Math104Hw14

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Exercise 1: Let $f(x) = x$ when x is rational an $f(x) = 0$ when x is irrational. Find $L(f)$ and $U(f)$ on $[0, 1]$. Show that f is not integrable on $[0, 1]$.

Proof. The definition of each is

$$L(f) = \sup\{L(f, P) : P \text{ is a partition}\}$$
$$U(f) = \inf\{U(f, P) : P \text{ is a partition}\}$$

Notice that no matter what partition we pick, each interval (t_{k-1}, t_k) will contain a rational because $\mathbb{Q}$ is dense in $\mathbb{R}$. So we have that since 1 is rational, $U(f) > 0$ because we have that $1(t_k - t_{k-1})$ is a summand and all summands are positive, as $f(x) \geqslant 0$. Also, we have that the infimum of f within any partition is always 0 because the irrationals are dense. So $U(f) \neq L(f)$ and f is not integrable on $[0, 1]$. □

Exercise 2: If f is integrable on $[a, b]$, then f is integrable on $[c, d] \subseteq [a, b]$.

Proof. Let P be a partition of $[a, b]$. We want to show that for any ε, there $\exists P'$ partition of $[c, d]$ such that

$$U(f, P') - L(f, P') < \varepsilon$$

Well, this is true for P:

$$U(f, P) - L(f, P) < \varepsilon$$

Then let $P' = P \cup \{c, d\}$ to which we see that

$$U(f, P') < U(f, P)$$

and

$$L(f, P') > L(f, P)$$

So

$$U(f, P') - L(f, P') < U(f, P) - L(f, P) < \varepsilon$$

Now we split P' into three partitions, P_1, P_2, P_3, where P_1 partitions, a to c, P_2, c to d, P_3, d to b. Then

$$(U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) + (U(f, P_3) - L(f, P_3)) < \varepsilon$$

Since each summand is positive, we see that

$$U(f, P_2) - L(f, P_2) < \varepsilon$$

So we have found a partition for the ε. □

Exercise 3: Suppose that f, g are continuous on $[0, 1]$ and $\int_0^1 f(x)\, dx = \int_0^1 g(x)\, dx$, show that $\exists x \in (0, 1)$ such that $f(x) = g(x)$.

Proof. Consider $\int_0^1 f(x)-g(x)\ dx$. If $f(x)-g(x) \geqslant 0$, we immediately get that $f(x)-g(x) = 0$. If $f(x) - g(x) \leqslant 0$, we get the same result. So we are done. We can also use the IVT for integrals. So there is an x_0 such that

$$f(x_0) - g(x_0) = \frac{1}{b-a}\int_a^b f(x) - g(x)\ dx$$

so

$$f(x_0) - g(x_0) = \int_0^1 f(x) - g(x)\ dx = 0$$

and we get

$$f(x_0) = g(x_0)$$

which is a cleaner proof. □

Exercise 4: Show $|\int_{-2\pi}^{2\pi} x^2 \sin^8 x e^x\ dx| \leqslant \frac{16\pi^3}{3}$.

Proof. We have

$$\left|\int_{-2\pi}^{2\pi} x^2 \sin^8 x e^x\ dx\right| \leqslant \int_{-2\pi}^{2\pi} \left|x^2 \sin^8 x e^x\right| dx$$

Since $|\sin^8 x| \leqslant 1$, $x^2 \geqslant 0$ we have

$$\int_{-2\pi}^{2\pi} |x^2 \sin^8 x e^2|\ dx \leqslant \int_{-2\pi}^{2\pi} x^2\ dx = \left(\frac{x^3}{3}\right)\Big|_{-2\pi}^{2\pi}$$

So

$$\frac{8\pi^3}{3} + \frac{8\pi^3}{3} = \frac{16\pi^3}{3}$$

and we are done. □

Exercise 5: Find $\lim\limits_{x\to 0} \frac{\int_0^x e^{t^2}\ dt}{x}$.

Proof. We have if $F(x) = \int_0^x e^{t^2}\ dt$, taking the limit of the numerator and denominator gives 0. Then $F'(x) = e^{x^2}$ and $x' = 1$. So

$$\lim_{x\to 0} F'(x) = 1$$

which is therefore the limit of the top thing by L'Hopital. On the other hand, you can also view the fraction as

$$\frac{F(x)}{x}$$

and see that it is the limit of $F'(x)$ as $x \to 0$. □

Exercise 6: Let f be a continuous function on $\mathbb{R}$ and define $F(x) = \int_{x-1}^{x+1} f(t)\ dt$. Prove that $F(x)$ is differentiable and find $F'(x)$.

Proof. $F(x)$ is differentiable because we have

$$\int_{x-1}^{x+1} f(t)\ dt = \int_c^{x+1} f(t)\ dt - \int_c^{x-1} f(t)\ dt$$

Since f continuous, by fundamental theorem, we get that $F(x)$ is differentiable. Also we let

$$F_1 = \int_c^{x+1} f(t)\ dt \text{ and } F_2 = \int_c^{x-1} f(t)\ dt$$

So $F_1'(x) = f(x+1)$ and $F_2'(x) = f(x-1)$. Then

$$F'(x) = F_1'(x) - F_2'(x) = f(x+1) - f(x-1)$$

□