Math104Hw13

Trustin Nguyen

December 2, 2023

Exercise 1: Find the limit $\lim_{x\to 0} \frac{e^{3x} - \cos x}{x}$ if it exists.

Proof. Using L'Hopital's rule, we have that $\lim_{x\to 0} e^{3x} - \cos x = \lim_{x\to 0} x = 0$ means that

$$\lim_{x\to 0} \frac{e^{3x} - \cos x}{x} = \lim_{x\to 0} \frac{3e^{3x} + \sin x}{1}$$

Then

$$\lim_{x\to 0} 3e^{3x} + \sin x = 3$$

is finite, so

$$\lim_{x\to 0} \frac{e^{3x} - \cos x}{x} = 3$$

$\square$

Exercise 2: Find the limit $\lim_{x\to 0} (1+2x)^{\frac{1}{x}}$ if it exists.

Proof. Using the Taylor Series of $(1+2x)^{\frac{1}{x}}$, we have:

$$\begin{aligned}(1+2x)^{\dfrac{1}{x}} &= \sum_{k\geqslant 0}^{\infty} \frac{\frac{1}{x}\left(\frac{1}{x} - 1\right)\cdots\left(\frac{1}{x} - k + 1\right)}{k!}(2x)^k \\ &= \sum_{k\geqslant 0} \frac{1(1-x)(1-2x)\cdots(1-(k-1)x)}{k!}(2^k)\end{aligned}$$

So

$$\lim_{x\to 0} (1+2x)^{\dfrac{1}{x}} = \sum_{k\geqslant 0} \frac{2^k}{k!}$$

But we have the Taylor Series of e^x as

$$e^x = \sum_{k\geqslant 0} \frac{x^k}{k!}$$

and therefore,

$$e^2 = \sum_{k\geqslant 0} \frac{2^k}{k!}$$

The limit is e^2. $\square$

Exercise 3: Find the limit $\lim_{x\to 0} (\frac{1}{\sin x} - \frac{1}{x})$ if it exists.

Proof. By L'Hopital,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} \\
&= \frac{0}{0} \\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x}
\end{aligned}$$

and since

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} = \frac{0}{0}$$

Check for $\lim\limits_{x\to 0}\frac{f'}{g'}$:

$$\lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = 0$$

so

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = 0$$

□

Exercise 4: Find the Taylor series of $\cos x$ and show that it converges to $\cos x$ for all $x \in \mathbb{R}$.

Proof. The Taylor Series is defined as

$$f(x) = \sum_{k\geqslant 0}\frac{f^{k}(0)}{k!}x^{k}$$

Then

$$\begin{aligned}
f(x) &= \cos x & f(0) &= 1 \\
f'(x) &= -\sin x & f'(0) &= 0 \\
f''(x) &= -\cos x & f''(0) &= -1 \\
f'''(x) &= \sin x & f'''(0) &= 0 \\
f''''(x) &= \cos x & f''''(0) &= 1
\end{aligned}$$

So for $f^{(k)}$, we have if $k = 2j$,

$$f^{(2j)}(0) = \begin{cases} 1 & \text{if } j = \text{ even} \\ -1 & \text{if } j = \text{ odd} \end{cases} = (-1)^{j}$$

and the Taylor series is

$$f(x) = \sum_{j\geqslant 0}\frac{(-1)^{j}}{(2j)!}x^{2j}$$

Since $|f^{(n)}(x)| \leqslant 1$, we have that $R_n(x) \to 0$. So the remainder converges to 0 and the Taylor series converges to the function $\cos x$. □

Exercise 5: Repeat the Q4 for the function $\sinh x = (e^{x} - e^{-x})$ (this is the def).

Proof. As before, the Taylor Series is

$$f(x) = \sum_{k\geqslant 0}\frac{f^{(k)}(0)}{k!}x^{k}$$

Now:

$$\begin{aligned} f(x) &= e^x - e^{-x} & f(0) &= 0 \\ f'(x) &= e^x + e^{-x} & f'(0) &= 2 \\ f''(x) &= e^x - e^{-x} & f''(0) &= 0 \\ \vdots &= \vdots & \vdots &= \vdots \end{aligned}$$

Then we have that for

$$\begin{aligned} f^k(0) &= \begin{cases} 0 & \text{if } k = \text{ even} \\ 2 & \text{if } k = \text{ odd} \end{cases} \\ &= 1 + (-1)^{k+1} \end{aligned}$$

So now we have

$$f(x) = \sum_{k \geqslant 0} \frac{1 + (-1)^{k+1}}{k!} x^k$$

Again, since $f^{(n)}(0)$ is bounded, we have that $R_n(x) \to 0$, so the Taylor series converges to $\sinh x$. □

Exercise 6: Repeat the Q4 for the function $\cosh x = (e^x + e^{-x})$ (this is the def).

Proof. As before, the Taylor Series is

$$f(x) = \sum_{k \geqslant 0} \frac{f^{(k)}(0)}{k!} x^k$$

Now:

$$\begin{aligned} f(x) &= e^x + e^{-x} & f(0) &= 2 \\ f'(x) &= e^x - e^{-x} & f'(0) &= 0 \\ f''(x) &= e^x + e^{-x} & f''(0) &= 2 \\ \vdots &= \vdots & \vdots &= \vdots \end{aligned}$$

Then we have that for

$$\begin{aligned} f^k(0) &= \begin{cases} 2 & \text{if } k = \text{ even} \\ 0 & \text{if } k = \text{ odd} \end{cases} \\ &= 1 + (-1)^{k} \end{aligned}$$

So now we have

$$f(x) = \sum_{k \geqslant 0} \frac{1 + (-1)^{k}}{k!} x^k$$

Again, since $f^{(n)}(0)$ is bounded, we have that $R_n(x) \to 0$, so the Taylor series converges to $\cosh x$. □