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1 a)

i) ① Assuming $[u] = [u]_B$

$\forall u, v \in V$

$$\langle u, v \rangle = [u]^T A [v]$$

$$= ([u]^T A [v])^T \quad (\text{for } \langle u, v \rangle \text{ is } 1 \times 1)$$

$$= [v]^T A^T [u]$$

$$= [v]^T A [u] \quad \text{for } A^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = A$$

$$= \langle v, u \rangle = \overline{\langle v, u \rangle}$$

② $\forall \alpha \in \mathbb{R}, u, v \in V$

$$\langle \alpha u, v \rangle = [\alpha u]^T A [v] = \alpha ([u]^T A [v])$$

$$= \alpha \langle u, v \rangle$$

③ $\forall u, v, w \in V$

$$\langle u+v, w \rangle = [u+v]^T A [w] = ([u]+[v])^T A [w]$$

$$= [u]^T A [w] + [v]^T A [w]$$

$$= \langle u, w \rangle + \langle v, w \rangle$$

④ $\forall u \in V$, $[u] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\langle u, u \rangle = [u]^T A [u] = [a\ b\ c] \begin{bmatrix} 2a+b \\ a+b \\ 4c \end{bmatrix}$$

$$= 2a^2 + ab + ab + b^2 + 4c^2$$

$$= a^2 + (a+b)^2 + 4c^2$$

$$\geq 0 \quad (\text{for } a^2, (a+b)^2, 4c^2 \geq 0)$$

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(4) continued.

if $\langle u,u\rangle = 0$ for $u \in V$, $[u]=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\Rightarrow a^2+(a+b)^2+4c^2=0$

$\Rightarrow a^2=(a+b)^2=4c^2=0$

so $a^2=0$, $a=0$

$4c^2=0$, $c=0$

$(a+b)^2=0$, $b^2=0$, $b=0$

And $u=\vec{0}$

so $\langle u,v\rangle$ is a <u>valid</u> innerproduct.

ii) consider $(x_1,x_2)=(-1,1)=(y_1,y_2)=u$

$\langle (x_1,x_2),(y_1,y_2)\rangle = 2(-1)(-1)+5(-1)(1)$
$+5(1)(-1)+(1)(1)$
$=2-5-5+1=-7$

so $\exists u \in V$, $\langle u,u\rangle<0$, Axiom 4 doesn't hold, so <u>not</u> an innerproduct.

b). $\|f\|^2 = \langle f,f\rangle = \int_0^1 (x^4+\frac{1}{3})^2\,dx$

$=\int_0^1 x^8+\frac{2}{3}x^4+\frac{1}{9}\,dx$

$=\left[\frac{x^9}{9}+\frac{2}{15}x^5+\frac{1}{9}x\right]_0^1$

$=\frac{1}{9}+\frac{2}{15}+\frac{1}{9}=\frac{16}{45}$

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$$\|f\| = \sqrt{\frac{16}{45}} = \frac{4}{3\sqrt{5}}$$

- $\|g\|^2 = \langle g, g \rangle = \int_0^1 (6x)^2 \, dx$

$$= \left[12x^3\right]_0^1 = 12$$

$$\|g\| = \sqrt{12} = 2\sqrt{3}$$

- $\langle f, g \rangle = \int_0^1 (x^4 + \frac{1}{3}) \, 6x \, dx = \int_0^1 6x^5 + 2x \, dx$

$$= \left[x^6 + x\right]_0^1$$

$$= 2$$

$$\cos\theta = \frac{\langle f, g \rangle}{\|f\| \, \|g\|} = \frac{2}{\frac{4}{3\sqrt{5}} \times 2\sqrt{3}} = \frac{3\sqrt{5}}{4\sqrt{3}}$$

$$= \frac{\sqrt{15}}{4}$$

$$\theta = \arccos\left(\frac{\sqrt{15}}{4}\right)$$

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2. a) $S = \{b_1, b_2\} = \{(-1,0,1,2), (0,1,0,1)\}$

$$u_1 = \frac{1}{\sqrt{6}}(-1,0,1,2)$$

$$w_2 = b_2 - \langle b_2, u_1\rangle u_1$$
$$= (0,1,0,1) - \frac{1}{6}(2)(-1,0,1,2)$$
$$= (0,1,0,1) - \frac{1}{3}(-1,0,1,2)$$
$$= \frac{1}{3}(1,3,-1,1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{w_2}{\sqrt{\frac{1}{9}(12)}} = \frac{w_2}{\sqrt{\frac{4}{3}}}$$
$$= \sqrt{\frac{3}{4}}\frac{1}{3}(1,3,-1,1)$$
$$= \frac{1}{2\sqrt{3}}(1,3,-1,1)$$

orthonormal basis

$$B = \left\{\frac{1}{\sqrt{6}}(-1,0,1,2), \frac{1}{2\sqrt{3}}(1,3,-1,1)\right\}$$

b) closest point on W from $(2,1,1,0) = u$

is $\text{proj}_W u = \langle u, b_1\rangle b_1 + \langle u, b_2\rangle b_2$

where $B = \{b_1, b_2\}$ from a

$$\text{proj}_W u = \frac{1}{6}(-1)(-1,0,1,2) + \frac{1}{12}(4)(1,3,-1,1)$$
$$= \frac{1}{6}(1,0,-1,-2) + \frac{1}{6}(2,6,-2,2)$$
$$= \frac{1}{6}(3,6,-3,0) = \frac{1}{2}(1,2,-1,0)$$

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$$d = \| u - \text{proj}_W u \| = \| \tfrac{1}{2}(4,2,2,0) - \tfrac{1}{2}(1,2,-1,0) \|$$

$$= \| \tfrac{1}{2}(3,0,3,0) \|$$

$$= \sqrt{\tfrac{1}{4}(18)}$$

$$= \sqrt{\tfrac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

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3 Notice that, $\forall u, v \in V$

$$\langle T(u+v), u+v\rangle = \langle T(u), u\rangle + \langle T(v), v\rangle + \langle T(u), v\rangle + \langle T(v), u\rangle$$

$$\langle T(u-v), u-v\rangle = \langle T(u), u\rangle + \langle T(v), v\rangle - \langle T(u), v\rangle - \langle T(v), u\rangle$$

so $\langle T(u+v), u+v\rangle - \langle T(u-v), u-v\rangle = 2\left(\langle T(u), v\rangle + \langle T(v), u\rangle\right)$ ①

$$\langle T(u+iv), u+iv\rangle = \langle T(u), u\rangle + \langle T(iv), iv\rangle + \langle T(u), iv\rangle + \langle T(iv), u\rangle$$

$$\langle T(u-iv), u-iv\rangle = \langle T(u), u\rangle + \langle T(iv), iv\rangle - \langle T(u), iv\rangle - \langle T(iv), u\rangle$$

so $\langle T(u+iv), u+iv\rangle - \langle T(u-iv), u-iv\rangle$ ②

$$= 2\left[\langle T(u), iv\rangle + \langle T(iv), u\rangle\right]$$

$$= 2\left[-i\langle T(u), v\rangle + i\langle T(v), u\rangle\right]$$

Hence ① + i ②

$$= 2\left[\langle T(u), v\rangle + \langle T(v), u\rangle\right] + 2\left[\langle T(u), v\rangle - \langle T(v), u\rangle\right]$$

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$= 4\langle T(u), v\rangle$

Therefore

$$4\langle T(u), v\rangle = \langle T(u+v), u+v\rangle - \langle T(u-v), u-v\rangle + i\left[\langle T(u+iv), u+iv\rangle - \langle T(u-iv), u-iv\rangle\right]$$

$\forall u, v \in V$

Hence because $\forall z \in V, \langle T(z), z\rangle = 0$

$$\begin{aligned} \forall u,v \in V,\ 4\langle T(u), v\rangle = &\ 0 \quad (z = u+v) \\ &- 0 \quad (z = u-v) \\ &+ 0 \quad (z = u+iv) \\ &- 0 \quad (z = u-iv) \\ = &\ 0 \end{aligned}$$

$\Rightarrow \langle T(u), v\rangle = 0 \quad (*)$

So for all $u \in V$, let $p = T(u) \in V$

let $W = \operatorname{span}\{p\}$

Notice, $W^{\perp} = V$, for $\forall v \in V, \forall z \in W$

$\langle v, z\rangle = \langle v, kp\rangle \quad (z = kp, \exists k \in \mathbb{C})$

$= \overline{\langle kT(u), v\rangle} = \bar{k}\,\overline{\langle T(u), v\rangle} = 0 \quad \left[\begin{array}{l} \text{via } * \\ \langle T(u), v\rangle = 0 \end{array}\right]$

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Then because

$$W \cap W^{\perp} = \{\vec{0}\} \quad \text{(exercise 272)}$$

with

$p \in \operatorname{span}\{p\} = W$, $p \in V = W^{\perp}$

$\Rightarrow p \in W \wedge p \in W^{\perp}$, so $p = T(u) = \vec{0}$

Therefore $\forall u \in V$ $T(u) = \vec{0}$,

so $T = 0$