Addendum - AMR L2 Projection

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1 Discontinuous Galerkin AMR L2 projection

In Discontinuous Galerkin (DG) methods, the solution within each element is approximated by a polynomial. When using Adaptive Mesh Refinement (AMR), elements are dynamically refined (split into smaller elements) or coarsened (merged into a larger element) to improve solution accuracy and computational efficiency. The L2 projection is a crucial operation for transferring solution data between different mesh levels (coarse and fine grids) while preserving the quality of the numerical solution, typically aiming to conserve quantities like mass or momentum locally. This section details the L2 projection process for DG methods employing hierarchical, Cartesian meshes where coarse cells are parents to refined child cells.

We consider a set of basis functions within each element. For a D-dimensional problem, these multi-dimensional basis functions $\phi_i(\xi)$ are constructed as tensor products of one-dimensional basis functions. A common choice, offering beneficial orthogonality properties, is the Legendre polynomials $P_k(z)$ defined on the interval $[-1, 1]$.

$$\phi_i(\xi) = \prod_{d=1}^{D} P_{i_d}(\xi_d)$$

Here, $\xi = (\xi_1, \dots, \xi_D)$ represents the coordinates in the D-dimensional reference element, typically $\Omega_{\text{ref}} = [-1, 1]^D$. The index i is a multi-index $(i_1, \dots, i_D)$, where P_{i_d} is the i_d-th degree Legendre polynomial in the d-th dimension. The total number of basis functions per element, N_p, depends on the maximum polynomial degree $p_{\max}$ used (e.g., $N_p = (p_{\max} + 1)^D$ if a complete tensor product basis up to degree $p_{\max}$ in each dimension is used).

Let $u_h(x)$ be the DG solution. Within any element Ω_k, it is represented as:

$$u_h(x)|_{\Omega_k} = \sum_{j=1}^{N_p} \hat{u}_k^j \phi_j^k(x)$$

where $\hat{u}_k^j$ are the degrees of freedom (coefficients) for element Ω_k, and $\phi_j^k(x)$ are the basis functions defined on element Ω_k (these are scaled and mapped versions of the reference basis functions $\phi_j(\xi)$).

1.1 Fine to Coarse Projection (Restriction)

When a refined region of the mesh (level $L + 1$) needs to be coarsened to level L, or when information needs to be transferred from child cells to a parent cell, we perform a fine-to-coarse projection. This operation takes the solution from a set of N_f fine cells Ω^l_{L+1} (where $l = 1, \ldots, N_f$, and $N_f = 2^D$ for standard quadtree/octree refinement) that collectively form a single coarse cell Ω_L, and projects it onto the basis functions of the coarse cell.

The L2 projection aims to find coefficients $\hat{u}^n_{k_L}$ for the coarse cell Ω_L such that the projection error is minimized in the L2 norm. This is achieved by enforcing:

$$\int_{\Omega_L} \left(\sum_{n=1}^{N_p} \hat{u}^n_{k_L} \phi^L_n(x) \right) \phi^L_j(x) dx = \int_{\Omega_L} u^{L+1}_h(x) \phi^L_j(x) dx$$

Since $u^{L+1}_h(x)$ is defined piecewise on the fine cells Ω^l_{L+1}, the right-hand side can be written as a sum over these fine cells:

$$\sum_{n=1}^{N_p} \hat{u}^n_{k_L} \int_{\Omega_L} \phi^L_n(x)\phi^L_j(x)dx = \sum_{l=1}^{N_f} \left[\sum_{m=1}^{N_p} \hat{u}^m_{k_{L+1},l} \int_{\Omega^l_{L+1}} \phi^{L+1}_m(x)\phi^L_j(x)dx \right]$$

for each coarse cell basis function $\phi^L_j(x)$, where $j = 1, \ldots, N_p$. Here, $\hat{u}^n_{k_L}$ are the unknown coefficients on the coarse cell Ω_L (denoted by k_L), and $\hat{u}^m_{k_{L+1},l}$ are the known coefficients on the l-th fine child cell (denoted by k_{L+1}, l) at level $L + 1$.

We then define the **coarse cell mass matrix** M^C and the **fine-to-coarse projection matrix** $P^{FC,l}$ for each child cell l:

The coarse cell mass matrix M^C has entries:

$$M^C_{jn} = \int_{\Omega_L} \phi^L_n(x)\phi^L_j(x)dx = \frac{|\Omega_L|}{2^D} \int_{[-1,1]^D} \phi^L_n(\xi)\phi^L_j(\xi)d\xi = \frac{|\Omega_L|}{2^D} M_{jn}$$

where $|\Omega_L|$ is the volume of the coarse cell Ω_L. The integral is transformed to the reference element $[-1, 1]^D$. The factor $1/2^D$ arises from the determinant of the Jacobian of the affine mapping from $[-1, 1]^D$ to a D-dimensional cube of side length 2. More generally, if $x = T_L(\xi)$ is the mapping from reference to physical coarse cell, then $dx = \det(J_L)d\xi = (|\Omega_L|/|\Omega_{\text{ref}}|)d\xi$. Since $|\Omega_{\text{ref}}| = 2^D$, this is correct.

The fine-to-coarse projection matrix $P^{FC,l}$ for the l-th fine child cell is:

$$\tilde{P}^{FC,l}_{jm} = \int_{\Omega^l_{L+1}} \phi^{L+1}_m(x)\phi^L_j(x)dx = \frac{|\Omega^l_{L+1}|}{2^D} \int_{[-1,1]^D} \phi^{L+1}_m(\xi')\phi^L_j(\Gamma^l_{FC}(\xi'))d\xi' = \frac{|\Omega^l_{L+1}|}{2^D} P^{FC,l}_{jm}$$

$$\approx \frac{|\Omega^l_{L+1}|}{2^D} \sum_{q=1}^{(p+1)^D} \phi^{L+1}_m(\xi'_q)\phi^L_j(\Gamma^l_{FC}(\xi'_q))w_{1,q}...w_{D,q} = \frac{|\Omega^l_{L+1}|}{2^D} \sum_{q=1}^{(p+1)^D} \phi^L_j(\Gamma^l_{FC}(\xi'_q))QM_{mq}$$

Here, Ω_{L+1}^l is the l-th fine cell, and $|\Omega_{L+1}^l|$ is its volume. The integral is over the reference element $[-1,1]^D$ corresponding to the fine cell (coordinates ξ'). $\phi_m^{L+1}(\xi')$ are the basis functions on the reference element for the fine cell. $\phi_j^L(\Gamma_{FC}^l(\xi'))$ represents the j-th coarse cell basis function evaluated at a point corresponding to ξ' in the fine cell's reference coordinates. The mapping $\Gamma_{FC}^l(\xi')$ transforms the fine cell reference coordinates ξ' to the coarse cell reference coordinates ξ. For a standard 2:1 refinement where a coarse cell is halved in each dimension to form 2^D children, this mapping is:

$$\xi = \Gamma_{FC}^l(\xi') = \frac{1}{2}\xi' + c_l = \frac{1}{2}\xi' \pm \frac{1}{2}$$

where c_l is a vector that shifts ξ' to the appropriate quadrant/octant within the coarse cell's reference domain. For example, in 1D, c_l would be $-1/2$ for the left child and $+1/2$ for the right child, so $\xi = (\xi' \mp 1)/2$. Your notation $\frac{1}{2}\xi' \pm \frac{1}{2}$ captures this; specifically, for a fine cell l occupying a quadrant/octant of the parent cell (when both are mapped to $[-1,1]^D$), the mapping from the child's local reference coordinates $\xi' \in [-1,1]^D$ to the parent's local reference coordinates $\xi \in [-1,1]^D$ is $\xi_d = (\xi_d' \pm 1)/2$ for each dimension d, depending on which child l it is.

The system of equations can then be written in matrix form:

$$M^C \hat{\mathbf{u}}_{k_L} = \sum_{l=1}^{N_f} P^{FC,l} \hat{\mathbf{u}}_{k_{L+1},l}$$

Solving for the coarse cell coefficients $\hat{\mathbf{u}}_{k_L}$:

$$\hat{\mathbf{u}}_{k_L} = (M^C)^{-1} \sum_{l=1}^{N_f} P^{FC,l} \hat{\mathbf{u}}_{k_{L+1},l}$$

$$\hat{u}_{k_L} = (M^C)^{-1} \sum_{l=1}^{N_f} \tilde{P}^{FC,l} \hat{u}_{k_{L+1},l} = \frac{2^D}{|\Omega_L|}(M)^{-1} \sum_{l=1}^{N_f} \frac{|\Omega_{L+1}^l|}{2^D} P^{FC,l} \hat{u}_{k_{L+1},l}$$

$$\hat{u}_{k_L} = \frac{|\Omega_{L+1}|}{|\Omega_L|}(M^C)^{-1} \sum_{l=1}^{N_f} P^{FC,l} \hat{u}_{k_{L+1},l}$$

$$\hat{u}_{k_L} = \frac{1}{2^D}(M)^{-1} \sum_{l=1}^{N_f} P^{FC,l} \hat{u}_{k_{L+1},l}$$

where $\hat{\mathbf{u}}_{k_L}$ is the vector of coefficients $\{\hat{u}_{k_L}^n\}$, and $\hat{\mathbf{u}}_{k_{L+1},l}$ is the vector of coefficients for the l-th fine cell.

1.2 Coarse to Fine Projection (Prolongation or Injection)

When a coarse cell Ω_L is refined into N_f fine cells Ω^l_{L+1}, or when boundary conditions for refined patches need to be set from a coarser solution, we perform a coarse-to-fine projection. This operation transfers the solution from the coarse parent cell Ω_L to each of its fine child cells Ω^l_{L+1}. This is typically done for each fine cell l individually.

For each fine cell Ω^l_{L+1}, we seek its coefficients $\hat{u}^n_{k_{L+1},l}$ by projecting the solution from the parent coarse cell Ω_L onto the basis functions of Ω^l_{L+1}:

$$\int_{\Omega^l_{L+1}} \left(\sum_{n=1}^{N_p} \hat{u}^n_{k_{L+1},l} \phi^{L+1}_n(x) \right) \phi^{L+1}_j(x) dx = \int_{\Omega^l_{L+1}} u^L_h(x) \phi^{L+1}_j(x) dx$$

Substituting the expansion for $u^L_h(x)$:

$$\sum_{n=1}^{N_p} \hat{u}^n_{k_{L+1},l} \int_{\Omega^l_{L+1}} \phi^{L+1}_n(x) \phi^{L+1}_j(x) dx = \sum_{m=1}^{N_p} \hat{u}^m_{k_L} \int_{\Omega^l_{L+1}} \phi^L_m(x) \phi^{L+1}_j(x) dx$$

for each fine cell basis function $\phi^{L+1}_j(x)$ where $j = 1, \ldots, N_p$.

We define the **fine cell mass matrix** $M^{F,l}$ for the l-th fine cell and the **coarse-to-fine projection matrix** $P^{CF,l}$:

The fine cell mass matrix $M^{F,l}$ for the l-th fine cell has entries:

$$M^{F,l}_{jn} = \int_{\Omega^l_{L+1}} \phi^{L+1}_n(x) \phi^{L+1}_j(x) dx = \frac{|\Omega^l_{L+1}|}{2^D} \int_{[-1,1]^D} \phi^{L+1}_n(\xi') \phi^{L+1}_j(\xi') d\xi' = \frac{|\Omega^l_{L+1}|}{2^D} M_{jn}$$

This is the standard mass matrix on the fine cell Ω^l_{L+1}, with the integral transformed to its reference element coordinates ξ'.

The coarse-to-fine projection matrix $P^{CF,l}$ for the l-th fine cell is:

$$\tilde{P}^{CF,l}_{jm} = \int_{\Omega^l_{L+1}} \phi^L_m(x) \phi^{L+1}_j(x) dx = \frac{|\Omega^l_{L+1}|}{2^D} \int_{[-1,1]^D} \phi^L_m(\Gamma^l_{FC}(\xi')) \phi^{L+1}_j(\xi') d\xi' = \frac{|\Omega^l_{L+1}|}{2^D} P^{CF,l}_{jm}$$

$$\approx \frac{|\Omega^l_{L+1}|}{2^D} \sum_{q=1}^{(p+1)^D} \phi^L_m(\Gamma^l_{FC}(\xi'_q)) \phi^{L+1}_j(\xi'_q) w_{1,q} ... w_{D,q} = \frac{|\Omega^l_{L+1}|}{2^D} \sum_{q=1}^{(p+1)^D} \phi^L_m(\Gamma^l_{FC}(\xi'_q)) Q M_{jq}$$

The integral is again over the fine cell Ω^l_{L+1} and transformed to its reference element ξ'. The coarse basis function $\phi^L_m(x)$ is evaluated within the domain of Ω^l_{L+1}. The mapping $\Gamma^l_{FC}(\xi')$ correctly transforms the fine cell reference coordinates ξ' to the coarse cell reference coordinates ξ, which is the appropriate argument for ϕ^L_m. Note that the matrix $P^{CF,l}$ is closely related to $(P^{FC,l})^T$. If the basis functions are normalized consistently, $P^{CF,l}_{jm}$ (integral of $\phi^L_m \phi^{L+1}_j$ over fine cell) would be the transpose of $P^{FC,l}_{mj}$ (integral of $\phi^{L+1}_j \phi^L_m$ over fine cell). So, $P^{CF,l} \approx (P^{FC,l})^T$.

The system of equations for the l-th fine cell becomes:

$$M^{F,l}\hat{\mathbf{u}}_{k_{L+1},l} = \tilde{P}^{CF,l}\hat{\mathbf{u}}_{k_L}$$

Solving for the fine cell coefficients $\hat{\mathbf{u}}_{k_{L+1},l}$:

$$\hat{\mathbf{u}}_{k_{L+1},l} = (M^{F,l})^{-1}\tilde{P}^{CF,l}\hat{\mathbf{u}}_{k_L}$$

$$\hat{u}_{k_{L+1},l} = (M^{F,l})^{-1}\tilde{P}^{CF,l}\hat{u}_{k_L} = \frac{2^D}{|\Omega^l_{L+1}|}\frac{|\Omega^l_{L+1}|}{2^D}(M)^{-1}P^{CF,l}\hat{u}_{k_L} = (M)^{-1}P^{CF,l}\hat{u}_{k_L}$$

This operation is performed for each child cell $l = 1, \ldots, N_f$ of the parent coarse cell Ω_L.

These L2 projection operators are fundamental for ensuring that the numerical solution remains consistent and conservative across different levels of refinement in an AMR framework with DG methods. The accurate computation of these mass and projection matrices is key to the stability and accuracy of the AMR scheme.

2 Flux Registers and Conservation

We begin with the differential (or local) form of a conservation law. This equation holds true at every single point in space and time:

$$\frac{\partial u}{\partial t} + \nabla \cdot F = 0$$

Next, we integrate this entire equation over a fixed (non-moving, non-deforming) control volume in space, which we call Ω:

$$\int_\Omega \frac{\partial u}{\partial t} dV = -\int_\Omega \nabla \cdot F dV$$

Rewriting via the Divergence Theorem:

$$\frac{\partial}{\partial t}\int_\Omega u dV = -\int_{\partial\Omega} F \cdot n dS$$

Now, we integrate over the timestep $[t^n, t^{n+1}]$:

$$\int_{t^n}^{t^{n+1}} \left(\frac{\partial}{\partial t}\int_\Omega u dV\right) dt = -\int_{t^n}^{t^{n+1}} \left(\int_{\partial\Omega} F \cdot n dS\right) dt$$

This simplifies to the fundamental conservation statement:

$$\int_\Omega u(t^{n+1}) dV - \int_\Omega u(t^n) dV = -\int_{t^n}^{t^{n+1}} \int_{\partial\Omega} F \cdot n dS dt$$

This equation shows that the change of a quantity inside a volume equals the total net flux that crossed its boundary over the time interval.

2.1 The Weak Form Correction

In our numerical scheme, we replace the exact flux with a numerical flux F^{num}. When local refinement occurs, a coarse face Γ_c may interface with multiple fine faces $\Gamma_{f,i}$. A mismatch arises between the flux computed on the coarse side and the composite fluxes computed on the fine side. While a Finite Volume scheme would simply calculate a scalar conservation error ΔF and redistribute it uniformly, the Modal Discontinuous Galerkin method is more sophisticated. We must ensure conservation in the "weak" sense to preserve the high-order accuracy of the solution. We seek a solution correction δu_h such that its projection onto the test space balances the projection of the flux mismatch. This leads to the weak form equation for the correction:

$$\int_{\Omega_L} \delta u_h \phi_j^L(x) dV = \int_{t^n}^{t^{n+1}} \int_{\Gamma_c} \left(F_c^{num} - F_{fine}^{composite} \right) \cdot n \, \phi_j^L(x) \, dS \, dt$$

This can be written as a linear system:

$$\mathbf{M}^C \delta \hat{\mathbf{u}}_{k_L} = \mathbf{r}_\Delta$$

where $\delta \hat{\mathbf{u}}_{k_L}$ is the vector of coefficients for the solution correction, and $\mathbf{M}^C$ is the standard mass matrix:

$$\mathbf{M}_{jn}^C = \int_{\Omega_L} \phi_j(x) \phi_n(x) dx = \frac{|\Omega_L|}{2^D} \int_{[-1,1]^D} \phi_j(\xi) \phi_n(\xi) d\xi$$

The Flux Register acts as an accumulator. Instead of storing a single scalar value for the face Γ_c, it stores a vector of flux moments, $\mathbf{f}_\Delta$. This vector captures the "shape" of the conservation error along the interface.The j-th component of this vector is defined as the projection of the flux difference onto the coarse basis function ϕ_j^c:

$$f_{\Delta,j} = \int_{t^n}^{t^{n+1}} \int_{\Gamma_c} \left(F_c^{num} - F_{fine}^{composite} \right) \cdot n \, \phi_j^L(x) \, dS \, dt$$

In practice, this is computed by accumulating contributions from the coarse step and subtracting contributions from the fine step. **Contribution from Coarse Flux:**

$$f_{\Delta,j} \leftarrow f_{\Delta,j} + \frac{\Delta t}{2} \frac{|\partial \Omega_c|}{2^{D-1}} \sum_q F_c^{num}(\xi_q) \cdot n \, \phi_j^c(\xi_q) w_q$$

Contribution from Fine Fluxes (with Mapping): For each fine face i, we map the quadrature points to the coarse frame using Ψ_i and subtract:

$$f_{\Delta,j} \leftarrow f_{\Delta,j} - \frac{\Delta t}{2} \frac{|\partial \Omega_{f,i}|}{2^{D-1}} \sum_q F_{f,i}^{num}(\xi_q) \cdot n \, \phi_j^c(\Psi_i(\xi_q)) w_q$$

At the end of this step, $\mathbf{f}_\Delta$ contains the complete high-order representation of the missing flux.

Once the flux mismatch vector $\mathbf{f}_\Delta$ is fully assembled, we must "lift" this surface error into the volume to correct the solution.We are solving for the solution correction vector $\delta\hat{\mathbf{u}}_{k_L}$ that satisfies the weak form:

$$\mathbf{M}^C \delta\hat{\mathbf{u}}_{k_L} = \mathbf{f}_\Delta$$

where $\mathbf{M}^C$ is the standard volume mass matrix. The update is explicitly calculated as:

$$\delta\hat{\mathbf{u}}_{k_L} = (\mathbf{M}^C)^{-1}\mathbf{f}_\Delta$$

Finally, the solution is updated:

$$\hat{\mathbf{u}}_{k_L} \leftarrow \hat{\mathbf{u}}_{k_L} + \delta\hat{\mathbf{u}}_{k_L}$$

This two-step procedure is mathematically identical to the direct projection, but it allows the Flux Register to be implemented as a simple storage container for the vector $\mathbf{f}_\Delta$, independent of the volume update logic.