Angle of Repose (θ)

When a plane is inclined to the horizontal by a certain angle, the body placed on it will remain at rest up to a certain angle of inclination, beyond which the body just begins to move. This maximum angle made by the inclined plane with the horizontal, when the body placed on that plane is just at the point of sliding down the plane, is known as the angle of repose. Repose means sleep which is disturbed at that particular angle of inclination.

Let us consider a body of weight W which is placed on an inclined plane as shown in Figure 8.4. The body is just at the point of sliding down the plane when the angle of inclination is θ. The various forces acting on the body are self-weight, normal reaction, and frictional force.

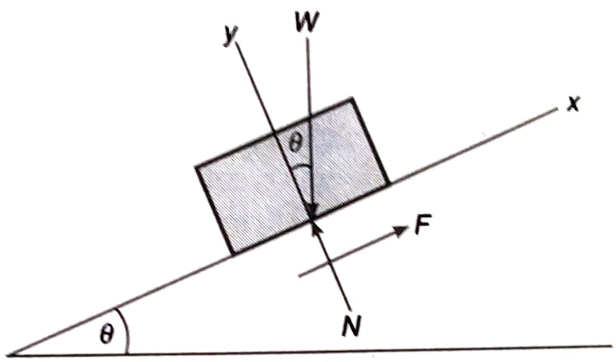

Figure 8.4 Angle of repose (θ).

Applying the conditions of equilibrium,

$$\Sigma F_x = 0;\ \Sigma F_y = 0$$

Resolving forces along the x-axis,

$$-F + W \sin\theta = 0$$

or

$$F = W \sin\theta \tag{8.2}$$

Resolving forces along the y-axis,

$$N - W\cos\theta = 0$$

or

$$N = W\cos\theta \tag{8.3}$$

We know that

$$\mu = \frac{F}{N}$$

⇒

$$\mu = \frac{W \sin\theta}{W\cos\theta} = \tan\theta \tag{8.4}$$

or

$$\tan\phi = \tan\theta$$

or

$$\phi = \theta$$

It is evident from Eqs. (8.1) and (8.4) that

Angle of friction = Angle of repose

Determination of Centroid by the Method of Moments

Let us consider a body of total weight W as shown in Figure 9.2. The centre of gravity of the whole figure is located at a distance $\bar{x}$ from the y-axis and at a distance $\bar{y}$ from the x-axis (the point through which the total weight W acts).

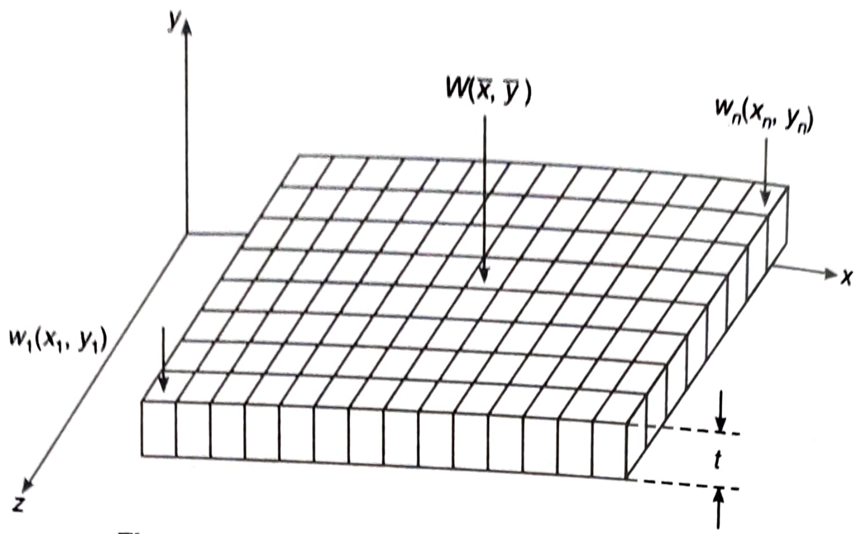

Figure 9.2 Centroid by the method of moments.

Let us divide the whole figure into a number of elemental strips of weights $w_1, w_2, w_3, w_4, \ldots, w_n$ whose centroids are located at distances $x_1, x_2, x_3, \ldots, x_n$ from the y-axis and $y_1, y_2, y_3, y_4, \ldots, y_n$ from the x-axis.

Applying the theorem of moments about the y-axis,

$$W\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 + \ldots + w_nx_n \quad \text{or} \quad \bar{x} = \frac{w_1x_1 + w_2x_2 + \ldots + w_nx_n}{W}$$

But if the body is homogeneous and isotropic, then the specific weight of the material is given by

$$\gamma = \frac{W}{V} = \frac{W}{A \times t}$$

where W is the weight of the body and V is the volume of the body, A is the cross-sectional area and t is the thickness which is constant.

Hence, we have

$$\bar{x} = \frac{\gamma a_1 t x_1 + \gamma a_2 t x_2 + \ldots + \gamma a_n t x_n}{\gamma a_1 t + \gamma a_2 t + \ldots + \gamma a_n t} = \frac{a_1x_1 + a_2x_2 + \ldots + a_nx_n}{a_1 + a_2 + \ldots + a_n} = \frac{\Sigma a_i x_i}{\Sigma a_i}$$

Similarly, $$\bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i}$$

If the width of the area is constant, then Area $= l \times b$, b is constant, then

$$\bar{x} = \frac{\Sigma(l_i\, b)x}{\Sigma(l_i\, b)} = \frac{\Sigma l_i . x}{\Sigma l_i} \quad \text{and} \quad \bar{y} = \frac{\Sigma(l_i b)y}{\Sigma(l_i b)} = \frac{\Sigma l_i . y}{\Sigma l_i}$$

The above two equations give the centroidal ordinates of the line element.

Axes of Reference

These are the axes with respect to which the centroid of a given fig

By considering a vertical strip, similarly, we can prove that

$$\bar{x} = \frac{b}{2}$$

Triangle

Consider a triangular lamina of area $(1/2) \times b \times d$ as shown in Figure 9.10.

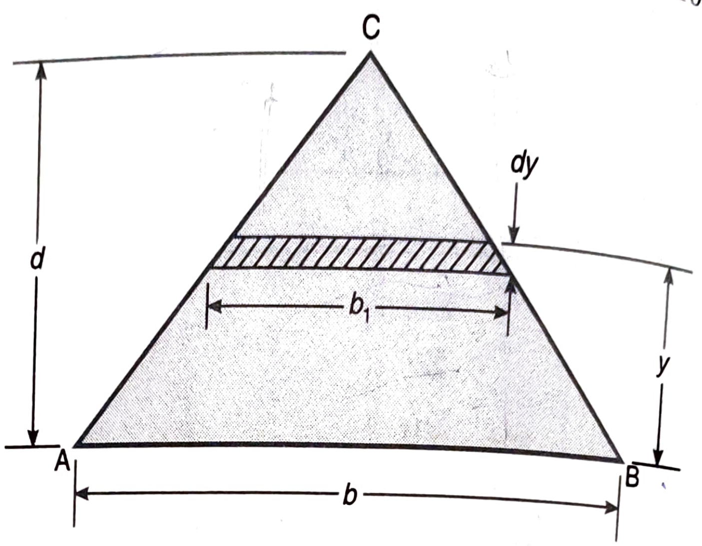

Figure 9.10 Triangular lamina.

Now consider an elementary strip of area $b_1 \times dy$ which is at a distance y from the reference axis AB.

Using the property of similar triangles, we have

$$\frac{b_1}{b} = \frac{d-y}{d}$$

or

$$b_1 = \frac{(d-y)b}{d}$$

$\frac{b_1}{(d-y)} = \frac{b}{d}$

$\frac{b_1}{b} = \frac{d-y}{d}$

$b_1 = \frac{(d-y)\,b}{d}.$

$$\text{Area of the elementary strip} = b_1 \times dy = \frac{(d-y)b \cdot dy}{d}$$

Moment of area of elementary strip about AB

$$= \text{area} \times y$$

$$= \frac{(d-y)b \cdot dy \cdot y}{d}$$

$$= \frac{b \cdot dy \cdot d \cdot y}{d} - \frac{by^2 \cdot dy}{d}$$

$$= by \cdot dy - \frac{by^2 \cdot dy}{d}$$

Sum of moments of such elementary strips is given by

$$\int_0^d by \cdot dy - \int_0^d \frac{by^2}{d} \cdot dy$$

$$= b \times \left[\frac{y^2}{2}\right]_0^d - \frac{b}{d}\left[\frac{y^3}{3}\right]_0^d$$

$$= \frac{bd^2}{2} - \frac{bd^3}{3d}$$

$$= \frac{bd^2}{2} - \frac{bd^2}{3}$$

$$= \frac{bd^2}{6}$$

Moment of total area about AB $= \frac{1}{2} bd \times \bar{y}$

Applying the principle of moments,

$$\frac{bd^2}{6} = \frac{1}{2} \times bd \times \bar{y}$$

$$\therefore \quad \bar{y} = \frac{d}{3}$$

Semicircle

Consider a semicircular lamina of area $\frac{\pi r^2}{2}$ as shown in Figure 9.11. Now consider a triangular elementary strip of area $\frac{1}{2} \times R \times R \times d\theta$ at an angle of θ from the x-axis, whose centre of gravity is at a distance of $\frac{2}{3} R$ from O and its projection on the x-axis $= \left(\frac{2}{3}\right) R \cos \theta$.

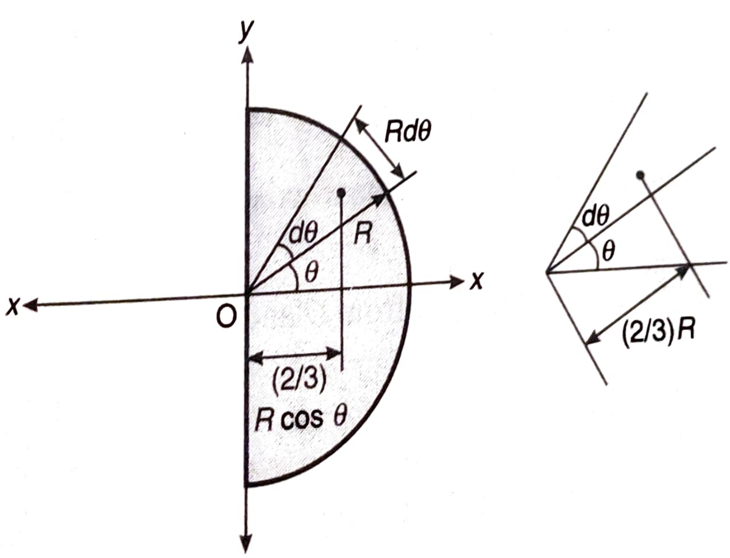

Figure 9.11 Semicircular lamina.

Moment of area of elementary strip about the y-axis $= \frac{1}{2} \times R^2 \cdot d\theta \cdot \left(\frac{2}{3}\right) R \cos \theta$

$$= \frac{R^3 \cdot \cos\theta \cdot d\theta}{3}$$

Sum of moments of such elementary strips about the y-axis

$$= \int_{-\pi/2}^{\pi/2} \frac{R^3}{3} \cos\theta \cdot d\theta$$

$$= \frac{R^3}{3} [\sin\theta]_{-\pi/2}^{\pi/2}$$

$$= \frac{R^3}{3}\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right] = \frac{2R^3}{3}$$

Moment of total area about the y-axis

$$= \frac{\pi R^2}{2} \times \bar{x}$$

Using the principle of moments

$$\frac{2R^3}{3} = \frac{\pi R^2}{2} \times \bar{x}$$

$\therefore$

$$\bar{x} = \frac{2R^3 \times 2}{3R^2 \pi}$$

or

$$\bar{x} = \frac{4R}{3\pi}$$

Quarter circle

Consider a quarter circular lamina of area $\frac{\pi R^2}{4}$ as shown in Figure 9.12. Consider a triangular elementary strip of area $\frac{1}{2} \times R \times R \times d\theta$ at an angle of θ from the x-axis, whose centre of gravity is at a distance of $\frac{2}{3} R$ from O and its projection on x-axis $= \frac{2}{3} R \cos\theta$.

Moment of area of elementary strip about the y-axis

$$= \frac{2}{3} R \cos\theta \times \frac{1}{2} \times R^2 \cdot d\theta = \frac{R^3 \cdot \cos\theta \cdot d\theta}{3}$$

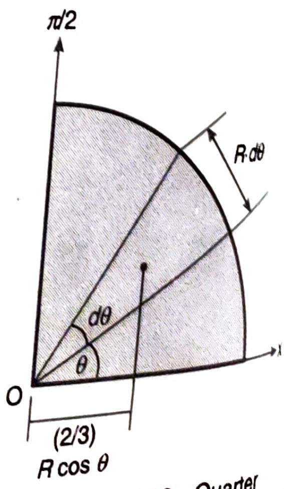

Figure 9.12 Quarter circular lamina

Sum of moments of such elementary strips about the y-axis

$$= \int_0^{\pi/2} \frac{R^3}{3} \cos\theta \cdot d\theta$$

$$= \frac{R^3}{3}\left[\sin\frac{\pi}{2}\right]$$

$$= \frac{R^3}{3}$$

Moment of total area about the y-axis

$$= \frac{\pi R^2}{4} \times \bar{x}$$

Using the principle of moments,

$$\frac{R^3}{3} = \frac{\pi R^2}{4} \times \bar{x}$$

$$\bar{x} = \frac{4R^3 \times 2}{3R^2\pi}$$

$\therefore$

$$\bar{x} = \frac{4R}{3\pi}$$

or

Similarly, we can prove that $\bar{y} = \dfrac{4R}{3\pi}$.

Sector of a circle

Consider a sector of a circular lamina as shown in Figure 9.13.

Consider a triangular elementary strip of area $\frac{1}{2} \times R \times R \times d\theta$ at an angle of θ from the x-axis, whose centre of gravity is at a distance of $\frac{2}{3}$ from O and its projection on x-axis $= \frac{2}{3} R\cos\theta$.

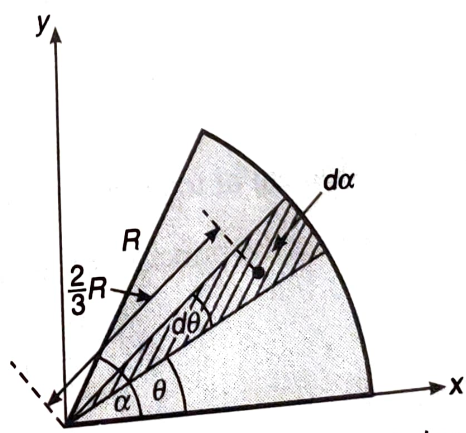

Figure 9.13 Sector of a circle.

$$\text{Area of strip} = \frac{1}{2} \times R^2 d\theta$$

Area of sector is given by

$$\int_0^{\alpha} \frac{1}{2} \times R^2 d\theta = \frac{1}{2} \times R^2 \alpha$$

Moment of area of elementary strip about y-axis

$$= \frac{2}{3} R\cos\theta \times \frac{1}{2} \times R^2 d\theta = \frac{R^3 \cdot \cos\theta \cdot d\theta}{3}$$

Sum of moments of such elementary strips about y-axis

$$= \int_0^\alpha \frac{R^3}{3} \cos\theta \cdot d\theta$$

$$= \frac{R^3}{3} (\sin\alpha)$$

Moment of total area about y-axis

$$= \frac{R^2\alpha}{2} \times \bar{x}$$

Using the principle of moments,

$$\frac{R^3}{3} \sin\alpha = \frac{R^2\alpha}{2} \times \bar{x}$$

or

$$\bar{x} = \frac{2R}{3}\left[\frac{\sin\alpha}{\alpha}\right]$$

Moment of area of elementary strip about x-axis

$$= \frac{2}{3} R \sin\theta \times \frac{1}{2} \times R^2 \cdot d\theta = \frac{R^3 \cdot \sin\theta \cdot d\theta}{3}$$

Sum of moments of such elementary strips about y-axis

$$= \int_0^\alpha \frac{R^3}{3} \sin\theta \cdot d\theta$$

$$= \frac{R^3}{3} [-\cos\theta]_0^\alpha = \frac{R^3}{3} (1 - \cos\alpha)$$

Moment of total area about y-axis $= \dfrac{R^2\alpha}{2} \times \bar{y}$

Using the principle of moments,

$$\frac{R^3}{3} (1 - \cos\alpha) = \frac{R^2\alpha}{2} \times \bar{y}$$

or

$$\bar{y} = \frac{2R}{3}\left[\frac{1 - \cos\alpha}{\alpha}\right]$$

about x-axis, $I_{xx} = Ay^2$

Least and greatest moment of inertia

$\bar{I}_x$ and $\bar{I}_y$ are the moment of inertia of a plane figure about x-axis and y-axis. If $\bar{I}_x$ is greater than $\bar{I}_y$ then $\bar{I}_x$ is known as the ***greatest moment of inertia*** and $\bar{I}_y$ is called the ***least moment of inertia***.

The unit of moment of inertia is mm^4 or m^4.

Radius of gyration (k)

It is the distance from the given axis where the whole area of a plane figure is assumed to be concentrated so as not to alter the moment of inertia about the given axis. For example, the moment of inertia about axis 1–1 (Figure 10.2) is

$$I = Ak^2$$

and hence
$$k = \sqrt{\frac{I}{A}}$$

That is,
$$k_x = \sqrt{\frac{\bar{I}_x}{A}} \quad \text{and} \quad k_y = \sqrt{\frac{\bar{I}_y}{A}}$$

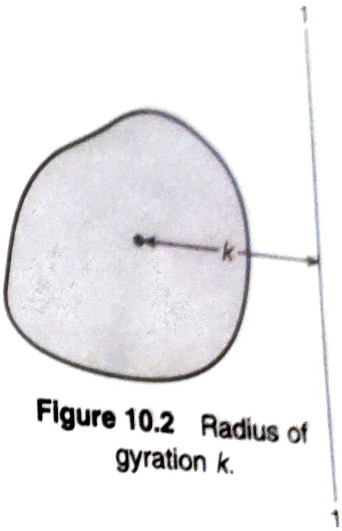

Figure 10.2 Radius of gyration k.

PARALLEL AXIS THEOREM

This theorem states that the moment of inertia of plane figure about an axis I_{1-1}, parallel to the centroidal axis, $\bar{I}_x$ is equal to sum of moment of inertia about centroidal axis, i.e. $\bar{I}_x$ and the product of area of the plane figure and square of the distance between the two axes.

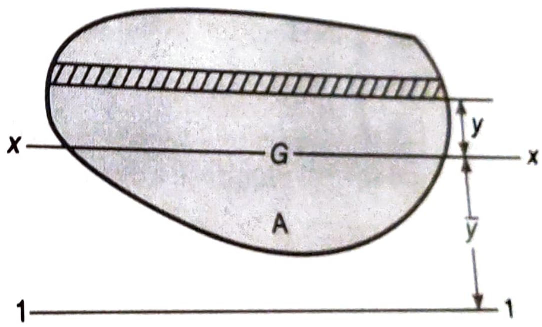

Figure 10.3 Illustration of parallel axis theorem for moment of inertia about an axis parallel to x–x axis.

Proof: Let us consider a plane figure of total area A as shown in Figure 10.3. Let $\bar{I}_x$ be the moment of inertia about the x-axis and I_{1-1} the moment of inertia about 1–1 axis.

Let us choose an elemental strip of area da at a distance y from the centroidal axis.

Moment of inertia of the strip about x–x axis $= da \cdot y^2$

Moment of inertia of the total area about the x–x axis $= \bar{I}_x = \Sigma da \cdot y^2$

Moment of inertia of the strip about 1–1 axis $= da(y + \bar{y})^2$

Moment of inertia of the total area about 1–1 axis

$$I_{1-1} = \Sigma da(y^2 + \bar{y}^2 + 2y\,\bar{y})$$

$$I_{1-1} = \Sigma day^2 + \Sigma da\,\bar{y}^2 + 2\,\bar{y}(\Sigma day)$$

As the distance of C.G. of whole area from the centroidal axis = 0, i.e. $y = 0$, we get

$$I_{1-1} = \bar{I}_x + A\,\bar{y}^2$$

Similarly the moment of inertia about an axis I_{2-2} as shown in Figure 10.4 is given by

$$I_{2-2} = \overline{I}_y + A\overline{x}^2$$

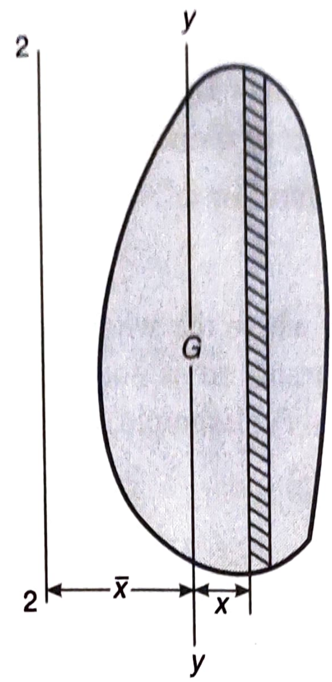

Figure 10.4 Illustration of parallel axis theorem for moment of inertia about an axis parallel to y–y axis.

PERPENDICULAR AXIS THEOREM

This theorem states that the moment of inertia of a plane figure about an axis which is perpendicular to the plane of the figure is equal to sum of moment of inertia about two mutually perpendicular axes.

Proof: Let us consider an irregular figure of total area A as shown in Figure 10.5. Let us choose an elemental strip of area da at a distance x from y-axis, y from x-axis and r from z-axis, respectively. Then,

$$r^2 = x^2 + y^2$$

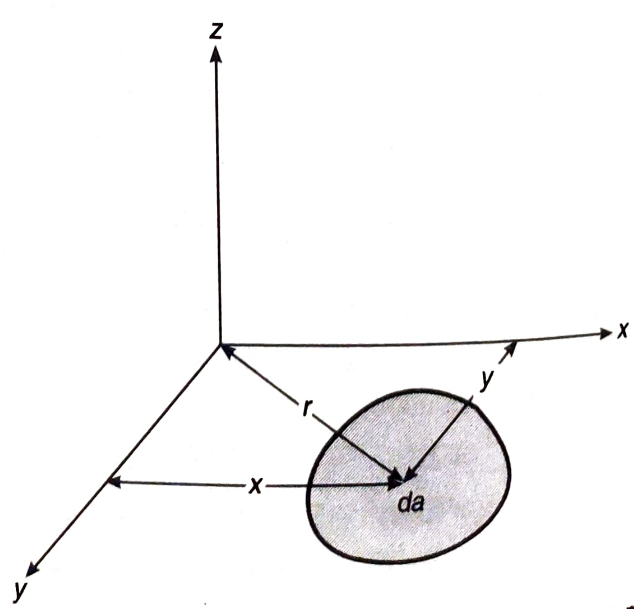

Figure 10.5 Illustration of perpendicular axis theorem.

Moment of inertia of the strip about x-axis $= da \times y^2$

Moment of inertia of the whole area about the x-axis $= \bar{I}_x = \Sigma da \cdot y^2$

Similarly, moment of inertia of the strip about y-axis $= da \times x^2$

Moment of inertia of the whole area about y-axis $= \bar{I}_y = \Sigma da \cdot x^2$

Moment of inertia of the strip about z-axis $= da \times r^2$

Moment of inertia of the whole area about z-axis $= \Sigma da \cdot r^2$

$$= \Sigma da(x^2 + y^2)$$

$$= \Sigma da \cdot x^2 + \Sigma da \cdot y^2$$

$$= \bar{I}_y + \bar{I}_x$$

That is,

$$\bar{I}_z = \bar{I}_x + \bar{I}_y$$

MOMENT OF INERTIA OF IMPORTANT FIGURES

Rectangle

Let us consider a rectangular lamina of breadth b and depth d whose moment of inertia is to be determined (Figure 10.6). Now consider an elementary strip of area $b \cdot dy$ at a distance y from the centroidal x–x axis. The moment of inertia of the strip about the x–x axis $= b \cdot dy \times y^2$.

Moment of inertia of the whole figure about the x–x axis

$$= \int_{-d/2}^{d/2} b \cdot dy \times y^2$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$= \frac{b \times d^3}{12}$$

That is,

$$\bar{I}_x = \frac{bd^3}{12}$$

Similarly,

$$\bar{I}_y = \frac{db^3}{12}$$

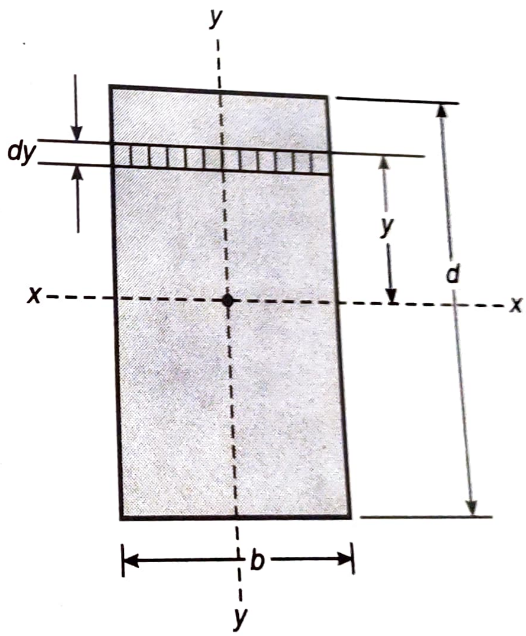

Figure 10.6 Rectangular lamina.

Triangle

Let us consider a triangular lamina of base b and depth d as shown in Figure 10.7. Let us consider an elementary strip of area $b_1 \times dy$ which is at a distance y from base AB.

Using the property of similar triangles,

$$\frac{b_1}{b} = \frac{d - y}{d}$$

or

$$b_1 = \frac{(d - y)b}{d}$$

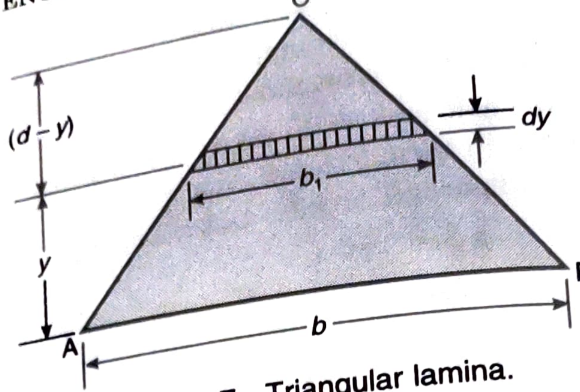

Figure 10.7 Triangular lamina.

$$\text{Area of the strip} = \frac{(d-y)b}{d} \cdot dy$$

$$\text{Moment of inertia of the strip about AB} = \frac{(d-y)b}{d} dy \times y^2$$

$$= \frac{bdy^2 \cdot dy}{d} - \frac{by^3 \cdot dy}{d}$$

$$= by^2 \cdot dy - \frac{by^3 \cdot dy}{d}$$

Moment of inertia of the whole area about AB,

$$I_{AB} = \int_0^d by^2 dy - \int_0^d \frac{b}{d} y^3 dy$$

$$= b\left[\frac{y^3}{3}\right]_0^d - \frac{b}{d}\left[\frac{y^4}{4}\right]_0^d$$

$$= \frac{bd^3}{3} - \frac{b}{d}\frac{d^4}{4}$$

$$= \frac{bd^3}{3} - \frac{bd^3}{4}$$

$$\therefore \qquad I_{AB} = \frac{bd^3}{12}$$

Moment of inertia about x–x axis is given by

$$I_{AB} = \bar{I}_x + Ay^2$$

i.e.

$$\bar{I}_x = I_{AB} - Ay^2$$

$$= \frac{bd^3}{12} - \frac{1}{2}bd\left(\frac{1}{3}d\right)^2 = \frac{bd^3}{36}$$

Therefore, the moment of inertia of the triangle about the centroidal y-axis $= \dfrac{db^3}{36}$.

Circle

Derive an expression of moment of inertia of a circle about its diametrical axis.

Let us consider a circular lamina of radius R as shown in Figure 10.8.

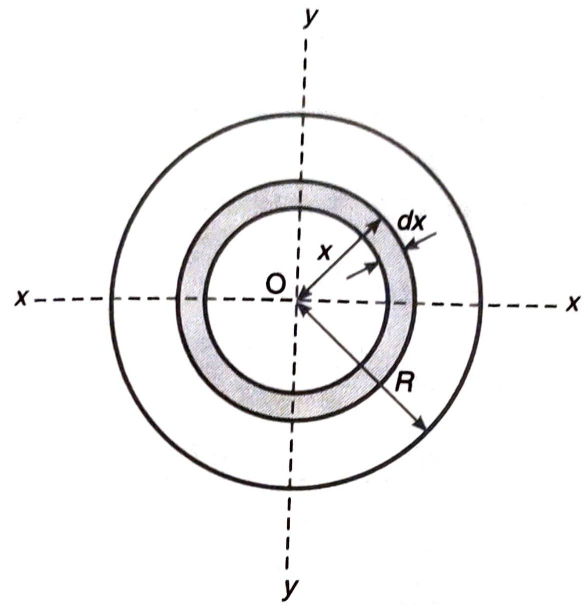

Figure 10.8 Circular lamina.

Let us choose a circular elementary strip of thickness dx at distance x from the centre.

Area of the strip = $2\pi x dx$.

Moment of inertia about the z–z axis = $2\pi x \cdot dx \cdot x^2$

Moment of inertia about the z–z axis for whole circle

$$= \bar{I}_z = \int_0^R 2\pi x^3 \cdot dx = 2\pi\left[\frac{x^4}{4}\right]_0^R$$

$$= \frac{2\pi R^4}{4} = \frac{\pi R^4}{2}$$

For the circular lamina,

$$\bar{I}_x = \bar{I}_y,$$

$\therefore$ Using the perpendicular axis theorem, we have

$$\bar{I}_z = \bar{I}_x + \bar{I}_y$$

or

$$\bar{I}_z = 2\bar{I}_x$$

or

$$\bar{I}_x = \frac{I_{zz}}{2}$$

$\therefore$

$$\bar{I}_x = \frac{\pi R^4}{2\times 2} = \frac{\pi R^4}{4} = \bar{I}_y$$

Semicircle

Let us consider a semicircular lamina of radius R as shown in Figure 10.9.

Moment of inertia of semicircle about the diametrical axis AB = $\frac{1}{2}\times\frac{\pi R^4}{4} = \frac{\pi R^4}{8}$

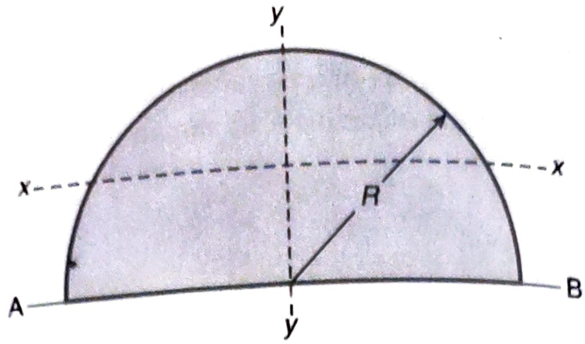

Figure 10.9 Semicircular lamina.

$$I_{AB} = \bar{I}_x + A\,\bar{y}^2$$

or

$$\bar{I}_x = I_{AB} - \frac{\pi R^2}{2}\left(\frac{4R}{3\pi}\right)^2$$

$$= \frac{\pi R^4}{8} - \frac{\pi R^2 \times 16R^2}{2 \times 9\pi^2}$$

$$= \frac{\pi R^4}{8} - \frac{8\pi R^4}{9\pi^2}$$

$$= \frac{\pi R^4}{8} - \frac{8R^4}{9\pi}$$

∴

$$\bar{I}_x = 0.11R^4$$

Moment of inertia about the y-axis,

$$\bar{I}_y = \frac{\pi R^4}{8} = \frac{1}{2} \times \frac{\pi R^4}{4}$$

Quarter Circle

For a quarter circle of radius R as shown in Figure 10.10,

$$\bar{I}_x = \bar{I}_y = \frac{0.11R^4}{2} = 0.055R^4$$

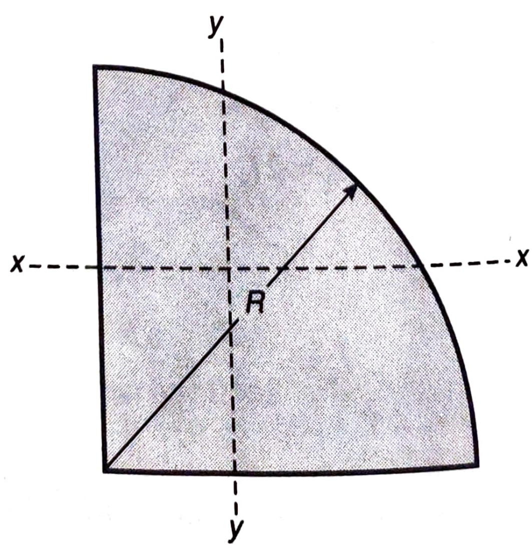

Figure 10.10 Quarter circle.