April 25, 2022 (Due: 08:00 May 9, 2022)

1. Derive the circular convolution theorem based on the following convention of DFT:

$$\widehat{u}_k = \sum_{k=0}^{n-1} \exp\Big(-\frac{2jk\pi\mathrm{i}}{n}\Big) u_j, \qquad (u \in \mathbb{C}^n).$$

2. Write a program to compute the product of two complex polynomials using fast convolution algorithms. You can make use of the MATLAB/Octave function `conv` to check the correctness of your implementation. Make a plot to demonstrate that the complexity of your implementation is $\Theta(n \log n)$. What is the complexity of `conv`?

3. You are given an audio file `DTMF_dialing.ogg`, which contains 80 touch tones from a DTMF keyboard. Try to determine the key corresponding to the kth tone in this audio file, where k is the unique integer in $\{1, 2, \ldots, 80\}$ satisfying

$$\text{[Your student ID]} \equiv k \pmod{80}.$$

4. Create a discontinuous function and smoothen it by convolving with Gaussian functions. Make plots to visualize the results.

FYI. There are also other choices on the kernel for the convolution. Especially, a class of functions, known as Friedrichs mollifiers, is frequently used.

5. Implement an FFT-based fast Poisson solver on unit square with Dirichlet boundary conditions.

6. (optional) Prove that

$$\delta(x) = \frac{1}{\pi} \lim_{\eta \to 0+} \frac{\eta}{x^2 + \eta^2}.$$

It suffices to show

$$\lim_{\eta \to 0+} \int_{-1}^{1} \frac{\eta f(x)}{x^2 + \eta^2} \, \mathrm{d}x = \pi f(0)$$

for any continuous function $f(x)$.

FYI. Other frequently used approximations to $\delta(x)$ include

$$\delta(x) = \lim_{\sigma \to 0+} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\Big(-\frac{x^2}{2\sigma^2}\Big) = \lim_{\eta \to 0+} \frac{1}{\pi x} \sin \frac{x}{\eta}.$$

7. (optional) Implement Radix-3 FFT and Radix-5 FFT. Make sure your implementations have complexity $\Theta(n \log n)$.

You may find the MATLAB/Octave function `fft` helpful for debugging purpose.