5.1.

Eg 3.1 $A=\begin{pmatrix}1&2\\0&2\end{pmatrix}$

<u>SVD</u>: $A^*A=\begin{pmatrix}1&0\\2&2\end{pmatrix}\begin{pmatrix}1&2\\0&2\end{pmatrix}=\begin{pmatrix}1&2\\2&8\end{pmatrix}$

$\sigma(A^*A)$: $(\lambda-1)(\lambda-8)-4=0$

$\lambda^2-9\lambda+4=0 \quad \Delta=\sqrt{81-16}=\sqrt{65}$

$\lambda_{1,2}=\dfrac{9\pm\sqrt{65}}{2}$

$\sigma_1=\sqrt{\lambda_1} \quad \sigma_2=\sqrt{\lambda_2}$

5.2 $\forall A\in\mathbb{C}^{m\times n}$. $A=U\Sigma V^*$ $U\in\mathbb{C}^{m\times m}$. $\Sigma\in\mathbb{C}^{m\times n}$ $V\in\mathbb{C}^{n\times n}$.

Define: $A_\varepsilon=U\underbrace{(\Sigma+\varepsilon I_{mn})}V^*$.

positive entries on diagonal.

$\mathrm{Rank}(A_\varepsilon)=\mathrm{Rank}(\Sigma+\varepsilon I_{mn})$ Full

$\|A-A_\varepsilon\|_2=\varepsilon\|UV^*\|_2=\varepsilon \xrightarrow{\varepsilon\to0} 0.$

5.3 $A=\begin{bmatrix}-2&11\\-10&5\end{bmatrix}$.

(a). $AA^*=\begin{bmatrix}-2&11\\-10&5\end{bmatrix}\begin{bmatrix}-2&-10\\11&5\end{bmatrix}=25\begin{bmatrix}5&3\\3&5\end{bmatrix}$

$\sigma(AA^*)$: $\lambda_1=50$. $u_1=\begin{pmatrix}a\\-a\end{pmatrix}$ $\sigma_1=\sqrt{\lambda_1}=5\sqrt{2}$.

$\lambda_2=200$. $u_2=\begin{pmatrix}b\\b\end{pmatrix}$ $\sigma_2=\sqrt{\lambda_2}=10\sqrt{2}$.

$\Sigma=\begin{pmatrix}5\sqrt{2}&\\&10\sqrt{2}\end{pmatrix}$. $U=[u_1\ u_2]=\begin{bmatrix}a&b\\-a&b\end{bmatrix}\xrightarrow{\text{unitary}} U=\frac{1}{\sqrt{2}}\begin{pmatrix}a&b\\-a&b\end{pmatrix}$ $a^2=b^2=1$.

$A=U\Sigma V^*\Rightarrow V^*=\Sigma^{-1}U^{-1}A=\frac{1}{5\sqrt{2}}\begin{pmatrix}1&\\&\frac{1}{2}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}a&-a\\b&b\end{pmatrix}\begin{bmatrix}-2&11\\-10&5\end{bmatrix}=\frac{1}{5}\begin{pmatrix}4a&3a\\-3b&4b\end{pmatrix}$.

$\Rightarrow V=\frac{1}{5}\begin{pmatrix}4a&-3b\\3a&4b\end{pmatrix}$

min# of minus $\Rightarrow$ $a=b=1$.

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5\sqrt{2} & \\ & 10\sqrt{2} \end{pmatrix} \begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix}$$

(b)

(c) $\|A\|_1 = 16$.

$\|A\|_2 = \|U\Sigma V^*\|_2 = \|\Sigma\|_2 = 10\sqrt{2}$.

$\|A\|_\infty = 15$.

$\|A\|_F = \sqrt{tr(AA^*)} = \sqrt{200+50} = 5\sqrt{10}$

(d). $A^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^* = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \frac{1}{5\sqrt{2}} \begin{pmatrix} 1 & \\ & 1/2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$= \frac{1}{100}\begin{pmatrix} 5 & -11 \\ 10 & -2 \end{pmatrix}$$

5(d)

$A = U\Sigma V^*$. $U, \Sigma, V^* \in \mathbb{C}^{m\times m}$

$$B = \begin{bmatrix} 0 & V\Sigma U^* \\ U\Sigma V^* & 0 \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & \Sigma U^* \\ \Sigma V^* & 0 \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix}\begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix}$$

$$B\begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix}\begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} I & I \\ I & \end{bmatrix}\begin{bmatrix} & \Sigma \\ \Sigma & \end{bmatrix}\begin{bmatrix} I & I \\ I & \end{bmatrix}$$

$$B\begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} I & I \\ I & \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} I & I \\ I & \end{bmatrix}\begin{bmatrix} \Sigma & I \\ \Sigma & \end{bmatrix}$$

$$B\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix} = \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\begin{bmatrix} \Sigma & \Sigma \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} \Sigma & \\ & -\Sigma \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

$$B\underline{\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}} = \underline{\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}}\begin{bmatrix} \Sigma & \\ & -\Sigma \end{bmatrix},$$