Math128aHw8

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Exercise Set 4.6

Exercise 6: Use Adaptive quadrature to approximate the following integrals to within 10^{-5}.

(d) $\int_0^{\pi/2} (6\cos 4x + 4\sin 6x)e^x \, dx$

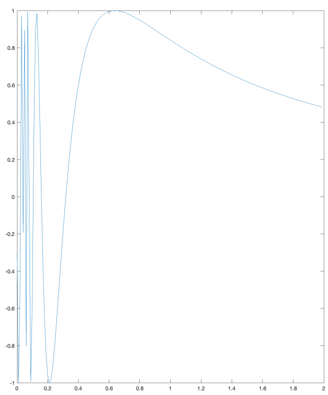
I got: 5.1138387475811616517766680276784. Here is the code:

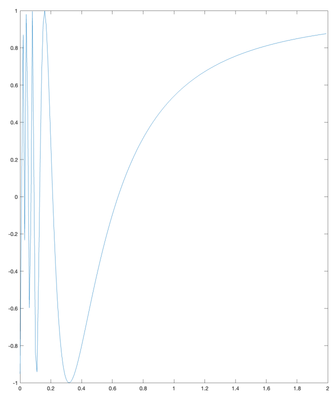
```
function APP = Simpson(f, a, b, tol, n)
APP = 0;
i = 1;
TOL = zeros(1, n); A = zeros(1, n); h = zeros(1, n); S = zeros(1, n);
TOL(i) = 10 * tol;
A(i) = a;
h(i) = (b - a) / 2;

FA = zeros(1, n); FC = zeros(1, n); FB = zeros(1, n);
FA(i) = f(a); FC(i) = f(a + h(i)); FB(i) = f(b);

% initial compute of Simpson
S(i) = h(i) * (FA(i) + 4 * FC(i) + FB(i)) / 3;

L = zeros(1, n);
L(i) = 1;

while i > 0
    % Simpson for l, r subinterval
    FD = f(A(i) + h(i) / 2);
    FE = f(A(i) + 3 * h(i) / 2);
    S1 = h(i) * (FA(i) + 4 * FD + FC(i)) / 6;
    S2 = h(i) * (FC(i) + 4 * FE + FB(i)) / 6;
    v1 = A(i); v2 = FA(i); v3 = FC(i); v4 = FB(i); v5 = h(i); v6 = TOL(i); v7 = S(i); v

    i = i - 1; % pop the node
    if abs(S1 + S2 - v7) < v6
        APP = APP + (S1 + S2);
    else
        if v8 >= n
            break
        end
        % Set left subinterval
        i = i + 1;
        A(i) = v1 + v5;
        FA(i) = v3; FC(i) = FE; FB(i) = v4;
```

```
        h(i) = v5 / 2;
        TOL(i) = v6 / 2;
        S(i) = S2;
        L(i) = v8 + 1;

        % Set right subinterval
        i = i + 1;
        A(i) = v1;
        FA(i) = v2; FC(i) = FD; FB(i) = v3;
        h(i) = h(i - 1);
        TOL(i) = TOL(i - 1);
        S(i) = S1;
        L(i) = L(i - 1);
    end
end

syms x
f(x) = (6 * cos(4 * x) + 4 * sin(6 * x)) * exp(x);

app = vpa(Simpson(f, 0, pi / 2, 10e-5, 10))
```

Exercise 9: Sketch the graphs of $\sin 1/x$ and $\cos 1/x$ on $[0, 1, 2]$. Use Adaptive quadrature to approximate the following integrals to within 10^{-3}.

Answer. For $\sin 1/x$, I got 1.3847974535053240059417414299214 and for $\cos 1/x$, I got 0.





Exercise 11: The differential equation

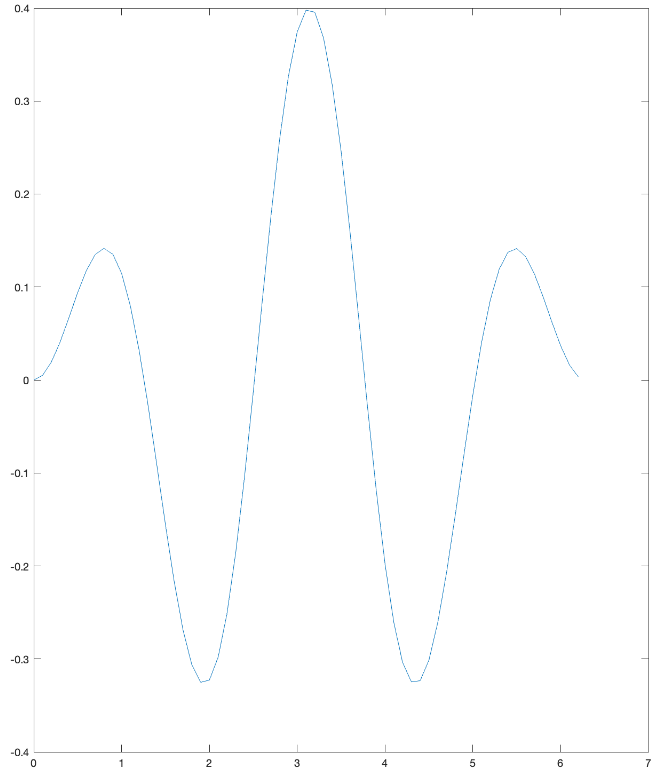
$$mu''(t) + ku(t) = F_0 \cos \omega t$$

describes a spring-mass system with mass m, spring constant k, and no applied damping. The term $F_0 \cos \omega t$ describes a periodic external force applied to the system. The solution to the equation when the system is initially at rest ($u'(0) = u(0) = 0$) is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t), \text{ where } \omega_0 = \sqrt{\frac{k}{m}} \neq \omega$$

Sketch the graph of u when $m = 1, k = 9, F_0 = 1, \omega = 2$, and $t \in [0, 2\pi]$. Approximate $\int_0^{2\pi} u(t)\, dt$ to within 10^{-4}.

Answer. Integrating yields: 0.00017441076341495540713583292051187.



Exercise 13: Let $T(a, b)$ and $T(a, \frac{a+b}{2}) + T(\frac{a+b}{2})$ be the single and double applications of the Trapezoidal rule to $\int_a^b f(x)\ dx$. Derive the relationship between

$$\left| T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right) \right|$$

and

$$\left| \int_a^b f(x)\ dx - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right) \right|$$

Answer. Using the error formula:

$$\int_a^b f(x)\ dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\mathcal{E}) = T(a,b) - \frac{h^3}{12} f''(\mathcal{E})$$

It can be seen that they differ by

$$\left| \frac{h^3}{12} f''(\mathcal{E}) \right|$$

Exercise Set 4.7

Exercise 2: Approximate the following integrals using Gaussian quadrature with $n = 2$ and compare your results to the exact values of the integrals.

(c) $\int_0^{3.5} \frac{x}{\sqrt{x^2-4}}\,dx$

Answer. I got: $-0.17681898945491190314095684640636$. Here is my code:

```
function app = gaussianQuad(f, a, b, n)
% Change of variables to integration on [-1, 1]
syms t;
CoV = ((b - a) * t + (b + a)) / 2;
g(t) = f(CoV);

L = [t, t^2 - 1/3, t^3 - 3 * t / 5, t^4 - 6 * t^2 / 7 + 3 / 35];
coeff = coeffs(L(n), "All");
r = roots(coeff);
c = zeros(size(r, 1), 1);
for i = 1:n
    h(t) = t / t;
    for j = 1:n
        if j ~= i
            h(t) = h(t) * (t - r(j)) / (r(i) - r(j));
        end
    end
    c(i) = int(h, -1, 1);
end

app = 0;
for i = 1:n
    app = app + c(i) * g(r(i));
end
app = app * (b - a) / 2;

end

syms x;
f(x) = 2 / (x^2 - 4);

app = gaussianQuad(f, 0, 0.35, 2)
```

Exercise 6: Repeat Exercise 2 with $n = 4$.

(c) $\int_0^{3.5} \frac{x}{\sqrt{x^2-4}}\,dx$

Answer. I got $-0.1768200201168954788441443888858$.

Exercise 11: Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)\,dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision three.

Answer. Evaluate the integral for $f(x) = 1, x, 3x^2, x^3$:

$$\int_{-1}^{1} 1 \, dx = 2$$
$$\int_{-1}^{1} x \, dx = 0$$
$$\int_{-1}^{1} 3x^2 \, dx = x^3 \Big|_{-1}^{1} = 2$$
$$\int_{-1}^{1} x^3 \, dx = 0$$

Now evaluate the expression

$$af(-1) + bf(1) + cf'(-1) + df'(1)$$

for $f(x) = 1, x, 3x^2, x^3$:

$$\begin{aligned} f(-1) &= 1, -1, 3, -1 \\ f(1) &= 1, 1, 3, 1 \\ f'(-1) &= 0, 1, -6, 3 \\ f'(1) &= 0, 1, 6, 3 \end{aligned}$$

So we have the linear system:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 3 & 3 & -6 & 6 \\ -1 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Solving this gives:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1/3 \\ -1/3 \end{bmatrix}$$

Exercise 14: Show that the formula $Q(P) = \sum_{i=1}^{n} c_i P(x_i)$ cannot have degree of precision greater than $2n - 1$ regardless of the choice of $c_1, \ldots, c_n$ and $x_1, \ldots, x_n$.

Exercise Set 4.8

Section 4.8: Problems 1bc, 2bc, 17