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On Tracking and Capture in Proportional-Control Bearing-Only Unicycle Pursuit

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***Abstract*— An agent with constant speed unicycle kinematics senses only the bearing towards a target and pursues it, guided by a steering control law proportional to the bearing-angle to the target. The target's speed is equal to that of the chasing agent while it moves at a constant velocity. This paper presents a comprehensive analysis of this unicycle pursuit problem and shows that with enough gain in the control law, the agent eventually either captures the target or follows its path while maintaining a finite distance from it, regardless of the initial conditions.**

***Index Terms*— Autonomous systems, Lyapunov methods, Nonholonomic systems.**

I. Introduction

PURSUIT problems are centuries old, yet still are the subject of much academic activity [1], [2]–[4]. Form Pierre Bouguer, who is credited with formulating the problem in the 18th century (as a pirate ship trying to intercept a merchantman), to mathematicians, control theorists and roboticists today, pursuit has fascinated generations of thinkers. This might be due to the simplicity and elegance of the problem statement, contrasted with the sophistication of the possible solutions and methods of reaching them. This is coupled with the endless variation options in the dynamics of the pursuing agent, its controller, its maneuverability compared to that of the target, the sensing and computing capabilities required of the pursuing agent's hardware, etc.

Pursuit still poses some interesting problems that the current literature has not fully addressed. The unicycle-model variant of pursuit, for instance, has yet to be completely solved, to the best of our knowledge. The unicycle model is a popular and useful model for representing complex robotic and vehicular systems and their behaviors in various tasks [5], [6], but it is an underactuated system, having three degrees of freedom (the unicycle's orientation, and two axes of its location on the plain) and only two actuators (forward speed and turn angle rate), which makes its control challenging.

The lack of a full analytic solution to all unicycle pursuit problems does not imply a lack of achievements towards this goal. Medagoda and Gibbens [7], for instance, propose a waypoint-following pursuit strategy wherein a virtual target moves linearly between path waypoints at a speed depending on parameters like the pursuer-target relative distance and the pursuer's speed. Tracking is achieved using an LQR based controller to pursue the virtual, linear agent. Following this work, Ratnoo et al. [8] present necessary conditions to achieve tracking (or tail chase) when the target is moving in a straight line. Resulting from a pure-pursuit guidance law that requires acquiring both bearing and distance to the target, their control policy aligns the pursuer with the target. Other versions of pure-pursuit and several other approaches have also been explored to address the tracking problem [9]–[14]. These versions of unicycle pursuit rely on sophisticated controllers, which in turn demand additional sensing and computing capabilities for their implementation and generally, also superior maneuverability of the pursuing agent compared to the target. Belkhouche and Belkhouche [15], for example, study capture conditions when the pursuer moves at a speed higher than that of the target.

In this work, we revisit the classic problem of pursuit when the motion of the pursuing agent is governed by unicycle kinematics. Instead of enhancing the pursuer's capabilities, we consider the case of an agent restricted to have a constant speed equal to that of the target. This hindrance has a double effect; by keeping the speed constant we underactuate the pursuer even more than it inherently is, since only the steering control can be used for the pursuit; by matching the speed to that of the target we keep the pursuer just barely capable of tracking it even in the case where the initial conditions put the pursuer in a good position to track the target. We further constrain our pursuing agent to only acquire information about the bearing towards the target, and act upon this information in a memory-less fashion; i.e. the pursuer does not remember the previous bearing readings, and has to control its parameters based on the instantaneous reading of the bearing only.

Our work contributes to the existing literature by guaranteeing convergence to either tracking or capture, under conditions we analyze in detail, from arbitrary initial conditions on the pursuer and the target using Lyapunov stability analysis [16]. We also present numerical simulations to illustrate the theoretical results, and make observations towards future directions of research.

II. Proportional-Control Bearing-Only Unicycle Pursuit

The classic pure pursuit problem involves a target (a merchantman) moving in a straight line with kinematics

$$p_m^T = (vt, 0)\,, \tag{1}$$

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and a pursuing agent (a pirate ship) with kinematics

$$\dot{p}_p = v_p \frac{p_m - p_p}{|p_m - p_p|} \tag{2}$$

where p_m, p_p are the target's and agent's locations, respectively, v is the target's speed and v_p is the pursuing agent's speed. The problem is to find the *curve of pursuit* $y(x)$ given $p_p^T = (x(t), y(t))$ and $(x(0), y(0)) = (x_0, 0)$. Nahin [2] dedicated a chapter in his book to the formulation, history and solution to this classic problem.

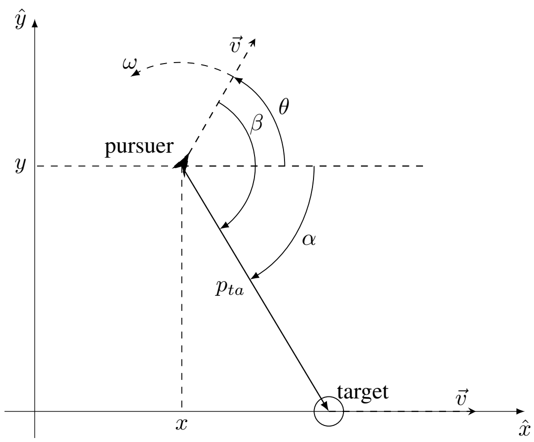

Fig. 1. The Unicycle Pursuit Problem

The work presented here replaces the pursuing agent's kinematics with Unicycle Kinematics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{3}$$

where $(x, y)^T$ is the agent's position, θ is the agent's orientation, $v > 0$ its speed, and ω its turning rate. The relationship between the agent and the target it pursues can be described by the distance between them r, the bearing angle towards the target as measured from the agent's frame β, and the bearing angle towards the agent from the target's frame, which is equivalent to $\alpha - \pi$, where α is the azimuth from the agent to the target in the global frame, see Fig. 1.

Since agents modeled as unicycles have an orientation, which must align with the bearing towards the target in order to mimic pure pursuit, we assign a proportional controller

$$\omega = \kappa\beta \tag{4}$$

where κ is a gain that amplifies the bearing angle β. Consequently, the pursuing agent applies the resulting signal ω as a steering command to its single controlled actuator, having no control over v. We define *capture* as reaching a distance r_c from the target, i.e. $\exists t_c \,|r(t_c) \leq r_c$, and *tracking* as reaching a trajectory which is very close to that of the target while remaining within a finite range from the target, i.e. $\exists R,\ 0 < R < \infty \ \mid r(t) < R\ ,\ \forall t$, and also given arbitrary $\varepsilon > 0$, $\exists t_\varepsilon \mid \forall t > t_\varepsilon\, , |\beta(t)| < \varepsilon$ and $|\alpha(t)| < \varepsilon$. The physical meaning of the distance r_c could be the dimension of either the target, the agent, or any combination of both, while ε could be the acceptable measure of observational error of the agent's sensors.

The analysis in the following section reveals that given $\kappa > \frac{2v}{r_c}$, the agent is guaranteed to eventually track or capture the target.

Problem Statement

A target with kinematics (1) is pursued by an agent with kinematics (3), (4), and a constant speed v equal to that of the target. Find κ_c such that if $\kappa > \kappa_c$, then either the agent captures the target or tracks it, i.e. $\exists t_c \,|r(t_c) \leq r_c$ or $\exists R,\ 0 < R < \infty \ \mid r(t) < R\ ,\ \forall t$, and $\forall \varepsilon > 0$, $\exists t_\varepsilon \mid \forall t > t_\varepsilon\, , |\alpha(t)| < \varepsilon$ and $|\beta(t)| < \varepsilon$.

III. Tracking and Capture of a Constant Velocity Target

Let α be the azimuth from the agent's location to the target's location,

$$\tan(\alpha) = \frac{y}{x - vt}; \tag{5}$$

and from Fig. 1,

$$\beta = \alpha - \theta. \tag{6}$$

Looking at the target's motion from a frame attached to the agent, with its real axis pointing in the direction of the target's velocity (see Fig. 1), we arrive at the complex representation of the position of the target as observed by the agent

$$p_{target}^{agent} = p_{ta} = re^{i\alpha}, \tag{7}$$

where $r = |p_{ta}|$ is the distance form target to agent. Taking the time derivative

$$\frac{d}{dt}p_{ta} = \frac{d}{dt}\left(re^{i\alpha}\right) = \left(\dot{r} + ir\dot{\alpha}\right)e^{i\alpha} = \left(\frac{\dot{r}}{r} + i\dot{\alpha}\right)p_{ta}, \tag{8}$$

and comparing it with the target's velocity as observed by the agent

$$\left(\frac{\dot{r}}{r} + i\dot{\alpha}\right)re^{i\alpha} = v\left(e^{i(0)} - e^{i(\theta)}\right) \tag{9}$$

$$\Downarrow$$

$$\dot{r} + ir\dot{\alpha} = v\left(e^{i(0-\alpha)} - e^{i(\theta-\alpha)}\right) = v\left(e^{-i\alpha} - e^{-i\beta}\right), \tag{10}$$

leaves us with an expression for the rate of rotation

$$r\dot{\alpha} = v\sin(-\alpha) - v\sin(-\beta) \tag{11}$$

$$\Downarrow$$

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right), \tag{12}$$

and the rate of change in distance

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right). \tag{13}$$

From (3), (4), (6), and (12),

$$\dot{\beta} = \dot{\alpha} - \dot{\theta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \omega \tag{14}$$

$$\Downarrow$$

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta. \tag{15}$$

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Before continuing to the main results, we define some useful phrases:

- the agent is *facing the target* when $|\beta| < \frac{\pi}{2}$, and
- the agent *falls behind the target* when $|\alpha| < \frac{\pi}{2}$.

Lemma 1: An agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), either captures the target or turns to face the target in finite time, such that $|\beta| \leq \frac{\pi}{3}$.

Proof: $|\sin(\beta) - \sin(\alpha)| \leq 2$; therefore from (15),

$$-2\frac{v}{r_c} - \kappa\beta \leq \dot{\beta} \leq 2\frac{v}{r_c} - \kappa\beta; \tag{16}$$

and the bounds on β are

$$\beta^+(t) = 2\frac{v}{\kappa r_c} + \left(\beta(t_0) - 2\frac{v}{\kappa r_c}\right) e^{-\kappa(t-t_0)}; \tag{17}$$

$$\beta^-(t) = -2\frac{v}{\kappa r_c} + \left(\beta(t_0) + 2\frac{v}{\kappa r_c}\right) e^{-\kappa(t-t_0)}; \tag{18}$$

i.e.

$$\beta^-(t) \leq \beta(t) \leq \beta^+(t). \tag{19}$$

If $\frac{\pi}{3} \leq \beta(t_0)$, then the upper bound on β decreases asymptotically to $\frac{2v}{\kappa r_c} < 1$, and reaches $\frac{\pi}{3} > 1$ by t_1,

$$\beta^+(t_1) = 2\frac{v}{\kappa r_c} + \left(\beta(t_0) - 2\frac{v}{\kappa r_c}\right) e^{-\kappa(t_1-t_0)} = \frac{\pi}{3} \tag{20}$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{\beta(t_0) - 2\frac{v}{\kappa r_c}}{\frac{\pi}{3} - 2\frac{v}{\kappa r_c}}\right). \tag{21}$$

Similarly, $\beta^-(t)$ reaches $-\frac{\pi}{3}$ if $\beta(t_0) \leq -\frac{\pi}{3}$, and if $\frac{\pi}{3} \leq |\beta(t_0)|$, then $\exists t_*,\ t_0 \leq t_* \leq t_1$, when either $r(t_*) = r_c$, or $|\beta(t_*)| \leq \frac{\pi}{3}$. ∎

Corollary 1: If an agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), faces the target, it will continue to face the target forever.

$$\exists t_* \ \Big|\ |\beta(t_*)| \leq \frac{\pi}{3} \Rightarrow |\beta(t)| \leq \frac{\pi}{3}\ \forall t > t_*. \tag{22}$$

Proof: β^+ and β^- from the previous proof are global bounds on $\beta(t)$, even for $|\beta| < \frac{\pi}{3}$, ∎

Notice that the higher the gain κ, the closer t_1 is to t_0 and the lower the actual upper bound on β is, at $\frac{2v}{\kappa r_c}$.

Lemma 2: If an agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), falls behind the target, it will remain behind the target forever.

$$\exists t_* \ \Big|\ |\alpha(t_*)| < \frac{\pi}{2} \Rightarrow |\alpha(t)| < \frac{\pi}{2}\ \forall t > t_*. \tag{23}$$

Proof: If $\exists t_* \ \big|\ |\alpha(t_*)| < \frac{\pi}{2}$, then $x(t_*) < x_t(t_*) = vt_*$, and from (1) and (3), $\dot{x} = v\cos\theta \leq v \Rightarrow x(t) - x(t_*) \leq v(t - t_*) = x_t(t) - x_t(t_*) \Rightarrow x(t) + x_t(t_*) \leq x_t(t) + x(t_*)$; but $x(t_*) < x_t(t_*) \Rightarrow x(t) < x_t(t)\ \forall t \geq t_*$. ∎

Lemma 3: An agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), either captures the target or falls behind it in finite time.

Proof: According to Lemma 1 and Corollary 1, if $|\beta(t_0)| \geq \frac{\pi}{3}$, then either the target has been captured by t_1, or $|\beta(t)| < \frac{\pi}{3}\ \forall t > t_1$; and if $|\beta(t_0)| \leq \frac{\pi}{3}$, $|\beta(t)| < \frac{\pi}{3}\ \forall t > t_0$, and (13) becomes $\dot{r} \leq -v\cos(\beta) < -\frac{v}{2}$, until $|\alpha(t_*)| < \frac{\pi}{2}$. Seeing that for the case where $|\beta(t_0)| > \frac{\pi}{3}$, $t_0 \leq t \leq t_1$, $\dot{r} \leq v$,

$$r(t) < r^+(t) = r(t_0) + v(t_1 - t_0) - \frac{v}{2}(t - t_1) \tag{24}$$

$$= r(t_0) + v\left(\frac{3}{2}t_1 - t_0 - \frac{1}{2}t\right). \tag{25}$$

By t_2, the agent must therefore capture its target, unless $|\alpha| < \frac{\pi}{2}$ sometime before;

$$r^+(t_2) = r_c \Rightarrow r(t_0) + v\left(\frac{3}{2}t_1 - t_0 - \frac{1}{2}t_2\right) = r_c \tag{26}$$

$$\Downarrow$$

$$t_2 = t_0 + 2\frac{r(t_0) - r_c}{v} + \frac{3}{\kappa}\ln\left(\frac{\beta(t_0) - 2\frac{v}{\kappa r_c}}{\frac{\pi}{3} - 2\frac{v}{\kappa r_c}}\right). \tag{27}$$

Similarly, if $|\beta(t_0)| < \frac{\pi}{3}$, $r(t) < r(t_0) - \frac{v}{2}(t - t_0)$, and $t_2 = t_0 + 2\frac{r(t_0)-r_c}{v}$. In any case,

$$t_2 \leq t_0 + 2\frac{r(t_0) - r_c}{v} + \frac{3}{\kappa}\ln\left(\frac{\pi - 2\frac{v}{\kappa r_c}}{\frac{\pi}{3} - 2\frac{v}{\kappa r_c}}\right), \tag{28}$$

and if the agent has not captured the target by t_2, then

$$\exists t_*,\ t_0 \leq t_* \leq t_2 \ \Big|\ |\alpha(t_*)| < \frac{\pi}{2}. \tag{29}$$

∎

Theorem 1: An agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), either captures the target or reaches a forward invariant configuration region $(\alpha, \beta) \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{3}, -\frac{\pi}{3}\right]$ in finite time.

Proof: Results directly from all previous statements. ∎

Lemma 4: If an agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), reaches the target's path with $\beta = 0$, then it tracks the target from that point on.

Proof: If $(\alpha, \beta) = (0, 0)$, then by (12), (13) and (15), $\left(\dot{\alpha}, \dot{\beta}, \dot{r}\right) = (0, 0, 0)$, and (α, β, r) become constant. ∎

Theorem 2: An agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), either captures the target or asymptotically reaches the target's path.

Proof: Consider the function

$$V(\alpha, \beta) = 2\left(\sin^2\left(\frac{\alpha(t)}{2}\right) + \sin^2\left(\frac{\beta(t)}{2}\right)\right) \geq 0. \tag{30}$$

$$\dot{V} = \sin(\alpha)\dot{\alpha} + \sin(\beta)\dot{\beta} \tag{31}$$

$$= \frac{v}{r}\left(\sin^2(\beta) - \sin^2(\alpha)\right) - \kappa\beta\sin(\beta) \tag{32}$$

$$\leq \left(\frac{v}{r} - \kappa\right)\beta\sin(\beta) - \frac{v}{r}\sin^2(\alpha). \tag{33}$$

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If the agent captures the target, then $\exists t_c < \infty \mid r(t_c) \leq r_c$, and the pursuit concludes; otherwise, $r(t) > r_c$, and since we choose the gain such that $\kappa > 2\frac{v}{r_c}$,

$$\dot{V} \leq \left(\frac{v}{r_c} - 2\frac{v}{r_c}\right)\beta \sin(\beta) - \frac{v}{r}\sin^2(\alpha) \leq 0. \tag{34}$$

We proceed to find the points in which $\dot{V} = 0$.

$$\dot{V} = 0 \Rightarrow \frac{v}{r}\left(\sin^2(\beta) - \sin^2(\alpha)\right) - \kappa\beta\sin(\beta) = 0 \tag{35}$$

$$\Downarrow$$

$$\sin^2(\alpha) = \beta\sin(\beta)\left(\frac{\sin(\beta)}{\beta} - \frac{\kappa r}{v}\right). \tag{36}$$

From Theorem 1, if the target was not captured by t_2, then $\forall t \geq t_2$, $(\alpha(t), \beta(t)) \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{3}, -\frac{\pi}{3}\right]$. Using this fact, we notice that $\frac{\kappa r}{v} > 2$, and that if $|\beta(t)| \leq \frac{\pi}{3}$, then $\frac{\sin(\beta)}{\beta} \leq 1$, $\beta\sin(\beta) \geq 0$, and the right hand side of the last equation becomes non-positive, while the left hand side non-negative, and the only point in the forward invariant region $(\alpha, \beta) \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{3}, -\frac{\pi}{3}\right]$ at which $\dot{V} = 0$ is the origin $(0, 0)$. To conclude the proof,

1) $\forall t > t_2$, $(\alpha(t), \beta(t)) \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{3}, -\frac{\pi}{3}\right]$,
2) $V(\alpha, \beta) > 0, \forall (\alpha, \beta) \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{3}, -\frac{\pi}{3}\right] \setminus (0, 0)$,
3) $\dot{V}(\alpha, \beta) < 0, \forall (\alpha, \beta) \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{3}, -\frac{\pi}{3}\right] \setminus (0, 0)$,
4) $V(0, 0) = \dot{V}(0, 0) = 0$;

V is therefore a Lyapunov function [17], and if the target was not captured in the process, then $\lim_{t\to\infty}(\alpha, \beta) = (0, 0)$. ∎

Though Theorem 2 asserts that $\dot{r}(t \to \infty) = 0$, it does not necessarily guarantee that $r(t \to \infty) < \infty$. If the rate at which $|\beta|$ and $|\alpha|$ shrink is too low, the target may slip away to infinity. The following theorem addresses this issue and affirms that even if the target does not get captured, it does not get away.

Theorem 3: The distance between an agent (3) governed by the bearing-only control law (4) with $\kappa > 2\frac{v}{r_c}$, in pursuit of a target moving in a straight line (1), is bounded from above by a finite value, i.e.

$$\exists R,\ 0 < R < \infty \mid r(t) < R\ ,\ \forall t. \tag{37}$$

Due to the length of the formal proof, which is available in our technical report [18], we present here a proof outline instead.

Proof: From (13), the distance r grows only when $\cos(\alpha) > \cos(\beta)$. After t_2 (27), this could happen only if $0 \leq \alpha < \beta < \frac{\pi}{2}$ or $0 \leq \alpha < -\beta < \frac{\pi}{2}$, and any time these conditions manifest, they remain valid only for a varying yet limited amount of time. Let ΔT_i denote the i^{th} time interval for which any one of the above conditions hold. During ΔT_i, the rate of contraction of an upper bound on α and β can be determined. The rate of increase in distance, $\dot{r}$, can then be bounded by a function of the upper bounds on $|\alpha|$ and $|\beta|$, which we denote $U_i(\alpha, \beta)$. This upper bound on $\dot{r}$, multiplied by the time period ΔT_i, results in L_i, the maximal possible increase in r during ΔT_i,

$$L_i = U_i \Delta T_i. \tag{38}$$

We conclude the proof by showing that

$$\sum_{i=1}^{\infty} L_i = L < \infty, \tag{39}$$

and that $r(t \to \infty) < r(t_2) + L = R$ ∎

IV. Simulation

We programmed our model in NetLogo [19], and made it available to the community on our laboratory's website[1]. Using this NetLogo model, we can illustrate our findings in the previous section. Fig. 2 shows a simulation run with 20 agents with randomly generated initial conditions $(\alpha(t_0), \beta(t_0), r(t_0))$ and uniform parameters $(v, r_c, \kappa) = \left(1, 30, \frac{1}{15}\right)$ in pursuit of a single target. The target is a circle with radius r_c. The simulation confirms our analysis, that those agents that do not capture the target fall behind it and gradually align their velocity with the target's velocity. The figure shows one simulation run at three consecutive moments. Fig. 2(a) shows the pursuers near their initial positions, scattered randomly on the plane. Fig. 2(b) shows the first agents capturing the target, while Fig. 2(c) shows the pursuers settling into tracking the target.

To investigate which initial conditions lead to capturing the target rather than tracking it, we generated the simulation presented in Fig. 3. In the simulation we sent 400 agents to chase after the same linearly moving target with uniform parameters $(v, r_c, \kappa) = \left(1, 30, \frac{1}{15}\right)$ and initial conditions $(\alpha(t_0), \beta(t_0), r(t_0))$, such that all agents share the same $\beta(t_0)$ and have varying $\alpha(t_0)$ and $r(t_0)$ according to their initial position. The darker regions mark the initial positions (determined by $\alpha(t_0)$ and $r(t_0)$) of agents that eventually captured the target, while lighter regions those of agents that ended up tracking the target instead. We noticed that $\beta(t_0)$ plays an important role in shaping the capture region, albeit to a lesser extent than $\alpha(t_0)$ and $r(t_0)$. For instance, in the simulation that resulted in Fig. 3(a), all the pursuing agents were initialized with $\beta(t_0) = 0$, meaning the target was straight ahead of every agent at $t = t_0$. Notice that the capture region for this experiment is smaller, perhaps counter-intuitively, than the capture region of the experiment in Fig. 3(c), where all agents were initialized with a heading opposite to where the target was. Arguably less surprising is the result shown in Fig. 3(b), where the initial bias of $90°$ caused the agents beginning their chase directly under the target to go in the opposite direction of the target, missing their chance to capture, while the agents starting directly above the target started their chase with their velocity parallel to that of the target, allowing them to adjust their alignment without falling far behind the target. The overall observed result in the $\beta(t_0) = \frac{\pi}{2}$ case is a bias of the entire capture region towards the upper part of the experiment arena. We explore these (α, β, r) capture regions further in a technical report [20].

V. Conclusion

In this paper we analyzed the pursuit problem to derive results on the stability of target tracking when the target is

[1] https://mars.cs.technion.ac.il/simulations/

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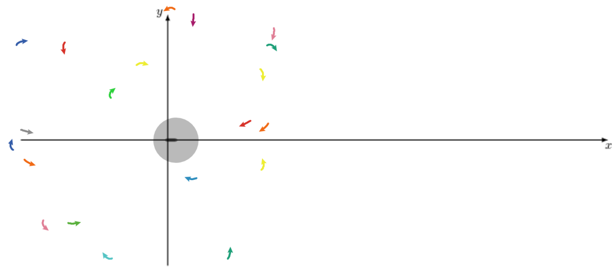
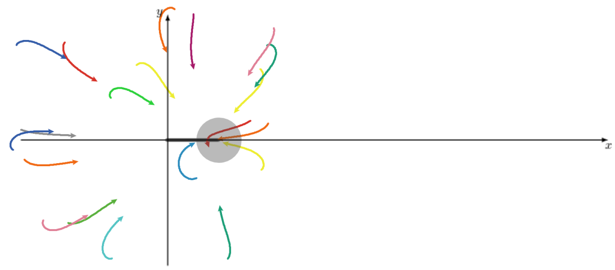
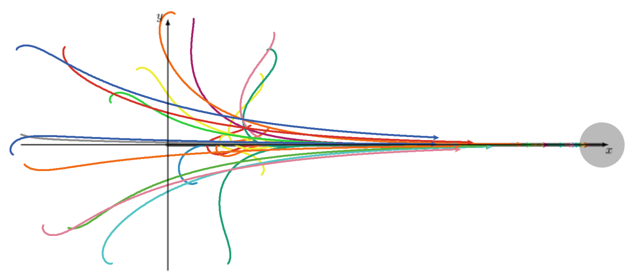


(a) The agents turn to face the target.

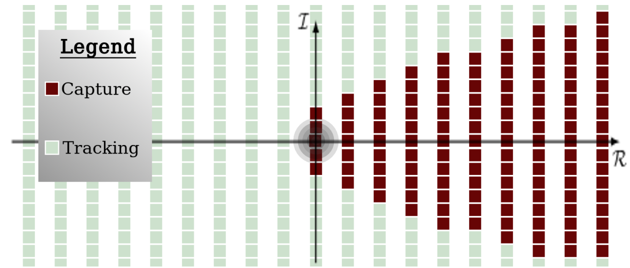


(b) Some agents capture the target.

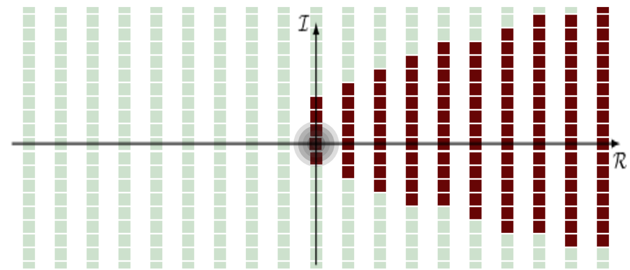


(c) Agents that fell behind the target asymptotically align their velocities with that of the target.

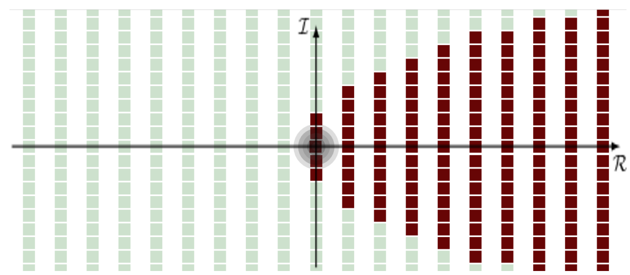
Fig. 2. Simulation results showing that with $\kappa = \frac{2v}{r_c}$, all agents either capture the target or track it.



(a) $\beta(t_0) = 0$.



(b) $\beta(t_0) = \frac{\pi}{2}$.



(c) $\beta(t_0) = \pi$.

Fig. 3. Capture Regions. The markers are drawn in the initial locations $p_{at}(t_0) = r(t_0)e^{i(\alpha(t_0)-\pi)} = -p_{ta}(t_0)$ of each of the agents participating in the experiment. Dark markers represent initial conditions that resulted in capture of the target, while light markers represent initial conditions that resulted in tracking of the target instead. The target's initial location is in the center of the arena.

moving in a constant velocity, and the pursuer is a unicycle with the same constant speed v. The pursuer's steering is governed by $\omega = \kappa\beta$, where β is the bearing angle towards the target, and κ is the gain amplifying it. We assume a capture radius r_c which represents the physical dimensions of the target or pursuer or both, and show that if $\kappa > \frac{2v}{r_c}$, then any initial configuration of target and pursuer results in either capture or tracking with a finite maximal distance between target and pursuer. We demonstrated these behaviors in simulation, and found indications that an analysis of guaranteed capture regions is within our reach. Future work involves employing the method described here on other controllers and models used for target interception and tracking to generate predictions in terms of the initial conditions for the time required for these models to attain capture.

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