Math143Hw3

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Exercise 1: Let R be a ring and let $I \subseteq R$ be an ideal.

(a) Show there is a natural bijection between ideals in R/I and ideals in R containing I.

Proof. Consider the homomorphism $\varphi : R \to R/I$ given by the projection. Let J be an ideal containing I, and consider the image of $\varphi(J)$ given by $j \in J$:

$$\varphi(j) = j + I$$

We will show that the image is an ideal. Since $J \supseteq I$, we must have a $j_0 \in I$ which implies that $j_0 \in J$. Then

$$\varphi(j_0) = 0 + I$$

The image of a homomorphism restricted to J is closed under addition, which can be seen. Furthermore, if $r + I \in R/I$, then $(r + I)(j + I) = rj + I = \varphi(rj)$. Since $rj \in J$, we have that the ideal generated by the image of J in this mapping is closed under multiplication from R/I. Notice that the image of $\varphi(J)$ is just J/I, since elements of the image are the cosets of I with representatives in J. Therefore, J/I is an ideal of R/I that we map J to.

(Injectivity) We will check for injectivity. Suppose that two ideals containing I from R, denoted J_1, J_2 map to the same ideal J/I. Suppose that $j \in J_1$ and $j \mapsto j + I$. We can find a $j' \in J_2$ such that $j' \mapsto j' + I$ and

$$j + I = j' + I$$

But we have the following conclusions:

$$\begin{aligned} j + I &= j' + I \\ 0 + I &= j' - j + I \end{aligned}$$

So $j' - j \in I$. But $I \subseteq J_2$, therefore, $j' - j \in J_2$ and so $j \in J_2$. We conclude $J_1 \subseteq J_2$. By the same argument, $J_2 \subseteq J_1$, so $J_1 = J_2$.

(Surjectivity) Consider an ideal of R/I which is generated by a number of cosets:

$$J = (a_1 + I, a_2 + I, \ldots) \in R/I$$

Consider the ideal generated by the elements of the union of the cosets:

$$J' = (a_1 + I \cup a_2 + I \cup \ldots)$$

($J \subseteq \varphi(J')$) Clearly, J' maps surjectively into J by φ. We just take $a_1 + i \mapsto a_1 + i + I = a_1 + I$. So we have a way to mapping to the generators.

($\varphi(J') \subseteq J$) Now consider an arbitrary element of J' which is of the form:

$$r_1(a_1 + i_1) + r_2(a_2 + i_2) + \ldots \mapsto r_1 a_1 + r_2 a_2 + \ldots + I$$

which is an element of J/I. Therefore, we have that $\varphi(J') = J$ by our double inclusion proof, showing that φ is surjective. $\square$

(b) Show that the bijection in part (a) induces a bijection between radical ideals in R and radical ideals in R/I.

Proof. We will show that radical ideals map to radical ideals. Consider the homomorphism given by the previous problem, where $\varphi(\sqrt{J}) \mapsto \sqrt{J}/I$. Suppose that $p^k + I = (p+I)^k \in \sqrt{J}/I$. We use the same construction which gave us the fact that φ was surjective in the previous problem. Consider the ideal generated by the elements of the union of the cosets:

$$J' = (a_1 + I \cup a_2 + I \cup \ldots)$$

We know that $p^k + I$ contains p^k. So J' contains p^k. From the injectivity inherited from problem (a), we have that $J' \mapsto \sqrt{J}/I$ and $\sqrt{J} \mapsto \sqrt{J}/I$, therefore, $p^k \in \sqrt{J}$. We conclude that $p \in \sqrt{J}$, therefore, $p + I \in \sqrt{J}/I$. So we have that radical ideals map to radical ideals. This part of the proof has also shown that the pre-image of a radical ideal must also be a radical ideal. Therefore, our map is surjective. □

(c) Show that the bijection in part (a) induces a bijection between maximal ideals in R and maximal ideals in R/I. Conclude that if there is a surjection $\varphi : R \to L$ where L is a field, then the kernel of φ is a maximal ideal.

Proof. We will show that maximal ideals map to maximal ideals. Let $I \subseteq J \subseteq R$ where J is a maximal ideal. Suppose that $b + I \in J/I$. We will show that $J/I + (b + I) = R/I$, which would allow us to conclude that J/I maximal. Since $b + I \notin J/I$, we must have $b \notin J$ otherwise, we would have, by the mapping we established in part (a):

$$\varphi(b) = b + I \in J/I$$

a contradiction. Since J is maximal, we have that the ideal $R' \supseteq J$ containing b must be the entire ring. We can write 1 as:

$$a_1 j + a_2 b = 1$$

To which we see:

$$R/I = 1 + I = \varphi(a_1 j + a_2 b) = a_1 j + I + a_2 b + I = J/I + (b + I)$$

Therefore showing J/I is maximal. Now to see surjectivity, suppose J/I is maximal in R/I. Then $R/I/J/I \cong R/J$ which is a field. So J is maximal and

$$\varphi(J) = J/I$$

So there is a maximal ideal that maps to J/I. Injectivity is inherited from φ. □

Exercise 2: Practice with maximal ideals:

(a) Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Show that I is radical if and only if it is equal to the intersection of all the maximal ideals containing I.

Proof. $(\to)$ Suppose $I = \sqrt{I}$. We will show that $I = \bigcap M_i$ for M_i maximal ideals containing I. We have that $I \subseteq \bigcap M_i$, which was given by the problem statement. Since k is algebraically closed, by one of the Weak Nullstellensatz's, we can conclude that maximal ideals correspond algebraic sets that are points. Consider the points that are killed by I, which can be extracted by the fact that

$$I = I(V(I)) = \sqrt{I}$$

Then we know that $V(I) = \{p_1, p_2, \ldots\}$. So we are considering the ideal that kills all p_i. But that is just the intersection of the ideal generated by each point as an algebraic set. So we have

$$\bigcap_i I(p_i) \subseteq I(V(I))$$

Since the $I(p_i)$'s are maximal ideals, we conclude that

$$\bigcap_i M_i \subseteq I(V(I)) = I$$

Note that it does not matter if there are more maximal ideals, since the intersection will be smaller and still a subset. We have shown a double inclusion, which finishes the proof.

($\leftarrow$) Suppose that $I = \bigcap M_i$ for M_i maximal ideals. Since maximal ideals are prime and prime ideals are maximal, we have that M_i's are radical. Since $I \subseteq M_i$, we must also have that if $p^k \in I$, then $p^k \in M_i$ and therefore, $p \in M_i$. But that means that $\sqrt{I} \subseteq M_i$. So $\sqrt{I} \subseteq \bigcap M_i$ We have the string of inclusions $I \subseteq \sqrt{I} \subseteq \bigcap M_i$. But since $\bigcap M_i = I$, we have $I = \sqrt{I}$. □

(b) Show that the radical of the ideal $I = (x^2 - 2xy^4 + y^6, y^3 - y) \subseteq \mathbb{C}[x, y]$ is the intersection of three maximal ideals.

Answer. To get the radical ideal, by the Nullstellensatz, we can take

$$I(V((x^2 - 2xy^4 + y^6, y^3 - y)))$$

So

$$\begin{aligned} V((x^2 - 2xy^4 + y^6, y^3 - y)) &= V((x^2 - 2xy^4 + y^6) + (y^3 - y)) \\ &= V((x^2 - 2xy^4 + y^6)) \cap V((y^3 - y)) \\ &= (0, 0) \cup (1, 1) \cup (1, -1) \end{aligned}$$

Now we have

$$I((0, 0) \cup (1, 1) \cup (1, -1)) = I((0, 0)) \cap I((1, 1)) \cap I((1, -1))$$

By one of the Weak Nullstellensatz theorems, we have that ideals generated by points as algebraic sets correspond to the maximal ideal of $\mathbb{C}[x, y]$. So we are done.

Exercise 3: Let $X = V(x^2 - yz, xz - x) \subseteq \mathbb{A}^3_{\mathbb{C}}$. Find the irreducible components of X and their corresponding prime ideals. Make sure you justify your solution.

Answer. We start by solving the equations for 0:

$$\begin{aligned} x^2 - yz &= 0 \\ xz - x &= 0 \end{aligned}$$

So we have that either $x = 0$ or $z = 1$. In the case of $x = 0$, we have

$$-yz = 0$$

So we have

$$\begin{aligned} V(x, yz) &= V(x) \cap V(yz) \\ &= V(x) \cap (V(y) \cup V(z)) \\ &= (V(x) \cap V(y)) \cup (V(x) \cap V(z)) \\ &= V(z) \cup V(y) \end{aligned}$$

The second case is when $z = 1$, so we get $V(z - 1, x^2 - y)$. But since $x^2 - y$ is irreducible in $\mathbb{A}^2_{\mathbb{C}}$, this is irreducible. Therefore, the decomposition is

$$X = V(z) \cup V(y) \cup V(z - 1, x^2 - y)$$

We have $I(V(z))$ is $\sqrt{(z)}$ which is just (z). The same reasoning gives us $I(V(y)) = (y)$. Finally, the last algebraic set is the parabola on the $z = 1$ xy-plane. We know that an algebraic set is irreducible if and only if its ideal is prime. Therefore, $I(V(z-1, x^2 - y))$ is prime. But that is just $\sqrt{(z-1, x^2 - y)}$. So we are done.

Exercise 4: Practice with field extensions:

(a) Let $k \subseteq L$ be a field extension. Show that the set of elements in L that are algebraic over k form a subfield of L containing k. (Hint: suppose $v^n + a_1 v^{n-1} + \ldots + a_n = 0$) with $a_n \neq 0$. Notice that $v(v^{n-1} + \ldots) = -a_n$.

Proof. Suppose that α is algebraic in k. Then we have some polynomial

$$f(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_1 x + k_0 = \sum_{i=0}^{n} k_i x^i$$

where $k_n \neq 0$ such that $f(\alpha) = 0$. Then we have

$$g(\alpha^{-1}) = \alpha^{-n} \cdot f(\alpha) = 0$$

where g is also a polynomial in $k[x]$. So if an element is algebraic, its multiplicative inverse is also algebraic. We can also consider the additive inverse $-\alpha$. If we take f' preserve the coefficients k_i of f but change the parity for odd powers of x, we have

$$f' = \sum_{i=0}^{n} k_i (-1)^i x^i$$

and

$$f'(-\alpha) = 0$$

So the additive inverse is also algebraic over k. To prove that it is a subring with inverses, we also need to show that the set is closed under addition/multiplication. If α, β are algebraic, we have that

$$1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$$

for some $n-1$ forms a basis of the extension $k[\alpha]$. And likewise for β:

$$1, \beta, \beta^2, \ldots, \beta^{n-1}$$

So we also notice that we will have a finite number of linearly independent and spanning basis elements for $k[\alpha, \beta]$:

$$\begin{array}{cccccc}
1 & \alpha & \alpha^2 & \alpha^3 & \ldots & \alpha^{n-1} \\
\beta & \alpha\beta & \alpha^2\beta & \alpha^3\beta & \ldots & \alpha^{n-1}\beta \\
\beta^2 & \alpha\beta^2 & \alpha^2\beta^2 & \alpha^3\beta^2 & \ldots & \alpha^{n-1}\beta^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta^{m-1} & \alpha\beta^{m-1} & \alpha^2\beta^{m-1} & \alpha^3\beta^{m-1} & \ldots & \alpha^{n-1}\beta^{m-1}
\end{array}$$

So we take the powers of $\alpha\beta$ or any other element of the array:

$$1, \alpha\beta, (\alpha\beta)^2, \ldots, (\alpha\beta)^{nm}$$

This list has $nm + 1$ elements, but our basis has nm elements. So our list must be linearly dependent. Therefore, we have a finite extension. Therefore, the product of algebraic elements of k is also algebraic. As for the sum, we define its powers by:

$$(\alpha + \beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k}$$

which can be written all can be written as a combination of elements of our basis. By the same argument, we take $nm + 1$ powers and get a linearly dependent list. So we can conclude that $\alpha + \beta$ is algebraic over k also. So we have that if $k \subseteq L$ is a field extension, then the set of elements of L that are algebraic over k is a subfield. □

(b) Suppose L is a finite extension of k and $k \subseteq R \subseteq L$ for a ring R. Prove that R is a field.

Proof. Finite extensions are algebraic. Therefore, every element in L is algebraic over k. But this means that R is a subset of L algebraic over k. By the previous problem, we have that there must be inverses for every algebraic element of R. So every element in R has an inverse except 0, therefore showing that R is a field. □

Exercise 5: Suppose $k \subseteq L$ is an algebraic extension, and $L \subseteq L'$ is an algebraic extension. Prove that $k \subseteq L'$ is an algebraic extension. (Hint: If $\alpha \in L'$ is algebraic over L, then there exist $c_i \in L$ such that $\alpha^n = c_0 + c_1\alpha + \ldots + c_{n-1}\alpha^{n-1}$. Then show that α and $c_0, \ldots, c_{n-1}$ are contained in a finite extension of k.)

Proof. Let $\alpha \in L'$ be arbitrary. Then we know that

$$\alpha^n = c_0 + c_1\alpha + \ldots + c_{n-1}\alpha^{n-1}$$

for some $c_i \in L$. Observe that since each c_i are algebraic over k, we have that a finite extension of k with any c_i induces a finite vector space over k since we have

$$c_i^m k_m + c_i^{m-1} k_{m-1} + \ldots + c_1 k_1 + k_0 = 0$$

for some $k_j \in k$, as c_i's are algebraic over k. Now we consider all combinations of the products of $\alpha, c_{n-1}, \ldots, c_0$ of the form

$$\alpha^{b_1} c_{n-1}^{b_2} \cdots c_0^{b_{n+1}}$$

There are a finite number of such products, the collection of which forms a basis over the extension $k[\alpha, c_{n-1}, \ldots, c_0]$. Therefore, we have that L' is an algebraic extension over k because any element is part of some finite and therefore algebraic extension of k. □