Math110Hw3

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Homework 3

Exercise 1: Let $U = \{p \in \mathcal{P}_3(\mathbb{R}) : \int_{-3}^{3} p(x)\, dx = 0\}$.

(a) Find a basis of U.

We start with an arbitrary polynomial in $\mathcal{P}(\mathbb{P})$:

$$a_3x^3 + a_2x^2 + a_1x + a_0$$

and evaluate the integral to solve for the restrictions on the coefficients a_0, a_1, a_2, a_3:

$$\begin{aligned}\frac{a_3x^4}{4} + \frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x\bigg|_{-3}^{3} &= 0\\ 2\left(\frac{81a_3}{4} + \frac{9a_1}{2}\right) &= 0\\ \frac{81a_3}{2} + 9a_1 &= 0\\ a_3 &= \frac{-2a_1}{9}\end{aligned}$$

Therefore, the basis is

$$U = \text{Span}\left\{x^2, 1, \frac{-2x^3}{9} + x\right\}$$

(b) Extend your basis in part (a) to a basis of $\mathcal{P}_3(\mathbb{R})$.

To extend, we can add the vector x^3 to the set, since the new linear combination will be

$$\begin{aligned}&\lambda_1x^2 + \lambda_2 + \lambda_3\left(\frac{-2x^3}{9} + x\right) + \lambda_4x^3\\ &= \lambda_1x^2 + \lambda_2 + \lambda_3\frac{-2x^3}{9} + \lambda_3x + \lambda_4x^3\\ &= \lambda_1x^2 + \lambda_2 + \lambda_3'x^3 + \lambda_3x\end{aligned}$$

which spans the space of polynomials of degree 3 or less.

(c) Find a subspace W of $\mathcal{P}_3(\mathbb{R})$ such that $\mathcal{P}_3(\mathbb{R}) = U \oplus W$.

By the previous item, $W = \{\lambda x^3 : \lambda \in \mathbb{R}\}$.

Exercise 2: Suppose $v_1, \ldots, v_m$ are linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{Span}(v_1 - w, v_2 - w, \ldots, v_m - w) \geq m - 1$$

Proof. Clearly, the span of $v_1 - w, v_2 - w, \ldots, v_m - w$ is a subset of the old set, since w is a linear combination of $v_1, \ldots, v_m$. Observe that in the process of making $v_1 - w, v_2 - w, \ldots, v_m - w$ a spanning set of $\operatorname{Span}(v_1, \ldots, v_m)$, we can just add w to our current set of vectors. So for any w, we can always create a spanning set of $\operatorname{Span}(v_1, \ldots, v_m)$ by adding w. $\square$

Exercise 3: Does the 'inclusion-exclusion formula' hold for three subspaces, i.e., is it always true that

$$\begin{aligned}\dim(U_1 + U_2 + U_3) &= \dim(U_1) + \dim(U_2) + \dim(U_3) \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3)\end{aligned}$$

Prove this formula or provide a counterexample.

Proof. In class, it was proven that

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) + \dim(U_1 \cap U_2)$$

To extend this to the third subspace, we start by generating the basis for

$$\begin{gathered}(U_1 \cup U_2) \cap U_3 \\ (U_1 \cap U_3) \cup (U_2 \cap U_3)\end{gathered}$$

its dimension is

$$\dim(U_1 \cap U_3 + U_1 \cap U_3) = \dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) - \dim(U_1 \cap U_2 \cap U_3)$$

We apply the inclusion-exclusion for two subspaces $U_1 + U_2$ and U_3:

$$\begin{aligned}\dim((U_1 + U_2) + U_3) &= \dim(U_1 + U_2) + \dim(U_3) - \dim((U_1 \cup U_2) \cap U_3) \\ &= \dim(U_1) + \dim(U_2) + \dim(U_3) \\ &- \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3)\end{aligned}$$

which concludes the proof. $\square$

Exercise 4: Let $a, b \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp := (2p(1) + 5p'(2) + ap(0)p(3), \int_{-1}^{2} x^3 p(x)\, dx + b\cos p(0)).$$

Show that T is linear if and only if $a = b = 0$.

Proof. We observe that $Tp = (f, g)$ for values f and g. For Tp to be linear,

$$Tp_1 + Tp_2 = (f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2) = Tp_1 + p_2$$

So it must be that the product of polynomials is linear:

$$\begin{aligned} ap_1(0)p_1(3) + ap_2(0)p_2(3) &= a(p_1(0) + p_2(0))(p_1(3) + p_2(3)) \\ &= a(p_1(0)p_1(3) + p_1(0)p_2(3) + p_1(3)p_2(0) + p_2(0)p_2(3)) \\ 0 &= a(p_1(0)p_2(3) + p_1(3)p_2(0)) \end{aligned}$$

So $a = 0$. We can also apply the same reasoning to the g component:

$$\begin{aligned} b(\cos p_1(0) + \cos p_2(0)) &= b\cos(p_1(0) + p_2(0)) \\ 0 &= b(\cos p_1(0) + \cos p_2(0) - \cos(p_1(0) + p_2(0))) \end{aligned}$$

Then it must either be that $b = 0$ or that

$$\cos p_1(0) + \cos p_2(0) = \cos(p_1(0) + p_2(0))$$

But that cannot be true, since if we take p_1, p_2 to be the constant functions, cosine is not linear. Therefore, $b = 0$. □

Exercise 5: Suppose $T \in \mathcal{L}(V, W)$, $v_1, \ldots, v_m \in V$ and the list $Tv_1, Tv_2, \ldots, Tv_m$ is linearly independent (in W). Prove that $v_1, \ldots, v_m$ must be linearly independent in V. What is the contrapositive of this statement?

Proof. Suppose that $Tv_1, \ldots, Tv_m$ are all linearly independent. Then

$$\lambda_1 Tv_1 + \ldots + \lambda_m Tv_m = 0 \tag{1}$$

implies that all $\lambda_i = 0$. Since T is linear,

$$T(\lambda_1 v_1 + \ldots \lambda_m v_m) = 0$$

is an equivalent statement. We also note that whenever $T(a) = 0$, then $a = 0$, since T satisfies the property:

$$T(a + v) = T(a) + T(v) = T(v)$$

So equation (1) is equivalent to

$$\lambda_1 v_1 + \ldots + \lambda_m v_m = 0$$

and conclude that all $\lambda_i = 0$. □

The contrapositive of the statement is that supposing $T \in \mathcal{L}(V, W), v_1, \ldots, v_m \in V$, if $v_1, \ldots, v_m$ are not linearly independent, then $Tv_1, \ldots, Tv_m$ are not linearly independent also.

Alternate proof:

Proof. Since $v_1, \ldots, v_m$ are not linearly independent, then we must have

$$\lambda_1 v_1 + \ldots + \lambda_m v_m = 0$$

where all $\lambda_i \neq 0$. But now, we check for linear independence of the vectors in W:

$$\begin{aligned} \sigma_1 T v_1 + \ldots + \sigma_m T v_m &= 0 \\ T(\sigma_1 v_1 + \ldots + \sigma_m v_m) &= 0 \end{aligned}$$

But we know that there are $\sigma_i \neq 0$ such that

$$\begin{aligned} \sigma_1 v_1 + \ldots + \sigma_m v_m &= 0 \\ T(0) &= 0 \end{aligned}$$

which concludes the alternate proof. $\square$