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1. let $P(n)$ be that $2^n < n!$

To show that $P(n)$ holds for all $n \geq 4$

Consider $P(4)$

$$(2^4 = 16) < (4! = 24)$$

hence $P(4)$ holds

Assume that $P(k)$ holds for some $k \geq 4$

therefore $2^k < k!$

Consider $P(k+1)$

$$2^{k+1} = 2 \times 2^k < 2k! \quad (\text{for } 2^k < k!)$$

and $2k! < (k+1)k! = (k+1)!$ $\left(\begin{array}{l}\text{for } k+1 > 2 \\ \text{due to } k \geq 4\end{array}\right)$

hence $2^{k+1} < (k+1)!$

so $P(k) \Rightarrow P(k+1)$

Therefore by principle of mathematical induction $P(4)$ holds, and if $P(k)$ holds, $P(k+1)$ also holds, so $P(n)$ holds for all $n \geq 4$

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2. $\begin{bmatrix} 1 & 2 & 1 & -k \\ 1 & 2+k & 1-k & 0 \\ 1 & 2 & 1-k+k^2 & -1 \end{bmatrix}$

$\downarrow R_2 - R_1,\ R_3 - R_1$

$\begin{pmatrix} 1 & 2 & 1 & -k \\ 0 & k & -k & k \\ 0 & 0 & -k+k^2 & k-1 \end{pmatrix}$

i) no solutions when pivot is on last column.

R3: $-k+k^2=0 \wedge k-1 \neq 0$

$\Rightarrow k(1-k)=0 \wedge k \neq 1$

so $k=0 \Rightarrow$ no solutions

ii) unique solution when all pivots are non-zero

R2: $k \neq 0 \wedge$ R3: $-k+k^2 \neq 0$

$\Rightarrow k \neq 0 \wedge k(1-k) \neq 0$

so $k \neq 0 \wedge k \neq 1$ leads to unique solution.

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2) continued

a) iii) more then one sol.

when a row of zeros

R3: $k^2 - k = 0 \wedge k - 1 = 0$

$\Rightarrow k(k-1) = 0 \wedge k = 1$

so $k = 1 \Rightarrow$ more than one solution

b) let $k = 1$

$$\begin{matrix} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

let $z = t$, $t \in \mathbb{R}$

$y - z = 1$, $y = 1 + z = 1 + t$

$x + 2y + z = -1$, $x = -2(1+t) - t - 1$

$= -3 - 3t$

Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \quad \forall t \in \mathbb{R}$$

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3. a) S spans $P_2(\mathbb{R})$ iff

$\forall v \in P_2(\mathbb{R})$, $\exists a, b, c, d \in \mathbb{R}$, s.t.

$a u_1 + b u_2 + c u_3 + d u_4 = v$

That is

$$\begin{pmatrix} 1 & 2 & 5 & -1 \\ 1 & -1 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Consider the left matrix

$$\Big\downarrow R_2 - R_1,\ R_3 R_1$$

$$\begin{pmatrix} 1 & 2 & 5 & -1 \\ 0 & -3 & -4 & 3 \\ 0 & -4 & -4 & 5 \end{pmatrix} \xrightarrow{R_3 - \frac{4}{3}R_2} \begin{pmatrix} 1 & 2 & 5 & -1 \\ 0 & -3 & -4 & 3 \\ 0 & 0 & \frac{4}{3} & 1 \end{pmatrix}$$

The system has more than one solution with parameter on d.

Hence there always exists a, b, c, d that solves the linear system. So S spans $P_2(\mathbb{R})$

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3 b)

$$\left(\begin{array}{cccc|c} 1 & 2 & 5 & -1 & 5 \\ 1 & -1 & 1 & 2 & -2 \\ 1 & -2 & 1 & 4 & 0 \end{array}\right)$$

$$\downarrow R_2 - R_1,\ R_3 - R_1$$

$$\begin{pmatrix} 1 & 2 & 5 & -1 & 5 \\ 0 & -3 & -4 & 3 & -7 \\ 0 & -4 & -4 & 5 & -5 \end{pmatrix}$$

$$\downarrow R_3 - \frac{4}{3}R_2$$

$$\begin{pmatrix} 1 & 2 & 5 & -1 & 5 \\ 0 & -3 & -4 & 3 & -7 \\ 0 & 0 & 4/3 & 1 & 13/3 \end{pmatrix}$$

let $d = 0$

$4/3 c + d = 13/3, \quad c = \frac{13}{4}$

$-3b - 4c = -7, \quad -3b = -7 + 13$

$b = -2$

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$$a + 2b + 5c = 5$$

$$a = 5 - 2(-2) - 5(13/4)$$

$$= -\frac{29}{4}$$

So $5 - 2x = -\frac{29}{4} u_1$

$+ -2 \; u_2$

$+ 13/4 \; u_3$

$+ 0 \; u_4$