Math250aHw13

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Exercise 1: Let K be a subfield of $\mathbb{C}$ and a, b elements in $K^\times$. Show equivalence between

- $K(\sqrt{a}) = K(\sqrt{b})$.
- There exists $c \in K^\times$ with $a = bc^2$.

Proof. ($2 \to 1$) Suppose that there is a $c \in K^\times$ such that $a = bc^2$. Then $c^{-1} \in K^\times$ and

$$K(\sqrt{a}) = K(\sqrt{bc^2}) = K(c\sqrt{b}) = K(\sqrt{b})$$

($1 \to 2$) Suppose that $K(\sqrt{a}) = K(\sqrt{b})$. Then we have that

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \in K(\sqrt{a})$$

Fields have inverses, so we have

$$\frac{\sqrt{a}}{\sqrt{b}}c = 1$$

Therefore,

$$ac^2 = b \text{ or } a = b(c^{-1})^2$$

□

Exercise 4: Let E/K be a finite field extension. Prove that if $E : K$ is a prime number, E/K has no proper intermediate fields, and for each $\alpha \in E$ such that $\alpha \notin K$ we therefore have $E = K(\alpha)$.

Proof. Suppose that $K \subseteq F \subseteq E$ is a field extension. Then

$$[E : K] = [E : F][F : K]$$

Since $[E : K]$ is prime/irreducible in $\mathbb{Z}$, we have that either $[E : F]$ or $[F : K]$ is 1. Then $F \cong K$ or $F \cong E$. □

Exercise 6: Let K be an infinite field and E an extension of degree $n > 1$ over K. Show that the quotient group $E^\times/K^\times$ of the multiplicative groups of E and K is infinite.

Proof. We have the extension $K \subseteq E$. Let $n = [E : K] > 1$ and therefore, $1, \alpha_1, \dots, a_{n-1}$ be a basis for E over K. Then any element of $E^\times$ can be represented as

$$\lambda_0 + \lambda_1\alpha_1 + \cdots + \lambda_{n-1}\alpha_{n-1}$$

For $\lambda_i \in K$. We send this under the quotient map to $E^\times/K^\times$. Notice that at least one of the λ_i are not 0, because $0 \notin E^\times$. Then wlog, say we can divide through by this λ_i:

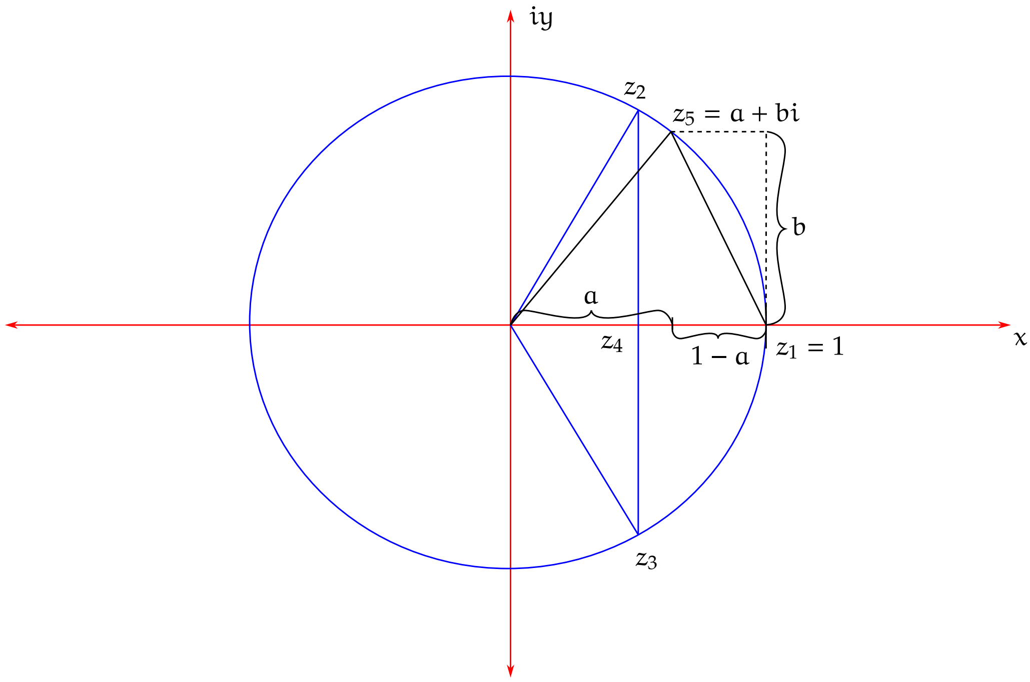
$$\left(\frac{\lambda_0}{\lambda_i} + \frac{\lambda_1}{\lambda_i}\alpha_1 + \cdots + \alpha_i + \cdots + \frac{\lambda_{n-1}}{\lambda_i}a_{n-1}\right) \cdot K^\times$$

is the image of the quotient map to $E^\times/K^\times$. Then each element of $E^\times/K^\times$ is in correspondence to a ratio between the coefficients, $\lambda_0, \dots, \lambda_{n-1}$, points in projective space. We fix an element to be 1 while varying another coordinate over elements of our field to get infinitely many points in projective space. So we are done. $\square$

Exercise 7: Let the circle of radius 1 centered at $z_1 = 1$ intersect S at z_2 and z_3. Let z_4 be the intersection of the line through z_2 and z_3 with the line through 0 and z_1. Beginning at z_1, mark off the distance $|z_4 - z_2|$ against the circle S, seven times in succession.

Does this mean that Gauss's statement (see F12 in chapter 5) that $e^{2\pi i/7} \notin \Delta\{0, 1\}$ is an error? Show that the points obtained according to the procedure above are the powers $z, z^2, \dots, z^7$ of the complex number $z = \frac{5}{8} + \frac{1}{8}\sqrt{39}i$. It follows that $65536z^7 = 65530 - 142\sqrt{39}i$.

Proof. Let $z_5 = a + bi$ be the first mark with distance $|z_4 - z_2|$ away from z_1.



We first have that $|z_4 - z_2| = \sqrt{3}/2$. This is because we have that the triangle formed by $0, z_1, z_2$ is equilateral, $z_2 - z_4$ is a perpendicular bisector, so it is the height of the triangle. We take $(\frac{1}{2})^2 + h^2 = 1$ and we get $h = \frac{\sqrt{3}}{2}$. Therefore, $||z_5|| = \frac{\sqrt{3}}{2}$. So we now have the system of equations:

$$\begin{aligned} a^2 + b^2 &= 1 \\ (1-a)^2 + b^2 &= \frac{3}{4} \end{aligned}$$

So

$$\begin{aligned} a^2 + b^2 &= 1 \\ 1 - 2a + a^2 + b^2 &= \frac{3}{4} \\ 2 - 2a &= \frac{3}{4} \\ -2a &= \frac{-5}{4} \\ a &= \frac{5}{8} \end{aligned}$$

Then using

$$a^2 + b^2 = 1$$

we have $\frac{25}{64} + b^2 = 1$, $b^2 = \frac{39}{64}$. Therefore, $b = \frac{\sqrt{39}}{8}$. So we have $z_5 = \frac{5}{8} + \frac{\sqrt{39}}{8}i$ and $65536z^7 = 65530 - 142\sqrt{39}i$. So $z^7 \neq 1$ and it is not a heptagon. $\square$