Part I

Value and Capital Budgeting

...in a Perfect Market under Risk Neutrality

The two primary goals of this first part of the book (Chapters 2–6) are to explain how to work with rates of return and how to decide whether to accept or reject investment projects. We assume in this part that there is a *perfect market*—one with no taxes, no transaction costs, no disagreements, and no limits as to the number of sellers and buyers in the market. This assumption will make it easier to understand finance first before applying it to the messy real world.

What You Want to Learn in this Part

- In Chapter 2, we start with the most basic possible scenario. In addition to a perfect market, we assume that there is no uncertainty: You know everything. We also assume that all rates of return in the economy are the same: A 1-year investment pays the same and perfectly known rate of return per annum as a 10-year investment. Under these assumptions, you learn how 1-year returns translate into multiyear returns and when you should accept or reject a project. The chapter introduces the important concept of "present value."

  Typical questions: If you earn 5% per year, how much will you earn over 10 years? If you earn 100% over 10 years, how much will you earn per year? What is the value of a project that will deliver $1,000,000 in 10 years? Should you buy this project if it costs you $650,000? What inputs do you need to make your decision?
- In Chapter 3, you learn how to value particular kinds of projects—perpetuities and annuities—if the economy-wide interest rate remains constant. You then learn how to apply the formulas to the valuation of stocks and bonds. The popular Gordon dividend growth model for valuing stocks assumes that dividends are a simple growing perpetuity cash flow stream, which makes it a perfect application of the perpetuity formula. Mortgages and other bonds are good applications of pricing using the annuities formulas.

  Typical questions: If a firm pays $1/share dividends next year, growing by 3% per year forever, then what should its stock price be? What is the monthly payment for a $300,000 mortgage bond if the interest rate is 4% per year?
- In Chapter 4, you learn more about capital budgeting methods. Although net present value (NPV) is the correct method, at least one other common method often comes to the correct result: the internal rate of return. In the real world, a number of other plainly incorrect methods are also widely used. You should know why you should be wary of them. This chapter also tells you what CFOs actually rely on.

  Typical questions: If a project has one investment outflow and two return inflows, how would you compute a "rate of return"? Can you accept projects whose rates of return are above their cost of capital? How bad is it when you use incorrect estimates—as you inevitably will—in your calculations? What are the big problems with a rule that accepts those projects that return money most quickly?
- In Chapter 5, you abandon the assumption that annual rates of return are the same for projects with different durations. For example, 1-year investments may pay 1% per year, while 30-year investments may pay 3% per year. The scenario of time-varying rates

of return is more realistic, but the questions that you want to answer still remain the same as those in Chapter 2. (The chapter then also explains more advanced aspects of bonds, such as the Treasury yield curve.)

Typical questions: If you earn 5% in the first year and 10% in the second year, how much will you earn over both years? What is the meaning of a 4% annualized interest rate? What is the meaning of a 4% yield-to-maturity? How can you value projects if appropriate rates of return depend on different time horizons?

- In Chapter 6, you abandon the assumption that you know the future. To be able to study uncertainty in the real world, you must first learn how to describe it. This is done with statistics, the necessary aspects of which are also explained here. The chapter then introduces risk neutrality, which is an assumption that can make it easier to understand some concepts in finance under uncertainty. Perhaps the two most important concepts are the difference between promised and expected rates of return and the difference between debt and equity. Under uncertainty, a project may not return the promised amount. ("Promised" can also be called "quoted" or "stated.") Because of the possibility of default, the *stated* rate of return must be higher than the *expected* rate of return. Although you are interested in the latter, it is almost always only the former that you are quoted (promised). It is important that you always draw a sharp distinction between promised=quoted=stated rates of return and expected rates of return. The second concept that this chapter explains is the difference between debt and equity—corporate claims that have a meaningful difference only under uncertainty.

  Typical questions: If there is a 2% chance that your borrower will not return the money, how much extra interest should you charge? From an investment perspective, what is the difference between debt and equity? What is financing priority? What is a residual claim?

Looking ahead, Part II will continue with uncertainty scenarios in which investors are risk-averse. Part III will explain what happens when financial markets or decision rules are not perfect.

2

Present Value

The Mother of All Finance

We begin with the concept of a rate of return—the cornerstone of finance. You can always earn an interest rate (and interest rates are rates of return) by depositing your money today into the bank. This means that money today is more valuable than the same amount of money next year. This concept is called the *time value of money* (TVM)—$1 in present value is better than $1 in future value.

Investors make up just one side of the financial markets. They give money today in order to receive money in the future. Firms often make up the other side. They decide what to do with the money—which projects to take and which projects to pass up—a process called *capital budgeting*. You will learn that there is one best method for making this critical decision. The firm should translate all *future* cash flows—both inflows and outflows—into their equivalent *present values* today. Adding in the cash flow today gives the *net present value*, or NPV. The firm should take all projects that have positive net present values and reject all projects that have negative net present values.

This all sounds more complex than it is, so we'd better get started.

2.1 The Basic Scenario

As promised, we begin with the simplest possible scenario. In finance, this means that we assume that we are living in a so-called **perfect market**:

We start with a so-called perfect market.

- There are no taxes.
- There are no transaction costs (costs incurred when buying and selling).
- There are no differences in information or opinions among investors (although there can be risk).
- There are so many buyers and sellers (investors and firms) in the market that the presence or absence of just one (or a few) individuals does not have an influence on the price.

The perfect market allows us to focus on the basic concepts in their purest forms, without messy real-world factors complicating the exposition. We will use these assumptions as our sketch of how financial markets operate, though not necessarily how firms' product markets work. You will learn in Chapter 11 how to operate in a world that is not perfect. (This will be a lot messier.)

In this chapter, we will make three additional assumptions (that are not required for a market to be considered "perfect") to further simplify the world:

In early chapters only, we add even stronger assumptions.

- The interest rate per period is the same.

- There is no inflation.
- There is no risk or uncertainty. You have perfect foresight.

Of course, this financial utopia is unrealistic. However, the tools that you will learn in this chapter will also work in later chapters, where the world becomes not only progressively more realistic but also more difficult. Conversely, if any tool does not give the right answer in our simple world, it would surely make no sense in a more realistic one. And, as you will see, the tools have validity even in the messy real world.

Q 2.1. What are the four perfect market assumptions?

2.2 Loans and Bonds

Finance jargon: interest, loan, bond, fixed income, maturity.

The material in this chapter is easiest to explain in the context of bonds and loans. A **loan** is the commitment of a borrower to pay a predetermined amount of cash at one or more predetermined times in the future (the final one called **maturity**), usually in exchange for cash upfront today. Loosely speaking, the difference between the money lent and the money paid back is the **interest** that the lender earns. A **bond** is a particular kind of loan, so named because it "binds" the borrower to pay money. Thus, for an investor, "buying a bond" is the same as "extending a loan." Bond buying is the process of giving cash today and receiving a binding promise for money in the future. Similarly, from the firm's point of view, it is "giving a bond," "issuing a bond," or "selling a bond." Loans and bonds are also sometimes called **fixed income** securities, because they promise a fixed amount of payments to the holder of the bond.

Why learn bonds first? Because they are easiest.

You should view a bond as just another type of investment project—money goes in, and money comes out. You could slap the name "corporate project" instead of "bond" on the cash flows in the examples in this chapter, and nothing would change. In Chapter 5, you will learn more about Treasuries, which are bonds issued by the U.S. Treasury. The beauty of such bonds is that you know exactly what your cash flows will be. (Despite Washington's dysfunction, we will assume that our Treasury cannot default.) Besides, much more capital in the economy is tied up in bonds and loans than is tied up in stocks, so understanding bonds well is very useful in itself.

Interest rates: limited upside. Rates of return: arbitrary upside.

You already know that the net return on a loan is called interest, and that the rate of return on a loan is called the **interest rate**—though we will soon firm up your knowledge about interest rates. One difference between an interest payment and a noninterest payment is that the former usually has a maximum payment, whereas the latter can have unlimited upside potential. However, not every rate of return is an interest rate. For example, an investment in a lottery ticket is not a loan, so it does not offer an interest rate, just a rate of return. In real life, its payoff is uncertain—it could be anything from zero to an unlimited amount. The same applies to stocks and many corporate projects. Many of our examples use the phrase "interest rate," even though the examples almost always work for any other rates of return, too.

Bond: defined by payment next year. Savings: defined by deposit this year.

Is there any difference between buying a bond for $1,000 and putting $1,000 into a bank savings account? Yes, a small one. The bond is defined by its future promised payoffs—say, $1,100 next year—and the bond's value and price today are based on these future payoffs. But as the bond owner, you know exactly how much you will receive next year. An investment in a bank savings account is defined by its investment today. The interest rate can and will change every day, so you do not know what you will end up with next year. The exact amount depends on future interest rates. For example, it could be $1,080 (if interest rates decrease) or $1,120 (if interest rates increase).

A bank savings account is like a sequence of 1-day bonds.

If you want, you can think of a savings account as a sequence of consecutive 1-day bonds: When you deposit money, you buy a 1-day bond, for which you know the interest rate this one day in advance, and the money automatically gets reinvested tomorrow into another bond with whatever the interest rate will be tomorrow.

Q 2.2. Is a deposit into a savings account more like a long-term bond investment or a series of short-term bond investments?

A Question of Principal

Who were the world's first financiers? Candidates are the Babylonian Egibi family (7th Century BCE), the Athenian Pasion (4th), or many Ancient Egyptians (1st). The latter even had a check-writing system! Of course, moneylenders were never popular—a fact that readers of the New Testament or the Koran already know. In medieval Europe, Genoa was an early innovator. In 1150, it issued a 400-lire 29-year bond, collateralized by taxes on market stalls. By the 15th Century, the first true modern banks appeared, an invention that spread like wildfire throughout Europe. *The Economist, Jan 10, 2009*

2.3 Returns, Net Returns, and Rates of Return

Defining return and our time. Our convention is that 0 means "right now."

The most fundamental financial concept is that of a return. The payoff or (dollar) **return** of an investment is simply the amount of cash (C) it returns. For example, an investment project that returns \$12 at time 1 has

$$C_1 = \text{Cash Return at Time 1} = \$12$$

This subscript is an instant in time, usually abbreviated by the letter t. When exactly time 1 occurs is not important: It could be tomorrow, next month, or next year. But if we mean "right now," we use the subscript 0.

Defining net return and rate of return.

The net payoff, or **net return**, is the difference between the return and the initial investment. It is positive if the project is profitable and negative if it is unprofitable. For example, if the investment costs \$10 today and returns \$12 at time 1 with nothing in between, then it earns a net return of \$2. Notation-wise, we need to use two subscripts on returns—the time when the investment starts (0) and when it ends (1).

$$\begin{aligned} \text{Net Return from Time 0 to Time 1} &= \$12 - \$10 = \$2 \\ \text{Net Return}_{0,1} &= C_1 - C_0 \end{aligned}$$

The double subscripts are painful. Let's agree that if we omit the first subscript on flows, it means zero. The **rate of return**, usually abbreviated r, is the net return expressed as a percentage of the initial investment.

$$\begin{aligned} \text{Rate of Return from Time 0 to Time 1} &= \frac{\$2}{\$10} = 20\% \\ r_{0,1} = r_1 &= \frac{\text{Net Return from Time 0 to Time 1}}{\text{Purchase Price at Time 0}} \end{aligned}$$

Here, I used our new convention and abbreviated $r_{0,1}$ as r_1. Often, it is convenient to calculate the rate of return as

$$r_1 = \frac{\$12 - \$10}{\$10} = \frac{\$12}{\$10} - 1 = 20\%$$

$$r_1 = \frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1 \tag{2.1}$$

Percent (the symbol %) is a unit of 1/100. So 20% is the same as 0.20.

Interest Rates over the Millennia

Historical interest rates are fascinating, perhaps because they look so similar to today's interest rates. Nowadays, typical interest rates range from 2% to 20% (depending on the loan). For over 2,500 years—from about the 30th century B.C.E. to the 6th century B.C.E.—normal interest rates in Sumer and Babylonia hovered around 10–25% per annum, though 20% was the legal maximum. In ancient Greece, interest rates in the 6th century B.C.E. were about 16–18%, dropping steadily to about 8% by the turn of the millennium. Interest rates in ancient Egypt tended to be about 10–12%. In ancient Rome, interest rates started at about 8% in the 5th century B.C.E. but began to increase to about 12% by the third century A.C.E. (a time of great upheaval and inflation). When lending resumed in the late Middle Ages (12th century), personal loans in continental Europe hovered around 10-20% (50% in England). By the Renaissance (16th Century), commercial loan rates had fallen to 5–15% in Italy, the Netherlands, and France. By the 17th century, even English interest rates had dropped to 6–10% in the first half, and to 3–6% in the second half (and mortgage rates were even lower). Most of the American Revolution was financed with French and Dutch loans at interest rates of 4–5%. *Homer and Sylla, A History of Interest Rates*

How to compute returns with interim payments. Capital gains versus returns.

Many investments have interim payments. For example, many stocks pay interim cash **dividends**, many bonds pay interim cash **coupons**, and many real estate investments pay interim **rent**. How would you calculate the rate of return then? One simple method is to just add interim payments to the numerator. Say an investment costs \$92, pays a dividend of \$5 (at the end of the period), and then is worth \$110. Its rate of return is

$$r = \frac{\$110 + \$5 - \$92}{\$92} = \frac{\$110 - \$92}{\$92} + \frac{\$5}{\$92} = 25\%$$

$$r_1 = \frac{C_1 + \text{All Dividends from 0 to 1} - C_0}{C_0} = \underbrace{\frac{C_1 - C_0}{C_0}}_{\text{Capital Gain, in \%}} + \underbrace{\frac{\text{All Dividends}}{C_0}}_{\text{Dividend Yield}}$$

When there are intermittent and final payments, then returns are often broken down into two additive parts. The first part, the price change or **capital gain**, is the difference between the purchase price and the final price, *not* counting interim payments. Here, the capital gain is the difference between \$110 and \$92, that is, the \$18 change in the price of the investment. It is often quoted in percent of the price, which would be \$18/\$92 or 19.6% here. The second part is the amount received in interim payments. It is the dividend or coupon or rent, here \$5. When it is divided by the price, it has names like **dividend yield**, **current yield**, **rental yield**, or **coupon yield**, and these are also usually stated in percentage terms. In our example, the dividend yield is $\$5/\$92 \approx 5.4\%$. Of course, if the interim yield is high, you might be experiencing a negative capital gain and still have a positive rate of return. For example, a bond that costs \$500, pays a coupon of \$50, and then sells for \$490, has a **capital loss** of \$10 (which comes to a –2% capital yield) but a rate of return of $(\$490 + \$50 - \$500)/\$500 = +8\%$. You will almost always work with rates of return, not with capital gains. The only exception is when you have to work with taxes, because the IRS treats capital gains differently from interim payments. (We will cover taxes in Section 11.4.)

➤ Corporate payouts and dividend yields, Chapter 20, Pg.555.

➤ Taxes on capital gains, Sect. 11.4, Pg.257.

Most of the time, people (incorrectly but harmlessly) abbreviate a rate of return or net return by calling it just a return. For example, if you say that the return on your \$10,000 stock purchase was 10%, you obviously do not mean you received a unitless 0.1. You really mean that your rate of return was 10% and you received \$1,000. This is usually benign, because your listener will know what you mean. Potentially more harmful is the use of the phrase *yield*, which, strictly speaking, means *rate of return*. However, it is often misused as a shortcut for dividend yield or coupon yield (the percent payout that a stock or a bond provides). If you say that the yield on your stock was 5%, then some listeners may interpret it to mean that you earned a total rate of return of 5%, whereas others may interpret it to mean that your stock paid a dividend yield of 5%.

People often use incorrect terms, but the meaning is usually clear, so this is harmless.

➤ Nominal, Sect. 5.2, Pg.82.

Interest rates should logically always be positive. After all, you can always earn 0% if you keep your money under your mattress—you thereby end up with as much money next period as you have this period. Why give your money to someone today who will give you less than 0% (less money in the future)? Consequently, interest rates are indeed almost always positive—the rare exceptions being both bizarre and usually trivial.

[Nominal] interest is [usually] nonnegative.

Here is another language problem: What does the statement "the interest rate has just increased by 5%" mean? It could mean either that the previous interest rate, say, 10%, has just increased from 10% to $10\% \cdot (1+5\%) = 10.5\%$, or that it has increased from 10% to 15%. Because this is unclear, the **basis point** unit was invented. A basis point is simply 1/100 of a percent. If you state that your interest rate has increased by 50 basis points, you definitely mean that the interest rate has increased from 10% to 10.5%. If you state that your interest rate has increased by 500 basis points, you definitely mean that the interest rate has increased from 10% to 15%.

Basis points avoid an ambiguity in the English language: 100 basis points equals 1%.

IMPORTANT

100 basis points constitute 1%. Somewhat less common, 1 point is 1%.
Points and basis points help with "percentage ambiguities."

Q 2.3. An investment costs \$1,000 and pays a return of \$1,050. What is its rate of return?

Q 2.4. An investment costs \$1,000 and pays a net return of \$25. What is its rate of return?

Q 2.5. Is 10 the same as 1,000%?

Q 2.6. You buy a stock for \$40 per share today. It will pay a dividend of \$1 next month. If you can sell it for \$45 right after the dividend is paid, what would be its dividend yield, what would be its capital gain (also quoted as a capital gain yield), and what would be its total rate of return?

Q 2.7. By how many basis points does the interest rate change if it increases from 9% to 12%?

Q 2.8. If an interest rate of 10% decreases by 20 basis points, what is the new interest rate?

2.4 Time Value, Future Value, and Compounding

Time Value of Money = Earn Interest.

Because you can earn interest, a given amount of money today is worth more than the same amount of money in the future. After all, you could always deposit your money today into the bank and thereby receive more money in the future. This is an example of the **time value of money**, which says that a dollar today is worth more than a dollar tomorrow. This ranks as one of the most basic and important concepts in finance.

The Future Value of Money

Here is how to calculate future payoffs given a rate of return and an initial investment.

➤ Rate of Return, Formula 2.1, Pg.14.

How much money will you receive in the future if the rate of return is 20% and you invest \$100 today? Turn around the rate of return formula (Formula 2.1) to determine how money will grow over time given a rate of return:

$$20\% = \frac{\$120 - \$100}{\$100} \Leftrightarrow \$100 \cdot (1 + 20\%) = \$100 \cdot 1.2 = \$120$$
$$r_1 = \frac{C_1 - C_0}{C_0} \Leftrightarrow C_0 \cdot (1 + r_1) = C_1$$

The \$120 next year is called the **future value** (**FV**) of \$100 today. Thus, future value is the value of a present cash amount at some point in the future. It is the time value of money that causes the future value, \$120, to be higher than its present value (PV), \$100. Using the abbreviations FV and PV, you could also have written the above formula as

$$r_1 = \frac{FV - PV}{PV} \Leftrightarrow FV = PV \cdot (1 + r_1)$$

➤ Apples and Oranges, Sect. 5.2, Pg.82.

(If we omit the subscript on the r, it means a 1-period interest rate from now to time 1, i.e., r_1.) Please note that the time value of money is not the fact that the prices of goods may change between today and tomorrow (that would be inflation). Instead, the time value of money is based exclusively on the fact that your money can earn interest. Any amount of cash today is worth more than the same amount of cash tomorrow. Tomorrow, it will be the same amount plus interest.

Q 2.9. A project has a rate of return of 30%. What is the payoff if the initial investment is \$250?

Compounding and Future Value

Interest on interest (or rate of return on rate of return) means rates cannot be added.

Now, what if you can earn the same 20% year after year and reinvest all your money? What would your two-year rate of return be? Definitely *not* 20% + 20% = 40%! You know that you will have \$120 in year 1, which you can reinvest at a 20% rate of return from year 1 to year 2. Thus, you will end up with

$$C_2 = \$100 \cdot (1 + 20\%)^2 = \$100 \cdot 1.2^2 = \$120 \cdot (1 + 20\%) = \$120 \cdot 1.2 = \$144$$
$$C_0 \cdot (1 + r)^2 = C_1 \cdot (1 + r) = C_2$$

This \$144—which is, of course, again a future value of \$100 today—represents a total two-year rate of return of

$$
\begin{aligned}
r_2 \;&=\; \frac{\$144 - \$100}{\$100} \;=\; \frac{\$144}{\$100} - 1 \;=\; 44\% \\
&\; \frac{C_2 - C_0}{C_0} \;=\; \frac{C_2}{C_0} - 1 \;=\; r_2
\end{aligned}
$$

This is more than 40% because the original net return of $20 in the first year earned an additional $4 in interest in the second year. You earn interest on interest! This is also called **compound interest**. Similarly, what would be your 3-year rate of return? You would invest $144 at 20%, which would provide you with

$$
\begin{aligned}
C_3 \;&=\; \$144 \cdot (1+20\%) \;=\; \$144 \cdot 1.2 \;=\; \$100 \cdot (1+20\%)^3 \;=\; \$100 \cdot 1.2^3 \;=\; \$172.80 \\
&\; C_2 \cdot (1+r) \;=\; C_0 \cdot (1+r)^3 \;=\; C_3
\end{aligned}
$$

The "one-plus" formula.

Your 3-year rate of return from time 0 to time 3 (call it r_3) would thus be

$$
\begin{aligned}
r_3 \;&=\; \frac{\$172.80 - \$100}{\$100} \;=\; \frac{\$172.80}{\$100} - 1 \;=\; 72.8\% \\
&\; \frac{C_3 - C_0}{C_0} \;=\; \frac{C_3}{C_0} - 1 \;=\; r_3
\end{aligned}
$$

This formula translates the three sequential 1-year rates of return into one 3-year **holding rate of return**—that is, what you earn if you hold the investment for the entire period. This process is called **compounding**, and the formula that does it is the "one-plus formula":

$$
\begin{aligned}
(1+72.8\%) \;&=\; (1+20\%) \cdot (1+20\%) \cdot (1+20\%) \\
(1+r_3) \;&=\; (1+r) \cdot (1+r) \cdot (1+r)
\end{aligned}
$$

or, if you prefer it shorter, $1.728 = 1.2^3$.

Exhibit 2.1 shows how your $100 would grow if you continued investing it at a rate of return of 20% per annum. The function is exponential—that is, it grows faster and faster as interest earns more interest.

IMPORTANT

The compounding formula translates sequential future rates of return into an overall holding rate of return:

$$
\underbrace{(1+r_t)}_{\text{Multiperiod Holding Rate of Return}} \;=\; \underbrace{(1+r)^t}_{\text{Multiperiod Holding Rate of Return}} \;=\; \underbrace{(1+r)}_{\text{Current 1-Period Spot Rate of Return}} \cdot \underbrace{(1+r)}_{\text{Next 1-Period Rate of Return}} \cdots \underbrace{(1+r)}_{\text{Final 1-Period Rate of Return}}
$$

The first rate is called the spot rate because it starts now (on the spot).

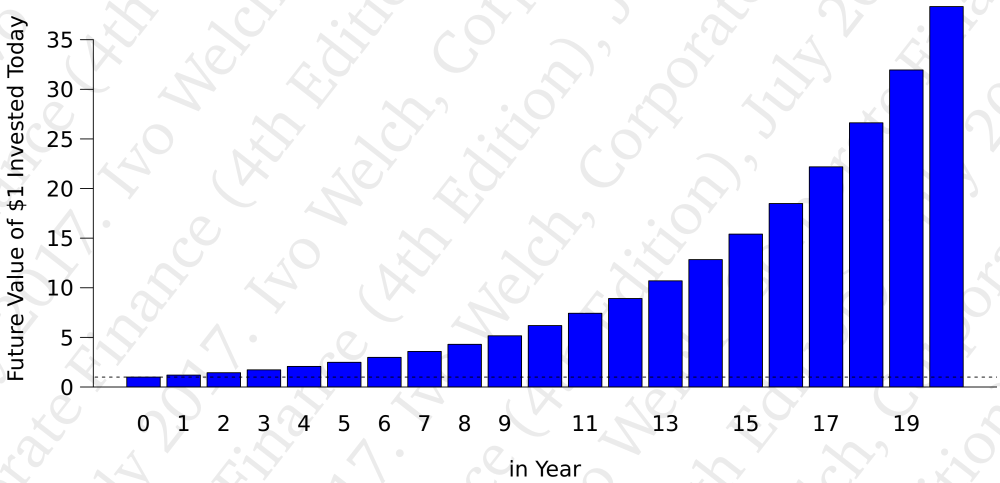
The compounding formula is so common that you must memorize it.

Another example of a payoff computation.

You can use the compounding formula to compute all sorts of future payoffs. For example, an investment project that costs $212 today and earns 10% each year for 12 years will yield an overall holding rate of return of

$$
\begin{aligned}
r_{12} \;&=\; (1+10\%)^{12} - 1 \;=\; (1.1^{12} - 1) \;\approx\; 213.8\% \\
&\; (1+r)^t - 1 \;=\; r_{12}
\end{aligned}
$$

Your $212 investment today would therefore turn into a future value of



Period	Start value	1 + one-year rate	End value	Total factor from time 0	Total rate of return $r_{0,t} = (1+r)^t - 1$
0 to 1	$100	$(1+20\%)$	$120.00	1.2	20.0%
1 to 2	$120	$(1+20\%)$	$144.00	$1.2 \cdot 1.2 = 1.44$	44.0%
2 to 3	$144	$(1+20\%)$	$172.80	$1.2 \cdot 1.2 \cdot 1.2 = 1.728$	72.8%
⋮					

Exhibit 2.1: *Compounding over 20 Years at 20% per Annum.* Money grows at a constant rate of 20% per annum. If you compute the graphed value at 20 years out, you will find that each dollar invested right now is worth $38.34 in 20 years. The money at first grows in a roughly linear pattern, but as more and more interest accumulates and itself earns more interest, the graph accelerates steeply upward.

$$\begin{array}{llllll} C_{12} & = \$212 \cdot (1+10\%)^{12} & = \$212 \cdot 1.1^{12} & \approx \$212 \cdot (1+213.8\%) & \approx \$665.35 \\ & \quad C_0 \cdot (1+r)^{12} & & & = \quad C_{12} \end{array}$$

"Uncompounding": Turn around the formula to compute individual holding rates.

Now suppose you wanted to know what constant two 1-year interest rates (r) would give you a two-year rate of return of 50%. The answer is not 25%, because $(1+25\%) \cdot (1+25\%) - 1 = 1.25^2 - 1 = 56.25\%$. Instead, you need to solve

$$(1+r) \cdot (1+r) = (1+r)^2 = 1 + 50\% = 1.50$$

The correct answer is

$$\begin{array}{rcl} r & = & \sqrt[2]{1+50\%} - 1 \approx 22.47\% \\ = & & \sqrt[t]{1+r_t} - 1 \quad = \quad r \end{array}$$

➤ Exponentiation, Book Appendix, Chapter A, Pg.621.

Check your answer: $(1+22.47\%) \cdot (1+22.47\%) = 1.2247^2 \approx (1+50\%)$. If the 12-month interest rate is 213.8%, what is the 1-month interest rate?

$$(1+r)^{12} \approx 1+213.8\%$$
$$\Leftrightarrow r = \sqrt[12]{1+213.8\%}-1 = (1+213.8\%)^{1/12}-1 \approx 10\%$$

Interestingly, compounding works even over fractional time periods. Say the overall interest rate is 5% per year, and you want to find out what the rate of return over half a year would be. Because $(1+r_{0.5})^2 = (1+r_1)$, you would compute

You can determine fractional time interest rates via compounding, too.

$$(1+r_{0.5}) = (1+r_1)^{0.5} = (1+5\%)^{0.5} \approx 1+2.4695\% = 1.024695$$

Check—compounding 2.4695% over two (6-month) periods indeed yields 5%:

$$(1+2.4695\%)\cdot(1+2.4695\%) = 1.024695^2 \approx (1+5\%)$$
$$(1+r_{0.5})\cdot(1+r_{0.5}) = (1+r_{0.5})^2 = (1+r_1)$$

Life Expectancy and Credit

Your life expectancy may be 80 years, but 30-year bonds existed even in an era when life expectancy was only 25 years—at the time of Hammurabi, around 1700 B.C.E. (Hammurabi established the Kingdom of Babylon and is famous for the Hammurabi Code, the first known legal system.) Moreover, four thousand years ago, Mesopotamians already solved interesting financial problems. A cuneiform clay tablet contains the oldest known interest rate problem for prospective students of the financial arts. The student must figure out how long it takes for 1 mina of silver, growing at 20% interest per year, to reach 64 minae. Because the interest compounds in an odd way (20% of the principal is accumulated until the interest is equal to the principal, and then it is added back to the principal), the answer to this problem is 30 years, rather than 22.81 years. This is not an easy problem to solve—and it even requires knowledge of logarithms!

William Goetzmann, Yale University

If you know how to use logarithms, you can also use the same formula to determine how long it will take at the current interest rate to double or triple your money. For example, at an interest rate of 3% per year, how long would it take you to double your money?

You need logs to determine the time needed to get x times your money.

$$(1+3\%)^x = (1+100\%) \Leftrightarrow x = \frac{\log(1+100\%)}{\log(1+3\%)} = \frac{\log(2.00)}{\log(1.03)} \approx 23.5$$
$$(1+r)^t = (1+r_t) \Leftrightarrow t = \frac{\log(1+r_t)}{\log(1+r)}$$

Compound rates of return can be negative even when the average rate of return is positive: think +200% followed by −100%. The average arithmetic rate of return in this example is (200% + (−100%))/2 = +50%, while the compound rate of return is −100%. Not a good investment! Thinking in arithmetic terms for wealth accumulation is a common mistake, if only because funds usually advertise their average rate of return. High-volatility funds (i.e., funds that increase and decrease a lot in value) look particularly good on this incorrect performance measure.

One more thing...

Errors: Adding or Compounding Interest Rates?

Adding rather than compounding can make forgivably small mistakes in certain situations—but don't be ignorant of what you are doing.

Unfortunately, when it comes to interest rates in the real world, many users are casual, sometimes to the point where they are outright wrong. Some people mistakenly add interest rates instead of compounding them. When the investments, interest rates, and time length are small, the difference between the correct and incorrect computation is often minor, so this practice can be acceptable, even if it is wrong. For example, when interest rates are 10%, compounding yields

$$\begin{aligned}
(1+10\%) \quad \cdot \quad (1+10\%) - 1 &= 1.1^2 - 1 = 21\% \\
(1+\mathrm{r}) \quad \cdot \quad (1+\mathrm{r}) - 1 &= \mathrm{r}_2 \\
= 1+\mathrm{r}+\mathrm{r}+\mathrm{r}\cdot\mathrm{r} \quad - \quad 1 &
\end{aligned}$$

which is not exactly the same as the simple sum of two r's, which comes to 20%. The difference between 21% and 20% is the "cross-term" $\mathrm{r}\cdot\mathrm{r}$. This cross-product is especially unimportant if both rates of return are small. If the two interest rates were both 1%, the cross-term would be 0.0001. This is indeed small enough to be ignored in most situations and is therefore a forgivable approximation. However, when you compound over many periods, you accumulate more and more cross-terms, and eventually the quality of your approximation deteriorates. For example, over 100 years, $1 million invested at 1% per annum compounds to $2.71 million, not to $2 million.

Q 2.10. If the 1-year rate of return is 20% and interest rates are constant, what is the 5-year holding rate of return?

Q 2.11. If you invest $2,000 today and it earns 25% per year, how much will you have in 15 years?

Q 2.12. What is the holding rate of return for a 20-year investment that earns 5%/year each year? What would a $200 investment grow into?

Q 2.13. A project lost one-third of its value each year for 5 years. What was its total holding rate of return? How much is left if the original investment was $20,000?

Q 2.14. If the 5-year holding rate of return is 100% and interest rates are constant, what is the (compounding) annual interest rate?

Q 2.15. What is the quarterly interest rate if the annual interest rate is 50%?

Q 2.16. If the per-year interest rate is 5%, what is the two-year total interest rate?

Q 2.17. If the per-year interest rate is 5%, what is the 10-year total interest rate?

Q 2.18. If the per-year interest rate is 5%, what is the 100-year total interest rate? How does this compare to 100 times 5%?

Q 2.19. At a constant rate of return of 6% per annum, how many years does it take you to triple your money?

IMPORTANT

When you compare your calculations to mine (not only in my exposition in the chapter itself but also in my answers to these questions in the back of the chapter), you will often find that they are slightly different. This is usually a matter of rounding precision—depending on whether you carry intermediate calculations at full precision or not. Such discrepancies are an unavoidable nuisance, but they are *not* a problem. You should check whether your answers are close, not whether they are exact to the x-th digit after the decimal point.

How Banks Quote Interest Rates

Banks add to the confusion, quoting interest rates using strange but traditional conventions.

Banks and many other financial institutions use a number of conventions for quoting interest rates that may surprise you. Consider the example of a loan or a deposit that has one flow of $1,000,000 and a return flow of $1,100,000 in six months. Obviously, the simple holding rate of return is 10%. Here is what you might see:

The effective annual rate (EAR) is what our book has called the real interest rate or holding rate of return. In this case, our only problem is to re-quote the six-month 10% rate into a twelve-month rate. This is easy,

$$\text{EAR} = (1+10\%)^{12/6} - 1 = 21\%$$

This 21% is usually a supplementary rate that any bank would quote you on both deposits and loans. The EAR is also sometimes called the **annual percentage yield (APY)**. And it is also sometimes (and ambiguously) called the **annual equivalent rate (AER)**.

The annual interest rate (stated without further explanations) is not really a rate of return, but just a method of quoting an interest rate. The true daily interest rate is this annual interest quote divided by 365 (or 360 by another convention). In the example, the 10% half-year interest rate translates into

$$\text{AIR} = \underbrace{(1.10^{1/(365/2)} - 1)}_{\text{Daily Interest Rate} \approx 0.0522384\%} \cdot 365 = 19.07\%$$

The annual interest rate is usually how banks quote interest rates on savings or checking accounts. Conversely, if the bank advertises a savings interest rate of 20%, any deposit would really earn an effective annual rate of $(1+20\%/365)^{365} - 1 \approx 22.13\%$ per year.

The annual percentage rate (APR) is a complete mess. Different books define it differently. Most everyone agrees that APR is based on monthly compounding:

$$\text{APR} = \underbrace{(1.10^{1/(12/2)} - 1)}_{\text{Monthly Interest Rate} \approx 1.6\%} \cdot 12 = 19.21\%$$

However, the APR is also supposed to include fees and other expenses. Say the bank charged $10,000 in application and other fees. This is paid upfront, so we should recognize that the interest rate is not 10%, but $\$1,100,000/\$990,000 - 1 \approx 11.1\%$. We could then "monthly-ize" this holding rate of return into an APR of $(1.1111^{2/12} - 1) \cdot 12 \approx 21.26\%$.

So far, so good—except different countries require different fees to be included. In the United States, there are laws that state how APR should be calculated—and not just one, but a few (the Truth in Lending Act of 1968 [Reg Z], the Truth in Savings Act of 1991, the Consumer Credit Act of 1980, and who knows what other Acts). Even with all these laws, the APR is still not fully precise and comparable. To add insult to injury, the APR is also sometimes abbreviated as AER, just like the EAR.

Interest rates are not intrinsically difficult, but they can be tedious, and definitional confusions abound. So if real money is on the line, you should ask for the full and exact calculations of all payments in and all payments out, and not just rely on what *you* think it is that the bank is really quoting you. Besides, the above rates are not too interesting (yet), because they don't work for loans that have multiple payments. You have to wait for that until we cover the yield-to-maturity.

➤ Yield-To-Maturity, Sect. 4.2, Pg.59.

Let's look at a **certificate of deposit (CD)**, which is a longer-term investment vehicle than a savings account deposit. If your bank wants you to deposit your money in a CD, do you think it will put the more traditional interest rate quote or the APY on its sign in the window? Because the APY of 10.52% looks larger and thus more appealing to depositors than the traditional 10%

interest rate quote, most banks advertise the APY for deposits. If you want to borrow money from your bank, do you think your loan agreement will similarly emphasize the APY? No. Most of the time, banks leave this number to the fine print and focus on the APR (or the traditional interest rate quote) instead.

Q 2.20. If you earn an (effective) interest rate of 12% per annum, how many basis points do you earn in interest on a typical calendar day? (Assume a year has 365.25 days.)

Q 2.21. If the bank quotes an interest rate of 12% per annum (not as an effective interest rate), how many basis points do you earn in interest on a typical day?

Q 2.22. If the bank states an *effective* interest rate of 12% per annum, and there are 52.2 weeks per year, how much interest do you earn on a deposit of $1,000 over 1 week? On a deposit of $100,000?

Q 2.23. If the bank quotes interest of 12% per annum, and there are 52.2 weeks, how much interest do you earn on a deposit of $1,000 over 1 week?

Q 2.24. If the bank quotes interest of 12% per annum, and there are 52.2 weeks, how much interest do you earn on a deposit of $1,000 over 1 year?

Q 2.25. If the bank quotes an interest rate of 6% per annum, what does a deposit of $100 in the bank come to after one year?

Q 2.26. If the bank quotes a loan APR rate of 8% per annum, compounded monthly, and without fees, what do you have to pay back in one year if you borrow $100 from the bank?

2.5 Present Value, Discounting, and Capital Budgeting

Capital budgeting: Should you budget capital for a project?

Now turn to the flip side of the future value problem: If you know how much money you will have next year, what does this correspond to in value *today*? This is especially important in a corporate context, where the question is, "Given that Project X will return $1 million in 5 years, how much should you be willing to pay to undertake this project today?" The process entailed in answering this question is called **capital budgeting** and is at the heart of corporate decision making. (The origin of the term was the idea that firms have a "capital budget," and that they must allocate capital to their projects within that budget.)

The "present value formula" is nothing but the rate of return definition—inverted to translate future cash flows into (equivalent) today's dollars.

Start again with the rate of return formula

$$r_1 = \frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1$$

You only need to turn this formula around to answer the following question: If you know the prevailing interest rate in the economy (r_1) and the project's future cash flows (C_1), what is the project's value to you *today*? In other words, you are looking for the **present value** (**PV**)—the amount a future sum of money is worth today, given a specific rate of return. For example, if the interest rate is 10%, how much would you have to save (invest) to receive $100 next year? Or, equivalently, if your project will return $100 next year, what is the project worth to you today? The answer lies in the present value formula, which translates future money into today's money. You merely need to rearrange the rate of return formula to solve for the present value:

$$C_0 = \frac{\$100}{1 + 10\%} = \frac{\$100}{1.1} \approx \$90.91$$

$$C_0 = \frac{C_1}{1 + r_1} = \text{PV}(C_1)$$

Check this—investing $90.91 at an interest rate of 10% will indeed return $100 next period:

$$10\% \approx \frac{\$100 - \$90.91}{\$90.91} = \frac{\$100}{\$90.91} - 1 \Leftrightarrow (1 + 10\%) \cdot \$90.91 \approx \$100$$

$$r_1 = \frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1 \Leftrightarrow (1 + r_1) \cdot C_0 = C_1$$

Discounting translates future cash into today's equivalent.

This is the **present value formula**, which uses a division operation known as **discounting**. (The term "discounting" indicates that we are reducing a value, which is exactly what we are doing when we translate future cash into current cash.) If you wish, you can think of discounting—the conversion of a future cash flow amount into its equivalent present value amount—as the *reverse* of compounding.

Present value varies inversely with the cost of capital.

Thus, the present value (PV) of next year's \$100 is \$90.91—the value today of future cash flows. Let's say that this \$90.91 is what the project costs. If you can borrow or lend at the interest rate of 10% elsewhere, then you will be indifferent between receiving \$100 next year and receiving \$90.91 for your project today. In contrast, if the standard rate of return in the economy were 12%, your specific project would not be a good deal. The project's present value would be

$$\text{PV}(C_1) = \frac{\$100}{1 + 12\%} = \frac{\$100}{1.12} \approx \$89.29$$

$$C_0 = \frac{C_1}{1 + r_1} = \text{PV}(C_1)$$

which would be less than its cost of \$90.91. But if the standard economy-wide rate of return were 8%, the project would be a great deal. Today's present value of the project's future payoff would be

$$\text{PV}(C_1) = \frac{\$100}{1 + 8\%} = \frac{\$100}{1.08} \approx \$92.59$$

which would exceed the project's cost of \$90.91. It is the present value of the project, weighed against its cost, that should determine whether you should undertake a project today or avoid it. The present value is also the answer to the question, "How much would you have to save at current interest rates today if you wanted to have a specific amount of money next year?"

The PV formula with two periods.

Let's extend the time frame in our example. If the interest rate were 10% per period, what would \$100 in two periods be worth today? The value of the \$100 is then

$$\text{PV}(C_2) = \frac{\$100}{(1 + 10\%)^2} = \frac{\$100}{1.21} \approx \$82.64$$

$$\text{PV}(C_2) = \frac{C_2}{(1 + r)^2} = C_0 \qquad (2.2)$$

Note the 21%. In two periods, you could earn a rate of return of $(1 + 10\%) \cdot (1 + 10\%) - 1 = 1.1^2 - 1 = 21\%$ elsewhere, so this is your appropriate comparable rate of return.

The interest rate can be called the "cost of capital."

This discount rate—the rate of return, r, with which the project can be financed—is often called the **cost of capital**. It is the rate of return at which you can raise money elsewhere. In a perfect market, this cost of capital is also the **opportunity cost** that you bear if you fund your specific investment project instead of the alternative next-best investment elsewhere. Remember—you can invest your money at this opportunity rate in another project instead of this one. When these alternative projects in the economy elsewhere are better, your cost of capital is higher, and the value of your specific investment project with its specific cash flows is relatively lower. An investment that promises \$1,000 next year is worth less today if you can earn 50% rather than 5% elsewhere. A good rule is to always mentally add the word "opportunity" before "cost of capital"—it is always your **opportunity cost of capital**. (In this part of our book, I will just tell you what the economy-wide rate of return is—here 10%—for borrowing or investing. In later chapters, you will learn how this opportunity cost of capital [ahem "rate of return"] is determined.)

IMPORTANT

Always think of the r in the present value denominator as your "opportunity" cost of capital. If you have great opportunities elsewhere, your projects have to be discounted at high discount rates. The discount rate, the cost of capital, and the required rate of return are really all just different names for the same thing.

The discount factor is a simple function of the cost of capital.

When you multiply a future cash flow by its appropriate **discount factor**, you end up with its present value. Looking at Formula 2.2, you can see that this discount factor is the quantity

$$\text{discount factor} \quad = \quad \left(\frac{1}{1+21\%}\right) \quad \approx \; 0.8264$$

In other words, the discount factor translates 1 dollar in the future into the equivalent amount of dollars today. In the example, at a two-year 21% rate of return, a dollar in two years is worth about 83 cents today. Because interest rates are usually positive, discount factors are usually less than 1—a dollar in the future is worth less than a dollar today. (Sometimes, people call this the **discount rate**, but the discount rate is really $r_{0,t}$ if you are a stickler for accuracy.)

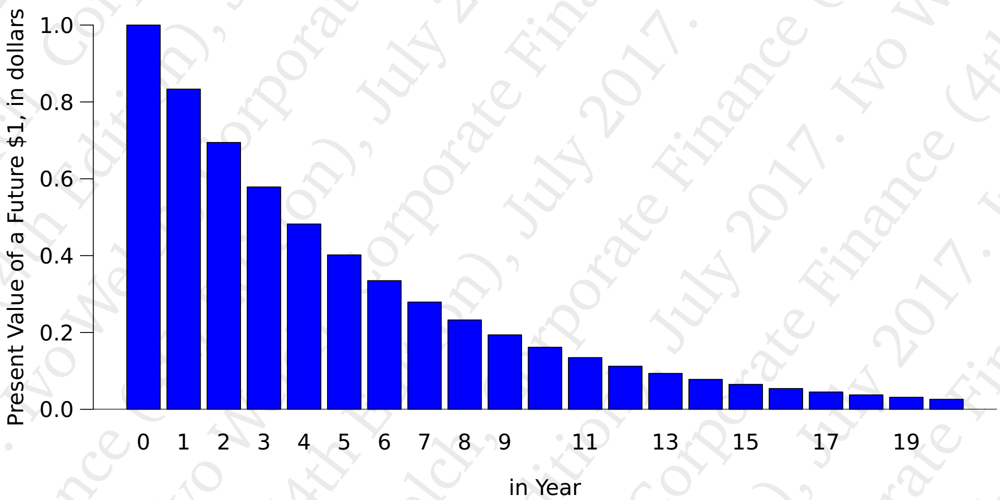


Exhibit 2.2: *Discounting over 20 Years at a Cost of Capital of 20% per Annum.* Each bar is $1/(1+20\%) \approx 83.3\%$ of the size of the bar to its left. After 20 years, the last bar is 0.026 in height. This means that $1 in 20 years is worth 2.6 cents in money today.

The discount rate is higher for years farther out, so the discount factor is lower.

Exhibit 2.2 shows how the discount factor declines when the cost of capital is 20% per annum. After about a decade, any dollar the project earns is worth less than 20 cents to you today. If you compare Exhibit 2.1 to Exhibit 2.2, you should notice how each is the "flip side" of the other.

IMPORTANT

The cornerstones of finance are the following formulas:

$$\text{Rate of Return:} \quad r_{0,t} = \frac{C_t - C_0}{C_0} = \frac{C_t}{C_0} - 1$$

Rearrange the formula to obtain the future value:

$$\text{Future Value:} \quad FV_t = C_t = C_0 \cdot (1 + r_t) = C_0 \cdot (1 + r)^t$$

The process of obtaining $r_{0,t}$ is called compounding, and it works through the "one-plus" formula:

$$\text{Compounding:} \quad \underbrace{(1 + r_{0,t})}_{\substack{\text{Total Holding}\\ \text{Rate of Return}}} = \underbrace{(1 + r)}_{\substack{\text{First Period}\\ \text{Rate of Return}}} \cdot \underbrace{(1 + r)}_{\substack{\text{Second Period}\\ \text{Rate of Return}}} \cdots \underbrace{(1 + r)}_{\substack{\text{Final Period}\\ \text{Rate of Return}}}$$

Rearrange the formula again to obtain the present value:

$$\text{Present Value:} \quad PV = C_0 = \frac{C_t}{(1 + r_{0,t})} = \frac{C_t}{(1 + r)^t}$$

The process of translating C_t into C_0—that is, the multiplication of a future cash flow by $1/(1 + r_{0,t})$—is called discounting. The discount factor is:

$$\text{Discount Factor:} \quad \frac{1}{(1 + r_{0,t})} = \frac{1}{(1 + r)^t}$$

It translates one dollar at time t into its equivalent value today.

Bonds' present values and the prevailing interest rates move in opposite directions.

Remember how bonds are different from savings accounts? The former is pinned down by its promised fixed future payments, while the latter pays whatever the daily interest rate is. This induces an important relationship between the value of bonds and the prevailing interest rates—*they move in opposite directions*. For example, if you have a bond that promises to pay \$1,000 in one year, and the prevailing interest rate is 5%, the bond has a present value of $\$1{,}000/1.05 \approx \$952.38$. If the prevailing interest rate suddenly increases to 6% (and thereby becomes your new opportunity cost of capital), the bond's present value becomes $\$1{,}000/1.06 \approx \$943.40$. You lose \$8.98, which is about 0.9% of your original \$952.38 investment. The value of your fixed-bond payment in the future has gone down, because investors can now do better than your 5% by buying new bonds. They have better opportunities elsewhere in the economy. They can earn a rate of return of 6%, not just 5%, so if you wanted to sell your bond now, you would have to sell it at a discount to leave the next buyer a rate of return of 6%. If you had delayed your investment, the sudden change to 6% would have done nothing to your investment. On the other hand, if the prevailing interest rate suddenly drops to 4%, then your bond will be more valuable. Investors would be willing to pay $\$1{,}000/1.04 \approx \$961.54$, which is an immediate \$9.16 gain. The inverse relationship between prevailing interest rates and bond prices is general and worth noting.

IMPORTANT

The price and the implied rate of return on a bond with fixed payments move in opposite directions. When the price of the bond goes up, its implied rate of return goes down. When the price of the bond goes down, its implied rate of return goes up.

Q 2.27. A project with a cost of capital of 30% pays off $250. What should it cost today?

Q 2.28. A bond promises to pay $150 in 12 months. The annual true interest rate is 5% per annum. What is the bond's price today?

Q 2.29. A bond promises to pay $150 in 12 months. The bank quotes you an interest rate of 5% per annum, compounded daily. What is the bond's price today?

Q 2.30. If the cost of capital is 5% per annum, what is the discount factor for a cash flow in two years?

Q 2.31. Interpret the meaning of the discount factor.

Q 2.32. What are the units on rates of return, discount factors, future values, and present values?

Q 2.33. Would it be good or bad for you, in terms of the present value of your liabilities, if your opportunity cost of capital increased?

Q 2.34. The price of a bond that offers a safe promise of $100 in one year is $95. What is the implied interest rate? If the bond's interest rate suddenly jumped up by 150 basis points, what would the bond price be? How much would an investor gain/lose if she held the bond while the interest rate jumped up by these 150 basis points?

2.6 Net Present Value

Present values are alike and thus can be added, subtracted, compared, and so on.

An important advantage of present value is that all cash flows are translated into the same unit: cash today. To see this, say that a project generates $10 in one year and $8 in five years. You cannot add up these different future values to come up with $18—it would be like adding apples and oranges. However, if you translate both future cash flows into their present values, you *can* add them. For example, if the interest rate was 5% per annum (so $(1+5\%)^5 = (1+27.6\%)$ over 5 years), the present value of these two cash flows together would be

$$\text{PV}\big(\$10 \text{ in 1 year}\big) = \frac{\$10}{1.05} \approx \$9.52$$

$$\text{PV}\big(\$8 \text{ in 5 years}\big) = \frac{\$8}{1.05^5} \approx \$6.27$$

$$\text{PV}(C_t) = \frac{C_t}{(1+r)^t}$$

Therefore, the total value of the project's future cash flows *today* (at time 0) is $15.79.

The definition of NPV.

The **net present value** (**NPV**) of an investment is the present value of all its future cash flows minus the present value of its cost. It is really the same as present value, except that the word "net" upfront reminds you to add and subtract *all* cash flows, including the *upfront* investment outlay today. The NPV calculation method is always the same:

1. Translate all future cash flows into today's dollars.
2. Add them all up. This is the present value of all future cash flows.
3. Subtract the initial investment.

A basic use example

NPV is the most important method for determining the value of projects. It is a cornerstone of finance. Let's assume that you have to pay $12 to buy this particular project with its $10 and $8 cash flows. In this case, it is a positive NPV project, because

$$\text{NPV} = -\$12 + \frac{\$10}{1.05} + \frac{\$8}{1.05^5} \approx \$3.79$$

$$C_0 + \frac{C_1}{1+r_1} + \frac{C_5}{(1+r)^5} = \text{NPV}$$

(For convenience, we omit the 0 subscript for NPV, just as we did for PV.)

Think about what NPV means, and how it can be justified.

There are a number of ways to understand net present value.

- One way is to think of the NPV of \$3.79 as the difference between the market value of the future cash flows (\$15.79) and the project's cost (\$12)—this difference is the "value added."
- Another way to think of your project is to compare its cash flows to an equivalent set of bonds that exactly *replicates* them. In this instance, you would want to purchase a 1-year bond that promises \$10 next year. If you save \$9.52—at a 5% interest rate—you will receive \$10. Similarly, you could buy a 5-year bond that promises \$8 in year 5 for \$6.27. Together, these two bonds exactly replicate the project cash flows. The **law of one price** tells you that your project should be worth as much as this bond project—the cash flows are identical. You would have had to put away \$15.79 today to buy these bonds, but your project can deliver these cash flows at a cost of only \$12—much cheaper and thus better than your bond alternative.
- There is yet another way to think of NPV. It tells you how your project compares to the alternative opportunity of investing in the capital markets. These opportunities are expressed in the denominator through the discount factor. What would you get if you took your \$12 and invested it in the capital markets instead of in your project? Using the future value formula, you know that you could earn a 5% rate of return from now to next year, and 27.6% from now to 5 years. Your \$12 would grow into \$12.60 by next year. You could take out the same \$10 cash flow that your project gives you and be left with \$2.60 for reinvestment. Over the next 4 years, at the 5% interest rate, this \$2.60 would grow into \$3.16. But your project would do better for you, giving you \$8. Thus, your project achieves a higher rate of return than the capital markets alternative.

Yet another way to justify NPV: opportunity cost.

The correct capital budgeting rule: Take all positive NPV projects.

The conclusion of this argument is not only the simplest but also the best capital budgeting rule: If the NPV is positive, as it is for our \$3.79 project, you should take the project. If it is negative, you should reject the project. If it is zero, it does not matter.

IMPORTANT

- The NPV formula is

$$\begin{aligned}
\text{NPV} &= C_0 + \text{PV}(C_1) + \text{PV}(C_2) + \text{PV}(C_3) + \text{PV}(C_4) + \cdots \\
&= C_0 + \frac{C_1}{1+r_1} + \frac{C_2}{1+r_2} + \frac{C_3}{1+r_3} + \frac{C_4}{1+r_4} + \cdots \\
&= C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \frac{C_4}{(1+r)^4} + \cdots
\end{aligned}$$

  The subscripts are time indexes, C_t is the net cash flow at time t (positive for inflows, negative for outflows), and r_t is the relevant interest rate for investments from now to time t. With constant interest rates, $r_t = (1+r)^t - 1$.
- The **NPV capital budgeting rule** states that you should accept projects with a positive NPV and reject those with a negative NPV.
- Taking positive NPV projects increases the value of the firm. Taking negative NPV projects decreases the value of the firm.
- NPV is definitively the best method for capital budgeting—the process by which you should accept or reject projects.

The NPV formula is so important that you must memorize it.

Let's work a project NPV example.

Let's work another NPV example. A project costs $900 today, yields $200/year for two years, then $400/year for two years, and finally requires a cleanup expense of $100. The prevailing interest rate is 5% per annum. These cash flows are summarized in Exhibit 2.3. Should you take this project?

First, determine your multiyear costs of capital.

1. You need to determine the cost of capital for tying up money for one year, two years, three years, and so on. The compounding formula is

$$(1 + r_t) = (1 + r)^t = (1.05)^t = 1.05^t$$

So for money right now, the cost of capital r_0 is $1.05^0 - 1 = 0$; for money in one year, r_1 is $1.05^1 - 1 = 5\%$; for money in two years, r_2 is $1.05^2 - 1 = 10.25\%$. And so on.

2. You need to translate the cost of capital into discount factors. Recall that these are 1 divided by 1 plus your cost of capital. A dollar in one year is worth $1/(1 + 5\%) = 1/1.05 \approx 0.9524$ dollars today. A dollar in two years is worth $1/(1 + 5\%)^2 = 1/1.05^2 \approx 0.9070$. And so on.
3. You can now translate the future cash flows into their present value equivalents by multiplying the payoffs by their appropriate discount factors. For example, the $200 cash flow at time 1 is worth about $0.9524 \cdot \$200 \approx \$190.48$.
4. Because present values are additive, you then sum up all the terms to compute the overall net present value. Make sure you include the original upfront cost as a negative.

➤ $68.14 or $68.15?: Rounding Error, Pg.20.

Consequently, the project NPV is about $68.15. Because $68.15 is a positive value, you should take this project.

		Interest Rate			
Time	Project Cash Flow	Annualized	Discount Holding	Present Factor	Value
t	C_t	r	r_t	$\frac{1}{(1+r)^t}$	$PV(C_t)$
Today 0	–$900	5.00%	0.00%	1.0000	–$900.00
Year +1	+$200	5.00%	5.00%	0.9524	+$190.48
Year +2	+$200	5.00%	10.25%	0.9070	+$181.41
Year +3	+$400	5.00%	15.76%	0.8638	+$345.54
Year +4	+$400	5.00%	21.55%	0.8227	+$329.08
Year +5	–$100	5.00%	27.63%	0.7835	–$78.35
			Net Present Value (Sum):		$68.15

Exhibit 2.3: *Hypothetical Project Cash Flow Table.* As a manager, you must provide estimates of your project cash flows. The appropriate interest rate (also called cost of capital in this context) is provided to you by the opportunity cost of your investors—determined by the supply and demand for capital in the broader economy, where your investors can invest their capital instead. The “Project Cash Flow” and the left interest rate column are the two input columns. The remaining columns are computed from these inputs. The goal is to calculate the final column.

However, if the upfront expense was $1,000 instead of $900, the NPV would be negative (–$31.84), and you would be better off investing the money into the appropriate sequence of bonds from which the discount factors were computed. In this case, you should have rejected the project.

If the upfront cost was higher, you should not take the project.

Q 2.35. Work out the present value of your tuition payments for the next two years. Assume that the tuition is $30,000 per year, payable at the start of the year. Your first tuition payment will occur in 6 months, and your second tuition payment will occur in 18 months. You can borrow capital at an effective interest rate of 6% per annum.

Q 2.36. Write down the NPV formula from memory.

Q 2.37. What is the NPV capital budgeting rule?

Q 2.38. Determine the NPV of the project in Exhibit 2.3, if the per-period interest rate were 8% per year, not 5%. Should you take this project?

Q 2.39. You are considering moving into for a building for three years, for which you have to make one payment now, one in a year, and a final one in two years.

1. Would you rather have a lease, paying $1,000,000 upfront, then $500,000 each in the following two years; or would you rather pay $700,000 rent each year?
2. If the interest rate is 10%, what equal payment amount (rather than $700,000) would leave you indifferent? (This is also called the equivalent annual cost (EAC).)

Q 2.40. Use a spreadsheet to answer the following question: Car dealer A offers a car for $2,200 upfront (first payment), followed by $200 lease payments over the next 23 months. Car dealer B offers the same lease at a flat $300 per month (i.e., your first upfront payment is $300). Which lease do you prefer if the interest rate is 0.5% per month?

Application: Are Faster-Growing Firms Better Bargains?

Let's work another NPV problem, applying to companies overall. Does it make more sense to invest in companies that are growing quickly rather than slowly? If you wish, you can think of this question loosely as asking whether you should buy stock of a fast-growing company like Google or stock of a slow-growing company like Procter & Gamble. Actually, you do not even have to calculate anything. In a perfect market, the answer is always that every publicly traded investment comes for a fair price. Thus, the choice does not matter. Whether a company is growing quickly or slowly is already incorporated in the firm's price today, which is just the present value of the firm's cash flows that will accrue to the owners. Therefore, neither is the better deal. Yet, because finance is so much fun, we will ignore this little nuisance and work out the details anyway.

The firm's price should incorporate the firm's attributes.

For example, say company "Grow" (G) will produce over the next 3 years

$$G_1 = \$100 \qquad G_2 = \$150 \qquad G_3 = \$250$$

and company "Shrink" (S) will produce

$$S_1 = \$100 \qquad S_2 = \$90 \qquad S_3 = \$80$$

Is G not a better company to buy than S?

Should you invest in a fast-grower or a slow-grower?

There is no uncertainty involved, and both firms face the same cost of capital of 10% per annum. The price of G today is its present value (PV)

Let's find out: Compute the values.

$$\text{PV}(G) = \frac{\$100}{1.1^1} + \frac{\$150}{1.1^2} + \frac{\$250}{1.1^3} \approx \$402.70 \tag{2.3}$$

and the price of S today is

$$\text{PV}(S) = \frac{\$100}{1.1^1} + \frac{\$90}{1.1^2} + \frac{\$80}{1.1^3} \approx \$225.39$$

Your investment dollar grows at the same 10% rate. Your investment's growth rate is disconnected from the cash flow growth rate.

What is your rate of return from this year to next year? If you invest in G, then next year you will have $100 cash and own a company with $150 and $250 cash flows coming up. G's value at time 1 (so PV now has subscript 1 instead of the usually omitted 0) will thus be

$$\text{PV}_1(G) = \$100 + \frac{\$150}{1.1^1} + \frac{\$250}{1.1^2} \approx \$442.98$$

Your investment will have earned a rate of return of $\$442.98/\$402.70 - 1 \approx 10\%$. If you invest instead in S, then next year you will receive $100 cash and own a company with "only" $90 and $80 cash flows coming up. S's value will thus be

$$\text{PV}_1(S) = \$100 + \frac{\$90}{1.1^1} + \frac{\$80}{1.1^2} \approx \$247.93$$

Your investment will have earned a rate of return of $\$247.93/\$225.39 - 1 \approx 10\%$. In either case, you will earn the fair rate of return of 10% from this year to next year. Whether cash flows are growing at a rate of +50%, -10%, +237.5%, or -92% is irrelevant: *The firms' market prices today already reflect their future growth rates*. There is no necessary connection between the growth rate of the underlying project cash flows or earnings and the growth rate of your investment money (i.e., your expected rate of return).

Any sudden wealth gains would accrue to existing shareholders, not to new investors.

Make sure you understand the thought experiment here: This statement that higher-growth firms do not necessarily earn a higher rate of return does not mean that a firm in which managers succeed in increasing the future cash flows at no extra investment cost will not be worth more. Such firms will indeed be worth more, and the current owners will benefit from the rise in future cash flows, but this will also be reflected immediately in the price at which you, an outsider, can buy this firm. This is an important corollary worth repeating. If General Electric has just won a large defense contract (like the equivalent of a lottery), shouldn't you purchase GE stock to participate in the windfall? Or if Wal-Mart managers do a great job and have put together a great firm, shouldn't you purchase Wal-Mart stock to participate in this windfall? The answer is that you cannot. The old shareholders of Wal-Mart are no dummies. They know the capabilities of Wal-Mart and how it will translate into cash flows. Why should they give you, a potential new shareholder, a special bargain for something to which you contributed nothing? Just providing more investment funds is not a big contribution—after all, there are millions of other investors equally willing to provide funds at the appropriate right price. It is competition—among investors for providing funds and among firms for obtaining funds—that determines the expected rate of return that investors receive and the cost of capital that firms pay. There is actually a more general lesson here. Economics tells you that you must have a scarce resource if you want to earn above-normal profits. Whatever is abundant and/or provided by many competitors will not be a tremendously profitable business.

An even more general lesson.

An even more general version of the question in this section (whether fast-growing or slow-growing firms are better investments) is whether good companies are better investments than bad companies. Many novices will answer that it is better to buy a good company. But you should immediately realize that the answer must depend on the price. Would you really want to buy a great company if its cost was twice its value? And would you really not want to buy a lousy company if you could buy it for half its value? For an investment, whether a company is a well-run purveyor of fine perfume or a poorly-run purveyor of fine manure does not matter by itself. What matters is only the company price relative to the future company cash flows that you will receive.

Q 2.41. Assume that company G pays no interim dividends, so you receive \$536 at the end of the project. What is G's market value at time 1, 2, and 3? What is your rate of return in each year? Assume that the cost of capital is still 10%.

Q 2.42. Assume that company G pays out the full cash flows (refer to the text example) in earnings each period. What is G's market value after the payout at time 1, 2, and 3? What is your rate of return in each year?

Q 2.43. One month ago, a firm suffered a large court award against it that will force it to pay compensatory damages of \$100 million next January 1. Are shares in this firm a bad buy until January 2?

Summary

This chapter covered the following major points:

- A perfect market assumes no taxes, no transaction costs, no opinion differences, and the presence of many buyers and sellers.
- A bond is a claim that promises to pay an amount of money in the future. Buying a bond is extending a loan. Issuing a bond is borrowing. Bond values are determined by their future payoffs.
- One hundred basis points are equal to 1%.
- The time value of money means that 1 dollar today is worth more than 1 dollar tomorrow because of the interest that it can earn.
- Returns must not be averaged, but compounded over time.
- Interest rate quotes are *not* interest rates. For example, stated annual rates are usually not the effective annual rates that your money will earn in the bank. If in doubt, ask!
- The discounted present value (PV) translates future cash values into present cash values. The net present value (NPV) is the sum of all present values of a project, including the investment cost (usually, a negative upfront cash flow today).
- The values of bonds and interest rates move in opposite directions. A sudden increase in the prevailing economy-wide interest rate decreases the present value of a bond's future payouts and therefore decreases today's price of the bond. Conversely, a sudden decrease in the prevailing economy-wide interest rate increases the present value of a bond's future payouts and therefore increases today's price of the bond.
- The NPV formula can be written as

$$\begin{aligned} \text{NPV} &= C_0 + \frac{C_1}{1+r_1} + \frac{C_2}{1+r_2} + \cdots \\ &= C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots \end{aligned}$$

  In this context, r is called the discount rate or cost of capital, and $1/(1+r)$ is called the discount factor.
- The net present value capital budgeting rule states that you should accept projects with a positive NPV and reject projects with a negative NPV.
- In a perfect market, firms are worth the present value of their assets. Whether firms grow quickly or slowly does not make them more or less attractive investments in a perfect market, because their prices always already reflect the present value of future cash flows.
- In a perfect market, the gains from sudden surprises accrue to old owners, not new capital providers, because old owners have no reason to want to share the spoils.

Keywords

Answers

Q 2.1 The four perfect market assumptions are no taxes, no transaction costs, no differences in opinions, and no large buyers or sellers.

Q 2.2 A savings deposit is an investment in a series of short-term bonds.

Q 2.3 $r = (\$1{,}050 - \$1{,}000)/\$1{,}000 = 5\%$

Q 2.4 $r = \dfrac{\$25}{\$1{,}000} = 2.5\%$

Q 2.5 Yes, $10 = 1{,}000\%$.

Q 2.6 The dividend yield would be $\$1/\$40 = 2.5\%$, the capital gain would be $\$45 - \$40 = \$5$, so that its capital gain yield would be $\$5/\$40 = 12.5\%$, and the total rate of return would be $(\$46 - \$40)/\$40 = 15\%$.

Q 2.7 $1\% = 100$ basis points, so an increase of 3% is 300 basis points.

Q 2.8 20 basis points are 0.2%, so the interest rate declined from 10.0% to 9.8%.

Q 2.9 $r = 30\% = (x - \$250)/\$250 \implies x = 1.3 \cdot \$250 = \$325$

Q 2.10 $1.20^5 - 1 \approx 148.83\%$

Q 2.11 $\$2{,}000 \cdot 1.25^{15} \approx \$56{,}843.42$

Q 2.12 The total holding rate of return is $1.05^{20} - 1 \approx 165.33\%$, so you would end up with $\$200 \cdot (1 + 165.33\%) \approx \$530.66$.

Q 2.13 Losing one-third is a rate of return of –33%. To find the holding rate of return, compute $[1 + (-1/3)]^5 - 1 \approx -86.83\%$. About $(1 - 86.83\%) \cdot \$20{,}000 \approx \$2{,}633.74$ remains.

Q 2.14 $(1 + 100\%)^{1/5} - 1 \approx 14.87\%$

Q 2.15 $(1 + r_{0.25})^4 = (1 + r_1)$. Thus, $r_{0.25} = \sqrt[4]{1 + r_1} - 1 = 1.5^{1/4} - 1 \approx 10.67\%$.

Q 2.16 $r_2 = (1 + r_{0,1}) \cdot (1 + r_{1,2}) - 1 = 1.05 \cdot 1.05 - 1 = 10.25\%$

Q 2.17 $r_{10} = (1 + r_1)^{10} - 1 = 1.05^{10} - 1 \approx 62.89\%$

Q 2.18 $r_{100} = (1 + r_1)^{100} - 1 = 1.05^{100} - 1 = 130.5 \approx 13{,}050\%$. In words, this is about 130 times the initial investment, and about 26 times more than the 500% (5 times the initial investment).

Q 2.19 Tripling is equivalent to earning a rate of return of 200%. Therefore, solve $(1 + 6\%)^x = (1 + 200\%)$, or $x \cdot \log(1.06) = \log(3.00)$ or $x = \log(3.00)/\log(1.06) \approx 18.85$ years.

Q 2.20 $(1 + r)^{365.25} = 1.12$. Therefore, $1.12^{(1/365.25)} - 1 \approx 0.000310 = 0.0310\% \approx 3.10$bp/day.

Q 2.21 The bank pays $12\%/365.25 \approx 3.28$bp/day.

Q 2.22 This question demonstrates a nuisance problem that is pervasive in this book: calculations often have rounding error, especially when intermediate results are shown. The following three routes are logically the same, but the precise number differs based on when and where you round:

- Based on 365.25 days per year (which is incidentally itself rounded from the more exact 365.2422 days), the true daily interest rate is 0.00031032517117.... If you use full precision in your calculations, your weekly interest comes to $1.00031032517117\ldots^7 - 1 \approx 0.002174300\ldots$.
- If you round the true daily interest rate to 0.00031, your weekly interest comes to $1.00031\ldots^7 - 1 \approx 0.002172\ldots$.
- Based on 52.2 weeks per year (itself rounded from 52.177 weeks), you could have computed $r = (1 + 12\%)^{(1/52.2)} - 1 \approx 0.002173406\ldots$.

In the \$1,000 case, all three methods give you the same answer of \$1,002.17. In the \$100,000 case, you would have ended up with slightly different numbers based on your route of calculation. All three methods would have been acceptably correct.

In any case, don't blame this book or yourself for small discrepancies in calculations.

Q 2.23 With 12% in nominal APR interest *quoted*, you earn $12\%/365 \approx 0.032877\%$ per day. Therefore, the weekly rate of return is $(1 + 0.032877\%)^7 - 1 \approx 0.23036\%$. Your \$1,000 will grow into \$1,002.30. Note that you end up with more money when the 12% is the quoted rate than when it is the effective rate.

Q 2.24 With 12% in nominal APR interest *quoted*, you earn $12\%/365 \approx 0.032877\%$ per day. Therefore, the annual rate of return is $(1+0.032877\%)^{365}-1 \approx 12.747462\%$. Your \$1,000 will grow into \$1,127.47.

Q 2.25 The bank quote of 6% means that it will pay an interest rate of $6\%/365 \approx 0.0164384\%$ per day. This earns an actual interest rate of $(1+0.0164384\%)^{365}-1 \approx 6.18\%$ per annum. Therefore, each invested \$100 grows to \$106.18, thus earning \$6.18 over the year.

Q 2.26 The bank quote of 8% means that you will have to pay an interest rate of $8\%/12 \approx 0.667\%$ per month. This earns an actual interest rate of $(1+0.667\%)^{12}-1 \approx 8.30\%$ per annum. You will have to pay \$108.30 in repayment for every \$100 you borrowed.

Q 2.27 $r = 30\% = (\$250 - x)/x$. Thus, $x = \$250/1.30 \approx \$192.31$.

Q 2.28 $\$150/(1.05) \approx \$142.86$

Q 2.29 $\$150/[1+(5\%/365)]^{365} \approx \$142.68$

Q 2.30 $1/[(1.05)\cdot(1.05)] \approx 0.9070$

Q 2.31 It is today's value in dollars for 1 future dollar, that is, at a specific point in time in the future.

Q 2.32 The rate of return and additional factors are unit-less. The latter two are in dollars (though the former is dollars in the future, while the latter is dollars today).

Q 2.33 Good. Your future payments would be worth less in today's money.

Q 2.34 The original interest rate is $\$100/\$95-1 \approx 5.26\%$. Increasing the interest rate by 150 basis points is 6.76%. This means that the price should be $\$100/(1.0676) \approx \$93.67$. A price change from \$95 to \$93.67 is a rate of return of $\$93.67/\$95-1 \approx -1.40\%$.

Q 2.35 The first tuition payment is worth $\$30{,}000/(1.06)^{1/2} \approx \$29{,}139$. The second tuition payment is worth $\$30{,}000/(1.06)^{3/2} \approx \$27{,}489$. Thus, the total present value is \$56,628.

Q 2.36 If you cannot write down the NPV formula by heart, do not go on until you have it memorized.

Q 2.37 Accept if NPV is positive. Reject if NPV is negative.

Q 2.38 $-\$900+\$200/(1.08)^1+\$200/(1.08)^2+\$400/(1.08)^3+\$400/(1.08)^4-\$100/(1.08)^5 \approx \$0.14$. The NPV is positive. Therefore this is a worthwhile project that you should accept.

Q 2.39 For the 3-years:

1. Your rent-vs-lease preference depends on the interest rate. If the interest rate is zero, then you would prefer the \$2 million sum-total lease payments to the \$2.1 million sum-total rent payments. If the prevailing interest rate is less than 21.5%, it is better to lease. If it is more than 21.5%, you prefer the rent. For example, if it is 40%, the net present cost of the lease is \$1.612 million, while the net present cost of the rent is \$1.557 million.
2. At a 10% interest rate, the total net present cost of the lease is $\$1+\$0.5/1.1+\$0.5/1.1^2 \approx \$1.868$ million. An equivalent rent contract must solve

$$x + \frac{x}{1.1} + \frac{x}{1.1^2} = \$1.868$$

Multiply by $1.1^2 = 1.21$

$$1.21\cdot x + 1.1\cdot x + x = \$1.868\cdot 1.21$$
$$\Leftrightarrow x\cdot(1.21+1.1+1) = \$2{,}260.28$$

Therefore, the equivalent rental cost would be $x \approx \$682.864$.

Q 2.40 Lease A has an NPV of –\$6,535. Lease B has an NPV of –\$6,803. Therefore, lease A is cheaper.

Q 2.41 For easier naming, let's use a specific year. Pretend it is the year 2000 now, and call 2000 your year 0. (Coincidence that the final digit is the same?!) The firm's present value in 2000 is $\$536/1.10^3 \approx \$402.70$—but you already knew this. If you buy this company, its value in 2001 depends on a cash flow stream that is \$0 in 2001, \$0 in year 2002, and \$536 in year 2003. It will be worth $\$536/1.10^2 \approx \$442.98$ in 2001. In 2002, your firm will be worth $\$536/1.10 \approx \$487.27$. Finally, in 2003, it will be worth \$536. Each year, you expect to earn 10%, which you can compute from the four firm values.

Q 2.42 Again, call 2000 your year 0. The firm's present value in 2000 is based on dividends of \$100, \$150, and \$250 in the next three years. The firm value in 2000 is the \$402.70 from Page 30. The firm value in 2001 was also worked out to be \$442.98, but you immediately receive \$100 in cash, so the firm is worth only $\$442.98-\$100 = \$342.98$. As an investor, you would have earned a rate of return of $\$442.98/\$402.70-1 \approx 10\%$. The firm value in 2002 is $PV_2(G) = \$250/1.1 \approx \$227.27$, but you will also receive \$150 in cash, for a total firm-related wealth of \$377.27. In addition, you will have the \$100 from 2001, which would have grown to \$110—for a total wealth of \$487.27. Thus, starting with wealth of \$442.98 and ending up with wealth of \$487.27, you would have earned a rate of return of $\$487.27/\$442.98-1 \approx 10\%$. A similar computation shows that you will earn 10% from 2002 (\$487.27) to 2003 (\$536.00).

Q 2.43 No! The market price will have already taken the compensatory damages into account in the share price a month ago, just after the information had become public.

End of Chapter Problems

Q 2.44. What is a perfect market? What were the assumptions made in this chapter that were not part of the perfect market scenario?

Q 2.45. In the text, I assumed you received the dividend at the end of the period. In the real world, if you received the dividend at the beginning of the period instead of the end of the period, could this change your effective rate of return? Why?

Q 2.46. Your stock costs $100 today, pays $5 in dividends at the end of the period, and then sells for $98. What is your rate of return?

Q 2.47. What is the difference between a bond and a loan?

Q 2.48. Assume an interest rate of 10% per year. How much would you lose over 5 years if you had to give up interest on the interest—that is, if you received 50% instead of compounded interest?

Q 2.49. The interest rate has just increased from 6% to 8%. How many basis points is this?

Q 2.50. Over 20 years, would you prefer 10% per annum, with interest compounding, or 15% per annum but without interest compounding? (That is, you receive the interest, but it is put into an account that earns no interest, which is what we call simple interest.)

Q 2.51. A project returned +30%, then –30%. Thus, its arithmetic average rate of return was 0%. If you invested $25,000, how much did you end up with? Is your rate of return positive or negative? How would your overall rate of return have been different if you first earned –30% and then +30%?

Q 2.52. A project returned +50%, then –40%. Thus, its arithmetic average rate of return was $(50\% + [-40\%])/2 = +5\%$. Is your rate of return positive or negative?

Q 2.53. An investment for $50,000 earns a rate of return of 1% in each month of a full year. How much money will you have at the end of the year?

Q 2.54. There is always disagreement about what stocks are good buys. A typical disagreement is whether a particular stock is likely to offer, say, a 10% (pessimistic) or a 20% (optimistic) annualized rate of return. For a $30 stock today, what does the difference in belief between these two opinions mean for the expected stock price from today to tomorrow? (Assume that there are 365 days in the year. Reflect on your answer for a moment—a $30 stock typically moves about ±$1 on a typical day. This unexplainable up-and-down volatility is often called noise. How big is the average move compared to the noise?)

Q 2.55. If the interest rate is 5% per annum, how long will it take to double your money? How long will it take to triple it?

Q 2.56. If the interest rate is 8% per annum, how long will it take to double your money?

Q 2.57. From Fibonacci's *Liber Abaci*, written in the year 1202: "A certain man gave 1 denaro at interest so that in 5 years he must receive double the denari, and in another 5, he must have double 2 of the denari and thus forever. How many denari from this 1 denaro must he have in 100 years?"

Q 2.58. A bank quotes you an annual loan interest rate of 14%, daily compounding, on your credit card. If you charge $15,000 at the beginning of the year, how much will you have to repay at the end of the year?

Q 2.59. Go to the website of a bank of your choice. What kind of quote does your bank post for a CD, and what kind of quote does your bank post for a mortgage? Why?

Q 2.60. What is the 1-year discount factor if the interest rate is 33.33%?

Q 2.61. You can choose between the following rent payments:

a. A lump sum cash payment of $100,000;
b. 10 annual payments of $12,000 each, the first occurring immediately;
c. 120 monthly payments of $1,200 each, the first occurring immediately. (Friendly suggestion: This is a lot easier to calculate on a computer spreadsheet.)

Now choose among them:

1. Which rental payment scheme would you choose if the interest rate was an effective 5% per year?
2. Spreadsheet question: At what interest rate would you be indifferent between the first and the second choice above? (Hint: Graph the NPV of the second project as a function of the interest rate.)

Q 2.62. A project has cash flows of $15,000, $10,000, and $5,000 in 1, 2, and 3 years, respectively. If the prevailing interest rate is 15%, would you buy the project if it costs $25,000?

Q 2.63. Consider the same project that costs $25,000 with cash flows of $15,000, $10,000, and $5,000. At what prevailing interest rate would this project be profitable? Try different interest rates, and plot the NPV on the y-axis, and the interest rate on the x-axis.

Q 2.64. Assume you are 25 years old. The IAW insurance company is offering you the following retirement contract (called an *annuity*): Contribute $2,000 per year for the next 40 years. When you reach 65 years of age, you will receive $30,000 per year for as long as you live. Assume that you believe that the chance that you will die is 10% per year after you will have reached 65 years of age. In other words, you will receive the first payment with probability 90%, the second payment with probability 81%, and so on. If the prevailing interest rate is 5% per year, all payments occur at year-end, and it is now January 1, is this annuity a good deal? (Use a spreadsheet.)

Q 2.65. A project has the following cash flows in periods 1 through 4: –$200, +$200, –$200, +$200. If the prevailing interest rate is 3%, would you accept this project if you were offered an upfront payment of $10 to do so?

Q 2.66. On January 1, 2016, Intel Corp's stock traded for $33.99. In 2012, it paid $0.21/quarter in dividends, then $0.225 in dividends until 2015 when it increased to $0.24, and finally to $0.26 in 2016. Assume Intel will pay $0.25/quarter in 2017. Further assume that the prevailing interest rate is 0.5% per quarter (i.e., 2.015% per annum). If you buy Intel stock on January 1, 2016, at what price would you have to be able to sell Intel stock at the end of 2017 in order to break even?

Q 2.67. If the interest rate is 5% per annum, what would be the equivalent annual cost (see Question 2.39) of a $2,000 lease payment upfront, followed by $800 for three more years?

Q 2.68. Assume that you are a real estate broker with an exclusive contract—the condo association rules state that everyone selling their condominiums must go through you or a broker designated by you. A typical condo costs $500,000 today and sells again every 5 years. Assume the first sale will happen in 5 years. This will last for 50 years, and then all bets are off. Your commission will be 3%. Condos appreciate in value at a rate of 2% per year. The interest rate is 10% per annum.

1. What is the value of this exclusivity rule for one condo? In other words, at what price should you be willing to sell the privilege of being the exclusive representation for one condo to another broker?
2. If free Internet advertising was equally effective and if it could replace all real-estate agents so that buyers' and sellers' agents would no longer earn the traditional 6% (3% each), what would happen to the value gain of the condo?

Q 2.69. The prevailing discount rate is 15% per annum. Firms live for three years. Firm F's cash flows start with $500 in year 1 and grow at 20% per annum for two years. Firm S's cash flows also start with $500 in year 1 but shrink at 20% per annum for two years. What are the prices of these two firms? Which one is the better "buy"?

3

Stock and Bond Valuation: Annuities and Perpetuities

Important Shortcut Formulas

The present value formula is the main workhorse for valuing investments of all types, including stocks and bonds. But these rarely have just two or three future payments. Stocks may pay dividends forever. The most common mortgage bond has 360 monthly payments. It would be possible but tedious to work with NPV formulas containing 360 terms.

Fortunately, there are some shortcut formulas that can speed up your PV computations if your projects have a particular set of cash flow patterns and the opportunity cost of capital is constant. The two most prominent are for projects called *perpetuities* (which have payments lasting forever) and *annuities* (which have payments lasting for a limited number of years). Of course, no firm lasts forever, but the perpetuity formula is often a useful "quick-and-dirty" tool for a good approximation. In any case, the formulas in this chapter are widely used and can help you understand the economics of corporate growth.

3.1 Perpetuities

A simple **perpetuity** is a project with a stream of constant cash flows that repeats forever. If the cost of capital (i.e., the appropriate discount rate) is constant and the amount of money remains the same or grows at a constant rate, perpetuities lend themselves to fast present value solutions—very useful when you need to come up with quick rule-of-thumb estimates. Though the formulas may seem intimidating at first, using them will quickly become second nature to you.

"Perpetuities" are projects with special kinds of cash flows, which permit the use of shortcut formulas.

The Simple Perpetuity Formula

At a constant interest rate of 10%, how much money do you need to invest today to receive the same dollar amount of interest of \$2 each year, starting next year, forever? Exhibit 3.1 shows the present values of all future payments for a perpetuity paying \$2 forever, if the interest rate is 10% per annum. Note how there is no payment at time 0, and that the individual payment terms become smaller and smaller the further out we go.

Here is an example of a perpetuity that pays \$2 forever.

To confirm the table's last row, which gives the perpetuity's net present value as \$20, you can spend from here to eternity to add up the infinite number of terms. But if you use a spreadsheet to compute and add up the first 50 terms, you will get a PV of \$19.83. If you add up the first 100 terms, you will get a PV of \$19.9986. Mathematically, the sum eventually converges to \$20 sharp. This is because there is a nice shortcut to computing the net present value of the perpetuity if the cost of capital is constant:

The shortcut perpetuity formula.

Time	Cash Flow	Discount Factor	Present Value	Cumul PV
0	Nothing! You have no cash flow here!			
1	\$2	$1/(1+10\%)^1 \approx 0.909$	\$1.82	\$1.82
2	\$2	$1/(1+10\%)^2 \approx 0.826$	\$1.65	\$3.47
3	\$2	$1/(1+10\%)^3 \approx 0.751$	\$1.50	\$4.97
⋮	⋮	⋮	⋮	⋮
50	\$2	$1/(1+10\%)^{50} \approx 0.0085$	\$0.02	\$19.83
⋮	⋮	⋮	⋮	⋮
			Net Present Value (Sum):	\$20.00

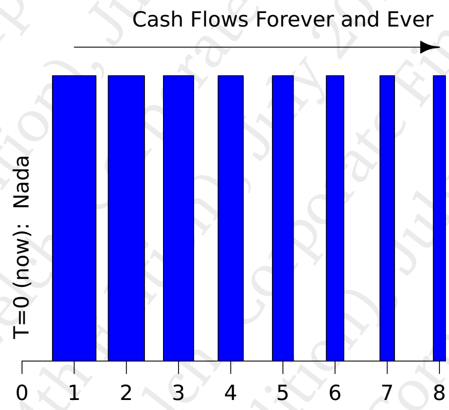


Exhibit 3.1: *Perpetuity Stream of \$2 with Interest Rate* $r = 10\%$. This exhibit shows cash flows, discount factors, and cumulative value. The height of the bars in the graph shows that the nominal cash flows are the same in every future period. Their widths (and thus their areas) indicate the present value of these cash flows. Each bar has less area than the preceding one. Otherwise, the cumulative sum could never be a finite number.

$$\text{Perpetuity PV} = \frac{\$2}{10\%} = \frac{\$2}{0.1} = \$20$$

$$PV_0 = \frac{C_1}{r}$$

The "1" time subscript in the formula is to remind you that the first cash flow occurs not now, but next year—the cash flows themselves will remain the same amount next year, the year after, and so on.

IMPORTANT

A stream of constant cash flows (C dollars each period and forever) beginning *next* period (i.e., time 1), which is discounted at the same per-period cost of capital r forever, is a special perpetuity worth

$$PV_0 = \frac{C_1}{r}$$

which is a shortcut for

$$PV_0 = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots + \frac{C_t}{(1+r)^T} + \cdots$$

C_2 and all other C_t are the same as C_1.

The Oldest Institutions and Perpetuities

Perpetuities assume that projects last forever. But nothing does. The oldest Western institution today may well be the Roman Catholic Church. Wikipedia lists the oldest existing company as the Keiunkan hotel in Japan, founded in 705. (A number of existing restaurants, hotels, and breweries in the West are also fairly old, dating from the late ninth century.) The oldest existing corporation in the United States is the Collegiate Reformed Protestant Dutch Church of the City of New York, formed in 1628 and granted a corporate charter by King William in 1696. The Canadian Hudson's Bay Company was founded in 1670 and claims to be the oldest continuously *incorporated* company in the world. The oldest U.S. companies are the Stroh's brewery and the Bowne printing firm, both of which were founded in 1885.

Guantanamo Naval Base was leased from Cuba in 1903 as a perpetuity by the United States in exchange for 2,000 pesos per annum in U.S. gold, equivalent to $4,085. In a speech, Fidel Castro redefined time as "whatever is indefinite lasts 100 years." In any case, the Cuban government no longer recognizes the agreement and does not accept the annual payments—but it has also wisely not yet tried to expel the Americans. Let's see what diplomacy will do. *Wikipedia*

The easiest way for you to get comfortable with perpetuities is to solve some problems.

Easier done than said.

Q 3.1. From memory, write down the perpetuity formula. Be explicit on when the first cash flow occurs.

Q 3.2. What is the PV of a perpetuity paying $5 each month, beginning *next* month, if the monthly interest rate is a constant 0.5%/month?

Q 3.3. What is the PV of a perpetuity paying $15 each month, beginning *next* month, if the *effective* annual interest rate is a constant 12.68% per year?

Q 3.4. Under what interest rates would you prefer a perpetuity that pays $2 million per year beginning next year to a one-time payment of $40 million?

Q 3.5. In Britain, there are **Consol** bonds that are perpetuity bonds. (In the United States, the IRS does not allow companies to deduct the interest payments on perpetual bonds, so U.S. corporations do not issue Consol bonds.) What is the value of a Consol bond that promises to pay $2,000 per year if the prevailing interest rate is 4%?

The Growing Perpetuity Formula

A growing perpetuity assumes that cash flows grow by a constant rate g forever.

➤ Growing Perpetuities, Exhibit 3.2, Pg.40.

What if, instead of the same amount of cash every period, the cash flows increase over time? The **growing perpetuity** formula allows for a constant rate g per period, provided it is less than the interest rate. Exhibit 3.2 shows a growing perpetuity that pays $2 next year, grows at a rate of 5%, and faces a cost of capital of 10%. The present value of the first 30 terms adds up to $30.09. The first 100 terms add up to $39.64. The first 200 terms add up to $39.98. Eventually, the sum approaches the formula

$$\text{PV of Growing Perpetuity} = \frac{\$2}{10\% - 5\%} = \$40$$

$$PV_0 = \frac{C_1}{r-g} \qquad (3.1)$$

Time	Cash Flow	Discount Factor	Present Value	Cumul PV
0	Nothing! You have no cash flow here!			
1	$(1+5\%)^0 \cdot \$2 = \$2.000$	$(1+10\%)^{-1} \approx 0.909$	$1.818	$1.82
2	$(1+5\%)^1 \cdot \$2 = \$2.100$	$(1+10\%)^{-2} \approx 0.826$	$1.736	$3.56
3	$(1+5\%)^2 \cdot \$2 = \$2.205$	$(1+10\%)^{-3} \approx 0.751$	$1.657	$5.22
⋮	⋮	⋮	⋮	⋮
30	$(1+5\%)^{29} \cdot \$2 \approx \$8.232$	$(1+10\%)^{-30} \approx 0.057$	$0.472	$30.09
⋮	⋮	⋮	⋮	⋮
		Net Present Value (Sum):		$40.00

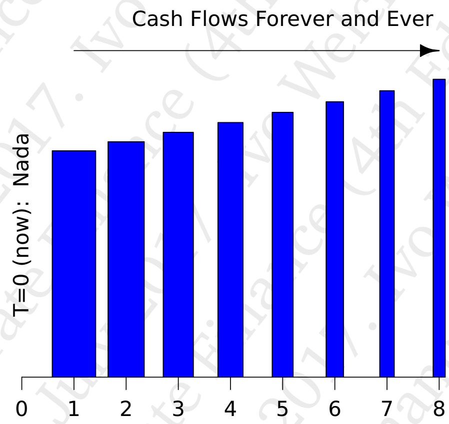


Exhibit 3.2: *Perpetuity Stream with* $C_1 = \$2$*, Growth Rate* $g = 5\%$*, and Interest Rate* $r = 10\%$*.* This exhibit shows cash flows, discount factors, and cumulative value. The height of the bars in the graph shows that the nominal cash flows are growing over time. However, their widths (and thus their areas) indicate the present value of these cash flows. Each bar has less area than the preceding one, which explains why the cumulative sum can be a finite number.

No cash flow at time 0. First growth is from time 1 to time 2.

As before, the "1" subscript indicates that cash flows begin next period, not this period, but here it is necessary because future cash flows will be different. The interest rate is r and it is reduced by g, the growth rate of your cash flows. Note how the table shows that the first application of the growth factor g occurs 1 period after the first application of the discount factor. For example, the cash flow at time 30 is discounted by $(1+r)^{30}$, but its cash flow is C multiplied by a growth factor of $(1+g)^{29}$. You will later encounter many applications of the growing perpetuity formula. For example, it is common to assume that cash flows grow by the rate of inflation. You will also later use this formula to obtain so-called terminal values in a chapter of this book, in which you design so-called pro formas.

➤ Terminal value, Sect. 21.2, Pg.585.

IMPORTANT

A stream of cash flows growing at a rate of g each period and discounted at a constant interest rate r is worth

$$PV_0 = \frac{C_1}{r - g}$$

The first cash flow, C_1, occurs next period (time 1), the second cash flow of $C_2 = C_1 \cdot (1 + g)$ occurs in two periods, and so forth, *forever*. For the formula to work, g can be negative, but r must be greater than g.

You need to memorize the growing perpetuity formula!

Be careful to use the cash flow *next* year in the numerator. The subscript "1" is there to remind you. For example, if you want to use this formula on your firm, and it earned \$100 million this year, and you expect it to grow at a 5% rate forever, then the correct cash flow in the numerator is $C_1 = \$105$ million, not \$100 million!

Although a subscript on C makes this seem more painful, it is a good reminder here.

What would happen if the cash flows grew faster than the interest rate ($g > r$)? Wouldn't the formula indicate a negative PV? Yes, but this is because the entire scenario would be nonsense. The present value in the perpetuities formulas is only less than infinity, because *in today's dollars*, each term in the sum is a little less than the term in the previous period. If g were greater than r, however, the cash flow 1 period later would be worth more even in today's dollars. For example, take our earlier example with a discount rate of 10%, but make the growth rate of cash flows $g = 15\%$. The first cash flow would still be \$2, which still discounts to \$1.818 today. But the second cash flow would be $\$2 \cdot 1.15 = \$2.30$, which discounts to \$1.901 today. The third cash flow would be $\$2 \cdot 1.15^2 = \$2.645$, which discounts to \$1.987 today. The present value of each cash flow is higher than that preceding it. Taking a sum over an infinite number of such increasing terms would yield infinity as the value. A value of infinity is clearly not sensible, as nothing in this world is worth an infinite amount of money. Therefore, the growing perpetuity formula yields nonsensical values if $g \geq r$—as it should!

The formula is nonsensical when $r < g$.

Q 3.6. From memory, write down the growing perpetuity formula.

Q 3.7. What is the PV of a perpetuity paying \$5 each month, beginning *this* month (in 1 second), if the monthly interest rate is a constant 0.5%/ month (6.2%/year) and the cash flows will grow at a rate of 0.1%/month (1.2%/year)?

Q 3.8. What is the PV of a perpetuity paying \$8 each month, beginning *this* month (in 1 second), if the monthly interest rate is a constant 0.5%/ month (6.2%/year) and the cash flows will grow at a rate of 0.8%/month (10%/year)?

Q 3.9. Here is an example of the most common use of the growing perpetuity model (called a pro forma). Your firm just finished the year, in which it had cash earnings of \$100 million. Excluding this amount, you want to determine the value of the firm. You forecast your firm to have a quick growth phase for 3 years, in which it grows at a rate of 20% per annum (ending year 1 with \$120 up to ending year 3 with \$172.8). Your firm's growth then slows down to 10% per annum for the next 3 years (ending year 4 with \$190.1, etc.). Finally, beginning in year 7, you expect it to settle into its long-term growth rate of 5% per annum. You also expect your cost of capital to be 10% in your 20% growth phase, 9% in your 10% growth phase, and 8% in your 5% growth phase. Excluding the \$100 million, what do you think your firm is worth today?

Q 3.10. An eternal patent contract states that the patentee will pay the patentor a fee of \$1.5 million next year. The contract terms state a fee growth with the inflation rate, which runs at 2% per annum. The appropriate cost of capital is 14%. What is the value of this patenting contract?

Q 3.11. How would the patent contract value change if the first payment did not occur next year, but tonight?

Application: Stock Valuation with A Gordon Growth Model

Perpetuities are imperfect approximations, but often give a useful upper bound.

With their fixed interest and growth rates and eternal payment requirements, perpetuities are rarely exactly correct. But they can be very helpful for quick back-of-the-envelope estimates. For example, consider a mature and stable business with profits of \$1 million next year. Because it is stable, its profits are likely to grow at the inflation rate of, say, 2% per annum. This means that it will earn \$1,020,000 in 2 years, \$1,040,400 in 3 years, and so on. The firm faces a cost of capital of 8%. The growing perpetuity formula indicates that this firm should probably be worth no more than

$$\text{Business Value} = \frac{\$1{,}000{,}000}{8\% - 2\%} \approx \$16{,}666{,}667$$

$$\text{Business Value} = \frac{C_1}{r - g}$$

because in reality, the firm will almost surely not exist forever. Of course, in real life, there are often even more significant uncertainties: Next year's profit may be different, the firm may grow at a different rate (or may grow at a different rate for a while) or face a different cost of capital for 1-year loans than it does for 30-year loans. Thus, \$16.7 million should be considered a quick-and-dirty useful approximation, perhaps for an upper limit, and not an exact number.

The Gordon growth model: constant eternal dividend growth.

The growing perpetuity model is sometimes directly applied to the stock market. For example, if you believe that a stock's dividends will grow by $g = 5\%$ forever, that the appropriate rate of return is $r = 10\%$, and that the stock market will earn and/or pay dividends of $D = \$10$ *next year*, then you would feel that a stock price today of

$$\text{Stock Price P Today} = \frac{\$10}{10\% - 5\%} = \$200$$

$$\text{Stock Price P Today} = \frac{\text{Dividends D Next Year}}{r - g} \tag{3.2}$$

would be appropriate. In this context, the growing perpetuity model is often called the **Gordon growth model**, after its inventor, Myron Gordon.

You could estimate the cost of capital for GE, based on its dividend yield and its expected dividend growth rate.

Let us explore the Gordon growth model a bit. In June 2016, YAHOO! FINANCE stated that General Electric (GE) had a dividend yield of 3.0%. This is the analysts' consensus forecast of next year's dividends divided by the stock price, D/P. This is called the **dividend yield**. Rearrange Formula 3.2:

$$\frac{\text{Dividends D Next Year}}{\text{Stock Price P Today}} = r - g = 3.0\%$$

Therefore, you can infer that the market believes that the appropriate cost of capital (r) for General Electric exceeds its growth rate of dividends (g) by about 3.0%. YAHOO! FINANCE further had a summary of GE's cash flow statement, which indicated that GE paid \$9.3 billion in dividends in 2015, up 5% from 2014's \$8.85 billion. Therefore, if you believe 5%/year is also a fair estimate of the *eternal* future growth rate of GE's dividends, then the financial markets valued GE as if it had a per-annum cost of capital of about

$$r = \frac{\text{Dividends D Next Year}}{\text{Stock Price P Today}} + g \approx 3\% + 5\% = 8\%$$

Don't take this estimate too seriously. It is an approximation that should be viewed just as a conversation starter.

Let's play another game that is prominent in the financial world. Earnings are, loosely speaking, cousins of the cash flows that corporate stockholders receive. You can then think of the value of the stock today as the value of the earnings stream the stock will produce. After all, recall from Chapter 1 that owners receive all dividends and all cash flows (earnings), presumably the former being paid out from the latter. (In Chapter 14, I will explain a lot of this in more detail as well as why earnings are only approximately but not exactly cash flows.)

Let's presume that the formula also applies to earnings.

Furthermore, it is common to assume that stock market values are capitalized as if corporate earnings were eternal cash flows that are growing at a constant rate g applicable to earnings (which is not necessarily the same as the growth rate applicable to dividends). This means that you would assume that the value of the firm is

You could also estimate the cost of capital for GE based on its price/earnings ratio and its earnings growth rate.

$$\text{Stock Price P Today} = \frac{\text{Earnings E Next Year}}{r - g}$$

Thus, to determine the rate of return that investors require (the cost of capital), all you need is a forecast of earnings, the current stock price, and the eternal growth rate of earnings. Again, YAHOO! FINANCE (Key Statistics and Analyst Estimates) gives you all the information you need. In June 2016, GE's "trailing P/E" ratio—calculated as the current stock price divided by historical earnings—was 41. More interestingly, the analysts predicted "forward P/E" ratios—calculated as the price divided by their expectations of *next* year's earnings—as 17. The growing perpetuity formula wants the earnings in *future* years, so the latter is closer to what you need. The analysts also expected GE's earnings to grow over the next 5 years at an average rate of 12%—the g in the formula if you are willing to assume that this is a long-term quasi-eternal growth rate. Therefore, all you have to do is rearrange the growing perpetuity formula, and out pops an appropriate rate of return:

$$r = \frac{\text{Earnings Next Year}}{\text{Stock Price Today}} + g = \frac{1}{P/E} + g \approx \frac{1}{17} + 12\% \approx 18\%$$

As a herd, analysts were quite optimistic on GE's earnings relative to its price and more so than they were with respect to how much it would pay out in dividends.

This formula is intuitive, but there are more complex versions. For example, analysts sometimes use one that contemplates that firms with higher earnings reinvestment rates (aka **plowback** ratios) should have higher earnings growth rates g.

Mention plowback extensions

It is important that you recognize these are just approximations that you should not take too seriously in terms of accuracy. GE will not last forever, earnings are not the cash flows you need, the discount rate is not eternally constant, earnings will not grow forever at 6.3%, and so on. However, the numbers are not uninteresting and may not even be too far off, either. GE is a very stable company that is likely to be around for a long time, and you could do a lot worse than assuming that the cost of capital (for investing in projects that are similar to GE stock ownership) is somewhere around 12% per annum—say, somewhere between 10% to 14% per annum.

Keep perspective! The model provides only a quick approximation.

➤ Price-earnings ratio, Sect. 15.3, Pg.392.

Q 3.12. A stock is paying a quarterly dividend of \$5 in 1 month. The dividend is expected to increase every quarter by the inflation rate of 0.5% per quarter—so it will be \$5.025 in the next quarter (i.e., paid out in 4 months). The prevailing cost of capital for this kind of stock is 9% per annum. What should this stock be worth?

Q 3.13. If a \$100 stock has earnings of \$5 per year, and the appropriate cost of capital for this stock is 12% per year, what does the market expect the firm's "as-if-eternal dividends" to grow at?

3.2 Annuities

An annuity pays the same amount for T years.

The second type of cash flow stream that lends itself to a quick formula is an **annuity**, which is a stream of equal cash flows for a given number of periods. Unlike a perpetuity, payments stop after T periods. For example, if the interest rate is 10% per period, what is the value of an annuity that pays \$5 per period for 3 periods?

By hand.

Let's first do this the slow way. You can hand-compute the net present value as

$$\text{PV} = \frac{\$5}{1.10} + \frac{\$5}{1.10^2} + \frac{\$5}{1.10^3} \approx \$12.4343$$

$$\text{PV} = \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_2)} + \frac{C_3}{(1+r_3)}$$

$$= \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3}$$

The annuity formula makes short work of this NPV calculation,

$$\text{PV} = \$5 \cdot \left\{ \frac{1-[1/(1+10\%)]^3}{10\%} \right\} \approx \$12.4343$$

$$\text{PV} = C_1 \cdot \left\{ \frac{1-[1/(1+r)]^T}{r} \right\} = \text{PV}$$

Is this really a shortcut? Maybe not for 3 periods, but try a 360-period annuity—which method do you prefer? Either works.

IMPORTANT

A stream of constant equal cash flows, beginning next period (time 1) and lasting for T periods, and discounted at a constant interest rate r, is worth

$$\text{PV}_0 = \frac{C_1}{r} \cdot \left[1 - \frac{1}{(1+r)^T} \right]$$

Q 3.14. How many years does it take for an annuity to reach three-quarters the value of a perpetuity if the interest rate is 5%? If the interest rate is r? To reach fraction f of the value?

Q 3.15. Recall from memory the annuity formula.

Q 3.16. What is the PV of a 360-month annuity paying \$5 per month, beginning at \$5 next month (time 1), if the monthly interest rate is a constant 0.5%/month (6.2%/year)?

Q 3.17. In *L'Arithmetique*, written in 1558, Jean Trenchant posed the following question: "In the year 1555, King Henry, to conduct the war, took money from bankers at the rate of 4% per fair [quarter]. That is better terms for them than 16% per year. In this same year before the fair of Toussaints, he received by the hands of certain bankers the sum of 3,945,941 ecus and more, which they called 'Le Grand Party' on the condition that he will pay interest at 5% per fair for 41 fairs after which he will be finished. Which of these conditions is better for the bankers?" Translated, the question is whether a perpetuity at 4% interest payout per quarter is better or worse than a 41-quarter annuity at 5% interest payout per quarter. (The answer will depend on the prevailing true interest rate, which you can assume to be constant.)

Q 3.18. Solve Fibonacci's annuity problem given in the anecdote on the next page: Compare the PV of a stream of quarterly cash flows of 75 bezants versus the PV of a stream of annual cash flows of 300 bezants. Payments are always at period-end. The interest rate is 2% per month. What is the relative value of the two streams? Compute the difference for a 1-year investment first.

Fibonacci and the Invention of Net Present Value

William Goetzmann argues that Leonardo of Pisa, commonly called Fibonacci, may have invented not only the famous "Fibonacci series" but also the concept of net present value.

Fibonacci's family were merchants in the Mediterranean in the thirteenth century, with trade relations to Arab merchants in Northern Africa. Fibonacci wrote about mathematics primarily as a tool to solve merchants' problems—in effect, to understand the pricing of goods and currencies relative to one another. Imagine how rich you could get if you were the only one who could quickly determine which goods were worth more than others! In fact, you should think of Fibonacci and other Pisan merchants as the "financial engineers" of the thirteenth century.

In 1202, the 30-year-old Fibonacci published his most famous treatise, *Liber Abaci*. We still are using its problems and answers today. One of his puzzles—which you solve in Q3.17—is called "On a Soldier Receiving 300 Bezants for His Fief":

> A soldier is granted an annuity by the king of 300 bezants per year, paid in quarterly installments of 75 bezants. The king alters the payment schedule to an annual year-end payment of 300. The soldier is able to earn 2 bezants on 100 per month (over each quarter) on his investment. How much is his effective compensation after the terms of the annuity changed?

To answer this problem, you must know how to value payments at different points in the future—you must understand the time value of money. What is the value of 75 bezants in one quarter, two quarters, and so forth? What is the value of 300 bezants in one year, two years, and so on? Yes, money sooner is usually worth more than money later—but you need to determine by exactly how much in order to determine how good or bad the change is for the king and the soldier. You must use the interest rate Fibonacci gives and then compare the two different cash flow streams—the original payment schedule and the revised payment schedule—in terms of a common denominator. This common denominator will be the two streams' present values.

William Goetzmann, Yale University

Annuity Application: Fixed-Rate Mortgage Payments

Most mortgages are **fixed-rate mortgage loans**, and they are basically annuities. They promise a specified stream of equal cash payments each month to a lender. A 30-year mortgage with monthly payments is really a 360-payment annuity. (The "annu-ity" formula should really be called a "month-ity" formula in this case.) What would be your monthly payment if you took out a 30-year mortgage loan for \$500,000 at a quoted interest rate of 7.5% per annum?

Mortgages and other loans are annuities, so the annuity formula is in common use.

Before you can proceed further, you need to know one more bit of institutional knowledge here: Mortgage providers—like banks—quote interest by just dividing the mortgage quote by 12, so the true monthly interest rate is $7.5\%/12 = 0.625\%$. (They do not compound; if they did, the monthly interest rate would be $(1+7.5\%)^{1/12} - 1 \approx 0.605\%$.)

Lenders quote interest rates using the same convention as banks.

A 30-year mortgage is an annuity with 360 equal payments with a discount rate of 0.625% per month. Its PV of \$500,000 is the amount that you are borrowing. You want to determine the fixed monthly cash flow that gives the annuity this value:

The mortgage payment can be determined by solving the annuity formula.

$$\$500{,}000 = \frac{C_1}{0.625\%} \cdot \left[1 - \frac{1}{(1+0.625\%)^{360}}\right] \approx C_1 \cdot 143.018$$

$$PV = \frac{C_1}{r} \cdot \left[1 - \frac{1}{(1+r)^T}\right]$$

Solving for the cash flow tells you that the monthly payment on your \$500,000 mortgage will be \$500,000/143.018 ≈ \$3,496.07 for 360 months, beginning next month (time 1).

Principal and Interest Components

Repayment and taxes are reasons to determine principal and interest components.

There are two reasons why you may want to determine how much of your \$3,496.07 payment should be called interest payment and how much should be called principal repayment. The first reason is that you need to know how much principal you owe if you want to repay your loan early. The second reason is that Uncle Sam allows mortgage borrowers to deduct the interest, but not the principal, from their tax bills.

Principal repayment is the sum left over after the interest payment from your monthly installment.

Here is how you can determine the split: In the first month, you pay $0.625\% \cdot \$500{,}000 = \$3{,}125$ in mortgage interest. Therefore, the principal repayment is \$3,496.07–\$3,125 = \$371.07 and the remaining principal is \$499,628.93. The following month, your interest payment is $0.625\% \cdot \$499{,}628.93 \approx \$3{,}122.68$ (note that your interest payment is now on the remaining principal), which leaves $\$3{,}496.07 - \$3{,}122.68 = \$373.39$ as your principal repayment, and \$499,255.54 as the remaining principal. And so on.

Q 3.19. Rental agreements are not much different from mortgages. For example, what would your rate of return be if you rented your \$500,000 warehouse for 10 years at a monthly lease payment of \$5,000? If you can earn 5% per annum elsewhere, would you rent out your warehouse?

Q 3.20. What is the monthly payment on a 15-year mortgage for every \$1,000 of mortgage at an effective interest rate of 6.168% per year (here, 0.5% per month)?

Application: A Level-Coupon Bond

Unlike zero-bonds, coupon bonds pay not only at the final time but also at interim points in time.

Let us exercise your newfound knowledge in a more elaborate example—this time with bonds. Recall that a bond is a financial claim sold by a firm or government. Bonds come in many varieties, but one useful classification is into coupon bonds and zero-bonds (short for **zero coupon bonds**). A **coupon bond** pays its holder cash at many different points in time, whereas a **zero-bond** pays only a single lump sum at the maturity of the bond with no interim coupon. Many coupon bonds promise to pay a regular coupon similar to the interest rate prevailing at the time of the bond's original sale, and then return a "principal amount" plus a final coupon at the end of the bond.

Bond naming conventions specify their promised payout patterns.

For example, think of a coupon bond that will pay \$1,500 each half-year (semi-annual payment is very common) for 5 years, plus an additional \$100,000 in 5 years. This payment pattern is so common that it has specially named features: A bond with coupon payments that remain the same for the life of the bond is called a **level-coupon bond**. These are the most common bonds today. The \$100,000 here would be called the **principal**, in contrast to the \$1,500 semiannual coupon. Level bonds are commonly named by just adding up all the coupon payments over 1 year (here, \$3,000) and dividing this sum of annual coupon payments by the principal. Thus, this particular bond would be called a "3% semiannual coupon bond" (\$3,000 coupon per year divided by the principal of \$100,000). Now, the "3% coupon bond" is just a naming convention for the bond with these specific cash flow patterns—it is not the interest rate that you would expect if you bought this bond. In Section 2.4, we called such name designations interest *quotes*, as distinct from interest *rates*.

➤ Compounding, Sect. 2.4, Pg.16.

Step 1: Write down the bond's payment stream.

What should this \$100,000, 3% semiannual level-coupon bond sell for today? First, you should write down the payment structure for a 3% semiannual coupon bond. This comes from its defined promised payout pattern:

Year	Due Date	Bond Payment	Year	Due Date	Bond Payment
0.5	Nov 2016	$1,500	3.0	May 2019	$1,500
1.0	May 2017	$1,500	3.5	Nov 2019	$1,500
1.5	Nov 2017	$1,500	4.0	May 2020	$1,500
2.0	May 2018	$1,500	4.5	Nov 2020	$1,500
2.5	Nov 2018	$1,500	5.0	May 2021	$101,500

Second, you need to determine the appropriate rates of return that apply to these cash flows. In this example, assume that the prevailing interest rate is 5% per annum. This translates into 2.47% for 6 months, 10.25% for 2 years, and so on.

Step 2: Find the appropriate cost of capital for each payment.

Year	Maturity	Discount Rate	Year	Maturity	Discount Rate
0.5	6 Months	2.47%	3.0	36 Months	15.76%
1.0	12 Months	5.00%	3.5	42 Months	18.62%
1.5	18 Months	7.59%	4.0	48 Months	21.55%
2.0	24 Months	10.25%	4.5	54 Months	24.55%
2.5	30 Months	12.97%	5.0	60 Months	27.63%

Third, compute the discount factors, which are just $1/(1+r_t) = 1/(1+r)^t$, and multiply each future payment by its discount factor. This will give you the present value (PV) of each bond payment. From there, you can compute the bond's overall value:

Step 3: Compute the discount factor $1/(1+r_t)$.

Year	Due Date	Bond Payment	Rate of Return	Discount Factor	Present Value
0.5	Nov 2016	$1,500	2.47%	0.9759	$1,463.85
1.0	May 2017	$1,500	5.00%	0.9524	$1,428.57
1.5	Nov 2017	$1,500	7.59%	0.9294	$1,394.14
2.0	May 2018	$1,500	10.25%	0.9070	$1,360.54
2.5	Nov 2018	$1,500	12.97%	0.8852	$1,327.76
3.0	May 2019	$1,500	15.76%	0.8638	$1,295.76
3.5	Nov 2019	$1,500	18.62%	0.8430	$1,264.53
4.0	May 2020	$1,500	21.55%	0.8277	$1,234.05
4.5	Nov 2020	$1,500	24.55%	0.8029	$1,204.31
5.0	May 2021	$101,500	27.63%	0.7835	$79,527.91
				Sum:	$91,501.42

You now know that you would expect this 3% semiannual level-coupon bond to be trading for $91,501.42 today in a perfect market. Because the current price of the bond is below its named final principal payment of $100,000, this bond would be said to trade at a **discount**. (The opposite would be a bond trading at a **premium**.)

Discount and premium bonds.

The bond's value can be calculated more quickly via the annuity formula. Let's work in half-year periods. You have 10 coupon cash flows, each $1,500, at a per-period interest rate of 2.47%. According to the formula, these 10 coupon payments are worth

Using the annuity formula to speed your calculations.

$$PV = C_1 \cdot \left\{ \frac{1-[1/(1+r)]^T}{r} \right\} = \$1{,}500 \cdot \left\{ \frac{1-[1/(1.0247)]^{10}}{2.47\%} \right\} \approx \$13{,}148.81$$

In addition, you have the $100,000 repayment of principal, which will occur in year 5 and is therefore worth

$$\text{PV} \;=\; \frac{\$100{,}000}{(1+5\%)^5} \;\approx\; \frac{\$100{,}000}{1+27.63\%} \;\approx\; \$78{,}352.62$$

$$\text{PV} \;=\; \frac{C_5}{(1+r)^5} \;=\; \frac{C_5}{(1+r_5)}$$

Together, the present values of the bond's cash flows again add up to $91,501.42.

The coupon rate is not the interest rate.

Important Reminder of Quotes versus Returns: Never confuse a bond designation with the interest it pays. The "3% semiannual coupon bond" is just a designation for the bond's payout pattern. The bond will not give you coupon payments equal to 1.5% of your $91,501.42 investment (which would be $1,372.52). The prevailing interest rate (cost of capital) has nothing to do with the quoted interest rate on the coupon bond. You could just as well determine the value of a 0% coupon bond, or a 10% coupon bond, given the prevailing 5% economy-wide interest rate. Having said all this, in the real world, many corporations choose coupon rates similar to the prevailing interest rate, so that at the moment of inception, the bond will be trading at neither a premium nor a discount. At least for this one brief at-issue instant, the coupon rate and the economy-wide interest rate may actually be fairly close. However, soon after issuance, market interest rates will move around, while the bond's payments will remain fixed, as designated by the bond's coupon name.

Q 3.21. You already learned that the value of one fixed future payment and the interest rate move in opposite directions (Page 25). What happens to the bond price of $91,501.42 in the level-coupon bond example if the economy-wide interest rates were to suddenly move from 5% per annum to 6% per annum?

Q 3.22. Assume that the 3% level-coupon bond discussed in this chapter has not just 5 years with 10 payments, but 20 years with 40 payments. Also, assume that the interest rate is not 5% per annum, but 10.25% per annum. What are the bond payment patterns and the bond's value?

Q 3.23. Check that the rates of return in the coupon bond valuation example on Page 47 are correct.

3.3 The Four Formulas Summarized

The growing annuity formula—it is used only rarely.

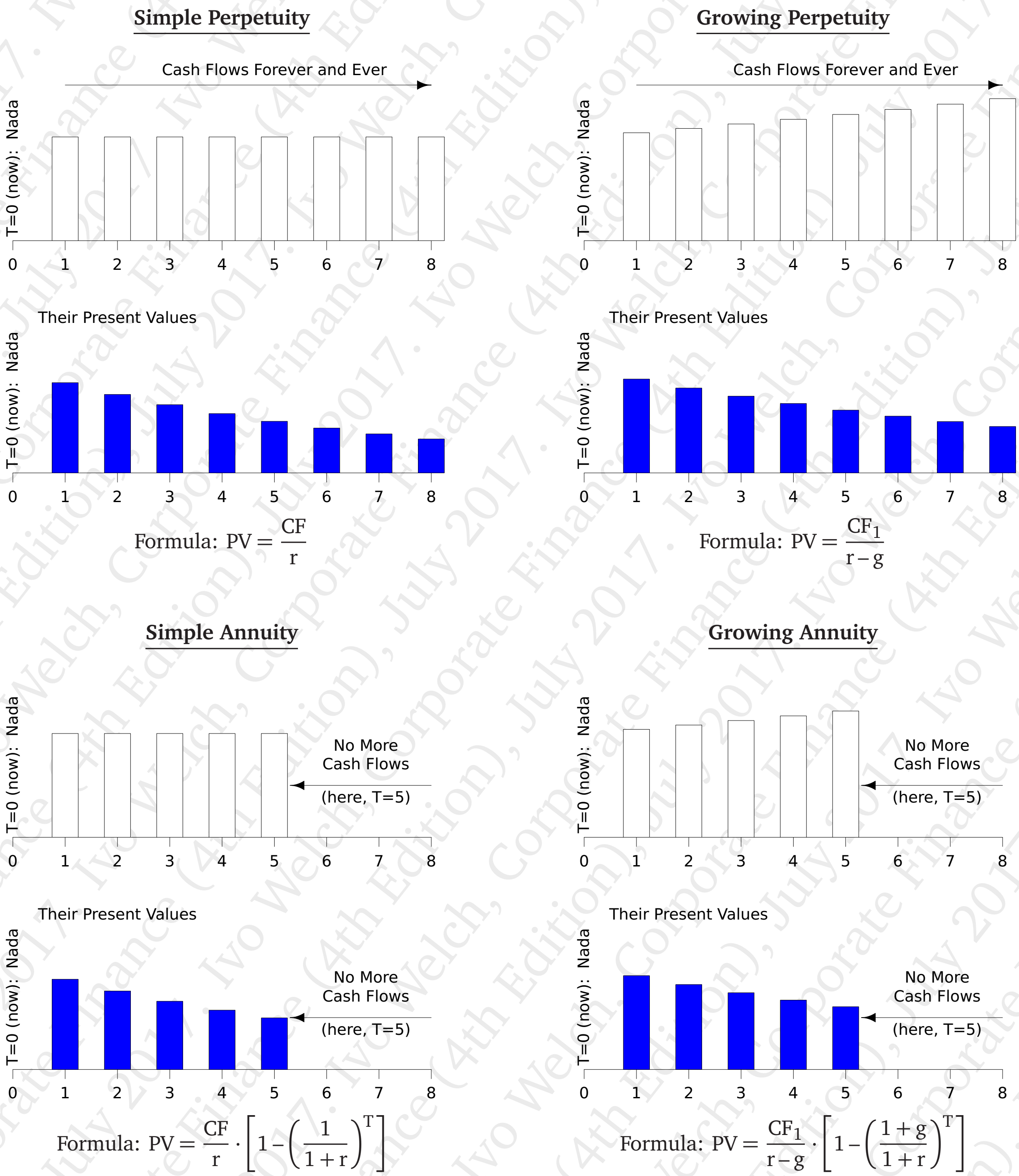
I am not a fan of memorization, but you must remember the growing perpetuity formula. You must also remember the annuity formula. They are used in many different contexts. There is also a **growing annuity** formula, which nobody remembers, but which you can look up if you need it:

$$\text{PV} = \frac{C_1}{r-g} \cdot \left[1 - \frac{(1+g)^T}{(1+r)^T}\right] \tag{3.3}$$

It is sometimes used in the context of pension cash flows, which tend to grow for a fixed number of time periods (T in the formula above) and then stop. However, even then it is not a necessary device. It is often more convenient and flexible to just work with the cash flows themselves within a spreadsheet.

A full summary.

Exhibit 3.3 summarizes the four special cash flows. The top graph shows the pattern of cash flows. For perpetuities, they go on forever. For annuities, they stop eventually. The bottom graph shows the present value of these cash flows. Naturally, these bars are shorter than those of their cash flows, which just means that there is a time value of money. The applicable formulas are below the graphs.



Simple Perpetuity — Formula: $PV = \frac{CF}{r}$

Growing Perpetuity — Formula: $PV = \frac{CF_1}{r-g}$

Simple Annuity — Formula: $PV = \frac{CF}{r} \cdot \left[1 - \left(\frac{1}{1+r}\right)^T\right]$

Growing Annuity — Formula: $PV = \frac{CF_1}{r-g} \cdot \left[1 - \left(\frac{1+g}{1+r}\right)^T\right]$

Exhibit 3.3: *The Four Payoff Streams and Their Present Values.*

Q 3.24. In many defined-contribution pension plans, the employer provides a fixed-percentage contribution to the employee's retirement. Assume that the employer must contribute \$4,000 per annum beginning next year (time 1), growing annually with the inflation rate of 2% per year. What is the present value of the pension cost of hiring a 25-year-old who will stay with the company for 35 years? Assume a discount rate of 8% per year. Note: Please look up the growing annuity formula to solve this problem.

Summary

This chapter covered the following major points:

- Exhibit 3.3 summarizes the four special cash flows and their quick valuation formulas.
- The PV of a simple perpetuity, which is a stream of constant cash flows that begin next period and that are to be discounted at the same annual cost of capital forever, is

$$PV = \frac{C_1}{r}$$

- The PV of a growing perpetuity—with constant growth g, cash flows C beginning next year (time 1), and constant per-period interest rate r—is

$$PV = \frac{C_1}{r - g}$$

- Stocks are often valued through an application of the growing perpetuity formula, called the Gordon dividend growth model.
- The PV of an annuity—T periods of constant C cash flows (beginning next year) and constant per-period interest rate r—is

$$PV = C_1 \cdot \left\{ \frac{1 - [1/(1+r)]^T}{r} \right\}$$

- Fixed-rate mortgages are annuities. Interest rate quoted on such bonds are computed with the annuity formula.

Preview of the Chapter Appendix in the Companion

The appendix to this chapter in the companion (not here)

- shows how the annuity and perpetuity formulas can be derived.
- explains "equivalent annual costs" (which you already briefly encountered in Question 2.39). These allow you to compare projects with different rental periods—such as an 8-year lease that charges \$1,000 per year and a 10-year lease that charges \$900 per year.

➤ Equivalent Annual Cost, Question 2.39, Pg.29.

Keywords

Answers

Q 3.1 C_1/r. The first cash flow occurs next period, not this period.

Q 3.2 $PV = C_1/r = \$5/0.005 = \$1{,}000$

Q 3.3 The interest rate is $1.1268^{(1/12)} - 1 \approx 1\%$ per month. Thus, $PV = C_1/r \approx \$15/0.01 \approx \$1{,}500$.

Q 3.4 Rearrange $P = C_1/r$ into $r = C_1/P = \$2/\$40 = 5\%$. At a 5% interest rate, you are indifferent. If the interest rate is above 5%, the immediate one-time payment is better, because future cash flows are less valuable. If the interest rate is below 5%, the perpetuity payment is better, because future cash flows are more valuable.

Q 3.5 $PV = \$2{,}000/4\% = \$50{,}000$

Q 3.6 $C_1/(r - g)$.

Q 3.7 You get $C_0 = \$5$ today, and next month you will receive a payment of $C_1 = (1+g) \cdot C_0 = 1.001 \cdot \$5 = \$5.005$. The growing perpetuity is worth $PV = C_1/(r-g) = \$5.005/(0.5\% - 0.1\%) = \$1{,}251.25$. The total value is \$1,256.25.

Q 3.8 This is a nonsensical question, because the value would be infinite if $g \geq r$.

Q 3.9 Your earnings will be as follows:

Yr	Earn Was	Grwth Rate	Earn Is	Yearly Dscnt R	Comp. Dscnt R	Present Value
0-1	\$100	20%	\$120.0	10%	10%	\$109.09
1-2	\$120	20%	\$144.0	10%	21%	\$119.01
2-3	\$144	20%	\$172.8	10%	33.1%	\$129.83
3-4	\$172.8	10%	\$190.1	9%	45.1%	\$131.02
4-5	\$190.1	10%	\$209.1	9%	58.1%	\$132.22
5-6	\$209.1	10%	\$230.0	9%	72.4%	\$133.43
6-7	\$230.0	5%	\$241.5	8%	86.2%	\$129.73
7-8	\$241.5	5%	\$253.6	8%	...	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮
				PV (Yr 0 through 7):		\$884.33
Subseq Perp in Yr 7:	\$8,452.38			PV Perp		\$4,540.44
				Total Project PV:		\$5,424.76

Standing in year 7, the growing perpetuity with cash flows of \$253.6 (projected for year 8) is worth $\$253.6/(8\% - 5\%) \approx \$8{,}452$. (If you want, you could round this number to \$8,500. If you are concerned about my rounding too aggressively, you have lost perspective—there is no firm in this world for which you can forecast the value in eight years with this much accuracy!) The \$8.453 billion is our assumption of what we will be able to sell the firm for at the end of year 7. It is our **terminal value**. All cash flows in year 7 (both the \$241.50 that we will still take home in year 7, plus the \$8,452) must then be discounted by 86.2%. Therefore, the PV is about \$884 million from cash flows that you computed explicitly (years 1 through 7), plus $\$8,452) \approx \$4.540$ billion from the cash flows that is the terminal value stand-in for all cash flows from year 8 to infinity. In sum, the estimate of this firm's present value is around \$5.4 billion. (Note: You could also calculate a terminal value in year 6 (for year 7 and beyond), and reach the same answer.)

Q 3.10 $\$1.5 \text{ million}/(14\% - 2\%) = \$12.5$ million.

Q 3.11 The immediate dividend would be worth \$1.5 million. In addition, you now have a growing perpetuity that starts with a payment of \$1.530 million. Therefore, the PV would be $\$1.500 + \$1.530/12\% = \$14.250$ million. Alternatively, you could multiply the \$12.5 million from your answer to Question 3.10 by $(1 + 14\%)$.

Q 3.12 First work out what the value would be if you stood at 1 month. The interest rate is $(1+9\%)^{1/12} - 1 \approx 0.7207\%$ per month, and $1.007207^3 - 1 \approx 2.1778\%$ per quarter. Thus, in 1 month, you will be entitled to a dividend stream of $\$5.025/(2.1778\% - 0.5\%) \approx \$299.50$. In addition, you get the \$5 for a total of \$304.50. Because this is your value in 1 month, discount \$304.50 at a 0.7207% interest rate to \$302.32 today.

Q 3.13 $g = r - E/P = 12\% - \$5/\$100 = 7\%$ per annum

Q 3.14 Compare the annuity and perpetuity formulas. The difference between them is the $1 - 1/(1+r)^t$ term. To be three-quarters of the value, this term has to be 3/4. So you must solve $1 - 1/(1+r)^t = 3/4$, or $1/(1+r)^t = 1 - 3/4 = 1/4$ or $(1+r)^t = 4$. Taking logs, $t = \log(4)/\log(1+r)$. In the main question, r was 5%, so $t = \log(4)/\log(1.05) \approx 28.41$ years. More generally, to reach a given fraction f of value, $t = \log[1/(1-f)]/\log(1+r)$. Think of this number of years as helping you judge the quality of the infinite-period approximation in the real world. If it is more realistic that you have fewer than 30 years of cash flows instead of an infinite stream, then the perpetuity formula may not be a great approximation of value when the interest rate is 5%.

Q 3.15 The annuity formula is $C_1 \cdot \left(\{1 - [1/(1+r)]^T\}/r\right)$.

Q 3.16 Your 360-month annuity is worth

$$C_1 \cdot \left\{ \frac{1 - [1/(1+r)]^T}{r} \right\} = \$5 \cdot \left\{ \frac{1 - [1/(1+0.005)]^{360}}{0.005} \right\}$$

$$\approx \$5 \cdot \left\{ \frac{1 - 0.166}{0.005} \right\} \approx \$833.96$$

Q 3.17 The unknown interest rate is r. For each 1e that you have lent out, if you have a choice of 0.04e payment forever or a 0.05e payment for 41 months, you would compare 0.04e/r to $(0.05e/r)\cdot[1-1/(1+r)^{41}]$. If $r < 0.0400352$, you would prefer the perpetuity. Otherwise, you prefer the annuity.

Q 3.18 For one year, the 300 bezants paid once at year-end are worth $300b/1.02^{12} \approx 236.55$ bezants today. Now for the quarterly payment schedule: The quarterly interest rate is $1.02^3 - 1 \approx 6.12\%$. Therefore, the 4-"quartity" is worth $75b/0.0612\cdot[1-1/1.0612^4] \approx 75b/1.0612^1 + 75b/1.0612^2 + 75b/1.0612^3 + 75b/1.0612^4 \approx 259.17$ bezants. The soldier would have lost 22.62 bezants in present value, which is 8.73% of what he was promised. (The same loss of $236.55/259.17 - 1 \approx 8.73\%$ would apply to longer periods.)

Q 3.19 To find the implicit cost of capital of the lease, you need to solve

$$\$500{,}000 = \frac{\$5{,}000}{r}\cdot\left[1-\frac{1}{(1+r)^{120}}\right]$$

The solution is $r \approx 0.31142\%$ per month, or 3.8% per annum. This is the implied rate of return if you buy the warehouse and then rent it out. You would be better off earning 5% elsewhere.

Q 3.20 For \$1,000 of mortgage, solve for C_1 in

$$\begin{aligned} PV &= C_1\cdot\left\{\frac{1-[1/(1+r)]^T}{r}\right\} \\ \$1{,}000 &= C_1\cdot\left\{\frac{1-[1/(1.005)]^{15\cdot 12=180}}{0.005}\right\} \approx C_1\cdot 118.504 \\ \iff & \quad C_1 \approx \$8.44 \end{aligned}$$

In other words, for every \$1,000 of loan, you have to pay \$8.44 per month. For other loan amounts, just rescale the amounts.

Q 3.21 The semiannual interest rate would now increase from 2.47% to

$$r = \sqrt[2]{1+6\%} - 1 = \sqrt{1.06} - 1 \approx 2.9563\%$$

To get the bond's new present value, reuse the annuity formula

$$\begin{aligned} PV &= C_1\cdot\left\{\frac{1-[1/(1+r)]^T}{r}\right\} + \frac{C_t}{1+r_t} \\ &\approx \$1{,}500\cdot\left\{\frac{1-[1/(1+2.9563\%)]^{10}}{2.9563\%}\right\} + \frac{\$100{,}000}{(1+2.9563\%)^{10}} \\ &\approx \$12{,}823.89 + \$74{,}725.82 \\ &\approx \$87{,}549.70 \end{aligned}$$

This bond would have lost \$3,951.72, or 4.3% of the original investment.

Q 3.22 The interest rate is 5% per half-year. Be my guest if you want to add 40 terms. I prefer the annuity method. The coupons are worth

$$\begin{aligned} PV(\text{Coupons}) &= C_1\cdot\left\{\frac{1-[1/(1+r)]^T}{r}\right\} \\ &= \$1{,}500\cdot\left\{\frac{1-[1/(1.05)]^{40}}{0.05}\right\} \approx \$25{,}738.63 \end{aligned}$$

The final payment is worth PV(Principal Repayment) $= \$100{,}000/(1.05)^{40} \approx \$14{,}204.57$. Therefore, the bond is worth about \$39,943.20 today.

Q 3.23 For 6 months, $(1+2.47\%)^2 - 1 \approx 5\%$. Now, define 6 months to be 1 period. Then, for t 6-month periods, you can simply compute an interest rate of $(1+2.47\%)^t - 1$. For example, the 30 months interest rate is $1.0247^5 - 1 \approx 12.97\%$.

Q 3.24 The solution is $\$4{,}000/(0.08-0.02)\cdot\left[1-\frac{1.02^{35}}{1.08^{35}}\right] \approx \$57{,}649.23$.

End of Chapter Problems

Q 3.25. A tall Starbucks coffee costs \$1.85 in 2017. If the bank's quoted interest rate is 6% per annum, compounded daily, and if the Starbucks price never changed, what would an endless, inheritable free subscription to one Starbucks coffee per day be worth today?

Q 3.26. If you could pay for your mortgage forever, how much would you have to pay per month for a \$1,000,000 mortgage, at a 6.5% annual interest rate? Work out the answer (a) if the 6.5% is a bank APR quote and (b) if the 6.5% is a true effective annual rate of return.

Q 3.27. What is the PV of a perpetuity paying \$30 each month, beginning *next* month, if the annual interest rate is a constant effective 12.68% per year?

Q 3.28. What is the prevailing interest rate if a perpetual bond were to pay \$100,000 per year *beginning next year* and costs \$1,000,000 today?

Q 3.29. What is the prevailing interest rate if a perpetual bond were to pay \$100,000 per year *beginning next year* (time 1) and payments grow with the inflation rate at about 2% per year, assuming the bond costs \$1,000,000 today?

Q 3.30. A tall Starbucks coffee costs \$1.85 a day. If the bank's quoted interest rate is 6% per annum and coffee prices increased at a 3% annual rate of inflation, what would an endless, inheritable free subscription to one Starbucks coffee per day be worth today?

Q 3.31. Economically, why does the growth rate of cash flows have to be less than the discount rate?

Q 3.32. Your firm just finished the year, in which it had cash earnings of \$400. You forecast your firm to have a quick growth phase from year 0 to year 5, in which it grows at a rate of 40% per annum. Your firm's growth then slows down to 20% per annum between year 5 to year 10. Finally, beginning in year 11, you expect the firm to settle into its long-term annual growth rate of 2% . You also expect your cost of capital to be 15% over the first 5 years, then 10% over the next 5 years, and 8% thereafter. What do you think your firm is worth today? (Advice: Use a computer spreadsheet program.)

Q 3.33. A stock pays an annual dividend of \$2. The dividend is expected to increase by 2% per year (roughly the inflation rate) forever. The price of the stock is \$40 per share. At what cost of capital is this stock priced?

Q 3.34. A tall Starbucks coffee costs \$1.85 a day. If the bank's quoted interest rate is 6% per annum, compounded daily, and if the Starbucks price never changed, what would a lifetime free subscription to one Starbucks coffee per day be worth today, assuming you will live for 50 more years? What should it be worth to you to be able to bequeath or sell it upon your departure?

Q 3.35. What maximum price would you pay for a standard 8% level-coupon bond (with semiannual payments and a face value of \$1,000) that has 10 years to maturity if the prevailing discount rate (your cost of capital) is an effective 10% per annum?

Q 3.36. If you have to pay off an effective 6.5% loan within the standard 30 years, then what are the per-month payments for the \$1,000,000 mortgage? As in Question 3.26, consider both an effective 6.5% interest rate per year, and a bank quote of 6.5% (APR) per year.

Q 3.37. Structure a mortgage bond for \$150,000 so that its monthly payments are \$1,000. The prevailing interest rate is quoted at 6% (APR) per year.

Q 3.38. (Advanced) You are valuing a firm with a "pro forma" (i.e., with your forward projection of what the cash flows will be). The firm just had cash flows of \$1,000,000 today. This year, it will be growing by a rate of 20% per annum. That is, at the end of year 1, the firm will have a cash flow of \$1.2 million. In each of the following years, the difference between the growth rate and the inflation rate of 2% will (forever) halve. Thus, from year 1 to year 2, the growth rate will be $2\% + (20\% - 2\%)/2 = 11\%$, so the next cash flow will be $\$1,200 \cdot 1.11 = \$1,332$ at the end of year 2. The following year, the growth rate will be $2\% + (11\% - 2\%)/2 = 6.5\%$, and the cash flow will be \$1,419 at the end of year 3. The growth will be less every year, but it will never reach the inflation rate of 2% perfectly. Next, assume that the appropriate discount rate for a firm this risky is a constant 12%/year. It is not time-varying. (The discount rate on the \$1.2 million cash flow is 12%. The total discount rate for the \$1,332 cash flow in year 2 is thus 25.4%, and so on.) What do you believe the value of this firm to be? (Hint: It is common in pro formas to project forward for a given number of years, say, 5 to 10 years, and then to assume that the firm will be sold for a terminal value, assuming that it has steady growth.)

4

A First Encounter with Capital-Budgeting Rules

The Internal Rate of Return, and More

This chapter elaborates on the ideas presented in the previous chapter. We still remain in a world of constant interest rates, perfect foresight, and perfect markets. Let's look a little more closely at capital budgeting—the possible decision rules that can tell you whether to accept or reject projects. You already know the answer to the mystery, though: NPV is best. Still, there is one very important alternative to NPV: the internal rate of return, which generalizes the rate of return concept and can often give you good recommendations, too. You will see how these approaches fit together.

One caveat—although you already know the concept of NPV, and although you will learn more about capital-budgeting rules in this chapter, most of the interesting and difficult issues in NPV's application are delayed until Chapter 13 (i.e., after we have covered uncertainty and imperfect markets).

4.1 Net Present Value

Recap: NPV is the most important building block in finance. You must be able to compute it in your sleep.

You have already learned how to use NPV in our perfect world. You first translate cash flows at different points in time into the same units—dollars today—before they can be compared or added. This translation between future values and present values—and its variant, net present value—ranks among the most essential concepts in finance.

A "free money" interpretation of NPV.

But why is NPV the right rule to use? The reason is that, at least in our perfect world with perfect information, a positive-NPV project is the equivalent of free money. For example, if you can borrow or lend money at 8% anywhere today, and you have an investment opportunity that costs \$1 and yields \$1.09, you can immediately contract to receive \$0.01 next year *for free*. (If you wish, discount it back to today, so you can consume it today.) Your rejecting this project would make no sense. Similarly, if you can sell someone an investment opportunity for \$1, which yields only \$1.07 next year, you can again earn \$0.01 *for free*. Again, rejecting this project would make no sense. (Remember that in our perfect world, you can buy or sell projects at will.) Only zero-NPV projects (\$1 cost for \$1.08 payoff) do not allow you to earn free money. Of course, I am using this argument not to show you how to get rich, but to convince you that the NPV rule makes sense and any rule that comes to a different conclusion does not.

IMPORTANT

In a perfect world, if you have all the right inputs to NPV, no other rule can make better decisions. Thus, it is the appropriate decision benchmark—and no other rule can beat it. This also means that information other than the NPV is redundant.

Positive-NPV projects are scarce.

In our perfect world with no uncertainty, logic dictates that positive-NPV projects should be scarce. If they were not scarce and could be found at will, you could get rich too easily. But not just you—everyone with access would want to take on cartloads of them. In real life, the economy would adjust. The "run" on positive-NPV projects would continue until the economy-wide appropriate rate of return (cost of capital) would be bid up to the level where positive-NPV projects are scarce again.

In the real world, NPV is very important, but other measures can provide useful information, too.

As you will find out in later chapters, despite its conceptual simplicity, the application of NPV in the real world is often surprisingly difficult. The primary reason is that you rarely know cash flows and discount factors perfectly. This means that you must estimate them. The secondary reason is that the world is never 100% perfect—that there are absolutely zero taxes, no transaction costs, no disagreements, and infinitely many buyers and sellers. Nevertheless, even in an imperfect market, NPV remains the most important benchmark. Yet other rules may then provide some additional useful information and potentially recommend alternative project choices.

Separating Investment and Consumption Decisions: Does Project Value Depend on When You Need Cash?

Who owns a project is not important in a perfect capital market.

In our perfect world, when you choose between NPV projects, should you let your preferences about the timing of cash flows influence your decisions? Perhaps you don't want to incur an upfront expense; perhaps you want money today; perhaps you want to defer your consumption and save for the future. Aren't these important factors in making your decision as to which project to choose? The answer is no—the value of any project is its net present value, regardless of your preferences.

The capital markets allow you to shift money across time periods—better than your investment projects can.

In a perfect market, how much cash the owner has also does not matter. Let me explain why. You already know about the time value of money, the fact that cash today is worth more than cash tomorrow. If you do not agree—that is, if you value money tomorrow more than you value money today—then just give it to me until you need it back. I can deposit it in my bank account to earn interest in the interim. In a perfect capital market, you can, of course, do better: You can always shift money between time periods at an "exchange rate" that reflects the time value of money.

Example: Even an "eager" consumer should take the positive-NPV project.

It is this shifting-at-will that explains why ownership does not matter. Assume that you have \$150 cash on hand and that you have exclusive access to a project that costs \$100, and returns \$200 next year. The appropriate interest rate (cost of capital) is 10%—but you *really* want to live it up today. How much can you consume? And, would you take the project? Here is the NPV prescription in a perfect market:

- Sell the project in the competitive market for its NPV:

$$-\$100 + \left(\frac{\$200}{1+10\%}\right) = -\$100 + \left(\frac{\$200}{1.10}\right) \approx \$81.82$$

- Spend the \$150 + (\$181.82 − \$100) ≈ \$231.82 today. You will be better off taking the project than consuming just your \$150 cash at hand.

Now, assume that you are Austin Powers, the frozen spy, who cannot consume this year. How much will you be able to consume next year? And, would you take the project? NPV tells you what you should do:

A "sleeper" consumer should also take the positive-NPV project.

- Sell the project in the competitive market for

$$-\$100 + \frac{\$200}{1 + 10\%} \approx \$81.82$$

- Put the \$81.82 into the bank for 10% today. Get \$90 next year.
- Also put your \$150 into the bank at 10% interest to receive \$165 next year.
- Next year, consume $\$90 + \$165 = \$255$.

Of course, an equally simple solution would be to take the project and just put your remaining \$50 into a bank account.

The moral of the story: Consumption and investment decisions can be separated in a perfect capital market.

The point of this argument is simple: Regardless of when you need or want cash (your consumption decision), you are better off taking all positive-NPV projects (your investment decision), and then using the capital markets to shift consumption to when you want it. It makes no sense to let your *consumption decisions* influence your *investment decisions*. This is called the **separation of decisions**: You can make investment decisions without concern for your consumption preferences. (However, this separation of investment and consumption decisions does not always hold in imperfect markets, in which you can face different borrowing and lending interest rates. You might take more projects if you have more cash.)

➤ Imperfect markets, lack of separation, Sect. 11.1, Pg.245.

Investing for the long-run is the same as investing for the short-run.

Here is a simple application of this simplest of insights. After they have lost their clients' money, many brokers like to muddle the truth by claiming that they invested their clients' money for the long term and not for the short term. This excuse presumes that, compared with short-term investments, long-term investments do worse in the short run but better in the long run. However, this makes no sense. See, if your broker had really known that the short-term asset would be better in the short-term, he should have bought it first, realized its higher rate of return over the short-run for you, and then bought you more of the long-term asset (which would now have been relatively cheaper). The fact is that no matter whether an investor needs money sooner or later, a broker should always buy the highest NPV investments. In the end, this is what is best for all clients.

Errors: Mistakes in Cash Flow versus Cost of Capital Estimates

In the real world, it is often impossible to get the NPV inputs perfectly correct.

Although it would be better to get everything perfect, it is often impossible to come up with perfect cash flow forecasts and appropriate interest rate estimates. Everyone makes errors when outcomes in the world are uncertain. How bad are estimation mistakes? Is it worse to commit an error in estimating cash flows or in estimating the cost of capital? To answer these questions, we will do a simple form of **scenario analysis**, in which we consider a very simple project to learn how changes in our estimates matter to the ultimate present value. Scenario analysis is also essential for managers, who need to learn how sensitive their estimated value is to reasonable alternative possible outcomes. Therefore, this method is also called a **sensitivity analysis**. (It becomes even more important when you work with real options in Chapter 13.)

The benchmark case: A short-term project, correctly valued.

Short-term projects: Assume that your project will pay off \$200 next year, and the proper interest rate for such projects is 8%. Thus, the correct project present value is

$$\text{Correct PV} = \frac{\$200}{1 + 8\%} \approx \$185.19$$

Committing an error in cash flow estimation.

If you make a 10% error in your cash flow, mistakenly believing it to return $220, you will compute the present value to be

$$\text{Cash Flow Error PV} = \frac{\$220}{1+8\%} \approx \$203.70$$

The difference between $203.70 and $185.19 is a 10% error in your present value.

Committing an error in interest rate estimation.

In contrast, if you make a 10% error in your cost of capital (interest rate), mistakenly believing it to require a cost of capital (expected interest rate) of 8.8% rather than 8%, you will compute the present value to be

$$\text{Discount Rate Error PV} = \frac{\$200}{1+8.8\%} \approx \$183.82$$

The difference between $183.82 and $185.19 is less than $2, which is an error of about 1%. In sum, discount rate errors tend to be less harmful than cash flow errors for short-run projects.

A long-term project, correctly valued and incorrectly valued.

Long-term projects: Now take the same example but assume the cash flow will occur in 30 years. The correct present value is now

$$\text{Correct PV} = \frac{\$200}{(1+8\%)^{30}} = \frac{\$200}{1.08^{30}} \approx \$19.88$$

The 10% "cash flow error" present value is

$$\text{Cash Flow Error PV} = \frac{\$220}{(1+8\%)^{30}} = \frac{\$220}{1.08^{30}} \approx \$21.86$$

and the 10% "interest rate error" present value is

$$\text{Discount Rate Error PV} = \frac{\$200}{(1+8.8\%)^{30}} = \frac{\$200}{(1.088\%)^{30}} \approx \$15.93$$

Both cash flow errors and cost of capital errors are important for long-term projects.

This calculation shows that cash flow estimation errors and interest rate estimation errors are now both important. For longer-term projects, estimating the correct interest rate becomes relatively more important. Yet, though correct, this argument may be misleading. Estimating cash flows 30 years into the future often seems more like voodoo than science. Your uncertainty usually explodes over longer horizons. In contrast, your uncertainty about the long-term cost of capital tends to grow very little with the time horizon—you might even be able to ask your investors today what they demand as an appropriate cost of capital for a 30-year investment! Of course, as difficult as cash flow estimation may be, you have no alternative. You simply must try to do your best at forecasting.

IMPORTANT

- For short-term projects, errors in estimating correct interest rates are less problematic in computing NPV than are errors in estimating future cash flows.
- For long-term projects, errors in estimating correct interest rates and errors in estimating future cash flows are both problematic in computing NPV. Nevertheless, in reality, you will tend to find it more difficult to estimate far-away future cash flows (and thus you will face more errors) than to estimate the appropriate discount rate demanded by investors today for far-away cash flows.

Q 4.1. What is the main assumption that allows you to consider investment (project) choices without regard to when you need wealth (or how much money you currently have at hand)?

Q 4.2. You have \$500 and really, really want to go to the Superbowl tonight (which would consume all your cash). You cannot wait until your project completes: This project would cost \$400 and offer a rate of return of 15%, although equivalent interest rates are only 10%. If the market is perfect, what should you do?

4.2 The Internal Rate of Return (IRR)

IRR ≈ NPV.

There is another common capital-budgeting method, which often leads to the same recommendations as the NPV rule. This method is useful because it does so through a different route and often provides good intuition about the project.

Our new capital-budgeting method compares the project's rate of return to the prevailing rate of return.

Let's assume that you have a project with cash flows that translate into a rate of return of 20% (e.g., \$100 investment, \$120 payoff), and the prevailing discount rate is 10%. Because your project's rate of return of 20% is greater than the prevailing discount rate of 10%, you should intuitively realize that it is a good one. It is also a positive-NPV project—in the example, $-\$100 + \$120/1.1 \approx \$9.10$.

We need a "sort-of average rate of return" that is implicit in future cash flows.

There is only one problem: How would you compute the rate of return on a project or bond that has many different payments? For example, say the investment costs \$100,000 and pays off \$5,000 in one year, \$10,000 in two years, and \$120,000 in three years. What is the rate of return of this project? Think about it. The rate of return formula works only if you have exactly one inflow and one outflow. This is not the case here. What you need now is a "kind of rate of return" (a "statistic") that can take many inflows and outflows and provide something similar to a rate of return. If there is only one of each, it should give the same number as the simple rate of return.

The IRR is this characteristic that describes multiple cash flows.

Such a measure exists. It is called the internal rate of return (IRR). The word "internal" is an indicator that the rate is intrinsic to your project, depending only on its cash flows.

IMPORTANT

- The **internal rate of return** (**IRR**) is the quantity, which, given a complete set of cash flows, solves the NPV formula set to zero,

$$0 = C_0 + \frac{C_1}{1 + \text{IRR}} + \frac{C_2}{(1 + \text{IRR})^2} + \frac{C_3}{(1 + \text{IRR})^3} + \cdots \quad (4.1)$$

- If there are only two cash flows, the IRR is the rate of return. Thus, the IRR generalizes the concept of rate of return to multiple cash flows. Every rate of return is an IRR, but the reverse is not the case.
- The IRR itself is best thought of as a characteristic of project cash flows.

YTM is the same as IRR.

The internal rate of return is such a common statistic in the context of bonds that it has acquired a second name: the **yield-to-maturity** (**YTM**). There is no difference between the IRR and the YTM.

IRR generalizes rates of return: A simple project's rate of return is its IRR.

Let's illustrate the IRR. First, if there is only one inflow and one outflow, the IRR is the simple rate of return. For example, if a simple project costs \$100 today and pays \$130 next year, the IRR is obtained by solving

$$-\$100 + \frac{\$130}{1+\text{IRR}} = 0 \iff \text{IRR} = \frac{\$130 - \$100}{\$100} = 30\%$$
$$C_0 + \frac{C_1}{1+\text{IRR}} = 0 \iff \text{IRR} = \frac{C_1 - C_0}{C_0}$$

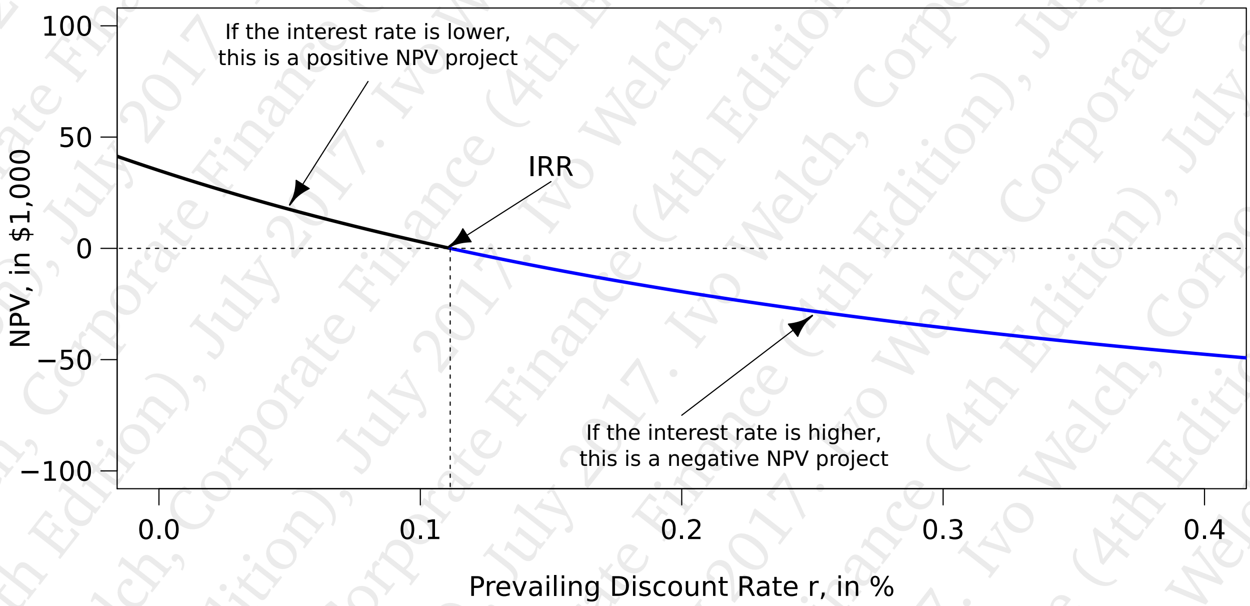


Exhibit 4.1: *NPV as a Function of the Interest Rate.* This figure draws the NPV for a project that costs $100,000 and pays $5,000, $10,000, and $120,000 in consecutive years. The IRR is the x-coordinate where the NPV function intersects the zero-line.

Here is an iteration method that shows how you can solve the IRR equation yourself.

Now consider an example where a simple rate of return won't work: What number would best characterize the implied rate of return for a project that costs $100,000 today and that will yield $5,000, $10,000, and $120,000? You cannot compute a simple rate of return with four cash flows. Exhibit 4.1 shows you the NPV of this project as a function of the prevailing interest rate. If the discount rate is very low, then the NPV is positive. IRR is the interest rate that makes the NPV exactly equal to zero. In this case, this means that you should solve

$$0 = -\$100{,}000 + \frac{\$5{,}000}{1+\text{IRR}} + \frac{\$10{,}000}{(1+\text{IRR})^2} + \frac{\$120{,}000}{(1+\text{IRR})^3}$$
$$0 = C_0 + \frac{C_1}{1+\text{IRR}} + \frac{C_2}{(1+\text{IRR})^2} + \frac{C_3}{(1+\text{IRR})^3}$$

What is the discount rate that sets the NPV equation to zero? If you do not want to draw the full figure to find out where your NPV function crosses the zero axis, then you can try to solve such equations by trial and error. Start with two values, say, 5% and 10%.

$$-\$100{,}000 + \frac{\$5{,}000}{1+10\%} + \frac{\$10{,}000}{(1+10\%)^2} + \frac{\$120{,}000}{(1+10\%)^3} \approx \$2{,}968$$

To reach zero, you need to slide above 10%. Try 11% and 12%,

$$-\$100,000 + \frac{\$5,000}{1+11\%} + \frac{\$10,000}{(1+11\%)^2} + \frac{\$120,000}{(1+11\%)^3} \approx \$364$$

$$-\$100,000 + \frac{\$5,000}{1+12\%} + \frac{\$10,000}{(1+12\%)^2} + \frac{\$120,000}{(1+12\%)^3} \approx -\$2,150$$

Okay, the solution is closer to 11%. A lucky trial reveals

$$-\$100,000 + \frac{\$5,000}{1+11.14252\%} + \frac{\$10,000}{(1+11.14252\%)^2} + \frac{\$120,000}{(1+11.14252\%)^3} \approx 0$$

Therefore, the answer is that this project has an IRR of about 11.14%. You can think of the internal rate of return as a sort-of average rate of return embedded in the project's cash flows.

Spreadsheets make it easy to find the IRR fast.

There is no easy general formula to compute the IRR if you are dealing with more than three cash flows. However, an automated function to compute an IRR is built into modern computer spreadsheets and usually precludes the need to solve algebraic equations by trial-and-error. Exhibit 4.2 (row 1) shows how you would find the IRR for this project in a spreadsheet.

	A	B	C	D	E	
1	–100,000	5,000	10,000	120,000	=IRR(A1:D1)	← E1 will become 11.142%
2	100,000	–5,000	–10,000	–120,000	=IRR(A2:D2)	← E2 will become 11.142%
3	–1,000	600	600	=IRR(A3:C3)		← D3 will become 13%

Exhibit 4.2: *IRR Calculations in a Computer Spreadsheet (Excel or OpenOffice).* The first line is the project worked out in the text. The second line shows that the negative of the project has the same IRR. The third line is just another example that you can check for yourself.

Multiplying all cash flows by the same factor does not change the IRR.

Note that the negative cash flow pattern in row 2 of Exhibit 4.2 has the same IRR. That is, receiving an inflow of \$100,000 followed by *payments* of \$5,000, \$10,000, and \$120,000 also has an 11.14252% internal rate of return. You can see that this must be the case if you look back at the IRR formula. Any multiplicative factor (like –1) simply cancels out and therefore has no impact on the solution.

$$\begin{aligned}
0 &= \text{Factor} \cdot C_0 + \frac{\text{Factor} \cdot C_1}{1+\text{IRR}} + \frac{\text{Factor} \cdot C_2}{(1+\text{IRR})^2} + \frac{\text{Factor} \cdot C_3}{(1+\text{IRR})^3} + \cdots \\
&= \text{Factor} \cdot \left[C_0 + \frac{C_1}{1+\text{IRR}} + \frac{C_2}{(1+\text{IRR})^2} + \frac{C_3}{(1+\text{IRR})^3} + \cdots \right] \\
&= C_0 + \frac{C_1}{1+\text{IRR}} + \frac{C_2}{(1+\text{IRR})^2} + \frac{C_3}{(1+\text{IRR})^3} + \cdots
\end{aligned}$$

Q 4.3. From memory, write down the equation that defines IRR.

Q 4.4. What is the IRR of a project that costs \$1,000 now and produces \$1,000 next year?

Q 4.5. What is the IRR of a project that costs \$1,000 now and produces \$500 next year and \$500 the year after?

Q 4.6. What is the IRR of a project that costs $1,000 now and produces $600 next year and $600 the year after?

Q 4.7. What is the IRR of a project that costs $1,000 now and produces $900 next year and $900 the year after?

Q 4.8. A project has cash flows of –$100, $55, and $70 in consecutive years. Use a spreadsheet to find the IRR.

Q 4.9. What is the YTM of an x% annual level-coupon bond whose price is equal to the principal paid at maturity? For example, take a 5-year bond that costs $1,000 today, pays 5% coupon ($50 per year) for 4 years, and finally repays $1,050 in principal and interest in year 5.

Q 4.10. What is the YTM of a 5-year zero-bond that costs $1,000 today and promises to pay $1,611?

Q 4.11. Compute the yield-to-maturity of a two-year bond that costs $25,000 today and pays $1,000 at the end of each of the 2 years. At the end of the second year, it also repays $25,000. What is the bond's YTM?

Projects with Multiple or No IRRs

Here is an example of a project with two IRRs.

When projects have many positive and many negative cash flows, they can often have multiple internal rates of return. For example, take a project that costs $100,000, pays $205,000, and has environmental cleanup costs of $102,000. Exhibit 4.3 shows that this project has two internal rates of return: $r = -15\%$ and $r = 20\%$. Confirm this:

$$-\$100{,}000 + \frac{\$205{,}000}{1 + (-15\%)} + \frac{-\$102{,}000}{[1 + (-15\%)]^2} = 0$$

$$-\$100{,}000 + \frac{\$205{,}000}{1 + 20\%} + \frac{-\$102{,}000}{(1 + 20\%)^2} = 0$$

Huh? So does this project have an internal rate of return of –15% or an internal rate of return of 20%? The answer is both—the fact is that both IRRs are valid according to the definition. And don't think the number of possible solutions is limited to two—with other cash flows, there could be dozens. What do computer spreadsheets do if there are multiple IRRs? You may never know. They usually just pick one for you. They don't even give you a warning.

Projects that have all negative or all positive cash flows have no IRRs—but so do some other projects.

While some projects have multiple IRRs, other projects have none. For example, what is the internal rate of return of a project that yields $10 today and $20 tomorrow (that is, it never demands an investment)? Such a project has no internal rate of return. The NPV formula is never zero, regardless of what the prevailing interest rate is. This makes sense, and the fact that there is no IRR is pretty obvious from the cash flows. After all, they both have the same sign. But what is the IRR of a project that has a cost of $10,000, then pays $27,000, and finally requires a cleanup cost of $20,000? Exhibit 4.3 shows that such a project also has no rate of return at which its NPV would turn positive. Therefore, it has no IRR. What do computer spreadsheets do if there are no IRRs? Thankfully, most of the time, they give an error message that will alert you to the problem.

The most common types of investment projects have a unique IRR, because they have one outflow followed only by inflows (or vice-versa).

Can you ever be sure that your project has one unique internal rate of return? Yes. It turns out that if you have one negative cash flow followed only by positive cash flows—which happens to be far and away the most common investment pattern—then your project has one and only one IRR. (Projects with cash flows with many different positive and negative signs can still have only one IRR, but it's not guaranteed.) Partly because bonds have such cash flow patterns, YTM is even more popular than IRR. Obviously, you also have a unique IRR if a project has the opposite cash flow pattern—that is, a positive cash inflow followed only by negative cash flows.

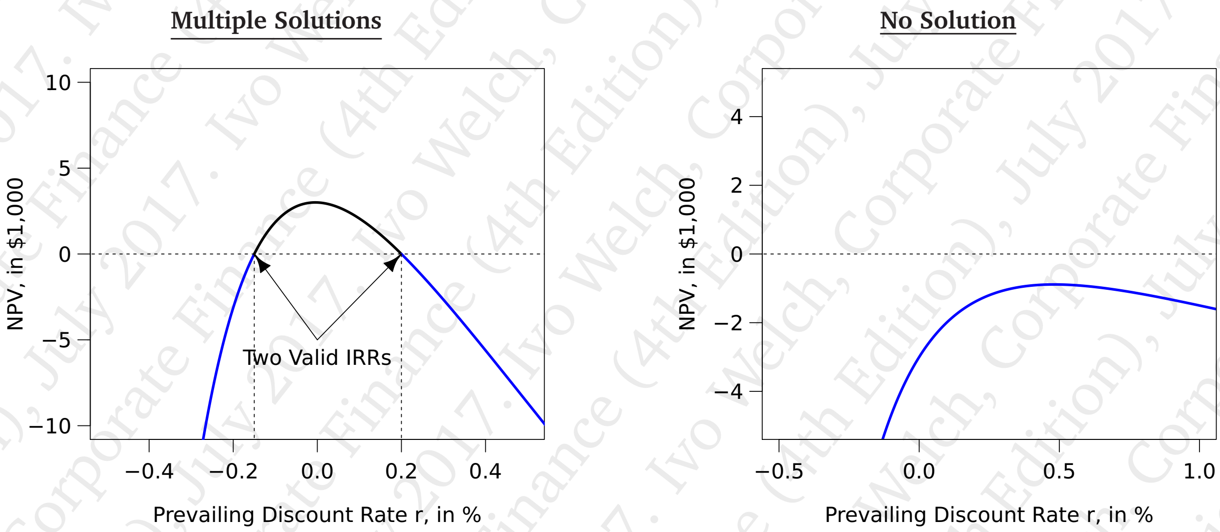


Exhibit 4.3: *Multiple and No IRR Solutions.* The left figure draws the NPV for a project that costs $100,000, pays $205,000, and then has cleanup costs of $102,000. The right figure draws the NPV for a project that costs $10,000, pays $27,000, and then requires a $20,000 cleanup cost.

Q 4.12. Give an example of a problem that has multiple IRR solutions.

Q 4.13. Give an example of a project that has no IRR.

Q 4.14. For the following projects A through G, plot the NPVs as a function of the prevailing interest rate and determine the appropriate IRRs.

	Y0	Y1	Y2	Y3	Y4
A	+$1,000	–$5,000	+$9,350	–$7,750	+$2,402.4
B	+$50,000	–$250,000	+$467,500	–$387,500	$120,120
C	+$100,000	–$250,000	+$200,000		
D	–$100	+$300	–$400	+$400	
E	+$100	–$300	+$400	–$400	
F	+$200	–$600	+$800	–$800	
G	–$100	+$300	–$200		

IRR as a Capital-Budgeting Rule

IRR rule often yields the same result as the NPV rule.

One important reason why IRR is so useful is that it can often substitute for NPV as an investment criterion.

IMPORTANT

- The IRR capital-budgeting rule states that if and only if an investment project's IRR (a characteristic of project cash flows) is above the appropriate discount rate (i.e., the cost of capital quoted like a required interest rate) for the project, then the project should be taken. In this context, the cost of capital is often called the **hurdle rate**.

In many cases, the IRR capital-budgeting rule gives the same correct answer as the NPV capital-budgeting rule. However, there are some delicate situations in which this is not the case. This will be explained below.

Confirm that the IRR and NPV capital-budgeting rules give the same recommendation.

Let me illustrate that you usually get the same answer. Return to our project that costs \$100,000 and yields \$5,000, \$10,000, and \$120,000 with its IRR of 11.14%. The IRR capital-budgeting rule states that if the prevailing cost of capital in the economy (i.e., the hurdle rate) to finance our project is 11.20%, then you should not take this project. If it is 11.10%, then you should take this project. Does NPV offer the same recommendation? Try it:

$$\text{NPV at } 11.10\% = -\$100{,}000 + \frac{\$5{,}000}{1+11.10\%} + \frac{\$10{,}000}{(1+11.10\%)^2} + \frac{\$120{,}000}{(1+11.10\%)^3} \approx +\$108$$

$$\text{NPV at } 11.20\% = -\$100{,}000 + \frac{\$5{,}000}{1+11.20\%} + \frac{\$10{,}000}{(1+11.20\%)^2} + \frac{\$120{,}000}{(1+11.20\%)^3} \approx -\$146$$

Indeed, you get the same recommendation.

If the cash flow is negative, the IRR stays the same, but the take-it-or-leave-it rule reverses.

If the cash flows are the exact opposite—that is, if you receive \$100,000 upfront and pay out \$5,000, \$10,000, and \$120,000—then this would not really be an investment project, but more like investment financing. You would now want to take this financing alternative if and only if the prevailing interest rate is *above* 11.14%. Be careful about whether you want your IRR to be above or below the hurdle rate! (My advice to avoid such errors is to always work out the NPV, too—it will never mislead you.)

IRR can be computed before the cost of capital is known.

IRR is a characteristic of a project's cash flows. (It is not an interest rate.)

Why use the IRR instead of the NPV investment criterion? The answer is that the former is often quite intuitive and convenient, provided that the project's cash flow stream implies one unique IRR. In this case, IRR is convenient because you can compute it without having looked at financial markets, interest rates, or costs of capital. This is IRR's most important advantage over NPV: *It can be calculated even before you know the appropriate interest rate (cost of capital).* Moreover, IRR can give you useful project information in and of itself. It is also helpful in judging project profitability and thereby allows you to judge the performance of a manager—it is often easier to hold her to her earlier promise of delivering an IRR of 20% than it is to argue with her about what the appropriate cost of capital for her project should be. And, finally, by comparing the IRR to the cost of capital, you can determine how much "buffer" you have in terms of getting your cash flow estimates wrong by a certain percentage and still be correct in your ultimate decision as to whether to take the project or not.

Q 4.15. A project has cash flows of –\$1,000, –\$2,000, +\$3,000, and +\$4,000 in consecutive years. Your cost of capital is 30% per annum. Use the IRR rule to determine whether you should take this project. Does the NPV rule recommend the same action?

Q 4.16. A project has cash flows of –\$1,000, –\$2,000, –\$3,000, +\$4,000, and +\$5,000 in consecutive years. Your cost of capital is 20% per annum. Use the IRR rule to determine whether you should take this project. Confirm your recommendation using the NPV rule.

Q 4.17. A project has cash flows of +\$200, –\$180, –\$40 in consecutive years. The prevailing interest rate is 5%. Should you take this project?

Q 4.18. You can invest in a project with diminishing returns. Specifically, the formula relating next year's payoff to your investment today is $C_1 = \sqrt{-C_0}$, where C_0 and C_1 are measured in millions of dollars. For example, if you invest \$100,000 in the project today, it will return $\sqrt{\$0.1} \approx \$0.316$ million next year. The prevailing interest rate is 5% per annum. Use a spreadsheet to answer the following two questions:

1. What is the IRR-maximizing investment choice? What is the NPV at this choice?
2. What is the NPV-maximizing investment choice? What is the IRR at this choice?

Problems with IRR as a Capital-Budgeting Rule

IRR is safe to use when there is only one positive or only one negative cash flow.

If you use IRR *correctly* and in the right circumstances, it can give you the same answer as the NPV rule. You cannot do better than doing it correctly, so it is always safer to use the NPV rule than the IRR rule. When does the IRR capital-budgeting rule work well? If there is only one unique IRR, it is often an elegant method. Of course, as just stated, you still have to make sure that you get the sign right. If your project requires an upfront outflow followed by inflows, you want to take the project if its IRR is *above* your cost of capital. If the project is financing (like debt, which has an upfront inflow followed by outflows), you want to take this project if its IRR is *below* your cost of capital. My advice is to use NPV as a check of your IRR calculations in any case.

IRR often fails in nonobvious ways when there are multiple negative or positive cash flows.

Unfortunately, if the IRR is not unique (and recall that there are projects with multiple IRRs or no IRR), then the IRR criterion becomes outright painful. For example, if your prevailing cost of capital is 9% and your project has IRRs of 6%, 8%, and 10%, should you take this project or avoid it? The answer is not obvious. In this case, to make an investment decision, you are better off falling back to drawing a part of the NPV graph in one form or another. My advice: just avoid IRR. (Yes, it is possible to figure out how to use IRR, depending on whether the NPV function crosses the 0-axis from above or below, but working with IRR under such circumstances only begs for trouble, i.e., mistakes. There is also a "modified IRR" [the so-called MIRR] measure that can sometimes eliminate multiple solutions. (MIRR is not worth the trouble.) If you have a project without any valid IRR, you again have to fall back to NPV, but using NPV will be simpler. Just work out whether the NPV function is above or below the 0-axis for any arbitrary discount rate (e.g., $r = 0$), and use this to decide whether to take or reject your project.

Two more problems: (1) IRR has no concept of scale; (2) there may not be an obvious hurdle rate to compare it to.

There are two more problems when using IRR that you need to be aware of:

1. **Project comparisons and scale:** The IRR criterion can be misleading when projects are mutually exclusive. For example, if you had to choose, would you always prefer a project with a 100% IRR to a project with a 10% IRR? Think about it.

   What if the first project is an investment opportunity of \$5 (returning \$10), and the second project is an investment opportunity of \$1,000 (returning \$1,100)? Take the case where the prevailing discount rate is 5% per annum. Then,

	Y0	Y1	IRR	NPV at 5%
A	–\$5	+\$10	100%	+\$4.52
B	–\$1,000	+\$1,100	10%	+\$47.62

   If you can only take one project, then you should take project B, even though its IRR is much lower than that of project A.

2. **Cost of capital comparison:** The next chapter will explain that long-term interest rates are often higher than short-term interest rates. For example, in mid-2016, a 1-year Treasury bond offered a rate of return of 0.5%, while a 30-year Treasury bond offered an annualized rate of return of 2.5%. Let's assume that your project is risk-free, too. Should you take a risk-free project that has an IRR of 1.5%? There is no clear answer.

These two problems may seem obvious when highlighted in isolation. But in the context of complex, real-world, multiple-project analyses, they are surprisingly often overlooked.

Q 4.19. What are the problems with the IRR computation and criterion?

Q 4.20. The prevailing interest rate is 25%. If the following two projects are mutually exclusive, which should you take?

	Y0	Y1	Y2	Y3	Y4
A	+$50,000	–$250,000	+$467,500	–$387,500	+$120,120
B	–$50,000	+$250,000	–$467,500	+$387,500	–$120,120

What does the NPV rule recommend? What does the IRR rule recommend?

Q 4.21. The prevailing interest rate is 25%. If the following two projects are mutually exclusive, which should you take?

	Y0	Y1	Y2	Y3
A	+$500,000	–$200,000	–$200,000	–$200,000
B	+$50,000	+$25,000		

What does the NPV rule recommend? What does the IRR rule recommend?

Q 4.22. The prevailing interest rate is 10%. If the following three projects are mutually exclusive, which should you take?

	Y0	Y1	Y2
A	–$500	+$300	+$300
B	–$50	+$30	+$30
C	–$50	+$35	+$35

What does the NPV rule recommend? What does the IRR rule recommend?

Q 4.23. The prevailing interest rate is 5% over the first year and 10% over the second year. That is, over two years, your compounded interest rate is $(1+5\%)\cdot(1+10\%)-1 = 15.5\%$. Your project costs $1,000 and will pay $600 in the first year and $500 in the second year. What does the IRR rule recommend? What does the NPV rule recommend?

4.3 The Profitability Index

A less prominent measure sometimes used in capital budgeting is the **profitability index**. It divides the present value of future cash flows by the project cost (the negative of the first cash flow). For example, if you have a project with cash flows

How the probability index is computed.

	Y0	Y1	Y2	Y3	PV(Y1 to Y3)
Project A Cash Flow	–$100	$70	$60	$50	$128.94

and the interest rate is 20% per annum, you would first compute the present value of future cash flows as

$$\text{PV} = \frac{\$70}{1.2} + \frac{\$60}{1.2^2} + \frac{\$50}{1.2^3} \approx \$128.94$$

$$= \text{PV}(\,C_1\,) + \text{PV}(\,C_2\,) + \text{PV}(\,C_3\,)$$

Subtract the $100 upfront cost, and the NPV is $28.94. The profitability index is

$$\text{Profitability Index} = \frac{\$128.94}{-(-\$100)} \approx 1.29$$

$$\text{Profitability Index} = \frac{\text{PV}(\,\text{Future Cash Flows}\,)}{\text{Original Cost}}$$

A positive-NPV project usually has a profitability index above 1—“usually” because the profitability index is meaningful only if the first cash flow is a cash outflow. When this is the case, you can use either NPV or the profitability index for a simple “accept/reject” decision: The statements “NPV > 0” and “profitability index > 1” are the same. That is, like IRR, the profitability index can give the correct answer in the most common situation of one negative cash flow upfront followed by all positive cash flows thereafter.

A profitability index-based capital-budgeting rule can give the same answer as IRR (and NPV).

Some managers like the fact that the profitability index gives information about relative performance and use of capital. For example,

Here it works nicely, and may even convey some information above and beyond IRR.

	Y0	Y1	Y2	Y3	PV(Y1 to Y3)
Project B Cash Flow	–$10.00	$21.14	$18.12	$15.10	$38.94

has the same NPV of $28.94 as the original project, but B’s profitability index is higher than 1.29 because it requires less capital upfront.

$$\text{Profitability Index} = \frac{\$38.94}{-(-\$10)} \approx 3.89$$

$$\text{Profitability Index} = \frac{\text{PV}(\,\text{Future Cash Flows}\,)}{\text{Original Cost}}$$

The reason is that the profitability index values the scale of the project differently. It is intuitively apparent that you would prefer the second project, even though it has the same NPV, because it requires less capital. It may even be less risky, but this can be deceiving, because we have not specified the risk of the future cash flows.

Unfortunately, this feature that you just considered as an advantage can also be a disadvantage. You cannot use the profitability index to choose among different projects. For example, assume that your first project returns twice as much in cash flow in all future periods, so it is clearly the better project now.

But here is where the profitability index can go wrong: Like IRR, it has no concept of scale.

	Y0	Y1	Y2	Y3	PV(Y1 to Y3)	NPV	Profitability Index
B	–$10	$21.14	$18.12	$15.10	$38.94	$28.94	$\frac{\$38.94}{-(-\$10)} \approx 3.89$
C	–$100	$140	$120	$100	$257.87	$157.87	$\frac{\$257.87}{-(-\$100)} \approx 2.58$

Note that the profitability index of project C is less than that of project B. The reason is that, when compared to NPV, the profitability index *really* "likes" lower-upfront investment projects. It can therefore indicate higher index values even when the NPV is lower. This is really the same scale problem that popped up when we tried to use IRR for comparing mutually exclusive projects. Both look at relative "percentage" performance, not at the dollar gain, like NPV does. You should really consider the profitability index in choosing among projects only if the NPVs of the two projects are equal (or at least very similar).

Q 4.24. The prevailing interest rate is 10%. If the following three projects are mutually exclusive, which should you take?

	Y0	Y1	Y2
A	–$500	+$300	+$300
B	–$50	+$30	+$30
C	–$50	+$35	+$35

You have already worked out the recommendations of the NPV and the IRR rule. What does the profitability rule recommend?

4.4 The Payback Capital-Budgeting Rule

The most common aberrant capital-budgeting rule in the real world is the payback rule.

What if you want something more "practical" than the eggheaded "theoretical" capital-budgeting methods? Aren't there easier methods that can help you make investment decisions? Yes, they exist—and they usually result in bad practical choices. Indeed, after IRR and NPV, the most commonly used capital-budgeting rule is a "practical" one, the **payback rule**. You need to know why you should not fall for it.

Three sample projects.

Under the payback rule, projects are assumed to be better if you can recover their original investment faster. For the most part, this is a stupid idea. Consider the following three projects:

	Y1	Y2	Y3	Y4	Payback Period
A	–$5	+$8			1 year
B	–$5	+$4	$100		2 years
C	–$5	+$4	$0	$100,000	3 years

Here is why choosing projects based solely on payback speed is dumb.

Project A has the shortest (best) payback period, but it is the worst of the three projects (assuming common discounting rates). Project B has the next shortest payback period, but it is the second-worst of the three projects (assuming reasonable interest rates). Project C has the longest (worst) payback period, but is the best project. There is also a version of payback in which future paybacks are discounted (**discounted payback**). This measure asks not how long it takes to get your money back, but how long it takes to get the *present value* of your money back. It is still a bad idea.

To be fair, payback can be an interesting number.

In fairness, the speed of payback can be an interesting statistic.

1. There is a beautiful simplicity to payback. Everyone will understand "you will get your money back within five years," but not everyone will understand "the NPV is $50 million."
2. Payback's emphasis on earlier cash flows helps firms set criteria when they don't trust their managers. For instance, if your department manager claims that you will get your money back within one year, and three years have already passed without your having seen a penny, then something is probably wrong and you may need a better manager.
3. Payback can also help if you are an entrepreneur with limited capital, faced with an imperfect capital market. In such cases, your cost of capital can be very high and getting your money back in a short amount of time is paramount. The payback information can help you assess your future "liquidity."

➤ Entrepreneurial finance, Sect. 11.5, Pg.263.

4. Finally, in many ordinary situations, in which the choice is a pretty clear-cut yes or no, the results of the payback rule may not lead to severe mistakes (as would a rule that would ignore all time value of money). If you have a project in which you get your money back within one month, chances are that it's not a bad one, even from an NPV perspective. If you have a project in which it takes fifty years to get your money back, chances are that it has a negative NPV.

Having said all this, if you use payback to make decisions, it can easily lead you to take the wrong projects and ruin your company. Why take a chance when you know better capital-budgeting methods? My view is that it is not a bad idea to work out the payback period and use it as "interesting supplemental information," but you should never base project choices on it—and you should certainly never compare different projects primarily on the basis of payback.

It is best to avoid payback as a primary decision rule.

4.5 How Do Executives Decide?

Method	CFO Usage	Yields Correct Answer	Main Explanation
Internal Rate of Return (IRR)	(76%)	Often	Chapter 4
Net Present Value (NPV)	(75%)	(Almost) Always	Chapter 2
Payback Period	(57%)	Rarely	Chapter 4
Earning Multiples (P/E Ratios)	(39%)	With Caution	Chapter 15
Discounted Payback	(30%)	Rarely	Chapter 4
Accounting Rate of Return	(20%)	Rarely	Chapter 15
Profitability Index	(12%)	Often	Chapter 4

Exhibit 4.4: *CFO Valuation Techniques.* Rarely means "usually no—often used incorrectly in the real world." NPV works *if correctly applied*, which is why I added the qualifier "almost" to always. Of course, if you are considering an extremely good or an extremely bad project, almost any evaluation criterion is likely to give you the same recommendation. (Even a stopped clock gives you the correct time twice a day.) Source: John Graham and Campbell Harvey, 2001.

A survey asked CFOs what they use. It found the good methods (NPV and IRR) are most important.

So what do managers really use for capital budgeting? In their 2001 survey (and regular updates thereafter), Graham and Harvey (from Duke University) asked 392 managers, primarily **chief financial officers (CFOs)**, what techniques they use when deciding on projects or acquisitions. The results are listed in Exhibit 4.4. The two most prominent measures are also the correct ones: They are the "internal rate of return" and the "net present value" methods. Alas, the troublesome "payback period" method and its cousin, the "discounted payback period," still remained surprisingly common. An updated 2016 paper by Mukhlynina and Nyborg finds that valuation practitioners nowadays usually use *both* multiples and discounted cash flow analysis, with frequencies in the mid-80s.

The two unexplained methods (P/E and accounting rate of return) in the table are based on accounting numbers.

➤ Financials, Chapter 14, Pg.355.

➤ Comparables, Chapter 15, Pg.387.

➤ ROE=Accounting rate of return, Formula 15.6, Pg.416.

➤ Warning about BV stock numbers, Sect. 14.7, Pg.380.

Of course, this is your first encounter with capital-budgeting rules, and there will be a lot more details and complications to come (especially for NPV). Let me also briefly explain the two methods mentioned in the table that you do not know yet: the "earnings multiples" and the "accounting rate of return" methods. They will be explained in great detail in Chapters 14 and 15. In a nutshell, the "earnings multiples" method tries to compare your project's earnings directly to the earnings of other firms in the market. If your project costs less and earns more than these alternative opportunities, then the multiples approach usually suggests you take it. It can often be useful, but considerable caution is warranted. The "accounting rate of return" method uses an accounting "net income" and divides it by the "book value of equity." This is rarely a good idea—financial accounting is not designed to accurately reflect firm value. (Accounting statements are relatively better in measuring flows [like earnings] than they are in measuring stocks [like book value].)

The survey unfortunately did not ask managers whether they select projects primarily to increase earnings—a pity.

Graham and Harvey did not allow respondents to select a third measure for project choice: a desire to maximize reported earnings. Managers care about earnings, especially in the short run and just before they are up for a performance evaluation or retirement. Thus, they may sometimes pass up good projects for which the payoff is far in the future.

Accounting-based rules are problematic.

As you will learn, rules that are based on accounting conventions and not on economics are generally not advisable. I almost always recommend against using them. I have no idea what kind of projects you will end up with if you were to follow their recommendations—except that in many cases, if the measures are huge (e.g., if the accounting rate of return is 190% per annum), then chances are that the project has positive NPV, too.

In real-life, the best projects are often not taken. Math may just be a rhetorical weapon to win the political fights to convince others to fund projects.

One view, perhaps cynical, is that all the capital-budgeting methods you have now learned give you not only the tools to choose the best projects but also the language to argue intelligently and professionally to get your favorite projects funded. In many corporations, "power" rules. The most influential managers get disproportionally large funding for their projects. This is of course not a good objective, much less a quantitative value-maximization method for choosing projects.

Summary

This chapter covered the following major points:

- If the market is perfect and you have the correct inputs, then net present value is the undisputed correct method to use.
- In a perfect market, projects are worth their net present values. This value does not depend on who the owner is or when the owner needs cash. Any owner can always take the highest NPV projects and use the capital markets to shift cash into periods in which it is needed. Therefore, consumption and investment decisions can be made independently.
- The internal rate of return, IRR, is computed from a project's cash flows by setting the NPV formula equal to zero.
- The internal rate of return does not depend on the prevailing cost of capital. It is a project-specific measure. It can be interpreted as a "sort-of-average" rate of return implicit in many project cash flows. Unlike

a simple rate of return, it can be computed when a project has more than one inflow and outflow.

- Projects can have multiple IRR solutions or no IRR solutions.
- Investment projects with IRRs above their costs of capital often, but not always, have positive net present values (NPV), and vice-versa. Investment projects with IRRs below their costs of capital often, but not always, have negative net present values (NPV), and vice-versa. If the project is a financing method rather than an ordinary investment project, these rules reverse.
- IRR suffers from comparison problems because it does not adjust for project scale. IRR can also be difficult to use if the cost of capital depends on the project cash flow timing.
- The profitability index is often acceptable, too. It rearranges the NPV formula. If used by itself, it often provides the same capital budgeting advice as NPV. But, like IRR, the profitability index can make projects with lower upfront costs and scale appear relatively more desirable.
- The payback measure is commonly used. It suggests taking the projects that return the original investment most quickly. It discriminates against projects providing very large payments in the future. Although it sometimes provides useful information, it is best avoided as a primary decision rule.
- The information that many other capital-budgeting measures provide can sometimes be "interesting." However, they often provide results that are not sensible and therefore should generally be avoided—or at least consumed with great caution.
- NPV and IRR are the methods most popular with CFOs. This makes sense. It remains a minor mystery as to why the payback method enjoys the popularity that it does.

Keywords

Answers

Q 4.1 The fact that you can use capital markets to shift money back and forth without costs allows you to consider investment and consumption choices independently.

Q 4.2 If you invest \$400, the project will give $\$400 \cdot 1.15 = \$460$ next period. The capital markets will value the project at $\$460/1.10 \approx \$418.18$. You should take the project and immediately sell it for \$418.18. Thereby, you will end up being able to consume $\$500 - \$400 + \$418.18 = \$518.18$.

Q 4.3 The equation that defines IRR is Formula 4.1 on Page 59.

Q 4.4 $-\$1,000 + \dfrac{\$1,000}{(1+\text{IRR})} = 0 \implies \text{IRR} = 0\%$

Q 4.5 $-\$1,000 + \dfrac{\$500}{(1+\text{IRR})} + \dfrac{\$500}{(1+\text{IRR})^2} = 0 \implies \text{IRR} = 0\%$

Q 4.6 $-\$1,000 + \dfrac{\$600}{(1+\text{IRR})} + \dfrac{\$600}{(1+\text{IRR})^2} = 0 \implies \text{IRR} \approx 13.07\%$

Q 4.7 $-\$1,000 + \dfrac{\$900}{(1+\text{IRR})} + \dfrac{\$900}{(1+\text{IRR})^2} = 0 \Rightarrow \text{IRR} = 50\%$

Q 4.8 The spreadsheet function is called IRR(). The answer pops out as 15.5696%. Check: $-\$100 + \$55/1.16 + \$70/1.16^2 \approx 0$.

Q 4.9 The coupon bond's YTM is 5%, because $-\$1,000 + \dfrac{\$50}{1.05} + \dfrac{\$50}{1.05^2} + \dfrac{\$50}{1.05^3} + \dfrac{\$50}{1.05^4} + \dfrac{\$1,050}{1.05^5} = 0$. The YTM of such a bond (annual coupons) is equal to the coupon rate when a bond is selling for its face value.

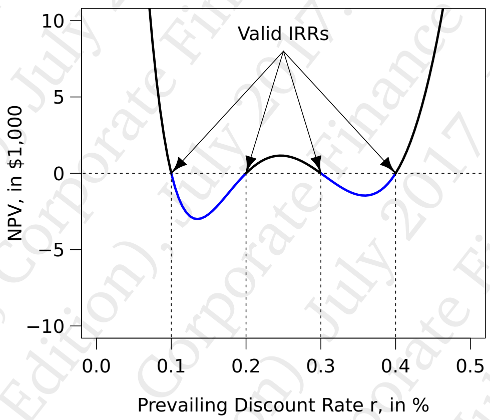
Q 4.10 The YTM is 10%, because $-\$1,000 + \$1,611/1.10^5 \approx 0$.

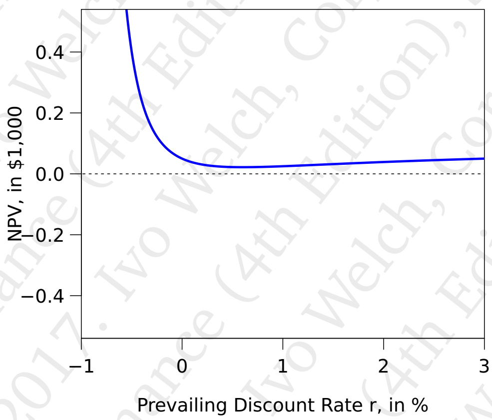
Q 4.11 You are seeking the solution to $-\$25,000 + \dfrac{\$1,000}{(1+\text{YTM})^1} + \dfrac{\$1,000}{(1+\text{YTM})^2} + \dfrac{\$25,000}{(1+\text{YTM})^2} = 0$. It is $\text{YTM} = 4\%$.

Q 4.12 For example, $C_0 = -\$100, C_1 = +\$120, C_2 = -\$140, C_3 = +\$160, C_4 = -\$20$. (The solutions are IRR ≈ –85.96% and IRR ≈ +$9.96%. The important aspect is that your example has multiple inflows and multiple outflows.)

Q 4.13 For example, $C_0 = -\$100, C_1 = -\$200, C_2 = -\$50$. No interest rate can make their present value equal to zero, because all cash flows are negative. This project should never be taken, regardless of cost of capital.

Q 4.14 For projects A and B, the valid IRRs are 10%, 20%, 30%, and 40%. The plot for A follows. The plot for B has a y-scale that is 50 times larger. For project C, there is no IRR, also shown in the plot below.





For projects D, E, and F, the IRR is 100%. For project G, the IRRs are 0% and 100%.

Q 4.15 The (unique) IRR is 56.16%. This is higher than your 30% cost of capital, so you should take this project. The NPV is +$1,057.35. Because this is positive, it gives the same recommendation—accept.

Q 4.16 The IRR is 19.73%. This is lower than your 20% cost of capital, so you should not take this project. The NPV is –$23.92. IRR and NPV agree on the reject recommendation.

Q 4.17 The IRR is 8.44%. This is above the prevailing interest rate. However, the cash flows are like that of a financing project. This means that it is a negative NPV project of –$7.71. You should not take it.

Q 4.18 (1) The IRR-maximizing investment choice of C_0 is an epsilon. The IRR is then close to infinity. The NPV is 0. (2) The NPV-maximizing (and best) choice is an investment of $226,757. This also happens to be the project's NPV. The IRR is 110%.

Q 4.19 The problems are (a) you need to get the sign right to determine whether you should accept the project above or below its hurdle rate; (b) you need to make sure you have only one unique IRR (or work with a more complicated version of IRR, which we have not done); (c) you cannot use it to compare different projects that have different scales; and (d) you must know your cost of capital.

Q 4.20 Project A has a positive NPV of

$$\$50,000 + \frac{-\$250,000}{1.25} + \frac{\$467,500}{1.25^2} + \frac{-\$387,500}{1.25^3} + \frac{\$120,120}{1.25^4} \approx \$1.15$$

Project B has an NPV of –$1.15. You should take project A, but not B. If you plot the NPV as a function of the interest, you will see that there are multiple IRRs for these projects, specifically at 10%, 20%, 30%, and 40%. With a cost of capital of 25%, you cannot easily determine which of these two projects you should take. Make your life easy, and just use NPV instead.

Q 4.21 Project A has an NPV of

$$+\$500,000 + \frac{-\$200,000}{1.25} + \frac{-\$200,000}{(1.25)^2} + \frac{-\$200,000}{(1.25)^3} = \$109,600$$

It has an IRR of 9.70%. Project B has an NPV of $70,000, and no IRR (it is always positive). Therefore, even though the second project should be taken for any interest rate—which is not the case for the first—the first project is better. Take project A.

Q 4.22 The first project (A) has an NPV of $20.66 and an IRR of 13.07%. The second project (B) has an NPV of $2.07 and the same IRR of 13.07%. The third project (C) has an NPV of $10.74 and an IRR of 25.69%. Even though project A does not have the highest IRR, you should take it.

Q 4.23 The IRR is 6.81%. This is between the one-year 5% and the two-year 10% interest rates. Therefore, the IRR capital-budgeting rule cannot be applied. The NPV rule gives you $-\$1,000 + \$600/1.05 + \$500/1.155 \approx \$4.33$, so this is a good project that you should take.

Q 4.24 The first project (A) has present values of future cash flows of $520.66; the second (B) of $52.07; the third (C) of $60.74. The profitability indexes are $\$520.66/\$500 \approx 1.04$, $\$52.07/\$50 \approx 1.04$, and $\$60.74/\$50 \approx 1.21$. Nevertheless, you should go with the first project, because it has the highest net present value. The discrepancy between the NPV and the profitability rule recommendations is because the latter does not take project scale into account.

End of Chapter Problems

Q 4.25. Given the same NPV, would you be willing to pay extra for a project that bears fruit during your lifetime rather than after you are gone?

Q 4.26. How bad a mistake is it to misestimate the cost of capital in a short-term project? Please illustrate.

Q 4.27. How bad a mistake is it to misestimate the cost of capital in a long-term project? Please illustrate.

Q 4.28. What is the difference between YTM and IRR?

Q 4.29. A project has cash flows of –\$1,000, +\$600, and +\$300 in consecutive years. What is the IRR?

Q 4.30. What is the YTM of a standard 6% level semiannual 10-year coupon bond that sells for its principal amount today (i.e., at par = \$100)?

Q 4.31. A coupon bond costs \$100, then pays \$10 interest each year for 10 years, and pays back its \$100 principal in 10 years. What is the bond's YTM?

Q 4.32. A project has cash flows –\$100 as of now, +\$55 next year, and +\$60.50 in the year after. How can you characterize the "rate of return" (loosely speaking) embedded in its cash flows?

Q 4.33. Under what circumstances is an IRR a rate of return? Under what circumstances is a rate of return an IRR?

Q 4.34. Give an example of a problem that has multiple IRR solutions.

Q 4.35. Your project has cash flows of –\$1,000 in year 0, +\$3,550 in year 1, –\$4,185 in year 2, and +\$1,638 in year 3. What is its IRR?

Q 4.36. Your project has cash flows of –\$1,000 in year 0, +\$3,550 in year 1, –\$4,185 in year 2, and –\$1,638 in year 3. What is its IRR?

Q 4.37. A project has cash flows of +\$400, –\$300, and –\$300 in consecutive years. The prevailing interest rate is 5%. Should you take this project?

Q 4.38. A project has cash flows of –\$100, +\$55, and +\$60.50 in consecutive years starting from right now. If the hurdle rate is 10%, should you accept the project?

Q 4.39. If a project has a cash inflow of \$1,000 followed by cash outflows of \$600 in two consecutive years, then under what discount rate scenario should you accept this project?

Q 4.40. You can invest in a project with returns that depend on the amount of your investment. Specifically, the formula relating next year's payoff (cash flow) to your investment today is $C_1 = \sqrt{-C_0 - \$0.1}$, where C_0 and C_1 are measured in millions of dollars. For example, if you invest \$500,000 in the project today, it will return $\sqrt{\$0.5 - \$0.1} \approx \$0.632$ million next year. The prevailing interest rate is 6% per annum. Use a spreadsheet to answer the following two questions:

1. What is the IRR-maximizing investment choice of C_0? What is the NPV at this level?
2. What is the NPV -maximizing investment choice of C_0? What is the IRR at this level?

Q 4.41. The prevailing interest rate is 10%. If the following three projects are mutually exclusive, which should you take?

	Y0	Y1	Y2
A	+\$500	–\$300	–\$300
B	+\$50	–\$30	–\$30
C	+\$50	–\$35	–\$35

What does the NPV rule recommend? What does the IRR rule recommend?

Q 4.42. What are the profitability indexes and the NPVs of the following two projects: project A that requires an investment of \$5 and gives \$20 per year for three years, and project B that requires an investment of \$9 and gives \$25 per year for three years? The interest rate is 10%. If you can invest in only one of the projects, which would you choose?

Q 4.43. Consider the following project:

Year	Y0	Y1	Y2	Y3	Y4	Y5	Y6
Cash Flow	–\$10	\$5	\$8	\$3	\$3	\$3	–\$6

1. What is the IRR?
2. What is the payback time?
3. What is the profitability index?

Q 4.44. Consider the following project:

	Y0	Y1	Y2	Y3	Y4	Y5	Y6	Y7
CF	$0	–$100	$50	$80	$30	$30	$30	–$60

1. What is the IRR?
2. What is the payback time?
3. What is the profitability index?

Q 4.45. The prevailing cost of capital is 9% per annum. What would various capital-budgeting rules recommend for the following projects?

	Y0	Y1	Y2	Y3	Y4
A	–$1,000	$300	$400	$500	$600
B	–$1,000	$150	$200	$1,000	$1,200
C	–$2,000	$1,900	$200		
D	–$200	$300			
E	–$200	$300	$0	–$100	

Q 4.46. What are the most prominent methods for capital budgeting in the real world? Which make sense?