Math104Hw10

Trustin Nguyen

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Exercise 1: Find the exact interval of convergence for the power series $\sum n^2x^n$.

Proof. We have that
$$\limsup_{n\to\infty}|n^2|^{\frac{1}{n}} = 1 = \beta$$
Therefore, $R = \frac{1}{\beta} = 1$. So it converges when $|x| < 1$. Now to check for the boundary points, we have for $x = 1$:
$$\sum n^2$$
and for $x = -1$:
$$\sum(-1)^n n^2$$
Both of these diverge. So the interval of convergence is $(-1, 1)$. □

Exercise 2: Find the exact interval of convergence for the power series $\sum \frac{n}{2^n}x^n$.

Proof. We can try the ratio test:
$$\lim_{n\to\infty}\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n} = \frac{1}{2} = \beta$$
Then $R = \frac{1}{\beta} = 2$. So now to test the endpoints:
$$\sum(-1)^n n \text{ and } \sum n$$
Both of these diverge, so the radius of convergence is $(-2, 2)$. □

Exercise 3: Let $f_n = \frac{1+\cos nx}{n}, x \in \mathbb{R}$. Find $f(x)$ so that $f_n \to f$ pointwise on $\mathbb{R}$, then check whether $f_n \to f$ uniformly or not on $\mathbb{R}$.

Proof. We have that $0 \leqslant \cos nx \leqslant 1$. So
$$\frac{1}{n} \leqslant \frac{1+\cos nx}{n} \leqslant \frac{2}{n}$$
Since $\lim\limits_{n\to\infty}\frac{2}{n} = 0$, by comparison test, we have that $f_n \to 0$.

Now we need to check uniform convergence or that $\forall\varepsilon > 0, \exists N > 0$ such that if $n > N$, we have:
$$|f_n(x) - f(x)| = \left|\frac{1+\cos nx}{n}\right| < \varepsilon$$
We have that
$$\left|\frac{1+\cos nx}{n}\right| \leqslant \left|\frac{2}{n}\right| < \varepsilon$$
$$\frac{2}{n} < \varepsilon$$
$$\frac{2}{\varepsilon} < n$$

So we require $N = \frac{2}{\varepsilon}$. Since N does not depend on the value of x, it converges uniformly. $\square$

Exercise 4: Let $f_n = \frac{nx}{1+n^2x^2}, x \in \mathbb{R}$. Prove that $f_n \to 0$ pointwise on $\mathbb{R}$, then check whether $f_n \to 0$ uniformly or not on $[0,1]$.

Proof. We find the convergence of f_n like so:

$$\begin{aligned}\lim_{n\to\infty} \frac{nx}{1+n^2x^2} &= \lim_{n\to\infty} \frac{\frac{1}{nx}}{\frac{1}{n^2x^2}+1} \\ &= \frac{\lim \frac{1}{nx}}{\lim \frac{1}{n^2x^2} \lim 1} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

So $f_n \to 0$.

Now to check for uniform convergence, we have to show that $\forall \varepsilon > 0, \exists N > 0$ such that $\forall n > N$, we have:

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2}\right| < \varepsilon$$

for $x \in [0,1]$.

Instead, it is equivalent to show that $\limsup\limits_{n\to\infty}\{|f_n(x)| : x \in [0,1]\} = 0$. First, the derivative:

$$\frac{d}{dx}\left(\frac{nx}{1+n^2x^2}\right) = \frac{(1+n^2x^2)n - nx(2n^2x)}{(1+n^2x^2)^2} = \frac{n-n^3x^2}{(1+n^2x^2)^2}$$

So we check:

$$n - n^3x^2 = 0 \implies x^2 = \frac{1}{n^2} \implies x = \pm\frac{1}{n}$$

We note that the derivative is positive on $[0, \frac{1}{n})$ and negative after $\frac{1}{n}$. So we obtain the supremum as

$$\frac{n\left(\frac{1}{n}\right)}{1+n^2\left(\frac{1}{n}\right)^2} = \frac{1}{2}$$

and the infimum as

$$0$$

So the $\limsup\limits_{n\to\infty}|\frac{nx}{1+n^2x^2}| \neq 0$ which means that it does not uniformly converge. $\square$

Exercise 5: Same f_n in Q4, check whether $f_n \to 0$ uniformly or not on $[1,\infty)$.

Proof. Recall from the previous problem that the function $\frac{nx}{1+n^2x^2}$ is increasing on the interval $(0, \frac{1}{n})$ and decreasing on the interval $(\frac{1}{n}, \infty)$. Then the sequence:

$$f_n(\frac{1}{n})$$

is increasing but the function $f_n(x) > f_n(y)$ for $x < y$. Therefore, the function assumes a maximum value in the interval $[1,\infty)$ for $x = 1$. So now we calculate the supremum, which is plugging in 1:

$$\frac{n}{1+n^2}$$

and the infimum calculating the limit $x \to \infty$

$$\lim_{x\to\infty} \frac{nx}{1+n^2x^2} = 0$$

Then we have:

$$\begin{aligned}\limsup_{n\to\infty}\{|f_n(x) : x \in [1,\infty)|\} &= \limsup_{n\to\infty} \frac{n}{1+n^2} \\ &= 0\end{aligned}$$

Since the limit is 0, we know that it uniformly converges on the interval. □

Exercise 6: Use the definition to show: if $f_n \to f$ uniformly on S and $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ uniformly on S.

Proof. If $f_n \to f$ uniformly, on S, then we know that $\forall \varepsilon > 0$, $\exists N_1 > 0$ such that $\forall n > N_1$, $x \in S$, we have that:

$$|f_n(x) - f(x)| < \varepsilon/2$$

Similarly, we know that there is an $N_2 > 0$ such that $\forall n > N_2$, $x \in S$, we have

$$|g_n(x) - g(x)| < \varepsilon/2$$

Then for all $\varepsilon > 0$, we take the maximum $\max(N_1, N_2)$ that exists for the first two equations. Then for all $x \in S$, we have:

$$\begin{aligned}|f_n(x) + g_n(x) - f(x) - g(x)| &\leqslant |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \varepsilon\end{aligned}$$

This means that $f_n(x) + g_n(x)$ uniformly converges to $f(x) + g(x)$ on S. □