European Journal of Operational Research 295 (2021) 378–393

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Interfaces with Other Disciplines

Diversification benefits in the cryptocurrency market under mild explosivity[☆]

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ARTICLE INFO

Article history:
Received 2 December 2019
Accepted 27 February 2021
Available online 3 March 2021

Keywords:
Finance
Stochastic spanning
Diversification
Cryptocurrencies
Mild explosivity
Bubbles

ABSTRACT

We investigate whether cryptocurrencies provide diversification benefits to risk averters via a stochastic spanning methodology. We avoid the conceptual and statistical problems of non-stationary returns by providing a modification of the second order stochastic dominance relation and of the related notion of stochastic spanning. These are compatible with a mildly explosive framework for the logarithm prices, along with conditions for asymptotic negligibility of bubbles. In the empirical application, we construct optimal portfolios, both with and without cryptocurrencies, and evaluate their comparative performance both in- and out-of-sample. A conservative modification of a t-test is presented to test the null hypothesis of non-dominance of an optimal portfolio that includes cryptocurrencies over the traditional portfolio of only stocks, bonds and cash. The augmented portfolio is found to be a good diversification option for some risk averse investors in the full sample period and in a sub-period of high cryptocurrency returns.

1. Introduction

One of the arguments commonly invoked to explain investors' interest in cryptocurrencies is their alleged diversification benefits. Cryptocurrency returns are expected to have low or negative correlation with the returns of proxies of traditional asset classes, like bonds and stocks. Their inclusion in traditional portfolios is expected to increase expected return per unit of risk. Briere, Oosterlinck, and Szafarz (2015), for example, use Mean-Variance spanning tests and find that investments in Bitcoin provide significant diversification benefits. Corbet, Lucey, and Yarovaya (2018a), in a time and frequency domain analysis, support the idea that cryptocurrencies constitute a new investment asset class as they are connected to each other, but disconnected from traditional financial assets. Liu and Tsyvinski (2018) established that the risk-return trade-offs in Bitcoin, Ripple and Ethereum are distinct from those found in stocks, currencies and precious metals. Bianchi (2020) finds that, except for a mild correlation with returns on precious metals, cryptocurrency returns are uncorrelated with traditional asset classes. In parallel, there has been a growing interest in research addressing the economics of cryptocurrencies. Some recent studies focus on the determinants of cryptocurrencies – largely Bitcoin returns (Demir, Gozgor, Lau, & Vigne, 2018) – and on prediction models for the Bitcoin market (Atsalakis, Pasiouras, & Zopounidis, 2019).

In our paper, we analyse whether cryptocurrencies provide diversification benefits for risk averse investors. To tackle our research question, we contribute to the aforementioned literature on cryptocurrencies by constructing optimal portfolios and assessing their performance in a non-parametric way. We employ a stochastic spanning approach (see Arvanitis, Hallam, Post, & Topaloglou, 2019) to construct optimal portfolios with and without cryptocurrencies, and to compare their performance both parametrically and non-parametrically. We find that both in- and out-of-sample portfolios with cryptocurrencies could be an optimal choice for some risk averse investors in the full sample period and in a sub-period of high cryptocurrency returns. This finding builds on Bianchi, Guidolin, and Pedio (2020), who established that cryptocurrencies appear to be able to generate considerable realised out-of-sample economic value when added to traditional portfolios. However, we

☆ We would like to thank the Editor, the Associate Editor and the referees for insightful comments and suggestions. We also thank Gibson Heather, Hiona Balfoussia, Athanasios Kontinopoulos, Panayiotis Andreou and Olivier Scaillet, as well as participants at the 2nd Endless Summer Conference on Financial Intermediation and Corporate Finance. All remaining errors are our own.

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[1] The views expressed in this article are those of the authors and not necessarily reflect those of the Bank of Greece or the Eurosystem.

https://doi.org/10.1016/j.ejor.2021.02.058
0377-2217/

also find evidence that for either high levels of risk aversion or for downturn periods the in-sample and out-of-sample results indicate that augmented portfolios i.e., those with cryptocurrencies, perform relatively worse than the traditional ones i.e., those without cryptocurrencies.

Our paper also makes a significant contribution to econometric theory. In order to avoid the conceptual and statistical problems of non-stationary returns, we extend the second order stochastic dominance relation (SSD), and the related notion of stochastic spanning, to be compatible with a mildly explosive framework for the logarithmic prices. This allows for the existence of multiple and possibly interconnected non-stationary bubbles, under appropriate limiting sparsity conditions that render them asymptotically negligible. In turn, this makes the related statistical machinery for stochastic spanning (see Arvanitis et al., 2019) usable in a more generalised setting.

Our statistical theory also includes a modification of the t-statistic of Davidson and Duclos (2013) for the null of non-dominance. We develop a non-parametric test involving empirical optimal portfolios that allows for a mildly explosive framework (as described below) and for potentially divergent data-dependent portfolios, typical to empirical applications in optimal portfolio analysis. The test employs an asymptotic rejection region based on the standard normal distribution and thus avoids the numerical burden associated with fine-tuning parameters in resampling procedures. In the empirical application, we use the test to non-parametrically compare the performance of optimal portfolios with and without cryptocurrencies.

The emergence of cryptocurrencies has drawn significant investment capital in recent years. The rate of increase in market capitalisation, in volumes traded, as well as in the price of certain cryptocurrencies has been exponential, indicating that changes in the cryptocurrency market can occur very rapidly. Yermack (2015) argues that Bitcoin appears to be a speculative investment rather than a currency, mainly due to its excessive volatility. In this context, the question of whether certain cryptocurrencies have been driven by speculative bubbles remains a significant topic of debate in the media and literature alike. To explore the topic of bubbles in more depth, Hencic and Gouriéroux (2015) apply a noncausal autoregressive process with Cauchy innovations to reproduce the bubble-like behaviour of Bitcoin over the period February to July 2013, while Fry and Cheah (2016) use econophysics models to identify speculative bubbles for Bitcoin and Ripple.

As there seems to be a consensus in the academic literature on the possibility of speculative bubbles in cryptocurrencies, their existence must first be tested. We apply a multiple bubble-testing and date-stamping approach, initially formulated by Phillips, Wu, and Yu (2011) (hereafter PWY) and later redefined by Phillips, Shi, and Yu (2015) (hereafter PSY), to the four largest cryptocurrencies by market capitalisation on the full sample period: 7 August 2015 to 30 April 2020. The testing algorithm of PWY and PSY allows for the detection and date-stamping of bubbles in the prices of an asset (see, for example, Contessi, De Pace, & Guidolin, 2020) and several papers have used the PWY or PSY framework to detect bubbles in cryptocurrencies. Corbet, Meegan, Larkin, Lucey, and Yarovaya (2018b) use the PWY methodology to examine the existence and periods of bubbles in Bitcoin and Ethereum. Hafner (2020) applies the PWY test for speculative bubbles to the eleven largest cryptocurrencies by market capitalisation up to December 2017. Bouri, Shahzad, and Roubaud (2019) apply the PSY test and find that the likelihood of explosive periods in one cryptocurrency generally depends on the presence of explosiveness in other cryptocurrencies.

We find evidence that there have been numerous bubble periods across the cryptocurrencies under investigation. This motivates a modification of the concept of stochastic spanning by Arvanitis et al. (2019) and of the methodology presented therein, in a framework of multivariate mildly explosive VAR(1) DGP processes. We impose a limiting sparsity and uniformly bounded intensity condition on the periods of explosivity and their characteristics, ensuring that the explosive periods become jointly asymptotically negligible. This allows for: (a) the evaluation of the employed statistic under potentially non-stationary returns, and (b) the invariance of the associated limit theories under a stationary and mixing framework. We note that the condition is consistent with strands of economic theory and the history of several financial bubbles. It is satisfiable when (a) the underlying technologies that affect the production, the market conditions that affect the distribution, and the technologies that characterise the regulation of the underlying assets improve, (b) investors improve their expectations on those characteristics, or (c) both. Consequently, investors may stop focusing on bubble-producing sunspot processes that are market irrelevant, or correct the parameters that correspond to market relevant processes. This economic reasoning is empirically reinforced: given the large sample period under study, it is seen that, for all four cryptocurrencies, the majority of the bubbles (those of significant duration) occurred somewhere in the period 2016 to 2018, and that the last bubble (of significant duration) appears to have burst over a year before the end of our sample period.

In the empirical application, we conduct both in- and out-of-sample tests. The S&P 500 Total Return Index, the Barclays US Aggregate Bond Index and the three-month T-bill rate are employed to proxy the traditional asset universe i.e., equities, bonds and cash. This portfolio is augmented by including four cryptocurrencies. Given an observed large return up to December 2017 and a subsequent fall for all four cryptocurrencies (see Section 3.1), we also perform our analysis for 7 August 2015 to 29 December 2017 (sub-period 1), and for 1 January 2018 to 30 April 2020 (sub-period 2).

In the in-sample analysis, we apply the Arvanitis et al. (2019) stochastic spanning test and find that the portfolios based on the traditional investment opportunity set do not span the corresponding portfolios that also include cryptocurrencies. Hence, some investors could benefit if their investment universe is augmented with cryptocurrencies. The focus is on the SSD criterion, which has a well-established economic interpretation in terms of expected utility theory and Yaari' s (1987) dual theory of risk.

In the out-of-sample analysis, daily optimal portfolios are constructed based on an asset universe comprising traditional assets, as well as on an asset universe augmented with cryptocurrencies, in a rolling-window fashion. We non-parametrically compare the two portfolios using our conservative modification of the t-statistic of Davidson and Duclos (2013) for the null of non-dominance (as described below). Additionally, a variety of well-known parametric performance measures are employed. Their use is justified in our non-stationary setting from the limiting sparsity condition.

Results show that under both the modified non-parametric stochastic (non-)dominance test and the parametric performance measures, the expanded investment universe empirically dominates the traditional investment universe – both in the totality of the period of analysis, and in the sub-period of high cryptocurrency returns – making some risk averse investors better off with respect to these performance criteria. However, for high levels of risk aversion and for the second sub-period, we find that the augmented portfolios perform relatively worse than the traditional ones. This is consistent to the empirical finding that cryptocurrencies constitute a separate asset class (see, for example, Bianchi, 2020). It also supports the claim that the sub-period of low returns is not adequate to destroy the investment opportunities created by the sub-period of high returns for certain investors, specifically, those with low levels of risk aversion.

Section 2 outlines our empirical methodology; it includes a description of the mildly explosive framework with multiple bubbles, a modification of the stochastic dominance relation and of the spanning methodology employed, limiting invariance compared to the stationary case under a limiting sparsity condition for the bubbles, and a derivation of the modification of the Davidson and Duclos (2013) (non-)dominance test. Section 3 provides results of the bubble tests applied to four cryptocurrencies, in-sample results of the stochastic spanning test of traditional portfolio spaces and of those augmented with cryptocurrencies, and results of the out-of-sample analysis. The final section provides concluding remarks, along with ideas for further research. Proofs and auxiliary results are given in the Appendix.

2. Methodology

Several empirical findings (see, for example, Hafner, 2020; Bouri et al. (2019); Contessi et al. (2020); Corbet et al. (2018b)), as well as those presented in the paper, report multiple - possibly interdependent - bubbles in the cryptocurrency prices. In order to employ in an empirical manner the Arvanitis et al. (2019) methodology, we develop in the following paragraphs a modification of the second order stochastic dominance relation and of the related notion of stochastic spanning, which are compatible with a mildly explosive framework for the logarithmic prices.

2.1. Mildly explosive framework with multiple bubbles

We work with a portfolio space defined as the set of positive convex combinations of d base assets, represented by the $d-1$ simplex $\{\boldsymbol{\lambda} \in \mathbb{R}^d_+ : \boldsymbol{\lambda}^T \mathbf{1}_d = 1\}$. In our empirical analysis the base assets are comprised by a collection of 'traditional' assets, that is a set of stocks and bonds, augmented with the a set of four cryptocurrencies. Notice that our mathematical framework simply considers the base assets as the vertices of the space, hence those need not necessarily be individual securities. Our theoretical constructions thus allow for base assets that are themselves constructed via complicated portfolio constraints on deeper underlying individual securities, like short sales, position limits, and restrictions on factor loadings.

We employ the following framework of multivariate mildly explosive AR(1) processes for the stochastic processes that describe the logarithmic prices. In what follows, and depending on the context, $\|\cdot\|$ denotes either the Euclidean norm on $\mathbb{R}^d$ or the Frobenius norm on the space of $d \times d$ real matrices. Moreover, c denotes a generic positive constant that may assume different values in different occurrences.

Assumption ME.

1. *$(\boldsymbol{\varepsilon}_t)_{t\in\mathbb{N}}$ is an $\mathbb{R}^d$-valued stationary and strong mixing process with mixing coefficient sequence $(\alpha_m)_{m\in\mathbb{N}}$ that satisfy $\alpha_m = O(m^r)$ for some $r > 1$. Furthermore, $\exists L, q > 0$ such that for large enough $\mathrm{t} > 0$, $\mathbb{E}[\exp(\mathrm{t}\|\boldsymbol{\varepsilon}_0\|)] \leq \exp(L\mathrm{t}^q)$.*
2. *For the sample size $T \in \mathbb{N}^\star$, $\{0, \ldots, T\}$ is partitioned in K mild-explosivity periods B_k, $k = 1, \ldots, K$ and the remaining stationary periods $\cap_{k=1}^K B_k^c$.*
3. *The logarithmic prices$\mathbb{R}^d$-valued process sequence satisfies the recursion $\boldsymbol{X}_t = (\mathrm{Id} + \sum_{k=1}^K \frac{C_k}{M(T,k)} \mathbb{I}\{t \in B_k\})\boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t$, $t > 0$, where C_k is a positive $d \times d$ explosivity coefficient matrix at the k^{th} explosive period, and $M(T, k) > 0$ and diverging to infinity as $T \to \infty$, which represents the rate at which the kth explosive behaviour vanishes as a function of T. The process is initiated by $\boldsymbol{X}_0$ and $\exists L^\star, q^\star > 0$ such that for large enough $\mathrm{t} > 0$, $\mathbb{E}[\exp(\mathrm{t}\|\boldsymbol{X}_0\|)] \leq \exp\left(L^\star \mathrm{t}^{q^\star}\right)$.*

Remark 1. Assumption ME.(1) is general enough to allow for a large variety of linear and/or conditionally heteroskedastic models typically used for the stationary parts of logarithmic returns in empirical finance (see, for example, Basrak, Davis, and Mikosch (2002); Drost and Nijman (1993). Hence, stationary, ergodic and geometrically mixing VARMA and/or multivariate GARCH and stochastic volatility types of temporal dynamics are allowed with parameter restrictions that are empirically relevant at least for moderate observation frequencies, compatible with our empirical analysis and innovation distributions that have densities (see for example Boussama, Fuchs, and Stelzer (2011) for a case of multivariate GARCH type models). ME.(2) allows for the existence of K sub-periods of non-stationary bubbles in parts of the base assets process. It is allowed that $K \to \infty$ as $T \to \infty$. This implies that the number of bubbles need not asymptotically stabilise, though the following assumption will specify bounds on the intensity of the bubbles as T grows. In ME.(3) at each bubble period k the structure of the explosivity coefficient matrix C_k is general enough to allow for intra-bubble dependence of currently explosive base assets on the dynamics of other currently and/or previously explosive assets, as well as on assets that are never explosive. Obviously, for the latter the relevant blocks of C_k are zero for all k. The moment generating function conditions that appear in ME.(1),(3) are compatible with sub-Gaussian and sub-exponential distributions for the random variables involved (see, for example, Chapter 2 of Vershynin (2018)). For example, they allow for multivariate Gaussianity for ε_0, $\mathbf{X}_0$, but they do not allow for marginal distributions that do not have tails that decay with fast enough rates, for example non-Gaussian stable distributions. They imply that the lower partial moments differentials that represent the second order stochastic dominance relations for the stationary parts of the net returns are well defined.

Notice that ME.(3) implies that the base assets net return process is thus defined by $\boldsymbol{R}_t := \exp^\star(\sum_{k=1}^K \frac{C_k}{M(T,k)} \mathbb{I}\{t \in B_k\}\boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t) - \mathbf{1}$, $t > 0$ with $\exp^\star : \mathbb{R}^d \to \mathbb{R}^d$ defined by $\exp^\star(\mathbf{y}) := (\exp(\mathbf{y}_1), \ldots, \exp(\mathbf{y}_d))^{\mathrm{T}}$ and $\mathbb{R}^d \ni \mathbf{1} := (1, \ldots, 1)^T$. Furthermore, $\mathcal{X} \subseteq [0, +\infty)$ is the maximal support of any of the portfolios constructed upon the base asset process, and it is by construction bounded from below.

Given that K is allowed to diverge, the following assumption prescribes restrictions between the singular values of the explosivity coefficient matrices C_k, the degree of the return to the random walk dynamics $M(T, k)$ and the maximal bubble time instance $\max_{t,k} B_k$.

Assumption AN. $\exists c > 0$, $\epsilon > 0$ such that $\max_k \frac{\|C_k\|(d+\epsilon)^{\max_{t,k} B_k}}{M(T,k)} \leq c$ for all $T \geq 0$. As $T \to \infty$, $\frac{\max_{t,k} B_k}{\sqrt{T}} = o(1)$.

Remark 2. AN is satisfied if for example $\|C_k\|$ is bounded in k, $\max_{t,k} B_k \sim c \ln T$ and $M(T, k) = \delta_k T^{a_k}$, with $\min_k \delta_k, \min_k a_k > 0$ (such choices are compatible with the bubble duration conditions of Phillips et al., 2015). The condition is not empirically identifiable since concerns among other the asymptotic behaviour of sequences. It is also forward looking as it refers to the future behaviour of bubbles. It allows for slowly diverging bubble durations compatible with future improvements in the technology and the regulation framework associated with the creation and circulation of cryptocurrencies. In fact, there have been numerous instances of investors overbidding new technology once it hits the market. After a bubble burst, concepts and technologies are expected to regrow while investors are expected to be more aware of the true potential of new technology. A series of severe bubbles could likely also lead to stricter regulation of cryptocurrencies, making it more difficult for another bubble to occur (or making future bubbles not to last too long). Similarly, the development of derivatives' market for cryptocurrencies, which are forward-looking by construction (see Fama and French (1987), for their relation with markets' expectations, and Fusari, Jarrow, and Lamichhane (2020) for bubbles associated with options data), and the existence of Futures for most of the cryptocurrencies - including Bitcoin, Litecoin, Ethereum and

Ripple - provides with some further justification for the validity of this condition. If those markets deter investors from forming expectations via sunspot processes that produce frequent and severe bubbles or force them to correct their parameters associated with fundamentals (see Diba & Grossman, 1988) then AN could be justifiable. Notice that AN does not preclude more complicated behaviours; for example, it allows for unbounded $\|C_k\|$ as k grows, thus intense intra-bubble feedbacks between explosive assets, that however get mitigated by stronger degrees of the return to the random walk dynamics due to learning mechanisms like the above. In any case, AN prescribes the exact conditions on the explosive dynamic parameters of the logarithmic prices process that imply asymptotic dominance of the stationary dynamics in the formation of the lower partial moments differentials employed in the SD relations.

2.2. Stochastic dominance and stochastic spanning

Stochastic dominance is non-parametric. This makes it a more appropriate tool to handle with the return distribution of cryptocurrencies than parametric approaches, especially since the return distribution is unknown, may be skewed and fat tailed, and the preferences may be more complex than the 'quadratic' utility assumption underlying the Mean-Variance assessments. SSD ranks investments based on conditions that characterise decision making under uncertainty with respect to the class of utilities that exhibit non-satiation and risk aversion (see Hadar & Russell, 1969; Hanoch & Levy, 1969; Rothschild & Stiglitz, 1970) and can be seen as a model-free alternative to Mean-Variance dominance (for overviews and bibliographies, see Levy (2015); Mosler and Scarsini (2012); Whang, 2019).

SSD and more generally stochastic dominance, is a central theme in a wide variety of applications in economics, finance and statistics; see, for example, Lozano and Gutiérrez (2008), Lizyayev and Ruszczyński (2012), Post and Kopa (2013), Christodoulakis, Mohamed, and Topaloglou (2018), Pinar, Stengos, and Topaloglou (2020). Representative applications include Constantinides, Jackwerth, and Perrakis (2009), Constantinides, Czerwonko, Carsten Jackwerth, and Perrakis (2011), and Hodder, Jackwerth, and Kolokolova (2015).

The concept of stochastic spanning, introduced in Arvanitis et al. (2019) with respect to SSD, can be viewed as a model-free alternative to Mean-Variance spanning (Huberman & Kandel, 1987). It amounts to comparing via SSD, two nested sets of securities. Spanning occurs if introducing new securities or relaxing investment constraints to the smaller set, thus transiting to the larger, does not improve the investment possibility set for the given class of risk averse investors/utilities. Hence, the notion is suitable in order to check whether portfolios with at least one cryptocurrency dominate the traditional ones. If we were to add cryptocurrencies to a portfolio of stocks, bonds and cash, and we fail to reject the spanning hypothesis, then this augmentation will be redundant for any risk averse investor, whereas if we reject spanning, then the inclusion of cryptocurrencies leads by definition to performance improvements at least for some insatiable and risk averse utility.

2.2.1. Stochastic dominance in the mildly explosive framework

SSD is by construction employed in stationary frameworks that are not generally compatible with Assumption ME. It is represented by sets of conditions in the form of lower partial moment (LPM) inequalities between the stationary distributions compared, that represent the risk properties relevant to the aforementioned class of utilities. Those conditions are definable by mild non-parametric restrictions on the distributions involved. In order to extend the definition so as to be compatible with Assumptions ME-AN, we employ the following functionals: for $z \in \mathcal{X}$ and $\boldsymbol{\kappa}, \boldsymbol{\lambda}$ elements of the unit $d-1$ simplex define:

$$D\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}_t\big) := \left(z - \boldsymbol{\kappa}^{\mathrm{T}}(\exp^{\star}(\boldsymbol{R}_t) - 1)\right)_+ - \left(z - \boldsymbol{\lambda}^{\mathrm{T}}(\exp^{\star}(\boldsymbol{R}_t) - 1)\right)_+,$$

$$D_T\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\big) := \frac{1}{T}\sum_{t=1}^{T} D\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}_t\big),\ D^{\star}\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\big) := \lim_{T\to\infty} \mathbb{E}\big[D_T\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\big)\big].$$

Notice that due to stationarity

$$D^{\star}\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\varepsilon}\big) = \mathbb{E}\Big[\left(z - \boldsymbol{\kappa}^{\mathrm{T}}(\exp^{\star}(\boldsymbol{\varepsilon}_0) - 1)\right)_+\Big] - \mathbb{E}\Big[\left(z - \boldsymbol{\lambda}^{\mathrm{T}}(\exp^{\star}(\boldsymbol{\varepsilon}_0) - 1)\right)_+\Big],$$

which is the standard LPM differential employed in SSD.

The assumption framework of Section 2.1 implies an asymptotic negligibility property for the totality of bubble periods, with respect to some of the functionals above when properly scaled. This is essentially represented by the following general result of central importance:

Proposition 3. *Suppose that Assumptions ME and AN hold. Then uniformly in $z, \boldsymbol{\lambda}$ as $T \to \infty$,*

$$\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\left(z - \boldsymbol{\lambda}^{\mathrm{T}}(\exp^{\star}(\boldsymbol{R}_t) - 1)\right)_+ - \frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\left(z - \boldsymbol{\lambda}^{\mathrm{T}}(\exp^{\star}(\boldsymbol{\varepsilon}_t) - 1)\right)_+\right|\right] = o(1).$$

The Lipschitz continuity property of $(\cdot)_+$, Assumption ME, Proposition 3 and dominated convergence imply that the $D^{\star}\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\big)$ functional is then well defined, bounded and continuous in $\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}\big)$. We thus modify the definition of SSD to be compatible with the framework of multiple non-stationary bubbles (MESSD) for the returns as follows.

Definition 4. $\boldsymbol{\kappa} \succeq_{\mathrm{MESSD}} \boldsymbol{\lambda}$ iff $\forall z \in \mathcal{X},\ D^{\star}\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\big) \leq 0$.

The Cezaro–limit based definition is similar to Definition 5.1 of Jin, Corradi, and Swanson (2016) that handles distributional heterogeneity in the context of forecast comparison. It corresponds to a limiting Lebesgue–Stieltjes (discrete) integration (across time) of the LPM differentials that collapses to the standard definition of SSD under stationarity. Then, the auxiliary Proposition 9 (see Appendix) directly implies that:

Corollary 5. *Under Assumptions ME and AN, $\boldsymbol{\kappa} \succeq_{\mathrm{MESSD}} \boldsymbol{\lambda}$ iff $\forall z \in \mathcal{X}$, $D^{\star}\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\varepsilon}\big) \leq 0$.*

Thus, Assumption AN ensures that MESSD is essentially SSD between the stationary parts of the returns. It essentially forces the non-stationary periods contributions to $D_T\big(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\big)$ to asymptotically vanish.

2.2.2. Stochastic spanning in the mildly explosive framework

Consider two non-empty subsets of the general portfolio space, $K \subset \Lambda$, which are also assumed to be closed and simplicial, to facilitate among others the invocation of properties of convex optimisation. The concept of stochastic spanning of Arvanitis et al. (2019) compares the two distinct portfolio sets via SSD. This is directly modified in our mild explosivity framework as follows:

Definition 6. $K \succeq \Lambda$ iff $\forall \boldsymbol{\lambda} \in \Lambda, \exists \boldsymbol{\kappa} \in K : \boldsymbol{\kappa} \succeq_{\mathrm{MESSD}} \boldsymbol{\lambda}$.

Continuity and compactness readily implies that $K \succeq_{\text{MESSD}} \Lambda$ iff

$$\eta^{\star} := \sup_{\Lambda} \inf_{K} \sup_{\mathcal{X}} D^{\star}\left(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\right) = 0.$$

Then auxiliary Proposition 9 (see Appendix) implies that MESSD spanning equivalently holds iff $\eta := \sup_{\Lambda} \inf_{K} \sup_{\mathcal{X}} D\left(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\right) = 0$, which is the standard definition of Arvanitis et al. (2019) employed on the stationary parts of the returns.

Suppose that we need to statistically test the hypothesis structure $\mathbf{H_0} : K \succeq_{\text{MESSD}} \Lambda$ vs. $\mathbf{H_1} : K \not\succeq_{\text{MESSD}} \Lambda$. Under an assumption framework involving stationarity and mixing for the base asset return process, a scaled by $\sqrt{T}$ empirical analogue of η is used in Arvanitis et al. (2019) as a Kolmogorov-Smirnov type test statistic for the null hypothesis. The fact that under Assumption AN the null hypothesis is equivalent to that $\eta^{\star} = \eta = 0$, along with auxiliary Proposition 10 (see Appendix) and the latency of the stationary part of the logarithmic returns process, $(\varepsilon_t)_t$, implies that along the lines of Assumptions ME and AN, the original spanning test statistic of Arvanitis et al. (2019) evaluated at the non-stationary returns sample is usable.

We thus employ a scaled by $\sqrt{T}$ empirical analogue of $\eta^{\star}$,

$$\eta_T^{\star} := \sup_{\Lambda} \inf_{K} \sup_{\mathcal{X}} \sqrt{T} D_T\left(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}\right).$$

The asymptotic decision rule is to reject $\mathbf{H}_0$ in favor of $\mathbf{H}_1$ iff $\eta_T^{\star} > q(\eta_{\infty}^{\star}, 1-\alpha)$, which is the $(1-\alpha)$ quantile of the distribution of the null limiting distribution of the statistic at a significance level $\alpha \in (0, 1)$. The quantile is expected to depend on latent parameters, like the dependence structure of $(\varepsilon_t)_t$ (see also Theorem 7 below and the auxiliary Proposition 11 in the Appendix). As in Arvanitis et al. (2019), we approximate the quantile via the use of a subsampling procedure.

Specifically, given the choice of the subsampling rate $1 \leq b_T < T$, this generates the maximally overlapping subsamples $(R_s)_{s=t}^{t+b_T-1}$, $t = 1, \cdots, T-b_T+1$, evaluates the test statistic on each subsample, thereby obtaining $\eta^{\star}_{b_T;T,t}$ for $t = 1, \ldots, T-b_T+1$, hence resulting to the evaluation of $q_{T,b_T}(1-\alpha)$, the $(1-\alpha)$ quantile of the empirical distribution of $\eta^{\star}_{b_T;T,t}$ across the subsamples. Using the above, the modified decision rule is to reject $\mathbf{H}_0$ in favor of $\mathbf{H}_1$ iff $\eta_T^{\star} > q_{T,b_T}(1-\alpha)$.

Auxiliary Propositions 10–11 (see Appendix) imply that the (first order) limit theory of the procedure is similar to that of Arvanitis et al. (2019) (see their Online Appendix) when employed to the stationary parts of the returns. This is exemplified in the following result. Under our assumption framework and a standard assumption in the subsampling literature we have that:

Theorem 7. *As $T \to \infty$, under Assumptions ME and AN, and under the assumption that the subsample length (b_T), possibly depending on $(R_t)_{t=1,\ldots,T}$, satisfies $\mathbb{P}(l_T \leq b_T \leq u_T) \to 1$, where (l_T) and (u_T) are real sequences such that $1 \leq l_T \leq u_T$ for all T, $l_T \to \infty$ and $\frac{u_T}{T} \to 0$ as $T \to \infty$:*

1. *if $\mathbf{H}_0 : K \succeq_{\text{MESSD}} \boldsymbol{\Lambda}$ holds,*
$$\eta_T^{\star} \rightsquigarrow \sup_{\Lambda} \inf_{K} \sup_{\mathcal{X}} \mathcal{L}\left(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}\right), \ \left(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}\right) \in \text{CS},$$
where the contact set $\text{CS} := \{(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}) : \boldsymbol{\lambda} \in \Lambda, \boldsymbol{\kappa} \in \mathbf{K}, \boldsymbol{\kappa} \succeq_{\text{MESSD}} \boldsymbol{\lambda}, z \in \mathcal{X}, D(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}) = 0\}$ and the limiting process $\mathcal{L}\left(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}\right)$ is defined in Proposition 11.
2. *if $\mathbf{H}_0 : K \succeq_{\text{MESSD}} \boldsymbol{\Lambda}$ and $\exists\left(z^{\star}, \boldsymbol{\kappa}^{\star}, \boldsymbol{\lambda}^{\star}\right) \in \text{CS} : \text{Var}\left(\mathcal{L}\left(z^{\star}, \boldsymbol{\kappa}^{\star}, \boldsymbol{\lambda}^{\star}\right)\right) > 0$ then the testing procedure is asymptotically exact if $\alpha < 0.5$.*
3. *if $\mathbf{H}_1 : K \not\succeq_{\text{MESSD}} \boldsymbol{\Lambda}$ then the testing procedure is consistent.*

Theorem 7 essentially generalises the analogous results of Proposition 4 and Proposition B.2 in Arvanitis et al. (2019), since (i) it allows for unbounded return supports, and (ii) it avoids strict extreme points comparisons between the parameter spaces K, Λ. This is accomplished via a mild assumption on the existence of a non-trivial contact and for empirically plausible significance levels. When this does not hold, arguments similar to Assumption 4.1.4 of Arvanitis, Post, and Topaloglou (2020) can be employed to ensure asymptotic conservatism.

Arvanitis et al. (2019) also propose a bias correction procedure for the quantile estimates $q_{T,b_T}(1-\alpha)$ to mitigate their sensitivity on the choice of b_T in finite samples of realistic time series and cross sectional dimensions. They propose to choose $b_T = \lfloor T^c \rfloor$, with c ranging from 0.6 to 0.9, then estimate a regression of the estimated critical values and the subsample length $(\lfloor T^c \rfloor)$ for several values of c in the aforementioned range, and finally use the estimated regression line evaluated at T in order to obtain the bias corrected critical value. They argue that this procedure does not affect the limit theory, and they provide Monte Carlo experiments that show that this method is efficient and powerful even in small samples. This procedure is used in the empirical application of the present study.

2.3. A conservative test for pairwise (non-) dominance

For the out-of-sample performance assessment of the empirical application, we provide with a modification of the Davidson and Duclos (2013) pairwise (non-) dominance test. In contrast to most of the tests that appear in the literature where the null hypothesis is dominance of one prospect over the other, the Davidson and Duclos (2013) test posits as null the less logically strict hypothesis of non-dominance. We retain this structure, yet the hypotheses are more composite as they concern the comparison between every possible cluster point of the portfolio sequences.

The procedure allows for processes that appear in the context of Assumptions ME-AN, and for stochastic portfolio weights that may not be consistent or convergent at all. This is generally expected to be the case for the empirically optimal portfolio emerging from the stochastic spanning criterion. This approximates its population analogue, whereas the latter need not have unique optimisers.

Furthermore, the test statistic is essentially a supremum of easily computable t statistics over $\mathcal{X}$, and the rejection region is based on the standard normal distribution. Thereby, the procedure is independent of the choice of numerical approximation parameters (like the subsampling length) as resampling based approximations of the limiting rejection region are avoided.

Formally, suppose that $\boldsymbol{\lambda}_T$, $\boldsymbol{\kappa}_T$ represent (potentially) stochastic (data dependent) portfolios. Our formulation is general enough to allow for their derivation to be based on some optimisation procedure on a part of the original sample that functions as an estimation window. We wish to test for pairwise dominance between them using empirical information provided by potentially another part of the sample reserved for some out-of-sample analysis of size T. The estimation window can be fixed, rolling or recursive. Our analysis assumes that both T and the estimation window size diverge to infinity. We abusively let $\boldsymbol{\lambda}_T$, $\boldsymbol{\kappa}_T$ to depend on T for notational brevity.

In what follows, we provide with a plausible-additional to ME-AN-assumption framework that allows for the description of the hypothesis structure in this general framework, the description of the testing procedure and the derivation of its asymptotic conservatism. Notice that the following contain trivially the case where the associated portfolios are non-stochastic and/or fixed.

Our first additional assumption essentially ensures the existence of non-stochastic cluster points for every subsequence of the stochastic portfolios. Those will constitute the form of the hypothesis structure.

Assumption CCP. $\boldsymbol{\lambda}_T$, $\boldsymbol{\kappa}_T$ almost surely belong to Λ which is compact. Their almost sure cluster points are non-stochastic.

The first part of Assumption CCP trivially holds in our framework as Λ is the convex hull of the augmented set of base assets. Due to it, $\boldsymbol{\lambda}_T$, $\boldsymbol{\kappa}_T$ possess almost sure cluster points. The second part posits that the latter are not stochastic. It can be a strong restriction, yet we can dispense with at the cost of heavier notation for the notions and the results that follow (see the penultimate paragraph of the current section for a more detailed discussion). The test properties that we later derive would be unaffected.

The composite hypothesis structure is $\mathbf{H_0} : \wedge \boldsymbol{\lambda} \succeq_{\text{MESSD}} \boldsymbol{\kappa}$ vs. $\mathbf{H_1} : \vee \boldsymbol{\lambda} \not\succeq_{\text{MESSD}} \boldsymbol{\kappa}$, where the union and intersection is with respect to every pair of almost sure cluster point of $(\boldsymbol{\lambda}_T, \boldsymbol{\kappa}_T)$. Hence from now on, $(\boldsymbol{\lambda}, \boldsymbol{\kappa})$ will denote any arbitrary pair of almost sure cluster points.

Our second additional assumption ensures the representation of the hypothesis structure by functionals on $D^\star$.

Assumption CS. *(i) If $\boldsymbol{\lambda} \succeq_{\text{MESSD}} \boldsymbol{\kappa} \vee \boldsymbol{\kappa} \succeq_{\text{MESSD}} \lambda$ then for all $\boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u$ in some neighborhood of $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$ respectively, $\forall z \in \text{int}\mathcal{X}$, $D^\star(z, \boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u, \boldsymbol{R}) \neq 0$, and (ii) $\exists \delta > 0$ such that, for all $\boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u$ in some neighborhood of $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$ respectively, $D^\star(\cdot, \boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u, \boldsymbol{R})$ is monotone on $A_\delta := [\inf \mathcal{X}, \inf \mathcal{X} + \delta]$ and iii) $\boldsymbol{\lambda}_u$ and $\boldsymbol{\kappa}_u$ have different means.*

Assumption CS implies that the only zeros of the LPMs for every pair of portfolios that are local to any cluster point pair occur only at $\underline{x}$. It can be verified by mild restrictions on the joint distribution of the portfolio returns by taking into account the convexity of $D^\star(\cdot, \boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u, \boldsymbol{R}) = D^\star(\cdot, \boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u, \varepsilon)$.

Let $A^s_\delta := \mathcal{X} - A_\delta$ Under Assumption CS the hypothesis structure above is equivalent to: $\mathbf{H_0} : \wedge \sup_{z \in A^s_\delta} D^\star(z, \boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u, \boldsymbol{R}) \geq 0$ vs. $\mathbf{H_1} : \vee \sup_{z \in A^s_\delta} D^\star(z, \boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u, \boldsymbol{R}) < 0$. Our final additional assumption concerns properties of the limiting process that is related to $D^\star$.

Assumption GL. *For the limiting process in auxiliary Proposition 11 in the Appendix,* $0 < \epsilon \leq \inf_{A^s_\delta, \boldsymbol{\lambda}, \boldsymbol{\kappa} \in \Lambda^2} \text{Var}[\mathcal{L}(z, \boldsymbol{\kappa}, \boldsymbol{\lambda})]$.

The boundedness away from zero of the variance is establishable via Assumption CS and by assuming that the stationary distribution of ε_0 is supported on $\mathcal{X}^d$.

Under the above, a plausible uniformly consistent estimator for $\text{Var}[\mathcal{L}(z, \boldsymbol{\kappa}, \boldsymbol{\lambda})]$ is the Newey and West (1987) kernel estimator applied on the original return sample, that we actually employ in the empirical application, i.e.,

$$V_{T,L}(z, \boldsymbol{\lambda}, \boldsymbol{\kappa}) := \frac{1}{T} \sum_{l=0}^{L} \sum_{t=l+1}^{T} \left(1 - \frac{l}{L+1}\right) \left(\begin{array}{c} D(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}_t) D(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}_{t-l}) \\ -\frac{1}{T} \sum_{t=1}^{T} D(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}_t) \frac{1}{T} \sum_{t=1}^{T} D(z, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{R}_{t-l}) \end{array} \right),$$

where the bandwidth parameter $L > 0$. Using this, we employ as test statistic the following:

$$t^\star_z := \frac{D_T(z, \boldsymbol{\lambda}_T, \boldsymbol{\kappa}_T, \boldsymbol{R})}{\sqrt{V_T(z, \boldsymbol{\lambda}_T, \boldsymbol{\kappa}_T)}}, \; z \in A^s_\delta$$

$$t^\star_T := \sup_{z \in A^s_\delta} t^\star_z,$$

and reject $\mathbf{H_0}$ if and only if $t^\star_T < q(\alpha, \zeta)$ and $\zeta \sim N(0, 1)$ for $0 < \alpha < \frac{1}{2}$.

The following result derives limiting properties of the procedure.

Theorem 8. *Under Assumptions ME, AN, CCP, CS and GL, if $\mathbf{H_0}$ holds and $L \to \infty$, $\frac{L}{T^{\frac{1}{4}}} \to 0$, then $\limsup_{T \to \infty} \mathbb{P}(t^\star_T < q(\alpha, \zeta)) \leq \alpha$. Under the same assumptions, if $\mathbf{H_1}$ holds then $\limsup_{T \to \infty} \mathbb{P}(t^\star_T < q(\alpha, \zeta)) = 1$. If moreover under $\mathbf{H_1}$, $\wedge \sup_{z \in A^s_\delta} D^\star(z, \boldsymbol{\lambda}_u, \boldsymbol{\kappa}_u, \boldsymbol{R}) < 0$, then $\lim_{T \to \infty} \mathbb{P}(t^\star_T < q(\alpha, \zeta)) = 1$.*

Under standard restrictions on the limiting behaviour of the bandwidth (see Newey & West (1987)), the procedure is asymptotically conservative. Under the alternative and the stronger condition that $\sup_{z \in A^s_\delta} D(z, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \boldsymbol{R}) < 0$ uniformly with respect to every pair of cluster points, the procedure is also consistent. Notice that this uniformity condition actually holds whenever the set of cluster points is closed, due to the compactness of A^s_δ, and the joint continuity of $D^\star(z, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \boldsymbol{R})$.

Had we allowed for stochastic cluster points, and accordingly modified the hypotheses structure above, an analogous result would hold whenever the criterion from which the potentially stochastic point arises, has a deterministic limit-possibly after rescaling (e.g. via multiplication by $\frac{1}{\sqrt{T}}$). This would directly imply that the cluster point becomes asymptotically independent from $\frac{D_T(z, \cdot, \cdot, \boldsymbol{R})}{\sqrt{V_T(z, \cdot, \cdot)}}$, and then conservativeness would follow by the additional use of the LIE in the current part of the proof, while consistency would readily follow by the conditionally almost sure divergence of the statistic (see also the proof of Theorem 4.2.1 in Arvanitis et al. (2020), for analogous arguments).

In practice, $t^\star_T$ is approximated by some finite stochastic discretisation/approximation of A^s_δ via the empirical observations of some base asset. Under further conditions ensuring appropriate limiting properties for the relevant quantile process, the results of Theorem 8 continue to hold (see, Molchanov (2006), Appendix B for the relevant notion of Painleve-Kuratowski set convergence).

3. Empirical application

Data on daily closing prices of several indices obtained from Bloomberg are used to proxy the traditional asset universe. The S&P 500 Total Return Index (S&P 500), the Barclays US Aggregate Bond Index (Bond Index), and the three-month T-bill rate (3-m Tbill) are employed for the equity market, the bond market and the risk-free rate, respectively. Additionally, to ensure robustness of the results to the choice of the traditional asset universe, we include the dynamic trading strategies SMB and HML, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds. Daily data on Bitcoin, Ethereum, Ripple and Litecoin US dollar closing prices are also extracted from the Bitfinex exchange market through the CoinMarketCap. The dataset spans the period from August 7, 2015 to April 30, 2020, a total of 1191 daily return observations, given that Ethereum was first publicly traded in July 2015. The choice of the four cryptocurrencies was based on the fact that these are the largest cryptocurrencies for the full sample in terms of market capitalisation.

Table 1 reports summary statistics. In terms of performance, the daily average return of the four cryptocurrencies is as expected higher than that of stocks and bonds; the same holds for the Sharpe ratio, although, cryptocurrencies show considerable standard deviation compared to the mean. Bitcoin, as the oldest cryptocurrency, has the lowest standard deviation. Unlike traditional assets, the skewness of cryptocurrencies is positive and the kurtosis is high, indicating under ME-AN, deviations from normality of their stationary part and providing motivation for the stochastic spanning methodology instead of the Mean-Variance.

3.1. Bubble tests

Phillips and Magdalinos (2007) propose that explosive behaviour in asset prices can be regarded as a signal of bubble behaviour. To test for the existence of speculative bubbles in the

Table 1
Descriptive statistics of daily returns.

	Mean	S.D.	Skewness	Kurtosis	Sharpe ratio
Asset					
S&P 500	0.00036	0.01212	−0.61457	21.25391	0.02564
Bond Index	0.00007	0.00292	−1.32057	55.29573	0.00701
3-m Tbill	0.00005	0.00003	0.07553	−1.46242	–
SMB	−0.00016	0.00597	0.53217	10.87851	−0.03426
HML	−0.00032	0.00676	−0.14575	6.69861	−0.05412
Russel 2000	0.00018	0.01450	−1.24475	18.47449	0.00889
Vanguard Value	0.00021	0.01196	−0.65239	21.25995	0.01381
RVanguard Small Cap	0.00006	0.01373	−1.35878	21.30487	0.00062
Bitcoin	0.00402	0.04733	0.04354	7.12097	0.08393
Ethereum	0.00731	0.08460	0.81035	13.60478	0.08587
Ripple	0.00588	0.08571	4.11833	39.22291	0.06808
Litecoin	0.00430	0.07016	2.47009	20.87801	0.06506

Entries report the descriptive statistics on daily returns from August 7, 2015 to April 30, 2020. The S&P 500 Total Return Index (S&P 500), the Barclays US Aggregate Bond Index (Bond Index), the three-month T-bill rate (3-m Tbill), the SMB and HML strategies, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds are employed for the equity market, the bond market and the risk-free rate, respectively. Bitcoin, Ethereum, Ripple and Litecoin US dollar closing prices are used to assess the cryptocurrency market. The average return, the standard deviation (S.D.), the skewness, the kurtosis, as well as the Sharpe ratio are reported.

cryptocurrencies prices, the PSY test for multiple bubbles, which includes the earlier PWY test procedure, is used.

Under the null hypothesis, logarithmic prices have a random walk unit root, and under the alternative hypothesis there is at least one sub-period with mild explosivity. The PSY test relies on a recursively estimation of right tailed Dickey-Fuller tests over rolling windows of increasing sizes, where r_0 is the smallest sample window width fraction (specified by the user to initialise computation) and 1 is the largest window fraction, i.e., the total sample size.

The test statistic (GADF) of PSY is defined as the supremum value of the ADF statistic sequence over all feasible values of r_1 and r_2:

$$GSADF(r_0) := \sup_{\substack{r_2\in[r_0,1]\\ r_1\in[0,r_2-r_0]}} ADF_{r_1,r_2}$$

where starting points r_1,are allowed to change within the range $[0, r_2 - r_0]$, and where r_2 is the respective end point of the estimation window.

If the null hypothesis is rejected, the bubble-test procedure can also be used under general regularity conditions as a date-stamping strategy for the estimation of the origination and termination of bubbles. Specifically, the strategy performs a supremum ADF test on a backward expanding sample sequence where the end point of each sample is r_2 and the start point varies within the range $[0, r_2 - r_0]$. The corresponding ADF statistic sequences is $\left\{ADF_{r_1}^{r_2}\right\}_{r_1\in[0,r_2-r_0]}$ and the backward statistic $BSADF_{r_2}(r_0)$ is the supremum value of the ADF statistic sequence over this interval.

The beginning of the bubble is estimated as

$$\hat{r}_{start} := \inf_{r_2\in[r_0,1]}\{r_2 : BSADF_{r_2}(r_0) > cv_{r_2}(\alpha_T)\}$$

and the burst is estimated as

$$\hat{r}_{end} := \inf_{r_2\in\left[\hat{r}_{start},1\right]}\{r_2 : BSADF_{r_2}(r_0) < cv_{r_2}(\alpha_T)\}\ ,$$

where $cv_{r_2}(\alpha_T)$ is the $100(1-\alpha_T)$ critical value of the PSY statistic based on $\lfloor Tr_2 \rceil$ observations. To eliminate Type I error there is a need for $\alpha_T \to 0$ as $T \to 0$.

The PSY test is applied to the set of the four cryptocurrencies using a starting proportion of 10% of the data (Phillips and Shi, 2020). Table 2 shows the test statistics along with the 10%, 5% and 1% critical values of the PSY distribution obtained by a Monte Carlo simulation. The shaded areas in Fig. 1 are the identified periods of bubbles obtained using the 95% bootstrap critical values (1,000 replications are used) for (a) Bitcoin, (b) Ethereum, (c) Ripple and (d) Litecoin. As is evident in the Figure, the procedure detects multiple bubble episodes for the four cryptocurrencies.

Table 3 reports the count of days per quarter in which the BSADF statistic overcomes the critical value, supporting evidence of an explosive price behaviour for each cryptocurrency. In general, the second, the third and the fourth quarter of 2017 were the periods characterised by the largest number of bubble episodes in the cryptocurrency market. It can be noticed that Bitcoin and Ethereum are the ones showing the highest number of explosivity days (308 and 312 respectively). However, the distribution of

Table 2
Empirical results of the PSY test.

	Bitcoin	Ethereum	Ripple	Litecoin
GSADF	3.380	3.878	6.929	4.565
Test critical values				
99%	2.779			
95%	2.202			
90%	1.953			

Empirical results of the PSY test applied to the four cryptocurencies, *p*-value is the simulated *p*-value. 10%, 5% and 1% are the critical values of the PSY distribution obtained by a Monte Carlo simulation.

Table 3
Number of explosivity days.

	Bitcoin	Ethereum	Ripple	Litecoin
2016 Q1	0	40	0	0
2016 Q2	17	6	0	4
2016 Q4	0	4	0	0
2017 Q1	14	21	0	0
2017 Q2	52	90	90	72
2017 Q3	33	57	82	77
2017 Q4	100	25	38	41
2018 Q1	33	31	68	59
2018 Q3	0	1	3	1
2018 Q4	21	28	0	9
2019 Q1	35	0	0	0
2019 Q2	3	3	0	0
2020 Q1	0	6	0	0

Entries report the number of days per quarter characterised by explosive price behaviour according to the BSADF test for the four analysed cryptocurrencies over the period from August 7, 2015 to April 30, 2020.

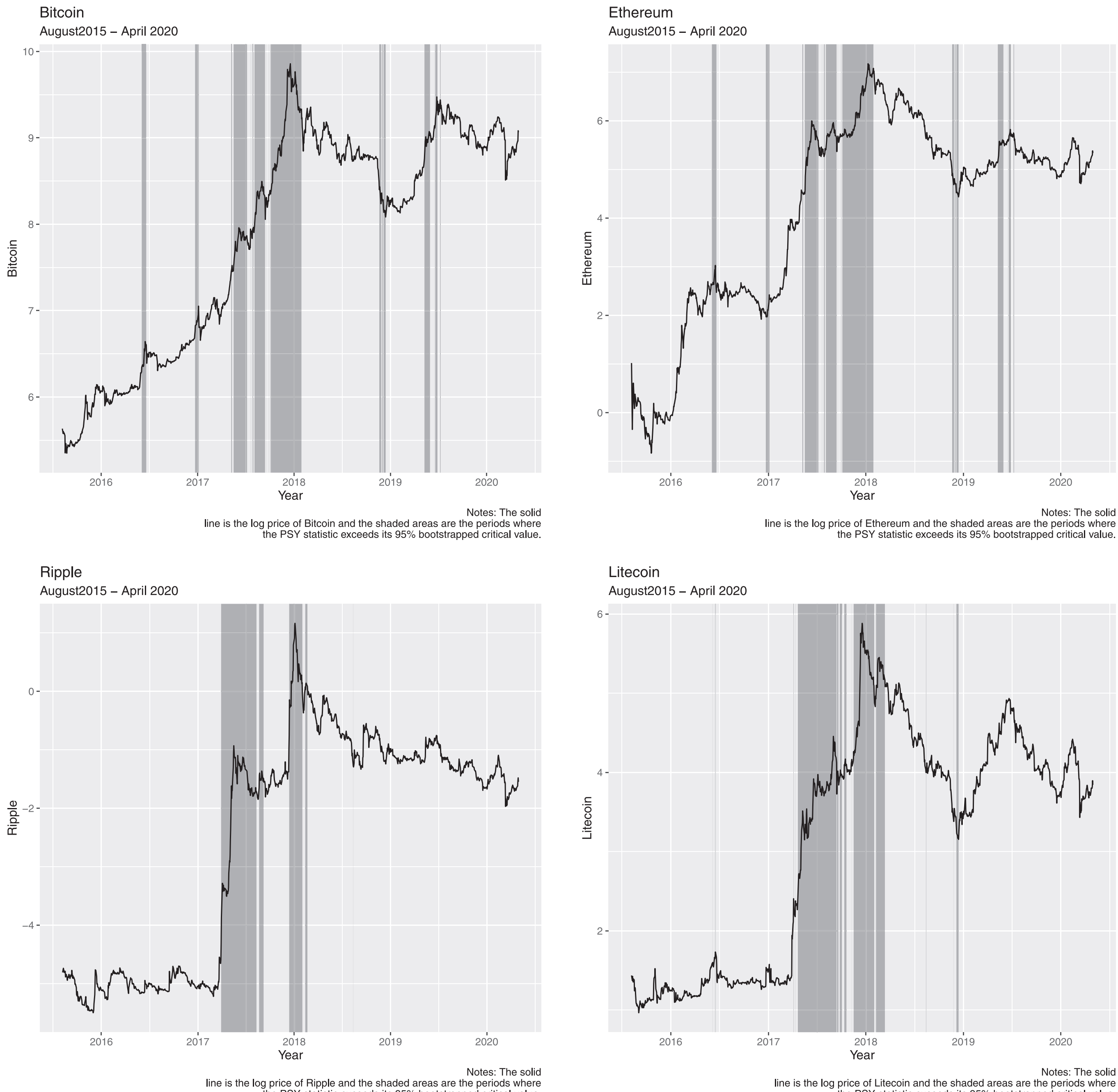

Fig. 1. Bubble periods in (a) Bitcoin (b) Etherum (c) Ripple (d) Litecoin for the sample period from August 7, 2015 to April 30, 2020.

bubble episodes of the two assets over the period was not so similar: for Bitcoin, the last quarter of 2017 was the sub-period showing the highest number of explosivity days, which was instead the second quarter of the same year for Ethereum. During the 2016–2018 period, 263 bubble days were reported for Litecoin, with the highest concentration in the third quarter of 2017. Over the 281 explosivity days of Ripple, 90 were observed in the second quarter of 2017.

Despite the non-identifiability, the bubble tests along with their time-stamping provide some evidence in favour of Assumption AN: given the large sample period under study it is seen that for all four cryptocurrencies, the majority of the bubbles (those of significant duration) occurred somewhere in the period 2016–2018 and that the last bubble (of significant duration) appears to have burst over a year before the end of our sample period.

3.2. In-sample analysis

In this section we test in-sample whether the traditional asset universe spans the one with cryptocurrencies. The null hypothesis is that the traditional asset class stochastically spans the augmented one under MESSD. We use the stochastic spanning in the mildly explosive framework developed in Section 2.

The computational strategy and the proofs for the Linear Programming (LP) problem are described in the Online Appendix of Arvanitis et al. (2019). Using their notation and the fact that the empirical application is based on the entire available history of daily investment returns to a standard set of benchmark assets ($N = 8,\ T = 1190$), we have that the empirical application uses $N_1 = 10$ and $N_2 = 5$, which gives $N_3 = \frac{1}{9!}\prod_{i=1}^{9}(4+i) = 715$ distinct utility functions and $2N_3 = 1430$ small LP problems, which is per-

Table 4
In-sample analysis: Test statistics and critical values.

	$\eta_T^{\star}$	q_T^{BC}	*Decision*
Full period	0.3481	0.3313	Reject Spanning
Sub-period 1	0.2395	0.2292	Reject Spanning
Sub-period 2	0.0	0.0763	Spanning

Stochastic spanning tests under MESSD of the traditional portfolio with respect to the portfolio with cryptocurrencies. The table exhibits the test statistics $\eta_T^{\star}$ and the bias corrected regression estimates of critical values q_T^{BC} from August 7, 2015 to April 30, 2020; and from August 7, 2015 to 29 December, 2017 (sub-period 1) and from January 1, 2018 to April 30, 2020 (sub-period 2). The S&P 500 Total Return Index (S&P 500), the Barclays US Aggregate Bond Index (Bond Index) the three-month T-bill rate (3-m T-bill), the SMB and HML strategies, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds are employed for the equity market, the bond market and the risk-free rate, respectively. Bitcoin, Ethereum, Ripple and Litecoin US dollar closing prices are used to assess the cryptocurrency market.

fectly manageable with modern-day computer hardware and solver software. The total run time of all computations for our application amounts to several working days on a standard desktop PC with a 2.93 GHz quad-core Intel i7 processor, 16GB of RAM and using MATLAB and GAMS with the Gurobi solver.

The subsampling distribution of the test statistic is derived for subsample size $b_T \in [T^{0.6}, T^{0.7}, T^{0.8}, T^{0.9}]$.

In order to analyse the performance of the portfolios with and without cryptocurrencies, before and after the spike observed in the last quarter of 2017 (see also the previous graphs), we also conduct the analysis for two sub-periods: from August 7, 2015 to December 29, 2017 (sub-period 1), and from January 1, 2018 to April 30, 2020 (sub-period 2). This does not contradict AN as long as we assume that the sub-period cut is of order cT for some $c \in (0, 1)$.

Table 4 summarises the test statistics $\eta_T^{\star}$. To avoid the specification of the subsample length and correct for bias in finite samples, the regression-based bias-correction method of Arvanitis et al. (2019) is used, and the regression estimates q_T^{BC} are given for significance level $\alpha = 0.05$.

The null hypothesis that the traditional asset class spans the augmented asset class under MESSD is rejected for the full period; the regression estimate $q_T^{BC} = 0.3313$ is lower than the value of the test statistic $\eta_T^{\star} = 0.3481$. Moreover, the results indicate that the performance of traditional portfolios can be enhanced by including cryptocurrencies in sub-period 1. Thus, some risk averse investors could benefit from the augmentation. However, an interesting feature of the data is that in sub-period 2 the null hypothesis of stochastic spanning is not rejected, reflecting the fact that during this period cryptocurrencies exhibited negative returns.

3.3. Out-of-sample analysis

In the in-sample analysis the tests tend to reject the null hypothesis of stochastic spanning for the full sample and for sub-period 1. This is an indication that the cryptocurrency market is segmented from equity and bond markets and exhibits characteristics of a unique asset class, and hence diversification benefits exist. However, we should also analyse whether the augmented portfolios outperform the traditional ones in an out-of-sample setting. We construct the optimal portfolios at time t based on the information prevailing at time t, and we calculate the realised return of these portfolios over the period $[t, t+1]$ (next day). The out-of-sample test is a real-time exercise mimicking the way that a real-time investor act.

Given the above, in the out-of-sample analysis we construct optimal portfolios separately for the two asset universes: one that includes only the traditional asset classes (hereafter traditional), i.e., equities, bonds and cash, and one that is also includes cryptocurrencies (hereafter augmented). Backtesting experiments are conducted on a rolling horizon basis. The rolling horizon simulations cover the 1190 working day period from August 7, 2015 to April 30, 2020. First, at each day, the data from the previous year (which consist of 269 daily observations) are used to calibrate the procedure. The resulting optimisation problem is solved for the stochastic spanning test and the optimal portfolios are recorded. The clock is advanced one day, and the realised returns of the optimal portfolios are determined from the actual returns of the various assets. This procedure is repeated for the next time period and so on. At the end, the ex post realised returns over the period from September 1, 2016 to April 30, 2020 (921 working days) are computed for both portfolios. As a robustness test, we also conducted an expanding window experiment and find that the results and general conclusions are similar (not shown here but are available upon request).

Fig. 2 illustrates the cumulative performance of the traditional versus the augmented optimal portfolios for the period from September 1, 2016 to April 30, 2020. The augmented optimal portfolio has more than 11 times higher value at the end of the holding period compared to the beginning, while the traditional portfolio has a small loss. Not surprisingly, the relevant performance of the portfolios with cryptocurrencies is 13 times higher compared to that of the traditional ones. Moreover, another interesting result is that the optimal augmented portfolios include large weights in cryptocurrencies (more than 90%).

The analysis is repeated in each of the two sub-periods. Fig. 3 illustrates the cumulative performance of the traditional and the augmented optimal portfolios for the two sub-periods. In the first sub-period (September 1, 2016 to December 29, 2017) the augmented optimal portfolio has more than 88 times higher value at the end of the holding period compared to the beginning while the traditional portfolio has only 1.32 times higher value. The relevant performance of the portfolios with cryptocurrencies is 78 times higher compared to that of the traditional portfolios. We obderve the opposite results for the second sub-period (January 1, 2018 to April 30, 2020). The augmented portfolio exhibits huge losses, and its final value is one seventh of the initial.

3.4. Out-of-sample performance assessment

The two optimal portfolios formed by the respective two asset universes, i.e., the traditional and the augmented, are compared in terms of their out-of-sample performance by using both the non-parametric stochastic (non-) dominance test presented previously as well as some well-known parametric performance measures.

3.4.1. Non-parametric stochastic dominance test

We apply our modification of the Davidson and Duclos (2013) pairwise (non-) dominance test on the two optimal portfolios derived in the previous out-of-sample analysis. According to the notation of Section 2.3, $\boldsymbol{\lambda}_T$ is the augmented optimal portfolio and $\boldsymbol{\kappa}_T$ is the traditional one.

Table 5 reports the quartile p-values from the distribution of daily portfolio returns, for the null hypothesis that the augmented portfolio does not stochastically dominate the traditional one by second order (see Davidson & Duclos, 2013). The results entail $T-1$ (820) overlapping periods for the in-sample fitting of the two portfolios with corresponding out-of-sample comparisons. The $T-1$ p-values are considered from September 1, 2016 to April 30, 2020, using overlapping periods of 100 daily returns both for the full sample and for the two sub-periods. The quartile p-values from the distribution of the $T-1$ modified t-test statistics are computed.

In the full sample, for the 25% (in 5%) and 50% (in 10%) quartile p-values, the null hypothesis that the augmented optimal portfolio

Fig. 2. Cumulative performance of the traditional optimal portfolio and of the optimal augmented portfolio with cryptocurrencies for the period from September 1, 2016 to April 30, 2020. The traditional set includes the S&P 500 Equity Index, the Barclays US Aggregate Bond Index, the three-month T-bill rate, the SMB and HML indices, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds. The augmented portfolio includes additionally Bitcoin, Ethereum, Ripple and Litecoin.

Table 5
Out-of-sample performance: Non-parametric stochastic dominance test.

	Traditional vs Augmented
Full Period	
Quartile	
25% Rejection rate	35.57%
50% Rejection rate	48.14%
75% Rejection rate	61.59%
Sub-period 1	
Quartile	
25% Rejection rate	62.95%
50% Rejection rate	75.82%
75% Rejection rate	91.17%
Sub-period 2	
Quartile	
25% Rejection rate	2.33%
50% Rejection rate	3.01%
75% Rejection rate	4.97%

Entries report quartile rejection rates from the distribution rejection rates across out-of-sample periods under the null hypothesis that the augmented cryptocurrencies optimal portfolio does not second order stochastically dominate the optimal traditional portfolio using a modification of the Davidson and Duclos (2013) test statistic, over the period from September 1, 2016 to April 20, 2020. Sub-period 1 is from September 1, 2016 to December 29, 2017 and sub-period 2 is from January 1, 2018 to April 30, 2020. The traditional set includes the S&P 500 Equity Index, the Barclays US Aggregate Bond Index, the three-month T-bill rate, the SMB and HML indices, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds. The augmented portfolio includes additionally Bitcoin, Ethereum, Ripple and Litecoin.

does not stochastically dominate the traditional one by second order is rejected in all cases. In sub-period 1, the null hypothesis is rejected even in the 5% confidence interval. However, in sub-period 2 the null hypothesis cannot be rejected.

3.4.2. *Parametric tests*

Finally, we compute a variety of commonly used parametric performance measures: the Sharpe ratio, the downside Sharpe ratio of Ziemba (2005) which uses the downside variance, the UP ratio which compares the upside potential to the shortfall risk over a specific target of Sortino and Van Der Meer (1991), and the portfolio turnover. Moreover, we use the opportunity cost, θ, of Simaan (1993), which is a measure for the economic significance of the performance difference of two optimal portfolios. It is defined as the return that needs to be added to (or subtracted from) the traditional portfolio return, so that the investor is indifferent (in utility terms) between the strategies imposed by the two different investment opportunity sets. We further assess the performance of the two portfolios under the risk-adjusted (net of transaction costs) returns measure of DeMiguel, Garlappi, and Uppal (2009), which is an indicator of how the proportional transaction cost generated by the portfolio turnover affects the portfolio returns. It is possible to prove a result of asymptotic negligibility for the bubble periods, similar to Proposition 1, for the aforementioned performance measures under Assumption AN.

For the transaction cost of stocks and bonds, 35 bps are typically used in the literature. In the cryptocurrency market, transaction

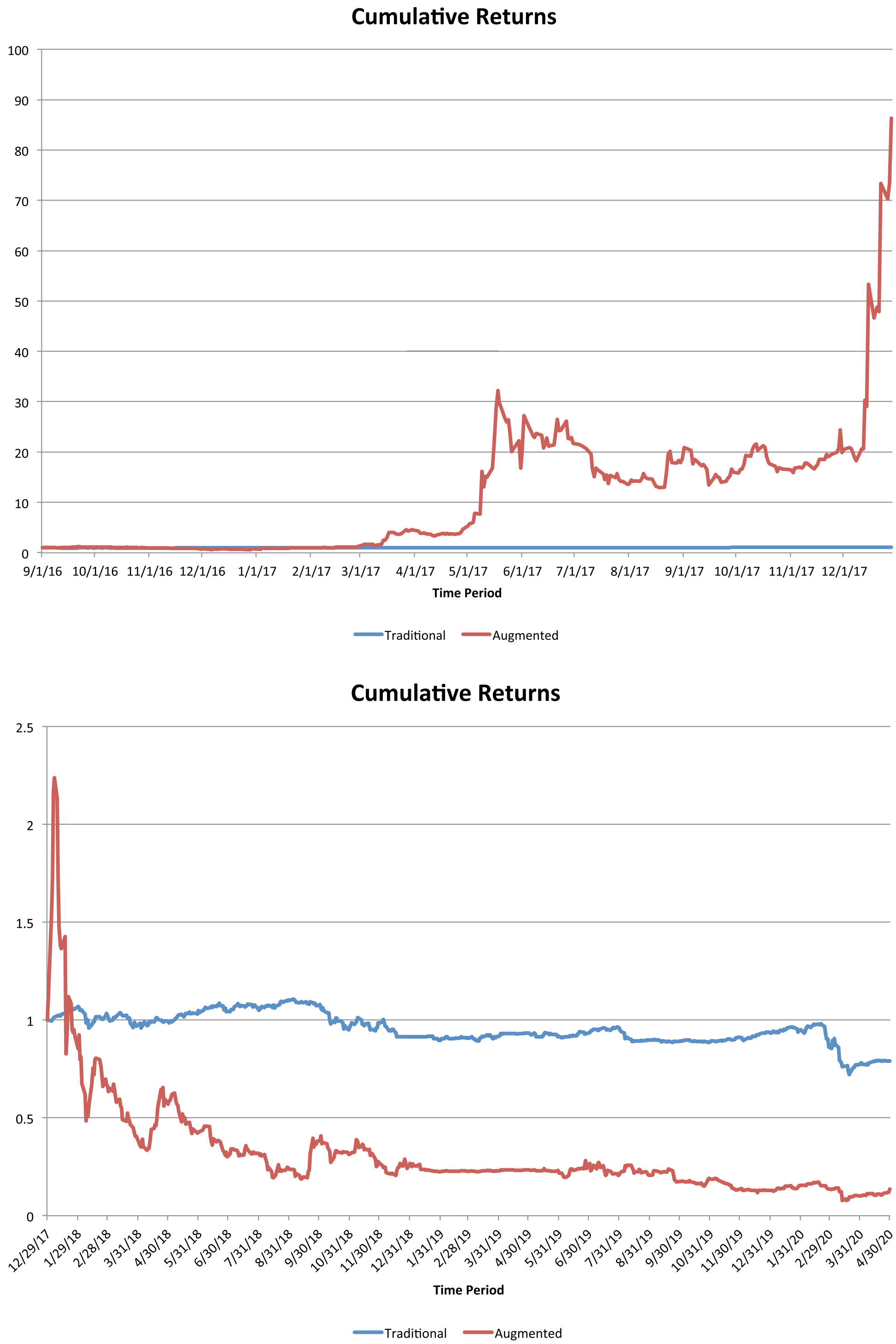

Fig. 3. Cumulative performance of the traditional optimal portfolio and of the augmented optimal portfolio for two sub-periods (a) September 1, 2016 to December 29, 2017; (b) January 1, 2018 to April 30, 2020. The traditional set includes the S&P 500 Equity Index, the Barclays US Aggregate Bond Index, the three-month T-bill rate, the SMB and HML indices, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds. The augmented portfolio includes additionally Bitcoin, Ethereum, Ripple and Litecoin.

costs occur in two ways: trading fees and bid-ask spread. Bid-ask spread is a type of risk premium to compensate market dealer for providing liquidity. To execute a transaction, investors should pay an additional premium to the exchange. Usually the exchange will ask for a high premium to reduce a potential loss by providing liquidity to the informed traders. Overall, the bid-ask spread is a minor issue compared to the normal price deviation. In contrast, other fees create more frictions for investors to make arbitrage. For example, BTC-E charges a 0.2 to 0.5 percent fee per transaction along with fees to deposit or withdraw traditional currency. According to CryptoCoins News, there is currently a $20 fee for a wire deposit. Bitstamp and Bitfinex also charge trading fees and deposit/withdrawal/fees. In order to confirm the out-of-sample superiority of the augmented portfolio, the transaction cost for the cryptos is set to 60 bps, higher than the one used for stocks and bonds.

Table 6
Out-of-sample performance: Full-sample parametric measures.

	Traditional	Augmented
Full Period		
Performance Measures		
Average return	−0.0001	0.0061
Standard Deviation	0.0089	0.0887
Sharpe ratio	−0.0154	0.0676
Downside Sharpe Ratio	−0.0139	0.967
UP ratio	0.4042	0.5999
Portfolio Turnover	9.53%	8.47%
Return Loss	0.069%	
Opportunity Cost		
Exponential Utility		
ARA=2	0.039%	
ARA=4	−0.582%	
ARA=6	−1.126%	
ARA=12	−1.935%	
Power Utility		
RRA=2	0.025%	
RRA=4	−0.616%	
RRA=6	−1.280%	
RRA=12	−1.992%	

Entries report the performance measures (Sharpe ratio, Downside Sharpe ratio, UP ratio, Portfolio Turnover, Return Loss and Opportunity Cost) for the traditional and the augmented optimal portfolios. The results for the opportunity cost are reported for different degrees of absolute risk aversion (ARA=2,4,6,12) and different degrees of relative risk aversion (RRA=2,4,6,12). The realised daily returns cover the period from September 1, 2016 to April 30, 2020. The traditional set includes the S&P 500 Equity Index, the Barclays US Aggregate Bond Index, the three-month T-bill rate, the SMB and HML indices, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds. The augmented portfolio includes additionally Bitcoin, Ethereum, Ripple and Litecoin.

Table 6 reports the parametric performance measures for the traditional and the augmented portfolios for the full sample. These performance measures, although parametric, will supplement the evidence obtained from the previously discussed non-parametric stochastic (non-) dominance tests. The higher the value of each one of these measures, the greater the investment opportunities for cryptocurrencies. The inclusion of cryptocurrencies into the opportunity set increases both the Sharpe ratios and the downside Sharpe ratios. This reflects an increase in the risk-adjusted performance (i.e., an increase in the expected return per unit of risk) and hence, expands the investment opportunities for some risk averse investors. The same is true for the UP ratio. Furthermore, the portfolios with only traditional assets induce more portfolio turnover than the ones with cryptocurrencies. Additionally, the return-loss measure that considers transaction costs is positive. The opportunity cost θ is found to be positive for low levels of absolute and relative risk aversion. This result indicates that an investor with low risk aversion needs an additional return equal to θ to be indifferent between investing into the traditional and the augmented portfolio. In contrast, this is not true for higher levels of risk aversion. The opportunity cost definition relies on the computation of the expected utility or equivalently on the probability density function of portfolio returns. Thus, the opportunity cost takes into account higher order moments in contrast to the Sharpe ratios.

Table 7 reports the parametric performance measures for the traditional and the augmented portfolios for each sub-period. The results are different in the two sub-periods. In the first one, the augmented portfolios clearly outperform the traditional portfolios in all parametric performance measures. The inclusion of cryptocurrencies leads to substantial improvement of these portfolios compared to the traditional ones. We also observe that risk averse investors with low average coefficients of risk aversion will benefit from the inclusion of cryptocurrencies in their portfolios, but this is not true for investors with high degrees of risk aversion. We observe the opposite picture in the second sub-period. The augmented portfolios exhibit the worst performance with respect to all parametric performance measures.

Table 7
Out-of-sample performance: Sub-period parametric measures.

	Sub-period 1		Sub-period 2	
	Traditional	Augmented	Traditional	Augmented
Performance Measures				
Average return	0.0004	0.0188	−0.0004	−0.0012
Standard Deviation	0.0073	0.1166	0.0097	0.0670
Sharpe ratio	0.0050	0.1610	−0.0438	−0.0188
Downside Sharpe Ratio	0.0501	0.3192	−0.0384	−0.0198
UP ratio	0.5698	0.9404	0.3515	0.4226
Portfolio Turnover	7.83%	7.14%	9.27%	8.43%
Return Loss	0.078%		−0.018%	
Opportunity Cost				
Exponential Utility				
ARA=2	0.811%		−0.515%	
ARA=4	0.107%		−0.964%	
ARA=6	−0.476%		−1.483%	
ARA=12	−0.854%		−2.164%	
Power Utility				
RRA=2	0.875%		−0.530%	
RRA=4	0.149%		−1.040%	
RRA=6	−0.472%		−1.720%	
RRA=12	−0.813%		−2.316%	

Entries report the performance measures (Sharpe ratio, Downside Sharpe ratio, UP ratio, Portfolio Turnover, Returns Loss and Opportunity Cost) for the traditional and the augmented optimal portfolios. The results for the opportunity cost are reported for different degrees of absolute risk aversion (ARA=2,4,6,12) and different degrees of relative risk aversion (RRA=2,4,6,12). The realised daily returns cover the period from September 1, 2016 to April 30, 2020. Sub-period 1 is from September 1, 2016 to December 29, 2017 and sub-period 2 is from January 1, 2018 to April 30, 2020. The traditional set includes the S&P 500 Equity Index, the Barclays US Aggregate Bond Index, the three-month T-bill rate, the SMB and HML indices, the Russell 2000 equity index, the Vanguard value and the Vanguard small-cap index funds. The augmented portfolio includes additionally Bitcoin, Ethereum, Ripple and Litecoin.

Table 8
Out-of-sample analysis: average portfolio composition.

	Full Period		Sub-period 1		Sub-period 2	
	Trad.	Augmt.	Trad.	Augmt.	Trad.	Augmt.
S&P 500	37.84%	4.22%	15.57%	0.0%	50.52%	6.63%
Bond Index	12.63%	2.23%	0.0%	0.0%	19.82%	3.50%
3-m Tbill	4.04%	1.62%	0.0%	0.0%	6.34%	2.54%
SMB	1.79%	0.28%	0.0%	0.0%	0.35%	0.44%
HML	0.54%	0.0%	1.5%	0.0%	0.0%	0.0%
Russell 2000	36.77%	1.04%	61.38%	0.0%	22.77%	1.63%
Vanguard V.	0.12%	0.01%	0.0%	0.0%	0.19%	0.02%
Vanguard s-c.	7.82%	0.0%	21.56%	0.0%	0.0%	0.0%
Bitcoin		21.05%		0%		33.02%
Ethereum		18.78%		50%		1.02%
Ripple		44.21%		49.40%		41.26%
Litecoin		6.55%		0.60%		9.94%

Entries report the average portfolio compositions for the full period and for the two sub-periods. Sub-period 1 is from September 1, 2016 to December 29, 2017 and sub-period 2 is from January 1, 2018 to April 30, 2020. The traditional set includes the S&P 500 Total Return Index (S&P 500), the Barclays US Aggregate Bond Index (Bond Index) and the three-month T-bill rate (3-m Tbill), the SMB and HML indices, the Russell 2000 equity index (Russell 2000), the Vanguard value (Vanguard V.) and the Vanguard small-cap index funds (Vanguard s-c.). The augmented portfolio includes additionally Bitcoin, Ethereum, Ripple and Litecoin.

Table 8 reports the average portfolio compositions for the full sample and for each sub-period. We observe that in the full period as well as in each sub-period, the optimal traditional portfolios consist mainly of the S&P 500 and the Russel 2000 indices, followed by the Bond Index. However, the optimal augmented portfolio includes mainly cryptocurrencies. In the full period, the optimal augmented portfolios are consisted by 90% of cryptocurrencies (Bitcoin, Ethereum and Ripple mostly), and 10% of traditional assets (mainly the S&P 500 and the Bond Index). In the first sub-period, the optimal portfolios include only Ethereum and Ripple, because of the huge increase in the price of these currencies. In the second sub-period, more than 15% of the augmented portfolios are invested in traditional assets, and the rest in Bitcoin, Ripple and Litecoin.

Overall, the modified stochastic (non-) dominance test and the employed parametric performance measures indicate that the inclusion of cryptocurrencies does not provide clear diversification benefits to all risk averse investors. In the full sample period and in the period of high cryptocurrency returns, the augmentation does provide investment opportunities to some risk averse investors making them better-off with respect to these performance criteria, but in the period of low cryptocurrency returns it does not. Investors with low and average degrees of risk aversion could benefit from the inclusion of cryptocurrencies in their portfolios, but for investors with high levels of risk aversion there are no clear diversification benefits. This is consistent to the empirical finding that cryptocurrencies constitute a separate asset class (see, for example, Bianchi, 2020).

The reported diversification benefits are justified by the in-sample tests presented previously, which provide evidence that cryptocurrency market is segmented from equity and bond markets, and exhibits characteristics of a unique asset class for the full sample period. This evidence is based on the argument that when markets are segmented, assets in one market are not spanned by assets in other markets and hence, diversification benefits arise. Further support to the later, is provided by the low correlations of cryptocurrencies with stocks and bonds (not reported here but available upon request). More importantly, in the first sub-period, the correlation coefficients for Ethereum are close to zero or negative. Large shocks in cryptocurrency prices exacerbate any market segmentation since risk-averse investors are less willing to trade across markets. Hence, optimal portfolios in this sub-period tend to hold large proportions of cryptocurrencies. In fact, as observed by the positive opportunity cost of these portfolios in the first sub-period, investors with low degrees of risk aversion benefit from the inclusion of cryptocurrencies in their portfolios. Our results also support the claim that the sub-period of low returns is not adequate to destroy the investment opportunities created by the sub-period of high returns for certain investors, specifically those with low levels of risk aversion.

4. Discussion and future research

Cryptocurrencies have drawn a significant level of critique on the one hand, and support on the other, both in increasing measures. We expanded analysis of the field by employing a stochastic spanning methodology to test whether cryptocurrencies offer diversification benefits to certain risk averse investors. We allowed for an empirically plausible framework of multiple – possibly interdependent – bubbles in cryptocurrencies by developing a modification of the notion of stochastic spanning compatible with a mildly explosive framework for the logarithmic prices, along with a condition of limiting bubble sparsity and uniformly bounded intensity that renders them asymptotically negligible. We conducted our analysis both in- and out-of-sample by constructing and comparing optimal portfolios derived from two respective asset universes: one that solely consists of traditional asset classes (equities, bonds and cash), and one that adds our four cryptocurrencies into the mix. The observed relative outperformance of the optimal portfolios with cryptocurrencies in the full sample period and in a sub-period of high cryptocurrency returns is justified using a modified consistent pairwise stochastic (non-)dominance test, where the null hypothesis is non-dominance.

Our sparsity and bounded intensity condition enabled the use of the original inferential procedures of Arvanitis et al. (2019) on the potentially non-stationary data, since it entails asymptotic stationarity at appropriate rates. We note, however, that since it is not empirically identifiable, its validity is based on theoretical arguments that posit, for example, that future bubbles on those assets will follow a historically established pattern for financial assets. If this does not hold and more generally the (essentially forward-looking) condition is not valid, then our results become ambiguous. Cryptocurrencies have a short history; at the same time, it is largely acknowledged that there are several issues to be addressed before cryptocurrencies can form a well-established asset class, for example, the technology of their production, their regulation by policymakers, etc. It may be thus of interest to extend the testing methodologies presented above in asymptotically persistent non-stationary frameworks for the returns. This is by no means trivial and presents an avenue for future research.

Under our assumption framework, the spanning test could also be used as a structural break kind of date-stamp, in order to trace consecutive maximal sub-periods where the spanning property alternates. Such information could help investors make better choices when forming their optimal portfolios. This is, however, another avenue for future research.

Appendix. Proofs and Auxiliary results

The appendix contains the proofs of the main results, as well as the derivation of several auxiliary results used in the proofs.

Proof of Proposition 1. In what follows CS in. abbreviates the Cauchy–Schwarz inequality, tr. in. the triangle inequality and norms in. the bounding from above of the max-norm by a constant multiple of the Euclidean norm in $\mathbb{R}^d$. $\odot$ denotes the Hadamard product. Due to the Lipschitz continuity property of $(\cdot)_+$,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\left(z-\boldsymbol{\lambda}^{\mathrm{T}}(\exp^{\star}(\mathbf{R}_t)-1)\right)_+-\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\left(z-\boldsymbol{\lambda}^{\mathrm{T}}(\exp^{\star}(\boldsymbol{\varepsilon}_t)-1)\right)_+\right|\right]\\
&\leq\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\mathbb{E}\left[\left|\boldsymbol{\lambda}^{\mathrm{T}}(\exp^{\star}(\mathbf{R}_t)-\exp^{\star}(\varepsilon_t))\right|\right]\\
&=\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\mathbb{E}\left[\left|\boldsymbol{\lambda}^{\mathrm{T}}\left(\left(\exp^{\star}\left(\sum_{k=1}^{K}\frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right)-\mathbf{1}\right)\odot\exp^{\star}(\varepsilon_t)\right)\right|\right]\\
&\underset{\text{CS in.}}{\leq}\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\mathbb{E}\left[\left\|\left(\exp^{\star}\left(\sum_{k=1}^{K}\frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right)-\mathbf{1}\right)\odot\exp^{\star}(\varepsilon_t)\right\|\right]\\
&\leq\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\mathbb{E}\left[\max_{j=1,\ldots,d}\left|\left(\exp^{\star}\left(\sum_{k=1}^{K}\frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right)-\mathbf{1}\right)_j\right|\|\exp^{\star}(\varepsilon_t)\|\right]\\
&\underset{\text{norms in.}}{\leq}\frac{c}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\mathbb{E}\left[\left\|\exp^{\star}\left(\sum_{k=1}^{K}\frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right)-\mathbf{1}\right\|\|\exp^{\star}(\varepsilon_t)\|\right]\\
&\underset{\text{tr. in.}}{\leq}\frac{c}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k}\mathbb{E}\left[\left(\left\|\exp^{\star}\left(\sum_{k=1}^{K}\frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right)\right\|+\sqrt{d}\right)\|\exp^{\star}(\varepsilon_t)\|\right],
\end{aligned}
\tag{1}
$$

where c in the final part of the previous display is independent of $z, \boldsymbol{\lambda}$ due to the Lipschitz continuity property of $(\cdot)_+$ and the fact that the simplex is bounded.

Furthermore notice that for any $\mathbf{x}\in\mathbb{R}^d$,

$$\|\exp^\star(\mathbf{x})\|^2 = \sum_{i=1}^d \exp^2(\mathbf{x}_i) \le \sum_{i=1}^d \exp^2(|\mathbf{x}_i|) \le d\exp^2\left(\max_i |\mathbf{x}_i|\right) \underset{\text{norms in.}}{\le} d\exp^2(c\|\mathbf{x}\|),$$

hence the final part of the previous display is less than or equal to

$$\frac{c\sqrt{d}}{\sqrt{T}}\sum_{t\in\cup_{k=1}^K B_k}\mathbb{E}\left[\left(\exp\left(\left\|\sum_{k=1}^K \frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right\|\right)+1\right)\exp(\|\varepsilon_t\|)\right]$$
$$\le \underbrace{\frac{c\sqrt{d}}{\sqrt{T}}\sum_{t\in\cup_{k=1}^K B_k}\mathbb{E}\left[\exp\left(\left\|\sum_{k=1}^K \frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right\|+\|\varepsilon_t\|\right)\right]}_{\text{A}}$$
$$+ \underbrace{\frac{c\sqrt{d}}{\sqrt{T}}\sum_{t\in\cup_{k=1}^K B_k}\mathbb{E}[\exp(\|\varepsilon_t\|)]}_{\text{B}}.$$

Due to ME.(1) we have that

$$0 \le \text{B} \le \frac{c\sqrt{d}}{\sqrt{T}}\sum_{t=0}^{\max_{t,k}B_k}\mathbb{E}[\exp(\|\varepsilon_t\|)] = c\sqrt{d}\exp(L)\frac{\max_{t,k}B_k}{\sqrt{T}} = o(1).$$

Now notice that under the usual conventions that $\prod_{j=0}^{-1} = 1$, $\sum_{j=0}^{-1} = 0$,

$$\sum_{k=1}^K \frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}$$
$$= \sum_{k=1}^K \frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\prod_{i=0}^{t-2}\left(\text{Id}+\sum_{k=1}^K\frac{C_k}{M(T,k)}\mathbb{I}\{t-1-i\in B_k\}\right)\boldsymbol{X}_0$$
$$+ \sum_{k=1}^K \frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\sum_{i=0}^{t-2}\prod_{j=0}^{i-1}\left(\text{Id}+\sum_{k=1}^K\frac{C_k}{M(T,k)}\mathbb{I}\{t-1-i\in B_k\}\right)\boldsymbol{\varepsilon}_{t-1-i}.$$

Therefore due to the triangle inequality and the Frobenius norm sub-multiplicativity

$$\left\|\sum_{k=1}^K \frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right\| + \|\varepsilon_t\|$$
$$\le \|\boldsymbol{X}_0\|\max_k\frac{\|C_k\|}{M(T,k)}\prod_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)$$
$$+ \max_k\frac{\|C_k\|}{M(T,k)}\sum_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^i\|\boldsymbol{\varepsilon}_{t-i-1}\| + \|\varepsilon_t\|,$$

and due to the previous and the Cauchy-Schwarz inequality

$$\mathbb{E}\left[\exp\left(\left\|\sum_{k=1}^K \frac{C_k}{M(T,k)}\mathbb{I}\{t\in B_k\}\boldsymbol{X}_{t-1}\right\|+\|\varepsilon_t\|\right)\right]$$
$$\le \sqrt{\mathbb{E}\left[\exp\left(2\max_k\frac{\|C_k\|}{M(T,k)}\prod_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)\|\boldsymbol{X}_0\|\right)\right]}$$
$$\times\mathbb{E}^{\frac14}\left[\exp\left(4\max_k\frac{\|C_k\|}{M(T,k)}\sum_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^i\|\boldsymbol{\varepsilon}_{t-i-1}\|\right)\right]$$
$$\times\mathbb{E}^{\frac14}[\exp(4\|\varepsilon_0\|)].$$

Due to ME.(1) the last term of the previous display is less than or equal to $\exp(4^{q-1}L)$. Under ME.(3) the first term is bounded from above by

$$\exp\left(2^{q^\star-1}L^\star\max_k\frac{\|C_k\|^{q^\star}}{M^{q^\star}(T,k)}\prod_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q^\star}\right).$$

Using the strong mixing inequality in Theorem 14.2 of Davidson (1994) for $p=q=1$, and the LIE, we have that the second term in the rhs of the previous inequality is less than or equal to

$$7^{\frac{\min\left(t-2,\max_{t,k}B_k\right)}{4}}\mathbb{E}^{\frac14}\left[\prod_{i\in\cup_k B_k\wedge t-2}\exp\left(4^qL\max_k\frac{\|C_k\|^q}{M^q(T,k)}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{qi}\right)\right]$$
$$=7^{\frac{\min\left(t-2,\max_{t,k}B_k\right)}{4}}\exp\left(4^{q-1}L\max_k\frac{\|C_k\|^q}{M^q(T,k)}\sum_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{qi}\right).$$

Putting the above together we have that A is less than or equal to

$$\frac{c\sqrt{d}\exp(4^{q-1}L)}{\sqrt{T}}\sum_{t\in\cup_{k=1}^K B_k}7^{\frac{\min\left(t-2,\max_{t,k}B_k\right)}{4}}$$
$$\exp\begin{pmatrix}c\max_k\frac{\|C_k\|^{q^\star}}{M^{q^\star}(T,k)}\prod_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q^\star}\\+\max_k\frac{\|C_k\|^q}{M^q(T,k)}\sum_{i\in\cup_k B_k\wedge t-2}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{qi}\end{pmatrix}$$
$$\le\frac{c\sqrt{d}\exp(4^{q-1}L)}{\sqrt{T}}\sum_{t\in\cup_{k=1}^K B_k}7^{\frac{t}{4}}$$
$$\exp\begin{pmatrix}c\max_k\frac{\|C_k\|^{q^\star}}{M^{q^\star}(T,k)}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q^\star\left(\max_{t,k}B_k-2\right)}\\+\max_k\frac{\|C_k\|^q}{M^q(T,k)}\frac{1-\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q(t-1)}}{1-\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^q}\end{pmatrix}$$
$$\le\frac{c\sqrt{d}\exp(4^{q-1}L)}{\sqrt{T}}$$
$$\exp\begin{pmatrix}c\max_k\frac{\|C_k\|^{q^\star}}{M^{q^\star}(T,k)}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q^\star\left(\max_{t,k}B_k-2\right)}\\+\max_k\frac{\|C_k\|^q}{M^q(T,k)}\frac{1-\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q\left(\max_{t,k}B_k-1\right)}}{1-\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^q}\end{pmatrix}\sum_{t=0}^{\max_{t,k}B_k}7^{\frac{t}{4}}$$
$$=\frac{c\sqrt{d}\exp(4^{q-1}L)}{\sqrt{T}}$$
$$\exp\begin{pmatrix}c\max_k\frac{\|C_k\|^{q^\star}}{M^{q^\star}(T,k)}\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q^\star\left(\max_{t,k}B_k-2\right)}\\+\max_k\frac{\|C_k\|^q}{M^q(T,k)}\frac{1-\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^{q\left(\max_{t,k}B_k-1\right)}}{1-\left(d+\max_k\frac{\|C_k\|}{M(T,k)}\right)^q}\end{pmatrix}\frac{1-7^{\frac{\max_{t,k}B_k+1}{4}}}{1-7^{\frac14}}$$

and the result follows due to Assumption AN. □

Proof of Theorem 1. (1) and (3) follow exactly as in the proofs of Propositions 4 and B.2 respectively in Arvanitis et al. (2019) (see their Online Appendix) given Proposition 11.

For (2) notice that if $\text{Var}\left(\mathcal{L}\left(z^\star,\boldsymbol{\kappa}^\star,\boldsymbol{\lambda}^\star\right)\right)>0$ and whenever $\mathcal{L}\left(z^\star,\boldsymbol{\kappa}^\star,\boldsymbol{\lambda}^\star\right)>0$, then $\sup_\Lambda\inf_K\sup_{\mathcal{X}}\mathcal{L}\left(z,\boldsymbol{\kappa},\boldsymbol{\lambda}\right)\ge\mathcal{L}\left(z^\star,\boldsymbol{\kappa}^\star,\boldsymbol{\lambda}^\star\right)$. Due to zero mean Gaussianity this occurs with probability at least 0.5. The rest follows as in the proof of Proposition B.2 in Arvanitis et al. (2019) (see their Online Appendix). □

Proof of Theorem 2. First notice that due to Propositions 10 and 12, as well as the arguments that lead to the proof of Theorem 2 of Newey and West (1987), $V_{T,L}\left(z,\boldsymbol{\lambda},\boldsymbol{\kappa}\right)$ is uniformly weakly consistent estimator of $\text{Var}\left[\mathcal{L}\left(z,\boldsymbol{\kappa},\boldsymbol{\lambda}\right)\right]$ when $L\to\infty$, $\frac{L}{T^{\frac14}}\to 0$. This, Proposition 10, and Assumption GL, imply that it suffices to consider $t_T:=\sup_{z\in A_\delta^s}\frac{D_T\left(z,\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T,\varepsilon\right)}{\sqrt{V_T\left(z,\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T\right)}}$. Due to Proposition 11, and

Assumptions CS and GL, if $\mathbf{H_0}$ holds then as $T\to\infty$,

$$t_z=\begin{array}{c}\frac{\sqrt{T}\left(D_T(z,\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T,\varepsilon)-D^\star(z,\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T,\varepsilon)\right)}{\sqrt{V_T(z,\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T)}}\\+\frac{\sqrt{T}D^\star(z,\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T,\varepsilon)}{\sqrt{V_T(z,\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T)}}\end{array}\rightsquigarrow\begin{cases}\frac{\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})}{\sqrt{\mathrm{Var}[\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})]}}, & \boldsymbol{\lambda}\sim\boldsymbol{\kappa}\\ +\infty, & \boldsymbol{\lambda}\prec\boldsymbol{\kappa}\end{cases},$$

where $\rightsquigarrow$ denotes weak convergence w.r.t. the set of pairs of cluster points of $(\boldsymbol{\lambda}_T,\boldsymbol{\kappa}_T)$, and thereby

$$t_T\rightsquigarrow t_\infty:=\begin{cases}\sup_{z\in A_\delta}\frac{\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})}{\sqrt{\mathrm{Var}[\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})]}}, & \boldsymbol{\lambda}\sim\boldsymbol{\kappa}\\ +\infty, & \boldsymbol{\lambda}\prec\boldsymbol{\kappa}\end{cases}.$$

Notice that $t_\infty\geq\frac{\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})}{\sqrt{\mathrm{Var}[\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})]}}\sim N(0,1)$ for any cluster point pair $(\boldsymbol{\lambda},\boldsymbol{\kappa})$ from which the first result follows. The second follows by noticing that under $\mathbf{H_1}$, $t_\infty:=\begin{cases}\sup_{z\in A_\delta}\frac{\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})}{\sqrt{\mathrm{Var}[\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})]}}, & \boldsymbol{\lambda}\sim\boldsymbol{\kappa}\\ +\infty, & \boldsymbol{\lambda}\prec\boldsymbol{\kappa}\\ -\infty, & \boldsymbol{\lambda}\succ\boldsymbol{\kappa}\end{cases}$. □

Auxiliary results

Proposition 9. *Suppose that Assumptions ME and AN hold. Then* $D^\star(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R})=D^\star(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}),\ \forall(z,\boldsymbol{\kappa},\boldsymbol{\lambda})$.

Proof. Due to stationarity for the $(\exp^\star(\boldsymbol{\varepsilon}_t)-1)_t$ process and Proposition 3, for any $z,\boldsymbol{\lambda}$,

$$\begin{aligned}&\left|\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\left(z-\boldsymbol{\lambda}^{\mathrm{T}}\exp^\star(\boldsymbol{R}_t)\right)_+\right]-\mathbb{E}\left[\left(z-\boldsymbol{\lambda}^{\mathrm{T}}(\exp^\star(\boldsymbol{\varepsilon}_0)-1)\right)_+\right]\right|\\&=\left|\frac{1}{T}\sum_{t=1}^{T}\left(\mathbb{E}\left[\left(z-\boldsymbol{\lambda}^{\mathrm{T}}\exp^\star(\boldsymbol{R}_t)\right)_+\right]-\mathbb{E}\left[\left(z-\boldsymbol{\lambda}^{\mathrm{T}}(\exp^\star(\boldsymbol{\varepsilon}_t)-1)\right)_+\right]\right)\right|\\&\leq\mathbb{E}\left[\left|\frac{1}{T}\sum_{t\in\cup_{k=1}^K B_k}\left(z-\boldsymbol{\lambda}^{\mathrm{T}}(\exp^\star(\boldsymbol{R}_t)-1)\right)_+\right.\right.\\&\left.\left.-\frac{1}{T}\sum_{t\in\cup_{k=1}^K B_k}\left(z-\boldsymbol{\lambda}^{\mathrm{T}}(\exp^\star(\boldsymbol{\varepsilon}_t)-1)\right)_+\right|\right]=o\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

And the result follows from the triangle inequality. □

Proposition 10. *Suppose that Assumptions ME and AN hold. Then as* $T\to\infty$,

$$\sup_{\Lambda,\mathrm{K},\mathcal{X}}\sqrt{T}\left|D_T(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R})-D^\star(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon})\right|=o_p(1).$$

Furthermore,

$$\left|\sup_{\Lambda}\inf_{K}\sup_{\mathcal{X}}\sqrt{T}D_T(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R})-\sup_{\Lambda}\inf_{K}\sup_{\mathcal{X}}\sqrt{T}D^\star(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon})\right|=o_p(1).$$

Proof. The first result follows from Proposition 3 and the fact that L_1 convergence implies convergence in probability. The second from that the processes involved have almost surely bounded paths, from the first result and the CMT. □

Proposition 11. *Suppose that Assumptions ME and AN hold. Then as* $T\to\infty$,

$$\begin{aligned}&\sqrt{T}\left(D_T(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R})-D^\star(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon})\right)\rightsquigarrow\mathcal{L}(z,\boldsymbol{\kappa},\boldsymbol{\lambda})\\&:=\int_{\mathbb{R}}\left(\left(z-\boldsymbol{\kappa}^{\mathrm{T}}\mathbf{x}\right)_+-\left(z-\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{x}\right)_+\right)d\mathcal{G}_{\mathrm{F}}(\mathbf{x}),\end{aligned}$$

in the space of $\mathbb{R}$ *bounded functions on* $K\times\Lambda\times\mathcal{X}$ *equipped with the sup norm, where* $\mathcal{G}$ *is a centered Gaussian process with covariance kernel given by*

$$\mathrm{Cov}(\mathcal{G}_{\mathrm{F}}(\mathbf{x}),\mathcal{G}_{\mathrm{F}}(\mathbf{y}))=\sum_{t\in\mathbb{Z}}\mathrm{Cov}(\mathbb{I}\{(\exp^\star(\boldsymbol{\varepsilon}_0)-1)\},\mathbb{I}\{(\exp^\star(\boldsymbol{\varepsilon}_t)-1)\})$$

and uniformly continuous sample paths on $\mathbb{R}^d$.

Proof. The result follows by the first part of Proposition 10, and Lemma A.1 in (the Technical Appendix of Arvanitis et al., 2020). □

Proposition 12. *Let* z *lie in a compact and non empty subset of the real line,* $\boldsymbol{\kappa},\boldsymbol{\lambda}$ *be elements of the* $d-1$ *simplex. Suppose that Assumptions ME and AN hold. Then uniformly in* $z,\boldsymbol{\kappa},\boldsymbol{\lambda}$ *as* $T\to\infty$, *for any fixed* $l\geq0$,

$$\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t,t-l\in\cup_{k=1}^K B_k}\left(D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l})-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l})\right)\right|\right]=o(1),$$

$$\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^K B_k,t-l\in\cap_{k=1}^K B_k^c}\left(D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l})-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l})\right)\right|\right]=o(1),$$

and

$$\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^K B_k,t-l\in\cap_{k=1}^K B_k^c}\left(D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l})-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l})\right)\right|\right]=o(1).$$

Finally, for $L=O(\sqrt{T})$,

$$\mathbb{E}\left[\left|\frac{1}{T}\sum_{l=0}^{L}\sum_{t=l+1}^{T}\left(1-\frac{l}{L+1}\right)\left(D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l})-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l})\right)\right|\right]=o(1).$$

Proof. Notice first that since the parameter $(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)$ is restricted on a compact set, $D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)$ and $D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)$ are almost surely uniformly (in the parameter and t) bounded. Then, for the first part we have that

$$\begin{aligned}&\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t,t-l\in\cup_{k=1}^K B_k}\left(D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l})-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l})\right)\right|\right]\\&\leq\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t,t-l\in\cup_{k=1}^K B_k}\left(D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l})-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\varepsilon_{t-l})\right)\right|\right]\\&+\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t,t-l\in\cup_{k=1}^K B_k}\left(D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l})-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l})\right)\right|\right]\\&\leq\frac{c}{\sqrt{T}}\sum_{t,t-l\in\cup_{k=1}^K B_k}\mathbb{E}\left[\left|D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t)-D(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t)\right|\right]=o(1),\end{aligned}\quad\text{(A.2)}$$

due to Proposition 3. Analogously for the second part

$$\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k,t-l\in\cap_{k=1}^{K}B_k^c}\Big(D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l}\big)\right.\right.$$
$$\left.\left.-D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l}\big)\Big)\right|\right]$$
$$=\mathbb{E}\left[\left|\frac{1}{\sqrt{T}}\sum_{t\in\cup_{k=1}^{K}B_k,t-l\in\cap_{k=1}^{K}B_k^c}\Big(D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l}\big)\right.\right.$$
$$\left.\left.-D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l}\big)\Big)\right|\right]$$
$$\leq\frac{c}{\sqrt{T}}\sum_{t,t-l\in\cup_{k=1}^{K}B_k}\mathbb{E}\Big[\Big|D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t\big)-D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t\big)\Big|\Big]=o(1).\quad\text{(A.3)}$$

The third part follows similarly. For the final part notice that due to the previous

$$\mathbb{E}\left[\left|\frac{1}{T}\sum_{l=1}^{L}\sum_{t=l+1}^{T}\left(1-\frac{l}{L+1}\right)\Big(D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l}\big)\right.\right.$$
$$\left.\left.-D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l}\big)\Big)\right|\right]$$
$$=\mathbb{E}\left[\left|\frac{1}{T}\sum_{l=1}^{L}\sum_{t,t-l\notin\cap_{k=1}^{K}B_k^c}\left(1-\frac{l}{L+1}\right)\Big(D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{R}_{t-l}\big)\right.\right.$$
$$\left.\left.-D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_t\big)D\big(z,\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\varepsilon}_{t-l}\big)\Big)\right|\right]$$
$$\leq\sum_{l=1}^{L}\left(1-\frac{l}{L+1}\right)o\left(\frac{1}{\sqrt{T}}\right)=o\left(\frac{L}{\sqrt{T}}\right)=o(1).$$

□

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