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1 a) let $w = 2222212$

consider

$$Hw = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

w is not a codeword as $Hw \neq \vec{0}$

To correct w, notice Hw is the fourth column of H.

Hence the corrected codeword w' is

$$w' = w - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2221212$$

b) H is in REF, solution space $H = C$

and REF has 4 non-pivot columns, so

a basis B for solspace H of 4 elements

The field $\mathbb{F}_3$ has 3 elements, so

$$|C| = |\text{span } B| = 3^4 = 81$$

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1 c) By lemma 18.15, 18.13

min weight C = min distance C

= smallest linearly dependent column set of H.

- For $\vec{0} \notin$ cols H, smallest lin dep col set $\neq 1$
- For there are no two columns that are multiples of each other, smallest lin dep col set $\neq 2$
- consider $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

  S is lin dep, for $1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \vec{0}$

  $|S| = 3$, so smallest lin dep set has size 3.

Hence min distance of C is 3

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2 a) $T: P_2(\mathbb{R}) \to M_{22}(\mathbb{R})$

is a function and

- $\forall p, q \in P_2(\mathbb{R})$, $p = a + bx + cx^2$, $q = \alpha + \beta x + \gamma x^3$

$$T(p+q) = T\left((a+\alpha) + (b+\beta)x + (c+\gamma)x^2\right)$$

$$= \begin{bmatrix} a+\alpha & a+\alpha+b+\beta+c+\gamma \\ a+\alpha+2(b+\beta)+4(c+\gamma) & a+\alpha+3(b+\beta)+9(c+\gamma) \end{bmatrix}$$

$$= \begin{bmatrix} a & a+b+c \\ a+2b+4c & a+3b+9c \end{bmatrix} + \begin{bmatrix} \alpha & \alpha+\beta+\gamma \\ \alpha+2\beta+4\gamma & \alpha+3\beta+9\gamma \end{bmatrix}$$

$$= \begin{bmatrix} p(0) & p(1) \\ p(2) & p(3) \end{bmatrix} + \begin{bmatrix} q(0) & q(1) \\ q(2) & q(3) \end{bmatrix}$$

$$= T(p) + T(q)$$

- $\forall \alpha \in \mathbb{R}$, $\forall p \in P_2(\mathbb{R})$, $p = a + bx + cx^2$

$$T(\alpha p) = T(\alpha a + \alpha bx + \alpha cx^2)$$

$$= \begin{bmatrix} \alpha a & \alpha a + \alpha b + \alpha c \\ \alpha a + 2\alpha b + 4\alpha c & \alpha a + 3\alpha b + 9\alpha c \end{bmatrix}$$

$$= \alpha \begin{bmatrix} p(0) & p(1) \\ p(2) & p(3) \end{bmatrix}$$

$$= \alpha T(p)$$

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2 b)

$$[T]_{C,B} = \left[[T(b_1)]_C \cdots [T(b_3)]_C \right]$$

where $B = \{b_1, b_2, b_3\} = \{1, x, x^2\}$

$$[T]_{CB} = \left[\left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right]_C \left[\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\right]_C \left[\begin{bmatrix} 0 & 1 \\ 4 & 9 \end{bmatrix}\right]_C \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

c) From exercise 170, 20.3

A basis $\{[b_1]_C \cdots\}$ for colspace$([T]_{CB})$

$\Rightarrow$ $B = \{b_1 \cdots\}$ is a basis for Im T

Hence consider REF of $[T]_{CB}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \xrightarrow[\substack{R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 3 & 9 \end{bmatrix} \xrightarrow{\substack{R_3 - 2R_2 \\ R_4 - 3R_2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{R_4 - 3R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivots at all columns.

$$[B]_C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \right\}$$

B for ImT $= \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 9 \end{bmatrix} \right\}$

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3 let $T: V \to W$ be an injective linear transformation

If W is finite m-dimensional, $m \in \mathbb{N}$

- because $\operatorname{Im} T \leq W$ (20.7 note)

$n = \dim(\operatorname{Im} T) \leq \dim(W) = m$ (exercise 125a as W is finite dim)

with $n \in \mathbb{N}$ and finite $\star$

- define $T': V \to \operatorname{Im} T$, $\forall u \in V$, $T'(u) = T(u)$

Because T is injective, T' is also injective

T' is surjective by definition of $\operatorname{Im} T$

$\forall v \in \operatorname{Im} T \Rightarrow T(u) = v$ for some $u \in V$ (def 20.7)

$\Rightarrow$ some $u \in V$, $T(u) = V$

Therefore T' is a bijective function.

So T' is invertible (exercise 185)

let $H: \operatorname{Im} T \to V$, $H = (T')^{-1}$

- H is also invertible $\Rightarrow$ H is bijective

So $\operatorname{Im} H = V$ for H is surjective

Because $\operatorname{Im} H \subseteq V$

And $\forall v \in V$, $\exists u \in \operatorname{Im} T$, $H(u) = v$ $\Rightarrow v \in \operatorname{Im} H \Rightarrow V \subseteq \operatorname{Im} H$

for H is surjective by Im defn

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3 cont.

so $\mathrm{Im}\,H = V$

- Suppose a basis $B = \{b_1, \ldots, b_n\}$ for $\mathrm{Im}\,T$, n from $*$

Because B is a basis for $\mathrm{Im}\,T$, and H is injective, $H(B)$ is a basis for $\mathrm{Im}\,H$ (exercise 170 c)

Hence $H(B)$ is also a basis for V, as $\mathrm{Im}\,H = V$

But $|H(B)| = |B| = n$

so $\dim V = n$, for n is finite $*$

$\Rightarrow$ V is finite dimensional