Masters in Finance
Toulouse Business School

Derivative Securities

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Course Syllabus

Objective:

The objective of this course is to provide a solid understanding of derivative securities. We study forward contracts, futures contracts and swaps. We examine options such as call and puts and examine the trading strategies they allow. Moving to the pricing of options, we derive the binomial model of option pricing. Taking a limit argument, we obtain the Black and Scholes formula. We then address the implementation of a pricing formula. We examine historical volatility and implied volatility. We then study option greeks. We examine how defaultable bonds, warrants and convertibles can be thought and valued as options. We study exotic options such as barrier, lookback and asian options. We examine how to value flexibility in projects using real options.

Course Contents:

1. Forwards, Futures and Swaps
   - Derivatives
   - Forward Contacts
     - Forward Price
   - Futures Contracts
     - Standardization of Contract Terms
     - Reversal of Positions
     - Hedging with Delivery Basis Risk
     - Margin Accounts
     - Futures Price
   - Swaps
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2. Options
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   - Put-Call Parity
   - Option Strategies
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   - American Options and Early Exercise
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   - Formal Derivation of the Binomial Model
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     - Risk Neutral Probability

        - Two Periods
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        - Binomial Model Pricing Formula
    - Limit Argument
        - Black-Scholes Pricing Formula

4. Historical, Implied Volatility and Option Greeks
    - Implementing a Pricing Model
        - Difficulty with Volatility
    - Historical Volatility
    - Implied Volatility
    - Option Greeks
        - Delta, Delta Neutral Position
        - Gamma, Target Position Gamma
        - Theta, Position Theta
        - Vega
        - Rho

5. Options in Corporate Securities and Exotic Options
    - Reminder of Option Pricing
    - Options in Corporate Securities
        - Zero-Coupon Bonds
        - Senior and Junior Zero Coupon Bonds
        - Warrants
        - Convertibles
    - Exotic Options
        - Barrier Options
        - Lookback Options
        - Asian Options

6. Real Options
    - Motivation
    - Real Options
    - Reminder of Option Pricing
    - Real Options Embedded in Industrial Projects
        - Case Study – Penelope's Personal Pocket Phones

Material on the Website:

– The entire material attached to this course is available in an "all-in-one" file:

_Derivative_Securities

This version is ideal if one wishes to print the material as a single leaflet. The file is designed to be printed with "two-pages-per-sheet".

– The material is also available as "detached" files. The files are:

The syllabus file:
_00_01_syllabus

Six files with lecture notes:
_01_01_Forwards_Futures_and_Swaps

_02_01_Options
_03_01_Option_Pricing
_04_01_Historical_Implied_Volatility_and_Option_Greeks
_05_01_Options_in_Corporate_Securities_and_Exotic_Options
_06_01_Real_Options

Four files with sets of problems and detailed solutions to these problems:
_03_02_Problem_Set_1
_03_03_Solutions_Problem_Set_1
_05_02_Problem_Set_2
_05_03_Solutions_Problem_Set_2

One background reading files with a press release from the Nobel committee describing related contributions:
_03_04_Nobel_MertonScholes_1997

One file with the case study "Penelope's Personal Pocket Phones":
_06_02_PenelopePP_Case_Study

1 – Forwards, Futures and Swaps

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Course Road Map

1. Forwards, Futures and Swaps
2. Options
3. Option Pricing
4. Historical, Implied Volatility and Option Greeks
5. Options in Corporate Securities and Exotic Options
6. Real Options

1 – Forwards, Futures and Swaps: Contents

Derivatives

- A derivative security is a financial security (asset, claim) whose value is derived from other, more primitive, variables such as:
  - stock prices
  - exchange rates
  - interest rates
  - commodity prices
- Futures and Forwards are agreements where two parties agree to a specified trade at a specified point in the future.
- Swaps are similar to Forwards except that the parties commit to multiple exchanges at different points in time.
- Options are contracts in which two parties agree to a trade in the future, but one party retains in the right to opt out of the trade.

Forward Contracts

A forward contract is an agreement between a buyer and a seller to trade in a specified quantity of a specified good at a specified date in the future at a specified price.

- The buyer is said to have a long position.
- The seller is said to have a short position.
- The specified good is known as the underlying asset.
- The specified date is known as the expiration or maturity date.
- The specified price is known as the delivery price.

Forward Contracts

- The forward contract is negotiated directly by the seller and the buyer.
- Terms of the contract can be "tailored."
- Neither party can walk away unilaterally from the contract after inception.
- There is possible default risk for both parties.
- Forward contracts are used on a variety of underlying assets such as currencies, commodities and interest rates.

Forward Contracts

- Forward contracts allow investors to lock–in a price for the underlying transaction.
- Thus, they facilitate hedging, the reduction of risk in cash flows associated with market commitments.
  - Example: Use of oil futures by Delta Airlines to hedge its fuel cost risk.
- After inception, either position could "lose" (leading to "ex–post regret") depending on price movements.
- A zero–sum game: ex–post, the buyer's gains are the seller's losses, and vice versa.

Forward Price

- By convention, at inception of the contract, the delivery price in a forward contract is chosen so that the contract has zero value to both parties.
- At a given point in time, the forward price for a certain contract is defined as the delivery price which would make the contract have zero value.
- It follows that at the time the contract is entered into, the forward price and the delivery price are equal.
- As time passes, the forward price is liable to change, whereas the delivery price remains fixed through the life of the contract.

Futures Contracts

- A futures contract is similar to a forward contract but is traded on an organized exchange.
- This results in some important differences:
  1. Buyers and sellers deal through the exchange, not directly.
  2. Contract terms are standardized (or *commoditized*).
  3. Either party can reverse its position at any time by closing out its contract.
  4. Default risk is borne by exchange, not by individual parties.
  5. "Margin accounts" are used to reduce default risk.

Standardization of Contract Terms

- Contract terms must be standardized, since buyer and seller do not interact directly.
- This most important task performed by the exchange is essential in promoting liquidity and improving quality of hedge.
- Involves three components:
  1. Quantity (size of contract).
  2. Quality (standard deliverable good).
  3. Delivery arrangements (cash or physical settlement, location of delivery).

Standardization of Contract Terms

Cash vs. Physical Settlement:

- The majority of financial futures do not lead to the actual delivery of the underlying asset, but are cash–settled:
- Instead of the long position receiving the underlying, it just receives its value (net of agreed delivery price) based on a pre-specified reliable price quote on the settlement day.
  - For example, crude oil futures prices on the Ney York Mercantile Exchange (NYMEX) are quoted in dollar per barrel to two decimal places.
- This eliminates the need to actually receive/deliver the underlying.

Reversal of Positions

- Unlike forward contracts, holders of futures contracts can unilaterally reverse (or "close out") their positions.
- Reversal involves taking the opposite position to the original.

Example:

- On 3rd February, an investor took a *long* position in 10 COMEX gold contracts for delivery on 1st December.
  Each COMEX gold contract is for the delivery of 100 troy ounces (1 oz.t. = 31.103477 grams) of 995 fineness gold.
- The long position in the 10 COMEX gold contracts was taken at the futures price of \$ 1900 per troy ounce.

Reversal of Positions

- On 10th November, the investor reverses this position taking a *short* position in 10 gold contracts for delivery on 1st December.
- The price at the time of close–out is up to \$ 1950 per t. oz.
- Then, effectively the investor agrees on 10th November to
  - buy at \$ 1900 per t. oz. on 1st December and
  - sell at \$ 1950 per t. oz. on 1st December

  for a net gain of $10 \times 100 \times (1950 - 1900) = \$\,50\,000$.
- Reversal of position is important because standardization of delivery dates creates "delivery basis risk": delivery dates on the contract (here 1st December) may not match the market commitment dates of the hedger (here 10th November).
- Allowing for reversal allows elimination of part of delivery basis risk.

Hedging with Delivery Basis Risk

- A company that knows it is due to buy (sell) an asset at a particular time in the future can hedge by taking a long (short) futures position.
  - If the price of the asset goes up,
    - the company takes a loss from the purchase (gains from the sale) of the asset,
    - but makes a gain on the long (takes a loss on the short) futures position.
  - If the price of the asset goes down,
    - the company gains from the purchase (takes a loss from the sale) of the asset,
    - but takes a loss on the long (makes a gain on the short) futures position.
- However, the hedge may require the futures contract to be closed out before its expiration date.
- This gives rise to delivery basis risk.

Hedging with Delivery Basis Risk

Example:

- It is 1st March.
- A shipping company knows it will need to purchase 20 000 barrels of crude oil at some uncertain time in October or November.
- Oil futures contracts are traded for delivery every month on NYMEX.
- Each contract is for the delivery of 1 000 barrels.
- The futures price is \$ 81 per barrel.

Hedging with Delivery Basis Risk

- Consider the following strategy.

  On 1st March:
  - Take a long position in 20 December oil futures contracts.

  The day in October or November it needs the 20 000 barrels:
  - Close out the long position by taking a short position in 20 December oil futures.
  - Purchase the 20 000 barrels on the spot market.
- The day the company purchases the oil, since the date (in October or November) will be "close" to the delivery date (December), the spot and the futures price of oil will be close to each other.
- Therefore, the short oil futures positions and the spot purchase of oil will approximately offset each other.
- If they perfectly offset each other, the company would purchase the oil for $20000 \times \$\,81 = \$\,1\,620\,000$.

Hedging with Delivery Basis Risk

- However,
  - the more the company needs the oil early,
  - the more the hedge requires the futures contract to be closed out before its expiration date, and
  - the more the spot purchase of oil and the short oil futures positions do not perfectly offset each other:
- Suppose the shipping company need the 20 000 barrels of oil on 10th November and at that date
  - the spot price is \$ 90 per barrel,
  - the futures price \$ 89 per barrel.
- The effective price paid is $\$\,81 + (\$\,90 - \$\,89) = \$\,82$ per barrel, or $20\,000 \times \$\,82 = \$\,1\,640\,000$ in total.

Margin Accounts

Since buyers and sellers do not interact directly, there is an incentive for either party to default if prices move adversely.

- Default risk can render market illiquid (asymmetric information).
- To inhibit default, futures exchanges use margin accounts.
- This is effectively the posting of collateral against default.
- The level at which margins are set is crucial for liquidity.
  - High levels eliminate default, but inhibit market participation.
  - However, too low levels increase default risk.
- In practice, margin levels are not set very high.

Margin Accounts

The margining procedure:

- Initial margin: amount initially deposited by investor into a margin account.
- Marking–to–market: daily adjustment of the customer's margin account reflect gains/losses from futures price movements over the day.
- Maintenance margin: floor level of margin account.
  - If balance falls below this, customer receives margin call.
  - If the margin call is not met, account is closed out immediately.

Margin Accounts

Example:

- On 1st September investor A takes a long position in 10 December wheat futures contracts on the Chicago Board of Trade (CBOT) at a futures price of \$ 2.60 per bushel.
  - One futures contract on the CBOT is for 5 000 bushels.
  - Therefore, futures price here is \$ 13 000 per contract.
- In these futures contracts,
  - the initial margin is set to \$ 1 000 per contract,
  - the maintenance margin is set at \$ 750 per contract.
- Then, on 1st September, investor A deposits \$ 10 000 into a margin account.

Margin Accounts

- Suppose 2nd September the price goes down to \$ 2.58 per bushel (or \$ 12 900 per contract). Settlement is as follows:
  - Investor A has effectively lost \$ 100 per contract from the price change.
  - This is deducted from A's margin account.
  - A's new margin account balance: \$ 9 000.
- Suppose on 3rd September the goes down again to \$ 2.54/bushel (or \$ 12 700 per contract). Settlement is as follows:
  - Effective loss from price change: \$ 200 per contract.
  - A's new margin account balance: \$ 7 000.
- But this is below the maintenance margin (of \$ 750 per contract), so margin call results.
- If investor meets the margin call, all is well; else the position is closed out.

Futures Price

- The futures price is defined in the same way as the forward price: it is the delivery price that will make the contract have zero value to both parties.
- However, there is an important difference between futures and forward contracts.
  - Unlike a forward contract, a futures contract is marked–to–market *daily*.
  - Therefore, the value of a futures contract is reset to zero every day.

1 – Forwards, Futures and Swaps: Contents

Swaps

- A swap is an agreement between two parties to exchange two streams of same or different assets over regular intervals until a terminal date.
- Specification of the swap contract: most follow the International Swaps and Derivatives Association (ISDA) Master Agreement.
  - Assets to be exchanged.
  - Dates of exchange.
  - Delivery instructions for both parties.
  - Close-out and netting.
  - Events of default and events of termination.
- As in forwards and futures, swap terms are set such that neither party pays the other to enter a swap, i.e., at inception, the swap is a "par swap."
- A swap can be treated as a portfolio of forward contracts.

Interest Rate Swaps

- An interest rate swap is an agreement between two parties A and B, where
  - A pays to B a fixed rate on a notional principal for a number of years.
  - B pays to A a floating rate on the same principal for the same period of time.
- The floating rate is most often the London Interbank Offer Rate (LIBOR).
  - LIBOR is the average rate of interest offered by one major global bank on deposits by another one.
    Exists on five currencies (U.S. dollar, euro, British pound, Japanese yen, and Swiss franc) and on seven different maturities (overnight, one week, and one, two, three, six, and 12 months).
  - Three-month LIBOR is the rate offered on three-month deposits.
  - LIBOR rates are determined by trading between banks and change continuously.
- Suppose the floating rate specified in the swap is the six-month LIBOR.
- Then at each payment date, the floating rate paid is the six-month LIBOR rate which prevailed six-months before the payment date.

Interest Rate Swaps

Example:

- 2-year swap, 5% fixed rate for the six-month LIBOR rate on a principal of $100M.
  - Payments to be exchanged every 6 months and that the 5% per annum rate is with semiannual compounding.
  - At the date the swap is initiated, the six-month LIBOR rate is 4.2%.
- The first exchange of payments will take place 6 months after the initiation of the agreement:
  - A will pay B $2.5M. This is the interest on the $100M principal for 6 months at 5%.
  - B will pay A interest on the $ 100 M principal at the six-month LIBOR rate prevailing at the date the swap is initiated.
    Hence B will pay $(0.042/2) \times \$\,100 = \$\,2.1$ M.

  There is no uncertainty about this first exchange of payments because it is determined by the LIBOR rate at the time the contract is entered into.

Interest Rate Swaps

- The second exchange of payments will take place 1 year after the initiation of the agreement:
  - A will pay again B \$ 2.5 M.
  - B will pay A interest on the \$ 100 M principal at the six-month LIBOR rate prevailing 6 months prior this payment date.
  - Suppose that the six-month LIBOR rate 6 months after initiation of the contract turns out to equal 4.8%.
  - Then B would pay $(0.048/2) \times \$100 = \$2.4$ M to A .

  There is uncertainty about the second and subsequent payments by B because they is determined by the LIBOR rate six months before the payment is made.

Currency Swaps

- A currency rate swap is an agreement between two parties A and B, where
  - A makes a series of payments in one currency to B for a number of years.
  - B makes a series of payments in another currency to A for the same period of time.

  Example:
  - A theme park company wishes to set up a big operation in Japan and hence requires Japanese Yen investments over the next few years.
  - The bond markets in Japan are not very liquid and hence investors are demanding an extra "illiquidity premium."
  - It is cheaper for the theme park company to raise funds in the relatively more liquid US bond and the Eurobond market if required.
  - The company approaches an investment bank and enters into a currency swap where the bank pays it Japanese Yen every period and in exchange the theme park company makes to the bank a Dollar payment each period.

Commodity Swaps

- A commodity swap is an agreement between two parties A and B, where
  - A makes a series of fixed payments to B for a number of years.
  - B makes a series of payments dependent on the price of an underlying commodity to A for the same period of time.

Example: 2–year, semi–annual commodity swap on 100 tonnes of garde #1 copper for $ 6 000 per tonne.

Payoffs to A from the swap:

Date (yr)	0	0.5	1.0	1.5	2.0
Outflow	-	$ 0.6 M	$ 0.6 M	$ 0.6 M	$ 0.6 M
Inflow	-	100 t.	100 t.	100 t.	100 t.

Commodity Swaps

- A commodity swap is usually used to hedge against price swings in the market for a commodity, such as oil and livestock (cattle, hog).

Example:

- A shipping company consumes 100 000 barrels of oil per year.
- The company enters into a commodity swap with an oil producer where
- the shipping company pays $ 4 M each year $t \in \{1, 2, \ldots 10\}$ and
- the oil producer pays in return $100\,000 \times S_t$ to the shipping company, where S_t is the price of oil per barrel at date t.
- This in effect locks-in the shipping company's oil cost at $ 40 a barrel.

Readings

Book:

- Chapters 4, 5, 6 and 7 of Hull.

2 – Options

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Course Road Map

2 – Options: Contents

Recap

This section is a recap of what was taught:

- in the M1 programme
- as part of the PSM - Finance
- in a course entitled Financing and Financial Securities.

Options – Some Definitions

There are *two* main classes of options:

Call Option

Gives the holder the right to purchase an asset (the underlying asset) for a given price (exercise price), on or before a given date.

Put Option

Gives the holder the right to sell an asset for a given price, on or before a give date.

Exercise Features:

- European Options: Gives the owner the right to exercise the option only at the expiration date.
- American Options: Gives the owner the right to exercise the option on or before the expiration date.

Options – Some Definitions

Key elements in defining an option:

- Underlying asset and its price, S.
- Exercise price, also known as strike price, K.
- Expiration date, also known as maturity date, T (today is 0).
- European or American.

Options – Payoff of Options

- The payoff of an option on the expiration date is determined by the price of the underlying asset at that date.

Example:

- Consider a European call option on IBM with exercise price \$100.
- This gives the owner (the buyer) of the right, which crucially is not an obligation, to buy one share of IBM at \$100 on the expiration date.

- Depending on the share price of IBM on the expiration date, the owners best action is clear and the resulting option payoff is as follows:

Options – Payoff of Options

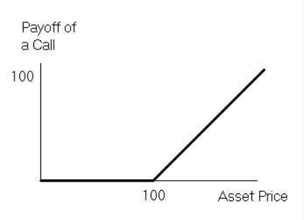
IBM Price, S_T	Action	Payoff at T
...	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
...	Exercise	$S_T - 100$

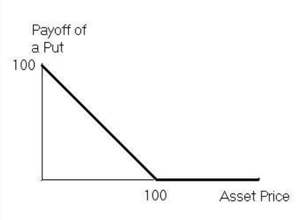
Note:

- The payoff of an option is never negative.
- Sometimes it is positive.
- The actual payoff depends on the price of the underlying asset.

Options – Payoff of Options

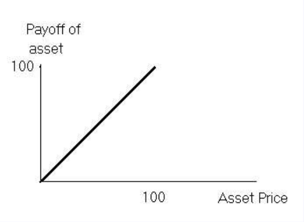
- Payoffs of calls and puts can then be described by plotting
  - their payoff at the expiration date
    as function of
  - the price of the underlying asset at the expiration date.

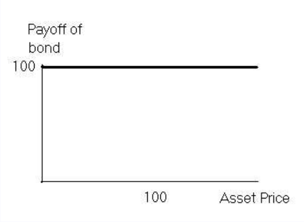




Options – Payoff of Options

- Notice that
  - the underlying asset itself, and
  - a discount bond (with face value $100)
    can also be visualized in terms of payoff diagrams





Options – Payoff of Options

- Using payoff diagrams, we can easily examine the payoff of a portfolio consisting of options and other assets.
- We will first use them to establish a very useful relation between the price of a put and that of a call: the Put-call parity.
- Options can in many situations be used strategically to modify ones exposure to risk, creating portfolios.
- We will secondly visualize this strategic exposure to risk using payoff diagrams:

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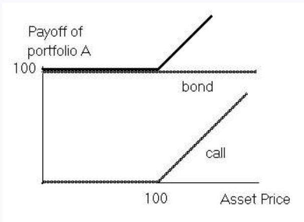
Put-Call Parity

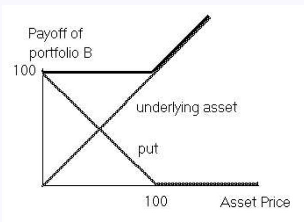
Put-Call parity is a very useful relation between the price of a European put and that of a European call. Consider the following two portfolios A and B:

- **Portfolio A**: Consists of
  - A European call with exercise price $K = \$100$ and
  - a bond with face value \$100 (the exercise price K).
- Portfolio As payoff is:



Put-Call Parity

- **Portfolio B**: Consists of
  - A European put with exercise price $K = \$100$ and
  - a share of the underlying asset.
- Portfolio Bs payoff is:



Put-Call Parity

Consider the following Tables.

Portfolio A	**Investment (date t)**	**Pay-off (date T)**	
		if $S_T \leq K$	if $S_T > K$
1 call option with exercise price K	c_t	0	$S_T - K$
1 bond with face value K	$PV[K]$	K	K
Total	$c_t + PV[K]$	K	S_T

Portfolio B	**Investment (date t)**	**Pay-off (date T)**	
		if $S_T \leq K$	if $S_T > K$
1 put option with exercise price K	p_t	$K - S_T$	0
1 underlying share	S_t	S_T	S_T
Total	$p_t + S_t$	K	S_T

The two portfolios have identical payoffs at the final date T.

Put-Call Parity

- Since the two portfolios have identical payoffs at the future date T, they must have the same value now (at date t).
- The following relation follows:

Put-Call Parity

The price of two European call and put options with same characteristics must satisfy

$$c_t(S_t|T,K) \; + \; K\, e^{-r(T-t)} \; = \; p_t(S_t|T,K) \; + \; S_t.$$

- That is, if you have the price of a call, you immediately can infer that of the put on similar terms, with the put-call parity.

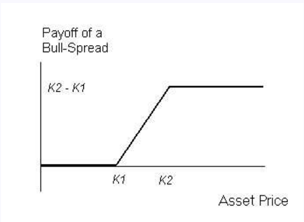
Options Strategies

Strategy 1: Optimism

- Options allow one to speculate on the price evolution of the underlying.
- You may have reasons to be optimistic and believe that the price of a stock will substantially increase from its present level.

Options Strategies

- Creating a "bull-spread" by
  - buying one call at a strike price, K_1 and
  - selling one call at a strike price, K_2, greater than K_1,
- This is initially costly as the call price C_1 is greater than C_2.
- At maturity:



- You will therefore profit if a positive price deviations occurs.
- Similarly a pessimist can create the reverse "bear-spread".

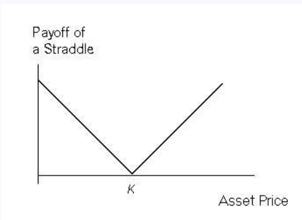
Options Strategies

Strategy 2: Speculating on Volatility

- Options allow one to speculate on the price volatility of the underlying.
- You may believe that as a result of upcoming drastic developments, the price of a stock will deviate from its present level, but do not know which way this development will go.
- This is often the case. Example: General elections, introduction of revolutionary technology to be (or not) adopted, heart-surgery on the firms founder.

Options Strategies

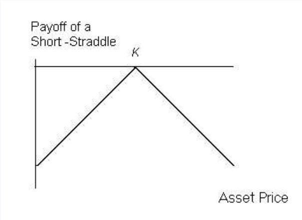
- Creating a "straddle" by
  - buying one call and
  - buying one put with a same strike price, K, close to the underlying asset price, S.
- This is initially costly.
- At maturity:



- You will therefore profit if substantial deviations occurs, whatever way it goes.

Options Strategies

- Conversely, if you anticipate a stable stock price, creating a "short (naked) straddle" by
  - selling one call and
  - selling one put with a same strike price, K, close to the underlying asset price, S.
- This is initially receive cash.
- Now, at maturity:

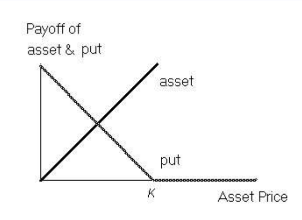


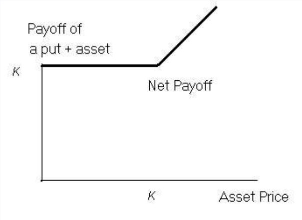
- You will therefore profit if substantial does not deviate much. These are essentially bets on volatility.

Options Strategies

Strategy 3: Hedging Downside while Keeping Upside

- One can hedge against the downside risk of an asset by
  - buying a stock and
  - buying a put
- This is initially costly.
- At maturity:





2 – Options: Contents

Comparative statics

- Let us examine the effect on the price of a stock option of increasing one variable while keeping all others fixed.
- There are six variable that affect option prices: There are six factors which affect the price of an option:
  - The current stock (underlying asset) price, S_t;
  - The strike price, K;
  - The time to expiration, $T - t$;
  - the risk-free interest rate, r;
  - the dividends expected during the life of the option, D;
  - The volatility of the stock price, σ.

Comparative statics

The following table presents a summary of the comparative statics of option prices:

Variable	European Call, c_t	American Call, C_t	European Put, p_t	American Put, P_t
Stock Price, S_t	$+$	$+$	$-$	$-$
Strike Price, K	$-$	$-$	$+$	$+$
Time to Maturity, $T-t$	?	$+$	?	$+$
Risk-free rate, r	$+$	$+$	$-$	$-$
Dividends, D	$-$	$-$	$+$	$+$
Volatility, σ	$+$	$+$	$+$	$+$

We now discuss each variable in turn:

Comparative statics: Stock, Strike Price and Maturity

- Stock and strike price: If exercised at time τ the pay-off of a *call* option is the amount by which the stock price ex- ceeds the strike price, $S_\tau - K$.
- As S_t rises and K falls the expected pay-off of a *call* option increases and with this its current value.
- Time to maturity: A shorter maturity American option is *embedded* in a longer maturity one. The long-life option has all the exercise opportunities open with the short-life option, and more.
- Conversely, with European options, the exercise opportunities of one option are not embedded in the other. In particular, if the stock pays dividends, the short-life option may be worth more than the log-life option.

Comparative statics: Risk-Free Interest Rate

- Risk-free interest rate: At first it might seem that the present value of both
  1. the *benefit* of of exercising (receiving the stock) and
  2. its *cost* (paying the strike price)

  will be lower, so that the overall effect is ambiguous.
- But the present value of a random future sum is simply the amount you would have to pay today, to secure today the ownership of that sum.
- No matter what the interest rate, to secure today the ownership of a random future stock price, you have to pay the current price of the stock, S_t. As we are only changing one variable at one time, the present value of the *benefits* remains constant.

Comparative statics: Risk-Free Interest Rate

- The higher the interest rate, the lower the present value of the strike price the call buyer has contracted to pay pay to exercise, which is the *cost* of exercising.
- Therefore, an increase in interest rate *increases* the price of a call option, and *decreases* the price of a put option.
- Dividends: When dividends are paid-out the stock price falls. Thus, their rise affects option prices *as* a fall in the stock price.

Comparative statics: Stock Price Volatility

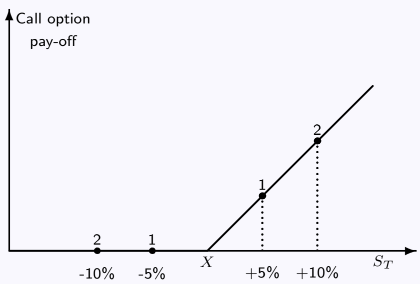
- Volatility: As the volatility of the stock price rises the chance that the stock price will be either very *large* or very *small* rises.
- The owner of *call* option *benefits* from an increase in the stock price but faces *no* downside risk if the stock price decreases (rises).

Illustration: Consider a European call option which is currently at the money ($S_t = K$). Let us consider two levels of volatility:

1. *Low* volatility scenario σ_1: There is a 50% chance the stock price rises by 5% and a 50% chance the stock price falls by 5% before maturity ;
2. *High* volatility scenario σ_2 (where $\sigma_2 > \sigma_1$): There is a 50% chance the stock price rises by 10% and a 50% chance the stock price falls by 10% before maturity.

Point made: *the option is worth more under the high volatility scenario.*

Comparative statics: Stock Price Volatility



- In the low stock price state the return on the option is *zero* (*no* downside risk) in both scenarios (σ_1 and σ_2).
- In the high stock price state the return on the option is larger in the *large* volatility scenario (σ_2).
- As the *expected* return on the option is larger in the *large* volatility scenario (σ_2) the option is worth more.

American Options and Early Exercise

Consider two portfolios:

- Portfolio 1: Consists of
  - A European call with exercise price K.

  The payoff of portfolio 1 at time T is $max[0\,;\, S_T \;-\; K\,]$.

- Portfolio 2: Consists of
  - holding one share and
  - borrowing $K\,e^{-r\,T}$ at the riskless rate until time T.

  The payoff of portfolio 2 at time T is $S_T \;-\; K$.

- Portfolio 1 dominate portfolio 2, hence at time t (today),

$$c(S_t|T,K) \;>\; S_t \;-\; K\,e^{-r\,T}. \tag{1}$$

American Options and Early Exercise

- Now an American options is *at least* as valuable as an European option,

$$C(S_t|T,K) \quad\geq\quad c(S_t|T,K). \tag{2}$$

  This is because exercising an American option *at* maturity pay-offs *at least* as high as its European counterpart.

- From (1) and (2),

$$C(S_t|T,K) \;\geq c(S_t|T,K) \;>\; S_t \;-\; K\,e^{-r\,T} \;>\; S_t \;-\; K. \tag{3}$$

  So the following inequality holds

$$C(S_t|T,K) \;>\; S_t \;-\; K\,.$$

Hence

Theorem

It is never optimal to exercise an American call option on a non-dividend-paying stock early.

American Options and Early Exercise

A similar results does *not* hold for American put options.

Claim

An American put option which is sufficiently deep in the money should be exercised early.

Example: Consider an American put option with strike price \$10
on a stock whose price is virtually zero.

- If exercised the holder makes an immediate gain of \$10.
- If she waits at most she can expect to gain \$10.

American Options and Early Exercise

In synthesis using simple arbitrage arguments we have shown that:

- For no-dividend-paying stocks corresponding American and European *call* options are worth *the same*,
$$c(S_t|T,K) \;=\; C(S_t|T,K).$$
- For no-dividend-paying stocks American *put* options are worth *more* than corresponding European *put* options,
$$p(S_t|T,K) \;\leq\; P(S_t|T,K).$$

Readings

Book:

- Chapters 3, 11 and 12 of Hull.

3 – Option Pricing

Derivative Securities
Masters in Finance
Toulouse Business School

Pierre Mella-Barral

Course Road Map

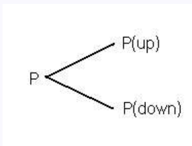
1. Forwards, Futures and Swaps $\surd$
2. Options $\surd$
3. Option Pricing
4. Historical, Implied Volatility and Option Greeks
5. Options in Corporate Securities and Exotic Options
6. Real Options

3 – Option Pricing: Contents

Option Pricing

What is the price of an option? Let us see how option pricing is done considering a European call on a share:

- The simplest possible method to capture price uncertainty is to consider that in one period of time, the price might go either up or down:



We say that the price follows a binomial process.

- The resulting model is known as the Binomial Option Pricing Model.

One Period Option Pricing

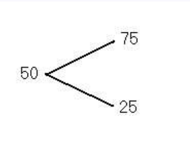
Consider the following example:

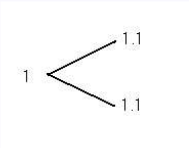
- The current share price is \$50.
  - This price will either
    - go "up" by 50% to \$75
    - or "down" by 50% to \$25.
- One period borrowing and lending rate is 10%.
- The European call we are trying to value is as follows:
  - The maturity of the option is in one period.
  - The strike price is \$50.
  - There are no cash dividends.

One Period Option Pricing

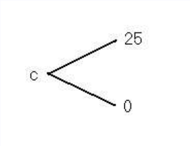
That is:

- The share and treasury bond present two investment opportunities:





- The payoff of the call at expiration (in one period) is known:



Pricing question: What is the value, c, of the call option today?

One Period Option Pricing

Crucial Trick: we can form a **portfolio** of shares and treasury bonds that gives identical payoffs as the call, at maturity.

- Consider a **portfolio** where
  - a is the number of shares of the stock held
  - b is the dollar amount invested in treasury bonds.
- so that (a, b) are such that
  - Payoff of this portfolio in the up state = 25:

$$\$75\, a \; + \; \$1.1\, b \quad = \quad 25$$

  - Payoff of this portfolio in the "down" state = 0:

$$\$25\, a \; + \; \$1.1\, b \quad = \quad 0$$

- Then whatever happens, this **portfolio** will gives identical payoffs as the call.

One Period Option Pricing

- The unique solution to this 2-equations 2-unknown problem is

$$a \;=\; 0.5 \quad \text{and} \quad b \;=\; -11.36\;.$$

- Therefore, in words, if you
  - buy half $(a = 0.5)$ a share of shares and
  - sell $\$1 \times 11.36$ $(b = -11.36)$ worth of bonds

  the payoff of your **portfolio** is identical to that of the call at maturity.
- But then, the present value of the call must equal the current cost of this replicating portfolio, which is:

$$\$50 \times 0.5 \; + \; \$1 \times (-11.36) \quad = \quad \$13.63\;.$$

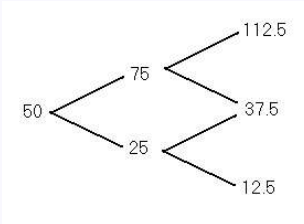
  Hence, the call is worth $c = \$13.63$.

Two Periods Option Pricing

Now we generalize the above example when there are two periods to go: period 1 and period 2.

Furthermore we will consider an American call option.

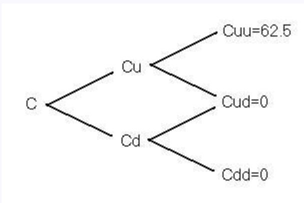
- The current share price is \$50.
  - Price will in each one of the following two periods either
    - go "up" by 50%
    - or "down" by 50%.
- The resulting stock process is:



Two Periods Option Pricing

- The calls payoff at expiration (in two period) is known.

  Denoting C_u and C_d the option value next period, when the stock price goes "up" and goes "down", respectively.
- The call price follows the process



- We then derive the current value of the call working the tree backwards in time: first compute its value next period, and then its current value.

Two Periods Option Pricing

Step A. Start with period 1:

Part 1 – Suppose the stock price goes "up" to \$75 in period 1:

- Construct the replicating portfolio at node ($t = 1$, "up"):

$$\begin{aligned} \$112.5\ a + \$1.1\ b &= \$62.5 \\ \$37.5\ a + \$1.1\ b &= \$0\ . \end{aligned}$$

- There is a unique solution

$$a = 0.833 \quad \text{and} \quad b = -28.4\ .$$

- The cost, in period 1, of this replicating portfolio (a, b) is

$$\$75 \times 0.833 + \$1 \times (-28.4) = \$34.075\ .$$

- Notice that the value of the option if exercised at $t = 1$ is

$$\$75 - \$50 = \$25 < \$34.075$$

- Hence early exercise is not worthwhile and $C_u = \$34.075$.

Two Periods Option Pricing

Part 2 – Suppose the stock price goes "down" to \$25 in period 1:

- Construct the replicating portfolio at node ($t = 1$, "down"):

$$\begin{aligned} \$37.5\ a + \$1.1\ b &= \$0 \\ \$12.5\ a + \$1.1\ b &= \$0\ . \end{aligned}$$

- There is a unique solution

$$a = 0 \quad \text{and} \quad b = 0\ .$$

- The cost, in period 1, of this replicating portfolio (a, b) is 0.
- Notice that the value of the option if exercised at $t = 1$ is

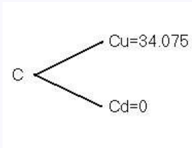
$$\$25 - \$50 = -\$25 < \$0\ .$$

- Hence early exercise is not worthwhile and $C_d = \$0$.

Two Periods Option Pricing

Step B. Now go backwards in time one period to period 0:

- The options value next period is either $C_u = \$34.075$ or $C_d = \$0$ depending upon whether the stock price goes "up" or "down":



- If we construct a portfolio of the stock and bond which replicates the value of the option next period, then the cost of this replicating portfolio must equal the options present value.

Two Periods Option Pricing

- Construct the replicating portfolio at node ($t = 0$):

$$\begin{aligned} \$75\ a \ + \ \$1.1\ b &= \$34.075 \\ \$25\ a \ + \ \$1.1\ b &= \$0\ . \end{aligned}$$

- There is a unique solution

$$a = 0.6815 \quad \text{and} \quad b = -15.48\ .$$

- The cost, in period 0, of this replicating portfolio (a, b) is

$$\$50 \times 0.6815 + \$1 \times (-15.48) = \$18.59\ .$$

- Notice that the value of the option if exercised at $t = 0$ is

$$\$50 - \$50 = \$0 < \$18.59$$

Two Periods Option Pricing

Summary of the replicating strategy: Let us "play forward" the replicating strategy we have just established.

- In period 0: **Step B**. Spend \$18.59 and borrow \$15.48 at 10% interest rate to buy 0.6815 shares of the stock.
- In period 1: **Step A**.
  - Part 1: When the stock price goes "up", the portfolio value becomes \$34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing \$28.4 at 10%. One period hence, the payoff of this portfolio exactly matches that of the call.
  - Part 2: When the stock price goes "down", the portfolio becomes worthless. Close out the position. One period hence, the payoff of this portfolio is zero.

Two Periods Option Pricing

Thus:

- Replicating strategy gives payoff identical to those of the call in every possible circumstance.
- Initial cost of the replicating strategy, \$18.59, must equal the call price, otherwise there would be an arbitrage opportunity.

Note: We have also check throughout for early exercise possibilities. We have seen that this American call option will not be exercised before maturity.

Remarks about the Binomial Model

What we have used to calculate options value:

1. Price of stock now, S_t.
2. Exercise price, X.
3. Time to maturity, T.
4. Stock price volatility, σ.
5. Interest rate, r.

Remarks about the Binomial Model

Most importantly, notice what we have not used:

- Probabilities of "upward" and "downward" movement.
- Characteristics of securities other than (a) the underlying stock and (b) the riskless bond.
- Investors attitude towards risk.

That is:

Investors may disagree on the probabilities of "upward" and "downward" movement and be more or less averse to risk, but they agree on the option price.

Remarks about the Binomial Model

Some possible objections to the the binomial model:

- What is the length of a period?
- Suppose that the length is one day, it seems silly to assume that at the end of a day the price can only take two possible values.
- The market is not open for trading only once a day - trading takes place continuously.

Response:

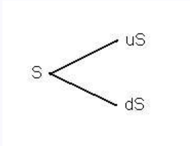
- the length of a period does not have to be a day. It can be anything we wish - an hour, a minute, a second.
- By letting the trading period be very small, we meet both objections.

3 – Option Pricing: Contents

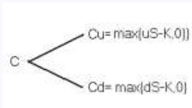
Formal Derivation: One Period Maturity

One period to go, $T = 1$:

- The current share price is S. This price will either
  - go "up" to $u\,S$
  - or "down" to $d\,S$.



- One period borrowing and lending rate is a constant r and let $R \equiv 1 + r$. No arbitrage requires $d < R < u$.
- American call, strike price X, no cash dividends.
  Call payoff at expiration is known:



Formal Derivation: One Period Maturity

- Find the replicating portfolio (a, b) where

$$\begin{aligned} u\,S\,a \;+\; R\,b \;&=\; C_u \\ d\,S\,a \;+\; R\,b \;&=\; C_d\,. \end{aligned}$$

- The unique solution is

$$a \;=\; \frac{C_u \;-\; C_d}{(u-d)\,S} \quad \text{and} \quad b \;=\; \frac{u\,C_d \;-\; d\,C_u}{(u-d)\,R}\,.$$

- The current call price equals the present value of the replicating portfolio

$$\begin{aligned} C(S|T=1,X) \;&=\; S\,a + b \;=\; \frac{C_u \;-\; C_d}{(u-d)} \;+\; \frac{u\,C_d \;-\; d\,C_u}{(u-d)\,R}\,, \\ &=\; \frac{1}{R}\left[\left(\frac{R\;-\;d}{u-d}\right)\,C_u \;+\; \left(\frac{u\;-\;R}{u-d}\right)\,C_d\right]\,. \end{aligned}$$

- Notice it is always greater than $S - X$, the value of early exercise.

Formal Derivation: One Period Maturity

- Define

$$q \equiv \frac{R - d}{u - d} .$$

- Then, $\frac{u-R}{u-d} = 1 - q$ and we can write C as

$$C(S|T = 1, X) = \frac{q\, C_u + (1 - q)\, C_d}{R} .$$

That is:

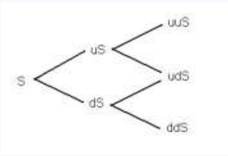
There is a probability q so that the value of the call equals its expected next period payoff discounted at the riskless interest rate.

- Under this probability, q, the expected return on the stock is $R - 1 = r$.
- Conversely, q is the unique probability under which the expected rate of return on the stock is r.
- This probability is called the "risk neutral" probability.

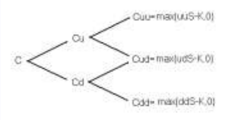
Formal Derivation: Two Period Maturity

Two periods to go, $T = 2$:

- Stock price follows the process:



- Call payoff at expiration is known:



Formal Derivation: Two Period Maturity

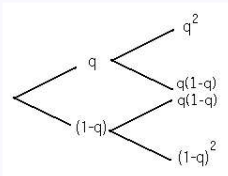
- Recursively solve the "replicating strategy" to get

$$C_u \;=\; \frac{q\,C_{uu} \,+\, (1-q)\;C_{ud}}{R} \quad \text{and} \quad C_d \;=\; \frac{q\,C_{ud} \,+\, (1-q)\;C_{dd}}{R}\;.$$

- Then,

$$\begin{aligned} C(S|T=2,X) \;&=\; \frac{q\,C_u \,+\, (1-q)\;C_u}{R} \\ &=\; \frac{q^2\,C_{uu} \,+\, 2\,q\,(1-q)\;C_{ud} \,+\, (1-q)^2\;C_{dd}}{R^2}\;. \end{aligned}$$

- Here, the tree for "risk neutral" probabilities is



Binomial Option Pricing Formula

- We have just developed a recursive procedure for finding the value at date t of a call whose maturity date is $T = t + n$.
  This, for any number of periods remaining until expiration, n (not just $n = 1, 2$).
- By starting at the expiration date and working backwards, we can write down the general valuation formula

$$C(S_t|T=t+n,K) \;=\; \frac{\sum_{j=0}^{n} prob(j|n)\;C_{u^j d^{n-j}}}{R^n}\;,$$

- where $prob(j|n)$ is the probability of observing j upwards movement during the n periods to expiration
- and $C_{u^j d^{n-j}} \;\equiv\; \max[0, u^j d^{n-j} S_t - K]$ is the option payoff at maturity (date T), if we observe j upwards movement during the n periods to expiration.

- From combinatorics, we have $prob(j|n) = \frac{n!}{j!(n-j)!}\; q^j\,(1-q)^{n-j}$,
  where $x! = x\,(x-1)\,(x-2)\ldots 2$ is the factorial of x.

Binomial Option Pricing Formula

- Consider a particular path the stock price could possibly take until the expiration date in a binomial tree constructed with $n = 5$ periods.
- That is, consider a particular sequence of five moves, say u, d, u, u, d.
- The resulting stock price, which we denote S_T, would then be $S_T = uduudS$.
- So $S_T/S_t = u^3d^2$, and $ln(S_T/S_t) = 3\,ln(u) + 2\,ln(d)$.

- More generally, if the tree is constructed breaking the overall time to maturity in n periods,

$$ln(S_T/S_t) = j\,ln(u) + (n-j)\,ln(d) = j\,ln(u/d) + n\,ln(d)\ ,$$

where j is the (random) number of upwards moves occurring during the n periods to expiration.

Binomial Option Pricing Formula

- Let j^* be the minimum number of upwards moves which the stock must take over the n periods, for the resulting stock price S_T to be greater than X.
- That is, j^* is the smallest integer such that $ln(S_T/S_t) \geq ln(X/S_t)$.
- So j^* is the smallest integer such that

$$j^*\,ln(u/d) + n\,ln(d) \geq ln(X/S_t)\ ,$$

- Hence j^* is the smallest integer such that

$$j^* \geq \frac{ln(X/S_t d^n)}{ln(u/d)}\ .$$

Binomial Option Pricing Formula

- j^* is the minimum number of upwards moves which the stock must take over the n periods, for the call to finish in the money. Then,

$$\begin{array}{lll} \text{for all} \quad j < j^* , & \max[0, u^j d^{n-j} S_t - K] & = 0 , \\ \text{and for all} \quad j \geq j^* , & \max[0, u^j d^{n-j} S_t - K] & = u^j d^{n-j} S_t - K . \end{array}$$

- Then, our prior expression of C,

$$\begin{aligned} C(S_t|T = t+n, K) &= \frac{\sum_{j=0}^{n} prob(j|n) \; C_{u^j d^{n-j}}}{R^n} \\ &= \frac{1}{R^n} \left[\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \; q^j (1-q)^{n-j} \; \max[0, u^j d^{n-j} S_t - K] \right] , \end{aligned}$$

can be written $\quad C(S_t|T = t+n, K) = \dfrac{1}{R^n} \left[\displaystyle\sum_{j=j^*}^{n} \frac{n!}{j!(n-j)!} \; q^j (1-q)^{n-j} \; \left[u^j d^{n-j} S_t - K \right] \right] .$

Binomial Option Pricing Formula

- We can break up this expression of C,

$$C(S_t|T = t+n, K) = \frac{1}{R^n} \left[\sum_{j=j^*}^{n} \frac{n!}{j!(n-j)!} \; q^j (1-q)^{n-j} \; \left[u^j d^{n-j} S_t - K \right] \right] ,$$

in two terms

$$\begin{aligned} C(S_t|T = t+n, K) &= S_t \left[\sum_{j=j^*}^{n} \frac{n!}{j!(n-j)!} \; q^j (1-q)^{n-j} \left(\frac{u^j d^{n-j}}{R^n} \right) \right] \\ &\quad - \frac{K}{R^n} \left[\sum_{j=j^*}^{n} \frac{n!}{j!(n-j)!} \; q^j (1-q)^{n-j} \right] . \end{aligned}$$

Binomial Option Pricing Formula

- Denoting $q' \equiv (u/R)\, q$, we have

$$q^j(1-q)^{n-j}\left(\frac{u^j d^{n-j}}{R^n}\right) = \left[\frac{u}{R}q\right]^j\left[\frac{d}{R}(1-q)\right]^{n-j} = q'^j(1-q')^{n-j}\,.$$

- So we can write C as follows

$$\begin{aligned} C(S_t|T=t+n,K) &= S_t\left[\sum_{j=j^*}^{n}\frac{n!}{j!(n-j)!}\; q'^j(1-q')^{n-j}\right] \\ &\quad - \frac{K}{R^n}\left[\sum_{j=j^*}^{n}\frac{n!}{j!(n-j)!}\; q^j(1-q)^{n-j}\right]\,. \end{aligned}$$

Binomial Option Pricing Formula

- The second bracketed expression gives the probability that there are at least j^* upwards moves in a binomial distribution when the probability of an upward move is q.
  Denote this probability $\Phi[j^*; n, q]$.
- The first bracketed expression gives the probability that there are at least j^* upwards moves in a binomial distribution when the probability of an upward move is q'.
  Denote this probability $\Phi[j^*; n, q']$.

Binomial Option Pricing Formula

- This is the

Binomial Option Pricing Formula

The value of a European call is

$$C(S_t|T = t+n, K) \;=\; S_t\ \Phi[j^*; n, q'] \;-\; \frac{K}{R^n}\ \Phi[j^*; n, q]\ ,$$
$$\text{where}\quad q \;\equiv\; \frac{R-d}{u-d} \qquad \text{and} \qquad q' \;\equiv\; \frac{u}{R}\, q\ ,$$
$$j^* \;\equiv\; \text{smallest non-negative integer} \;>\; \frac{\ln(K/S_t\, d^n)}{\ln(u/d)}\ .$$

- $\Phi[j^*; n, q]$ is the complementary binomial distribution function with probabilities p, $1-q$ and number of trials n, evaluated at j^*.

3 – Option Pricing: Contents

- Option Pricing √
  - One Period √
  - Two Periods √
- Formal Derivation of the Binomial Model √
  - One Period √
  - Risk-Neutral Probability √
  - Two Periods √
  - n Periods √
  - Binomial Model Pricing Formula √
- Limit Argument
  - Black-Scholes Pricing Formula

Limit Argument

- Increasing the number of steps, n, can be interpreted as increasing the frequency of trading.
- As the step size, $h \equiv T/n$, approaches zero, we reach the limit of continuous trading.
- In this limit, prices changes over a finite period can take many values (actually infinite), not just two or several; as in the binomial model.
- Consider again

$$ln(S_T/S_t) \quad = \quad j\, ln(u/d) \;+\; n\, ln(d)\ .$$

- The expected value and variance of $ln(S_T/S_t)$ are

$$\begin{aligned} E\,[\,ln(S_T/S_t)\,] &= E[j]\, ln(u/d) \;+\; n\, ln(d)\ , \\ Var\,[\,ln(S_T/S_t)\,] &= Var[j]\,[ln(u/d)]^2\ . \end{aligned}$$

Limit Argument

- Each of the n possible upward moves has (actual) probability p. Thus, $E[j] = np$.
- Also, since the variance each period is

$$p(1-p)^2 \;+\; (1-p)(0-p)^2 \quad = \quad p(1-p)\ ,$$

then $Var[j] = np(1-p)$.

- Combining with our expressions of the expected value and variance of $ln(S_T/S_t)$ we obtain

$$\begin{aligned} E\,[\,ln(S_T/S_t)\,] &= [\,p\, ln(u/d) \;+\; ln(d)\,]\, n \;\equiv\; \hat{\mu} n\ , \\ Var\,[\,ln(S_T/S_t)\,] &= p(1-p)\,[ln(u/d)]^2\, n \;\equiv\; \hat{\sigma}^2 n\ . \end{aligned}$$

Limit Argument

- Consider that our estimates of the annual mean and variance of the rate of return on the stock price are μ and σ.
- Suppose we set (u, d, p) as the following function of (μ, σ):

$$\begin{aligned} u &= e^{\sigma\sqrt{(T-t)/n}} , \\ d &= e^{-\sigma\sqrt{(T-t)/n}} = 1/u , \\ \text{and} \quad p &= \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{(T-t)/n} . \end{aligned}$$

Limit Argument

- Replacing these expressions in $\hat{\mu}n$ and $\hat{\sigma}^2 n$, we obtain that for any n,

$$\hat{\mu}n = \mu(T-t) , \quad \text{and} \quad \hat{\sigma}^2 n = [\sigma^2 - \mu^2(T-t)/n](T-t) .$$

- So as $n \to +\infty$, $\hat{\sigma}^2 n \to \sigma^2(T-t)$, while $\hat{\mu}n = \mu(T-t)$ for all values of n.
- With this choice of (u, d, p) from (μ, σ):
  - $\lim_{n\to+\infty} \hat{\mu}n$ and $\lim_{n\to+\infty} \hat{\sigma}^2 n$, the mean and variance of the continuously compounded rate of return on the stock price assumed in the binomial model coincides with
  - $\mu(T-t)$ and $\sigma^2(T-t)$, those of the actual stock price.

Black-Scholes Option Pricing Formula

- Then, as $n \to +\infty$ (hence stepsize $h \equiv T/n \to 0$), the binomial option pricing formula converges in the limit to the

Black-Scholes Option Pricing Formula

A European call with exercise price X maturing at T, is worth

$$C(S_t|T,X) = S_t \, \Phi[\, d_1 \,] - X \, e^{-r(T-t)} \, \Phi[\, d_2 \,]$$

$$\text{where} \qquad d_2 = a(X) \equiv \frac{ln(S_t/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \,,$$

$$d_1 \equiv a(X) + \sigma\sqrt{T-t} \,.$$

- $\Phi[x] \equiv \int_{-\infty}^{x} \phi[u] du$, where $\phi[u] = \frac{1}{\sqrt{2\pi}} e^{-(u)^2/2}$, is the cumulative standard normal distribution, i.e. $\Phi[x]$ is the probability that the variable is less than x.

Readings

Book:

- Chapters 13, 14 and 15 of Hull.

Papers:

– Cox, J.C., S.A. Ross, and M. Rubinstein, (1979) "Options Pricing: A Simplified Approach," *Journal of Financial Economics*, Vol. 47, pp. 229-263.

– Black, F., and M. Scholes, (1973) "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, Vol. 81, pp. 637-659.

– Merton, R.C., (1973) "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, Vol. 4, pp. 141-183.

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Problem Set 1

1. A stock price is currently 50. Over each of the next two three-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding.
   (a) What is the value of a six-month European call option with a strike price of 51?
   (b) What is the value of a six-month European put option with a strike price of 51?
   (c) Verify that the European call and European put satisfy put-call parity.
   (d) If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the three.

2. A stock price is currently 80. Over this coming year it is expected to go up by 50% or down by 50%. Over the following year it is expected to go up by 70% or down by 30%. The risk-free interest rate is 10% per annum over these two years.
   (a) Using the binomial model, what is the value of a two years European option to sell with a strike price of 80 ?
   (b) Discuss whether you could use the Black-Scholes model to give a more accurate estimate.
   (c) Would there be a difference in value if the option was American ?

3. A stock price is currently 25. It is known that at the end of two months it will be either 23 or 27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose that S_T is the stock price at the end of the two months. What is the value of a derivative that pays of S_T^2 at that time?

4. Black-Scholes gives the value of a European call with exercise price K maturing at T as

$$
\begin{aligned}
C_t \quad = \quad & S_t \; \Phi \left[\frac{ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right] \\
& \qquad - K \, e^{-r(T-t)} \; \Phi \left[\frac{ln(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right] .
\end{aligned}
$$

   Show that the Black-Scholes formula for a call option, gives a price which tends to $\max[0, S-K]$ as $t \to T$.

5. Read the enclosed article, “Hedge Fund Guru Suffers Wipeout.” Based on this report, answer the following questions.

(a) Draw a picture illustrating the payoff profile of the strategy Niederhoffer executed.

(b) Why did Niederhoffer execute the strategy described above; that is, how did he expect to make money given his view of the price of S&P index?

(c) Why did the strategy lose so much money?

6. Read the enclosed article, "Losses At Barings Grow to $1.24 Billion." Based on the article, answer the following questions.

   (a) Describe the strategy in Nikkei-options that Leeson was executing. In particular, describe the 'strangle.' (Hint: the payoff from a long position in stock-index futures is similar to a long position in stock.)

   (b) Draw a payoff chart to show for what values of the Nikkei the strategy will make a profit and when it will lose money.

Hedge Fund Guru Suffers Wipeout

David Henry, USA Today (This article has been edited for class use.)

NEW YORK – Financial futures traders were buzzing Wednesday over the spectacle and possible consequences of the biggest publicly known casualty of Monday's stock plunge, the hedge funds of Victor Niederhoffer.

Niederhoffer is one of the best-known names in the business and author of the best-selling book *Education of a Speculator*. On Wednesday Niederhoffer told investors in three hedge funds he runs that their stakes had been "wiped out" Monday by losses that culminated from three days of falling stocks and big hits earlier this year in Thailand. The funds had **sold put options** on the Standard & Poor's 500 index, betting the market would go up.

Losses At Barings Grow to $1.24 Billion

British Authorities Begin Sale of Assets;
Missing Trader Had Gone to Singapore To Help Solve Backroom Woes.

Jeremy Mark in Singapore, Michael R. Sesit in London, And Laura Jereski in New York
The Wall Street Journal (This article has been edited for class use.)

In the early 1990s, a pallid, twenty-something settlements specialist from Britain named Nicholas William Leeson joined Baring Securities in Singapore to help unravel some back-room problems.

Within a year, he had joined the Barings trading team on the floor of the Singapore International Monetary Exchange. By day, he executed trades in the Nikkei Stock Average futures contract under the tutelage of Barings traders in Japan. By night, he partied in the yuppie bars along the Singapore River.

Eventually, two former colleagues say, he became chief trader —while continuing to oversee settlement operations for his own trades. ‘Once you win the trust of top management, you are left on your own,’ said a broker who once worked alongside Mr. Leeson. ‘Maybe he was left with too much discretion. When you have someone taking charge of both settlement and trading, it’s just a bomb waiting to explode.’

The bomb exploded over the weekend, shattering a 233-year-old financial institution with losses now estimated at $1.24 billion. As regulators and auditors scrambled to pick up the pieces of Barings PLC and its subsidiaries yesterday, the whereabouts of the 28-year-old trader — last reported staying at the deluxe Regent Hotel in Kuala Lumpur, Malaysia, on Thursday — remained unknown.

Mr. Leeson’s primary job at Baring Securities was to arbitrage Nikkei futures contracts in Singapore and Osaka, making relatively small amounts of money on the differences in the prices of two similar contracts. In recent months, Mr. Leeson began taking unauthorized long positions, exposing Barings to big losses if the Tokyo market fell. For awhile, he was able to fully hedge his positions by taking offsetting contracts in the Singapore and Osaka markets.

But Mr. Leeson ran into problems when the Tokyo market began sliding after the Jan. 17 earthquake in Kobe, Japan. Rather than cut his losses, Mr. Leeson compounded his risk by adding to his positions.

Stunned money men were left wondering how a trader could have amassed such huge losses, apparently without drawing the attention of his own firm and Singapore’s regulators. That question is particularly puzzling because the markets had known for weeks that Barings was building up huge obligations in the Nikkei average stock-index futures contract, leaving its mounting positions open day after day. Stock-index futures allow traders to bet on the level at which a market index will stand at a specified date; while often used for hedging risks, they also provide a potent tool for speculation.

‘I find it inconceivable that Barings was unaware of the situation,’ said a former Barings executive who once was a senior futures trader in Japan. ’Japanese houses have been out there [in the futures market] telling clients to go short for weeks because of Barings’ long positions.’

Mr. Leeson’s strategy involved not only buying stock-index futures but selling options on the Nikkei-225 index. Together, these trades obligated Barings to sell the Nikkei-225 index when it neared 20,000 and to buy it when it fell close to 18,000, positions that would generate huge losses if the market moved sharply in either direction.

This so-called ‘strangle’ strategy fell apart when the Japanese stock market began tumbling after the Kobe earthquake. Instead of cutting his losses by closing out positions as the market declined, Mr. Leeson appears to have compounded them by doubling up on his positions. From its 19331.17 level on Jan. 16, the day before the Kobe earthquake, the Nikkei index has plunged 13% to a 16808.70 close in Tokyo last night – representing staggering losses to Barings.

‘What Barings did was buy futures, increasing their liability. Once the market moves outside those [18,000 and 20,000] levels, you begin to lose money fast,’ says a trader who adds that his firm has been ’tracking Barings’ for several weeks.

By the time Barings collapsed, Mr. Leeson had accumulated Nikkei-225 index positions that effectively amounted to a $7 billion ‘long’ bet on the Tokyo stock market, according to Eddie George, governor of the Bank of England.

‘Basically every 1% the market went down, they [Barings] were losing another $70 million,’ says another trader.

At their height, Mr. Leeson’s trades accounted for about half of the open interest, or outstanding positions, in the Nikkei-225 futures, according to a dealer whose firm clears trades on both Singapore and Osaka exchanges. That ‘should have set off a lot of bells, but didn’t,’ he says of exchange regulators and Barings management.

Meanwhile, as the losses accelerated, Barings’ London executives and international traders alike continued to think Mr. Leeson was acting on behalf of clients. First they thought it was big commodity traders Tudor Investment Corp. But after that firm made it known it wasn’t involved, they turned their suspicions to Ross Capital, a Bermuda-based hedge fund. The firms couldn’t be reached for comment.

‘No one in the market could conceive that these were Barings positions,’ says a London trader. ‘We assumed it could only be big hedge funds or aggressive central banks like [Malaysia’s Bank] Negara and the Monetary Authority of Singapore’ and that Barings was acting on their behalf, he says.

As the losses mounted, Mr. Leeson apparently realized that his situation had become untenable. On Thursday, he left Singapore with his wife. On Friday, a Barings executive said, Mr. Leeson sent a fax to the firm announcing his resignation.

Glenn Whitney in London also contributed to this article.

Solutions to Problem Set 1

1. (a) A tree describing the behavior of the stock price is shown in the diagram below

$$\begin{tikzcd}
 & & 56.18 \\
 & 53 \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
50 \arrow[ur, "q"] \arrow[dr, "1-q"'] & & 50.35 \\
 & 47.5 \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
 & & 45.12
\end{tikzcd}$$

The risk-neutral probability of an up move, q is given by

$$q \quad = \quad \frac{e^{0.05\times 0.25} \ - \ 0.95}{1.06 \ - \ 0.95} \ = \ 0.569 \ .$$

The payoffs from the six-month European call with a strike price of 51 are

$$\begin{tikzcd}
 & & C_{uu} \ = \ \max[0, u^2S - K] \ = \ 5.18 \\
 & C_u \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
C \arrow[ur, "q"] \arrow[dr, "1-q"'] & & C_{ud} \ = \ \max[0, udS - K] \ = \ 0 \\
 & C_d \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
 & & C_{dd} \ = \ \max[0, d^2S - K] \ = \ 0
\end{tikzcd}$$

The value of the option is therefore

$$5.18 \ \times \ 0.569^2 \ \times \ e^{-0.05\times 0.5} \quad = \quad 1.635 \ .$$

(b) The payoffs from the six-month European put with a strike price of 51 are

$$\begin{tikzcd}
 & & P_{uu} \ = \ \max[0, K - u^2S] \ = \ 0 \\
 & P_u \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
P \arrow[ur, "q"] \arrow[dr, "1-q"'] & & P_{ud} \ = \ \max[0, K - udS] \ = \ 0.65 \\
 & P_d \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
 & & P_{dd} \ = \ \max[0, K - d^2S] \ = \ 5.875
\end{tikzcd}$$

The value of the option is therefore

$$\left[0.65 \times 2 \times 0.569 \times 0.431 + 5.875 \times 0.431^2\right] e^{-0.05\times 0.5} = 1.376\ .$$

(c) The value of the put plus the stock price is

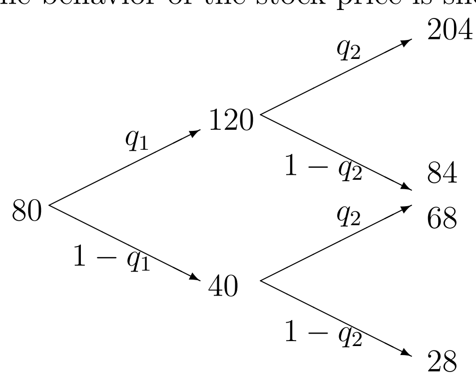
$$1.376 + 50 = 51.376\ .$$

The value of the call plus the present value of the strike price is

$$1.635 + 51\, e^{-0.05\times 0.5} = 51.376\ .$$

(d) To test whether it is worth exercising the option early we compare the value calculated for the option at each node with the payoff from early exercise. At the node P_d the payoff from immediate exercise $K - dS = 51 - 47.5 = 3.5$. Since this is greater than 2.8664 the option should be exercised at this node. The option should not be exercised at either node P or node P_u.

2. (a) A tree describing the behavior of the stock price is shown in the diagram below



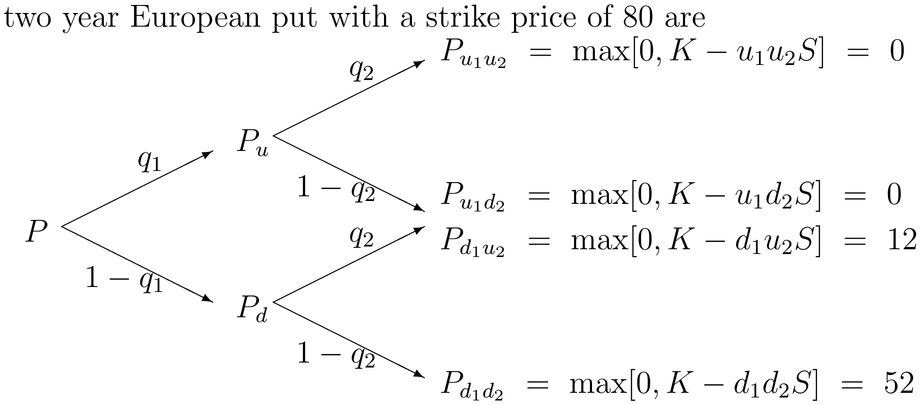
The risk-neutral probability of an up move *in the first year*, q_1 is given by

$$q_1 = \frac{1.1 - 0.5}{1.5 - 0.5} = 0.6\ .$$

The risk-neutral probability of an up move *in the second year*, q_2 is given by

$$q_2 = \frac{1.1 - 0.7}{1.7 - 0.7} = 0.4\ .$$

The payoffs from the two year European put with a strike price of 80 are



The value of the European put option is therefore

$$[12 \times 0.4 \times 0.4 + 52 \times 0.4 \times 0.6] \; / \; (1.1)^2 \quad = \quad 11.90 \; .$$

(b) Given that the pair $(u_1; d_1)$ differs from the pair $(u_2; d_2)$, we expect the volatility in the first year to differ from that of the second year. Then, Black-Scholes simply cannot be used because it assumes a constant volatility throughout the life of the option.

However, calculate the mean of returns for one period

$$\begin{aligned}
\bar{r}_t &= E\left[\frac{S_{t+1}}{S_t}\right] , \\
&= q\,u \; + \; (1-q)\,d \\
&= \left(\frac{R-d}{u-d}\right) u \; + \; \left(\frac{u-R}{u-d}\right) d \\
\bar{r}_t &= R \; .
\end{aligned}$$

Then calculate the volatility of returns for one period

$$\begin{aligned}
Var_t &= E\left[\left(\frac{S_{t+1}}{S_t} - E\left[\frac{S_{t+1}}{S_t}\right]\right)^2\right] , \\
&= q\,(u-\bar{r}_t)^2 \; + \; (1-q)\,(\bar{r}_t - d)^2 \\
&= \left(\frac{R-d}{u-d}\right)(u-R)^2 \; + \; \left(\frac{u-R}{u-d}\right)(R-d)^2 \\
&= (R-d)\,(u-R)\; \frac{(u-R) \; + \; (R-d)}{u-d} \\
Var_t &= (R-d)\,(u-R) \; .
\end{aligned}$$

In our case, in the first year,

$$Var_1 \quad = \quad (1.1 \; - \; 0.5) \; \times \; (1.5 \; - \; 1.1) \; = \; 0.6 \; \times \; 0.4 \; ,$$

and in the second year,

$$Var_2 \quad = \quad (1.1 \; - \; 0.7) \; \times \; (1.7 \; - \; 1.1) \; = \; 0.4 \; \times \; 0.6 \; .$$

So the variance turns out to be the same in the first and second year (just because of the numbers). So you can use Black-Scholes.

The correct answer consists of showing that the variance is the same in both years

$$Var_1 \quad = \quad Var_2 \; ,$$

hence showing that using Black-Scholes is not "simply wrong", although it does not bring any improvement.

(c) The value of a two year American put with a strike price of 80 is different if it is worth exercising the option early. We compare the value calculated for the option at each node with the payoff from early exercise. At the node P_d the payoff from immediate exercise $K - dS = 80 - 40 = 40$. This should be compared from its continuation value which is

$$[12 \times 0.4 + 52 \times 0.6] \;/\; 1.1 = 32.73 \; .$$

At this node, the option should therefore be exercised. The option should not be exercised at either node P or node P_u. The value of the American option is therefore

$$[40 \times 0.4] \;/\; 1.1 = 14.55 \; .$$

3. At the end of the two months the value of the power option will be either 729 (if the stock price is 27) or 529 (if the stock price is 23). Consider a portfolio consisting of

$$\begin{array}{rcl} +\Delta & : & \text{shares} \\ -1 & : & \text{power option} \; . \end{array}$$

The value of the portfolio is either $27\Delta - 729$ (if the stock price is 27) or $23\Delta - 529$ (if the stock price is 23) in two months.

$$\begin{array}{rrcl} \text{If} & 27\,\Delta - 729 & = & 23\,\Delta - 529 \; , \\ \text{i.e.,} & \Delta & = & 50 \; , \end{array}$$

the value of the portfolio is certain to be 621. For this value of Δ the portfolio is therefore riskless. The present value of the portfolio is

$$50 \times 25 - f$$

where f denotes the value of the power option. Since the portfolio must earn a risk-free rate of interest,

$$\begin{array}{rcl} [50 \times 25 - f] \; e^{0.10 \times 0.1667} & = & 621 \\ \text{i.e.,} \qquad f & = & 639.3 \; . \end{array}$$

Instead of constructing the replicating portfolio, the value of the power option can also be directly obtained calculating

$$u = \frac{uS}{S} = \frac{27}{25} = 1.08 \; , \qquad d = \frac{dS}{S} = \frac{23}{25} = 0.92 \; ,$$

so that

$$q = \frac{e^{0.10 \times 0.1667} - 0.92}{1.08 - 0.92} = 0.605 \; .$$

Then,

$$f = [0.605 \times 729 + 0.395 \times 529] \; e^{-0.10 \times 0.1667} = 639.3 \; .$$

4. Consider the term in the first cumulative normal, which is often denoted d_1:

$$d_1 \equiv \frac{ln(S_t/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{ln(S_t/X)}{\sigma\sqrt{T-t}} + \frac{(r+\sigma^2/2)}{\sigma}\sqrt{T-t} \; .$$

As $t \to T$, the second term on the right hand side tends to zero. The first term tends to $+\infty$ if $ln(S_t/X) > 0$, hence when $S > X$. Conversely this first term tends to $-\infty$ if $ln(S_t/X) < 0$, hence when $S > X$. Overall,

$$\begin{aligned} & d_1 \to +\infty \text{ as } t \to T \quad \text{when } S > X \ , \\ \text{and} \quad & d_1 \to -\infty \text{ as } t \to T \quad \text{when } S < X \ . \end{aligned}$$

Similarly consider the term in the second cumulative normal, which is often denoted d_2:

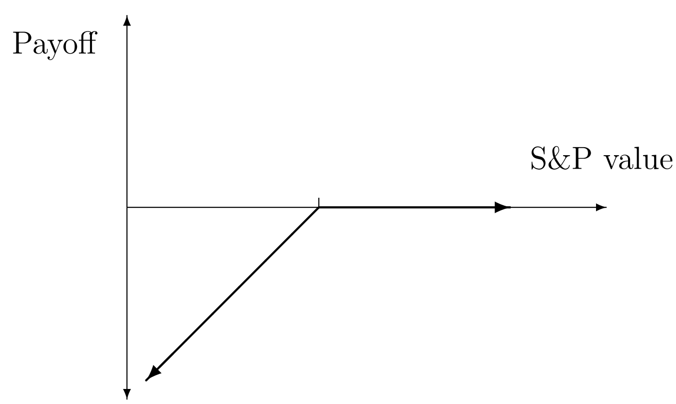
$$d_2 \equiv \frac{ln(S_t/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{ln(S_t/X)}{\sigma\sqrt{T-t}} + \frac{(r-\sigma^2/2)}{\sigma}\sqrt{T-t} \ .$$

With a similar reasoning we obtain

$$\begin{aligned} & d_2 \to +\infty \text{ as } t \to T \quad \text{when } S > X \ , \\ \text{and} \quad & d_2 \to -\infty \text{ as } t \to T \quad \text{when } S < X \ . \end{aligned}$$

From the above results, when $S > X$, the terms $\Phi[d_1] \to 1$ and $\Phi[d_2] \to 1$ as $t \to T$, so that $C \to S - X$. Also, when $S < X$, the terms $\Phi[d_1] \to 0$ and $\Phi[d_2] \to 0$ as $t \to T$, so that $C \to 0$. These results show that $C \to \max[0, S - X]$ as $t \to T$.

5. (a) The report describes Niederhoffer's position as one where put options were sold.



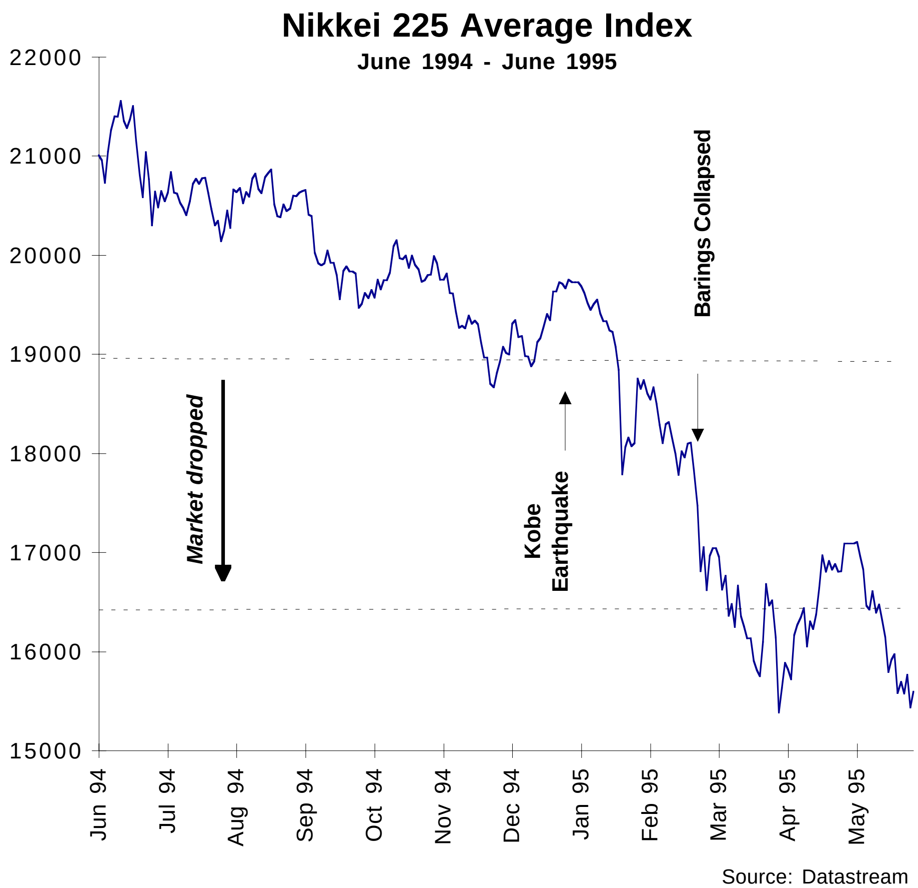
(b) Niederhoffer's view was that the value of the S&P would increase. Thus, he planned to make money by selling put options (and receiving the option premium from this sale), with the expectation that the payoff on the options would be zero, and hence, his future obligation would be zero.

(c) Niederhoffer lost a lot of money because he took a big speculative position and his view turned out to be wrong.

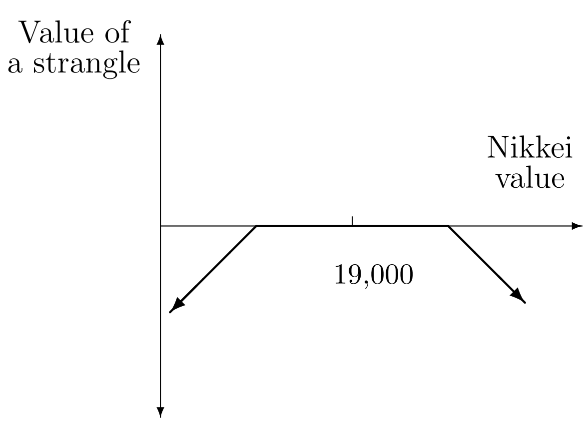
6. (a) The option strategy described in the article is one where, "these trades obligated Barings to sell the Nikkei-225 index when it neared 20,000 and to buy it when it fell close to 18,000, positions that would generate huge losses if the

market moved sharply in either direction." This position is a "short strangle," which entails selling calls and puts at different strikes. (This is in contrast to a straddle where the calls and puts are at the same strike.)

The reason for executing a short strangle position is to generate income from the sale of the options, with the expectation that the underlying (in this case, Nikkei) will not move sharply. As the figure below shows, this is not what happened.



(b) The payoff profile of a 'short strangle' is given below. From the figure, we see that losses will mount as the Nikkei index moves outside the range 18,000–20,000.



Robert Merton and Myron Scholes

Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel
October 14, 1997
(This article has been edited for class use.)

The Royal Swedish Academy of Sciences has decided to award the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel, 1997, to Professor Robert C. Merton, Harvard University, Cambridge, USA and Professor Myron S. Scholes, Stanford University, Stanford, USA for a new method to determine the value of derivatives.

Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.

In a modern market economy it is essential that firms and households are able to select an appropriate level of risk in their transactions. This takes place on financial markets which redistribute risks towards those agents who are willing and able to assume them. Markets for options and other so-called derivatives are important in the sense that agents who anticipate future revenues or payments can ensure a profit above a certain level or insure themselves against a loss above a certain level. (Due to their design, options allow for hedging against one-sided risk - options give the right, but not the obligation, to buy or sell a certain security in the future at a prespecified price.) A prerequisite for efficient management of risk, however, is that such instruments are correctly valued, or priced. A new method to determine the value of derivatives stands out among the foremost contributions to economic sciences over the last 25 years.

This year's laureates, Robert Merton and Myron Scholes, developed this method in close collaboration with Fischer Black, who died in his mid-fifties in 1995. These three scholars worked on the same problem: option valuation. In 1973, Black and Scholes published what has come to be known as the Black-Scholes formula. Thousands of traders and investors now use this formula every day to value stock options in markets throughout the world. Robert Merton devised another method to derive the formula that turned out to have very wide applicability; he also generalized the formula in many directions.

Black, Merton and Scholes thus laid the foundation for the rapid growth of markets for derivatives in the last ten years. Their method has more general applicability, however, and has created new areas of research - inside as well as outside of financial economics. A similar method may be used to value insurance contracts and guarantees, or the flexibility of physical investment projects.

The problem: Attempts to value derivatives have a long history. As far back as 1900, the French mathematician Louis Bachelier reported one of the earliest attempts in his doctoral dissertation, although the formula he derived was flawed in several ways. Subsequent researchers handled the movements of stock prices and interest rates more successfully. But all of these attempts suffered from the same fundamental shortcoming: risk premia were not dealt with in a correct way.

The value of an option to buy or sell a share depends on the uncertain development of the share price to the date of maturity. It is therefore natural to suppose - as did earlier researchers - that valuation of an option requires taking a stance on which risk premium to use, in the same way as one has to determine which risk premium to use when calculating present values in the evaluation of a future physical investment project with uncertain returns. Assigning a risk premium is difficult, however, in that the correct risk premium depends on the investor's attitude towards risk. Whereas the attitude towards risk can be strictly defined in theory, it is hard or impossible to observe in reality.

The method: Black, Merton and Scholes made a vital contribution by showing that it is in fact not necessary to use any risk premium when valuing an option. This does not mean that the risk premium disappears; instead it is already included in the stock price.

The idea behind their valuation method can be illustrated as follows:

Consider a so-called European call option that gives the right to buy one share in a certain firm at a strike price of \$ 50, three months from now. The value of this option obviously depends not only on the strike price, but also on today's stock price: the higher the stock price today, the greater the probability that it will exceed \$ 50 in three months, in which case it pays to exercise the option. As a simple example, let us assume that if the stock price goes up by \$ 2 today, the option goes up by \$ 1. Assume also that an investor owns a number of shares in the firm in question and wants to lower the risk of changes in the stock price. He can actually eliminate that risk completely, by selling (writing) two options for every share that he owns. Since the portfolio thus created is risk-free, the capital he has invested must pay exactly the same return as the risk-free market interest rate on a three-month treasury bill. If this were not the case, arbitrage trading would begin to eliminate the possibility of making a risk-free profit. As the time to maturity approaches, however, and the stock price changes, the relation between the option price and the share price also changes. Therefore, to maintain a risk-free option-stock portfolio, the investor has to make gradual changes in its composition.

One can use this argument, along with some technical assumptions, to write down a partial differential equation. The solution to this equation is precisely the Black-Scholes' formula. Valuation of other derivative securities proceeds along similar lines.

The Black-Scholes formula: Black and Scholes' formula for a European call option can be written as

$$C \quad = \quad S\,N(d) \quad - \quad L\,e^{-rt}\,N(d - \sigma\sqrt{t})$$

where the variable d is defined by

$$d \quad = \quad \frac{\ln(S/L) + (r + \sigma^2/2)t}{\sigma\sqrt{t}} \; .$$

According to this formula, the value of the call option C, is given by the difference between the expected share value - the first term on the right-hand side - and the expected cost - the second term - if the option right is exercised at maturity. The formula says that the option value is higher the higher the share price today S, the higher the volatility of the share price (measured by its standard deviation) sigma, the higher the risk-free interest rate r, the longer the time to maturity t, the lower the strike price L, and the higher the probability that the option will be exercised (the probability is evaluated by the normal distribution function N).

Other applications: Black, Merton and Scholes' method has become indispensable in the analysis of many economic problems. Derivative securities constitute a special case of so-called contingent claims and the valuation method can often be used for this wider class of contracts. The value of the stock, preferred shares, loans, and other debt instruments in a firm depends on the overall value of the firm in essentially the same way as the value of a stock option depends on the price of the underlying stock. The laureates had already observed this in their articles published in 1973, thereby laying the foundation for a unified theory of the valuation of corporate liabilities.

A guarantee gives the right, but not the obligation, to exploit it under certain circumstances. Anyone who buys or is given a guarantee thus holds a kind of option. The same is true of an insurance contract. The method developed by this year's laureates can therefore be used to value guarantees and insurance contracts. One can thus view insurance companies and the option market as competitors.

Investment decisions constitute another application. Many investments in equipment can be designed to allow more or less flexibility in their utilization. Examples include the ease with which one can close down and reopen production (in a mine, for instance, if the metal price is low) or the ease with which one can switch between different sources of energy (if, for instance, the relative price of oil and electricity changes). Flexibility can be viewed as an option. To choose the best investment, it is therefore essential to value flexibility in a correct way. The Black-Merton-Scholes' methodology has made this feasible in many cases.

Banks and investment banks regularly use the laureates' methodology to value new financial instruments and to offer instruments tailored to their customers' specific risks. At the same time such institutions can reduce their own risk exposure in financial markets.

4 – Historical, Implied Volatility and Option Greeks

Derivative Securities
Masters in Finance
Toulouse Business School

Pierre Mella-Barral

Course Road Map

1. Forwards, Futures and Swaps $\surd$
2. Options $\surd$
3. Option Pricing $\surd$
4. Historical, Implied Volatility and Option Greeks
5. Options in Corporate Securities and Exotic Options
6. Real Options

4 – Historical, Implied Volatility and Option Greeks: Contents

- Implementing a Pricing Model
  - Difficulty with Volatility
- Historical Volatility
- Implied Volatility
- Option Greeks
  - Delta, Delta Neutral Position
  - Gamma, Target Position Gamma
  - Theta, Position Theta
  - Vega
  - Rho

Implementing a Pricing Model

- To illustrate our discussion of the implementation of a pricing formula, we will consider an example.
- Suppose the following data is available concerning the European Call we are trying to price.

  Today's stock price: \$40.75; Todays date: January 2^{nd}
  The strike price: $X = \$40$ Expiration date: May 16^{th}
  Interest rate on a 3-month Treasury bill: 3.81% per year.

Implementing a Pricing Model

To implement the Black-Scholes option pricing formula or a binomial model built in a computer programme, the following inputs are necessary:

1. The stock price S_t, which is \$ 40.750.
2. The strike price X, which is \$ 40.
3. The time to expiration.
   To do this, we need to count the number of calendar days between today's date and the expiration date. Here, there are 133 days between today's date, January 2^{nd}, and the expiration date, May 16^{th}.
   Here the time to expiration is $T - t = 133/365$ years.
4. The safe interest rate, which is 0.0381.
5. The volatility of the underlying stock price, σ.

Difficulty with Volatility

- The volatility of the underlying stock price, σ is the one parameter entering the pricing formula which cannot be observed.
- Nevertheless, this parameter affects very substantially the price of the option one is trying to calculate.
- Overall, volatility is difficult to assess, but it is important to assess it accurately.

  There are two ways to assess the input parameter σ:
- One approach consists of estimating the historical volatility from past observations of the stock price.
- The alternative consists of calculating the implied volatility.

Historical Volatility

- The historical volatility is an estimate of the volatility from past observations of the stock price (historical data). Suppose past weekly closing prices are

7/3/19	26.375	7/11/19	27.125
7/18/19	28.875	7/25/19	29.625
8/1/19	32.250	8/8/19	35.000
8/15/19	36.000	8/22/19	38.625
8/29/19	38.250	9/5/19	40.250
9/12/19	36.250	9/19/19	41.500
9/26/19	38.250	10/3/19	41.125
10/10/19	42.250	10/17/19	41.500
10/24/19	39.250	10/31/19	37.500
11/7/19	37.750	11/14/19	42.000
11/21/19	44.000	11/28/19	49.750
12/5/19	42.750	12/12/19	42.000
12/19/19	38.625	12/26/19	41.000
1/2/20	40.750		

Historical Volatility

- The binomial model is based on the assumption that, if at the beginning of a period the stock price is $S_t = S$, it will be either $S_{t+1} = uS$ or $S_{t+1} = dS$ at the end of the period.
- It therefore assumes that, over any period (here, a week), the price relative (final stock price divided by initial stock price) can take two values: $S_{t+1}/S_t = u$ and $S_{t+1}/S_t = d$.
- The Black-Scholes formula is therefore based on the assumption that the logarithm of the price relative has normal distribution, with mean and variance proportional to the length of the period (stock prices are lognormally distributed).

Historical Volatility

- For example, the first four prices are 26.375, 27.125, 28.875 and 29.625.
- So the first three price relatives are $27.125/26.375 = 1.028436$, 1.064516, and 1.025974.
- The natural logs are therefore respectively $ln[1.028436] = 0.028039$, 0.062520, and 0.025642.
- If the basic assumption underlying the formula is correct, then these are three independent samples from a normal distribution whose standard deviation is the weekly volatility of the stock.
- Using the rest of our data, we have twenty-six independent samples from the distribution:

Date j	S_j	$R_j \equiv S_j/S_{j-1}$	$ln[R_j]$
7/3/19	26.375		
7/11/19	27.125	1.028436	0.028039
7/18/19	28.875	1.064516	0.062520
7/25/19	29.625	1.025974	0.025642
8/1/19	32.250	1.088608	0.084899
8/8/19	35.000	1.085271	0.081830
8/15/19	36.000	1.028571	0.028171
8/22/19	38.625	1.072917	0.070381
8/29/19	38.250	0.990291	-0.009756
9/5/19	40.250	1.052288	0.050966
9/12/19	36.250	0.900621	-0.104671
9/19/19	41.500	1.144828	0.135254
9/26/19	38.250	0.921687	-0.081550
10/3/19	41.125	1.075163	0.072473
...	...	...	...
12/12/19	42.000	0.982456	-0.017700
12/19/19	38.625	0.919643	-0.083770
12/26/19	41.000	1.061489	0.059672
1/2/20	40.750	0.993902	-0.06117

Historical Volatility

- Hence, we can apply standard statistical techniques for estimating the parameters of a normal distribution with unknown mean μ and variance σ^2.
- In our example, the sample mean and variance are

$$\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{j=1}^{n} ln[R_j] \\
&= \frac{0.435039}{26} = 0.016732 \ , \quad \text{and} \\
\hat{\sigma}^2 &= \frac{1}{n}\sum_{j=1}^{n} (ln[R_j] - \hat{\mu})^2 \\
&= \frac{0.130407}{26} = 0.005011 \ .
\end{aligned}$$

where n is the number of price relatives in the series, which is 26.

Date j	S_j	$R_j \equiv S_j/S_{j-1}$	$ln[R_j]$	$(ln[R_j] - \hat{\mu})^2$
7/3/19	26.375			
7/11/19	27.125	1.028436	0.028039	0.000128
7/18/19	28.875	1.064516	0.062520	0.002097
7/25/19	29.625	1.025974	0.025642	0.000079
8/1/19	32.250	1.088608	0.084899	0.004647
8/8/19	35.000	1.085271	0.081830	0.004238
8/15/19	36.000	1.028571	0.028171	0.000131
8/22/19	38.625	1.072917	0.070381	0.002878
8/29/19	38.250	0.990291	-0.009756	0.000702
9/5/19	40.250	1.052288	0.050966	0.001172
9/12/19	36.250	0.900621	-0.104671	0.014739
9/19/19	41.500	1.144828	0.135254	0.014048
9/26/19	38.250	0.921687	-0.081550	0.009659
10/3/19	41.125	1.075163	0.072473	0.003107
...	...	...	...	...
12/12/19	42.000	0.982456	-0.017700	0.001186
12/19/19	38.625	0.919643	-0.083770	0.010100
12/26/19	41.000	1.061489	0.059672	0.001844
1/2/20	40.750	0.993902	-0.06117	0.000522
Sum			0.435039	0.130407

Historical Volatility

- Our estimators of the standard deviation and variance have desirable properties except for one feature - they are biased.
- This bias is however easily corrected for: We have to multiply our original (biased) estimators by a correction factor which depends on the sample size.
- The correction factor for the variance is $n/n-1$.

Historical Volatility

- So our unbiased estimator, $\hat{\sigma}^2_{ub}$, of the variance is

$$\begin{aligned}\hat{\sigma}^2_{ub} &= \frac{1}{n-1}\sum_{j=1}^{n}(ln[R_j]-\hat{\mu})^2\\ &= \left(\frac{26}{25}\right)\frac{0.130407}{26} = 0.005216 .\end{aligned}$$

Given that this is weekly data, our unbiased estimate of the annual variance is

$$\hat{\sigma}^2_{ub} = 52 \times 0.005216 = 0.271232 ,$$

and our estimate of the annual volatility (standard deviation) is therefore

$$\hat{\sigma}_{ub} = \sqrt{0.271232} = 0.520799 .$$

Implementing the Black-Scholes Option Pricing Formula

- Recall the Black-Scholes option pricing formula:

$$C(S_t|T,X) \;=\; S_t\ \Phi\,[\,d_1\,] - \ X\,e^{-r(T-t)}\ \Phi\,[\,d_2\,]$$

$$\text{where} \qquad d_2 \;\equiv\; \frac{ln(S_t/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\ ,$$

$$d_1 \;\equiv\; d_2 + \sigma\sqrt{T-t}\ ,$$

and $\Phi[x]$ is the cumulative standard normal distribution, defined as $\Phi[x] = \int_{-\infty}^{x} \phi[u]du$, where $\phi[x] = \frac{1}{\sqrt{2\pi}} e^{-(x)^2/2}$ is the standard normal density.

Implementing the Black-Scholes Option Pricing Formula

With a figure for volatility (here $\sigma = 0.52$), it is easy to implement the Black-Scholes option pricing formula:

- First, calculate d_2:

$$\begin{aligned} d_2 &= \frac{ln(S_t/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\ , \\ &= \frac{ln(40.75/40) + (0.0381 - (0.52)^2/2)(133/365)}{0.52\sqrt{133/365}}\ , \\ &= -0.05354\ . \end{aligned}$$

Black-Scholes Option Pricing Formula

- Second, calculate d_1:

$$\begin{aligned} d_1 &= d_2 + \sigma\sqrt{T-t} \\ &= -0.05354 + 0.52\sqrt{133/365} \\ &= 0.26035\ . \end{aligned}$$

- Finally, apply the formula:

$$\begin{aligned} C(S_t|T,X) &= S_t\ \Phi\left[\,d_1\,\right] - \ X\,e^{-r(T-t)}\ \Phi\left[\,d_2\,\right]\ , \\ &= 40.75\ \Phi\left[0.26035\right] \\ &\qquad -\ 40\,e^{-0.0381(133/365)}\ \Phi\left[-0.05354\right]\ , \\ &= \$\,5,6781\ . \end{aligned}$$

Implementing a Binomial Model

To implement the binomial model in a computer programme:

- First, select the number of steps n the loop in your programme should iterate over. This number you choose can be quite high for increased accuracy.
- Second, calculate the "up" and "down" factors u and d:

$$\begin{aligned} u &= e^{\sigma\sqrt{(T-t)/n}}\ , \\ &= \exp\left[0.52\,\sqrt{(133/365)/n}\right]\ , \\ d &= e^{-\sigma\sqrt{(T-t)/n}}\ , \\ &= 1/u\ . \end{aligned}$$

4 – Historical, Implied Volatility and Option Greeks: Contents

Implied Volatility

- You may not be satisfied with your historical volatility estimate, because you consider past observations of the stock price run too deep in the past to be relevant.
- You may then turn to an implied volatility estimate.
- The implied volatility is the volatility that is implied by an option price observed in the market at the time.
- The implied volatility of an option is then used as an estimate of the volatility one needs as input to calculate the price of another option on the same underlying stock.

Implied Volatility

- Consider another European Call on the same stock, which has a different exercise price than the one we were trying to price.
- This option has strike price, X, of \$ 45 (instead of \$ 40).
- The observed price of this other European call is $C_t = \$4.375$.
- The implied volatility is the one input value of σ, which if used in the Black-Scholes equation generates a value for the Call equal to the observed Call price $C_t = \$4.375$.

Implied Volatility

- There is no explicit expression of the implied volatility:
  - We have seen how to implement the Black-Scholes formula.
  - With an input σ you can obtain the call price C_t.
  - But there is no expression of σ as a function of a call price C_t.
- The implied volatility cannot be explicitly derived, but it can be obtained through an iterative process:
  - Here, a value of $\sigma = 0.40$ generates a Black-Scholes value of 3.224 for the Call, which is too low.
  - Trying next a higher value $\sigma = 0.60$ yields a Black-Scholes value of 5.188, which is too high.
  - In the end, a value of $\sigma = 0.5171$ gives a calculated value for the Call which is very close to 4.375, the actual Call price.
- In this case, the option is said to have an implied volatility of $\sigma = 0.5171$.

Implied Volatility

- However, if one calculates the implied volatility of a first option and that of a second option, they usually differ.
- Which one should then be used to calculate the value of a third option on the same stock?
- What practitioners often do is to compute a composite implied volatility:
- This is a weighted average of individual implied volatilities, where the weights reflect the sensitivity of the option price to the volatility.
- The idea being that the implied volatility of an option whose price is very sensitive to volatility is more likely to be accurate, hence should be given more weight.

Implied Volatility

Suppose that observing three different Call options, three implied volatility estimates are available:

- The first one is the one we have just considered. It is based on an At-The-Money option (the exercise price $X = \$45$ is close to the current stock price $S_t = \$40.75$, i.e. $S_t \simeq X$) and we have seen how to calculate its implied volatility which is $\sigma_{at} = 0.5171$.
- The second implied volatility estimate is $\sigma_{out} = 0.5070$ and is based on a deep Out-of-the-Money option (one with high exercise price X relative to $S_t = \$40.75$, i.e. $S_t \ll X$) with same maturity.
- The third one is $\sigma_{in} = 0.5530$ and is based on a deep In-the-Money option (one with small exercise price X relative to $S_t = \$40.75$, i.e. $S_t \gg X$) with same maturity.

Implied Volatility

- The price of the At-the-Money (ATM) option is more sensitive to volatility than the price of either the deep In-the-Money (ITM) or deep Out-of-the-Money (OTM) options.
- Its accuracy is therefore expected to be greater.
- In this situation a greater weight is given to the first implied volatility estimate.
- The weighted implied volatility could then be

$$\sigma_{impl} = 0.8\,\sigma_{at} + 0.1\,\sigma_{out} + 0.1\,\sigma_{in} = 0.5197 \; .$$

4 – Historical, Implied Volatility and Option Greeks: Contents

Option Greeks

- The Black–Scholes model assumptions takes account of possible changes in
  1. Price of the underlying asset: S_t.
  2. Passage of time: t.
- Correspondingly, there are three important sensitivities or Greeks of option prices:
  1. Delta (Δ): impact of small change in underlying asset price.
  2. Gamma (Γ): impact of large change in underlying asset price.
  3. Theta (Θ): impact of passage of time.
- In addition, it is often of interest to know how sensitive are option prices to volatility input and interest rate input giving rise to:
  1. Vega (V): impact of change in volatility.
  2. Rho (ρ): impact of change in interest rate.

Option Greeks

- Option greeks are used to design sophisticated hedging schemes, which option traders use to manage their portfolio.
- Using the Black and Scholes formula we can derive the greeks of positions.
- For expositional convenience we will only consider positions consisting of at most two different securities.
  Its extension to more complex positions should however be apparent.
- Let V_t^a and V_t^b denote the current price of securities a and b.
  Securities a and b can be calls and/or puts: $a, b = call, put, Call, Put$.
  Let n^a and n^b denote the number of units purchased of securities a and b.
- The position value is the sum of the value of the two associated securities, each weighted by the number bought or sold,

$$\text{Position Value} \quad = \quad n^a\, V_t^a \; + \; n^b\, V_t^b \; .$$

Delta

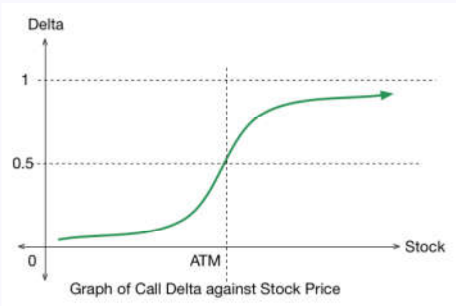
- The delta of an option tells us how much the option value will change with a small change in the price of the underlying stock, other things being equal:

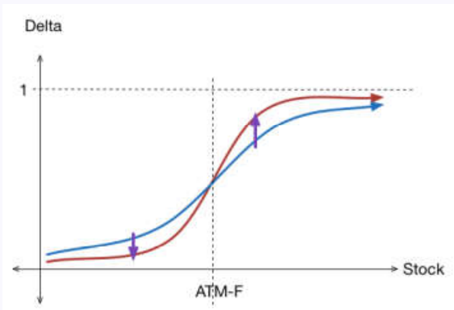
$$\Delta = \frac{\partial V_t}{\partial S_t} .$$

- For a call, Δ^{call} is always a positive number between zero and one.
- For a put, Δ^{put} is always a negative number between zero and one.
- Under the Black-Scholes option pricing formula, it can be shown that:

$$\Delta^{call} = \Phi[d_1] ,$$
$$\Delta^{put} = -(1 - \Phi[d_1]) .$$

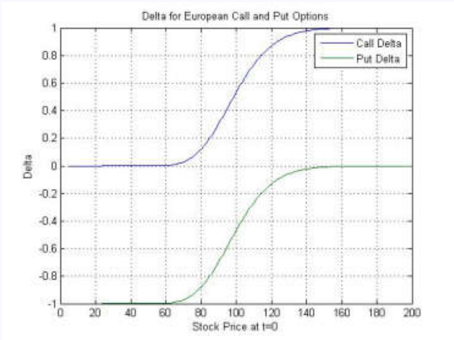
Delta

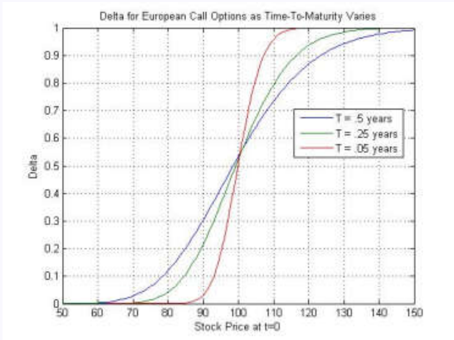




Delta

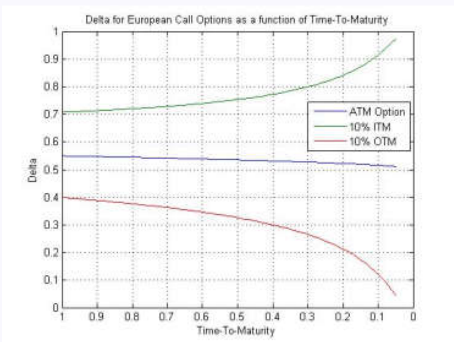
- Delta becomes steeper around the strike price, X. In particular as time approaches the maturity date, T.





Delta

- The delta of an In-The-Money call option ($S_t \gg X$) increases as time elapses.
- The delta of an Out-of-The-Money call option ($S_t \ll X$) decreases as time elapses.



(in this graph, time flows from left to right, i.e as $T - t$ decreases)

Delta

- Similarly the position delta tells us how much the position value will change for a small change in the price of its underlying stock, other things being equal.
- Therefore for two associated securities,

$$\begin{aligned}\text{Position Delta} &= \frac{\partial\,\text{Position Value}}{\partial\, S_t} \\ &= n^a\,\frac{\partial\, V_t^a}{\partial\, S_t} + n^b\,\frac{\partial\, V_t^b}{\partial\, S_t} = n^a\,\Delta^a + n^b\,\Delta^b\,.\end{aligned}$$

- The position delta is then the sum of the weighted deltas of its constituent securities.

Delta Neutral Position

- The position delta measures how exposed our position is to movements in the stock price. If we are uncertain about the direction of the stock price movement (positive or negative) and we want to insulate ourselves from the uncertainty, we want our position delta to be zero.
- By definition a neutral position has a zero delta.
- The neutral position ratio can be determined by setting the position delta to zero. For two associated securities, this involves

$$\frac{n^a}{n^b} = -\,\frac{\Delta^b}{\Delta^a}\,.$$

Gamma

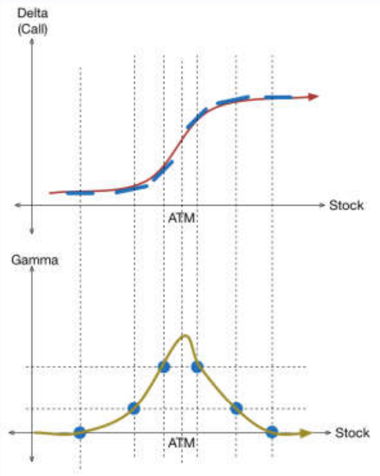
- The gamma of an option measures the how much the option delta will change for a small change in the stock price, other things being equal:

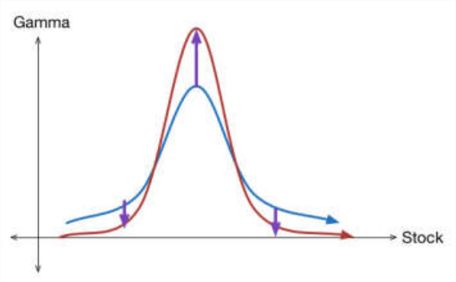
$$\Gamma \;=\; \frac{\partial^2 V_t}{\partial S_t^2} \;=\; \frac{\partial \Delta}{\partial S_t}\,.$$

- For both calls and puts, Γ is a strictly positive number.
- Under the Black-Scholes option pricing formula, it can be shown that:

$$\Gamma^{call} \;=\; \Gamma^{put} \;=\; \frac{e^{-(d_1)^2/2}}{S_t\;\sigma\sqrt{2\,\pi\,(T-t)}}\,.$$

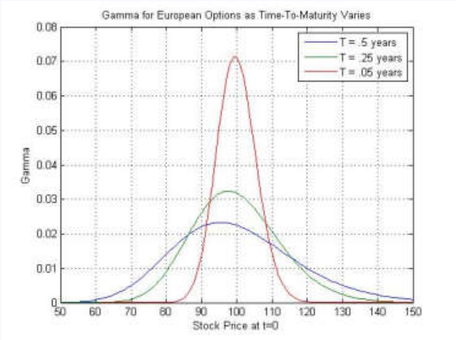
Gamma

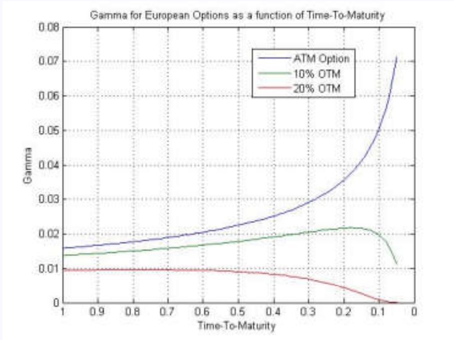




Gamma

- Gamma is largest for At-The-Money options ($S_t \simeq X$). In particular as time approaches the date of maturity, T.





Target Position Gamma

- A reinforcing strategy is to select initial positions that, in addition to coinciding with the target delta, have deltas that are relatively insensitive to movements in the stock price.
- The position gamma is the sum of the weighted gammas of its constituent securities. For two associated securities,

$$\begin{aligned}\text{Position Gamma} &= \frac{\partial\,\text{Position Delta}}{\partial\,S_t}\\ &= n^a\,\frac{\partial\,\Delta^a}{\partial\,S_t} + n^b\,\frac{\partial\,\Delta^b}{\partial\,S_t} = n^a\,\Gamma^a + n^b\,\Gamma^b\,.\end{aligned}$$

- The absolute magnitude of the position gamma, measured at the target delta position ratio, indicates how fast changes in the stock price will push the position delta past the critical distance and force revision of the position ratio.

Target Position Gamma

- For a delta-neutral position, we can express the position gamma in terms of the option deltas and the size of the position. Since $\frac{n^a}{n^b} = -\frac{\Delta^b}{\Delta^a}$, then the

$$\text{Position Gamma} \quad = \quad n^a\,\Delta^a \left[\frac{\Gamma^a}{\Delta^a} - \frac{\Gamma^b}{\Delta^b}\right] .$$

- n^a, the number of shares and options that benefit from a rise in the stock price, represents the size of the position.
- For two securities,
  - while a position delta of zero only determines the ratio ($\frac{n^a}{n^b}$) in which they are held,
  - an additional fixed gamma target determines the size of the position (n^a).

Theta

- Even if the stock price remains unchanged, as the maturity date approaches, the mere passing of time creates profits or losses in option positions.
- If, while the stock price remains unchanged, decreasing time to expiration decreases (increases) the value of the option, we say that the option has negative (positive) time bias.
- Since the pure influence of time is not captured by delta and gamma, a third measure is needed to characterize the option positions's sensitivity to time adequately.

Theta

- This time bias is measured by the option's theta, defined as

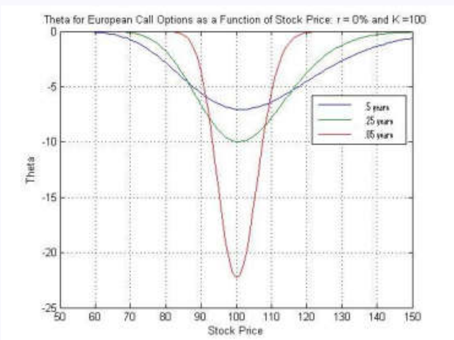
$$\Theta \quad = \quad - \frac{\partial V}{\partial (T - t)} \; .$$

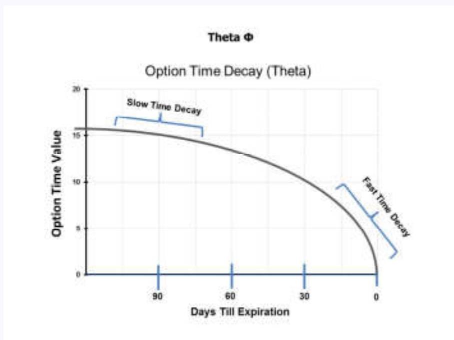
- For a call, Θ^{call} is always negative.
- For a put, Θ^{put} is typically negative but can occasionally be positive.
- Under the Black-Scholes option pricing formula, it can be shown that:

$$\begin{aligned} \Theta^{call} &= -X\, r\, e^{-r\,(T-t)}\, N[d_2] - \frac{S_t\, \sigma\, e^{-(d_1)^2/2}}{2\sqrt{2\,\pi\,(T-t)}} \; , \\ \Theta^{put} &= X\, r\, e^{-r\,(T-t)}\, N[-d_2] - \frac{S_t\, \sigma\, e^{-(d_1)^2/2}}{2\sqrt{2\,\pi\,(T-t)}} \; . \end{aligned}$$

Theta

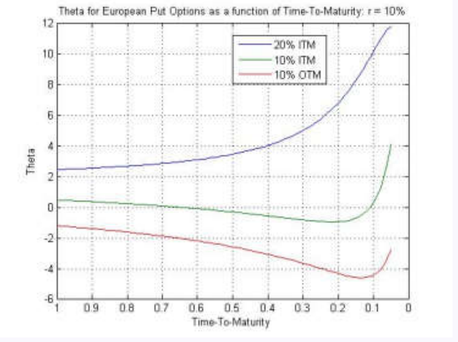
- Theta is most substantial for At-The-Money options ($S_t \simeq X$).
- Theta of is most substantial approaching the date of maturity, T.

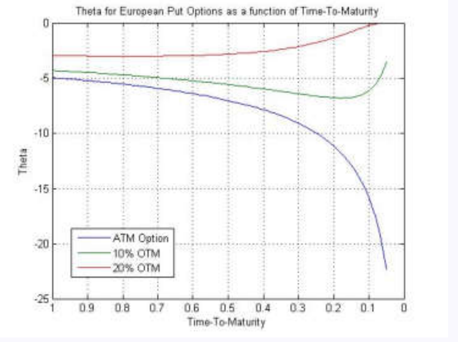




Theta

- Theta of In-The-Money put options ($S_t \ll X$) can be positive.





Position Theta

- The position theta measures how much the position value will change as time to expiration decreases, other things being equal.
- For two associated securities,

$$\begin{aligned} \text{Position Theta} &= -\frac{\partial \, \text{Position Value}}{\partial (T-t)} \\ &= n^a \, \Theta^a + n_b \, \Theta^b \, . \end{aligned}$$

Vega

- The vega of an option measures the how much the option price will change for a small change in the volatility, other things being equal:

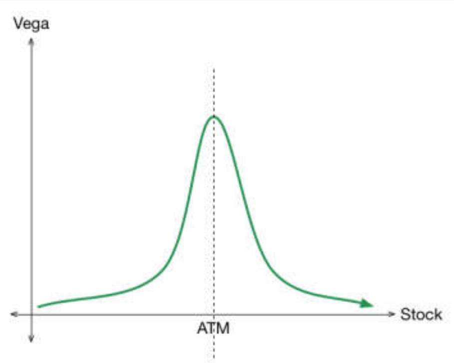
$$\nu \quad = \quad \frac{\partial V_t}{\partial \sigma} \, .$$

- For both calls and puts, ν is a strictly positive number.
- Under the Black-Scholes option pricing formula, it can be shown that:

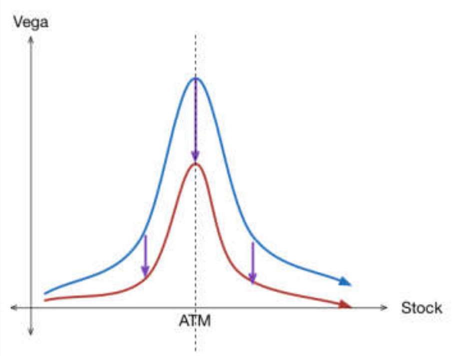
$$\nu^{call} \quad = \quad \nu^{put} \quad = \quad \frac{S_t \, \sqrt{(T-t)} \; e^{-(d_1)^2/2}}{\sqrt{2\,\pi}} \, .$$

Vega

- Vega peaks around the strike price, X.



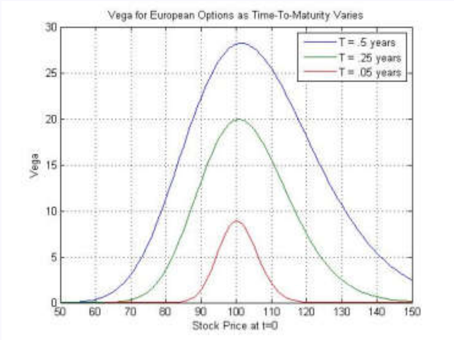
Graph of Vega against Stock Price

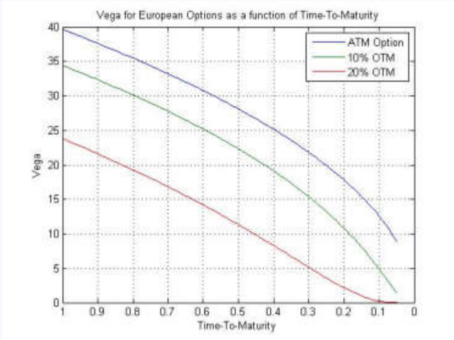


Vega as time passes

Vega

- Vega is largest for At-The-Money options ($S_t \simeq X$). In particular well before the date of maturity, T.





Rho

- The rho of an option measures the how much the option price will change for a small change in interest rate, other things being equal:

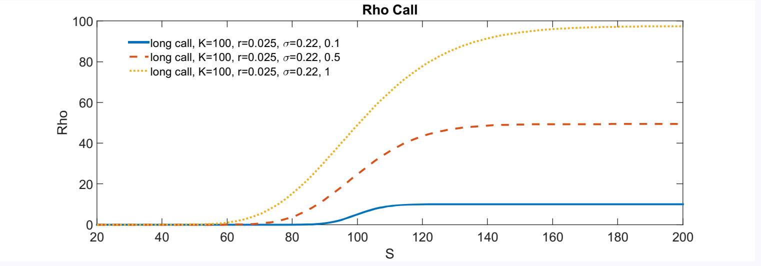
$$\rho \quad = \quad \frac{\partial V_t}{\partial r} \; .$$

- For a call, ρ^{call} is always positive.
- For a put, ρ^{put} is always negative.
- Under the Black-Scholes option pricing formula, it can be shown that:

$$\begin{aligned} \rho^{call} &= (T-t)\,X\,e^{-r\,(T-t)}\;\Phi[d_2] \; , \\ \rho^{put} &= -(T-t)\,X\,e^{-r\,(T-t)}\Phi[-d_2] \; . \end{aligned}$$

Rho

- Rho is largest for In-The-Money options ($S_t \gg X$ for calls and $S_t \ll X$ for puts).



Readings

Book:

- Chapter 19 of Hull.

5 – Options in Corporate Securities and Exotic Options

Derivative Securities
Masters in Finance
Toulouse Business School

Pierre Mella-Barral

Course Road Map

1. Forwards, Futures and Swaps $\surd$
2. Options $\surd$
3. Option Pricing $\surd$
4. Historical, Implied Volatility and Option Greeks $\surd$
5. Options in Corporate Securities and Exotic Options
6. Real Options

5 – Options in Corporate Securities and Exotic Options: Contents

Reminder of Option Pricing

Black-Scholes Option Pricing Formula

A European call with exercise price K maturing at T, is worth

$$C(S_t|T,X) \quad = \quad S_t\ \Phi\,[\,d_1\,] - \ X\,e^{-r(T-t)}\ \Phi\,[\,d_2\,]$$

$$\text{where} \qquad d_2 \;\equiv\; \frac{ln(S_t/X)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\ ,$$

$$d_1 \;\equiv\; d_2+\sigma\sqrt{T-t}\ ,$$

and $\Phi[x]=\int_{-\infty}^{x}\phi[u]du$ is the cumulative standard normal distribution, with $\phi[x]=\frac{1}{\sqrt{2\pi}}e^{-(x)^2/2}$ the standard normal density.

Options in Corporate Securities and Exotic Options

- We can readily use the Black-Scholes pricing formula to price corporate securities such as
  - Zero-Coupon Bonds
  - Senior and Junior (Subordinated) Zero-Coupon Bonds
  - Warrants
  - Convertible Bonds
- We examine each one in turn:

Zero-Coupon Bonds

A zero-coupon bond is a promise to receive a single payment, P (principal), at a future maturity date, T.

- Consider a firm which has issued zero-coupon bonds (debt) worth B_t, and has n outstanding shares, each worth S_t, at the initial date t.
- Denote V_t the total value of the firm's securities, or more briefly, the value of the firm,

$$V_t \quad \equiv \quad n\,S_t \; + B_t \; .$$

Zero-Coupon Bonds

- The payoff diagram of the shares and the debt at the date of maturity, T, is as follows:

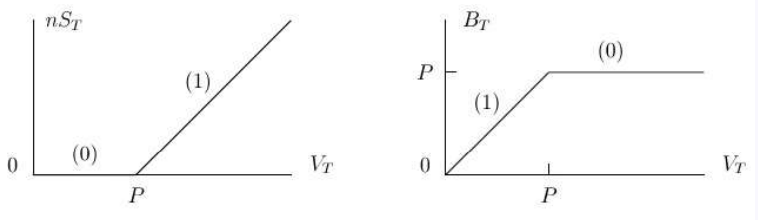


Figure 1: Stock and Ordinary Bond Payoffs at Maturity

- The value of the zero-coupon bonds at the date of maturity is $B_T = \min[V_T; P]$.

Zero-Coupon Bonds

Intuitively:

- The payoff to the bondholders, $B_T = \min[V_T; P]$, is exactly the same as that received by someone
  - who owns a *default-free* zero-coupon bond paying P at date T, and
  - who has written a European put on the value of the firm, with a strike price of P.

- Alternatively, shareholders hold a *limited liability* call option to default on their debt obligation. The value of the n shares at maturity is $nS_T = \max[0; V_T - P]$.

Zero-Coupon Bonds

- Therefore, the values of the n shares and the m bonds at the initial date t are

$$\begin{aligned} n\,S_t &= C(V_t|T,P) \\ B_t &= V_t - C(V_t|T,P)\ . \end{aligned}$$

where $C(S|T,K)$ denotes the Black-Scholes formula.

Senior and Junior Zero-Coupon Bonds

- Suppose now that the debt consists of senior zero-coupon bonds promising P^s and junior zero-coupon bonds promising P^j, both at the date of maturity T.
- According to the indentures,
  - the junior bondholders can only be paid
  - after the senior bondholders have been paid in full.

Senior and Junior Zero-Coupon Bonds

The payoff diagrams of the three classes of securities are as follows:

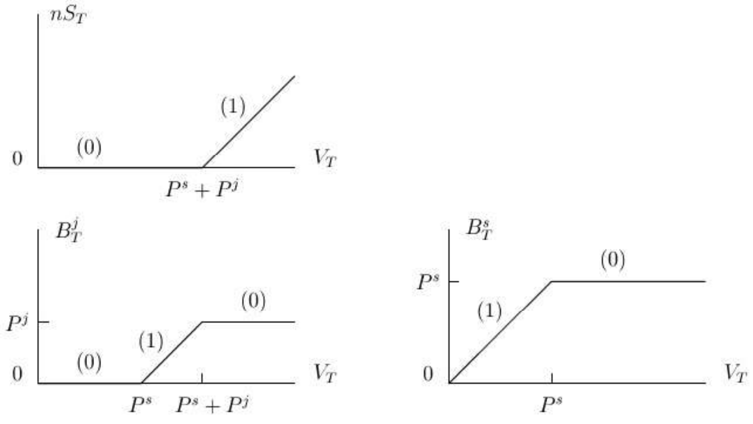


Figure 2: Stock, Junior and Senior Bond Payoffs at Maturity

Senior and Junior Zero-Coupon Bonds

- The value of the senior, junior zero-coupon bonds, and the n shares at the date of maturity, T, are respectively

$$\begin{aligned} B_T^s &= \min[V_T; P^s] \ , \\ B_T^j &= \min[\max[0; V_T - P_s] ; P^j] \ , \\ nS_T &= \max[0; V_T - P_s - P_j] \ . \end{aligned}$$

- Therefore, the values of the three securities at the initial date t are

$$\begin{aligned} n\,S_t &= C(V_t|T, P^s + P^j) \ , \\ B_t^s &= V_t \ - \ C(V_t|T, P^s) \ , \\ B_t^j &= C(V_t|T, P^s) \ - \ C(V_t|T, P^s + P^j) \ . \end{aligned}$$

Warrants

A warrant gives the owner the right to buy a fixed number of newly created shares at a specified price, by a given date.

- A warrant is similar to a call option.
- However, when a warrant is exercised new shares are created, which dilutes their value.

Warrants

- Consider a firm that, at an initial date t, has
  1. n outstanding shares, and
  2. m European warrants.
- Each warrant can be converted at the maturity date T into one share of newly issued share, upon payment of an exercise price K.
- Denote W_t and S_t the values at date t of a warrant and a share, respectively.
- The value of the firm is then

$$V_t \; = \; n\, S_t \; + \; m\, W_t .$$

Warrants

- If at maturity warrant-holders exercise their warrants:
- Because exercise price have been paid, the value of the firm increases to $V_T + mX$.
- The number of outstanding shares becomes $n + m$.
- The price of one share therefore becomes

$$S_T \;=\; \frac{V_T \;+\; m\,X}{n+m}.$$

- However, the benefit to a warrant holder is only

$$\frac{V_T \;+\; m\,X}{n+m} \;-\; X \;=\; \frac{V_T \;-\; n\,X}{n+m}.$$

- Clearly, warrant-holder will *only* exercise if this benefit is *positive*, hence if

$$V_T \;>\; n\,X.$$

Warrants

So at the date of maturity:

- If $V_T > n\,X$, the m warrants are exercised,
  1. the n pre-existing shares are worth,

$$n\,S_T \quad = \quad \left(\frac{n}{n+m}\right)(V_T + mX)\ ,$$

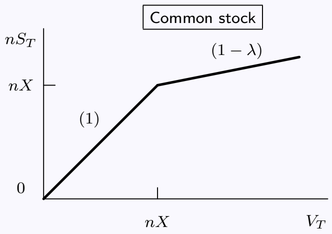
  2. and the m warrants are worth,

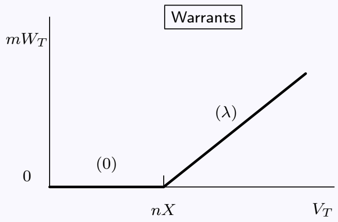
$$m\,W_T \quad = \quad \left(\frac{m}{n+m}\right)(V_T - nX)\ .$$

- If $V_T < n\,X$, the m warrants are not exercised and worthless.

Warrants

- Denoting $\lambda \equiv m/(n+m)$ the dilution factor of the stock, the final pay-offs of shares and warrants are





- The values of the n shares and the m warrants at the initial date t are therefore,

$$
\begin{aligned}
n\,S_t &= V_t - \lambda\, C(V_t|T, nX), \\
m\,W_t &= \lambda\, C(V_t|T, nX)\ .
\end{aligned}
$$

Convertible Bonds

Convertible bonds combine many of the features of ordinary bonds and warrants.

- Like ordinary bonds, they are entitlements to fixed coupon and principal payments, and have priority over the stock in the event of bankruptcy.
- Like warrants, they can be exchanged for a specified number of newly issued shares.

- Whereas warrants also require the payment of an exercise price when the exchange is made, convertible bonds rarely do.

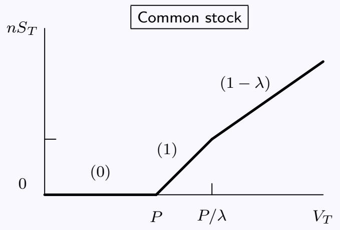
Convertible Bonds

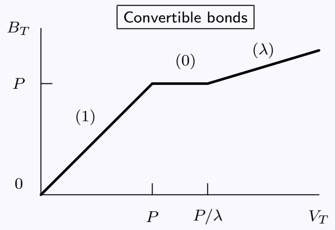
- Consider a firm that, at an initial date t, has
  1. n outstanding shares and
  2. m convertible zero-coupon bonds.
- Assume that
  1. such bonds promise to pay an overall principal, P, at the maturity date T,
  2. and each of them can be converted into k shares of newly is- sued shares immediately before maturity, T.
- Denote B_t and S_t the values at date t of a share and the convertible bonds, respectively.
- The value of the firm is then

$$V_t \;=\; n\,S_t \;+\; B_t.$$

Convertible Bonds

- If conversion is chosen, then the (ex-)convertible holders would own the fraction $\lambda \equiv mk/(n+mk)$ of the firm.
- Convertible holders will therefore convert when $\lambda V_T > P$.
- The pay-offs at maturity of the shares and convertible bonds are





Convertible Bonds

- Most convertible bonds can be converted *any* time before the maturity date.
- However in the *absence* of anticipated dividend payments, as for American call options, *early* conversion is *never* optimal.
- The values of the n shares and the m convertible bonds at the initial date t are therefore,

$$
\begin{aligned}
n\,S_t &= C(V_t|T,P) \;-\; \lambda\; C(V_t|T,P/\lambda), \\
B_t &= V_t \;-\; C(V_t|T,P) \;+\; \lambda\; C(V_t|T,P/\lambda).
\end{aligned}
$$

5 – Options in Corporate Securities and Exotic Options: Contents

- Reminder of Option Pricing $\surd$
- Options in Corporate Securities $\surd$
  - Zero-Coupon Bonds $\surd$
  - Senior and Junior Zero Coupon Bonds $\surd$
  - Warrants $\surd$
  - Convertibles $\surd$
- Exotic Options
  - Barrier Options
  - Lookback Options
  - Asian Options

Exotic Options

- The term exotic option is very broad.
- It refers to derivatives whose payoffs are more intricate than combinations of American or European calls and puts.
- Classifying them is sometimes difficult, and the following is just the few most common types.

Barrier Options

- A barrier option is a derivative such that part of the option contract is triggered if and only if, any time prior to expiry, the underlying asset value, S_t, hits some pre-established barrier.
- One can distinguish four types of single barrier options:
  1. Down-and-out: the option becomes worthless if a barrier, $\underline{S}$, is hit from above;
  2. Up-and-out: the option becomes worthless if a barrier, $\overline{S}$, is hit from below;
  3. Down-and-in: the option remains worthless unless a barrier, $\underline{S}$, is hit from above;
  4. Up-and-in: the option remains worthless unless a barrier, $\overline{S}$, is hit from below.

Barrier Options

- Here, the right to exercise can "disappear" or "appear".
- Reaching this barrier can either terminate (out) or initiate (in) the existence of the option.
- Furthermore, the barrier can only be smaller or greater than the current value of the underlying, S_t, hence it can only be hit either form above (down) or from below (up).
- Although they are not common, the structure of barrier options can involve up to two barriers: a lower barrier, $\underline{S}$, and an upper barrier, $\overline{S}$.

Barrier Options

- The contract may also specify a rebate, R, which is a lump-sum to be paid if the barrier is reached (not reached) in the case of out-barriers (in-barriers).
- The stipulated timing of exercise may be of American or European type.
- Importantly, the value of such options evolve only with S_t and t.
- It will not involve variables representing the specific path followed by the underlying process.
- In this respect, the valuation of a barrier option is only a weakly path-dependent problem.

Barrier Options

- Let us examine the case of a down-and-out call.
- This derivative is identical to a European call, with the additional feature that the contract is canceled if the stock price reaches a pre-specified lower boundary.
- It is an option to buy the stock for an exercise price X at time T, if and only if the stock price does not fall to a lower boundary $\underline{S}$ before then.

Barrier Options

Down-and-Out Call Option Pricing Formula

A down-and-out call with exercise price X and lower barrier $\underline{S}$, maturing at T, is worth

$$
\begin{aligned}
G_t &= S_t\left(\Phi\left[a(S_t/X)+\sigma\sqrt{T-t}\right]-\left(\frac{\underline{S}}{S_t}\right)^{2r/\sigma^2+1}\Phi\left[a(\underline{S}^2/XS_t)+\sigma\sqrt{T-t}\right]\right)\\
&+ e^{-r(T-t)}\,X\left(\Phi\left[a(S_t/X)\right]-\left(\frac{\underline{S}}{S_t}\right)^{2r/\sigma^2-1}\Phi\left[a(\underline{S}^2/XS_t)\right]\right)
\end{aligned}
$$

$$\text{where}\qquad a(X)\equiv\frac{ln(X)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,.$$

- Notice that in this expression, if the lower barrier, $\underline{S}$, is set equal to zero, the formula reduces to the Black-Scholes formula.

Lookback Options

- A lookback option is a derivative whose payoff depends on the maximum (or minimum) value, $\hat{S}_T$ (or $\check{S}_T$), realized by the underlying asset over the life of the option.
- Here, if the option was initiated at date 0,

$$\hat{S}_t \;\equiv\; \max_{0\leq\tau\leq t} S_\tau \;, \qquad \text{and} \qquad \check{S}_t \;\equiv\; \min_{0\leq\tau\leq t} S_\tau \;.$$

Lookback Options

- For example, a European lookback put is an option to sell an asset at time T, for the highest value it reached throughout the period $[0; T]$.
- The payoff at expiry is the difference between the maximum value achieved by the underlying asset during the life of the option, and its final value, i.e. the payoff of the option at expiry is,

$$\max\left\{ \hat{S}_T \;-S_T \;;\; 0 \right\}.$$

- Similarly, a European lookback call is an option to buy at the lowest value reached during the life of the option. The payoff at expiry of such an option is

$$\max\{ S_T \;-\check{S}_T \;;\; 0 \}.$$

Lookback Options

- Here, the value of such options involves not only S_t and t, but also $\hat{S}_t$ (or $\check{S}_t$).
- Clearly, if the maximum (or minimum) is updated continuously, at the expiry date T, it can only be greater (or smaller) than the value of the underlying asset, and the option will always be exercised.
- Therefore a lookback option with a continuously sampled maximum (or minimum) is not really an option, as exercise is certain.

Lookback Options

Lookback Put Option Pricing Formula

A lookback put initiated at date 0 with maturing at T, is worth at date $t \in [0; T]$

$$\begin{aligned} K_t &= S_t \left((1 + \frac{\sigma^2}{2r}) \, \Phi \left[a(S_t/\hat{S}_t) + \sigma\sqrt{T-t} \right] \; - \; 1 \right) \\ &+ e^{-r(T-t)} \, \hat{S}_t \left(\Phi \left[-a(S_t/\hat{S}_t) \right] \; - \; \frac{\sigma^2}{2r} \left(\frac{\hat{S}_t}{S_t} \right)^{2r/\sigma^2 - 1} \Phi \left[-a(\hat{S}_t/S_t) \right] \right) \end{aligned}$$

$$\text{where} \qquad a(X) \equiv \frac{ln(X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \, .$$

Lookback Options

- In order to reduce the costs a contract requiring constant monitoring and recording of the historical maximum (or minimum) would involve, most lookbacks contracts do not involve measuring continuously the maximum (or minimum).
- The payoff of a discretely sampled European lookback put, over a set of sampling dates, $\mathcal{D}_T \equiv \{0, 1, 2, ..T\}$, becomes

$$\max\left\{\max_{i\in\mathcal{D}_T} S_i \; - S_T \; ; \; 0\right\}.$$

- Now, the underlying asset value can exceed a discretely sampled maximum. This brings back the possibility that the option will not be exercised at expiry.
- The valuation of such options involves S_t, t, and $\max_{i\in\mathcal{D}_t} S_i$ and/or $\min_{i\in\mathcal{D}_t} S_i$.
- Again the option may be of American or European type.

Asian Options

- An Asian option is a derivative whose payoff depends on a period of the history of the value of the underlying asset process, via some sort of average.
- For example, an average strike call gives the right to buy an asset for its average price over some past period, Δ.
- Its payoff is the difference, if positive, between the underlying asset value, S_t, and its average prior to the exercise date, $A(t \mid \Delta)$.
- The average may be arithmetic, which is the mean of the underlying asset value

$$A(t \mid \Delta) \quad \equiv \quad \frac{1}{\Delta}\int_{t-\Delta}^{t} S_\tau d\tau \; .$$

- Alternatively, it may be geometric, in which case

$$A(t \mid \Delta) \quad \equiv \quad \exp\left[\frac{1}{\Delta}\int_{t-\Delta}^{t} ln(S_\tau) d\tau\right] \; .$$

Asian Options

- The average may involve a weighted averaging, if for example, greater weight is contractually given to recent realizations.
- The contract may stipulate a discrete sampling scheme of the underlying asset value.
- It is certainly easier to respect a contract which involves the mean of a small number of asset values, rather than the average of all realizations.
- The option can be European, if it stipulates the time of potential exercise.

Asian Options

- Next, we highlight that the payoff of Asian options is sensitive to the specifications of the averaging procedure established in the contract, like
  - the kind of average,
  - the weighting scheme and
  - the frequency of averaging.
- Consider a European average strike call on the underlying stock S with time to maturity $T = 10$ days. The payoff at maturity of a this Asian option is:

$$\max(S_T - A, 0)$$

where A is a specific average of stock prices over specific dates of observation.

- Suppose the stock price takes the following path in the 10 days to maturity: $S_{t_1} = 100$; $S_{t_2} = 114$; $S_{t_3} = 115$; $S_{t_4} = 102$; $S_{t_5} = 112$; $S_{t_6} = 116$; $S_{t_7} = 108$; $S_{t_8} = 112$; $S_{t_9} = 115$; $S_{t_{10}} = 116$ (with $t_1 = 0$ and $t_n = T$).

Asian Options - Kind of Averaging

- With an equally-weighted arithmetic average over the last 10 days:

$$A^A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} = 111 \,,$$

and the payoff of the option is:

$$\text{Payoff at mat.} = \max\left(S_T - A^A, 0\right) = \max\left(116 - 111, 0\right) = 5 \,.$$

- An equally-weighted geometric average over the last 10) days gives instead:

$$A^G = \left(\prod_{i=0}^{1} S_{t_i}\right)^{1/n} = 110.85 \,,$$

and the payoff of the option becomes:

$$\text{Payoff at mat.} = \max\left(S_T - A^G, 0\right) = \max\left(116 - 110.85, 0\right) = 5.15 \,.$$

Asian Options - Kind of Averaging

- Using risk-neutral valuation, the price of the option is the present value, discounted at riskless interest rate, of the expected payoff at maturity, where the expectation is computed under risk-neutrality.
- Under the assumption that the underlying asset price follows a lognormal distribution (as in Black and Scholes), the price of an Asian option can be derived analytically, but only if the averaging is geometric.
- When the averaging is arithmetic, the risk-neutral probability distribution for an Asian option is difficult to calculate. Two routes are used by practitioners.
  1. In the previous example, we have seen that the value of the geometric average is very similar to the value of the arithmetic average.
     Good analytical approximations of Asian options with arithmetic averaging, based on this similarity, simply assume that the averaging is geometric.
  2. Alternatively, use numerical methods, like Monte Carlo simulations.

Asian Options - Weighting Scheme

- So far, averaging gave equal importance to all observations.
- Some contracts however attribute more weight to recent realizations.
- Consider the following progressive scheme of weights: $w_1 = 0.01$; $w_2 = 0.03$; $w_3 = 0.04$; $w_4 = 0.05$; $w_5 = 0.06$; $w_6 = 0.14$; $w_7 = 0.15$; $w_8 = 0.16$; $w_9 = 0.17$; $w_{10} = 0.19$.
- The arithmetic average with different weighting becomes:

$$A^{wA} = \sum_{i=1}^{n} w_i S_{t_i} \ , \quad \text{where} \quad \sum_{i=1}^{n} w_i = 1 \ .$$

(the previous case of equal weights corresponds to $w_i = \frac{1}{n} = 0.1$, for each i).

Asian Options - Weighting Scheme

- The arithmetic average with such weighting scheme equals $A^{w\,A} = 112.79$
- This turns out here to be higher than with equal-weighting ($A^A = 111$). Clearly, this results from the positive tendency of the price.
- With an upward trending price, an average which gives more importance to the latest observations is typically higher than an equally weighted average.
- The payoff of the option will be therefore lower:

$$\text{Payoff at mat.} \quad = \quad \max\left(S_T - A^{wA}, 0\right) \quad = \quad \max\left(116 - 112.79, 0\right) = 3.21 \ .$$

Asian Options - Frequency of Averaging

- In some contracts, the averaging is over a specific subset of dates prior to maturity (a subset of the observed path of prices in the period).
- For example, suppose the average is only over the subset:

$$D_T' \equiv \{t_1, t_3, t_5, t_7, t_9\}$$

  The arithmetic average over such subset is $A^{A'} = 110$.

- This turn out here to be lower than that of the average over the full set of dates (recall $A^A = 111$). The payoff of the option is then higher:

$$\text{Payoff at mat.} = \max\left(S_T - A^{A'}, 0\right) = \max(116 - 110, 0) = 6 \ .$$

- In this example, averaging over a smaller set of realizations happens to eliminate some high realizations from the computation of the average.
- In general however, beforehand, no specific relationship between frequency of averaging and value of average is to be expected.

Readings

Book:

- Chapters 24 and 25 of Hull.

Papers:

– Black, F., and J. Cox, (1976) "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance*, Vol. 31, pp. 351-367.

– Leland, H., (1994) "Risky Debt, Bond Covenants and Optimal Capital Structure," *Journal of Finance*, Vol. 49, No. 4, September, pp. 1213-1252.

– Goldman, B., H. Sosin, and M.A. Gatto, (1979) "Path-Dependent Options: Buy at the Low, Sell at the High," *Journal of Finance*, Vol. 34, pp. 1111-1127.

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Problem Set 2

1. Suppose that the observations on a stock price at the end of each of 15 consecutive weeks are as follows: 30.25, 32, 31.125, 30.125, 30.25, 30.375, 30.625, 33, 32.875, 33, 33.5, 33.5, 33.75, 33.5, 33.25. Estimate the stock price volatility.

2. A lookback call option is an option to buy an underlying asset for the minimum price realized between the date the option is issued and the date it is exercised.

   Use a two-time-step tree to value at the date of issue, t, a two-month American lookback call option (which pays $S_\tau - \check{S}_\tau$ if exercised at date τ, where $\check{S}_\tau \equiv \min_{t \le u \le \tau} S_u$) on a stock price with current value 30 and volatility 24% per annum. Assume a continuously compounded risk-free interest rate of 10%. Remember in your calculations to employ Cox, Ross and Rubinstein's approach, which by letting

   $$u = e^{\sigma\sqrt{(T-t)/n}} , \qquad \text{and} \quad d = 1/u ,$$

   ensures that the mean and variance of the continuously compounded rate of return of the stock price movement assumed in a binomial model, as the number of steps $n \to +\infty$, coincide with that of the actual stock price.

3. Use a three-time-step tree to value an American put option on the geometric average of the price of a non-dividend-paying stock (which therefore pays $X - A^G_\tau$ if exercised at date τ) when the stock price is \$40, the strike price is \$40, the continuously compounded risk-free interest rate is 10% per annum, the volatility is 35% per annum, and the time to maturity is three months. The geometric average, A^G, is measured from today until option matures.

4. Average options are options whose payoffs depend on the average price $\overline{S}$ of the underlying asset over the option's life. Amongst these:
   – An average price call with strike price X gives the holder the right to receive the average price, $\overline{S}$, against the strike price, X;
   – An average price put with strike price X gives the holder the right to receive the strike price, X, against the average price, $\overline{S}$;
   – An average strike call gives the holder the right to receive the underlying asset price against the average price, $\overline{S}$;
   – An average strike put gives the holder the right to receive the average price, $\overline{S}$ against the underlying asset price.

   Suppose that c_1 and p_1 are the prices of a European average price call and a European average price put with strike X and maturity T , c_2 and p_2 are the prices of a European average strike call and European average strike put with maturity T, and c_3 and p_3 are the prices of a regular European call and a regular European put with strike price X and maturity T. Show that $c_1 + c_2 - c_3 = p_1 + p_2 - p_3$.

Solutions to Problem Set 2

1. Denote S_j where $j \in \{1,..,15\}$ the weekly stock price observations: 30.25, 32, 31.125, 30.125, 30.25, 30.375, 30.625, 33, 32.875, 33, 33.5, 33.5, 33.75, 33.5, 33.25.

   The corresponding returns series $u_j \equiv ln(S_j/S_{j-1})$ is: 0.05624, -0.02772, -0.03266, 0.00414, 0.00412, 0.00820, 0.07469, -0.00380, 0.00380, 0.01504, 0.0000, 0.00743, -0.00743, -0.00749. Hence,

$$\sum_{j=1}^{n} u_j = 0.09456 \quad \text{and} \quad \sum_{j=1}^{n} u_j^2 = 0.01110 ,$$

   where n is the number of price relatives in the series, which is 14.

   Now, the sample mean and variance of this series are

$$\begin{aligned} \hat{\mu} &= \frac{1}{n}\sum_{j=1}^{n} u_j , \quad \text{and} \\ \hat{\sigma}^2 &= \frac{1}{n}\sum_{j=1}^{n} (u_j - \hat{\mu})^2 . \end{aligned}$$

   However,

$$\begin{aligned} \hat{\sigma}^2 &= E\left[(u_j - E[u_j])^2\right] \\ &= E\left[u_j^2\right] - (E[u_j])^2 . \end{aligned}$$

   Therefore,

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n} u_j^2 - \left(\frac{1}{n}\sum_{j=1}^{n} u_j\right)^2 .$$

   Consequently, a biased estimate of the variance of weekly returns is therefore:

$$\frac{0.01110}{14} - \left(\frac{0.09456}{14}\right)^2 .$$

   A unbiased estimate of the variance of weekly returns is therefore obtained multiplying the biased estimate by 14/13:

$$\frac{0.01110}{13} - \frac{(0.09456)^2}{14 \times 13} ,$$

   hence an unbiased estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01110}{13} - \frac{(0.09456)^2}{14 \times 13}} = 0.02837 .$$

   The volatility per annum is therefore $0.02837\sqrt{52} = 0.2046$, or 20.46%.

2. $S = 30$; $r = 0.1$; $\sigma = 0.24$; $T - t = 1/6 = 0.1666$; $\Delta t = 1/12 = 0.08333$. Following the Cox, Ross and Rubinstein approach,

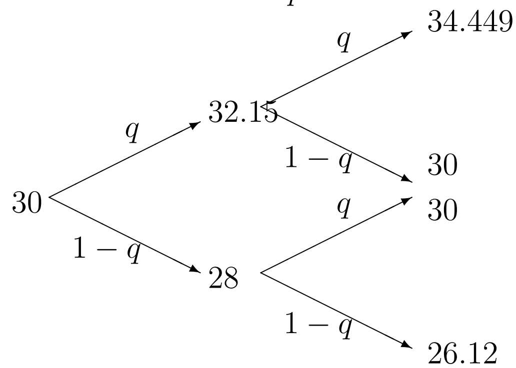
$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} = e^{0.24\sqrt{0.08333}} = 1.0716 , \\ d &= 1/u = 0.9331 . \end{aligned}$$

The risk neutral probability, q, of an up move is

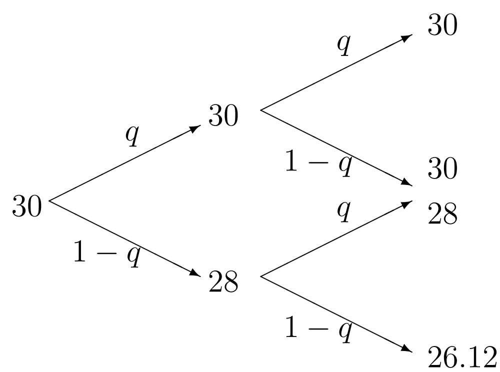
$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.1\times 0.08333} - 0.9331}{1.0716 - 0.9331} = 0.543 ,$$

and consequently, $1 - q = 0.457$.

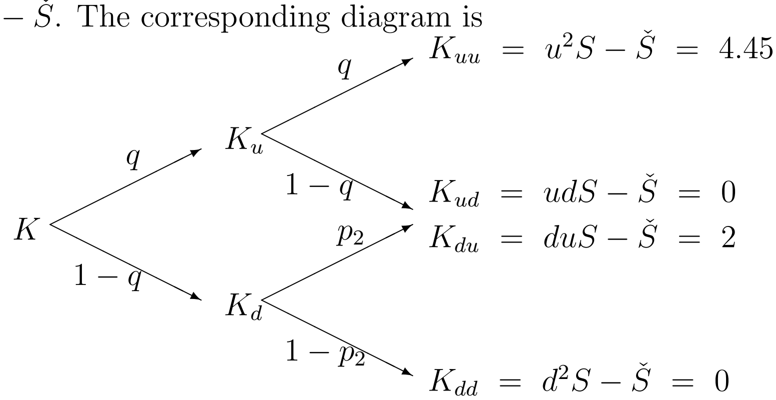
A tree describing the behavior of the *stock price* is shown in the diagram below



A tree describing the behavior of the *historical minimum* is shown in the diagram below



The option pays off $S - \check{S}$. The corresponding diagram is



Calculate first the value at node K_u:

$$[4.45 \times q \; + \; 0 \times (1 - q)] \times e^{-0.1\times 0.08333} = 2.396 ,$$

given that it is an American option, compare with the exercise value: $32.15 - 30 = 2.15$. Therefore at node 1 it is not worthwhile exercising.

Then, calculate the value at node K_d:

$$[2.00 \times q + 0 \times (1-q)] \times e^{-0.1 \times 0.08333} = 1.079 ,$$

compare it again with the exercise value: $28 - 28 = 0$. At node 2 it is certainly not worthwhile exercising.

Working backwards in time, the current value of the option, K, is

$$[2.396 \times q + 1.079 \times (1-q)] \times e^{-0.1 \times 0.08333} = 1.779 .$$

3. Today's stock price (date t) equals the exercise price, $S_t = X = 40$; $r = 0.1$; $\sigma = 0.35$; The maturity date T is in three months, so $T - t = 1/4 = 0.25$ year; The tree is a three steps tree, so each period lasts $\Delta = 1/12 = 0.08333$ year. Following the Cox, Ross and Rubinstein approach,

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} = e^{0.35\sqrt{0.08333}} = 1.1063 , \\ d &= 1/u = 0.9039 . \end{aligned}$$

The risk neutral probability, q, of an up move is

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.1 \times 0.08333} - 0.9039}{1.1063 - 0.9039} = 0.5161 ,$$

and consequently, $1 - q = 0.4839$

First, expand a tree describing the behavior of the stock price, S_τ where $\tau \in \{t, t+\Delta, t+2\Delta, t+3\Delta\}$. Second, develop the resulting geometric average, $A_\tau^G = \left(\prod_{t_i=t}^{\tau} S_{t_i}\right)^{1/(\frac{\tau - t}{\Delta})}$, over time. The results are is shown in the diagram below.

$$S_{uuu} = 54.16 \quad A_{uuu} = 46.54$$

$$S_{uu} = 48.96 \quad A_{uu} = 44.27$$

$$S_{uud} = 44.25 \quad A_{uud} = 44.25$$

$$S_u = 44.25 \quad A_u = 42.07$$

$$S_{udu} = 44.25 \quad A_{udu} = 42.07$$

$$S_{ud} = 40 \quad A_{ud} = 41.37$$

$$S_{udd} = 36.16 \quad A_{udd} = 40.00$$

$$S = 40$$

$$S_{duu} = 44.25 \quad A_{duu} = 40.00$$

$$S_{du} = 40 \quad A_{du} = 38.68$$

$$S_{dud} = 36.16 \quad A_{dud} = 38.03$$

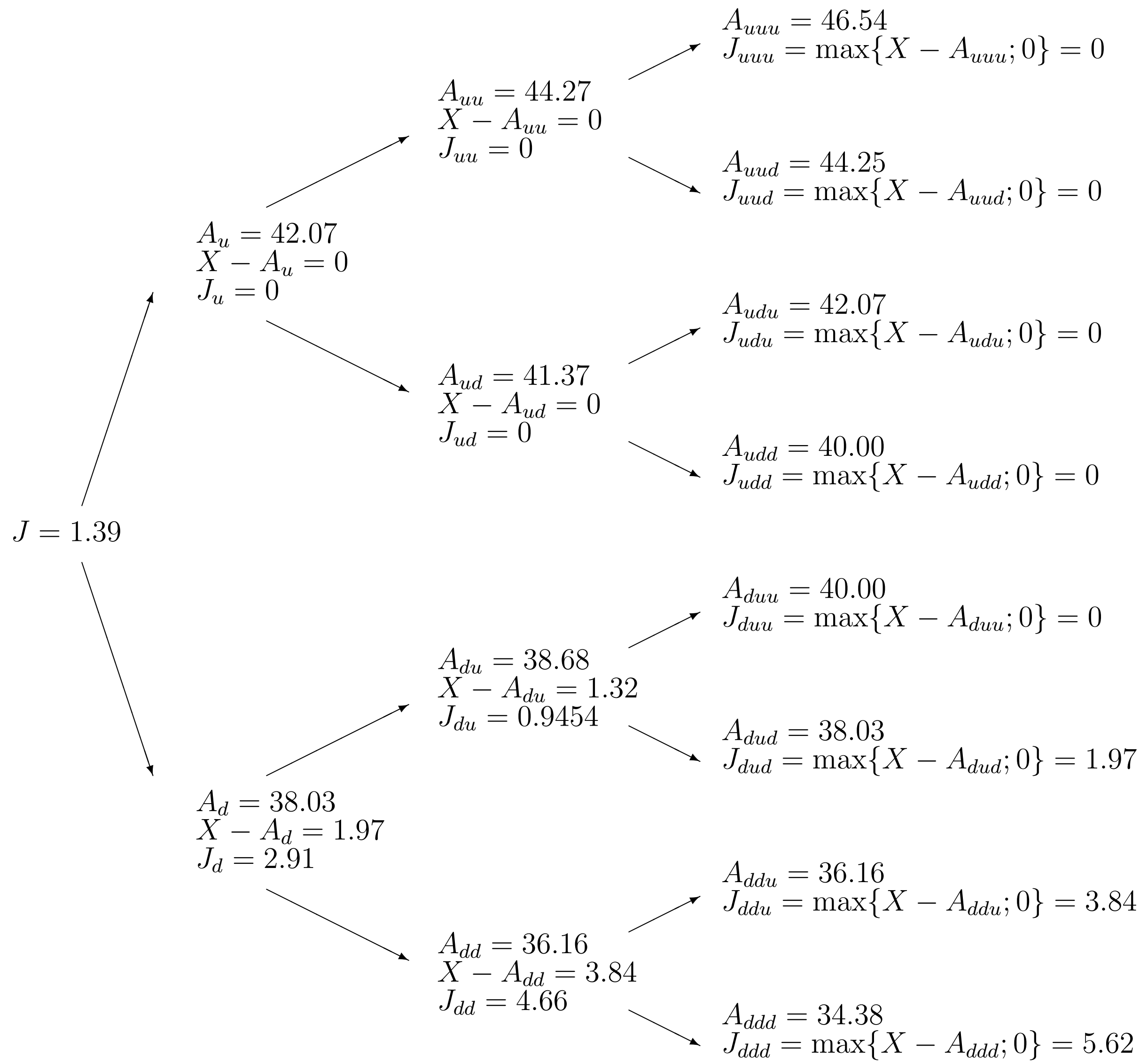
$$S_d = 36.16 \quad A_d = 38.03$$

$$S_{ddu} = 36.16 \quad A_{ddu} = 36.16$$

$$S_{dd} = 32.68 \quad A_{dd} = 36.16$$

$$S_{ddd} = 29.54 \quad A_{ddd} = 34.38$$

The following diagram first recalls the geometric average established in the previous diagram. Second, the diagram shows the value of the option at the date of maturity, which is simply the payoff $J_T = \max\{X - A_T; 0\}$. Third, the diagram shows the option's pay off $X - A^G_\tau$ if exercised at earlier dates $\tau \in \{t+\Delta, t+2\Delta\}$. Working backwards, the option value before maturity J_τ where $\tau \in \{t+\Delta, t+2\Delta\}$, is computed according to the risk-neutral valuation, considering the different possible paths the price can follow.



At node J_{du} the discounted expected value of the option if it is kept alive is

$$[0 \times q \; + \; 1.97 \times (1-q)] \; e^{-0.1 \times 0.0833} \quad = \quad 0.9454 \; .$$

However, given that the option is an American-style option, its actual value is

$$max \left\{ X - A_{du} \; ; \; [J_{duu} \, q \; + \; J_{dud} \, (1-q)] \; e^{-r \, \Delta t} \right\} \quad = \quad max \{ 1.32 \; ; \; 0.9454 \} \; = \; 1.32 \; .$$

It turns out that this is the only case where it is optimal to exercise early. Working backwards the entire tree, we obtain the current value of the option, which is

$$[2.91 \times 0.4839] \; e^{-0.1 \times 0.0833} \quad = \quad 1.39 \; .$$

4. The associated payoffs are as follows:

$$
\begin{array}{lll}
c_1: & max(\overline{S} - X, 0) & \text{(Average Price Call)} \\
c_2: & max(S_T - \overline{S}, 0) & \text{(Average Srike Call)} \\
c_3: & max(S_T - X, 0) & \text{(Call)} \\
p_1: & max(X - \overline{S}, 0) & \text{(Average Price Put)} \\
p_2: & max(\overline{S} - S_T, 0) & \text{(Average Strike Put)} \\
p_3: & max(X - S_T, 0) & \text{(Put)}\ .
\end{array}
$$

The payoff from $c_1 - p_1$ is always $\overline{S} - X$; the payoff from $c_2 - p_2$ is always $S_T - \overline{S}$; the payoff from $c_3 - p_3$ is always $S_T - X$; it follows that

$$
\begin{array}{l}
(c_1 - p_1) + (c_2 - p_2) = (c_3 - p_3)\ , \qquad or \\
c_1 + c_2 - c_3 = p_1 + p_2 - p_3\ .
\end{array}
$$

6 – Real Options

Derivative Securities
Masters in Finance
Toulouse Business School

Pierre Mella-Barral

Course Road Map

1. Forwards, Futures and Swaps $\surd$
2. Options $\surd$
3. Option Pricing $\surd$
4. Historical, Implied Volatility and Option Greeks $\surd$
5. Options in Corporate Securities and Exotic Options $\surd$
6. Real Options

6 – Real Options: Contents

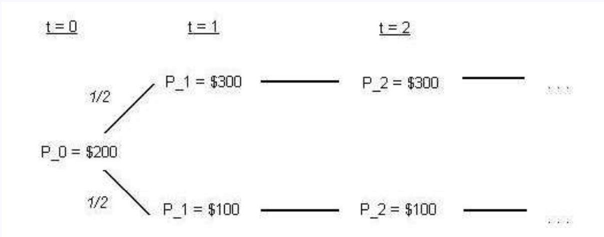
Motivation

Example 1:

- Consider a firm that is trying to decide whether to invest in a widget factory.
- Assume the firm should discount future cash flows at $r = 10\%$.
- Assume that the factory can be built instantaneously, at a cost $I = \$1.6M$ and will produce 1000 widgets per year forever, with zero operating cost.

Motivation

- Currently the price of a widget is \$200, but next year the price will change.
  - With probability $1/2$ it will rise to \$300 and
  - with probability $1/2$ it will fall to \$100.
- The price will then remain at this new level forever:



- Should the firm invest in this widget factory?

Motivation

- Calculating the NPV of this investment in the traditional way we get:

$$\begin{aligned} \text{NPV} &= -\$1.6M + \$0.2M + \frac{\$0.2M}{1+0.1} + \frac{\$0.2M}{(1+0.1)^2} + \ldots \\ &= -\$1.6M + \$2.2M = \$0.6M \ . \end{aligned}$$

- The NPV of this project is positive. The current value of the widget factory is $\$2.2M$ which exceeds the $\$1.6M$ it costs. Hence it would seem that we should go ahead with the investment.

Motivation

- The conclusion is incorrect, however, because it ignores the *value* of the possibility *of not investing* should the price fall.
- To see this, let us calculate the NPV of this project a second time, this time assuming that instead of investing now, we will wait one year and then invest *only* if the price of the widgets goes up.

$$
\begin{aligned}
\text{NPV} &= \frac{1}{2}\left[-\frac{\$1.6M}{1+0.1}+\frac{\$0.3M}{1+0.1}+\frac{\$0.3M}{(1+0.1)^2}+\ldots\right]+\frac{1}{2}0 \\
&= \frac{\$0.85M}{1+0.1} \quad = \quad \$0.773M\ .
\end{aligned}
$$

Motivation

- If we wait a year before deciding whether to invest in the factory, the projects NPV is $\$0.773M$,
  whereas it is only $\$0.6M$ if we invest in the factory now.
- Clearly it is better to wait than to invest right away. The option to wait before deciding is here worth

$$\$0.173M \qquad (= \$0.773M - \$0.6M).$$

- Here, by calculating the NPV in the traditional way, the nature of the decision (investing) is still correct, but the timing of this decision is wrong.
- Importantly, forgetting the option to wait one underestimates the value of the project by about $30\%(= \$0.173M/\$0.6)$!

Motivation

Example 2:

- Consider now the same example, but with the cost of building the factory is $I = \$2.3M$, instead of $I = \$1.6M$. Similar calculations yield:
- If we wait a year before deciding whether to invest in the factory, the projects NPV is $\$0.455M$,
  whereas it is negative $-\$0.1M$ if we invest in the factory now.
- Here, by calculating the NPV in the traditional way, not only the timing of this decision (investing) is wrong, but also the nature of the decision is wrong!
- The option to wait before deciding is then worth $\$0.555M$ and constitutes most of the projects value.

Motivation

More generally:

- Business opportunities involve various options.
- When managers take decisions, including the timing of these decisions, they essentially choose among several options.
- Evaluation of these opportunities require understanding and evaluating the *imbedded* options.

◦ We are going to use our knowledge of financial options. It will enable us to examine real options.

Real Options

- A Real option is essentially an opportunity, but not the obligation, to take a deferred action/decision.
  - Hence all decisions available to corporations can be viewed as real options.
  - Their value can be assessed using option pricing.

  Decision making can be drastically affected.

Areas in which it employed:

- Oil Industry: Value of a field.
- Mining: Value of switching on and off production.
- Ecology: Assessing the cost of polluting

Real Options

- Real option are however everywhere.

An admittedly stupid example for you to understand:

- When you marry you get an option to divorce.
- Some rich divorcees knew about divorce option value when they married.

Real Options

One can attempt to classify Real options by the type of flexibility they correspond to:

- A deferral option is an call option found in most projects where one has the possibility to delay the start of the project;
- The option to abandon a project for a certain price is essentially a put option;
- The option to expand a project by paying to scale up the operations is a call option.
- The option to extend the life of a project by paying a supplement is also a call option.
- A switching option is the possibility to stop (or start) operations with the further possibility to re-start (or re-stop).

Real Options

- By nature, most of these options are American rather that European options. These are usually decisions available up to a certain time, rather than decisions one is only allowed to take at a given future date.
- Many decisions are however options on other options, called compound options.
  - A factory can be built in phases: (1) design phase, (2) engineering phase, and (3) construction. You have the option to defer or stop the project at each phase.
  - Later phases (2) and (3) are options that are contingent on the exercise of earlier options.

  Other example: Star Wars 9 is a valuable compound option on the Star Wars 8 option.

Real Options

- Is NPV wrong?
  Short answer: **No**, but calculating it the traditional implicitly assumes that projects present a single decision point and that decision point is now.
- However:
  - Many investment decisions can be postponed.
  - Many investments are partially reversible, meaning that if cash flows deteriorate, we can abandon, cut back and recover some capital, or temporarily stop operations.
  - Many projects can be expanded if future conditions make the incremental investment worthwhile.
- If this is the case, calculating NPV the traditional way amounts to assume that all these options are worth zero.

Real Options

- In this Section, we will examine real options embedded in industrial projects:
  - Like our motivation example.
  - We will examine a Case Study in portable telephony.
- In terms of pricing technology, in most cases, the Black-Scholes pricing formula is sufficient.

Reminder of Option Pricing

Black-Scholes Option Pricing Formula

A European call with exercise price K maturing at T, is worth

$$C(S_t|T,X) = S_t\,\Phi[\,d_1\,] - X\,e^{-r(T-t)}\,\Phi[\,d_2\,]$$

$$\text{where} \qquad d_2 \equiv \frac{ln(S_t/X)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,,$$

$$d_1 \equiv d_2+\sigma\sqrt{T-t}\,,$$

and $\Phi[x]=\int_{-\infty}^{x}\phi[u]du$ is the cumulative standard normal distribution, with $\phi[x]=\frac{1}{\sqrt{2\pi}}e^{-(x)^2/2}$ the standard normal density.

Case Study – Penelope's Personal Pocket Phones

- Penelope Phillips tries to determine whether she should start a company focussed on the next generation of wireless phone technology.
- In order to get the first generation to the market, she would have to invest $\$10M$ in the first year and the cash flow forecasts are as follows:

	2001	2002	2003	2004	2005	2006
Net Earnings	-4000	0	2405	2285	-65	-900
Depreciation	900	900	900	900	900	900
Inv. Net Work. Capital	1500	0	0	0	0	-1500
Capital Expenditure	5400	0	0	0	0	0
Free Cash Flow	-10000	900	3305	3185	835	1500

Case Study – Penelope's Personal Pocket Phones

- Market Conditions are as follows:
  (a) Comparable firms in the industry have an unlevered betas of around 1.2.
  (b) The term structure of interest rates is flat. All Treasury bonds yield 10 %.
  (c) The expected return on the market portfolio is $E[r_m] = 15\%$.

- Penelope wonders:

Question 1

Shall she invest in the first-generation phone?

Case Study – Penelope's Personal Pocket Phones

- To answer we need to calculate the NPV of the first-generation phone.
- To do this, we need to estimate the expected rate of return for this project, $E[r_a]$.
- This is an all-equity firm, therefore $E[r_a] = E[r_e]$.
- We then use the CAPM

$$E[r_a] = r_f + \beta_e \left(E[r_m] - r_f \right) .$$

- Similar firms have an unlevered beta of 1.2. Therefore

$$E[r_a] = 10.0\% + 1.2 \left(15.0\% - 10.0\% \right) = 16.0\% .$$

- Using this to discount the future free cash flows we obtain an NPV = -\$3.552M.
- Penelope can conclude that this investment is not worthwhile, hence seeking financing for it is pointless.

Case Study – Penelope's Personal Pocket Phones

- Penelope however knows that by starting the company today, she would have the opportunity to invest in the subsequent generation of phones.
- She expects this generation of phones would require an investment of $\$100M$.
- This investment would have to be entirely done in 4 years.
- She wonders:

Question 2 (a)

How large would her current expected value of second-generation phone have to be to justify the initial investment in the first-generation phone?

Case Study – Penelope's Personal Pocket Phones

- The investment in the first-generation phone will be justified if the option on the expected value of the second-generation is worth at least as much.
- The investment will be justified if this real call option is worth at least \$3.552M.
- Use the Black-Scholes formula to value this option, $c(S, X, T, r_f, \sigma)$.
- Search for the minimum current expected value of second stage project, S, which generates a call value c greater than $\$3.552M$.
  i.e. the S such that $c(S, K, T, r_f, \sigma) = \$3.552M$.

Case Study – Penelope's Personal Pocket Phones

- Inputs are as follows:
  - The "exercise price" is the cost of developing and producing the second generation of phone, $K = \$100$M.
  - The time to maturity of this option is 4 years, $T = 4$.
  - The riskless rate is $r_f = 10\%$.
- Penelope needs an assessment of the volatility, σ. She recons:
  (a) Cash flows from the second-generation phones are as uncertain as those from the first-generation phones.
  (b) The expected value of cash flows from the first-generation phones either increases by $u \equiv 64.9\%$ or decreases by $d \equiv 39.3\%$ each year. This essentially implies that the instantaneous standard deviation rate on these assets is $\sigma = \ln(1+u) = 0.5$.

Case Study – Penelope's Personal Pocket Phones

- Solving using the attached (to be downloaded) spreadsheet *penelope.xls*, the current expected value of second stage project, S, must be greater than \$26.3M:

PV of cash-flows ($\$M$), S	26.0	26.1	26.2	**26.3**	26.4	26.5
Real Call Option ($\$M$), c	3.46	3.49	3.53	**3.56**	3.59	3.62
Investment Justified?	No	No	No	**Yes**	Yes	Yes

- Notice interestingly, that even when this option is deep out of the money (that is when expected value of second stage project is currently, at $S = \$26.3M$, far from the cost of developing and producing the second generation of phone, $K = \$100$M) it still has enough value to induce investment in the unprofitable first generation of phones.

Case Study – Penelope's Personal Pocket Phones

- Penelope has second thoughts concerning here assessment of the volatility, σ. She actually thinks:
  - (a) Cash flows from the second-generation phones are more uncertain than those from the first-generation phones.
  - (b) This implies that the instantaneous standard deviation rate on these assets is more than $\sigma = 0.5$.
- So she wonders:

Question 2 (b)

What happens to the value of the opportunity to invest in the second project if the volatility of expected value increases?

Case Study – Penelope's Personal Pocket Phones

- Increasing the instantaneous standard deviation rate of the second-generation project value to $\sigma = 0.6$, we see that to be worthwhile, the current expected value of second stage project, S, must only be greater than \$21.2M:

PV of cash-flows (\$$M$), S	20.9	21.0	21.1	**21.2**	21.3	21.4
Real Call Option (\$$M$), c	3.46	3.49	3.53	**3.56**	3.60	3.64
Investment Justified?	No	No	No	**Yes**	Yes	Yes

- The value of the opportunity to invest in the second project increases as if the volatility of expected value increases, because there is no downside risk.

Case Study – Penelope's Personal Pocket Phones

- Penelope is told that she could sell her equipment at the end of the second year, for \$4M.
- So Penelope would like to know:

Question 3

Ignoring the second-generation phone, how valuable is the ability to sell the equipment in year 2?

Case Study – Penelope's Personal Pocket Phones

- Ignoring the possibility of investing in the second-generation phone, the option to sell the equipment can be valued like a put option, i.e. it is just an insurance.
- Use the Black-Scholes formula to value this option. Inputs are as follows:
  - The exercise price, $K = \$4M$, the expected price at which the equipment could be sold in two years.
  - The time to maturity of this option is 2 years, $T = 2$.
  - The instantaneous standard deviation rate of the project value is $\sigma = 0.5$.
  - The riskless rate is $r_f = 10\%$. The implicit investment horizon is here 2 years.

Case Study – Penelope's Personal Pocket Phones

- The item which is most difficult to determine is the current underlying asset price, S.
- This is the value of the assets that will be then sold.
- Someone buying the assets in two years, only gets the free cash flows generated by the first-generation phones from then onwards:

	2003	2004	2005	2006
Net Earnings	2405	2285	-65	-900
Depreciation	900	900	900	900
Inv. Net Work. Capital	0	0	0	-1500
Capital Expenditure	0	0	0	0
Free Cash Flow	3305	3185	835	1500

  The NPV of these expected cash flows is \$5.672M.
- Note that this NPV already incorporates the residual value of the assets. Therefore the current underlying asset price we should input is, $S = \$5.672$.

Case Study – Penelope's Personal Pocket Phones

- Computing the Black-Scholes formula we obtain the value of a call with similar terms. We then use the put-call parity

$$p = c + PV(K) - S ,$$

  to obtain the value of this real put option, which is $p = \underline{\$0.364M}$.
- Notice that this value is fairly sensitive to the expected volatility:

Volatility, σ	0.4	0.5	0.6
Put (Real) Option, p	\$0.217M	\$0.364M	\$0.508M

Readings

Book:

- Chapter 36 of Hull.

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PAUL GOMPERS

Penelope's Personal Pocket Phones

Penelope Phillips sat in her laboratory at the University of the North and tried to determine whether she should start a company focussed on the next generation of wireless phone technology. Her work in electrical engineering and the 15 patents she held told her that she could enter the market with a new generation of phones. The problem was, however, that the market was quite competitive and she knew that it would therefore be difficult to succeed. Penelope understood that getting into the market today might lead to much bigger opportunities in the future.

Penelope looked at her projections. In order to get the first generation to market she would have to invest $10 million in the first year. The cash flow forecasts in **Exhibit 1** show what she expected to earn on this first product. Comparable firms in the industry had unlevered betas of around 1.2 and annual standard deviation of returns of 50%, so she set out to see if the investment was worth the time and energy. The 10-year Treasury bond was yielding 10.0% at the time.

Penelope also knew that by starting the company today, she would have the opportunity to invest in the subsequent generation of phones. Given the expectations about future costs, this opportunity would take $100 million to bring to market. She estimated, however, that she would have to make the investment four years from now when the entire $100 million would have to be invested. She wondered how big the current expected value on the second-generation phone would have to be in order to justify investing in the proposed project. She set about trying to calculate that value.

Thirty minutes into her calculations, Jay Thomas called to tell her that she would be able to start the project using equipment that could easily be sold for $4 million in year two if demand was not high for her phones. By year two, she could be reasonably confident of what the value of her first generation of phones would be; that is, she assumed that the value would be known with certainty at that time. If that were the case, Penelope wondered what the value of the first project would be. She decided to ignore the second-generation phones for a while and focus on this new problem. Did the possibility of selling the equipment at the end of year two make the first project worth it even if there were no follow-on project? If she modeled the annual change in value, Penelope figured that the expected value of cash flows from the first-generation phones would either increase by 64.9% or decrease by 39.3% each year. She wondered how to proceed with her analysis.

Professor Paul Gompers prepared this case. HBS cases are developed solely as the basis for class discussion. Cases are not intended to serve as endorsements, sources of primary data, or illustrations of effective or ineffective management.

Exhibit 1 Pro forma projections for Penelope's Personal Pocket Phones

	2001	2002	2003	2004	2005	2006
INCOME STATEMENT						
Net Sales	$0	$8,600	$14,000	$18,000	$14,500	$8,000
COGS	0	3,500	5,300	7,100	6,500	3,200
Gross Profit	0	5,100	8,700	10,900	8,000	4,800
SG&A	1,900	2,300	3,000	3,700	4,200	4,000
R&D	2,100	2,800	3,000	3,500	3,900	2,000
EBIT	(4,000)	0	2,700	3,700	(100)	(1,200)
Income Tax[a]	0	0	295	1,415	(35)	(300)
Net earnings	0	0	2,405	2,285	(65)	(900)
Depreciation	900	900	900	900	900	900
Investment in Net Working Capital	1,500	0	0	0	0	(1,500)

Source: Casewriter calculations

[a]If a firm makes a loss but has paid taxes in previous years it receives a refund on previous taxes.