March 28, 2022 (Due: 08:00 April 4, 2022)

1. Chebyshev polynomials can be expressed as $T_n(x) = \cos(n \arccos x)$ for $x \in [-1, 1]$. Can you find an explicit expression of $T_n(x)$ for $x \in \mathbb{R} \setminus [-1, 1]$ in terms of algebraic functions?

2. Find

$$\min_{a,b,c \in \mathbb{R}} \int_0^{\pi/2} |\sin x - ax^2 - bx - c|^2 \, \mathrm{d}x$$

without programming.

3. Find

$$\min_{a,b \in \mathbb{R}} \max_{-1 \le x \le 2} |x^3 + ax + b|$$

without programming.

4. Find

$$\min_{a,b,c} \max_{0 \le x \le 1} |\ln(1 + x) + ax^2 + bx + c|$$

using Remez algorithm. Visualize the approximation error and the (nonuniform) alternating set by taking a few snapshots. (E.g., you may plot the error curve for the initial guess, two intermediate solutions, as well as the final solution.)

5. (optional) Show that in each step of Remez algorithm, the linear system has a unique solution.

6. (optional) Let $p(x)$ be a real polynomial such that $\deg p(x) \le n$ and

$$\max_{-1 \le x \le 1} |p(x)| \le 1.$$

Show that $|p(t)| \le |T_n(t)|$ for any $t \in \mathbb{R} \setminus [-1, 1]$.