Math143Hw7

Trustin Nguyen

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Exercise 1: Given a variety X, recall that we defined the field of rational functions on X to be $k(X) = \operatorname{Frac}\Gamma(X)$. Let $X = V(xw - yz) \subseteq \mathbb{A}^4$ and let $f = \overline{x}/\overline{y} \in k(X)$.

(a) Prove that the poles of f are exactly $V(y, w) \subseteq X$, equivalently the open set where f is defined is the compliment of $V(y, w) \subseteq X$.

Proof. We have that in $k(X)$, $\frac{x}{y} = \frac{z}{w}$ because of the fact that:

$$xw - zy = 0 \implies xw = zy \implies \frac{x}{y} = \frac{z}{w}$$

Therefore, we look for when the denominators for both of these vanish. This is just $V(y) \cap V(w) = V(y, w)$. □

(b) Show it is impossible to write $f = a/b$ for $a, b \in \Gamma(X)$ where $b(P) \neq 0$ for every P where f is defined.

Proof. Suppose that $f = a/b$. Then by the relation:

$$\frac{a}{b} = \frac{x}{y},$$

we get:

$$ay = bx$$

Now for a point in $V(y) - V(x) - V(w)$, which is non-empty, we have

$$ay(p) = 0 = bx(p)$$

But p does not vanish on x, so it must vanish on b. □

Exercise 2: Practice with the local ring

(a) Let X be a variety. In class we defined $\mathcal{O}_P(X) \subseteq k(X)$ as the subset of rational functions that are defined at $P \in X$. Prove that $\mathcal{O}_P(X)$ is in fact a subring.

Proof. Let $f, g \in \mathcal{O}_P(X)$. We have commutativity, associativity. We need to prove that the identity exists, for both multiplication, addition, and that it is closed under these operations.

– Because $\Gamma(X) \subseteq \mathcal{O}_P(X)$, we have that 1, 0 as polynomials are in $\mathcal{O}_P(X)$. We see that $1 \in \mathcal{O}_P(X)$ because $1(p) = 1 \neq 0$. So therefore, a denominator of 1 is always possible, and we just choose our numerator to be elements of $\Gamma(X)$.

– If $\frac{f(x)}{g(x)}, \frac{f'(x)}{g'(x)} \in \mathcal{O}_P(X)$, we have:

$$\frac{f(x)}{g(x)} + \frac{f'(x)}{g'(x)} = \frac{f(x)g'(x) + f'(x)g(x)}{g(x)g'(x)}$$

and because both $g(p) \neq 0$, $g'(p) \neq 0$, we are over a field, which is an integral domain, so $g(p)g'(p) \neq 0$. So this sum is in $\mathcal{O}_P(X)$.

- If $\frac{f(x)}{g(x)}, \frac{f'(x)}{g'(x)} \in \mathcal{O}_P(X)$, we have:

$$\frac{f(x)}{g(x)} \cdot \frac{f'(x)}{g'(x)} = \frac{f(x)f'(x)}{g(x)g'(x)}$$

  where again, $g(p)g'(p) \neq 0$, so the product is in $\mathcal{O}_P(X)$.

- We also have that additive inverses exist, because -1 is in $\Gamma(X)$, which means that $-f(x)/g(x) \in \mathcal{O}_P(X)$.

So $\mathcal{O}_P(X)$ is a ring. □

(b) Let

$$R = \left\{\frac{a}{b} \in k(x) : a, b \in k[x] \text{ and } b(0) \neq 0\right\}$$

Prove that R is a local ring (i.e. that R has a unique maximal ideal, or equivalently that the non-units form an ideal).

Proof. The non-units are elements a/b where $x \mid a$. If $x \nmid a$, the a has a constant term, non-zero, so b/a exists in R. Also if $x \mid a$, then it is a non-unit because b/a has $a(0) = 0$ so $b/a \notin R$. So we have found the non-units. Now we just need to show that they form an ideal. So if

$$N = \{\text{non-units of } R\}$$

Then suppose that $f/g, f'/g' \in N$. Then:

$$\frac{f}{g} + \frac{f'}{g'} = \frac{fg' + f'g}{gg'}$$

But $x \mid f$, $x \mid f'$, so $x \mid fg' + f'g$. Additionally, $gg'(0) \neq 0$ because we are over a field. So therefore, the sum is a non-unit also and therefore, is in N. Now if $a/b \in R$, $f/g \in N$, then:

$$\frac{a}{b} \cdot \frac{f}{g} = \frac{af}{bg}$$

and by the same idea, $x \mid af$, $bg(0) \neq 0$ so it is closed under multiplication from the ring R. □

Exercise 3: Let k be algebraically closed. Let $\mathcal{O}_P(X)$ be the local ring of a variety X at a point P. Show that there is a one-to-one correspondence between the prime ideals in $\mathcal{O}_P(X)$ and the subvarieties of X that pass through P.

Proof. Let I be a prime ideal in $\mathcal{O}_P(X)$. We have that:

$$\Gamma(X) \cap I = \{f \in \mathcal{O}_p(X) : \frac{f}{g} \in I\}$$

or the intersection is the set of numerators of I. Then if $f_1 f_2 \in \Gamma(X) \cap I$, then we have:

$$\frac{f_1}{1} \cdot \frac{f_2}{1} \in I$$

and since I is prime, wlog, $f_1/1 \in I$. Then $f_1 \in \Gamma(X) \cap I$, so $\Gamma(X) \cap I$ is prime also. Prime ideals of $\Gamma(X)$ are radical ideals. We know that there is a bijection between radical ideals of $k[x_1, \dots, x_n]$ and radical ideals of $k[x_1, \dots, x_n]/I(X)$. Furthermore, there is a bijection between radical ideals and algebraic sets by taking the vanishing. So if I is prime of $\mathcal{O}_P(X)$, then we map it to

$$I \cap \Gamma(X) \to I \cap \Gamma(X) + I(X) \to V(I \cap \Gamma(X) + I(X))$$

And since
$$I \cap \Gamma(X) + I(X) \supseteq I(X)$$
then
$$V(I \cap \Gamma(X) + I(X)) \subseteq V(I(X)) = X$$
To go backwards, we can try:
$$Y \subseteq X \mapsto J \subseteq \mathcal{O}_P(Y) \subseteq \mathcal{O}_P(X)$$
if $Y \subseteq X$, then we have $I(Y) \supseteq I(X)$ which means $\Gamma(Y) \subseteq \Gamma(X)$ so indeed $\mathcal{O}_P(Y) \subseteq \mathcal{O}_P(X)$. As a local ring, we can only map it to the ideal of non-units, which is prime, because if $f_1 f_2 \in J$, then f_1, f_2 cannot both be units, otherwise, we have a unit in J. So there is an inverse map, and notice that the composition is the identity:
$$J \cap \Gamma(X) = J \cap \Gamma(Y) = I(Y)$$
since $I(Y)$ is the pole we then have:
$$\begin{tikzcd}
I(Y) \arrow[r] & I(Y) + I(X) \arrow[r] & V(I(Y) + I(X)) = V(I(X)) \cap V(I(Y)) = Y \\
\Gamma(X) & k[x_1, \dots, x_n] & \mathbb{A}^n
\end{tikzcd}$$
which concludes the proof. □

Exercise 4: Let $k = \mathbb{C}$. For each polynomial f below, find all singular points of $V(f)$. For each singular point of $V(f)$, find the multiplicity and the tangent cone. Write the tangent cone as a union of lines.

(a) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$

Answer. We have:
$$\begin{aligned} f_x &= 3x^2 - 2x + 3y^2 + 6xy + 2y \\ f_y &= 3y^2 - 2y + 6xy + 3x^2 + 2x \end{aligned}$$
so when $f_x = 0$, we have:
$$3x^2 + 3y^2 + 6xy = 2x - 2y$$
So that also means:
$$f_y = 3y^2 + 6xy + 3x^2 - 2y + 2x = 2x - 2y - 2y + 2x$$
So if $f_y = 0$ also,
$$-4y + 4x = 0 \implies 4x = 4y \implies x = y$$
Plugging this back into f, we get:
$$y^3 - y^2 + y^3 - y^2 + 3y^3 + 3y^3 + 2y^2 = 8y^3$$
Now if $f = 0$, we have:
$$8y^3 = 0$$
which means that $y = 0$, $x = 0$. Then $(0, 0)$ is the only singular point. The multiplicity is the minimal degree of the terms which is 2. The tangent cone is $V(f_2) = V(x^2 - 2xy + y^2) = V(x - y)$. So the tangent cone is $y = x$.

(b) $x^4 + y^4 - x^2y^2$

Proof. Same process as above:

$$f_x = 4x^3 - 2y^2$$
$$f_y = 4y^3 - 2x^2$$

and

$$f_x = 0 \implies 2y^2 = 4x^3$$

so we get:

$$f_y = 2x^3y - 2x^2$$

Now if $f_y = 0$ also:

$$2x^3y - 2x^2 = 2x^2(xy - 1) = 0$$

So $x = 0$ or $xy = 1$. The singular points are the points in $V(x^4 + y^4 - 1)$ and $(0,0)$. The multiplicity of the point $(0,0)$ is 2 while it is 1 for $V(x^4+y^4-1)$ after performing some shift of coordinates I think. And the tangent cone is $V(x^2y^2) = V(x) \cup V(y)$. So the lines $y = 0$ and $x = 0$ make up the tangent cone. □

(c) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$.

Proof. We have:

$$f_x = 3x^2 - 6x + 3y$$
$$f_y = 3y^2 - 6y + 3x$$

And if $f_x = 0$:

$$3y = 6x - 3x^2 \implies y = 2x - x^2 \implies y = 1 - (x-1)^2$$

so

$$f_y = 6xy - 3x^2y - 9x + 6x^2$$

and if $f_y = 0$:

$$3x(y - xy - 3 + 2x) = 0$$

Didn't Finish □

Exercise 5: Let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial map and let $0 \neq f \in k[x,y]$ be a nonzero polynomial with no repeated factors. Let $P \in \mathbb{A}^2$ and let $Q = \varphi(P)$.

(a) Suppose $Q \in V(f)$. Show that $P \in V(\varphi^* f)$.

Proof. Since $Q \in V(f)$, $f(Q) = 0$. Then $f(\varphi(P)) = 0$. So $(f\varphi)(P) = 0$ and $\varphi^* f(P) = 0$. So $P \in V(\varphi^* f)$. □

(b) Prove that if φ is a translation, then the multiplicity of $V(\varphi^* f)$ at P equals the multiplicity of $V(f)$ at Q

Proof. Suppose that the multiplicity of $V(f)$ at Q is m. Let $Q = (q_1, q_2)$. Then consider the pullback of f:

$$\psi : \mathbb{A}^2 \to \mathbb{A}^2$$
$$\psi(x,y) = (x - q_1, y - q_2)$$

and

$$\psi^* : k[x,y] \to k[x,y]$$
$$\psi^* f(x,y) = f(x - q_1, y - q_2)$$

Then if f' is such that $\varphi^* f = f'$, Then the multiplicity of Q at f is the multiplicity of $(0,0)$ at f'. So denote:

$$f' = f_m + f_{m+1} + \cdots + f_n$$

Then now we take the translation of f' which preserves multiplicity, but this time, we send $(0,0) \to P = (p_1, p_2)$.

$$\pi : \mathbb{A}^2 \to \mathbb{A}^2$$
$$\pi(x,y) = (x + p_1, y + p_2)$$

Then

$$\pi^* : k[x,y] \to k[x,y]$$
$$\pi^* f(x,y) = f(x + p_1, y + p_2)$$

So $\pi^* f'$ gives us that the multiplicity of $(0,0)$ in $V(f')$ is the multiplicity of P in $V(\pi^* f')$. But this is just:

$$V(\pi^* \psi^* f) = V(\varphi^* f)$$

where φ translates $Q \to P$. So multiplicity is preserved under translation. □

(c) Prove that if φ is any polynomial map, then the multiplicity of $V(\varphi^* f)$ at P is greater than or equal to the multiplicity of $V(f)$ at Q.

Proof. If φ is a polynomial map, we have:

$$\varphi(p) = (\varphi_1(p), \varphi_2(p), \ldots, \varphi_m(p))$$

Then φ_i are all polynomials. Since translation preserves multiplicity, we can take

$$\varphi'(p) = (\varphi'_1(p), \ldots, \varphi'_m(p))$$

where φ'_i have no constant terms. So

$$\varphi'^*(f(x_1, x_2, \ldots, x_m)) = f(\varphi'_1(p), \ldots, \varphi'_m(p))$$

Now if we consider the pullback of map of the translation map $\psi : Q \to 0$, and consider the pullback of f, let that have multiplicity m = multiplicity of $V(f)$ at Q. Then if we take $\varphi'^* \psi^* f$, and another pullback, this time sending $\pi : P \to 0$, we have $\pi^* \varphi'^* \psi^* f$ with the same multiplicity as $V(\varphi^* f)$ at P. We know that the lowest non-zero homogeneous term of the polynomial $\pi^* \varphi'^* \psi^* f$ is at least greater than or equal to that of $\psi^* f$ because $\varphi'^* \psi^* f$ preserves the degree of the polynomial f. □