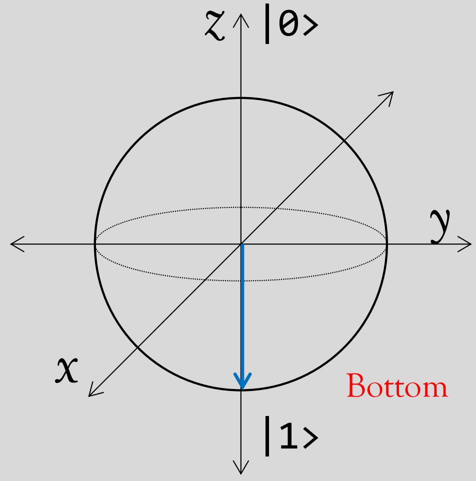


A different reference (basis) can make state description more clear

The Computational Basis



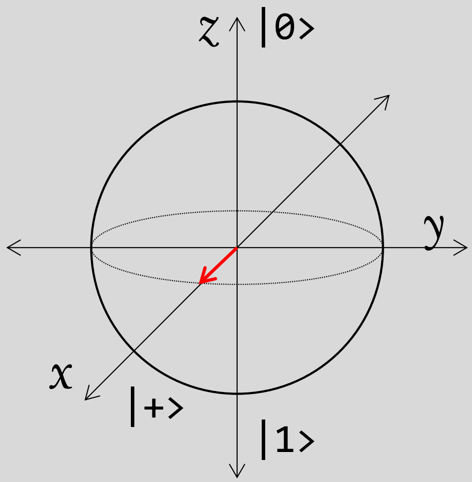
$$1\,|0\rangle + 0\,|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



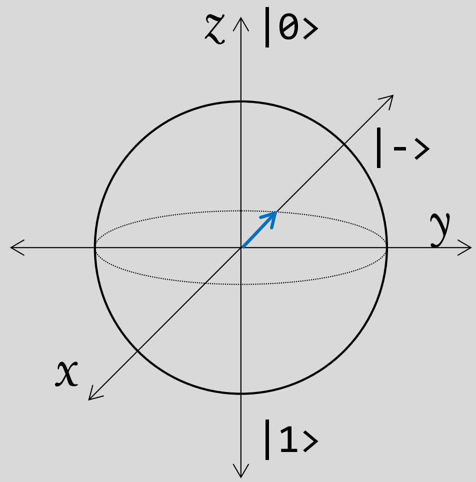
$$0\,|0\rangle + 1\,|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The computational basis rests on the *z*-axis

The Hadamard Basis



$$|+\rangle = \frac{1\,|0\rangle + 1\,|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$



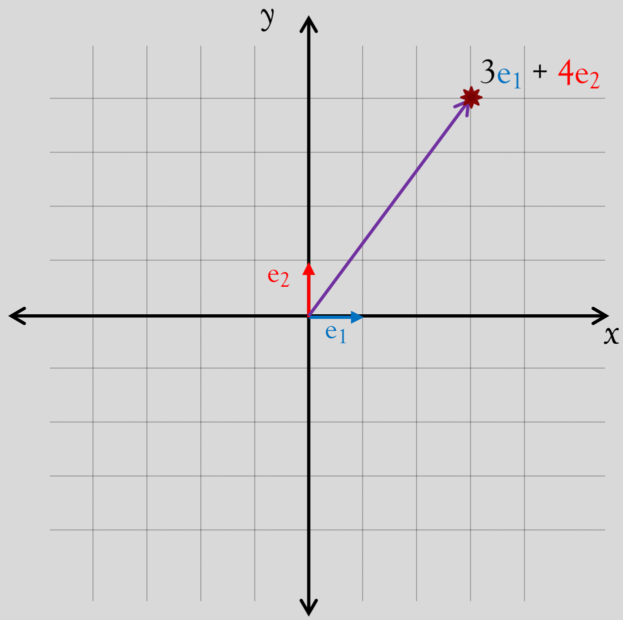
$$|-\rangle = \frac{1\,|0\rangle - 1\,|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

The hadamard basis rests on the *x*-axis

Basis of a Vector Space

- A basis, $X=\{x_1,x_2,\ldots,x_n\}$, is a set of vectors that express any point in a vector space V, as a linear combination,
- Different basis can be chosen for V, and choice of basis can make a problem easier to solve,
- The standard basis, E, is used frequently to describe vectors in the xy plane:

$$E = \{e_1 = (1,0), e_2 = (0,1)\}$$



Change of Basis: (Computational ⇄ Hadamard)

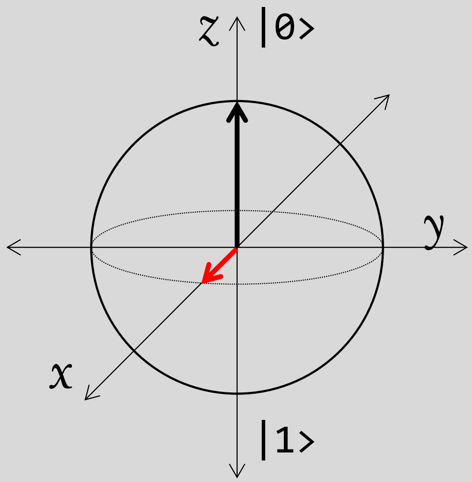
- Linear transform with a matrix, `M`, allows base conversion

$$\mathbf{M}_{C\to H} = \mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

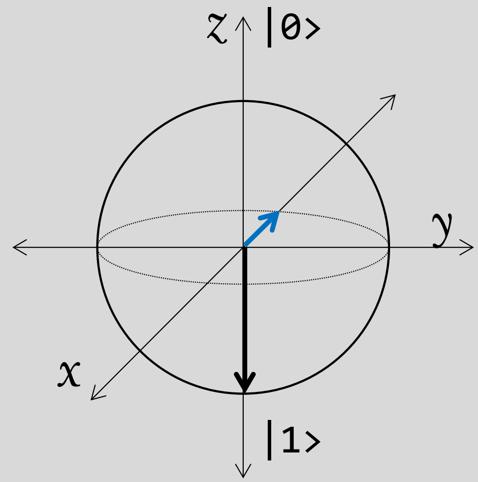
$$\mathbf{M}_{H\to C} = \mathbf{H}^{-1} = \mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

Mapping: Computational to Hadamard

$|0\rangle \rightarrow |+\rangle$



$|1\rangle \rightarrow |-\rangle$



$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$