Unfoldings of 4D-Hypercube Unfoldings that tile $\mathbb{R}^2$

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Abstract

We prove that every face-unfolding of the 4D hypercube has an edge-unfolding that tiles the plane. Hence the hypercube is a "complete dimension-descending tiler". We also illustrate the algorithm used to manifest this result.

1 Introduction

It is a well-known fact that the cube has 11 distinct (incongruent) edge-unfoldings each of which tiles $\mathbb{R}^2$. So the cube has the delightful property that it itself tiles $\mathbb{R}^3$, and all of its unfoldings tile $\mathbb{R}^2$. The cube is therefore a "complete dimension-descending tiler" (see section 2).

We consider this property in a higher dimensional cube: the 4D cube (which we also refer to as the *hypercube*). The 4D cube tiles $\mathbb{R}^4$ and admits 261 distinct face-unfoldings to order-8 polycubes [1]. A natural question would then be, do all of these unfoldings tile 3D space.

In 2015, G. Diaz and J. O' Rourke [2] explored this very question; and 4 (of the 261) face-unfoldings of the hypercube were shown to tile $\mathbb{R}^3$. Later in 2021, whuts.org [3], M. Firsching [4], and G. Papoutsis [5] independently showed that in fact all 261 unfoldings tile $\mathbb{R}^3$.

Interestingly, G. Diaz and J. O' Rourke [2] also proved that one of the face-unfoldings had an edge-unfolding that then tiled $\mathbb{R}^2$. Likewise, S. Langerman and A. Winslow [6] proved that another face-unfolding, the *dali-cross,* also admits an edge-unfolding that tiles $\mathbb{R}^2$.

As far as we are aware, *only* 2 face-unfoldings of the hypercube were known to have an edge-unfolding that tiles the plane. Therefore, we consider the following question posed by S. Langerman [7]:

Question 1 *Does every face-unfolding of the hypercube have an edge-unfolding that tiles space?*

We answer this positively in Section 3 and illustrate the algorithm adopted in the brute-force search to prove this. We, in a way, complete the journey initiated in 2015.

2 Definitions

Dual Graph. The *dual graph* of a polycube has a vertex for each *square face* and edges between edge-adjacent squares.

Complete Dimension-descending Tiler (c-DDT). A polytope that monohedrally tiles $\mathbb{R}^d$ is a c-DDT if all of its facet-unfoldings tile $\mathbb{R}^{d-1}$, and every $\mathbb{R}^{d-1}$ polytope has a facet-unfolding that tiles $\mathbb{R}^{d-2}$. The $\mathbb{R}^{d-2}$ polytope must then have a facet-unfolding that tiles $\mathbb{R}^{d-3}$ and so on till an edge-unfolding that tiles $\mathbb{R}^2$.

Linear Unfolding [8] Let T be a hamiltonian path on the dual graph of the polycube. The unfolding corresponding to T is linear if direction x is used and direction –x is not.

3 Results

We briefly describe the exhaustive search algorithm used to prove Theorem 3.1 and then demonstrate this in Example 3.2 on the *Dali-cross* unfolding of the hypercube.

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Previous approaches that attempted unfolding the face-unfoldings relied on finding spanning trees, on their *dual graphs,* to generate edge-unfoldings [2]. The algorithm we employ generates hamiltonian paths to enumerate unfoldings and then filters *linear edge-unfoldings*. Any linear unfolding tiles the plane. Jelmer Firet [8] proved that every linear unfolding will tile space (and used a similar algorithm to show that linear *face-unfoldings* of the hypercube tile 3D space).

Theorem 3.1 *Every face-unfolding of the hypercube has an edge-unfolding that tiles* $\mathbb{R}^2$.

Example 3.2 *Consider the Dali-cross face-unfolding of the hypercube (see Figure 1 (a)). A hamiltonian path (blue) on its dual graph (black) is shown in Figure 1 (b). The corresponding unfolding happens to be linear and is therefore a monohedral tile. (see Figure 2).*

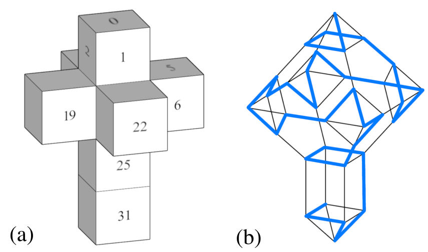

Figure 1: (a) *Dali-cross* unfolding of the hypercube (the numbers correspond to those in Figure 2). (b) A hamiltonian path on the polycube's dual tree *mapping* to the unfolding in Figure 2.

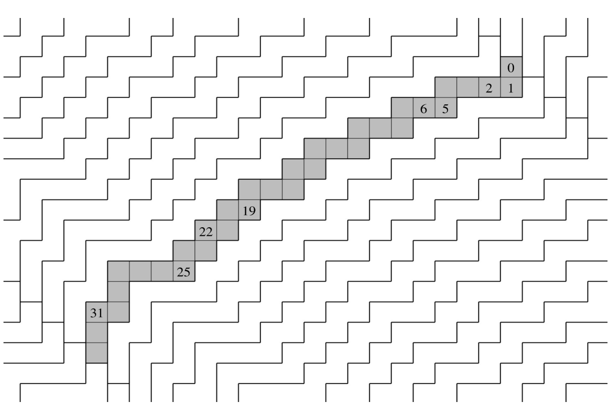

Figure 2: *Linear unfolding* of the *dali-cross* tiling space. Faces are numbered according to position in the unfolding and correspond to Figure 1 (a).

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The following repository contains edge-unfolding to every face-unfolding of the hypercube:
T. Ramteke. HyperCube Unfoldings Data, 2022. https://github.com/TrunInGitHub/HyperCube_Unfoldings_Data.