Tianrui Qi 1473217

1 a) consider C_A

$$C_A = \begin{vmatrix} x-5 & -4 & 0 \\ 1 & x-1 & 0 \\ -2 & -3 & x-4 \end{vmatrix}$$

$$= (x-4) \begin{vmatrix} x-5 & -4 \\ 1 & x-1 \end{vmatrix}$$

$$= (x-4)\left((x-5)(x-1) + 4 \right)$$

$$= (x-4)\left(x^2-6x+9\right) = (x-4)(x-3)(x-3)$$

$\lambda = 4, 3$

consider E_4

$$\left[\begin{array}{ccc|c} 5-4 & 4 & 0 & 0 \\ -1 & 1-4 & 0 & 0 \\ 2 & 3 & 4-4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right]$$

$\downarrow R_2+R_1,\ R_3-2R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow[R_3+5R_2]{R_1-4R_2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right]$$

$$B_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

consider E_3

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$$\left[\begin{array}{ccc|c} 5-3 & 4 & 0 & 0 \\ -1 & 1-3 & 0 & 0 \\ 2 & 3 & 4-3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow \tfrac{1}{2}R_1,\ R_2+\tfrac{1}{2}R_1,\ R_3-R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

for $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in sol space

- c is free var
- $b = c$
- $a = -2b = -2c$

$\therefore B_3 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Because the union of eigenspace basis

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

has size 2, while A is 3×3, not enough eigenvectors for a basis, so <u>not diagonalizable</u>.

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1 b) Consider C_B

$$= \begin{vmatrix} x-1 & 1 & -1 \\ -2 & x+1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} \overset{(F_5)}{=} \begin{vmatrix} x+4 & 1 & 4 \\ 3 & x+1 & 1 \\ 1 & 1 & x+4 \end{vmatrix}$$

$$\downarrow R_2+R_1$$

$$= (x+2) \begin{vmatrix} x+4 & 1 & 4 \\ 1 & 1 & 0 \\ 1 & 1 & x+4 \end{vmatrix} \leftarrow \begin{vmatrix} x+4 & 1 & 4 \\ x+2 & x+2 & 0 \\ 1 & 1 & x+4 \end{vmatrix}$$

$$\downarrow C_1-C_2$$

$$= (x+2) \begin{vmatrix} x+3 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & x+4 \end{vmatrix} = (x+2) \begin{vmatrix} x+3 & 4 \\ 0 & x+4 \end{vmatrix}$$

$$= (x+2)(x+3)(x+4)$$

$\lambda = 3, 2, 1$

<u>Diagonalizable</u> for there are 3 unique eigenvalues, for a 3×3 matrix.

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consider E_3

$$\left[\begin{array}{ccc|c} 1-3 & -1 & 1 & 0 \\ 2 & -1-3 & -1 & 0 \\ -1 & -1 & 1-3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right]$$

$\downarrow R_2+R_1$, $2R_3-R_1$

$$\left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

- c is free var
- $b = 0$
- $-2a = b - c = -c$

$a = 3c$

$B_3 = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Consider E_2

$$\left[\begin{array}{ccc|c} 1-2 & -1 & 1 & 0 \\ 2 & -1-2 & -1 & 0 \\ -1 & -1 & 1-2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right]$$

$\downarrow R_2+2R_1$, R_3-R_1

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_3+2R_2} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

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$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\cdot\ c = 0$, b is free variable

$\cdot\ a = -b = 4b$

$B_2 = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \right\}$

consider E_1

$$\left[\begin{array}{ccc|c} 1-1 & -1 & 1 & 0 \\ 2 & -1-1 & -1 & 0 \\ -1 & -1 & 1-1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 2 & -2 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$$

$$\downarrow R_2 \leftrightarrow R_1,\ 2R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -4 & -1 & 0 \end{array}\right]$$

$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, c is free var.

$b = c$

$2a = 2b + c = 3c$

$a = 4c$

$B_1 = \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$

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$$B = B_1 \cup B_2 \cup B_3 = \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 4 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Such that $B = PDP^{-1}$

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Q2 let $A = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$, then $M = \frac{1}{2}A$

consider $c_A(x)$

$C_A = x^2 - x - 2 = (x-2)(x+1)$

$\lambda = 2, -1$

consider E_2

$$\left[\begin{array}{cc|c} 3-2 & 2 & 0 \\ -2 & -2-2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -2 & -4 & 0 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$B_2 = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

consider E_{-1}

$$\left[\begin{array}{cc|c} 3+1 & 2 & 0 \\ -2 & -2+1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 4 & 2 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow{2R_2+R_1} \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$B_{-1} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

$B = B_2 \cup B_{-1} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

therefore

$P = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ $P^{-1} = \frac{-1}{3} \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

$A^n = PD^nP^{-1} = P \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} P^{-1}$

$= -\frac{1}{3} P \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

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$$A^n = -\frac{1}{3} P \begin{bmatrix} 2^{n+1} & 2^n \\ (-1)^{n+1} & -2(-1)^n \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} -2 \cdot 2^{n+1} + (-1)(-1)^{n+1} & (-2) 2^n + 2(-1)^n \\ 2^{n+1} + 2(-1)^{n+1} & 2^n - 4(-1)^n \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2^{n+2} + (-1)^{n+1} & 2^{n+1} - 2(-1)^n \\ -2^{n+1} - 2(-1)^{n+1} & -2^n + 4(-1)^n \end{bmatrix}$$

$$= \frac{1}{3} 2^n \begin{bmatrix} 2^2 & 2 \\ -2 & -1 \end{bmatrix} + \frac{1}{3}(-1)^n \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$$

$$M^n = \frac{1}{2^n} A^n = \frac{1}{3} \begin{bmatrix} 2^2 & 2 \\ -2 & -1 \end{bmatrix} + \frac{1}{3} \frac{(-1)^n}{2^n} \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$$

$$\lim_{n\to\infty} M^n = \lim_{n\to\infty} \frac{1}{3} \begin{bmatrix} 2^2 & 2 \\ -2 & -1 \end{bmatrix} \text{ for } \lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$$

$$= \begin{bmatrix} 4/3 & 2/3 \\ -2/3 & -1/3 \end{bmatrix}$$

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3 a) let some $k \in \mathbb{N}$ (so $k \neq 0$) such that $A^k = 0$

let $P, D \in M_{nn}(\mathbb{F})$ such that

$A = PDP^{-1}$, and D is

diagonal $= \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

then

$$0 = A^k = (PDP^{-1})^k = P D^k P^{-1}$$

then $P^{-1} 0 P = P^{-1}(P D^k P^{-1})P$

and $0 = D^k$

therefore $\begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix} = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$

so for all $i \in [0, n]$, $d_i^k = 0$

$\Rightarrow d_i = 0$ (because only $0^k = 0$, $k \neq 0$)

$\Rightarrow D = 0$

therefore $A = PDP^{-1} = P 0 P^{-1} = 0$

b) consider $c_B(x)$ for $B \in M_{n,n}(\mathbb{C})$

because the field is $\mathbb{C}$, there will be exactly n (non-distinct) eigenvalues:

$\lambda_1 \dots \lambda_n$

so, $c_B(x) = (x - \lambda_1) \cdots (x - \lambda_n)$

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If 0 is the only eigenvalue, then

$$c_B(x) = (x-0) \cdots (x-0) \quad (\forall i, \lambda_i = 0)$$
$$= x^n$$

By the Cayley-Hamilton theorem

$$c_B(B) = 0$$

but $c_B(B) = B^n$, so $B^n = 0$

Therefore B is nilpotent with $k = n$, where

$$B^k = 0$$