In general, a statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every x in D.

By the way, in ordinary language the words *in general* mean that something is usually, but not always, the case. (In general, I take the bus home, but today I drove.) In mathematics, the words *in general* are used quite differently. When they occur just after discussion of a particular example (as in the paragraph above), they are a signal that what is to follow is a generalization of some aspect of the example that always holds true.

EXERCISE SET 2.1*

1. A menagerie consists of seven brown dogs, two black dogs, six gray cats, ten black cats, five blue birds, six yellow birds, and one black bird. Determine which of the following statements are true and which are false.
   a. There is an animal in the menagerie that is red.
   b. Every animal in the menagerie is a bird or a mammal.
   c. Every animal in the menagerie is brown or gray or black.
   d. There is an animal in the menagerie that is neither a cat nor a dog.
   e. No animal in the menagerie is blue.
   f. There are a dog, a cat, and a bird in the menagerie that all have the same color.

2. Find the truth set of each predicate below.
   a. predicate: $x > 1/x$, domain: $\mathbf{R}$
   b. predicate: $n^2 \leq 30$, domain: $\mathbf{Z}$

3. Let $\mathbf{R}$ be the domain of the predicates "$x > 1$," "$x > 2$," "$|x| > 2$," and "$x^2 > 4$." Which of the following are true and which are false?
   a. $x > 2 \Rightarrow x > 1$ b. $x > 2 \Rightarrow x^2 > 4$
   c. $x^2 > 4 \Rightarrow x > 2$ d. $x^2 > 4 \Leftrightarrow |x| > 2$

Find counterexamples to show that the statements in 4–7 are false.

4. $\forall x \in \mathbf{R}, x > 1/x$.

5. $\forall a \in \mathbf{Z}, (a-1)/a$ is not an integer.

6. $\forall$ positive integers m and n, $m \cdot n \geq m + n$.

7. $\forall$ real numbers x and y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

8. Consider the following statement:

$$\forall \text{ basketball players } x, x \text{ is tall.}$$

Which of the following are equivalent ways of expressing this statement?
   a. Every basketball player is tall.
   b. Among all the basketball players, some are tall.
   c. Some of all the tall people are basketball players.
   d. Anyone who is tall is a basketball player.
   e. All people who are basketball players are tall.
   f. Anyone who is a basketball player is a tall person.

9. Consider the following statement:

$$\exists x \in \mathbf{R} \text{ such that } x^2 = 2.$$

Which of the following are equivalent ways of expressing this statement?
   a. The square of each real number is 2.
   b. Some real numbers have square 2.
   c. The number x has square 2, for some real number x.
   d. If x is a real number, then $x^2 = 2$.
   e. Some real number has square 2.
   f. There is at least one real number whose square is 2.

10. Rewrite the following statements informally in at least two different ways without using variables or the symbols $\forall$ or $\exists$:
   a. $\forall$ squares x, x is a rectangle.
   b. $\exists$ a set A such that A has 16 subsets.

11. Rewrite each of the following statements in the form "$\forall$ _____ x, _____."
   a. All dinosaurs are extinct.
   b. Every real number is positive, negative, or zero.
   c. No irrational numbers are integers.
   d. No logicians are lazy.

12. Rewrite each of the following in the form "$\exists$ _____ x such that _____":
   a. Some exercises have answers.
   b. Some real numbers are rational.

*Exercises with blue numbers or letters have solutions in Appendix B. The symbol *H* indicates that only a hint or a partial solution is given. The symbol ◆ signals that an exercise is more challenging than usual.

13. Consider the following statement:

$\forall$ integers n, if n^2 is even then n is even.

Which of the following are equivalent ways of expressing this statement?
a. All integers have even squares and are even.
b. Given any integer whose square is even, that integer is itself even.
c. For all integers, there are some whose square is even.
d. Any integer with an even square is even.
e. If the square of an integer is even, then that integer is even.
f. All even integers have even squares.

14. Rewrite the following statement informally in at least two different ways without using variables or the symbols $\forall$ or $\exists$:

$\forall$ students S, if S is in CSC 310 then S has taken MAT 140.

15. Rewrite each of the following statements in the form "$\forall$ _____, if _____ then _____."
a. All COBOL programs have at least 20 lines.
b. Any valid argument with true premises has a true conclusion.
c. The sum of any two even integers is even.
d. The product of any two odd integers is odd.

16. Rewrite each of the following statements in the two forms "$\forall x$, if _____ then _____" and "$\forall$ _____ x, _____" (without an if–then).
a. The square of any even integer is even.
b. Every computer science student needs to take assembly language programming.

17. Rewrite the following statements in the two forms "$\exists$ _____ x such that _____" and "$\exists x$ such that _____ and _____."
a. Some hatters are mad.
b. Some questions are easy.

18. Find an example in any mathematics or computer science text of a statement that is universal but is implicitly quantified. Copy the statement as it appears and rewrite it making the quantification explicit. Give a complete citation for your example including title, author, publisher, year, and page number.

19. Which of the following is a negation for "Every polynomial function is continuous"? More than one answer may be correct.
a. No polynomial function is continuous.
b. Some polynomial functions are not continuous.
c. Every polynomial function fails to be continuous.
d. There is a noncontinuous polynomial function.

In 20–23 write negations for each statement in the referenced exercise.

H20. exercise 11 H21. exercise 12

H22. exercise 15 H23. exercise 16

In each of 24–27 determine whether the proposed negation is correct. If it is not, write a correct negation.

24. *statement:* The sum of any two irrational numbers is irrational.
proposed negation: The sum of any two irrational numbers is rational.

25. *statement:* The product of any irrational number and any rational number is irrational.
proposed negation: The product of any irrational number and any rational number is rational.

26. *statement:* For all integers n, if n^2 is even then n is even.
proposed negation: For all integers n, if n^2 is even then n is not even.

27. *statement:* For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 = x_2$.
proposed negation: For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 \neq x_2$.

28. Let $D = \{-48, -14, -8, 0, 1, 3, 16, 23, 26, 32, 36\}$. Determine which of the following statements are true and which are false. Provide counterexamples for those statements that are false.
a. $\forall x \in D$, if x is odd then $x > 0$.
b. $\forall x \in D$, if x is less than 0 then x is even.
c. $\forall x \in D$, if x is even then $x \leq 0$.
d. $\forall x \in D$, if the ones digit of x is 2, then the tens digit is 3 or 4.
e. $\forall x \in D$, if the ones digit of x is 6, then the tens digit is 1 or 2.

Write negations for each of the statements in 29–36.

29. $\forall$ real numbers x, if $x > 3$ then $x^2 > 9$.

30. $\forall$ computer programs P, if P is correct then P compiles without error messages.

31. $\forall x \in \mathbf{R}$, if $x(x+1) > 0$ then $x > 0$ or $x < -1$.

32. $\forall n \in \mathbf{Z}$, if n is prime then n is odd or $n = 2$.

33. $\forall$ integers a, b, and c, if $a - b$ is even and $b - c$ is even, then $a - c$ is even.

34. $\forall$ animals x, if x is a cat then x has whiskers and x has claws.

35. If an integer is divisible by 2, then it is even.

Note that Prolog can find the solution $X = b_1$ by merely searching the original set of given facts. However, Prolog must *infer* the solution $X = g$ from the following statements:

$$\text{isabove}(g, b_1),$$
$$\text{isabove}(b_1, w_1),$$
$$\text{isabove}(X, Z) \text{ if isabove}(X, Y) \text{ and isabove}(Y, Z).$$

Write the answers Prolog would give if the following questions were added to the program above.

a. ?isabove(b_2, w_1) b. ?color(w_1, X) c. ?color(X, blue)

Solution
a. The question means "Is b_2 *above* w_1?"; so the answer is "No."
b. The question means "For what colors X is the predicate 'w_1 is colored X' true?"; so the answer is "X = white."
c. The question means "For what blocks is the predicate 'X is colored blue' true?"; so the answer is "$X = b_1$," "$X = b_2$," and "$X = b_3$." ■

EXERCISE SET 2.2

1. The following statement is true; "∀ nonzero real numbers x, ∃ a real number y such that $x \cdot y = 1$." For each x given below, find a y to make the predicate "$x \cdot y = 1$" true.
a. $x = 2$ b. $x = -1$ c. $x = 3/4$

2. The following statement is true: "∀ real numbers x, ∃ an integer n such that $n > x$."* For each x given below, find an n to make the predicate "$n > x$" true.
a. $x = 15.83$ b. $x = 10^8$ c. $x = 10^{10^{10}}$

In each of 3–8, (a) rewrite the statement in English without using the symbols ∀ or ∃ and expressing your answer as simply as possible, and (b) write a negation for the statement.

3. ∀ colors C, ∃ an animal A such that A is colored C.

4. ∃ a book b such that ∀ people p, p has read b.

5. ∀ odd integers n, ∃ an integer k such that $n = 2k + 1$.

a 6. $\forall r \in \mathbf{Q}$, ∃ integers a and b such that $r = a/b$.

7. $\forall x \in \mathbf{R}$, ∃ a real number y such that $x + y = 0$.

8. $\exists x \in \mathbf{R}$ such that for all real numbers y, $x + y = 0$.

9. Consider the statement "Everybody is older than somebody." Rewrite this statement in the form "∀ people x, ∃ ______."

*This is called the Archimedean principle because it was first formulated (in geometric terms) by the great Greek mathematician Archimedes of Syracuse, who lived from about 287 to 212 B.C.

10. Consider the statement "Somebody is older than everybody." Rewrite this statement in the form "∃ a person x such that ∀ ______."

In 11–17, (a) rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement.

11. Everybody trusts somebody.

12. Somebody trusts everybody.

a 13. Any even integer equals twice some other integer.

14. The number of rows in any truth table equals 2^n for some integer n.

15. Every action has an equal and opposite reaction.

16. There is a program that gives the correct answer to every question that is posed to it.

17. There is a prime number between every integer and its double.

For each of the statements in 18 and 19, (a) write a new statement by interchanging the symbols ∀ and ∃, and (b) state which is true: the given statement, the version with interchanged quantifiers, neither, or both.

18. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $x < y$.

19. $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}^-$ (the set of negative real numbers), $x > y$.

20. This exercise refers to Example 2.2.4. Determine whether each of the following statements is true or false.
a. ∀ students S, ∃ a dessert D such that S chose D.
b. ∀ students S, ∃ a salad T such that S chose T.

c. $\exists$ a dessert D such that $\forall$ students S, S chose D.
d. $\exists$ a beverage B such that $\forall$ students D, D chose B.
e. $\exists$ an item I such that $\forall$ students S, S did not choose I.
f. $\exists$ a station Z such that $\forall$ students S, $\exists$ an item I such that S chose I from Z.

21. How could you determine the truth or falsity of the following statements for the students in your discrete mathematics class? Assume that students will respond truthfully to questions that are asked of them.

a. There is a student in this class who has dated at least one person from every residence hall at this school.
b. There is a residence hall at this school with the property that every student in this class has dated at least one person from that residence hall.
c. Every residence hall at this school has the property that if a student from this class has dated at least one person from that hall, then that student has dated at least two people from that hall.

Give the contrapositive, converse, and inverse of each statement in 22–29.

22. $\forall x \in \mathbf{R}$, if $x > 3$ then $x^2 > 9$.

23. $\forall$ computer programs P, if P is correct then P compiles without error messages.

24. If an integer is divisible by 6, then it is divisible by 3.

25. If the square of an integer is even, then the integer is even.

26. $\forall x \in \mathbf{R}$, if $x(x+1) > 0$ then $x > 0$ or $x < -1$.

27. $\forall n \in \mathbf{Z}$, if n is prime then n is odd or $n = 2$.

28. $\forall$ integers a, b, and c, if $a - b$ is even and $b - c$ is even, then $a - c$ is even.

29. $\forall$ animals A, if A is a cat then A has whiskers and A has claws.

30. Give an example to show that a universal conditional statement is not logically equivalent to its inverse.

Rewrite each statement of 31–34 in if–then form.

31. Earning a grade of C$-$ in this course is a sufficient condition for it to count toward graduation.

32. Being divisible by 6 is a sufficient condition for being divisible by 3.

33. Being on time each day is a necessary condition for keeping this job.

34. A grade-point average of at least 3.7 is a necessary condition for graduating with honors.

Use the facts that the negation of a $\forall$ statement is a $\exists$ statement and that the negation of an if–then statement is an *and* statement to rewrite each of statements 35–38 without using the words *sufficient* or *necessary*.

35. Divisibility by 4 is not a necessary condition for divisibility by 2.

36. Having a large income is not a necessary condition for a person to be happy.

37. Having a large income is not a sufficient condition for a person to be happy.

38. Being continuous is not a sufficient condition for a function to be differentiable.

39. The following statement is from *An Introduction to Programming.** Rewrite it without using the words *necessary* or *sufficient*.

> The absence of error messages during translation of a computer program is only a necessary and not a sufficient condition for reasonable [program] correctness.

40. Find the answers Prolog would give if the following questions were added to the program given in Example 2.2.9:
a. ?isabove(b_1, w_1) b. ?isabove(w_1, g)
c. ?color(w_2, blue) **d.** ?color(X, white)
e. ?isabove(X, b_1) **f.** ?isabove(X, b_3)
g. ?isabove(g, X)

41. Write the negation of the definition of limit of a sequence given in Example 2.2.3.

42. The notation $\exists!$ stands for the words "there exists a unique." Thus, for instance, "$\exists!\ x$ such that x is prime and x is even" means that there is one and only one even prime number. Which of the following statements are true and which are false? Explain.
a. $\exists!$ real number x such that $\forall$ real numbers y, $xy = y$.
b. $\exists!$ integer x such that $1/x$ is an integer.
c. $\forall$ real numbers x, $\exists!$ real number y such that $x + y = 0$.

◆ 43. Suppose that $P(x)$ is a predicate and D is the domain of x. Rewrite the statement "$\exists!\ x \in D$ such that $P(x)$" without using the symbol $\exists!$. (See exercise 42 for the meaning of $\exists!$.)

*Richard Conway and David Gries, *An Introduction to Programming,* 2d ed. (Cambridge, Massachusetts: Winthrop, 1975), p. 224.