$= [z_3 : z_4 : z_1 : z_3]$

$= [z_4 : z_3 : z_2 : z_1]$

① $[z_1 : z_2 : z_3 : z_4] = [z_2 : z_1 : z_4 : z_3] = \alpha$

$$\frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4} = \frac{z_1 z_2 - z_2 z_3 - z_1 z_4 + z_3 z_4}{z_1 z_2 - z_2 z_4 - z_1 z_3 + z_3 z_4}$$

$[z_2 : z_1 : z_3 : z_4]$

$$= \frac{z_2 - z_3}{z_2 - z_4} \Big/ \frac{z_1 - z_3}{z_1 - z_4} = \frac{(z_2 - z_3)(z_1 - z_4)}{(z_2 - z_4)(z_1 - z_3)} = \frac{1}{\alpha}$$

$$[z_1 : z_3 : z_2 : z_4] = \frac{z_1 - z_2}{z_1 - z_4} \Big/ \frac{z_3 - z_2}{z_3 - z_4}$$

$$= \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{z_1 z_3 - z_1 z_4 - z_2 z_3 + z_2 z_4}{z_1 z_3 - z_1 z_2 - z_3 z_4 + z_2 z_4} = \frac{N}{D}$$

$D = -$ denominator of $[z_1 : z_2 : z_3 : z_4]$

$N = -$ denom of $[z_1 : z_2 : z_3 : z_4]$ +
numerator of $[z_1 : z_2 : z_3 : z_4]$

$N = -1 + \alpha$ $\qquad [z_1 : z_3 : z_2 : z_4] = 1 - \alpha$

$D = -1$

$$\boxed{\text{Values: } \alpha, \frac{1}{\alpha}, 1-\alpha, \frac{1}{1-\alpha}, 1 - \frac{1}{\alpha}, \frac{\alpha}{\alpha - 1}}$$

$$1 - \frac{1}{1-\alpha} = \frac{1-\alpha}{1-\alpha} - \frac{1}{1-\alpha} = \frac{-\alpha}{1-\alpha} = \frac{\alpha}{\alpha - 1}$$

② z_0, $z = kre^{i\theta} + z_0$, $z' = mkre^{i\theta} + z_0$

$$|z' - z_0||z - z_0| = |kre^{i\theta}||mkre^{i\theta}| = r^2$$

$$= mk^2 r^2 \qquad = r^2$$

$$m = \frac{1}{k^2}$$

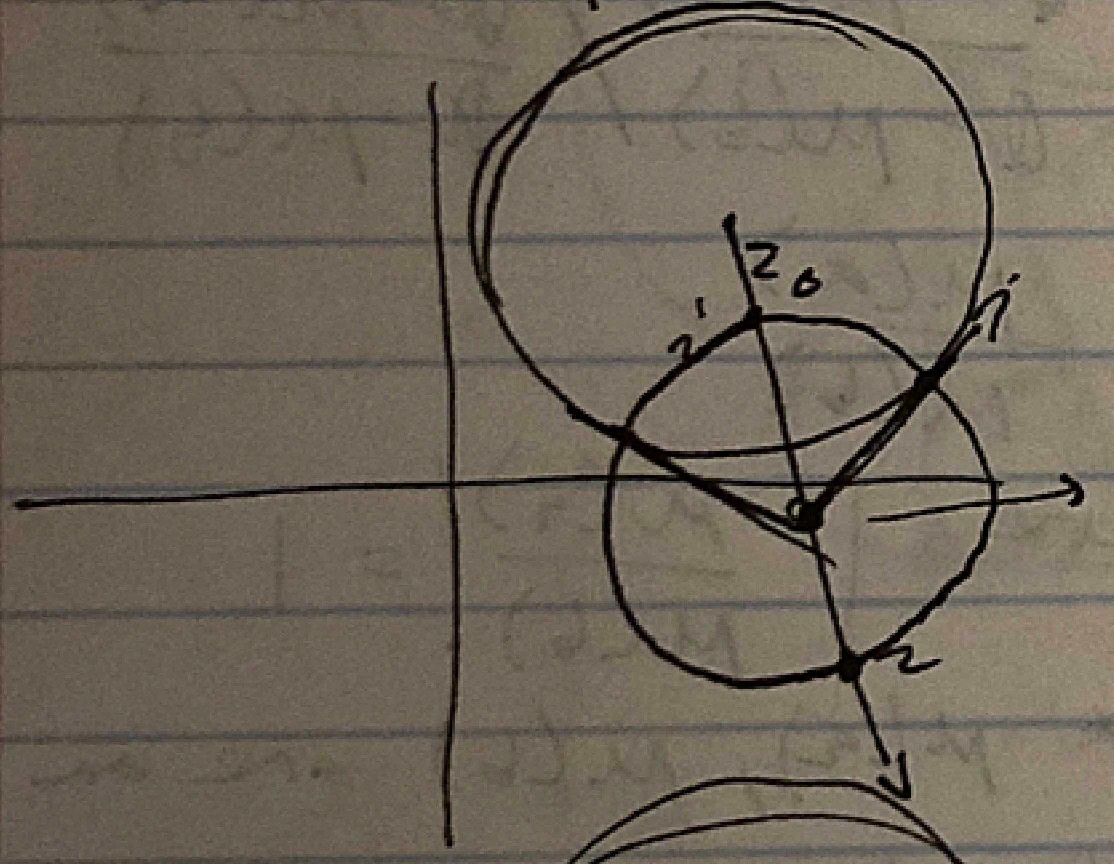
$$z' = \frac{1}{k} re^{i\theta} + z_0$$

$$z - z_0 = kre^{i\theta} \qquad \frac{z - z_0}{kr} = e^{i\theta}$$

$$|z - z_0| = kr$$

$$\frac{r}{|z - z_0|} = \frac{1}{k} \qquad \frac{z - z_0}{|z - z_0|} = e^{i\theta}$$

a) $$z' = \frac{r^2}{|z - z_0|^2}(z - z_0) + z_0$$



c) By b, we see that choosing any other value either increases the angle at intersection or decreases it

~~z, z'~~

b) Recall $|z'| = \frac{r}{k}$, $|z| = rk$

Show that

$$(|z| - |r'|)^2 - |r'|^2 = r^2$$

$$|z|^2 - 2|z|r' + \cancel{|r'|^2 - |r'|^2}$$

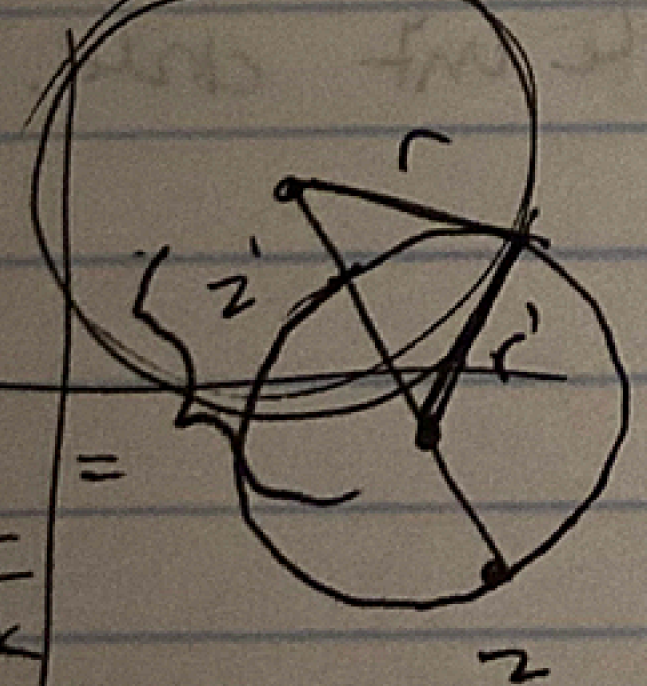
$$|z|^2 - 2|z|r'$$

$$= r^2k^2 - 2rk \cdot \frac{rk^2 - r}{2k}$$

$$= r^2k^2 \cancel{- r^2k^2} + r^2$$

$$= r^2 \checkmark$$

so circles meet at right angles



$$\frac{rk - \frac{r}{k}}{2} + \frac{r}{k} = \frac{\frac{rk^2}{k} - \frac{r}{k}}{2} + \frac{r}{k} = \boxed{\frac{rk^2 - r}{2k}} + \frac{r}{k} = \frac{rk^2 + r}{2k}$$

$= r'$

$$\frac{z-a}{z-b} \Big/ \frac{z'-a}{z'-b} = \frac{(z-a)(z'-b)}{(z-b)(z'-a)}$$

$$= \frac{zz' - bz - az' + ab}{zz' - az - bz' + ab}$$

3 $$[z : z' ; a : b] = [0 : \infty : \mu(a) : \mu(b)]$$

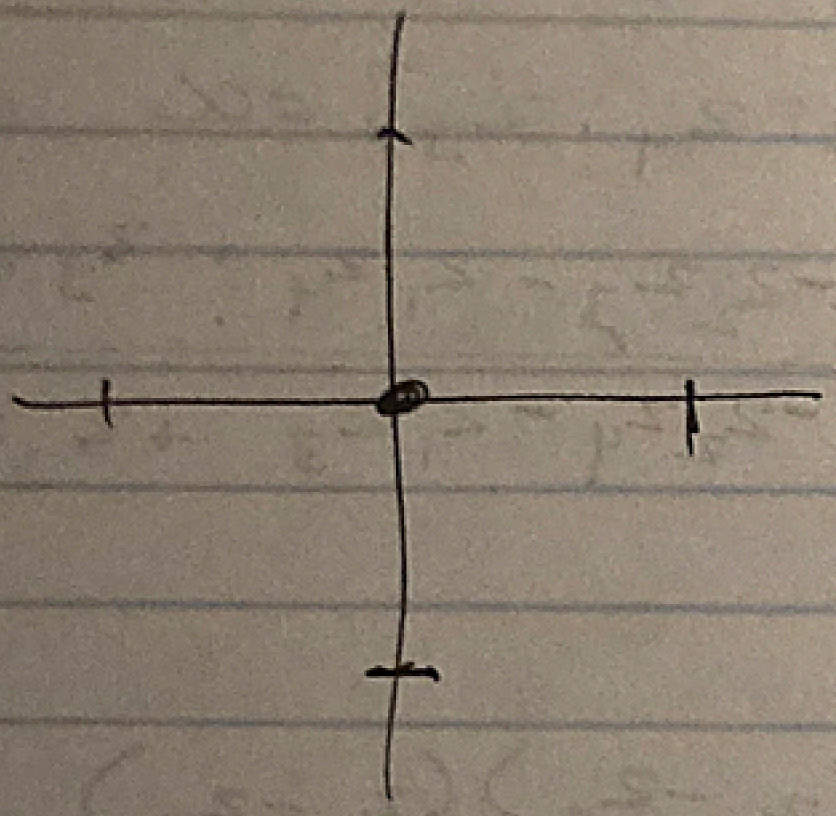
$$= \frac{0 - \mu(a)}{0 - \mu(b)} \Big/ \frac{\infty - \mu(a)}{\infty - \mu(b)}$$

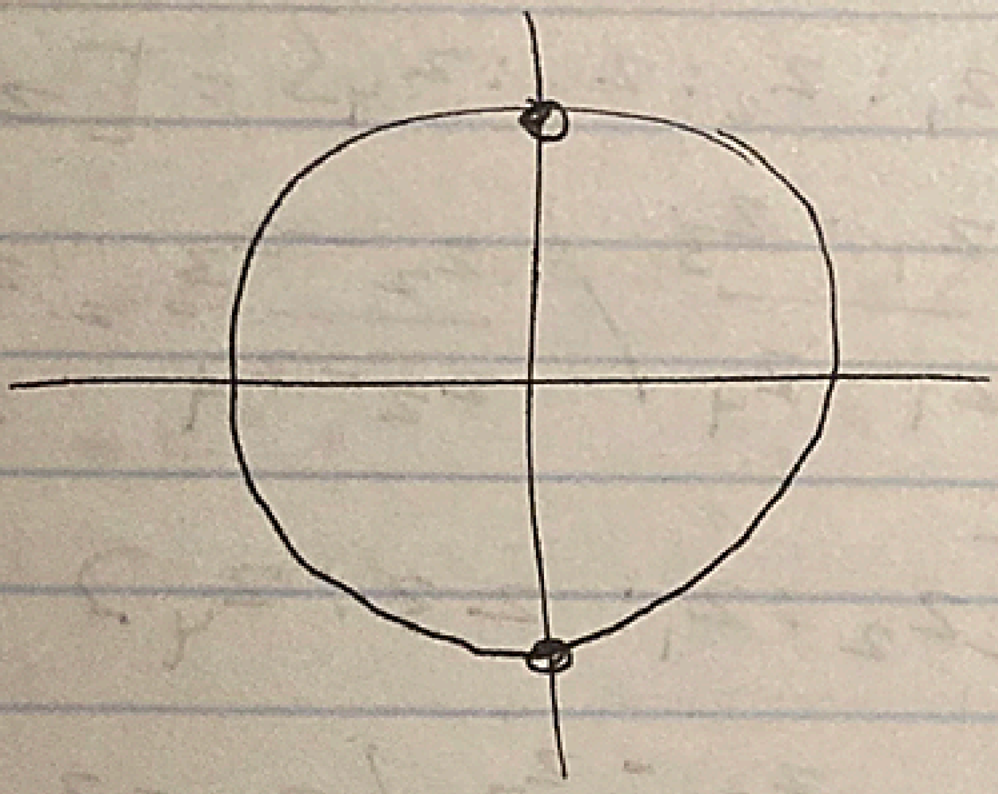
$$= \frac{\mu(a)}{\mu(b)}$$

where $\left| \frac{\mu(a)}{\mu(b)} \right| = 1$

since $\mu(a), \mu(b)$ are on the unit circle.

(4)





$\infty \longmapsto \pm i \qquad 0 \longmapsto \pm i$

$$\frac{Az+B}{Cz+D} \qquad \frac{A}{C} = \pm i$$

$$\frac{B}{D} = \pm i$$

$A = Ci$

$B = -Di$

$$\frac{iCz - iD}{Cz+D} = \frac{i(Cz-D)}{Cz+D}$$

If z real, then $\frac{Cz - D}{Cz+D}$ is real

so $\arg Cz + D = -\arg Cz - D$

$C = r_1 e^{i\theta_1}$

$D = r_2 e^{i\theta_2}$

$$\frac{r_1 z e^{i\theta_1} - r_2 e^{i\theta_2}}{r_1 z e^{i\theta_1} + r_2 e^{i\theta_2}}$$

$$\frac{\frac{r_1 z e^{i\theta_1}}{r_2 e^{i\theta_2}} - 1}{\frac{r_1 z e^{i\theta_1}}{r_2 e^{i\theta_2}} + 1}$$

$\frac{r_1}{r_2} = r'$

$$\frac{r' z e^{i(\theta_1 - \theta_2)} - 1}{r' z e^{i(\theta_1 - \theta_2)} + 1}$$

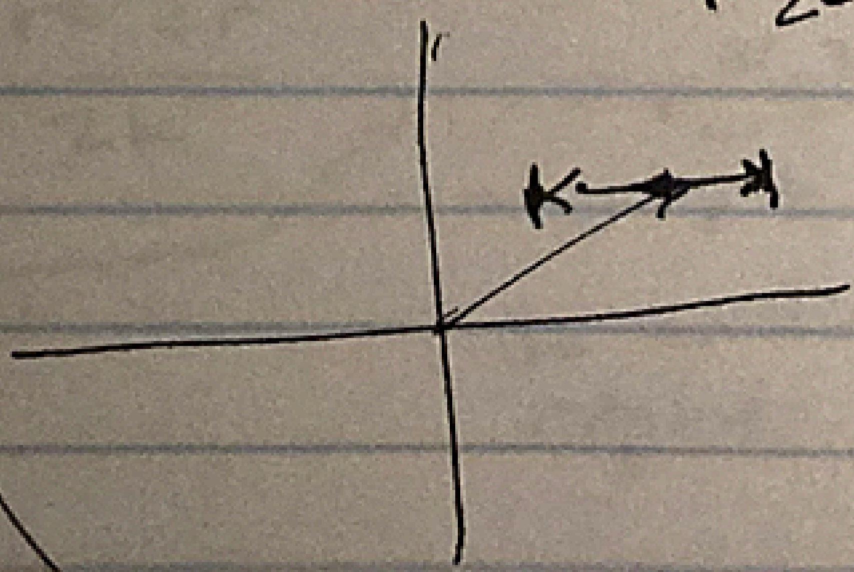
So the Möbius transformations are of the form

$$i\left(\frac{Cz - D}{Cz + D}\right)$$

for C, D real and

$$-i\left(\frac{Cz-D}{Cz+D}\right)$$

for C, D real



so $\theta_1 - \theta_2 = 0$

$\theta_1 = \theta_2$

or $\theta_1 = -\theta_2$

$\to C, D$ real