The Number e

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base b is influenced by the way the graph of $y = b^x$ crosses the y-axis. Figures 13 and 14 show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point $(0, 1)$. (Tangent lines will be defined precisely in Section 2.7. For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.) If we measure the slopes of these tangent lines at $(0, 1)$, we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

Figure 13

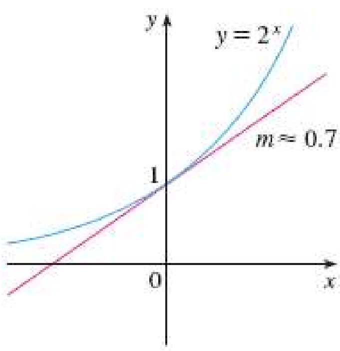
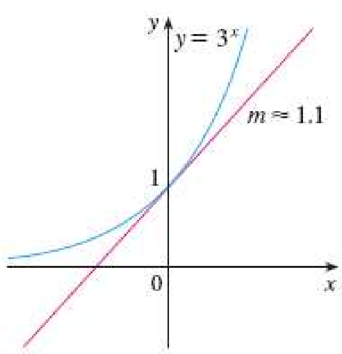


Figure 14



It turns out, as we will see in Chapter 3, that some of the formulas of calculus will be greatly simplified if we choose the base b so that the slope of the tangent line to $y = b^x$ at $(0, 1)$ is *exactly* 1. (See Figure 15.) In fact, there is such a number and it is denoted by the letter e. (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.) In view of Figures 13 and 14, it comes as no surprise that the number e lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. (See Figure 16.) In Chapter 3 we will see that the value of e, correct to five decimal places, is

$$e \approx 2.71828$$

Figure 15

The natural exponential function crosses the y-axis with a slope of 1.

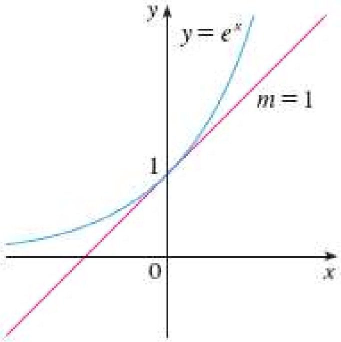
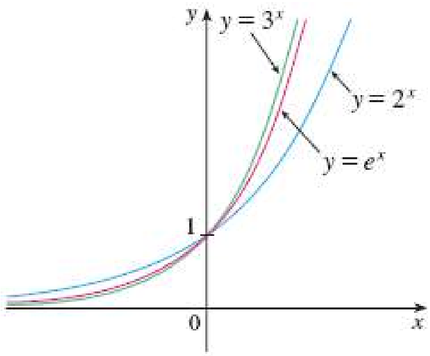


Figure 16



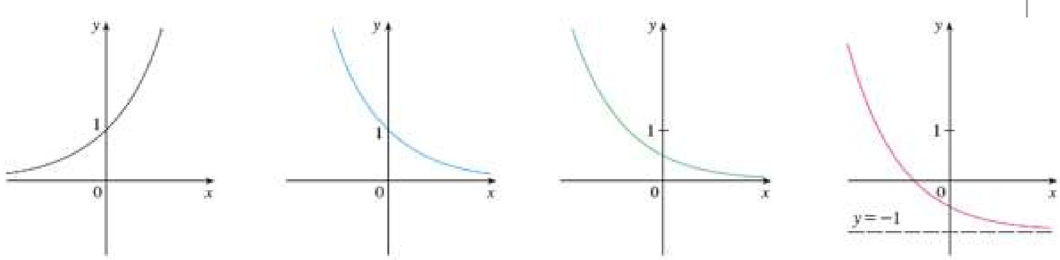
We call the function $f(x) = e^x$ the **natural exponential function**.

Example 4

Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

Solution We start with the graph of $y = e^x$ from Figures 15 and 17(a) and reflect about the y-axis to get the graph of $y = e^{-x}$ in Figure 17(b). (Notice that the graph crosses the y-axis with a slope of -1). Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure 17(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 17(d). The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.

Figure 17



(a) $y = e^x$ (b) $y = e^{-x}$ (c) $y = \frac{1}{2}e^{-x}$ (d) $y = \frac{1}{2}e^{-x} - 1$

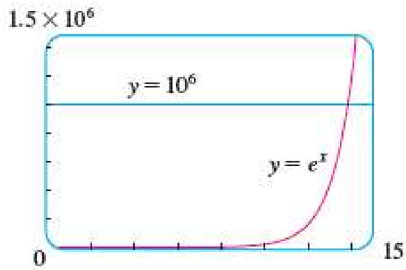
How far to the right do you think we would have to go for the height of the graph of $y = e^x$ to exceed a million? The next example demonstrates the rapid growth of this function by providing an answer that might surprise you.

Example 5

Use a graphing device to find the values of x for which $e^x > 1{,}000{,}000$.

Solution In Figure 18 we graph both the function $y = e^x$ and the horizontal line $y = 1{,}000{,}000$. We see that these curves intersect when $x \approx 13.8$. Thus $e^x > 10^6$ when $x > 13.8$. It is perhaps surprising that the values of the exponential function have already surpassed a million when x is only 14.

Figure 18



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