Symmetric Chain Decompostions of Peck Posets

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August 1, 2023

1 Introduction

Partially Ordered Sets (Posets) are a fundamental mathematical object that I began exploring during my research in the Summer of 2022 with MathILy-EST These structures help us represent systems where some elements are related in terms of being 'greater than' or 'less than,' while other elements have no specific relationship.

A classic example is a family tree, where parents are 'greater than' their children in terms of generations, but siblings have no defined order between them. Each Poset has different hierarchical levels we call "ranks." Posets are often used to organize and understand complex data.

In my research, I focus on 'Peck Posets.' These posets maintain a significant presence in the existing literature due to their symmetrical properties.

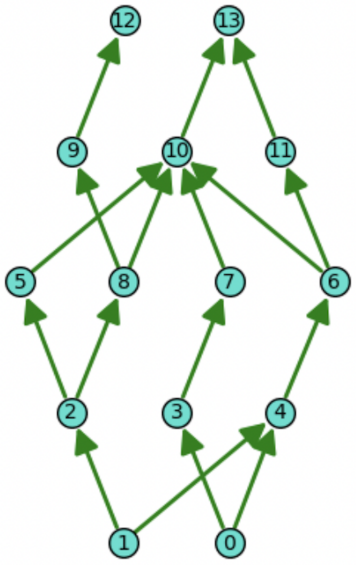

Figure 1: A Peck Poset

A central part of my research is to further understand these Peck Posets through 'Symmetric Chain Decompositions' (SCDs). This method decomposes a Poset into chains of elements, allowing us to view the structure not merely in terms of its individual elements, but also in terms of these well-defined groups.

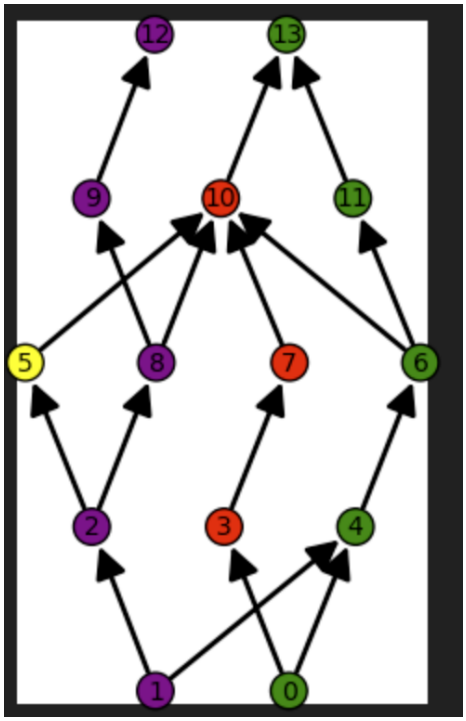

Figure 2: An SCD of a Peck Poset

The displayed Symmetric Chain Decomposition of the previously observed Poset consists of four chains: one single-element chain [5] (in yellow), a three-element chain $[3, 7, 10]$ (in red), and two five-element chains, $[1, 2, 8, 9, 12]$ (in purple) and $[0, 4, 6, 11, 13]$ (in green).

Formally, a Symmetric Chain Decomposition of a ranked Poset P breaks down P into symmetric, saturated, and disjoint chains.

- **Symmetry:** For any chain, if a horizontal line were drawn through the middle of the Poset, the split parts on both sides would mirror each other.
- **Saturation:** A chain is saturated if doesn't skip over any rank in its span, hence cannot be internally extended.
- **Disjointness:** In the SCD, Each vertex (node) appears in only one chain.

It turns out that applying SCDs to certain Poset classes could potentially resolve a significant conjecture in my field of research, combinatorial representation theory.

2 Brief Algorithm Overview

The objective of this process is to examine an arbitrary "Peck Poset" and determine when it has an SCD. There are three steps:

- **Generation:** First, we generate random Peck Posets.
- **Noise Reduction:** In this phase, the Poset undergoes a "noise reduction" process. Here, elements that do not contribute to the fundamental structure are removed to simplify the Poset.
- **Final Decomposition:** Finally, the script computes the Symmetric Chain Decomposition (SCD) for the simplified Poset.

Currently, there is no known algorithm that can find the SCDs of Posets in sub-exponential time (without checking every possible scenario). This makes the preprocessing step, the "noise-reduction" function, essential. It aims to eliminate pitfalls in the brute-force exponential search for an SCD.

My future plans include conducting a full time-complexity analysis of the algorithm as well as refining it in order to tackle some edge cases as well as improve the search process.

This work was done in VSCode using a Juypter Notebook.

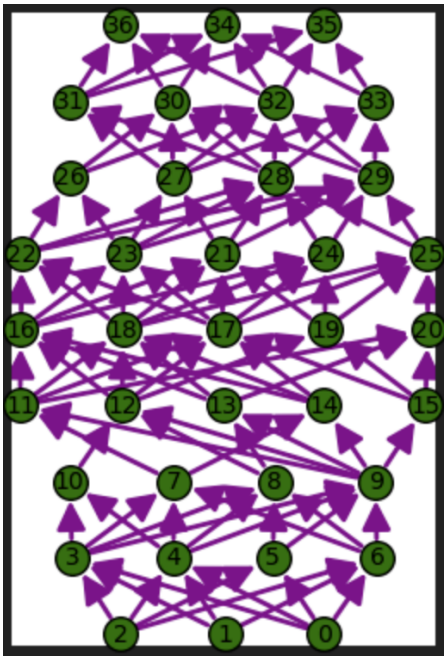

Figure 3: Motivation