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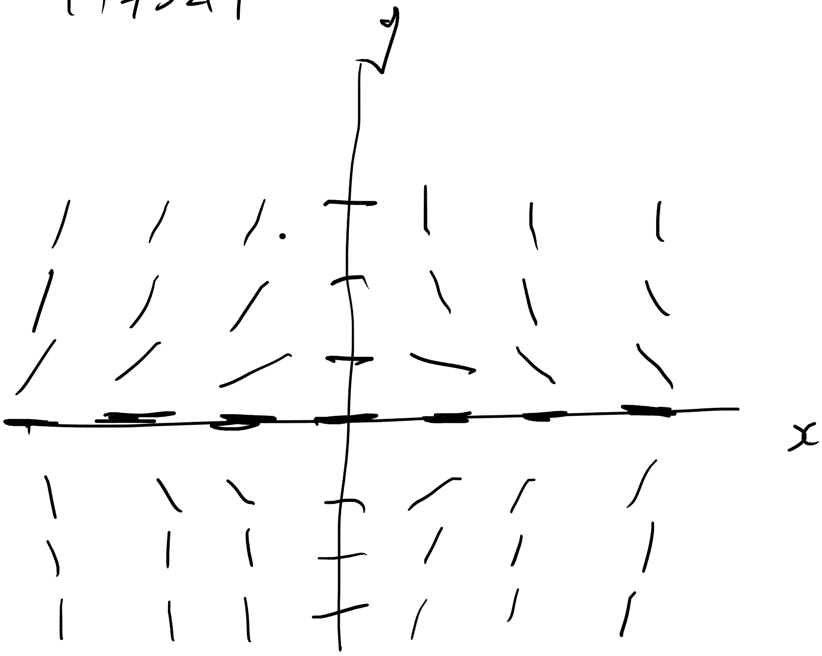
1 i)



ii) If φ is a solution

$$\varphi' = \begin{cases} -x\sqrt{\varphi} & \text{if } \varphi \geq 0 \\ x\sqrt{-\varphi} & \text{if } \varphi < 0 \end{cases}$$

Then let $f = -\varphi$

$$f' = (-\varphi)' = -\varphi' = \begin{cases} x\sqrt{\varphi} & \text{if } \varphi \geq 0 \\ -x\sqrt{-\varphi} & \text{if } \varphi < 0 \end{cases}$$

$$= \begin{cases} x\sqrt{-f} & \text{if } -f \geq 0 \\ -x\sqrt{f} & \text{if } -f < 0 \end{cases}$$

$$= \begin{cases} x\sqrt{-f} & \text{if } f \leq 0 \\ -x\sqrt{f} & \text{if } f > 0 \end{cases} \quad \text{(Inequality theorem)}$$

As $f=0 \Rightarrow f'=0 = -x\sqrt{f}\Big|_{f=0} = x\sqrt{f}\Big|_{f=0}$

$$f' = \begin{cases} x\sqrt{-f} & \text{if } f < 0 \\ -x\sqrt{f} & \text{if } f \geq 0 \end{cases}$$

Therefore $f = -\varphi$ is also a solution

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1 iii) when $y \geq 0$

$$y' = -x\sqrt{y}$$

- if $\varphi = 0$, $\varphi' = 0 = -x\sqrt{\varphi}\,\big|_{\varphi=0}$

  Hence $\varphi_c = 0$ is a valid solution $\forall x \in \mathbb{R}$

- if $\varphi > 0$, use separable diff eq. theorem

$$y' = -x\sqrt{y} = f(x)g(y)$$

where $f = -x$, $g = \sqrt{y}$

$$\Rightarrow \int \frac{1}{g} g' = \int f$$

$$\Rightarrow \int \frac{1}{\sqrt{y}} y' = \int -x$$

$$\Rightarrow 2\sqrt{y} = -\frac{x^2}{2} + c_1$$

$$\Rightarrow \sqrt{y} = -\frac{x^2}{4} + c \quad \text{where } c - \frac{x^2}{4} > 0$$

$$\Rightarrow y = \left(c - \frac{x^2}{4}\right)^2$$

Therefore

$$\varphi_c(x) = \left(c - \frac{x^2}{4}\right)^2, \quad c > \frac{x^2}{4} > 0$$

are solutions.

- The domain $c > \frac{x^2}{4} \Rightarrow |x| < \sqrt{2c}$

  But because $\lim_{x \to \sqrt{2c}^-} \varphi_c(x) = 0 = \lim_{x \to -\sqrt{2c}^+} \varphi_c(x)$

- The family of non-negative solutions is

$$\varphi_c = \begin{cases} \left(c - \frac{x^2}{4}\right)^2 & \text{if } |x| < \sqrt{2c} \\ 0 & \text{if } |x| \geq \sqrt{2c} \end{cases}, \quad \varphi_c \in C^1(\mathbb{R})$$

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1 iv) For all $y_0 \in \mathbb{R}$

The IVP $y(0) = y_0$ is uniquely solvable

If $y_0 = 0$, $\varphi = 0$

If $y_0 > 0$ $\varphi = \begin{cases} (\sqrt{y_0} - \frac{x^2}{4})^2 & \text{if } \sqrt{y_0} - \frac{x^2}{4} > 0 \\ 0 & \text{otherwise} \end{cases}$

If $y_0 < 0$ $\varphi = \begin{cases} -(\sqrt{-y_0} - \frac{x^2}{4})^2 & \text{if } \sqrt{-y_0} - \frac{x^2}{4} > 0 \\ 0 & \text{otherwise} \end{cases}$

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2. i) let $u = x+y$, $y' = u^2$

$u' = 1 + y' = 1 + u^2 = f(x)g(u)$

where $f = 1$, $g = 1+u^2$

ii) For $g(u) = 1+u^2 \geq 1 > 0$, this is separable

$u' = 1+u^2 \Rightarrow \frac{1}{1+u^2}u' = 1$

$\Rightarrow \int \frac{1}{1+u^2} u' = \int 1$

$\int 1 = x$, $\int \frac{1}{1+u^2} = \arctan u$

Hence

$\arctan u = x + C$, $|x+C| < \frac{\pi}{2}$

$u = \tan(x+C)$

For $u = x+y$, $y = u - x$

$y = \tan(x+C) - x$ is the solution, given $|x+C| < \frac{\pi}{2}$

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3 i) $y' + \sin x \, y = 0$

By the homogeneous theorem:

All solutions are of the form

$\varphi = e^{-\int P}$, where $P(x) = \sin x$

$\int P = \int \sin x = -\cos x + C$

$\Rightarrow \varphi_c(x) = e^{\cos x} e^{C}$

ii) the inhomogeneous diff eq is

$y' + y\sin x = \sin^3 x$

- Guess $\varphi = \varphi_0 e^{\cos x}$, for solution φ

where $\varphi_0(x)$ is the variation of the constant

e^c in the homogeneous solution

Then $\varphi' = \varphi_0' e^{\cos x} + \varphi_0 e^{\cos x}(-\sin x)$

$= \varphi_0' e^{\cos x} + \varphi(-\sin x)$

$\Rightarrow \varphi' + \varphi \sin x = \varphi_0' e^{\cos x}$

$\sin^3 x = \varphi_0' e^{\cos x}$

$\varphi_0' = \sin^3 x \, e^{-\cos x}$

$\varphi_0 = \int \sin^3 x \, e^{-\cos x}$

Because $\sin^3 x = \sin x (1 - \cos^2 x)$

$= \sin x - \sin x \cos^2 x$

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3 ii) $\varphi_0 = \int (\sin x - \sin x \cos^2 x) e^{-\cos x}$

$= \int \sin x \, e^{-\cos x} - \int \sin x \cos^2 x \, e^{-\cos x}$

$\int \sin x e^{-\cos x} = e^{-\cos x}$

$\int \sin x \cos^2 x \, e^{-\cos x} = \int (-u)^2 e^u$ if $u = -\cos x$, $du = \sin x \, dx$

$= \int u^2 e^u$

By integration by parts

D	I
$+ \; u^2$	e^u
$- \; 2u$	e^u
$+ \; 2$	e^u
$- \; 0$	e^u

$= u^2 e^u - 2u e^u + 2e^u$

$\varphi_0 = e^u - u^2 e^u + 2u e^u - 2e^u$

$= e^u (1 - u^2 + 2u - 2)$

$= e^{-\cos x} (-\cos^2 x - 2\cos x - 1)$

Hence, a particular solution is

$\varphi = \varphi_0 e^{\cos x}$

$= -\cos^2 x - 2\cos x - 1$

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3 iii) The general solution where $y(1) = y_0$, is

$$\varphi = \left[c + e^{-\cos x}(-\cos^2 x - 2\cos x - 1)\right) e^{\cos x}$$

$$= ce^{\cos x} + (-\cos^2 x - 2\cos x - 1)$$

$$\varphi(1) = ce^1 + (-1-2-1)$$

$$= ce - 4 = y_0$$

$$\Rightarrow \quad c = \frac{y_0 + 4}{e}$$

Hence $\varphi(x) = \frac{y_0+4}{e} e^{\cos x} + (-\cos^2 x - 2\cos x - 1)$.

$$= (y_0+4)e^{\cos x - 1} + \left[-\cos^2 x - 2\cos x - 1\right]$$

with $\varphi(0) = y_0$

is the general solution

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1* • If $\forall x \in (0,1]$, $y(x) \neq 0$

Then by the theorem of separable equations

$$F = \int_1^x f \quad G = \int_1^y \frac{1}{y} = \ln|y|$$

$$\Rightarrow \ln|y| = F + C$$

$$\Rightarrow y = \pm e^C e^F$$

If for some solution $\varphi = \pm e^C e^F$

and $\lim_{x \to 0^+} \varphi = 0$

then $\lim_{x \to 0^+} \pm e^C e^{F(x)} = 0$

$\Leftrightarrow \lim_{x \to 0^+} F(x) = -\infty$ (for $\pm e^C$ is constant and $\lim_{x \to -\infty} e^x = 0$)

so $\lim_{x \to 0^+} \int_1^x f = -\infty$

— If within $(0,1]$, $f(x)$ is bounded by

$$|f| \leq N$$

for some $N \geq 0$

then $|\lim_{x \to 0^+} \int_1^x f| \leq 2N$, which does not

head to infinity

Therefore f must not be bounded within $(0,1]$

namely at $x=0$ — as $f(x)$ is continuous in $(0,1]$

so $\lim_{x \to 0^+} f(x) = -\infty$

which is a restriction of $f(x)$.

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1 * • if for some $x' \in (0,1]$, the solution φ
$\varphi(x') = 0$

then if for some other $\alpha \in (0,1]$
$\varphi(\alpha) \neq 0$
set $(\alpha, \varphi(\alpha))$ as the IVP for
the non-zero differential equation solution
⇒ But that unique solution has no roots
therefore such α cannot exist.

So $\varphi(x) = 0$ $\forall x \in (0,1]$, solving $y' = y f(x)$
By which for any $f(x)$
$\lim_{x \to 0^+} \varphi(x) = 0$, with no $f(x)$ restrictions

• Hence the only restriction on $f(x)$ is

$$\lim_{x \to 0^+} f(x) = -\infty$$

for any solution φ
$\lim_{x \to 0^+} \varphi = 0$