Hypothesis Test for Linear Regession

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Statistical analysis and document mining
Complementary course, MSIAM

Outline

1. Simple linear regression
   - Estimation of the parameters by least squares
   - Motivation: advertising data
   - Assessing the accuracy of the coefficient estimates
2. Hypothesis tests on the coefficients
   - Review of hypothesis testing and p-values
   - The t-test versus Wald test
   - Applying for simple linear regression
   - Assessing the overall accuracy of the model

Simple linear regression

- We assume a model
$$Y = \beta_0 + \beta_1 X + \epsilon,$$
where ϵ is the error term, and two unknown constants (also known as coefficients or parameters)
  - β_0: intercept,
  - β_1: slope.
- The **hat** symbol denotes an estimated value. Given some estimates $\widehat{\beta}_0$ and $\widehat{\beta}_1$ for β_0 and β_1, respectively, we define a prediction of Y based on the basis of $X = x$ as follows
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- Given n independent observations $\left(x_{[N]}, y_{[N]}\right) \equiv \{(x_n, y_n)\}_{n\in[N]}, [N] \equiv \{1, \ldots, N\}$, our goal is to obtain coefficient estimates $\widehat{\beta}_0$ and $\widehat{\beta}_1$ such that $y_n \approx \widehat{\beta}_0 + \widehat{\beta}_1 x_n, n \in [N]$.

Outline

Estimation of the parameters by least squares

- The least squares approach chooses $\widehat{\beta}_0$ and $\widehat{\beta}_1$ to minimize the residual sum of squares (RSS)

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- By using calculus, $\left(\frac{\partial \, \mathrm{RSS}}{\partial \widehat{\beta}_1}, \frac{\partial \, \mathrm{RSS}}{\partial \widehat{\beta}_0} \right) = (0, 0)$, the minimizing values can be shown to be (see for example chapter 3 from [Hastie et al., 2009, James et al., 2021])

$$\widehat{\beta}_1 = \frac{\sum_{n=1}^{N} (x_n - \bar{x})(y_n - \bar{y})}{\sum_{n=1}^{N} (x_n - \bar{x})^2}, \quad \widehat{\beta}_0 = \bar{y} - \widehat{\beta}_1 \bar{x},$$

where $\bar{x} = \frac{1}{N} \sum_{n=1}^{N} x_n$ and $\bar{y} = \frac{1}{N} \sum_{n=1}^{N} y_n$ are the sample means.

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  where $\bar{x} = \frac{1}{N} \sum_{n=1}^{N} x_n$ and $\bar{y} = \frac{1}{N} \sum_{n=1}^{N} y_n$ are the sample means.
- This is a minimum (and not a maximum or saddle point): RSS is a quadratic function and has positive coefficients of the squared term of $\widehat{\beta}_0$ and $\widehat{\beta}_1$.

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Motivation: advertising data

- **Description:** set consists of the sales of that product in 200 different markets, along with advertising budgets for the product in each of those markets for three different media: TV, radio, and newspaper [James et al., 2021, Chapter 2].

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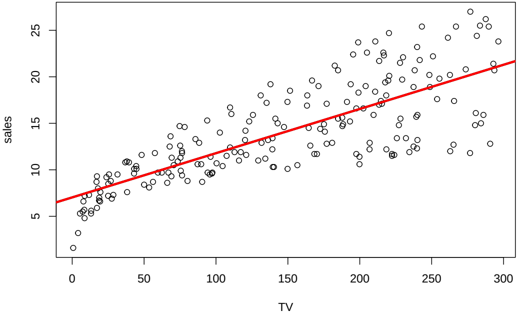
- **Description:** set consists of the sales of that product in 200 different markets, along with advertising budgets for the product in each of those markets for three different media: TV, radio, and newspaper [James et al., 2021, Chapter 2].
- **Goal:** develop an accurate model that can be used to predict sales on the basis of the three media budgets $\leftarrow$ Linear regression in $\boldsymbol{R}$.



Goal: understand how linear regression works in R

```
Call:
lm(formula = sales ~ TV)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.3860 -1.9545 -0.1913  2.0671  7.2124 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 7.032594   0.457843   15.36   <2e-16 ***
TV          0.047537   0.002691   17.67   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.259 on 198 degrees of freedom
Multiple R-squared:  0.6119,    Adjusted R-squared:  0.6099 
F-statistic: 312.1 on 1 and 198 DF,  p-value: < 2.2e-16
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Assessing the accuracy of the coefficient estimates

1. Note that the estimated parameters $\widehat{\beta}_0$ and $\widehat{\beta}_1$ are unbiased (TD1), we wonder how close $\widehat{\beta}_0$ and $\widehat{\beta}_1$ are to the true values β_0 and β_1 $\rightarrow$ computing the standard error, $\mathrm{SE}(\widehat{\beta}_i) = \mathrm{var}\left(\widehat{\beta}_i\right)^{1/2}, i = 0, 1$.

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2. When $\epsilon_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2), n \in [N]$, we can show that (TD1)

$$\mathrm{var}\left(\widehat{\beta}_0\right) = \frac{\sigma^2}{N}\left(1 + \frac{\bar{x}^2}{s_X^2}\right), \quad \mathrm{var}\left(\widehat{\beta}_1\right) = \frac{\sigma^2}{N}\frac{1}{s_X^2}.$$

where $\sigma^2 = \mathrm{var}\left(\epsilon\right)$, $s_X^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \bar{x})^2$.

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where $\sigma^2 = \mathrm{var}\left(\epsilon\right)$, $s_X^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \bar{x})^2$.
3. These standard errors can be used to compute confidence intervals. A 95% confidence interval is defined as a range of values such that with 95% probability, the range will contain the true unknown value of the parameter. Using normal-based confidence interval [Wasserman, 2004, Theorem 6.16], for $i = 0, 1$, it has the form
$$[\widehat{\beta}_i - 2 \times \mathrm{SE}(\widehat{\beta}_i), \widehat{\beta}_i + 2 \times \mathrm{SE}(\widehat{\beta}_i)] \text{ since } \widehat{\beta}_i \sim \mathcal{N}\left(\beta_i, \mathrm{SE}(\widehat{\beta}_i)\right).$$

Recall the normal-based confidence interval

Recall the normal-based confidence interval [Wasserman, 2004]:

6.16 Theorem (Normal-based Confidence Interval). *Suppose that* $\widehat{\theta}_n \approx N(\theta, \widehat{\mathsf{se}}^2)$. *Let* Φ *be the* CDF *of a standard Normal and let* $z_{\alpha/2} = \Phi^{-1}(1-(\alpha/2))$, *that is,* $\mathbb{P}(Z > z_{\alpha/2}) = \alpha/2$ *and* $\mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1-\alpha$ *where* $Z \sim N(0,1)$. *Let*

$$C_n = (\widehat{\theta}_n - z_{\alpha/2}\,\widehat{\mathsf{se}},\ \widehat{\theta}_n + z_{\alpha/2}\,\widehat{\mathsf{se}}). \tag{6.10}$$

Then

$$\mathbb{P}_\theta(\theta \in C_n) \to 1-\alpha. \tag{6.11}$$

PROOF. Let $Z_n = (\widehat{\theta}_n - \theta)/\widehat{\mathsf{se}}$. By assumption $Z_n \rightsquigarrow Z$ where $Z \sim N(0,1)$. Hence,

$$\begin{aligned}
\mathbb{P}_\theta(\theta \in C_n) &= \mathbb{P}_\theta\left(\widehat{\theta}_n - z_{\alpha/2}\,\widehat{\mathsf{se}} < \theta < \widehat{\theta}_n + z_{\alpha/2}\,\widehat{\mathsf{se}}\right) \\
&= \mathbb{P}_\theta\left(-z_{\alpha/2} < \frac{\widehat{\theta}_n - \theta}{\widehat{\mathsf{se}}} < z_{\alpha/2}\right) \\
&\to \mathbb{P}\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) \\
&= 1-\alpha. \quad \blacksquare
\end{aligned}$$

For 95 percent confidence intervals, $\alpha = 0.05$ and $z_{\alpha/2} = 1.96 \approx 2$ leading to the approximate 95 percent confidence interval $\widehat{\theta}_n \pm 2\,\widehat{\mathsf{se}}$.

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Hypothesis tests on the coefficients

Standard errors can also be used to perform hypothesis tests on the coefficients. The most common hypothesis test involves testing the null hypothesis of

- $\mathcal{H}_0$: There is no relationship between X and Y versus the alternative hypothesis
- $\mathcal{H}_1$: There is some relationship between X and Y.

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- Let X be a random variable and let $\mathcal{X}$ be the range of X. Given rejection region R, then
  - $X \in R \implies$ reject $\mathcal{H}_0$,
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  - $X \in R \Longrightarrow$ reject $\mathcal{H}_0$,
  - $X \notin R \Longrightarrow$ retain (do not reject) $\mathcal{H}_0$.
- Usually, the rejection region R is of the form $R = \{x : T(x) \geq c\}$, where T is a test statistic and c is a critical value.
  $\Longrightarrow$ Hypothesis testing $\longleftrightarrow$ find appropriate T and c.

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- The Wald test: $\mathcal{H}_0 : \theta = \theta_0$ versus $\mathcal{H}_1 : \theta \neq \theta_0$.
  - Assume that $\hat{\theta}$ is asymtotically Normal: $\frac{\hat{\theta}-\theta_0}{\mathsf{SE}(\hat{\theta})} \rightsquigarrow \mathcal{N}(0,1)$, where $\hat{\theta}$ and $\mathsf{SE}(\hat{\theta})$ are estimate of θ and estimated standard error of $\hat{\theta}$, respectively.

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  - We can show that, asymptotically, the Wald test has size α. Indeed, by using asymptotically Normal,

$$\mathbb{P}_{\theta_0}\left(|W| > z_{\alpha/2}\right) = \mathbb{P}_{\theta_0}\left(\frac{|\hat{\theta} - \theta_0|}{\mathsf{SE}(\hat{\theta})} > z_{\alpha/2}\right) \to \mathbb{P}(|Z| \geq z_{\alpha/2}) = \alpha.$$

Theorem (Scientific significance versus statistical significance)

The size α Wald test rejects $\mathcal{H}_0 : \theta = \theta_0$ (say statistically significant) versus $\mathcal{H}_1 : \theta \neq \theta_0$ if and only if $\theta_0 \notin C$ where $C = \left(\hat{\theta} - \mathsf{SE}(\hat{\theta}) z_{\alpha/2}, \hat{\theta} + \mathsf{SE}(\hat{\theta}) z_{\alpha/2}\right)$ is $1 - \alpha$ asymptotic confidence interval.

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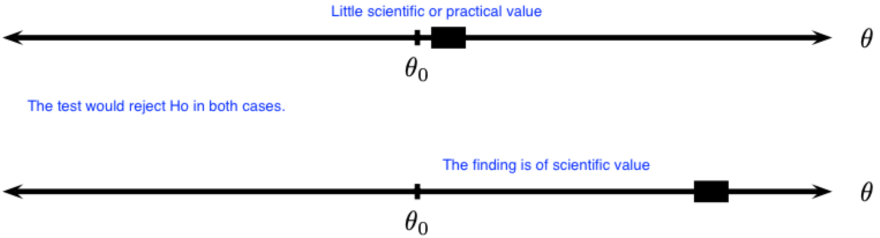
Therefore, testing the hypothesis $\Longleftrightarrow$ checking whether the null value is in the confidence interval.

Statistical significance $\nrightarrow$ scientific importance.
Confidence intervals are often more informative than tests.



Definition (p-values)

Suppose that for every $\alpha \in (0,1)$, we have a size α test with rejection region $\mathbb{R}_\alpha$. Then. p-value $= \inf \{\alpha : T(x) \in R_\alpha\}$. That is, the p-value is the smallest level at which we can reject $\mathcal{H}_0$.

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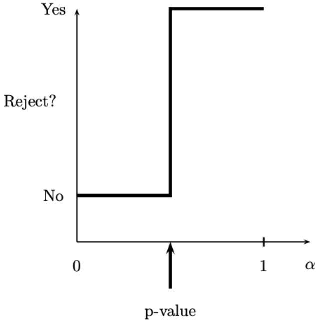
Informally, the smaller the p-value, the stronger the evidence against $\mathcal{H}_0$.
BUT, large p-value is not strong evidence in favor of $\mathcal{H}_0$: (i) $\mathcal{H}_0$ is true or (ii) $\mathcal{H}_0$ is false but the test has low power.
DO NOT CONFUSE: p-value $\neq \mathbb{P}(\mathcal{H}_0|\text{Data})$.



Theorem (Compute the p-values)

Suppose that the size α test is of the form reject $\mathcal{H}_0$ if and only if $T(X_{[N]}) \geq c_\alpha$. Then, given the observed value $x_{[N]}$ of random sample $X_{[N]}$,

$$p\text{-}value = \sup_{\theta \in \Theta_0} \mathbb{P}_{\theta_0}\left(T(X_{[N]}) \geq T(x_{[N]})\right).$$

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Let $w = \hat{\theta} - \theta_0 / \operatorname{SE}(\hat{\theta})$ denote the observed value of the Wald statistic W,

$$p\text{-}value = \mathbb{P}_{\theta_0}(|W| \geq |w|) \approx \mathbb{P}(|Z| \geq |w|) = 2\Phi(-|w|), Z \sim \mathcal{N}(0,1).$$

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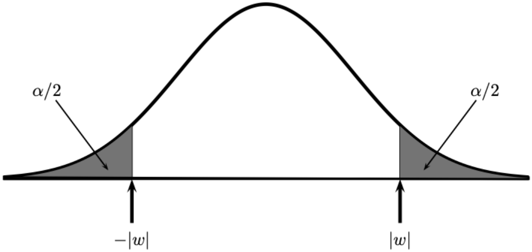
Informally, p-value = the probability (under H0) of observing a value of the test statistic the same as or more extreme than what was actually observed.



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Let $X_n \sim \mathcal{N}(\mu, \sigma^2), n \in [N]$ where μ, σ^2 are both unknown. Suppose we want to test $\mathcal{H}_0 : \mu = \mu_0$ versus $\mathcal{H}_1 : \mu \neq \mu_0$. We choose

$$T = \frac{\sqrt{N}(\bar{X}_N - \mu_0)}{S_n} \sim t_{N-1} \text{ under } \mathcal{H}_0,$$

where $S_n^2 = \frac{1}{N-1} \sum_{n=1}^{N} (X_n - \bar{X}_N)^2$ is the sample variance and t_{N-1} is Student's t-distributiont with $N - 1$ degrees of freedom.

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When N is moderately large, $T \approx \mathcal{N}(0,1)$ under $\mathcal{H}_0$: the t-test is essentially identical to the Wald test.

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- Standard errors can also be used to perform hypothesis tests on the coefficients. The most common hypothesis test involves testing the null hypothesis of
  - $\mathcal{H}_0$: There is no relationship between X and Y versus the alternative hypothesis
  - $\mathcal{H}_1$: There is some relationship between X and Y.
- Mathematically, this corresponds to testing

$$\mathcal{H}_0 : \beta_1 = 0 \text{ versus } \mathcal{H}_1 : \beta_1 \neq 0.$$

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- To test the null hypothesis, we compute a t-statistics, given by

$$t = \frac{\widehat{\beta}_1 - 0}{\mathrm{SE}(\widehat{\beta}_1)} \sim t_{N-2} \text{ assuming } \beta_1 = 0.$$

- Using statistical software, it is easy to compute the probability of observing any value equal to $|t|$ or larger, p-value $= \mathbb{P}(|T| \geq |t|)$.

Outline

1. Simple linear regression
   - Estimation of the parameters by least squares
   - Motivation: advertising data
   - Assessing the accuracy of the coefficient estimates
2. Hypothesis tests on the coefficients
   - Review of hypothesis testing and p-values
   - The t-test versus Wald test
   - Applying for simple linear regression
   - Assessing the overall accuracy of the model

Assessing the overall accuracy of the model

- Given the Residual sum-of-squares $\mathrm{RSS} = \sum_{n=1}^{N}(y_n - \hat{y}_n)^2$, we compute the Residual Standard Error

$$\mathrm{RSE} = \sqrt{\frac{1}{N-2}\mathrm{RSS}} = \sqrt{\frac{1}{N-2}\sum_{n=1}^{N}(y_n - \hat{y}_n)^2}.$$

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- R-squared or fraction of variance explained is

$$R^2 = \frac{\text{TSS} - \text{RSS}}{\text{TSS}} = 1 - \frac{\text{RSS}}{\text{TSS}}.$$

where the total sum of square is $\text{TSS} = \sum_{n=1}^{N}(y_n - \bar{y})^2$.

Goal: understand how linear regression works in R

```
Call:
lm(formula = sales ~ TV)

Residuals:
    Min      1Q  Median      3Q     Max
-8.3860 -1.9545 -0.1913  2.0671  7.2124

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 7.032594   0.457843   15.36   <2e-16 ***
TV          0.047537   0.002691   17.67   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.259 on 198 degrees of freedom
Multiple R-squared:  0.6119,    Adjusted R-squared:  0.6099
F-statistic: 312.1 on 1 and 198 DF,  p-value: < 2.2e-16
```

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