24.1 $A \in \mathbb{C}^{m\times m}$. ew. ev.

(a). ✓

$$p_{A-\mu I}(\lambda-\mu) = \det[(\lambda-\mu)I - A + \mu I] = p_A(\lambda) = 0.$$

(b). ✗

$$p_A(-\lambda) = \det(-\lambda I - A) = (-1)^m \det(\lambda I + A) = (-1)^m p_{-A}(\lambda).$$

$A = \mathrm{diag}(\lambda_1 \cdots \lambda_n)$. $p_A(z) = \prod^n (z-\lambda_i)$ $\lambda_i \in \mathbb{R}^+$.

$-A = \mathrm{diag}(-\lambda_1, \cdots, -\lambda_n)$ $p_{-A}(z) = \prod^n (z+\lambda_i)$

$\Rightarrow p_A(-\lambda_k) = (-1)^m p_{-A}(\lambda_k) \neq 0.$

(c). ✓

$$Ax = \lambda x \Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$$

(d) ✓.

$$Ax = \lambda x. \Rightarrow. A^{-1}x = \lambda^{-1}x.$$

(e). ✗.

$p_A(z) = \det(zI - A)$

$p_A(0) = \det(-A) = (-1)^m \det(A) = 0 \Rightarrow A$ singular

Eg: $A = \begin{bmatrix} 0 & \cdots & 0 \\ I_{m-1} & & 0 \end{bmatrix}_{m\times m} \neq 0.$

$p_A(\lambda) = \det(\lambda I - A) = \lambda^m \Rightarrow. \sigma(A) = \{0\}.$

(f). ✓.

- $Ax = \lambda x \Leftrightarrow x^* A = \bar{\lambda} x^* \Leftrightarrow x^* A x = \bar{\lambda} x^* x \Leftrightarrow \lambda |x|^2 = \bar{\lambda}|x|^2 \Leftrightarrow \underline{\underline{\lambda \in \mathbb{R}}}$
- $\begin{cases} Ax = \lambda x \\ Ay = \mu y \\ \lambda \neq \mu \end{cases}$ $x^* y = \frac{1}{\mu} x^*(\mu y) = \frac{1}{\mu} x^* A y = \frac{1}{\mu}\bar{\lambda} x^* y = \frac{1}{\mu}\lambda x^* y$

  $\Rightarrow. (\mu - \lambda) x^* y = 0. \Rightarrow \underline{\underline{x^* y = 0.}}$

- $A = Q \Lambda Q^*$ $\Lambda = \mathrm{diag}$ (ew) Q: corresponding ev.

  $= \underbrace{Q}_{U} |\Lambda| \underbrace{\mathrm{sign}(\Lambda) Q^*}_{V.}$

(g)

Diagonalizable $\iff$ nondefective.

$\iff \lambda \in \sigma(A)$. Geo Multi = Alg Multi.

All ew equal $\Rightarrow p_A(z) = (z-\lambda)^m$.

$\Rightarrow \dim(E_\lambda) = m$ (RK: $E_\lambda = \{0\} + \{ev\}$.)

$\Rightarrow \dim(\ker(\lambda I - A)) = m-1$.

$\Rightarrow \operatorname{rank}(\lambda I - A) = 1$

$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. $p_A(\lambda) = \lambda^2$. $\lambda = 0$.

24.2.

(a) i.e. WTS. $\sigma(A) \subseteq \bigcup_{i=1}^{m} D_i$. $D_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}$.

$\forall \lambda \in \sigma(A)$. $\exists x \neq 0$. s.t. $Ax = \lambda x$.

ith row: $a_{ii} x_i + \sum_{j \neq i}^{m} a_{ij} x_j = \lambda x_i$.

$\lambda - a_{ii} = \sum_{j \neq i}^{m} a_{ij} \frac{x_j}{x_i}$.

let k be: $|x_k| = \|x\|_\infty$

$|\lambda - a_{kk}| \leq \sum_{j \neq k}^{m} |a_{kj}|$

$\Rightarrow \lambda \in D_k \subseteq \bigcup_{i=1}^{m} D_i$

(b) $A(\varepsilon) = D + \varepsilon B$. $D = \operatorname{Diag}(A)$. $B = A - D$. $\varepsilon \in [0,1]$.

$\left.\begin{array}{l} A(0) = D \\ A(1) = A. \end{array}\right\}$ $A(\varepsilon)$. cts. w.r.t. ε.

By (a). $\sigma(A(\varepsilon)) \subseteq \bigcup_{j=1}^{m} D_i^{\varepsilon}$ $D_i^{\varepsilon} = \{z \in \mathbb{C} : |z - a_{ii}| \leq \varepsilon \sum_{j \neq i}^{m} |a_{ij}|\}$.

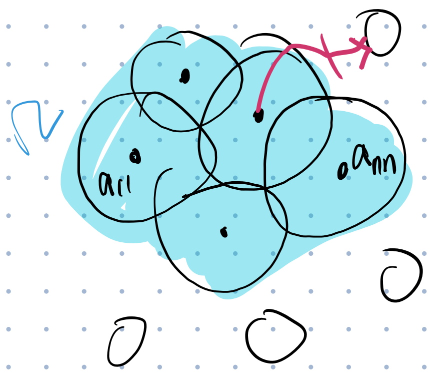
$\varepsilon \in [0,1]$. $\sigma(A(\varepsilon)) \subseteq \bigcup_{j=1} D_i$.

if n of D_i form a connected domain. disjoint with other $m-n$.

WLOG $\Omega = \bigcup_{i=1}^{n} D_i$ connected Domain.

$\{D_i, i = n+1, \cdots, m\}$. disjoint with Ω.

$\{a_{11}, \cdots, a_{nn}\} \subset \Omega$. $\Rightarrow$ $A(0)$ has n ew lies in Ω.



$\lambda(A_\varepsilon) = \lambda(\varepsilon)$ cts w.r.t. ε.

so $A(1) = A$ has n ew lies in Ω

otherwise. contradict to the disjointness

(c)

$A = \begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix}$ $|\varepsilon| < 1$.

$\begin{cases} |\lambda_1 - 8| \le 1 \\ |\lambda_2 - 4| \le 1 + |\varepsilon| \\ |\lambda_3 - 1| \le |\varepsilon| \end{cases} \Rightarrow \begin{cases} 7 \le \lambda_1 \le 9 \\ 3 - |\varepsilon| \le \lambda_2 \le 5 + |\varepsilon| \\ 1 - |\varepsilon| \le \lambda_3 \le 1 + |\varepsilon| \end{cases} \Rightarrow \begin{cases} 7 \le \lambda_1 \le 9 \\ 2 < \lambda_2 < 6 \\ 0 < \lambda_3 < 2 \end{cases}$ Disjoint

(d)

$\left(\begin{array}{c|c} \bar{A} & \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \\ \hline (0, \varepsilon) & 1 \end{array}\right) = X^{-1} B X$

$\left(\begin{array}{c|c} \bar{X} & y \\ \hline w^T & c \end{array}\right) \left(\begin{array}{c|c} \bar{A} & \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \\ \hline (0, \varepsilon) & 1 \end{array}\right) = \left(\begin{array}{c|c} \bar{A} & \alpha \\ \hline (0, \varepsilon^2) & 1 \end{array}\right) \left(\begin{array}{c|c} \bar{X} & y \\ \hline w^T & c \end{array}\right)$

$\begin{pmatrix} \bar{X}\bar{A} + y(0, \varepsilon) & \bar{X}\begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} + y \\ w^T \bar{A} + (0, c\varepsilon) & w^T \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} + c \end{pmatrix} = \begin{pmatrix} \bar{A}\bar{X} + \alpha w^T & \bar{A} y + c\alpha \\ (0, \varepsilon^2)\bar{X} + w^T & (0, \varepsilon^2) y + c \end{pmatrix}$

let $\bar{X} = I_2$ $y = w = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \bar{A} & \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \\ [0, c\varepsilon] & c \end{pmatrix} = \begin{pmatrix} \bar{A} & c\alpha \\ [0, \varepsilon^2] & c \end{pmatrix}$$

$\Rightarrow$ $c = \varepsilon$. $\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\Rightarrow$ $A = X^{-1} B X$ $\quad X = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \varepsilon \end{bmatrix}$. $\quad B = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & \varepsilon^2 & 1 \end{bmatrix}$.

$\Rightarrow$ A, B have same ews.

$$\Rightarrow B: \begin{cases} |\lambda_1 - 8| \le 1 \\ |\lambda_2 - 4| \le 2 \\ |\lambda_3 - 1| \le \varepsilon^2 \end{cases} \Rightarrow \begin{cases} 7 \le \lambda_1 \le 9. \\ 2 \le \lambda_2 \le 6. \\ |\lambda_3 - 1| \le \varepsilon^2 \end{cases}$$