Chapter Review

Principles of Problem Solving

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

1Understand the Problem

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?

What are the given quantities?

What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2Think of a Plan

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x-5=7$, we suppose that x is a number that satisfies $3x-5=7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x=4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q, we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n, it is frequently helpful to use the following principle.

Principle of Mathematical Induction

Let S_n be a statement about the positive integer n. Suppose that

1. S_1 is true.
2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n.

This is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3Carry out the Plan

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4Look Back

Having completed our solution, it is wise to look back over it, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, “Every problem that I solved became a rule which served afterwards to solve other problems.”

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these Principles of Problem Solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

Example 1

Express the hypotenuse h of a right triangle with area $25\ \mathrm{m}^2$ as a function of its perimeter P.

Solution Let’s first sort out the information by identifying the unknown quantity and the data:

PS Understand the problem

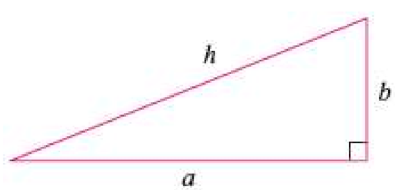
Unknown: hypotenuse h

Given quantities: perimeter P, area $25\ \mathrm{m}^2$

It helps to draw a diagram and we do so in Figure 1.

PS Draw a diagram

Figure 1



In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which are the lengths of the other two sides of the

PS Connect the given with the unknown

PS Introduce something extra

triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$25 = \frac{1}{2}ab \qquad P = a + b + h$$

Since P is given, notice that we now have three equations in the three unknowns a, b, and h:

1

$$h^2 = a^2 + b^2$$

2

$$25 = \frac{1}{2}ab$$

3

$$P = a + b + h$$

PS Relate to the familiar

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the left sides of Equation 1, Equation 2, and Equation 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a + b)^2$ in two ways. From Equation 1 and Equation 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4(25)$$

From Equation 3 we have

$$\begin{aligned}(a+b)^2 = (P-h)^2 &= P^2 - 2Ph + h^2 \\ h^2 + 100 &= P^2 - 2Ph + h^2 \\ 2Ph &= P^2 - 100 \\ h &= \tfrac{P^2-100}{2P}\end{aligned}$$

This is the required expression for h as a function of P.

As the next example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

Example 2

Solve the inequality $|x-3|+|x+2|<11$.

Solution Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$\begin{aligned}|x-3| &= \begin{cases} x-3 & \text{if } x-3 \geqslant 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} \\ &= \begin{cases} x-3 & \text{if } x \geqslant 3 \\ -x+3 & \text{if } x < 3 \end{cases}\end{aligned}$$

Similarly

$$\begin{aligned}|x+2| &= \begin{cases} x+2 & \text{if } x+2 \geqslant 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases} \\ &= \begin{cases} x+2 & \text{if } x \geqslant -2 \\ -x-2 & \text{if } x < -2 \end{cases}\end{aligned}$$

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leqslant x < 3 \qquad x \geqslant 3$$

PS Take cases

Case I If $x < -2$, we have

$$\begin{aligned}|x-3|+|x+2| &< 11 \\ -x+3-x-2 &< 11 \\ -2x &< 10 \\ x &> -5\end{aligned}$$

Case II If $-2 \leqslant x < 3$, the given inequality becomes

$$\begin{aligned}-x+3+x+2 &< 11 \\ 5 &< 11 \qquad \text{(always true)}\end{aligned}$$

Case III If $x \geqslant 3$, the inequality becomes

$$x - 3 + x + 2 < 11$$

$$2x < 12$$

$$x < 6$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$.

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove our conjecture by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps:

Step 1

Prove that S_n is true when $n = 1$.

Step 2

Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

Step 3

Conclude that S_n is true for all n by the Principle of Mathematical Induction.

Example 3

If $f_0(x) = x/(x + 1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

Solution We start by finding formulas for $f_n(x)$ for the special cases $n = 1, 2,$ and 3.

PS Analogy: Try a similar, simpler problem

PS Look for a pattern

$$\begin{aligned} f_1(x) &= (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right) \\ &= \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{2x+1} \\ f_2(x) &= (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right) \\ &= \frac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \frac{\frac{x}{2x+1}}{\frac{3x+1}{2x+1}} = \frac{x}{3x+1} \\ f_3(x) &= (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right) \\ &= \frac{\frac{x}{3x+1}}{\frac{x}{3x+1}+1} = \frac{\frac{x}{3x+1}}{\frac{4x+1}{3x+1}} = \frac{x}{4x+1} \end{aligned}$$

We notice a pattern: The coefficient of x in the denominator of $f_n(x)$ is $n+1$ in the three cases we have computed. So we make the guess that, in general,

4

$$f_n(x) = \frac{x}{(n+1)x+1}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that (4) is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$f_k(x) = \frac{x}{(k+1)x+1}$$

Then

$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$$

$$= \frac{\frac{x}{(k+1)x+1}}{\frac{x}{(k+1)x+1}+1} = \frac{\frac{x}{(k+1)x+1}}{\frac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that (4) is true for $n = k+1$. Therefore, by mathematical induction, it is true for all positive integers n.

Problems

1. One of the legs of a right triangle has length 4 cm. Express the length of the altitude perpendicular to the hypotenuse as a function of the length of the hypotenuse.
2. The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.
3. Solve the equation $|2x-1| - |x+5| = 3$.
4. Solve the inequality $|x-1| - |x-3| \geqslant 5$.

5. Sketch the graph of the function $f(x) = \left|x^2 - 4\left|x\right| + 3\right|$.

6. Sketch the graph of the function $g(x) = \left|x^2 - 1\right| - \left|x^2 - 4\right|$.

7. Draw the graph of the equation $x + |x| = y + |y|$.

8. Sketch the region in the plane consisting of all points (x, y) such that

$$|x - y| + |x| - |y| \leqslant 2$$

9. The notation $\max\{a, b, \ldots\}$ means the largest of the numbers $a, b, \ldots$. Sketch the graph of each function.

   a. $f(x) = \max\{x, 1/x\}$

   b. $f(x) = \max\{\sin x, \cos x\}$

   c. $f(x) = \max\{x^2, 2 + x, 2 - x\}$

10. Sketch the region in the plane defined by each of the following equations or inequalities.

    a. $\max\{x, 2y\} = 1$

    b. $-1 \leqslant \max\{x, 2y\} \leqslant 1$

    c. $\max\{x, y^2\} = 1$

11. Evaluate $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32)$.

12. a. Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.

    b. Find the inverse function of f.

13. Solve the inequality $\ln(x^2 - 2x - 2) \leqslant 0$.

14. Use indirect reasoning to prove that $\log_2 5$ is an irrational number.

15. A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of $30\ \mathrm{mi/h}$; she drives the second half at $60\ \mathrm{mi/h}$. What is her average speed on this trip?

16. Is it true that $f \circ (g + h) = f \circ g + f \circ h$?

17. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.

18. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

19. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

20. a. If $f_0(x) = \dfrac{1}{2 - x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.

b. Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.