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Active Target Defense Differential Game

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Abstract— **A pursuit-evasion differential game involving three agents is discussed. This scenario considers an Attacker missile pursuing a Target aircraft. The Target is however aided by a Defender missile launched by, say, its wingman, to intercept the Attacker before it reaches the Target aircraft. Thus, a team is formed by the Target and the Defender which cooperate to maximize the distance between the Target aircraft and the point where the Attacker missile is intercepted by the Defender missile, and the Attacker which tries to minimize said distance. The solution to this differential game provides optimal heading angles for the Target and the Defender team to maximize the terminal separation between Target and Attacker and it also provides the optimal heading angle for the Attacker to minimize the said distance.**

I. INTRODUCTION

Multi-agent pursuit-evasion scenarios represent important and challenging types of problems in aerospace, control, and robotics. In these types of problems one or more pursuers try to maneuver and reach a relatively small distance with respect to one or more evaders, which strive to escape the pursuers. Several types of pursuit-evasion scenarios involving many agents have been studied. This problem is usually posed as a dynamic game [1], [2], [3]. A dynamic Voronoi diagram has been used in problems with several pursuers in order to capture an evader within a bounded domain [2], [4]. Cooperation between two agents with the goal of evading a single pursuer has been addressed in [5]. Scott and Leonard [6] presented a scenario where two evaders employ coordinated strategies to evade a single pursuer, but also to keep them close to each other.

In this paper we consider a zero-sum three-agent pursuit-evasion differential game. The two-agent team consists of a Target (T) and a Defender (D) who cooperate; the Attacker (A) is the opposition. The goal of the Attacker is to capture the Target while the Target tries to evade the Attacker and avoid capture. The Target cooperates with the Defender which pursues and tries to intercept the Attacker before the latter captures the Target. Cooperation between the Target and the Defender is such that the Defender will capture the Attacker before the latter reaches the Target. Such a scenario of active target defense has been analyzed in the context of cooperative optimal control [7], [8]. Indeed, sensing capabilities of missiles and aircraft allow for implementation of complex pursuit and evasion strategies [9], [10], and recent work has proposed different guidance laws for the agents A and D. Thus, in [11] the authors addressed the case where the Defender implements Command to the Line of Sight (CLOS) guidance to pursue the Attacker which requires the Defender to have at least the same speed as the Attacker. A different guidance law for the Target-Attacker-Defender (TAD) scenario was given by Yamasaki *et.al.* [12], [13]. These authors investigated an interception method called Triangle Guidance (TG), where the objective is to command the defending missile to be on the line-of-sight between the attacking missile and the aircraft for all time while the aircraft follows some predetermined trajectory. The authors show, through simulations, that TG provides better performance in terms of Defender control effort than a number of variants of Proportional Navigation (PN) guidance laws, that is, when the Defender uses PN to pursue the Attacker instead of TG. These approaches constrain and limit the level of cooperation between the Target and the Defender by implementing Defender guidance laws without regard to the Target's trajectory.

Rubinsky and Gutman [14], [15] presented an analysis of the end-game TAD scenario based on the Attacker/Target miss distance for a *non-cooperative* Target/Defender. The authors develop linearization-based Attacker maneuvers in order to evade the Defender and continue pursuing the Target.

Different types of cooperation have been recently proposed in [16], [17], [18], [19], [20], [21], [22] for the TAD scenario. In these papers the Target represents an aircraft trying to evade a missile homing on it. The Defender represents another missile launched by the aircraft (or a wingman) in order to intercept and destroy the Attacker in order to guarantee the survival of the aircraft. Thus, in [18] optimal policies (lateral acceleration for each agent including the Attacker) are provided for the case of an aggressive Defender, that is, the Defender has a definite maneuverability advantage. A linear quadratic optimization problem is posed where the Defender's control effort weight is driven to zero to increase its aggressiveness. The work [19] provided a game theoretical analysis of the TAD problem using different guidance laws for both the Attacker and the Defender. The cooperative strategies in [20] allow for a maneuverability disadvantage for the Defender with respect to the Attacker and the results show that the optimal Target maneuver is either constant or arbitrary. Shaferman and Shima [21] implemented a Multiple Model Adaptive Estimator (MMAE) to identify the guidance law and parameters of the incoming missile and optimize a Defender strategy to minimize its control effort. In the recent

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paper [22] the authors analyze different types of cooperation assuming the Attacker is oblivious of the Defender and its guidance law is known. Two different one-way cooperation strategies were discussed: when the Defender acts independently, the Target knows its future behavior and cooperates with the Defender, and vice versa. Two-way cooperation where both Target and Defender communicate continuously to exchange their states and controls is also addressed, and it is shown to have a better performance than the other types of cooperation - as expected.

Our preliminary work [23], [24] considered the cases when the Attacker implements typical guidance laws of Pure Pursuit and PN, respectively. In these papers, the Target-Defender team solves an *optimal control* problem that returns the optimal strategy for the $T-D$ team so that D intercepts the Attacker and at the same time the separation between Target and Attacker at the instant of interception of A by D is maximized.

In this paper, the cooperative optimal guidance approach is extended to consider a differential game where also the Attacker missile solves an optimal control problem in order to minimize the final separation between itself and the Target. Assuming that the Attacker knows the position of the Defender, this strategy provides better performance for the Attacker than using CLOS or PN. From the Attacker's point of view, it is better to bring the Defender-Attacker interception point closer to the Target's position (and hopefully produce some damage), even though the Attacker is then captured by the Defender.

We also obtain the analytical solutions of the differential game, and give special attention to the case where the Target starts closer to the Attacker than to the Defender. For this scenario we also provide the critical minimal speed of the Target for it to avoid capture; that is, when the Target starts closer to the Attacker than to the Defender, its speed must be bounded from below; otherwise the Target will be captured by the Attacker before the Defender can get in the way of the Attacker and intercept it.

The paper is organized as follows. Section II describes the engagement scenario. In Section III a numerical method is described to address the differential game. An analytical solution of the differential game for the case of point capture is provided in Section IV. Examples are given in Section V and concluding remarks are made in Section VI.

II. Problem Statement

The Target-Attacker-Defender engagement is illustrated in Figure 1. The speeds of the Target, Attacker, and Defender are denoted by V_T, V_A, and V_D, respectively, which are assumed to be constant.

The dynamics of the three vehicles in the realistic game space are given by:

$$\dot{x}_T = V_T \cos\hat{\phi}, \qquad \dot{y}_T = V_T \sin\hat{\phi} \tag{1}$$

$$\dot{x}_A = V_A \cos\hat{\chi}, \qquad \dot{y}_A = V_A \sin\hat{\chi} \tag{2}$$

$$\dot{x}_D = V_D \cos\hat{\psi}, \qquad \dot{y}_D = V_D \sin\hat{\psi} \tag{3}$$

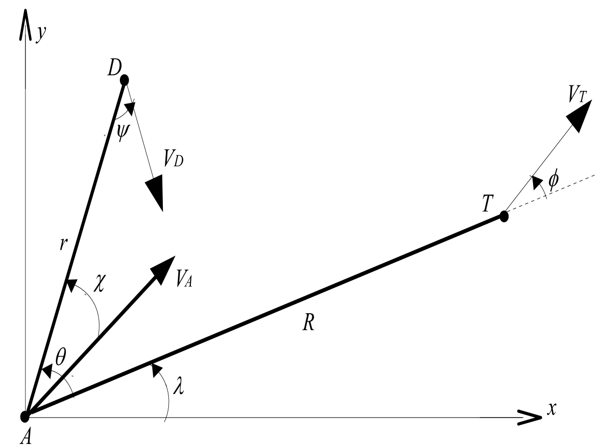

Fig. 1. Reduced state space

where the headings of T, D, and A are, respectively, $\hat{\phi} = \phi + \lambda$, $\hat{\psi} = \psi + \theta + \lambda - \pi$, and $\hat{\chi} = \lambda + \theta - \chi$.

Define the speed ratio problem parameter $\alpha = V_T/V_A$. In general, we have that the Attacker missile is faster than the Target aircraft, so that $\alpha < 1$. In this work we assume the Attacker and Defender missiles are somewhat similar, so $V_D = V_A$. The variables R and r represent the separation between the Attacker and the Target and between the Attacker and the Defender, respectively. The parameter R_c represents the Attacker capture radius and the parameter r_c represents the Defender capture radius. In this game the Attacker pursues the Target and tries to capture it. The Target and the Defender cooperate in order for the Defender to intercept the Attacker before the latter captures the Target. Thus, the Target-Defender team search for a cooperative optimal strategy to maximize the distance between the Target and the Attacker at the time instant of the Defender-Attacker collision. The Attacker will search for its corresponding optimal strategy in order to minimize the terminal $A-T$ distance.

III. Numerical treatment of the differential game

In this section, the corresponding dynamics of the three-body engagement will be modeled using the reduced state space formed by the ranges R and r, and by the angle between them, denoted by θ, see Fig. 1. The objective of the Target-Defender team is to determine their optimal heading angles ϕ and ψ in this reduced state space such that the distance $R(t_f)$ is maximized at the time instant t_f where the separation $r(t_f) = r_c$. The interception time t_f is free. The objective of the Attacker is to determine its optimal heading angle, denoted by χ, such that the distance $R(t_f)$ is minimized. Note that the relative heading angles can be easily transformed to heading angles with respect to the fixed coordinate axis x using the line of sight angle from the Attacker to the Target, denoted by λ. The use of the reduced state space provides a compact representation of the dynamics of this three-agent differential game.

The dynamics in the reduced state space are

$$\dot{R} = \alpha\cos\phi - \cos(\theta-\chi), \qquad R(t_0) = R_0 \tag{4}$$

$$\dot{r} = -\cos\chi - \cos\psi, \qquad r(t_0) = r_0 \tag{5}$$

$$\begin{aligned}\dot{\theta} = &-\frac{\alpha}{R}\sin\phi + \frac{1}{R}\sin(\theta-\chi)\\ &-\frac{1}{r}\sin\psi + \frac{1}{r}\sin\chi, \qquad \theta(t_0) = \theta_0\end{aligned} \tag{6}$$

for $0 \leq t \leq t_f$.

The objective of the Target-Defender team is to maximize the separation between the Target and the Attacker at the interception time $R(t_f)$, where the terminal time t_f is free, such that $r(t_f) = r_c$. The objective of the Attacker is to minimize the same distance $R(t_f)$. This can be expressed as

$$\max_{\phi,\psi}\min_{\chi} J = \int_{t_0}^{t_f} \dot{R}dt. \tag{7}$$

Then, the Hamiltonian is given by

$$\begin{aligned}H = &\cos(\theta-\chi) - \alpha\cos\phi\\ &+ (\alpha\cos\phi - \cos(\theta-\chi))\lambda_R\\ &- (\cos\chi + \cos\psi)\,\lambda_r\\ &+ (-\tfrac{\alpha}{R}\sin\phi + \tfrac{1}{R}\sin(\theta-\chi) - \tfrac{1}{r}\sin\psi + \tfrac{1}{r}\sin\chi)\lambda_\theta\end{aligned} \tag{8}$$

and the co-state dynamics are given by:

$$\dot{\lambda_R} = \frac{\lambda_\theta}{R^2}\left(\sin(\theta-\chi) - \alpha\sin\phi\right) \tag{9}$$

$$\dot{\lambda_r} = \frac{\lambda_\theta}{r^2}\left(\sin\chi - \sin\psi\right) \tag{10}$$

$$\dot{\lambda_\theta} = (1-\lambda_R)\sin(\theta-\chi) - \frac{\lambda_\theta}{R}\cos(\theta-\chi). \tag{11}$$

The terminal conditions for this free terminal time problem are as follows. The terminal state $r(t_f)$ is fixed and equal to r_c. Because the terminal states $R(t_f)$, and $\theta(t_f)$ are free, we have $\lambda_R(t_f) = \lambda_\theta(t_f) = 0$. The final terminal condition for optimality for this problem requires that $H(x^*(t_f), u^*(t_f), \lambda^*(t_f), t_f) = 0$. In summary, the terminal conditions are:

$$\begin{aligned}r(t_f) &= r_c\\ \lambda_R(t_f) &= 0\\ \lambda_\theta(t_f) &= 0\\ \alpha^2 + 2(\alpha + \cos\theta(t_f))\lambda_r(t_f) - 1 &= 0.\end{aligned} \tag{12}$$

Proposition 1: The Target and Defender optimal control headings that maximize the separation between the Target and the Attacker and achieve $r(t_f) = r_c$ are given by

$$\sin\psi^* = \frac{\lambda_\theta}{r\sqrt{\lambda_r^2 + \lambda_\theta^2/r^2}} \tag{13}$$

$$\cos\psi^* = \frac{\lambda_r}{\sqrt{\lambda_r^2 + \lambda_\theta^2/r^2}} \tag{14}$$

$$\sin\phi^* = \frac{\lambda_\theta}{R\sqrt{(1-\lambda_R)^2 + \lambda_\theta^2/R^2}} \tag{15}$$

$$\cos\phi^* = \frac{1-\lambda_R}{\sqrt{(1-\lambda_R)^2 + \lambda_\theta^2/R^2}}. \tag{16}$$

The Attacker optimal control heading that minimizes the separation between itself and the Target at $t = t_f$ is given by

$$\sin\chi^* = \frac{\chi_s}{\sqrt{\chi_s^2 + \chi_c^2}} \tag{17}$$

$$\cos\chi^* = \frac{\chi_c}{\sqrt{\chi_s^2 + \chi_c^2}} \tag{18}$$

where $\chi_s = (1-\lambda_R)\sin\theta - \frac{\lambda_\theta}{R}\cos\theta + \frac{\lambda\theta}{r}$ and $\chi_c = (1-\lambda_R)\cos\theta + \frac{\lambda_\theta}{R}\sin\theta - \lambda_r$.

The optimal heading angles expressions (13)-(18) are used to numerically solve the Two-Point Boundary Value Problem (TPBVP) (4)-(7), (9)-(18). The numerical solution is found by substituting the optimal control headings into the state equations (4)-(6), and the co-state equations (9)-(11), with the terminal conditions given by (12).

IV. Differential Game with Point Capture

A. Target Defense Differential Game

We now undertake analysis of the Target Defense differential game. In this section we confine our attention to point capture, that is, we assume $r_c \to 0$. The Target (T), the Attacker (A), and the Defender (D) have "simple motion" a la Isaacs. We emphasize that T, A, and D have constant speeds of V_T, V_A, and V_D, respectively. We assume that $V_A = V_D$ and the speed ratio $\alpha = \frac{V_T}{V_A} < 1$. T and D form a team to defend from A. Thus, A strives to close in on T while T and D maneuver such that D intercepts A before the latter reaches T and the distance at interception time is maximized, while A strives to minimize the separation between T and A at the instant of interception.

In Fig. 2 the points A and D represent the positions of the Attacker and the Defender, respectively. A Cartesian frame is attached to the points A and D in such a way that the extension to infinity of $\overline{AD}$ in both directions represents the X-axis and the orthogonal bisector line of $\overline{AD}$ represents the Y-axis.

With respect to Fig. 2 we note that the Attacker aims at minimizing the distance between the Target at the time instant when the Defender intercepts the Attacker, point T', and the point I on the orthogonal bisector of $\overline{AD}$ where the Defender intercepts the Attacker. The points T and T' represent the initial and terminal positions of the Target, respectively.

B. Critical Speed Ratio for Target Survival

Assume that the speed ratio $0 < \alpha < 1$ and $x_T > 0$. The Target needs to be able to break into the Left Half Plane (LHP) before being intercepted by the Attacker for the Defender to be able to assist the Target to escape, by intercepting the Attacker who is on route to the Target. Thus, a solution to the differential game exists if and only if the Apollonius circle, which is based on the segment $\overline{AT}$ and the speed ratio α, intersects the orthogonal bisector of $\overline{AD}$. This imposes a lower limit $\bar{\alpha}$ on the speed ratio, that is, we need $\bar{\alpha} < \alpha < 1$. The critical speed ratio $\bar{\alpha}$ corresponds to the case where the Apollonius circle is tangent to the orthogonal

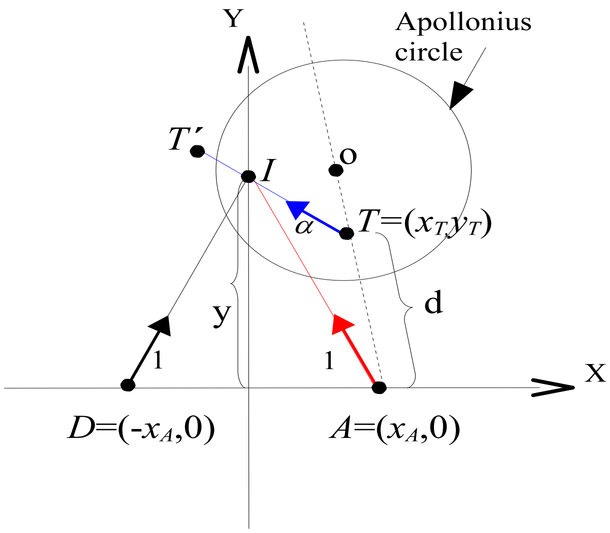

Fig. 2. Target Defense Differential Game

bisector of $\overline{AD}$. Note that if the speed ratio $\alpha \geq 1$ the Target always escapes and there is no need for a Defender missile, that is, there is no Target Defense Differential Game.

The Attacker's initial position, the Target's initial position, and the center O of the Apollonius circle are collinear and lie on the dotted line in Fig. 2 which can be represented as

$$y = -\frac{y_T}{x_A - x_T}x + \frac{x_A y_T}{x_A - x_T}$$

The geometry of the Apollonius circle is as follows: The center of the circle, denoted by O, is at a distance of $\frac{\alpha^2}{1-\alpha^2}d$ from T and its radius is $\frac{\alpha}{1-\alpha^2}d$, where d is the distance between A and T and is given by

$$d = \sqrt{(x_A - x_T)^2 + y_T^2}. \tag{19}$$

Hence, the following holds

$$\begin{aligned} &\left(\frac{x_T y_T}{x_A - x_T} - \frac{y_T}{x_A - x_T}x_0\right)^2 + (x_0 - x_T)^2 \\ &= \frac{\alpha^4}{(1-\alpha^2)^2}[(x_A - x_T)^2 + y_T^2] \end{aligned} \tag{20}$$

and we calculate the coordinates of the center of the Apollonius circle

$$\begin{aligned} x_O &= \tfrac{1}{1-\alpha^2}x_T - \tfrac{\alpha^2}{1-\alpha^2}x_A \\ y_O &= \tfrac{1}{1-\alpha^2}y_T. \end{aligned} \tag{21}$$

Consequently, the critical speed ratio $\bar{\alpha}$ is the positive solution of the quadratic equation

$$x_T - \alpha^2 x_A = \alpha\sqrt{(x_A - x_T)^2 + y_T^2} \tag{22}$$

and is given by

$$\bar{\alpha} = \frac{\sqrt{(x_A + x_T)^2 + y_T^2} - \sqrt{(x_A - x_T)^2 + y_T^2}}{2x_A}. \tag{23}$$

In the special case where $x_T = x_A$, the critical speed ratio

$$\bar{\alpha} = \frac{\sqrt{4x_A^2 + y_T^2} - y_T}{2x_A},$$

as expected. Since $y_T > 0$ we have $\bar{\alpha} < 1$.

In the special case when $y_T = 0$, we have that $\bar{\alpha} = x_T/x_A < 1$ if $x_A > x_T$, and $\bar{\alpha} = 1$ if $x_A \leq x_T$. Also note that when $x_T = 0$, $\bar{\alpha} = 0$, as expected.

In general, it can be seen from Fig. 2 that if $x_T < 0$ then $\bar{\alpha} = 0$ as well. The case when $x_T < 0$, that is, when the Target is on the Defender's side of the orthogonal bisector of $\overline{AD}$ is also of interest but will not be addressed here because of space constraints.

We will assume $\bar{\alpha} < \alpha < 1$, so that a solution to the Target Defense Differential Game exists; otherwise, if $\alpha \leq \bar{\alpha}$, the Defender will not be able to help the Target by intercepting the Attacker before the latter inevitably captures the Target. And if $\alpha \geq 1$ then the Target cannot be intercepted by the Attacker and there is no need for the Defender.

C. *Optimal Strategies*

The Attacker will be intercepted by the Defender on the orthogonal bisector of $\overline{AD}$. Thus, the Attacker chooses his aimpoint, denoted by I, on the orthogonal bisector of $\overline{AD}$ in order to minimize the cost function

$$J(y) = \alpha\sqrt{x_A^2 + y^2} - \sqrt{(y - y_T)^2 + x_T^2} \tag{24}$$

which represents the final separation between Target and Attacker, and where y represents the coordinate of the aimpoint I on the orthogonal bisector of $\overline{AD}$. This is so because the Target will head away from I.

In order to find the minimum of (24) we differentiate eq. (24) in y

$$\frac{dJ(y)}{dy} = \frac{\alpha y}{\sqrt{x_A^2 + y^2}} - \frac{y - y_T}{\sqrt{(y - y_T)^2 + x_T^2}} = 0 \tag{25}$$

which can also be written as follows

$$\frac{\alpha^2 y^2}{x_A^2 + y^2} = \frac{(y - y_T)^2}{(y - y_T)^2 + x_T^2} \tag{26}$$

and we obtain the quartic equation in $y \geq 0$

$$\begin{aligned} &(1-\alpha^2)y^4 - 2(1-\alpha^2)y_T y^3 \\ &+\big((1-\alpha^2)y_T^2 + x_A^2 - \alpha^2 x_T^2\big)y^2 \\ &-2x_A^2 y_T y + x_A^2 y_T^2 = 0 \end{aligned} \tag{27}$$

When $\alpha = 1$, (27) can be reduced to the following quadratic equation

$$\left(1 - \frac{x_T^2}{x_A^2}\right)y^2 - 2y_T y + y_T^2 = 0 \tag{28}$$

and the solutions, when $|x_T| \neq x_A$, are given by

$$y = \frac{x_A y_T}{x_A + x_T}, \quad y = \frac{x_A y_T}{x_A - x_T}. \tag{29}$$

However, the case $\alpha = 1$ is outside the scope of the Target defense differential game. Nevertheless, if $\alpha = 1$ and $x_T = x_A$ then the solution $y^* = \frac{1}{2}y_T$ makes no good sense, as pictured in Fig. 3. In this case the Target is better off to run

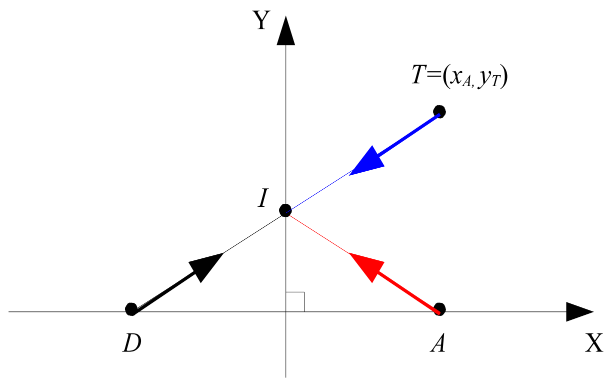

Fig. 3. “Solution” when $\alpha = 1$ and $x_T = x_A$ - Incorrect

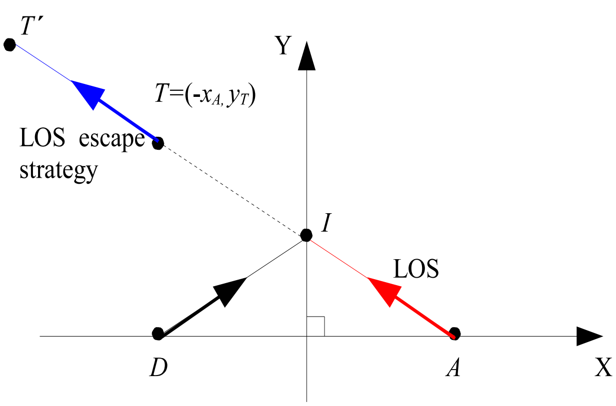

Fig. 4. Solution when $\alpha = 1$ and $x_T = -x_A$

North. Now, if $\alpha = 1$ and $x_T = -x_A = x_D$ then the solution $y^* = \frac{1}{2} y_T$ makes sense: the initial separation between Target and Attacker, $J^* = 2\sqrt{x_A^2 + y_T^2/4}$, is maintained, as expected. This situation is shown in Fig. 4 where it can be seen that the Target runs away from the Attacker and there is no need for a Defender missile since the Target will not be captured. Therefore, there is no Target defense differential game in this particular situation.

Remark. Writing eq. (27) as $f(y) = 0$ we can see that $f(0) = x_A^2 y_T^2 > 0$, $f(y_T) = -\alpha^2 x_T^2 y_T^2 < 0$, and $f(\infty) = +\infty$. Therefore, equation (27) has two real solutions. Equation (27) has a real solution $0 < y < y_T$ and an additional real solution $y_T < y$, provided that $x_T \neq 0$. If $x_T = 0$, then y_T is a repeated solution of (27) because this equation can then be written as follows

$$\begin{aligned} 0 &= (1-\alpha^2)(y^4 - 2y_T y^3 + y_T^2 y^2) \\ &\quad + x_A^2 (y^2 - 2y_T y + y_T^2) \\ &= (y^2 - 2y_T y + y_T^2)\big((1-\alpha^2)y^2 + x_A^2\big) \\ &= (y - y_T)^2 \big((1-\alpha^2)y^2 + x_A^2\big). \end{aligned} \tag{30}$$

Hence, y_T is a repeated root of (27). In addition, there are two complex roots: $y = \pm i \frac{1}{\sqrt{1-\alpha^2}} x_A$.

Note that (27) is parameterized by x_T^2, so whether $x_T > 0$ or $x_T < 0$ makes no difference in terms of the solutions to the quartic equation (27).

When $x_T > 0$, by choosing his heading, the Target (and the Defender) thus determine the coordinate y to maximize $J(y)$; that is, y is the Target's (and Defender's) choice. Then the payoff is given by eq. (24) and the expression for $\frac{dJ(y)}{dy}$ was shown in (25). We also have that

$$\frac{d^2 J(y)}{dy^2} = \frac{\alpha x_A^2}{(x_A^2 + y^2)^{3/2}} - \frac{x_T^2}{\big((y - y_T)^2 + x_T^2\big)^{3/2}}. \tag{31}$$

Since the Target is choosing y to maximize the cost $J(y)$, then the optimal coordinate y^* is the solution of (27) such that $\frac{d^2 J(y)}{dy^2} < 0$. In view of (25) we know that

$$\frac{1}{\sqrt{(y - y_T)^2 + x_T^2}} = \alpha \frac{y}{y - y_T} \frac{1}{\sqrt{x_A^2 + y^2}}. \tag{32}$$

Then, inserting (32) into (31) yields

$$\frac{d^2 J(y)}{dy^2} = \frac{\alpha}{(x_A^2 + y^2)^{3/2}} \left(x_A^2 - \alpha^2 \left(\frac{y}{y - y_T} \right)^3 x_T^2 \right) \tag{33}$$

and we have that $\frac{d^2 J(y)}{dy^2} < 0$ if and only if

$$\frac{1}{\alpha^2} \left(\frac{x_A}{x_T} \right)^2 < \left(\frac{y}{y - y_T} \right)^3. \tag{34}$$

Hence, the solution $y < y_T$ of (27) does not fulfill the role of yielding a maximum and the second solution $y > y_T$ of (27) is the optimal solution.

Inserting (32) into (24) yields the Target and Defender payoff

$$\begin{aligned} J^*(y) &= \alpha \sqrt{x_A^2 + y^2} - \tfrac{1}{\alpha} \tfrac{y - y_T}{y} \sqrt{x_A^2 + y^2} \\ &= \tfrac{1}{\alpha} \sqrt{x_A^2 + y^2} \big(\tfrac{y_T}{y} - (1 - \alpha^2) \big). \end{aligned} \tag{35}$$

When $\alpha > \bar{\alpha}$ we have that $J^*(y) > 0$. Hence, the solution $y > y_T$ of (27) must satisfy

$$y_T < y < \tfrac{1}{1-\alpha^2} y_T. \tag{36}$$

This situation is shown in Fig. 5 where the three points T, I, and T' are collinear. Concerning expression (34), we also need the solution of the quartic equation to satisfy

$$y < \frac{1}{1 - \alpha^{2/3} (\frac{x_T}{x_A})^{2/3}} y_T. \tag{37}$$

D. Discussion

When $x_T > 0$ the Attacker and the Target are faced with a maxmin optimization problem: the Target chooses y_1 and the Attacker chooses y_2, see Fig. 6.

The decision variables y_1 and y_2 jointly determine the distance $d(A', T')$, that is, the function

$$\tilde{J}(y_1, y_2) = d(A', T'). \tag{38}$$

Thus, the Attacker and the Target solve the following problem

$$\max_{y_1} \min_{y_2} \tilde{J}(y_1, y_2). \tag{39}$$

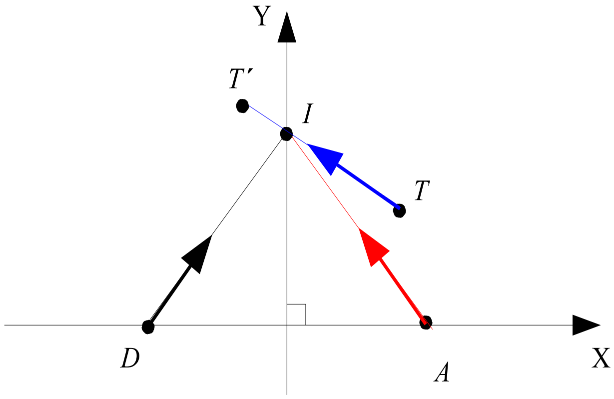

Fig. 5. Optimal solution when $x_T > 0$

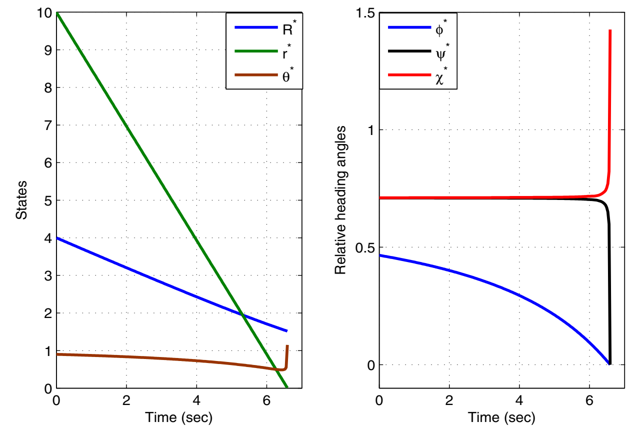

Fig. 7. Numerical solution of Example 1

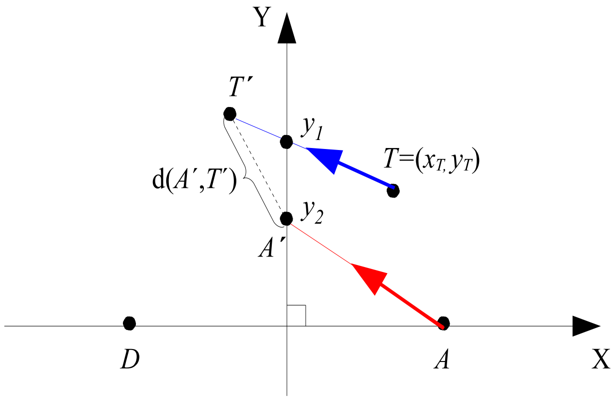

Fig. 6. Maxmin optimization problem

Should the Target choose y_1, the Attacker would respond with

$$y_2^* = f^*(y_1) = \arg\ \min_{y_2} \tilde{J}(y_1, y_2) \tag{40}$$

and knowing this, the Target chooses

$$y_1^* = \arg\ \max_{y_1} \tilde{J}(y_1, f^*(y_1)) \tag{41}$$

which yields the payoff $\tilde{J}(y_1^*, f^*(y_1^*))$.

Thus the choices of the Target and the Attacker which solve the maxmin optimization problem at hand are y_1^* and $y_2^* = f^*(y_1^*)$. The Defender's choice is then determined by y_2^*.

The function $\tilde{J}(y_1, y_2)$ is calculated as follows. The decision variable of the Target is y_1 and the decision variable of the Attacker is y_2. The terminal Attacker coordinates are $A' = (0, y_2)$. We now calculate the coordinates of the point $T' = (x_{T'}, y_{T'})$ as follows

$$\frac{y_1 - y_T}{x_T} = \frac{y_{T'} - y_T}{x_T - x_{T'}} \tag{42}$$

and

$$(x_T - x_{T'})^2 + (y_{T'} - y_T)^2 = \alpha^2(x_A^2 + y_2^2). \tag{43}$$

Inserting (42) into (43) yields

$$\left(1 + \left(\frac{y_1 - y_T}{x_T}\right)^2\right)(x_T - x_{T'})^2 = \alpha^2(x_A^2 + y_2^2) \tag{44}$$

and we have that

$$x_{T'} = \left(1 - \alpha\sqrt{\frac{x_A^2 + y_2^2}{x_T^2 + (y_1 - y_T)^2}}\right)x_T \tag{45}$$

$$y_{T'} = y_T + \alpha(y_1 - y_T)\sqrt{\frac{x_A^2 + y_2^2}{x_T^2 + (y_1 - y_T)^2}}. \tag{46}$$

Finally, the distance between the terminal points A' and T' is given by

$$\begin{aligned} & d^2(A', T') = \\ & \left(1 - \alpha\sqrt{\tfrac{x_A^2+y_2^2}{x_T^2+(y_1-y_T)^2}}\right)^2 x_T^2 \\ & + \left(y_T - y_2 + \alpha(y_1 - y_T)\sqrt{\tfrac{x_A^2+y_2^2}{x_T^2+(y_1-y_T)^2}}\right)^2. \end{aligned} \tag{47}$$

Proposition 2: . Given the functions $\tilde{J}(y_1, y_2)$, the solution y_1^* and y_2^* of the optimization problem $\max_{y_1}\ \min_{y_2} \tilde{J}(y_1, y_2)$ and the function $f^*(\bullet)$, where $y_2^* = f^*(y_1)$ are such that the function $f^*(\bullet)$ has a fixed point and

$$y_1^* = y_2^*. \tag{48}$$

Moreover, the function $f^*(y_1) = y_1$ so that when $x_T > 0$ it suffices to solve the optimization problem $\max_y J(y)$ where

$$J(y) = \alpha\sqrt{x_A^2 + y^2} - \sqrt{(y - y_T)^2 + x_T^2}.$$

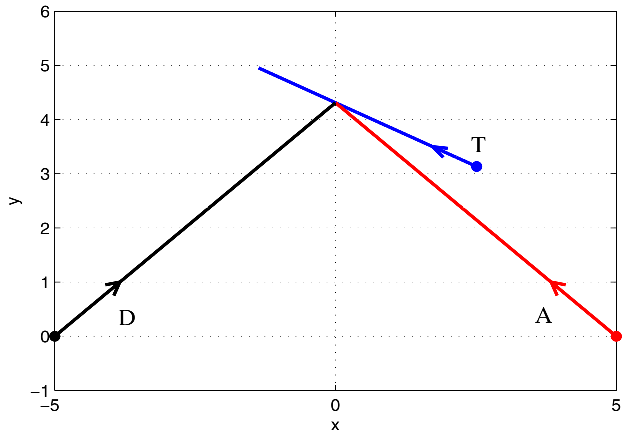

Fig. 8. Optimal trajectories for Example 1

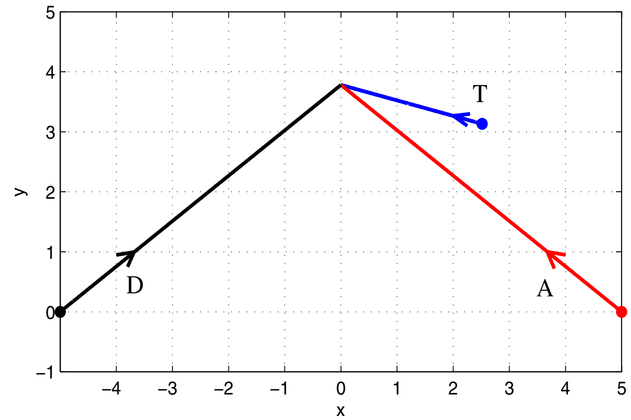

Fig. 9. Optimal trajectories for Example 2

V. EXAMPLES

Example 1. We use analytical and numerical methods to solve the Target Defense Differential Game. We consider an Attacker and a Defender missile with unit speed and with initial positions given by $A = [5, 0]$ and $D = [-5, 0]$, respectively. The Target has a speed $\alpha = 0.65$ and it is initially located at $T = [2.514, 3.133]$. In terms of the variables used in Section III we have the following initial conditions: $R_0 = 4$, $r_0 = 10$, $\theta_0 = 0.9\ rad$. The numerical solution of the differential game described in Section III (when $r_c \to 0$) is shown in Fig. 7; the left plot of this figure shows the optimal states and the right plot shows the optimal relative heading angles. The relative heading angles can be transformed to absolute angles with respect to a fixed frame which were denoted by $\hat{\phi}$, $\hat{\psi}$, and $\hat{\chi}$. For the Target Defense Differential Game the absolute optimal angles are always constant. For this example, we have that: $\hat{\phi} = 2.7033\ rad$, $\hat{\psi} = 0.7115\ rad$, and $\hat{\chi} = 2.4301\ rad$. The interception point's coordinate is $y = 4.311$ which was corroborated by solving the quartic equation (27) which was derived in Section IV. Both methods provide the same solutions, as expected, and we have that $t_f = 6.6\ sec$ and the final separation is $R(t_f) = 1.515$. An important additional piece of information is the minimum relative speed to guarantee Target survival which, for this example, is given by $\bar{\alpha} = 0.4141$. Obviously, the numerical and the analytical solutions provide the same trajectories and here it suffices to show one set of trajectories. Analytically derived optimal trajectories are shown in Fig. 8.

Example 2. Limit $\alpha \to \bar{\alpha}$. Consider the same initial positions for the Target, Attacker, and Defender as in Example 1. The difference in this example is that the speed of the Target is $\alpha = 0.415$ which is greater than the critical speed but very close to that critical value $\bar{\alpha} = 0.4141$. In this case the interception point is $y = 3.783$, the interception time is $t_f = 6.34\ sec$, and the final separation is $R(t_f) = 0.0058$. The final separation is positive but very close to zero. This is expected since the value of α is almost equal to $\bar{\alpha}$. The trajectories for this example are shown in Fig. 9.

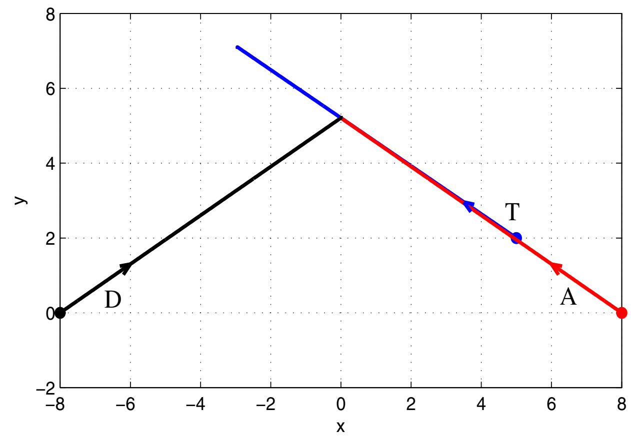

Fig. 10. Optimal trajectories for Example 3

Example 3. Limit $\alpha \to 1$. Consider an Attacker and a Defender missile with unit speed and with initial positions given by $A = [8, 0]$ and $D = [-8, 0]$, respectively. The Target has a speed $\alpha = 0.99$ and it is initially located at $T = [5, 2]$. This value of α is very close to 1; recall that when $\alpha = 1$ we expect a LOS pursuit and escape strategy. The interception point is calculated for this example and it is found to be $y = 5.21$, the interception time is $t_f = 9.55\ sec$, and the final separation is $R(t_f) = 3.51$. The trajectories for this example are shown in Fig. 10. Note that the Target's escape trajectory and the Attacker's pursuit trajectory are similar (but not exactly) to LOS trajectories, as expected. The absolute heading angles of the Target and the Attacker are close, but not the same, and they are given by $\hat{\phi} = 2.5709\ rad$ and $\hat{\chi} = 2.5643\ rad$, which show that the Target and the Attacker do not exactly follow a LOS escape and pursuit strategy.

VI. Conclusions

A differential game for the cooperative aircraft defense scenario was formulated and analyzed. In this differential game of perfect information the Attacker is aware of the Defender's position and of the cooperation between Target and Defender. The Attacker computes and implements the optimal heading angle that minimizes the final Target-Attacker separation at the time instant when the Defender intercepts the Attacker. The Target strives to maximize said distance. For the case of point capture, an analytical solution of the Target Defense Differential Game is provided. For completeness, critical Target speeds were also examined in order to determine a lower bound on the speed of the Target that guarantees, with the help of the Defender, its escape from the Attacker.

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