Exercise 1. Central Limit Theorem

Objective: Consider the problem of throwing dice multiple times and calculate the resulting probability distribution.

1 Task

1.1 Figure 1

A die, thrown once generates an integer number in the range from 1 to 6.

Calculate the mean value μ,

$$\mu = \langle x \rangle = \frac{1}{6} \sum_{i=1}^{6} x_i \tag{1}$$

with $x_i = i$ and the variance σ

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{2}$$

where

$$\langle x^2 \rangle = \frac{1}{6} \sum_{i=1}^{6} x_i^2 \tag{3}$$

Use a random number generator to simulate dice throwing and calculate the resulting probability distribution p_i of obtaining each numbers i from 1 to 6. Use discrete normalization

$$\sum_i p_i = 1 \tag{4}$$

The random variable of interest is

$$x = \text{rand}(6) \tag{5}$$

Plot the resulting probability distribution function p_i and compare it with uniform distribution prediction.

1.2 Figure 2

Assume that one random even consists in throwing a die twice and calculating the average value,

$$x = \frac{\text{rand}(6) + \text{rand}(6)}{2} \tag{6}$$

Calculate the probability distribution of the outcome and show it on the figure.

1.3 Figure 3

Consider the case when one random even consists in throwing a die N times and calculating the average value

$$x = \frac{\sum_i^N \text{rand}(6)}{N} \tag{7}$$

Calculate the probability distribution $p(x)$ and assume that for large N, the spacing dx between two allowed subsequent values of x is small and that the random value x can be considered as a continuous variable. Use continuous variable convention to normalize the PDF,

$$\int p(x)dx = 1 \tag{8}$$

Compare the obtained result with the prediction of the Central Limit Theorem

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma'} \exp\left(-\frac{(x-\mu)^2}{2\sigma'^2}\right) \tag{9}$$

where the reduced variance is given by $\sigma' = \sigma/\sqrt{N}$.

Find a large enough value of N so that the CLT applies and show the resulting plot.

1.4 Error estimation

Assume that multiple dice throwing is used to estimate the mean value μ. Estimate μ by throwing $N = 10$ and $N = 100$ dice. Estimate the statistical error by σ'. Report the obtained numbers.