25.1

a)

$$A = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{m-1} \\ & & b_{m-1} & a_m \end{bmatrix}$$

suppose $\lambda \in \sigma(A) \subseteq \mathbb{R}$. $A-\lambda I$ singular.

$$A-\lambda I = \begin{bmatrix} a_1-\lambda & b_1 & & & & \\ b_1 & a_2-\lambda & b_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{i-1} & a_i-\lambda & b_i & \\ & & & \ddots & \ddots & \ddots \\ & & & & b_{m-1} & a_m-\lambda \end{bmatrix} = \begin{bmatrix} c_1 & b_1 & & & & \\ \boxed{b_1} & c_2 & b_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{i-1} & c_i & b_i & \\ & & & \ddots & \ddots & b_{m-1} \\ & & & & b_{m-1} & c_m \end{bmatrix}$$

V: diagonal with entries on diagonal $\neq 0$.

$\Rightarrow \det(V) \neq 0$

$\Rightarrow V$ nonsingular.

$\Rightarrow \text{Rank}(V) = m-1$.

$\Rightarrow \text{Rank}(A-\lambda I) \geq m-1$.

- if $\text{Rank}(A-\lambda I) = m$. $\Rightarrow$ Nonsingular. $\Rightarrow$ X Eigenvec.
- if $\text{Rank}(A-\lambda I) = m-1$ $\Rightarrow \dim(\ker(A-\lambda I)) = m-(m-1) = 1$
  - $\Rightarrow$ Only have 1 eigenvec.
  - $\Rightarrow \dim(E_\lambda) = 1$.
  - $\Rightarrow$ Geometric Multiplicity 1.

Hermitian $\xRightarrow{\text{Thm 24.7}}$ Diagonalizable.

$\Leftrightarrow$ Nondefective.

$\Leftrightarrow$ Geo Multi = Algebraic Multi

Thus. for each λ. Algebraic Multi $= 1$. $\Rightarrow$ Simple.

$\Rightarrow$ all λ are distinct & Simple b). A : upper-Hessenberg :

$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $\det(\lambda I - A) = (\lambda - 1)^3$, Not Distinct

25.3 let $A = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$

a) $B = \begin{bmatrix} \times & \times & 0 \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$.

(i). if $B = QA$.

then $[b_1\ b_2\ b_3] = [Qa_1\ Qa_2\ Qa_3]$

$b_1^T b_3 = 0 \iff (Qa_1)^T(Qa_3) = 0$

$\iff a_1^T a_3 = 0$. Not necessary holds

$\Rightarrow$ (i) $\times$

(ii). $A = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} \xrightarrow[Q_1]{\text{Householder}} B_1 = QA = \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$

$\downarrow$

$B_1^* = A^* Q_1^* = \begin{pmatrix} \times & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$

Q_2^* $\downarrow$ Household

$B_2^* = Q_2^* B_1^* = \begin{pmatrix} \times & 0 & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}$

$\downarrow$

$B_2 = B_1 Q_2 = \begin{pmatrix} \times & \times & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$

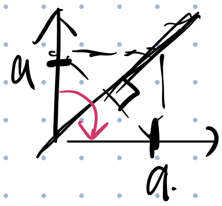
Rk: $\begin{pmatrix} \times \\ \times \\ \times \end{pmatrix} \xrightarrow{\text{Householder } H = \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}} \begin{pmatrix} \times \\ \times \\ 0 \end{pmatrix}$

$B = Q_3 B_2 = \begin{pmatrix} \times & \times & 0 \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix} \xleftarrow[Q_3]{\text{householder}} B_2$

$B = Q_3 B_1 Q_2 = Q_3 Q_1 A Q_2$. $\Rightarrow$ (ii) Applicable b). $B=\begin{pmatrix} x & x & 0 \\ x & 0 & x \\ 0 & x & x \end{pmatrix}$

Housholder. Q_1. Q_2. s.t. $\begin{pmatrix} x & x & 0 \\ x & x & x \\ 0 & 0 & x \end{pmatrix} = Q_1 A Q_2$.

$\begin{pmatrix} x & x \\ 0 & x \end{pmatrix} \xrightarrow{\text{House}} \begin{pmatrix} 0 & x \\ x & x \end{pmatrix}$ eg: $\begin{pmatrix} a & x \\ 0 & x \end{pmatrix} \to \begin{pmatrix} 0 & x \\ a & x \end{pmatrix}$



$Q_3 = I - 2\frac{vv^*}{v^*v}$, $v = \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix}$.

(ii) ✓

(i) X. let $A = [a\ a\ a]$.

$B = \begin{bmatrix} x & x & 0 \\ x & 0 & x \\ 0 & x & x \end{bmatrix} = [Qa\ Qa\ Qa] = QA$.

$\Rightarrow Qa = 0$.

$\Rightarrow QA = 0$. contradict.

c). $B = \begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$ $\det(B) = 0$. $\Rightarrow$ B singlular.

$\Rightarrow \exists x \neq 0$. s.t. $Bx = 0$

(i). X. since. $B = QA$

$QAx = 0$.

$Ax = 0$. A is singular.

(ii) X since $B = Q_1 A Q_2$.

$Q_1 A Q_2 x = 0$

Not Necessary

$\Rightarrow AQ_2X = 0. \Rightarrow A(Q_2X) = 0$. A singular.
$Q_2X \neq 0$.

Thus (iii) $\checkmark$.