Logarithmic Functions

If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base b** and is denoted by $\log_b$. If we use the formulation of an inverse function given by (3),

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

then we have

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$$\log_b x = y \quad \Leftrightarrow \quad b^y = x$$

Thus, if $x > 0$, then $\log_b x$ is the exponent to which the base b must be raised to give x. For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The cancellation equations (4), when applied to the functions $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, become

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$$\begin{aligned} \log_b(b^x) &= x \qquad \text{for every } x \in \mathbb{R} \\ b^{\log_b x} &= x \qquad \text{for every } x > 0 \end{aligned}$$

The logarithmic function $\log_b$ has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = b^x$ about the line $y = x$.

Figure 11 shows the case where $b > 1$. (The most important logarithmic functions have base $b > 1$.) The fact that $y = b^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_b x$ is a very slowly increasing function for $x > 1$.

Figure 11

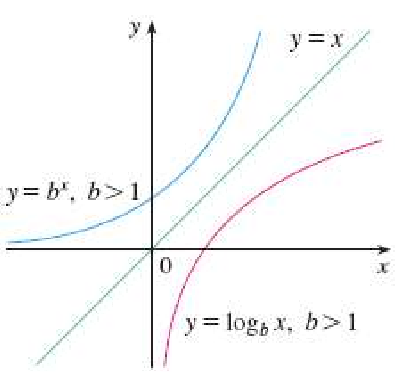
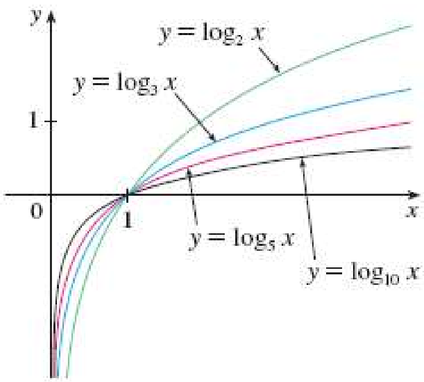


Figure 12 shows the graphs of $y = \log_b x$ with various values of the base $b > 1$. Since $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

Figure 12



The following properties of logarithmic functions follow from the corresponding properties of exponential functions given in Section 1.4.

Laws of Logarithms

If x and y are positive numbers, then

1. $\log_b (xy) = \log_b x + \log_b y$
2. $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b (x^r) = r \log_b x$ (where r is any real number)

Example 6

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Solution Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2 \left(\frac{80}{5}\right) = \log_2 16 = 4$$

because $2^4 = 16$.