ECMT3150: The Econometrics of Financial Markets

1b. Linear Time Series Analysis

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Outline

Moving-Average (MA) Model

$MA(q)$:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q},$$

where $\{\varepsilon_t\} \sim wn(0, \sigma_\varepsilon^2)$. Noting that $E(y_t) = \mu$, we can demean the $MA(q)$ model into

$$\begin{aligned} u_t &= \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \\ &= \left[1 + \theta\left(L\right)\right] \varepsilon_t, \end{aligned}$$

where $\theta\left(x\right) = \sum_{i=1}^{q} \theta_i x^i$.

Ex: Show that for an $MA(q)$ model, the ACF is given by, for $j > 0$,

$$\begin{aligned} \rho_j &\equiv Corr(u_t, u_{t-j}) = Corr(y_t, y_{t-j}) \\ &= \begin{cases} \frac{\theta_j + \sum_{i=1}^{q-j} \theta_{j+i}\theta_i}{1 + \sum_{i=1}^{q} \theta_i^2} & \text{if } j \leq q, \\ 0 & \text{if } j > q, \end{cases} \end{aligned}$$

and $\rho_{-j} = \rho_j$. Plot the ACF against lag order.

MA Model Estimation

Given the time series data $\{y_t\}_{t=1}^T$, we can estimate an $MA(q)$ model by either *conditional* or *exact maximum likelihood estimation* (MLE).

- Conditional MLE: Set the initial errors (ε_t for $t = 0, -1, \ldots, -q+1$) to zero. Then assume a distribution on $\{\varepsilon_t\}_{t=1}^T$ (usually iid normal). The joint likelihood function is obtained in terms of the MA parameters $\mu, \theta_1, \ldots, \theta_q$. Maximize the log-likelihood w.r.t. $\mu, \theta_1, \ldots, \theta_q$.
- Exact MLE: Treat the initial errors (ε_t for $t = 0, -1, \ldots, -q+1$) as extra parameters. Maximize the log-likelihood w.r.t. $\mu, \theta_1, \ldots, \theta_q, \varepsilon_0, \varepsilon_{-1}, \ldots, \varepsilon_{-q+1}$.

MA Model Selection

- Use sample ACF. The asymptotic variance of $\hat{\rho}_j$ is $\frac{1}{T}$. For large T, the 95% confidence interval is roughly given by $\hat{\rho}_j \mp \frac{1.96}{\sqrt{T}}$.
- Use information criteria.

Forecasting with MA model

Suppose $\{y_t\} \sim MA(q)$ with errors $\{\varepsilon_t\} \sim wn(0, \sigma_\varepsilon^2)$. For $\ell = 1$,

$$\begin{aligned}\hat{y}_t(1) &= E[y_{t+1}|\mathcal{F}_t] \\ &= E[\mu + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \cdots + \theta_q \varepsilon_{t-q+1}|\mathcal{F}_t] \\ &= \mu + \theta_1 \varepsilon_t + \cdots + \theta_q \varepsilon_{t-q+1},\end{aligned}$$

as $E[\varepsilon_{t+1}|\mathcal{F}_t] = 0$, and $\varepsilon_t, \ldots, \varepsilon_{t-q+1}$ are measurable w.r.t. $\mathcal{F}_t$. The forecast error is $e_t(1) = y_{t+1} - \hat{y}_t(1) = \varepsilon_{t+1}$, with variance $Var[e_t(1)] = \sigma_\varepsilon^2$.

Forecasting with MA model

For $\ell = 2$,

$$\begin{aligned}\hat{y}_t(2) &= E[y_{t+2}|\mathcal{F}_t] \\ &= E[\mu + \varepsilon_{t+2} + \theta_2\varepsilon_{t+1} + \cdots + \theta_q\varepsilon_{t-q+2}|\mathcal{F}_t] \\ &= \mu + \theta_2\varepsilon_t + \cdots + \theta_q\varepsilon_{t-q+2}.\end{aligned}$$

The forecast error is $e_t(2) = y_{t+2} - \hat{y}_t(2) = \varepsilon_{t+2} + \theta_1\varepsilon_{t+1}$, with variance

$$\begin{aligned}Var[e_t(2)] &= Var(\varepsilon_{t+2}) + \theta_1^2 Var(\varepsilon_{t+1}) + 2\theta_1 Cov(\varepsilon_{t+2}, \varepsilon_{t+1}) \\ &= \sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2 + 0 \\ &= (1+\theta_1^2)\sigma_\varepsilon^2.\end{aligned}$$

Forecasting with MA model

Q: What is the ℓ-step ahead forecast of an $MA(q)$ model? What are the forecast error and its variance? What happens to the forecast and its variance when ℓ increases beyond q?

A: The ℓ-step ahead forecast is

$$\hat{y}_t(\ell) = \begin{cases} \mu + \theta_\ell \varepsilon_t + \cdots + \theta_q \varepsilon_{t-q+\ell} & \text{if } \ell \le q, \\ \mu & \text{if } \ell > q. \end{cases} \tag{1}$$

The forecast error is

$$e_t(\ell) = \begin{cases} \varepsilon_{t+\ell} + \theta_1 \varepsilon_{t+\ell-1} + \cdots + \theta_{\ell-1} \varepsilon_{t+1} & \text{if } \ell \le q, \\ \varepsilon_{t+\ell} + \theta_1 \varepsilon_{t+\ell-1} + \cdots + \theta_q \varepsilon_{t+\ell-q} & \text{if } \ell > q. \end{cases} \tag{2}$$

The forecast variance is

$$Var[e_t(\ell)] = \begin{cases} (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_{\ell-1}^2)\sigma_\varepsilon^2 & \text{if } \ell \le q, \\ (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_\varepsilon^2 & \text{if } \ell > q, \end{cases} \tag{3}$$

which is equal to $Var(y_t)$.
As ℓ increases beyond q, $\hat{y}_t(\ell)$ stays at the mean level μ, and the forecast variance remains at $Var(y_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_\varepsilon^2$.

Autoregressive Moving-Average Model

$\{y_t\} \sim ARMA(p, q)$ if

$$\begin{aligned} y_t &= \phi_0 + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \\ [1 - \phi(L)]\, y_t &= \phi_0 + [1 + \theta(L)]\, \varepsilon_t, \end{aligned}$$

where $\{\varepsilon_t\} \sim wn(0, \sigma_\varepsilon^2)$.

After demeaning ($y_t = \mu + u_t$), the expressions become

$$\begin{aligned} u_t &= \phi_1 u_{t-1} + \cdots + \phi_p u_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \\ [1 - \phi(L)]\, u_t &= [1 + \theta(L)]\, \varepsilon_t. \qquad (4) \end{aligned}$$

Ex: What is the ACF of a stationary $ARMA(1, 1)$ model?

Stationarity Condition

$\{y_t\}$ is said to be *stationary* if it can be expressed as an $MA(\infty)$ process of the form $u_t = [1 + \psi(L)]\,\varepsilon_t$.

Stationarity condition: The $ARMA(p, q)$ model is stationary iff all the roots of the polynomial equation $1 - \phi(x) = 0$ lie outside the unit circle.
In this case, $1 + \psi(L) := \frac{1+\theta(L)}{1-\phi(L)}$ is a well-defined infinite-order polynomial.

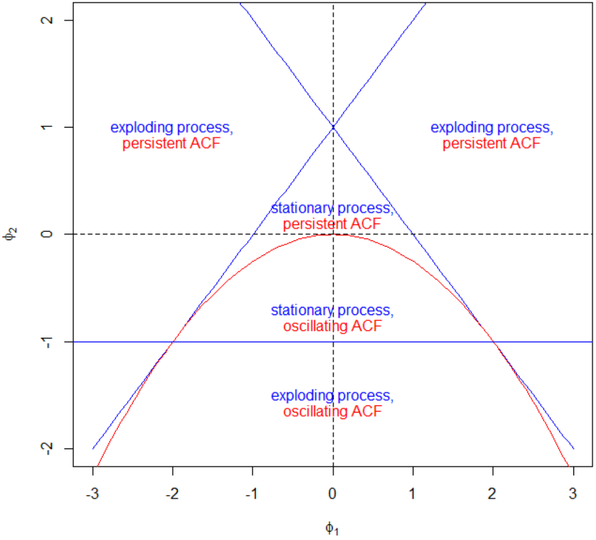
Q: Does the stationarity condition imply weak (covariance) stationarity?
Hint: $ARMA(p, q)$ can be expressed as $MA(\infty)$ under the stationarity condition. The Yule-Walker equations yield a unique set of autocovariances as solution: $\gamma_0, \gamma_1, \gamma_2, \ldots$. Verify that the mean (by definition), the variance γ_0 and autocovariances γ_j are all time-invariant; hence the model is weakly stationary.

Stationarity Condition

Q: What is the stationarity condition for an $AR(1)$ model?
A: For $AR(1)$, we want the root of the equation $1 - \phi_1 x = 0$ to have magnitude greater than one, i.e., $|x| = \frac{1}{|\phi_1|} > 1$, so the stationarity condition is $|\phi_1| < 1$.

Q: How about $AR(2)$?
Hint: The polynomial equation is $1 - \phi_1 x - \phi_2 x^2 = 0$. Let $z = \frac{1}{x}$. The polynomial equation then becomes $z^2 - \phi_1 z - \phi_2 = 0$. Now impose the condition $|x| > 1$, which is equivalent to $|z| < 1$. Let the two roots be z_1 and z_2, where $z_1 < z_2$. Then from the conditions $z_1 > -1$, $z_2 < 1$ and $|z_1 z_2| < 1$, we obtain $\phi_2 - \phi_1 < 1$, $\phi_1 + \phi_2 < 1$ and $|\phi_2| < 1$.

Stationarity Condition of AR(2)



Invertibility Condition

$\{y_t\}$ is said to be *invertible* if it can be expressed as an $AR(\infty)$ process of the form $[1 - \pi(L)]\, u_t = \varepsilon_t$.

Invertibility condition: The $ARMA(p, q)$ model is invertible iff all the roots of the polynomial equation $1 + \theta(x) = 0$ lie outside the unit circle.
In this case, $1 - \pi(L) := \frac{1-\phi(L)}{1+\theta(L)}$ is a well-defined infinite-order polynomial.

Q: What is the invertibility condition for an $MA(1)$ model?
A: For $MA(1)$, the invertibility condition is $|\theta_1| < 1$.

Ex: What is the invertibility condition of an $MA(2)$ model?
Ex: Is a stationary $AR(p)$ invertible?
Ex: Is an invertible $MA(q)$ stationary?

AR Representation

Given an $ARMA(p,q)$ model:

$$\begin{aligned} y_t &= \mu + u_t, \\ [1-\phi(L)]\,u_t &= [1+\theta(L)]\,\varepsilon_t. \end{aligned}$$

Suppose the invertibility condition holds, so that $\frac{1}{1+\theta(L)}$ is a well-defined infinite-order polynomial. Define $1+\pi(L)=\frac{1-\phi(L)}{1+\theta(L)}$. Then we can rewrite the model as $AR(\infty)$:

$$\begin{aligned} \varepsilon_t &= \frac{1-\phi(L)}{1+\theta(L)}u_t \\ &= [1+\pi(L)]u_t \\ &= u_t + \pi_1 u_{t-1} + \pi_2 u_{t-2} + \cdots . \end{aligned}$$

MA Representation

Given an $ARMA(p, q)$ model:

$$\begin{aligned} y_t &= \mu + u_t, \\ [1 - \phi(L)]\, u_t &= [1 + \theta(L)]\, \varepsilon_t. \end{aligned}$$

Suppose the stationarity condition holds, so that $\frac{1}{1-\phi(L)}$ is a well-defined infinite-order polynomial. Define $1 + \psi(L) = \frac{1+\theta(L)}{1-\phi(L)}$. Then we can rewrite the model as $MA(\infty)$:

$$\begin{aligned} u_t &= \frac{1 + \theta(L)}{1 - \phi(L)} \varepsilon_t \\ &= [1 + \psi(L)]\varepsilon_t \\ &= \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots . \end{aligned} \tag{5}$$

Interpretation: ψ_j as a function of j is the *impulse response function*. ψ_j is the marginal impact of the lag j shock ε_{t-j} on the current observation y_t.

MA Representation

Q: Let $\{y_t\}$ be a weakly stationary process. What happens to the ℓ-step ahead forecast and its variance as $\ell \longrightarrow \infty$?

A: By weak stationarity, $\{y_t\}$ can be expressed as an $MA(\infty)$ process (5), so that

$$\begin{aligned} Var(y_t) &= Var(\varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots) \\ &= \sigma_\varepsilon^2 \left(1 + \sum_{i=1}^{\infty} \psi_i^2\right) < \infty. \end{aligned}$$

This implies that $\psi_i \longrightarrow 0$ as $i \longrightarrow \infty$.
It follows from (1) with $q = \infty$ that the ℓ-step ahead forecast is

$$\begin{aligned} y_t(\ell) &= \mu + \psi_\ell \varepsilon_t + \psi_{\ell+1} \varepsilon_{t-1} + \cdots \\ &\longrightarrow \mu \text{ as } \ell \longrightarrow \infty. \end{aligned}$$

This is the *mean-reverting property*. The forecast variance is, by (3) with $q = \infty$,

$$\begin{aligned} Var[e_t(\ell)] &\longrightarrow (1 + \psi_1^2 + \psi_2^2 + \cdots)\sigma_\varepsilon^2 \\ &= Var(y_t) \text{ as } \ell \longrightarrow \infty. \end{aligned}$$

Model Checking

If a model is correctly specified, the residual process $\{\hat{\varepsilon}_t\}$ should look like a white noise.

To test $H_0 : \rho_\ell = 0$ vs $H_a : \rho_\ell \neq 0$, we get the sample ACF $\hat{\rho}_\ell$ of $\{\hat{\varepsilon}_t\}$.

- Suppose $\rho_j = 0$ for all $j > \ell$ under H_0. Then,

$$\sqrt{T}\hat{\rho}_\ell \xrightarrow{d} N\left(0, 1 + 2\sum_{j=1}^{\ell-1} \rho_j^2\right) \text{ as } T \to \infty.$$

  The t ratio converges in distribution to standard normal:

$$t = \frac{\hat{\rho}_\ell}{\sqrt{\left(1 + 2\sum_{j=1}^{\ell-1} \hat{\rho}_j^2\right) / T}} \xrightarrow{d} N(0,1) \text{ as } T \to \infty.$$

- Suppose $\{\varepsilon_t\} \sim wn(0, \sigma_\varepsilon^2)$ under H_0. Then all $\rho_j = 0$ for all $j \neq 0$, and so

$$\sqrt{T}\hat{\rho}_\ell \xrightarrow{d} N(0,1) \text{ as } T \to \infty.$$

This is a two-sided test (reject H_0 at α level if $|t| > z_{\alpha/2}$).

Portmanteau Tests

Set a maximum lag m. We want to test $H_0 : \rho_1 = \cdots = \rho_m = 0$ vs $H_a : \rho_j \neq 0$ for some $j = 1, \ldots, m$.
Assume $\{y_t\} \sim ARMA(p, q)$, with $\{\varepsilon_t\} \sim iid$ and some moment conditions.
Box-Pierce test on $\{\hat{\varepsilon}_t\}$:

$$Q^*(m) = T \sum_{\ell=1}^{m} \hat{\rho}_\ell^2 \xrightarrow{d} \chi^2(m - p - q) \text{ as } T \to \infty.$$

Ljung-Box test on $\{\hat{\varepsilon}_t\}$:

$$Q(m) = T(T+2) \sum_{\ell=1}^{m} \frac{\hat{\rho}_\ell^2}{T - \ell} \xrightarrow{d} \chi^2(m - p - q) \text{ as } T \to \infty.$$

They are one-sided tests (reject H_0 at α level if $Q(m) > \chi^2_\alpha$).