Math185Hw2

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Exercise 1: Split the polynomial $x^4 + 1$ into four linear factors (with complex coefficients). Then, by combining pairs of complex-conjugate factors, find a splitting of the same into two quadratic real factors.

Proof. We have that

$$x^4 = -1 = e^{i\pi}, e^{3\pi i}, e^{5i\pi}, e^{7i\pi}$$

and therefore, the roots are

$$e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}}, e^{\frac{5i\pi}{4}}, e^{\frac{7i\pi}{4}}$$

Now we multiply the conjugates together:

$$(x - e^{\frac{i\pi}{4}})(x - e^{\frac{7i\pi}{4}}) = x^2 - (e^{\frac{i\pi}{4}} + e^{\frac{7i\pi}{4}})x + 1 = x^2 - \sqrt{2}x + 1$$

while

$$(x - e^{\frac{3i\pi}{4}})(x - e^{\frac{5i\pi}{4}}) = x^2 - (e^{\frac{3i\pi}{4}} + e^{\frac{5i\pi}{4}})x + 1 = x^2 + \sqrt{2} + 1$$

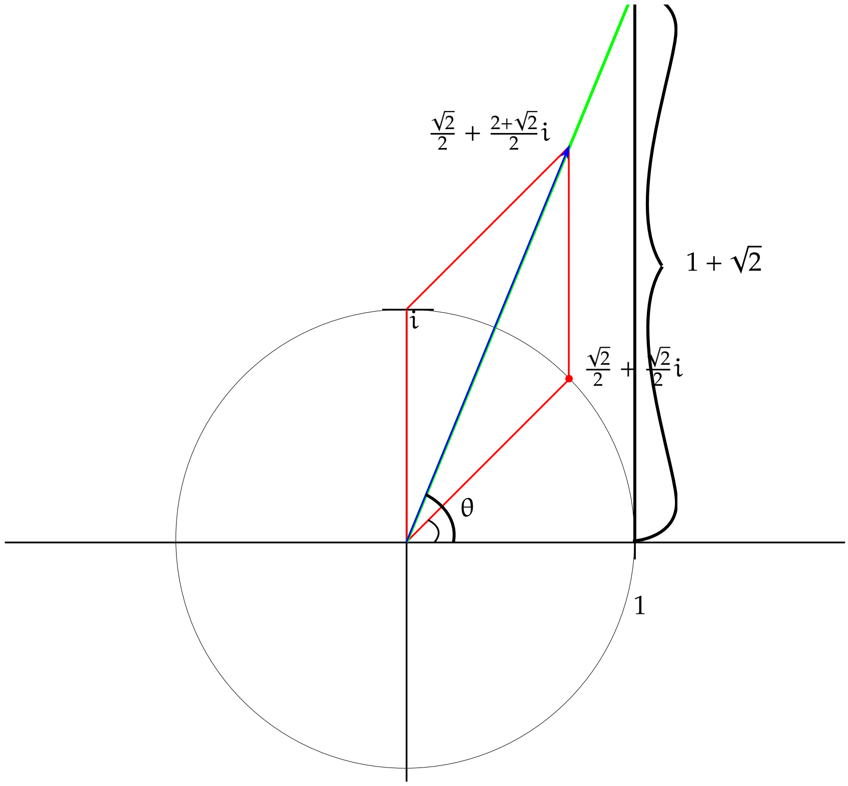
So the factorization is

$$(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$$

□

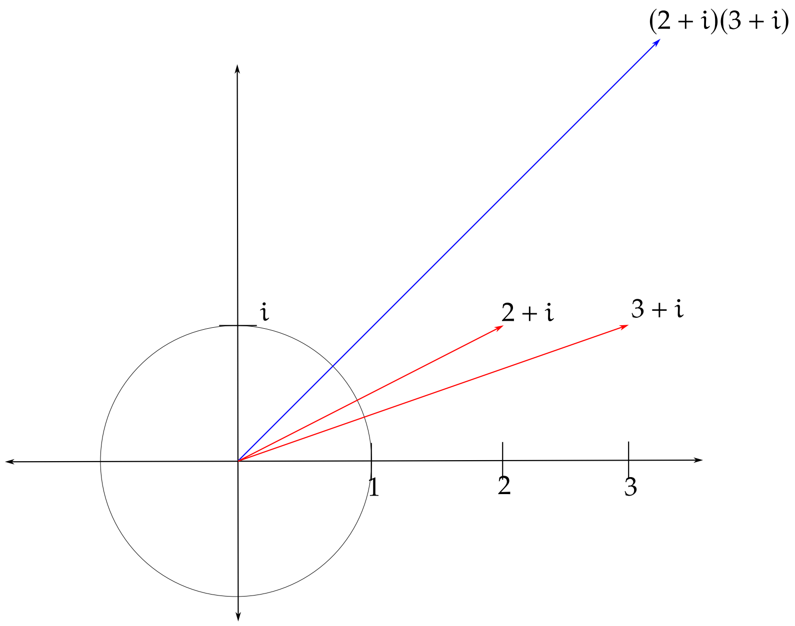
Exercise 2: Represent $\exp\left(\frac{\pi i}{4}\right), \exp\left(\frac{\pi i}{2}\right)$ and their sum in the complex plane. By expressing each of them as $x + iy$, deduce that $\tan\frac{3\pi}{8} = 1 + \sqrt{2}$. By considering $(2 + i)(3 + i)$, show that $\frac{\pi}{4} = \tan^{-1} 1/2 + \tan^{-1} 1/3$.

Proof. The vectors $\exp\left(\frac{i\pi}{4}\right)$ and $\exp\left(\frac{i\pi}{2}\right)$ are $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and i respectively. Their sum is $\frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i$. Drawn on the complex plane:



So we see that by similar triangles, the ratio of the y to x value of the vector $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}+2}{2}i$ is the same as the ratio of 1 to $1+\sqrt{2}$. So $\tan\theta = 1+\sqrt{2}$. We know that the blue vector bisects the rhombus's angle. The angle is therefore $\frac{\frac{\pi}{4}+\frac{\pi}{2}}{2} = \frac{3\pi}{8}$. Therefore, $\tan\frac{3\pi}{8} = 1+\sqrt{2}$.

For the second part, we can draw a picture:



Notice that $(2+i)(3+i) = 5(1+i) = 5e^{i\frac{\pi}{4}}$. Then we know that the angle $\frac{\pi}{4}$ is the sum of

$arg(3+i)+arg(2+i)$. Using $\tan^{-1}$, we can get these:

$$\begin{aligned} arg(3+i) &= \tan^{-1} 1/3 \\ arg(2+i) &= \tan^{-1} 1/2 \end{aligned}$$

Therefore, $\frac{\pi}{4} = \tan^{-1} 1/3 + \tan^{-1} 1/2$. □

Exercise 3: Show that, in polar coordinates (r, θ), the Cauchy-Riemann equations for the differentiable function $(r, \theta) \mapsto u + iv$ read as follows, when $r > 0$.

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof. Suppose we have $(r, \theta) \to r(\cos\theta + i\sin\theta)$. Then splitting it into the real and imaginary component, we have $u(r, \theta) = r\cos\theta$ and $v(r, \theta) = r\sin\theta$. Now compute the Jacobian:

$$\begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

From this, we see that

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

and

$$-r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta}$$

□

Exercise 4: Which of the following functions are holomorphic functions of $z = x + iy = r(\cos\theta + i\sin\theta)$?

$$e^{-y}(\cos x + i\sin x);\ \cos x - i\sin y;\ r^3 + 3i\theta;\ re^{r\cos\theta}(\cos(\theta + r\sin\theta) + i\sin(\theta + r\sin\theta))$$

Proof. Check:

- $e^{-y}(\cos x + i\sin x)$. This is holomorphic because it is a composition of holomorphic functions: $z \mapsto iz \mapsto \exp(iz)$. In other words:

  $$x + iy \mapsto -y + ix \mapsto e^{-y}(\cos x + i\sin x)$$

- $\cos x - i\sin y$. This not holomorphic. We see that the Jacobian is $\begin{bmatrix} -\sin x & 0 \\ 0 & -\cos x \end{bmatrix}$.

- $r^3 + 3i\theta$. By exercise 3, we need that

  $$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

  Let $u(r, \theta) = r^3$ and $v(r, \theta) = 3\theta$. Then

  $$\frac{\partial u}{\partial r} = 3r^2, \frac{\partial u}{\partial \theta} = 0, \frac{\partial v}{\partial r} = 0, \frac{\partial v}{\partial \theta} = 3$$

  We see that it is holomorphic for $r = 1$.

- $re^{r\cos\theta}(\cos(\theta + r\sin\theta) + i\sin(\theta + r\sin\theta))$. We have

  $$\begin{aligned} re^{r\cos\theta}(\cos(\theta + r\sin\theta) + i\sin(\theta + r\sin\theta)) &= re^{r\cos\theta}(e^{i\theta + ri\sin\theta}) \\ &= re^{i\theta}e^{re^{i\theta}} \\ &= ze^{z} \end{aligned}$$

  The product of holomorphic functions is holomorphic.

$\square$

Exercise 5: Let the function f be holomorphic in an open disk $D \subseteq \mathbb{C}$. Show that each of the following conditions forces f to be constant.

(a) $f' \equiv 0$ in D

(b) f is real-valued in D

Proof. If f is real-valued, then $f = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ where $v(x,y) = 0$. So the Jacobian is $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ 0 & 0 \end{bmatrix}$. By the CR equations, we see that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} = 0$. So $u(x,y)$ has no x or y terms. So $u(x,y) = c$ where c is a constant. $\square$

(c) $|f|$ is constant in D

(d) $\arg f$ is constant in D

(e) $\overline{f(z)}$ is also holomorphic.

Exercise 6: Find all the complex solutions of the following equations (log is the multi-valued function):

(a) $\log(z) = \frac{\pi i}{2}$

Answer. We have that $\log(z) = \log(re^{i\theta}) = \log(r) + i\theta = \frac{\pi i}{2}$. So

$$\log(r) = 0$$

and $r = 1$. Now

$$i\theta = \frac{\pi i}{2}$$

This gives $\theta = \frac{\pi}{2}$. So the solution is $e^{i\pi/2}$.

(b) $\exp(z) = \pi i$

Answer. Expand out $\exp(z)$ in terms of x, y:

$$\exp(z) = e^x(\cos y + i \sin y) = \pi i$$

Then $e^x = \pi$, as $\sqrt{\|\pi i\|} = \pi$. So $x = \log(\pi)$. Now since there is not real component, $\cos y = 0$. Also, $\sin y = 1$. This is true for $y = \frac{\pi}{2}$. So $z = \log(\pi) + i\frac{\pi}{2}$.

(c) $\sin z = \cos z$

Answer. Using the formula in terms of e:

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2}$$

So

$$e^{iz} - e^{-iz} = ie^{iz} + ie^{-iz}$$

Reordering:

$$\begin{aligned} e^{iz} - ie^{iz} &= ie^{-iz} + e^{-iz} \\ (1-i)e^{iz} &= (i+1)e^{-iz} \\ \frac{e^{iz}}{e^{-iz}} &= \frac{(i+1)}{(1-i)} \\ e^{2iz} &= \frac{2i}{2} \\ e^{2iz} &= i \end{aligned}$$

Using $\exp(2ix - 2y) = e^{-2y}(\cos 2x + i\sin 2x)$. Since $\sqrt{\|i\|} = 1$, we have $y = 0$. Then this says that $\cos z = \sin z$ at real values of z. These are known as $\frac{\pi}{4} + k\pi$ for $k \in \mathbb{Z}$.

(d) $\overline{\exp(iz)} = \exp(i\overline{z})$

Answer. Expand both sides:

$$\begin{aligned}\overline{\exp(iz)} &= \overline{\exp(-y + ix)}\\ &= \overline{e^{-y}(\cos x + i\sin x)}\\ &= e^{-y}(\cos x - i\sin x)\end{aligned}$$

and

$$\begin{aligned}\exp(i\overline{z}) &= \exp(i(x - iy))\\ &= \exp(y + ix)\\ &= e^{y}(\cos x + i\sin x)\end{aligned}$$

Now set them equal and simplify:

$$\begin{aligned}e^{-y}(\cos x - i\sin x) &= e^{y}(\cos x + i\sin x)\\ \cos x - i\sin x &= e^{2y}(\cos x + i\sin x)\end{aligned}$$

Let $w = \cos x + i\sin x$. Then

$$\overline{w} = e^{2y}w$$

Since $w \neq 0$, we can divide:

$$e^{2y} = \frac{\overline{w}}{w} = 1$$

So $y = 0$. So there are only real solutions. Going back to one of the equations above, we continue:

$$\begin{aligned}\cos x - i\sin x &= \cos x + i\sin x\\ 0 &= 2i\sin x\\ 0 &= \sin x\end{aligned}$$

which we know has solutions at $x = k\pi$ for $k \in \mathbb{Z}$.

Exercise 7: Establish the identities ($x = \mathrm{Re}(z), y = \mathrm{Im}(z)$)

$$\begin{aligned}|\cos z|^2 &= \cos^2 x + \sinh^2 y\\ |\sin z|^2 &= \sin^2 x + \sinh^2 y\end{aligned}$$

Proof. Using the fact that

$$\begin{aligned}\cos x + iy &= \cos x\cosh y - i\sin x\sinh y\\ \sin x + iy &= \sin x\cosh y + i\cos x\sinh y\end{aligned}$$

So

$$\begin{aligned}\|\cos x + iy\| &= \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y\\ &= \cos^2 x\cosh^2 y + (1 - \cos^2 x)\sinh^2 y\\ &= \sinh^2 x + \cos^2 x(\cosh^2 y - \sinh^2 y)\end{aligned}$$

Now using the formulas:

$$\begin{aligned}\cos iy &= \cosh y\\ \sin iy &= i\sinh y\end{aligned}$$

Now we see that $\cos^2 iy + \sin^2 iy = \cosh^2 y - \sinh^2 y$. Plugging it in above, we get

$$\|\cos x + iy\| = \sinh^2 x + \cos^2 x$$

as desired.

For the other one, we use the other formula shown at the top:

$$\begin{aligned}
\|\sin x + iy\| &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\
&= \sinh^2 y + \sin^2 x(\cosh^2 y - \sinh^2 y) \\
&= \sinh^2 y + \sinh^2 x
\end{aligned}$$

which concludes the proof. □