March 14, 2022 (Due: 08:00 March 21, 2022)

1. Interpolate the function $f(x) = |x|$ over $[-1, 1]$ using polynomials. Use equispaced nodes and Chebyshev nodes and plot the results. What do you observe?
2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function. Show that

$$\lim_{(x_1,\ldots,x_k)\to(x_*,\ldots,x_*)} f[x_1, x_2, \ldots, x_k] = \frac{f^{(k-1)}(x_*)}{(k-1)!}$$

for any $x_* \in \mathbb{R}$.
3. Try to simplify Newton's interpolation polynomial for equally spaced interpolation nodes $x_1 < x_2 < \cdots < x_n$ (with $x_i = x_1 + (i-1)h$).
4. Approximate the sine function over the closed interval $[0, 2\pi]$ using piecewise cubic Hermite interpolation, and visualize your result. You are recommended to partition the interval with n equally spaced interpolation nodes for $n = 2$, 3, 5, 9.
5. (optional) This exercise is about an atypical approach for two-dimensional interpolation.

Interpolating a data set $\{(x_i, y_i, z_i)\}_{i=1}^n \subset \mathbb{R}^3$ can be understood as interpolating $\{(x_i + \mathrm{i}y_i, z_i)\}_{i=1}^n \subset \mathbb{C} \times \mathbb{R}$, where the interpolation nodes $x_i + \mathrm{i}y_i$'s are complex numbers. The polynomial interpolation techniques we have learned from this course theoretically carry over to complex inputs, while the resulting interpolation polynomial is in general complex-valued. Nevertheless, we can take the real part of the output.

Use this approach to interpolate the following data set over the unit disk and visualize the result.

x_i	y_i	z_i
1.00000	0.00000	−1.0000
0.80902	0.58779	−2.6807
0.30902	0.95106	5.6161
−0.30902	0.95106	5.6161
−0.80902	0.58779	−2.6807
−1.00000	0.00000	−1.0000
−0.80902	−0.58779	−2.6807
−0.30902	−0.95106	5.6161
0.30902	−0.95106	5.6161
0.80902	−0.58779	−2.6807

(If you use MATLAB/Octave, the functions `imagesc` and `colorbar` are useful for visualizing a bivariate function.)