1 Sigma Algebra

A measure is the generalized volume or length of a subset. For instance, measure on the real line provides a generalized notion of length for subsets of the real numbers. The normal length measure of a closed interval on the real line $[a, b]$ is $b - a$.

We will consider a abstract measure theory. For a set X, let the set of its subsets (powerset) be denoted as $P(X)$.

Definition 1.1 (Measurable subsets)**.** *For a set X, let set $\mathcal{A} \subseteq P(X)$ contain some subsets of X. The set $\mathcal{A}$ is a σ-algebra if it fulfills*

1. *The empty and whole set must be in $\mathcal{A}$, where $\{\}, X \in \mathcal{A}$.*
2. *The set complement of any set in $\mathcal{A}$ must also be in $\mathcal{A}$, where*

$$A \in \mathcal{A} \implies A^c = X \setminus A \in \mathcal{A}$$

3. *The union of a countably sized number of sets in $\mathcal{A}$ is also in $\mathcal{A}$, where for a countable selection of A_i*

$$i \in \mathbb{N}, A_i \in \mathcal{A} \implies \bigcup_{i=0}^{\infty} A_i \in \mathcal{A}$$

For such a σ-algebra $\mathcal{A}$ on the set X, the subsets of X contained in $\mathcal{A}$ (or just members of the σ-algebra) are called measurable subsets.

Measurable sets are closed in the σ-algebra under complements and countable unions.

Some examples of σ-algebras on the set X are

- $\mathcal{A} = \{\emptyset, X\}$, this is the smallest σ-algebra
- $\mathcal{A} = P(X)$, this is the largest σ-algebra

2 Borel Sigma Algebras

Theorem 2.1 (Intersection of σ-algebras)**.** *For an arbitrary amount of sigma algebras $i \in I, \mathcal{A}_i$ on X, the intersection of them is also a sigma algebra on X,*

$$\bigcap_{i \in I} \mathcal{A}_i$$

Proof. Consider the sigma algebras $\mathcal{A}_i$ on X and their intersections $\mathcal{B} = \bigcap \mathcal{A}_i$, we need to show that $\mathcal{B}$ is also a sigma algebra.

For 1, note that because both the empty set and full set are in all sigma algebras, then they are also in $\mathcal{B}$

$$\forall i \in I, \emptyset, X \in \mathcal{A}_i \implies \emptyset, X \in \mathcal{B}$$

For 2, if a set $B \in \mathcal{B}$, then B is in all sigma algebras as well as B^c, so B^c is in $\mathcal{B}$

$$B \in \mathcal{B} \implies \forall i \in I, B \in \mathcal{A}_i \implies B^c \in \mathcal{A}_i \implies B^c \in \mathcal{B}$$

For 3, for some countable number of elements $B_k \in \mathcal{B}$, each is also in all sigma algebras $B_k \in \mathcal{A}_i$, hence the union of B_k is in all sigma algebras, and thus also in $\mathcal{B}$

$$k \in \mathbb{N}, B_k \in \mathcal{B} \implies \forall i \in I, B_k \in \mathcal{A}_i \implies \forall i \in I, \bigcup_{k=0}^{\infty} B_k \in \mathcal{A}_i \implies \bigcup_{k=0}^{\infty} B_k \in \mathcal{B}$$

□

Definition 2.1 (Smallest σ-algebra)**.** *For any set $M \in P(X)$ that is a set of subsets in X, the smallest σ-algebra that contains the elements of M is the intersection of all σ-algebra that contains the elements of M.*

$$M \subseteq \bigcap_{M \subseteq \mathcal{A}} \mathcal{A} = \sigma(M)$$

and $\mathcal{A}$ are all σ-algebras.

The smallest σ-algebra of M is denoted by $\sigma(M)$, or called the σ-algebra generated by M.

To practically construct the smallest σ-algebra of a set M, we progressively apply the definitions of σ-algebras and append new subsets into the set.

Definition 2.2 (Borel Sigma Algebra)**.** *For the set X which is a metric space (consists of a set and a metric, like $\mathbb{R}^n$ and Euclidean distance), we define the Borel Sigma algebra to be the sigma algebra generated by all the open sets in X.*

An open set S in X is a subset where for every element s in S, there exists a distance $\epsilon > 0$ such that $\forall x, d(x, s) < \epsilon \implies x \in S$. That is, for each element in S there exists a small disc around which is also contained in S.

The Borel Sigma algebra is in essence

$$B(X) = \sigma(M)$$

where M contains all the open sets in X.

The Borel σ-algebra on space X is importantly not the power set of the space, $P(X)$. We require such structure to define a measure function with appropriate features.

3 Measure

Definition 3.1 (Measurable Space)**.** *A measurable space consists of an ordered-tuple* $(X, \mathcal{A})$ *where* X *is a set and* $\mathcal{A}$ *is a* σ*-algebra on the set* X*.*

Definition 3.2 (Measure)**.** *For a measurable space* $(X, \mathcal{A})$*, a mapping* μ *from the* σ*-algebra to the zero and positive real numbers* $\mu \colon \mathcal{A} \to [0, \infty]$ *is called a measure if it satisfies:*

1. *For the empty set* $\emptyset \in \mathcal{A}$ *from the* σ*-algebra* $\mathcal{A}$*, its volume/measure should be zero*

$$\mu(\emptyset) = 0$$

2. *The* σ*-additive rule. For any countable number of mutually disjointed measurable sets in our sigma algebra, the volume of their union is the sum of their individual volumes. So given*

$$i \in \mathbb{N}, A_i \in \mathcal{A}, i \neq j \implies A_i \cap A_j = \emptyset$$

then we have

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

Note that the union of the measurable sets has a measure because of their closure under the σ*-algebra.*

The co-domain of the measure includes both the positive reals and a symbol infinity, where $[0, \infty] = [0, \infty) \cup \{\infty\}$*. This space has some special rules:*

1. $\forall x \in [0, \infty], x + \infty = \infty$
2. $\forall x \in (0, \infty], x \times \infty = \infty$
3. $0 \times \infty = 0$ *for some cases only in measure theory.*

Definition 3.3 (Measure Space)**.** *A measure space is a ordered set* $(X, \mathcal{A}, \mu)$ *where* X *is a set,* $\mathcal{A}$ *is a* σ*-algebra on* X*, and* μ *is a measure on the measurable space.*

3.1 Examples of Measures

The following measures works on all sets X and any of their their σ-algebras $\mathcal{A}$.

The counting measure is defined by the mapping

$$\mu(A) = \begin{cases} |A| & A \text{ is finite sized} \\ \infty & \text{otherwise} \end{cases}$$

where $A \in \mathcal{A}$ is a member of the σ-algebra. The measure axioms for the finite cases is obvious to show. The counting measure maps subsets to their sizes as generalized volumes.

The Dirac measure on a point $p \in X$ maps subsets containing p to 1 and subsets not containing it to 0, it is defined by

$$\delta_p(A) = \begin{cases} 1 & p \in A \\ 0 & p \notin A \end{cases}$$

To search for a measure that generalizes volume in Euclidean space $X = \mathbb{R}^n$, we require that our measure adhere to some basis axioms:

1. The measure of a unit volume/cube (set of points from $[0, 1]$ in all dimensions) is always 1

$$\mu([0,1]^n) = 1$$

2. The measure is invariant under translation by a vector

$$\forall x \in \mathbb{R}^n, \forall A \in \mathcal{A}, \mu(x + A) = \mu(A)$$

One measure that works is the Lebesgue measure, but it only works on σ-algebras that are not simply the powerset — it operates on the Borel σ-algebras on a metric space.

4 Lebesgue Measures On the Real Powerset Proof

Definition 4.1 (Equivalence class)**.** *An equivalence class within the set X defined by the equivalence relation $x \sim y$ on $a \in X$ is*

$$[a] = \{x \in X : x \sim a\} \subseteq X$$

Definition 4.2 (Axiom of Choice)**.** *For an indexed set of subsets (of set X), $i \in I, S_i \subseteq X$, there exists a set A containing a single element from each set*

$$i \in I, a_i = f(S_i), A = \{a_0, a_1, \dots\}$$

where $f: P(X) \to X$ is a choice function on the subsets S_i.

Theorem 4.1 (Monotonic Measures)**.** *Consider a measure μ on a measurable space $(X, \mathcal{A})$. If for some $A, B \in \mathcal{A}$, $A \subseteq B$, then*

$$\mu(A) \leq \mu(B)$$

Proof. For the given sets $A, B \in \mathcal{A}$ with $A \subseteq B$, let $C = B \setminus A$ so that $A \cup C = B$. Then

$$\mu(B) = \mu(A \cup C) = \mu(A) + \mu(C) \geq \mu(A)$$

for $\mu(C) \geq 0$. □

Lemma 4.2 (Powerset Zero Measure)**.** *For all measures μ on the measurable space $(\mathbb{R}, P(\mathbb{R}))$ that satisfies*

1. $\mu((0,1]) < \infty$
2. $\forall x \in \mathbb{R}, A \in P(\mathbb{R}), \mu(x + A) = \mu(A)$

the measure must be the zero measure

$$\forall A \in P(\mathbb{R}), \mu(A) = 0$$

Proof. Within the interval $I = (0, 1]$, consider the equivalence relation defined by

$$\forall x, y \in I, x \sim y \iff x - y \in \mathbb{Q}$$

We can partition the interval I with some infinite equivalence relations that may or may not be countable

$$I = \bigcup_i [a_i], a_i \in I$$

and notice that each equivalence class is disjoint with all others.

Using the Axioms of Choice, we create a set A by picking one unique element from each equivalence class

$$A = \{a_0, a_1, \dots\} \subseteq I$$

namely, the set A has the property that

1. For any equivalence class, there is an element in A that is in that class.

$$\forall [a_i], \exists a \in A, a \in [a_i]$$

2. That the element in (1) is unique in A.

$$\forall [a_i], \forall a, b \in A, a, b \in [a_i] \implies a = b$$

Consider the enumerated rational translations of such set A, where $r_n, n \in \mathbb{N}$ is an enumeration of rational numbers within $(-1, 1]$

$$A_n = A + r_n$$

We note that these translated sets are disjoint, namely

$$n \neq m \implies A_n \cap A_m = \emptyset$$

this can be proved by contraposition

$$\begin{aligned}
A_n \cap A_m \neq \emptyset &\implies \exists x, x \in A_n \land x \in A_m \\
&\implies \exists a_n, a_m \in A, x = r_n + a_n = r_m + a_m \\
&\implies a_n - a_m = r_m - r_n \in \mathbb{Q} \\
&\implies a_n \sim a_m \\
&\implies a_n = a_m \\
&\implies r_m = r_n \implies n = m
\end{aligned}$$

Then notice that both

$$(0, 1] \subseteq \bigcup_n A_n \subseteq (-1, 2]$$

as

$$\begin{aligned}
x \in (0, 1] &\implies \exists a \in A, a \in [x] \\
&\implies \exists r \in \mathbb{Q}, a + r = x \\
&\implies \exists n, r_n = r, x \in A_n \\
&\implies x \in \bigcup_n A_n \\
x \in \bigcup_n A_n &\implies \exists n, x \in A_n \\
&\implies x \in r_n + A \\
r_n \in (-1, 1], A \subseteq (0, 1] &\implies -1 < x \leq 2 \\
&\implies x \in (-1, 2]
\end{aligned}$$

Because measures are monotonic,

$$\mu((0, 1]) \leq \mu(\bigcup_n A_n) \leq \mu((-1, 2])$$

Suppose that $\mu((0,1]) = c < \infty$

$$\mu((-1,2]) = \mu((-1,0] \cup (0,1] \cup (1,2]) = 3c$$

Then because $\mu(A_n) = \mu(r_n + A) = \mu(A)$ we have

$$c \leq \sum_n \mu(A) \leq 3c$$

As c is finite, the middle summation must also be finite. For n is countably infinite, the only way this identity holds is if $\mu(A) = 0$ as anything else results in infinity.

Hence $c = 0$, with

$$\mu(\mathbb{R}) = \mu(\bigcup_{z \in \mathbb{Z}} (0,1] + z) = \sum_z \mu((0,1]) = 0$$

As any subset of $\mathbb{R}$ must also have zero measure (otherwise the measure for $\mathbb{R}$ is non-zero)

$$\forall A \in P(\mathbb{R}) \implies A \subseteq \mathbb{R} \implies \mu(A) = 0$$

□

Theorem 4.3 (Non Lebesgue Measures)**.** *For the measurable space $(X, P(X))$ where the set is the real number line, $X = \mathbb{R}$, there does not exist a Lebesgue measure — a measure that follows geometric intuition $\mu \colon P(X) \to [0, \infty]$ where*

1. $\mu([0,1]) = 1$
2. $\forall x \in X, A \in P(X), \mu(x + A) = \mu(A)$

Proof. By the previous lemma, if such measure μ exists,

$$1 = \mu([0,1]) = \mu(\{0\} \cup (0,1]) \implies \mu((0,1]) = 1 - \mu(\{0\}) < \infty$$

Therefore $\mu = 0$, which is a contradiction to the fact that $\mu([0,1]) = 1 \neq 0$. Hence a measure μ can't exist. □

5 Measurable Maps

Definition 5.1 (Measurable Maps)**.** *Given two measurable spaces* $(\Omega_1, \mathcal{A}_1)$ *and* $(\Omega_2, \mathcal{A}_2)$*, a mapping* $f\colon \Omega_1 \to \Omega_2$ *is a measurable map when the pre-image of measurable sets are also measurable sets*

$$\forall A \in \mathcal{A}_2, f^{-1}(A) \in \mathcal{A}_1$$

This definition of measurable maps is important in defining an integral of a function f. If the function is measurable, then the pre-image of any measurable set of its range is also measurable on its domain, so we can take the measure of the pre-image and multiply it against the height (summing for every height) to compute the integral.

Theorem 5.1 (Composition of Measurable Mappings)**.** *For the measurable spaces* $(\Omega_1, \mathcal{A}_1), (\Omega_2, \mathcal{A}_2), (\Omega_3, \mathcal{A}_3)$ *and the mappings* $f\colon \Omega_1 \to \Omega_2$ *and* $g\colon \Omega_2 \to \Omega_3$*. If both* f *and* g *are measurable with respect to the* σ*-algebras, then their composition* $g \circ f\colon \Omega_1 \to \Omega_3$ *is also measurable with the same* σ*-algebras.*

Proof. Consider the measurable set $S \in \mathcal{A}_3$. Using the identity of pre-images

$$(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$$

As g is measurable, $g^{-1}(S) \in \mathcal{A}_2$ is also a measurable set; As f is measurable, $f^{-1}(g^{-1}(S)) \in \mathcal{A}_1$ is also a measurable set. Hence the pre-image of S is a measurable set and the composition is also measurable. $\square$

5.1 Examples of Measurable Maps

Consider the measurable spaces $(\Omega, \mathcal{A})$ and $(\mathbb{R}, B(\mathbb{R}))$, the characteristic map on set $A \subseteq \Omega$ is defined to be

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

The characteristic map is measurable when A is measurable $A \in \mathcal{A}$, for let $B = B(\mathbb{R})$:

1. if B contains neither 0 or 1, then its pre-image is the empty set, which is in $\mathcal{A}$
2. if B contains only 0 or 1, then its pre-image is either A or A^c, which are both in $\mathcal{A}$
3. if B contains both 0 and 1, then its pre-image is Ω which is in $\mathcal{A}$

The map is measurable because all pre-images of measurable sets in the co-domain is also a measurable set in the domain with respect to $\mathcal{A}$.

Theorem 5.2 (Measurable Real Number Functions)**.** *For the measurable spaces* $(\Omega, \mathcal{A})$ *and* $(\mathbb{R}, B(\mathbb{R}))$*. If the functions* $f, g\colon \Omega \to \mathbb{R}$ *are both measurable, then*

$$f + g, f - g, f \cdot g, |f|$$

are all measurable functions.

Proof. This proof depends on the properties of the Borel σ-algebra on the real numbers, which is really difficult. Take this theorem as given please. $\square$

Continuous functions in the real numbers are always measurable.

6 Lebesgue Integral