April 11, 2022 (Due: 08:00 April 18, 2022)

1. Develop a quadrature rule for the integral $\int_a^b \cos(mx) f(x)\,\mathrm{d}x$ such that it provides exact results for polynomials of degree up to three.

2. Determine the degrees of exactness of the following 2-D quadrature rules:

$$\int_0^1 \int_0^{1-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{6}\left(f\left(\frac{1}{2}, 0\right) + f\left(0, \frac{1}{2}\right) + f\left(\frac{1}{2}, \frac{1}{2}\right)\right),$$
$$\int_0^1 \int_0^{1-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{6}\left(f\left(\frac{2}{3}, \frac{1}{6}\right) + f\left(\frac{1}{6}, \frac{2}{3}\right) + f\left(\frac{1}{6}, \frac{1}{6}\right)\right).$$

Hint: Check whether the quadrature rules provide exact results for bivariate polynomials 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, $\ldots$

3. Let $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \colon x + y \le 1,\ x \ge 0,\ y \ge 0\}$. Estimate

$$\iint_{\mathcal{D}} \mathrm{e}^x \sin y \,\mathrm{d}x\,\mathrm{d}y$$

by partitioning $\mathcal{D}$ with a triangular mesh and applying a composite quadrature rule. Compare your result with the exact one.

4. Use a linear combination of nine function values $f(x+ih, y+jh)$ (for i, $j \in \{-1, 0, 1\}$) to approximate

$$\frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y)$$

(as accurate as you can). Estimate the truncation error.

5. Use Richardson extrapolation to estimate the derivative of $f(x) = x^3\mathrm{e}^x$ at $x = 1$. Keep a record for intermediate results. What happens if you iterate for many steps?