Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth and radioactive decay. In later chapters we will pursue these and other applications in greater detail.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then we have

$$p(1) = 2p(0) = 2 \times 1000$$
$$p(2) = 2p(1) = 2^2 \times 1000$$
$$p(3) = 2p(2) = 2^3 \times 1000$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)\,2^t$$

This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figures 2 and 7. Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

Table 1

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450

t (years since 1900)	Population (millions)
90	5280
100	6080
110	6870

Figure 8

Scatter plot for world population growth

The image consists of a visual representation of a graph. A dotted curve is graphed on the x-y coordinate plane and only the first quadrant is shown here. The curve starts from the bottom of the y-axis on a point in between the points (0, 10) and (0, 20) on the y-axis and it goes up towards the right and ends at the top end of the right side of the graph. The x-axis is labeled t and is indicated as 'years since 1990' and the y-axis is labeled as P and is indicated as 5 X 10^9.

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (1436.53) \cdot (1.01395)^t$$

where $t = 0$ corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

Figure 9

Exponential model for population growth

The image consists of a visual representation of a graph. Twelve solid points are plotted on P t coordinate plane, y-axis shows years since 1900, values from left to right 0. 20, 40, 60, 80, 100, 120 respectively, P-axis shows population, values from bottom to top 0, 1*10^9, 2*10^9, 3*10^9, 4*10^9, 5*10^9, 6*10^9, 7*10^9 respectively. Plotted points are (0, 1.7*10^9), (10, 1.75*10^9), (20, 1.8*10^9), (30, 1.85*10^9), (40, 2*10^9), (50, 2*10^9), (60, 2.8*10^9), (70, 3.2*10^9), (80, 4*10^9), (90, 5*10^9), (100, 6*10^9), (110, 6.3*10^9). Here (t, P) shows, t years since 1900, the

In 1995 a paper appeared detailing the effect of the protease inhibitor ABT-538 on the human immunodeficiency virus HIV-1. * Table 2 shows values of the plasma viral load $V(t)$ of patient 303, measured in RNA copies per mL, t days after ABT-538 treatment was begun. The corresponding scatter plot is shown in Figure 10.

Table 2

t (days)	$V(t)$
1	76.0
4	53.0
8	18.0
11	9.4
15	5.2
22	3.6

Figure 10

Plasma viral load in patient 303

The image consists of a visual representation of a graph. Six solid points are plotted in t V graph, t-axis shows the values of days, values from left to right are 0, 10, 20, 30 and V-axis shows the values of RNA copies/ mL, values from bottom to top are 0, 20, 40, 60. The solid points are (1, 76.0), (4, 53.0), (8, 18.0), (11, 9.4), (15, 5.2), (22, 3.6). Here (t, V) means in t days they measured V RNA copies per ml.

The rather dramatic decline of the viral load that we see in Figure 10 reminds us of the graphs of the exponential function $y = b^x$ in Figures 3 and 4(a) for the case where the base b is less than 1. So let's model the function $V(t)$ by an exponential function. Using a graphing calculator or computer to fit the data in Table 2 with an exponential function of the form $y = a \cdot b^t$, we obtain the model

$$V = 96.39785 \cdot (0.818656)^t$$

In Figure 11 we graph this exponential function with the data points and see that the model represents the viral load reasonably well for the first month of treatment.

Figure 11

Exponential model for viral load

The image consists of a visual representation of a graph. Six solid points are plotted in t V graph, t-axis shows the values of days, values from left to right are 0, 10, 20, 30 and V-axis shows the values of RNA copies/ mL, values from bottom to top are 0, 20, 40, 60. The solid points are (1, 76.0), (4, 53.0), (8, 18.0), (11, 9.4), (15, 5.2), (22, 3.6). Here (t, V) means in t days they measured V RNA copies per ml. A curve is graphed in the graph starting from (1, 76.0) and goes down to the right with decreasing steepness connecting all twelve solid points and ends just above the point (30, 0) on the x-axis.

We could use the graph in Figure 11 to estimate the **half-life** of V, that is, the time required for the viral load to be reduced to half its initial value (see Exercise 33). In the next example we are given the half-life of a radioactive element and asked to find the mass of a sample at any time.

Example 3

The half-life of strontium-90, $^{90}\mathrm{Sr}$, is 25 years. This means that half of any given quantity of $^{90}\mathrm{Sr}$ will disintegrate in 25 years.

(a) If a sample of $^{90}\mathrm{Sr}$ has a mass of 24 mg, find an expression for the mass $m(t)$ that remains after t years.

(b) Find the mass remaining after 40 years, correct to the nearest milligram.

(c) Use a graphing device to graph $m(t)$ and use the graph to estimate the time required for the mass to be reduced to 5 mg.

Solution

(a) The mass is initially 24 mg and is halved during each 25-year period, so

$$\begin{aligned} m(0) &= 24 \\ m(25) &= \tfrac{1}{2}(24) \\ m(50) &= \tfrac{1}{2}\cdot\tfrac{1}{2}(24) = \tfrac{1}{2^2}(24) \\ m(75) &= \tfrac{1}{2}\cdot\tfrac{1}{2^2}(24) = \tfrac{1}{2^3}(24) \\ m(100) &= \tfrac{1}{2}\cdot\tfrac{1}{2^3}(24) = \tfrac{1}{2^4}(24) \end{aligned}$$

From this pattern, it appears that the mass remaining after t years is

$$m(t) = \frac{1}{2^{t/25}}(24) = 24\cdot 2^{-t/25} = 24\cdot\left(2^{-1/25}\right)^t$$

This is an exponential function with base $b = 2^{-1/25} = 1/2^{1/25}$.

(b) The mass that remains after 40 years is

$$m(40) = 24\cdot 2^{-40/25} \approx 7.9 \text{ mg}$$

(c) We use a graphing calculator or computer to graph the function $m(t) = 24\cdot 2^{-t/25}$ in Figure 12. We also graph the line $m = 5$ and use the cursor to estimate that $m(t) = 5$ when $t \approx 57$. So the mass of the sample will be reduced to 5 mg after about 57 years.

Figure 12

The image consists of a visual representation of a graph. One straight line and a curve are graphed on a closed coordinate plane. The values of horizontal axis are 0, 20, 40, 60, 80, 100 and the values vertical axis are 0, 5, 10, 15, 20, 25, 30. The blue line is labeled as m=5, and it starts from the point (0, 5) and is exactly parallel to the horizontal axis ends in the point (100, 5). The red line is labeled as m= 24*2^minus t/25 and it starts from (0, 24) goes down to the right of the viewing display, ends in just above the point (100, 0). The curve is said to be concave up. Both the line and the curve is intersected in one point in the same quadrant.