THE UNIVERSITY OF MELBOURNE

Department of Finance
FNCE10002 Principles of Finance
Teaching Note 1
Introduction to Financial Mathematics*

Outline

This teaching note deals with the area of Principles of Finance that is central to the understanding of the investment and financing decisions of investors and companies, namely financial mathematics. After reading this note you should be able to:

- Calculate the future and present values of a series of cash flows
- Calculate the future and present values of ordinary annuities
- Calculate the future and present values of annuities due
- Calculate the present value of ordinary and deferred perpetuities
- Calculate the future and present values of growing annuities

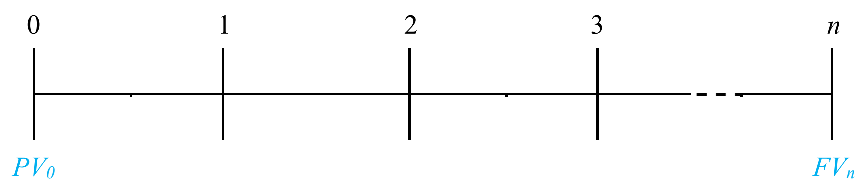
1. Future and present values of a single cash flow

Given the time value of money, a cash flow received today (that is, time 0) is more valuable than the same cash flow received some time in the future. Accordingly, a cash flow of PV_0 today that earns interest at a rate of r percent per period for n periods has a future value of:

$$FV_n = PV_0(1+r)^n, \tag{1}$$

where $(1+r)^n$ is the future dollar value of $1 today earning an interest rate of r percent per period for n time periods. This amount is then multiplied by PV_0 to obtain its future value at the end of time period n. On a timeline, the cash flow can be represented as follows:

Figure 1: Present and Future Values of Single Cash Flows



Note that in the above timeline we're assuming that the cash flow occurs at the end of a particular period. For example, PV_0 occurs at the end of period 0 while FV_n occurs at the end of period n. This is the convention that we use throughout this subject to simplify the calculations. Where a cash flow does not occur at the end of a period it would need to be specified as such. (For example, in our discussion of annuities due later in this note we will assume that the cash flows occur at the beginning of the period.)

Example 1: Future value of a single cash flow

What is the future value of $1,000 invested at an interest rate of 10 percent per annum at the end of: (a) 3 years and (b) 20 years?

Solution

a) The future value at the end of 3 years is:

FV_3 = \$1,000(1 + 0.10)^3 = \$1,331.00.

b) The future value at the end of 20 years is:

FV_{20} = \$1,000(1 + 0.10)^{20} = \$6,727.50.

A similar principle applies in the determination of a present value equivalent of an expected cash flow in a future period. Formally, a cash flow of FV_n that is due in n periods and is discounted at a rate of r

percent per period has a present value today of:

$$PV_0 = \frac{FV_n}{(1+r)^n}, \tag{2}$$

where $1/(1+r)^n$ is the present dollar value of $1 to be received n time periods from today at an interest rate of r percent per period. Note that the term $1/(1+r)$ is referred to as the one-period *discount factor*. It is the dollar value today of $1 occurring at the end of one period discounted at an interest rate of r.

Example 2: Present value of a single cash flow

Calculate the present value today of $1,331.00 occurring at the end of year 3 assuming an interest rate of 10 percent per annum. Calculate the present value of $6,727.50 occurring at the end of year 20 assuming an interest rate of 10 percent per annum.

Solution

The present value today of $1,331 at the end of year 3 is:

$PV_0 = \$1{,}331/(1 + 0.10)^3 = \$1{,}000.00$.

The present value today of $6,727.50 at the end of year 20 is:

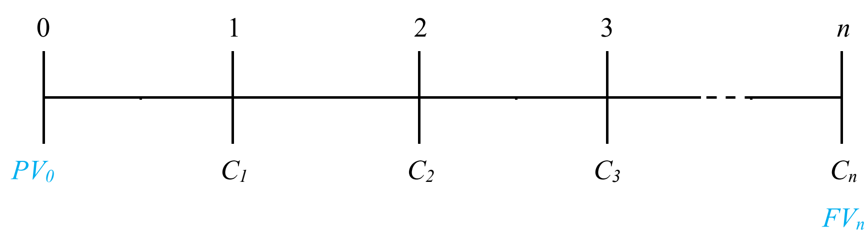
$PV_0 = \$6{,}727.50/(1 + 0.10)^{20} = \$1{,}000.00$.

Note that, as expected, the present values calculated here are the same as those in the previous example.

2. Future and present values of a series of cash flows

If we are given a series of cash flows that occur over different time periods their future value can be calculated as the sum of the future values of the individual cash flows. This is also known as the *value additivity principle*. On a timeline, the cash flow can be represented as follows:

Figure 2: Present and Future Values of Multiple Cash Flows with no Cash Flow at Time 0

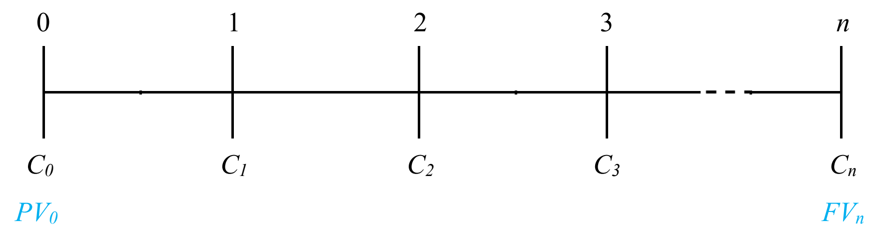


As the future value of a sum of a series of cash flows earning an interest rate of r percent per period at the end of n periods is equal to the sum of their individual future values, we have:

$$FV_n = C_1(1+r)^{n-1} + C_2(1+r)^{n-2} + \ldots + C_n. \tag{3a}$$

Note that in the above expression we assume that the first cash flow occurs at the end of time 1 and the last cash flow occurs at the end of time n. Clearly, the cash flow occurring at the end of time n will not earn any interest. If we assume that the first cash flow occurs at the end of time 0 (that is, today) then the cash flows will look a little different, as follows:

Figure 3: Present and Future Values of Multiple Cash Flows with a Cash Flow at Time 0



The future value at the end of time n will now be:

$$FV_n = C_0(1+r)^n + C_1(1+r)^{n-1} + C_2(1+r)^{n-2} + \ldots + C_n. \tag{3b}$$

This amount is higher than what we got in expression (3a) because there is now one additional cash flow at time 0 (C_0) which earns interest over n time periods.

Example 3: Future value of a series of cash flows

Calculate the future value at the end of year 4 of investing the following cash flows: C_1 = \$1,000, C_2 = \$2,000, C_3 = \$3,500, C_4 = \$3,000. Assume that the applicable interest rate is 10 percent per annum. Recalculate the future value above if an additional cash flow of \$2,000 were invested today (that is, C_0 = \$2,000).

Solution

In the first case, we have:

FV_4 = \$1,000(1.10)^3 + \$2,000(1.10)^2 + \$3,500(1.10)^1 + \$3,000.

FV_4 = \$1,331 + \$2,420 + \$3,850 + \$3,000 = \$10,601.00.

In the second case, we have:

FV_4 = \$2,000(1.10)^4 + \$1,000(1.10)^3 + \$2,000(1.10)^2 + \$3,500(1.10)^1 + \$3000.

FV_4 = \$2,928.20 + \$1,331 + \$2,420 + \$3,850 + \$3,000 = \$13,529.20.

If we are given a series of cash flows occurring over different time periods their present value can be calculated as the sum of the present values of each individual cash flow. That is, the present value of a sum of a series of cash flows discounted at r percent per period over n periods is equal to the sum of their individual present values, which is:

$$PV_0 = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n}. \tag{4a}$$

Note again that in the above expression we assume that the first cash flow occurs at the end of time 1 and the last cash flow occurs at the end of time n (see the timeline in figure 2). If we assume that the first cash flow occurs at the end of time 0 (that is, today, as in the timeline in figure 3) then the present value today will be:

$$PV_0 = C_0 + \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n}. \tag{4b}$$

Note that the difference between the present value in expression (4b) and (4a) is the additional cash flow at time 0 (C_0) which does not need to be discounted as it occurs at time 0.

Example 4: Present value of a series of cash flows

Calculate the present value of the following future cash flows: C_1 = \$1,000, C_2 = \$2,000, C_3 = \$3,500, C_4 = \$3,000. Assume that the applicable interest rate is 10 percent per annum. Recalculate the present value if an additional cash flow of \$2,000 were invested today (that is, C_0 = \$2,000). What is the relation between the future values calculated in each case of the previous example and the present values calculated here?

Solution

In the first case, we have:

$PV_0 = \$1,000/(1.10)^1 + \$2,000/(1.10)^2 + \$3,500/(1.10)^3 + \$3,000/(1.10)^4.$

$PV_0 = \$909.10 + \$1,652.89 + \$2,629.60 + \$2,049.04 = \$7,240.63.$

In the second case, we have:

$PV_0 = \$2,000 + \$1,000/(1.10)^1 + \$2,000/(1.10)^2 + \$3,500/(1.10)^3 + \$3,000/(1.10)^4.$

$PV_0 = \$2,000 + \$909.10 + \$1,652.89 + \$2,629.60 + \$2,049.04 = \$9,240.63.$

Note the relation between the future values calculated in each case of the previous example and the present values calculated above. Given the interest rate of 10 percent per annum, if we know the future value at the end of year 4 we can directly calculate the present value today using the future value amount, as follows:

In the first case, we have:

$PV_0 = FV_n/(1 + r)^n = 10601/(1 + 0.10)^4 = \$7,240.63.$

In the second case, we have:

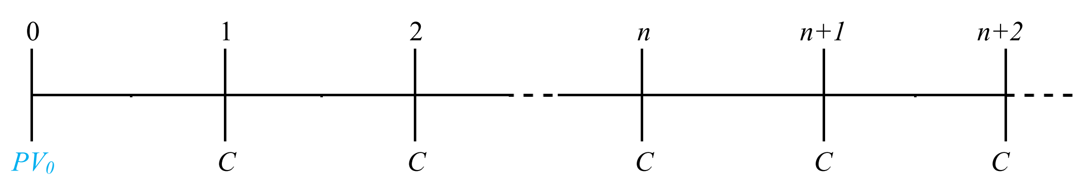
$PV_0 = \$13,529.20/(1 + 0.10)^4 = \$9,240.63.$

3. Present and future values of equal, periodic cash flows

3.1 Present value of a perpetuity

The simplest type of equal, periodic cash flow is a perpetuity where the cash flow recurs forever. On a timeline, a perpetual cash flow (C) can be shown as follows:

Figure 4: Present Value of a Perpetuity



The present value of a perpetuity is calculated by discounting each cash flow to time period 0 as follows:

$$PV_0 = C/(1+r) + C/(1+r)^2 + \ldots + C/(1+r)^n + C/(1+r)^{n+1} + \ldots$$

The above expression can be rewritten as:

$$PV_0 = C[1/(1+r) + 1/(1+r)^2 + \ldots + 1/(1+r)^n + 1/(1+r)^{n+1} + \ldots]$$

As n approaches infinity, the right-hand side expression $[1/(1+r) + 1/(1+r)^2 + \ldots + 1/(1+r)^n + 1/(1+r)^{n+1} + \ldots]$ approaches $1/r$. So, in the limit, the present value of a perpetuity is:

$$PV_0 = \frac{C}{r}. \tag{5}$$

Note that in the above expression the **first cash flow occurs at the end of time 1**, and not time 0.

Example 5: Present value of a perpetuity

Your company can lease a computer system for equal annual payments of \$2,000 forever or purchase it today for \$23,000. The first payment is to be made at the end of year 1 with subsequent payments being made at the end of each year. Ignoring taxes and other complications, what should the company do if the interest rate is 10 percent per annum?

Solution

We need to compare the purchase price today of \$23,000 with the present value of equal annual payments of \$2000 forever. The present value of this perpetuity is:

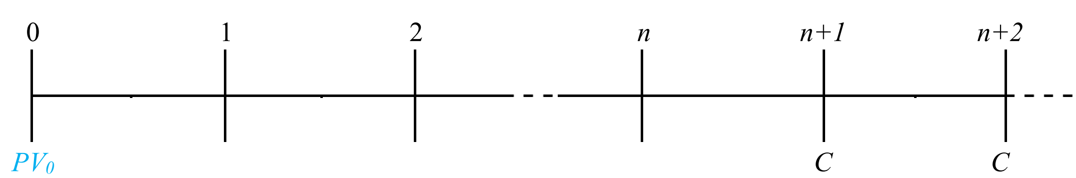
$$PV_0 = \frac{\$2{,}000}{0.10} = \$20{,}000.$$

So, the company would prefer to lease the computer system as it has the lower present value of cost.

3.2 Present value of a deferred perpetuity

A deferred perpetuity is a series of equal, periodic cash flows that recur forever but with the first cash flow occurring at some point in the future. For example, the following timeline shows a perpetual cash flow which is *deferred* until the end of year n+1.

Figure 5: Present Value of a Deferred Perpetuity



The present value of a deferred perpetuity can be calculated by first obtaining the present value of the perpetuity at time n using expression (5), as follows:

$$PV_n = \frac{C}{r}.$$

Next, we get the present value of the above cash flow at time 0 by discounting it over n time periods, as follows:

$$PV_0 = \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^n}\right). \tag{6}$$

Again, note that in the above expression we assume that the first cash flow of the deferred annuity occurs at the end of time *n* + *1*, and not time *n*. The first term in the above expression (*C*/*r*) is the present value of this deferred perpetuity at the end of time *n*. We then discount this future value over *n* time periods to get the present value at time 0, as shown in expression (6).

Example 6: Present value of a deferred perpetuity

Your company can lease a computer system for equal annual payments of $2,000 forever, or purchase it today for $14,500. The company has been able to enter a deal with the supplier where the first lease payment has been deferred to the end of year 4 with subsequent payments being made at the end of each of the following years forever. Ignoring taxes and other complications, what should the company do if the interest rate is 10 percent per annum?

Solution

We need to compare the purchase price today of $14,500 with the present value of the deferred perpetuity of $2,000 forever where the cash flow is deferred until the end of year 4. The present value of this deferred perpetuity is:

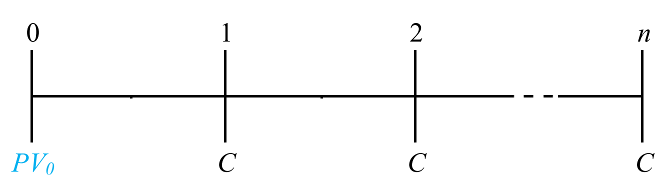
$$PV_0 = \left(\frac{\$2{,}000}{0.10}\right)\left(\frac{1}{(1+0.10)^3}\right) = \$15{,}026.30.$$

Again, note that the first amount ($2,000/0.10 = $20,000) is the present value of the deferred perpetuity at the end of year 3. This amount is then discounted over 3 years to get the present value at the end of year 0. The company would prefer to purchase the computer system as it has the lower present value of cost.

3.3 Present value of an ordinary annuity

An annuity is a series of equal, periodic cash flows occurring over *n* periods. *Ordinary annuities* occur at the *end* of each period. In valuing an ordinary annuity, it is assumed that the first cash flow of the annuity occurs at the end of the *first* period, and the last cash flow occurs at the end of period *n*. On a timeline, an *n*-period ordinary annuity is as follows:

Figure 6: Present Value of an Ordinary Annuity



Note that the present value of an *n*-period annuity can be obtained as the difference between the present value of a perpetuity that starts at the end of time period 1, or expression (5) above, and the present value of a deferred perpetuity that starts at the end of time period *n*+1, or expression (6) above. That is:

$$PV_0 = \left(\frac{C}{r}\right) - \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^n}\right).$$

Simplifying the above expression, we get:

$$PV_0 = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right). \tag{7}$$

Example 7: Present value of an annuity

You have won a contest and have been given the choice between accepting \$32,000 today or an equal annual cash flow of \$5,000 per year at the end of each of the next 10 years. What should you do if the interest rate is 10 percent per annum?

Solution

We need to compare the lump sum amount of \$32,000 available today with the present value of the ten-year annuity of \$5,000 per year. The present value of this annuity is:

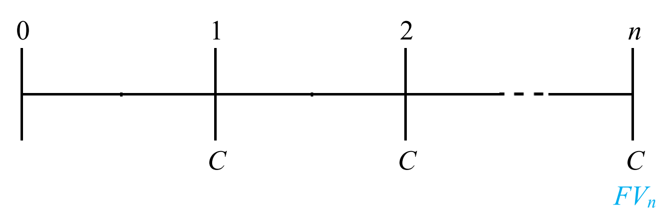
$$PV_0 = \left(\frac{\$5{,}000}{0.10}\right)\left(1 - \frac{1}{(1+0.10)^{10}}\right) = \$30{,}722.84.$$

So, you would prefer the lump sum amount today as it has a higher present value compared to the annuity of \$5,000 per year for ten years.

3.4 Future value of an ordinary annuity

The future value at the end of period n of an n-period ordinary annuity is the sum of the future values of each of these n cash flows. On a timeline, the future value of an n-period ordinary annuity can be depicted as follows:

Figure 7: Future Value of an Ordinary Annuity



The future value of an n-period ordinary annuity is easily obtained from its present value in expression (7) by compounding the present value over n time periods, as follows:

$$FV_n = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r)^n.$$

Simplifying the above expression, we get:

$$FV_n = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right]. \tag{8}$$

Example 8: Future value of an annuity

You have won a contest which pays an equal annual cash flow of $5,000 per year at the end of each of the next 10 years. What is the future value of this prize if the interest rate is 10 percent per annum?

Solution

Using the future value of an ordinary annuity expression, we get:

$$FV_{10} = \left(\frac{\$5{,}000}{0.10}\right)[(1 + 010)^{10} - 1] = \$79{,}687.12.$$

Note that we had calculated the present value of this cash flow in the previous example at $30,722.84. So, we could also calculate the future value of this cash flow at the end of ten years as:

$FV_{10} = \$30{,}722.84(1 + 0.10)^{10} = \$79{,}687.12$.

3.5 Present and future values of annuities due

So far, the series of cash flows we have valued have been assumed to occur at the end of each period. In some cases, these cash flows may occur at the beginning, rather than the end, of each period. Such an annuity is referred to as an *annuity due.* Using the convention that we have been using above and treating cash flows as occurring at the end of a particular time period this implies that if a cash flow now occurs at the beginning of time n then that is the same as the cash flow occurring at the end of time $n - 1$. This is because the beginning of time n is the same as the end of time $n - 1$. For example, a cash flow occurring at the beginning of 2022 (that is, 1 January 2022) is the same as if the cash flow occurs at the end of 2021 (that is, 31 December 2021) and so on, as shown below.

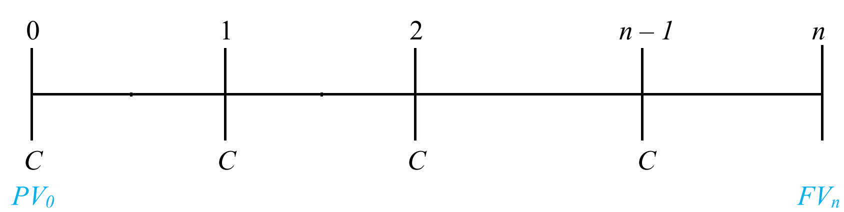
Beginning of year	End of previous year
1 Jan 2022	31 Dec 2021
1 Jan 2023	31 Dec 2022
1 Jan 2024	31 Dec 2023
1 Jan 2025	31 Dec 2024

If the cash flows were monthly, we would have the following equivalence:

Beginning of month	End of previous month
1 Sep 2022	31 Aug 2022
1 Oct 2022	30 Sep 2022
1 Nov 2022	31 Oct 2022
1 Dec 2022	30 Nov 2022

On a timeline, an n-period annuity due can be shown as follows:

Figure 8: Present and Future Values of Annuities Due



As the figure shows, an annuity due is essentially the same as an ordinary annuity with each cash flow "moved back" by one time period. So, the first cash flow occurs at the end of time 0 (which is the same as the beginning of time 1) while the last cash flow occurs at the end of time $n-1$ (which is the same as the beginning of time n).

Assume that we need the present value of an annuity due at the end of time 0 and the future value at the end of time n, as shown in the timeline above. As we have already calculated the present and future values of an ordinary annuity, we can obtain the present and future values of an annuity due by simply compounding these values by a factor of $(1 + r)$. So, the present and future values of an annuity due are obtained as:

$$PV_0 = \left(\frac{C}{r}\right)\left(1-\frac{1}{(1+r)^n}\right)(1+r), \textit{ and} \tag{9a}$$

$$FV_n = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right](1+r). \tag{9b}$$

The reason this simple adjustment works is because the present value *without* the adjustment would occur at the end of "time –1" and not time 0. So, compounding this present value by a factor of $(1 + r)$ moves the present value to the correct point in time, which is the end of time 0. Similarly, *without* the adjustment, the future value is calculated at the end of time $n-1$ and the adjustment moves the future value to the correct point in time, which is the end of time n.

We could, of course, have calculated the present value of the annuity due using a two-step process, that is, by discounting the $n-1$ *ordinary* annuity and then adding the cash flow at time 0. That is:

$$PV_0 = C + \left(\frac{C}{r}\right)\left(1-\frac{1}{(1+r)^{n-1}}\right). \tag{9c}$$

While it may not look like it, expressions (9a) and (9c) are the same as we can get from (9a) to (9c) as follows. Starting with expression (9a), we have:

$$PV_0 = \left(\frac{C}{r}\right)\left(1-\frac{1}{(1+r)^n}\right)(1+r), \tag{9a}$$

which gives:

$$PV_0 = \left(\frac{C}{r}\right)\left((1+r)-\frac{1}{(1+r)^{n-1}}\right),$$

and,

$$PV_0 = \left(\frac{C}{r}\right) + \left(\frac{C}{r}\right)r - \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^{n-1}}\right),$$

which simplifies to expression (9c):

$$PV_0 = C + \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^{n-1}}\right). \tag{9c}$$

Example 9: Present and future values of an annuity due

You have won a contest and have been given the choice of between accepting \$32,000 today or an equal annual cash flow of \$5,000 per year at the *beginning* of each of the next 10 years. What should you do if the interest rate is 10 percent per annum? What is the future value of this prize at the end of ten years?

Solution

We need to compare the lump sum amount available today with the present value of the ten-year annuity due of \$5,000 per year. The present value of this annuity due is:

$$PV_0 = \left(\frac{\$5{,}000}{0.10}\right)\left(1 - \frac{1}{(1+0.10)^{10}}\right)(1+0.10) = \$33{,}795.12.$$

So, you would prefer the annuity due of \$5,000 per year since it has a higher present value than the lump sum of \$32,000 today.

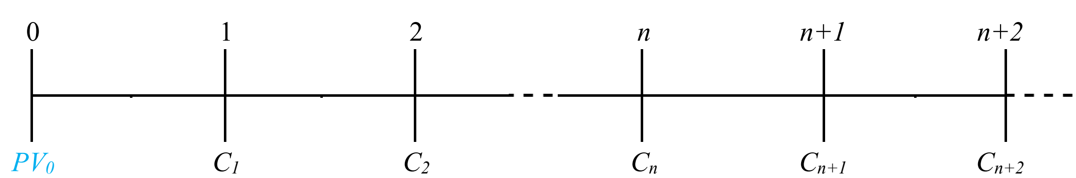
The future value of this annuity due is:

$$FV_n = \left(\frac{\$5{,}000}{0.10}\right)[(1+0.10)^{10} - 1](1+0.10) = \$87{,}655.84.$$

3.6 Present value of a growing perpetuity

A growing perpetuity is a series of periodic cash flows occurring at the end of each period and growing at a constant rate forever. On a timeline, a growing perpetual cash flow can be shown as follows:

Figure 9: Present Value of Growing Perpetuity



A constant growth rate of g percent per annum implies the following cash flows over time:

$C_2 = C_1(1 + g)$.

$C_3 = C_2(1 + g) = C_1(1 + g)^2$.

$C_n = C_1(1 + g)^{n-1}$.

Note that the first cash flow is assumed to occur at the end of time 1, not time 0. That is, the present value at time 0 of a growing perpetuity does not include time 0's cash flows. The present value of a growing perpetuity is calculated as:

$PV_0 = C_1/(1 + r) + C_1(1 + g)/(1 + r)^2 + C_1(1 + g)^2/(1 + r)^3 + \ldots + C_1(1 + g)^{n-1}/(1 + r)^n \ldots$

$PV_0 = [C_1/(1 + r)][1 + (1 + g)/(1 + r) + (1 + g)^2/(1 + r)^2 + \ldots + (1 + g)^{n-1}/(1 + r)^{n-1} \ldots]$

As n approaches infinity the second expression in the square brackets:

$[1 + (1 + g)/(1 + r) + (1 + g)^2/(1 + r)^2 + \ldots + (1 + g)^{n-1}/(1 + r)^{n-1} \ldots]$

approaches $1/[1 - (1 + g)/(1 + r)]$.

Substituting this expression in the expression for the present value of a growing perpetuity results in the following simplified expression for the present value of a growing perpetuity:

$$PV_0 = \frac{C_1}{r - g}. \tag{10}$$

Note that the subscript 1 to the cash flow highlights the fact that the first cash flow occurs at the end of time 1, not time 0. Also note that for the above expression to be correctly defined we also need to assume that $r > g$. The growth rate itself can be positive or negative (that is, declining rather than growing cash flows). Finally, if the growth rate of the cash flows is zero we revert to the simple perpetual cash flow case discussed earlier.

Example 10: Present value of a growing perpetuity

Your company can lease a computer system for an annual lease payment of \$2,000 next year with lease payments increasing at a constant annual rate of 2 percent forever, or purchase it today for \$23,000. Assume an interest rate of 10 percent per annum and end of the year cash flows. Ignoring taxes and other complications, what should the company do? How does your answer change if the lease payments were \$2,400 next year but *declining* at an annual rate of 1 percent forever?

Solution

The cash flow at the end of year 1 is \$2,000 which is expected to grow at 2% per annum. The present value of this growing perpetuity is:

$$PV_0 = \frac{\$2{,}000}{0.10 - 0.02} = \$25{,}000.$$

So, the company would prefer to purchase the computer system because the cost (in present value terms) is lower compared with leasing the system.

In the second case, we a have higher lease payment but with that cash flow declining at a constant rate over time (that is, g = –1% p.a.):

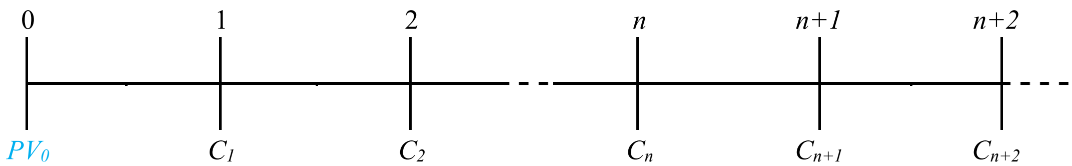
$$PV_0 = \frac{\$2{,}400}{0.10 - (-0.01)} = \$21{,}818.18.$$

The company would now prefer to lease the computer system because the leasing cost is lower in present value terms.

3.7 Present and future values of a growing annuities

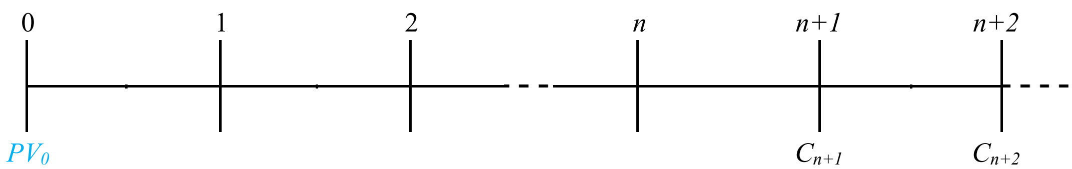
Another common type of cash flow is a growing ordinary annuity which is a series of periodic cash flows occurring at the end of each period and lasting for n periods where the cash flows grow at a constant rate g percent until time n. We can calculate the present and future values of such cash flows using the same method used to calculate the present and future values of ordinary annuities. The difference, of course, is that in this case the cash flows are growing over time. Graphically, what we're doing is taking the difference of a growing perpetual cash flow with the first cash flow occurring at the end of time 1 and another growing perpetual cash flow where the first cash flows is *deferred* until the end of time $n + 1$. Recall that on a timeline, a growing perpetual cash flow with the first cash flow occurring at the end of time 1 was as follows:

Figure 9: Present Value of Growing Perpetuity



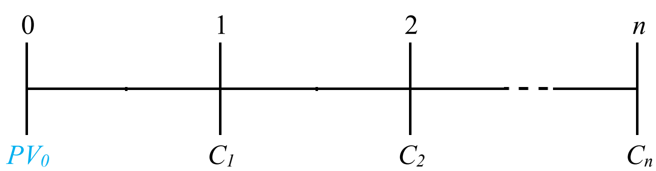
A growing perpetual cash flow with the first cash flow *deferred* until at the end of time $n + 1$ is as follows:

Figure 10: Present Value of Growing Deferred Perpetuity



The difference between the first and the second set of cash flows is a growing ordinary annuity, as follows:

Figure 11: Present Value of Growing Annuity



We have already obtained the present value of a growing perpetuity as:

$$PV_0 = \frac{C_1}{r-g}. \tag{10}$$

For the growing perpetuity deferred until the end of time $n + 1$ we first get its present value at the end of time n as:

$$PV_n = \frac{C_{n+1}}{r-g} = \frac{C_1(1+g)^n}{r-g}.$$

The above expression shows that the cash flow at the end of time $n + 1$ is equal to the cash flow at time 1 growing at g percent per period over n periods, that is, $C_{n+1} = C_1(1 + g)^n$.

Next, we calculate the present value of this cash flow at time 0 by discounting what is now a future value equal to PV_n by discounting it at r percent over n time periods as:

$$PV_0 = \left[\left(\frac{C_1(1+g)^n}{r-g} \right) \Big/ (1+r)^n \right]. \tag{11}$$

Finally, the present value of an annuity growing at *g* percent per period over *n* periods can be calculated as the difference between expressions (10) and (11) as:

$$PV_0(GA) = \frac{C_1}{r-g} - \left[\left(\frac{C_1(1+g)^n}{r-g} \right) \Big/ (1+r)^n \right].$$

Simplifying the above expression, we get the present value of a growing annuity as:

$$PV_0(GA) = \left(\frac{C_1}{r-g} \right) \left(1 - \left(\frac{1+g}{1+r} \right)^n \right). \tag{12}$$

Note that the only time the above expression cannot be used is when $r = g$. In that case, one would need to value the cash flows individually. The future value of a growing annuity is simply calculated as the future value at *r* percent over *n* time periods of the right-hand side in expression (12) as:

$$FV_n(GA) = \left(\frac{C_1}{r-g} \right) \left(1 - \left(\frac{1+g}{1+r} \right)^n \right) (1+r)^n . \tag{13}$$

Note that in the above expressions if we set the growth rate equal to 0 we revert to ordinary annuities and get expressions (7) and (8) for the present and future values respectively. Also, in expression (12), if the growth in cash flows is perpetual, that is *n* approaches infinity, the second term converges to 1 giving us expression (10) which is the present value of a growing perpetuity.

Example 11: Present and future value of a growing annuity

You have won a contest and have been given the choice between accepting $32,000 today or $5,000 in year 1 which will then grow at 2 percent per annum until the end of year 10. What should you do if the interest rate is 10 percent per annum? What is the future value of this prize at the end of year 10?

Solution

We need to compare the lump sum amount of $32,000 available today with the present value of the ten-year growing annuity of $5,000. The present value of this growing annuity is:

$$PV_0(GA) = \left(\frac{\$5{,}000}{0.10 - 0.02} \right) \left(1 - \left(\frac{1 + 0.02}{1 + 0.10} \right)^{10} \right) = \$33{,}126.56.$$

Note that in example 7 we had the same cash flows but with zero growth. The difference between the present value in that example and this one is $2,403.72. This is the present value that we can attach to the 2 percent growth in cash flows over the ten-year period. As a result of this growth in cash flows, you would now prefer the growing cash flows over time rather than the lump sum today.

The value of the prize at the end of year 10 is the future value of this growing annuity, which is:

$$FV_{10}(GA) = \left(\frac{\$5{,}000}{0.10 - 0.02} \right) \left(1 - \left(\frac{1 + 0.02}{1 + 0.10} \right)^{10} \right) (1 + 0.10)^{10} = \$85{,}921.75.$$

Note that we could have also calculated the future value using the present value already calculated, as:

$FV_{10} = 33126.56(1 + 0.10)^{10} = \$85,921.75$.

Practice Problem 1

You have just won a lottery and have been offered the following alternative ways of receiving the prize money. Assume that each alternative is risk free and the interest rate is 8 percent per annum.

a) $110,000 immediately.

b) $140,000 at the end of year 3.

c) $28,000 at the end of each of the next 5 years.

d) $9,000 at the end of each year (starting at the end of year 1) in perpetuity.

e) $6,500 at the end of the first year growing at an annual rate of 2 percent in perpetuity.

Assuming end of the year cash flows, which is the *best* way to receive the prize money?

Practice Problem 2

Assume that the interest rate is 10 percent per annum and that all cash flows occur at the end of each year. *Round off your final answers to the nearest dollar*.

a) Suppose you decide to invest $50,000 today. Calculate the future value of your investment at the end of 10 years.

b) Your friend also decides to invest $50,000 but plans to do so in installments. Specifically, she will invest $5,000 now, $10,000 at the end of year 1, $15,000 at the end of year 2, and $20,000 at the end of year 3. Calculate the future value of her investment at the end of 10 years.

c) Another friend of yours also decides to invest $50,000 but to defer the investment until the end of year 3. Calculate the future value of his investment at the end of 10 years.

d) What factor(s) account for the differences in the future values in parts (a) through (c)?

e) Now assume that you and your friends have an investment time horizon of 10 years. What *additional* amount would your friends need to invest *today* so that, at the end of year 10, the future value of their respective investments is the same as the future value of your investment?

f) Suppose that instead of investing the additional amounts today in part (e) your friends decide to invest funds in equal annual amounts over the ten-year period. Calculate the equal annual amounts your two friends would now need to invest.

g) Now suppose that you and your friends decide to invest funds in equal annual amounts over the ten-year period so each of you has the same total value at the end of ten years as calculated in part (a). What would this equal annual amount be?

Practice Problem 3

You are running a hot internet company. Market analysts estimate that its earnings will grow at 30% per annum for the next five years (that is, years 1 – 5). After that, as competition increases, earnings growth is expected to slow down to 2% per annum and continue at that level forever. Your company has just announced earnings of $1 million today. Assuming end of the year cash flows, what is the present value of all **future** earnings if the interest rate is 8 percent per annum?

Practice Problem 4

Your second cousin is about to start kindergarten at Melbourne's third most prestigious private school. The tuition is $20,000 per year, payable at the *beginning* of the school year. You expect your second cousin to remain in private school through high school. What is the present value of the tuition payments if the interest rate is 5 percent per year? How does your answer change if you expect the tuition fees to increase at a rate of 5 percent per year over the 13 years of her schooling?

4. Some applications of financial mathematics

4.1. Valuing home mortgages

In a standard, fixed-rate mortgage, a lump sum cash flow (i.e., the loan amount) is exchanged for equal, periodic payments over the duration of the mortgage. The payments are typically made on a monthly basis where each payment equals the sum of the interest payment and principal repayment. Calculating the value of such a mortgage at any point over the loan's time horizon is a specific application of the present value of an ordinary annuity.

Calculating the periodic loan payments

Assume PV_0 is borrowed today at an interest rate of r percent per period to be repaid in n equal payments of C dollars per period. Assuming that the periodic payment is made at the end of each period we have an n-period ordinary annuity. The amount borrowed today is the present value of this annuity as shown in expression (9c):

$$PV_0 = C[1 - (1 + r)^{-n}]/r. \tag{9c}$$

Re-arranging the above expression we can obtain the periodic payment as:

$$C = r \times PV_0/[1 - (1 + r)^{-n}]. \tag{9d}$$

Example 12: Valuing a home mortgage

A 25-year home loan for $100,000 has a monthly interest rate of 1 percent and payments are to be made at the end of each month. Calculate the monthly payment on this loan. Calculate the principal balance remaining at the end of the (a) first and (b) fifth year of the loan.

Solution

In this case, $n = 25 \times 12 = 300$ months.

Using expression (9d) we can obtain the monthly payment as:

Monthly payment, $C = 0.01(\$100{,}000)/(1 - 1.01^{-300}) = \$1{,}053.22$.

The process of discounting the monthly payment essentially "strips off" the interest component from each month's payment. So, we can use expression (9c) to calculate the principal balance remaining on the loan at the end of *any* time period.

The principal balance remaining at the end of the first year of the loan (i.e., after 12 months) is the present value of the *remaining* 288 (= 300 – 12) monthly payments, which is:

$PV_{12} = \$1{,}053.22[1 - (1.01)^{-288}]/0.01 = \$99{,}324.59$.

Similarly, the principal balance remaining at the end of the fifth year of the loan (i.e., after 60 months) is the present value of the *remaining* 240 (= 300 – 60) monthly payments, which is:

PV_{60} = \$1,053.22[1 – $(1.01)^{-240}$]/0.01 = \$95,652.83.

Obtaining the loan amortization schedule

A loan amortization schedule shows the total payments on a loan and separates this amount into the interest paid in a particular period, the principal balance repaid in that period and the principal balance outstanding at the end of that period. The *interest paid* in any period is calculated as the per-period interest rate multiplied by the principal balance at the start of that period which is equal to the principal balance at the end of the previous period. The *principal repaid* in any period is calculated as the loan payment minus the interest paid in that period. Finally, the *principal remaining* in any period is the previous period's principal less the principal repaid in that period. That is:

Interest paid = Previous period's principal × Interest rate per period,

Principal repaid = Loan payment – Interest paid, and

Principal balance remaining = Previous period's principal – Principal repaid.

The following example illustrates a typical loan amortization schedule.

Example 13: Amortization schedule for a home mortgage

Consider Example 12 relating to a 25-year home loan for \$100,000 with a monthly interest rate of 1 percent and payments are to be made at the end of each month. What is the interest paid and principal repaid in the first month? Develop the loan's amortization schedule for the first three months.

Solution

The monthly payments on the loan were calculated in Example 12 as \$1,053.22.

The interest paid in month 1 = 0.01(\$100,000) = \$1,000.

So, the principal repaid in month 1 = \$1,053.22 – \$1,000 = \$53.22.

The principal balance at the end of month 1 = \$100,000 – \$53.22 = \$99,946.78.

The loan's amortization schedule for the next two months can be similarly obtained as shown in the following table.

Amortization schedule of a standard, fixed-rate mortgage

Month (1)	*Total payment (2)*	*Interest (3) = (5) × 1%*	*Principal repaid (4) = (2) – (3)*	*Previous period's principal (5)*	*Principal remaining (6) = (5) – (4)*
0	–	–	–	–	\$100,000.00
1	\$1,053.22	\$1,000.00	\$53.22	\$100,000.00	\$99,946.78
2	\$1,053.22	\$999.47	\$53.75	\$99,946.78	\$99,893.03
3	\$1,053.22	\$998.93	\$54.29	\$99,893.03	\$99,838.74

Note that of the first month's payment of \$1,053.22, about 95 percent goes towards paying interest and only 5 percent goes towards repayment of principal. As the principal remaining *reduces* over time, the proportion of the monthly payment going towards paying the monthly interest *decreases* and that going towards repaying the principal *increases*.

Practice Problem 5

Consider, once again, the 25-year home loan for $100,000 with a monthly interest rate of 1 percent and payments are to be made at the end of each month. Calculate the following:

a) The principal balance remaining on this loan at the end of the first year.

b) The total principal repaid over the first year of the loan.

c) The total interest paid over the first year of the loan.

4.2. Valuing debt securities

The basic valuation process for financial securities, including debt and equity securities, involves obtaining the present value of the future cash flows that the security is expected to generate. Thus, the valuation process essentially involves applying the concepts discussed above.

In general, debt securities are securities where the issuer borrows funds from investors with a contractual obligation to make regular interest payments to these investors as well as repay the funds borrowed when the contract matures in the future. The simplest type of debt security is one that pays a fixed amount (the face or maturity value) at some time in the future and no other cash flow over its life. Examples of this type of security are Treasury bills and bank bills which mature in less than a year and zero coupon bonds which typically have maturity dates well into the future.

Example 14: Valuing discount securities

Consider a Treasury bill with a face (or maturity) value of $100,000 and which matures in 180 days. Assuming that the market yield on this security is 6% p.a. what is its price today?

Solution

Since the security matures in 180 days we first need to calculate the appropriate market yield over the security's life. As there are 365 days in the year, the market yield appropriate over the 180-day period is:

180-day market yield = $(n/365) \times r = (180/365) \times 0.06 = 0.02959$ or 2.959%.

The price can then be calculated as the present value of the face value, which is:

$P_0 = \$100{,}000/[1 + 0.02959] = \$97{,}126.13$.

Example 15: Valuing a zero coupon bond

A zero coupon bond matures in 5 years and has a face value of $1,000. Calculate the price of the bond today if the market yield on the bond is 8% p.a. If tomorrow the price of the bond changes to: (a) $690.00 or (b) $675.00 what has happened to the market yield on the bonds?

Solution

The price of the bond today is the present value of the face value which investors will receive at maturity 5 years from now. That is:

$P_0 = \$1{,}000/(1 + 0.08)^5 = \$680.58$.

If the price *rises* to \$690.00 tomorrow the new market yield (r) will *fall* to:

$P_0 = \$690 = \$1{,}000/(1 + r)^5$.

So, $(1 + r)^5 = \$1{,}000/\$690$.

That is, $r = (\$1{,}000/\$690)^{1/5} – 1 = 7.70\%$.

If the price *falls* to \$675.00 tomorrow the new market yield (r) will *rise* to:

$P_0 = \$675 = \$1{,}000/(1 + r)^5$.

So, $r = (\$1{,}000/\$675)^{1/5} – 1 = 8.18\%$.

Valuing coupon paying debt securities is a little more complex because it involves calculating the present value of the face value at maturity and the promised periodic coupon (or interest) payments, which are stated as a percentage of the face value. The coupon payments are typically made on a semi-annual basis, but their valuation can be approximated quite well assuming annual coupon payments. So, the value of such securities can be calculated as the sum of the present value of the face value and the interest annuity, as shown in the following example.

Example 16: Valuing a coupon paying bond

Consider a bond which pays an annual coupon of 10 percent with 5 years to maturity and a face value of \$1,000. If the bond has a market yield of 8 percent what price should it be selling for today? What would the price of this bond be if the coupons were paid on a semi-annual basis?

Solution

The price of the security is equal to the present value of the annuity of coupon payments on the face value of \$1000 and the face value paid at the end of year 5. If the coupons are paid on an annual basis the price for the bond should be:

$P_0 = \$100[(1 – (1 + 0.08)^{–5})/0.08] + \$1{,}000/(1 + 0.08)^5$.

$P_0 = \$399.27 + \$680.58 = \$1{,}079.85$.

If the coupons are paid semi-annually, the coupon payment every six months will be:

Semi-annual coupon payments = $0.10/2 \times \$1{,}000 = \$50$ every six months.

There are 10 (= 5×2) coupon payments over the life of the security. Using the appropriate semi-annual market yield of 4% (= 0.08/2), the price of the bond today should be:

$P_0 = \$50[(1 – (1 + 0.04)^{–10})/0.04] + \$1{,}000/(1 + 0.04)^{10}$.

$P_0 = \$405.54 + \$675.56 = \$1{,}081.10$.

4.3. Valuing equity securities

The two most common types of equity securities are preference shares and ordinary shares. Unlike debt securities, there are no obligations for payments to be made to such shareholders. That said, preference shares *always* have a preference over dividend payments over ordinary shares. That is, dividends on ordinary shares cannot be made if the promised dividends are not paid to preference shareholders. In that sense, preference shares have debt-like characteristics that ordinary shares do not have.

The dividends paid to preference shareholders are based on the face value of the shares on issue and the shares are valued as the present value of a perpetuity, as shown in the following example.

Example 17: Valuing preference shares

ACB Ltd has 8%, $100 preference shares on issue and on which investors require a return of 10 percent per annum. At what price should these shares be selling for today? How does the price change if the return required by investors falls from 10% to 6%?

Solution

The price of the preference shares is calculated as the present value of the expected perpetual dividend stream. In this example, the dividends paid to preference shareholders is 8% of the face value of $100, that is:

$D_p = 0.08 \times \$100 = \$8.00$.

The price of the preference shares is:

$P_0 = D_p/r_p = (0.08 \times \$100)/0.10 = \$80.00$.

If the return required by investors *falls* to 6% you would expect the price to *rise* above the face value, and vice versa. That is:

$P_0 = D_p/r_p = (0.08 \times \$100)/0.06 = \$133.33$.

This again illustrates the *inverse* relation between prices and required returns. As the return required rises (falls) the price of the preference shares falls (rises). The economic rationale behind this is relatively straightforward. When investors required a return of 10%, the preference shares were trading at $80 per share. If investors now require a lower return of 6% *and* if the price were *not* to change from $80 everyone would rush to buy these preference shares because investors would be expecting to earn a return that is 4% higher than what they are happy with now (that is, 10% versus 6%). As investors rush to buy these shares, the share price will be bid up to $133.33 at which point everyone will earn the lower required return of 6%. The *same* reasoning essentially applies when valuating any type of financial security.

Dividends to ordinary shareholders need not follow any particular pattern although companies with a policy of paying regular dividends on ordinary shares generally like to maintain either a constant dividend payout or with dividends growing over time. So, ordinary shares can often be valued as the present value of a perpetual stream of expected future dividends or a growing perpetuity over time, as shown in the following example.

Example 18: Valuing ordinary shares

You are valuing the ordinary shares of four companies which are expected to pay the following dividends over time:

- BCA Ltd is expected to pay a dividend of $1.00 next year with this dividend maintained forever.
- DCA Ltd is expected to pay a dividend of $1.00 next year with this amount expected to grow at a constant rate of 5% per annum forever.

- FCA Ltd is not expected to pay a dividend for the next 3 years and then pay a dividend of $1.00 per year forever.

- HCA Ltd is not expected to pay a dividend for the next 3 years and then pay a dividend of $1.00 per year which is expected to grow at a constant rate of 5% per annum forever.

What are your price estimates for each company if investors require a return of 10 percent per annum on all these companies' shares?

Solution

BCA's shares can be valued as the present value of a perpetuity. That is:

BCA's price today, $P_0 = D_1/r$ = $1.00/0.10 = $10.00.

DCA's shares can be valued as the present value of a *growing* perpetuity. That is:

DCA's price today, $P_0 = D_1/(r - g)$ = $1.00/(0.10 – 0.05) = $20.00.

FCA's shares can be valued as the present value of a *deferred* perpetuity. Here, the dividend is *deferred* until the end of year 4 after which it becomes a perpetual cash flow stream of $1.00 per year. The price of the shares at the end of year 3 is:

FCA's price at the end of year 3, $P_3 = D_4/r$ = $1.00/0.10 = $10.00.

FCA's price today, $P_0 = P_3/(1 + r)^3$ = $10.00/(1 + 0.10)^3 = $7.51.

Alternatively, in one step, we have:

FCA's price today, $P_0 = (D_4/r)/(1 + r)^3 = (1.00/0.10)/(1 + 0.10)^3$ = $7.51.

HCA's shares can be valued as the present value of a *deferred* perpetuity that *grows* at a constant rate. Here, the dividend is also deferred until the end of year 4 after which it becomes a growing perpetual cash flow stream. The price of the shares at the end of year 3 is:

HCA's price at the end of year 3, $P_3 = D_4/(r - g)$ = $1.00/(0.10 – 0.05) = $20.00.

HCA's price today, $P_0 = P_3/(1 + r)^3$ = $20.00/(1 + 0.10)^3 = $15.03.

Alternatively, in one step, we have:

HCA's price = $[D_4/(r - g)]/(1 + r)^3$ = [$1.00/(0.10 – 0.05)]/(1 + 0.10)^3 = $15.03.

5. Effective interest rates

The above discussion assumes that interest is paid or received annually. However, this is often not the case. In the mortgage example above, interest is paid on the monthly principal balance outstanding. Although the monthly interest rate in our mortgage example was 1 percent, the effective annual interest rate is *not* 12 percent because interest on the loan is calculated (i.e., compounded) more frequently than once a year.

The *effective* annual interest rate will differ from the nominal (or quoted) annual interest rate as long as interest is calculated more frequently than once a year. If the stated annual interest rate is r percent and interest is calculated m times a year, the *per-period* interest rate is r/m percent and the *effective* annual interest rate is calculated as:

$$r_e = (1 + r/m)^m - 1. \quad (14)$$

The above expression means that if we invest \$1 at the quoted annual interest rate of r percent, but where interest is compounded m times during the year, our initial \$1 investment will amount to \$$(1 + r/m)^m$ by the end of the year. The *effective* annual interest rate would be as in expression (14). Note that if interest is calculated once a year ($m = 1$), then $r_e = r$. In all other cases, $r_e > r$.

Continuous compounding

The compounding interval becomes continuous as m approaches infinity; the expression $(1 + r/m)^m$ approaches e^r, where e is the exponential constant taking the value 2.718281. With continuous compounding, the effective interest rate is calculated as:

$$r_e = e^r - 1. \quad (15)$$

Example 19: Comparing effective interest rates

You are considering depositing \$10,000 in one of the following banks. The interest rates offered by these banks and the compounding intervals are as follows:

	Stated annual interest rate	*Compounding interval*
Bank A	8.10%	Annual
Bank B	8.00%	Semi-annual
Bank C	7.95%	Monthly
Bank D	7.90%	Daily
Bank E	7.93%	Continuous

Which bank offers the *best* interest rate? If you were *borrowing* funds from one of these banks, which bank would offer the best interest rate?

Solution

Clearly we cannot compare the stated interest rates here because the compounding intervals differ among the five banks. The appropriate comparison can be made by calculating the effective annual interest rates which consider the different compounding intervals. These are as follows:

	Effective annual interest rates
Bank A	8.100%
Bank B	$(1 + 0.0800/2)^2 - 1 = 8.160\%$
Bank C	$(1 + 0.0795/12)^{12} - 1 = 8.246\%$
Bank D	$(1 + 0.0790/365)^{365} - 1 = 8.219\%$
Bank E	$e^{0.0793} - 1 = 8.253\%$

So, Bank E offers the best effective annual interest rate. Of course, if you were *borrowing* funds from one of these banks then the bank offering the *lowest* effective annual interest rate would be the best as, in that case, you want to borrow funds at the lowest cost possible, that is, at Bank A.

Note that in Australia the market convention for daily compounding is to use 365 days in calculating the effective annual interest rate. This differs from some other markets, such as the United States, where the market convention for daily compounding is to assume that there are 360 days in the year. In such cases, the divisor for the daily compounding calculation above would be 360 and not 365. The exponent, of course, remains 365 because we need to compound the daily interest rate *over the year*, which has 365 days. For example, in the case of Bank D above, using 360-day compounding gives us an effective annual interest rate of:

$r_e = (1 + 0.0790/360)^{365} - 1 = 8.338\%$.

The obvious question is, which basis would one use for daily compounding? The simple answer is that the default for us (in Australia) is 365 days. If 360-day basis compounding is required then the question needs to state that daily compounding is on a 360-day basis.

In calculating the present and future values above we assumed annual cash flows with interest compounded annually. In general, this is not going to be the case with cash flows and/or interest rates. The main thing to keep in mind is that the periodicity of the cash flows being valued should match the periodicity of the interest rate being used to value these cash flows. That is, if the cash flows are quarterly then the quarterly interest rate should be used to value the cash flows. Similarly, if the cash flows are monthly then the monthly interest rate should be used to value the cash flows. It is possible that the cash flows are annual, but interest is compounded (say) monthly. In this case, the appropriate interest rate would be effective annual interest rate because this rate would match the periodicity of the cash flows and take into account the compounding interval within the year. For example, the future and present values of a single cash flow where the stated annual interest rate r is compounded m times a year are:

$$FV_n = P_0(1 + r/m)^{n \times m}, \text{ and} \tag{16}$$

$$PV_0 = F_n/(1 + r/m)^{n \times m}. \tag{17}$$

To obtain the future and present values of an ordinary annual annuity where the stated annual interest rate r is compounded m times a year we first need to calculate the effective annual interest rate. This is the appropriate interest rate to use since the cash flows are annual while the stated interest rate is compounded m times a year. That is:

$$r_e = (1 + r/m)^m - 1.$$

The future and present values of an ordinary annual annuity over n years can then be calculated as:

$$FV_n = C \times \frac{(1+r_e)^n - 1}{r_e}, \text{ and} \tag{18}$$

$$PV_0 = C \times \frac{1-(1+r_e)^{-n}}{r_e}. \tag{19}$$

Example 20: Future and present values with compounding more than once a year

Calculate the future or present values of the following cash flows assuming that the interest rate is 12 percent p.a. with (i) annual, (ii) quarterly and (iii) monthly compounding. Assume end of the year cash flows.

a) The future value at the end of year 5 of $50,000 now.

b) The present value of $50,000 at the end of year 5.

c) The future value at the end of year 5 of $5,000 per year for 5 years with the first cash flow occurring at the end of year 1.

d) The present value of $5,000 per year for 5 years with the first cash flow occurring at the end of year 1.

Solution

a) Annual compounding: FV_5 = \$50,000(1 + 0.12/1)$^{5 \times 1}$ = \$88,117.

Quarterly compounding: FV_5 = \$50,000(1 + 0.12/4)$^{5 \times 4}$ = \$90,306.

Monthly compounding: FV_5 = \$50,000 (1 + 0.12/12)$^{5 \times 12}$ = \$90,835.

b) Annual compounding: PV_0 = \$50,000/(1 + 0.12/1)$^{5 \times 1}$ = \$28,371.

Quarterly compounding: PV_0 = \$50,000/(1 + 0.12/4)$^{5 \times 4}$ = \$27,684.

Monthly compounding: PV_0 = \$50,000/(1 + 0.12/12)$^{5 \times 12}$ = \$27,522.

c) Annual compounding: $FV_5 = \$5{,}000 \times \frac{(1+0.12)^5-1}{0.12} = \$31{,}764$.

With quarterly compounding and annual cash flows, we first need to calculate the effective annual interest rate, as follows:

$r_e = (1 + r/m)^m - 1 = (1 + 0.12/4)^4 - 1 = 12.551\%$.

The future value at the end of year 5 is:

$FV_5 = \$5{,}000 \times \frac{(1+0.12551)^5-1}{0.12551} = \$32{,}114$.

Similarly, with monthly compounding and annual cash flows, we calculate the effective annual interest rate, as follows:

$r_e = (1 + 0.12/12)^{12} - 1 = 12.6825\%$.

The future value at the end of year 5 now is:

$FV_5 = \$5{,}000 \times \frac{(1+0.126825)^5-1}{0.126825} = \$32{,}198$.

d) Annual compounding: $P_0 = \$5{,}000 \times \frac{1-(1+0.12)^{-5}}{0.12} = \$18{,}024$.

Using the effective annual interest rate from part (c), we can get the present value today as:

$PV_0 = \$5{,}000 \times \frac{1-(1+0.12551)^{-5}}{0.12551} = \$17{,}781$.

Using the effective annual interest rate from part (c), we can get the present value today as:

$PV_0 = \$5{,}000 \times \frac{1-(1+0.126825)^{-5}}{0.126825} = \$17{,}723$.

Practice Problem 6

Suppose you decide to invest \$120,000 today for a five year period at an interest rate of 8 percent per annum.

a) Calculate the value of your investment at the end of year 5 assuming that interest is compounded annually.

b) Suppose that the amount invested earned interest compounded on a monthly basis. If you wanted the value of your investment at the end of year 5 to be the same amount as that calculated in part (a) then what amount would you need to invest today?

c) What is the effective annual interest rate that you are earning on your investment in part (b)?

d) Suppose you decide to invest $24,000 at the end of each year for the next five years rather than the $120,000 today. Calculate the value of your investment at the end of year 5 if interest is compounded on an (i) annual and (ii) monthly basis.

6. Suggested answers to practice problems

Practice Problem 1

We can calculate and compare the present values of each prize as follows:

b) $PV_0 = \frac{\$140{,}000}{(1+0.08)^3} = \$111{,}137.$

c) $PV_0 = \left(\frac{\$28{,}000}{0.08}\right)\left(1 - \frac{1}{(1+0.08)^5}\right) = \$111{,}796.$

d) $PV_0 = \left(\frac{\$9{,}000}{0.08}\right) = \$112{,}500.$

e) $PV_0 = \left(\frac{\$6{,}500}{0.08-0.02}\right) = \$108{,}333.$

Based on the present values of the prizes you would prefer the prize of $9,000 at the end of each year in perpetuity.

Practice Problem 2

a) The future value at the end of *n* years of your investment is given by:

$FV_n = PV_0(1 + r)^n$.

So, $FV_{10} = \$50{,}000(1 + 0.10)^{10} = \$129{,}687$.

b) The future value at the end of 10 years of this friend's investment can be calculated as the sum of the future values of the individual cash flows. The first cash flow will earn interest over 10 years, the second over 9 years and so on, as follows:

$FV_{10} = \$5{,}000(1.10)^{10} + \$10{,}000(1.10)^9 + \$15{,}000(1.10)^8 + \$20{,}000(1.10)^7 = \$107{,}676$.

c) In this case your friend earns no return for the first 3 years and the value of the $50,000 invested over the remaining 7 years is as follows:

$FV_{10} = \$50{,}000(1.10)^7 = \$97{,}436$.

d) As expected, the future values in parts (b) and (c) are lower than in part (a). This is because in part (b) all the funds are not being invested immediately so the funds that are invested later earn a lower compounded return over the time horizon. In part (c) all the funds are invested at the same time, but this investment is deferred until year 4 so no compounded return is earned over the first three years.

e) From part (a), the future value of your investment at the end of 10 years is $129,687.

Let the *additional* amounts invested today by your friends be $\$X$ and $\$Y$, respectively.

We need the future value of the first friend's investment to be worth \$129,687 once the additional amount of $\$X$ invested today is included in the answer from part (b) above. That is:

$\$129{,}687 = \$107{,}676 + X(1.10)^{10}$.

So, $X = (\$129{,}687 – \$107{,}676)/1.10^{10} = \$8{,}486$.

Similarly, for the second friend, we need the future value of his investment to be worth \$129,687 once the additional amount of $\$Y$ invested today is included in the answer from part © above. That is:

$\$129{,}687 = \$97{,}436+ Y(1.10)^{10}$.

So, $Y = (\$129{,}687 – \$97{,}436)/1.10^{10} = \$12{,}434$.

f) From part (e) we know that the amounts your two friends need to invest today are \$8,486 and \$12,434, respectively. To obtain the equal annual amounts that they should invest rather than these lump sum amounts we need to convert these amounts today into annuities using the present value of an annuity expression, which is:

$$PV_0 = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right).$$

Rewriting the above expression in terms of the unknow cash flow (C), we get:

$$C = \frac{r \times PV_0}{\left(1 - \frac{1}{(1+r)^n}\right)}.$$

So, the equal annual amounts that the first friend needs to invest can be calculated as:

$$C_{Mate1} = \frac{r \times PV_0}{\left(1 - \frac{1}{(1+r)^n}\right)}.$$

$$C_{Mate1} = \frac{0.10 \times \$8{,}486}{\left(1 - \frac{1}{(1+0.10)^{10}}\right)} = \$1{,}381.$$

Similarly, the equal annual amounts that the second friend needs to invest can be calculated as:

$$C_{Mate2} = \frac{0.10 \times \$12{,}434}{\left(1 - \frac{1}{(1+0.10)^{10}}\right)} = \$2{,}024.$$

g) The future value at the end of 10 years that we need is \$129,687 from part (a). To obtain the equal annual amounts that should be invested over the ten-year period we need to use the future value of an annuity, which is:

$$FV_n = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right].$$

So, the equal annual amounts that needs to be invested (C) can be calculated as:

$$FV_{10} = \$129{,}687 = \left(\frac{C}{0.10}\right)[(1+0.10)^{10} - 1].$$

$$C = \frac{0.10 \times \$129{,}687}{[(1+0.10)^{10} - 1]} = \$8{,}137.$$

Practice Problem 3

The timeline for this scenario is as follows where the $1 million earnings today are expected to grow by 30% per annum for five years and then at 2% per annum forever.

0	1	2	3	4	5	6	7
	$\$1(1.3)$	$\$1(1.3)^2$	$\$1(1.3)^3$	$\$1(1.3)^4$	$\$1(1.3)^5$	$\$1(1.3)^5(1.02)$	$\$1(1.3)^5(1.02)^2$

This scenario consists of two parts:

1. A growing annuity over years 1 – 5.

2. A growing perpetuity after 5 years (that is, year 6 onwards).

We calculate the present value of part 1 as:

$$PV(\text{GA}) = \left(\frac{\$1m(1+0.30)}{0.08-0.30}\right)\left(1 - \left(\frac{1.30}{1.08}\right)^5\right) = 9.02million.$$

Next, we calculate the present value of part 2. The cash flow at the end of year 6 is $\$1(1.3)^5(1.02)$. So, the present value at the end of year 5 of this growing perpetuity is:

$$PV_5 = \frac{\$1m(1.30)^5(1+0.02)}{0.08-0.02} = \$63.12million.$$

The present value of this amount at time 0 is:

$$PV_0 = \frac{\$63.12m}{(1.08)^5} = \$42.96million.$$

Finally, adding the present value of parts 1 and 2 together gives the present value of future earnings as:

PV_0 = $9.02 million + $42.96 million = $51.98 million.

Practice Problem 4

In the first case, we have an annuity due and the timeline (assuming end-of-the-year cash flows) is as follows:

0	1	2	3	….	12	13
\$20,000	\$20,000	\$20,000	\$20,000		\$20,000	0

The present value of an annuity due is:

$$PV_0 = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r).$$

In this case, we get:

$$PV_0 = \left(\frac{\$20{,}000}{0.05}\right)\left(1 - \frac{1}{(1+0.05)^{13}}\right)(1+0.05) = \$197{,}265.$$

Note that we could have also calculated the present value of the 12-year ordinary annuity (from years 1 – 12) and then added the cash flow of \$20,000 in year 0 to get the same answer, as follows:

$$PV_0 = \$20{,}000 + \left(\frac{\$20{,}000}{0.05}\right)\left(1 - \frac{1}{(1+0.05)^{12}}\right) = \$197{,}265.$$

In the second case, the timeline (assuming end-of-the-year cash flows) is as follows:

0	1	2	3	….	12	13
\$20,000	\$20,000(1.05)	\$20,000$(1.05)^2$	\$20,000$(1.05)^3$		\$20,000$(1.05)^{12}$	0

The first payment is already at time 0. The remaining 12 payments is a growing annuity cash flow which we would normally value using the following expression:

$$PV_0(GA) = \left(\frac{C_1}{r-g}\right)\left(1 - \left(\frac{1+g}{1+r}\right)^n\right).$$

Note, however, that in this case we **cannot** use the present value of a growing annuity expression above because $r = g$! So, we need to calculate the present value of each of the remaining 12 payments and add them up to get the present value today, as follows:

$$PV_0 = \frac{\$20{,}000(1.05)}{(1.05)} + \frac{\$20{,}000(1.05)^2}{(1.05)^2} + \frac{\$20{,}000(1.05)^3}{(1.05)^3} + \cdots + \frac{\$20{,}000(1.05)^{12}}{(1.05)^{12}}$$

Because the growth rate in the tuition fees equals the interest rate used to evaluate the cash flows the above set of calculations essentially is just adding up the cash flows:

PV_0 = \$20,000 + \$20,000 + \$20,000 + … + \$20,000 = \$20,000 × 12 = \$240,000

Adding the initial tuition payment today gives a total present value of \$240,000 + \$20,000 = \$260,000.

Suggested answers to practice problems 5 and 6

Practice Problem 5

a) We have already computed the monthly payment in this loan as \$1,053.22. The principal balance remaining at the end of the first year can be calculated as the present value of the remaining 288 (= 300 – 12) payments, which is:

Principal at the end of the first 12 months, PV_{12} = \$1,053.22[1 – $(1.01)^{-288}$]/0.01 = \$99,324.59.

b) The principal repaid in the first year can be calculated as the difference between the beginning principal balance and the principal balance remaining at the end of the first year, which is:

Principal repaid at the end of month 12 = \$100,000 – \$99,324.59 = \$675.41.

c) The total interest paid during the first year can be calculated as the difference between the total payments made during the year and the principal repaid during that year, as follows:

Total payments made during the first 12 months = 12 × \$1,053.22 = \$12,638.64.

Total interest paid during the first 12 months = \$12,638.64 – \$675.41 = \$11,963.23.

Practice Problem 6

a) The future value of the amount invested at the end of year 5 is:

$FV_5 = \$120{,}000(1 + 0.08)^5 = \$176{,}319$.

b) With monthly compounding, the amount you need to invest today will be less than \$120,000 because interest is compounded on a monthly, rather than annual, basis. The amount you would need to invest today is the present value of \$176,319 *taking into account* monthly compounding where the monthly interest rate is 0.6667% (= 8%/12) and the total time horizon is 60 months (= 5 × 12). That is:

$PV_0 = \$176{,}319/(1 + 0.08/12)^{5 \times 12} = \$118{,}347$.

c) The effective annual interest rate is calculated as:

$r_e = (1 + r/m)^m - 1 = (1 + 0.08/12)^{12} - 1 = 8.3\%$.

d) (i) The future value of the \$24,000 annual annuity where interest is compounded on an annual basis involves using the future value of an ordinary annuity, which is:

$F_n = C[(1 + r)^n - 1]/r$.

$F_5 = \$24{,}000\left[\frac{(1+0.08)^5-1}{0.08}\right] = \$140{,}798$.

(ii) To calculate the future value of the \$24,000 annual annuity where interest is compounded on a monthly basis we need to use the effective annual interest rate of 8.3 percent from part (c) rather than the stated interest rate of 8 percent because interest is compounded on a monthly basis rather than on an annual basis while the annuity occurs annually. The future value is:

$F_5 = \$24{,}000 \times \frac{(1+0.083)^5-1}{0.083} = \$141{,}643$.