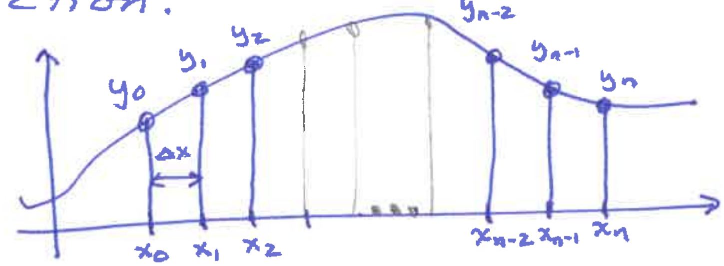
7.7 Approximate (Numerical) Integration

What to use when you can't find an anti-derivative or when you have data points but not a function.



Versions we already know:

$L_n = \Delta x(y_0 + y_1 + y_2 + \cdots + y_{n-1})$

$R_n = \Delta x(y_1 + y_2 + y_3 + \cdots + y_n)$

$M_n = \Delta x$(sum of heights at midpoints of the rectangles)

New: Trapezoid Rule

$T_n = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$ ← each trapezoid area $= \Delta x\left(\frac{y_a + y_{a+1}}{2}\right)$

also: $T_n = \frac{1}{2}(L_n + R_n)$ = the average of L_n and R_n.

New: Simpson's Rule

S_n = approximation with $\frac{n}{2}$ parabolas (so, n must be even)

$S_n = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$

or

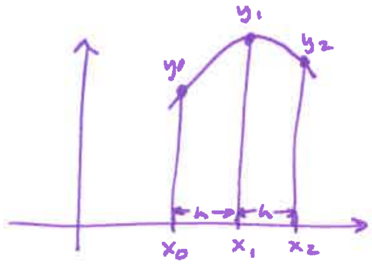
S_n = width of a parabola section (weighted average of the heights to make the parabola)

$= 2\Delta x\left(\frac{y_0 + 4y_1 + y_2}{6} + \frac{y_2 + 4y_3 + y_4}{6} + \cdots + \frac{y_{n-2} + 4y_{n-1} + y_n}{6}\right)$

How do these methods compare?

	L_n, R_n	T_n	M_n	S_n	Actual Integral
worst	iffy	good	better	even better	perfect

__Theorem__



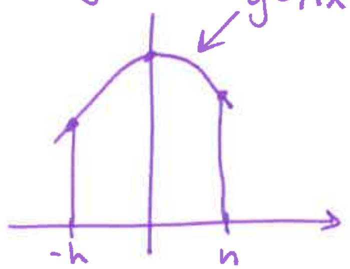
Area under parabola $= \frac{h}{3}(y_0 + 4y_1 + y_2)$

Try an example?

Wow! how are we going to prove that?

$\boxed{\text{Step 1}}$ Move over so centered at the origin:

$y = Ax^2 + Bx + C$



$$\text{Area} = \int_{-h}^{h} Ax^2 + Bx + C\,dx$$

$$= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \Big]_{-h}^{h}$$

$$= \frac{A}{3}(h)^3 + \cancel{\frac{B}{2}(h)^2} + C(h) - \left(\frac{A}{3}(-h)^3 + \cancel{\frac{B}{2}(-h)^2} + C(-h)\right)$$

$$= \frac{2}{3}Ah^3 + 2Ch$$

they're the same!

$\boxed{\text{Step 2}}$ Compare to:

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

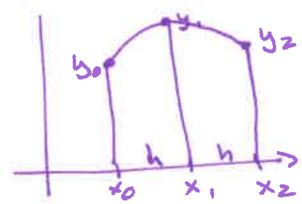
$$= \frac{h}{3}\left(A(-h)^2 + B(-h) + C + 4(C) + Ah^2 + Bh + C\right)$$

$$= \frac{h}{3}\left(Ah^2 - \cancel{Bh} + C + 4C + Ah^2 + \cancel{Bh} + C\right)$$

$$= \frac{h}{3}\left(2Ah^2 + 6C\right)$$

$$= \frac{2}{3}Ah^3 + 2Ch$$

__Conclusion:__ Area under

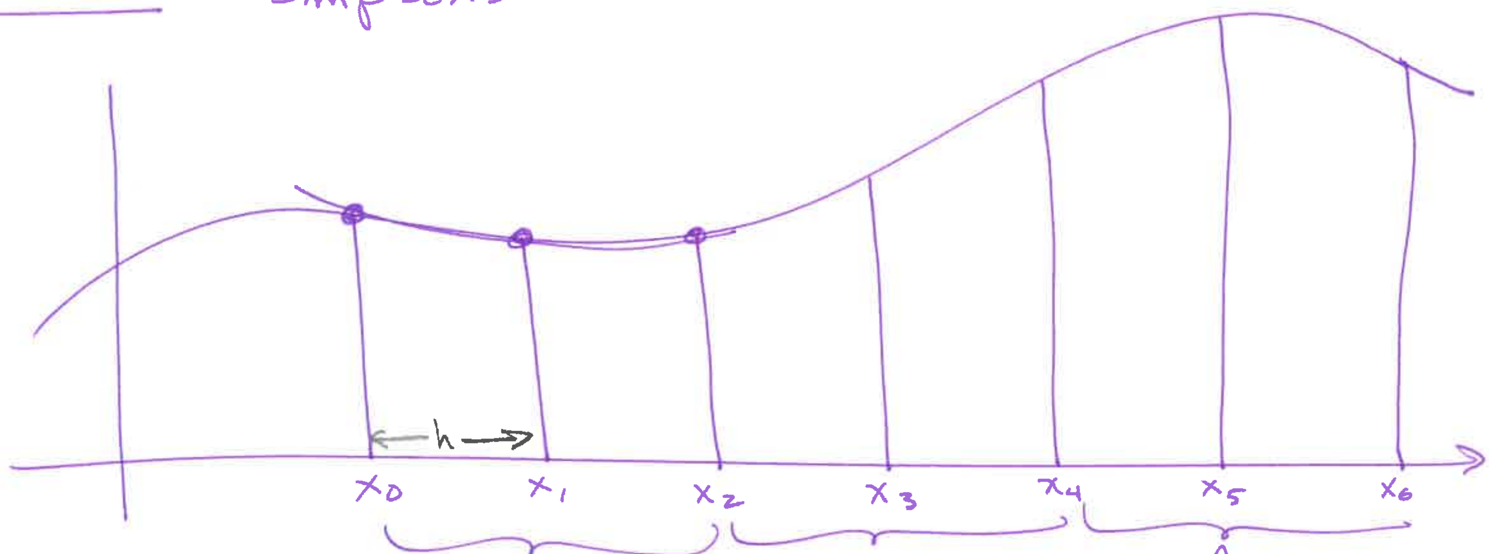


$= \frac{h}{3}(y_0 + 4y_1 + y_2)$

__OR?__ width (weighted average of heights)

$= 2h\left(\frac{y_0 + 4y_1 + y_2}{6}\right)$

<u>So what?</u> Simpson's Rule...



Area = Area = Area =

$$\frac{h}{3}(y_0+4y_1+y_2) + \frac{h}{3}(y_2+4y_3+y_4) + \frac{h}{3}(y_4+4y_5+y_6)$$

$$S_6 = \text{Total Area} = \frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+1y_6\right)$$