Feb. 28, 2022 (Due: 08:00 Mar. 7, 2022)

1. The Lambert W-function, $y = W(x)$, is the inverse function of $x = y\exp(y)$ for $y \in [-1, +\infty)$. Make a plot of the Lambert W-function, with at least 100 equally spaced sampling points over x. Verify your plot using the graph of $x = y\exp(y)$.
2. Use bisection and *regula falsi*, respectively, over the interval $[0, 1]$ to find the root of $x^{64} - 0.1 = 0$ with absolute accuracy 10^{-12}. Visualize the convergence history of these methods in one figure.
3. Using Newton's method to find the root of $\arctan x = 0$ is an overkill, since the unique solution, $x_* = 0$, is trivial. However, this is a good example to see that the convergence of Newton's method relies on the initial guess. The set of real initial guesses such that Newton's method converges to x_* is of the form $(-\alpha, \alpha)$, where $\alpha > 0$. Try to calculate α with at least 10 significant decimal digits. What happens if α is used as the initial guess?
4. When applying Newton's method to solve the equation $f(x) = 0$, we usually require that $f'(x_*) \neq 0$, i.e., the root x_* is a simple one. Without such a condition, Newton's method is still applicable to find x_* while the convergence is no longer quadratic.
(a) Use Newton's method to solve $1 + \cos x = 0$ around $x_0 = 3$ and plot the convergence history.
Parts (b) and (c) are optional. Let us assume that $f(x)$ is sufficiently smooth to avoid complications in theoretical analysis.
(b) Let x_* be a root of $f(x)$ with multiplicity higher than one, i.e.,

$$h(x_*) = f'(x_*) = 0.$$

Show that Newton's method converges (locally) linearly around x_*.
(c) Let x_* be a root of $f(x)$ with multiplicity $m > 1$, i.e.,

$$f(x_*) = f'(x_*) = \cdots = f^{(m)}(x_*) = 0 \neq f^{(m+1)}(x_*).$$

We can modify Newton's method as

$$x_{k+1} = x_k - \frac{(m+1)f(x_k)}{f'(x_k)}$$

to achieve local quadratic convergence. Try to explain why such a modification improves the convergence.
5. (optional) Visualize the curve $y = (x - 2)^9$ around $x = 2$, where the function $f(x) = (x - 2)^9$ is evaluated through an expanded form. What is the attainable accuracy if bisection is used to find the root of this function?