Math 231 – HW 7.5 Name: KEY

Remember -- FORMAT is as important as CONTENT – get them both right!

3.3 11, 29 3.6 10, 23, 6.

Warm-up Questions -- Negations of Quantified Statements.

The original quantified statement, in words:	*The negation of the original statement, in words:*
Any rational number is also a real number.	There is a rational number that is not real.
The original statement, in symbols:	*The negation of the original statement, in symbols:*
$\forall x \in \mathbb{Q},\ x \in \mathbb{R}$	$\exists x \in \mathbb{Q}$ s.t. $x \notin \mathbb{R}$

The original quantified statement, in words:	*The negation of the original statement, in words:*
There is a rational number that is also an integer.	No rational number is an integer.
The original statement, in symbols:	*The negation of the original statement, in symbols:*
$\exists x \in \mathbb{Q}$ s.t. $x \in \mathbb{Z}$	$\forall x \in \mathbb{Q},\ x \notin \mathbb{Z}$

3.3 (11) Do a formal proof of the theorem:

Theorem: If n = 4k+1 (for some integer k), then n^2-1 is divisible by 8.

Proof: Let $n = 4k+1$ for some integer k.

Then $n^2 - 1 = (4k+1)^2 - 1$

$= 16k^2 + 8k + 1 - 1$

$= 16k^2 + 8k$

$= 8(2k^2 + k)$

$= 8(\text{integer})$ ← bcs products and sums of integers are integers.

So, n^2-1 is divisible by 8.

$\square$.

3.3 (29) Use the sieve of Eratosthenes to find all the prime numbers less than 100. (read the rest of the problem in the book!)

The Sieve of Eratosthenes

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Here's the list of prime numbers less than 100:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29,

31, 37, 41, 43, 47, 53, 59, 61, 67,

71, 73, 79, 83, 89, 97

3.6 (10) Prove the theorem in two ways -- by contraposition and by contradiction.

Theorem: If the square of an integer is odd, then the original integer is odd.

By contraposition:

Exploration: Rewrite the theorem as an if-then conditional:

$\forall n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

Write the contraposition of your if-then statement:

$\forall n \in \mathbb{Z}$, if n is even, then n^2 is even.

Now, let's do the proof by contraposition:

Theorem: If the square of an integer is odd, then the original integer is odd.

Proof: It is sufficient to show that: if n is even, then n^2 is even
(insert your contrapositive statement).

Since n is even, $n = 2k$ for some $k \in \mathbb{Z}$.

$$\Rightarrow n^2 = (2k)^2$$
$$= 4k^2$$
$$= 2(2k^2)$$
$$= 2(\text{integer})$$ ← bcs products of integers are integers.
$$= \text{even}. \quad \square.$$

By contradiction:

Exploration: Write the negation of the theorem:

$\exists n \in \mathbb{Z}$ s.t. n^2 is odd but n is even.
OR: There is some even integer whose square is odd.

Now, let's do the proof by contradiction:

Theorem: If the square of an integer is odd, then the original integer is odd.

Proof: Suppose not. In other words there exists an even integer whose square is odd.
(insert your negation of the theorem).

Let n be our even integer.

Since n is even, $n = 2k$ for some $k \in \mathbb{Z}$

Then $n^2 = (2k)^2$

$$= 4k^2$$
$$= 2(2k^2)$$
$$= 2(\text{integer})$$ bcs products of integers are integers.
$$= \text{even}.$$

But we assumed that n^2 was odd. $\Rightarrow\Leftarrow$

$\square$.

3.6 (23) Prove the theorem in two ways -- by contraposition and by contradiction.

> Theorem: If r is any non-zero rational number, and s is any irrational number, then $\frac{r}{s}$ is irrational.

By contraposition:

Exploration: Rewrite the theorem as an if-then conditional:

$\forall r \in \mathbb{Q}$ and $r \neq 0$, if s is irrational, then $\frac{r}{s}$ is irrational.

Write the contraposition of your if-then statement:

$\forall r \in \mathbb{Q}$ and $r \neq 0$, if $\frac{r}{s}$ is rational, then s is rational.

Now, let's do the proof by contraposition:

Theorem: If r is any non-zero rational number, and s is any irrational number, then $\frac{r}{s}$ is irrational.

Proof: It is sufficient to show that: if $\frac{r}{s}$ is rational, then s is rational. (insert your contrapositive statement).

r and $\frac{r}{s}$ are non-zero rationals, so...

$r = \frac{a}{b}$ and $\frac{r}{s} = \frac{c}{d}$ where $a,b,c,d \in \mathbb{Z}^{\text{non-zero}}$.

Also, $s = r \div \left(\frac{r}{s}\right) = \frac{a}{b} \div \left(\frac{c}{d}\right) = \frac{a}{b}\left(\frac{d}{c}\right) = \frac{ad}{bc} = \frac{\text{integer}}{\text{non-zero integer}}$

(bcs products of non-zero integers are non-zero integers)

So, s = rational. $\square$

By contradiction:

Exploration: Write the negation of the theorem:

$\forall r \in \mathbb{Q}^{\text{non-zero}}$, $\exists$ an irrational number s such that $\frac{r}{s}$ is rational.

Now, let's do the proof by contradiction:

Theorem: If r is any rational number, and s is any irrational number, then $\frac{r}{s}$ is irrational.

Proof: Suppose not. In other words $\forall r \in \mathbb{Q}^{\text{non-zero}}$, $\exists$ irrational s s.t. (insert your negation of the theorem). $\frac{r}{s}$ is rational.

r and $\frac{r}{s}$ are non-zero rationals, so...

$r = \frac{a}{b}$ and $\frac{r}{s} = \frac{c}{d}$ where $a,b,c,d \in \mathbb{Z}^{\text{non-zero}}$

Also $s = r \div \left(\frac{r}{s}\right) = \frac{a}{b} \div \left(\frac{c}{d}\right) = \frac{ad}{bc} = \frac{\text{integer}}{\text{non-zero integer}}$

(bcs products of non-zero integers are non-zero integers)

So s is rational, but we assumed it was irrational.

$\Rightarrow\Leftarrow$

$\square$.

3.6 (6) Finish the following proof.

Theorem: The difference of any rational and any irrational number is irrational.

Proof: Suppose not. In other words, suppose that there is a rational number x and an irrational number y such that $z = x - y$ is rational.

Extra question: Rewrite the above statement using symbols instead of words.

$\exists x, y \in \mathbb{R}$ s.t. $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ and $z = x - y$ is rational

OR? if $z = x - y$, then $z \in \mathbb{Q}$.

Since x is rational, $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$.

Since z is rational, $z = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ and $d \neq 0$.

then $z = x - y$

implies $\frac{c}{d} = \frac{a}{b} - y$

implies $y = \frac{a}{b} - \frac{c}{d}$

$= \frac{ad - cb}{bd}$

$= \frac{\text{integer}}{\text{non-zero integer}}$ bcs products and sums of integers are integers and products of non-zero integers are non-zero.

= a rational number.

But, we assumed that y was irrational!

Contradiction.

$\square$.