Math185Hw3

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Exercise 1: Let f be a holomorphic function on an open set $D \subseteq \mathbb{C}$. Assuming that f is twice continuously differentiable, show that the complex derivative df/dz is also holomorphic in D.

Proof. Using the definition:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

and that

$$f = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$$

We can compute the real and imaginary components of $\frac{\partial f}{\partial z}$:

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\begin{bmatrix} u_x \\ v_x \end{bmatrix} - i\begin{bmatrix} u_y \\ v_y \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\begin{bmatrix} u_x \\ v_x \end{bmatrix} + \begin{bmatrix} v_y \\ -u_y \end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix} u_x + v_y \\ v_x - u_y \end{bmatrix}
\end{aligned}$$

Now we check CR:

$$J = \begin{bmatrix} u_{xx} + v_{xy} & u_{xy} + v_{yy} \\ v_{xx} - u_{xy} & v_{xy} - u_{yy} \end{bmatrix}$$

Since f is holomorphic, $u_x = v_y$, $u_y = -v_x$. So:

$$u_{xx} + v_{xy} = v_{xy} + v_{xy} = v_{xy} - u_{yy}$$

and

$$u_{xy} + v_{yy} = u_{xy} + u_{xy} = -v_{xx} + u_{xy} = -(v_{xx} - u_{xy})$$

so the new partials satisfy the CR equations and $\frac{\partial f}{\partial z}$ is holomorphic. $\square$

Exercise 2: If u is twice continuously differentiable in $D \subseteq \mathbb{C}$, show that the complex function $u_x - i \cdot u_y$ is holomorphic in D if and only if u is harmonic.

Proof. ($\longrightarrow$) Suppose that $u_x - i \cdot u_y$ is holomorphic. Then by CR,

$$\begin{aligned} u_{xx} &= -u_{yy} \\ u_{xy} &= -(-u_{xy}) \end{aligned}$$

From the top equation, we immediately get: $u_{xx} + u_{yy} = 0$, so u is harmonic.

($\longleftarrow$) Suppose that u is harmonic. Then $u_{xx} + u_{yy} = 0$. Check CR:

$$J = \begin{bmatrix} u_{xx} & u_{xy} \\ -u_{xy} & -u_{yy} \end{bmatrix}$$

Indeed, it satisfies CR as $u_{xy} = -(-u_{xy})$ and $u_{xx} = -u_{yy}$ from the harmonic constraint.
$\square$

Exercise 3: Show that a (homogeneous) linear isomorphism from $\mathbb{R}^n$ to itself which preserves orthogonality of vectors (that is, sends a pair of mutually orthogonal vectors to a mutually orthogonal pair) must be the composite of an orthogonal transformation with a scaling. In particular, it preserves absolute values of angles.

Hint: Check what happens to the standard basis, and to the diagonals of the squares built on vectors in the standard basis.

Remark: Reflections also preserve the absolute values of angles, but in $\mathbb{R}^2$, reflection about a line changes the sign of the angles so is not conformal in the sense we defined.

Proof. Suppose that our isomorphism is given by a matrix:

$$A = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{bmatrix}$$

Then looking at the image of e_i, e_j, we see that

$$Ae_i \cdot Ae_j = 0$$

because orthogonality is preserved. So that means that v_i, v_j are orthogonal, as $Ae_i = v_i$, $Ae_j = v_j$. Now we need to show that each v_i has the same norm. Consider orthogonal vectors $w_{i,j,-}, w_{i,j,+}$, where

$$w_{i,j,-} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix}, w_{i,j,-} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Where $w_{i,j,-}$ has all 0's except the i, j-th entry, and 1 in the i-th and -1 in the j-th entry. Also, say that $w_{i,j,+}$ has all 0's except the i, j-th entry, which will contain 1 instead. These vectors are orthogonal. Furthermore,

$$Aw_{i,j,-} \cdot Aw_{i,j,+} = 0$$

because orthogonality is preserved. So $Aw_{i,j,-} = v_i - v_j$ and $Aw_{i,j,+} = v_i + v_j$. So

$$(v_i - v_j) \cdot (v_i + v_j) = \|v_i\| - \|v_j\| = 0$$

And therefore,

$$\|v_i\| = \|v_j\|$$

which concludes the proof. □

Exercise 4: Show that the image under $z \mapsto \cos z$ of a horizontal line is an ellipse, and the image of a vertical line is a hyperbola. There are some exceptional cases: discuss those.

Proof. (Horizontal line) Any point on a horizontal line is of the form $x + ic$ for c constant and $x \in \mathbb{R}$. So if $z = x + ic$, then

$$\cos z = \cos x + ic = \cos x \cosh c - i \sin x \sinh c$$

Since $\cosh c, \sinh c$ are real number, constants, let:

$$\alpha = \cosh c$$
$$\beta = \sinh c$$

Then

$$\cos z = \alpha \cos x - i\beta \sin x$$

Let u denote the real axis and v denote the imaginary. Then we have that:

$$u = \alpha \cos x$$
$$v = -\beta \sin x$$

and therefore the relation:

$$\frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} = 1$$

which is the equation of an ellipse.

(Vertical line) Any point on the vertical line is of the form $c + iy$ for c constant and $y \in \mathbb{R}$. So if $z = c + iy$,

$$\begin{aligned}\cos z = \cos c + iy &= \cos c \cosh y - i \sin c \sinh y \\ &= \alpha \frac{e^y + e^{-y}}{2} - i\beta \frac{e^y - e^{-y}}{2}\end{aligned}$$

Now setting $u = \alpha \frac{e^y + e^{-y}}{2}$ and $v = -\beta \frac{e^y - e^{-y}}{2}$, after some computation:

$$e^{-y} = \frac{2v}{\beta} + e^y$$

and solving the quadratic using $y = \ln(x)$:

$$\begin{aligned}1 &= \frac{2v}{\beta} e^y + e^{2y} \\ 0 &= e^{2y} + \frac{2v}{\beta} e^y - 1 \\ &= x^2 + \frac{2v}{\beta} x - 1 \\ x &= -\frac{v}{\beta} + \sqrt{\frac{v^2}{\beta^2} + 1}\end{aligned}$$

Plugging this into u:

$$\begin{aligned}
u &= \frac{\alpha}{2}\left(e^{y} + e^{-y}\right) \\
&= \frac{\alpha}{2}\left(e^{\ln(x)} + \frac{2v}{\beta} + e^{\ln(x)}\right) \\
&= \alpha\sqrt{\frac{v^2}{\beta^2} + 1} \\
\frac{u}{\alpha} &= \sqrt{\frac{v^2}{\beta^2} + 1} \\
\frac{u^2}{\alpha^2} - \frac{v^2}{\beta^2} &= 1
\end{aligned}$$

which is the equation of a hyperbola.

The other cases are when $\cos c = 0, \sin c = 0, \sinh c = 0$, or $\cosh c = 0$. The first two cases concern vertical lines, while the second two are tied to horizontal lines. If $\cos c = 0$ or $\sin c = 0$, then $c \in \frac{\pi}{4}\mathbb{Z}$. On the other hand, $\cosh c$ is never 0 from the formula $\frac{e^x + e^{-x}}{2}$. We have $\sinh c$ is 0 when $c = 0$. □

Exercise 5: Compute $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$.

Answer. First partial fraction decomposition:

$$\frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2}$$

Then compute:

$$\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} = 1$$

Exercise 6: Does the series $\sum_{n=1}^{\infty} \frac{i^n}{n}$ converge? Is the convergence absolute? Explain why or why not.

Answer. The series can be expanded to:

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = i - \frac{1}{2} - \frac{i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} - \cdots$$

And collecting the real and imaginary parts:

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} + i\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

Since the series for the real component $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges and the series for the imaginary component $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ converges by the alternating series test, the series converges.

Exercise 7: For which real values of x do the following power series converge? (You must also check the borderline cases which the ratio test fails to settle.)

(a) $\sum_{n\geqslant 0} \frac{x^n}{n^2}$;

Answer. Using the ratio test, we require:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x(n^2)}{(n+1)^2}\right| < 1$$

So $|x| < \frac{(n+1)^2}{n^2}$. So for any x, we just take a large enough n. So this converges for all $x \in \mathbb{R}$.

(b) $\sum_{n\geqslant 0} \frac{x^n}{2^n}$;

Answer. Using the ratio test, we require:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x}{2}\right| < 1 \implies |x| < 2$$

Now check the boundary cases:

$$\sum_{n\geqslant 0} 1^n = \text{diverges}$$
$$\sum_{n\geqslant 0} (-1)^n = \text{diverges}$$

So the interval of convergence is $(-2, 2)$.

(c) $\sum_{n\geqslant 0} \frac{x^n}{\sqrt{n!}}$;

Answer. Using the ratio test, we require that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|x\sqrt{\frac{1}{n+1}}\right| < 1$$

So $|x| < \sqrt{n+1}$. So if we choose a large enough n, we have that the condition is satisfied. So it converges for all $x \in \mathbb{R}$.

(d) $\sum_{n\geqslant 0} \sqrt{n!} \cdot x^n$.

Answer. Using the ratio test, we require that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|x\sqrt{n}\right| < 1$$

So $|x| < 1/\sqrt{n}$. We see that for any x we choose, for n sufficiently large, the inequality does not hold. So it does not converge for any x except for $x = 0$.

Exercise 8: Find the radii of convergence of $\sum n^n z^n$ and of $\sum n^2 z^n$. Sum the second series in closed form.

Proof. The radius of convergence of $\sum n^n z^n$, using the geometric series test is when

$$|nz| < 1$$

So $|z| < \frac{1}{n}$. This holds for all n only when $z = 0$. So the radius of convergence is 0. For $\sum n^2 z^n$, using the ratio test, we require:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z(n+1)^2}{n^2}\right| < 1$$

So $|z| < \frac{n^2}{(n+1)^2}$. Since the limit of the RHS is 1, the radius of convergence is 1. It does not converge on the boundary points. We can get the closed form by computation:

$$S = \sum n^2 z^n = z + 4z^2 + 9z^3 + 16z^5 + \cdots$$

Subtract:

$$S - zS = z + 3z^2 + 5z^3 + 7z^4 + \cdots$$

Subtract again:

$$S - zS - z(S - zS) = z + 2z^2 + 2z^3 + 2z^4 + \cdots$$

So

$$S - zS - zS + z^2 S = z + 2z^2 \sum_{n \geqslant 0} z^n$$

So we solve for S:

$$\begin{aligned} S(1 - 2z + z^2) &= z + \frac{2z^2}{1 - z} \\ S(1 - 2z + z^2) &= \frac{z + z^2}{1 - z} \\ S &= \frac{z(1 + z)}{(1 - z)(1 - 2z + z^2)} \\ S &= \frac{z(1 + z)}{(1 - z)^3} \end{aligned}$$

And we are done. $\square$