Proofs

Problem 1 (EG 7.70). Determine for each of the following expressions whether it is a tautology, a contradiction, or neither.

(a) $(\neg q \wedge p) \vee (p \rightarrow q)$

(b) $(p \wedge q) \vee (\neg q \rightarrow q) \vee p$

(c) $(p \wedge q) \wedge \neg((p \leftrightarrow q) \vee (p \vee q))$

(d) $((p \vee q) \leftrightarrow q) \wedge (\neg q \wedge (p \rightarrow q))$

(e) Convert the expression that is a tautology to 1, using the rules for rewriting, and the one that is a contradiction to 0.

Predicates and Sets

Predicates

Problem 2 (EG 7.47). Write "There is more than one object having property P" using predicate-logical symbols.

Problem 3 (EG 7.48). We have a universe consisting of the letters a-f. $C(x)$ represents "x is a consonant" and $V(x)$ "x is a vowel". Which of the following statements are true?

(a) $C(a)$

(b) $V(a)$

(c) $C(b)$

(d) $V(b)$

(e) $\forall x C(x)$

(f) $\exists y C(y)$

(g) $\forall z V(z)$

(h) $\exists t V(t)$

(i) $\forall u(C(u) \vee V(u))$

(j) $\exists v(C(v) \wedge V(v))$

Problem 4 (EG 7.56). Rewrite $\exists x P(x) \rightarrow \forall x Q(x)$ so that all the quantifiers end up in front.

Problem 5 (EG 7.58). Find some examples where $\forall x \exists y P(x, y)$ is true while $\exists y \forall x P(x, y)$ is false.

Problem 6 (EG 7.59). Find some examples where $\forall x \exists y P(x, y)$ and $\exists y \forall x P(x, y)$ actually mean the same thing.

Problem 7 (EG 7.60). Is it possible to find an example where $\forall x \exists y P(x, y)$ is false while $\exists y \forall x P(x, y)$ is true?

Problem 8 (EG 7.78). We have the set $A = \{1, 2, \{1, 3\}\}$. Start by making a list of all elements in A and another one listing all subsets of A. Then determine whether the following statements are true or false:

(a) $1 \in A$.

(b) $1 \subseteq A$.

(c) $\{1\} \in A$.

(d) $\{1\} \subseteq A$.

(e) $\{1,2\} \in A$.

(f) $\{1,2\} \subseteq A$.

(g) $\{1,3\} \in A$.

(h) $\{1,3\} \subseteq A$.

Sets

Problem 9 (EG 2.52)**.** We have the set $A = \{1,2,\{1,3\}\}$. Start by making a list of all elements in A and another one listing all subsets of A. Then determine whether the following statements are true or false:

(a) $1 \in A$.

(b) $1 \subseteq A$.

(c) $\{1\} \in A$.

(d) $\{1\} \subseteq A$.

(e) $\{1,2\} \in A$.

(f) $\{1,2\} \subseteq A$.

(g) $\{1,3\} \in A$.

(h) $\{1,3\} \subseteq A$.

Problem 10 (EG 2.54)**.** Determine the following sets:

(a) $\{2,3,5,7\} \cup \{1,3,5,7,9\}$

(b) $\{2,3,5,7\} \cap \{1,3,5,7,9\}$

(c) $\{2,3,5,7\} \setminus \{1,3,5,7,9\}$

(d) $\{1,3,5,7,9\} \setminus \{2,3,5,7\}$

(e) $\{x \mid x \text{ is an even natural number}\} \cup \{x \mid x \text{ is an odd natural number}\}$

(f) $\{x \mid x \text{ is an even natural number}\} \cap \{x \mid x \text{ is an odd natural number}\}$

(g) $\{x \mid x \text{ is an even natural number}\} \setminus \{x \mid x \text{ is an odd natural number}\}$

Problem 11 (EG 2.55)**.** The following equality holds:

$$(A \cup B) \cap (A \cap B)^c = (A \cap B^c) \cup (A^c \cap B)$$

(a) Show the equality using the rules. Note in every step which rule you are using.

(b) Illustrate the equality using Venn diagrams.

(c) Describe in words the set in question.

(d) Prove the equality using a formal set-theoretical argument.

Problem 12 (EG 2.56)**.** We have the set $A = \{1, 2, \dots, 10\}$ and the set $B = \{a, b, \dots, j\}$. C is the subset of $A \times B$ where the number is odd, D is the subset of $A \times B$ where the letter is a vowel. Calculate $|C \cup D|$.

Functions and Relations

Relations

Problem 13 (EG 8.56)**.** Here is a relation:

$$R = \{(a,a),(a,b),(a,b),(a,c),(a,f),(b,a),(b,b),(b,c),(b,f),(c,a),\\ (c,b),(c,c),(d,d),(d,e),(e,d),(e,e),(f,a),(f,b),(f,f)\}$$

.

(a) Draw the graph of the relation.

(b) Write down the matrix of the relation.

(c) Is the relation reflexive?

(d) Is the relation symmetric?

(e) Is the relation anti-symmetric?

(f) Is the relation transitive?

(g) Is the relation an equivalence relation.

Problem 14 (EG 8.57)**.** Answer the same questions as in the previous exercise for the relation $R = \{(a,a),(a,b),(b,b),(b,c),(c,c),(d,c),(d,e),(e,e),(e,f),(f,a),(f,c),(f,f)\}$.

Functions

Problem 15 (EG 8.64)**.** We are studying the function $f(x) = 2x$, $f : \mathbb{Z} \to \mathbb{Z}$.

(a) Is f injective?

(b) Is f surjective?

(c) What does the range look like?

Now assume that we are calculating modulo 9 instead, that is to say that f is defined as $f : \mathbb{Z}_9 \to \mathbb{Z}_9$.

(a) Is f injective?

(b) Is f surjective?

(c) What does the inverse of f look like?

Problem 16 (EG 8.69)**.**

(a) How many different relations are there in total on a set with n elements?

(b) How many out of these are reflexive?

(c) How many are symmetric?

(d) How many are anti-symmetric?

(e) How many are transitive?

(f) How many are equivalence relations?

(g) How many are partial orders?

Induction

Problem 17 (EG 4.2)**.** Write a recursive definition of integer exponentiation.

(a) What is a suitable base case? (That is, is there any power that can be found without calculations?)

(b) What is a suitable recursive part? (That is: How can you figure out a "difficult" power with the help of a somewhat simpler one?)

(c) Put the parts together to make a complete definition.

(d) Calculate 5^3 using this.

Problem 18 (EG 4.34)**.** The *factorial function* is defined as $n! = \prod_{k=1}^{n} k$. ("$n!$" is pronounced "n-factorial".)

Find an n such that $n! > 2^n$, and prove that the inequality holds for every number from this point.

Problem 19 (EG 4.38)**.** Write a recursive expression giving the number of binary strings of length n that do not include any consecutive zeros.

Problem 20 (EG 4.55)**.** A number sequence is defined as follows:

$$\begin{cases} a_3 = 4 \\ a_{n+1} = a_3 + a_4 + \cdots + a_n, \quad n \geq 3 \end{cases}$$

Statement: All numbers in the sequence are divisible by 4.

(a) If you want to prove this, what does the induction hypothesis have to look like?

(b) Prove the statement.

Graphs and Trees

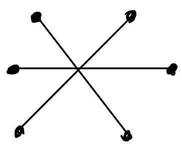
Problem 21 (EG 6.9)**.** The shortest walk between two points is always a path. Explain why!

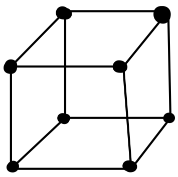
Problem 22 (EG 6.11)**.** Explain why the number of connected components has to be less than or equal to the number of vertices, but the number of edges can be both greater and smaller than both these numbers.

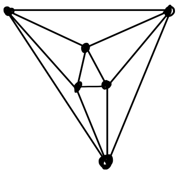
Problem 23 (EG 6.12)**.** Draw K_1, K_2, and $K_{1,2}$.

Problem 24 (EG 6.13)**.** Explain why the number of edges in the complete graph K_n is $\binom{n}{2}$.

Problem 25 (EG 6.15)**.** Here are three graphs. Determine whether they are bipartite!

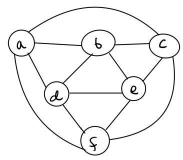
(a) 

(b) 

(c) 

Problem 26 (EG 6.18)**.** If a graph describes a social network, that is to say: which people know which, what does the complement graph then describe?

Problem 27 (EG 6.19)**.** What does it mean if a social network can be described with a complete graph? When do such situations occur?

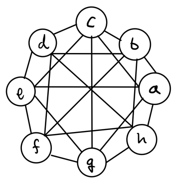


Problem 28 (EG 6.31)**.** Find at least two different Eulerian circuits in the following graph.

Problem 29 (EG 6.32)**.** Model the bridges of Königsberg using a multigraph and explain why the problem is equivalent to finding an Eulerian circuit in this graph.

Problem 30 (EG 6.56)**.** Explain why a cycle will always be formed if an edge is added between two nodes in a tree.

Problem 31 (EG 6.105)**.**



(a) Write down the adjacency matrix of the following graph.

(b) Write down the adjacency matrix of the complement of the graph.

(c) In general: how do you find the adjacency matrix of the complement of a graph, the adjacency matrix of the graph being given?

(d) What will the solution of this problem look like if you have the incidence matrix instead?

Problem 32 (EG 6.117)**.** Let $\lceil x \rceil$ denote the rounding upwards of the number x. Explain why $\lceil \log_2(n+1) \rceil - 1$ is the lowest possible height of a binary tree having n nodes.

Combinatorics

Problem 33 (EG 5.77)**.** Show the equality

$$\binom{n+2}{k} = \binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$$

where n and k are at least 2.

(a) using the formula for binomial numbers (algebraic proof).

(b) by analysing the meaning of the expressions (combinatorial proof).

Problem 34 (EG 5.80)**.** A *binary string* of length n is a sequence of n zeros and/or ones.

(a) How many binary strings of length 8 are there?

(b) How many of those include exactly two ones?

(c) How many include an even number of ones?

1. Number Theory

Problem 35 (EG 3.3)**.** Calculate the quotient and the principal remainder when 35 is divided by 4 and when 40 is divided by -3. Write down all steps in the calculations.

Problem 36 (EG 3.5)**.** Which numbers are divisors of zero, and why? And which numbers does zero itself divide? Which numbers have one as a divisor, and which numbers divide one?

Problem 37 (EG 3.7)**.** Which is the smallest prime number?

Problem 38 (EG 3.8)**.** Which is the smallest composite number?

Problem 39 (EG 3.15)**.** Use the Euclidean algorithm on the numbers 100 and 70 to check that it really gives the answer 10.

Problem 40 (EG 3.62)**.** Draw a number circle representing $\mathbb{Z}_8$. Then find $6+5$ and $2-7$ using the circle.