Math 231 – HW 7 **Name:** KEY

Remember -- FORMAT is as important as CONTENT – get them both right!

3.4 18, 25 3.6 1, 2, 9, 10, 6.

Warm-up Questions -- Negations of Quantified Statements.

The original quantified statement, in words:	*The negation of the original statement, in words:*
All dinosaurs like movies.	There is a dinosaur who doesn't like movies.
The original statement, in symbols:	*The negation of the original statement, in symbols:*
$\forall d \in$ dinosaurs, d likes movies.	$\exists d \in$ dinosaurs, s.t. d doesn't like movies.

The original quantified statement, in words:	*The negation of the original statement, in words:*
There is a dinosaur who is purple.	No dinosaur is purple.
The original statement, in symbols:	*The negation of the original statement, in symbols:*
$\exists d \in$ dinosaurs s.t. d is purple.	$\forall d \in$ dinosaurs, d is not purple.

The original quantified statement, in words:	*The negation of the original statement, in words:*
The square of any real number is positive.	There exists a real number whose square is not positive.
The original statement, in symbols:	*The negation of the original statement, in symbols:*
$\forall x \in \mathbb{R}, x^2 > 0$.	$\exists x \in \mathbb{R}$ s.t. $x^2 \leq 0$.

The original quantified statement, in words:	*The negation of the original statement, in words:*
There is a real number that can be expressed as a fraction (of integers!)	No real number can be expressed as a fraction. OR: All real numbers cannot be expressed as fractions.
The original statement, in symbols:	*The negation of the original statement, in symbols:*
$\exists x \in \mathbb{R}$ s.t. $x \in \mathbb{Q}$.	$\forall x \in \mathbb{R}, x \notin \mathbb{Q}$.

3.4

(18) Give a formal proof of the theorem:

Theorem: The product of any two consecutive integers is even.

Proof: Given any two consecutive integers n and n+1, either n is even or n is odd.

Case 1: Assume n is even.

Therefore n = 2k, and n+1 = 2k+1, for some integer k.

So, n(n+1) = $(2k)(2k+1)$

$= 2(k(2k+1))$

$= 2(\text{integer})$ bcs products and sums of integers are integers.

$=$ even.

Case 2: Assume n is odd.

Therefore $n = 2k+1$ for some integer k.

Therefore $n+1 = (2k+1)+1 = 2k+2$.

So, $n(n+1) = (2k+1)(2k+2)$

$= (2k+1)(2)(k+1)$

$= 2((2k+1)(k+1))$

$= 2(\text{integer})$ bcs products and sums of integers are integers.

$=$ even.

$\square$.

3.4

(25) Give a formal proof of the theorem:

4m or 4m+1 $\quad$ m ← Typo!

Theorem: The square of any integer has the form ~~4k or 4k+1~~ for some integer ~~k.~~

Proof: For any integer n, either n is even or n is odd.

Case 1: Suppose n is even.

Therefore n = 2k, for some integer k.

So, $n^2 = (2k)^2$

$= 4k^2$

$= 4(\text{integer})$ bcs products of integers are integers.

$= 4m$ for some $m \in \mathbb{Z}$

this line is optional, but nice!

Case 2: Suppose n is odd.

therefore $n = 2k+1$, for some integer k.

So, $n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 4(k^2+k) + 1$

$= 4(\text{integer}) + 1$ bcs products and sums of integers are integers.

$= 4m + 1$ for some $m \in \mathbb{Z}$.

this line is optional, but nice!

$\square$.

3.6 (1) ***First, let's explore the idea that " There is no positive real number that is smaller than all other positive real numbers."***

Pick any really small positive real number: ____________________

Could it possibly be the smallest positive real number? Let's see:

Take half of your number: __________

Is it still positive? __________

Is it smaller than your original number? ____________

Ok, let's try again -- maybe your new number is the smallest possible real number. Let's check:

Take half of your new number: __________

Is it still positive? __________

Is it smaller than your original number? ____________

Now, explain why you believe that there is no smallest possible real number:

fill this in!

Let's do the formal proof:

Theorem: There is no positive real number that is smaller than all other positive real numbers.

Proof: Suppose not. In other words, assume that there is some real number x such that x is positive and for all positive real numbers y, $x < y$.

Extra question: Rewrite the above statement using symbols instead of words:

$\exists x \in \mathbb{R}^{pos}$ s.t. $\forall y \in \mathbb{R}^{pos}$, $x < y$.

Consider the number $y = \frac{x}{2}$.

We know that $y = \frac{x}{2}$ is positive because x was positive.

We know that $x > \frac{x}{2}$ because half of any positive number is less than the original number.

So, now we know that $x > y$.

But wait! We assumed that x was smaller than all positive real numbers, and now we've shown that y is positive and smaller than x.

Contradiction.

Therefore, there is no smallest positive real number. □.

3.6 (2) Finish the following proof that there is no greatest (largest) positive even integer.

Theorem: There is no greatest even integer.

Proof: Suppose not. In other words, suppose that there is some positive, even integer n such that n is greater than all even integers.

> *Extra question: Rewrite the above statement using symbols instead of words. Use the variable m for your extra variable.*
>
> $\exists n \in \mathbb{Z}^{\text{even}}$ s.t. $\forall m \in \mathbb{Z}^{\text{even}}$, $n > m$.

Consider the number $m = 2n$.

We know that m is positive because n was positive.

We know that m is an integer because the product of integers is an integer.

We know that m is even because it equals twice an integer.

We know that $2n > n$ because twice any positive number is bigger than the original number.

So, now we know that $m > n$.

But wait! We assumed that n was greater than all positive even integers, and now we've shown that m is an even integer and greater than n.

Contradiction.

Therefore, there is no greatest positive even. integer.

□.

3.6 (9) Prove the theorem in two ways -- by contraposition and by contradiction.

Theorem: The negative of any irrational number is irrational.

By contraposition:

Exploration: Rewrite the theorem as an if-then conditional:

$\forall$ x$\in\mathbb{R}$, if x is irrational, then $-x$ is irrational.

Write the contraposition of your if-then statement:

$\forall$ x$\in\mathbb{R}$, if $-x$ is rational, then x is rational.

Now, let's do the proof by contraposition:

Theorem: The negative of any irrational number is irrational.

Proof: It is sufficient to show that: $\forall x \in \mathbb{R}$, if $-x$ is rational, then x is rational.
(insert your contrapositive statement).

Since $-x$ is rational, $-x = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$.

Therefore $x = \frac{-a}{b}$

$= \frac{\text{integer}}{\text{non-zero integer}}$ bcs products of integers are integers.

$=$ rational. $\square$.

By contradiction:

Exploration: Write the negation of the theorem:

TYPO! → ~~Suppose~~ There is some irrational number whose negative is rational.

Now, let's do the proof by contradiction:

Theorem: The negative of any irrational number is irrational.

Proof: Suppose not. In other words there is some irrational number
(insert your negation of the theorem).
whose negative is rational.

Let x be our irrational number.

But $-x$ is rational, so $-x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$.

Therefore $x = \frac{-a}{b}$

$= \frac{\text{integer}}{\text{non-zero integer}}$ bcs products of integers are integers.

$=$ rational.

But we assumed that x was irrational! Contradiction.

$\square$.

3.6 (10) Prove the theorem in two ways -- by contraposition and by contradiction.

Theorem: If the square of an integer is odd, then the original integer is odd.

By contraposition:

Exploration: Rewrite the theorem as an if-then conditional:

$\forall\, n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

Write the contraposition of your if-then statement:

$\forall\, n \in \mathbb{Z}$, if n is even (not odd), then n^2 is even (not odd).

Now, let's do the proof by contraposition:

Theorem: If the square of an integer is odd, then the original integer is odd.

Proof: It is sufficient to show that: $\forall n \in \mathbb{Z}$, if n is even, then n^2 is even.
(insert your contrapositive statement).

Since n is even, $n = 2k$ for some integer k.

Therefore, $n^2 = (2k)^2$
$= 4k^2$
$= 2(2k^2)$
$= 2(\text{integer})$ bcs products of integers are integers.
$= \text{even}.$

$\square$.

By contradiction:

Exploration: Write the negation of the theorem:

Suppose there is some even integer whose square is odd.

Now, let's do the proof by contradiction:

Theorem: If the square of an integer is odd, then the original integer is odd.

Proof: Suppose not. In other words there exists some even integer whose
(insert your negation of the theorem).
square is odd.

Let n be our even integer.
Since n is even, $n = 2k$ for some integer k.

Then, $n^2 = (2k)^2$
$= 4k^2$
$= 2(2k^2)$
$= 2(\text{integer})$ bcs products of integers are integers.
$= \text{even}.$

But we assumed that n^2 was odd! Contradiction.

$\square$.

3.6 (6) Finish the following proof.

Theorem: The difference of any rational and any irrational number is irrational.

Proof: Suppose not. In other words, suppose that there is a rational number x and an irrational number y such that $z = x - y$ is rational.

Extra question: Rewrite the above statement using symbols instead of words.

$\exists\, x, y \in \mathbb{R}$ s.t. $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ and $z = x - y$ is rational.

OR? if $z = x - y$, then $z \in \mathbb{Q}$.

Since x is rational, $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$.

Since z is rational, $z = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ and $d \neq 0$.

then $z = x - y$

implies $\frac{c}{d} = \frac{a}{b} - y$

implies $y = \frac{a}{b} - \frac{c}{d}$

$= \frac{ad - cb}{bd}$

$= \frac{\text{integer}}{\text{non-zero integer}}$ bcs products and sums of integers are integers and products of non-zero integers are non-zero.

$=$ a rational number.

But, we assumed that y was irrational!

Contradiction.

$\square$.