Math128aHw2

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Exercise Set 2.2

Exercise 1: Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

c. $g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{1/2}$.

Answer. If $f(p) = 0$, then

$$0 = p^4 + 2p^2 - p - 3 \implies p + 3 = p^4 + 2p^2$$

Then

$$g(p) = \left(\frac{p+3}{p^2+2}\right)^{1/2} = \left(\frac{p^4+2p^2}{p^2+2}\right)^{1/2} = (p^2)^{1/2} = p$$

Exercise 8: Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$

Answer. We first have

$$g(x) = x + hf(x)$$

and require the two conditions:

$$g(x) \in [1,2] \text{ if } x \in [1,2]$$
$$|g'(x)| \leqslant k < 1 \text{ if } x \in [1,2]$$

We first see if $g(x)$ has achieves a max or min in $(1, 2)$:

$$g'(x) = 3hx^2 - h + 1$$

and solving for the roots:

$$\begin{aligned} 3hx^2 &= h - 1 \\ x^2 &= \frac{h-1}{3h} \\ x &= \sqrt{\frac{h-1}{3h}} \end{aligned}$$

We will show that $h - 1 < 0$ and therefore, no minimum or maximum is achieved in the interval $(1, 2)$. We have that

$$\begin{aligned} g(1) &= 1 - h \\ g(2) &= 5h + 2 \end{aligned}$$

Since $g(c) \in [1,2]$, we can solve inequalities to get: $\frac{-1}{5} \leqslant h \leqslant 0$. So $h - 1 < 0$. The derivative $3hx^2 - h + 1$ is strictly decreasing on the interval $[1,2]$. So we need to check that the endpoints are $\leqslant k < 1$:

$$g'(1) = 2h + 1$$
$$g'(2) = 11h + 1$$

Let $h = \frac{-1}{10}$. Then clearly, the second condition for fixed point iteration to converge is satisfied. Here is the code:

```
function y = myfunc1(x)
    y = x^3 - x - 1;
end

function x = fixed_point(f, h, p0, tol)

x = p0;
while 1
    x = x - h * f(x);
    if abs(f(x)) < tol
        break
    end

end

x = vpa(fixed_point(@myfunc1, 1/10, 1, 0.01))
```

and the answer is 1.3225067925568798621327459841268.

Exercise 19: Let $g \in C^1[a,b]$ and p be in (a,b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so that fixed-point iteration does not converge if $p_0 \neq p$.

Answer. From $|g'(p)| > 1$, we get:

$$\left| \lim_{x \to p} \frac{g(x) - g(p)}{x - p} \right| = k > 1$$
$$\lim_{x \to p} \frac{|g(x) - g(p)|}{|x - p|} = k > 1$$

This means that for any sequence p_n that converges to p, we know that there exists an N such that for all $n > N$,

$$\left| \left| \frac{g(x) - g(p)}{x - p} \right| - k \right| < \delta$$

for some $\delta > 0$. Therefore:

$$-\delta < \left| \left| \frac{g(x) - g(p)}{x - p} \right| - k \right| < \delta$$
$$k - \delta < \left| \frac{g(x) - p}{x - p} \right| < k + \delta$$
$$(k - \delta)|x - p| < |g(x) - p|$$

If we choose δ such that $k - \delta > 1$, which is possible because $k > 1$, then we have that:

$$|x - p| < |g(x) - p|$$

where $x = p_n$. So there is a ε such that if $|x - p| < \varepsilon$, then $|x - p| < |g(x) - p|$, which completes the proof.

Exercise 20: Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

Answer. We see that if $f(x) = Ax^2 - x$, then $g(x) = x - f(x)$. Since fixed point iteration converges:

$$\lim_{x\to p} g(x) = \lim_{x\to p} x - f(x) = p - f(p) = g(p)$$

But $g(p) = p$, so $f(p) = 0$. Then $Ap^2 - p = 0, p(Ap - 1) = 0$. So $p = 0, \frac{1}{A}$. Since $g(p) \neq 0$ and therefore $p \neq 0$, $p = 1/A$.

b. Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Answer. We first require that $|g'(x)| \leqslant k < 1$ in the interval. Let us denote the interval as $[\frac{1}{A} - \delta, \frac{1}{A} + \delta]$. Then

$$g'(x) = 2 - 2Ax$$
$$g'\left(\frac{1}{A} - \delta\right) = 2A\delta$$
$$g'\left(\frac{1}{A} + \delta\right) = -2A\delta$$

Since g' is either strictly increasing or decreasing on the interval, we just require the endpoints to be bounded:

$$|2A\delta| \leqslant k < 1$$
$$|\delta| \leqslant \frac{k}{2A}$$

Now we plug in $\delta = \frac{k}{2A}$ and require that $g(c) \in [\frac{1}{A} - \delta, \frac{1}{A} + \delta]$:

$$g\left(\frac{2+k}{2A}\right) = \frac{2+k}{A} - \frac{4+4k+k^2}{4A}$$
$$= \frac{4-k^2}{4A}$$
$$g\left(\frac{2-k}{2A}\right) = \frac{2-k}{A} - \frac{4-4k+k^2}{4A}$$
$$= \frac{4-k^2}{4A}$$

g achieves its max/min at the endpoints and $1/A$. At $1/A$, $g = 1/A$which is in the interval. Now for the endpoints:

$$\frac{2-k}{2A} \leqslant \frac{4-k^2}{4A} \leqslant \frac{2+k}{2A}$$
$$4 - 2k \leqslant 4 - k^2 \leqslant 4 + 2k$$
$$2 \geqslant k \geqslant -2$$

This means that k can be anything in $(1, 0)$. We can let $k = \frac{1}{2}$, the interval for the iteration be $[\frac{3}{4A}, \frac{5}{4A}]$.

Exercise Set 2.3

Exercise 6: Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

c. $2x\cos 2x - (x-2)^2 = 0$ for $2 \leqslant x \leqslant 3$ and $3 \leqslant x \leqslant 4$.

Answer. Here is the matlab code:

```
function x = newton(f, p0, metric, tol, x_true)
    if ~exist('x_true')
        x_true = 0;
    end
    x = p0;
    syms c;
    df = matlabFunction(diff(f(c), c));
    while 1
        if metric(x, x_true, f) < tol
            break;
        end
        x = x - f(x) / df(x);
    end

end

function y = myfunc1(x)
    y = 2*x*cos(2*x) - (x-2)^2;
end

y = newton(@myfunc1, 2.5, @out_error, 1e-5)
z = newton(@myfunc1, 3.5, @out_error, 1e-5)
```

And this gives:

$$x_1 = 2.370687825747464$$
$$x_2 = 3.722112773106613$$

Exercise 8: Repeat Exercise 6 using the Secant method.

Answer. Here is the matlab code:

```
function x = secant(f, p0, tol)
    x = f(p0);
    while 1
        if abs(f(x)) < tol
            break;
        end
        x_new = f(x);
        inv_df = (x - p0) / (x_new - x);
        x = x - x_new * inv_df;
    end
end

function y = myfunc1(x)
    y = 2*x*cos(2*x) - (x-2)^2;
end

y = secant(@myfunc1, 2.5, 1e-5)
z = secant(@myfunc1, 3.5, 1e-5)
```

And this gives:

$$x_1 = 2.370686187519918$$
$$x_2 = 3.722113027769073$$

Exercise 16: The equation $x^2 - 10\cos x = 0$ has two solutions, ± 1.3793646. Use Newton's method to approximate the solutions to within 10^{-5} with the following values of p_0.

a. $p_0 = -100$

Answer.

```
function y = myfunc2(x)
    y = x^2 - 10*cos(x);
end

y = newton(@myfunc2, -100, @abs_error, 1e-5, -1.3793646)
z = newton(@myfunc2, 25, @abs_error, 1e-5, -1.3793646)
```

which gave:

$$x = -1.379370269507622$$

d. $p_0 = 25$

Answer. From the same part above, I got:

$$x = -1.379368417659703$$

Exercise Set 2.4

Exercise 2: Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

c. $\sin 3x + 3e^{-2x}\sin x - 3e^{-x}\sin 2x - e^{-3x} = 0$, for $3 \leqslant x \leqslant 4$.

Answer.

```
function x = newton(f, p0, metric, tol, x_true)
    if ~exist('x_true')
        x_true = 0;
    end
    x = p0;
    syms c;
    df = matlabFunction(diff(f(c), c));
    while 1
        if metric(x, x_true, f) < tol
            break;
        end
        x = x - f(x) / df(x);
    end

end

function y = myfunc3(x)
    y = sin(3*x) + 3*exp(-2*x)*sin(x) - 3*exp(-x)*sin(2*x) - exp(-3*x);
end

y = newton(@myfunc3, 3.5, @out_error, 1e-5)
```

which gave:

$$x = 3.141567877980774$$

Exercise 8:

a. Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to 0.

Answer. Since the limit is 0, we must show that:

$$\lim_{n\to\infty}\left|\frac{p_{n+1}}{(p_n)^2}\right| = \lambda$$

for $\lambda > 0$. Substitute:

$$\lim_{n\to\infty}\left|\frac{10^{-2^{n+1}}}{(10^{-2^n})^2}\right| = \lim_{n\to\infty}\left|\frac{10^{-2^{n+1}}}{10^{-2^{n+1}}}\right| = 1$$

So the convergence is quadratic.

b. Show that the sequence $p_n = 10^{-n^k}$ does not converge quadratically to 0 regardless of the size of the exponent $k > 1$.

Answer. The setup:

$$\lim_{n\to\infty}\left|\frac{10^{-(n+1)^k}}{10^{-2*n^k}}\right| = \lim_{n\to\infty}\left|10^{-(n+1)^k+2n^k}\right| = \lim_{n\to\infty}|10^{n^k+\dots}|$$

which diverges. So it does not converge quadratically.

Exercise 9:

a. Construct a sequence that converges to 0 of order 3.

Answer. Taking inspiration from the previous question, let $p_n = 10^{-3^n}$. This converges to 0. Also:

$$\lim_{n\to\infty}\left|\frac{10^{-3^{n+1}}}{(10^{-3^n})^3}\right| = \lim_{n\to\infty}\left|\frac{10^{-3^{n+1}}}{10^{-3^{n+1}}}\right| = 1$$

b. Suppose $\alpha > 1$. Construct a sequence that converges to 0 of order α.

Answer. We can generalize to $p_n = 10^{-\alpha^n}$

$$\lim_{n\to\infty}\left|\frac{10^{-\alpha^{n+1}}}{(10^{-\alpha^n})^\alpha}\right| = \lim_{n\to\infty}\left|\frac{10^{-\alpha^{n+1}}}{10^{-\alpha^{n+1}}}\right| = 1$$

Extra:

- The speed at which the sequence generated by an iterative method converges is called the methods ORDER of convergence. There are many types of orders of convergence: linear, superlinear, sublinear, quadratic, cubic, and so on. Discuss how a linearly convergent sequence could be accelerated.

  Answer. One idea is to run the linear method for a couple iterations before switching to a faster method, since order of convergence is measured as a limit. Another would be to make a better guess at the solution at each iteration, for example, using the previous sequence terms to approximate a 'derivative' (where p_n is itself a function wrt n) and perform a line search.

- Show that the sequence $\{p_n\}$ for $p_n = 1/n^2$, converges sublinearly to $p = 0$, in that

  $$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \infty \text{ for } \alpha > 1$$

  and that there does not exist a $0 < \lambda < 1$ such that

  $$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda$$

  How large must n be before $|p_n - p| \leqslant 5 \times 10^{-2}$?

  Answer. We have:

  $$\lim_{n\to\infty} \frac{|1/(n+1)^2|}{|1/n^2|^\alpha} = \lim_{n\to\infty} \frac{n^{2\alpha}}{(n+1)^2} = \infty$$

  since $2\alpha > 2$. If $\alpha = 1$, then the limit is 1, because $2\alpha = 2$. So there is no $0 < \lambda < 1$. Finally,

  $$\left|\frac{1}{n^2}\right| \leqslant 5 \times 10^{-2}$$

  when $n^2 \geqslant \frac{1}{5\times 10^{-2}} = \frac{100}{5} = 20$. So $n \geqslant 5$.