Math172Ex11

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Exercise 6: Let $G = (V, E)$ be a simple connected graph with n vertices. Number its vertices using numbers from $[n]$ and let $x_1, \ldots, x_n$ be n real variables. Define the function $f_G(x_1, \ldots, x_n)$ as follows:

$$f_G(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{(i,j) \in E} x_i x_j$$

In other words, each vertex i contributes x_i^2 and each edge (i, j) contributes $-x_i x_j$ to this function $f_G(x_1, \ldots, x_n)$ is called *non-negative definite* if $f_G(x_1, \ldots, x_n) \geqslant 0$ for all real numbers $x_1, \ldots, x_n$, and it is called *positive definite* if in addition $f_G(x_1, \ldots, x_n) > 0$ for all choice of $x_1, \ldots, x_n$ with the only exception $x_1 = x_2 = \cdots = x_n = 0$.

- Classify, up to isomorphism, all connected graphs G such that f_G is positive definite.
- Classify, up to isomorphism, all connected graphs G such that f_G is non-negative definite.

Proof. Using the formula:

$$(x_1 + x_2 + \cdots + x_n)^2 \geqslant 0$$

we get

$$x_1^2 + \cdots + x_n^2 \geqslant \sum_{1 \leqslant i < j \leqslant n} -2x_i x_j$$

□