Math185Notes

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Contents

Chapter 1

Week 1

1.1 Complex Numbers

Complex numbers are defined by pairs: $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. Addition:

$$(x, y) + (u, v) = (x + u, y + v)$$

and multiplication:

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu)$$

Theorem 1.1.1 Addition is commutative, associative and $\mathbb{R}$- linear. Multiplication is commutative, associative and distributive for addition.

Notation, definitions, and observations:

- 0 stands for $(0, 0)$
- 1 stands for $(1, 0)$
- $\mathbb{R}$ can be identified with $\mathbb{R} \times 0 \subseteq \mathbb{C}$, operations match.
- i stands for $(0, 1)$. Note that $i^2 = (-1, 0)$
- Can write $(x, y) = x + iy$, $(u, v) = u + iv$ with multiplication as:

$$(x + iy)(u + iv) = xu + i^2uv + ixv + iuy$$

- If $z = x + iy$, then $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.
- We have the conjugate $\bar{z} = x - iy$.

Proposition:

- $\mathrm{Re}(\bar{z}w)$ is the real dot product of $z = (x, y)$ and $w = u, v$
- $\mathrm{Im}(\bar{z}w)$ is the cross product also.
- $\bar{z}z$ is the squared length of (x, y).

Modulus and argument

Definition 1.1.1 The length of $z = x + iy$ or $\sqrt{z\bar{z}}$. The argument, $\arg(z)$, is the angle formed with the x axis and undefined when $z = 0$.

Corollary: Every complex number is invertible.

Geometric Interpretation:

$$|z_1 \cdot z_2| = |z_1||z_2|$$
$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

and

$$z = (x, y) = x + iy = r(\cos\theta + i\sin\theta)$$

with $r = |z|$. Notice that

$$(\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) = \cos(\varphi + \psi) + i\sin(\varphi + \psi)$$

So angles are added for unit vectors.

More Facts:

- Triangle Inequality: $|z + w| \leqslant |z| + |w|$ or $|\sum_{i=1}^{k} z_i| \leqslant \sum_{i=1}^{k} |z_i|$.
- Parallelogram law: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

Complex exponential: Define the function $\exp() : \mathbb{C} \to \mathbb{C}$.

$$\exp(z) = e^x(\cos y + i\sin y)$$

for $z = x + iy$. It follows that $|\exp(z)| = e^{\mathrm{Re}(z)}$ and $\arg(\exp(z)) = \mathrm{Im}(z)$.

Theorem 1.1.2 $\exp(z + w) = \exp(z) \cdot \exp(w)$

1.2 Complex Numbers: Algebra, Geometry, and Applications

Integral powers of complex numbers from geometry: If $z = r \cdot e^{i\theta}$, so $|z| = r$ and $\arg(z) = \theta$. We get

$$z^n = r^n \cdot (e^{i\theta})^n = r^n \cdot e^{in\theta}$$

so

$$|z|^n = |z|^n, \arg(z^n) = n \cdot \arg(z)$$

If we consider the map $z \mapsto z^n$, then 0 and 1 are fixed points. This map takes polar grid to polar grid.

Example 1.2.1: $z \mapsto z^2$ in the cartesian grid. Look at action on real and imaginary axis:

$$z = x + 0 \cdot i$$

then

$$(x, 0) \mapsto (x^2, 0)$$

while

$$z = 0 + iy$$

and

$$(0, y) \mapsto (0, -y^2)$$

Consider a line that does not pass through the origin: $z = a + iy \mapsto a^2 - y^2 + 2ay \cdot i$.

$$(a, y) \mapsto (a^2 - y^2, 2ay)$$

In the image, we have (u, v), so $v = 2ay$ and $u = a^2 - \frac{v^2}{4a^2}$.

n-th roots of unity are solutions to $w^n = z$. Modulus $\sqrt[n]{|z|}$ and n possible values for modulus.

De Moivre formulas:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proposition:

$$x^n - 1 = (x-1)(x-\zeta_n)(x-\zeta_n^2)\cdots(x-\zeta_n^{n-1})$$

into linear terms.

Primitive roots of unity are roots of unity ζ_n^k such that $\zeta_n^{kj} \neq 1$ for any $1 \leqslant j < n$

Theorem 1.2.1 The set of n-th root of unity break up into primitive d-th roots of unity for the group of divisors d of n.

Theorem 1.2.2 Grouping the factors of

$$x^n - 1 = (x-1)(x-\zeta_n)(x-\zeta_n^2)\cdots(x-\zeta_n^{n-1})$$

according to divisors of n with primitive roots gives a factorization

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

and $\Phi_d(x)$ are the d-th cyclotomic polynomials which has integer coefficients.

Example 1.2.2: $n = 3$, we have

$$x^3 - 1 = (x-1)(x^2+x+1)$$

then

$$\Phi_1(x) = (x-1)$$
$$\Phi_3(x) = x^2+x+1$$

Example 1.2.3: $n = 6$,

$$x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$$

Aside: Cyclotomic integers and fake proofs of FLT. Cyclotomic integers are $\mathbb{Z}[\zeta_2, \zeta_3, \ldots]$, adjoin all roots of unity of order to $\mathbb{Z}$. Recall FLT:

$$x^n + y^n = z^n \text{ for } n > 2 \text{ has no integer solutions}$$

or

$$x^n - y^n = z^n$$

Factor:

$$(x - y)(x - \zeta_n \cdot y)(x - \zeta_n^2 \cdot y) \cdots (x - \zeta_n^{n-1} y) = z^n$$

Relies on $\mathbb{Z}[\zeta_n]$ being a UFD.

Chapter 2

Week 2

2.1 Applications Continued

Cardano's formula: $f(x) = x^3 - 3px - 2q = 0$ has solutions

$$(q + \sqrt{q^2 - p^3})^{1/3} + (q - \sqrt{q^2 - p^3})^{1/3}$$

where

$$\Delta = \sqrt{q^2 - p^3}$$

Cases:

- $p < 0$ there is a unique real root
- $p > 0$ then there is extrema at $3x^2 - 3p = 0$, $x = \pm\sqrt{p}$, value : $\pm 2p\sqrt{p} - 2q$.If $|q| > p\sqrt{p}$, there is a unique solution. If $|q| < p\sqrt{p}$, there are 3 solutions.

Geometry:

- Translation: $a \in \mathbb{C}$: $z \mapsto z + a$
- Rotation: Multiplication by $e^{i\theta}$: $z \mapsto e^{i\theta}z$
- Scaling by $r \in \mathbb{R}$: $z \mapsto r \cdot z$.

Observation: $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ acts as rotation on $\mathbb{R}^2$ by θ where $p \pm iq = r \cdot e^{i\theta}$.

Areas of Polygons: The area of a polygon with vertices located at $z_1, \dots, z_n$ ordered anti-clockwise is

$$\frac{1}{2}(\mathrm{Im}(\overline{z_1}z_2 + \overline{z_2}z_3 + \cdots + \overline{z_{n-1}}z_n + \overline{z_n}z_1))$$

Proof. The formula is invariant under translations and rotations:

$$z_k \mapsto e^{i\theta} \cdot z_k$$

and

$$\overline{z_k}z_{k+1} \mapsto \overline{z_k}e^{-i\theta} \cdot e^{i\theta}z_{k+1} = \overline{z_k}z_{k+1}$$

Do the same for translation. Additive check:

$$(\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1) + (\overline{z_1}z_3 + \overline{z_3}z_4 + \overline{z_4}z_1)$$

□

Complex Inversion: This is the map

$$z \mapsto \frac{1}{z}$$

Bijection of $\mathbb{C}^{\times}$. Real inversion sends $(r, \theta) \mapsto (\frac{\frac{1}{r}}{\theta})$. Complex inversion: $(r, \theta) \mapsto (\frac{1}{r}, -\theta)$. Fixed points: $z = \frac{1}{z}, z^2 = 1$.

Theorem 2.1.1 **Inversion**

Inversions maps clircles to clircles. Clircles are circles or lines.

Note: Equation of a line in $\mathbb{C}$ is $\alpha z + \overline{\alpha z} = c$, where α is in $\mathbb{C} \setminus \{0\}$ and c is a real number. Equation of a circle is $|z - p|^2 = r^2$.

Definition 2.1.1 **Mobius Transformations**

The map

$$z \mapsto \frac{az + b}{cz + d} = \mu(z)$$

where $ad - bc \neq 0$.

Theorem 2.1.2

- μ is a bijection from $\mathbb{C} \setminus \{-\frac{d}{c}\}$ to $\mathbb{C} \setminus \{\frac{a}{c}\}$.
- Composition of maps μ is matrix multiplication:
- Every Mobius transformation is a composition of rotations, scalings, translations and inversions. So these take clircles to clircles.

Definition 2.1.2 **Riemann sphere**

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This has a topology as

- $\mathbb{C} \cong \mathbb{R}^2$
- neighborhood of ∞ sets containing the exterior of some large disk in $\mathbb{C}$.

Corollary: We say that a sequence of $F_{\ltimes}$ diverges to ∞ if $\lim |z_n| = \infty$.

We get $\hat{\mathbb{C}}$ as a unit sphere in $\mathbb{R}^3$ by stereographic projection. The point at the north pole corresponds to ∞.

2.2 Complex Differentiation

Functions are defined on open sets in $\mathbb{C}$ which will be continuous in this course. Functions might have singularities $z \mapsto \frac{1}{z}$.

Definition 2.2.1

Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \longrightarrow \mathbb{R}$ is differentiable at a point $x_0 \in I$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. $f(x_0) + (x - x_0) \cdot f'(x_0)$ is the best linear approximation of f for x_0.

Definition 2.2.2

Let $U \subseteq \mathbb{R}^2$ be an open subset. A function $f : U \to \mathbb{R}$ is differentiable at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in U$ if there exists a linear map $Df : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{|f(x) - f(p) - Df(x-p)}{\|x - p\||} \to 0$$

Remark: If $f = \begin{bmatrix} u \\ v \end{bmatrix}$, then Df is the Jacobian matrix: $\begin{bmatrix} U_x & U_y \\ V_x & V_y \end{bmatrix}$.

$$f \begin{bmatrix} u \\ v \end{bmatrix} (x, y) = \begin{bmatrix} u \\ v \end{bmatrix} (x, y) + \begin{bmatrix} U_x & U_y \\ V_x & V_y \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \text{(error)}$$

Theorem 2.2.1 **Differentiability Theorem**

If the partial derivatives are everywhere defined and continuous in U, then f is differentiable in U.

Remark: For limits and convergence in $\mathbb{C}$, use $\mathbb{R}^2$. The formula $z \mapsto \frac{1}{z}$ can be rewritten as $z \mapsto \frac{\bar{z}}{\|z\|}$ which is continuous away from 0. So algebra and limits interact similar to real case.

Definition 2.2.3

f is complex differentiable at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

$f(z) = z^n$,

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z^1 z_0^{n-2} + z_0^{n-1}$$

Right hand side goes to nz^{n-1}, so $f'(z) = nz^{n-1}$.

Theorem 2.2.2

If f, g are complex differentiable at z_0, so are their products, sum, quotient.

Corollary: We can differentiable rational functions also:

$$z \mapsto \frac{P(z)}{Q(z)}$$

for P, Q polynomials, when $Q(z) \neq 0$.

$f(z) = \frac{1}{z}$, $f'(z) = \frac{-1}{z^2}$

Definition 2.2.4 **Holomorphic**

A function $f : U \to \mathbb{C}$ is holomorphic if it is complex differentiable everywhere in U.

2.3 Complex vs Real Differentiability

Proposition: f is complex differentiable at $z_0 = x_0 + iy_0$ means the map $\begin{bmatrix} \mathrm{Re}(f) \\ \mathrm{Im}(f) \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Theorem 2.3.1 **Cauchy-Riemann**

$z \mapsto f(z)$ is complex differentiable at z_0 iff the map $(x, y) \mapsto (u, v)$ is real differentiable at (x_0, y_0) and satisfies $u_x = v_y$, $u_y = -v_x$.

Example 2.3.1: The complex exponential map is holomorphic in $\mathbb{C}$.

$$z \mapsto \exp(z)$$
$$(x, y) \mapsto (e^x \cos y, e^x \sin y)$$

and

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$$

The derivative is $p + iq = e^x(\cos y + i \sin y) = e^z$.

Example 2.3.2: The map $(x, y) \mapsto (x^2 + y^2, 2xy)$ is differentiable at $y = 0$ but not holomorphic in $\mathbb{C}$.

Example 2.3.3: $(x, y) \mapsto (\frac{1}{2} \log(x^2 + y^2), \arctan \frac{y}{x})$ for $x > 0$.

General properties of holomorphic maps:

The set of holomorphic maps $f : U \to \mathbb{C}$ is an algebra. Set is closed under addition and multiplication. Also closed under multiplication by inverse.

Theorem 2.3.2

If $f : U \to \mathbb{C}$ is holomorphic and twice real differentiable, then f' is also holomorphic.

Proof. We have

$$f' = u_x + iv_x$$

and checking CR for f', we need $(u_x)_x = (v_x)_y$ and $(u_x)_y = -(v_y)_x$. Flip the subscripts by Clairant theorem and we see that f' is holomorphic. □