Vector-Space Geometric Algebra

Me

1 Basis and Operations

The algebra focuses on the operations and representations of sets of extended vector-like objects. While it applies to a generalized vector space, we'll instead focus on real vectors in the form of

$$\mathbf{v} \in \mathbb{R}^n, n \in \mathbb{N}$$

particular in the 3D case where $n = 3$.

1.1 Extend Vectors

We first define the bivectors (in 3D) as so

Definition 1.1 (Bivectors) *Consider the orthonormal real basis vectors*

$$\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \subseteq \mathbb{R}^3$$

Let the basis bivectors be

$$\{\mathbf{xy}, \mathbf{yz}, \mathbf{zx}\}$$

and a decomposed bivector be

$$\mathbf{v} = v_{xy}\mathbf{xy} + v_{yz}\mathbf{yz} + v_{zx}\mathbf{zx}$$

Intuitively, let the bivector be any plane (in 3D), with each basis bivector component dictating the signed area of the plane projected onto the basis planes (the xy-yz-zx planes).

Like vectors, two bivectors are equivalent if they have the same components. This implies that a bivector is invariant under translation, as well as shape-shifting that preserves areas.

Therefore, we can visually represent the basis bivectors as a square of sides 1 originated at $\mathbf{0}$ in the 3 basis planes.

Similarly, a trivector (and beyond) can be defined the same way

Definition 1.2 (Trivector) *Under the real orthonormal basis vectors* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$*, define the basis trivector as*

$$\{\mathbf{xyz}\}$$

And a decomposed trivector be

$$\mathbf{v} = v_{xyz}\mathbf{xyz}$$

A trivector can be similarly imaged as a volume, with its components be the volume's signed projection onto the xyz volume. Like bivectors, a trivector is invariant under translations, and warpings if it preserves the volume.

The basis trivector is a cube of sides 1 centered at $\mathbf{0}$.

Common names of bivectors and trivectors under regular vectors are

VGA Name	Linear Algebra Name
Bivectors	Pseudo-Vectors
Trivectors	Pseudo-Scalars
$\{1, \mathbf{xy}, \mathbf{yz}, \mathbf{zx}\}$	Quaternions
$\{1, \mathbf{xy}\}$	Complex Numbers

Figure 1: VGA Names

1.2 Operations

The main axiom underlying the definitions of VGA vectors and operations is

$$\mathbf{v}^2 = \|\mathbf{v}\|^2$$

where, of course, the norm of a vector (or bivector) is often defined by the Euclidean distance — the square root of the sum of its squared components.

Definition 1.3 (Dot product) *Let the dot product be the regular inner product in real number vector spaces*

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i$$

The dot product measures the similarity in angles between the two vectors.

Definition 1.4 (Geometric Product) *Let the geometric product be the naive expansion of the vector multiplication in component notation. For the 2D case under the orthonormal basis* $\{1, \mathbf{x}, \mathbf{y}, \mathbf{xy}\}$*, it is*

$$\begin{aligned}\mathbf{ab} &= (a_x\mathbf{x} + a_y\mathbf{y})(b_x\mathbf{x} + b_y\mathbf{y}) \\ &= a_x b_x + a_y b_y + (a_x b_y - b_x a_y)\mathbf{xy}\end{aligned}$$

Notice that the geometric product of two vectors is a scalar plus bivector entity.

Notice some identities of the geometric product

- $\mathbf{xx} = 1$
- $\mathbf{xy} = \mathbf{xy}$
- $\mathbf{xy} = -\mathbf{yx}$
- $(c\mathbf{a})\mathbf{b} = c\mathbf{ab}$

Definition 1.5 (Wedge Product) *Let the wedge product be the bivector component under a geometric product, namely define*

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{ab} - \mathbf{a} \cdot \mathbf{b}$$

The wedge product takes two vectors and produces a bivector interpreted as the oriented area formed by the parallelogram between the two vectors. The norm of that bivector equals the absolute area of the parallelogram.

Some identities of the wedge product include

- $\mathbf{x} \wedge \mathbf{x} = \mathbf{0}$
- $\mathbf{x} \wedge \mathbf{y} = \mathbf{xy}$
- $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$
- $(c\mathbf{a}) \wedge \mathbf{y} = c(\mathbf{a} \wedge \mathbf{b})$

We can thus define the geometric product by the dot and wedge product

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

1.3 Complex Analogy

To showcase the VGA vector-like elements and their operations, we'll use an analogy with the complex numbers.

There is an isomorphism between the complex number space $\mathbb{C}$ and the 2D rotor space of basis $\{1, \mathbf{xy}\}$. Namely due to that

$$\mathbf{xy}^2 = \mathbf{xyxy} = -\mathbf{xxyy} = -1 = i^2$$

Consider the four arithmetic operations on both spaces (where the vector multiplication is the geometric product)

$$\begin{aligned}
(a + bi) + (c + di) &= a + c + (b + d)i \\
(a + b\mathbf{xy}) + (c + d\mathbf{xy}) &= a + c + (b + d)\mathbf{xy} \\
(a + bi)(c + di) &= ac - bd + (ad + bc)i \\
(a + b\mathbf{xy})(c + d\mathbf{xy}) &= ac - bd + (ad + bc)\mathbf{xy}
\end{aligned}$$

and notice that they are identical.

2 Geometric Interpretation

The various fundamental multivectors and operations can be constructed to structure a few geometric operations. Namely, we can express the actions of projection and rejection, reflection, rotation, and so on in the language of geometric algebra.

2.1 Projection and Rejection