Math104Midterm2

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November 4, 2023

Exercise 1: Define $f(x) = \frac{x^2}{x-1}$ when $x < 0$ and $f(x) = \sin x$ when $x > 0$.

- Use $\varepsilon - \delta$ to prove that $f(x)$ is continuous on $(0, \infty)$; you can directly use the facts that $|\sin x| \leqslant |x|$ and $\sin x - \sin y = \cos\frac{x+y}{2}\sin\frac{x-y}{2}$ for any x, y.

  Proof. We want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if for $x, x_0 \in (0, \infty)$,

  $$|x - x_0| < \delta$$

  then

  $$|\sin x - \sin x_0| < \varepsilon$$

  First take $\delta = x_0$. Then $x > 0$. Now consider the hint:

  $$\begin{aligned} |\sin x - \sin x_0| &= |\cos\frac{x+y}{2}\sin\frac{x-y}{2}| \\ &= 2|\cos\frac{x+y}{2}||\sin\frac{x-y}{2}| \\ &\leqslant 2|\sin\frac{x-y}{2}| \\ &\leqslant 2|\frac{x-y}{2}| \\ &= \delta \end{aligned}$$

  Then if we take $\delta = \min(x_0, \varepsilon)$, then we have as desired. So $f(x)$ is continuous on $(0, \infty)$. $\square$

- Can you define $f(0)$ so that f is continuous at 0? Explain why. Yes, since the limits

  $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin x = 0$$

  and

  $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2}{x-1} = 0$$

  are equal and exist, then for f to be continuous at 0, we just require $\lim\limits_{x\to 0} f(x) = f(0)$. So the limit must be equal to the RHS and LHS limits which is 0. So $f(0) = 0$.

Exercise 2: Consider a subset E on the x-coordinate of $\mathbb{R}^2 = (x, y)$, $E = \{(x, 0) : x \in (-1, 0) \cup (0, 1)\}$.

- Prove that E is disconnected.

  Proof. Consider the open sets $U_1 = (-1, 0)$ and $U_2 = (0, 1)$. Then we note that $U_1 \cap E \neq \emptyset$, $U_2 \cap E \neq \emptyset$. We also see that

  $$(U_1 \cap E) \cap (U_2 \cap E) = \emptyset$$

and

$$(U_1 \cap E) \cup (U_2 \cap E) = E$$

So E is disconnected.
(Alternate Definition) Consider $A = (-1, 0)$ and $B = (0, 1)$. We note that they are non empty. Also, we have

$$A^- \cap B = [-1, 0] \cap (0, 1) = \emptyset$$

and

$$A \cap B^- = (-1, 0) \cap [0, 1] = \emptyset$$

Since $A \cup B = E$, we have that E is disconnected. □

- Find a real-valued function $f : E \to \mathbb{R}$ which is continuous on E but not uniformly continuous on E; no proof required.

  Answer. We can choose $y = \frac{1}{x}$. Not uniformly continuous because it grows too fast for small x. We can take a Cauchy sequence $(\frac{1}{n})$ which converges to 0 but we have that $f((\frac{1}{n})) = n$ which diverges. Uniformly continuous functions take cauchy sequences to cauchy ones.

Exercise 3: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1/2$.

- Find a sequence $(x_n) \subseteq (0, 1)$ such that $\sum f(x_n)$ converges absolutely.

  Proof. Consider the sequence $(\frac{1}{n^2})$ where the series $\sum \frac{1}{n^2}$ converges absolutely. We can take the sequence $x_n \subseteq (0, 1)$ such that $f(x_n) \leqslant (\frac{1}{n^2})$ pointwise. This will give us a convergent series by the comparison test, because $|f(x_n)| \leqslant \frac{1}{n^2}$ for each n. So the series converges absolutely. □

- Prove that $f([0, 1])$ is bounded and closed interval.

  Proof. Continuous functions take compact sets to compact sets. It also takes connected sets to connected ones, which means that the image is an interval. □

- Prove that there exists $s \in (0, 1)$ such that $f(x) = s - 1/4$.

  Proof. Take $g(s) = f(s) - s + \frac{1}{4}$. We know that $g(0) = \frac{1}{4}$ and $g(1) = -\frac{1}{4}$. By the intermediate value theorem, there is some value $s \in (0, 1)$ such that $f(s) = 0$. □

Exercise 4: True or False. No proof is needed.

- Assume a sequence (s_n) converges to 0 absolutely, then $\sum(-1)^n s_n$ converges.

  Answer. False. Take $(-1)^n \frac{1}{n}$ which converges to 0 absolutely, but $\sum \frac{1}{n}$ does not converge.

- Assume $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$, and E is a compact subset of $\mathbb{R}$, then $f^{-1}(E)$ must be compact.

  Answer. False. Take any constant function. Then we know that E is compact because $[c]$ is closed, bounded. But $f^{-1}(E) = \mathbb{R}$ which is not bounded.

- If E is a connected subset of a metric space (S, d), then E is path-connected.

  Answer. False. Counterexample shown in lecture.

- Any function from integers to real numbers, $f : \mathbb{Z} \to \mathbb{R}$ is uniformly continuous.

Answer. True. Take $\delta = 1$. Then we have

$$|x - y| < \delta$$

means

$$|f(x) - f(y)| < \varepsilon$$

which is true because $x - y$ is less than delta when $x = y$ in the integers. So indeed $|f(x) - f(y)| = 0 < \varepsilon$.

- $x^2 + x^5 - 1 = 0$ has a solution on $(0, 1)$.

  Answer. True. The sum of continuous functions is continuous. We know the functions takes a value of -1 at 0 and 1 at 1. By IVT, it takes on the value 0 somewhere in between.