Examen ALN

I Méthode de Richardson

$\alpha > 0$

$$\begin{cases} x_0 \in \mathbb{R}^n \\ x_{k+1} = x_k + \alpha (b - Ax_k) \end{cases}$$

1) Posons $r_k = b - Ax_k$

on a $x_{k+1} = x_k + \alpha r_k$

~~On~~ pose $\boxed{M = \frac{1}{\alpha} I_n}$ inversible

donc

$$x_{k+1} = x_k + M^{-1} r_k$$

D'où $A = M - N \Rightarrow N = M - A$

$$\boxed{N = \frac{1}{\alpha} I_n - A}$$

2) $\lambda \in \mathbb{C}$

$$M^{-1}N = \alpha \left(\frac{1}{\alpha} I_n - A\right)$$

$$= I_n - \alpha A$$

D'où $\det(M^{-1}N - \mu I_n) = 0 \Leftrightarrow \det(I_n - \alpha A - \mu I_n) = 0$

$\Leftrightarrow \det((1-\mu) I_n - \alpha A) = 0$

$\Leftrightarrow \alpha^n \det\left(\left(\frac{1-\mu}{\alpha}\right) I_n - A\right) = 0$

$\Leftrightarrow (-\alpha)^n \det\left(A - \frac{1-\mu}{\alpha} I\right) = 0$

$\Leftrightarrow \frac{1-\mu}{\alpha}$ vp de A

$\Rightarrow \exists \lambda$ vp de A tq

$\lambda = \frac{1-\mu}{\alpha}$

$\Rightarrow \exists \lambda$ vp de A tq $\mu = 1 - \alpha\lambda$

D'où $\boxed{\lambda \text{ vp de } A \Leftrightarrow 1-\alpha\lambda \text{ vp de } M^{-1}N}$

3) $\forall \lambda$ vp de $A \Rightarrow \lambda \in \mathbb{R}$

$\forall x_0 \in \mathbb{R}^n$ convergence $\Leftrightarrow \rho(M^{-1}N) < 1$

$\Rightarrow |1-\alpha\lambda| < 1$

$\forall \lambda$ vp de A

$\Rightarrow -1 \leq 1-\alpha\lambda \leq 1$

$\forall \lambda$ vp de A

$$\boxed{\forall x_0 \in \mathbb{R}^n \text{ convergence} \Rightarrow 0 \leq \alpha\lambda \leq 2 \quad \forall \lambda \text{ vp de } A}$$

4) $A \in S_n(\mathbb{R})$ déf positive

• Posons $f(x) = \frac{1}{2} x^T A x - b^T x$

on a $\nabla f(x) = Ax - b$

D'où le schéma s'écrit :

$$\begin{cases} x_0 \in \mathbb{R}^n \\ x_{k+1} = x_k - \alpha \nabla f(x_k) \end{cases}$$

• Steepest descent? $\alpha_k = \dfrac{\|\nabla f(x_k)\|^2}{\|\nabla f(x_k)\|_A^2}$

II Préconditionnement

$P^{-1}Ax = P^{-1}b$

5) $x_{k+1} = x_k + \alpha(P^{-1}b - P^{-1}Ax_k)$

$$\boxed{x_{k+1} = x_k + P^{-1}(b - Ax_k)}$$

$$x_{k+1} = x_k + P^{-1} r_k$$

D'où $\boxed{M = P}$ inversible

On a alors $N = M - A$

$$\boxed{N = P - A}$$

7) a) $P = D = \operatorname{diag}(A) \rightsquigarrow$ Jacobi

inversible si $a_{ii} \neq 0 \ \forall i$

on a $x_{k+1} = x_k + D^{-1}(b - A x_k)$

D'où

$$D x_{k+1} = D x_k + b - A x_k$$

or $A = D - E - F$

D'où $D x_{k+1} = (D - D + E + F) x_k + b$

$$\boxed{D x_{k+1} = (E + F) x_k + b}$$

Jacobi

b) $P = D - E$ inversible si $a_{ii} \neq 0 \ \forall i$

D'où

$$\begin{aligned}(D - E) x_{k+1} &= (D - E) x_k + b - A x_k \\ &= (D - E - D + E + F) x_k + b \\ &= F x_k + b\end{aligned}$$

$$\boxed{(D - E) x_{k+1} = F x_k + b}$$

Gauss-Seidel