# Math55Hw10

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## 6.1: 28, 35bc, 36, 40, 41, 46, 48, 52x

**Exercise 28**: How many license plates can be made using either three digits followed by three upper-case English letters or three upper-case English letters followed by three digits?

Construct a series of choices for three digits followed by three letters:

$C_1$ = choose a digit from $0-9$ for the first entry.

$C_2$ = choose a digit from $0-9$ for the second entry.

$C_3$ = choose a digit from $0-9$ for the third entry.

$C_4$ = choose a letter from $a-z$ for the fourth entry.

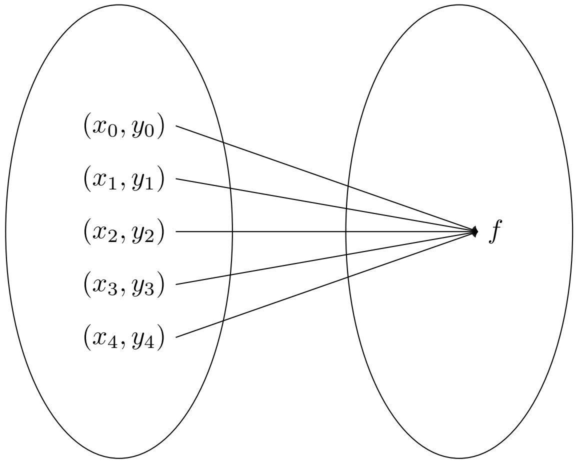
$C_5$ = choose a letter from $a-z$ for the fifth entry.

$C_6$ = choose a letter from $a-z$ for the sixth entry.

The number of difference $C_1, C_2, C_3$ is 10 and the number of different $C_5, C_5, C_6$ is 26. The number of ways to create plates with three digits followed by three letters is $10^3 \cdot 26^3$. Create a bijection to license plates beginning in three letters followed by 3 digits by taking the first group of three elements of the string and shifting it to the back of the string. So the total number of license plates is $2 \cdot 10^3 \cdot 26^3$.

**Exercise 35bc**: How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

**b)** 5




Notice that for every set of $\{(x_0, y_0), \ldots, (x_4, y_4)\}$ there is a 5-to-1 function from that set to a one-to-one function. We can construct $f$ through a series of choices:

$C_1$ = For one element $X$ choose an element of $Y$. Remove them and call the new sets $X'$ and $Y'$.

$C_2$ = For one element from $X'$ choose an element of $Y'$. Remove them and call the new sets $X''$ and $Y''$

$\vdots$

$C_5$ = For one element from $X''''$ choose an element of $Y''''$.

Then there is a bijection from the set of pairs of elements from each choice to a one-to-one function $f$. There are 5 ways to carry out $C_1$, 4 for $C_2$, 3 for $C_3$, 2 for $C_4$, and 1 for $C_5$. Multiply them by the product rule to get 5!.

**d)** 7

By the same reasoning as the first we have

| Choices | Number of ways |
|---|---|
| $C_1$ | 7 |
| $C_2$ | 6 |
| $C_3$ | 5 |
| $C_4$ | 4 |
| $C_5$ | 3 |

Which gives us $\frac{7!}{2!}$.

**Exercise 36**: How many functions are there from the set $\{1, 2, \ldots, n\}$, where $n$ is a positive integer to the set $\{0, 1\}$?

The functions must map each element of the domain $\{1, 2, \ldots, n\}$ to an element in the range $\{0, 1\}$. We can construct all possible functions $f$ through a series of choices:

$C_1$ = Set an element in $\{1, 2, \ldots, n\}$ called $k_1$ and choose an element of $\{0, 1\}$ called $j_1$ and create the coordinate $(k_1, j_1)$.

$C_2$ = Set an element in $\{1, 2, \ldots, n\} - \{k_1\}$ called $k_2$ and choose an element of $\{0, 1\}$ called $j_2$ and create the coordinate $(k_2, j_2\}$.

$\vdots$

$C_n$ = Set an element in $\{1, 2, \ldots, n\} - \{k_1, k_2, \ldots, k_{n-1}\}$ called $k_n$ and choose an element of $\{0, 1\}$ called $j_n$ and create the coordinate $(k_n, j_n)$.

Observe that these choices run through every element of the domain, so the set of coordinates generated by a sequence of choices $C_1, C_2, \ldots, C_n$ can be mapped to a function. There are 2 ways to carry out $C_1, \ldots, C_n$. By product rule, the number of functions from $\{1, 2, \ldots, n\}$ to $\{0, 1\}$ is $2^n$.

**Exercise 40**: How many subsets of a set with 100 elements have more than one element?

Let $K$ be the set of 100 elements. Count the number of subsets that have one or less elements:

$$\begin{aligned} S_1 = \{\{x\} : x \in K\} &\text{ and } S_0 = \{\emptyset\} \\ |S_1| = 100 &\text{ and } |S_0| = 1 \\ |S_1| + |S_0| &= 101 \end{aligned}$$

To compute the opposite, take all possible subsets of $K$ and subtract 101: The number of subsets of $K$ is $2^{100}$ since each element can be either in the subset or out of it. So the answer is $2^{100} - 101$.

**Exercise 41**: A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length $n$ are palindromes?

Observe that every palindrome is uniquely specified by the first $\left\lfloor \frac{n}{2} \right\rfloor$ bits. Since we have $\left\lfloor \frac{n}{2} \right\rfloor$ bits and we can make a series of choices: choosing a 0 or 1 for each position, the number of palindromes is $2^{\lfloor n/2 \rfloor}$.

**Exercise 46**: How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right partner.

There are $\binom{10}{4}$ ways to choose 4 people out of 10. First let us find all possible seatings of 4 people:

$C_1$ = choose a person for seat number 1

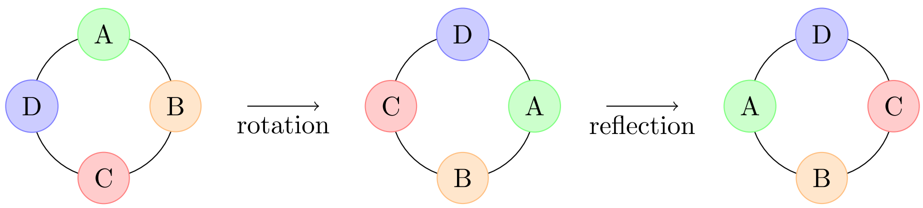
$C_2$ = choose a person for seat number 2

$C_3$ = choose a person for seat number 3

$C_4$ = choose a person for seat number 4

We end up with 4! ways to seat 4 people.

Now consider the number of seatings of 4 people that are equivalent. Let $A, B, C, D$ be four different people at a table and observe that adjacency is preserved by rotation or reflection of the table:




Notice that for a person $p_0 \in P = \{A, B, C, D\}$, if their adjacent neighbors are $p_1, p_2 \in P$, then we can do some combination of reflection and rotation to obtain all table configurations (broken into cases where $p_1$ is to the left of $p_0$ and $p_1$ is to the right of $p_0$ for all possible seat positions of $p_0$). So rotation and reflection are two choices that cover all similar seatings of 4 people:

Rotation = there are 4 ways to rotate a table given a seating.

Reflection = there are 2 ways to reflect a table given a seating.

By the division rule, the number of unique seatings for 4 people is $\frac{4!}{8} = 3$. Since each choice of 4 people has 3 unique seatings, our final answer is $\binom{10}{4}3$.

**Exercise 48**: In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and groom are among these 10 people, if

**a)** the bride must be in the picture?

If the bride must be in the picture, we are choosing 5 other people for the picture out of the set of 9 people which is $\binom{9}{5}$. Then there is the arrangement of the people where order matters. From the 6 people, there are 6! ways to order them. So by our two steps of counting, the number of ways to take the picture is $\binom{9}{5}6!$.

**b)** both the bride and groom must be in the picture?

Using the same method as in (**a)**), we have to choose 4 other people from a set of 8 people which gives us $\binom{8}{4}$. But there are also 6! ways to order

6 people in a line. By our two steps of counting, we get $\binom{8}{4}6!$.

**c)** exactly one of the bride and the groom must be in the picture?

First, start by choosing the bride to be in the picture. That means that the next 5 we choose will be from the set of 9 people with the groom removed: $\binom{8}{5}$. There are 6! ways to place them in a line. Notice that there is a bijection from the set of photos with the bride in the photo only to the set with the groom in the photo only. We just replace each instance of the bride with the groom. So the count is $\binom{8}{5}6! \cdot 2$

**Exercise 52x**: How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Let us count the number of bit strings with 5 0s first. If our bit string has 5 0s, let the bit string 00000 $= A$ and observe that in our 10-bit string, $A$ can occupy 6 different positions, where each empty underlined position represents a bit that can either be 0 or 1:

| | | | | | | |
|---|---|---|---|---|---|---|
| $S_0$ : | A | ___ | ___ | ___ | ___ | ___ |
| $S_1$ : | ___ | A | ___ | ___ | ___ | ___ |
| $S_2$ : | ___ | ___ | A | ___ | ___ | ___ |
| $S_3$ : | ___ | ___ | ___ | A | ___ | ___ |
| $S_4$ : | ___ | ___ | ___ | ___ | A | ___ |
| $S_5$ : | ___ | ___ | ___ | ___ | ___ | A |

Observe that each type of string $S_0, S_1, \ldots, S_6$ are disjoint. Each type of $S_i$ can be determined by its five empty positions. We choose from 0,1 for the first empty position, 0,1 for the second, and so on for five empty positions. That gives us a total of $2^5$ unique strings for each of $S_i$. We add $\sum_{i=1}^{6} |S_i|$ which gives us $6(2^n)$ types of bit strings with 5 consecutive 0's. Now there is a bijection to unique bit strings with 5 consecutive 1's by replacing $A = 00000$ with $A' = 11111$. Therefore, the answer is $6(2^7)$.

## 6.3: 12, 22ace, 24, 30, 36

**Exercise 12**: How many bit strings of length 12 contain

**a)** exactly three 1's?

We can make the choice to choose three positions to place our 1's in our 12-bit string. Since order does not matter, the answer is $\binom{12}{3}$.

**b)** at most three 1's?

Let $S_i = \{(b_1, \ldots, b_{12} : b_1, \ldots, b_{12} \in \{0,1\}$ and $i$ bits are $1\}$

We would need to count this for all $i \leq 3$ which is $\sum_{i=0}^{3} \binom{12}{i}$.

**c)** at least three 1's?

Subtract the total number of 12-bit strings with the bit strings that have less than 3 1's: $2^{12} - \binom{12}{2} - \binom{12}{1} - \binom{12}{0}$.

**d)** and equal number of 0's and 1's?

We just need to count the number with exactly 6 0's, since they should also have 6 1's, which is what we want. Using the idea from **(a)**, it is $\binom{12}{6}$.

**Exercise 22ace**: How many permutations of the string $ABCDEFGH$ contain

**a)** the string $ED$?

Start by grouping $ED$ together. Then the string we wish to count has 7 elements arranged in any order. But that is just 7!.

**c)** the strings $BA$ and $FGH$?

We first count the number of strings let's call Type A which contain a $BA$ grouping. Using the same process as **(a)**), it is 7!. Let Type B strings be strings with the $FGH$ grouping which number to be 6!. Notice that the set of Type A strings and Type B strings are not disjoint. So let us count their union which contains both the $BA$ grouping and $FGH$ grouping. This string contains 5 elements: $BA, FGH, C, D, E$ so there are 5! strings that are both Type A and Type B. By the subtraction rule, the number of strings with both $BA$ and $FGH$ strings is $7! + 6! - 5!$.

**e)** the strings $CAB$ and $BED$?

We repeat the same process as above where Type A strings contain $CAB$ and number to be 6! while Type B strings contain $BED$ and number to be 6!. Notice that if a string is both Type A and Type B, then it contains the $CABED$ grouping. That string has 4 elements: $CABED, F, G, H$. By subtraction rule, we get: $2(6!) - 4!$.

**Exercise 24**: How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other?

We first line up all 10 women. Observe that there are 11 ways to place a man somewhere in the line:

Line: $\underline{W}$ $\underline{W}$ $\underline{W}$ $\underline{W}$ $\underline{W}$ $\underline{W}$ $\underline{W}$ $\underline{W}$ $\underline{W}$ $\underline{W}$

$\underline{0}$ $\underline{1}$ $\underline{2}$ $\underline{3}$ $\underline{4}$ $\underline{5}$ $\underline{6}$ $\underline{7}$ $\underline{8}$ $\underline{9}$ $\underline{10}$

For the next one, there are 10 positions, since we have taken up one of the possible positions labelled $0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$ or 10. For the third, 9, fourth, 8, and fifth, 7 such that no two are next to each other. So there are $\frac{11!}{6!}$ different types of lines that satisfy the condition that no two men are next to each other.

**Exercise 30**: A professor writes 40 discrete mathematics true/false questions. Out of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

There is a bijection to a 40-bit string with exactly 17 1's and 23 0's. This gives us $\binom{40}{17}$ since we are picking a 17 subset of 40 and for this subset of 1's, there is a 17!-to-1 function from the number of ways to order the elements to the output of the ordering: 11111111111111111.

**Exercise 36**: Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

We can break this into choices:

$C_0 = 0$ men $= \binom{15}{6}$ ways to choose the 6 women for the committee.

$C_1 = 1$ man $= \binom{15}{5}$ ways to choose the 5 women for the committee.

$C_2 = 2$ men $= \binom{15}{4}$ ways to choose the 4 women for the committee.

Since there is one way to choose 0 men, there are $\binom{15}{6}$ ways for choice 0. For choice 1, there are $\binom{10}{1}$ ways to choose one man so there are $\binom{10}{1}\binom{15}{5}$ ways for choice 1. Then for choice 2, there are $\binom{10}{2}$ ways to choose two men so there are $\binom{10}{2}\binom{15}{4}$ ways for choice 2. We add them all up:

$$\binom{10}{0}\binom{15}{6} + \binom{10}{1}\binom{15}{5} + \binom{10}{2}\binom{15}{4}$$

## 6.4: 5, 12a, 14, 18, 25, 26a, 28, 31x, 35

**Exercise 5**: How many terms are there in the expansion of $(x + y)^{100}$ after all the terms are collected. Each term has an $x$ with degree $0 \leq d \leq 100$ and there is a bijection from the number of unique degrees of $x$ to the number of terms. There are 101 total possible degrees, so 101 terms.

**Exercise 12a**: Use the binomial theorem to find the coefficient of $x^a y^b$ in the expansion $(5x^2 + 2y^3)^6$, where

**a)** $a = 6, b = 9$. Using the formula:

$$(5x^2 + 2y^3)^6 = \sum_{k=0}^{6} \binom{6}{k} (5x^2)^k (2y^3)^{6-k}$$

We are looking for when $2k = 6$ and $3(6-k) = 9$ which gives us $k = 3$. The term of the expansion when $k = 3$ is:

$$\binom{6}{3} (5x^2)^3 (2y^3)^{6-3}$$

The coefficient is $\binom{6}{3}(125)(8)$.

**Exercise 14**: Give a formula for the coefficient of $x^k$ in the expansion of $(x + 1/x)^{100}$, where $k$ is an integer.

By the binomial theorem,

$$\begin{aligned}\sum_{k=0}^{100} \binom{100}{k} (x^k) \left(\frac{1}{x}\right)^{100-k} &= \sum_{k=0}^{100} \binom{100}{k} (x^k)(x)^{k-100} \\ &= \sum_{k=0}^{100} \binom{100}{k} x^{2k-100}\end{aligned}$$

If our degree is $a = 2k - 100$, then we solve for $k$ for $\binom{100}{k}$: $k = \dfrac{100 + a}{2}$. So the coefficient is $C(k) = \binom{100}{(100+a)/2}$

**Exercise 18**: Show that if $n$ is a positive integer then $1 = \binom{n}{0} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$.

*Proof.* We will proceed by induction.

Basis Step: For $n = 1$, it is indeed true that $\binom{1}{0} = \binom{1}{1}$.

Inductive Step: Suppose that $1 = \binom{n}{0} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$ for some arbitrary $n$. We know that $\binom{n+1}{p} = \binom{n+1}{n-p}$, so we only need to prove $1 = \binom{n+1}{0} < \cdots < \binom{n+1}{\lfloor n + 1/2 \rfloor}$. If we wish to count the combinations of $k$ elements in an $n+1$ element set, we would have $\binom{n}{k} + \binom{n}{k-1}$. If we wish to count the number of $k+1$ element subsets of and $n+1$ element set, it would be $\binom{n+1}{k+1} + \binom{n+1}{k}$.

If $k < \lfloor {}^{(n+1)}/_{2} \rfloor$, we have

$$\binom{n}{k} + \binom{n}{k-1} < \binom{n}{k} + \binom{n}{k+1}$$

by the inductive hypothesis or $\binom{n+1}{k} < \binom{n+1}{k+1}$, as desired. $\square$

**Exercise 25**: Prove that if $n$ and $k$ are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$,

**a)** using a combinatorial proof.

*Proof.* Suppose there is an $n$ element set $N$ where a $k$ element subset $Q$ is chosen containing the "qualifiers". Suppose that there can only be one "winner" $w$ in $N$ and $w \in Q$ is necessary. We wish to count the number of combinations of both winners and qualifiers.

Method 1: Count the number of ways to have a set of qualifiers: $\binom{n}{k}$ then the number of ways for each subset to have a winner: $k$. We mulitply them to get $k\binom{n}{k}$.

Method 2: Assume that there is a predetermined winner: $n$ ways. Then construct the $Q$ subset by choosing $k-1$ elements from the set $N - \{w\}$: $\binom{n-1}{k-1}$. Mulitplying them, we get $n\binom{n-1}{k-1}$.

Since both methods counted the same thing, $k\binom{n}{k} = n\binom{n-1}{k-1}$ as desired. $\square$

**b)** using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 6.3.

*Proof.* Observe the following computations:

$$k\binom{n}{k} = \frac{n!}{(n-k)!(k-1)!} = n\frac{(n-1)!}{(n-k)!(k-1)!} = n\binom{n-1}{k-1}$$

which proves the equality. $\square$

**Exercise 26a**: Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever $n, r$, and $k$ are nonnegative integers with $r \leq n$ and $k \leq r$

**a)** using a combinatorial argument.

*Proof.* Suppose there is a tree containing a set of apples $A$ with $|A| = n$. A gardener wishes to pick apples only $r \leq n$ apples. Once in the house, $k$ apples are discarded into the trash and only the "good" apples are "kept". We wish to count the number of combinations of apples where they can be in the trash, tree, or kept by the gardener.

Method 1: Choose an $r$ apple subset from $A$. Then choose $k$ apples from that $r$ subset to keep. The number of combinations is $\binom{n}{r}\binom{r}{k}$.

Method 2: Choose $k$ apples to throw in the trash first: $\binom{n}{k}$. Then go back to the tree and choose $r-k$ of the good apples from the remaining apples to keep: $\binom{n-k}{r-k}$. There are $\binom{n}{k}\binom{n-k}{r-k}$ possibilities.

Since both methods count the same thing, the quantities are equal. $\square$

**Exercise 28**: Show that if $p$ is a prime and $k$ is an integer such that $1 \leq k \leq p-1$, then $p$ divides $\binom{p}{k}$.

*Proof.* Observe that since $k \leq p-1$, then $1, 2, \ldots, k$ does not divide $p$. So $k! \nmid p$. Also, $p-k \geq 1$ so $(p-k)! \nmid p$. We have $(p-k)!k! \nmid p$. Since $\binom{p}{k}$ is an integer, $(p-k)!k!|p(p-1)!$. By Euclid's Lemma, $(p-k)!k!|(p-1)!$. Therefore, $\binom{p}{k} = p(m)$ where $m = \dfrac{(p-1)!}{(p-k)!k!} \in \mathbb{Z}$. $\square$

**Exercise 35**: Show that a nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

*Proof.* We will do some casework for a set $S$ where $|S| = 1$ and $|\geq| 2$.

Case 1: Suppose $|S = \{s\}| = 1$. Then we verify that the subsets of $S$ are $\emptyset$ and $\{s\}$. It follows that there are the same number of odd number of element subsets as there are even number of element subsets of S.

Case 2: Suppose $|S| \geq 2$. We remove an arbitrary element from $S$ called $s_0$ and call that new set $S'$.

Let

$$\begin{aligned}
E &= \{\text{subsets of } S \text{ with an even number of elements}\} \\
O &= \{\text{subsets of } S \text{ with an odd number of elements}\} \\
E' &= \{\text{subsets of } S' \text{ with an even number of elements}\} \\
O' &= \{\text{subsets of } S' \text{ with an odd number of elements}\}
\end{aligned}$$

We construct all subsets of $S$ by the process:

(1) If $e \in E'$, $e \cup \emptyset \in E$

(2) If $o \in O'$, $o \cup \{s_0\} \in E$

(3) If $e \in E'$, $e \cup \{s_0\} \in O$

(4) If $o \in O'$, $o \cup \emptyset \in O$

If we add up the number of subsets modified in (1) and (2), we get $|E|$. If we add up the number of subsets modified in (3) and (4), we get $|O|$. Thus,

$$|E| = |E'| + |O'| = |O'| + |E'| = |O|$$

as desired. $\square$

## 6.5: 10acd, 14, 20, 22, 28

**Exercise 10acd**: A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose

**a)** a dozen croissants?

There are 6 types of croissants. We have 6 distinguished boxes and 12 non distinguished tokens. The number is $\binom{12+6-1}{6-1}$.

**c)** two dozen croissants with at least two of each kind?

We start with 6 boxes with 2 in each box. That leaves 12 croissants left to assign to a box flavor. This gives us $\binom{12+6-1}{6-1}$.

**d)** two dozen croissants with no more than two broccoli croissants?

Case 1: We have 2 broccoli croissants. We assign the rest of 22 croissants to 5 flavors which gives us: $\binom{22+5-1}{5-1}$.

Case 2: We have 1 broccoli croissant. We assign the rest of 23 croissants to 5 flavors which gives us: $\binom{23+5-1}{5-1}$.

Case 3: We have 0 broccoli croissants. We assign the rest of 24 croissants to 5 flavors which gives us:$\binom{24+5-1}{5-1}$.

Since the cases are disjoint and are exhaustive, the number is $\binom{22+5-1}{5-1} + \binom{23+5-1}{5-1} + \binom{24+5-1}{5-1}$.

**Exercise 14**: How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where $x_1, x_2, x_3, x_4$ are nonnegative integers?

This is just how many ways to assign 17 1's to 4 distinguishable groups. The value of $x_1, x_2, x_3, x_4$ will be equal to the number of 1's they are assigned. The answer is $\binom{17+3-1}{3-1}$.

**Exercise 20**: How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where $x_1, x_2, x_3$ are nonnegative integers?

We can create a "discard" group other than $x_1, x_2, x_3$, where we assign a 1. This will decrease the overall sum of $x_1, x_2, x_3$. If we let 0 be the divider between the groups $x_1, x_2, x_3$ and the discard group, we will have 3 0's. This is simply: $\binom{11+3}{3}$.

**Exercise 22**: In how many ways can an airplane pilot be scheduled for five days of flying in October if he cannot work on consecutive days?

We will break this into cases:

Case 1: The pilot does not get assigned to the first or last day. Then there are 29 days possible to choose the first day: $\binom{29}{1}$. Now we remove 3 days, the day chosen itself, the day before it, and the day after. Choose the next day: $\binom{26}{1}$. Repeat this process 3 more times: $\binom{23}{1}, \binom{20}{1}, \binom{17}{1}$. By product rule, the number of ways is $29 \cdot 26 \cdot 23 \cdot 20 \cdot 17$.

Case 2: the pilot gets assigned to either the first or last day and not both. Using the same reasoning, we get $29 \cdot 26 \cdot 23 \cdot 20$ if we fix the first day of the month. This is the same number of ways for if the last day of the month was chosen instead.

Case 3: The pilot gets assigned to both the first and last day. We get $27 \cdot 24 \cdot 21$.

Add up all the cases to get the answer, since they are disjoint.

**Exercise 28**: How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

If it is a number less than 1,000,000, it will have 6 digits. That means we can consider 6 positions and each position has at most 9 assignments of 1's to it:

$$P_1 : \quad \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}$$

$$P_1 : \quad \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}$$

$$P_2 : \quad \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}$$

$$P_3 : \quad \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}$$

$$P_4 : \quad \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}$$

$$P_5 : \quad \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}$$

$$P_6 : \quad \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}$$

We choose out of 56 positions, 19 1's to add. This gives us $\binom{56}{19}$.

**Additional Exercise**: Count the number of graphs with vertex set 1, 2, ..., n.

We break the count down into steps:

(1) Find the number of unordered pairs of vertices there are. This is $\binom{n}{2}$.

(2) For each pair, we determine if it is in our graph or not. If it is, then the corresponding edge is in the graph. If no, the edge is not in the graph. There is a bijection to a bit string of length $\binom{n}{2}$. So we have $2^{\binom{n}{2}}$.