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1

$$G(x)=\int_a^x f(u)(x-u)\,du = x\int_a^x f(u)\,du - \int_a^x f(u)u\,du$$

$$G'(x) = x f(x) + \int_a^x f(u)\,du - f(x)x$$ (By the fund. theorem of calc)

$$= \int_a^x f(u)\,du$$

let $x=u$, $u=t$

$G'(u) = \int_a^u f(t)\,dt$. For $G'(u)$ is continous,

$$\Rightarrow \int_a^x \int_a^u f(t)\,dt\,du = \int_a^x G'(u)\,du$$

$$= G(x) - G(a)$$ (By the fund. theorem of calc)

With $G(a) = \int_a^a f(u)(x-u)\,du = 0$

Hence

$$\int_a^x \int_a^u f(t)\,dt\,du = G(x) = \int_a^x f(u)(x-u)\,du$$

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2 i) • For even n, $n = 2k$, $k \in \mathbb{N}$

let $\epsilon > 0$

$|x^n e^x - 0| = |x^n||e^x|$ (absolute law)

$= x^n e^x$ (for $x^{2k} \geq 0$, $e^x > 0$).

$= \frac{x^n}{e^{-x}}$.

From lectures, we know that $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$ (*) for $x > 1$

$\Rightarrow \lim_{x \to -\infty} \frac{e^{-x}}{(-x)^n} = \infty$, for $x < -1$

But n is even, $(-x)^n = x^n$

So let $N = \frac{1}{\epsilon}$

$\exists a < -1, \forall x < a \Rightarrow \frac{e^{-x}}{x^n} > N = \frac{1}{\epsilon}$ (for (*))

$\Rightarrow \frac{x^n}{e^{-x}} < \epsilon$ (Ineq. law)

Hence $\lim_{x \to -\infty} x^n e^x = 0$ for even n

• If n is odd, $n = 2k - 1$, for $k \in \mathbb{N}$

$\lim_{x \to -\infty} x^n e^x = \lim_{x \to -\infty} x^{2k} e^x x^{-1}$ (exponential law)

$= \lim_{x \to -\infty} e^{2k} e^x \lim_{x \to -\infty} \frac{1}{x}$ (limit product law)

$= 0 \times 0$

$= 0$

Tianrui Qi 1473217

2 ii) suppose $p_n(x) = ax^n + p_{n-1}(x)$, $a \neq 0$

$$(*)\ \lim_{x \to \infty} p_n(x) = a \lim_{x \to \infty} x^n + p_{n-1}(x)$$

$$= a \times \infty \qquad \text{(from lecture)}$$

$$= \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$$

- If $a < 0$,

let $N > 0$, because $\lim_{x \to \infty} \frac{e^x}{x} = \infty$ (by lecture)

$\exists k_1 > 1$, $\forall x > k_1$

$\Rightarrow \frac{e^x}{x} > N$

And as $\lim_{x \to \infty} p_n(x) = \infty$, by $(*)$

$\exists k_2 > 0$, $\forall x > k_2$

$\Rightarrow p_n(x) > k_1 \Rightarrow \frac{e^{p_n(x)}}{p_n(x)} > N$

Hence $\forall x > k_2 > 0$

$\Rightarrow \frac{e^{p_n(x)}}{p_n(x)} > N$,

so $\lim_{x \to \infty} \frac{e^{p_n(x)}}{p_n(x)} = \infty$ when $a > 0$

Tianrui Qi 1473217

2) iii) continued

- if $a < 0$

Notice that

$$\lim_{x\to\infty} \frac{e^{-x}}{-x} = \lim_{x\to\infty} \frac{-1}{xe^{x}}$$
$$= 0 \quad . \quad (\text{for } \lim_{x\to\infty} xe^{x} = \infty \text{ from lecture})$$

Here let $q_n(x) = -p_n(x)$

thus $\lim_{x\to\infty} q_n(x) = -\lim_{x\to\infty} p_n(x) = \infty$, by (*)

$$\lim_{x\to\infty} \frac{e^{p_n(x)}}{p_n(x)} = \lim_{x\to\infty} \frac{e^{-q_n(x)}}{-q_n(x)}$$
$$= \lim_{x\to k} \frac{e^{-x}}{-x} \quad \left(\begin{array}{l}\text{where } k = \lim_{x\to\infty} q_n(x) \\ \qquad = \infty \\ \text{using the continuous comp. theorem}\end{array}\right)$$
$$= 0$$

- Hence, if $p_n(x) = ax^n + \cdots$

$$\lim_{x\to\infty} \frac{\exp(p_n(x))}{p_n(x)} = \begin{cases} \infty & \text{if } a > 0 \\ 0 & \text{if } a < 0 \end{cases}$$

Tianrui Qi 1473217

3 i) let $N > 0$, define $f(a) = g(a) = 0$

- for $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$,

$\exists \delta > 0\ \forall x, 0 < |x - a| < \delta$

$\Rightarrow \frac{f'(x)}{g'(x)} > N \Rightarrow g'(x) \neq 0$. imply that $f'(x)$ and $g'(x)$ are continuous in $(a-\delta, a+\delta)$

- for $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, take $\varepsilon = 1$

$\exists \delta_2, \delta_3 > 0$, where $\forall x. |x - a| < \min(\delta_2, \delta_3) = 2b$

where $|f(x)| < \varepsilon$, $|g(x)| < \varepsilon$

imply that $f(x), g(x)$ are continuous in $[a-b, a+b]$

- Take $d = \min\{b, \delta\}$

$\forall x, 0 < |x - a| < d$

[$g(x) - g(a) = g(x) \neq 0$ and $g'(\alpha) \neq 0$ as the same reason as lecture notes]

$\Rightarrow \frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)}$ (for we've defined $f(a) = g(a) = 0$ for continuity)

$= \exists \alpha \in (a, x)\ \frac{f'(\alpha)}{g'(\alpha)}$ (By cauchy's mean value theorem and $f(x), g(x), f'(x), g'(x)$ continuity within $[a, a+d]$ $(a, a+d)$)

But $|\alpha - a| < d < \delta$

$\Rightarrow \frac{f'(\alpha)}{g'(\alpha)} > N$

$\Rightarrow \frac{f(x)}{g(x)} > N$ ($\forall x, 0 < |x - a| < d$)

Hence $\lim_{x \to a} \frac{f(x)}{g(x)} = \infty$

Tianrui Qi 1473217

3 ii) $f(x) = x$, $\lim_{x \to 0} f(x) = 0$, $a = 0$

$g(x) = x^3$, $\lim_{x \to 0} g(x) = 0$

And, $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{1}{3x^2} = \infty$

Hence $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x}{x^3} = \lim_{x \to 0} \frac{1}{x^2} = \infty$

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4.

i) $y'' = -v^2 y$

As $f'' = -Av^2\cos(vt+\theta) = -v^2 f$

ii) suppose y is a potential solution, $v \neq 0$

Consider

$$g = y - y(0)\cos(vt) - \frac{y'(0)\sin(vt)}{v}$$

Notice that

$g(0) = 0$, $g'(0) = 0$

And g is also a solution, for

$$g'' = y'' + y(0)v^2\cos(vt) + y'(0)v\sin(vt)$$

$$= -v^2 y - (-v^2)\left[y(0)\cos(vt) + \frac{y'(0)}{v}\sin(vt)\right] = -v^2 g$$

consider

$$E = v^2 g^2 + (g')^2$$

$$\partial_t E = 2v^2 g g' + 2g'g''$$

$$= 2g'(v^2 g + g'')$$

$$= 0$$

Hence by the mean value theorem

for some c, $E = c = (vg)^2 + (g')^2$, But $E(0) = (vg(0))^2 + (g'(0))^2 = 0 \Rightarrow c = 0$

Therefore, $vg(t) = 0$, as $v \neq 0$

$\Rightarrow g(t) = 0$, $\forall t$, thus $y = y(0)\cos(vt) + \frac{y'(0)}{v}\sin(vt)$

$a = y(0)$, $b = y'(0)/v$

Tianrui Qi 1473217

4 iii) • $a = y(0) = A\cos(\theta)$

$b = \dfrac{y'(0)}{v} = -Av\sin(vt+\theta)/v$

$= -A\sin(\theta)$

• Vice versa

v does not depend on a, b

$a^2+b^2 = A^2\cos^2\theta + A^2\sin^2\theta$

$= A^2$ (as $\sin^2\theta + \cos^2\theta = 1$)

$\Rightarrow A = \pm\sqrt{a^2+b^2}$

For ~~either~~ $A = \sqrt{a^2+b^2}$ or $-\sqrt{a^2+b^2}$

$\Rightarrow \theta$ is the solution of both

$$\begin{cases} \cos(\theta) = \dfrac{a}{A} = \dfrac{a}{\pm\sqrt{a^2+b^2}} \\ \sin(\theta) = -\dfrac{b}{A} = \dfrac{-b}{\pm\sqrt{a^2+b^2}} \end{cases}$$

which is unique

i.e. $\theta = \arccos\left(\dfrac{a}{\sqrt{a^2+b^2}}\right) = \arcsin\left(\dfrac{-b}{\sqrt{a^2+b^2}}\right)$

when the RHS agrees.

Tianrui Qi 1473217

5 i) • As $\lim_{x\to 0} \ln(1+x) - x + x^2/2 - x^3/3 = 0$

and $\lim_{x\to 0} x^3 = 0$

use l'hopital's rule

$$l = \lim_{x\to 0} \frac{\ln(1+x) - x + x^2/2 - x^3/3}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x} - 1 + x - x^2}{3x^2} \quad \left(\text{for } \frac{d}{dx}\ln(1+x) = \frac{1}{1+x}\right)$$

• Again, $\lim_{x\to 0} \frac{1}{1+x} - 1 + x - x^2 = 0$

$\lim_{x\to 0} 3x^2 = 0$

use l'hopital's rule

$$l = \lim_{x\to 0} \frac{\frac{-1}{(1+x)^2} + 1 - 2x}{6x}$$

• Again, $\lim_{x\to 0} \frac{-1}{(1+x)^2} + 1 - 2x = 0$

$\lim_{x\to 0} 6x = 0$

use l'hopital's rule

$$l = \lim_{x\to 0} \frac{\frac{2}{(1+x)^3} - 2}{6} = \frac{2-2}{6} = 0$$

Tianrui Qi 1473217

5 ii) $l = \lim_{x \to \infty} x \ln(1 + 1/x)$

$$= \lim_{x \to \infty} \frac{\ln(1 + 1/x)}{x^{-1}}$$

For $\lim_{x \to \infty} \ln(1 + 1/x) = 0$

$\lim_{x \to \infty} \frac{1}{x} = 0$

- Apply l'hopital's rule

$$l = \lim_{x \to \infty} \frac{1}{1 + 1/x} \times \frac{-1}{x^2} \Big/ \frac{-1}{x^2}$$

$$= \lim_{x \to \infty} \frac{1}{1 + 1/x}$$

$$= 1$$