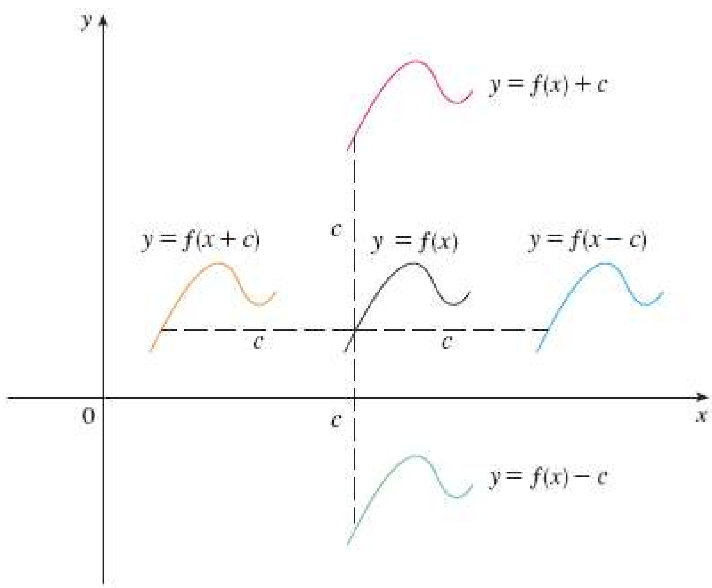
Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

Figure 1

Translating the graph of f



Vertical and Horizontal Shifts

Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

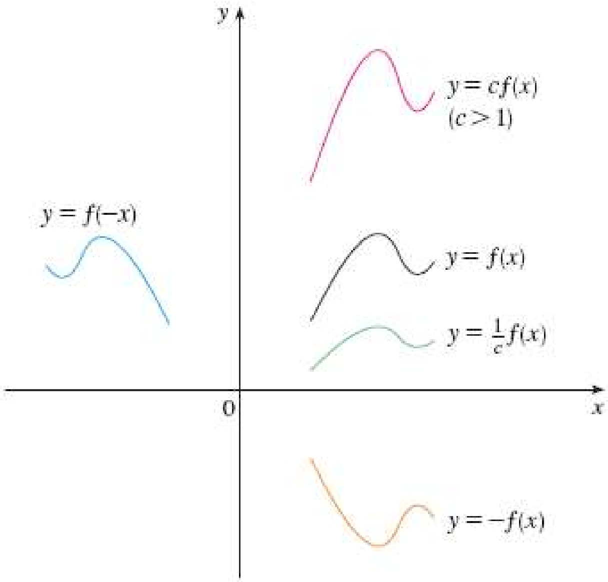
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)

Figure 2

Stretching and reflecting the graph of f



Vertical and Horizontal Stretching and Reflecting

Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

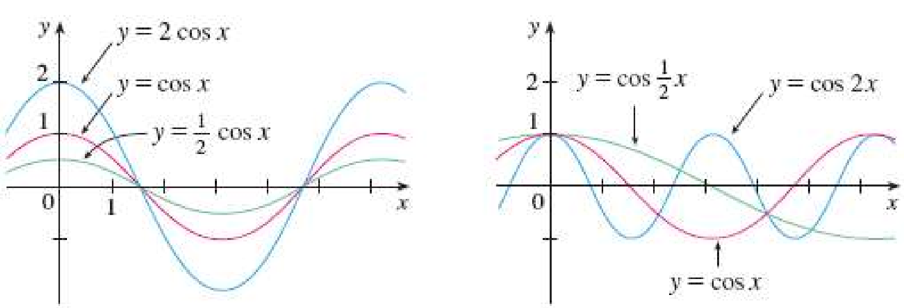
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2\cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

Figure 3

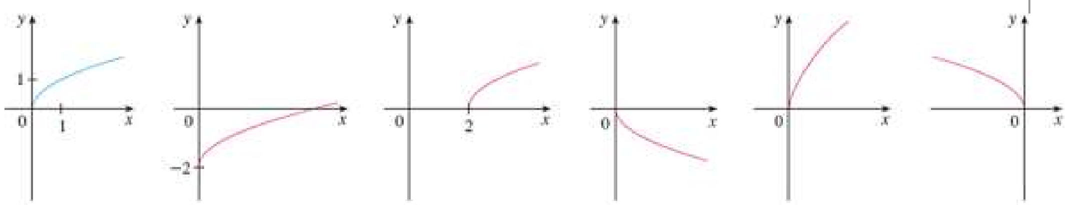


Example 1

Given the graph of $y=\sqrt{x}$, use transformations to graph $y=\sqrt{x}-2$, $y=\sqrt{x-2}$, $y=-\sqrt{x}$, $y=2\sqrt{x}$, and $y=\sqrt{-x}$.

Solution The graph of the square root function $y=\sqrt{x}$, obtained from Figure 1.2.13(a), is shown in Figure 4(a). In the other parts of the figure we sketch $y=\sqrt{x}-2$ by shifting 2 units downward, $y=\sqrt{x-2}$ by shifting 2 units to the right, $y=-\sqrt{x}$ by reflecting about the x-axis, $y=2\sqrt{x}$ by stretching vertically by a factor of 2, and $y=\sqrt{-x}$ by reflecting about the y-axis.

Figure 4



(a) $y=\sqrt{x}$ (b) $y=\sqrt{x}-2$ (c) $y=\sqrt{x-2}$ (d) $y=-\sqrt{x}$ (e) $y=2\sqrt{x}$ (f) $y=\sqrt{-x}$

Example 2

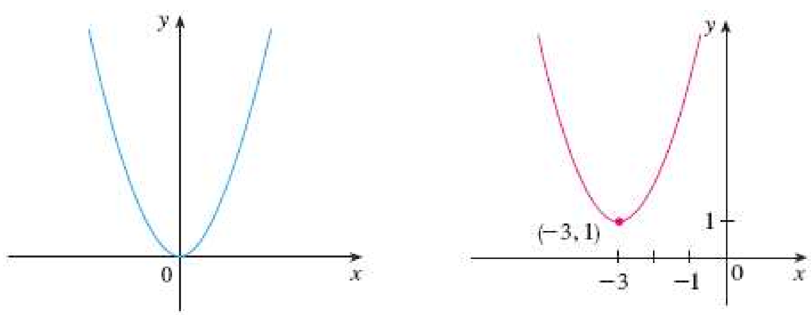
Sketch the graph of the function $f(x)=x^2+6x+10$.

Solution Completing the square, we write the equation of the graph as

$$y=x^2+6x+10=(x+3)^2+1$$

This means we obtain the desired graph by starting with the parabola $y=x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

Figure 5



(a) $y=x^2$ (b) $y=(x+3)^2+1$

Example 3

Sketch the graphs of the following functions.

(a) $y=\sin 2x$

(b) $y=1-\sin x$

Solution

(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2. (See Figures 6 and 7.) Thus, whereas the period of $y = \sin x$ is 2π, the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

Figure 6

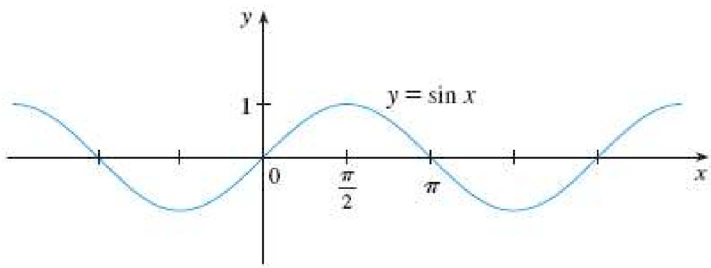
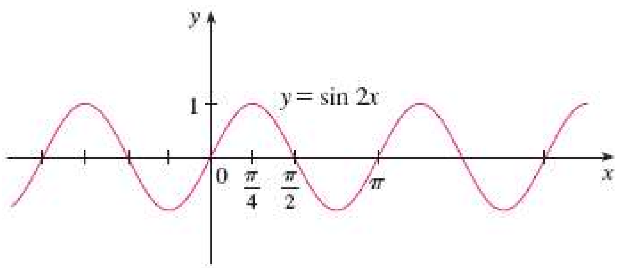
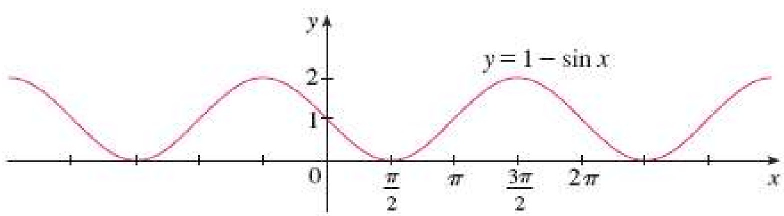


Figure 7



(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x-axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

Figure 8

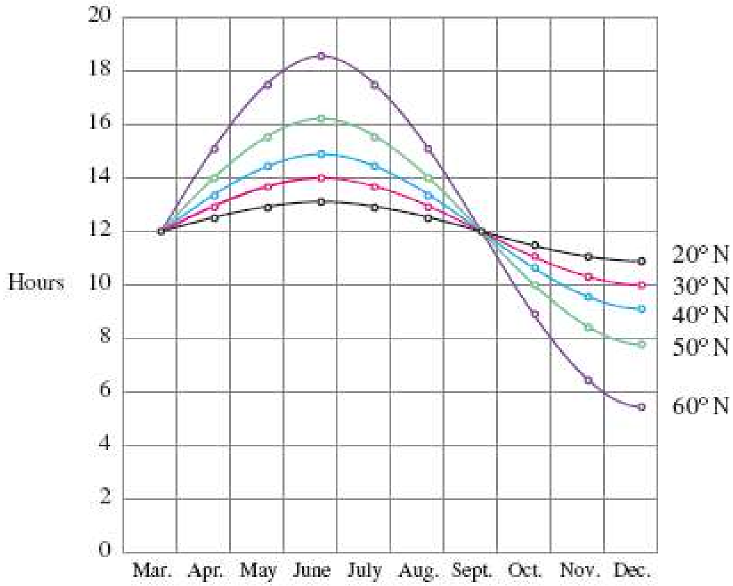


Example 4

Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately $40°\mathrm{N}$ latitude, find a function that models the length of daylight at Philadelphia.

Figure 9

Graph of the length of daylight from March 21 through December 21 at various latitudes



Source: Adapted from L. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935), 40.

Solution Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π, so the horizontal stretching factor is $2\pi/365$.

We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore we model the length of daylight in Philadelphia on the tth day of the year by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geqslant 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x-axis remains the same; the part that lies below the x-axis is reflected about the x-axis.

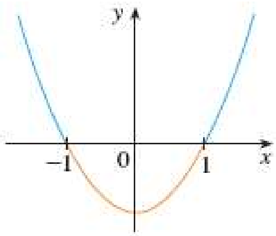
Example 5

Sketch the graph of the function $y = \left|x^2 - 1\right|$.

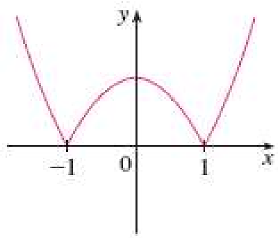
Solution We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x-axis when

$-1 < x < 1$, so we reflect that part of the graph about the x-axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b).

Figure 10



(a) $y = x^2 - 1$



(b) $y = |x^2 - 1|$

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