4.1

$A = U\Sigma V^*$

$A^*A = V\Sigma^* U^* U \Sigma V^* = V\Sigma^*\Sigma V^*$

$\Sigma^*\Sigma = V^* A^* A V = V^{-1} A^* A V$

$A^* A V = V \Sigma^* \Sigma$

$\Rightarrow$ ew of A^*A are σ_i^2 with $V = [$ev of $A^*A]$.

(a) • $A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$. $A^*A = AA^* = \begin{pmatrix} 9 & \\ & 4 \end{pmatrix}$. $\sigma(A^*A) \begin{cases} \lambda_1 = 9. & v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 = 4. & v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

• $\sigma_i = \sqrt{\lambda_i} \Rightarrow \Sigma = \begin{pmatrix} 3 & \\ & 2 \end{pmatrix}$.

• $V = [v_1 \ v_2] \Rightarrow V = I$

• $A = U\Sigma V^* \Rightarrow U\Sigma = AV \Rightarrow U = AV\Sigma^{-1}$.

$= A\,[v_1 \ v_2] \begin{pmatrix} \frac{1}{\sigma_1} & \\ & \frac{1}{\sigma_2} \end{pmatrix}$

$= [\frac{1}{\sigma_1} A v_1 \ \ \frac{1}{\sigma_2} A v_2]$

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{matrix} 0 & 0 \\ 0 & -2 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 0 \\ -2 \end{matrix}$

$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & \\ & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4.2

$A \in \mathbb{C}^{m\times n}$ $\quad B \in \mathbb{C}^{n\times m}$. $\quad A^T \in \mathbb{C}^{n\times m}$

$A = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{nm} \end{bmatrix} \quad B = \begin{bmatrix} a_{1m} & \cdots & a_{11} \\ \vdots & & \vdots \\ a_{nm} & \cdots & a_{n1} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$

$B = A^T \underbrace{P_1 \cdots P_r}_{\text{a sequence of Permutation Matrix}}$. $\quad \triangleq A^T P$: P unitary.

$A = U\Sigma V^*$. $\Rightarrow A^T = V\Sigma U^*$ $\Rightarrow B = V\Sigma U^* P = V\Sigma \tilde{U}^*$ $(\tilde{U} = P^* U)$

4.4

"$\Rightarrow$". $A = QBQ^*$.

$A = U\Sigma V^*$.

$B = Q^*AQ = \underline{Q^*U}\Sigma \underline{V^*Q}$.

$\Sigma_A = \Sigma_B$.

"$\nLeftarrow$". A. $-A$. have same singular vals. $A \neq 0$.

suppose $\exists$ unitary Q s.t. $A = Q(-A)Q^* = -QAQ^*$

$$\underset{\det(A)}{\underset{\parallel}{\det(-QAQ^*)}} = (-1)^m \det(QAQ^*) = (-1)^m \det(A)$$

contradiction.

4.5.

$A \in \mathbb{R}^{m\times n}$.

$A^*A = A^TA \in \mathbb{R}^{n\times n}$.

$\forall x \neq 0$. $x^TA^TAx \geq 0$. Positive-semidefinite.

$\forall \lambda \in \sigma(A^TA)$.

$A^TAv = \lambda v \Rightarrow v^TA^TA = \lambda v^T \Rightarrow v^TA^TAv = \lambda v^Tv \geq 0 \Rightarrow \lambda \geq 0$

$\Rightarrow$. $A^TA = VDV^T$. $D = \operatorname{diag}\{ev\}$. $V = \{ev\}$.

$\Rightarrow$. $D = V^TA^TAV = (AV)^TAV$

A $m\times n$. A^TA $n\times n$. V $n\times n$. AV $m\times n$.

choose $Q \in \mathbb{C}^{m\times n}$ unitary s.t. $AV = QD^{\frac{1}{2}}$.

Extend Q to $U = (Q\ \tilde{Q})$ $m\times m$. (Assume $m \geq n$)

$$\Sigma = \begin{pmatrix} D_{n\times n} \\ 0 \end{pmatrix}_{m\times n}$$

$A = U\Sigma V^T$