Inverse Trigonometric Functions

When we try to find the inverse trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 17 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x, -\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one (see Figure 18). The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

Figure 17

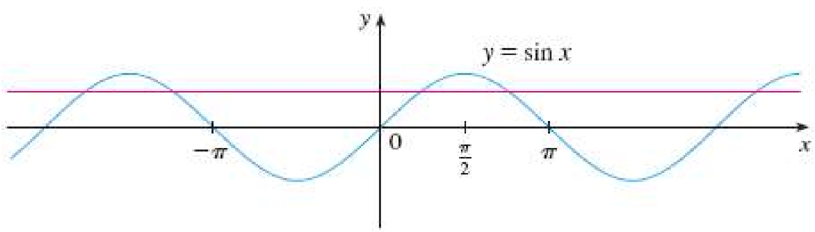
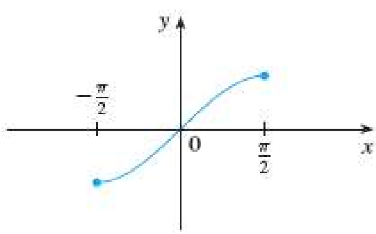


Figure 18

$y = \sin x, -\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$



Since the definition of an inverse function says that

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

we have

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Thus, if $-1 \leqslant x \leqslant 1$, $\sin^{-1} x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

Example 12

Evaluate

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ and

(b) $\tan\left(\arcsin\frac{1}{3}\right)$.

Solution

(a) We have

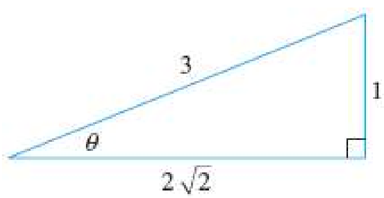
$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin\frac{1}{3}$, so $\sin\theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 19 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-1} = 2\sqrt{2}$. This enables us to read from the triangle that

$$\tan\left(\arcsin\frac{1}{3}\right) = \tan\theta = \frac{1}{2\sqrt{2}}$$

Figure 19



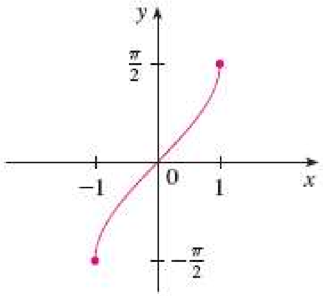
The cancellation equations for inverse functions become, in this case,

$$\begin{aligned} \sin^{-1}(\sin x) &= x \quad \text{for } -\tfrac{\pi}{2} \leqslant x \leqslant \tfrac{\pi}{2} \\ \sin(\sin^{-1} x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 20, is obtained from that of the restricted sine function (Figure 18) by reflection about the line $y = x$.

Figure 20

$y = \sin^{-1} x = \arcsin x$

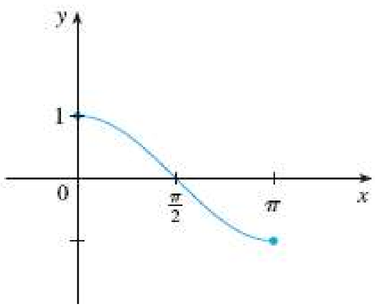


The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x, 0 \leqslant x \leqslant \pi$, is one-to-one (see Figure 21) and so it has an inverse function denoted by $\cos^{-1}$ or $\arccos$.

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

Figure 21

$y = \cos x, 0 \leqslant x \leqslant \pi$



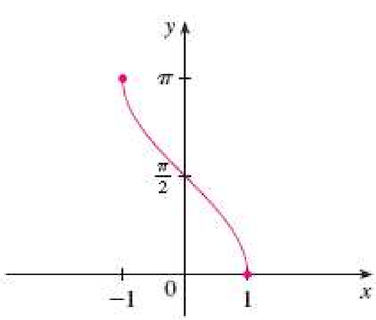
The cancellation equations are

$$\begin{aligned} \cos^{-1}(\cos x) &= x \qquad \text{for } 0 \leqslant x \leqslant \pi \\ \cos(\cos^{-1} x) &= x \qquad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$. Its graph is shown in Figure 22.

Figure 22

$y = \cos^{-1} x = \arccos x$

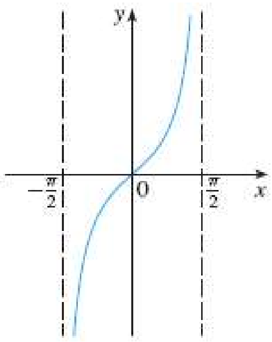


The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 23.) It is denoted by $\tan^{-1}$ or $\arctan$.

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Figure 23

$y = \tan x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$



Example 13

Simplify the expression $\cos\left(\tan^{-1} x\right)$.

Solution 1 Let $y = \tan^{-1} x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\begin{aligned} \sec^2 y &= 1 + \tan^2 y = 1 + x^2 \\ \sec y &= \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi < y < \pi/2) \end{aligned}$$

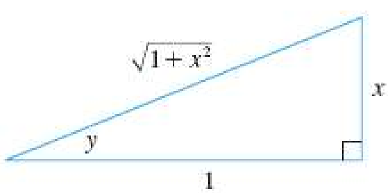
Thus

$$\cos\left(\tan^{-1} x\right) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

Solution 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1} x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$

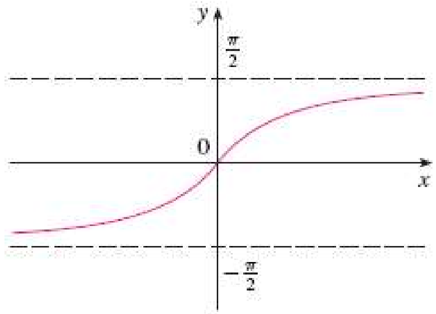
Figure 24



The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 25.

Figure 25

$y = \tan^{-1} x = \arctan x$



We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of $\tan$. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

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$$\begin{aligned}
y &= \csc^{-1} x \ (|x| \geqslant 1) & \Leftrightarrow \quad & \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2] \\
y &= \sec^{-1} x \ (|x| \geqslant 1) & \Leftrightarrow \quad & \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2) \\
y &= \cot^{-1} x \ (x \in \mathbb{R}) & \Leftrightarrow \quad & \cot y = x \quad \text{and} \quad y \in (0, \pi)
\end{aligned}$$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 26 that both this choice and the one in (11) will work.]

Figure 26

$y = \sec x$