Ex:

$$f_X(x) = \begin{cases} 0 & \text{otherwise} \\ 2x & 0 < x < 1 \end{cases}$$

$$f_Y(x) = \begin{cases} 1 & 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Then joint pdf is

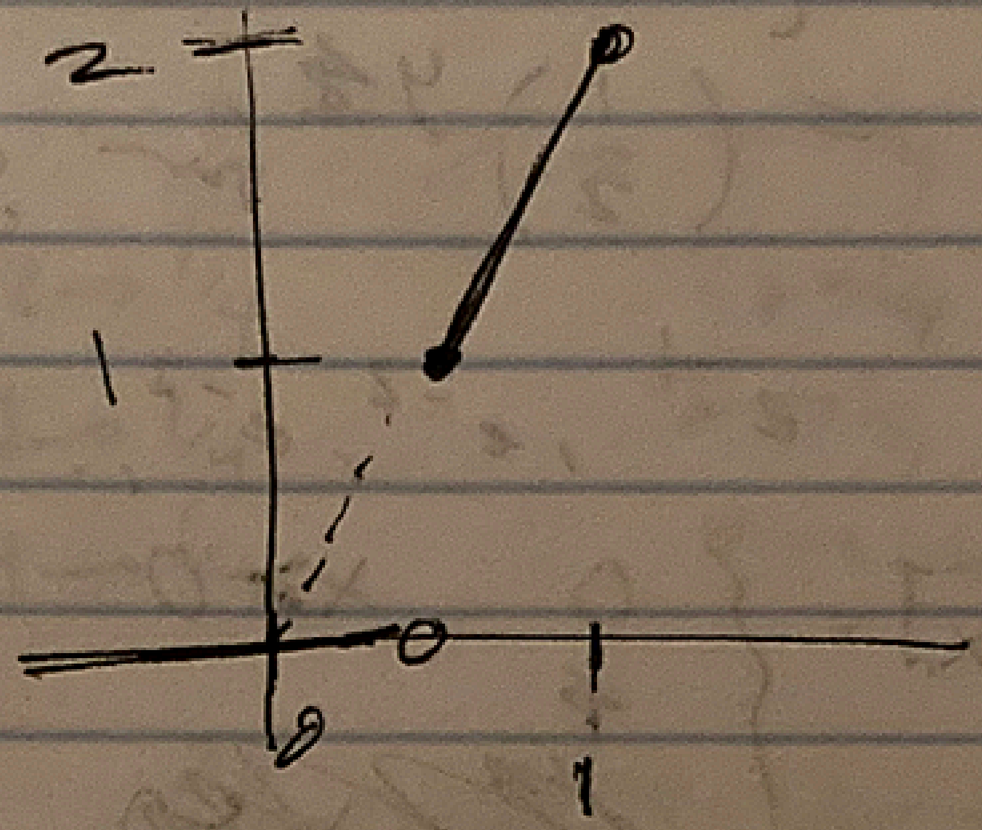
$$f(x,y) = \begin{cases} 2x & 0 < x < 1,\ 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$P(Y \geq \frac{3}{2} + X)$

$$\cancel{= \int_{\frac{3}{2}+X}^{2} \int_0^1 2x \, dx \, dy}$$

$$\cancel{= \int_{\frac{3}{2}+X}^{2} 1 \, dy}$$

$$\cancel{= (y)\Big|_{\frac{3}{2}+X}^{2}}$$



$$\int_0^1 \int_{\frac{3}{2}+X}^{2} 2x \, dy \, dx$$

$$= \int_0^1 (2xy)\Big|_{\frac{3}{2}+x}^{2} dx = \int_0^1 4x - 3x + 2x^2 \, dx$$

$$= \int_0^1 x + 2x^2 \, dx$$

$$= \left(\frac{x}{2} + \frac{2x^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{2} + \frac{2}{3} - 0 = \boxed{\frac{5}{6}}$$

Ex2;

$$P_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P_Y(y) = \left\{ \left(\tfrac{1}{2}\right)^{y-1}\left(\tfrac{1}{2}\right) \quad y \in \mathbb{Z}_{\geq 1} \right.$$

$$= \left(\tfrac{1}{2}\right)^{y} \text{ for } y \geq 1$$

$\lfloor x \rfloor$ $\quad e^{-1}, e^{-2}, e^{-3}, \ldots$

$$\lfloor x \rfloor = \begin{cases} 0 & x > 0 \\ \cancel{1} & \\ \cancel{\quad x \leq 0} & \end{cases}$$

$x = 0 \to 1$

$x < 0 \to 0$

$$\lfloor x \rfloor = \begin{cases} 0 & x > 0 \\ 1 & x = 1 \\ 0 & x < 0 \end{cases}$$

$$P(\lfloor x \rfloor = Y) = P(\lfloor x \rfloor = 0, Y = 0) + P(\lfloor x \rfloor = 1, Y = 1)$$

$P(\lfloor x \rfloor = 0) \cdot P(Y = 0) = 0$

$P(\lfloor x \rfloor = 1) \cdot P(Y = 1) = 1 \cdot P(Y = 1) = \frac{1}{2}$

$$\boxed{P(\lfloor x \rfloor = Y) = \frac{1}{2}}$$

$\mathbb{E}(|XY|) =$

$\mathbb{E}(|X||Y|) = \mathbb{E}(|X|) \cdot \mathbb{E}(|Y|)$

$= \mathbb{E}(X) \cdot \mathbb{E}(Y) = 0$

$X \sim \begin{cases} 1 \to 1/6 \\ 2 \to 1/6 \\ 3 \to 1/6 \\ 4 \to 1/6 \\ 5 \to 1/6 \\ 6 \to 1/6 \end{cases}$ $\quad Y \sim \begin{matrix} 1 \to 1/4 \\ 2 \to 1/4 \\ 3 \to 1/4 \\ 4 \to 1/4 \end{matrix}$ $\quad Z \to \begin{matrix} 1 \to 1/4 \\ 2 \to 1/4 \\ 3 \to 1/4 \\ 4 \to 1/4 \end{matrix}$

$\mathrm{Var}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

$= \frac{101}{6} - \left(\frac{21}{6}\right)^2$

$= \frac{101}{6} - \frac{49}{4}$

$= \frac{202}{12} - \frac{147}{12} = \frac{55}{12}$

$$\begin{array}{r} 1 \\ 36 \\ +25 \\ \hline 61 \\ +16 \\ \hline 87 \\ +9 \\ \hline 96 \\ +4 \\ \hline 100 \\ +1 \\ \hline 101 \end{array}$$

$\cancel{\mathrm{Var}\,Y}$ $\mathrm{Var}\,Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$

$= \frac{30}{4} - \left(\frac{10}{4}\right)^2$

$= \frac{30}{4} - \frac{25}{4} = \frac{5}{4}$

$\mathrm{Var}\,Z = \frac{5}{4}$

Since X, Y, Z ind, $\mathrm{Var}(X+Y+Z) = \mathrm{Var}\,X + \mathrm{Var}\,Y + \mathrm{Var}\,Z$

$= \frac{55}{12} + \frac{10}{4} = \frac{55}{12} + \frac{30}{12}$

$= \boxed{\frac{85}{12}}$

(5)

Let I_i represent $\begin{cases} 1 & \text{if } i-1, i, i+1 \text{ is six} \\ 0 & \text{otherwise} \end{cases}$

for $2 \leq i \leq 97$

$$\begin{array}{r} 3 \\ 36 \\ \times\ 6 \\ \hline 216 \end{array}$$

Then $\mathbb{E}(\sum I_i) = \sum \mathbb{E} I_i$

$\mathbb{E}(Y) =$

We know for arbitrary i, $\mathbb{E} I_i$

$= P(i-1, i, i+1 \text{ is six})$

$= \left(\frac{1}{6}\right)^3 = \frac{1}{216}$

Then $\mathbb{E}(Y) = \sum_{2 \leq i \leq 97} \frac{1}{216} = \frac{96}{216} = \frac{48}{108} = \frac{24}{54} = \frac{12}{27} = \boxed{\frac{4}{9}}$

~~(6) Let ~~$\cancel{I_i}$~~ there be I_1, I_2, I_3, I_4.~~

~~$I_1 = 1$~~

~~$I_2 = \begin{cases} 1 & \text{if 2nd diff from prev} \\ 0 & \text{else} \end{cases}$~~

~~$I_3 = \begin{cases} 1 & \text{if 3rd diff from 1st, 2nd} \\ 0 & \text{else} \end{cases}$~~

~~$I_4 = \begin{cases} 1 & \text{if 4th diff from 1, 2, 3} \\ 0 & \text{else} \end{cases}$~~

~~Then let X = # of different rolls. Then $X = \sum I_i$~~

~~So $\mathbb{E} X = \mathbb{E} \sum I_i = \sum \mathbb{E} I_i$~~

~~We know $\mathbb{E} I_1 = 1$~~

~~$\mathbb{E} I_2 = 5/6$~~

$\mathbb{E} I_3 =$

(2) Let us have 4 ind functions I_1, I_2, I_3, I_4

where $I_i = \begin{cases} 1 & \text{if there are } i \text{ distinct outcomes} \\ 0 & \text{else} \end{cases}$

Then $\mathbb{E}X = \mathbb{E}\Sigma_i I_i = \Sigma\, \mathbb{E}_i I_i = \Sigma_i\, \mathbb{E} I_i$

We compute each $\mathbb{E}I_i$: $\mathbb{E}I_1 = 6\left(\frac{1}{6}\right)^4 = \frac{1}{6^3} = \frac{1}{216}$

$$\mathbb{E}I_2 = \frac{\binom{6}{2}\binom{4}{2}}{6^4} = \frac{15\cdot 6}{6^4}$$

$$\mathbb{E}I_3 = \frac{\binom{6}{1}\binom{5}{2}\cdot 4!}{2!\,6^4} = \frac{6\cdot 12\cdot 10}{6^4}$$

$$\mathbb{E}I_4 = \frac{\binom{6}{4}\cdot 4!}{6^4} = \frac{24\cdot 15}{6^4}$$

Then $\mathbb{E}X = \cancel{\frac{1}{6^3} + 2\cdot\frac{80}{6^4}}$

$$\mathbb{E}I_1 + 2\cdot\mathbb{E}I_2 + 3\cdot\mathbb{E}I_3 + 4\cdot\mathbb{E}I_4$$

given the calculations above.

(3) $\mathbb{E}(|XY|) = \mathbb{E}(|X|\cdot|Y|) = \mathbb{E}(|X|)\cdot\mathbb{E}(|Y|) = (\mathbb{E}(|X|))^2$

$\cancel{\mathrm{Var}(|X|) = \mathbb{E}(X^2) - (\mathbb{E}(|X|))^2}$

$\cancel{\mathrm{Var}(|X|) =}$

$$\mathbb{E}(|X|) = \int_{-\infty}^{\infty} |x|\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$= 2\int_0^{\infty} x\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$= \frac{2}{\sqrt{2\pi}}\left(-e^{-\frac{1}{2}x^2}\right)\Big|_0^{\infty}$$

$$= \frac{2}{\sqrt{2\pi}}(0+1) = \frac{2}{\sqrt{2\pi}}$$

$(\mathbb{E}(|X|))^2 = \left(\frac{2}{\sqrt{2\pi}}\right)^2 = \frac{4}{2\pi} = \frac{2}{\pi}$

$\mathbb{E}(|XY|) = \frac{2}{\pi}$