Linear maps or linear transformations

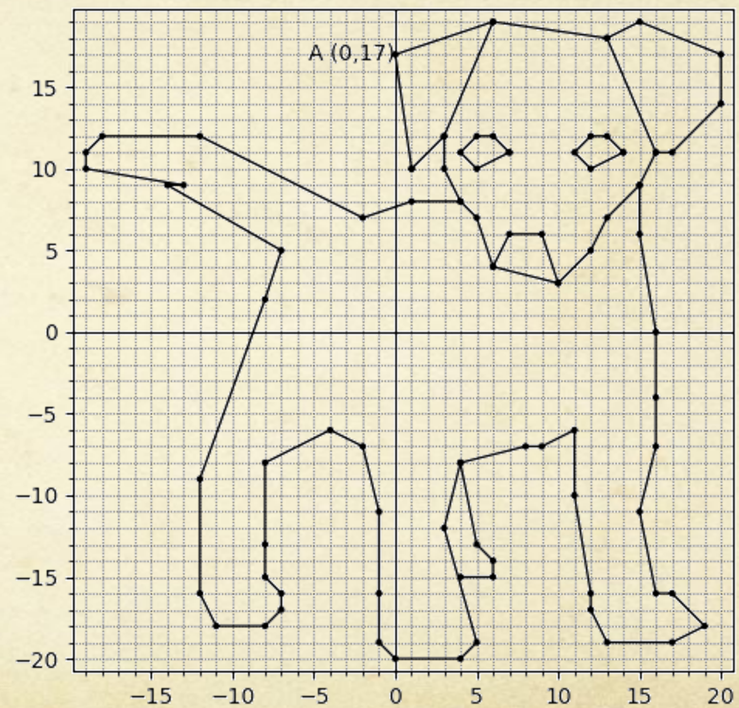
Applications of linear algebra for computer graphics

Linear maps

- A linear map (aka. a linear mapping, linear transformation, vector space homomorphism, or in some contexts linear function) is a mapping $V \to W$ between two vector spaces that preserves the operations of vector addition and scalar multiplication,
- Ummmm, let's check what that means…

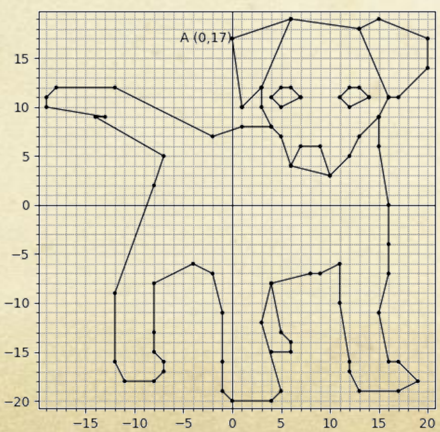
A 2D figure
(let me introduce my dog Descartes)

- Descartes is made by 91 points and lines,
  - Well, actually vectors
    $$\mathbf{P_0} = \begin{bmatrix} 0 \\ 17 \end{bmatrix}, \quad \mathbf{P_1} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad \ldots, \quad \mathbf{P_{90}} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}.$$
  - Or even a matrix $\mathbf{P} = \begin{bmatrix} 0 & 1 & \ldots & 13 \\ 17 & 10 & \ldots & 18 \end{bmatrix}.$
- You can operate over these points using matrix operations:
  - Linear transformations on each points.

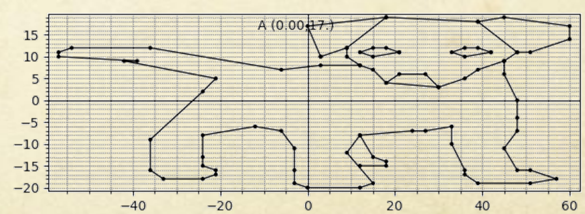


Descartes' stretch on X

- Stretch on X axis with the matrix $\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}.$
- For example, with $\mathbf{A} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}.$



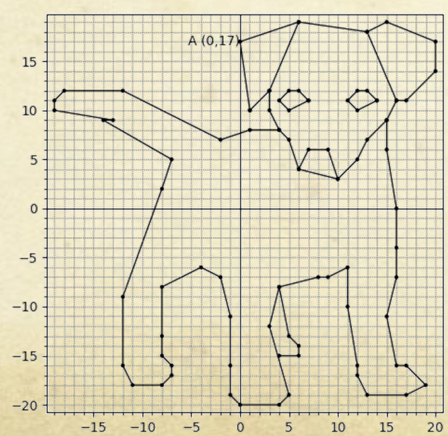
$$\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 17 \end{bmatrix} = \begin{bmatrix} 0 \\ 17 \end{bmatrix}.$$



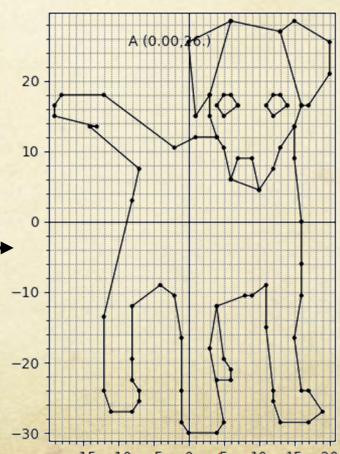
not-to-scale

Descartes' stretch on Y

- Stretch on Y axis with the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}.$
- For example, with $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}.$



$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 17 \end{bmatrix} = \begin{bmatrix} 0 \\ 25.5 \end{bmatrix}.$$



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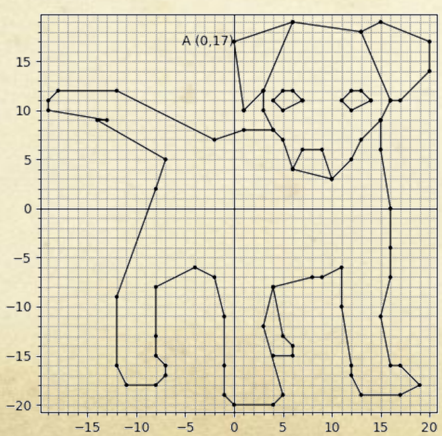
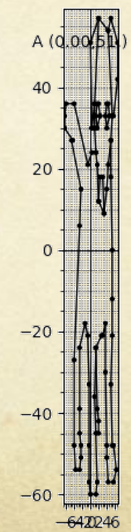
Descartes' squeeze on $\mathcal{X}$

- Squeeze with the matrix $\mathbf{A} = \begin{bmatrix} \frac{1}{p} & 0 \\ 0 & p \end{bmatrix}$
- For example, with $\mathbf{A} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix}$.



$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 17 \end{bmatrix} = \begin{bmatrix} 0 \\ 51 \end{bmatrix}.$$



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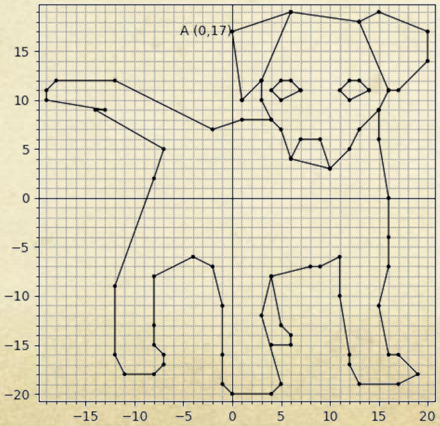
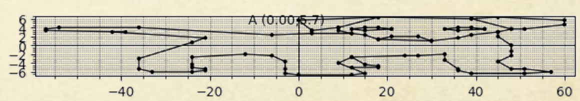
Descartes' squeeze on Y

- Squeeze with the matrix $\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & \frac{1}{p} \end{bmatrix}.$
- For example, with $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$



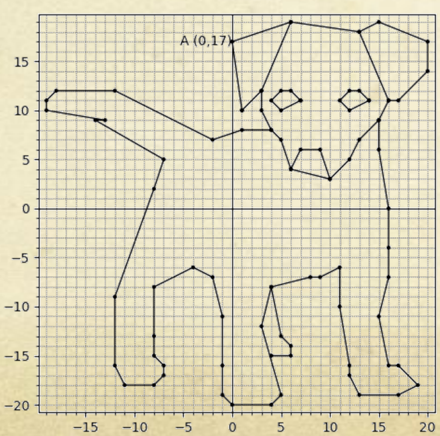
$$\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 17 \end{bmatrix} = \begin{bmatrix} 0 \\ 5.66 \end{bmatrix}.$$



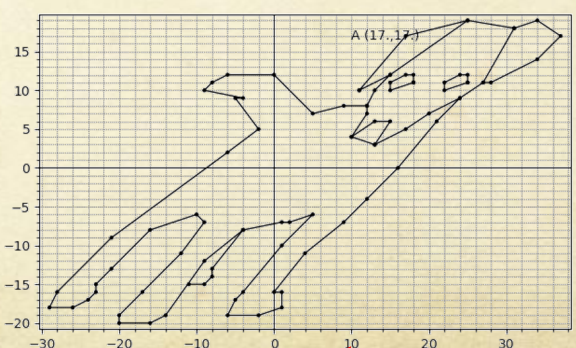
not-to-scale

Descartes' shear on X

- Shear on X with the matrix $\mathbf{A} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}.$
- For example, with $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 17 \end{bmatrix} = \begin{bmatrix} 17 \\ 17 \end{bmatrix}.$$



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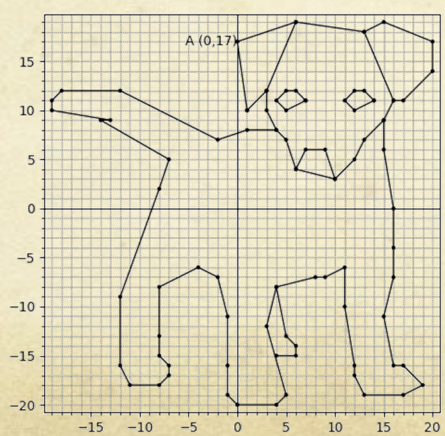
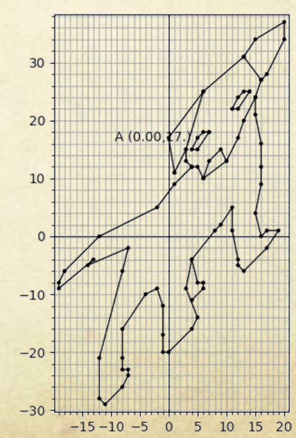
Descartes' shear on Y

- Shear on Y with the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}.$
- For example, with $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$



$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 17 \end{bmatrix} = \begin{bmatrix} 0 \\ 17 \end{bmatrix}.$$



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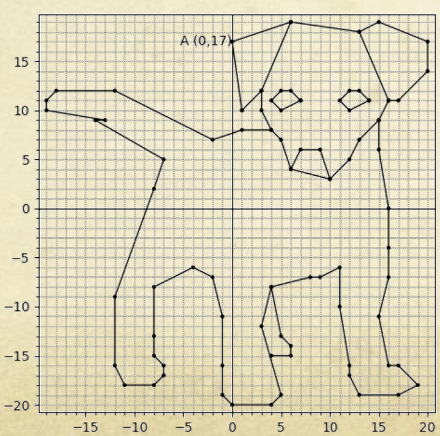
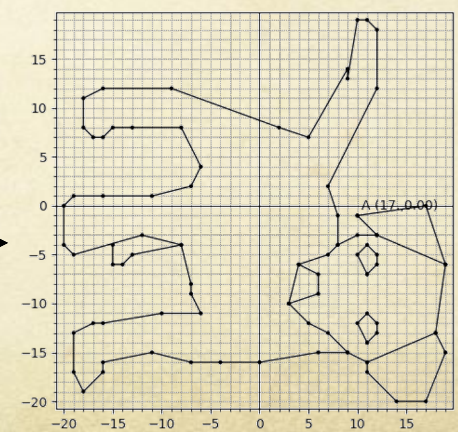
Rotate clockwise θ rads

- Rotate with the matrix $\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$
- For example, rotate $\frac{\pi}{2}$ rads with $\mathbf{A} = \begin{bmatrix} \cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) \\ -\sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix}.$



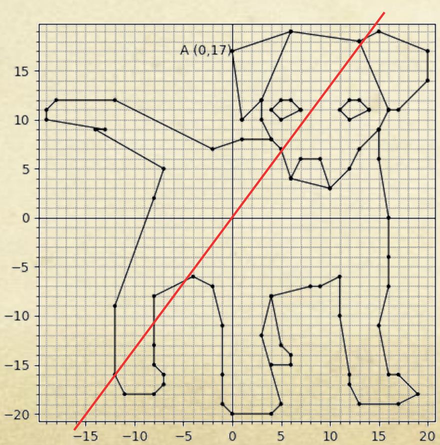
$$\begin{bmatrix} \cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) \\ -\sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 0 \\ 17 \end{bmatrix} = \begin{bmatrix} 17\sin(\frac{\pi}{2}) \\ 17\cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 17 \\ 0 \end{bmatrix}.$$



to-scale

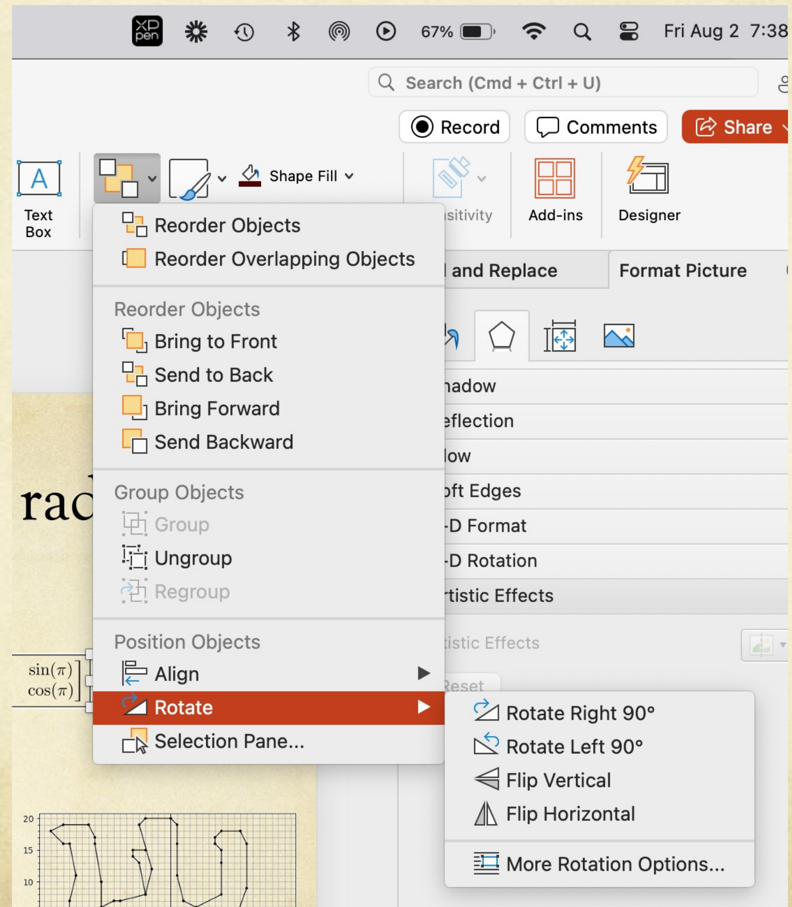
Reflect around axis $\vec{l} = (l_x, l_y)$

- Reflect around with the matrix $\mathbf{A} = \frac{1}{|\vec{l}|^2}\begin{bmatrix} l_x^2 - l_y^2 & 2l_x l_y \\ 2l_x l_y & l_y^2 - l_x^2 \end{bmatrix}.$
  - For example, for the axis $\vec{l} = (3, 4)$ with $\mathbf{A} = \frac{1}{25}\begin{bmatrix} 9-16 & 2\times 3\times 4 \\ 2\times 3\times 4 & 16-9 \end{bmatrix}.$



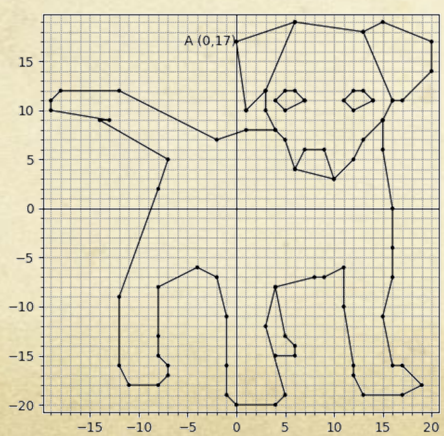
$$\frac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ 17 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 408 \\ 119 \end{bmatrix} = \begin{bmatrix} 16.3 \\ 4.7 \end{bmatrix}.$$

In PowerPoint®

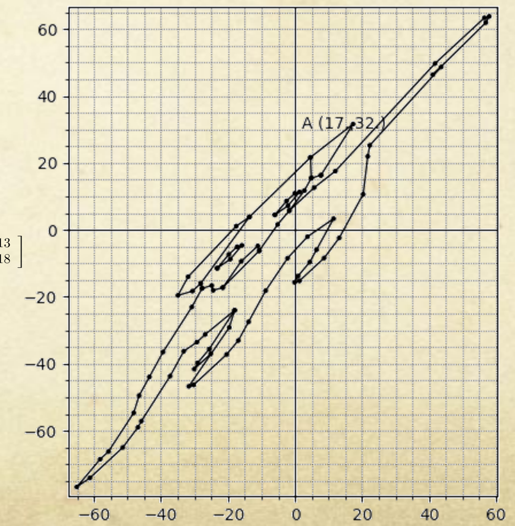


Descartes' whole enchilada

- Sequential operations on points are equivalent to sequential products on the whole matrix:



$$\frac{1}{25}\overset{\text{Reflejo}}{\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}}\overset{\text{Shear Y}}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}\overset{\text{Shear X}}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}\overset{\text{Rotación}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}\overset{\text{Squezze Y}}{\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}}\overset{\text{Squezze X}}{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix}}\overset{\text{Stretch Y}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}}\overset{\text{Stretch X}}{\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}}\overset{\text{puntos}}{\begin{bmatrix} 0 & 1 & \dots & 13 \\ 17 & 10 & \dots & 18 \end{bmatrix}}$$

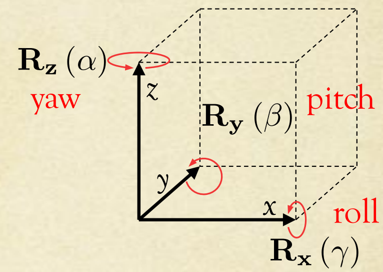


not-to-scale

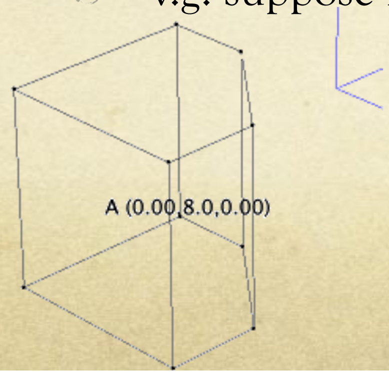
We can even rotate in 3D

- 3D rotation: first roll γ rads, pitch β rads, and yaw α rads, is defined with the matrix:

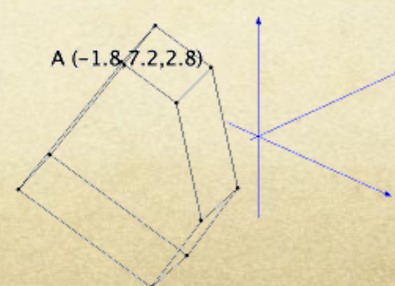
$$\mathbf{R} = \mathbf{R_z}(\alpha)\mathbf{R_y}(\beta)\mathbf{R_x}(\gamma) = \overset{\text{yaw}}{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{pitch}}{\begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}} \overset{\text{roll}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}}$$



- Every vector is transformed: $\vec{v_{\text{new}}} = \mathbf{R}(\alpha,\beta,\gamma)\,\vec{v_{\text{old}}} = \mathbf{R_z}(\alpha)\mathbf{R_y}(\beta)\mathbf{R_x}(\gamma)\,\vec{v_{\text{old}}}$
  - v.g. suppose roll of $\pi/7$, then pitch $\pi/5$ and then yaw $\pi/6$:



$$\underbrace{\overset{\text{yaw}}{\begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 0 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{pitch}}{\begin{bmatrix} \cos(\frac{\pi}{5}) & 0 & \sin(\frac{\pi}{5}) \\ 0 & 1 & 0 \\ -\sin(\frac{\pi}{5}) & 0 & \cos(\frac{\pi}{5}) \end{bmatrix}} \overset{\text{roll}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{7}) & -\sin(\frac{\pi}{7}) \\ 0 & \sin(\frac{\pi}{7}) & \cos(\frac{\pi}{7}) \end{bmatrix}} \overset{\text{point}}{\begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}} = \overset{\text{new point}}{\begin{bmatrix} -1.8 \\ 7.2 \\ 2.8 \end{bmatrix}}}_{\longrightarrow}$$



Let's back to 2D

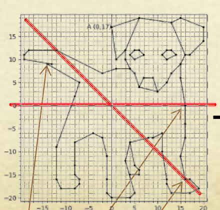
- Suppose the transformation: $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$
$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

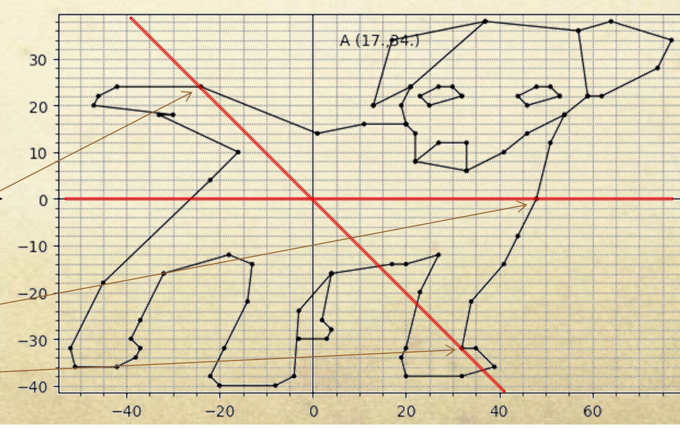
stretch *3* on *x*-axis, *2* on *y*-axis (slant to the right)

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Vectors span-out on their own span



$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \longrightarrow$$



to-scale

Check out these points

Let's try transformation

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

eigen-values

eigen-vectors

$$\begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

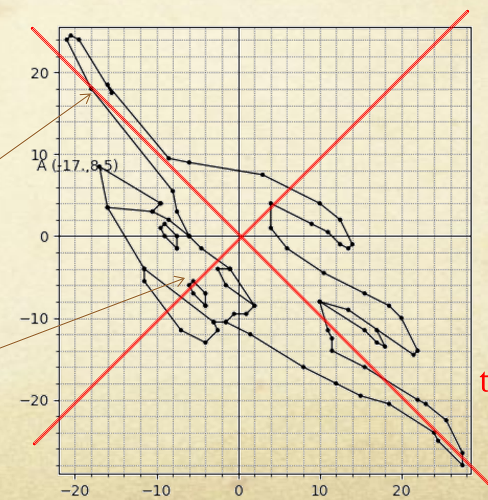
$$\begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}.$$

$$\xrightarrow{\mathbf{A} = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}}$$



to-scale

Check out these points

Let's try transformation

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

eigen-values

eigen-vectors

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

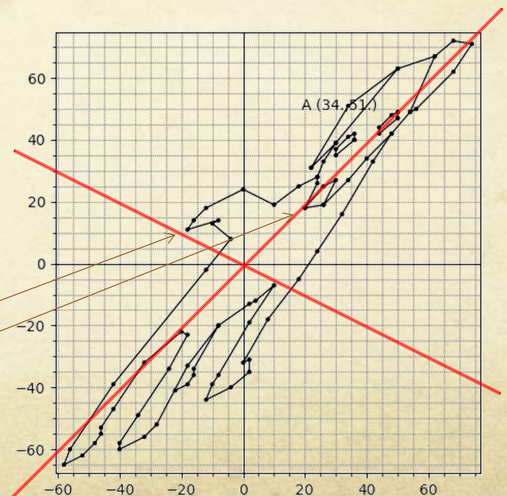
$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \longrightarrow$$



Check out these points

not-to-scale

Descartes's stretch on X revisited

- Stretch on X axis with the matrix

$$\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$

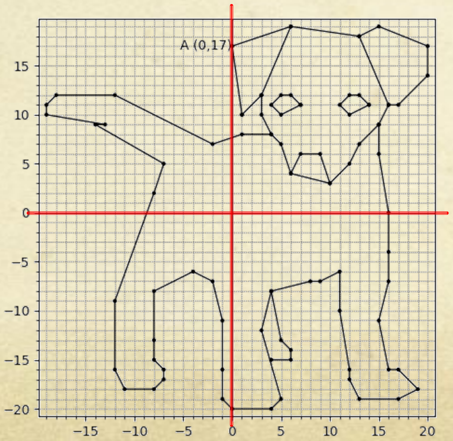
eigen-values

eigen-vectors

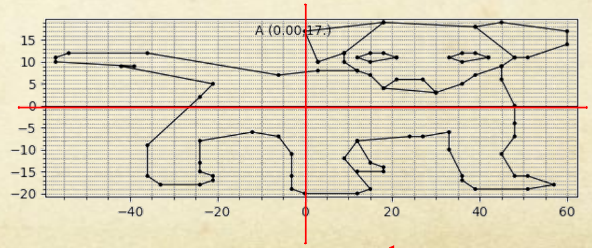
$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$



$$\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$



not-to-scale

Infinite lines... Infinite eigen-vectors: just multiply the eigen vector for any scalar

Descartes's stretch on Y revisited

- Stretch on Y axis with the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$

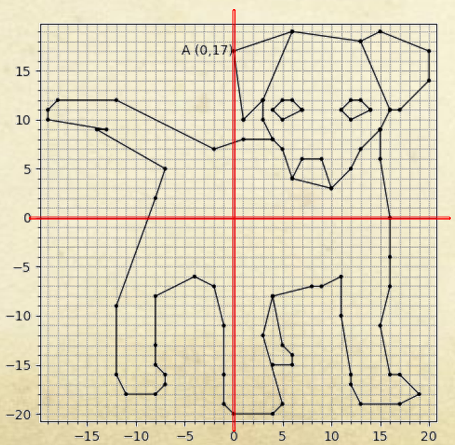
eigen-values

eigen-vectors

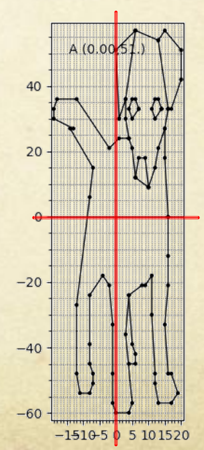
$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix} = p \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$



$$\xrightarrow{\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}}$$



not-to-scale

Infinite lines... Infinite eigen-vectors: just multiply the eigen vector for any scalar

Descartes's Squeeze

- Squeeze with the matrix $\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & \frac{1}{p} \end{bmatrix}$

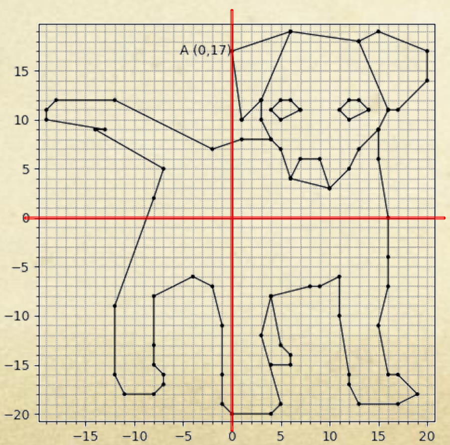
eigen-values

eigen-vectors

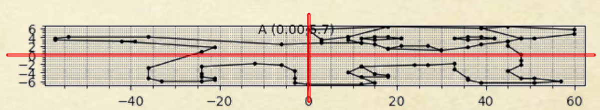
$$\begin{bmatrix} p & 0 \\ 0 & \frac{1}{p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{p} \end{bmatrix} = \frac{1}{p} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} p & 0 \\ 0 & \frac{1}{p} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$



$$\xrightarrow{\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & \frac{1}{p} \end{bmatrix}}$$



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Infinite lines... Infinite eigen-vectors: just multiply the eigen vector for any scalar

Descartes' shear on $\mathcal{X}$ revisited

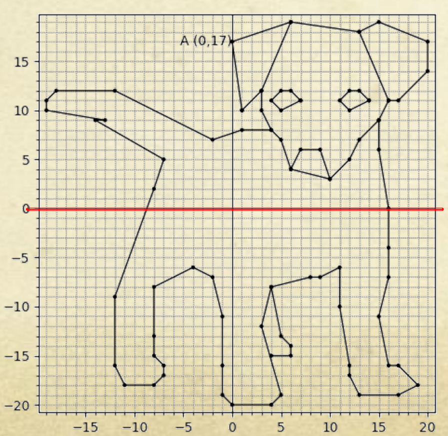
- Shear on X with the matrix $\mathbf{A} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}$

Infinite eigen-values

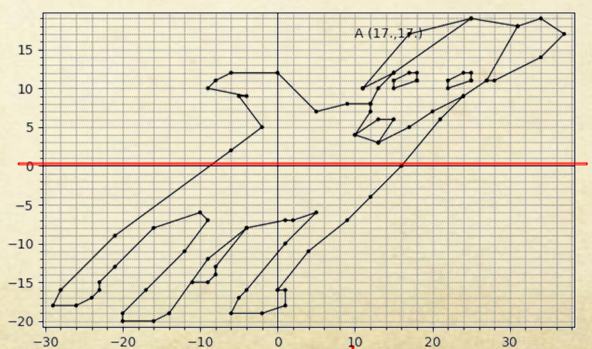
Infinite eigen-vectors

$$\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$



$$\mathbf{A} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \longrightarrow$$



not-to-scale

Infinite lines… … Infinite eigen-vectors: just multiply the eigen vector for any scalar

Descartes' shear on Y revisited

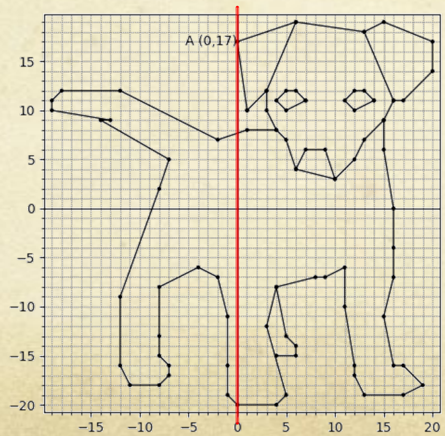
- Shear on Y with the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}$

Infinite eigen-values

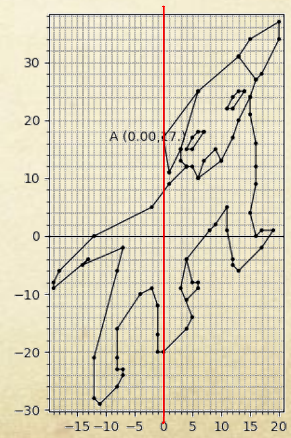
Infinite eigen-vectors

$$\begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$



$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}$



not-to-scale

Infinite lines... ... Infinite eigen-vectors: just multiply the eigen vector for any scalar

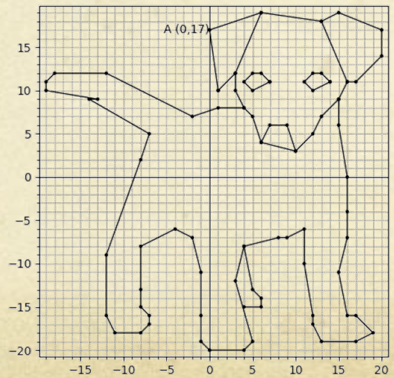
Let's try transformation

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

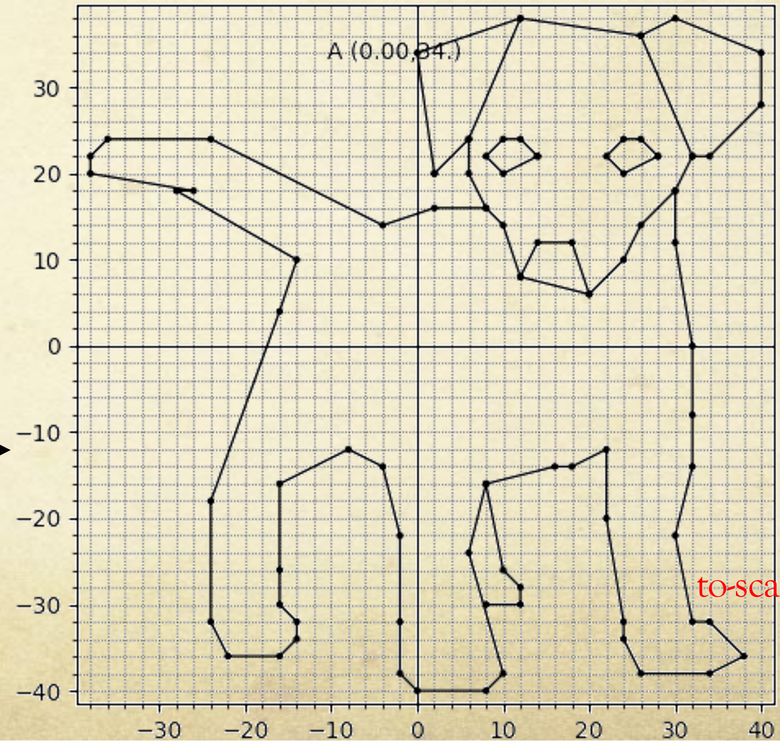
Only one eigen-value

Anything is an eigen-vector

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}.$$



$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



to-scale

Reflect around axis $\vec{l} = (l_x, l_y)$ revisited

- Reflect around with the matrix
  - For example, for the axis $\vec{l} = (3,4)$ with

$$\mathbf{A} = \frac{1}{|\vec{l}|^2}\begin{bmatrix} l_x^2 - l_y^2 & 2l_xl_y \\ 2l_xl_y & l_y^2 - l_x^2 \end{bmatrix}.$$

$$\mathbf{A} = \frac{1}{25}\begin{bmatrix} 9-16 & 2*3*4 \\ 2*3*4 & 16-9 \end{bmatrix}.$$

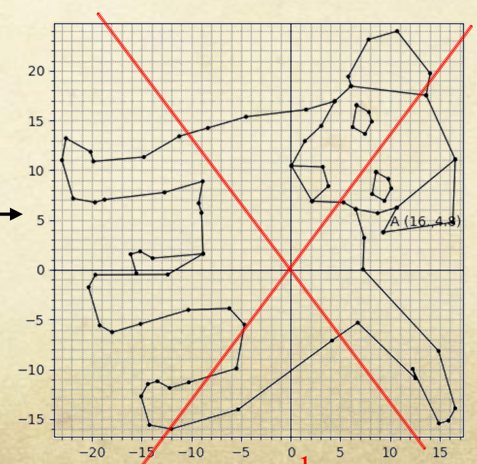
eigen-values eigen-vectors

$$\frac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -\frac{21}{4}+24 \\ 18+7 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} \frac{75}{4} \\ 25 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}.$$

$$\frac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} \frac{28}{3}+24 \\ -32+7 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} \frac{100}{3} \\ -25 \end{bmatrix} = -\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

$$\mathbf{A} = \frac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$$



to-scale

Let's try transformation

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$3\pi/2$ rotation

Eigen-values

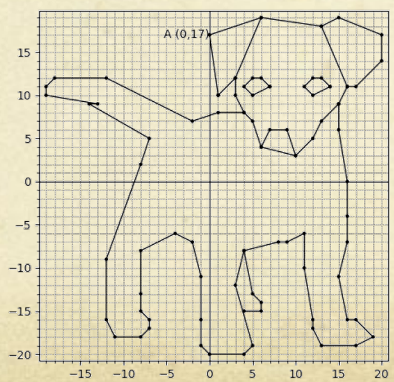
Eigen-vectors

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \imath \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \imath \end{bmatrix} = \begin{bmatrix} \imath^2 \\ \imath \end{bmatrix} = \imath \begin{bmatrix} \imath \\ 1 \end{bmatrix}.$$

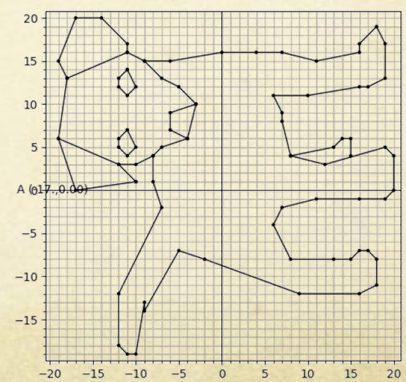
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -\imath \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -\imath \end{bmatrix} = \begin{bmatrix} \imath^2 \\ -\imath \end{bmatrix} = -\imath \begin{bmatrix} -\imath \\ 1 \end{bmatrix}.$$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

Every vector rotates, nothing stay put...
no real values for eigen-vectors / eigen-values...



$$\xrightarrow{\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$



to-scale

Eigen-values and Eigen-vectors

- For a matrix $\mathbf{A}$ are the values such that: $\mathbf{A}\vec{v} = \lambda\vec{v}$
- But, how to calculate them?

$$\mathbf{A}_{n\times n}\ \vec{v}_{n\times 1} = \lambda\ \vec{v}_{n\times 1} = \lambda\ \mathbf{I}_{n\times n}\ \vec{v}_{n\times 1}.$$

$$\lambda\ \mathbf{I}_{n\times n} = \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

$$\mathbf{A}_{n\times n}\ \vec{v}_{n\times 1} - \lambda\ \mathbf{I}_{n\times n}\ \vec{v}_{n\times 1} = \mathbf{0}_{n\times 1}.$$

$$\left(\mathbf{A}_{n\times n} - \lambda\ \mathbf{I}_{n\times n}\right)\ \vec{v}_{n\times 1} = \mathbf{0}_{n\times 1}.$$

$$\det\left(\mathbf{A}_{n\times n} - \lambda\ \mathbf{I}_{n\times n}\right) = 0.$$

Eigen-values and Eigen-vectors (an example)

- Calculate eigen-values and eigen-vectors for $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$$\det\left(\mathbf{A}_{n\times n} - \lambda\,\mathbf{I}_{n\times n}\right) = 0.$$

$$\det\left(\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = 0.$$

$$(3-\lambda)\,(2-\lambda) - 0 \times 1 = 0.$$

$$\lambda^2 - 3\lambda - 2\lambda + 6 = \lambda^2 - 5\lambda + 6 = 0. \quad \lambda_1 = 3, \quad \lambda_2 = 2.$$

$$\lambda_1 = 2\ldots \text{ So } \begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x + y = 0\ldots \text{ Any vector of the form } a\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ is an eigen-vector.}$$

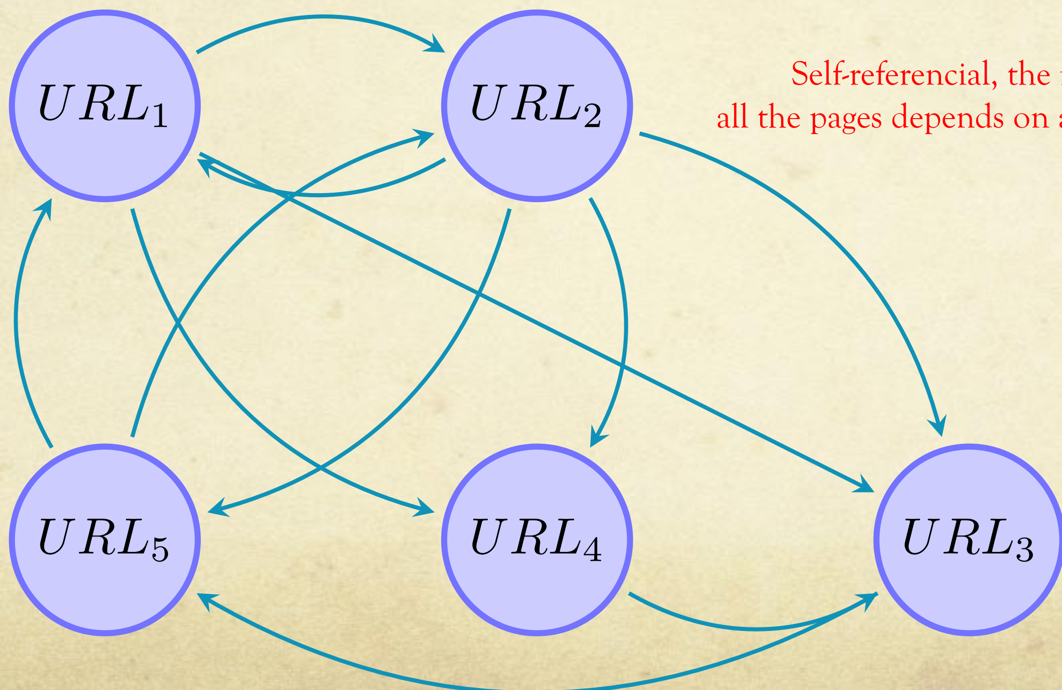
$$\lambda_2 = 3\ldots \text{ So } \begin{bmatrix} 3-3 & 1 \\ 0 & 2-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad y = 0. \text{ Any vector of the form } a\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is an eigen-vector.}$$

Eigen-values and Eigen-vectors (epilogue)

- Geometrically 2D, the eigenvectors are the vectors that the matrix merely elongates or shrinks by an amount given by its corresponding eigenvalue, without changing the vector's direction,
- The expression $\det\left(\mathbf{A}_{n\times n} - \lambda\, \mathbf{I}_{n\times n}\right) = 0$ yields an equation for the eigenvalues,
- The characteristic polynomial is $p(\lambda) = \det\left(\mathbf{A}_{n\times n} - \lambda\, \mathbf{I}_{n\times n}\right) = 0$
  - An n_{th}-order polynomial equation in the unknown λ with n solutions:
    - The set of solutions (i.e. the eigenvalues) is the matrix's **spectrum**.

Eigen-values and Eigen-vectors (applications)

- Which website will have the most traffic?



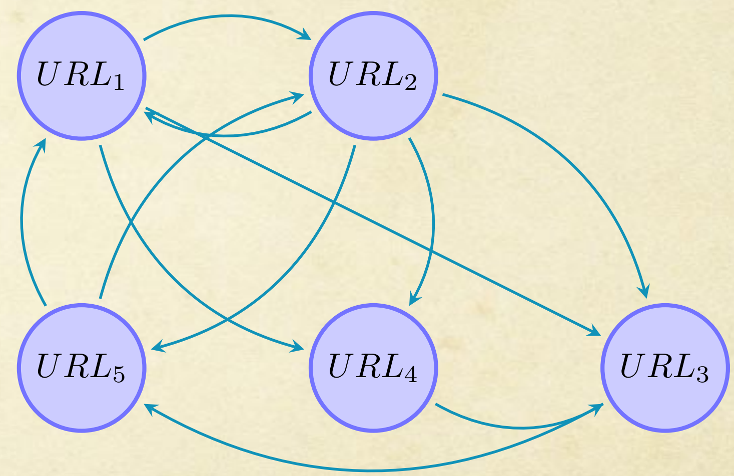
Self-referencial, the rank of
all the pages depends on all the pages

What a drunk guy would do?

The Google Matrix
(What a drunk guy would do?)

No link from URL_1 to URL_1

$$
\begin{array}{c c}
 & \begin{array}{ccccc} URL_1 & URL_2 & URL_3 & URL_4 & URL_5 \end{array} \\
\begin{array}{c} URL_1 \\ URL_2 \\ URL_3 \\ URL_4 \\ URL_5 \end{array} &
\begin{bmatrix}
0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\
\frac{1}{3} & \frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 1 & 0 & 0
\end{bmatrix} \\
 & \begin{array}{ccccc} \sum = 1 & \sum = 1 & \sum = 1 & \sum = 1 & \sum = 1 \end{array}
\end{array}
$$



drunken walk model

$M_{i,j}$: probability of a user navigating from url_j to url_i

M is a really sparse matrix

The Google *pagerank* algorithm

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

$$\begin{array}{c|ccccc} & URL_1 & URL_2 & URL_3 & URL_4 & URL_5 \\ \hline URL_1 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} \\ URL_2 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ URL_3 & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ URL_4 & \frac{1}{3} & \frac{1}{4} & 0 & 0 & 0 \\ URL_5 & 0 & \frac{1}{4} & 1 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 \\ 1.067 \\ 1.2 \\ 0.6 \\ 1.467 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1.067 \\ 1.2 \\ 0.6 \\ 1.467 \end{bmatrix}$$

The largest eigen-value gives us the dominant eigen-vector

$$\begin{bmatrix} 1 \\ 0.741 \\ 0.680 \\ -0.907 \\ -1.514 \end{bmatrix} = -0.572 \begin{bmatrix} 1 \\ 0.741 \\ 0.680 \\ -0.907 \\ -1.514 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1.197 \\ 0.293 \\ -0.118 \\ 0.023 \end{bmatrix} = -0.288 \begin{bmatrix} 1 \\ -1.197 \\ 0.293 \\ -0.118 \\ 0.023 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1.028 + 0.111\imath \\ -1.253 + 0.539\imath \\ -0.121 + 0.822\imath \\ -0.655 - 1.471\imath \end{bmatrix} = (-0.070 - 0.708\imath) \begin{bmatrix} 1 \\ 1.028 + 0.111\imath \\ -1.253 + 0.539\imath \\ -0.121 + 0.822\imath \\ -0.655 - 1.471\imath \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1.028 - 0.111\imath \\ -1.253 - 0.539\imath \\ -0.121 - 0.822\imath \\ -0.655 + 1.471\imath \end{bmatrix} = (-0.070 + 0.708\imath) \begin{bmatrix} 1 \\ 1.028 - 0.111\imath \\ -1.253 - 0.539\imath \\ -0.121 - 0.822\imath \\ -0.655 + 1.471\imath \end{bmatrix}$$

This is the website that will have the most traffic

Eigen-vector for the biggest eigen-value

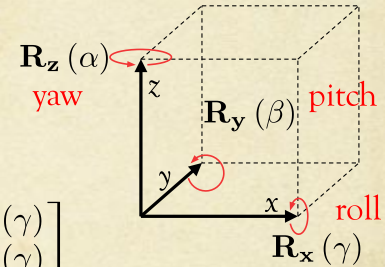
	4th	3rd	2nd	5th	1st
$[1, 1.067, 1.2, 0.6, 1.467] \xrightarrow{\times \frac{1}{1.467}}$	$[0.682,$	$0.727,$	$0.818,$	$0.409,$	$1]$
	URL_1	URL_2	URL_3	URL_4	URL_5

Let's back to 3D

- 3D rotation: first roll γ rads, pitch β rads, and yaw α rads, is defined with the matrix:

$$\mathbf{R} = \mathbf{R_z}(\alpha)\mathbf{R_y}(\beta)\mathbf{R_x}(\gamma) = \overset{\text{yaw}}{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{pitch}}{\begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}} \overset{\text{roll}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}}$$

$$\mathbf{R}(\alpha,\beta,\gamma) = \begin{bmatrix} \cos(\alpha)\cos(\beta) & \cos(\alpha)\sin(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) & \cos(\alpha)\sin(\beta)\cos(\gamma) + \sin(\alpha)\sin(\gamma) \\ \sin(\alpha)\cos(\beta) & \sin(\alpha)\sin(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \sin(\alpha)\sin(\beta)\cos(\gamma) - \cos(\alpha)\sin(\gamma) \\ -\sin(\beta) & \cos(\beta)\sin(\gamma) & \cos(\beta)\cos(\gamma) \end{bmatrix}$$



Let's back to 3D (.)

- Suppose a 3×3 rotation matrix ***R***,
  - what is the rotation axis and the rotation angle?

Better results using quaternions (no gimbal lock)

- A vector $\vec{u}$ parallel to the rotation axis must satisfy: $\mathbf{R}\vec{u} = \vec{u}$

Vector over the rotation axis don't change at all

  - It's an eigen-vector of ***R***, with eigen-value $\lambda=1$,
    - One eigen-value real ($\lambda=1$), the other two are complex conjugates of each other,

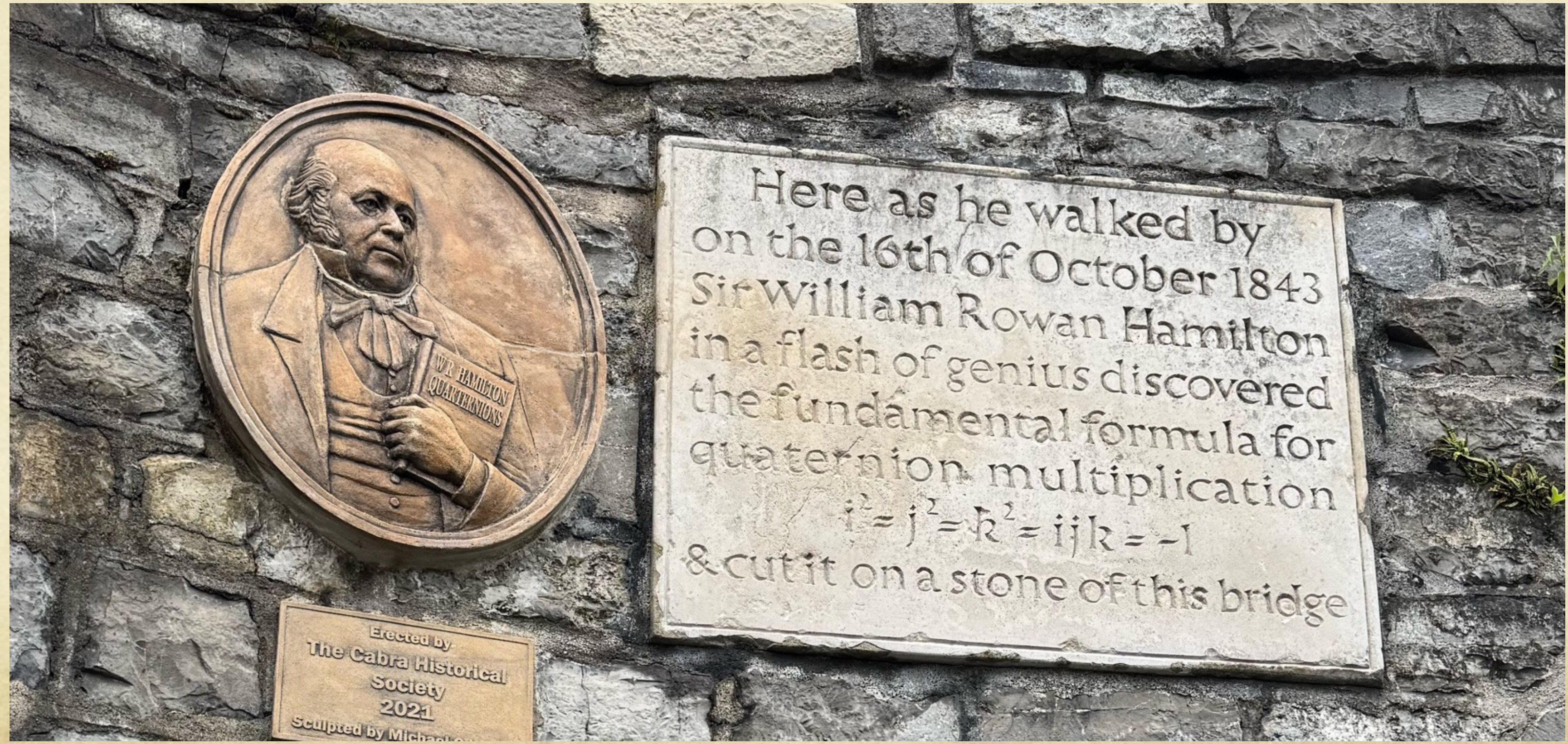
Rotation only, no stretch, no squeeze, no shear

- For a 3D rotation $\mathbf{R} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}$, rotation axis: $\vec{u} = \begin{bmatrix} r_{3,2} - r_{2,3} \\ r_{1,3} - r_{3,1} \\ r_{2,1} - r_{1,2} \end{bmatrix}$, rotation angle: $|\vec{u}| = 2\sin\theta$
  - Caveat: if ***R*** is symmetric the angle of rotation is 0 or π,
- Example

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad |\vec{u}| = 2\sin\theta = \sqrt{1+1+1} = \sqrt{3}, \quad \sin\theta = \frac{\sqrt{3}}{2}, \qquad \theta = \frac{\pi}{3}$$

Quaternions – Broom’s bridge - Dublin

Quaternions



Quaternions

- A complex number ($a+bi$) is made by a real part and one imaginary part i
    - $i^2=1$,
- A quaternion is an algebraic structure with a real part and three imaginary parts i,j,k:
    - $i^2 = j^2 = k^2 = ijk = -1$,

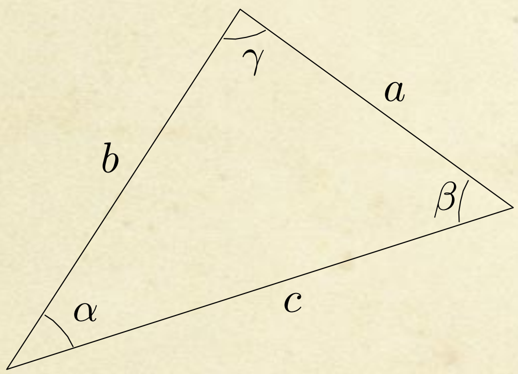
Quaternions

- The set of complex number is defined as: $\mathbb{C} = \{a + \imath b \ \mid \ a, b \in \mathbb{R}, \imath^2 = -1\}$
- A quaternion can be written as: $q = q_0 + q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k}$
  - a sum of one real part and three imaginary parts
  - Also written as: $q = (q_0, q_1, q_2, q_3) = (q_0, \vec{q})\,, \quad \vec{q} = \hat{\imath}q_0 + \hat{\jmath}q_1 + \hat{k}q_2 \qquad \hat{\imath} = (1,0,0)\,, \quad \hat{\jmath} = (0,1,0)\,, \quad \hat{k} = (0,0,1)\,.$

$$\begin{cases} \hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1 \\ \hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k} \\ \hat{\jmath}\hat{k} = -\hat{k}\hat{\jmath} = \hat{\imath} \\ \hat{k}\hat{\imath} = -\hat{\imath}\hat{k} = \hat{\jmath} \end{cases}$$

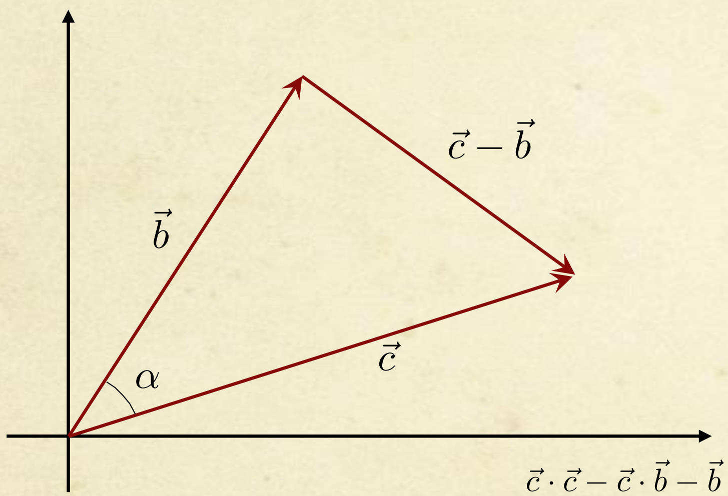
- The set of quaternions is defined as: $\mathbb{H} = \{q = q_0 + q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k} \ \mid \ q_t \in \mathbb{R}, \ \ \hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1\}$
  - A superset of complex numbers: $q_2 = q_3 = 0$
  - A superset of real numbers: $q_1 = q_2 = q_3 = 0.$

Remember trigonometry



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

The angle between vectors



- By cosine law

$$|\vec{c} - \vec{b}|^2 = |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos\alpha$$

- The dot product...

$$\vec{v} \cdot \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2 = |\vec{v}|^2$$

- So:

$$|\vec{c} - \vec{b}|^2 = \left(\vec{c} - \vec{b}\right) \cdot \left(\vec{c} - \vec{b}\right) = |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos\alpha$$

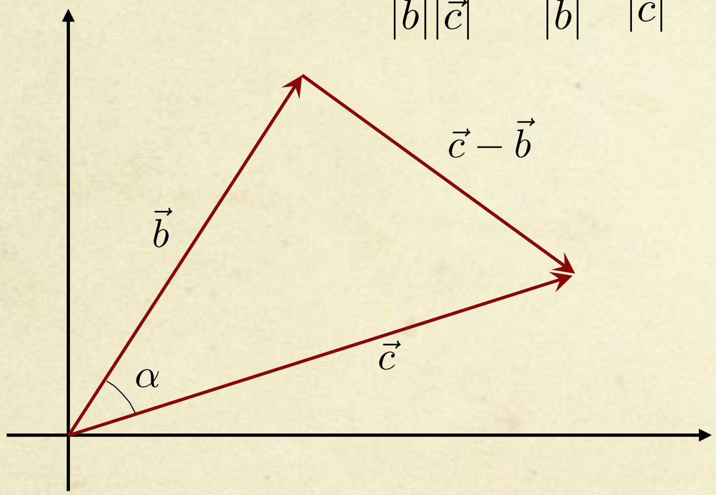
$$\vec{c} \cdot \vec{c} - \vec{c} \cdot \vec{b} - \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{b} = \cancel{|\vec{c}|^2} + \cancel{|\vec{b}|^2} - 2\vec{b} \cdot \vec{c} = \cancel{|\vec{b}|^2} + \cancel{|\vec{c}|^2} - 2|\vec{b}||\vec{c}|\cos\alpha$$

$$-2\vec{b} \cdot \vec{c} = -2|\vec{b}||\vec{c}|\cos\alpha$$

$$\cos\alpha = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|}$$

Cosine similarity and the angle between vectors

$$\cos\alpha = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{\vec{b}}{|\vec{b}|}\cdot\frac{\vec{c}}{|\vec{c}|}$$



- What if α is close to $0°$?
  - $\cos(\alpha)$ is close to *1*
    - Both vectors are very 'similar'
- What if α is close to $90°$?
  - $\cos(\alpha)$ is close to *0*
    - Both vectors are 'dissimilar'
- So $\cos(\alpha)$ is a fine measurement for similarity between vectors,

What about *n*-dimensions?

- What is the angle α (remember $\cos\alpha = \dfrac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|}$) between the vector:

$$\vec{q} = (0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0), \quad |\vec{q}| = \sqrt{1^2+1^2+1^2} = \sqrt{3},$$

and the vectors:

$$\vec{a} = (2,4,1,0,0,1,0,1,0,1,0,0,1,1,0,0,5,1,2), \quad |\vec{a}| = \sqrt{2^2+4^2+1^2+1^2+1^2+1^2+1^2+1^2+5^2+1^2+2^2} = \sqrt{56}$$
$$\vec{b} = (0,0,0,1,3,0,1,1,1,0,0,0,0,0,0,0,0,0,0), \quad |\vec{b}| = \sqrt{1^2+3^2+1^2+1^2+1^2} = \sqrt{13}$$
$$\vec{c} = (0,0,1,0,0,0,0,2,0,0,4,0,0,0,1,4,0,0,0), \quad |\vec{c}| = \sqrt{1^2+2^2+4^2+1^2+4^2} = \sqrt{38}$$

- Let's calculate:

$$\cos\left(\alpha_{\measuredangle \vec{q}-\vec{a}}\right) = \frac{1\cdot 1+0}{\sqrt{3}\sqrt{56}} = \frac{1}{\sqrt{3}\sqrt{56}} = 0.07715167,$$
$$\cos\left(\alpha_{\measuredangle \vec{q}-\vec{b}}\right) = \frac{1\cdot 1+0}{\sqrt{3}\sqrt{13}} = \frac{1}{\sqrt{3}\sqrt{13}} = 0.16012815,$$
$$\cos\left(\alpha_{\measuredangle \vec{q}-\vec{c}}\right) = \frac{1\cdot 1+1\cdot 4+1\cdot 4+0}{\sqrt{3}\sqrt{38}} = \frac{9}{\sqrt{3}\sqrt{38}} = 0.84292723.$$

More similar

Corpus Reuters

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STANDARD OIL SRD TO FORM FINANCIAL UNIT
CLEVELAND February 26 Standard Oil Co and BP North America Inc said they plan to form a venture to manage the money market borrowing and investment activities of both companies
BP North America is a subsidiary of British Petroleum Co Plc BP which also owns a 55 pct interest in Standard Oil
The venture will be called BP Standard Financial Trading and will be operated by Standard Oil under the oversight of a joint management committee Reuter

<REUTERS ID="3">
TEXAS COMMERCE BANCSHARES TCB FILES PLAN
HOUSTON February 26 Texas Commerce Bancshares Inc's Texas Commerce Bank Houston said it filed an application with the Comptroller of the Currency in an effort to create the largest banking network in Harris County
The bank said the network would link 31 banks having 13.5 billion dlrs in assets and 7.5 billion dlrs in deposits Reuter

...

<REUTERS ID="1218">
JAGUAR JAGRY FEBRUARY U.S. SALES FALL
LEONIA N.J. March 3 Jaguar PLC's Jaguar Cars Inc U.S. subsidiary said February sales were 1,466 down from 1,673 a year before
The British company said it expects a resumption of U.S. sales growth towards the latter half of 1987
Jaguar said year to date U.S. sales were 2,523 down from 2,684 a year before Reuter

America	2
BP	4
British	1
County	0
Commerce	0
Committee	1
Currency	0
February	1
Harris	0
Interest	1
Jaguar	0
Leonia	0
Management	1
Petroleum	1
Resumption	0
Sales	0
Standard	5
Trading	1
Venture	2

$\vec{a} = (2, 4, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 5, 1, 2)$

America	0
BP	0
British	0
County	1
Commerce	3
Committee	0
Currency	1
February	1
Harris	1
Interest	0
Jaguar	0
Leonia	0
Management	0
Petroleum	0
Resumption	0
Sales	0
Standard	0
Trading	0
Venture	0

$\vec{b} = (0, 0, 0, 1, 3, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

America	0
BP	0
British	1
County	0
Commerce	0
Committee	0
Currency	0
February	2
Harris	0
Interest	0
Jaguar	4
Leonia	0
Management	0
Petroleum	0
Resumption	1
Sales	4
Standard	0
Trading	0
Venture	0

$\vec{c} = (0, 0, 1, 0, 0, 0, 0, 2, 0, 0, 4, 0, 0, 0, 1, 4, 0, 0, 0)$

query: british jaguar sales

$\vec{q} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0)$

America	0
BP	0
British	1
County	0
Commerce	0
Committee	0
Currency	0
February	0
Harris	0
Interest	0
Jaguar	1
Leonia	0
Management	0
Petroleum	0
Resumption	0
Sales	1
Standard	0
Trading	0
Venture	0

Search Engines

- The “Oxford English Dictionary” includes 171,476 words,
  - “Diccionario de la RAE” includes 93,000 words,
  - Assuming 20 characters per word, it will give a lexicon of a couple of MBytes:
    - Not a big deal in storage requirements
- Each document in Internet is a bag of words,
  - We can associate a vector to each document:
    - A vector of ~200,000 positions:
      - We are talking about really sparse vectors
    - A VSM – Vector Space Model.
- Use cosine similarity to calculate closeness of a query to a document.

Thinking about words as vectors…

- Assign to each word a long list (embeddings) of numbers:
  - From 4,096 to 12,288 features in each embedding,
- For example, the word 'dog' will have:
  - A very high value for 'furry' but low value for 'metallic',
  - Embeddings for 'dog' and 'cat' will be more similar than the embeddings between 'dog' and 'car',
    - There are semantic associations related to embeddings similarity,
- Nowadays embeddings are found using neural networks,
  - Not anymore by hand,
    - Spiders travels the Internet getting content for these neural networks,

Writing about words as vectors…

- Embeddings can be used for predicting the next word in a sequence of words:
  - ‘I hired a pet sitter to feed my ___’,
    - the next word might be ‘dog’ or ‘cat’, but it’s probably not ‘car’,
  - What about ‘The current president of France is named ___’,
    - It’s not intelligence is just statistics