Chapter 4 Random Processes

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4 Random Processes

4.1 Definition of a Random Process

From chapter 2 and 3 we leaned that an experiment is specified by the three tuple $(S, BF, P(\cdot))$,where S is a countable or noncountable set representing the **outcomes** of the experiment, BF is a **Borel field** specifying the set of events for which consistent probabilities exist, and $P(\cdot)$ is a probability measure that allows the calculation of the probability of all the events in the Borel field. A **real random variable** $X(e)$ was defined as a real-valued function of S subject to (a) the event specified by $\{e : X(e)\} \le x$ is a member of the BF for all e, thus guaranteeing the existence of the cumulative distribution, and (b) $P\{e : X(e) = \infty\} = 0$ or $P\{e : X(e) = -\infty\} = 0$, or both.

If, instead of assigning a real value to each e, a time function $X(t,e)$ is defined for each e, we say a **random process** is specified. Roughly speaking, a random process is a family of time functions together with probability measure. For a finite sample space S we can visualize the random process as in Figure 4.1, that of a mapping from the sample space to a space of time waveforms.

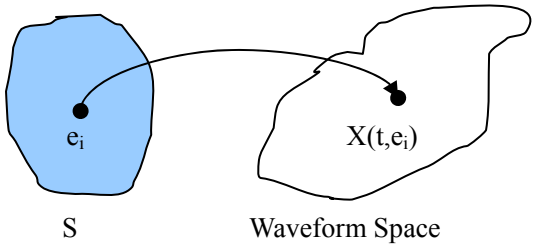


Figure 4.1 A Random process viewed as a functional mapping

A random process can also be viewed as shown in Figure 4.2. For a particular outcome e_i, with probability P_i, the time waveform shown $X(t, e_i)$ occurs. The n time signals represent an ensemble of time waveforms.

If we evaluate these time waveforms at t_0, the values on the second column from the right are obtained . Coming down that column of values, we have a mapping from the outcomes on the far left, and this mapping describes a particular random variable $X(t_0)$. Similarly a random variable $X(t_1)$ is defined for the mapping shown at time t_1 as in the last column. A simple example of random process will now be given.

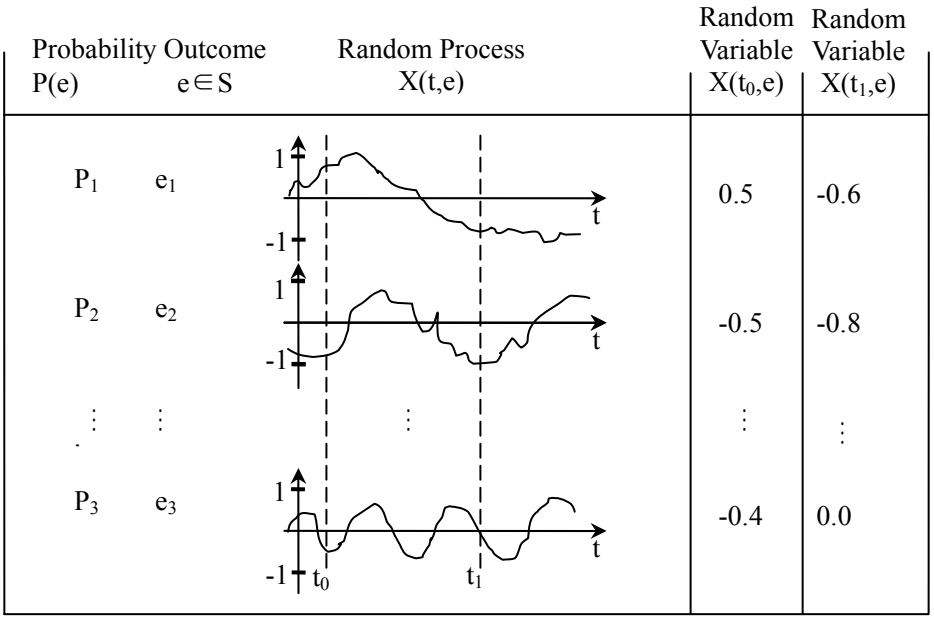


Figure4.2 Illustration of a random process as an ensemble of time waveforms and a related sample space and probability measure.

It should be noted that a **random process** $X(t,e)$ is a **function** of two variables, t and e, where t is the time variable and e is the outcome variable. For a particular time $t=t_0$, $X(t_0,e)$ specifies a **random variable.** If $e=e_i$, or the outcome is fixed, then $X(t,e_i)$ is a particular **time function** called a sample function or realization of the random process. With both t and e fixed, say at t_0 and e_i, $X(t_0,e)$ becomes a **real number** for a real random process or a complex value for a complex random process.

As the notation for a random variable is usually a capital letter X, Y or Z rather than the more formal $X(e), Y(e), Z(e)$, a random process is usually denoted as $X(t)$ rather than $X(t,e)$. Also, it is common to use a small x or $x(t)$, to indicate a particular value of the random variable X or a particular time function of a random process $X(t)$, respectively. Such an $x(t)$, or $X(t,e_i)$, is called a **sample function** or a **realization** of the process.

Also as $X(t,e)$ is a particular random variable for each t, a common alternative definition of a random process is an indexed set of random variables where $t\in\gamma$ the indexing set. If the index set γ can be put into a one to one correspondence with the set of integers, the random process is called a random sequence or a **discrete time random process** if t is time. While, if the indexing set takes on a continuum of values, we will call the resulting process a continuous random process or **continuous time random process**.

4.2 Characterization of a Random Process

We saw in Chapter 2 that partial characterizations for a random variable include its mean, variance, moments, and the like, and similarly, useful partial characterizations for a random process include the mean, variance, and moments. However, all will, in general, be functions of time. For random processes new types of characterizations like the *n*th-order densities, autocorrelation function, and spectral density will be defined that will be useful in analyzing random processes. These characterizations give different statistical properties of the process and different amounts of information concerning the process. A hierarchical relationship between the various types of characterizations exists and is shown in Figure 4.3. Each of the characterizations will now be defined and their relationships explained.

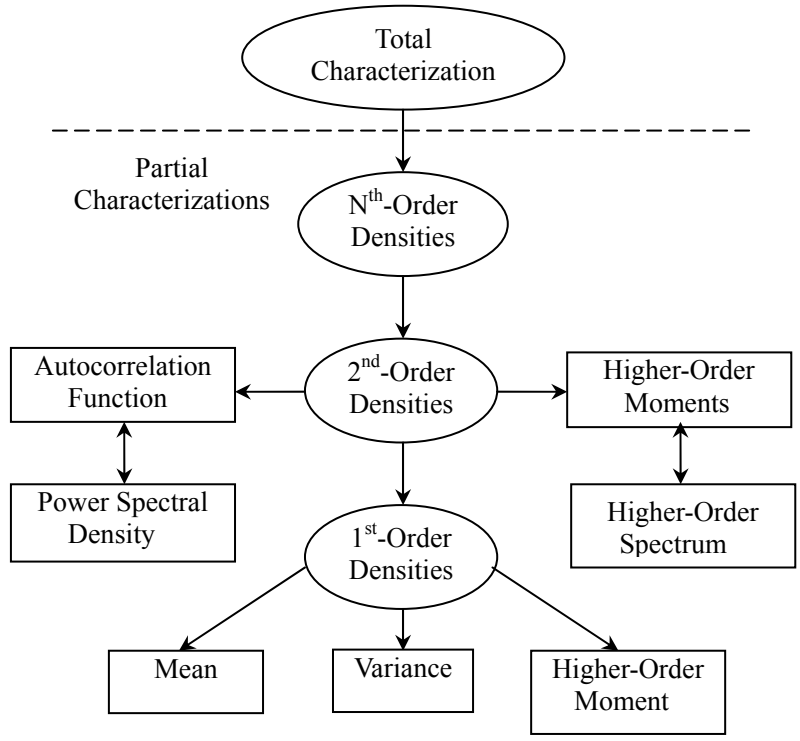


Figure 4.3 Relationships between various characterizations for random process

4.2.1 Total Characterization of a Random Process

Remember that for every t, $X(t)$ is a random variable. This gives us a countably infinite or infinite number of random variables described for the random process. A random process is defined to be completely or **totally characterized** if the joint densities for the random variables $X(t_1), X(t_2),..., X(t_n),$ are known for all times $t_1, t_2, \ldots, t_n$ and all n.

4.2.2 First-Order Densities of a Random Process

The **first-order probability density functions** of the random variables $X(t)\overset{\Delta}{=}X_t$ defined for all time t will be denoted by $f_{X_t}(x_t)$ or $f(x;t)$. If they are all the same, then $f(x;t)$ does not depend on t and we call the resulting density the first-order density of the process; otherwise, we have a family of first-order densities. An example of such a family of first-order densities is shown in Fig. 4.4.

The first-order densities are only a partial characterization as they do not contain information that specifies the joint densities of the random variables defined at two or more different times.

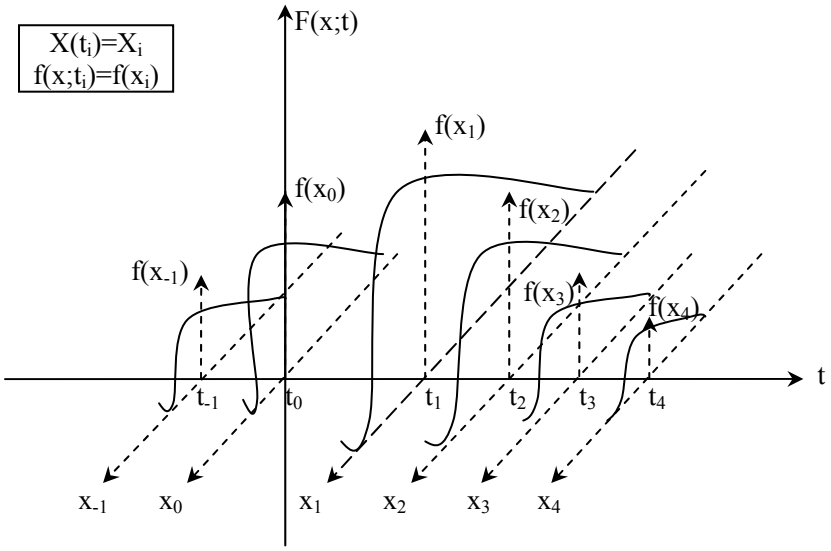


Figure 4.4 Family of first order densities for different times t_i.

4.2.3 Mean of a Random Process

The **mean** of a random process, $\eta_X(t)$, is thus a function of time specified by

$$\eta_X(t) = E[X(t)] = E[X_t] = \int_{-\infty}^{+\infty} x_t f_{X_t}(x_t)dx_t \tag{4.1}$$

For the case where the mean of X_t does not depend on t, we use the notation $E[X(t)] = \eta_X$, where η_X is a constant.

4.2.4 Variance of a Random Process

The **Variance of a Random Process** $X(t)$ is similarly defined by

$$\sigma_X(t) = E\left[(X(t)-\eta_x(t))^2\right] = E[X_t^2] - \eta_X^2(t) \tag{4.2}$$

4.2.5 Second-Order Densities of a Random Process

For any two times t_1 and t_2, the random variables $X(t_1)=X_1$ and $X(t_2)=X_2$ are defined. The second-order densities of the random process $X(t)$ are the family of densities specified for all t_1 and t_2 where t_1 and t_2 are not equal. These densities can be denoted as $f(x_1,x_2)$ or $f(x_1,x_2;t_1,t_2)$.

4.2.6 Autocorrelation and Autocovariance Functions of Random Process

Given the two random variables $X(t_1)$ and $X(t_2)$ we know that a measure of linear relationships between them is specified by the correlation $E[X(t_1)X(t_2)]$.As t_1 and t_2 go through all possible values，this correlation can change and is thus a function of t_1 and t_2. The **autocorrelation function** of a real process is, in general, a two-dimensional of the variables t_1 and t_2 defined by

$$R_{XX}(t_1,t_2)=E[X(t_1)X(t_2)] \tag{4.3}$$

Since $X(t_1)X(t_2)=X(t_2)X(t_1)$, $R_{XX}(t_1,t_2)$ is seen to be symmetrical in t_1 and t_2 as

$$R_{XX}(t_1,t_2)=R_{XX}(t_2,t_1) \tag{4.4}$$

In certain cases, what we will define latter as wide sense stationarity, the autocorrelation function is seen to depend only on the time difference $\tau=t_1-t_2$. For this case, we define a one-dimensional autocorrelation function $R_{XX}(\tau)$ by

$$R_{XX}(\tau)=E[X(t+\tau)X(t)] \tag{4.5}$$

Since the $X(t+\tau)$ can be interchanged with $X(t)$, we see that $R_{XX}(\tau)$ is an even function of τ ,which is written as

$$R_{XX}(-\tau)=R_{XX}(\tau) \tag{4.6}$$

An **autocovariance function**, $C_{XX}(t_1,t_2)$, is defined by

$$C_{XX}(t_1,t_2)=E[(X(t_1)-\eta_X(t_1))(X(t_2)-\eta_X(t_2))]=R_{XX}(t_1,t_2)-\eta_X(t_1)\eta_X(t_2) \tag{4.7}$$

If $C_{XX}(t_1,t_2)$ is normalized, we obtain the **normalized autocovariance** function defined by

$$\rho_{XX}(t_1,t_2)=\frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_1)C_{XX}(t_2,t_2)}} \tag{4.8}$$

In general, the **autocorrelation function** for a **complex random process** $X(t)$ is defined by

$$R_{XX}(t_1,t_2)\overset{\Delta}{=}E[X(t_1)X^*(t_2)] \tag{4.9}$$

Where * stands for complex conjugate. By writing $X(t)$ as $X_R(t)+jX_I(t)$, we can easily show the autocorrelation function to have the properties

$$R_{XX}(t_1,t_2)=R^*_{XX}(t_2,t_1)$$
$$R_{XX}(-\tau)=R^*_{XX}(\tau) \tag{4.10}$$

An **autocovariance function** $C_{XX}(t_1,t_2)$ **for a complex process** $X(t)$ is defined by

$$C_{XX}(t_1,t_2)=E\left[(X(t_1)-\eta_X(t_1))(X(t_2)-\eta_X(t_2))^*\right]=R_{XX}(t_1,t_2)-\eta_X(t_1)\eta_X{}^*(t_2) \tag{4.11}$$

If $C_{XX}(t_1,t_2)$ for a complex process is normalized, the **normalized autocovariance** function is defined by

$$\rho_{XX}(t_1,t_2)=\frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_1)C^*_{XX}(t_2,t_2)}} \tag{4.12}$$

4.2.7 Power Spectral Density of a Random Process

A very important class of random processes are those for which the autocovariance function $R_{XX}(t_1,t_2)$ can be written as a function of the time difference $t_1-t_2=\tau$ as $R_{XX}(\tau)$.This type of process will be described later as being stationary in autocorrelation. This characterization is called a **power spectral density** $S_{XX}(\omega)$ and defined as the Fourier transform of the autocorrelation function as follows:

$$S_{XX}(\omega)=\int R_{XX}(\tau)e^{-j\omega\tau}d\tau \tag{4.13}$$

This formulation roughly describes the region in the frequency domain where power of the process exists and the relative proportion of power at each frequency.

The power spectral density of a given real ransom process has the following properties:

$S_{XX}(-w)=S_{XX}(w)$ Even function of w

$S_{XX}(w)\geq 0$ Real and nonnegative

$S_{XX}(w)=F(R_{XX}(\tau))$
$R_{XX}(\tau)=F^{-1}(S_{XX}(w))$ Fourier Transform Pairs

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(w)dw=R_{XX}(0) \tag{4.14}$$

Total average power in the process

$$\frac{1}{\pi}\int_{w_1}^{w_2}S_{XX}(w)dw=\frac{1}{2\pi}\int_{-w_2}^{-w_1}S_{XX}(w)dw+\frac{1}{2\pi}\int_{w_1}^{w_2}S_{XX}(w)dw$$

Average power in frequency band (w_1,w_2)

4.2.8 Higher-order Moments

For the n random variables $X(t_1),X(t_2),\ldots,X(t_n)$ the **higher-order moments** include $E[X_1X_2X_3]$

$E[X_1X_2X_3X_4]$, $E[X_1^2, X_2^2]$, and so on. In liner systems we will find that these higher-order characterization are not necessary and that the mean and autocorrelation are sufficient to discuss input-output characteristics. On the other hand, for nonlinear systems the order of densities and moments required depends entirely on the type of system considered.

4.2.9 Higher-order Spectra

For certain random process, especially those generated as the output of a nonlinear system, it become conventions to define **higher-order spectra**. Precise definition and properties of higher-order spectra are presented in Chapter 6.

4.2.10 *N*th-Order Densities

The ***n*th order density function** for the random process $X(t)$ at times $t_1, t_2, \ldots, t_n$, are given by $f(x_1, x_2, \ldots, x_n)$ or $f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$ to emphasize the times.

Each of the characterizations of a random process contains statistical information concerning the process, and the various relationships shown in Figure 4.3 provides a way to view the many different statistical characterizations for random processes. Relationships between forms of statistical stationarity exits and will be found useful for describing further statistical properties of random variables.

4.3 Stationarity of Random Process

A random process $X(t)$ is called **strong sense or strict sense stationary** if the sets of random variables $X(t_1), X(t_2), \ldots, X(t_n)$ and $X(t_1+\varepsilon), X(t_2+\varepsilon), \ldots, X(t_n+\varepsilon)$ have the same probability density function for all t_i, all n and all ε. That is for all $t_1, t_2, \ldots, t_n, \varepsilon$ and n, we have

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = f(x_1, x_2, \ldots, x_n; t_1+\varepsilon, t_2+\varepsilon, \ldots, t_n+\varepsilon) \quad (4.15)$$

A slightly weaker form of stationarity is **stationarity of order N** where the conditions are true for all $n \le N$ for N a fixed integer. If the conditions are true for the $n = N$, the joint density may be integrated to give one less variable, making the conditions true for all n less than N. Stationarity of order N does not preclude that the process could be stationary of an order greater than N. However, if it is not stationary of order N, it will be not stationary of any order greater than N.

4.3.1 Wide sense Stationary Random Process

A process $X(t)$ is **weak sense or wide sense stationary** if

(1) The expected value of the process $E[X(t)]$ is a constant, thus independent of the time variable, and

(2) The autocorrelation function $R_{XX}(t_1,t_2)$ is a function of time difference t_2-t_1 only.

When $R_{XX}(t_1,t_2)$ can be written as a function of time difference, $\tau=t_2-t_1$ only, we will denote the autocorrelation of the process as $R_{XX}(\tau)$ or sometimes as just $R_X(\tau)$.

If condition (2) above is satisfied, the process has the lowest form of **stationary in autocorrelation** .If condition (1) is satisfied, the process has the lowest form of stationarity called **stationarity in mean**. A random process $X(t)$ is **stationarity in mean** if $E[X(t)]=\eta_X$, a constant for all time. If a process is stationary of order one, then it must be stationary in mean. However, a random process can be stationary in mean without being stationary of order one or stationary in autocorrelation. In Figure 4.6 the relationships between the various forms of stationarity are shown. A solid arrow means implication, no arrow means no implication, while a dotted arrow with the further information given by the arrow means implication with that information.

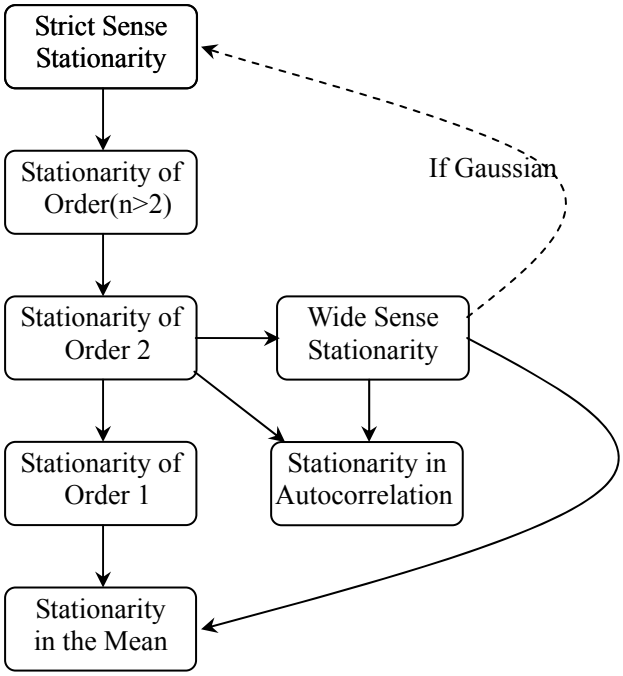


Figure 4.6 Conceptual relationship between some common forms of stationarity.

A random process is called **asymptotically stationary** of a particular sense if conditions for that sense are satisfied in the limit as all times approach infinite. We say that a process is **cyclo stationary** of a particular sense if the sense is satisfied for times that are periodically displaced by an amount T.

It has been shown that if a random process is Gaussian and weak sense stationary, it is strict sense stationary. This is truly a remarkable result as it means that the autocorrelation function and the mean of a weak sense stationary process must contain all the information necessary to determine the *n*th-order density function. This property will be examined further in Section 4.7 on Gaussian random process.

4.3.2 Properties for Wide Sense Stationary Random Processes

The autocorrelation function for a wide sense stationary process is a function of the time difference and can thus be written in terms of that variable as $R_{XX}(\tau)$. For a real process the following properties can be shown to be true:

$$R_{XX}(-\tau)=R_{XX}(\tau) \qquad \text{even function of } \tau$$

$$|R_{XX}(\tau)|\le R_{XX}(0) \qquad \text{bounded function of } \tau$$

$$R_{XX}(\tau)=F^{-1}S_{XX}(w),\quad S_{XX}(w)=F(R_{XX}(\tau)) \qquad \text{Fourier transform pairs}$$

$$F(R_{XX}(\tau))\ge 0 \quad for\ all\ \omega \qquad \text{real and positive} \tag{4.16}$$

$$R_{XX}(0)=\frac{1}{2}\int_{\infty}^{\infty}S_{XX}(w)dw \qquad \text{average power in the process}$$

$$R_{XX}(\tau)=C_{XX}(\tau)+\eta_X^2 \qquad \text{relationship to autocovariance function of process}$$

4.4 Examples of Random Processes

4.4.1 Straight Line Process

Given a random process $X(t)$ defined as

$$X(t)=P+Qt \tag{4.17}$$

This expression is an abbreviation for

$$X(t,e)=P(e)+Q(e)t \tag{4.18}$$

For every e_i, $X(t,e_i)$ is a straight line with intercept $P(e_i)$ and slope $Q(e_i)$. Some realizations of the process are presented in Figure 4.7.

Example 4.2

We are asked to find the following for the straight line process:

(a) First-order densities of the process.

(b) Second-order densities $f(x_1,x_2;t_1,t_2)$ of the process.

(c) Mean $\eta_X(t)$ of the process.

(d) Autocorrelation function $R_{XX}(t_1,t_2)$ of the process.

(e) The different type of stationarity for this process.

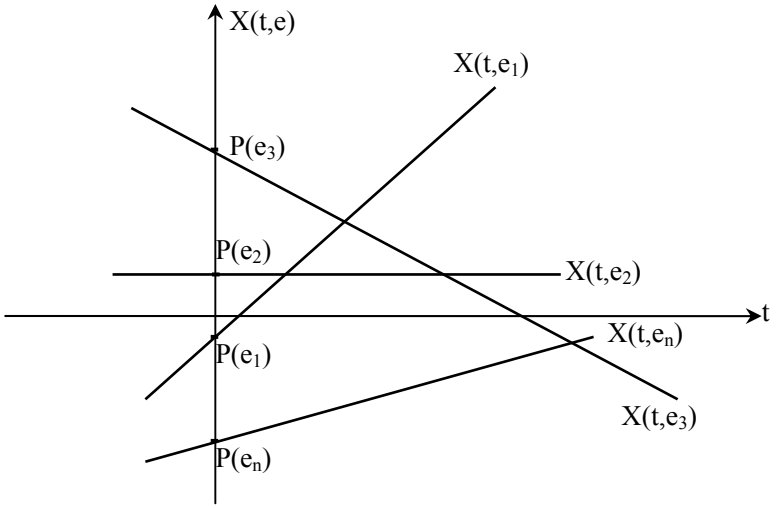


Figure 4.7 Four Realizations of the straight line process

Solution

(a) The first-order density $f_X(x_t)$ of the process can be obtained from the transformational theorem as X_t can be written as $X_t = P + Qt$, the sum of the two random variables P and Qt. By defining an auxiliary random variable $Y_t = P$, the joint density for X_t and Y_t becomes

$$f_{X_t Y_t}(x_t, y_t) = \sum_{(p_i, q_i)} \frac{f_{PQ}(p,q)}{|J(p,q)|}\bigg|_{(p_i, q_i)} \tag{4.19}$$

Where p_i and q_i are the roots of $x_t = p + qt$ and $y_t = p$. The only solution to these equations is

$$p = y_t, \qquad q = \frac{x_t - y_t}{t} \tag{4.20}$$

The Jacobian $J(p,q)$ is easily calculated as

$$J(p,q) = \begin{bmatrix} 1 & t \\ 1 & 0 \end{bmatrix} = -t \tag{4.21}$$

Since the desired marginal density is obtained by integrating out the auxiliary variable, we have

$$f_{X_t}(x_t) = \int_{-\infty}^{\infty} \frac{f_{PQ}(y_t, (x_t - y_t/t))}{|t|} dy_t \tag{4.22}$$

Noticed that if P and Q are statistically independent, $f_p(p,q)$ can be written as a product and $f_X(x_t)$ can be determined by the convolution of the two densities as X_t is a sum of two independent random variables.

(b) The second-order densities of the process can be obtained by using the transformational theorem for two functions of two random variables, since $X(t_1) = X_1$ and $X(t_2) = X_2$ are given by

$$X_1 = P + Qt_1, X_2 = P + Qt_2 \tag{4.23}$$

The set of equations

$$x_1 = p + qt_1, x_2 = p + qt_2 \tag{4.24}$$

has only one solution given by

$$p = \frac{t_2 x_1 - t_1 x_2}{t_2 - t_1}, q = \frac{x_2 - x_1}{t_2 - t_1} \tag{4.25}$$

The Jocabian is easily calculated as $(t_2 - t_1)$, so the joint probability density function for X_1 and X_2 can be shown to be

$$f_{X_1 X_2}(x_1, x_2) = \frac{f_{PQ}\left((t_2 x_1 - t_1 x_2)/(t_2 - t_1), (-x_1 + x_2)/(t_2 - t_1)\right)}{|t_2 - t_1|} \tag{4.26}$$

(c) The mean of the process, $\eta_x(t)$, is computed by finding $E[X(t)]$. One way to get the mean is to use the first-order density of $X(t)$ determined in part (a), that is

$$\eta_x(t) = \int_{-\infty}^{\infty} x_t f_{X_t}(x_t) dx_t \tag{4.27}$$

An alternative approach is to notice that the random variable X_t is really a function of the two random variables P and Q. In this way $\eta_x(t)$ can be written as

$$\eta_X(t) = E[X_t] = E[P + Qt] = E[P] + E[Q]t \tag{4.28}$$

Therefore, the mean of the process is a straight line that has intercept of $E[P]$ and a slope of $E[Q]$. It is not necessary that the mean be one of he realizations of sample functions of the process.

(d) To obtain the autocorrelation function for the random process, it is necessary to determine

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] \tag{4.29}$$

Again, this function can be found in several different ways. First, it can be found by using the joint density of the random variables $X(t_1) = X_1$ and $X(t_2) = X_2$ determined in part (b) as follows:

$$R_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X_1 X_2}(x_1 x_2) dx_1 dx_2 \tag{4.30}$$

If you do not have the second-order density, you must determine it before the integral above can be evaluated. An alternative approach is to notice that the product $X(t_1)X(t_2)$ is a function of the two random variables P and Q and times t_1 and t_2, from which the excepted value can be caculated as

$$R_{XX}(t_1, t_2) = E[(P + Qt_1)(P + Qt_2)] = E[P^2] + E[PQ](t_1 + t_2) + E[Q^2]t_1 t_2 \tag{4.31}$$

Therefore, once $f_{PQ}(p, q)$ is specified, the moments and joint moments given above can be calculated and $R_{XX}(t_1, t_2)$ can be determined.

(e) The mean of the process is given by $E[X(t)] = E[P] + E[Q]t$. If the $E[Q] = 0$, the mean is not a function of time and the random process $X(t)$ is stationary in the mean. Otherwise, it will not be stationary in the mean. The first-order density given is inherently a function of time. Therefore the process

$X(t)$ is not stationary of order 1, and consequently not stationary of any higher-order densities.

For the process to be stationary in autocorrelation $R_{XX}(t_1,t_2)$ must be a function of time difference only. Except for the degenerative case of $E[PQ]=0$ and $E[Q^2]=0$, the process will not be stationary in autocorrelation, and thus it cannot be wide sense stationary well.

4.4.2 Semirandom Binary Transmission process

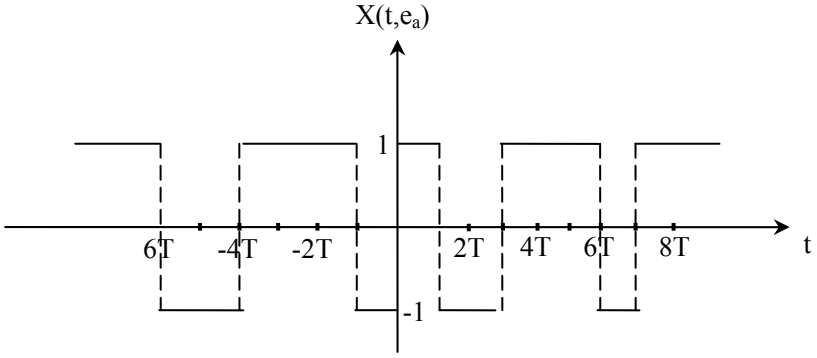
The h and t could represent the outcome of flipping an honest coin. It will be assumed that the probability of getting a head on a single trial is 1/2 and that tosses at different times are independent. The sample space consists of all possible doubly infinite sequences of heads and tails :

$$S=\{e:e=(...,q_{-1},q_0,q_1,q_2,...) \qquad for \qquad q_i\in\{h,t\} \tag{4.32}$$

With the above sample space, probability measure, and the Borel field equal to the power set, the semirandom binary transmission process, $X(t)$, using 1 and -1 as values, is described on each subinterval of the time axis as

$$X(t,e)=\begin{cases}1 & if\ q_i=h\\ -1 & if\ q_i=t\end{cases}, \qquad for \qquad iT\le t\le(i+1)T \tag{4.33}$$

A realization of the process $X(t,e_a)$ is shown in Figure 4.8 for the particular outcome e_a of the experiment described by $e_a=(\ldots,h,t,t,h,h,t,h,t,t,h,h,h,t,h,h,\ldots)$.



e_a=(…, h, t, t, h, h, h, t, h, t, t, h, h, h, t, h, h,…)

Figure 4.8 A Realization of the Semirandom Binary Transmission Process.

The defined process $X(t)$ is a collection or ensemble of time waveforms that change from 1 to -1, change from -1 to 1, or remain the same at possible transition time points that are equally spaced a distance T apart along the time axis. We are unable for the process. We can, however, characterize the process by specifying certain statistical properties. In the following paragraphs the first-order density, the mean, the second-order density function, and the autocorrelation function of the process will be found.

First-Order Density. By cutting across the process at an arbitrary time t, a random variable is defined. Since the valued of $X(t)$ for all t are either 1 or -1, this random variables are the same. From the honest coin assumption the probability of getting either a head or a tail (1 or -1 for the process during each interval) is 1/2. Therefore the first-order density is as follows for all t:

$$f_{X_t}(x_t)=\frac{1}{2}\delta(x_t+1)+\frac{1}{2}\delta(x_t-1) \tag{4.34}$$

Mean. The mean, $E[X(t)]$, of the process can be found directly from the first-order density by

$$\begin{aligned}\eta_X(t)=E[X(t)]&=\int_{-\infty}^{\infty}x_t f_{X_t}(x_t)dx_t\\&=\int_{-\infty}^{\infty}x_t\left[\frac{1}{2}\delta(x_t+1)+\frac{1}{2}\delta(x_t-1)\right]dx_t\\&=\frac{1}{2}(-1)+\frac{1}{2}(1)=0\end{aligned} \tag{4.35}$$

Second-order Density. X_1 and X_2 representing $X(t_1)$ and $X(t_2)$ for arbitrary times t_1 and t_2 are discrete random variables which take on values of +1 and -1. Their joint density function is given by

$$\begin{aligned}f(x_1,x_2)=&P\{X_1=1,X_2=1\}\delta(x_1-1,x_2-1)\\&+P\{X_1=1,X_2=-1\}\delta(x_1-1,x_2+1)\\&+P\{X_1=-1,X_2=1\}\delta(x_1+1,x_2-1)\\&+P\{X_1=-1,X_2=-1\}\delta(x_1+1,x_2+1)\end{aligned} \tag{4.36}$$

If t_1 and t_2 are in different intervals, X_1 and X_2 are independent random variables as the tosses were independent. If t_1 and t_2 are in the same intervals, they are the same random variable. The second-order density thus depends on whether t_1 and t_2 are in the same or different intervals.

For t_1 and t_2 in different intervals,

$$P\{X_1=1,X_2=1\}=P\{X_1=1\}\cdot P\{X_2=1\}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4} \tag{4.37}$$

Similarly $P\{X_1=1,X_2=-1\}$, $P\{X_1=-1,X_2=1\}$, and $P\{X_1=-1,X_2=-1\}$ are seen to be equal to 1/4. Thus the second-order density from equation (4.36) becomes

$$f(x_1,x_2)=\frac{1}{4}\left[\left(\delta(x_1-1,x_2-1)+\delta(x_1-1,x_2+1)+\delta(x_1+1,x_2-1)+\delta(x_1-1,x_2+1)\right)\right] \tag{4.38}$$

For t_1 and t_2 in same intervals, the probabilities needed can be obtained by using the conditional probabilities as follows:

$$P\{X_1=1,X_2=1\}=P\{X_2=1|X_1=1\}P\{X_1=1\} \tag{4.39}$$

In the same interval $P\{X_2=1|X_1\}=1$. Therefore

$$P\{X_1=1,X_2=1\}=P\{X_1=1\}=\frac{1}{2} \tag{3.40}$$

Similarly $P\{X_1=-1,X_2=-1\}=1/2$. Since X_1 and X_2 in the same interval cannot changing sign, $P\{X_2=1,X_1=-1\}$ and $P\{X_2=-1,X_1=1\}$ are both zero. Thus

$$P\{X_1=1,X_2=-1\}=P\{X_2=-1|X_1=1\}=0 \tag{4.41}$$

Using these results in Eq. (4.36) gives the second-order density of the random process for times t_1 and t_2 in the same interval as

$$f(x_1,x_2)=\frac{1}{2}[(\delta(x_1-1,x_2-1)+\delta(x_1-1,x_2+1))] \tag{4.42}$$

Autocorrelation Function. The autocorrelation function $R_{xx}(t_1,t_2)$ is found by computing the expected value of the product of $X(t_1)$ and $X(t_2)$ as

$$R_{XX}(t_1,t_2)=E[X(t_1)X(t_2)]=E[X_1X_2]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2f_{X_1X_2}(x_1,x_2)dx_1dx_2 \tag{4.43}$$

As the density $f(x_1,x_2)$ is different for the cases of t_1 and t_2 in the same and different intervals, the autocorrelation function is computed as follows:

For t_1 and t_2 in different intervals,

$$\begin{aligned}R_{XX}(t_1,t_2) &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2\frac{1}{4}[\delta(x_1-1,x_2-1)+\delta(x_1-1,x_2+1)+\delta(x_1+1,x_2-1)+\delta(x_1+1,x_2+1)]dx_1dx_2\\ &=\frac{1}{4}[(1)(1)+(1)(-1)+(-1)(1)+(-1)(-1)]=0\end{aligned} \tag{4.44}$$

For t_1 and t_2 in the same interval,

$$\begin{aligned}R_{XX}(t_1,t_2) &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}[\delta(x_1-1,x_2-1)+\delta(x_1+1,x_2+1)]dx_1dx_2\\ &=(1)(1)\frac{1}{2}+(-1)(-1)\frac{1}{2}=1\end{aligned} \tag{4.45}$$

Therefore $R_{XX}(t_1,t_2)$ is the two-dimensional "snake of blocks" function along the line $t_1=t_2$ shown in Figure 4.9.

Stationarity. The semirandom binary transmission process is stationary in the mean since $E[X(t)]=0$, which is not a function of time t, and stationary of order 1 since its first-order densities are not a function of time. However, the semirandom process is not stationary of order 2, and thus all higher orders, because the second-order densities are not the same for shifted versions of time differing by a set amount. Similarly the process is seen to be not stationary in autocorrelation and thus not wide sense stationary also.

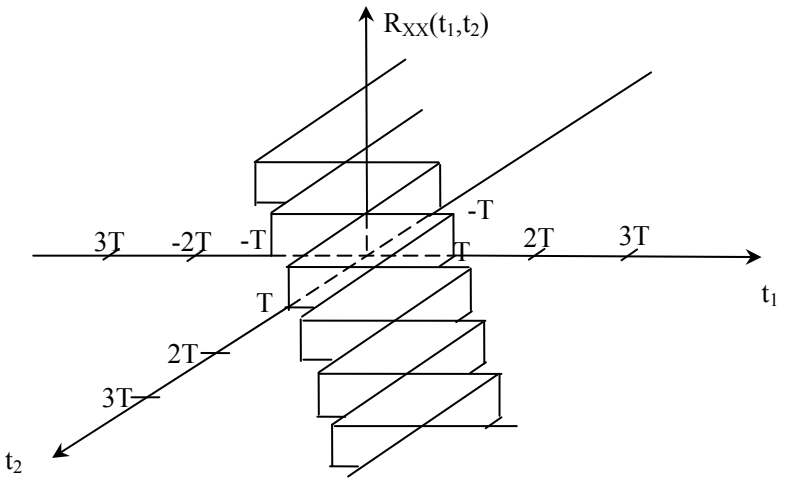


Figure 4.9 The autocorrelation function $R_{XX}(t_1,t_2)$ for the semitandom binary transmission process.

4.4.3 Random Binary Transmission Process

The **random binary transmission process**, $Y(t)$, is defined by

$$Y(t) = X(t - D) \tag{4.46}$$

Where $X(t)$ is the semirandom binary transmission process and D is a uniform random variable on $[0,T]$ that is independent of $X(t)$.

First Order Densities of Random Binary Transmission Proccess. The first-order density for this process can be obtained by first finding the first-order density function for $X(t-d)$ conditioned on a particular value of D and then integrating this conditional density with the probability density function for D. The first-order density function for $X(t-d)$ is obtained similarly to (4.34), and since d plays no part,the conditional density is the same. Since the result is not a function of d, the integration gives the first-order density as the same as that for the semirandom binary process:

$$f(x_t) = \frac{1}{2}\delta(x_t - 1) + \frac{1}{2}\delta(x_t + 1) \tag{4.47}$$

Mean of Random Binary Transmission Process. Since the first-order density is the same for the random binary transmission process, the expected value will also be the same, that is, zero. Alternatively, we can find the mean by using the iterated expected value formula as follows:

$$E[Y(t)] = E\left[E[Y(t-d)]_{d=D}\right] = 0 \tag{4.48}$$

Second Order Densities of Random Binary Transmission Process. To obtain the second-order densities, it will be convenient to use the conditional probability density function for $X(t-d)$, where d is any value between 0 and T. Thus the second-order density conditioned on d is as follows: For values of t_1

and t_2 inside and outside the doubly shaded area shown in Figure 4.10, the density becomes:

inside:

$$f(x_1,x_2|D=d)=\frac{1}{2}\delta(x_1-1,x_2-1)+\frac{1}{2}\delta(x_1+1,x_2+1) \tag{4.49a}$$

outside:

$$f(x_1,x_2|D=d)=\frac{1}{4}\delta(x_1-1,x_2-1)+\frac{1}{4}\delta(x_1-1,x_2+1)+\frac{1}{4}\delta(x_1+1,x_2-1)+\frac{1}{4}\delta(x_1+1,x_2+1) \tag{4.49b}$$

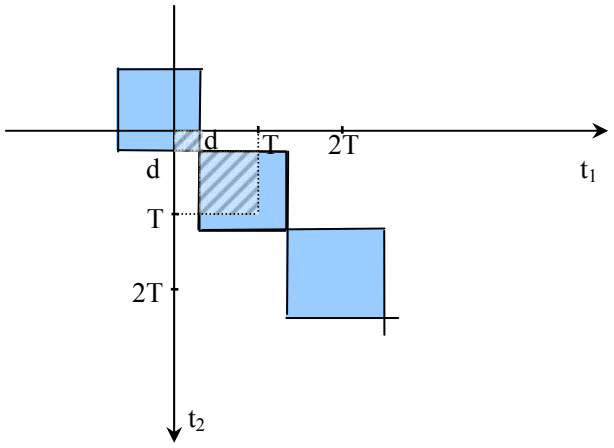


Figure 4.10 Regions for the conditional densities

For $t_1 > t_2$ multiplying (4.49a) and (4.49b) by the appropriate probabilities and adding gives

$$\begin{aligned} f(x_1,x_2)=&\left[\frac{1}{2}-\frac{(t_1-t_2)}{4T}\right]\left[\delta(x_1-1,x_2-1)+\delta(x_1+1,x_2+1)\right] \\ &+\frac{(t_1-t_2)}{4T}\left[\delta(x_1-1,x_2+1)+\delta(x_1+1,x_2-1)\right] \end{aligned} \tag{4.50}$$

Autocorrelation Function of Random Binary Transmission Process. The autocorrelation function for the random binary transmission process can be found by using the iterated expected value approach as follows:

$$\begin{aligned} R_{YY}(t_1,t_2) &= E[Y(t_1)Y(t_2)]=E[X(t_1-D)X(t_2-D)] \\ &= E\left[R_{XX}(t_1-d,t_2-d)\big|_{d=D}\right] \\ &= \frac{1}{T}\int_0^T R_{XX}(t_1-d,t_2-d)dd \end{aligned} \tag{4.51}$$

where $R_{XY}(t_1,t_2)$, is known to be 1 for (t_1,t_2) in the same box as shown in Figure 4.10 and zero outside that region. So the $R_{XX}(t_1-d,t_2-d)$ will be one provided that the point (t_1-d,t_2-d) falls in one of the intervals. Assume that $t_1 > t_2$, that $t_1-t_2=\tau$, and that $0<\tau\le t$. Then the point (t_1-d,t_2-d) will appear as in Figure 4.11. It will be in a proper interval provided that $t_1-d<T$ and $t_2-d>0$.

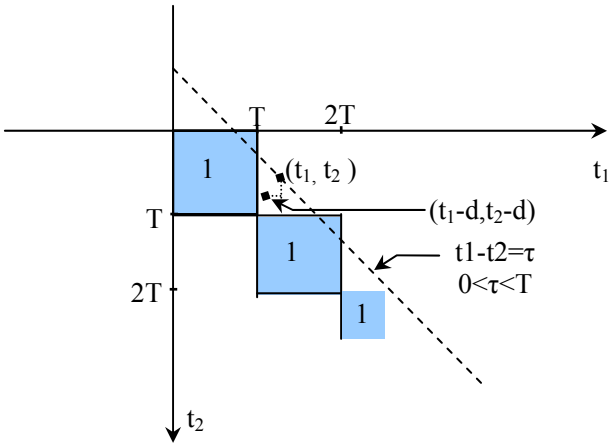


Figure 4.11 Regions for the random binary transmission process

Rearranging these conditions to be put as a region for d gives $d > t_1 - T$ and $d < t_2$ to be in the proper interval for the value of the $R_{XX}(t_1 - d, t_2 - d)$ to be one. Thus from (4.51) we have

$$\begin{aligned} R_{YY}(t_1,t_2) &= \frac{1}{T}\int_{t_1-T}^{t_2} 1 dd = \frac{t_2 - (t_1 - T)}{T} \\ &= 1 - \frac{t_1 - t_2}{T} \end{aligned} \tag{4.52}$$

Repeating the steps above for $t_1 < t_2$, we find that $R_{YY}(t_1,t_2) = 1 - (t_1 - t_2)/T$. If $t_1 - t_2 = \tau$ where $\tau > t$, the point will never fall in an interval because d is always $\le T$, so $R_{YY}(t_1,t_2) = 0$ for those values of t_1 and t_2. Thus the autocorrelation function for the random binary transmission procress can be expressed as

$$R_{YY}(t_1,t_2) = \begin{cases} 1 - |t_2 - t_1|/T, & -T \le (t_1 - t_2) \le T \\ 0, & otherwise \end{cases} \tag{4.53}$$

Stationary of the Random Binary Transmission Process. The random binary random transmission wave is thus seen to be stationary of mean, stationary in autocorrelation, stationary of order one, and wide sense stationary.

4.4.4 Semirandom Telegragh Waves Process

Consider the experiment of a random selection of times at a uniform rate λ per unit time such that the probability of k points in the interval $[0,t]$ is given by

$$P(k \quad point s \in [0,t]) = \frac{(\lambda k)^k e^{-\lambda t}}{k!} \tag{4.54}$$

The outcome of a single trial of the experiment results in a sequence of times

$$e_i = [\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots] \tag{4.55}$$

Define the **semirandom telegraph process** as follows:

$$X(t_1, e_i) = \begin{cases} 1 \ at \ time \ t \ if \ e_i \ has \ an \ even \ number \ of \ points \ in [0,t] \\ 1 \ at \ time \ t \ if \ e_i \ has \ an \ odd \ number \ of \ points \ in [0,t] \end{cases} \tag{4.56}$$

First-Order Density. At any given time t, X_t is a discrete random variable denoted by X_t which takes on only the values 1 or -1. The probability density function for X_t can be written as

$$f_{X_t}(x_t) = P\{x_t = 1\}\delta\{x_t - 1\} + P\{X_t = -1\}\delta\{x_t + 1\} \tag{4.57}$$

Where

$$\begin{aligned} P\{X_t = 1\} &= P\{the \ \ number \ \ of \ \ points \in [0,t] \ \ is \ \ even\} \\ &= P\{\ \}\{exactly \ \ 0 \ points \in [0,t] \\ &\qquad or \ \ exactly \ \ 2 \ points \in [0,t] \\ &\qquad \ldots or \ \ exactly \ \ 2k \ points \in [0,t] \\ &\qquad \ldots or \ldots\} \end{aligned} \tag{4.58}$$

Since the events exactly $2k$ and $2j$ points are mutually exclusive for $k \neq j$, the probability of the union of all the events can be written as the sum of probabilities. Substituting the probability of exactly $2k$ points from Eq. (4.54) into (4.58) gives the following for $t > 0$:

$$P\{X_t = 1\} = \sum_{k=0}^{\infty} P\{2k \ \ points \ \ in [0,t]\} = \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k} e^{-\lambda k}}{(2k)!} \tag{4.59}$$

Factoring out the $e^{-\lambda t}$ and recognizing the sum as $\cosh(\lambda t)$, $P\{X_t = 1\}$ simplifies to

$$P\{X_t = 1\} = e^{-\lambda t}\cosh(\lambda t) \tag{4.60}$$

In a similar fashion the $P\{X_t = -1\}$ can be written as

$$P\{X_t = -1\} = \sum_{k=0}^{\infty} P\{2k+1 \ points \ in [0,t]\} = \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1} e^{-\lambda t}}{(2k+1)!} = e^{-\lambda t}\sinh(\lambda t) \tag{4.61}$$

From these results the first-order density of the semirandom telegraph wave become

$$f_{X_t}(x_t) = e^{-\lambda t}\cosh(\lambda t)\delta(x_t - 1) + e^{-\lambda t}\sinh(\lambda t)\delta(x_t + 1) \tag{4.62}$$

For $t < 0$, the density is obtained by replacing λt by $-\lambda t$.

Mean of $X(t)$. Using the above obtained first-order density function for the process $X(t)$, we calculate the mean $E[X(t)]$ as

$$\begin{aligned} E[X(t)] &= \int_{-\infty}^{\infty} x_t f_{x_t}(x_t) dx_t \\ &= \int_{-\infty}^{\infty} x_t \left[e^{-\lambda t}\cosh(\lambda t)\delta(x_t - 1) + e^{-\lambda t}\sinh(\lambda t)\delta(x_t + 1) \right] dx_t \end{aligned} \tag{4.63}$$

The sampling property of the delta function can be used to simplify $E[X(t)]$ as follows:

$$E[X(t)] = e^{-\lambda t}\cosh(\lambda t) - e^{-\lambda t}\sinh(\lambda t) = e^{-2\lambda t} \tag{4.64}$$

For $t<0$, $E[X(t)]$ becomes $e^{2\lambda}$,and thus $E[X(t)]$ can finally be written as

$$E[X(t)] = e^{-2\lambda t} \tag{4.65}$$

A plot of the mean $E[X(t)]$ is shown in Figuring 4.13. It is noticed that for positive time it starts at 1 and goes to zero as $t \to \infty$. Close to the origin tiis almost 1.since each realization of the process starts at 1.

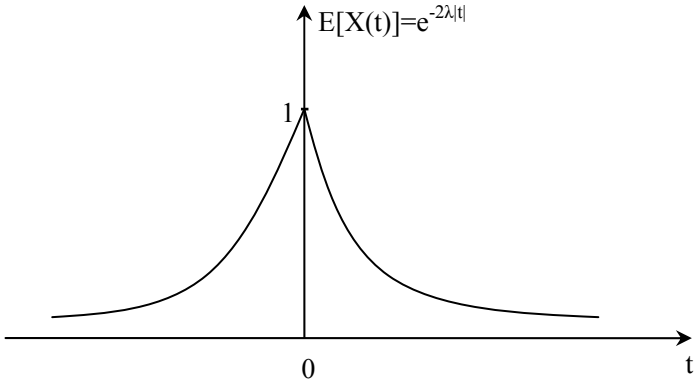


Figure 4.13 Mean of the semirandom telegraph wave process

Second-order Density. The random variables $X(t_1)$ and $X(t_2)$ denoted by X_1 and X_2 ,respectively, are discrete random variables taking on only values 1 and -1 with joint probability density function $f_{X_1X_2}(x_1,x_2)$ as

$$\begin{aligned} f_{X_1X_2}(x_1,x_2) = {} & P\{X_1=+1, X_2=+1\}\delta(x_1-1,x_2-1) \\ & + P\{X_1=+1, X_2=-1\}\delta(x_1-1,x_2+1) \\ & + P\{X_1=-1, X_2=+1\}\delta(x_1+1,x_2-1) \\ & + P\{X_1=-1, X_2=+1\}\delta(x_1+1,x_2+1) \end{aligned} \tag{4.66}$$

The weights of the delta functions are found by written the random variables joint probability in terms of a conditional probability and marginal probability as follows:

$$P\{X_1=+1, X_2=-1\} = P\{X_1=-1 \,|\, X_2=+1\}P\{X_1=1\} \tag{4.67}$$

The conditional probability that an odd number of points occur in interval $[t_1,t_2]$ is for $t_1>t_2$ the following

$$P\{X_2=-1 \,|\, X_1=+1\} = e^{\lambda(t_2-t_1)}\sinh(\lambda(t_2-t_1)) \tag{4.68}$$

The $P\{X_1=+1\}$ was found earlier as

$$P\{X_1=+1\} = e^{-\lambda t_1}\sinh(\lambda t_1) \tag{4.69}$$

Using the conditional and marginal probabilities just determined gives the joint probability as

$$P\{X_1=+1, X_2=-1\} = e^{-\lambda t_2}\cosh(\lambda t_1)\sinh(\lambda(t_2-t_1)) \tag{4.70}$$

In a similar way the other joint probabilities can be determined, resulting in the joint probability density function

$$\begin{aligned} f_{X_1X_2}(x_1,x_2) &= e^{-\lambda t_2}\cosh(\lambda t_1)\cosh(\lambda(t_2-t_1))\delta(x_1-1,x_2-1) \\ &+ e^{-\lambda t_2}\cosh(\lambda t_1)\cosh(\lambda(t_2-t_1))\delta(x_1-1,x_2+1) \\ &+ e^{-\lambda t_2}\sinh(\lambda t_1)\sinh(\lambda(t_2-t_1))\delta(x_1+1,x_2-1) \\ &= e^{-\lambda t_2}\sinh(\lambda t_1)\cosh(\lambda(t_2-t_1))\delta(x_1+1,x_2+1) \end{aligned} \tag{4.71}$$

Autocorrelation Function. From the $f_{X_1X_2}(x_1,x_2)$ above the autocorrelation function

$$R_{XX}(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2f_{X_1X_2}(x_1x_2)dx_1dx_2 \tag{4.72}$$

Carrying out the integration and using the sampling property of the two-dimensional delta function, $R_{XX}(t_1,t_2)$ can be simplified to give

$$R_{XX}(t_1,t_2)=e^{-2\lambda(t_2-t_1)} \qquad for \qquad t_2\geq t_1 \tag{4.73}$$

By reversing the roles of t_1 and t_2 and repeating the procedure described above, $R_{XX}(t_1,t_2)$ can be shown to be

$$R_{XX}(t_1,t_2)=e^{-2\lambda|(t_2-t_1)|} \qquad for\ all\ \ t_2\ and\ t_1 \tag{4.74}$$

A plot of $R_{XX}(t_1,t_2)$ as a function of t_1 and t_2 is shown in Figure 4.14a and is a "trough" of absolute exponentials. In Figure 4.14b, $R_{XX}(t_1,t_2)$ is plotted as a function of the time difference. Since $R_{XX}(t_1,t_2)$ is a function of time difference only, it can be written as $R_{XX}(\tau)$ where $\tau=t_1-t_2$. Thus the semirandom telegraph wave process is stationary in autocorrelation. However, it is not wide sense stationary since $E[X(t)]$ is a function of t as shown in Figure 4.13.

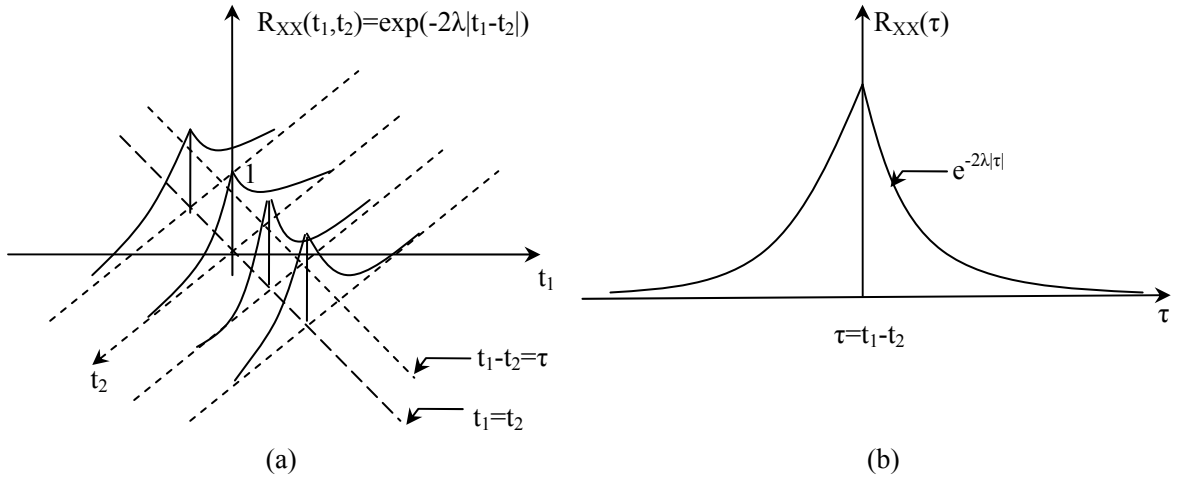


Figure 4.14 Autocorrelation function for the semirandon telegraph wave process.
(a) plot of $R_{XX}(t_1,t_2)$ as function of t_1 and t_2, (b) plot of $R_{XX}(\tau)$.

4.4.5 Random Telegraph Process

Let $X(t)$ be the semirandom telegraph random process. The **random telegraph wave** can be defined as

$$Y(t) = AX(t) \tag{4.75}$$

Where A is discrete random variable with probability density as

$$f_A(a) = \frac{1}{2}\delta(a+1) + \frac{1}{2}\delta(a-1) \tag{4.76}$$

A is assumed to be independent of the random process $X(t)$. A random variable is independent of a random process if it is group independent of all random variables defined across the process.

The expected value or mean of the process $Y(t)$ can be determined as follows since the expected value of A is zero:

$$E[Y(t)] = E[AX(t)] = E[A]E[X(t)] = 0 \tag{4.77}$$

The autocorrelation function for $Y(t)$ can be found as follows:

$$\begin{aligned} R_{YY}(t_1,t_2) &= E[AX(t_1)AX(t_2)] = E[A^2]E[X(t_1)X(t_2)] \\ &= 1 \cdot R_{XX}(t_1,t_2) = e^{-2\lambda|t_2-t_1|} \end{aligned} \tag{4.78}$$

The expected value of the product above can be written as the product of the expected values because A^2 is also independent of the process (function of independent processes are independent), and since $E[A^2] = 1$, the autocorrelation function can be written in terms of the time difference τ as

$$R_{YY}(t_1,t_2) = e^{-2\lambda|\tau|} \tag{4.79}$$

Since the mean is a finite constant, (zero), and the autocorrelation function is a function of time difference only, we have shown hat the random telegraph wave is a wide sense stationary random process.

A logical question at this time is: Is the random telegraph wave stationary of any other variety? For example, is it stationary of order 1 ? In order to answer that question the first-order density for the semirandom binary transmission must be found. One way of obtaining the density is to integrate out the conditional density for Y_t:

$$f_{Y_t}(y_t) = \int_{-\infty}^{\infty} f(y_t|a) f_A(a) da \tag{4.80}$$

Where $f_A(a)$ is given by Eq.(4.76) and $f_{Y_t}(y_t)$ can be determined by modifying Eq.(4.62), which for the semirandom telegraph first-order density, at amplitude $A = a$, is

$$f(y_t|a) = e^{-\lambda t}\cosh(\lambda t)\delta(y_t - a) + e^{-\lambda t}\sinh(\lambda t)\delta(y_t + a) \tag{4.81}$$

Substituting the two densities into the equation for $f_{Y_t}(y_t)$ and integrating using the samping property of the delta function fives, after simplification, the density as

$$f_{Y_t}(y_t) = 0.5\delta(y_t + 1) + 0.5\delta(y_t - 1) \tag{4.82}$$

The first-order density is thus shown to be independent of time t, and hence the random telegraph wave is stationary of order one.

The answer to the question of stationarity of order two is saved for the reader's homework.

4.4.6 Random Sinusoidal Signals

In many situations collections of these signals have different radian frequencies, different phases, and different phases, and different amplitudes. If these parameters can be modeled as random variables, then the **random sinusoidal signal random process** can be defined by

$$x(t)=A\cos(\Omega t+\Phi) \tag{4.83}$$

Where A, Ω and Φ are random variables.

Example 4.3

Define random process $X(t)$ as $x(t)=A\cos(\Omega t+\Phi)$, where A and Φ are independent random variables with known probability density functions $f_A(a)$ and $f_\Phi(\phi)$ and where Ω_0 is deterministic and known. This is the case of a known sinusoidal frequency signal with random amplitude and phase. Further it is assumed that A is a Rayleigh random variable and Φ is a uniform random variable with the following probability density functions:

$$f_A(a)=ae^{-a^2/2}\mu(a) \qquad f_\Phi(\phi)=\begin{cases}\dfrac{1}{2\pi}, & 0<\phi<2\pi\\ 0, & elsewhere\end{cases} \tag{4.84}$$

For this defined random process determine (a) the mean of the process, (b) the autocorrelation function of the process, (c) the first-order densities of the process, (d) the second order density functions, and (e) the types of stationarity of the process.

Solution

(a) The mean of the process, $E(X(t)$, is

$$E(X(t)=E[A\cos(\Omega_0 t+\Phi)]=E(A)E[A\cos(\Omega_0 t+\Phi)] \tag{4.85}$$

The second step can be made since the expected value of a product of independent random variables is the product of the expected values. Since Φ is uniform on $[0,2\pi]$, the second expected value above is zero, thus giving for all t,

$$E[X(t)]=0 \tag{4.86}$$

(b) The autocorrelation function $R_{XX}(t_1,t_2)$ is

$$R_{XX}(t_1,t_2)=E[A\cos(\Omega_0 t_1+\Phi)A\cos(\Omega_0 t_2+\Phi)]=E[A^2]E[\cos(\Omega_0 t_1+\Phi)\cos(\Omega_0 t_2+\Phi)] \tag{4.87}$$

Again the second step can be done since functions of independent random variables are independent. The first term is easily obtained from the density $f_A(a)$, whereas the second term takes slightly more work. Using the product identity for cosines the second term becomes

$$E[\cos(\Omega_0 t_1+\Phi)\cos(\Omega_0 t_2+\Phi)]=E\left[\frac{1}{2}\cos(\Omega_0(t_1+t_2)+2\Phi)\right]+E\left[\frac{1}{2}\cos(\Omega_0(t_1-t_2)\right] \tag{4.88}$$

The first expected value on the right side of the equation above is zero since the integration of a cosine over two periods is zero, while the second expected value gives the bracketed term since it is nonrandom. Therefore the auto correlation function becomes

$$R_{XX}(t_1,t_2)=\frac{1}{2}E[A^2]\cos(\Omega_0(t_2-t_1) \tag{4.89}$$

(c) The first-order densities $f(x;t)$ can be obtained in several different ways for the random variable X_t defined by

$$X_t(t)=A\cos(\Omega_0 t+\Phi) \tag{4.90}$$

From the equation above, X_t is one function of the two random variables A and Φ. Defining a random variable B by $B=\cos(\Omega_0 t+\Phi)$, X_t is seen as the product of A and B. The density for the product X_t has been shown to be

$$f_{X_t}(x_t)=\int_{-\infty}^{\infty} f_{X_t}(x_t\mid b)f_B(b)db \tag{4.91}$$

Where $f_B(b)$ is known to be

$$f_B(b)=\begin{cases}\dfrac{1}{\pi\sqrt{1-b^2}}, & |b|<1\\ 0, & elsewhere\end{cases} \tag{4.92}$$

And $f_A(a)$ is given in the problem statement. Strange as it may seem, the result of performing the integration gives $f(x_t)$

$$f_{X_t}(x_t)=\frac{1}{\sqrt{2\pi}\sigma_{X_t}}\exp\left\{\frac{-x_t^2}{2\sigma_{x_t}^2}\right\} \tag{4.93}$$

Therefore the first order density for the process is Gaussian with variance given by evaluating Eq.(4.89) for $t_1=t_2=t$ as

$$\sigma_{X_t}^2=\frac{1}{2}E[A^2]=\frac{1}{2}\left(2-\frac{\pi}{2}\right)=0.2146 \tag{4.94}$$

(d) The second-order density for $X(t_1)$ and $X(t_2)$ defined by $X(t_1)$ and $X(t_2)$ as functions of the random variables A and Φ,, respectively, can be found using the two functions of two random variables transformation theorem on the following transformation:

$$\begin{aligned} X(t_1) \overset{\Delta}{=} X_1 &= A\cos(\Omega_0 t_1 + \Phi) \\ X(t_2) \overset{\Delta}{=} X_2 &= A\cos(\Omega_0 t_2 + \Phi) \end{aligned} \tag{4.95}$$

Another way to obtain the density uses the result obtained in Chapter 2, that $X(t_1)$ and $X(t_2)$ are Gaussian random variables since Φ is a Rayleigh distributed random variable independent of the random variable A which is uniformly distributed. By this result, the problem of characterizing the two Gaussian random variables $X(t_1)$ and $X(t_2)$ is equivalent to finding the means, variances, and correlation for those variables. It becomes expedient in finding those moments that we expand out the cosine functions from above to give

$$\begin{aligned} X_1 &= A\cos\Phi\cos(\Omega_0 t_1) + A\sin\Phi\sin(\Omega_0 t_1) \\ X_2 &= A\cos\Phi\cos(\Omega_0 t_2) + A\sin\Phi\sin(\Omega_0 t_2) \end{aligned} \tag{4.96}$$

Since $A\cos\Phi$ and $A\sin\Phi$ are independent and Gaussian, the random variables X_1 and X_2 are jointly Gaussian as they are linear functions of Gaussian random variables. By taking expected values, we can show the moments to be

$$\begin{aligned} &E[X_1] = 0 \\ &E[X_1^2] = \frac{1}{2}E[A^2] \\ &E[X_1 X_2] = E[A^2]R_{XX}(t_1, t_2) \\ &E[X_2] = 0 \\ &E[X_2^2] = \frac{1}{2}E[A^2] \end{aligned} \tag{4.97}$$

These five parameters determine the second-order Gaussian density function for the random sinusoidal described in this example.

(e) Since the mean of $X(t)$ is zero for all time, the process is stationary in mean. As the autocorrelation function is a function of time difference only, the process is stationary on auto correlation, and since it is stationary in mean, it is also wide sense stationary. Further, since $X(t)$ is a Gaussian random process and wide sense stationary, it is strict sense stationary.

4.4.7 Random Walk Process

Consider the experiment of tossing a coin countably infinite number of times. The outcome of the experiment is a cartesian product of single tosses that are either heads or tails. The samplr space for the random process is then the set of all countably infinite sequences of heads and tails. The Borel field is defined as the power set of the sample space that is the set of all possible subsets of this sample space. Assume that the probability of getting a head at each toss is equal to p and that each toss is "independent" of the others.

The **random walk process** is defined along uniform segments of the time axis for each outcome $e_a = (...,h,h,t,t,...)$ as follows:

$$X(t,e_a) = cS, \qquad (n-1)T \le t \le nT, \qquad n = 1,2,....$$
$$\text{C} = \text{number of heads - number of tails in first elements of } \text{e}_\text{a} \tag{4.98}$$

The T and S are time duration and step size, respectively, and fixed. If there are k heads in the first n elements, then the number of tails is $(n-k)$ and the number of heads minus the number of tails is $k-(n-k)$ or $2k-n$. Therefore the process can be defined as follows for each outcome:

$$X(t,e) = (2K-n)S, \qquad (n-1)T \le t < nT, \qquad n = 1,2,\ldots \tag{4.99}$$

where K is a random variable representing the number of heads. A few realizations of the process are shown in Figure 4.15.

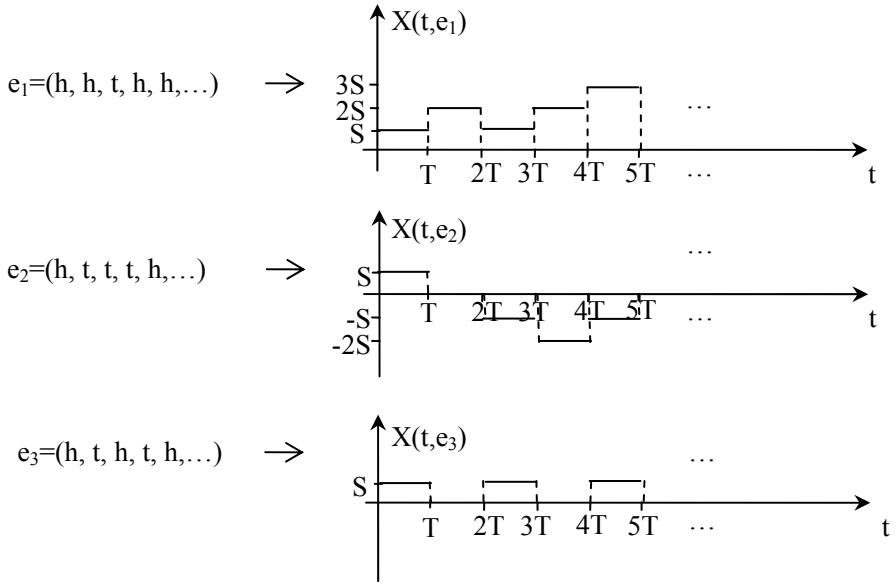


Figure 4.15 Realization of the random walk process.

The first-order density, mean, second-order density, and autocorrelation function for the random walk are now presented.

First Order Density. By the independent assumption, K has a binomial density ad follows:

$$f_K(k) = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \delta(k-j) \tag{4.100}$$

Where p is the probability of a head in each toss and $(1-p)$ is the probability of a tail. Since $X_t = (2K-n)S$ or a transformation of K, the probability density of X_t for $(n-1)T < t < nT$ becomes

$$f_{X_t}(x_t) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{nk} \delta(x_t - (2k-n)S) \tag{4.101}$$

The result above was obtained by using the procedure for transformation of a discrete random variable. If

we fix t in the first interval $0 \le t < T$, the density becomes

$$f_{X_t}(x_t) = (1-p)\delta(x_t + S) + \delta(x_t - S) \tag{4.102}$$

and for any $1 \le t < 2$, the second interval, we have

$$f_{X_t}(x_t) = (1-p)^2 \delta(x_t + S) + 2p(1-p)\delta(x_t) + p^2 \delta(x_t - S) \tag{4.103}$$

Representative probability density function for several t's are shown in Figure 4.16. As t becomes large, the envelope of the density functions approaches a Gaussian function centered at np and with a variance parameter of $np(p-1)$.

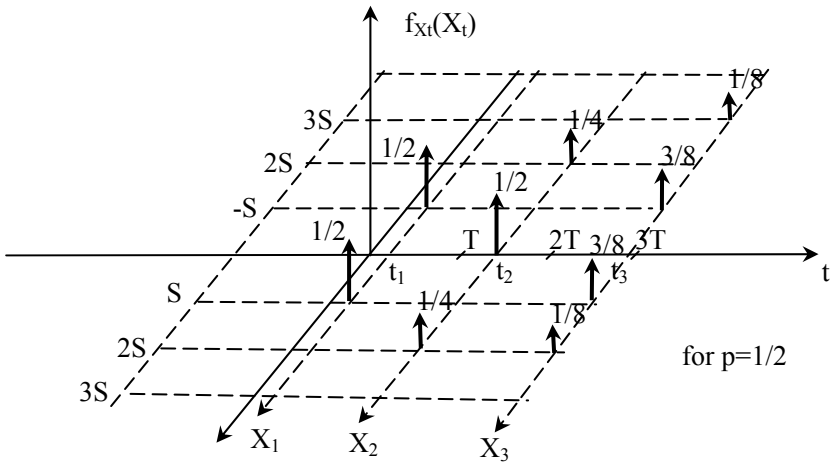


Figure 4.16 Representative first order probability density functions for the random walk process

Mean. Using the density function for X_t given in Eq. (4.101), we can write the mean of X_t as

$$\begin{aligned} E[X(t)] &= E[X_t] = \int_{-\infty}^{\infty} x_t f_{X_t}(x_t) dx_t \\ &= \int_{-\infty}^{\infty} x_t \left(\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x_t - (2k-n)S) \right) dx_t \end{aligned} \tag{4.104}$$

Using the sampling property of the delta function, we can write the expected value of the random walk process for $(n-1)T < t < nT$, is

$$E[X(t)] = \sum_{k=0}^{n} (2k-n)S \binom{n}{k} p^k (1-p)^{n-k} \tag{4.105}$$

The first part of the sum is just the average value the binomial density times 2 and the second is just n times 1, since it is a sum over the binomial density. Therefore $E[X(t)]$, for $(n-1)T < t < nT$, is

$$E[X(t)] = (np)2S - (n)S = n(2p-1)S \tag{4.106}$$

A plot of the expected value is shown in Figure 4.17 for $p > 1/2, p = 1/2$, and $p < 1/2$. For all p other than 1/2 the line $E[X(t)]$ as t. Thus, unless $p = 1/2$, the random walk process is not stationary in the mean.

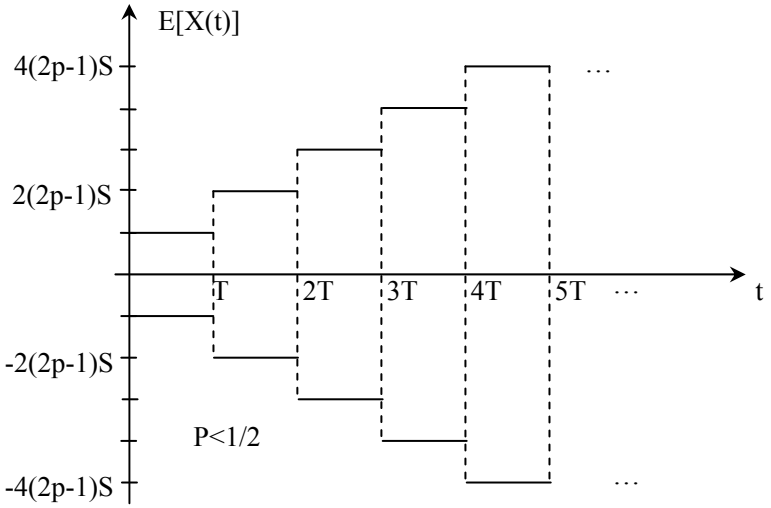


Figure 4.17 The mean of the random walk process for various of p, the Probability of a heads.

Autocorrelation Function. One approach that can be used to find the autocorrelation is to first find the second-order density and then compute the expected value using that density.

An alternative approach, which for this particular problem is easier, is to compute the expected value directly as a function of other random variables. To do this, we have to view the random process as the sum of independent identically distributed random variables:

$$X(t)=\sum_{i=1}^{n}X_i,\qquad (n-1)T<t<nT,\qquad n=1,2,\ldots \tag{4.107}$$

where X_i are discrete independent random variables with identical probability density functions

$$f_{X_i}(x_i)=(1-p)\delta(x_i+S)+p\delta(x_i-S) \tag{4.108}$$

Reminiscent of the random telegraph wave development, the autocorrelation function $R_{XX}(t_1,t_2)$ will take on different intervals along the t axis.

Assuming $t_1\geq t_2$, where t_1 is in the n th intervals, $(n-1)T<t<nT$, the autocorrelation function becomes for $m>n$,

$$\begin{aligned}R_{XX}(t_1,t_2)&=E[X(t_1)X(t_2)]=E\left[\sum_{i=1}^{m}X_i\sum_{j=1}^{n}X_j\right]\\&=E\left[\left(\sum_{i=1}^{n}X_i+\sum_{i=n+1}^{m}X_i\right)\sum_{j=1}^{n}X_j\right]\\&=E\left[\left(\sum_{i=1}^{m}X_i\right)\left(\sum_{j=1}^{n}X_j\right)\right]+E\left[\left(\sum_{i=n+1}^{m}X_i\right)\left(\sum_{j=1}^{n}X_j\right)\right]\end{aligned} \tag{4.109}$$

Because the X_i are independent and identically distributed, the first expected value can be written as

$nE\left[X_i^2\right]+\left(n^2-n\right)E^2\left[X_i\right]$ and the second becomes $(m-n)nE^2\left[X_i\right]$, since products contain all different random variables. Therefore $R_{XX}(t_1,t_2)$ becomes

$$R_{XX}(t_1,t_2)=nE\left[X_i^2\right]+\left(n^2-n\right)E^2\left[X_i\right]+(m-n)nE^2\left[X_i\right] \tag{4.110}$$

Since the are identically distributed with densities, Eq. (4.108), their first- and second-order moments are

$$\begin{aligned} E\left[X_i\right]&=(1-p)(-S)+pS=(2p-1)S \\ E\left[X_1^2\right]&=(1-p)(-S)^2+p(S)^2=S^2 \end{aligned} \tag{4.111}$$

Substituting these moment into Eq.(4.110) and simplifying, gives $R_{XX}(t_1,t_2)$, for $m>n$, as follows:

$$R_{XX}(t_1,t_2)=nS^2+n(m-1)(2p-1)^2S^2 \tag{4.112}$$

Interchanging m and n gives us the following for $m<n$:

$$R_{XX}(t_1,t_2)=mS^2+m(n-1)(2p-1)^2S^2 \tag{4.113}$$

If we set $p=1/2$, then $R_{XX}(t_1,t_2)$ becomes as shown in Figure 4.18.

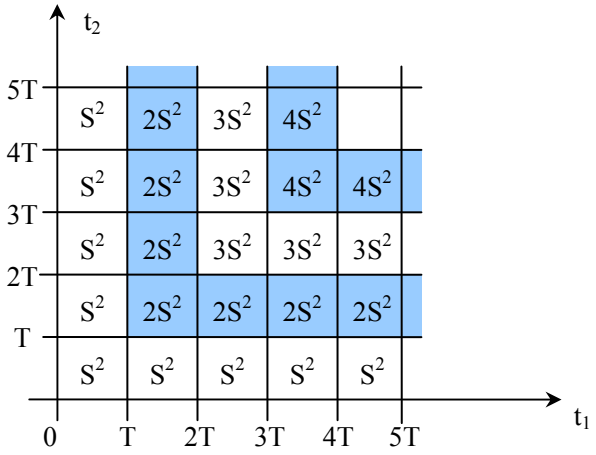


Figure 4.18 Autocorrelation function for the random walk when *p*=5

4.5 Definite In tegrate of Random Processes

For a given random process $X(t,e)$ and a given e, say e_1, we have the definite integral of that particular waveform by

$$I(e_1)=\int_a^b X(t,e_1)dt \tag{4.114}$$

As e goes over the sample space, $I(e)$ specifies a random variable. For simplicity, we will write that random variable $I(e)$ and denote it by I:

$$I=\int_a^b X(t)dt \tag{4.115}$$

It is logical at this point to ask what is the mean and the variance of such a random variable and

what information about the random process do we need to find them. The mean of I is given as

$$E[I]=E\left[\int_a^b[X(t)]dt\right] \tag{4.116}$$

It can be shown in most cases that the expected value and the integral can be interchanged to give

$$E[I]=\int_a^b E[X(t)]dt=\int_a^b \eta_X(t)dt \tag{4.117}$$

In a similar fashion the second moment of I can be found as

$$\begin{aligned} E[I^2]&=E\left(\int_a^b X(t)dt\int_a^b X(u)du\right)=\int_a^b\int_a^b E[X(t)X(u)]dtdu \\ &=\int_a^b\int_a^b R_{XX}(t,u)dtdu \end{aligned} \tag{4.118}$$

Given $E[I]$ and $E[I^2]$, the variance of I is easily seen to be

$$\sigma_I^2=E[I^2]-E^2[I]=\int_a^b\int_a^b C_{XX}(t,u)dtdu \tag{4.119}$$

Therefore the mean and variance of I are determined by the second-order characterization of the process, namely its mean $\eta_X(t)$ and autocovariance function $C_X(t,u)$. Higher-order moments require higher-order characterizations of the process $X(t)$ itself. In order to obtain the probability density function for I, a total characterization of the process $X(t)$ on the intervial of integration is required.

If the process $X(t)$ is a wide sense stationary process with mean η_X and autocorrelation function $R_{XX}(\tau)$ the formulas for $E[I]$ and $\sigma_I^{\,2}$ can be simplified. The expected value of I from equation then becomes

$$E[I]=E\int_a^b[X(t)]dt=\int_a^b\eta_X(t)dt=(b-a)\eta_X \tag{4.120}$$

Since the autocorrelation function $R_{XX}(\tau)$ is a function of the time difference $\tau=t-u$, the two-dimensional integral given in the equation can be changed to a one-dimensional integral . The incremental area required in this change is shown in Figure 4.19, and the resulting integral for the variance becomes

$$\begin{aligned} &\sigma_I^2=\int_{-(b-a)}^{b-a}[(b-a)-|\tau|][R_{XX}(\tau)-\eta_X^2]d\tau \\ &or \\ &\sigma_I^2=\int_{-(b-a)}^{b-a}[(b-a)-|\tau|]C_{XX}(\tau)d\tau \end{aligned} \tag{4.121}$$

An example illustrating the procedure for finding the mean and variance of a definite integral of a wide sense stationary process is now presented.

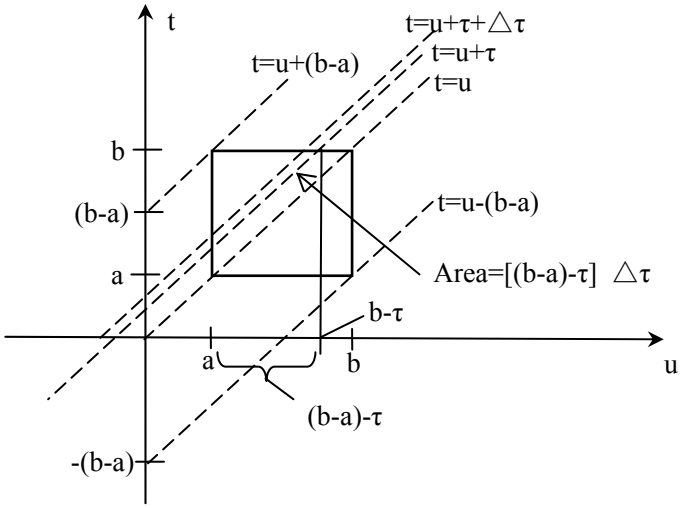


Figure 4.19 Incremental area for the change of variables t-u=τ.

Example 4.4

A wide sense stationary random process $X(t)$ is characterized by its mean and autocorrelation function as follows: $E[X(t)]=2$ and $R_{XX}(\tau)=4+e^{-|\tau|}$. Define a random variable I and $I=\int_1^3 X(t)dt$. Determine $E[I]$ and $\sigma_I^{\,2}$.

Solution

The expected value of the random variable I is easily obtained from (4.120) as

$$E[I]=(b-a)\eta_X=(3-1)2=4$$

From the given mean and autocorrelation function, the variance of I can be determined using (4.121) as follows:

$$\begin{aligned}\sigma_I^2&=\int_{-(b-a)}^{b-a}[(b-a)-|\tau|][R_{XX}(\tau)-\eta_X{}^2]d\tau\\&=\int_{-2}^{2}[2-|\tau|][4+e^{-|\tau|}-4]\\&=2\int_0^2(2-\tau)e^{-\tau}d\tau=4e^{-2}\end{aligned}$$

4.6 Joint Characterizations of Random Process

Two random process $X(t)$ and $Y(t)$ are defined to be **independent** if any set of random variables defined across the random process $X(t)$ is group independent of any set of random variables defined across the other process $Y(t)$. Two random process are called **uncorrelated** if their cross-correlation function can be written as the product

$$E[X(t_1)Y(t_2)]=E[X(t_1)]E[Y(t_2)]=\eta_X(t_1)\eta_Y(t_2)\qquad for\ \ all\ \ t_1,t_2 \tag{4.122}$$

which implies that any random variable defined across the process $X(t)$ at time is uncorrelated with any random variable across the process $Y(t)$ at time t_2.

4.6.1 First-Order Joint Densities

Define the random variables $X(t_1)$ and $Y(t_2)$, notation X_1 and Y_2 respectively. The second-order joint density is given as the joint probability density function for these two random variables as $f(x_1, y_2; t_1, t_2)$. The notation $f(x_1, y_2)$ is commonly used. The first-order joint densities consist of all joint densities from all choices of times t_1 and t_2.

4.6.2 Cross-correlation Function

The **cross-correlation function**, denoted as $R_{XY}(t_1, t_2)$, is defined as

$$R_{XY}(t_1, t_2) \stackrel{\Delta}{=} E[X(t_1)Y(t_2)] \tag{4.123}$$

If the random processes are complex process, a useful definition for the cross-correlation function can be obtained by taking the complex conjugate of the second process as

$$R_{XY}(t_1, t_2) = E[X(t_1)Y^*(t_2)] \tag{4.124}$$

4.6.3 Cross-covariance Function

Similar to the auto correlation function except now defined across two processes the **cross-covariance function** $C_{XY}(t_1, t_2)$ is defined by

$$\begin{aligned} C_{XY}(t_1, t_2) &= E\left[\left(X(t_1) - E[X(t_1)]\right)\left(Y(t_2) - E[Y(t_2)]\right)\right] \\ &= E\left[\left(X(t_1) - \eta_X(t_1)\right)\left(Y(t_2) - \eta_Y(t_2)\right)\right] \end{aligned} \tag{4.125}$$

If the process are complex, then the following definition is used for the cross-correlation function:

$$C_{XY}(t_1, t_2) = E\left[\left(X(t_1) - \eta_X(t_1)\right)\left(Y(t_2) - \eta_Y(t_2)\right)^*\right] \tag{4.126}$$

The cross covariance function and cross correlation function are related by

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \eta_X(t_1)\eta_Y^*(t_2) \tag{4.127}$$

where $\eta_X(t_1)$ and $\eta_Y(t_2)$ represent the means of process $X(t)$ and $Y(t)$ as functions of t_1 and t_2, respectively.

4.6.4 Joint Stationarity

The random processes are **jointly wide sense stationary** if each process $X(t)$ and $Y(t)$ are wide sense

stationary themselves and the cross-covariance function is only a function of time difference and $E[X(t_1)Y(t_2)]$ is a constant and not a function of t_1 or t_2. When this is true, the cross-correlation function and cross-covariance function are written in terms of the time difference τ only:

$$C_{XY}(\tau) = R_{XY}(\tau) - \eta_X \eta_Y \tag{4.128}$$

Joint strict sense stationarity would imply that for every time translate of amount ε for any n, the joint probability densities are the same. This can be expressed as

$$\begin{aligned} &f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m; t_1 + \varepsilon, t_2 + \varepsilon, \ldots t_n + \varepsilon, t_{n+1} + \varepsilon, t_{n+2} + \varepsilon, \ldots, t_{n+m} + \varepsilon) \\ &= f(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m}; t_1, t_2, \ldots t_n, t_{n+1}, t_{n+2}, \ldots, t_{n+m}) \end{aligned} \tag{4.129}$$

4.6.5 Cross-spectral Density

When the random process $X(t)$ and $Y(t)$ are jointly wide sense stationary, it is convenient to define a cross-spectral density that gives spectral content information to the joint process. The **cross–spectral density** for processes $X(t)$ and $Y(t)$ is defined as the Fourier transform of the cross-correlation function $R_{XY}(\tau)$ as

$$\begin{aligned} S_{XY}(\omega) &= \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau \\ S_{YX}(\omega) &= \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j\omega\tau} d\tau \end{aligned} \tag{4.130}$$

4.7 Gaussian Random Processes

A random process $X(t)$ is a **Gaussian random process** if the random variables $X(t_1), X(t_2), \ldots, X(t_n)$, are jointly Gaussian for all choices of $t_1, t_2, \ldots, t_n$, and for all n.

There are a number of properties and theorems related to Gaussian random process. Three of the most useful follow.

(1) A random process that is wide sense stationary and Gaussian is strict sense stationary.

(2) Integrals of Gaussian random process over a given time interval are Gaussian random variables and random variables over many different intervals are jointly Gaussian random variables.

(3) The process defined by the integral of a Gaussian random process over the interval $-\infty$ to t, as t varies from $-\infty$ to ∞, is a Gaussian random process, which can be written as

$$Y(t) = \int_{-\infty}^{t} X(t) dt \qquad for -\infty < t < \infty \tag{4.131}$$

4.7.1 First-Order Densities for Gaussian Random Processes

If a Gaussian random process is characterized by its mean $\eta_X(t)$ and autocorrelation function

$R_{XX}(t_1,t_2)$, its first- and second-order density functions can be written in terms of the mean and autocovariance function $C_{XX}(t_1,t_2)$ .The first-order density is

$$f(x;t)=\frac{1}{\sqrt{2\pi C_{XX}(t,t)}}\exp\left(-\frac{(x-\eta_X(t))^2}{2C_{XX}(t,t)}\right) \tag{4.132}$$

4.7.2 Second-Order Densities for Gaussian Random Processes

The second-order density for the random variables $X(t_1)$ and $X(t_2)$ defined, respectively, as X_1 and X_2 is

$$f(x_1,x_2;t_1,t_2)=\frac{1}{2\pi\sqrt{|K|}}\exp\left\{\frac{1}{2}(x-m)^T K^{-1}(x-m)\right\}$$

Where

$$x=\begin{bmatrix}x_1\\x_2\end{bmatrix},\quad m=\begin{bmatrix}\eta_X(t_1)\\\eta_X(t_2)\end{bmatrix}\quad K=\begin{bmatrix}C_{XX}(t_1,t_1) & C_{XX}(t_1,t_2)\\C_{XX}(t_2,t_1) & C_{XX}(t_2,t_2)\end{bmatrix}, \tag{4.133}$$

When the Gaussian random process is wide sense stationary, the first- and second- order densities can be written in terms of the mean η_X and autocovariance function $C_{XX}(\tau)$ as

$$f(x;t)=\frac{1}{\sqrt{2\pi C_{XX}(0)}}\exp\left(-\frac{(x-\eta_X)^2}{2C_{XX}(0)}\right) \tag{4.134}$$

And

$$f(x_1,x_2;t_1,t_2)=\frac{1}{2\pi\sqrt{|K|}}\exp\left\{-\frac{1}{2}(x-m)^T K^{-1}(x-m)\right\}$$

Where

$$X=\begin{bmatrix}x_1\\x_2\end{bmatrix},\quad m=\begin{bmatrix}\eta_X\\\eta_X\end{bmatrix},\quad K=\begin{bmatrix}C_{XX}(0) & C_{XX}(t_1-t_2)\\C_{XX}(t_2-t_1) & C_{XX}(0)\end{bmatrix} \tag{4.135}$$

4.8 White Random Processes

For case in analysis, it is convenient to define the while noise process. The name comes from the fact that the spectral density is assumed to be flat and contain all frequencies. A useful definition is that a **white noise random process** is characterized by its power spectral density $S_{XX}(\omega)$as follows:

$$S_{XX}(\omega)=\frac{1}{2}N_0 \qquad for \qquad -\infty\le\omega\le\infty \tag{4.136}$$

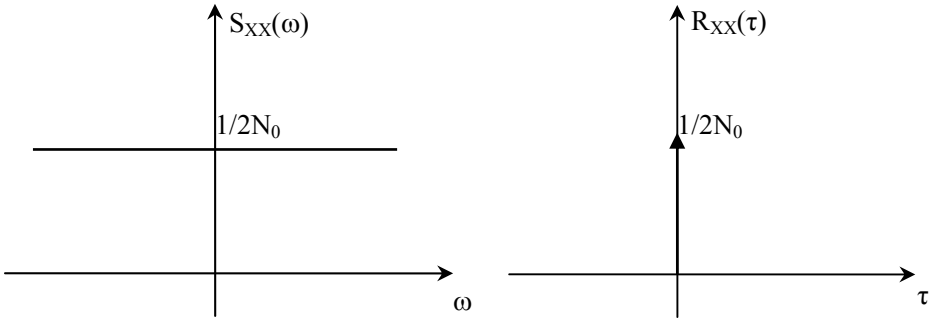
The autocorrelation function for the white noise process is the inverse Fourier transform of the power spectral density, which becomes

$$R_{XX}(\omega)=\frac{1}{2}N_0\delta(\tau) \tag{4.137}$$

The autocorrelation function and power spectral density for the while noise process are given in Figure 4.20. This particular definition implies the following:

(1) $E[X(t)]=0$

(2) $E[X(t_1)X(t_2)]=0$ for t_1 not equal t_2. This means that the random variables defined at two not equal times are orthogonal and because of the zero mean, they are also uncorrelated. Furthermore, if the white noise process is Gaussian, they are independent random variables.



(a) White Noise PSD (b) White Noise Autocorrelation

Figure 4.20 Power spectral density and autocorrelation function for the white noise random process.

(3) The average power is infinite. Thus the white noise process is not physically realizable, but it can be used as a basis for generating realizable processes by passing white noise through filter. White noise should not be thought of as an observable process; only its output after filtering is observable. Nonetheless, it is convenient to make white noise assumptions to simplify analysis.

No other characterizations are given in the definition, so it is possible to have any first-order and higher-order probability density functions. In many problems it is convenient to use densities that are Gaussian, and thus the words **Gaussian white noise** are commonly used to identify a white noise random process with the same autocorrelation as given

And the nth order densities being Gaussian.

In some problems the power is not constant but varies with time. A white noise process is defined as a **nonstationary white random process** if its autocorrelation function is written as

$$R_{XX}(t,u)=\frac{1}{2}N_0(T)\delta(t-u) \tag{4.138}$$

4.9 ARMA Random Processes

In working with continuous linear time-varying and time-invariant systems, a useful system model is a differential equation in terms of the input and output. For time-invariant systems the coefficients of the differential equation are constant, and the Laplace transform is convenient and expedient for analysis. For discrete time shift invariant linear systems, a difference equation in terms of input and output is a useful

model, and the Z-transform plays a significant role in analyzing such systems. Further discussions on these relationships will be presented in Chapter 5, while in this section we investigate a special class of random process resulting from driving a discrete time system with a white noise process.

If $w(n)$ represents the input to a linear time-invariant causal system and $x(n)$ the output, the input-output relationship can be written as

$$x[n]=-\sum_{k=1}^{p}a_k x[n-k]+\sum_{k=0}^{p}b_k w[n-k] \tag{4.139}$$

It is usually assumed that $p>q$ and that the equation above is defined for either all n or for $n\geq 0$.

If $w(n)$ is a white sequence with $E[w[n]]=0$ and $E[w[j]w[k]]=\sigma^2\delta_{jk}$ for all j and k, the resulting process $x(n)$ is called an **autoregressive moving average(ARMA)** random process of order p, q, which is written as ARMA(p,q).

If all the $a_k=0$ for all $k=1,2,...,p$, the resulting random process $x(n)$ is called a **moving average (MA)** process of order q abbreviated as MA(q) and governed by

$$x[n]=\sum_{k=0}^{p}b_k w[n-k] \tag{4.140}$$

If $b_k=0$ for all $k=1,2,...,p$, the resulting random process $x(n)$ is called an **auto regressive(AR)** process of order p, written as AR(P) and governed by

$$x[n]=-\sum_{k=1}^{p}a_k x[n-k]+b_0 w[n] \tag{4.141}$$

The word "autoregressive" comes form the statistical community and implies that the $x(n)$ regresses on itself, which means that it can be written as a linear function of its own previous values.

Depending on the statistical properties assumed for $w(k)$, the ARMA processes defined by (4.139) have varying properties with respect to characterizations like mean, variance, autocorrelation, first-order densities, and so on. Also different results are obtained if the $w(n)$ is defined for all time indexes or just for indexes greater than or equal zero. If $w(n)$ and $x(n)$ are defined for all n, the system will have reached a steady state output. However, if they are defined for only time indexes $n\geq 0$, the beginning samples will contain a transient portion reaching steady state only for large values of the time index n.

4.9.1 Moving Average Process, MA(*q*)

For this discussion let the governing equation for a moving average process of order q, MA(q) in Eq. (4.140), be defined as

$$x[n]=\sum_{k=0}^{p}b_k w[n-k], \qquad n\geq 0 \tag{4.142}$$

Where $w(n)$ is a white stationary process, not necessarily Gaussian, with zero mean, $E\{w[n]\}=0$,and variance given by $E\{w^2[n]\}=\sigma^2$ for all $n\geq 0$.

Mean of MA(q) Process. The mean of the moving average process of order q can be found by taking the expected value of Eq. (4.142) as

$$E\{X[n]\}=E\left\{\sum_{k=0}^{p}b_k w[n-k]\right\}=\sum_{k=0}^{p}b_k E\{w[n-k]\}, \qquad n\geq 0 \tag{4.143}$$

Since $E\{w[n-k]\}=0$ for $k=0$ to q, the mean of the MA(q) process is given by

$$E\{x[n]\}=0, \qquad n\geq 0 \tag{4.144}$$

Variance of the Moving Average Process, MA(q). The variance of the MA(q) process is given by

$$\sigma_X^2[n]=E\{(x(n)-E\{x(n)\})^2\}, \qquad n\geq 0 \tag{4.145}$$

The variance $\sigma_X^2[n]$ is zero by definition for all $n<0$, From(4.144) we know that $E\{x[n]\}=0$.For all $n\geq 0$,so the variance is just the second moment. After substituting the $x(n)$ of Eq. (4.142) into Eq. (4.145), the variance becomes

$$\sigma_X^2[n]=E\{x^2[n]\}=E\left\{\left(\sum_{k=0}^{q}b_k w[n-k]\right)^2\right\}=E\left\{\sum_{k=0}^{q}b_k w[n-k]\sum_{j=0}^{q}b_j w[n-j]\right\} \tag{4.146}$$

Multiplying out the terms in the brackets and taking the expected value through the summations gives for $n\geq p$,

$$\sigma_X^2[n]=\sum_{k=0}^{q}\sum_{j=0}^{q}b_k b_j E\{w[n-k]w[n-j]\}=\sigma^2\sum_{k=0}^{q}b_k^2, \qquad n\geq q \tag{4.147}$$

The last step is possible because the only terms where $E\{w[n-k]w[n-j]\}$ are nonzero are those for which $j=k$. Notice that this variance is thus a constant for $n\geq p$.

If $0\leq n<p$, the result above is modified to include only those $w(n-k)$ and $w(n-j)$ that have arguments ≥ 0, and is thus a function of n:

$$\sigma_X^2[n]=\sigma^2\sum_{k=0}^{n}b_k{}^2, \qquad 0\leq n<q \tag{4.148}$$

The variance $\sigma_X^2[n]$ is thus seen to be a function of n and a part of the initial transient period until n reaches q yielding a steady state after that.

Autocorrelation Function for the Moving Average Process. MA(q). For k and j less than zero the autocorrelation function $R_{XX}(j,k)$ for the MA(q) process is defined to be zero ,while in general, it will

depend on two time indexes $k+m$ and k ,where k and m are positive integers and can be written as

$$E\{x[k+m]x[k]\}=E\left\{\sum_{i=0}^{q}b_k w[k+m-i]\sum_{j=0}^{q}b_j w[k-j]\right\},\qquad k,m\geq 0 \tag{4.149}$$

Let $k>q$ to begin with and $0\leq m\leq q$. For this range of m, Eq. (4.149) can be written as

$$R_{XX}(k+m,k)=E\left\{\begin{matrix}\left(b_0 w[k+m]+b_1 w[k+m-1]+\cdots+b_q w[k+m-q]\right)\\ \cdot\left(b_0 w[k]+b_1 w[k-1]+\cdots+b_q w[k-q]\right)\end{matrix}\right\} \tag{4.150}$$

Looking at all cross product terms, we see that the only ones that give a nonzero value of σ^2 are those with equal indexes, so Eq.(4.150) reduces to

$$\begin{aligned}R_{XX}[k+m,k]&=\sum_{r=0}^{q-m}b_r b_{r+m}E\{w^2[k-r]\}\\&=\sigma^2\sum_{r=0}^{q-m}b_r b_{r+m},\qquad 0\leq m\leq q\end{aligned} \tag{4.151}$$

For $m>q$ there are no overlaps in the products, so we have

$$R_{XX}[k+m,k]=0 \tag{4.152}$$

Clearly, the autocorrelation function for the MA(q) process is independent of the staring time k and a function of the time difference m, provided that $k>q$. Thus after the time q, the process becomes wide sense stationary since the mean was previously shown to be a constant, namely 0, and the autocorrelation has been shown to be a function of time difference only.

Example 4.5

Find the mean, autocorrelation function, and variance for the MA(1), MA(2), and MA(3),random processes assuming that a steady state has been reached, $k>q$, for the models given and with the $w(n)$ as a zero mean white random process.

Solution

For the MA(q) random process the means are all zero because of the assumption that $w(n)$ is a white random process; $E\{x(n)\}=0$ for $n\geq 0$, and defined to be zero for $n<0$.

MA(1) For this process we have the defining equation

$$x[n]=b_0 w[n]+b_1 w[n-1]\qquad for\quad n\geq 0 \tag{4.153}$$

The autocorrelation function evaluated at lags of 0,1, and 2 after steady state are given from (4.151) as

$$\begin{aligned}R_{XX}[0]&=\sigma^2\sum_{r=0}^{1}b_r b_r=\sigma^2\left(b_0{}^2+b_1{}^2\right)\\R_{XX}[1]&=\sigma^2\sum_{r=0}^{0}b_r b_{r-1}=\sigma^2 b_0 b_1\\R_{XX}[k]&=0,\qquad k>1\end{aligned} \tag{4.154}$$

The steady state variance $\sigma_X^2[k]$ is given by $\sigma_X^2[k]=R_{XX}(0)$, since the mean is zero. The autocorrelation function for negative time difference is obtained by symmetry $R_{XX}[-k]=R_{XX}[k]$.

In a similar fashion we obtain the autocorrelation functions for the MA(2) and MA(3) as follows:

MA(2)

$$x[n]=b_0w[n]+b_1w[n-1]+b_2w[n-2] \qquad for \quad n\geq 0 \tag{4.155}$$

The autocorrelation function after steady state is easily found from (4.151) to be

$$\begin{aligned} R_{XX}[0]&=\sigma^2\left(b_0{}^2+b_1{}^2+b_2{}^2\right)\\ R_{XX}[1]&==\sigma^2\left(b_0b_1+b_1b_2\right)\\ R_{XX}[2]&==\sigma^2b_0b_2\\ R_{XX}[k]&=0, \qquad k>2\\ R_{XX}[-k]&=R_{XX}[k] \end{aligned} \tag{4.156}$$

The steady state variance $\sigma_X^2[k]$ is given by $\sigma_X^2[k]=R_{XX}(0)$, since the mean is zero.

MA(3)

$$x[n]=b_0w[n]+b_1w[n-1]+b_2w[n-2]+b_3w[n-3] \qquad for \quad n\geq 0 \tag{4.157}$$

The autocorrelation function after steady state is easily determined from (4.151) as

$$\begin{aligned} R_{XX}[0]&=\sigma^2\left(b_0{}^2+b_1{}^2+b_2{}^2+b_3{}^2\right)\\ R_{XX}[1]&==\sigma^2\left(b_0b_1+b_1b_2+b_2b_3\right)\\ R_{XX}[2]&==\sigma^2\left(b_0b_2+b_1b_3\right)\\ R_{XX}[3]&==\sigma^2b_0b_3\\ R_{XX}[k]&=0, \qquad k>3\\ R_{XX}[-k]&=R_{XX}[k] \end{aligned} \tag{4.158}$$

The steady state variance $\sigma_X^2[k]$ is given by $\sigma_X^2[k]=R_{XX}(0)$, since the mean is zero.

4.9.2 Autoregressive Processes, AR(*p*)

An autoregressive process, AR(p), is a special case of the ARMA(p,q) process. For this discussion it is assumed that AR(p) is governed for $n\geq 0$ by

$$x[n]=-\sum_{k=1}^{p}a_kx[n-k]+b_0w[n], \qquad n\geq 0 \tag{4.159}$$

The excitation process $w(n)$, for $n\geq 0$, is a white stationary random process with $E\{w[j]w[k]\}=\sigma^2\delta_{jk}$. We would like to determine the mean, variance, and autocorrelation for this AR(p) random process.

Mean of AR(p) Process. The mean of the AR(p) process is obtained by taking the expected value of the

governing Eq. (4.159) to yield the following difference equation:

$$E\{X[n]\} = -\sum_{k=1}^{p} a_k E\{x[n-k]\} + b_0 E\{w[n]\}, \qquad n \geq 0 \tag{4.160}$$

Since $E\{w[n]\} = 0$, and $x(n) = 0$, for $n < 0$, the solution of the difference equation yields

$$E\{x[n]\} = 0, \qquad n \geq 0 \tag{4.161}$$

Variance of AR(p) Process. The variance of the defined AR(p) process can be found directly from the definition of the variance and the process $x(n)$ given in Eq. (4.159). Using the zero mean assumption, we write variance as

$$\sigma_X^2[n] = E\{x^2[n]\} = E\left\{\left(-\sum_{k=0}^{p} a_k x[n-k] + b_0 w[n]\right)\left(-\sum_{j=0}^{p} a_j x[n-j] + b_0 w[n]\right)\right\} \tag{4.162}$$

The expected value of all cross products of $w(n)$ with the summations of the $x(n-k)$, given by

$$E\left\{\left(-\sum_{k=0}^{p} a_k x[n-k]\right) w[n]\right\} = 0 \tag{4.163}$$

are all zero, since $x[n-k]$ are not functions of $w(n)$ as they are from a time preceding $w(n)$

Thus the variance reduces to

$$\sigma_X^2[n] = E\{x^2[n]\} = E\left\{\left(-\sum_{k=0}^{p} a_k x[n-k] w[n]\right) \cdot \left(-\sum_{j=0}^{p} a_j x[n-j]\right)\right\} + b_0^2 \sigma^2 \tag{4.164}$$

Multiplying the product terms out, taking the expected value and rearranging, we find the variance from Eq, (4.164) to be

$$\sigma_X^2[n] = \sum_{k=1}^{p} \sum_{j=1}^{p} a_j a_k R_{XX}[n-k, n-j] + b_0{}^2 \sigma^2 \tag{4.165}$$

The variance at time index n is thus seen to be a function of the variances at the p previous times, plus a function of various preceding autocorrelation functions at various time indexes, plus a weighted sum of the white sequence variance.

An alternative form for the variance can be obtained by multiplying the $x(n)$ of Eq. (4.159) by $x(n)$ and taking the expected value as

$$\sigma_{XX}[n] = E\{x^2[n]\} = E\left\{\left(-\sum_{k=1}^{p} a_k x[n-k] + b_0 w[n]\right) x[n]\right\} \tag{4.166}$$

Multiplying out and taking the expected value of the terms results in the following equation for the variance which, because of the zero mean, is the autocorrelation function at n and n:

$$R_{XX}[n,n] = -\sum_{k=1}^{p} a_k R_{XX}[n, n-k] + b_0{}^2 \sigma^2 \tag{4.167}$$

In the form the variance is written in terms of the autocorrelation functions for indexes n, $n-k$ as k

goes from 1 to p. This same approach will now be used to determine the autocorrelation function for the autoregressive process at other lags.

Autocorrelation of AR(p) Process. The autocorrelation function, in general, goes through a transient period since the signal starts at zero with zero initial conditions. To derive the autocorrelation function, it is assumed that $n > p$. The derivation begins with the single step difference by taking the expected value of $x(n)x(n-1)$ as follows:

$$E\{x[n]x[n-1]\} = E\left\{\left(-\sum_{k=0}^{p} x[n-k] + b_0 w[n]\right)x[n-1]\right\} \tag{4.168}$$

Multiplying out, taking the expected value through the sum, and using the fact that $E\{w[n]x[n-1]\} = 0$,since $x\{n-1\}$ is not a function of $w(n)$ and $w(n)$ is a white process, yields the equation

$$R_{XX}[n, n-1] = -\sum_{k=1}^{p} a_k R_{XX}[n-1, n-k] \tag{4.169}$$

Similarly multiplying $x(n)$ of Eq. (4.159) by $x[n-j]$ for $j = 2$ to p, taking the expected value, and simplifying leads to

$$R_{XX}[n, n-j] = -\sum_{k=1}^{p} a_k R_{XX}[n-j, n-k], \qquad j = 2,3,\ldots,p \tag{4.170}$$

Thus Eqs. (4.167), (4.169), and (4.170) gives a set of equations that determines the autocorrelation functions at lags 0 to p in terms of various other autocorrelation functions, and these are the key equations in solving for the steady state. If $j > p$, Eq. (4.170) holds as well.

If we assume that a steady state has been reached, the $R_{XX}(r,s)$ are functions of time difference $r-s$ only. So we can rewrite Eqs. (4.169) and (4.170) as

$$R_{XX}[j] = -\sum_{k=1}^{p} a_k R_{XX}[k-j], \qquad j = 1,2,\ldots,p \tag{4.171}$$

By the symmetry property of the autocorrelation function for a stationary random process, $R_{XX}(-j) = R_{XX}(j)$, these equations can be put into the following matrix form:

$$\begin{bmatrix} R_{XX}[0] & R_{XX}[1] & \cdots & R_{XX}[p-1] \\ R_{XX}[1] & R_{XX}[0] & \cdots & R_{XX}[p-2] \\ \vdots & \vdots & \vdots & \vdots \\ R_{XX}[p-1] & R_{XX}[p-2] & \cdots & R_{XX}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = -\begin{bmatrix} R_{XX}[1] \\ R_{XX}[2] \\ \vdots \\ R_{XX}[p] \end{bmatrix} \tag{4.172}$$

The equations represented in this matrix form are called the **Yule-Walker equations** for the autoregressive process defined in Eq.(4.159). If we know the $R_{XX}(0)$ through $R_{XX}(p)$, or their estimates, these equations may be solves for the AR parameters $a_1, a_2, \ldots, a_p$, and thus be used for system identification.

The Levinson recursion algorithm [11] can be used to expedite the solution, since the coefficient matrix is a Toepliz matrix.

However, if we have the $a_1, a_2, \ldots, a_p$ values, meaning the system is known, and we want to solve for the autocorrelation function, we must use Eq.(4.167) for $R_{XX}(0)$ and rearrange the equations of (4.172) so that the $R_{XX}[k]$ are the unknown autocorrelations at the lags 0 through p :

$$\begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{p-2} & a_{p-1} & a_p \\ a_1 & 1+a_2 & a_3 & \cdots & a_{p-1} & a_p & 0 \\ a_2 & a_1+a_3 & 1+a_4 & \cdots & a_p & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_p & a_{p-1} & \cdots & a_3 & 1+a_2 & a_1 \\ a_p & a_{p-1} & a_{p-2} & \cdots & a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} R_{XX}[0] \\ R_{XX}[1] \\ R_{XX}[2] \\ \vdots \\ R_{XX}[p-1] \\ R_{XX}[p] \end{bmatrix} = - \begin{bmatrix} b_0\sigma^2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{4.173}$$

Examples 4.6 and 4.7 illustrate this procedure for finding the autocorrelation function for AR(1) and AR(2).

Example 4.6

Find the mean, autocorrelation, and variance for the autoregressive process AR(1) random process defined by

$$x[n] = -a_1 x[n-1] + b_0 w[n], \qquad n \geq 0 \tag{4.174}$$

Where a_1 is such that a stable system is represented.

Solution

From (4.167) and (4.169) the autocorrelation function at 0 and 1 must satisfy the two equations

$$\begin{aligned} R_{XX}[n,n] &= -a_1 R_{XX}[n,n-1] + b_0{}^2\sigma^2 \\ R_{XX}[n,n-1] &= -a_1 R_{XX}[n-1,n-1] \end{aligned} \tag{4.175}$$

If we assume that a steady state has been reached, then the autocorrelation function can be written in terms of time difference only and with the symmetry property of the autocorrelation functions (4.175) becomes

$$\begin{aligned} R_{XX}[0] &= -a_1 R_{XX}[1] + b_0{}^2\sigma^2 \\ R_{XX}[1] &= -a_1 R_{XX}[0] \end{aligned} \tag{4.176}$$

Solving these two equations for $R_{XX}(0)$ and $R_{XX}(1)$ gives

$$R_{XX}(0) = \frac{b_0^2\sigma^2}{1-a_1^2}, \qquad R_{XX}(1) = \frac{-a_1 b_0^2\sigma^2}{1-a_1^2} \tag{4.177}$$

To obtain the autocorrelation function at lags greater that 1,Eq. (4.170) and $j > 1$, we have

$$R_{XX}[n,n-j] = -a_1 R_{XX}[n-j,n-1] \tag{4.178}$$

and in the steady state we obtain

$$R_{XX}[j]=-a_1R_{XX}[j-1]=(-a_1)^{j-1}R_{XX}[1]=\frac{b_0^2\sigma^2}{1-a_1^2}(-a_1)^j,\qquad j>1 \tag{4.179}$$

4.9.3 Autoregressive Moving Average Process, ARMA (*p*, *q*)

For this discussion it will be assumed that the governing equation for a moving average process of order q, ARMA (p,q) given in Eq. (4.139) is defined as

$$x[n]=-\sum_{k=1}^{p}a_kx[n-k]+\sum_{k=0}^{q}b_kw[n-k],\qquad n\geq 0 \tag{4.189}$$

for positive time index n from 0 to ∞ and that $w(n)$ is a white stationary process, not necessarily Gaussian, with $E\{w(n)\}=0$ for all $n\geq 0$, and variance given by $E\{w^2(n)\}=\sigma^2$ for all $n\geq 0$, We would like to partially characterize the resulting ARMA (p,q) process by finding its mean, variance, and autocorrelation function.

Mean of ARMA (p, q). The mean of the ARMA (p,q) process is easily found by taking the expected value of Eq. (4.189) to give

$$E\{x[n]\}=-\sum_{k=1}^{p}a_kE\{x[n-k]\}+\sum_{k=0}^{q}b_kE\{w[n-k]\},\qquad n\geq 0 \tag{4.190}$$

Since $E\{w(n-k)\}=0$ for all $k=0,1,\ldots,q$, Eq. (4.190) can be written as a difference equation for $E\{x(n)\}$ with a zero driving function:

$$E\{x[n]\}=-\sum_{k=1}^{p}a_kE\{x[n-k]\},\qquad n\geq 0 \tag{4.191}$$

Since all initial conditions are assumed equal to zero, the mean $E\{x(n)\}=0$ for $n\geq 0$.

Variance of ARMA (p, q). Since the mean is zero for all n, the variance can be written in terms of the autocorrelation function evaluated at time indexes n and n as

$$\sigma^2[n]=E\{x^2[n]\}=R_{XX}[n,n]=E\left\{\left(-\sum_{k=0}^{p}a_kx[n-k]+b_0\sum_{k=0}^{q}w[n-k]\right)x[n]\right\} \tag{4.192}$$

On taking the expected value in Eq. (4.192), we can show the variance to be

$$\sigma^2[n]=-\sum_{k=1}^{p}a_kR_{XX}[n,n-k]+\sum_{k=0}^{q}b_kE\{w[n-k]x[n]\} \tag{4.193}$$

This result can be written as

$$\sigma^2[n]=-\sum_{k=1}^{p}a_kR_{XX}[n,n-k]+\sum_{k=0}^{q}b_kR_{XW}[n,n-k] \tag{4.194}$$

Thus the variance is written in terms of the weighted sum of the autocorrelation function evaluated at different times and the weighted sum of various cross correlations at different times. These terms will be determined in the following section on the determination of the autocorrelation function for a general ARMA process.

Autocorrelation for ARMA (p, q). The autocorrelation function, in general, goes though a transient period since the signal starts at zero initial conditions. To derive the autocorrelation function, it is assumed that $n > p$. The derivation begins with the expected value of the product of $x(n)$ and $x(n-1)$, where $x(n)$ is from (4.189), as follows:

$$E\{x[n]x[n-1]\} = E\left\{\left(-\sum_{k=1}^{p} x[n-k] + \sum_{k=0}^{q} b_k w[n-k]\right)x[n-1]\right\} \tag{4.195}$$

Multiplying out and taking the expected value through the sum yields the equation

$$R_{XX}[n,n-1] = -\sum_{k=1}^{p} a_k R_{XX}[n-1,n-k] + \sum_{k=0}^{q} b_k E\{w[n-k]x[n-1]\} \tag{4.196}$$

Similarly multiplying $x(n)$ of Eq. (4.189) by $x(n-j)$ for $j=2$ to p, taking the expected value, and simplifying leads to

$$R_{XX}[n,n-j] = -\sum_{k=1}^{p} a_k R_{XX}[n-j,n-k] + \sum_{k=0}^{q} b_k E\{w[n-k]x[n-j]\}, \qquad j = 2,3,....,p \tag{4.197}$$

Thus Eqs. (4.196) and (4.197) give a set of equations that determine the autocorrelation function at lags 1 to p in terms of various other autocorrelation functions. If we assume that a steady state has been reached, the $R_{XX}(r,s)$ are functions of time difference $r-s$ only. So we can rewrite Eqs. (4.196) and (4.197) as

$$R_{XX}[j] = -\sum_{k=1}^{p} a_k R_{XX}[k-j] + f_j(a,b) \qquad j = 1,2,....,p \tag{4.198}$$

Where the $f_j(a,b)$ are the second sums shown in Eqs. (4.196) and (4.197).

These $f_j(a,b)$ in the simultaneous equations above are rather complex nonlinear function of the ARAM(p,q) model parameter vectors **a** and **b.** Using the symmetry property of the autocorrelation function for a stationary random process, $R_{XX}(-j) = R_{XX}(j)$, allows the equations of (4.198) to be put into the following matrix form:

$$\begin{bmatrix} R_{XX}[0] & R_{XX}[1] & \cdots & R_{XX}[p-1] \\ R_{XX}[1] & R_{XX}[0] & \cdots & R_{XX}[p-2] \\ \vdots & \vdots & \vdots & \vdots \\ R_{XX}[p-1] & R_{XX}[p-2] & \cdots & R_{XX}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} + \begin{bmatrix} f_1(a,b) \\ f_2(a,b) \\ \vdots \\ f_p(a,b) \end{bmatrix} = -\begin{bmatrix} R_{XX}[1] \\ R_{XX}[2] \\ \vdots \\ R_{XX}[p] \end{bmatrix} \tag{4.199}$$

Thus the solution for the autocorrelation function $R_{XX}(k)$ at all lags is no longer a solution of simultaneous linear equations but a solution of a set simultaneous nonlinear equations. The following

example illustrates the overall procedure in finding the mean, variance, and autocorrelation function of the simplest ARMA process, namely the ARMA(1,1) process.

Example 4.8

Find the (a) mean, (b) autocorrelation function, and (c) variance for the ARMA(1,1) random process given by the difference equation

$$x[n] = -a_1 x[n-1] + b_0 w[n] + b_1 w[n-1], \qquad n \geq 0 \tag{4.200}$$

Assume that $w(n)$ is a zero mean white random process with variance $E[w(j)w(k)] = \sigma^2 \delta_{jk}$. Assume all initial conditions for negative time index to be zero and that a steady state has been obtained.

Solution

(a) It was shown in Eq. (4.191) that the mean of ARMA process is zero. Therefore $E\{x[n]\} = 0$ for all $n \geq 0$.

(b) The variance for n is the same as the autocorrelation function at a zero lag since the mean is zero. Therefore we have using (4.193) that

$$\begin{aligned} \sigma^2[n] &= E\{x^2[n]\} = R_{XX}[n,n] \\ &= E\{x[n](-a_1 x[n-1]) + b_0 w[n] + b_1 w[n-1]\} \\ &= -a_1 R_{XX}[n,n-1] + b_0 E\{x[n]w[n]\} + b_1 E\{x[n]w[n-1]\} \end{aligned} \tag{4.201}$$

The last two terms in the expression above will be evaluated separately. Substituting the difference equation for $x(n)$, multiplying out, and taking the expected value gives

$$\begin{aligned} E\{x[n]w[n-1]\} &= E\{(-a_1 x[n-1] + b_0 w[n] + b_1 w[n-1])w[n-1]\} \\ &= -a_1 E\{x[n-1]W[n-1]\} + b_1\sigma^2 \\ &= -a_1 E\{(-a_1 x[n-2] + b_0 w[n-1] + b_1 w[n-2])w[n-1]\} + b_1\sigma^2 \\ &= -a_1 b_0 \sigma^2 + b_1\sigma^2 \end{aligned} \tag{4.202}$$

The first expected value in the expression above is zero because $x(n-1)$ is not a function of $w(n)$, and the last term is zero because the $w(n)$ is a white sequence.

The other expected value is given as

$$E\{x[n]w[n]\} = E\{(-a_1 x[n-1] + b_0 w[n] + b_1 w[n-1])w[n]\} = b_0\sigma^2 \tag{4.203}$$

Substituting the results from (4.202) and (4.203) into (4.201), we finally obtain the variance as

$$\sigma^2[n] = -a_1 R_{XX}[n,n-1] + b_0^2\sigma^2 - a_1 b_1 b_0 \sigma^2 + b_1^2\sigma^2 \tag{4.204}$$

The $R_{XX}[n,n-1]$ can be calculated by substituting the definition of the ARMA(1,1) process into the expected value;

$$\begin{aligned} R_{XX}(n,n-1) &= E\{x[n]x[n-1]\} = E\{(-a_1 x[n-1] + b_0 w[n] + b_1 x[n-1])x[n-1]\} \\ &= -a_1 R_{XX}[n-1,n-1] + b_0 E\{w[n]x[n-1]\} + b_1 E\{w[n-1]x[n-1]\} \end{aligned} \tag{4.205}$$

The $E\{x[n]x[n-1]\}$ term is zero because $x[n-1]$ is not a function of $w[n]$. The third term above

$E\{w[n-1]x[n-1]\}$ is determined by replacing $x[n-1]$ by its equivalent from the defining equation:

$$\begin{aligned} E\{w[n-1]x[n-1]\} &= E\{w[n-1](-a_1x[n-2]+b_0w[n-1]+b_1w[n-2])\} \\ &= b_0\,\sigma^2 \end{aligned} \tag{4.206}$$

Thus, after substituting (4.206) into (4.205), the single lag autocorrelation function of the ARMA(1,1) process becomes

$$R_{XX}[n,n-1] = -a_1R_{XX}[n-1,n-1]+b_0b_1\sigma^2 \tag{4.207}$$

To obtain the steady state solution, we assume that the autocorrelation function can be written in terms of time difference only. Thus (4.204) and (4.207) are written as

$$\begin{aligned} R_{XX}[0] &= -a_1R_{XX}[1]+b_0^2\sigma^2-a_1b_0b_1\sigma^2+b_1^2\sigma^2 \\ R_{XX}[1] &= -a_1R_{XX}0+b_0b_1\sigma^2 \end{aligned} \tag{4.208}$$

Solving these two equations for $R_{XX}[0]$ and $R_{XX}[1]$ yields the following values for the autocorrelation function at 0 to 1:

$$\begin{aligned} R_{XX}[0] &= \frac{(b_0^2+b_1^2-2a_1b_0b_1)\sigma^2}{1-a_1^2} \\ R_{XX}[1] &= \frac{(a_1^2b_0b_1-a_1b_0^2-a_1b_1^2+b_0b_1)\sigma^2}{1-a_1^2} \end{aligned} \tag{4.209}$$

To find the autocorrelation function for higher lads, $j=2$, we have that $R_{XX}[n,n-2]$ with $E\{w[n]x[n-2]\}=0$ and $E\{w[n-1]x[n-2]\}=0$ is determined as

$$\begin{aligned} R_{XX}[n,n-2] &= E\{x[n]x[n-2]\} \\ &= E\{(-a_1x[n-1]+b_0w[n]+b_1w[n-1])x[n-2]\} \\ &= -a_1R_{XX}[n-1,n-2] \end{aligned} \tag{4.210}$$

In a similar fashion it is easy to show that $R_{XX}[n,n-j]=-a_1R_{XX}[n,n-j+1]$ for lags $j>2$. This property is general for ARMA(p,q) processes whenever j is greater than q, evaluating this result in the steady state yields the autocorrelation function for lags greater than one for the ARMA(1,1) process as follows;

$$R_{XX}[j] = (-a_1)^{j-1}R_{XX}[1], \qquad j\geq 2 \tag{4.211}$$

(c) The steady state variance is just the $R_{XX}[0]$ given in (4.209), since the mean is zero. Thus

$$\sigma_{XX}^2(n) = R_{XX}[0] = \frac{(b_0^2+b_1^2-2a_1b_0b_1)\sigma^2}{1-a_1^2} \tag{4.212}$$

It becomes increasingly more difficult to obtain closed form solutions for the autocorrelation function and variance as the order p and q of the ARMA(p,q) process is increased.

4.10 Periodic Random Processes

A wide sense stationary random process is defined as **periodic** if its autocorrelation function is periodic.

The random sinusoidal process $x(t)=A\sin(\Omega_0 t+\Phi)$ (defined in Example 4.3) has an autocorrelation function from (4.89):

$$R_{XX}(\tau)=K\cos(\Omega_0\tau) \tag{4.213}$$

This process is a periodic random process since its autocorrelation function is periodic.

In a similar fashion a random process $X(t)$ defined by a Fourier series with independent zero mean random variables A_k and B_k as amplitudes given by

$$X(t)=A_0+\sum_{k=0}^{\infty}A_k\cos(k\Omega_0 t)+\sum_{k=0}^{\infty}B_k\sin(k\Omega_0 t) \tag{4.214}$$

can be shown to have a periodic autocorrelation function and thus be periodic.

4.11 Sampling of Continuous Random Processes

The purpose of this section is to determine the statistical properties of a discrete time random process $X[k]$ that is generated by sampling uniformly, with spacing T, a continuous time random process and thus defined by $X[k]=X(kT)$.

The mean of $X[k]$ can be easily seen to be

$$E[X[n]]=E[X(nT)]=\eta_X(nT) \tag{4.215}$$

The autocorrelation function $R_{XX}[k_1,k_2]$ is easily determined as

$$R_{XX}[k_1,k_2]=E[X(k_1T)X(k_2T)]=R_{XX}(k_1T,k_2T) \tag{4.216}$$

Therefore the mean and autocorrelation function of a process generated by sampling a continuous process are just the sampled versions of the mean and autocorrelation function of the continuous process.

It is also easy to see that is $X(t)$ is a wide sense stationary continuous time process, the random sequence generated by uniform sampling is a wide sense stationary discrete time sequence, that is , its mean is independent of n and its autocorrelation function, depends only on the time difference, k_1-k_2. The converse of this statement is not necessarily true; that is, if a discrete time process that is a sampled version of a continuous time process is wide sense stationary, the continuous time process is mot necessarily wide sense stationary. Similarly, if the sampled version of a random process is in some sensse stationary, this does not imply that the continuous time process is the same sense stationary.

The first- and higher-order densities are easily seen to be just a sampled version of the respective densities, which can be expressed for $X(t_1),X(t_2),\ldots,X(t_n)$, as

$$f[x_1,x_2,\ldots,x_n;k_1,k_2,\ldots,k_n]=f[x_1,x_2,\ldots,x_n;k_1T,k_2T,\ldots,k_nT] \tag{4.217}$$

where $t_1 = k_1 T, t_2 = k_2 T, \ldots, t_n = k_n T$.

4.12 Ergodic Random Processes

Say we are given a random process $X(t)$ with a known mean $E[X(t)]$. A realization of the process $X(t, e_i)$ is a deterministic waveform and has a time average

$$N_T(e_i) = \frac{1}{2T}\int_{-T}^{T} X(t, e_i)dt$$

The time average is a real value and possibly different for each outcome e_i, thus defining a random variable N_T . We will define N as the limit of N_T as $T \to \infty$ as

$$N = \lim_{T\to\infty} N_T$$

As $T \to \infty$ the random variable N_T may converge in some cense to another random variable. The definition of convergence for the deterministic or nonrandom case must be modified to be able to discuss convergence of random variables as $T \to \infty$. There are many different forms of convergence that have been defined, including mean squared convergence, convergence in the mean, convergence in probability, and convergence with probability 1. These forms of convergence and theorems relating to ergodicity are introduced in Papoulis [1], and a thorough treatment of them would take us far afield of our intent of introducing basic concepts of random process.

A random process $X(t)$ is defined as **ergodic in the mean** if the N_T converges to degenerate random variable with deterministic value equal to the statistical average given by $E[X(t)] = \eta_X$. Notice that this definition implies that a random process $X(t)$ that is not stationary in the mean cannot be ergodic in the mean.

Similarly, if a random variable $R_T(\tau)$ is defined by

$$R_T(\tau) = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(\tau)dt$$

And we define $R(\tau)$ as the limit of $R_T(\tau)$ as $T \to \infty$, we can define a process to be **ergodic in autocorrelation** if $R_T(\tau)$ for all τ converges to a degenerate random variable with deterministic value equal to the statistical average given by $E[X(t+\tau)X(t)]$, which is the autocorrelation function evaluated at τ . Again we see that if the process $X(t)$ is not wide sense stationary, then it cannot possibly be ergodic in autocorrelation.

The statistical means and autocorrelation functions for ergodic processes are thus interchangeable with their time determined counterparts.

4.13 Summary

The main objective of this chapter was to present the basic background material in random processes including process formulation and analysis. The definition of a random process in terms of an indexed set of random variables where the indexing set could be continuous or discrete links the concepts of random variables and random processes. A random process could also be viewed as a mapping from the sample space of a defined experiment to a set of output functions. Evaluating a random process at a particular time gives a random variable, at a particular outcome gives a function of time, and evaluation of a random variable at a time and an outcome gives a real value.

Although random processes can always be described in terms of an underlying experiment that gives a total characterization with respect to the calculation of probabilities of events, various partial characterization prove to be useful. These include the mean, variance, autocorrelations function, first- and second-order densities, as well as higher-order densities and moments.

Various forms of stationatity of random process were defined including stationarity in the mean, variance, auto correlation function, first- and second-order densities, higher-order densities and moments. Perhaps most important here is the concept of wide sense stationarity. A random process is wide sense stationary provided that its mean is a constant, independent of time, and its autocorrelation function is a function of time difference only and does not depend on the particular times. The partial characterization of mean and autocorrelation function for wide sense stationarity random process will be shown to be enough information about the process to perform the various optimum filtering operations explored in Chapters 7 and 8.

Next a series of important examples of random processes were presented, including the straight line process, semirandom and random transmission processes, semirandom and random telegraph processes, the random sinusoidal process, and the random walk process.

Important special cases of a white noise random process were discussed and processes generated by passing white noise through a discrete time linear system were shown to lead to MA, AR, and ARMA random processes.

----------This is the end of Chapter04----------