Tianrui Qi 1473217

1 a) $P_{S,B} = [id]_{S,B}$

$$= \left[[x+x^2]_S \; [1-2x-x^2]_S \; [1-x+x^2] \right]$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$P_{B,S} = (P_{S,B})^{-1}$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\downarrow R_2 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{array}\right] \xleftarrow[\text{then } R_2 \times -1]{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\downarrow R_1 + R_3,\; R_2 + 2R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{array}\right] \xrightarrow[R_3 \times -1]{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & -1 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$P_{B,S} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

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b) Because $P_{B,S} = [\,[1]_B \;\cdots\,]$

$$[1]_B = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$ first col of $P_{B,S}$

c) $[T]_{B,C} = P_{B,S}\,[T]_{S,S}\,P_{S,C}$

• $P_{S,C} = \left[\, [-1+x+2x^2]_S \; [1-x-x^2]_S \; [2-x-x^2]_S \,\right]$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

$$[T]_{B,C} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -6 & 2 \\ 3 & 8 & -2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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2 a) i) $c_A(x) = \det(xI - A)$

$$= \begin{vmatrix} x-2 & 1 & 1 \\ -1 & x-4 & -3 \\ 0 & 0 & x-2 \end{vmatrix}$$

$C_2 \leftrightarrow C_3$

$R_2 \leftrightarrow R_3$

$$= \begin{vmatrix} x-2 & 1 & 1 \\ 0 & x-2 & 0 \\ -1 & -3 & x-4 \end{vmatrix}$$

$$= (x-2) \begin{vmatrix} x-2 & 1 \\ -1 & x-4 \end{vmatrix}$$

$$= (x-2)((x-2)(x-4)+1)$$

$$= (x-2)(x^2 - 6 + 9)$$

$$c_A(x) = (x-2)(x-3)(x-3)$$

ii) $\lambda = 2, 3$ for $c_A(\lambda) = 0$

iii) • for $\lambda = 2$

eigenspace $\cong$ solution space $(A - 2I)$

$$\Rightarrow \left[\begin{array}{ccc|c} 2-2 & -1 & -1 & 0 \\ 1 & 4-2 & 3 & 0 \\ 0 & 0 & 2-2 & 0 \end{array}\right]$$

$\downarrow$

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$$\downarrow$$

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

using $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, c is free varrable

$b = -c$

$a = -2b - 3c = -c$

$$B = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} \text{ for } E_2$$

- for $\lambda = 3$

consider sol space $A - 3I$

$$\Rightarrow \left[\begin{array}{ccc|c} 2-3 & -1 & -1 & 0 \\ 1 & 4-3 & 3 & 0 \\ 0 & 0 & 2-3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\downarrow R_2 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow[-1 \times R_1]{R_3 + \frac{1}{2}R_2} \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

using $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $c = 0$

b is free var

$a = -b - c = -b$

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ for } E_3$$

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2 b)

i) $C_B(x) = \det(xI - B)$

$$= \begin{vmatrix} x-2 & -1 & -2 \\ -1 & x-1 & -1 \\ 0 & -1 & x-1 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix}$$

$$+ 1 \times \begin{vmatrix} -1 & -1 \\ 0 & x-1 \end{vmatrix}$$

$$- 2 \times \begin{vmatrix} -1 & x-1 \\ 0 & -1 \end{vmatrix}$$

$$= (x-2)((x-1)^2 - 1)$$

$$+ (-1)(x-1) - 2(1)$$

$$= (x-2)(x^2-2x) - x - 1$$

(mod 3) $= (x+1)(x^2-2x) - (x+1)$

$$= (x+1)(x^2-2x-1)$$

$C_B(x) = (x+1)(x^2+x+2)$. for no more $\mathbb{F}_3$ roots

ii) $\lambda = 2$ for $C_B(2) = 0$

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iii) consider sol space of $A-2I$

$$\Rightarrow \left[\begin{array}{ccc|c} 2-2 & 1 & 2 & 0 \\ 1 & 1-2 & 1 & 0 \\ 0 & 1 & 1-2 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\downarrow R_2+R_1,\ R_3-R_1$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

using $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $a=0$
c is free variable

$b = -2c = c$

$B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ for E_2

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8 Suppose E is the eigenspace assiociated with eigenvalue $\lambda \in \mathbb{F}$

$\forall u \in E$, $T(u) = \lambda u$ (eigenspace defn)

$\Rightarrow S(T(u)) = S(\lambda u)$

$\Rightarrow T(S(u)) = \lambda S(u)$ ($ST = TS$ and S is linear)

$\Rightarrow S(u) \in E$ (eigenspace defn)

So because $\forall u \in E$, $S(u) \in E$

$\Rightarrow \forall v \in S(E)$, $\exists u \in E$ s.t. $S(u) = v$
(for $v \in \operatorname{Im} S|_E$)

therefore $S(u) \in E$ and $v \in E$

Hence

$S(E) \subseteq E$