Math104Hw11

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Exercise 1: Show that $f_n = \frac{x^n}{n}$ converges uniformly on $[0, 1]$.

Proof. We will show that $f_n \to 0$ uniformly by $\lim_{n\to\infty} \sup\{|f_n(x)| : x \in [0, 1]\} = 0$. Notice that for each f_n, the derivative $f'_n = x^{n-1}$, which is positive on $[0, 1]$, so f_n achieves its max at $x = 1$. It achieves its minimum at $x = 0$. We have that

$$0 = |f_n(0)| < |f_n(1)| = \frac{1}{n}$$

And indeed, $\lim_{n\to\infty} \frac{1}{n} = 0$. So it converges uniformly on $[0, 1]$. □

Exercise 2: Assume that $\sum |a_k| < \infty$, prove that $\sum a_k x^k$ converges uniformly on $[-1, 1]$.

Proof. By the Weierstrass M-Test, we know that $\sum a_k x^k$ converges uniformly on S if $|a_k x^k| \leqslant |a_k|$ for $x \in S$. So we have:

$$\begin{aligned} |a_k||x^k| &\leqslant |a_k| \\ |x^k| &\leqslant 1 \\ -1 \leqslant x^k &\leqslant 1 \end{aligned}$$

which is true exactly when $x \in [-1, 1]$, so we are done. □

Exercise 3: Show that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.

Proof. We have that $\sum_{n\geqslant 0} x^n = \frac{1}{1-x}$. Then we use the ratio test:

$$\beta = \lim_{n\to\infty} \frac{1}{1} = 1$$

So $R = \frac{1}{\beta} = 1$. The series does not converge on $-1, 1$. So $\sum_{n\geqslant 0} x^n$ is differentiable on $(-1, 1)$. Taking the derivative:

$$\begin{aligned} \frac{d}{dx} \sum_{n\geqslant 0} x^n &= \frac{d}{dx} \frac{1}{1-x} \\ \sum_{n\geqslant 1} nx^{n-1} &= \frac{1}{(1-x)^2} \end{aligned}$$

Then multiply by x on both sides:

$$\sum_{n\geqslant 1} nx^n = \frac{x}{(1-x)^2}$$

□

Exercise 4: Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Proof. Recall that $\sum_{n\geqslant 0} y^n = \frac{1}{1-y}$. Substituting $y = \frac{1}{2}x$, we get:

$$\sum_{n\geqslant 0} \frac{1}{2^n}x^n = \frac{1}{1-\frac{1}{2}x} = \frac{1}{\frac{2-x}{2}} = \frac{2}{2-x}$$

To make sure we can take the derivative, find the radius of convergence. Use the ratio test:

$$\beta = \lim_{n\to\infty}\left|\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{2^n}{2^{n+1}}\right| = \frac{1}{2}$$

Then $R = \frac{1}{\beta} = 2$. We can take the derivative in the interval $[-1, 1]$, so it is fine.

Taking the derivative of both sides we get:

$$\sum_{n\geqslant 1} \frac{n}{2^n}x^{n-1} = \frac{2}{(2-x)^2}$$

Substituting $x = 1$, we find:

$$\sum_{n\geqslant 1} \frac{n}{2^n} = \frac{2}{(2-1)^2} = 2$$

□

Exercise 5: Use Q3 to find the explicit formula for $\sum_{n=1}^{\infty} n^2x^n$ when $|x| < 1$.

Proof. We have $\sum_{n\geqslant 1} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$. We can take the derivative again. Notice that radius of convergence is preserved on derivatives. So:

$$\begin{aligned}
\frac{d}{dx}\sum_{n\geqslant 1} nx^n &= \frac{d}{dx}\frac{x}{(1-x)^2} \\
\sum_{n\geqslant 2} n^2x^{n-1} &= \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} \\
\sum_{n\geqslant 2} n^2x^{n-1} &= \frac{1-2x+x^2+2x-2x^2}{(1-x)^4} \\
\sum_{n\geqslant 2} n^2x^{n-1} &= \frac{1-x^2}{(1-x)^4} \\
\sum_{n\geqslant 2} n^2x^{n} &= \frac{x(1-x^2)}{(1-x)^4}
\end{aligned}$$

So that is the formula. □

Exercise 6: Let $f(x) = |x|$ on $\mathbb{R}$, prove that there is no (a_n) such that $\sum_{n=0}^{\infty} a_nx^n = f(x)$ for any $x \in \mathbb{R}$.

Proof. Suppose for contradiction $f(x) = \sum_{n\geqslant 0} a_nx^n$ for some sequence (a_n). Then the radius of convergence contains 0, and we know that $\sum_{n=0}^{\infty} a_nx^n$ is differentiable. Then $f'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$ for $x = 0$. Now we show that $|x|$ is not differentiable at 0. Consider

$$\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{|x|}{x} = 1$$

and

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{|x|}{x} = -1$$

Since the limits are not equal, the limit does not exist for

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

which is a contradiction. $\square$