$\operatorname{Res}_{z=i} e^{iz}/(z^2+1)^5 \longrightarrow$ Laurent Expand around $z=i$

$\hookrightarrow w = z-i$, $h(z) = \dfrac{e^{iz}}{(z+i)^5}$

$$h(w+i) = \frac{e^{i(w+i)}}{(w+2i)^5} = \frac{e^{-1}}{(2i)^5}\cdot\frac{e^{iw}}{(1+\frac{w}{2i})^5}$$

$$= \frac{e^{-1}}{32i}\cdot\frac{e^{iw}}{(1+\frac{w}{2i})^5}$$

$$= \frac{e^{-1}}{32i}\left(\sum_{n\geq 0}\frac{(iw)^n}{n!}\right)\left(\sum_{m\geq 0}\left(-\frac{w}{2i}\right)^m\right)^5$$

$\downarrow$ Extract coefficients

$\sum_{m\geq 0}\left(-\frac{w}{2i}\right)^m \longrightarrow 1, -1, 1, -1, 1$

$\left(\sum_{m\geq 0}\left(-\frac{w}{2i}\right)^m\right)^2 \longrightarrow 1, -2, 3, -4, 5$

$\left(\sum_{m\geq 0}\left(\frac{-w}{2i}\right)^m\right)^3 \longrightarrow 1, -3, 6, -10, 15$

$\left(\sum_{m\geq 0}\left(\frac{-w}{2i}\right)^m\right)^4 \longrightarrow 1, -4, 10, -20, 35$

$\left(\sum_{m\geq 0}\left(\frac{-w}{2i}\right)^m\right)^5 \longrightarrow 1, -5, 15, -35, 70$

Get w^4 coeff from $\dfrac{e^{-1}}{32i}\left(1 + iw + \dfrac{(iw)^2}{2!} + \dfrac{(iw)^3}{3!} + \dfrac{(iw)^4}{4!}\right)$

$$\left(1 - 5\left(\frac{w}{2i}\right) + 15\left(\frac{w}{2i}\right)^2 - 35\left(\frac{w}{2i}\right)^3 + 70\left(\frac{w}{2i}\right)^4\right)$$

$$\rightarrow \frac{e^{-1}}{32i}\left(\frac{(iw)^4}{4!} - 5\left(\frac{w}{2i}\right)\frac{(iw)^3}{3!} + 15\left(\frac{w}{2i}\right)^2\frac{(iw)^2}{2!} - 35\left(\frac{w}{2i}\right)^3 iw + 70\left(\frac{w}{2i}\right)^4\right)$$

$$\rightarrow \frac{e^{-1}}{32i}\left(\frac{1}{24} + \frac{5}{12} + \frac{15}{8} + \frac{35}{8} + \frac{70}{16}\right)$$

$$\rightarrow \frac{e^{-1}}{32i}\left(\frac{1}{24} + \frac{10}{24} + \frac{45}{24} + \frac{105}{24} + \frac{105}{24}\right) = \frac{e^{-1}}{32i}\left(\frac{266}{24}\right)$$

$$= \frac{133}{e\,384i} = \operatorname{Res}_{z=i} e^{iz}/(z^2+1)^5$$

$$\int_{H_R}\frac{e^{iz}}{(z^2+1)^5}dz = \int_{R}\frac{e^{iz}}{(z^2+1)^5}dz \rightarrow 0$$

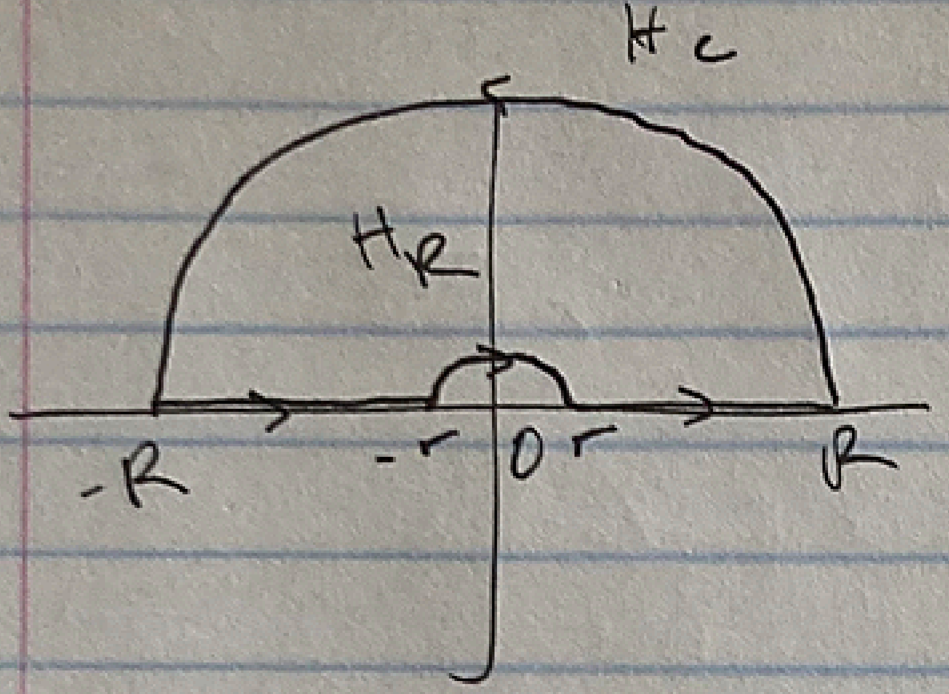
$$+ \int_{-R}^{R}\frac{e^{ix}}{(x^2+1)^5}dx$$

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+1)^5}dx = 2\pi \cancel{i}\,\frac{133}{e\,384\cancel{i}} = \frac{133\pi}{e\,192}$$

$$2\int_0^{\infty}\frac{\cos x}{(x^2+1)^5}dx =$$

$$\int_0^{\infty}\frac{\cos x}{(x^2+1)^5}dx = \boxed{\frac{133\pi}{384e}}$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$



$$(i\sin z + \cos z)^2 = \sin^2 z + 2i\cos z\sin z + \cos^2 z$$

$$\cancel{e^{2iz} = 1 + 2\cos z\sin z}$$

$$\cancel{e^{2iz} - 1 = 2\cos z\sin z}$$

$$\cancel{= \sin 2z/2}$$

$$e^{2iz} = 1 + 2i\cos z\sin z - 2\sin^2 z$$

$$\cancel{= 1 + 2\sin z(-\sin z + i\cos z)}$$

$$e^{2iz} - 1 = 2i\cos z\sin z - 2\sin^2 z$$

$$e^{2iz} - 1 = 2i\sin 2z/2 - 2\sin^2 z$$

$$\frac{e^{2iz}-1}{z^2} = \frac{2i\sin 2z/2}{z^2} - \frac{2\sin^2 z}{z^2}$$

$$\int_{H_R}\frac{e^{2iz}-1}{z^2}\,dz = \int_{H_R}\frac{2i\sin 2z/2}{z^2}\,dz - 2\int_{H_R}\frac{\sin^2 z}{z^2}\,dz$$

$$\operatorname{Res}_{z=0}\frac{e^{2iz}-1}{z^2}$$

$$\operatorname{Res}_{z=0}\frac{\sum\frac{(2iz)^n}{n!} - 1}{z^2} = \operatorname{Res}_{z=0}\frac{\left(1 + 2iz - \frac{4z^2}{2!} + \frac{(2iz)^3}{3!} + \dots\right)}{z^2}$$

$$= 2i$$

$$\int_{H_R}\frac{e^{2iz}-1}{z^2}\,dz = 2\pi i\cdot 2i = -4\pi$$

$$2i\int_{H_R}\frac{\sin 2z/2}{z^2}\,dz = \operatorname{Res}_{z=0}\frac{\sin 2z/2}{z^2} = \operatorname{Res}_{z=0}\frac{\left(\frac{z}{2} - \frac{(z/2)^3}{3!} + \frac{(z/2)^5}{5!}\dots\right)}{z^2}$$

$$= \frac{1}{2}$$

$$2i\cdot 2\pi i\cdot\frac{1}{2} = -2\pi = -2\pi$$

$$-2\int_{H_R}\frac{\sin^2 z}{z^2}\,dz = -4\pi + 2\pi = -2\pi$$

$$\int_{H_R}\frac{\sin^2 z}{z^2}\,dz = \pi = \int_r^\infty\frac{\sin^2 x}{x^2}\,dx + \int_{H_C}\frac{\sin^2 z}{z^2}\,dz \to 0$$

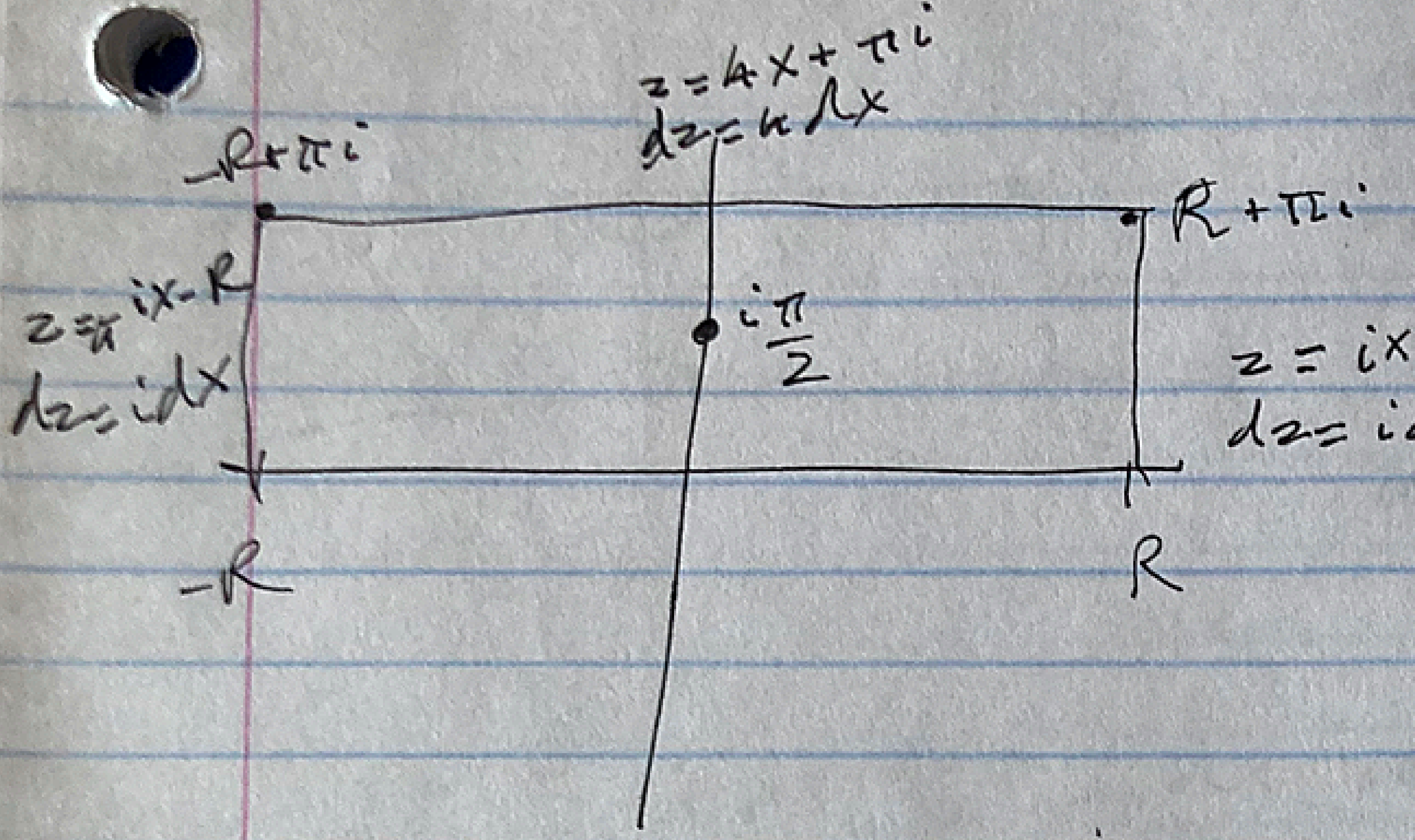
$$+ \int_{-\infty}^{-r}\frac{\sin^2 x}{x^2}\,dx + \int_{H_r}\frac{\sin^2 z}{z^2}\,dz$$

$$2\int_0^\infty\frac{\sin^2 x}{x^2}\,dx + \int_{H_r}\frac{\sin^2 z}{z^2}\,dz = \pi$$

$= 0$ since Taylor exp has even powers

$$\boxed{\int_0^\infty\frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}}$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$
$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$



$$\int_{Rect} \frac{\cos pz}{\cosh z}\,dz = \int_{R}^{R+\pi i} \frac{\cos pz}{\cosh z}\,dz = i\int_0^{\pi} \frac{\cos p(ix+R)}{\cosh ix+R}\,dx \longrightarrow 0$$
$$+\int_{R+\pi i}^{-R+\pi i} \frac{\cos pz}{\cosh z}\,dz = -\int_{-R}^{R} \frac{\cos p(\cancel{x}+\pi i)}{\cosh \cancel{x}+\pi i}\,dx$$
$$+\int_{-R+\pi i}^{-R} \frac{\cos pz}{\cosh z}\,dz = -i\int_0^{\pi} \frac{\cos p(ix-R)}{\cosh ix-R}\,dx \longrightarrow 0$$
$$+\int_{-R}^{R} \frac{\cos px}{\cosh x}\,dx = 2\int_0^{R} \frac{\cos px}{\cosh x}\,dx$$

$$\operatorname{Res}_{z=\frac{i\pi}{2}} \frac{\cos pz}{\cosh z} = \lim_{z\to\frac{i\pi}{2}} \frac{\left(z-\frac{i\pi}{2}\right)\cos pz}{\cos iz} = \frac{0}{0}$$
$$= \lim_{z\to\frac{i\pi}{2}} \frac{\cos pz - p\sin z\cdot\left(z-\frac{i\pi}{2}\right)}{-i\sin iz} = \frac{i\cosh\frac{p\pi}{2}}{\sin -\frac{\pi}{2}} = -i\cosh\frac{p\pi}{2}$$

$$\int_{Rect} \frac{\cos pz}{\cosh z}\,dz = 2\pi i\cdot \operatorname{Res}_{z=\frac{i\pi}{2}} \frac{\cos pz}{\cosh z} = 2\pi\cosh\frac{p\pi}{2}$$

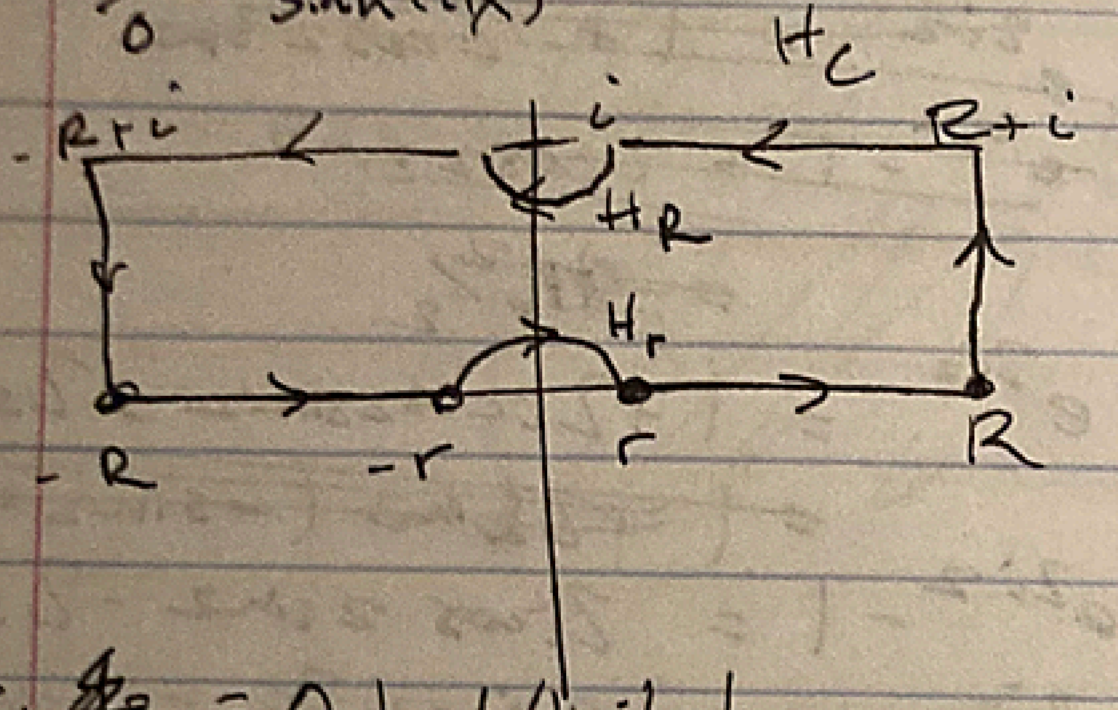
$$-\int_{-R}^{R} \frac{\cos p(x+\pi i)}{\cosh(x+\pi i)} = -\int_{-R}^{R} \frac{\cos px\cos p\pi i \cancel{+\sin px\sin p\pi i}}{\cos ix\cos -\pi \cancel{+\sin ix\sin -\pi}} = \int_{-R}^{R} \frac{\cos px\cosh p\pi}{\cosh x}\,dx$$

$$2\pi\cosh\frac{p\pi}{2} = (2+2\cosh p\pi)\int_0^{R} \frac{\cos px}{\cosh x}\,dx$$
$$\frac{\pi\cosh\frac{p\pi}{2}}{1+\cosh p\pi} = \int_0^{R} \frac{\cos px}{\cosh x}\,dx$$

Using $\frac{1}{2}\theta$: $\cosh\theta/2 = \sqrt{\frac{1+\cosh\theta}{2}} \rightarrow 2\cosh^2\theta/2 = 1+\cosh\theta$

$$\frac{\pi\cosh p\pi/2}{2\cosh^2 p\pi/2} = \int_0^{R} \frac{\cos px}{\cosh x}\,dx$$
$$\boxed{\frac{\pi}{2\cosh p\pi/2}} =$$

$$\int_0^\infty \frac{x\,dx}{\sinh(\pi x)}\,dx$$



$$\int_{H_C} \frac{z}{\sinh(\pi z)}\,dz$$
$$= \int_{H_r} \frac{z}{\sinh(\pi z)}\,dz$$
$$+ 2\int_r^R \frac{x}{\sinh(\pi x)}\,dx$$
$$+ \cancel{\int_R^{R+i} \frac{z}{\sinh(\pi z)}\,dz} \to 0$$
$$+ \int_{R+i}^{r+i} \frac{z}{\sinh(\pi z)}\,dz$$
$$+ \cancel{\int_{H_R} \frac{z}{\sinh(\pi z)}\,dz}$$
$$+ \int_{-r+i}^{-R+i} \frac{z}{\sinh(\pi z)}\,dz$$
$$+ \cancel{\int_{-R+i}^{-R} \frac{z}{\sinh(\pi z)}\,dz} \to 0$$

$$1 = 4\int_r^R \frac{x}{\sinh \pi x}\,dx$$

$$\boxed{\frac{1}{4} = \int_0^\infty \frac{x}{\sinh \pi x}\,dx}$$

$\operatorname{Res}_{z=0} \frac{z}{\sinh(\pi z)}\,dz = 0$, by L'Hôpital

$$\operatorname{Res}_{z=i} \frac{z}{\sinh(\pi z)} = \frac{i}{\pi}$$

$$\lim_{z\to i} \frac{z}{\sinh(i\pi z)} = \lim_{z\to i} \frac{1}{i\pi \cos(i\pi z)} = -\frac{1}{i\pi}$$

$$\text{LHS: } \int_{H_C} \frac{z}{\sinh(\pi z)} = \frac{2\pi i\left(\frac{i}{\pi}\right)}{2} = 1$$

$$\cancel{\lim_{R\to 0} \int_{H_R} \frac{z}{\sinh(\pi z)}\,dz \le \int_{H_R} \left|\frac{z}{\sinh(\pi z)}\right| |dz|}$$
$$\cancel{\le \left|\frac{z}{\sinh(\pi z)}\right| |r\pi|}$$
$$\cancel{\le \frac{r}{\left|\frac{e^{r\pi}-e^{-\pi r}}{2}\right|} |r\pi|} \to 0$$

$$\text{L'Hôpital} \to \lim_{r\to 0} \frac{2\pi r}{\pi \cosh(\pi r)} = \frac{2\pi r}{\left|\frac{e^{\pi r}+e^{-\pi r}}{2}\right|} = 1$$

$$\int_{R+i}^{r+i} \frac{z}{\sinh(\pi z)}\,dz = \int_R^r \frac{x+i}{\sinh(\pi x + i\pi)}\,dx = \int_R^r \frac{x+i}{\sinh i\pi x - \pi}\,dx = \int_R^r \frac{x+i}{\sin i\pi x \cos(\pi) - \cos i\pi x \sin \pi}\,dx$$
$$= -\int_R^r \frac{x+i}{\sinh \pi x}$$

$z = x+i$
$dz = dx$

$$\int_{-r+i}^{-R+i} \frac{z}{\sinh(\pi z)}\,dz = \int_{-r}^{-R} \frac{x+i}{\sinh(\pi x + i\pi)}\,dx = \int_{-r}^{-R} \frac{x+i}{\sin i\pi x - \pi}\,dx = \int_{-r}^{-R} \frac{x+i}{\sin i\pi x \cos \pi - \cos i\pi x \sin \pi}\,dx$$

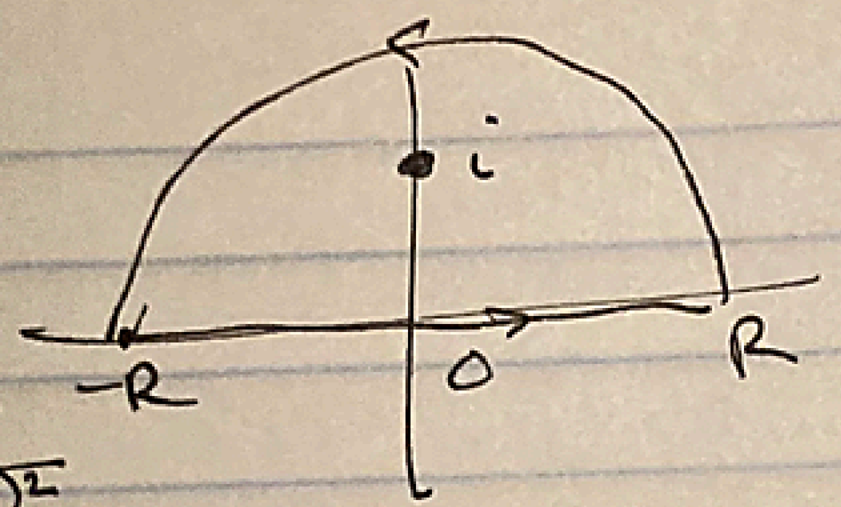
$z = x+i$
$dz = dx$

$$= -\int_{-r}^{-R} \frac{x+i}{\sinh \pi x}$$
$$= \int_{-R}^{-r} \frac{x+i}{\sinh \pi x}\,dx$$
$$= \int_R^r \frac{-x+i}{\sinh \pi x}\,dx$$

$+$

$$= -2\int_R^r \frac{x}{\sinh \pi x}\,dx$$
$$= 2\int_r^R \frac{x}{\sinh \pi x}\,dx$$

$$\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx$$



$$\int_{H_c} \frac{z^\alpha}{(z^2+1)^2}\,dz = \int_{H_R} \frac{z^\alpha}{(z^2+1)^2} + \int_{-R}^{R} \frac{x^\alpha}{(x^2+1)^2}\,dx$$

$$h(z) = \frac{z^\alpha}{(z+i)^2}$$

$$h'(z) = \frac{(z+i)^2 \alpha z^{\alpha-1} - z^\alpha 2(z+i)}{(z+i)^4}$$

$$\operatorname{Res}_{z=i} \frac{z^\alpha}{(z^2+1)^2} = h'(i) = \frac{-4\alpha z^{\alpha-1} - z^\alpha 4i}{16} = \frac{-\alpha z^{\alpha-1} - z^\alpha i}{4}$$

$$= \frac{z^{\alpha-1}(-zi - \alpha)}{4}$$

$$= \frac{i^{\alpha-1}(1-\alpha)}{4}$$

$\cancel{i^\alpha = e^{i\pi/2}}$

$$\int_{H_c} \frac{z^\alpha}{(z^2+1)^2} = 2\pi i \cdot \frac{i^{\alpha-1}(1-\alpha)}{4} = \frac{\pi i^\alpha (1-\alpha)}{2}$$

$$(-1)^\alpha = e^{i\pi\alpha}$$

$\int_{H_R} \frac{z^\alpha}{(z^2+1)^2} \to 0$ by magnitude check.

$$\int_{-R}^{R} \frac{x^\alpha}{(x^2+1)^2}\,dx \to \int_{-\infty}^{\infty} \frac{x^\alpha}{(x^2+1)^2}\,dx = -\int_0^{\infty} \frac{(-1)^\alpha x^\alpha}{(x^2+1)^2} + \int_0^\infty \frac{x^\alpha}{(x^2+1)^2}$$

$$= (-e^{i\pi\alpha} + 1)\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}$$

$$\frac{\pi i^\alpha (1-\alpha)}{2} = \frac{\pi e^{i\pi/2\,\alpha}(1-\alpha)}{2} = \cancel{\frac{\pi(1-\alpha)}{2\cos\frac{\pi}{2}\alpha - 2i\sin\frac{\pi}{2}\alpha}}$$

$$\int_0^\infty \frac{x^\alpha}{(x^2+1)^2} = \frac{\pi e^{i\pi/2\,\alpha}(1-\alpha)}{2(-e^{i\pi\alpha}+1)}$$

$$= \frac{\pi(1-\alpha)}{2(-e^{i\pi\alpha - i\pi/2\,\alpha} + e^{i\pi/2\,\alpha})}$$

$$= \frac{\pi(1-\alpha)}{2(-e^{\frac{i\pi}{2}\alpha} + e^{i\pi/2\,\alpha})} = \frac{\pi(1-\alpha)}{4\cos\left(\frac{\pi}{2}\alpha\right)}$$

$$\oint_C \frac{z^k f'(z)}{f(z)}\,dz$$

f holomorphic $\to \exists$ Taylor exp around any $s_i \in \mathbb{C}$.

Let $T_f(s_i)$ be Taylor exp of f around s_i, and $T_f^m(s_i)$ be the poly such that $T_f(s_i) = (x-s_i)^m T_f^m(s_i)$

Calculate $\operatorname{Res}_{z=s_i}$

$$\frac{z^k f'(z)}{f(z)} = \frac{h(z)}{(z-s_i)^{m_i}}$$ where $h(z)$ is holomorphic.

If $f(z) = T_f(s_i) = (\cancel{z}-s_i)^m T_f^m(s_i)$,

$$f'(z) = m(\cancel{z}-s_i)^{m-1} T_f^m(s_i) + (\cancel{z}-s_i)^m \left(T_f^m(s_i)\right)'$$

So $$h(z) = \frac{z^k\left(m(\cancel{z}-s_i)^{m-1} T_f^m(s_i) + (\cancel{z}-s_i)^m\left(T_f^m(s_i)\right)'\right)}{T_f^m(s_i)}$$

$$= z^k\left[m(\cancel{z}-s_i)^{m-1} + \underbrace{\frac{(\cancel{z}-s_i)^m\left(T_f^m(s_i)\right)'}{T_f^m(s_i)}}\right]$$

We want $h^{m_i-1}(s_i)$. This means that

$$\frac{(z-s_i)^m\left(T_f^m(s_i)\right)'}{T_f^m(s_i)} \to 0$$

since exponent of $(z-s_i)^m$ is too large.

So we have $h^{m-1}(s_i) = \left(z^k m(z-s_i)^{m-1}\right)^{''''\cdots\, m-1 \text{ times}}\Big|_{z=s_i}$

$$= z^k \cdot m!$$

Then $$\operatorname{Res}_{z=s_i} = \frac{1}{(m-1)!} h^{m-1}(s_i) = \frac{s_i^k \cdot m!}{(m-1)!} = m \cdot s_i^k$$

So $$\oint_C \frac{z^k f'(z)}{f(z)}\,dz = 2\pi i\left(m_1 s_1^k + \dots + m_n s_n^k\right)$$

where $s_1, \dots, s_n$ distinct, and $m_1, \dots, m_n$ are multiplicities of $s_1, \dots, s_n$ respectively.