Natural Logarithms

Of all possible bases b for logarithms, we will see in Chapter 3 that the most convenient choice of a base is the number e, which was defined in Section 1.4. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

Note

Notation for Logarithms

Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

If we put $b = e$ and replace $\log_e$ with "ln" in (6) and (7), then the defining properties of the natural logarithm function become

8

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

9

$$\begin{aligned} \ln(e^x) &= x \qquad x \in \mathbb{R} \\ e^{\ln x} &= x \qquad x > 0 \end{aligned}$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

Example 7

Find x if $\ln x = 5$.

Solution 1 From (8) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

Solution 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (9) says that $e^{\ln x} = x$. Therefore $x = e^5$.

Example 8

Solve the equation $e^{5-3x} = 10$.

Solution We take natural logarithms of both sides of the equation and use (9):

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \frac{1}{3}(5 - \ln 10) \end{aligned}$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution: to four decimal places, $x \approx 0.8991$.

Example 9

Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

Solution Using Laws 3 and 1 of logarithms, we have

$$\begin{aligned} \ln a + \frac{1}{2}\ln b &= \ln a + \ln b^{1/2} \\ &= \ln a + \ln \sqrt{b} \\ &= \ln(a\sqrt{b}) \end{aligned}$$

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

10 Change of Base Formula

For any positive number b $(b \neq 1)$, we have

$$\log_b x = \frac{\ln x}{\ln b}$$

Proof

Let $y = \log_b x$. Then, from (6), we have $b^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln b = \ln x$. Therefore

$$y = \frac{\ln x}{\ln b}$$

Scientific calculators have a key for natural logarithms, so Formula 10 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 10 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 43 and 44).

Example 10

Evaluate $\log_8 5$ correct to six decimal places.

Solution Formula 10 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$