7.1. $A=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}$ $B=\begin{bmatrix}1&2\\0&1\\1&0\end{bmatrix}$.

(a). Reduced QR: $A=\hat{Q}\hat{R}$. $\hat{Q}\in\mathbb{C}^{3\times2}$. $\hat{R}\in\mathbb{C}^{2\times2}$.

$[a_1\ a_2]=[q_1\ q_2]\begin{bmatrix}r_{11}&r_{12}\\0&r_{22}\end{bmatrix}=[r_{11}q_1.\ r_{12}q_1+r_{22}q_2]$.

$q_1^Tq_1=\frac{1}{r_{11}^2}a_1^Ta_1=\frac{2}{r_{11}^2}=1.\Rightarrow r_{11}=\sqrt{2}\ .\ q_1=\frac{1}{\sqrt{2}}a_1$.

$q_2=a_2.\Rightarrow(1-r_{22})q_2=r_{12}q_1\Rightarrow r_{22}=1.\ r_{12}=0$,

$\hat{Q}=\begin{pmatrix}1/\sqrt{2}&0\\0&1\\1/\sqrt{2}&0\end{pmatrix}$ $\hat{R}=\begin{pmatrix}\sqrt{2}&0\\0&1\end{pmatrix}$.

Full QR: $A=QR$. $Q\in\mathbb{C}^{3\times3}$. $R\in\mathbb{C}^{3\times2}$.

$[a_1\ a_2]=[q_1\ q_2\ q_3]\left[\begin{array}{c|c}r_{11}&r_{12}\\0&r_{22}\\\hline 0&0\end{array}\right]$.

$q_3=\begin{pmatrix}a\\b\\c\end{pmatrix}$. $\left.\begin{array}{l}q_1^Tq_3=0\Rightarrow a+c=0\\q_2^Tq_3=0\Rightarrow b=0.\end{array}\right\}$ $q_3=\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}$

$Q=\begin{pmatrix}1/\sqrt{2}&0&1/\sqrt{2}\\0&1&0\\1/\sqrt{2}&0&-1/\sqrt{2}\end{pmatrix}$ $R=\begin{pmatrix}\sqrt{2}&0\\0&1\\0&0\end{pmatrix}$

(b) $B=\begin{bmatrix}1&2\\0&1\\1&0\end{bmatrix}=[b_1\ b_2]$.

GS: $v_1=b_1.\ r_{11}=\|v_1\|_2=\sqrt{2}\cdot q_1=v_1/r_{11}=\frac{1}{\sqrt{2}}b_1$.

$v_2=b_2.\ r_{12}=q_1^Tb_2=\frac{1}{\sqrt{2}}b_1^Tb_2=\sqrt{2}.\ v_2=v_2-r_{12}q_1.=b_2-b_1=\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

$r_{22}=\|v_2\|_2=\sqrt{3}.\ q_2=v_2/\sqrt{3}=\frac{1}{\sqrt{3}}(1.\ 1.\ -1)^T$.

Reduced. $\hat{Q}=\begin{pmatrix}1/\sqrt{2}&1/\sqrt{3}\\0&1/\sqrt{3}\\1/\sqrt{2}&-1/\sqrt{3}\end{pmatrix}$. $\hat{R}=\begin{pmatrix}\sqrt{2}&\sqrt{2}\\0&\sqrt{3}\end{pmatrix}$.

Full. $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{2} \\ 0 & 1/\sqrt{3} & 0 \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$. $R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$

1c) $[a_1 \cdots a_n] = [q_1 \cdots q_n] \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ & r_{22} & \vdots \\ & & \ddots \; r_{nn} \end{pmatrix}$.

$$q_k = \frac{a_k - \sum_{i=1}^{k-1} r_{ik} q_i}{r_{kk}}$$

$k=1$. $q_1 = a_1 / r_{11}$ $\quad a_i^* q_1 \neq 0$

$\forall j$. with $j+1$ odd.

$$a_j^* q_1 = a_j^* a_1 = 0.$$

Assume $k-1$ holds. i.e. if $j+(k-1)$ is odd. then. $a_j^* q_{k-1} = 0$

$k=k$. $j+k$ is odd.

$$a_j^* q_k = \underbrace{a_j^* a_k}_{0} - \sum_{i=1}^{k-1} a_i^* q_k (a_j^* q_i).$$

For each i.

if. $i+j$ odd.

then. $a_j^* q_i = 0$

$i+k$ even.

$= 0$

7.3.

$A = QR$. $Q, R \in \mathbb{C}^{m\times m}$

$$\det(A) = \det(Q)\det(R) = \det(R) = \prod_{j=1}^{m} r_{jj}.$$

$$|\det(A)| \le \prod_{j=1}^{m} |r_{jj}|$$

$$= \prod_{j=1}^{m} \|a_j - \sum_{i=1}^{j-1} r_{ij} q_i\|_2.$$

$$\|a_j - \sum_{i=1}^{j-1} r_{ij} q_i\|_2^2 = \|a_j\|_2^2 - 2 a_j^* \sum_{i=1}^{j-1} r_{ij} q_i + \|\sum_{i=1}^{j-1} r_{ij} q_i\|_2^2.$$

$r_{ij} = q_i^* a_j$

$= a_j^* q_i$.

$$= \|a_j\|_2^2 - 2\sum_{i=1}^{j-1} r_{ij}^2 + \sum_{i=1}^{j-1} r_{ij}^2$$

$$\le \|a_j\|_2^2.$$

$$\Rightarrow |\det(A)| \le \prod_{j=1}^{m} \|a_j\|_2.$$

7.4.

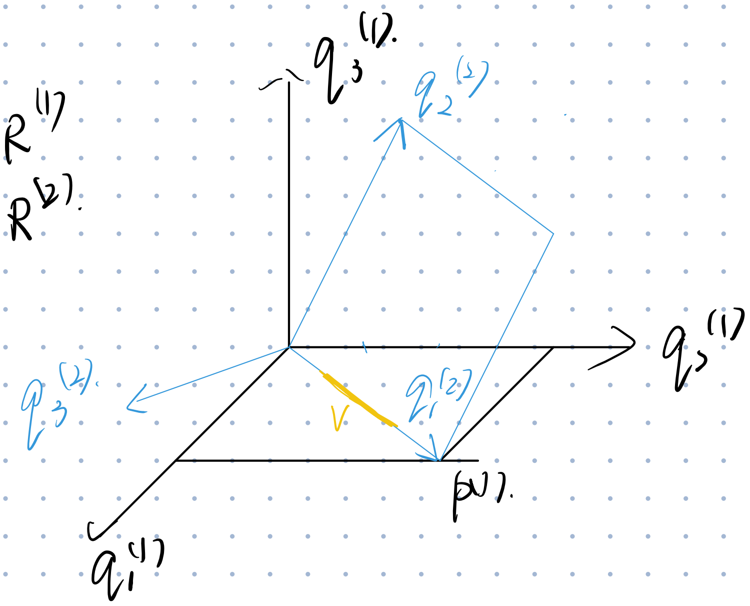
$v \in P = P^{(1)} \cap P^{(2)}$.

$P^{(1)} = [q_1^{(1)}\ q_2^{(1)}\ q_3^{(1)}]\, R^{(1)}$

$P^{(2)} = [q_1^{(2)}\ q_2^{(2)}\ q_3^{(2)}]\, R^{(2)}$.

so $v \perp q_3^{(1)}$ $v \perp q_3^{(2)}$

$v \perp (q_3^{(1)} \times q_3^{(2)})$.



7.5

(a). $\Rightarrow$ $A = \hat{Q}\hat{R} = (q_1 \cdots q_n)\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ & & r_{nn} \end{pmatrix}$

Assume $\exists k$ s.t. $r_{kk} = 0$. $|r_{kk}| = \|a_k - \sum^{k-1} r_{ik} q_i\|_2$.

$\Rightarrow a_k = \sum^{k-1} r_{ik} q_i \in \operatorname{span}(a_1 \cdots a_{k-1})$,

$\Rightarrow \operatorname{rank}(A) < n$.

$\Leftarrow$. Assume $\operatorname{rank}(A) < n$.

$\exists k$. s.t. $a_k = \sum^{k-1} \beta_i a_i$. $\beta_i \neq 0$.

$r_{kk} = q_k^* a_k = q_k^* \sum^{k-1} \beta_i a_i = 0$,

(b).

k Nonzero entries. $k < n$.

$\operatorname{rank}(A) \geq \operatorname{rank}(\langle a_1 \cdots a_k \rangle) = \operatorname{rank}(\langle q_1 \cdots q_k \rangle) = k$.