Stat134Hw4

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Exercise 1: Drop a uniformly random point inside the triangle with vertices at $(0,0),(5,0)$ and $(5,2)$. Let X be the x-coordinate of this random point. Find the cumulative distribution function and probability density function of X.

Answer. First calculate the area of the triangle:

$$5 * 2 * \frac{1}{2} = 5$$

Then the probability that a chosen point $P(x,y)$ where $X \leqslant x$ given by the area of the sample space over the event area. We have $Area(\Omega) = 5$ as calculated above. Now if W is the event that $X \leqslant x$. Then $Area(W) = x * \frac{2}{5}x * \frac{1}{2} = \frac{x^2}{5}$. So our cumulative density function is

$$F(X \leqslant x) = \begin{cases} \dfrac{x^2}{25} & \text{if } 0 \leqslant x \leqslant 5 \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 5 \end{cases}$$

Exercise 2: Let X be a uniform random variable on $[-1,2]$. Show that $Y = X^2$ is a continuous random variable and find its density.

Answer. Y is a continuous random variable because it has uncountably many values for which $p(x) \neq 0$, where p is the probability density function. Now the probability density function for X is

$$p_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ 0 & \text{if } x > 2 \\ \dfrac{1}{3} & \text{if } -1 \leqslant x \leqslant 2 \end{cases}$$

Now the definition of pdf for Y is that $\mathbb{P}(Y = k) = \mathbb{P}(X = Y^{-1}(k))$. So we have:

$$p_Y(y) = \begin{cases} \frac{2}{3} & \text{if } 0 \leqslant y \leqslant 1 \\ \frac{1}{3} & \text{if } 1 \leqslant y \leqslant 2 \\ 0 & \text{if } y > 2 \end{cases}$$

Exercise 3: Suppose that Y is a discrete random variable whose probability mass function is:

x	1	2	3
$p_Y(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(a) What is $\mathbb{P}(Y \geqslant 2)$?

Answer. There are three values for $Y : 1, 2, 3$. Then

$$\mathbb{P}(Y \geqslant 2) = \mathbb{P}(Y = 2) + \mathbb{P}(Y = 3)$$

So the answer is

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

(b) Compute $\mathbb{E}[\frac{1}{Y}]$

Answer. We have by definition that:

$$\mathbb{E}\left[\frac{1}{Y}\right] = \sum_{k \in \mathbb{R} \setminus \{0\}} k\mathbb{P}\left(\frac{1}{Y} = k\right)$$

We only have 3 possible values for $\frac{1}{Y} : 1, \frac{1}{2}, \frac{1}{3}$, so the expectation is

$$\frac{1}{3}\mathbb{P}\left(\frac{1}{Y} = \frac{1}{3}\right) + \frac{1}{2}\mathbb{P}\left(\frac{1}{Y} = \frac{1}{2}\right) + \mathbb{P}\left(\frac{1}{Y} = 1\right)$$

So the answer is

$$\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} = \frac{1}{12} + \frac{1}{4} + \frac{1}{4} = \frac{7}{12}$$

Exercise 4: Let X be a uniformly chosen element of the set $\{1, 2, 4, 8, \dots, 2^{99}\}$. Find the expected value of the random variable X.

Answer. Let $S = \{1, 2, 4, 8, \dots, 2^{99}\}$. Then $|S| = 100$. Each element has a $\frac{1}{100}$ probability of being chosen. The formula for expectation is

$$\mathbb{E}(X) = \sum_{k \in \mathbb{R} \setminus \{0\}} k \cdot \mathbb{P}(X = k)$$

We know that each probability is $\frac{1}{100}$, so we have that

$$\mathbb{E}(X) = \sum_{k=0}^{99} 2^k \frac{1}{100} = \frac{1}{100} \sum_{k=0}^{99} 2^k$$

Now

$$\begin{aligned} S &= \sum_{k=0}^{99} 2^k \\ 2S &= \sum_{k=1}^{100} 2^k \\ S - 2S &= 1 - 2^{100} \\ S &= \frac{1 - 2^{100}}{1 - 2} \end{aligned}$$

So the answer is

$$\frac{2^{100} - 1}{100}$$

Exercise 5: Let (X, Y) be a uniformly chosen random point on the unit circle. Show that $Z = X/Y$ is a continuous random variable and find its probability density function.