Math104Hw6

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Exercise 1: Prove that $\sum \frac{n^2}{2^n}$ converges.

Proof. We need to find when $2^n > n^4$. Let $n = 2^m$:

$$2^{2^m} > 2^{4m}$$

which is true when $2^m > 4m$ so $2^{m-2} > m$. So we require $m \geqslant 5$ or $n \geqslant 32$. So for $n > 32$, $\frac{n^2}{n^4} = \frac{1}{n^2} > \frac{n^2}{2^n}$. So if we start our sum at 32, $|\frac{n^2}{2^n}| = \frac{n^2}{2^n} \leqslant \frac{1}{n^2}$ so by the comparison test, our series is cauchy and converges. □

Exercise 2: Assume that $\sum a_k$ converges and (b_n) is bounded. Prove that $\sum a_k b_k$ converges.

Proof. Consider $b' = \max(|\inf b_n|, |\sup b_n|)$. Then we see that

$$b'\left|\sum_{k=m}^{n} |a_k|\right| = \left|\sum_{k=m}^{n} |a_k||b'|\right| \geqslant \left|\sum_{k=m}^{n} |a_k||b_k|\right| = \left|\sum_{k=m}^{n} |a_k b_k|\right|$$

But we see that the series $\sum a_k$ converges. So we have:

$$b'\left|\sum_{k=m}^{n} |a_k|\right| < \varepsilon$$

is cauchy. Then since $|a_k b_k| \leqslant |b' a_k|$, we know that by comparison test, $\sum |a_k b_k|$ converges. But absolutely converging series imply that $\sum a_k b_k$ converges also. So we are done. □

Exercise 3: Assume $\liminf |a_n| = 0$, then there is a subsequence (a_{n_k}) of (a_n) so that $\sum a_{n_k}$ converges absolutely.

Proof. Take $(a_{n_k}) = \inf\{a_j : j > k\}$. Since $\liminf |a_n| = 0$, we have that $\forall \varepsilon > 0, \exists N$ such that $\forall n_k > N$,

$$|a_{n_k}| < \frac{\varepsilon}{h}$$

for any $h > 1 \in \mathbb{N}$. Then we have:

$$\left|\sum_{n_k=m}^{m+h-1} a_{n_k}\right| \leqslant |a_{n_m}| + |a_{n_{m+1}}| + \cdots + |a_{n_{m+h-1}}| < \frac{\varepsilon}{h} + \cdots + \frac{\varepsilon}{h} = \varepsilon$$

So our series satisfies the cauchy criterion and is therefore convergent. □

Exercise 4: Show that $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges to ∞.

Proof. We know that $n > \log n$. This means that $\frac{1}{n} < \frac{1}{\log n}$. By the comparison test, since $\frac{1}{n} > 0$ and $\frac{1}{\log n} > \frac{1}{n}$ for all $n > 1$, since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\log n}$ diverges also. $\square$

Exercise 5: Prove that $\sum \frac{1}{n(n+1)} = 1$

Proof. We have that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Then:

$$\sum_{k=1}^{r} \frac{1}{n(n+1)} = \sum_{k=1}^{r} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{r+1}$$

As $r \to \infty$, we have $1 - \frac{1}{r+1} \to 1$. So $\sum \frac{1}{n(n+1)} = 1$. $\square$

Exercise 6: Give examples of

- $\sum a_k$ converges but $\sum a_k^2$ diverges

  Answer. By theorem 15.3, we have that in the series $\sum \frac{1}{\sqrt{n}}$, $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$ so the series $\sum (-1)^n \frac{1}{\sqrt{n}}$ converges. But we have that $((-1)^k \frac{1}{\sqrt{k}})^2 = \frac{1}{k}$ which diverges.

- $\sum a_k$ diverges but $\sum a_k^2$ converges.

  Answer. We know that $\sum \frac{1}{k}$ diverges but $\sum \frac{1}{k^2}$ converges which was proved in class.