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1 Consider the REF of A

$$\begin{bmatrix} 2 & 0 & 0 & 3 & 2 \\ -1 & 0 & 11 & -7 & 11 \\ -2 & 0 & 0 & -3 & -2 \\ -1 & 0 & 3 & -3 & 2 \end{bmatrix}$$

$$\downarrow R_2 \times 2,\ R_4 \times 2$$

$$\begin{bmatrix} 2 & 0 & 0 & 3 & 2 \\ -2 & 0 & 22 & -14 & 22 \\ -2 & 0 & 0 & -3 & -2 \\ -2 & 0 & 6 & -6 & 4 \end{bmatrix}$$

$$\downarrow R_2 + R_1,\ R_3 + R_1,\ R_4 + R_1$$

$$\begin{bmatrix} 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 22 & -11 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -3 & 6 \end{bmatrix}$$

$$\downarrow R_4 - \frac{6}{22} R_2$$

$$\begin{bmatrix} 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 22 & -11 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{6}{11} \end{bmatrix}$$

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$$\downarrow R_3 \Leftrightarrow R_4$$

$$\begin{bmatrix} 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 22 & -11 & 24 \\ 0 & 0 & 0 & 0 & -6/11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In REF

a) basis for rowspace are the non-zero rows

$$B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 22 \\ -11 \\ 24 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -6/11 \end{bmatrix} \right\}$$

b) basis for colspace are the pivot colums in A, index 1, 3, 5

$$B = \left\{ \begin{bmatrix} 3 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 11 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 11 \\ -2 \\ 2 \end{bmatrix} \right\}$$

c) use solution space algorithm.

$$[A \mid \vec{0}] \sim [\text{REF} \mid \vec{0}]$$

$$= \left[\begin{array}{ccccc|c} 2 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 22 & -11 & 24 & 0 \\ 0 & 0 & 0 & 0 & -6/11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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suppose $A\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \vec{0}$, $a, b, c, d, e \in \mathbb{R}$

- $-\frac{6}{11} e = 0$, $e = 0$
- d is parameter
- $22c = 11d - 24e$, $c = 1/2 d$
- b is parameter
- $2a = -3d - 2e$, $a = -3/2 d$

$$B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

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2 a) consider the basis

$$B_W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

for W.. Also, $\dim W = |B_W| = 3$

- using spanning algorithm under B_W (17.3)

$$A = \left[[u_1]_{B_W} \cdots [u_5]_{B_W} \right]$$

$$= \begin{bmatrix} 2 & 2 & 2 & 4 & 4 \\ 2 & 4 & 1 & 3 & 3 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

$$\downarrow R_2 - R_1, \quad 2R_3 - R_1$$

$$\begin{bmatrix} 2 & 2 & 2 & 4 & 4 \\ 0 & 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

- $\operatorname{rank} A = 3 = \dim W$

hence S spans W

- use subset for span algo (17.5)

AS $\operatorname{span} S = W$, so basis for span = basis for W

Pivot columns at 1, 2, 5

$$\Rightarrow C = \{ u_1, u_2, u_5 \}$$

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2 b) let $[w]_C = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

So $a u_1 + b u_2 + c u_5 = w$

Under basis B_W, this system is

$a[u_1]_{B_W} + b[u_2]_{B_W} + c[u_5]_{B_W} = [w]_{B_W}$

$$\left(\begin{array}{ccc|c} 2 & 2 & 4 & 3 \\ 2 & 4 & 3 & 4 \\ 1 & 1 & 0 & 3 \end{array}\right)$$

$\downarrow R_2 - R_1,\ 2R_3 - R_1$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 3 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -4 & 3 \end{array}\right]$$

Then, under mod 5

- $-4c = 3, \quad c = 3$
- $2b = 1 + c, \quad b = 2$
- $2a = 3 - 2b - 4c, \quad a = 1$

$[w]_C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

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2 c) using superset basis algo. (17.7)

- under standard basis B for $M_{2,2}$

$$A = [\,[u_1][u_2][u_5][b_1][b_2][b_3][b_4]\,]$$

where $B = \{b_1 \; \dots \; b_4\}$

$$A = \begin{bmatrix} 2 & 2 & 4 & 1 & 0 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ 2R_4 - R_1}]{R_2 - R_1} \begin{bmatrix} 2 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -4 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$$\downarrow R_3 - R_2$$

$$\begin{bmatrix} 2 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -4 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \xleftarrow{R_3 \Leftrightarrow R_4} \begin{bmatrix} 2 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -4 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Pivots at 1, 2, 3, 5th

hence

$$B = \left\{ u_1, u_2, u_5, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

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3) Consider a minimal spanning set $B \subseteq U$,
by definition it spans V, $\text{span } B = V$

To see that it is lin indep,
consider the contrapositive :

If B is linearly dependent,
By exercise 115, $\exists u \in B$
such that $\text{span}(B \setminus \{u\}) = \text{span } B$
Hence there exists a proper subset $B \setminus \{u\}$
that also spans U, so B is not a
minimal spanning set of V.

Hence B must be lin indep.

Because a minimal spanning set B
- spans V
- linearly independent

B is a basis for U

In the other direction, consider the
basis B for V.

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3. cont.

By definition, span $B = V$, so B is a spanning set for V.

For any proper subset of B, say $A \subset B$, $|A| < |B|$. By theorem 14.4, in having less than $|B|$ elements (less than $\dim V$ elements), A cannot span V, so B has no proper spanning subsets.

B must be a minimal spanning set.

Hence B is a basis iff B is a minimal spanning set.