Math110Hw

Trustin Nguyen

January 2023

Homework 1

Problem 1: Suppose $a \in \mathbb{F}, v, w \in V$ (vector space of $\mathbb{F}$), and $av = aw$. Prove that $a = 0$ or $v = w$.

Proof. Suppose for contradiction that $a \neq 0$ and $v \neq w$. Then since $av = aw$,

$$\begin{aligned}
(av)(a^{-1}) &= (aw)(a^{-1}) \\
aa^{-1}v &= aa^{-1}(w) \\
1 \cdot v &= 1 \cdot w \\
v &= w
\end{aligned}$$

which gives us a contradiction. $\square$

Problem 2: Is $\mathbb{R}^{\mathbb{Z}}$ a vector space over $\mathbb{Z}$? Over $\mathbb{Q}$? Over $\mathbb{R}$? Over $\mathbb{C}$? Explain.

Yes it is a vector space over $\mathbb{Z}$. First check the properties of addition:

Commutative:

$$\begin{aligned}
(r_1, \dots, r_j), (s_1, \dots, s_j) &\in \mathbb{R}^{\mathbb{Z}} \\
(r_1, \dots, r_j) + (s_1, \dots, s_j) &= (r_1 + s_1, \dots, r_j + s_j) \\
&= (s_1 + r_1, \dots, s_j + r_j) \\
&= (s_1, \dots, s_j) + (r_1, \dots, r_j)
\end{aligned}$$

Associative:

$$\begin{aligned}
(r_1, \dots, r_j), (s_1, \dots, s_j), (t_1, \dots, t_j) &\in \mathbb{R}^{\mathbb{Z}} \\
((r_1, \dots, r_j) + (s_1, \dots, s_j)) + (t_1, \dots, t_j) &= (r_1 + s_1, \dots, r_j + s_j) + (t_1, \dots, t_j) \\
&= (r_1 + s_1 + t_1, \dots, r_j + s_j + t_j) \\
&= (r_1, \dots, r_j) + (s_1 + t_j, \dots, s_j + t_j) \\
&= (r_1, \dots, r_j) + ((s_1, \dots, s_j) + (t_1, \dots, t_j))
\end{aligned}$$

Identity:

$$\begin{aligned}(r_1, \ldots, r_j) &\in \mathbb{R}^{\mathbb{Z}} \\ (0, \ldots, 0) + (r_1, \ldots, r_j) &= (r_1 + 0, \ldots, r_j + 0) \\ &= (r_1, \ldots, r_j)\end{aligned}$$

Additive Inverse:

$$\begin{aligned}(r_1, \ldots, r_j) &\in \mathbb{R}^{\mathbb{Z}} \\ (r_1, \ldots, r_j) + (-r_1, \ldots, -r_j) &= (r_1 + (-r_1), \ldots, r_j + (-r_j)) \quad = (0, \ldots, 0)\end{aligned}$$

The vector space is closed under addition. For multiplication, the proof for commutativity, associativity, and the identity property is analogous to that of addition. It is also closed under scalar multiplication:

$$\begin{aligned}(r_1, r_2, \ldots, r_j) &\in \mathbb{R}^{\mathbb{Z}} \\ z &\in \mathbb{Z} \\ z \cdot (r_1, r_2, \ldots, r_j) &= (zr_1, zr_2, \ldots, zr_j) \in \mathbb{R}^{\mathbb{Z}}\end{aligned}$$

The property of distributivity holds also (trust me). $\mathbb{R}^{\mathbb{Z}}$ is also a vector space over $\mathbb{Q}, \mathbb{R}$ since it will still be closed under addition and addition and scalar multiplication still operates in the same way. As for $\mathbb{C}$, it is no longer closed under scalar multiplication:

Suppose $r_1, r_2, \ldots, r_j \neq 0$.

$$\begin{aligned}1 + i &\in \mathbb{C} \\ (r_1, r_2, \ldots, r_j) &\in \mathbb{R}^{\mathbb{Z}} \\ (1 + i) \cdot (r_1, \ldots, r_j) &= (r_1 + r_1 i, \ldots, r_j + r_j i)\end{aligned}$$

Clearly, each component is in $\mathbb{C}$ since the imaginary component is non-zero.

Problem 3: Suppose $\{0, 1, x\}$ is a field with exactly three elements. What do the addition and multiplication tables have to be in that case? Based on the addition and multiplication tables that you get, check that this is indeed a field. What is the natural way to think of this field (and of x)?

$+$	0	1	x
0	0	1	x
1	1	0	1
x	x	1	0

$\times$	0	1	x
0	0	0	0
1	0	1	x
x	0	x	1

Since we have

$$1 + x = 1 \qquad\qquad 1 + 0 = 1$$

Then we conclude that

$$\begin{aligned} 1+x &= 1+0 \\ 1+1+x &= 1+1+0 \\ 0+x &= 0+0 \\ x &= 0 \end{aligned}$$

This tells us that the field addition is regular addition mod 2 and that multiplication is multiplication mod 2. The other table shows values for when x is equal to 1. So there might not be a field with exactly 3 elements and if we were to consider one such as $\{0, 1, x\}$, then $x = 0, 1$.

Problem 4: Prove that any field $\mathbb{F}$ is also a vector space over itself, with the field addition used as vector addition, and the field multiplication used as scalar multiplication.

Proof. By definition, $\mathbb{F}$ is an abelian group under addition and multiplication is commutative, associative, contains an identity element, and is distributive. Since it is closed under addition and multiplication, $\mathbb{F}$ would be a vector space over itself. $\square$

Problem 5: For which values of a is the set of all real-valued twice differentiable functions f on the interval $(0, \infty)$ such that $f''(2) - af(0) = a$ a vector space over $\mathbb{R}$?

For the set to be a vector space, 0 must also be in the vector space, therefore,

$$\begin{aligned} 0 - a(0) &= a \\ 0 &= a \end{aligned}$$

But now since $a = 0$, we have

$$f''(2) = 0$$

Observe that the vector space is closed under addition and scalar multiplication since the property $f''(2) = 0$ is conserved. This automatically implies the existence of additive inverses in the vector space since the multiplication of a function by the scalar -1 gives an additive inverse. Functions are also commutative and associative since addition/multiplication is commutative and associative when dealing with functions in $\mathbb{R}$. The function $f(x) = 1$ is in the vector space since $f''(x) = 0$. Clearly, it also satisfies distributivity if we treat the function as a real number for each $x \in (0, \infty)$. It is reduced to seeing that distributivity within the field $\mathbb{R}$ holds, which is by definition true.