Review 1

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Review

1. **Groups and Homomorphisms**: Binary operations, Definition of groups, Order of Group, Aberlian Group, Subgroups

   Results:

   (a) Identity is unique, Inverses are unique, $(a^{-1})^{-1} = a$, $(ab)^{-1} = b^{-1}a^{-1}$.

   (b) Subgroup criteria I and II

   (c) Subgroups of $(\mathbb{Z}, +)$ are in $n\mathbb{Z}$.

2. **Homomorphisms**: Functions, Composition, injective, surjective, bijective, definition of homomorphisms, isomorphisms, image, and kernel.

   Results:

   (a) $f(e_G) = e_H$, $f(a^{-1}) = f(a)^{-1}$

   (b) Compositions of homomorphisms is a homomorphism

   (c) Inverse of an Isomorphism is an Isomorphsim

   (d) Image and kernel are subgroups

   (e) If $a \in G$, $k \in \ker f$, then $ak^{-1}a \in \ker f$.

   (f) Injective iff $\ker f = \{e\}$

   (g) Surjective iff $\operatorname{Im}\{f\} = H$ where $f : G \to H$

3. **Cyclic Groups**: Definiton of a cyclic, C_n, order of an element, exponent of a group,

   Results:

   (a) $\forall a \in G : \operatorname{ord}(a) = |\langle a \rangle|$

   (b) Cyclic $\to$ Abelian.

4. **Dihedral Groups**: Definition of dihedral groups D_{2n} (Symmetries of a regular n-gon).

5. **Direct Product of Groups**: Definition of direct products
   (a) $C_m \times C_n \cong C_{nm}$ iff $\gcd(n, m) = 1$
   (b) Direct Product Theorem
6. **Symmetric groups**: Permutations, Symmetric group of a set X, Row and cycle notation, $k-$cycles and transpositions, cycle type/shape, sign of permutations, Alternating subgroups.

   Results:
   (a) Sym X is a group
   (b) Disjoint cycles commute
   (c) Any $\sigma \in S_n$ is uniquely a product of disjoint cycles.
   (d) $\text{ord}(\sigma)$, $\sigma \in S_n$ is the lcm of the lengths in the disjoint cycle representation of σ.
   (e) Every $\lambda \in S_n$ is a product of transpositions.
   (f) The number of transpositions is always either even or odd in the result above.
   (g) $\forall n \geq 2$, $\text{sgn} : S_n \to \{\pm 1\}$ is a homomorphism.
   (h) σ is an even permutation $\text{sgn}(\sigma) = 1$ iff the number of cycles of even length is even.
   (i) Every subgroup of S_n contains either no odd permutations or exactly half.
7. **Lagrange**: Cosets, Partitions of a set, Index of subgroups, equivalence relations and equivalence classes, Euler totient funciton

   Results:
   (a) Lagrange's Theorem
   (b) Left cosets partition G and all cosets have the same size
   (c) $\text{ord}(a) \mid |G|$
   (d) $\forall a \in G : a^{|G|} = e$
   (e) Groups of prime ordere are cyclic
   (f) Fermat-Euler Theorem
   (g) Every group of order 4 is either C_4 or $C_2 \times C_2$
   (h) Any group of order 6 is either cyclic or dihedral.
8. **Quotient Groups**: Normal subgroups, quotient groups, simple groups

   Results:

(a) Index of 2 implies that the group is a normal subgroup

(b) Subgroups of abellian groups are normal

(c) Kernals are normal

(d) If $K \triangleleft G$, left cosets of K form a group

(e) Natrual projection $G \to G/K$ is a surjective group homomorphism

(f) Quotient of cyclic is cyclic

(g) Isomorphism theorem: $G/\ker f \cong \mathrm{Im}\{f\}$

(h) Any cyclic group is $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$

9. **Group Actions**: Group action, Kernel of action, faithful action, orbit, stabilizer, transitive action, conjugation of an element, conjugacy classes, centralizers, center, normalizer.

   Results:

   (a) Criteria for group actions

   (b) Stabilizer of X is a subgroup

   (c) Orbits partition your set X

   (d) Orbit-Stablizer Theorem: $|\mathrm{Orb}(x)||\mathrm{Stab}(x)| = |G|$

   (e) Important Actions: Left regular action, Conjugation action, Cayley's Theorem, Normal subgroups are unions of conjugacy classes, G acts on its subgroups

   (f) Stabilizers of elements in the same orbit are conjugate

   (g) Cauchy's Theorem