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Math 231 – HW 6 – Spring 2017 **Name:** KEY

Remember -- FORMAT is as important as CONTENT – get them both right!

3.2 8, 13, 17, 19, 21, 24
3.3 3, 4, 7, 14, 16

3.2

(8) The zero product property says that if a product of two real numbers is zero, then one of the numbers must be zero.

(a) Write this property formally using quantifiers and variables.

$\forall x, y \in \mathbb{R}$, if $x \cdot y = 0$, then $x = 0$ or $y = 0$ ✓

(b) Write the contrapositive of your answer to part a.

$\forall x, y \in \mathbb{R}$, if $x \neq 0$ and $y \neq 0$, then $x \cdot y \neq 0$ ✓

(c) Write the contrapositive informally – no quantifiers, no variables.

The product of any two nonzero real numbers is non-zero. ✓

(13) Give a formal proof of the statement: The product of any two rational numbers is a rational number.

Thm: The product of any two rational numbers is rational.

✓ Proof: Let x and y be rational.

Therefore, $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.

Also, $y = \frac{c}{d}$, where $c, d \in \mathbb{Z}$ and $d \neq 0$. ✓

So, the product is:

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d}$$

$$= \frac{ac}{bd}$$

$$= \frac{\text{integer}}{\text{non-zero integer}}$$

bcs products of integers are integers, and bcs product of non-zero numbers is non-zero. ✓

$= $ a rational number. ✓

$\square$.

(17) Given any two distinct rational numbers r and s, with r<s, find a rational number x such that r<x<s. You do not need to give a formal proof, but show me how x is rational.

Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$

Choose $x = \frac{r+s}{2}$ (the average of r and s)

So, $x = \dfrac{\frac{a}{b} + \frac{c}{d}}{2}$

$x = \dfrac{ad + cb}{2bd}$

$x = \dfrac{\text{integer}}{\text{non-zero integer}}$ ← bcs prod's and sums of integers are integers.

it's rational!

✓✓✓

(19) Give a formal proof that the square of any odd integer is odd.

Thm: The square of any odd integer is odd.

Proof: Let $n = 2k+1$ $(k \in \mathbb{Z})$ be any odd integer.

Then: $n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2(\text{integer}) + 1$ bcs products and sums of integers are integers.

= an odd number!

$\square$.

✓✓✓

(21) Give a formal proof that if n is an odd integer, then n^2+n is even.

Thm: if n is any odd integer, n^2+n will be even.

Prf: Let $n = 2k+1$ $(k \in \mathbb{Z})$ be any odd integer.

Then: $n^2 + n = (2k+1)^2 + (2k+1)$

$= 4k^2 + 4k + 1 + 2k + 1$

$= 4k^2 + 6k + 2$

$= 2(2k^2 + 3k + 1)$

$= 2(\text{integer})$ bcs products and sums of integers are integers.

= an even number!

$\square$.

✓✓✓

(24) Suppose a, b, c, and d are integers, and a is not equal to c. Suppose also that x is a real number that satisfies the given equation. Must x be rational? If so, express x a ratio of two integers.

$$\frac{ax+b}{cx+d}=1$$

$$\Rightarrow ax+b = cx+d$$
$$ax-cx = d-b$$
$$x(a-c) = d-b$$
$$x = \frac{d-b}{a-c}$$
$$x = \frac{\text{integer}}{\text{non-zero integer}}$$

bcs differences of integers are integers and $a \neq c$.

3.3

(3) Is (3k+1)(3k+2)(3k+3) divisible by 3, if k is an integer? Explain!

Yes! $(3k+1)(3k+2)(3k+3)$
$$= (3k+1)(3k+2)\,3(k+1)$$
$$= 3\big((3k+1)(3k+2)(k+1)\big)$$
$$= 3(\text{integer})$$
bcs products and sums of integers are integers.

(4) Is 2m(2m+4) divisible by 4, if m is an integer? Explain!

Yes! $2m(2m+4)$
$$= 2\cdot m\cdot 2\cdot(m+2)$$
$$= 4\cdot m(m+2)$$
$$= 4(\text{integer})$$
bcs prod's and sums of integers are integers.

(7) Is 6a(a+b) a multiple of 3a, if a and b are integers? Explain!

Yes! $6a(a+b)$
$$= 3\cdot 2\cdot a\cdot(a+b)$$
$$= 3a\big(2(a+b)\big)$$
$$= 3a(\text{integer})$$
bcs prod's and sums of integers are integers.

(14) Give a formal proof of the statement. For all integers a, b, and c, if a|b and a|c, then a|(b+c).

Thm: $\forall a,b,c \in \mathbb{Z}$, if $a|b$ and $a|c$, then $a|(b+c)$.

Proof: Since $a|b$, we know that $b = a \cdot k$ for some $k \in \mathbb{Z}$.
Since $a|c$, we know that $c = a \cdot k'$ for some $k' \in \mathbb{Z}$.

Then:
$$\begin{aligned} b+c &= a \cdot k + a \cdot k' \\ &= a(k+k') \\ &= a(\text{some integer}) \end{aligned}$$
bcs sums of integers are integers.

So, b+c is divisible by a!

$\square$.

(16) Give a formal proof of the statement. The sum of any three consecutive integers is divisible by 3. Use n as your first integer.

Thm: The sum of any three consecutive integers is divisible by 3.

Proof: Let our three integers be n, $n+1$, and $n+2$.

So, their sum is
$$\begin{aligned} & n + (n+1) + (n+2) \\ &= 3n+3 \\ &= 3(n+1) \\ &= 3(\text{integer}) \end{aligned}$$
bcs sums of integers are integers.

so, it's divisible by 3!

$\square$.