Homework 13
1.3-1.6

a)

p	q	$p \vee q$	$\neg p$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

→ Tautology

b)

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T
F	T	T	T	T	T	T
T	F	T	F	T	F	T
F	F	T	T	T	T	T
T	T	F	T	F	F	T
F	T	F	T	F	F	T
T	F	F	F	T	F	T
F	F	F	T	T	T	T

c)

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

d)

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r) \wedge (q \to r)$	$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$
T	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	T	T	F	T
T	T	F	T	F	F	F	T
T	F	F	T	F	~~F~~ T	F	T
F	T	F	T	T	F	F	T
F	F	F	F	T	T	F	T

$\therefore$ Contradiction.

(14) Show that if hypothesis is true $\rightarrow$ conclusion is true.

Supp. $\neg p \wedge (p \vee q)$

$(\neg p \wedge p) \vee (\neg p \wedge q)$

$\therefore (\neg p \wedge q)$

q is true ✓

Supp. $(p \rightarrow q) \wedge (q \rightarrow r)$

Supp. p

$\therefore q \wedge r$

$\therefore p \rightarrow r$ ✓

Supp. $p \wedge (p \rightarrow q)$

$\therefore q$ ✓

Supp. $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$

Case 1: P is true. From $p \rightarrow r$, r is true

Case 2: q is true. From $q \rightarrow r$, r is true

Case 3: $p \wedge q$ is true. From $(p \rightarrow r) \wedge (q \rightarrow r)$, r is true

$\therefore r$ is true ✓

16) a) $[\neg p \wedge (p \vee q)] \to q \equiv [(\neg p \wedge p) \vee (\neg p \wedge q)] \to q \equiv (\neg p \wedge q) \to q$
$\equiv p \vee (\neg q \vee q) \equiv T$

b) $[(p \to q) \wedge (q \to r)] \to (p \to r) \equiv (p \to r) \to (p \to r) \equiv T$

c) $[p \wedge (p \to q)] \to q \equiv q \to q \equiv T$

d) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$
$[(p \vee q) \wedge [(p \vee q) \to r]] \to r \equiv r \to r \equiv T$

18) a) $C(x) \wedge D(x) \wedge F(x) \to \boxed{\exists x (C(x) \wedge F(x) \wedge D(x))}$

b) $\forall x (C(x) \vee D(x) \vee F(x))$

c) $\exists x (C(x) \wedge F(x) \wedge \neg D(x))$

d) $\neg \exists x (C(x) \wedge F(x) \wedge D(x))$

e) $\exists x (C(x)) \wedge \exists x (P(x)) \wedge \exists x (F(x))$

16) a) $\exists x (x^2 = 2)$
$P(x) \equiv x^2 = 2$
$P(\sqrt{2}) : 2 = 2 \to T$
$\bigvee_{i=0}^{\infty} P(x_i) \to T$
$\exists x (x^2 = 2) \to T$

b) $\exists x (x^2 = -1)$
$x^2 = -1$
$x = \pm\sqrt{-1} = \pm i$
False

c) $\forall x (x^2 + 2 \geq 1)$
$x^2 \geq 0$
$x^2 + 2 \geq 2 > 1$
~~False~~ True

d) $\forall x (x^2 \neq x)$
$\exists x (x^2 = x)$
$x^2 = x$
$x^2 - x = 0$
$x(x-1) = 0$
$x = 0, 1$
True $\longrightarrow$ False

36) a) $\neg \forall x (-2 < x < 3) \equiv \cancel{\exists x (-2 \geq x \geq 3)}$
$\cancel{\equiv \exists x} \equiv \neg \forall x ((-2 < x) \wedge (x < 3))$
$\equiv \exists x ((-2 \geq x) \vee (x \geq 3))$

b) $\neg \forall x (0 \leq x < 5)$
$\equiv \neg \forall x ((0 \leq x) \wedge (x < 5))$
$\equiv \exists x ((0 > x) \vee (x \geq 5))$

38) a) $\forall x (x^2 \neq x)$
$x = 0, 1 \to 1^2 = 1$
$0^2 = 0$

b) $\forall x (x^2 \neq 2)$
$x = \pm\sqrt{2} \quad (\pm\sqrt{2})^2 = 2$

c) $\forall x (|x| > 0)$
$x = 0 \to |0| = 0$

46) $\forall x (P(x) \leftrightarrow Q(x)) \equiv \forall x ((P(x) \to Q(x)) \wedge (P(x) \leftarrow Q(x)))$
$\equiv \forall x (P(x) \to Q(x)) \wedge \forall x (P(x) \leftarrow Q(x))$
$\hookrightarrow$ Show that $\forall x (P(x) \to Q(x)) \equiv \forall x P(x) \to \forall x Q(x)$

Suppose $\forall x (P(x) \leftrightarrow Q(x))$ is true

$P(x_i)$	$Q(x_i)$
T	T
F	F

☆ x_i is arbitrary

These truth values make $\forall x P(x) \to \forall x Q(x)$ true also

Suppose $\forall x P(x) \leftrightarrow \forall x Q(x)$ is true

$\forall x P(x)$	$\forall x Q(x)$
T	T
F	F

These truth values make $\forall x (P(x) \leftrightarrow Q(x))$ true also.

$\therefore$ ☆ Implication is shown both ways & statements are equivalent.

(52) Suppose $(\forall x\, P(x) \vee \forall x\, Q(x)) \equiv (\forall x (P(x) \vee Q(x)))$.
Then $(\forall x (P(x) \vee Q(x))) \rightarrow (\forall x\, P(x) \vee \forall x\, Q(x))$.
Let the hypothesis be true. In the case that $P(a)$ is true and ~~Q(b) is false~~ $Q(a)$ is false and $P(b)$ is ~~true~~ false while $Q(b)$ is true, $\forall x P(x) \vee \forall x Q(x)$ is false. For arbitrary values of a and b. Since the implication $(\forall x (P(x) \vee Q(x))) \rightarrow (\forall x\, P(x) \vee \forall x\, Q(x))$ is false, the statements are not equivalent.

(20) a) ~~$\forall x \in \mathbb{Z}($~~ ~~$\forall a,b \in \mathbb{Z}($~~ $\forall a,b \in \mathbb{N}(-a \cdot -b \geq 0)$
b) ~~$\forall a,b($~~ $\forall a,b \in \mathbb{Z}((a \geq 0 \wedge b \geq 0) \rightarrow \frac{a+b}{2} \geq 0)$
c) $\exists x,y \in \mathbb{N}(-x-(-y) \geq 0)$
d) $\forall x,y \in \mathbb{Z}(|x+y| \leq |x|+|y|)$

(24) a) For every real number, there exists another such that ~~then~~ their sum is equal to the original #.
b) The ~~differen~~ ~~distance~~ difference between a positive # and a negative one is always positive.
d) Two numbers are not equal to 0 if and only if their product is not 0

(44) $\forall a,b,c \in \mathbb{R}\, \exists x_1,x_2 \in \mathbb{R}(ax_1^2+bx_1^2+c=0 \wedge ax_2^2+bx_2+c=0 \wedge$
$(\exists x_3 \in \mathbb{R}(ax_3^2+bx_3+c=0) \rightarrow (x_3=x_1) \vee (x_3=x_2))$

#10

Hw1.6 a) p: I play hockey

q: I am sore

r: I use the whirlpool

Propositions

$p \to q$ $\quad q \to r$

~~$\neg q$~~

$\neg r$

$$\begin{array}{l} q \to r \\ \neg r \\ \hline \therefore \neg q \end{array} \qquad \begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

*used Modus ~~ponens~~ tollens

Conclusions: I am not sore and I did not play hockey.

b) ~~p: I worked last Monday~~

~~q: I worked last Friday~~

~~r: It was sunny on Tuesday~~

~~s: It was partly sunny on Friday~~

~~Propositions~~

~~$(p \vee q) \to ($~~

p(x): I worked on ~~day~~ x

q(x): It was sunny on x

r(x): It was partly sunny on x

Propositions

~~$\forall x(p(x) \to (q(x) \vee r(x)))$~~

$p(\text{Monday}) \vee p(\text{Friday})$

$\neg q(\text{Tuesday})$

$\neg r(\text{Friday})$

$(p(x) \to q(x)) \vee (p(x) \to r(x))$

$(p(\text{Friday}) \to q(\text{Friday})) \vee (p(\text{Friday}) \to r(\text{Friday}))$ substitution

Since $\neg r(\text{Friday})$

$(p(\text{Friday}) \to q(\text{Friday})) \vee \neg p(\text{Friday})$ Proposition Contrapositive (Modus Tollens)

$\equiv q(\text{Friday}) \vee \neg p(\text{Friday})$

$\equiv$ $p(\text{Friday}) \to q(\text{Friday})$

$p(\text{Monday}) \to (q(\text{Monday}) \vee r(\text{Monday}))$

$p(\text{Monday}) \vee p(\text{Friday})$

$\therefore q(\text{Friday}) \vee q(\text{Monday}) \vee r(\text{Monday})$

c) I(x): x is an insect

S(x): x has six legs

Premises

$\forall x(I(x) \to S(x))$

$I(\text{Dragonfly})$

$\neg S(\text{Spider})$

$$\begin{array}{l} I(x) \to S(x) \\ I(\text{Dragonfly}) \\ \hline \therefore S(\text{Dragonfly}) \end{array} \qquad \begin{array}{l} I(x) \to S(x) \\ \neg S(\text{Spider}) \\ \hline \therefore \neg I(\text{Spider}) \end{array}$$

Modus Ponens $\qquad$ Modus Tollens

14 a) Universal modus ponens → ~~Universal~~ existential generalization

b) Universal modus ponens → Universal generalization

c) Universal modus ponens → Existential generalization

d) Existential instantiation → Universal modus ponens → Existential generalization a)

16 $\forall x(E(x) \rightarrow D(x))$ Premise — Argument is correct

$E(Mia) \rightarrow D(Mia)$ Universal Instantiation

$\neg D(Mia)$ Premise

$\therefore \neg E(Mia)$ Modus Tollens

b) $\forall x(C(x) \rightarrow F(x))$ Premise

$C(\text{Isaac's Car}) \rightarrow F(\text{Isaac's Car})$ Universal Instantiation

$\neg C(\text{Isaac's Car})$ Premise

$\therefore \neg F(\text{Isaac's Car})$ Fallacy of denying hypothesis

↳ Argument is incorrect.

c) ~~$\forall m(Q(m))$~~ Premise

$Q(\text{Eight Men Out})$ ~~Universal Instantiation~~ Premise

$\forall m(A(m) \rightarrow Q(m))$ Premise

$Q(\text{Eight Men Out})$ Premise

$A(\text{Eight Men Out}) \rightarrow Q(\text{Eight Men Out})$ Universal Instantiation

$\therefore A(\text{Eight Men Out})$ Fallacy of affirming the conclusion

↳ Argument is incorrect

d) $\forall x(L(x) \rightarrow T(x))$ Premise

$L(\text{Hamilton})$ Premise

$L(\text{Hamilton}) \rightarrow T(\text{Hamilton})$ Universal Instantiation

$\therefore T(\text{Hamilton})$ Modus ponens

↳ Argument is correct

18 The step from $\exists s\, S(s, Max)$ to $S(Max, Max)$ is incorrect. If we were to use existential instantiation, then $s \neq Max$ since a person cannot be shorter than themself.

24 Error at step 3 and 5. Simplification cannot be used on a disjunction.

Error at step 4 and 6. Since c is not arbitrary, universal generalization can't be used.

~~26~~

28)
1. $\forall x((\neg P(x) \land Q(x)) \to R(x))$ Premise
2. $(\neg P(c) \land Q(c)) \to R(c)$ Universal instantiation (c is arbitrary)
3. $\neg R(c)$ Premise
4. $P(c) \lor \neg Q(c)$ ~~Mor~~ Modus Tollens
5. $\forall x(P(x) \lor Q(x))$ Premise
6. $P(c) \lor Q(c)$ Universal Instantiation (c is arbitrary)
7. $(P(c) \lor \neg Q(c)) \land (P(c) \lor Q(c))$ From (4 and 6)
8. $P(c) \lor P(c)$ Resolution
9. $P(c)$ Equivalent to (8)
10. $\neg R(c) \to P(c)$ From (3 and 9)
11. $\forall x(\neg R(x) \to P(x))$ Universal generalization

Hw 1.7 1)

Definition: An odd integer n can be written in the form 2k+1 for some integer k.

~~$\exists k(2k+1=n)$ n is an integer~~

$\forall n(\text{odd}(n) \leftrightarrow \exists k(2k+1=n))$

If m is an odd integer, $m = 2j+1$ for some integer j.

~~$\exists j(2j+1=m)$ m is an integer~~

$\forall m(\text{odd}(m) \leftrightarrow \exists j(2j+1=m))$

Observe: $m+n = (2k+1)+(2j+1)$
$= 2k+2j+2$
$= 2(k+j+1)$

Definition: An even integer i can be written as $i=2p$ for some p integer.

$\forall i(\text{even}(i) \leftrightarrow \exists p(2p=i))$

Let $k+j+1 = h$
Since $m+n = 2h$
Therefore, even(m+n) which is what was desired.

Scanned with CamScanner

(8) Premise: n is a perfect square: $\forall(x)(\text{perfectsquare}(x) \to \exists y\,(x = y^2))$

Prop A: The product of two perfect squares is a perfect square:

$$a = b^2,\ c = d^2 \to ac = b^2 d^2 = (bd)^2$$

- With $n \geq 0$, $(n+1)^2$ is the next greatest square from n^2

~~From Prop A,~~

Suppose for contradiction $n+2$ is also a perfect square.

By Prop A, $n(n+2)$ is also a perfect square.

- But Observe: $n^2 < n(n+2) < (n+1)^2$

$$n^2 \leq n^2 + 2n < n^2 + 2n + 1$$

Case 1: $n = 0$: $n+2 \Rightarrow 2 = (\sqrt{2})^2$, $\sqrt{2}$ is not an integer, so contradiction.

Case 2: $n > 0$: $n^2 < n^2 + 2n < n^2 + 2n + 1$

- Since $n^2 + 2n + 1$ is the next greatest square after n^2, $n^2 + 2n$ is not a perfect square. Contradiction.

Which is what was desired.

(2) ~~Ex1~~

$\forall x,y((\text{rational}(x) \land x \neq 0 \land \text{irrational}(y)) \to \text{irrational}(xy))$

Def: A rational number x can be written as the division of two integers a, b, $a \neq 0, b \neq 0$

$$x = \frac{a}{b}$$

Suppose for contradiction that their product is rational. Then for irrational y,

$$xy = \frac{m}{n}$$

For integers $m, n \neq 0$. Observe,

$$xy = \frac{m}{n} \to \frac{a}{b}y = \frac{m}{n} \to y = \frac{mb}{na}$$

Since mb, na are integers that are non-zero, y is rational. Contradiction as desired. The product of a nonzero rational # and an irrational number is irrational.

(18) $\forall m,n(\text{even}(mn) \to (\text{even}(m) \lor \text{even}(n)))$

Proof by contrapositive, $\forall m,n((\text{odd}(m) \land \text{odd}(n)) \to \text{odd}(mn))$

Suppose two integers m, n are odd. Then there exists two integers k, j such that

$$m = 2k+1 \quad \text{and} \quad n = 2j+1$$

It follows that their product is

$$mn = (2k+1)(2j+1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$$

Since $2kj+k+j$ is an integer, by definition of an odd #: $\forall x(\text{odd}(x) \to \exists y(2y+1=x))$ that mn is odd as desired.

(26) Chosen(25) $\rightarrow \exists x_1, x_2, x_3 (x_1 \neq x_2 \neq x_3 \wedge x_1, x_2, x_3 \in M_i)$

M_i for $1 \leq i \leq 12$ ~~are~~ is the set of days for month i.

Suppose for contradiction that less than 3 days fall in the same month. Denote the number of days that fall in month i as a_i. We know that $0 \leq a_i \leq 2$, so the total # of days we have chosen is

$$0 \leq a_1 + a_2 + \ldots + a_{12} \leq 24$$

for $1 \leq i \leq 12$. But the premise ~~is~~ that 25 days were chosen and $25 \notin [0, 24]$. Contradiction as desired.

(28) $\forall n (n > 0 \rightarrow (\text{even}(n) \leftrightarrow (7n+4)\text{even})$

Let n be arbitrary and greater than 0.

Definition: $\forall x (\text{even}(x) \rightarrow \exists y (2y = x))$

We can write n in the form of

$$n = 2a$$

Observe that

$$7n+4 = 7(2a)+4 = 2 \cdot (7a+2)$$

Since $7a+2$ is an integer, $7n+4$ is even. Therefore,

$\forall n (n > 0 \rightarrow (\text{even}(n) \rightarrow (7n+4)\text{even})$

Now suppose $7n+4$ is even

$$7n+4 = 2b$$

for some integer b. So

$$7n = 2b - 4 \rightarrow (2(3)+1)n = 2(b-2)$$

Prop: ~~The product of two odd integers is odd.~~

If $m, n \in \mathbb{Z}$ and mn is even, then either m is even or n is even.

So for $m = (2(3)+1)$, $n = n$, and $mn = 2(b-2)$,

$\neg \text{even}(m)$, therefore, n is even:

$\forall n (n > 0 \rightarrow (\text{even}(n) \leftarrow (7n+4)\text{even}))$

as desired.