Math104Hw4

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Exercise 1: Let $s_n = (-1)^n + \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that (s_n) is bounded and find a monotone convergent subsequence.

Proof. We will show that the bound of a sum of sequences is the sum of their bounds. Suppose that s_n and (t_n) are bounded. Then:

$$l_1 \leqslant s_n \leqslant u_1$$
$$l_2 \leqslant t_n \leqslant u_2$$

which is true for all n. Then we have:

$$l_1 + l_2 \leqslant s_n + t_n \leqslant u_1 + u_2$$

so therefore, we know that $(-1)^n$ is bounded below by -1 and above by 1. We know that $\frac{1}{n}$ is bounded below by 0 and above by 1. So therefore, $(-1)^n + \frac{1}{n}$ is bounded below by -1 and above by 2. □

Exercise 2: Let S be the set of all subsequential limits of a real number sequence (s_n). Prove that $S \cap \mathbb{R}$ is closed in $\mathbb{R}$. (Hint use Thm 11.9 and Prop 13.9)

Proof. We know that if we take a sequence t_n on $S \cap \mathbb{R}$, then $\lim(t_n) = t \in S \cap \mathbb{R}$ by Thm 11.9. Then by Proposition 13.9, we know that this implies that $S \cap \mathbb{R}$ is bounded. □

Exercise 3: Prove that (s_n) is bounded $\iff$ $\limsup(|s_n|) < \infty$. (Hint: see the proof of Thm 9.1)

Proof. $(\rightarrow)$ We know that if (s_n) is bounded, then:

$$|s_n| \leqslant M$$

for some $M = \max(|u|, |l|)$ for u, l upper and lower bounds. So we know that:

$$\sup\{s_n : n > N\}$$

for every n is $\leqslant M$ because the supremum is the least lower bound. So we can say that for some $L \leqslant M$, we have for every $\varepsilon > 0$, we can find an N for all $n > N$ such that:

$$|\sup\{s_n : n > N\} - L| < \varepsilon$$

so $\limsup(|s_n|) = L \leqslant M < \infty$.

$(\leftarrow)$ Suppose that $\limsup(|s_n|) < \infty$. Then we know that for every $\varepsilon > 0$, there is an N such that $\forall n > N$:

$$|\sup\{s_n : n > N\} - L| < \varepsilon$$

for $L < \infty$. Now let $\varepsilon = 1$. So we can find an N_0 such that for all $n > N_0$,

$$|\sup\{s_n : n > N\}| < \varepsilon + L$$

and

$$-\varepsilon - L < \sup\{s_n : n > N\} < \varepsilon + L$$

Now we see that for $\varepsilon = 1$, $\max(L+1, s_1, s_2, \dots, s_{N_0})$ is the upper bound and $\min((-1 - L, s_1, s_2, \dots, s_{N_0}))$ is the lower bound. So we are done. $\square$

Exercise 4: Prove that the subsequence of a subsequence of a given sequence is a subsequence of the given sequence. Equivalently, let (s_{n_k}) be a subsequence of (s_n), and (s_{k_l}) be a subsequence of (s_{n_k}), show that $(s_{n_{k_l}})$ is a subsequence of (s_n).

Proof. Suppose that s_{n_k} is a subsequence of s_n. Then we know that

$$n_1 < n_2 < n_3 < \cdots$$

where $n_k \in \{1, 2, 3, \dots\}$. Now suppose that $s_{n_{k_l}}$ is a subsequence of s_{n_k}. Then

$$n_{k_1} < n_{k_2} < n_{k_3} < \cdots$$

where $n_{k_l} \in \{n_1, n_2, n_3, \dots\}$. By the above inequality chain and because $n_{k_l} \in \{n_1, n_2, n_3, \dots,\} \subseteq \{1, 2, 3, \dots\}$, then we have that $s_{n_{k_l}}$ is a subsequence of s_n. $\square$

Exercise 5: Define $s_1 = 1$ and $s_{n+1} = \frac{s_n}{n+1}$ for any $n \in \mathbb{N}$ (actually $s_n = \frac{1}{n!}$). Prove that $\lim(s_n)^{\frac{1}{n}} = 0$. (hint: cor 12.3)

Proof. By corollary 12.3, if the limit of $\frac{s_{n+1}}{s_n}$ exists, then is equal to the limit of $s_n^{\frac{1}{n}}$. In this case, we have that

$$\frac{s_{n+1}}{s_n} = \frac{\frac{s_n}{n+1}}{s_n} = \frac{1}{n+1}$$

which we know converges to 0. Therefore, $\lim(s_n)^{\frac{1}{n}} = 0$. $\square$

Exercise 6: Assume that (s_n) and (t_n) are bounded sequences. Prove that:

$$\limsup(s_n + t_n) \leqslant \limsup(s_n) + \limsup(t_n)$$

Proof. We see that:

$$\sup\{s_n + t_n : n > N\} \leqslant \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

because we have that

$$\{s_n + t_n : n > N\} \subseteq \{s_n : n > N\} + \{t_n : n > N\} \ni \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

If we interpret $s'_n = \sup\{s_n + t_n : n > N\}$ as a sequence and $t'_n = \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$ as a sequence, because the limit exists and $s_n \leqslant t_n$ for all $n > N$, by the previous homework, we have that:

$$\lim(s'_n) \leqslant \lim(t'_n)$$

as desired. $\square$