Math185Hw5

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February 27, 2024

Exercise 1: Verify Stokes' formula in the plane for the vector field $(x, y) \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$ and the region bounded by the circle of radius R, centered at zero.

Verify Green's formula for the vector field $(x, y) \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$ and the same region in the plane.

Proof. (Stokes') We have:

$$\int_\gamma \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \cdot \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \, d\theta = \int_0^{2\pi} r^2(\sin^2\theta + \cos^2\theta) \, d\theta = 2\pi r^2$$

Now compute over area:

$$\int\int_D \nabla \times f \, dx \, dy = \int_0^r \int_0^{2\pi} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \times \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \, dx \, dy = \int_0^r \int_0^{2\pi} 2 \, d\theta \, dr = 2\pi r^2$$

(Green's) We have:

$$\int_\gamma \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \, d\theta = \int_0^{2\pi} r^2(\cos^2\theta + \sin^2\theta) \, d\theta = 2\pi r^2$$

And for over the area:

$$\int\int_D \nabla \cdot f \, dx \, dy = \int_0^r \int_0^{2\pi} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \, dx \, dy = \int_0^r \int_0^{2\pi} 2 \, d\theta \, dr = 4\pi r$$

□

Exercise 2: Prove Green's formula for the closed path described by the right-angled triangle with vertices at $0, a, bi$ ($a, b \in \mathbb{R}$) and a general continuously differentiable vector field $f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$; do so by reducing to the Fundamental Theorem of Calculus in 1 variable.

Note: The hypotenuse will need more care. Assume $a, b > 0$ for a definite picture.

Exercise 3: Evaluate $\int(3z^2 + z)\,dz$ along

(a) the circular arc $|z| = 2$, from 2 to $2i$;

Answer. Let $z = 2e^{i\theta}$. Then we have

$$\int_0^{\pi/2}(3(2e^{i\theta})^2+2e^{i\theta})i2e^{i\theta}\ d\theta = \int_0^{\pi/2} 24ie^{3i\theta}+4ie^{2i\theta}\ d\theta = (8e^{2i\theta}+2e^{2i\theta})\Big|_0^{\pi/2} = -12-8i$$

(b) the straight line from 2 to $2i$; Using the parametrization $\gamma(t) = 2(1-t)+2it = 2+(2i-2)t$, our integral turns into:

$$\begin{aligned}
\int_\gamma f\ d\gamma &= (2i-2)\int_0^1 3[2(1-t)+2it]^2\ dt + (2i-2)\int_0^1 2(1-t)+2it\ dt\\
&= (2i-2)\int_0^1 3[2-2t+2it]^2\ dt + 2(2i-2)\int_0^1 (1-t)+it\ dt\\
&= (2i-2)\int_0^1 [2-(2t-2it)]^2\ dt + 2(2i-2)(t-\frac{t^2}{2}+i\frac{t^2}{2})\Big|_0^1\\
&= 3(2i-2)\int_0^1 4-4(2t-2it)+(2t-2it)^2\ dt + 2(2i-2)-(2i-2)+i(2i-2)\\
&= 12(2i-2)+24(2i-2)\int_0^1 -t+it-it^2\ dt + (2i-2)+i(2i-2)\\
&= 12(2i-2)+24(2i-2)(\frac{-t^2}{2}+i\frac{t^2}{2}-\frac{it^3}{3})\Big|_0^1 + (2i-2)+i(2i-2)\\
&= 12(2i-2)-12(2i-2)+12i(2i-2)-8i(2i-2)-4\\
&= 4i(2i-2)-4\\
&= -12-8i
\end{aligned}$$

(c) the straight lines from 2 to $2+2i$ and then $2+2i$ to $2i$.

Answer. Using the parametrizations $\gamma_1(t) = 2+2it$ and $\gamma_2(t) = 2-2t+2i$, we have

$$\begin{aligned}
\int_\gamma f\ d\gamma &= \int_{\gamma_1} f\ d\gamma_1 + \int_{\gamma_2} f\ d\gamma_2\\
&= 2i\int_0^1 3(2+2it)^2+2+2it\ dt - 2\int_0^1 3(2-2t+2i)^2+2-2t+2i\ dt\\
&= 2i\int_0^1 3(4+8it-4t^2)+2+2it\ dt - 2\int_0^1 3((2-2t)^2+8i-8it-4)+2-2t+2i\ dt\\
&= 2i\int_0^1 14+26it-12t^2\ dt - 2\int_0^1 3(4-8t+4t^2)+26i-24it-10-2t\ dt\\
&= 2i(14t+\frac{i13t^2}{2}-4t^3)\Big|_0^1 - 2\int_0^1 12-26t+12t^2+26i-24it\ dt\\
&=
\end{aligned}$$

Exercise 4: By choosing convenient parametrizations, evaluate the following integrals:

(a) $\int z^{-1}\, dz$, around the square with vertices at $\pm 1 \pm i$.

Proof. We can parametrize starting from the bottom right counterclockwise:

$$\gamma_1(t) = 1 - i + 2it$$
$$\gamma_2(t) = i + 1 - 2t$$
$$\gamma_3(t) = i - 2it - 1$$
$$\gamma_4(t) = -i - 1 + 2t$$

Then our integral is:

$$\int_{\gamma_1} z^{-1}\, d\gamma_1 + \int_{\gamma_2} z^{-1}\, d\gamma_2 + \int_{\gamma_3} z^{-1}\, d\gamma_3 + \int_{\gamma_4} z^{-1}\, d\gamma_4$$

which is

$$\left(\log \frac{1}{\gamma_1(t)}\right)\Big|_0^1 + \left(\log \frac{1}{\gamma_2(t)}\right)\Big|_0^1 + \left(\log \frac{1}{\gamma_3(t)}\right)\Big|_0^1 + \left(\log \frac{1}{\gamma_4(t)}\right)\Big|_0^1$$

We get:

$$\log \frac{1}{1-i} - \log \frac{1}{1+i} + \log \frac{1}{1+i} - \log \frac{1}{-1+i} + \log \frac{1}{-1+i} - \log \frac{1}{-1-i} + \log \frac{1}{-1-i} - \log \frac{1}{1-i} = 0$$

So the total integral is 0. □

(b) $\int z^m\, dz$ around the unit circle, m an integer. (You should get 0, if $m \neq -1$.)

Answer. We can parametrize with $z = e^{i\theta}$:

$$\int_{\gamma} e^{mi\theta}\, d\gamma = \int_0^{2\pi} e^{mi\theta} i e^{i\theta}\, d\theta = i \int_0^{2\pi} e^{(m+1)i\theta}\, d\theta = \begin{cases} 0 & \text{if } m+1 \neq 0 \\ 2\pi & \text{if } m+1 = 0 \end{cases}$$

So it is 0 for $m \neq -1$.

Exercise 5: Derive the *Wallis formula*

$$\int_0^{2\pi} \cos^{2n} \theta \ d\theta = 2\pi \frac{(2n)!}{2^{2n}(n!)^2}$$

by integrating $\frac{1}{z}(z + \frac{1}{z})^{2n}$ around the unit circle, using the binomial formula and invoking Q.4.2.

Answer. We have that $(z + \frac{1}{z})^{2n} = (2\cos\theta)^{2n}$. Then

$$\int_0^{2\pi} \frac{1}{z}(z + \frac{1}{z})^{2n} \ dz = \int_0^{2\pi} i(2\cos n)^{2n} \ d\theta = i2^{2n} \int_0^{2\pi} \cos^{2n} \theta \ d\theta$$

Now by the binomial expansion:

$$(z + \frac{1}{z})^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-k} z^{-k} = \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-2k}$$

By the last question, we have:

$$\int_\gamma \frac{1}{z}(z + \frac{1}{z})^{2n} \ dz = \int_\gamma \binom{2n}{n} z^{-1} \ dz = \int_0^{2\pi} \binom{2n}{n} i \ d\theta = 2\pi \binom{2n}{n} i$$

Overall, we have:

$$i2^{2n} \int_0^{2\pi} \cos^{2n} \theta \ d\theta = 2\pi \binom{2n}{n} i$$

so

$$\int_0^{2\pi} \cos^{2n} \theta \ d\theta = 2\pi \binom{2n}{n} \cdot \frac{1}{2^{2n}}$$

which is what we wanted.

Exercise 6: For the vector field f below, show that $\int_\gamma f \cdot d\gamma = 0$ for any simple (not self-intersecting) closed curve γ.

$$f = \begin{bmatrix} y^2 \cos x - 2e^y \\ 2y \sin x - 2xe^y \end{bmatrix}$$

Compute the same integral along the arc of parabola $y = x^2$ form $(0, 0)$ to (π, π^2).

Proof. By Stokes' Theorem, we have that

$$\begin{aligned} \int_\gamma f \cdot d\gamma &= \int \int_D \nabla \times f \; dx \; dy = \int \int_D \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \times \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} dx \; dy \\ &= \int \int_D 2y \cos x - 2e^y - (2y \cos x - 2e^y) \; dx \; dy = 0 \end{aligned}$$

We have that $f = \begin{bmatrix} u(x, y)_x \\ u(x, y)_y \end{bmatrix}$. And since $f = u_x + iu_y$, $f = y^2 \sin x - 2xe^y$. Taking the endpoints, we get $2\pi e^{\pi^2}$. ◻