FIXED INCOME SECURITIES

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FIXED INCOME INSTRUMENTS

What makes fixed-income instruments special?

- Known (or almost) cash flows associated to the instrument
- Government bonds: liquid, diverse market – lots of information – with extremely low default risk
- Implies remaining risks can be better identified and managed

What are risks of fixed-income instruments?

- Credit risk – default, downgrade, credit spread
- Capital risk – if not held until maturity, changes in interest rates affect bond's resale price
- Inflation – bond cash flows typically stated in nominal terms, so their real value is depleted by inflation
- Exchange-rate risk if investment across world markets

Fixed income security markets

- Used to determine interest rates and price other securities

FIXED INCOME INSTRUMENTS (1)

Zero-coupon bonds

- Pay face value ("par") at maturity

Example. A T-bill (US govt. zcb) with maturity in 1 year costs 99.97% right now and has a face value of $1000. This means you pay $999.7 right now and receive a cash flow of $1000 in one year.

Coupon bonds

- Pay face value at maturity, plus regular interests

Example. A 2-year T-note (US govt. short term coupon bond) with semi-annual coupon of 2% is sold today for 101.53% and has face value of $1000. This means that you pay $1015.3 right now, you receive $10 in 6 months, $10 in 1 year, $10 in 1.5 years, and $1010 in two years from now.

Some pricing conventions:

- Price & coupons listed as percentage of par – thus, prices & coupons are always on base 100
- Coupon rates are *annualized*, even if paid *n* times per year. E.g., 5% coupon, semiannual, means 2.5% of face value paid every 6 months

U.S. TREASURY SECURITIES

Treasury bills, notes, and bonds

- T-bills are zero-coupon bonds with maturity up to 1 year
- T-notes are coupon bonds with maturity up to 10 years
- T-bonds are coupon bonds with longer maturities, up to 30 years

Others:

- TIPS - inflation protected treasury securities
- FRNs – floating rate notes
- STRIPS - artificial zero-coupon bonds: separate coupon from principal

FIXED INCOME INSTRUMENTS (2)

Money market

- Short-term loans, mostly between banks, uncollateralized

Example. The effective Federal Funds rate on Oct. 10, 2020, was 0.09%. This is the average of all rates charged between banks to lend each other money for deposit in the Federal Reserve. On the same day, the LIBOR overnight rate was 0.08263%, and the 1-month rate was 0.147%, meaning this is on average what banks charged each other for, respectively, overnight and 1-month loans in the London market.

Repo market

- Collateralized loans. Formally, a *repurchase agreement* (repo) is an agreement to sell securities to another party, and to buy them back at a fixed date, for a fixed amount (loan + repo rate)
- A reverse repo is an agreement to buy securities now and resell at a fixed date for a fixed price

Example. On October 10, 2020, the *broad general collateral rate* (BGCR) was 0.06%. This means that to borrow money overnight with a collateral of US Treasury securities, a bank had to pay a repo rate of 0.06%.

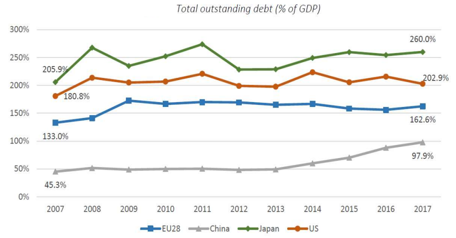
- Repo rates close to zero may lead to many *failures to deliver*: the "keeper" of the security does not hand them back at maturity. Regulation to prevent failures may artificially increase bond prices.

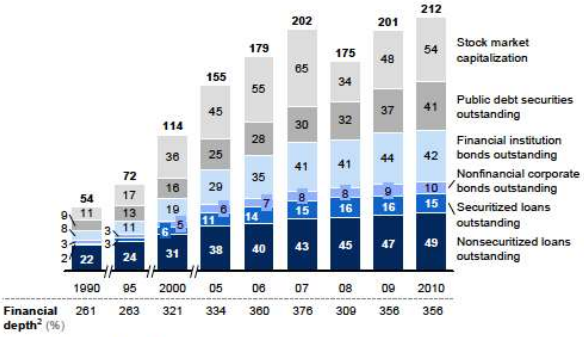
Others: subordinate debt (w/ M. Pech), derivatives (later)

SOME NUMBERS

Source: Gleisner & Thomadakis (2018). *European Capital Markets Institute →*

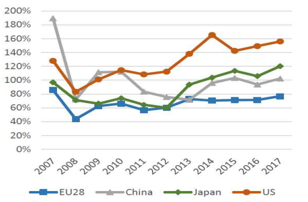
Source: BIS





1 Based on a sample of 79 countries.
2 Calculated as global debt and equity outstanding divided by global GDP.

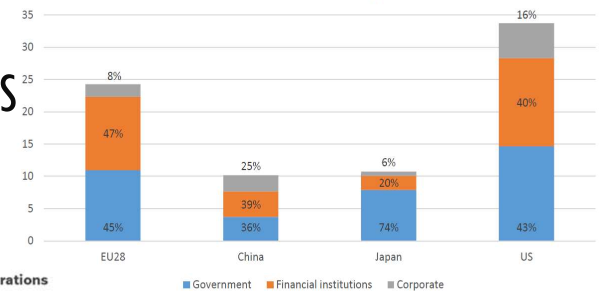
Total outstanding equity (% of GDP)



SOME NUMBERS

Source: Gleisner & Thomadakis (2018).
European Capital Markets Institute →

Figure 6. Total outstanding debt per issuer (€ trillion, end-2017)



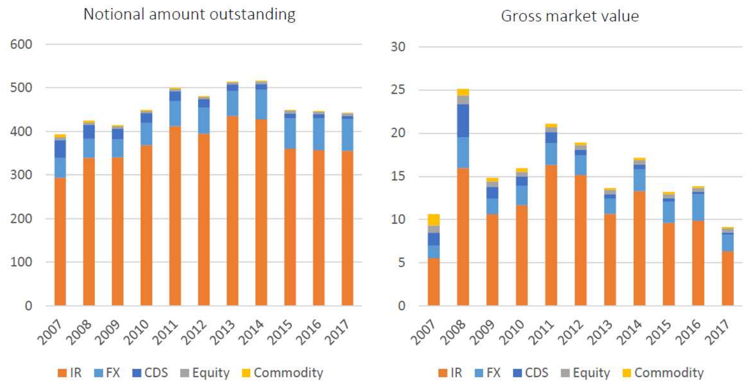
Source: 2018 ECMI Statistical Pa

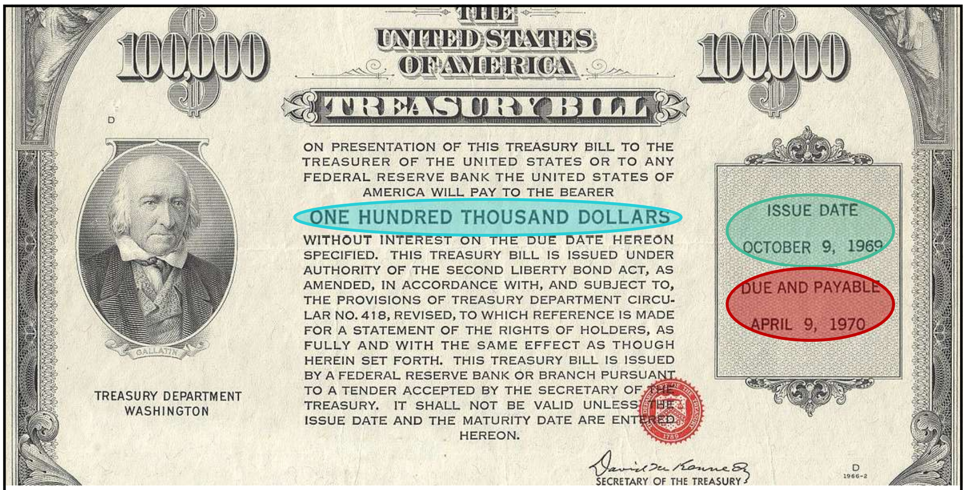
Source: SIFMA 2019 Outlook

SOME NUMBERS - DERIVATIVES

Source: Gleisner & Thomadakis (2018).
European Capital Markets Institute

Figure 12. Notional amount outstanding and gross market value of OTC derivatives (€ trillion)





100,000

THE UNITED STATES OF AMERICA

TREASURY BILL

100,000

D

ON PRESENTATION OF THIS TREASURY BILL TO THE TREASURER OF THE UNITED STATES OR TO ANY FEDERAL RESERVE BANK THE UNITED STATES OF AMERICA WILL PAY TO THE BEARER

ONE HUNDRED THOUSAND DOLLARS

WITHOUT INTEREST ON THE DUE DATE HEREON SPECIFIED. THIS TREASURY BILL IS ISSUED UNDER AUTHORITY OF THE SECOND LIBERTY BOND ACT, AS AMENDED, IN ACCORDANCE WITH, AND SUBJECT TO, THE PROVISIONS OF TREASURY DEPARTMENT CIRCULAR NO. 418, REVISED, TO WHICH REFERENCE IS MADE FOR A STATEMENT OF THE RIGHTS OF HOLDERS, AS FULLY AND WITH THE SAME EFFECT AS THOUGH HEREIN SET FORTH. THIS TREASURY BILL IS ISSUED BY A FEDERAL RESERVE BANK OR BRANCH PURSUANT TO A TENDER ACCEPTED BY THE SECRETARY OF THE TREASURY. IT SHALL NOT BE VALID UNLESS THE ISSUE DATE AND THE MATURITY DATE ARE ENTERED HEREON.

ISSUE DATE

OCTOBER 9, 1969

DUE AND PAYABLE

APRIL 9, 1970

GALLATIN

TREASURY DEPARTMENT
WASHINGTON

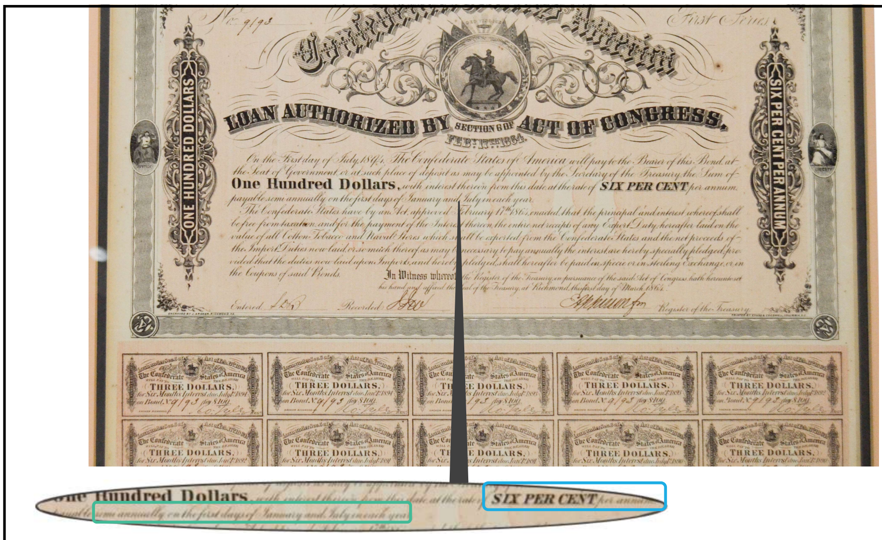
SECRETARY OF THE TREASURY

D

1966-2

ZCB ELEMENTS

- Face value: how much you receive at maturity
- Issue date
- Due date (*maturity*): when the face value is paid



No. 9193

First Series

LOAN AUTHORIZED BY SECTION 6 OF ACT OF CONGRESS, FEB. 17th 1864.

ONE HUNDRED DOLLARS

SIX PER CENT PER ANNUM

On the First day of July, 1874, The Confederate States of America will pay to the Bearer of this Bond at the Seat of Government, or at such place of deposit as may be appointed by the Secretary of the Treasury the Sum of **One Hundred Dollars**, with interest thereon from this date at the rate of **SIX PER CENT** per annum, payable semi annually on the first days of January and July in each year.

The Confederate States have by an Act, approved February 17th 1864, enacted that the principal and interest whereof shall be free from taxation, and for the payment of the Interest thereon, the entire net receipts of any Export Duty, hereafter laid on the value of all Cotton, Tobacco and Naval Stores which shall be exported from the Confederate States and the net proceeds of the Import Duties now laid, or so much thereof as may be necessary to pay annually the interest, are hereby specially pledged, provided that the duties now laid upon Imports, and hereby pledged, shall hereafter be paid in specie or in sterling Exchange, or in the Coupons of said Bonds.

In Witness whereof the Register of the Treasury, in pursuance of the said Act of Congress, hath hereunto set his hand and affixed the Seal of the Treasury, at Richmond, this first day of March 1864.

Entered

Recorded

Register of the Treasury

The Confederate States of America will pay to the bearer THREE DOLLARS, for Six Months Interest due July 1st 1874 on Bond No. 9193 for $100.

COUPON BOND ELEMENTS

- ZCB elements, plus
- Coupon (rate, annualized)
- Coupon frequency (and maturity dates)

PRICE?

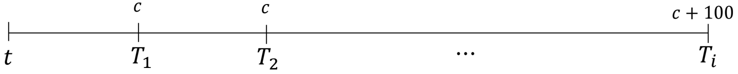
The bond's price is not part of its definition. Determined by market

- primary or secondary
- organized as auction, continuous market, over the counter, etc.

What should a bond's price be?

- Bond = series of cash flows
- Present value of cash flows is bond's "value" today = bond's fair price
- Discounting cash flows requires using a cost of capital (discount rate, interest rate…)

Discounted cash flows principle



$$P_t = Z(t, T_1) \times c + Z(t, T_2) \times c + \cdots + Z(t, T_i) \times [c + 100]$$

$Z(t, T_j)$ is the discount factor between dates t and T_j, when the j-th coupon is paid

YIELD TO MATURITY - YTM

The *yield to maturity* (YTM) is a fixed rate with which bond cash flows are discounted to determine present value/price of bond

- Required return by investors given available investment opportunities at the same risk
- Fixed rate of return implicit given a bond's cash flows and current market price
- The given YTM is "annualized"

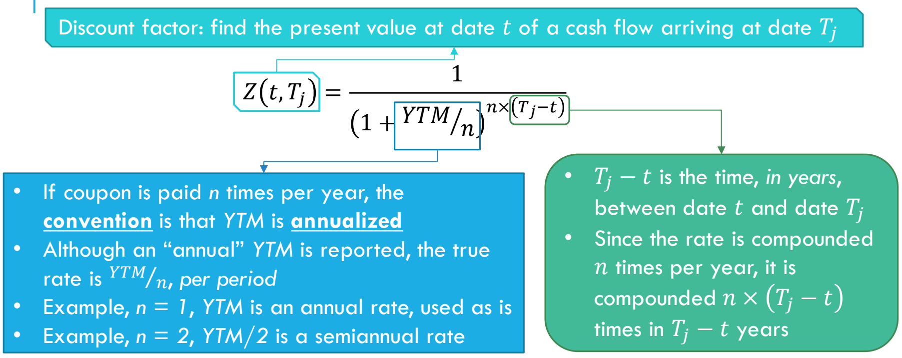
The yield to maturity equals the expected fixed rate of return for a bond

- if the bond is held until its maturity,
- if there are no defaults on payments, and
- if all coupons are reinvested at the same rate

In this case, the discount factor between dates t and T_j

$$Z(t, T_j) = {}^{1}\Big/\left(1 + \frac{YTM}{n}\right)^{n \times (T_j - t)}$$

ANATOMY OF A FORMULA



Discount factor: find the present value at date t of a cash flow arriving at date T_j

$$\boxed{Z(t,T_j)} = \frac{1}{\left(1+\boxed{{}^{YTM}/_{n}}\right)^{\boxed{n\times(T_j-t)}}}$$

- If coupon is paid n times per year, the **convention** is that YTM is **annualized**
- Although an "annual" YTM is reported, the true rate is ${}^{YTM}/_{n}$, *per period*
- Example, $n = 1$, YTM is an annual rate, used as is
- Example, $n = 2$, $YTM/2$ is a semiannual rate

- $T_j - t$ is the time, *in years*, between date t and date T_j
- Since the rate is compounded n times per year, it is compounded $n \times (T_j - t)$ times in $T_j - t$ years

$$P_t = \frac{c}{\left(1+\frac{YTM}{n}\right)^{n\times(T_j-t)}} + \frac{c}{\left(1+\frac{YTM}{n}\right)^{n\times(T_{j+1}-t)}} + \cdots + \frac{c+100}{\left(1+\frac{YTM}{n}\right)^{n\times(T_i-t)}}$$

General formula for the price of a bond at any date (using YTM)

BOND PRICE CALCULATION (YTM)

Simple examples: annual coupons, price at bond's issue time

Exercise. A T-bill with maturity in 1 year is issued today. If the relevant YTM is 2%, what is the fair price of this zero-coupon bond?

$$P = \frac{100}{1+0.02} = 98.039$$

Answer: the price today is 98.039% of its face value

Exercise. A 3-year bond is issued, with annual coupon of 5%. If investors require a YTM of 2%, what is the bond's fair price?

$$P = \frac{5}{1+0.02} + \frac{5}{(1+0.02)^2} + \frac{105}{(1+0.02)^3} = 108.652$$

Answer: the price today is 108.652% of its face value

BOND PRICE CALCULATION (YTM)

Simple example: *n* coupons per year, price at issue

- With *n* payments per year, time between two coupons ($T_{j+1} - T_j$) is always $^1/_n$ years

Exercise. What is the price of a 10-year bond with semiannual coupon of 3% if the YTM is 4%?

$$P = \underbrace{\left[\frac{1.5}{1.02} + \frac{1.5}{(1.02)^2} + \cdots + \frac{1.5}{(1.02)^{20}}\right]}_{\text{PV of regular annuity}} + \frac{100}{(1.02)^{20}} = 1.5 \times \left[\frac{1 - \frac{1}{(1.02)^{20}}}{0.02}\right] + \frac{100}{(1.02)^{20}} = 91.8243$$

Answer: the price today is 91.8243% of its face value

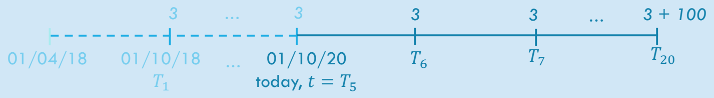
In general, the price of a bond with *n* coupons per year, at issue, is

$$\boxed{P = c \times \left[\frac{1 - \frac{1}{\left(1 + {}^{YTM}/_{n}\right)^{i}}}{{}^{YTM}/_{n}}\right] + \frac{100}{\left(1 + {}^{YTM}/_{n}\right)^{i}}}$$

BOND PRICE CALCULATION (YTM)

Example: *n* coupons per year, price at a *later* date than issue, but still a coupon-payment date

Example. A 10-year bond with 6% semiannual coupon was issued on April 1, 2018. Today is October 1, 2020 (a coupon has just been paid!). What is the price of this bond today, if the required YTM is 5%?



- Coupon of 3 paid every six months; YTM of 2.5% compounded every 6 months
- 15 coupon payments remaining, first arrives in exactly 6 months from now

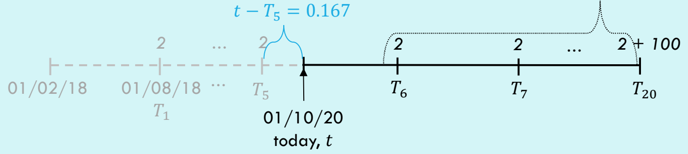
$$P = 3 \times \frac{1 - \frac{1}{(1.025)^{15}}}{0.025} + \frac{100}{(1.025)^{15}} = 106.191$$

The fair price today of this bond is 106.191% of its face value

BOND PRICE CALCULATION (YTM)

Example: *n* coupons per year, price at a *later* date than issue, *not* a coupon-payment date

Exercise. A 10-year bond with 4% semiannual coupon was issued on February 1, 2018. Today is October 1, 2020 (last coupon was paid on August 1, 2020; next coupon is due on February 1, 2021). What is the price of this bond today, if the required YTM is 5%?



- The current date is located between coupon 5 and 6. Thus, $T_{j-1} = T_5$ =August 1, 2020.
- The last coupon is T_{20}, thus $20 - 5 = 15$ coupons remain
- Time elapsed since the last coupon is $t - T_5 = \frac{61}{365} = 0.167$ (time always measured in years!)
- Find PV of cash flows at time T_5 and then "move it" to time t by multiplying by $(1.025)^{2\times0.167}$

$$P = (1.025)^{2\times0.16} \times \left\{ 2 \times \left[\frac{1 - \frac{1}{(1.025)^{15}}}{0.025} \right] + \frac{100}{(1.025)^{15}} \right\} = 94.586$$

BOND PRICE CALCULATION (YTM)

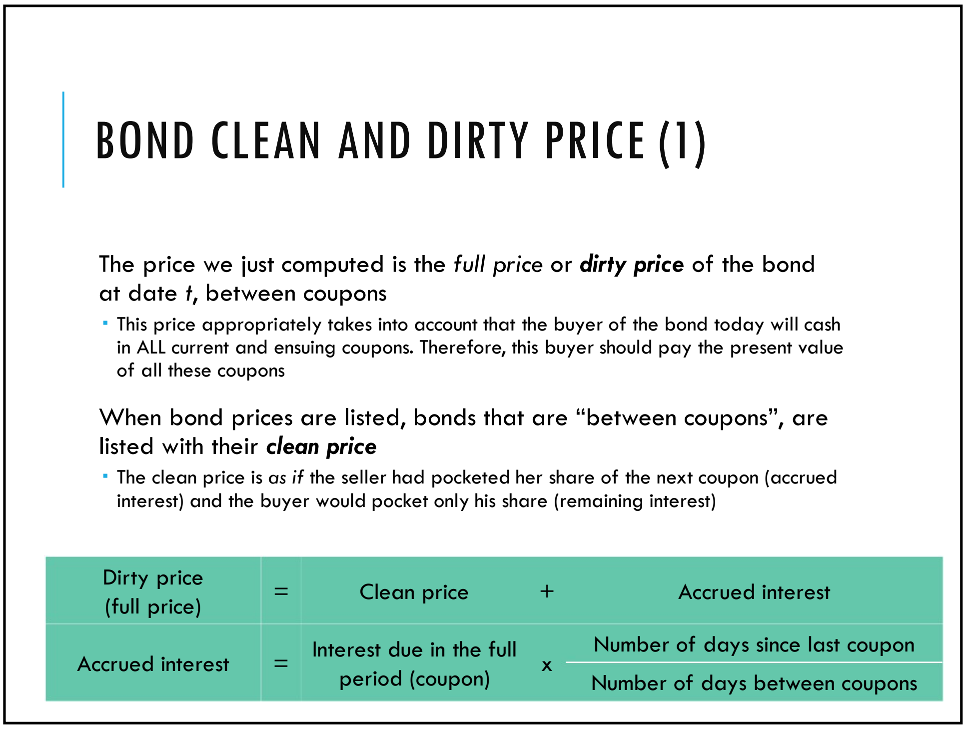
In previous example, we managed to use formula of *PV* of ordinary annuity

In general, if located at a date t, $T_{j-1} \leq t < T_j$,

- $i - j + 1$ coupon payments left until maturity→ find PV of these coupons at date t
- To use annuities formula in this case, one must be aware that the PV will be "located" at date T_{j-1} → bring it forward in time to date t after calculation

Thus,

$$P_t = \left(1 + \frac{YTM}{n}\right)^{n\times(t-T_{j-1})} \times c \times \left[\frac{1 - \frac{1}{\left(1 + {}^{YTM}/_{n}\right)^{i-j+1}}}{{}^{YTM}/_{n}} \right] + \frac{100}{\left(1 + {}^{YTM}/_{n}\right)^{i-j+1}}$$



BOND CLEAN AND DIRTY PRICE (1)

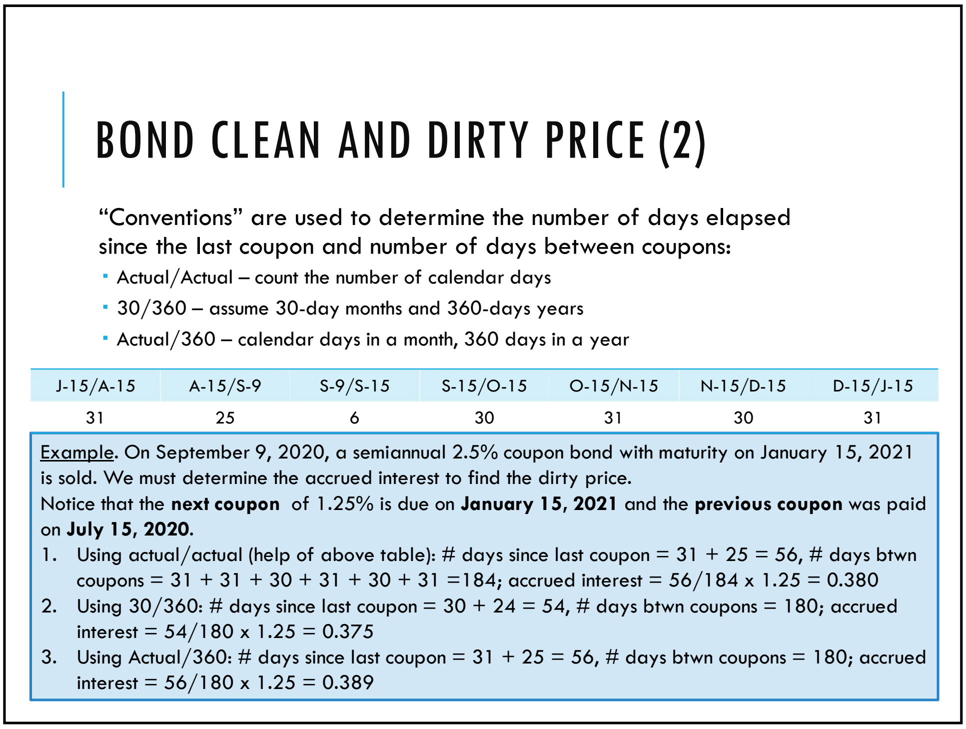
The price we just computed is the *full price* or ***dirty price*** of the bond at date *t*, between coupons

- This price appropriately takes into account that the buyer of the bond today will cash in ALL current and ensuing coupons. Therefore, this buyer should pay the present value of all these coupons

When bond prices are listed, bonds that are "between coupons", are listed with their ***clean price***

- The clean price is *as if* the seller had pocketed her share of the next coupon (accrued interest) and the buyer would pocket only his share (remaining interest)

Dirty price (full price)	=	Clean price	+	Accrued interest
Accrued interest	=	Interest due in the full period (coupon)	x	Number of days since last coupon Number of days between coupons



BOND CLEAN AND DIRTY PRICE (2)

"Conventions" are used to determine the number of days elapsed since the last coupon and number of days between coupons:

- Actual/Actual – count the number of calendar days
- 30/360 – assume 30-day months and 360-days years
- Actual/360 – calendar days in a month, 360 days in a year

J-15/A-15	A-15/S-9	S-9/S-15	S-15/O-15	O-15/N-15	N-15/D-15	D-15/J-15
31	25	6	30	31	30	31

Example. On September 9, 2020, a semiannual 2.5% coupon bond with maturity on January 15, 2021 is sold. We must determine the accrued interest to find the dirty price.
Notice that the **next coupon** of 1.25% is due on **January 15, 2021** and the **previous coupon** was paid on **July 15, 2020.**

1. Using actual/actual (help of above table): # days since last coupon = 31 + 25 = 56, # days btwn coupons = 31 + 31 + 30 + 31 + 30 + 31 =184; accrued interest = 56/184 x 1.25 = 0.380
2. Using 30/360: # days since last coupon = 30 + 24 = 54, # days btwn coupons = 180; accrued interest = 54/180 x 1.25 = 0.375
3. Using Actual/360: # days since last coupon = 31 + 25 = 56, # days btwn coupons = 180; accrued interest = 56/180 x 1.25 = 0.389

POSTED BOND PRICES

The table contains T-bond* prices posted on the WSJ on September 9, 2020 (the WSJ's source is Tullett Prebon)

- BID is the highest price a buyer in the markets was willing to pay
- ASK is the lowest price a seller in the markets was willing to accept
- CHG is how much the price changed w.r.t. previous trading day
- ASKED YIELD is a YTM computed using the current ask price (soon!)
- When posted prices are used to compute interest rates, the *mid price* is used (average of ask and bid)

MATURITY	COUPON	BID	ASKED	CHG	ASKED YIELD
9/15/2020	1.375	100	100	-0	-0.54
9/30/2020	1.375	100	100	unch.	-0.12
9/30/2020	2	100	100	unch.	-0.31
…					
1/15/2022	2.5	103.1	103.1	0.002	0.138
1/31/2022	1.375	101.2	101.2	0.002	0.138
1/31/2022	1.5	101.3	101.3	0.002	0.134
…					
11/15/2023	2.75	108.1	108.1	0.02	0.171
11/30/2023	2.125	106.1	106.1	0.02	0.175
11/30/2023	2.875	108.2	108.2	0.018	0.176
2/15/2030	1.5	107.3	107.3	0.1	0.63
5/15/2030	0.625	99.2	99.21	0.092	0.661
5/15/2030	6.25	152.3	152.3	0.13	0.61
8/15/2030	0.625	99.14	99.15	0.122	0.679
2/15/2031	5.375	147.2	147.2	0.154	0.65
2/15/2036	4.5	151.2	151.3	0.93	0.899
…					
2/15/2049	3	137.2	137.3	1.138	1.387
5/15/2049	2.875	134.3	135	1.118	1.391
8/15/2049	2.25	119.3	119.3	1.086	1.407
11/15/2049	2.375	123	123.1	1.766	1.405
2/15/2050	2	114	114.1	1.756	1.41
5/15/2050	1.25	95.24	95.26	1.04	1.424
8/15/2050	1.375	98.26	98.28	1.058	1.421

* T-bonds pay semi-annual coupons and use the actual/actual convention for pricing

Exercise. On 9/9/2020, what would you pay to immediately buy the first T-bond with maturity 11/30/2023?

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SUMMARY OF “CONVENTIONS”

Prices and coupons are posted as percentages of the bond's face value. Therefore, they are always on base 100.

Coupons and YTMs are “annualized” versions of their values per-period. Suppose coupons are paid *n* times per year:

- To know what coupon is paid each period, divide the annualized coupon by *n*
- To know what rate to use for discounting (*n*-compounded rate), divide annualized YTM by *n*

Between coupons, **clean** bond prices are posted. The full price of the bond, called **dirty** price, must be computed – this is what you pay!

- Dirty price = sum of discounted cash flows of bond (discounted cash flows principle)
- Dirty price = clean price + accrued interest

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* SKIP

OTHER PRICING CONVENTIONS

Sometimes, for zcb, instead of posting the price, an implied interest rate is used:

- US T-bills (US treasury zero-coupon bonds, maturity up to 1 year). The implied <u>*discount rate*</u> is posted: If today is date *t* and the T-bill's maturity is at *T*,

$$d = \frac{100 - P}{100} \times \frac{1}{T - t} \Rightarrow P = 100 \times (1 - (T - t) \times d)$$

<u>Exercises</u>.

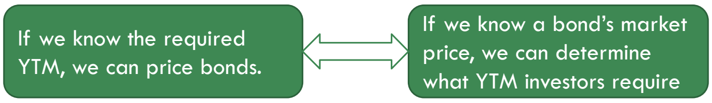
1. If a 26-weeks T-bill is trading for 99.23% of face value, what discount rate will be posted?
2. On January 28, 2020, the 4-weeks T-bill was quoted at a discount of 1.50%. What was the price of this bond on January 28, 2020?

- French government BTF (*bons du tresor à taux fixe,* zcb, up to 1 year). The implied yield or <u>rate of return</u> is posted: If today is *t* and the BTF's maturity is at *T*, with distance between dates measured in years,

$$y = \frac{100 - P}{P} \times \frac{1}{T - t} \Rightarrow P = \frac{100}{[1 + (T - t) \times y]}$$

<u>Exercise</u>. On January 08, 2020, the French Treasury Agency (AFT) issued 13-weeks BTF at an average rate of -0.610%. What price did investors pay for these bonds?

CALCULATING YTM (1)


If we know the required YTM, we can price bonds.

If we know a bond's market price, we can determine what YTM investors require

<u>Example</u>. A 13-weeks BTF issued by the French Treasury Agency is priced at 100.1527% of par. We can obtain the 13-weeks YTM from

$$100.1527 = \frac{100}{1 + {YTM}/_{4}} \Rightarrow \frac{YTM}{4} = \frac{100}{100.1527} - 1 = -0.001525,$$

When annualized, this means $YTM = -0.001525 \times 4 = -0.00609 \approx -0.610\%$

- In general, knowing *P* and the coupons, solve for *YTM*/*n* from equation

$$P = \frac{c}{1 + {YTM}/_{n}} + \frac{c}{\left(1 + {YTM}/_{n}\right)^{2}} + \cdots + \frac{c + 100}{\left(1 + {YTM}/_{n}\right)^{i}},$$

then multiply by *n* to obtain annualized YTM

- Solving the above equation for *YTM*/*n* is not trivial → Excel or financial calculator (IRR)
- [Notice that BTF's *return rate* is exactly the same as its YTM! (see previous exercise)]

CALCULATING YTM (2)

Exercises. Find the YTM for the following bonds:

1. A T-bill with maturity 1 year, currently sold at 96.6184%
2. A T-note with maturity 1 year and 3% semi-annual coupon, currently sold at 99.4158%
3. A T-note with maturity 4 years and 2.5% semi-annual coupon, currently sold at 101.9129% (Excel)

Solution.

1. Set up the equation: $96.6184 = \frac{100}{1+YTM} \Rightarrow YTM = \frac{100}{96.6184} - 1 = 3.5\%$ (annual compounding)
2. Set up the equation: $99.4158 = \frac{1.5}{1+\frac{YTM}{2}} + \frac{101.5}{\left(1+\frac{YTM}{2}\right)^2}$. Let $x = \frac{1}{1+\frac{YTM}{2}}$, to rewrite

$$99.4158 = 1.5x + 101.5x^2 \Rightarrow 101.5x^2 + 1.5x - 99.4158 = 0$$

Apply the quadratic formula, $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, to find $x = 0.982318$, and solve

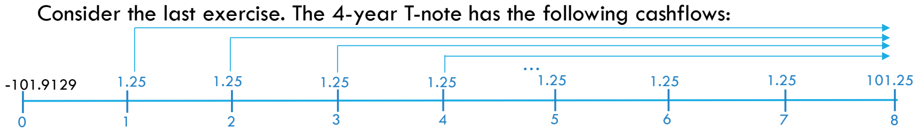
$$YTM = 2 \times \left(\frac{1}{x} - 1\right) = 2 \times \left(\frac{1}{0.982318} - 1\right) = 3.6\%$$

3. Must use Excel or a financial calculator to solve it. Using Excel's IRR (TRI), we obtain 1%, semiannually → 2% (annualized) YTM

INTERPRETATION OF YTM

Remember that *"The yield to maturity equals the expected fixed rate of return for a bond if the bond is held until its maturity, if there are no defaults on payments, and if all coupons are reinvested at the same rate"*

Consider the last exercise. The 4-year T-note has the following cashflows:



Using the YTM calculated before (2%), we can find the future value of the investment in this bond *if we hold it until maturity* and *reinvest all coupons at YTM*

- First coupon earns an interest of 1% per period, during 7 periods, so its final value is $1.25 \times (1.01)^7 = 1.340$. Second coupon earns 1% during 6 periods, $\mathrm{FV} = 1.25 \times (1.01)^6 = 1.327\ldots$
- Face value is received at final date, so it earns no interest…
- FV of all cash flows: $FV = 1.25 \times \left[\frac{(1.01)^8-1}{0.01}\right] + 100 = 110.3571$
- What rate applied to initial investment (101.9129) for 8 periods leads to final value (110.3571)?

$101.9129 \times (1+r)^8 = 110.3571 \Rightarrow r = \sqrt[8]{110.3571/101.9129} - 1 = 1\%$

The **YTM**!
(annualized 2%)

INTEREST-RATE RISK

Version 1: using YTM

PRICE-YTM RELATION (1)

Example. . Consider a three-year government bond with a semi-annual 2% coupon.

1. If investors require a 3% YTM on this bond, its price today is

$$P_c(0,3) = \frac{1}{1.015} + \frac{1}{(1.015)^2} + \frac{1}{(1.015)^3} + \cdots + \frac{101}{(1.015)^6} = \frac{1-(1.015)^{-6}}{0.015} + \frac{\mathbf{100}}{(1.015)^6} = 97.15141\%$$

2. If investors require a 1.5% YTM on this bond, its price today is

$$P_c(0,3) = \frac{1-(1.0075)^{-6}}{0.0075} + \frac{100}{(1.0075)^6} = 101.4614\%$$

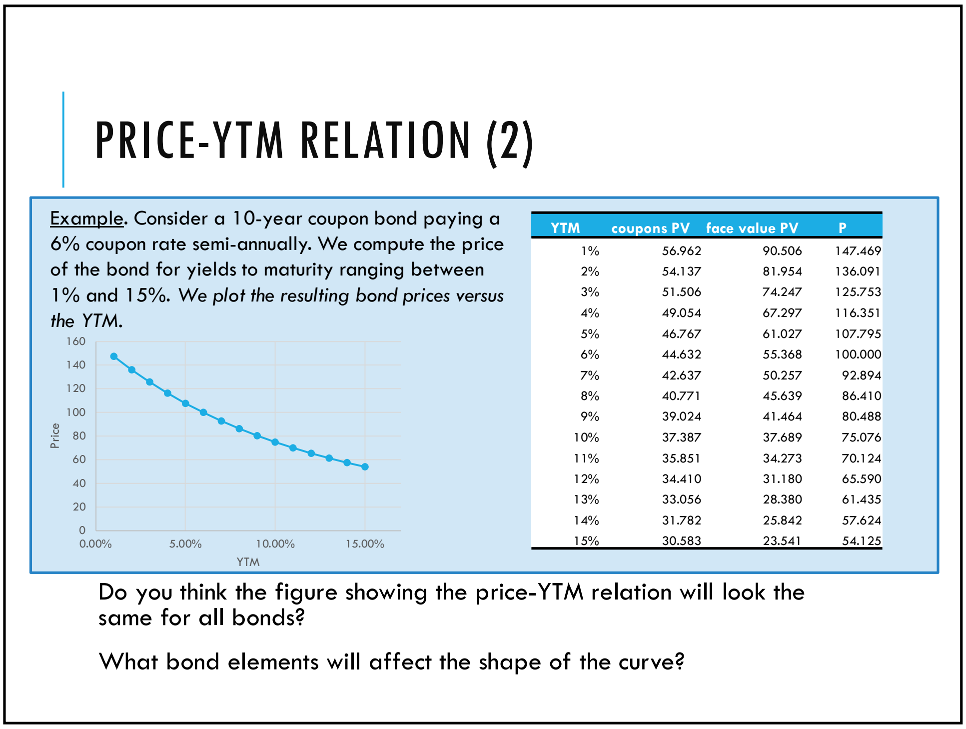
3. If investors require a 2% YTM on this bond, its price today is

$$P_c(0,3) = \frac{1-(1.01)^{-6}}{0.01} + \frac{100}{(1.01)^6} = 100\%$$

The higher the required YTM, the lower the price

Moreover, coupon-YTM relation:

- If coupon < YTM, price < 100
- If coupon > YTM, price > 100
- If coupon = YTM, price = 100



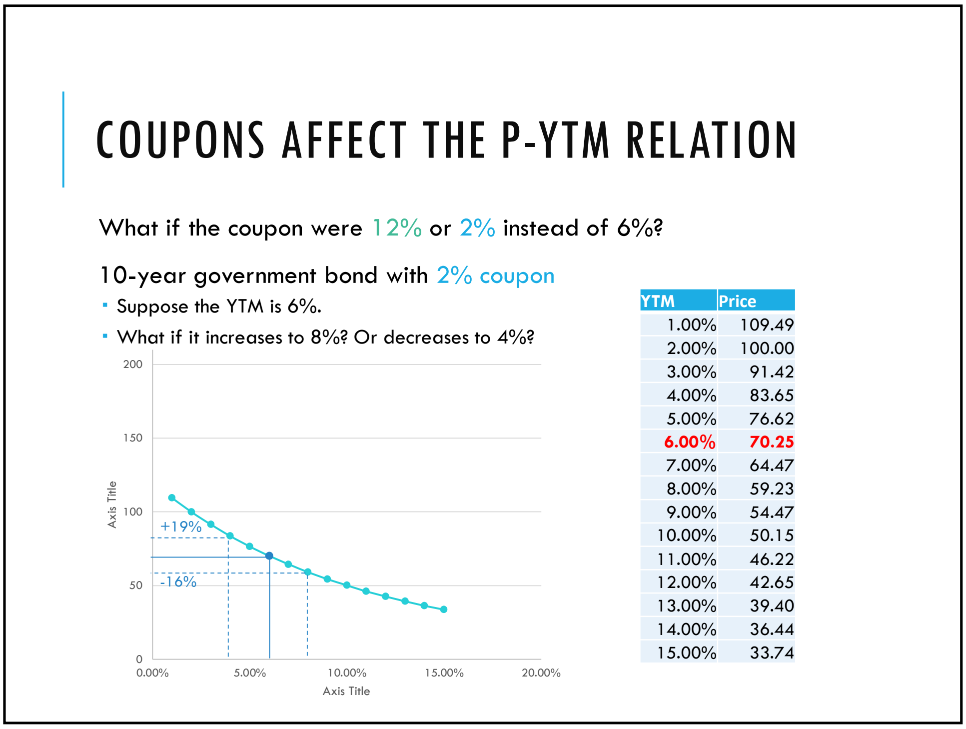
PRICE-YTM RELATION (2)

Example. Consider a 10-year coupon bond paying a 6% coupon rate semi-annually. We compute the price of the bond for yields to maturity ranging between 1% and 15%. *We plot the resulting bond prices versus the YTM.*

YTM	coupons PV	face value PV	P
1%	56.962	90.506	147.469
2%	54.137	81.954	136.091
3%	51.506	74.247	125.753
4%	49.054	67.297	116.351
5%	46.767	61.027	107.795
6%	44.632	55.368	100.000
7%	42.637	50.257	92.894
8%	40.771	45.639	86.410
9%	39.024	41.464	80.488
10%	37.387	37.689	75.076
11%	35.851	34.273	70.124
12%	34.410	31.180	65.590
13%	33.056	28.380	61.435
14%	31.782	25.842	57.624
15%	30.583	23.541	54.125

Do you think the figure showing the price-YTM relation will look the same for all bonds?

What bond elements will affect the shape of the curve?



COUPONS AFFECT THE P-YTM RELATION

What if the coupon were 12% or 2% instead of 6%?

10-year government bond with 2% coupon

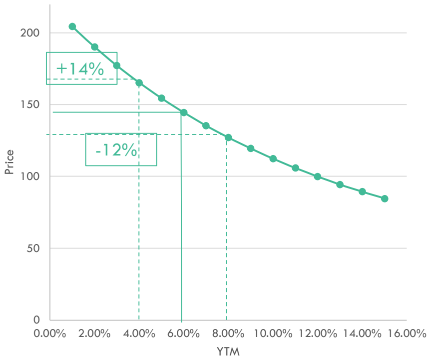
- Suppose the YTM is 6%.
- What if it increases to 8%? Or decreases to 4%?

YTM	Price
1.00%	109.49
2.00%	100.00
3.00%	91.42
4.00%	83.65
5.00%	76.62
6.00%	**70.25**
7.00%	64.47
8.00%	59.23
9.00%	54.47
10.00%	50.15
11.00%	46.22
12.00%	42.65
13.00%	39.40
14.00%	36.44
15.00%	33.74

COUPONS AFFECT THE P-YTM RELATION

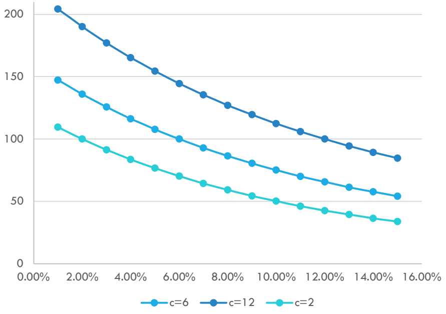
10-year government bond with 12% coupon

- Suppose the YTM is 6%
- What if it increases to 8%? Or decreases to 4%?



YTM	Price
1.00%	204.43
2.00%	190.23
3.00%	177.26
4.00%	165.41
5.00%	154.56
6.00%	144.63
7.00%	135.53
8.00%	127.18
9.00%	119.51
10.00%	112.46
11.00%	105.98
12.00%	100.00
13.00%	94.49
14.00%	89.41
15.00%	84.71

COUPONS AFFECT THE P-YTM RELATION

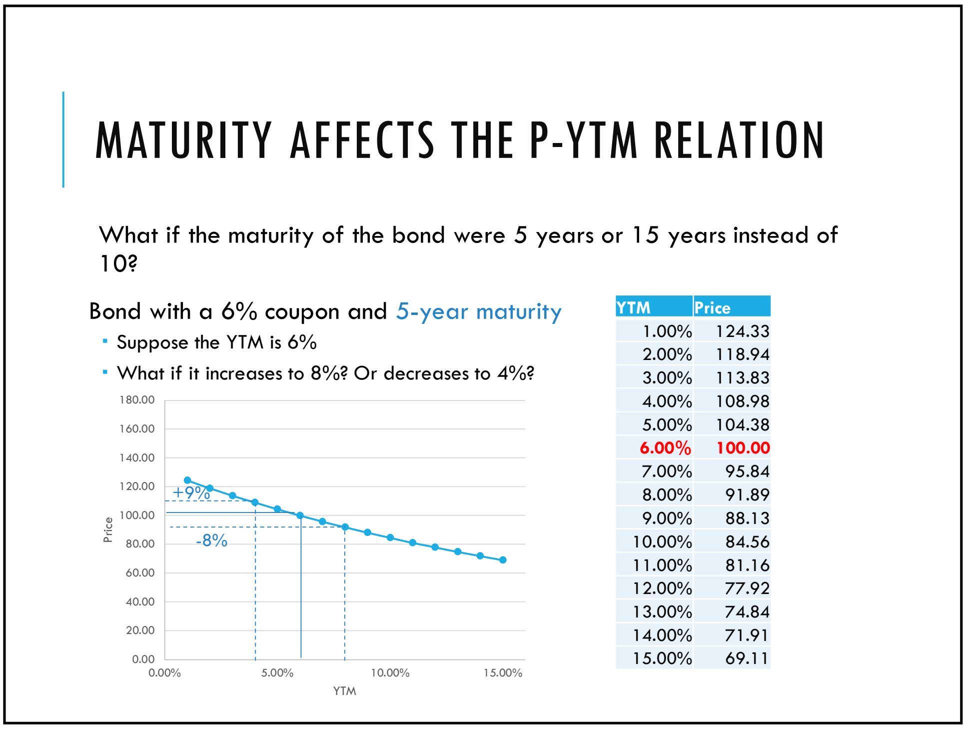


c = 2			c = 12		
YTM	Price	% change	YTM	Price	% change
1.00%	109.49	55.87%	1.00%	204.43	41.35%
2.00%	100.00	42.36%	2.00%	190.23	31.52%
3.00%	91.42	30.14%	3.00%	177.26	22.56%
4.00%	83.65	19.08%	4.00%	165.41	14.36%
5.00%	76.62	9.07%	5.00%	154.56	6.87%
6.00%	70.25		6.00%	144.63	
7.00%	64.47	-8.22%	7.00%	135.53	-6.29%
8.00%	59.23	-15.68%	8.00%	127.18	-12.07%
9.00%	54.47	-22.45%	9.00%	119.51	-17.37%
10.00%	50.15	-28.61%	10.00%	112.46	-22.24%
11.00%	46.22	-34.20%	11.00%	105.98	-26.73%
12.00%	42.65	-39.28%	12.00%	100.00	-30.86%
13.00%	39.40	-43.91%	13.00%	94.49	-34.67%

✓ The smaller coupon bond is more sensitive (in percentage) to changes in *YTM*

✓ Why?

- High coupon bond: big part of investment's cash flows arrives every six months
- Low coupon bond: bulk of the investment's cash flows arrives at maturity
- Change in *YTM* → change in time value of money → mainly affects cash flows arriving far in time



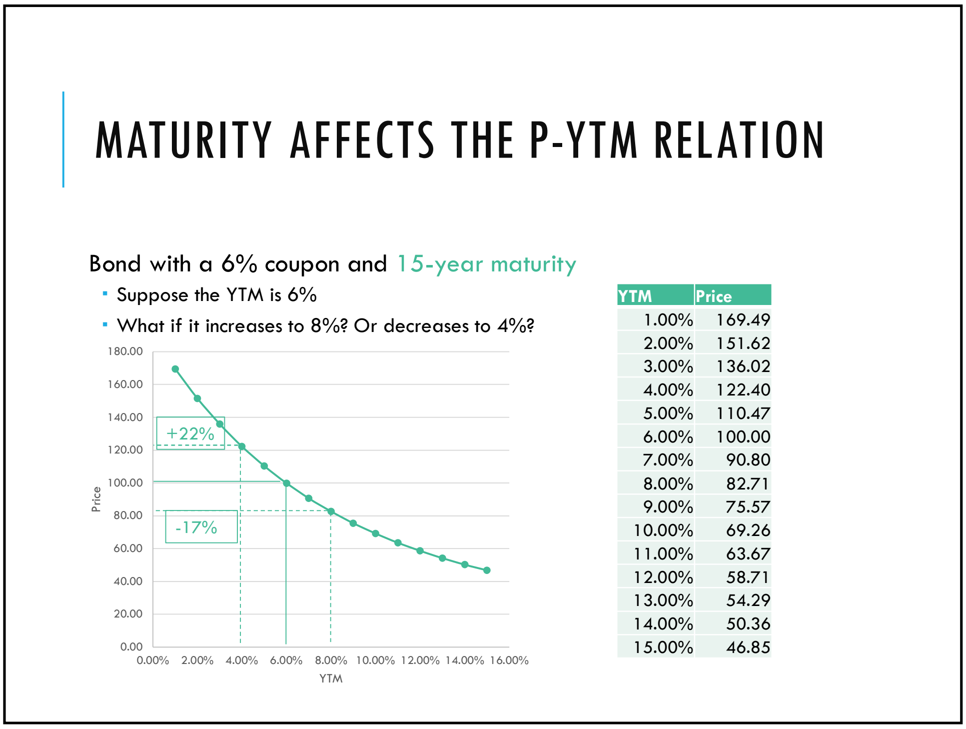
MATURITY AFFECTS THE P-YTM RELATION

What if the maturity of the bond were 5 years or 15 years instead of 10?

Bond with a 6% coupon and 5-year maturity

- Suppose the YTM is 6%
- What if it increases to 8%? Or decreases to 4%?

YTM	Price
1.00%	124.33
2.00%	118.94
3.00%	113.83
4.00%	108.98
5.00%	104.38
6.00%	100.00
7.00%	95.84
8.00%	91.89
9.00%	88.13
10.00%	84.56
11.00%	81.16
12.00%	77.92
13.00%	74.84
14.00%	71.91
15.00%	69.11



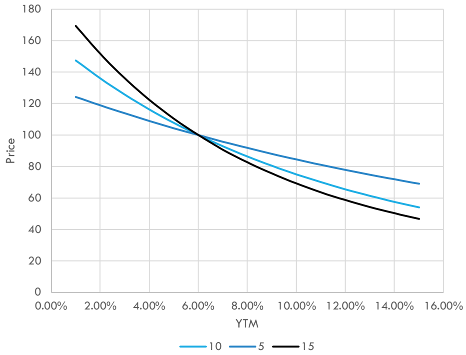
MATURITY AFFECTS THE P-YTM RELATION

Bond with a 6% coupon and 15-year maturity

- Suppose the YTM is 6%
- What if it increases to 8%? Or decreases to 4%?

YTM	Price
1.00%	169.49
2.00%	151.62
3.00%	136.02
4.00%	122.40
5.00%	110.47
6.00%	100.00
7.00%	90.80
8.00%	82.71
9.00%	75.57
10.00%	69.26
11.00%	63.67
12.00%	58.71
13.00%	54.29
14.00%	50.36
15.00%	46.85

MATURITY AFFECTS THE P-YTM RELATION



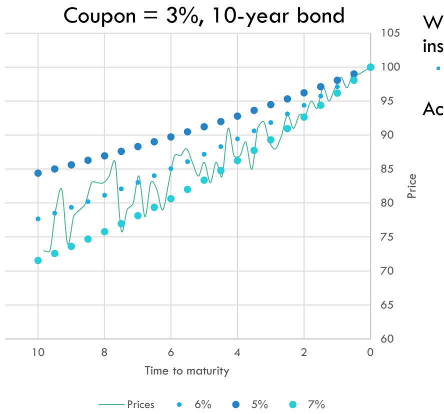
	5 years			15 years	
YTM	Price	% change	YTM	Price	% change
1.00%	124.33	24.33%	1.00%	169.49	69.49%
2.00%	118.94	18.94%	2.00%	151.62	51.62%
3.00%	113.83	13.83%	3.00%	136.02	36.02%
4.00%	108.98	8.98%	4.00%	122.40	22.40%
5.00%	104.38	4.38%	5.00%	110.47	10.47%
6.00%	100.00		6.00%	100.00	
7.00%	95.84	-4.16%	7.00%	90.80	-9.20%
8.00%	91.89	-8.11%	8.00%	82.71	-17.29%
9.00%	88.13	-11.87%	9.00%	75.57	-24.43%
10.00%	84.56	-15.44%	10.00%	69.26	-30.74%
11.00%	81.16	-18.84%	11.00%	63.67	-36.33%
12.00%	77.92	-22.08%	12.00%	58.71	-41.29%
13.00%	74.84	-25.16%	13.00%	54.29	-45.71%

✓ The coupon bond with longer maturity is more sensitive to changes in *YTM*

✓ Why?

- Change in *YTM* → change in time value of money → affects cash flows arriving far in time the most
- Long maturity → more cash flows arriving far in time

P-YTM RELATION IN TIME

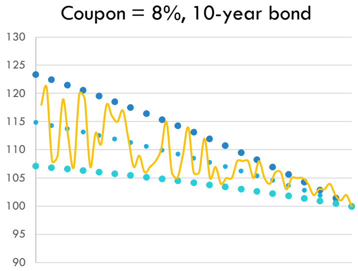


Dashed line: P when YTM = 6%. With a coupon of 3%, this price is below par, increasing up to par at maturity

Window of prices if *YTM* is 7% (lower) or is 5% (upper) instead of 6%

- As maturity approaches, the window narrows → less risk

Actual prices can be very different over bond lifetime



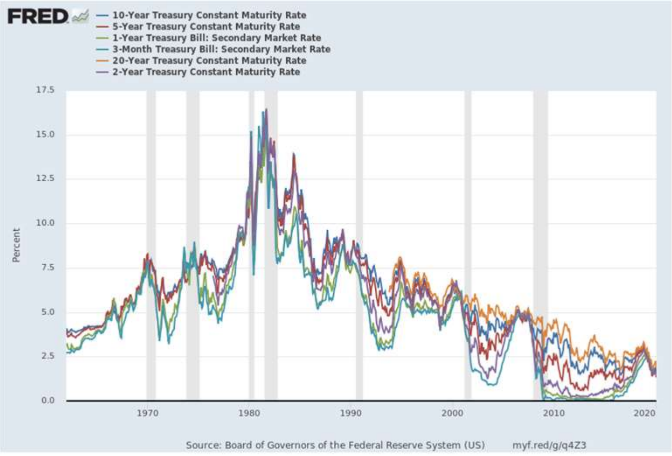
With an 8% coupon, and *YTM* of either 5%, 6%, or 7%, prices start out above par when the bond is issued and decrease to par in year 10

PRICE-YTM RELATIONSHIP

1. Price–YTM relationship is **downward sloping,** with the exact shape being specific to the bond: coupon and maturity matter
2. Fix the **coupon rate.** Bonds with longer maturity have greater price volatility
3. Fix the **time to maturity**. Bonds with a lower coupon rate have greater price volatility
4. Since the price of a bond changes depending on the *YTM*, the holder of a bond is exposed to *YTM-risk*, or interest-rate risk

INTEREST RATE RISK

How important is interest rate risk?



Source: Board of Governors of the Federal Reserve System (US)   myf.red/g/q4Z3

- Wild shifts in levels over time for all maturity interest rates
- (By the way: different maturities have different rates… Next chapter)

BOND DURATION

Can we capture interest rate risk with one number?

- Remember that a bond's price varies differently with yield depending on its coupon and time to maturity
- Take a specific bond (time to maturity and coupon) and a given current yield to maturity,

What percentage change will the bond's price incur if the *YTM* increases/falls by an amount *dYTM*?

If the *YTM* changes to $\overline{YTM} = YTM + dYTM$,

- The price of the bond changes to $\bar{P} = P + dP$
- In percentage, the price of the bond changes by $\frac{dP}{P}$
- The percentage change in the bond's price **due to a change of interest rate**, is $\boxed{\frac{1}{P}\frac{dP}{dYTM}}$

One number that captures interest rate risk

BOND DURATION (YTM)

The **modified duration** of a bond is the (negative of the) percent sensitivity of the price *P* to a small change in the YTM:

$$MD = -\frac{1}{P}\frac{dP}{dYTM}$$

The modified duration is very closely linked to the **Macaulay duration** of a bond

- Recall that the price of a bond at a date $T_{j-1} < t < T_j$ is given by

$$P_t = \frac{c}{\left(1+\frac{YTM}{n}\right)^{n(T_j-t)}} + \frac{c}{\left(1+\frac{YTM}{n}\right)^{n(T_{j+1}-t)}} + \cdots + \frac{c+100}{\left(1+\frac{YTM}{n}\right)^{n(T_i-t)}}$$

The **Macaulay duration** of a bond is given by

$$MacD_t = \frac{c}{P_t\left(1+\frac{YTM}{n}\right)^{n(T_j-t)}}(T_j-t) + \frac{c}{P_t\left(1+\frac{YTM}{n}\right)^{n(T_{j+1}-t)}}(T_{j+1}-t) + \cdots + \frac{c+100}{P_t\left(1+\frac{YTM}{n}\right)^{n(T_i-t)}}(T_i-t)$$

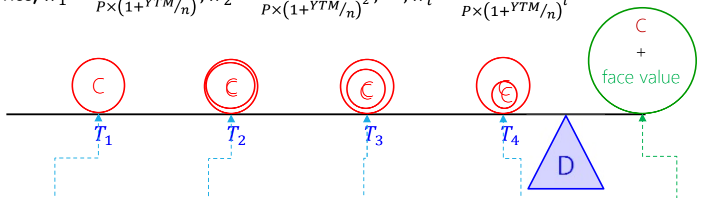
- If *YTM* or continuously-compounded term structure is used, modified duration will be a simple transformation of the Macaulay duration.

MACAULAY DURATION (YTM – FORMULA)

Consider the simplest case of MacD at the issue date of the bond ($t = 0$)

MacD is a weighted average of the times to maturity of each cashflow of the bond

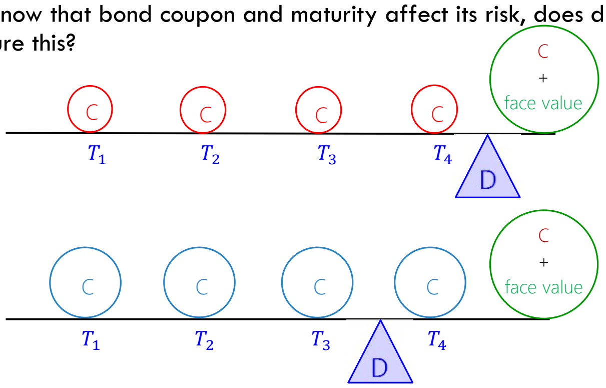
- Times to maturity $T_1, T_2, \cdots, T_i$
- Weights: present value of the cash flow arriving at each time, divided by the bond's price, $w_1 = \frac{c}{P\times(1+{}^{YTM}/_{n})}, w_2 = \frac{c}{P\times(1+{}^{YTM}/_{n})^2}, \cdots, w_i = \frac{c+100}{P\times(1+{}^{YTM}/_{n})^i}$



$$MacD = \frac{c}{P\left(1+\frac{YTM}{n}\right)}T_1 + \frac{c}{P\left(1+\frac{YTM}{n}\right)^2}T_2 + \frac{c}{P\left(1+\frac{YTM}{n}\right)^3}T_3 + \frac{c}{P\left(1+\frac{YTM}{n}\right)^4}T_4 + \frac{c+100}{P\left(1+\frac{YTM}{n}\right)^5}T_5$$

MACAULAY DURATION (1)

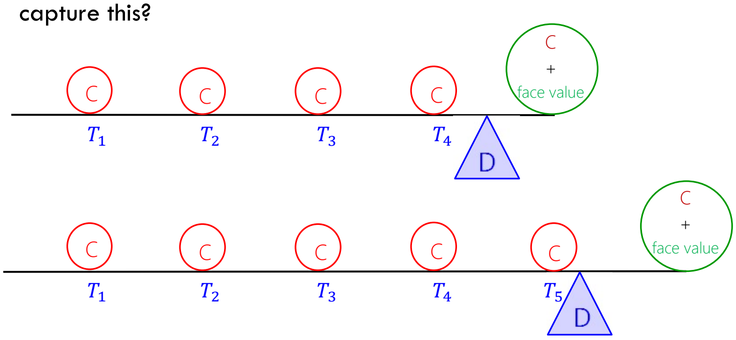
We know that bond coupon and maturity affect its risk, does duration capture this?



Larger coupons decrease the Macaulay duration of a bond

MACAULAY DURATION (2)

We know that bond coupon and maturity affect its risk, does duration capture this?



Longer time to maturity increases the Macaulay duration of a bond

MACAULAY DURATION (3)

Exercises. Consider a 3-year bond with 2% semiannual coupon and current YTM of 1.5%.

1. What is the price of this bond?
2. What is the Macaulay duration of this bond?

We can use a table to compute all terms. At each coupon date, T_j, the discount factor is

$$Z(0,T_j) = \frac{1}{(1+0.0075)^{2\times T_j}}$$

T	$(1.0075)^{2T}$	$Z(0,T_j)$	Cash flow	$CF \times Z(0,T_j)$	w_i	$w_i \times T$
0.5	1.0075	0.992556	1	0.992556	0.009783	0.0049
1	1.015056	0.985167	1	0.985167	0.00971	0.0097
1.5	1.022669	0.977833	1	0.977833	0.009637	0.0145
2	1.030339	0.970554	1	0.970554	0.009566	0.0191
2.5	1.038067	0.963329	1	0.963329	0.009495	0.0237
3	1.045852	0.956158	101	96.57196	0.95181	2.8554
			Price:	**101.4614**	**MacD:**	**2.9274**

Summing all discounted cash flows, we obtain a Price of 101.4614%. Using the price we can compute weights and, finally, the Macaulay duration = 2.9274 years.

MODIFIED DURATION

The modified duration is our estimate of the percentage change of bond price if the YTM changes by some amount *dYTM*

- It is calculated from the Macaulay duration as follows

$$MD = \frac{1}{1 + {}^{YTM}/_{n}} \times MacD$$

- If known, the modified duration can be used to estimate expected bond prices after a change in YTM

$$\bar{P} \approx P \times [1 - (MD \times dYTM)]$$

<u>Exercise</u>. For the bond in the previous exercise, determine the modified duration. Suppose the YTM right after bond issue increases by 20 basis points (1bp = 0.01%), what will be, approximately, the new price of the bond?

The modified duration is found using the MacD previously computed:

$$MD = \frac{2.927355}{1.0075} = 2.90556$$

We next use the MD to find an approximate new price given *dYTM* = 0.002. We obtain

$$\bar{P} \approx 101.4614 \times [1 - (2.90556 \times 0.002)] = 100.871795\%$$

MACAULAY DURATION (4)

How do we determine interest rate risk for a bond that is not just issued?

- We use duration, being aware of the date, t, at which we are currently located
- Only remaining coupons must be taken into account

<u>Example</u>. A 10-year bond with 4% semiannual coupon was issued on February 1, 2018. Today is October 1, 2020. What is the price, *MacD*, and *MD* of this bond today, if the required YTM is 5%?

- Between October 1 and the next coupon on February 1, we must wait 123/365 = 0.337 years; for the second coupon, 0.837 years, etc.
- Use these time frames to construct our price/duration table:

T_j	$T_j - t$	$1.025^{2(T_j-t)}$	$Z(t,T_j)$	CF	$Z(t,T_j) \times CF$	w_j	$w_j \times (T_j - t)$
01/02/21	0.337	1.016782	0.983495	2	1.96699	0.0208	0.00701
01/08/21	0.837	1.042202	0.959507	2	1.919014	0.020293	0.016985
01/02/22	1.337	1.068257	0.936105	2	1.872209	0.019798	0.026469
…							
01/08/27	6.837	1.401645	0.713447	2	1.426894	0.015089	0.103161
01/02/28	7.337	1.436686	0.696046	102	70.9967	0.750752	5.508264
				P	**94.5675**	**MacD***	**6.352412**

SUMMARY

Yield to maturity of a bond

- Return expected by investors from holding a bond until maturity
- If characteristics of a bond are known, the YTM can be calculated from these and the bond's market price

If YTM known:

- "Fair" price of a bond can be determined
- YTM-risk of a bond can be determined → Macaulay duration and modified duration

Fair price of a bond:

- Present value of all remaining cash flows
- This is also known as the dirty price of a bond
- Listed price of a bond between coupon payments, is the clean price. To recover dirty price from clean price, must add accrued interest

VARYING DISCOUNT RATES

Interest rates, prices, and interest rate risk

OBJECTIVE

Understand bond pricing using interest rates

- Different rate for each maturity → each coupon is discounted with different rate
- Continuously-compounded interest rate

Interest-rate risk

- Macaulay duration and modified duration revisited
- Money duration, effective duration, and key-rate duration

Floating-rate bonds

- Different approach to security pricing (backwards induction)
- Interest-rate risk of a floating-rate bond

YTM AND IMPLIED INTEREST

Exercise. On t = January 9, 2020, the US treasury issued T-bills with maturity T_1 = July 9, 2020 at a price of 99.231556%. On the same day, it issued T-bills with maturity T_2 = April 9, 2020 at a price of 99.615778%.

1. What is the YTM of each of these bonds?
2. Which one is a better investment? (Careful: how do you compare them?)
3. Why, if they're both T-bills, with equal default risk, do they provide different returns?

1. Remember that $P = \frac{100}{1+{}^{YTM}/_{n}}$. We know that for bond 1, $n = 2$, for bond 2, $n = 4$. Therefore:

$$YTM_1 = 2 \times \left(\frac{100}{99.213556} - 1\right) = 1.5488\%; \text{ and } YTM_2 = 4 \times \left(\frac{100}{99.615778} - 1\right) = 1.5428\%$$

2. We cannot compare the above rates because they have different compounding. E.g., a 4% rate compounded every quarter yields $(1.01)^2 = 1.0201$ in one semester → annualized rate of 4.02% > 4% per semester. We must equate our bonds' compounding. Some standards are semiannual, annual, or continuous compounding. Use semiannual:

- Bond 1 is already a semiannual rate, $YTM_1 = r_2 = 1.5488\%$
- Bond 2: annualized 1.5428% means 0.3857% per quarter. The return after 1 semester (2 quarters), is $\left(1 + {}^{YTM_2}/_{4}\right)^2 - 1 = (1.003857)^2 - 1$ =0.7729%. Annualized: 2 x 0.7729% = 1.5458%.
- Now we can compare! Bond 1 is a "better" investment since 1.5488% > 1.5458%

INTEREST RATES

Notice that when compounding is more frequent, you need a lower rate to achieve the same return after, for example, 1 year

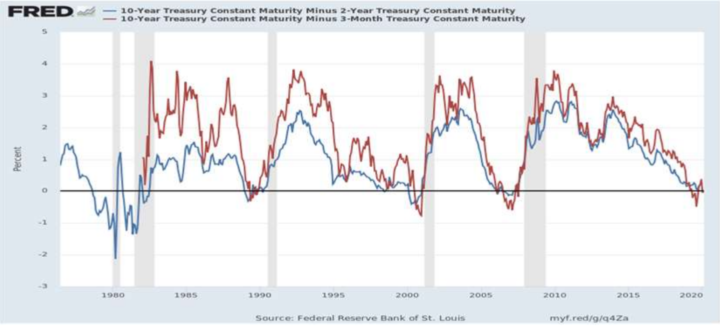
Notice that the US Treasury offers you a higher rate of return for lending them money for 1 semester than 1 quarter

- Think of bank loan interest rates for 10-y loan vs. 30-y loan…

The YTM of zero-coupon bonds is equivalent to the *interest rate* from today to the date of the bond's maturity

Even when converted to same compounding frequency, interest rates for different maturities are <u>different</u>

INTEREST RATE GAPS



Source: Federal Reserve Bank of St. Louis

- Gap between long- and short-term rates changes over time and is not the same for all short-term bonds

YIELD CURVE

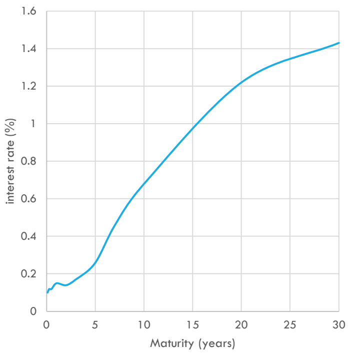
In general, rates are different for different maturities

The *yield curve* or *term structure of interest rates* shows interest rates for each maturity for a given class of bonds

- If zcb available for all maturities, these would be the YTMs implied by zcb prices

The yield curve can be used to price other bonds of the same or similar class

US Treasury yield curve on September, 10, 2020



Source: FRED

BOND PRICING WITH YIELD CURVE

An investor may know the YTM they demand for a certain bond

…But, typically, investors learn about discount rates from markets

- If discount rates are determined from market information, you learn the yield curve

YTM interpreted as the *implied* return from a given investment

- After you know the price of a bond, you can determine its YTM (= its internal rate of return if held until maturity)
- Rarely will you know a bond's YTM *before* its price → rarely is YTM used for pricing

When yield curve is used for pricing, each cash flow of the bond (coupon, principal) is discounted with a different rate

Makes a lot of intuitive sense!

- Depending on the yield curve's shape, it may be better to hold short/medium/ or long-term investments…

BOND PRICING WITH YIELD CURVE (1)

When working with interest rates, it is fundamental that all rates (for all maturities), be of the same compounding frequency!

Standard compounding frequencies used for yield curves are semiannual, annual, or continuous – we will work with all three

- Figure on previous slide was the yield curve of **annually-compounded** rates

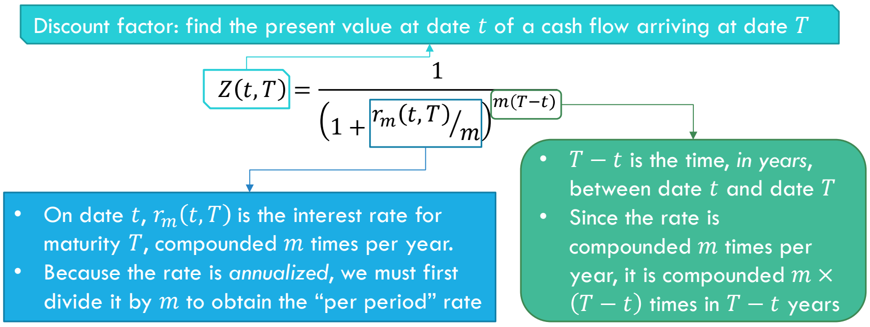
Remember the "discounted cash flows principle"

$$P_t = Z(t,T_1) \times c + Z(t,T_2) \times c + \cdots + Z(t,T_i) \times [c + 100]$$

$Z\left(t,T_j\right)$ is the discount factor between dates t and T_j

- When using YTM for pricing, $Z\left(t,T_j\right) = \frac{1}{\left(1+{}^{YTM}/_{n}\right)^{n\times\left(T_j-t\right)}}$
- When we use *m-compounding* interest rates, $Z\left(t,T_j\right) = \frac{1}{\left(1+{}^{r_m\left(t,T_j\right)}/_{m}\right)^{m\left(T_j-t\right)}}$

ANATOMY OF A FORMULA



Discount factor: find the present value at date t of a cash flow arriving at date T

$$\boxed{Z(t,T)} = \frac{1}{\left(1 + \boxed{{}^{r_m(t,T)}/_{m}}\right)^{\boxed{m(T-t)}}}$$

- On date t, $r_m(t,T)$ is the interest rate for maturity T, compounded m times per year.
- Because the rate is *annualized*, we must first divide it by m to obtain the "per period" rate

- $T - t$ is the time, *in years*, between date t and date T
- Since the rate is compounded m times per year, it is compounded $m \times (T-t)$ times in $T - t$ years

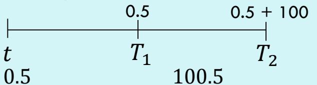
Example. On October 1, 2020 (t), the semiannually-compounded interest rate for maturity February 1, 2021 (T), is 1.3%. What discount factor must be used to discount a cash flow arriving on Feb 01, 2021?

- The interest rate is annualized, so the per-*semester* rate is ${}^{r_2(t,T)}/_{2} = 0.65\%$
- Between t and T, there are 4 months: $T - t = {}^{1}/_{3}$ years and $m \times (T-t) = {}^{2}/_{3}$ semesters

$$Z(t,T) = {}^{1}\Big/_{(1+0.0065)^{2/3}} = 0.99569$$

BOND PRICING WITH YIELD CURVE (2)

Exercise. On t = January 9, 2020, you know (see previous exercise) that the semiannually-compounded interest rates with maturities T_1 = April 9, 2020, and T_2 = July 9, 2020, are, respectively, 1.5458% and 1.5488%. A special bond with quarterly coupons and maturity July 9, 2020, is issued on that same day. If it has a 2% coupon, what is its price today?



$$P = \frac{0.5}{\left(1+\frac{0.015458}{2}\right)^{\frac{1}{2}}} + \frac{100.5}{\left(1+\frac{0.015488}{2}\right)^{1}} = 100.2258$$

*Another way to think about it, is to first find the *equivalent* quarterly compounded rates:

$$r_4(t,T_1)/_4 = \left[1 + {r_2(t,T_1)}/_2\right]^{\frac{1}{2}} - 1 = 0.3857\%\,; {r_4(t,T_2)}/_4 = \left[1 + {r_2(t,T_2)}/_2\right]^{\frac{1}{2}} - 1 = 0.3865\%$$

And use this quarterly rate to discount the first cash flow 1 period and the second 2 periods:

$$P = \frac{0.5}{1+0.003857} + \frac{100.5}{(1+0.003865)^2} = 100.2257$$

BOND PRICING WITH YIELD CURVE (3)

Most of the time, you will have a yield curve with compounding as frequent as your coupon payments

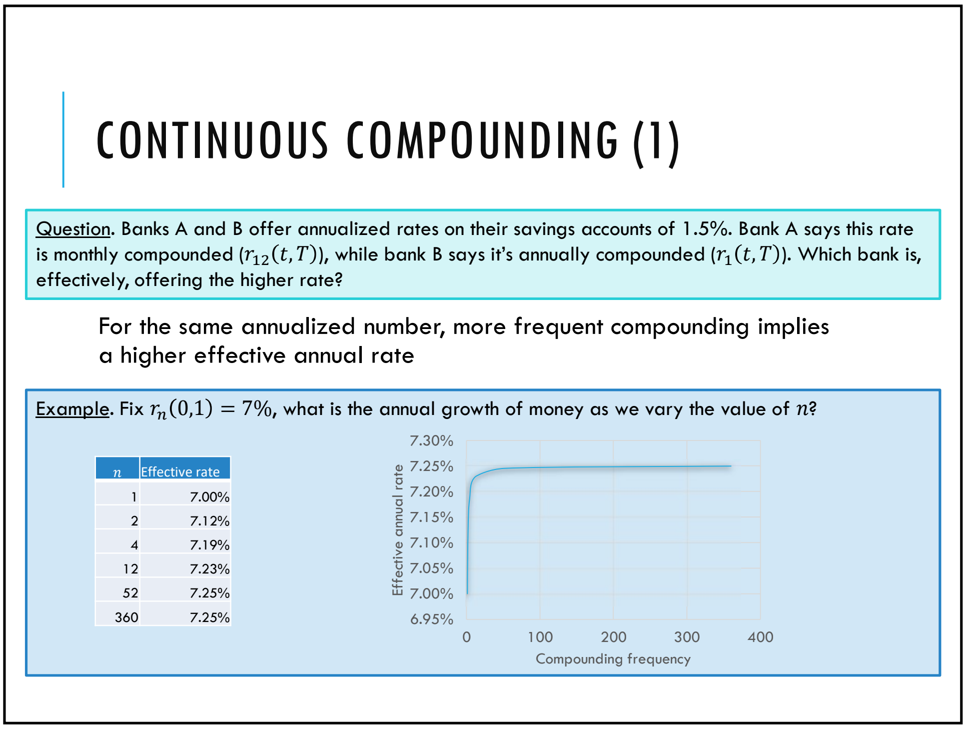
- Typically, semiannually-compounded rates, and semiannual coupons

Example. On January 31, 2020, semi-annually compounded interest rates with maturities T_1 = July 31, 2020, T_2 = January 31, 2021, and T_3 = July 31, 2021 are given by $r_2(t,T_1) = 1.487\%$, $r_2(t,T_2) = 1.495\%$, and $r_2(t,T_3) = 1.503\%$. A T-note is issued on January 31, 2020, with coupon 2% and maturity July 31, 2021. We can compute discount factors:

$$Z(t,T_1) = \left(1+\tfrac{0.01487}{2}\right)^{-1} = 0.99262;\ Z(t,T_2) = \left(1+\tfrac{0.01495}{2}\right)^{-2} = 0.98522;\ Z(t,T_3) = \left(1+\tfrac{0.01503}{2}\right)^{-3} = 0.97779$$

The bond price is thus: $P_c(t,T_3) = 0.99262 + 0.98522 + 101 \times 0.97779 = 100.7346\%$

…Or a yield curve of continuously-compounding interest rates, which can be easily adapted to any frequency of coupon payments!



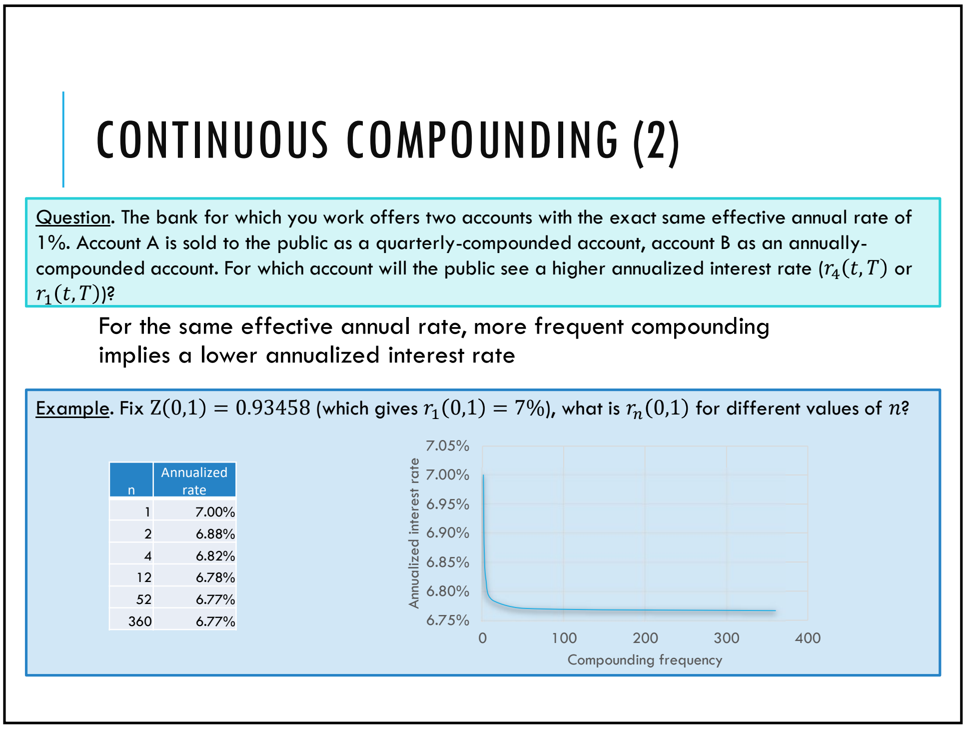
CONTINUOUS COMPOUNDING (1)

Question. Banks A and B offer annualized rates on their savings accounts of 1.5%. Bank A says this rate is monthly compounded ($r_{12}(t,T)$), while bank B says it's annually compounded ($r_1(t,T)$). Which bank is, effectively, offering the higher rate?

For the same annualized number, more frequent compounding implies a higher effective annual rate

Example. Fix $r_n(0,1) = 7\%$, what is the annual growth of money as we vary the value of n?

n	Effective rate
1	7.00%
2	7.12%
4	7.19%
12	7.23%
52	7.25%
360	7.25%



CONTINUOUS COMPOUNDING (2)

Question. The bank for which you work offers two accounts with the exact same effective annual rate of 1%. Account A is sold to the public as a quarterly-compounded account, account B as an annually-compounded account. For which account will the public see a higher annualized interest rate ($r_4(t,T)$ or $r_1(t,T)$)?

For the same effective annual rate, more frequent compounding implies a lower annualized interest rate

Example. Fix $Z(0,1) = 0.93458$ (which gives $r_1(0,1) = 7\%$), what is $r_n(0,1)$ for different values of n?

n	Annualized rate
1	7.00%
2	6.88%
4	6.82%
12	6.78%
52	6.77%
360	6.77%

CONTINUOUS COMPOUNDING (3)

In general, an *m*-compounded interest rate (annualized), $r_m(0,1)$ makes €1 of investment grow to $\left(1+\frac{r_m(0,1)}{m}\right)^m$ in one year

- What is this number if *m* goes to infinity (continuous compounding)?
- Fix a value for the annualized rate, say $\bar{r}$, the annual growth becomes

$$\lim_{m\to\infty}\left(1+\frac{\bar{r}}{m}\right)^m = e^{\bar{r}}$$

If investment period is not 1 year, compound annual growth over appropriate time, $(e^{\bar{r}})^{T-t} = e^{\bar{r}(T-t)}$

If you know that the continuously-compounded interest rate between t and T is $r(t,T)$,

- the value at T of an investment of PV at t is $FV = PV \times e^{r(t,T)(T-t)}$
- The value at t of a future cash flow FV arriving at T is $PV = {}^{FV}\!/_{e^{r(t,T)(T-t)}} = FV \times e^{-r(t,T)(T-t)}$

BOND PRICING WITH CC-YIELD CURVE (1)

Remember the “discounted cash flows principle”

$$P_t = Z(t,T_1)\times c + Z(t,T_2)\times c + \cdots + Z(t,T_i)\times[c+100]$$

$Z(t,T_j)$ is the discount factor between dates t and T_j

- When using YTM for pricing, $Z(t,T_j) = \frac{1}{\left(1+{}^{YTM}\!/_{n}\right)^{n(T_j-t)}}$
- When we use *m-compounding* interest rates, $Z(t,T_j) = \frac{1}{\left(1+{}^{r_m(t,T_j)}\!/_{m}\right)^{m(T_j-t)}}$
- When we use continuously-compounding rates, $Z(t,T_j) = e^{-r(t,T_j)(T_j-t)}$

Notation: we use $r_m(t,T)$ to denote an *m*-compounding interest rate, and $r(t,T)$ (no subscript!) to denote a continuously-compounding interest rate

BOND PRICING WITH CC-YIELD CURVE (2)

Maturity	$r(t,T)$
0.25	6.33%
0.5	6.49%
0.75	6.62%
1	6.71%
1.25	6.79%
1.5	6.84%
1.75	6.87%
2	6.88%
2.25	6.89%
2.5	6.88%
2.75	6.86%
3	6.83%
3.25	6.80%
3.5	6.76%
3.75	6.72%
4	6.67%

Exercise. The table on the right contains the term structure of *continuously-compounded* interest rates on $t =$ March 15, 2000. Maturities are given in years, starting on date t. What is the price of a 4-year bond with semiannual coupon of 7% issued on that day?

- A coupon of 3.5 is paid every 6 months. The relevant rates for the question are those at maturity 0.5, 1, 1.5, 2, 2.5, 3, 3.5, and 4.
- We use the discounted cash flows principle with the discount factor appropriate for continuously-compounded rates:

$$P = 3.5 \times [e^{-0.0649\times0.5} + e^{-0.0671\times1} + e^{-0.0684\times1.5} + e^{-0.0688\times2} + e^{-0.0688\times2.5} + e^{-0.0683\times3} + e^{-0.0676\times3.5}] + 103.5 \times e^{-0.0667\times4} = 100.694$$

What if you want to price a bond that was not issued today?

- You discount cash flows only until today (fraction of coupon period), just like before
- Careful! You must have the appropriate interest rates in your yield curve!

BOND PRICING WITH CC-YIELD CURVE (3)

Maturity	$r(t,T)$
0.25	6.33%
0.5	6.49%
0.75	6.62%
1	6.71%
1.25	6.79%
1.5	6.84%
1.75	6.87%
2	6.88%
2.25	6.89%
2.5	6.88%
2.75	6.86%
3	6.83%
3.25	6.80%
3.5	6.76%
3.75	6.72%
4	6.67%

Exercise. Using the same term structure of interest rates as before, located at $t =$ March 15, 2000, determine the price of a 3-year coupon bond issued on December 15, 1998, with a semiannual coupon of 4%.

- The next coupon payment of this bond is June 15, 2000 (in 3 months). Another 3 coupons and the principal are paid in 9, 15, and 21 months.
- We have the appropriate rates! $r(0, 0.25)$, $r(0, 0.75)$, $r(0, 1.25)$, & $r(0, 1.75)$

$$P = 2 \times [e^{-0.0633\times0.25} + e^{-0.0662\times0.75} + e^{-0.0679\times1.25}] + 102 \times e^{-0.0687\times1.75} = 96.155$$

What if you wanted to price a bond issued 4 months ago (next coupon in 2 months)?

- You would need interest rates with maturities 0.167, 0.667, 1.167, etc.
- You cannot do it with the yield curve you have at your disposal!

MACAULAY DURATION REVISITED

Also when we use a yield curve for pricing, we can use <u>duration</u> to assess the interest-rate risk of a bond

We can easily extend the formula of Macaulay duration to the situation in which we use the yield curve to price a bond

- Each maturity has a weight, which is the appropriately-discounted cash flow arriving at that maturity: “appropriate” means “using the interest rate for that maturity”

At date t, with i remaining coupons, the Macaulay duration of a bond is given by

$$MacD_t = \frac{c \times Z(t,T_1)}{P_t}(T_1 - t) + \frac{c \times Z(t,T_2)}{P_t}(T_2 - t) + \cdots + \frac{(c+100) \times Z(t,T_i)}{P_t}(T_i - t),$$

With

- $Z(t,T_j) = \frac{1}{\left(1 + {}^{YTM}/_{n}\right)^{n\left(T_j - t\right)}}$ if we are pricing the bond with YTM
- $Z(t,T_j) = \frac{1}{\left(1 + {}^{r_m\left(t,T_j\right)}/_{m}\right)^{m\left(T_j - t\right)}}$ if we are pricing the bond with an *m*-compounded yield curve
- $Z(t,T_j) = e^{-r\left(t,T_j\right)\left(T_j - t\right)}$ if we are pricing the bond with a continuously-compounded yield curve

MACAULAY DURATION WITH YIELD CURVE

<u>Exercise</u>. Consider a 3-year bond with 2% semiannual coupon and consider the term structure of *semiannually-compounded interest rates* given in the table to the right.

1. What is the price of the bond today?
2. What is the Macaulay duration of the bond?

- Use the formula $Z(0,T_j) = \frac{1}{\left(1 + {}^{r_2\left(0,T_j\right)}/_{2}\right)^{2T_j}}$ to find discount factors

T	r_2(0,T)
0.5	3.00%
1	3.20%
1.5	3.30%
2	3.39%
2.5	3.42%
3	3.46%

T_j	$Z(0,T)$	CF	CF x $Z(0,T)$	w_j	$w_j \times T_j$
0.5	0.9852	1	0.9852	0.0103	0.0051
1	0.9688	1	0.9688	0.0101	0.0101
1.5	0.9521	1	0.9521	0.0099	0.0149
2	0.9350	1	0.9350	0.0098	0.0195
2.5	0.9187	1	0.9187	0.0096	0.0240
3	0.9022	101	91.1228	0.9504	2.8511
		Price:	**95.8826**	**MacD:**	**2.9247**

MACAULAY DURATION WITH YIELD CURVE

Exercise. Consider a 3-year bond with 6% semiannual coupon and consider the term structure of *continuously-compounded interest rates* given in the table to the right.

1. What is the price of the bond today?
2. What is the Macaulay duration of the bond?

T_j	$r(0,T_j)$
0.5	6.49%
1	6.71%
1.5	6.84%
2	6.88%
2.5	6.88%
3	6.83%

- Use the formula $Z(0,T_j) = e^{-r(0,T_j)\times T_j}$ to find discount factors

T_j	$r(0,T_j)$	$Z(0,T_j)$	*CF*	$CF \times Z(0,T_j)$	w_j	$w_j \times T_j$
0.5	6.49%	0.9681	3	2.9042	0.0298	0.0149
1	6.71%	0.9351	3	2.8053	0.0288	0.0288
1.5	6.84%	0.9025	3	2.7075	0.0278	0.0417
2	6.88%	0.8714	3	2.6143	0.0268	0.0536
2.5	6.88%	0.8420	3	2.5259	0.0259	0.0648
3	6.83%	0.8147	103	83.9171	0.8609	2.5827
			Price:	**97.4743**	**MacD:**	**2.7865**

- This way we obtain the *MacD* in years: 2.7865

MODIFIED DURATION (YIELD CURVE)

Modified duration was our measure of bond “yield risk”: how sensitive is the bond’s value to small changes in the YTM

- Mathematically, *MD* is the derivative of bond price w.r.t. YTM, divided by (relative to) the bond’s price
- We computed *MD* from MacD: we divided *MacD* by (1+ YTM/n)

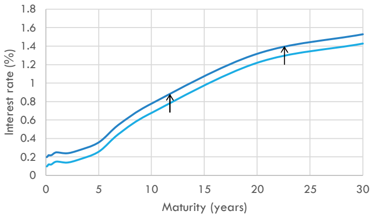
Similar measures of interest-rate risk can be computed when we use the yield curve to value bonds

The *modified duration* will tell us how sensitive the bond’s value is to a small change in the yield curve

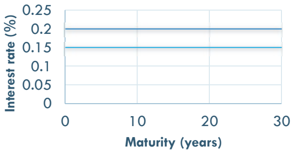
- There are many ways in which the yield curve may change!
- Mathematically, *MD* is the derivative of bond price w.r.t. a fixed number added to all interest rates in yield curve, divided by (relative to) price
- **Only for continuously-compounded interest rates is there a link between *MD* and *MacD***

MODIFIED DURATION (YIELD CURVE)

The **modified duration** of a bond is the (negative of the) percent sensitivity of the price *P* to a small <u>parallel</u> shift of the yield curve



The definition with YTM is equivalent to having a flat yield curve!



The modified duration for **continuously-compounded** interest rates* can be computed from the Macaulay duration, since $MD = MacD$ in this case

*That is: "the percent sensitivity of a bond's price to a small parallel shift of the term structure of continuously-compounded interest rates"

The modified duration for *m*-compounded interest rates **cannot** be computed from *MacD*

MODIFIED DURATION (YIELD CURVE)

To compute modified duration for *m*-compounded interest rates, we must use a different formula

$$MD_{Qt} = \frac{1}{P_t} \times \left[\frac{c}{\left[1 + {}^{r_m(t,T_1)}/_m\right]^{m(T_1-t)+1}} \times (T_1 - t) + \frac{c}{\left[1 + {}^{r_m(t,T_2)}/_m\right]^{m(T_2-t)+1}} \times (T_2 - t) + \cdots + \frac{c+100}{\left[1 + {}^{r_m(t,T_i)}/_m\right]^{m(T_i-t)+1}} \times (T_i - t) \right]$$

- The little +1 in the exponent of the denominators, makes this different from *MacD*
- The weight given to each maturity is now $w_j = \frac{c}{P_t\left[1+{}^{r_m(t,T_j)}/_m\right]^{m(T_j-t)+1}}$. These weights **do not** add up to 1
- MD_Q stands for "quasi"-modified duration, a name given to this formula

...Or we can compute the *approximate modified duration* or **effective duration**

$$AMD = ED = \frac{P_- - P_+}{2 \times dr \times P}$$

- P_- is the price computed using current rates minus dr
- P_+ is the price computed using current rates plus dr

MODIFIED DURATION (YIELD CURVE)

Example. Remember the exercise where we computed the Macaulay duration of a 3-year bond with semiannually-compounded 6% coupon using the term structure of *continuously-compounded interest rates*. We found that *MacD = 2.7865*. The modified duration of this bond is *MD = MacD = 2.7865*

Example. Consider a 3-year bond with 2% semiannual coupon, and consider the term structure of *semiannually-compounded interest rates* given in the table to the right. We previously computed its price, *P = 95.8826%*, and *MacD = 2.9247*. We now compute its modified duration:

T_j	$r_2(0,T_j)$
0.5	3.00%
1	3.20%
1.5	3.30%
2	3.39%
2.5	3.42%
3	3.46%

Using the quasi-modified duration formula

T_j	$2T_j+1$		$\left[1+{r_2(0,T_j)}/{2}\right]^{2T_j+1}$	CF	$\frac{CF}{\left[1+{r_2(0,T_j)}/{2}\right]^{2T_j+1}}$	w_j	$w_j \times T_j$
0.5	2	1.50%	1.0302	1	0.9707	0.0101	0.0051
1	3	1.60%	1.0488	1	0.9535	0.0099	0.0099
1.5	4	1.65%	1.0677	1	0.9366	0.0098	0.0147
2	5	1.70%	1.0877	1	0.9194	0.0096	0.0192
2.5	6	1.71%	1.1071	1	0.9033	0.0094	0.0236
3	7	1.73%	1.1276	101	89.5732	0.9342	2.8026
							2.8750

Using the approximate modified duration formula, assuming 1bp parallel shift, *dr = 0.01%*

We compute P, P-, and P+ in a table:

T_j	$r_2(0,T_j)$	$r_2(0,T_j)+dr$	$r_2(0,T_j)-dr$	$Z(0,T_j)$	$Z_+(0,T_j)$	$Z_-(0,T_j)$	CF	$CF \times Z$	$CF \times Z_+$	$CF \times Z_-$
0.5	3.00%	3.01%	2.99%	0.9852	0.9852	0.9853	1	0.9852	0.9852	0.9853
1	3.20%	3.21%	3.19%	0.9688	0.9687	0.9688	1	0.9688	0.9687	0.9688
1.5	3.30%	3.31%	3.29%	0.9521	0.9519	0.9522	1	0.9521	0.9519	0.9522
2	3.39%	3.40%	3.38%	0.9350	0.9348	0.9352	1	0.9350	0.9348	0.9352
2.5	3.42%	3.43%	3.41%	0.9187	0.9185	0.9189	1	0.9187	0.9185	0.9189
3	3.46%	3.47%	3.45%	0.9022	0.9019	0.9025	101	91.1228	91.0959	91.1497
								95.8826	**95.8550**	**95.9101**

…And use the formula $AMD = \frac{P_- - P_+}{2 \times dr \times P} = \frac{95.9101 - 95.8550}{2 \times 0.0001 \times 95.8826} = 2.8750$. **The same as with MD_Q formula!**

IMPORTANT CASE: ZERO-COUPON BOND

The Macaulay duration of a **zero-coupon bond** is just the time left to maturity

- Remember that the price at t of a zcb with maturity T, is $P = Z(t,T) \times 100$
- Using the formula of MacD we have $\frac{100 \times Z(t,T)}{P} \times (T-t) = \frac{100 \times Z(t,T)}{100 \times Z(t,T)} \times (T-t) = T-t$
- This is true regardless of whether we are using YTM or yield curves

The modified duration of a zcb can always be computed from *MacD*

- $MD = {}^{MacD}/_{1+{}^{YTM}/_{n}}$ if we're considering YTM
- $MD = \frac{MacD}{1+{}^{r_m(t,T)}/_{m}}$ if we're considering the m-compounding term structure of interest rates
- $MD = MacD$ if we're considering the term structure of continuously-compounded rates

Example. Three months before maturity, the Macaulay duration of a zcb is 0.25 (measured in years). If the current quarterly-compounded interest rate with maturity in 3 months is $r_4(t, t+0.25) = 3\%$, the modified duration of this zcb w.r.t. a *parallel shift of the term structure of quarterly-compounded rates* is 0.25/(1+0.0075) = 0.2481

IMPORTANT CASE: PORTFOLIO

Suppose you have a portfolio of fixed-income securities containing securities 1, 2, …, S (*s* denotes a generic security in the portfolio)

- Each security has a weight, w_s, in your portfolio. That is the amount of money currently invested in that security divided by the total value of the portfolio
- Each security has a Macaulay duration $MacD_s$ and a modified duration MD_s

The Macaulay duration of the portfolio equals the *weighted sum* of Macaulay durations of all assets

$$MacD_{port} = w_1 \times MacD_1 + w_2 \times MacD_2 + \cdots + w_S \times MacD_S$$

The modified duration of the portfolio equals the *weighted sum* of the modified durations of all assets

$$MD_{port} = w_1 \times MD_1 + w_2 \times MD_2 + \cdots + w_S \times MD_S$$

Example. You manage a fixed income portfolio with 40% in a 10-year bond with semiannual coupons and *MacD* = 9.138 and YTM = 5%, 15% in a 1-year T-bill with YTM = 3%, and 45% in a 6-months T-bill with YTM = 1.4%. The *MacD* of this portfolio is $MacD_{port} = 0.4 \times 9.138 + 0.15 \times 1 + 0.45 \times 0.5 = 4.0302$.
Exercise: What is the modified duration of the portfolio?

OTHER MEASURES: MONEY DURATION (1)

Remember that modified duration is the change in bond value caused by a change in YTM or *r*, **relative** to the investment value

$$MD = -\frac{1}{P}\frac{dP}{dr}$$

If, instead, we wish to know how much **money** we would really gain/lose due to a change in YTM or *r*, it suffices to not divide by P

Money duration measures how much money is lost/gained per unit change of YTM or r

$$D^{\$} = -\frac{dP}{dr}$$

- For nonzero valued positions,

$$D^{\$} = P \times MD$$

OTHER MEASURES: MONEY DURATION (2)

Money duration of a portfolio with N_s units of each security, *s*, is

$$D^{\$} = \sum_{s=1}^{S} N_s \times D_s^{\$}$$

New value of portfolio (price), after change in interest rate, can be approximated with money duration:

$$\bar{V} = V - D^{\$} \times dr$$

Exercise. What is the money duration with respect to a change in its YTM, of an investment with face value of $1million in a 1-year zcb that is sold for 98.039% at issue?

We can use the formula $D^{\$} = P \times MD$, and $MD = {}^{MacD}/_{1+YTM}$ (we don't divide YTM by n because it's a 1-year bond, so YTM compounds once per year). To know the current value of YTM, notice that $P = \frac{100}{1+YTM} \Rightarrow YTM = \frac{100}{98.039} - 1 = 2\%$. Now we have all elements:

- $YTM = 2\%, MacD = 1 \Rightarrow MD = \frac{1}{1.02} = 0.9804$
- $MD = 0.9804, P = 980\ 390$ (the value of the investment today) $\Rightarrow D^{\$} = 980\ 390 \times 0.9804 = 961\ 174.356$

If the YTM increases by 100bps (1%), the value of the investment goes down to 970 778.256

OTHER MEASURES: KEY RATE DURATION (1)

Deal with **non-parallel** shifts of the yield curve

- Take only one or a few maturities considered "key"

Key duration is defined in a similar way as effective duration, but using only key maturities to compute new prices

- Compute the new bond price if interest rates for "key" maturities increase, P_+^*
- Compute the new bond price if interest rates for "key" maturities decrease, P_-^*

$$D^* = \frac{P_-^* - P_+^*}{2 \times dr \times P}$$

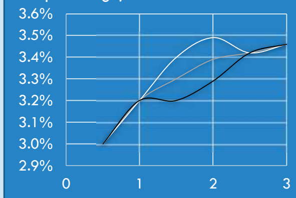
If you change only one maturity at a time, the sum of key durations for all maturities equals the effective duration of the bond (duration for parallel shift)

OTHER MEASURES: KEY RATE DURATION (2)

Example. Consider a 3-year bond with 2% semiannual coupon, and consider the term structure of *semiannually-compounded interest rates* given in the table to the right. We previously computed its price, $P = 95.8826$, its $MacD = 2.9247$, and its modified duration (both with the quasi-mod formula and the effective duration formula), $MD = 2.8750$.

T_j	$r_2(0, T_j)$
0.5	3.00%
1	3.20%
1.5	3.30%
2	3.39%
2.5	3.42%
3	3.46%

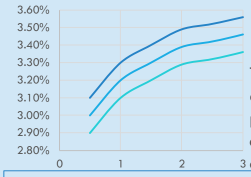
Consider the middle maturities (1.5 and 2 years) as key. Under this assumption, the new yield curves assuming a 10bps change, are:



T_j	$r_2(\cdot) + dr$	$r_2(\cdot) - dr$
0.5	3.00%	3.00%
1	3.20%	3.20%
1.5	3.40%	3.20%
2	3.49%	3.29%
2.5	3.42%	3.42%
3	3.46%	3.46%

The prices P_-^* and P_+^* are $P_-^* = 95.8858$ and $P_+^* = 95.8793$, so that key-rate duration in this case is

$$D^* = \frac{95.8858 - 95.8793}{2 \times 0.001 \times 95.8826} = 0.0338$$



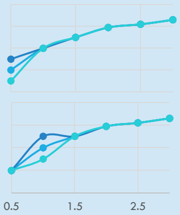
To compute effective duration we assumed a parallel shift of the yield curve (all rates increased or decreased by 1bp).

Consider changing interest rates one at a time (graphs show 6-months and 1-year rate cases).

Six different key-rate durations are computed

- $D_{0.5}^* = 0.005062$
- $D_1^* = 0.009944$
- $D_{1.5}^* = 0.014653$
- $D_2^* = 0.019178$
- $D_{2.5}^* = 0.023552$
- $D_3^* = 2.802597$

If all are added up, we obtain 2.875, which is effective duration of this bond



USING DURATION

We can approximate the new price of a bond if the YTM or the term structure of interest rates change, using duration:

- Remember that the NEW price of the bond is $\bar{P} = P + dP$
- Remember that modified duration is $MD = -\frac{1}{P}\frac{dP}{dr}$, so that, for small values of *dr*, we have

$$dP = -MD \times P \times dr$$

- The new price of a bond after a small change in the term structure of interest rates (or the *YTM*) is, therefore:

$$\bar{P} \approx P \times [1 - (MD \times dr)]$$

This approximation allows for many applications of duration:

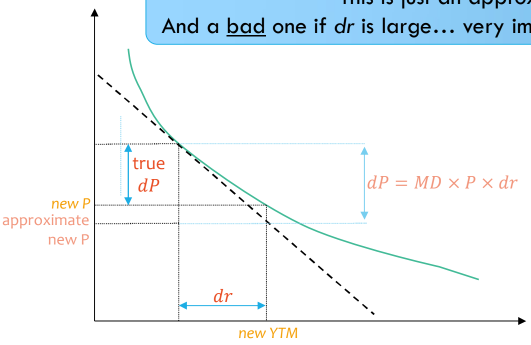
- Estimate downside of an investment
- Estimate value at risk (if you know probabilities of different interest rates)
- Create portfolios with low interest rate exposure (immunization)

WARNING ON APPLICATIONS

Applications of duration to real-world problems is based on

$$dP = -MD \times P \times dr$$

This is just an approximation!!
And a <u>bad</u> one if *dr* is large… very important for VaR application



By using duration, we move on a straight line to find new price (value) of the bond (portfolio).

True new price is on a curve, not a line!

The farther we move, the more the curvature matters → poor approximation

The notion of bond ***convexity*** will improve the approximation

CASE: FLOATING RATE BONDS

"different" bond pricing

FLOATING RATE BOND

A *floating rate bond* or "floater" is defined by

- A fixed maturity at date T_i,
- Fixed coupon payment dates, called *reset dates*, $T_1, T_2, \cdots, T_i$
- A variable part of the coupon, which **changes**, but is known one period in advance

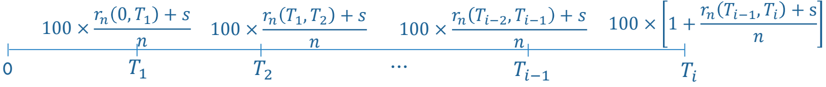
Example. A 3-year floater in US$ paying semiannual coupons is pegged to the LIBOR. If it is issued today, the first coupon, paid in six months, equals today's six-months LIBOR rate.

- A fixed part of the coupon, called *spread*

Example. If the above floater has a spread of 2bps, its coupon in six months equals today's 6-months LIBOR rate + 0.02%/2

When we buy a floating-rate bond, we only know the value of the first coupon. Remaining cash flows are unknown.

- Suppose, for simplicity, coupon frequency = rate compounding frequency. Only $r_n(0, T_1)$ is known today, other $r_n(T_j, T_{j+1})$ unknown today



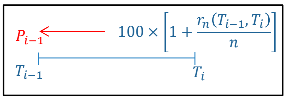
FLOATING RATE BOND: PRICE (1)

→ Assumption: The interest rate to which the floating-rate bond's coupons are pegged is also the relevant discount rate for this bond's cash flows

Using this assumption let's price a floater with <u>zero</u> spread:

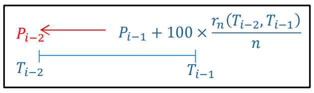
- Suppose today is date T_{i-1}, **one period before final coupon payment**
- At T_{i-1} we <u>know</u> all remaining cash flows: $100 \times \left[1 + \frac{r_n(T_{i-1}, T_i)}{n}\right]$
- If we want to know the value of this bond at T_{i-1} (ex-coupon), we discount this single cash flow at the prevailing rate, $r_n(T_{i-1}, T_i)$, for one period:

$$P_{i-1} = \frac{100 \times \left[1 + {r_n(T_{i-1}, T_i)}/{n}\right]}{1 + {r_n(T_{i-1}, T_i)}/{n}} = \mathbf{100}$$



- Now place yourself at T_{i-2}:
  - You know $r_n(T_{i-2}, T_{i-1})$ but not $r_n(T_{i-1}, T_i)$
  - However, you know that if you resold the bond at date T_{i-1}, you would receive 100. In this scenario, you <u>know all remaining cash flows</u>

$$P_{i-2} = \frac{100 \times \left[1 + {r_n(T_{i-2}, T_{i-1})}/{n}\right]}{1 + {r_n(T_{i-2}, T_{i-1})}/{n}} = \mathbf{100}$$



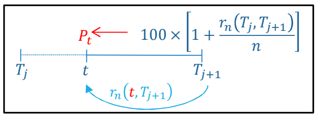
FLOATING RATE BOND: PRICE (2)

Repeat the above until date 0 to discover the following

At any *reset date* (0, T_1, …, T_{i-1}) a zero-spread floating rate bond's price equals 100% of its face value

What if today is not a reset date?

- The coupon was set at the last reset date, using the prevailing rate, $r_n(T_j, T_{j+1})$
- Meanwhile, between T_j and today, t, the rate may have changed!
  - The rate you use for discounting may differ from the rate paid as coupon



- At any date that is not a reset date, the bond's price is

$$P_t = \frac{100 \times \left[1 + \frac{r_n(T_j, T_{j+1})}{n}\right]}{\left[1 + {r_n(t, T_{j+1})}/{n}\right]^{n(T_j - t)}}$$

FLOATING-RATE BOND: PRICE (3)

Example. On October 1, 2020, the semiannually-compounded Treasury rate for a 3-months maturity was 0.1%. On that date you wished to price a US Treasury FRN (“floating-rate note”) with semiannual coupon (pegged to the US-Treasury semiannual rate) that was issued on July 1, 2020 (next coupon due on Jan 1, 2021). The semiannually-compounding 6-months rate on July 1, 2020, was 0.175%.

- The coupon that will be paid on January 1, 2021, was set on July 1, 2020, using the 6-months rate at that time. This means it equals 0.175/2=0.0875
- If the bond is resold *ex-coupon* at the next reset date, you will receive 100. Previously you also receive the coupon
- These cash flows, on October 1, 2020, have a present value of

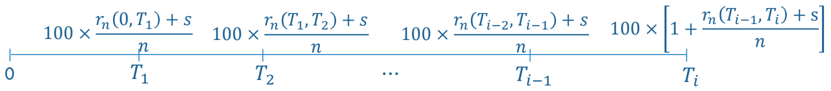
$$P_t = \frac{100 + 0.0875}{\left[1 + {}^{0.001}\!/_{2}\right]^{1/2}} = 100.0625$$

Notice that the change in interest rates since the last reset date (lower rates) plays in your favor! If the yield curve had been flat and had not changed since July, you would use a semiannually-compounded rate of 0.175% to discount the cash flows. In that case, the bond’s price would have been 100.0437.

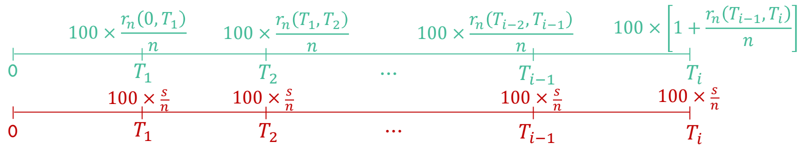
Notice that even with a flat, unchanging yield curve, the price at t is larger than 100 (the price at a reset date). This is because a person buying the bond on October 1 will cash in the full coupon by waiting only 3 months, while the person who bought it on July 1 (at a price of 100), would have had to wait a full 6 months. It is only fair that the October buyer pay more!

FLOATING-RATE BOND: PRICE (4)

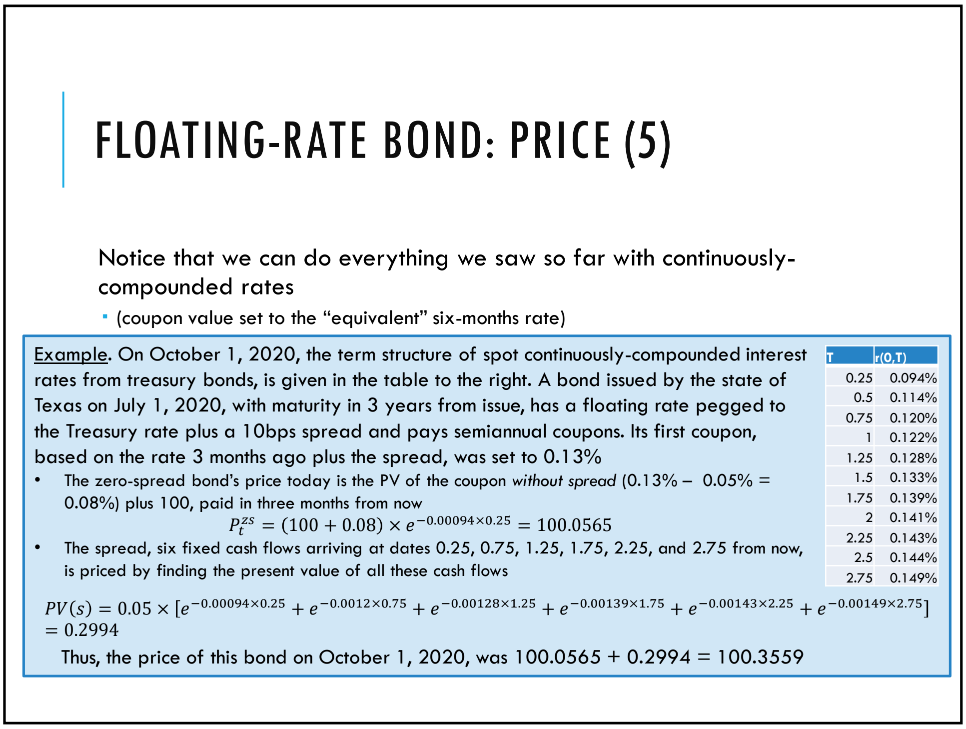
So far, we ignored the spread. How to incorporate it in the price?



- Simple! Treat the cash flows of this bond as two separate investments: a floating rate bond with zero spread, and a set of fixed payments (the spread)



- We find the price of the floater with **zero spread**
- We find the present value of the spreads using the appropriate yield curve
- We sum the two



FLOATING-RATE BOND: PRICE (5)

Notice that we can do everything we saw so far with continuously-compounded rates

- (coupon value set to the "equivalent" six-months rate)

Example. On October 1, 2020, the term structure of spot continuously-compounded interest rates from treasury bonds, is given in the table to the right. A bond issued by the state of Texas on July 1, 2020, with maturity in 3 years from issue, has a floating rate pegged to the Treasury rate plus a 10bps spread and pays semiannual coupons. Its first coupon, based on the rate 3 months ago plus the spread, was set to 0.13%

T	r(0,T)
0.25	0.094%
0.5	0.114%
0.75	0.120%
1	0.122%
1.25	0.128%
1.5	0.133%
1.75	0.139%
2	0.141%
2.25	0.143%
2.5	0.144%
2.75	0.149%

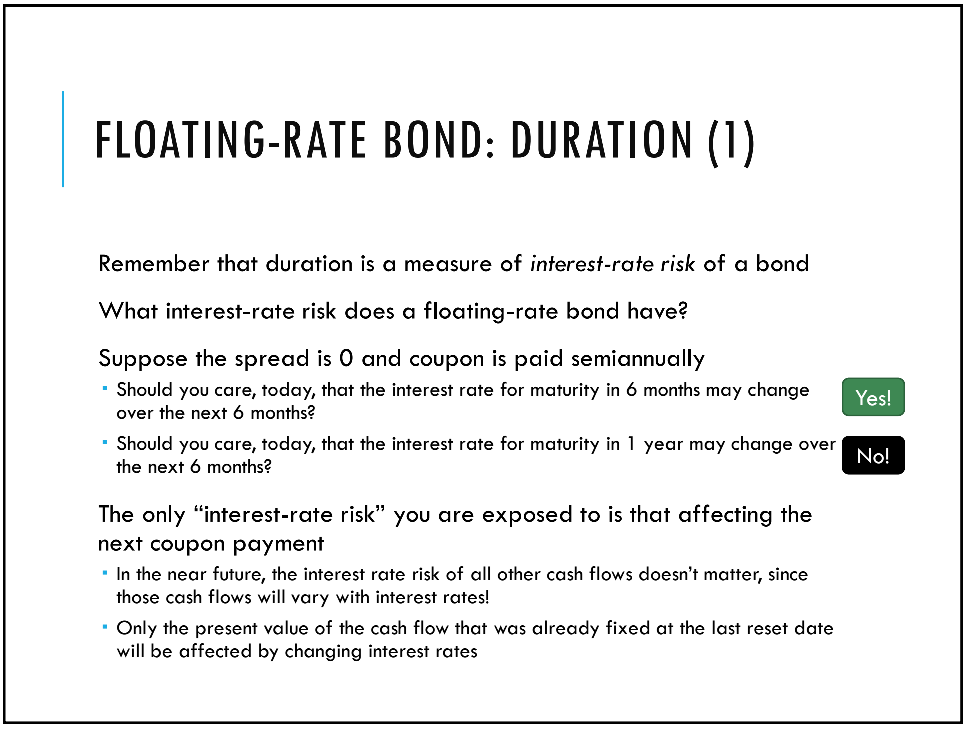
- The zero-spread bond's price today is the PV of the coupon *without spread* (0.13% – 0.05% = 0.08%) plus 100, paid in three months from now

$$P_t^{zs} = (100 + 0.08) \times e^{-0.00094 \times 0.25} = 100.0565$$

- The spread, six fixed cash flows arriving at dates 0.25, 0.75, 1.25, 1.75, 2.25, and 2.75 from now, is priced by finding the present value of all these cash flows

$$PV(s) = 0.05 \times [e^{-0.00094 \times 0.25} + e^{-0.0012 \times 0.75} + e^{-0.00128 \times 1.25} + e^{-0.00139 \times 1.75} + e^{-0.00143 \times 2.25} + e^{-0.00149 \times 2.75}]$$
$$= 0.2994$$

Thus, the price of this bond on October 1, 2020, was 100.0565 + 0.2994 = 100.3559



FLOATING-RATE BOND: DURATION (1)

Remember that duration is a measure of *interest-rate risk* of a bond

What interest-rate risk does a floating-rate bond have?

Suppose the spread is 0 and coupon is paid semiannually

- Should you care, today, that the interest rate for maturity in 6 months may change over the next 6 months? Yes!
- Should you care, today, that the interest rate for maturity in 1 year may change over the next 6 months? No!

The only "interest-rate risk" you are exposed to is that affecting the next coupon payment

- In the near future, the interest rate risk of all other cash flows doesn't matter, since those cash flows will vary with interest rates!
- Only the present value of the cash flow that was already fixed at the last reset date will be affected by changing interest rates

FLOATING-RATE BOND: DURATION (2)

The Macaulay duration of a floater with **zero spread** at time t, between coupons $j-1$ and j, $T_{j-1} \leq t \leq T_j$, equals the time left until the next reset date (next coupon):

$$MacD = T_j - t$$

- At a reset date, *ex-coupon*, the MacD is maximal, equal to $T_j - T_{j-1}$

The modified duration of a floater with zero spread is

- $MD = \frac{T_j - t}{1 + {r_m(t,T_j)}/{m}}$, to assess risk w.r.t. a parallel shift in the term structure of m-compounded interest rates
- $MD = T_j - t$, to assess risk w.r.t. a parallel shift in the term structure of continuously-compounded interest rates

Exercise. A coupon has just been paid for a note with semiannual floating rate. The semiannually-compounded LIBOR used to set its coupon for next semester was 2.4%. What is the Macaulay duration of this bond? What is its modified duration?

The Macaulay duration is the time left until next coupon: six months, $MacD = 0.5$. The modified duration is the MacD divided by the rate for the period of time left until the next coupon, which, as we know, is $r_2(0{,}0.5) = 2.4\%$. Thus, $MD = {}^{0.5}/_{1+0.012} = 0.4941$

FLOATING-RATE BOND: DURATION (3)

If the floater has a spread, we treat it as a portfolio

- The duration of the portfolio is the weighted sum of each component's duration
- We find duration for the spread (a series of cash flows) and the zero-spread floater separately, then sum (with weights)

Example. On October 1, 2020, the term structure of spot continuously-compounded interest rates from treasury bonds, is given in the table to the right. A bond issued by the state of Texas on July 1, 2020, with maturity in 3 years from issue, has a floating rate pegged to the Treasury rate plus a 5bps spread and pays semiannual coupons. Its first coupon, based on the rate 3 months ago plus the spread, was set to 0.13%.

- We found that the price of the spread-free part of this bond was 100.0565, the price of its spread was 0.2994, and, thus, its total price was 100.3559
- The weight of the zero-spread in this bond is 100.0565/100.3559 = 0.997; and that of the spread is 0.2994/100.3559=0.003.
- The MacD of the zero-spread part is the remaining time to reset = 0.25; the MacD of the spread is computed with MacD formula (Excel) and gives 1.499
- The MacD of the entire bond – treated as a portfolio – is *MacD = 0.997 x 0.25 + 0.003 x 1.499* = ***0.254***

T	r(0,T)
0.25	0.094%
0.5	0.114%
0.75	0.120%
1	0.122%
1.25	0.128%
1.5	0.133%
1.75	0.139%
2	0.141%
2.25	0.143%
2.5	0.144%
2.75	0.149%

SUMMARY SO FAR

Price bonds

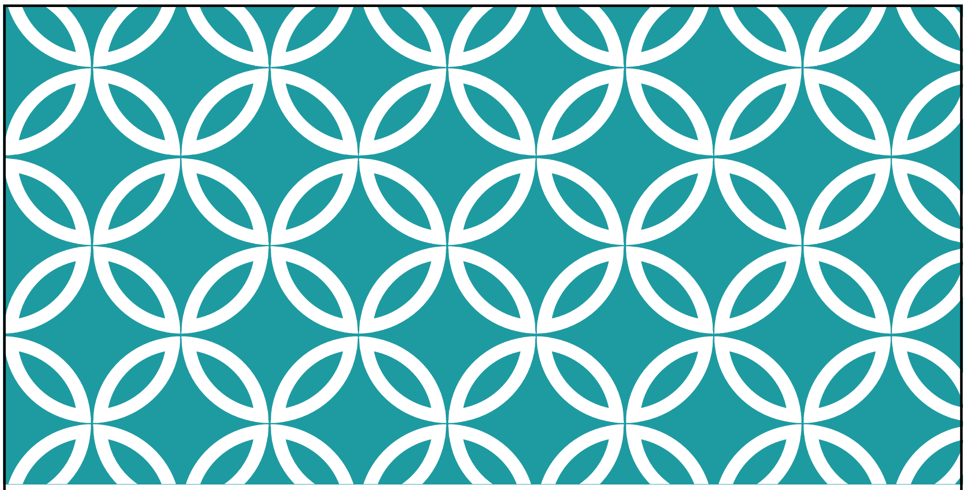
- Knowing the required YTM
- Using a reference yield curve or *term structure of interest rates* from markets
  - m-compounding interest rates
  - Continuous-compounding interest rates

Measure bond *interest-rate risk*

- Macaulay duration & modified duration
- Price approximation with modified duration
- "Special" durations: money, key-rate, effective

Floating-rate bonds – *price, risk*

Where does the yield curve come from?



VARYING DISCOUNT RATES

Interest rates, prices, and interest rate risk

OBJECTIVE

Understand bond pricing using interest rates
- Different rate for each maturity → each coupon is discounted with different rate
- Continuously-compounded interest rate

Interest-rate risk
- Macaulay duration and modified duration revisited
- Money duration, effective duration, and key-rate duration

Floating-rate bonds
- Different approach to security pricing (backwards induction)
- Interest-rate risk of a floating-rate bond

YTM AND IMPLIED INTEREST

Exercise. On $t =$ January 9, 2020, the US treasury issued T-bills with maturity $T_1 =$ July 9, 2020 at a price of 99.231556%. On the same day, it issued T-bills with maturity $T_2 =$ April 9, 2020 at a price of 99.615778%.

1. What is the YTM of each of these bonds?
2. Which one is a better investment? (Careful: how do you compare them?)
3. Why, if they're both T-bills, with equal default risk, do they provide different returns?

1. Remember that $P = \frac{100}{1+{}^{YTM}/_{n}}$. We know that for bond 1, $n = 2$, for bond 2, $n = 4$. Therefore:

$$YTM_1 = 2 \times \left(\frac{100}{99.213556} - 1\right) = 1.5488\%; \text{ and } YTM_2 = 4 \times \left(\frac{100}{99.615778} - 1\right) = 1.5428\%$$

2. We cannot compare the above rates because they have different compounding. E.g., a 4% rate compounded every quarter yields $(1.01)^2 = 1.0201$ in one semester → annualized rate of 4.02% > 4% per semester. We must equate our bonds' compounding. Some standards are semiannual, annual, or continuous compounding. Use semiannual:

- Bond 1 is already a semiannual rate, $YTM_1 = r_2 = 1.5488\%$
- Bond 2: annualized 1.5428% means 0.3857% per quarter. The return after 1 semester (2 quarters), is $\left(1 + {}^{YTM_2}/_{4}\right)^2 - 1 = (1.003857)^2 - 1 =$0.7729%. Annualized: 2 x 0.7729% = 1.5458%.
- Now we can compare! Bond 1 is a "better" investment since 1.5488% > 1.5458%

INTEREST RATES

Notice that when compounding is more frequent, you need a lower rate to achieve the same return after, for example, 1 year

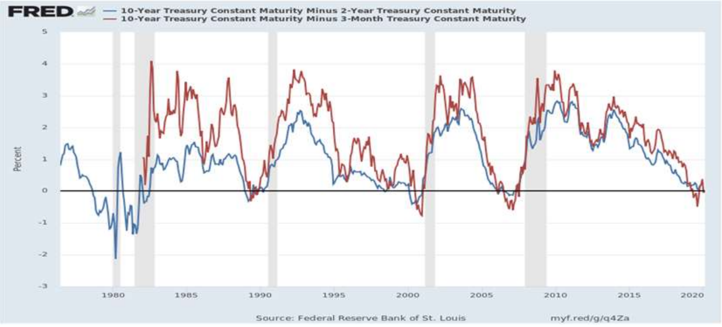
Notice that the US Treasury offers you a higher rate of return for lending them money for 1 semester than 1 quarter

- Think of bank loan interest rates for 10-y loan vs. 30-y loan…

The YTM of zero-coupon bonds is equivalent to the *interest rate* from today to the date of the bond's maturity

Even when converted to same compounding frequency, interest rates for different maturities are <u>different</u>

INTEREST RATE GAPS



Source: Federal Reserve Bank of St. Louis myf.red/g/q4Za

- Gap between long- and short-term rates changes over time and is not the same for all short-term bonds

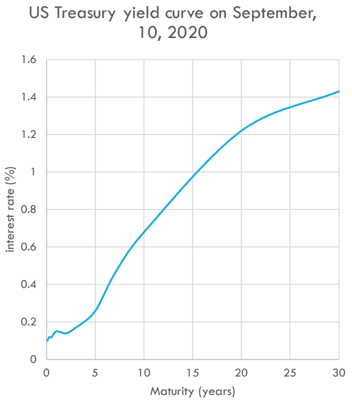
YIELD CURVE

In general, rates are different for different maturities

The *yield curve* or *term structure of interest rates* shows interest rates for each maturity for a given class of bonds

- If zcb available for all maturities, these would be the YTMs implied by zcb prices

The yield curve can be used to price other bonds of the same or similar class



Source: FRED

BOND PRICING WITH YIELD CURVE

An investor may know the YTM they demand for a certain bond

…But, typically, investors learn about discount rates from markets

- If discount rates are determined from market information, you learn the yield curve

YTM interpreted as the *implied* return from a given investment

- After you know the price of a bond, you can determine its YTM (= its internal rate of return if held until maturity)
- Rarely will you know a bond's YTM *before* its price → rarely is YTM used for pricing

When yield curve is used for pricing, each cash flow of the bond (coupon, principal) is discounted with a different rate

Makes a lot of intuitive sense!

- Depending on the yield curve's shape, it may be better to hold short/medium/ or long-term investments…

BOND PRICING WITH YIELD CURVE (1)

When working with interest rates, it is fundamental that all rates (for all maturities), be of the same compounding frequency!

Standard compounding frequencies used for yield curves are semiannual, annual, or continuous – we will work with all three

- Figure on previous slide was the yield curve of **annually-compounded** rates

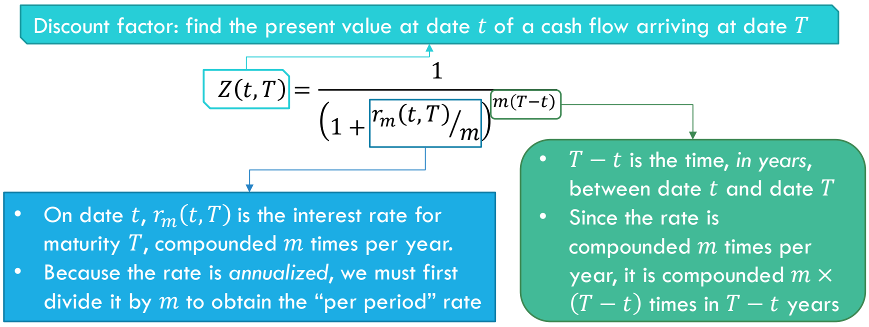
Remember the “discounted cash flows principle”

$$P_t = Z(t, T_1) \times c + Z(t, T_2) \times c + \cdots + Z(t, T_i) \times [c + 100]$$

$Z(t, T_j)$ is the discount factor between dates t and T_j

- When using YTM for pricing, $Z(t, T_j) = \frac{1}{(1 + {}^{YTM}/_{n})^{n \times (T_j - t)}}$
- When we use *m-compounding* interest rates, $Z(t, T_j) = \frac{1}{\left(1 + {}^{r_m(t, T_j)}/_{m}\right)^{m(T_j - t)}}$

ANATOMY OF A FORMULA



Discount factor: find the present value at date t of a cash flow arriving at date T

$$Z(t, T) = \frac{1}{\left(1 + {}^{r_m(t, T)}/_{m}\right)^{m(T - t)}}$$

- On date t, $r_m(t, T)$ is the interest rate for maturity T, compounded m times per year.
- Because the rate is *annualized*, we must first divide it by m to obtain the “per period” rate

- $T - t$ is the time, *in years*, between date t and date T
- Since the rate is compounded m times per year, it is compounded $m \times (T - t)$ times in $T - t$ years

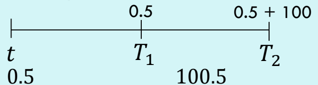
Example. On October 1, 2020 (t), the semiannually-compounded interest rate for maturity February 1, 2021 (T), is 1.3%. What discount factor must be used to discount a cash flow arriving on Feb 01, 2021?

- The interest rate is annualized, so the per-*semester* rate is ${}^{r_2(t,T)}/_{2} = 0.65\%$
- Between t and T, there are 4 months: $T - t = {}^{1}/_{3}$ years and $m \times (T - t) = {}^{2}/_{3}$ semesters

$$Z(t, T) = {}^{1}\Big/_{(1 + 0.0065)^{2/3}} = 0.99569$$

BOND PRICING WITH YIELD CURVE (2)

Exercise. On $t =$ January 9, 2020, you know (see previous exercise) that the semiannually-compounded interest rates with maturities $T_1 =$ April 9, 2020, and $T_2 =$ July 9, 2020, are, respectively, 1.5458% and 1.5488%. A special bond with quarterly coupons and maturity July 9, 2020, is issued on that same day. If it has a 2% coupon, what is its price today?



$$P = \frac{0.5}{\left(1 + \frac{0.015458}{2}\right)^{\frac{1}{2}}} + \frac{100.5}{\left(1 + \frac{0.015488}{2}\right)^{1}} = 100.2258$$

*Another way to think about it, is to first find the *equivalent* quarterly compounded rates:

$$r_4(t,T_1)/_4 = \left[1 + r_2(t,T_1)/_2\right]^{\frac{1}{2}} - 1 = 0.3857\% \,; r_4(t,T_2)/_4 = \left[1 + r_2(t,T_2)/_2\right]^{\frac{1}{2}} - 1 = 0.3865\%$$

And use this quarterly rate to discount the first cash flow 1 period and the second 2 periods:

$$P = \frac{0.5}{1 + 0.003857} + \frac{100.5}{(1 + 0.003865)^2} = 100.2257$$

BOND PRICING WITH YIELD CURVE (3)

Most of the time, you will have a yield curve with compounding as frequent as your coupon payments

- Typically, semiannually-compounded rates, and semiannual coupons

Example. On January 31, 2020, semi-annually compounded interest rates with maturities $T_1 =$ July 31, 2020, $T_2 =$ January 31, 2021, and $T_3 =$ July 31, 2021 are given by $r_2(t,T_1) = 1.487\%$, $r_2(t,T_2) = 1.495\%$, and $r_2(t,T_3) = 1.503\%$. A T-note is issued on January 31, 2020, with coupon 2% and maturity July 31, 2021. We can compute discount factors:

$$Z(t,T_1) = \left(1 + \tfrac{0.01487}{2}\right)^{-1} = 0.99262;\ Z(t,T_2) = \left(1 + \tfrac{0.01495}{2}\right)^{-2} = 0.98522;\ Z(t,T_3) = \left(1 + \tfrac{0.01503}{2}\right)^{-3} = 0.97779$$

The bond price is thus: $P_c(t,T_3) = 0.99262 + 0.98522 + 101 \times 0.97779 = 100.7346\%$

…Or a yield curve of continuously-compounding interest rates, which can be easily adapted to any frequency of coupon payments!

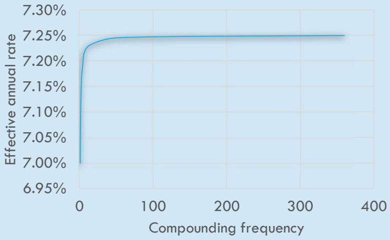
CONTINUOUS COMPOUNDING (1)

Question. Banks A and B offer annualized rates on their savings accounts of 1.5%. Bank A says this rate is monthly compounded ($r_{12}(t,T)$), while bank B says it's annually compounded ($r_1(t,T)$). Which bank is, effectively, offering the higher rate?

For the same annualized number, more frequent compounding implies a higher effective annual rate

Example. Fix $r_n(0,1) = 7\%$, what is the annual growth of money as we vary the value of n?

n	Effective rate
1	7.00%
2	7.12%
4	7.19%
12	7.23%
52	7.25%
360	7.25%



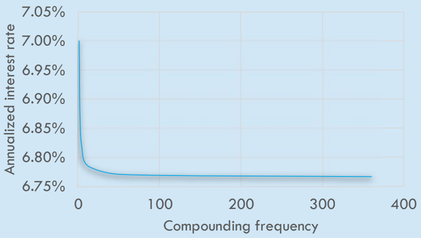
CONTINUOUS COMPOUNDING (2)

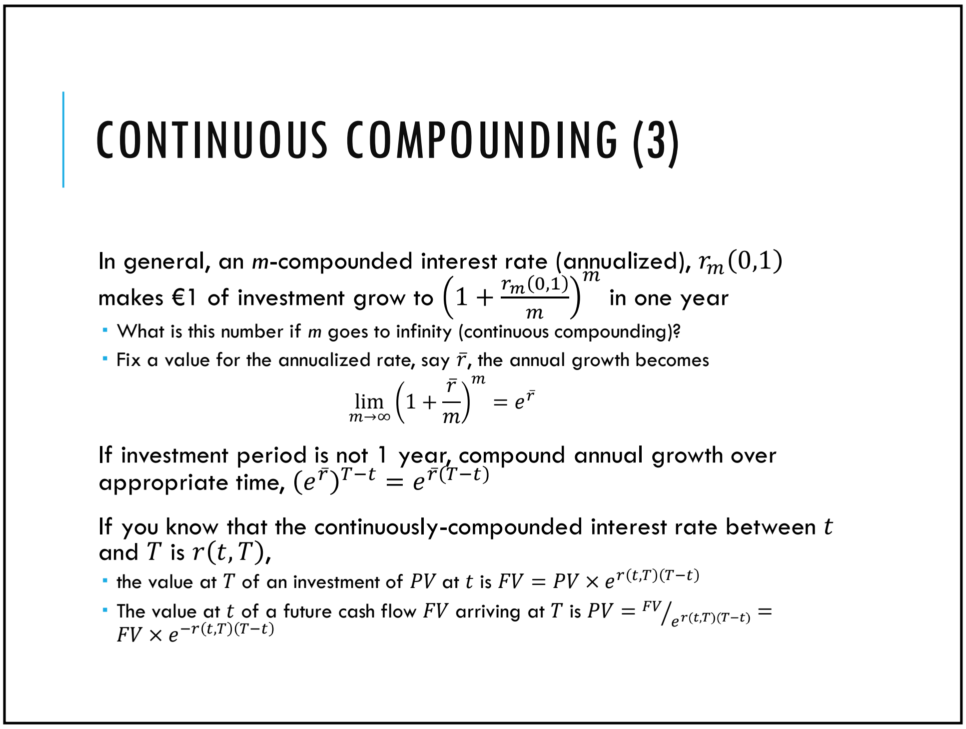
Question. The bank for which you work offers two accounts with the exact same effective annual rate of 1%. Account A is sold to the public as a quarterly-compounded account, account B as an annually-compounded account. For which account will the public see a higher annualized interest rate ($r_4(t,T)$ or $r_1(t,T)$)?

For the same effective annual rate, more frequent compounding implies a lower annualized interest rate

Example. Fix $Z(0,1) = 0.93458$ (which gives $r_1(0,1) = 7\%$), what is $r_n(0,1)$ for different values of n?

n	Annualized rate
1	7.00%
2	6.88%
4	6.82%
12	6.78%
52	6.77%
360	6.77%





CONTINUOUS COMPOUNDING (3)

In general, an *m*-compounded interest rate (annualized), $r_m(0,1)$ makes €1 of investment grow to $\left(1+\frac{r_m(0,1)}{m}\right)^m$ in one year

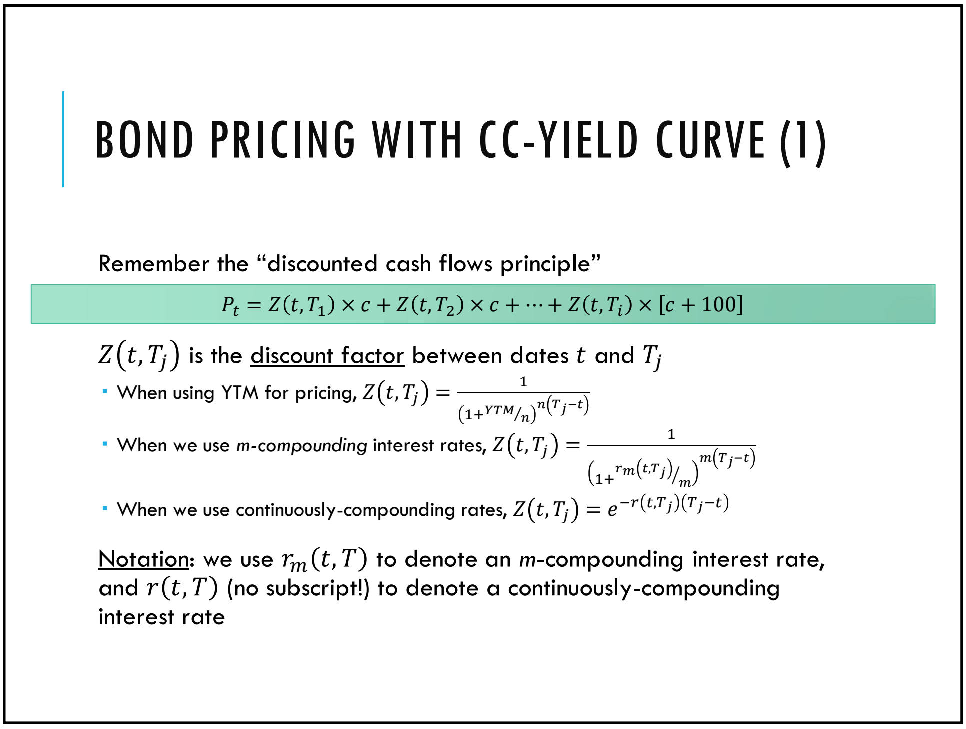
- What is this number if *m* goes to infinity (continuous compounding)?
- Fix a value for the annualized rate, say $\bar{r}$, the annual growth becomes

$$\lim_{m\to\infty}\left(1+\frac{\bar{r}}{m}\right)^m = e^{\bar{r}}$$

If investment period is not 1 year, compound annual growth over appropriate time, $(e^{\bar{r}})^{T-t} = e^{\bar{r}(T-t)}$

If you know that the continuously-compounded interest rate between t and T is $r(t,T)$,

- the value at T of an investment of PV at t is $FV = PV \times e^{r(t,T)(T-t)}$
- The value at t of a future cash flow FV arriving at T is $PV = {}^{FV}\!/_{e^{r(t,T)(T-t)}} = FV \times e^{-r(t,T)(T-t)}$



BOND PRICING WITH CC-YIELD CURVE (1)

Remember the “discounted cash flows principle”

$$P_t = Z(t,T_1) \times c + Z(t,T_2) \times c + \cdots + Z(t,T_i) \times [c+100]$$

$Z(t,T_j)$ is the discount factor between dates t and T_j

- When using YTM for pricing, $Z(t,T_j) = \frac{1}{(1+{}^{YTM}\!/_{n})^{n(T_j-t)}}$
- When we use *m-compounding* interest rates, $Z(t,T_j) = \frac{1}{\left(1+{}^{r_m(t,T_j)}\!/_{m}\right)^{m(T_j-t)}}$
- When we use continuously-compounding rates, $Z(t,T_j) = e^{-r(t,T_j)(T_j-t)}$

Notation: we use $r_m(t,T)$ to denote an *m*-compounding interest rate, and $r(t,T)$ (no subscript!) to denote a continuously-compounding interest rate

BOND PRICING WITH CC-YIELD CURVE (2)

Exercise. The table on the right contains the term structure of *continuously-compounded* interest rates on $t =$ March 15, 2000. Maturities are given in years, starting on date t. What is the price of a 4-year bond with semiannual coupon of 7% issued on that day?

- A coupon of 3.5 is paid every 6 months. The relevant rates for the question are those at maturity 0.5, 1, 1.5, 2, 2.5, 3, 3.5, and 4.
- We use the discounted cash flows principle with the discount factor appropriate for continuously-compounded rates:

$$P = 3.5 \times \left[e^{-0.0649\times 0.5} + e^{-0.0671\times 1} + e^{-0.0684\times 1.5} + e^{-0.0688\times 2} + e^{-0.0688\times 2.5} + e^{-0.0683\times 3} + e^{-0.0676\times 3.5}\right] + 103.5 \times e^{-0.0667\times 4} = 100.694$$

Maturity	$r(t,T)$
0.25	6.33%
0.5	6.49%
0.75	6.62%
1	6.71%
1.25	6.79%
1.5	6.84%
1.75	6.87%
2	6.88%
2.25	6.89%
2.5	6.88%
2.75	6.86%
3	6.83%
3.25	6.80%
3.5	6.76%
3.75	6.72%
4	6.67%

What if you want to price a bond that was not issued today?

- You discount cash flows only until today (fraction of coupon period), just like before
- Careful! You must have the appropriate interest rates in your yield curve!

BOND PRICING WITH CC-YIELD CURVE (3)

Exercise. Using the same term structure of interest rates as before, located at $t =$ March 15, 2000, determine the price of a 3-year coupon bond issued on December 15, 1998, with a semiannual coupon of 4%.

- The next coupon payment of this bond is June 15, 2000 (in 3 months). Another 3 coupons and the principal are paid in 9, 15, and 21 months.
- We have the appropriate rates! $r(0, 0.25)$, $r(0, 0.75)$, $r(0, 1.25)$, & $r(0, 1.75)$

$$P = 2 \times \left[e^{-0.0633\times 0.25} + e^{-0.0662\times 0.75} + e^{-0.0679\times 1.25}\right] + 102 \times e^{-0.0687\times 1.75} = 96.155$$

Maturity	$r(t,T)$
0.25	6.33%
0.5	6.49%
0.75	6.62%
1	6.71%
1.25	6.79%
1.5	6.84%
1.75	6.87%
2	6.88%
2.25	6.89%
2.5	6.88%
2.75	6.86%
3	6.83%
3.25	6.80%
3.5	6.76%
3.75	6.72%
4	6.67%

What if you wanted to price a bond issued 4 months ago (next coupon in 2 months)?

- You would need interest rates with maturities 0.167, 0.667, 1.167, etc.
- You cannot do it with the yield curve you have at your disposal!

MACAULAY DURATION REVISITED

Also when we use a yield curve for pricing, we can use duration to assess the interest-rate risk of a bond

We can easily extend the formula of Macaulay duration to the situation in which we use the yield curve to price a bond

- Each maturity has a weight, which is the appropriately-discounted cash flow arriving at that maturity: “appropriate” means “using the interest rate for that maturity”

At date t, with i remaining coupons, the Macaulay duration of a bond is given by

$$MacD_t = \frac{c \times Z(t,T_1)}{P_t}(T_1 - t) + \frac{c \times Z(t,T_2)}{P_t}(T_2 - t) + \cdots + \frac{(c+100) \times Z(t,T_i)}{P_t}(T_i - t),$$

With

- $Z(t,T_j) = \frac{1}{\left(1+{}^{YTM}/_{n}\right)^{n(T_j - t)}}$ if we are pricing the bond with YTM
- $Z(t,T_j) = \frac{1}{\left(1+{}^{r_m(t,T_j)}/_{m}\right)^{m(T_j - t)}}$ if we are pricing the bond with an *m*-compounded yield curve
- $Z(t,T_j) = e^{-r(t,T_j)(T_j - t)}$ if we are pricing the bond with a continuously-compounded yield curve

MACAULAY DURATION WITH YIELD CURVE

Exercise. Consider a 3-year bond with 2% semiannual coupon and consider the term structure of *semiannually-compounded interest rates* given in the table to the right.

1. What is the price of the bond today?
2. What is the Macaulay duration of the bond?

- Use the formula $Z(0,T_j) = \frac{1}{\left(1+{}^{r_2(0,T_j)}/_{2}\right)^{2T_j}}$ to find discount factors

T	r_2(0,T)
0.5	3.00%
1	3.20%
1.5	3.30%
2	3.39%
2.5	3.42%
3	3.46%

T_j	$Z(0,T)$	CF	CF x Z(0,T)	w_j	$w_j \times T_j$
0.5	0.9852	1	0.9852	0.0103	0.0051
1	0.9688	1	0.9688	0.0101	0.0101
1.5	0.9521	1	0.9521	0.0099	0.0149
2	0.9350	1	0.9350	0.0098	0.0195
2.5	0.9187	1	0.9187	0.0096	0.0240
3	0.9022	101	91.1228	0.9504	2.8511
		Price:	**95.8826**	**MacD:**	**2.9247**

MACAULAY DURATION WITH YIELD CURVE

Exercise. Consider a 3-year bond with 6% semiannual coupon and consider the term structure of *continuously-compounded interest rates* given in the table to the right.

1. What is the price of the bond today?
2. What is the Macaulay duration of the bond?

T_j	$r(0,T_j)$
0.5	6.49%
1	6.71%
1.5	6.84%
2	6.88%
2.5	6.88%
3	6.83%

- Use the formula $Z(0,T_j) = e^{-r(0,T_j)\times T_j}$ to find discount factors

T_j	$r(0,T_j)$	$Z(0,T_j)$	CF	$CF \times Z(0,T_j)$	w_j	$w_j \times T_j$
0.5	6.49%	0.9681	3	2.9042	0.0298	0.0149
1	6.71%	0.9351	3	2.8053	0.0288	0.0288
1.5	6.84%	0.9025	3	2.7075	0.0278	0.0417
2	6.88%	0.8714	3	2.6143	0.0268	0.0536
2.5	6.88%	0.8420	3	2.5259	0.0259	0.0648
3	6.83%	0.8147	103	83.9171	0.8609	2.5827
			Price:	97.4743	MacD:	2.7865

- This way we obtain the *MacD* in years: 2.7865

MODIFIED DURATION (YIELD CURVE)

Modified duration was our measure of bond "yield risk": how sensitive is the bond's value to small changes in the YTM

- Mathematically, *MD* is the derivative of bond price w.r.t. YTM, divided by (relative to) the bond's price
- We computed *MD* from MacD: we divided *MacD* by (1+ YTM/n)

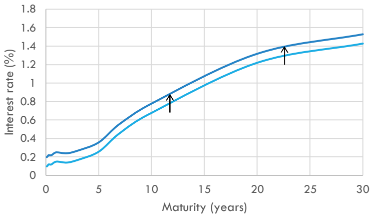
Similar measures of interest-rate risk can be computed when we use the yield curve to value bonds

The *modified duration* will tell us how sensitive the bond's value is to a small change in the yield curve

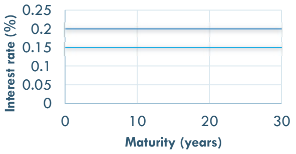
- There are many ways in which the yield curve may change!
- Mathematically, *MD* is the derivative of bond price w.r.t. a fixed number added to all interest rates in yield curve, divided by (relative to) price
- **Only for continuously-compounded interest rates is there a link between *MD* and *MacD***

MODIFIED DURATION (YIELD CURVE)

The **modified duration** of a bond is the (negative of the) percent sensitivity of the price *P* to a small parallel shift of the yield curve



The definition with YTM is equivalent to having a flat yield curve!



The modified duration for **continuously-compounded** interest rates* can be computed from the Macaulay duration, since $MD = MacD$ in this case

*That is: "the percent sensitivity of a bond's price to a small parallel shift of the term structure of continuously-compounded interest rates"

The modified duration for *m*-compounded interest rates **cannot** be computed from *MacD*

SKIP

MODIFIED DURATION (YIELD CURVE)

To compute modified duration for *m*-compounded interest rates, we must use a different formula

$$MD_{Qt} = \frac{1}{P_t} \times \left[\frac{c}{\left[1 + {r_m(t,T_1)}/{m}\right]^{m(T_1-t)+1}} \times (T_1 - t) + \frac{c}{\left[1 + {r_m(t,T_2)}/{m}\right]^{m(T_2-t)+1}} \times (T_2 - t) + \cdots + \frac{c+100}{\left[1 + {r_m(t,T_i)}/{m}\right]^{m(T_i-t)+1}} \times (T_i - t) \right]$$

- The little +1 in the exponent of the denominators, makes this different from *MacD*
- The weight given to each maturity is now $w_j = \frac{c}{P_t\left[1+{r_m(t,T_j)}/{m}\right]^{m(T_j-t)+1}}$. These weights **do not** add up to 1
- MD_Q stands for "quasi"-modified duration, a name given to this formula

…Or we can compute the *approximate modified duration* or **effective duration**

$$AMD = ED = \frac{P_- - P_+}{2 \times dr \times P}$$

- P_- is the price computed using current rates minus dr
- P_+ is the price computed using current rates plus dr

SKIP

MODIFIED DURATION (YIELD CURVE)

Example. Remember the exercise where we computed the Macaulay duration of a 3-year bond with semiannually-compounded 6% coupon using the term structure of *continuously-compounded interest rates*. We found that *MacD = 2.7865*. The modified duration of this bond is *MD = MacD = 2.7865*

SKIP

Example. Consider a 3-year bond with 2% semiannual coupon, and consider the term structure *semiannually-compounded interest rates* given in the table to the right. We previously computed its price, *P = 95.8826%*, and *MacD = 2.9247*. We now compute its modified duration:

T_j	$r_2(0,T_j)$
0.5	3.00%
1	3.20%
1.5	3.30%
2	3.39%
2.5	3.42%
3	3.46%

Using the quasi-modified duration formula

T_j	$2T_j+1$		$\left[1+{}^{r_2(0,T_j)}/_2\right]^{2T_j+1}$	CF	$\frac{CF}{\left[1+{}^{r_2(0,T_j)}/_2\right]^{2T_j+1}}$	w_j	$w_j \times T_j$
0.5	2	1.50%	1.0302	1	0.9707	0.0101	0.0051
1	3	1.60%	1.0488	1	0.9535	0.0099	0.0099
1.5	4	1.65%	1.0677	1	0.9366	0.0098	0.0147
2	5	1.70%	1.0877	1	0.9194	0.0096	0.0192
2.5	6	1.71%	1.1071	1	0.9033	0.0094	0.0236
3	7	1.73%	1.1276	101	89.5732	0.9342	2.8026
							2.8750

Using the approximate modified duration formula, assuming 1bp parallel shift, *dr = 0.01%*

We compute P, P-, and P+ in a table:

T_j	$r_2(0,T_j)$	$r_2(0,T_j)+dr$	$r_2(0,T_j)-dr$	$Z(0,T_j)$	$Z_+(0,T_j)$	$Z_-(0,T_j)$	CF	$CF \times Z$	$CF \times Z_+$	$CF \times Z_-$
0.5	3.00%	3.01%	2.99%	0.9852	0.9852	0.9853	1	0.9852	0.9852	0.9853
1	3.20%	3.21%	3.19%	0.9688	0.9687	0.9688	1	0.9688	0.9687	0.9688
1.5	3.30%	3.31%	3.29%	0.9521	0.9519	0.9522	1	0.9521	0.9519	0.9522
2	3.39%	3.40%	3.38%	0.9350	0.9348	0.9352	1	0.9350	0.9348	0.9352
2.5	3.42%	3.43%	3.41%	0.9187	0.9185	0.9189	1	0.9187	0.9185	0.9189
3	3.46%	3.47%	3.45%	0.9022	0.9019	0.9025	101	91.1228	91.0959	91.1497
								95.8826	**95.8550**	**95.9101**

…And use the formula $AMD = \frac{P_{-}-P_{+}}{2\times dr\times P} = \frac{95.9101-95.8550}{2\times 0.0001\times 95.8826} = 2.8750$. **The same as with MD_Q formula!**

IMPORTANT CASE: ZERO-COUPON BOND

The Macaulay duration of a **zero-coupon bond** is just the time left to maturity

- Remember that the price at t of a zcb with maturity T, is $P = Z(t,T) \times 100$
- Using the formula of *MacD* we have $\frac{100 \times Z(t,T)}{P} \times (T-t) = \frac{100 \times Z(t,T)}{100 \times Z(t,T)} \times (T-t) = T - t$
- This is true regardless of whether we are using YTM or yield curves

The modified duration of a zcb can always be computed from *MacD*

- $MD = {}^{MacD}/_{1+{}^{YTM}/_{n}}$ if we're considering YTM
- $MD = \frac{MacD}{1+{}^{r_m(t,T)}/_{m}}$ if we're considering the *m*-compounding term structure of interest rates
- $MD = MacD$ if we're considering the term structure of continuously-compounded rates

Example. Three months before maturity, the Macaulay duration of a zcb is 0.25 (measured in years). If the current quarterly-compounded interest rate with maturity in 3 months is $r_4(t, t+0.25) = 3\%$, the modified duration of this zcb w.r.t. a *parallel shift of the term structure of quarterly-compounded rates* is 0.25/(1+0.0075) = 0.2481

IMPORTANT CASE: PORTFOLIO

Suppose you have a portfolio of fixed-income securities containing securities 1, 2, …, S (*s* denotes a generic security in the portfolio)

- Each security has a weight, w_s, in your portfolio. That is the amount of money currently invested in that security divided by the total value of the portfolio
- Each security has a Macaulay duration $MacD_s$ and a modified duration MD_s

The Macaulay duration of the portfolio equals the *weighted sum* of Macaulay durations of all assets

$$MacD_{port} = w_1 \times MacD_1 + w_2 \times MacD_2 + \cdots + w_S \times MacD_S$$

The modified duration of the portfolio equals the *weighted sum* of the modified durations of all assets

$$MD_{port} = w_1 \times MD_1 + w_2 \times MD_2 + \cdots + w_S \times MD_S$$

Example. You manage a fixed income portfolio with 40% in a 10-year bond with semiannual coupons and *MacD* = 9.138 and YTM = 5%, 15% in a 1-year T-bill with YTM = 3%, and 45% in a 6-months T-bill with YTM = 1.4%. The *MacD* of this portfolio is $MacD_{port} = 0.4 \times 9.138 + 0.15 \times 1 + 0.45 \times 0.5 = 4.0302$.
Exercise: What is the modified duration of the portfolio?

OTHER MEASURES: MONEY DURATION (1)

Remember that modified duration is the change in bond value caused by a change in YTM or *r*, **relative** to the investment value

$$MD = -\frac{1}{P}\frac{dP}{dr}$$

If, instead, we wish to know how much **money** we would really gain/lose due to a change in YTM or *r*, it suffices to not divide by P

Money duration measures how much money is lost/gained per unit change of YTM or r

$$D^{\$} = -\frac{dP}{dr}$$

- For nonzero valued positions,

$$D^{\$} = P \times MD$$

OTHER MEASURES: MONEY DURATION (2)

Money duration of a portfolio with N_s units of each security, *s*, is

$$D^{\$} = \sum_{s=1}^{S} N_s \times D_s^{\$}$$

New value of portfolio (price), after change in interest rate, can be approximated with money duration:

$$\bar{V} = V - D^{\$} \times dr$$

Exercise. What is the money duration with respect to a change in its YTM, of an investment with face value of $1million in a 1-year zcb that is sold for 98.039% at issue?

We can use the formula $D^{\$} = P \times MD$, and $MD = {}^{MacD}/_{1+YTM}$ (we don't divide YTM by n because it's a 1-year bond, so YTM compounds once per year). To know the current value of YTM, notice that $P = \frac{100}{1+YTM} \Rightarrow YTM = \frac{100}{98.039} - 1 = 2\%$. Now we have all elements:

- $YTM = 2\%, MacD = 1 \Rightarrow MD = \frac{1}{1.02} = 0.9804$
- $MD = 0.9804, P = 980\ 390$ (the value of the investment today) $\Rightarrow D^{\$} = 980\ 390 \times 0.9804 = 961\ 174.356$

If the YTM increases by 100bps (1%), the value of the investment goes down to 970 778.256

OTHER MEASURES: KEY RATE DURATION (1)

Deal with **non-parallel** shifts of the yield curve

- Take only one or a few maturities considered "key"

Key duration is defined in a similar way as effective duration, but using only key maturities to compute new prices

- Compute the new bond price if interest rates for "key" maturities increase, P_+^*
- Compute the new bond price if interest rates for "key" maturities decrease, P_-^*

$$D^* = \frac{P_-^* - P_+^*}{2 \times dr \times P}$$

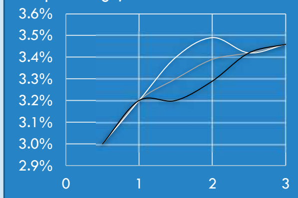
If you change only one maturity at a time, the sum of key durations for all maturities equals the effective duration of the bond (duration for parallel shift)

OTHER MEASURES: KEY RATE DURATION (2)

Underlined Example. Consider a 3-year bond with 2% semiannual coupon, and consider the term structure of *semiannually-compounded interest rates* given in the table to the right. We previously computed its price, *P = 95.8826*, its *MacD = 2.9247*, and its modified duration (both with the quasi-mod formula and the effective duration formula), *MD = 2.8750*.

T_j	$r_2(0, T_j)$
0.5	3.00%
1	3.20%
1.5	3.30%
2	3.39%
2.5	3.42%
3	3.46%

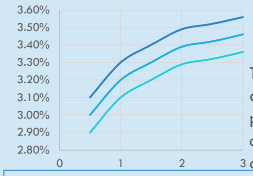
Consider the middle maturities (1.5 and 2 years) as key. Under this assumption, the new yield curves assuming a 10bps change, are:



T_j	$r_2(\cdot) + dr$	$r_2(\cdot) - dr$
0.5	3.00%	3.00%
1	3.20%	3.20%
1.5	3.40%	3.20%
2	3.49%	3.29%
2.5	3.42%	3.42%
3	3.46%	3.46%

The prices P_-^* and P_+^* are $P_-^* = 95.8858$ and $P_+^* = 95.8793$, so that key-rate duration in this case is

$$D^* = \frac{95.8858 - 95.8793}{2 \times 0.001 \times 95.8826} = 0.0338$$



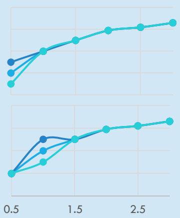
To compute effective duration we assumed a parallel shift of the yield curve (all rates increased or decreased by 1bp).

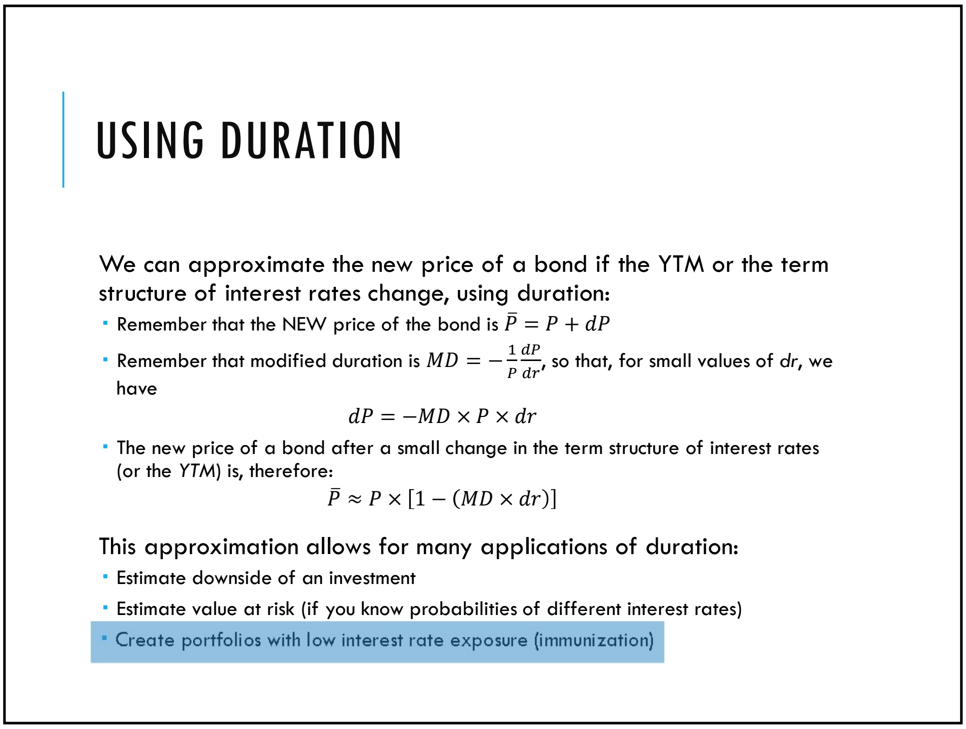
Consider changing interest rates one at a time (graphs show 6-months and 1-year rate cases).

Six different key-rate durations are computed

- $D_{0.5}^* = 0.005062$
- $D_1^* = 0.009944$
- $D_{1.5}^* = 0.014653$
- $D_2^* = 0.019178$
- $D_{2.5}^* = 0.023552$
- $D_3^* = 2.802597$

If all are added up, we obtain 2.875, which is effective duration of this bond





USING DURATION

We can approximate the new price of a bond if the YTM or the term structure of interest rates change, using duration:

- Remember that the NEW price of the bond is $\bar{P} = P + dP$
- Remember that modified duration is $MD = -\frac{1}{P}\frac{dP}{dr}$, so that, for small values of *dr*, we have

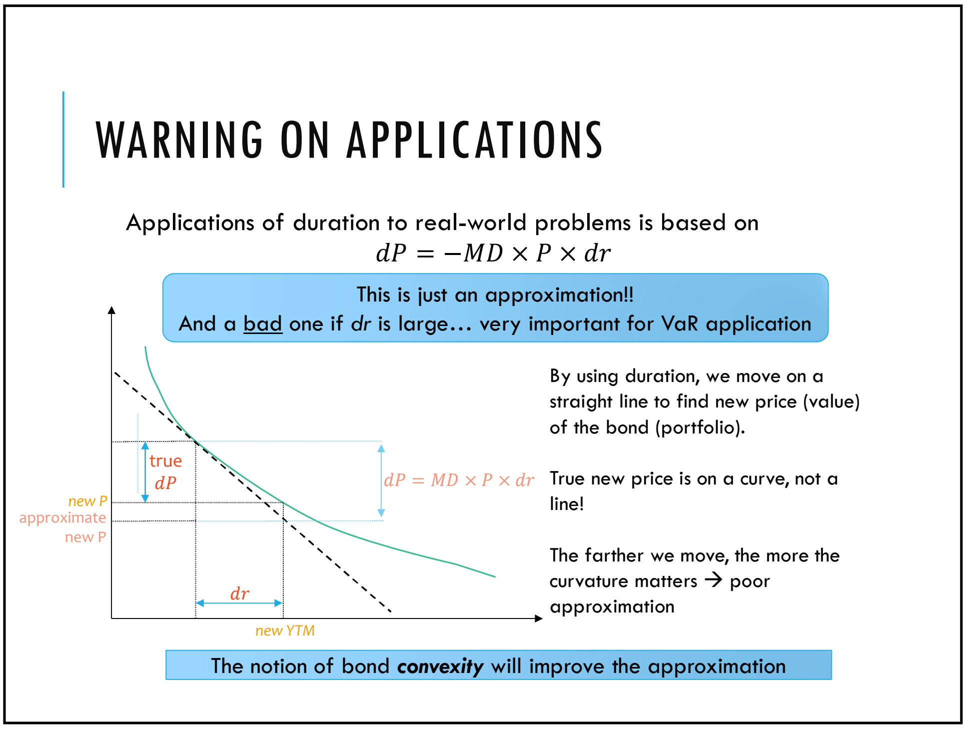
$$dP = -MD \times P \times dr$$

- The new price of a bond after a small change in the term structure of interest rates (or the *YTM*) is, therefore:

$$\bar{P} \approx P \times [1 - (MD \times dr)]$$

This approximation allows for many applications of duration:

- Estimate downside of an investment
- Estimate value at risk (if you know probabilities of different interest rates)
- Create portfolios with low interest rate exposure (immunization)



WARNING ON APPLICATIONS

Applications of duration to real-world problems is based on

$$dP = -MD \times P \times dr$$

This is just an approximation!!
And a bad one if *dr* is large… very important for VaR application

By using duration, we move on a straight line to find new price (value) of the bond (portfolio).

True new price is on a curve, not a line!

The farther we move, the more the curvature matters → poor approximation

The notion of bond ***convexity*** will improve the approximation

CASE: FLOATING RATE BONDS

"different" bond pricing

FLOATING RATE BOND

A *floating rate bond* or "floater" is defined by

- A fixed maturity at date T_i,
- Fixed coupon payment dates, called *reset dates*, $T_1, T_2, \cdots, T_i$
- A variable part of the coupon, which **changes**, but is known one period in advance

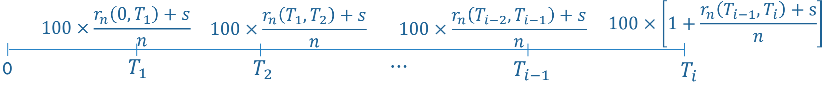
Example. A 3-year floater in US$ paying semiannual coupons is pegged to the LIBOR. If it is issued today, the first coupon, paid in six months, equals today's six-months LIBOR rate.

- A fixed part of the coupon, called *spread*

Example. If the above floater has a spread of 2bps, its coupon in six months equals today's 6-months LIBOR rate + 0.02%/2

When we buy a floating-rate bond, we only know the value of the first coupon. Remaining cash flows are unknown.

- Suppose, for simplicity, coupon frequency = rate compounding frequency. Only $r_n(0, T_1)$ is known today, other $r_n(T_j, T_{j+1})$ unknown today



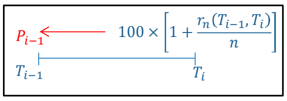
FLOATING RATE BOND: PRICE (1)

→ Assumption: The interest rate to which the floating-rate bond's coupons are pegged is also the relevant discount rate for this bond's cash flows

Using this assumption let's price a floater with <u>zero</u> spread*:*

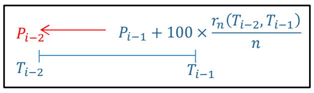
- Suppose today is date T_{i-1}, **one period before final coupon payment**
- At T_{i-1} we <u>know</u> all remaining cash flows: $100 \times \left[1 + \frac{r_n(T_{i-1}, T_i)}{n}\right]$
- If we want to know the value of this bond at T_{i-1} (ex-coupon), we discount this single cash flow at the prevailing rate, $r_n(T_{i-1}, T_i)$, for one period:

$$P_{i-1} = \frac{100 \times \left[1 + {r_n(T_{i-1}, T_i)}/{n}\right]}{1 + {r_n(T_{i-1}, T_i)}/{n}} = \mathbf{100}$$



- Now place yourself at T_{i-2}:
  - You know $r_n(T_{i-2}, T_{i-1})$ but not $r_n(T_{i-1}, T_i)$
  - However, you know that if you resold the bond at date T_{i-1}, you would receive 100. In this scenario, you <u>know</u> <u>all remaining cash flows</u>

$$P_{i-2} = \frac{100 \times \left[1 + {r_n(T_{i-2}, T_{i-1})}/{n}\right]}{1 + {r_n(T_{i-2}, T_{i-1})}/{n}} = \mathbf{100}$$



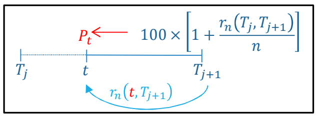
FLOATING RATE BOND: PRICE (2)

Repeat the above until date 0 to discover the following

At any *reset date* (0, T_1, …, T_{i-1}) a zero-spread floating rate bond's price equals 100% of its face value

What if today is not a reset date?

- The coupon was set at the last reset date, using the prevailing rate, $r_n(T_j, T_{j+1})$
- Meanwhile, between T_j and today, t, the rate may have changed!
  - The rate you use for discounting may differ from the rate paid as coupon



- At any date that is not a reset date, the bond's price is

$$P_t = \frac{100 \times \left[1 + \frac{r_n(T_j, T_{j+1})}{n}\right]}{\left[1 + {r_n(t, T_{j+1})}/{n}\right]^{n(T_j - t)}}$$

FLOATING-RATE BOND: PRICE (3)

<u>Example</u>. On October 1, 2020, the semiannually-compounded Treasury rate for a 3-months maturity was 0.1%. On that date you wished to price a US Treasury FRN ("floating-rate note") with semiannual coupon (pegged to the US-Treasury semiannual rate) that was issued on July 1, 2020 (next coupon due on Jan 1, 2021). The semiannually-compounding 6-months rate on July 1, 2020, was 0.175%.

- The coupon that will be paid on January 1, 2021, was set on July 1, 2020, using the 6-months rate at that time. This means it equals 0.175/2=0.0875
- If the bond is resold *ex-coupon* at the next reset date, you will receive 100. Previously you also receive the coupon
- These cash flows, on October 1, 2020, have a present value of

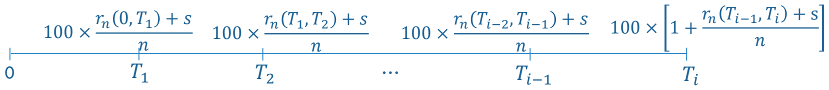
$$P_t = \frac{100 + 0.0875}{\left[1 + {}^{0.001}/_{2}\right]^{1/2}} = 100.0625$$

Notice that the change in interest rates since the last reset date (lower rates) plays in your favor! If the yield curve had been flat and had not changed since July, you would use a semiannually-compounded rate of 0.175% to discount the cash flows. In that case, the bond's price would have been 100.0437.

Notice that even with a flat, unchanging yield curve, the price at t is larger than 100 (the price at a reset date). This is because a person buying the bond on October 1 will cash in the full coupon by waiting only 3 months, while the person who bought it on July 1 (at a price of 100), would have had to wait a full 6 months. It is only fair that the October buyer pay more!

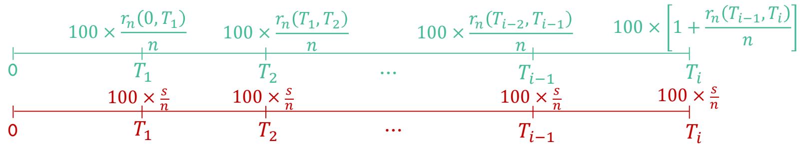
FLOATING-RATE BOND: PRICE (4)

So far, we ignored the <u>spread</u>. How to incorporate it in the price?



$$100\times\frac{r_n(0,T_1)+s}{n} \quad 100\times\frac{r_n(T_1,T_2)+s}{n} \quad 100\times\frac{r_n(T_{i-2},T_{i-1})+s}{n} \quad 100\times\left[1+\frac{r_n(T_{i-1},T_i)+s}{n}\right]$$

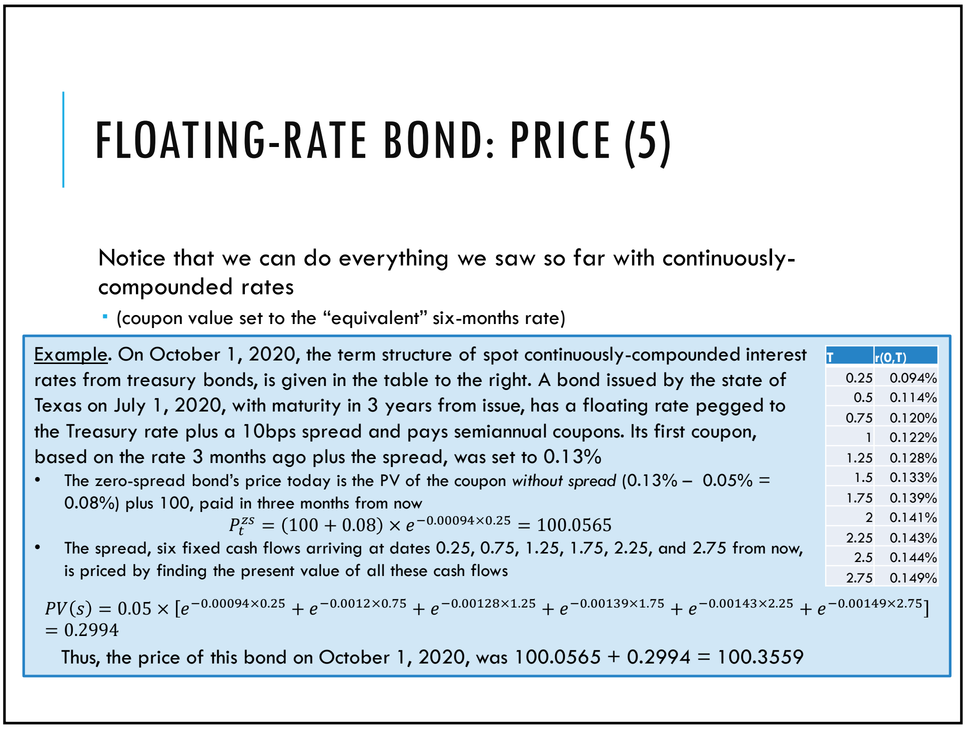
- Simple! Treat the cash flows of this bond as two separate investments: a floating rate bond with zero spread, and a set of fixed payments (the spread)



$$100\times\frac{r_n(0,T_1)}{n} \quad 100\times\frac{r_n(T_1,T_2)}{n} \quad 100\times\frac{r_n(T_{i-2},T_{i-1})}{n} \quad 100\times\left[1+\frac{r_n(T_{i-1},T_i)}{n}\right]$$

$$100\times\frac{s}{n} \quad 100\times\frac{s}{n} \quad 100\times\frac{s}{n} \quad 100\times\frac{s}{n}$$

- We find the price of the floater with **zero spread**
- We find the present value of the spreads using the appropriate yield curve
- We sum the two



FLOATING-RATE BOND: PRICE (5)

Notice that we can do everything we saw so far with continuously-compounded rates

- (coupon value set to the “equivalent” six-months rate)

Example. On October 1, 2020, the term structure of spot continuously-compounded interest rates from treasury bonds, is given in the table to the right. A bond issued by the state of Texas on July 1, 2020, with maturity in 3 years from issue, has a floating rate pegged to the Treasury rate plus a 10bps spread and pays semiannual coupons. Its first coupon, based on the rate 3 months ago plus the spread, was set to 0.13%

T	r(0,T)
0.25	0.094%
0.5	0.114%
0.75	0.120%
1	0.122%
1.25	0.128%
1.5	0.133%
1.75	0.139%
2	0.141%
2.25	0.143%
2.5	0.144%
2.75	0.149%

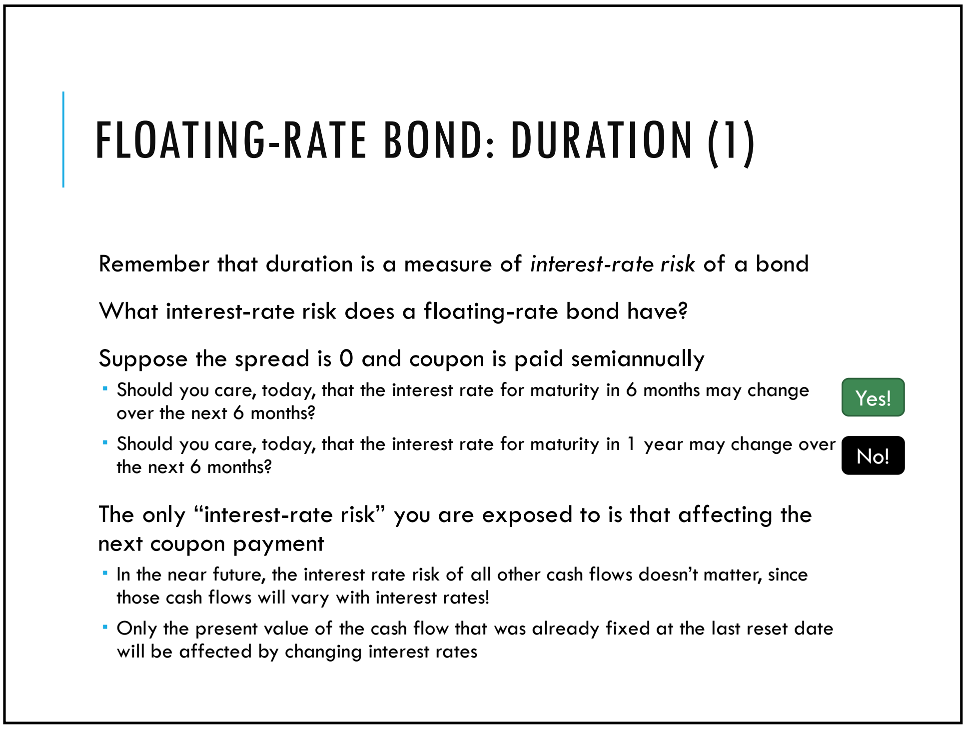
- The zero-spread bond’s price today is the PV of the coupon *without spread* (0.13% – 0.05% = 0.08%) plus 100, paid in three months from now

$$P_t^{zs} = (100 + 0.08) \times e^{-0.00094\times0.25} = 100.0565$$

- The spread, six fixed cash flows arriving at dates 0.25, 0.75, 1.25, 1.75, 2.25, and 2.75 from now, is priced by finding the present value of all these cash flows

$$PV(s) = 0.05 \times [e^{-0.00094\times0.25} + e^{-0.0012\times0.75} + e^{-0.00128\times1.25} + e^{-0.00139\times1.75} + e^{-0.00143\times2.25} + e^{-0.00149\times2.75}]$$
$$= 0.2994$$

Thus, the price of this bond on October 1, 2020, was 100.0565 + 0.2994 = 100.3559



FLOATING-RATE BOND: DURATION (1)

Remember that duration is a measure of *interest-rate risk* of a bond

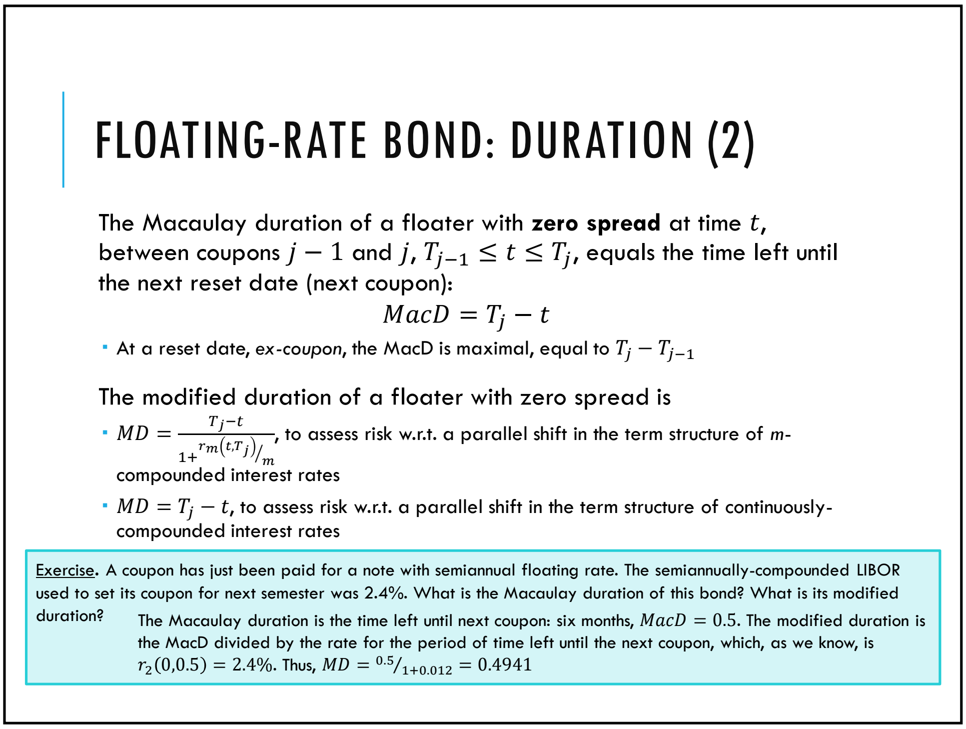
What interest-rate risk does a floating-rate bond have?

Suppose the spread is 0 and coupon is paid semiannually

- Should you care, today, that the interest rate for maturity in 6 months may change over the next 6 months? Yes!
- Should you care, today, that the interest rate for maturity in 1 year may change over the next 6 months? No!

The only “interest-rate risk” you are exposed to is that affecting the next coupon payment

- In the near future, the interest rate risk of all other cash flows doesn’t matter, since those cash flows will vary with interest rates!
- Only the present value of the cash flow that was already fixed at the last reset date will be affected by changing interest rates



FLOATING-RATE BOND: DURATION (2)

The Macaulay duration of a floater with **zero spread** at time t, between coupons $j-1$ and j, $T_{j-1} \leq t \leq T_j$, equals the time left until the next reset date (next coupon):

$$MacD = T_j - t$$

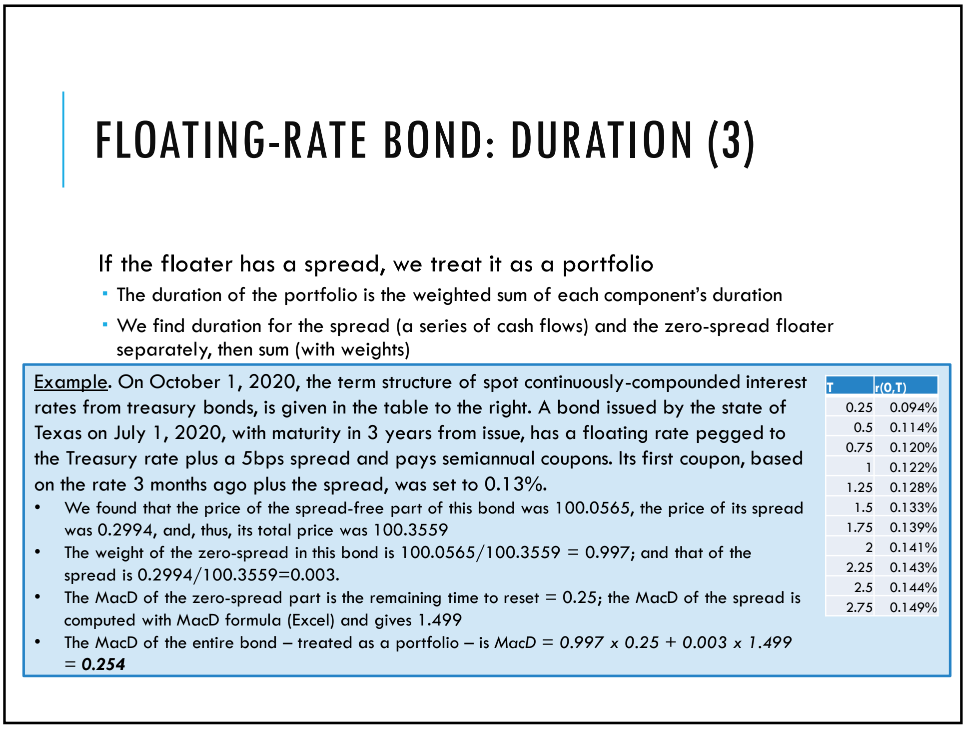
- At a reset date, *ex-coupon*, the MacD is maximal, equal to $T_j - T_{j-1}$

The modified duration of a floater with zero spread is

- $MD = \frac{T_j - t}{1 + {r_m(t, T_j)}/{m}}$, to assess risk w.r.t. a parallel shift in the term structure of *m*-compounded interest rates
- $MD = T_j - t$, to assess risk w.r.t. a parallel shift in the term structure of continuously-compounded interest rates

Exercise. A coupon has just been paid for a note with semiannual floating rate. The semiannually-compounded LIBOR used to set its coupon for next semester was 2.4%. What is the Macaulay duration of this bond? What is its modified duration?

The Macaulay duration is the time left until next coupon: six months, $MacD = 0.5$. The modified duration is the MacD divided by the rate for the period of time left until the next coupon, which, as we know, is $r_2(0{,}0.5) = 2.4\%$. Thus, $MD = {}^{0.5}/_{1+0.012} = 0.4941$



FLOATING-RATE BOND: DURATION (3)

If the floater has a spread, we treat it as a portfolio

- The duration of the portfolio is the weighted sum of each component's duration
- We find duration for the spread (a series of cash flows) and the zero-spread floater separately, then sum (with weights)

Example. On October 1, 2020, the term structure of spot continuously-compounded interest rates from treasury bonds, is given in the table to the right. A bond issued by the state of Texas on July 1, 2020, with maturity in 3 years from issue, has a floating rate pegged to the Treasury rate plus a 5bps spread and pays semiannual coupons. Its first coupon, based on the rate 3 months ago plus the spread, was set to 0.13%.

- We found that the price of the spread-free part of this bond was 100.0565, the price of its spread was 0.2994, and, thus, its total price was 100.3559
- The weight of the zero-spread in this bond is 100.0565/100.3559 = 0.997; and that of the spread is 0.2994/100.3559=0.003.
- The MacD of the zero-spread part is the remaining time to reset = 0.25; the MacD of the spread is computed with MacD formula (Excel) and gives 1.499
- The MacD of the entire bond – treated as a portfolio – is *MacD = 0.997 x 0.25 + 0.003 x 1.499 =* ***0.254***

T	r(0,T)
0.25	0.094%
0.5	0.114%
0.75	0.120%
1	0.122%
1.25	0.128%
1.5	0.133%
1.75	0.139%
2	0.141%
2.25	0.143%
2.5	0.144%
2.75	0.149%

SUMMARY SO FAR

Price bonds

- Knowing the required YTM
- Using a reference yield curve or *term structure of interest rates* from markets
  - m-compounding interest rates
  - Continuous-compounding interest rates

Measure bond *interest-rate risk*

- Macaulay duration & modified duration
- Price approximation with modified duration
- “Special” durations: money, key-rate, effective

Floating-rate bonds – *price, risk*

Where does the yield curve come from?