Math143Hw11

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Exercise 1: Let P_1, P_2, P_3 (resp. Q_1, Q_2, Q_3) be three points in $\mathbb{P}^2$ not lying on a line. Show that there is a projective change of coordinates $T : \mathbb{P}^2 \to \mathbb{P}^2$ such that $T(P_i) = Q_i$, for $i = 1, 2, 3$.

Proof. We want to solve for a transformation matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$ such that:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} | & | & | \\ P_1 & P_2 & P_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ Q_1 & Q_2 & Q_3 \\ | & | & | \end{bmatrix}$$

Suppose that $P = \begin{bmatrix} | & | & | \\ P_1 & P_2 & P_3 \\ | & | & | \end{bmatrix}$ is not invertible. Then we have a linear dependence of the rows of P. Let R_1, R_2, R_3 be the rows of P. Then

$$a_1 R_1 + a_2 R_2 + a_3 R_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

and P_1, P_2, P_3 all satisfy the solution to the equation $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$ which is the equation of a line. So they lie on the same line. Then since the points are not on the same line, P is invertible. So $A = QP^{-1}$. It is a change of coordinates because A is invertible also as $A^{-1} = PQ^{-1}$ since Q is invertible by the same reasoning as above. □

Extra Credit: Extend this to $n + 1$ points in $\mathbb{P}^n$, not lying on a hyperplane.

Proof. This can be extended to $n + 1$ points by solving for A in the system:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n+1} \end{bmatrix} \begin{bmatrix} | & | & & | \\ P_1 & P_2 & \cdots & P_{n+1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Q_1 & Q_2 & \cdots & Q_{n+1} \\ | & | & & | \end{bmatrix}$$

We find that the rows are linearly independent if P is not invertible, so

$$a_1 R_1 + \cdots + a_{n+1} R_{n+1} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

and this shows that $P_1, \dots, P_{n+1}$ satisfy the solution to the hyperplane $a_1 x_1 + \cdots + a_{n+1} x_{n+1} = 0$. Therefore, by contrapositive, P, Q are invertible, $A = QP^{-1}$, $A^{-1} = PQ^{-1}$. □

Exercise 2: Duals. Let $\Lambda = \mathbb{V}(a_1 x_1 + \cdots + a_{n+1} x_{n+1}) \subseteq \mathbb{P}^n$ with $a_1, \dots, a_{n+1} \in k$ not all zero. Recall that we call Λ a *hyperplane*. Note that $(a_1, \dots, a_{n+1})$ is determined by Λ up to rescaling. As discussed in class, assigning $[a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$ to Λ sets up a one-to-one correspondence between {hyperplanes in $\mathbb{P}^n$} and $\mathbb{P}^n$.

(a) Given $P = [a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$, write $P^* = \mathbb{V}(a_1x_1 + \cdots + a_{n+1}x_{n+1})$ for the corresponding hyperplane; if Λ is a hyperplane, let Λ^* denote the corresponding point. Prove that $(P^*)^* = P$ and $(\Lambda^*)^* = \Lambda$.

Proof. First, we have

$$\begin{aligned}(P^*)^* &= (\mathbb{V}(a_1x_1 + \cdots + a_{n+1}x_{n+1}))^* \\ &= [a_1 : \cdots : a_{n+1}]\end{aligned}$$

and then

$$\begin{aligned}(\Lambda^*)^* &= ((\mathbb{V}(a_1x_1 + \cdots + a_{n+1}x_{n+1}))^*)^* \\ &= ([a_1 : \cdots : a_{n+1}])^* \\ &= \mathbb{V}(a_1x_1 + \cdots + a_{n+1}x_{n+1}) \\ &= \Lambda\end{aligned}$$

which completes the proof. □

(b) Show that $P \in \Lambda$ if and only if $\Lambda^* \in P^*$.

Proof. ($\rightarrow$) Suppose that $P \in \Lambda$. Then $P \in \mathbb{V}(a_1x_1 + \cdots + a_{n+1}x_{n+1})$. So we have $P = [p_1 : \cdots : p_{n+1}]$ such that

$$a_1p_1 + \cdots + a_{n+1}p_{n+1} = 0$$

We then have $\Lambda^* = [a_1 : \cdots : a_{n+1}]$ and $P^* = \mathbb{V}(p_1x_1 + \cdots + p_{n+1}x_{n+1})$. Then we have that $\Lambda^* \in P^*$ because

$$p_1a_1 + \cdots + p_{n+1}a_{n+1} = a_1p_1 + \cdots + a_{n+1}p_{n+1} = 0$$

($\leftarrow$) Suppose that $\Lambda^* \in P^*$. Then by the previous direction, we have that $(P^*)^* \in (\Lambda^*)^*$. Also by part (a), we have $P \in \Lambda$ which concludes the proof. □

Exercise 3: Assume k infinite. Show that for any finite set of points $P_1, \ldots, P_r \in \mathbb{P}^2$, there exists a line $L \subseteq \mathbb{P}^2$ that does not pass through any of them.

Proof. We require that for all i, $P_i \notin \Lambda$, which would indicate that the line Λ does not pass through P_i. By the previous problem, this is equivalent to saying that $\Lambda^* \notin P_i^*$ for all i. So we require

$$P_1^* \cup P_2^* \cup \cdots \cup P_r^* \neq \mathbb{P}^2$$

Since k is infinite, $\mathbb{A}^2 \subseteq \mathbb{P}^2$ is infinite. Each P_i^* contains only one point, because we have:

$$P_i^* = \mathbb{V}(a_1x_1 + a_2x_2)$$

So we have:

$$\{[x_1 : x_2] : a_1x_1 + a_2x_2 = 0\}$$

Then

$$x_2 = \begin{cases} \dfrac{-a_2}{a_1} & \text{if } a_1 \neq 0 \\ 0 & \text{if } a_1 = 0 \end{cases}$$

So

$$P_i^* = \{[1 : \frac{-a_2}{a_1}]\} \text{ or } \{[0 : 1]\}$$

Therefore,

$$P_1^* \cup P_2^* \cup \cdots \cup P_r^* \neq \mathbb{P}^2$$

Therefore, we can find some $\Lambda^* \notin P_i^*$. So $P_i \notin \Lambda$ as desired. □

Exercise 4: Veronese embeddings.

(a) Prove that Veronese embedding $\nu_{1,3} : \mathbb{P}^1 \to \mathbb{P}^3$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$ is an isomorphism onto its image.

Proof. Consider the mapping:

$$[x_1 : x_2 : x_3 : x_4] \mapsto \begin{cases} [x_1 : x_2] & \text{if } x_1 \neq 0 \text{ over } (U_1 \cap \varphi(\mathbb{P}^1)) \\ [x_3 : x_4] & \text{if } x_4 \neq 0 \text{ over } (U_4 \cap \varphi(\mathbb{P}^1)) \end{cases}$$

We first need to check that our morphism uses all the points in $\varphi(\mathbb{P}^1)$. Consider the complement of $(U_1 \cap \varphi(\mathbb{P}^1)) \cup (U_4 \cap \varphi(\mathbb{P}^1))$. These are elements in the image where $x_1 = 0$ and $x_4 = 0$. Then that would mean that for $[s^3 : s^2t : st^2 : t^3] \in \varphi(\mathbb{P}^1)$, we have $s = 0$ and $t = 0$. Then that is the empty set because $[0 : 0 : 0 : 0] \notin \mathbb{P}^3$. So this is a morphism of projective algebraic sets.

Well defined: Suppose that we had a point $[s^3 : s^2t : st^2 : t^3]$ such that $s \neq 0$ and $t \neq 0$. Then

$$\begin{aligned} \varphi^{-1}([s^3 : s^2t : st^2 : t^3]) &= [s^3 : s^2t] = [s : t] \\ &= [st^2 : t^3] = [s : t] \end{aligned}$$

$(\varphi(\varphi^{-1}) = \mathrm{id}_{\mathbb{P}^1})$ We have

$$\begin{aligned} \varphi(\varphi^{-1}([s^3 : s^2t : st^2 : t^3])) &= \varphi([s : t]) \\ &= [s^3 : s^2t : st^2 : t^3] \end{aligned}$$

$(\varphi^{-1}(\varphi) = \mathrm{id}_{\mathbb{P}^3})$ We have

$$\begin{aligned} \varphi^{-1}(\varphi([s : t])) &= \varphi^{-1}([s^3 : s^2t : st^2 : t^3]) \\ &= [s : t] \end{aligned}$$

Which finishes the proof. □

(b) The Veronese embedding $\nu_{3,2} : \mathbb{P}^3 \to \mathbb{P}^9$ is given by

$$[x_1 : x_2 : x_3 : x_4] \mapsto [x_1^2 : x_1x_2 : x_1x_3 : x_1x_4 : x_2^2 : x_2x_3 : x_2x_4 : x_3^2 : x_3x_4 : x_4^2]$$

Let $z_1, \dots, z_{10}$ be the coordinates on $\mathbb{P}^9$. Find a matrix M whose entries are polynomials in the z_i so that the image of $\nu_{3,2}$ is the set of points in $\mathbb{P}^9$ where the matrix M has rank $\leqslant k$ for some integer k. Justify your answer. (This implies that the equations that define the image are the $(k+1) \times (k+1)$ minors of M.)

Proof. First is calculating the inverse morphism:

$$[z_1 : z_2 : \cdots : z_9 : z_{10}] \mapsto \begin{cases} [z_1 : z_2 : z_3 : z_4] & \text{if } z_1 \neq 0 \text{ over } U_1 \cap \Im\nu_{3,2} \\ [z_2 : z_5 : z_6 : z_7] & \text{if } z_5 \neq 0 \text{ over } U_5 \cap \Im\nu_{3,2} \\ [z_3 : z_6 : z_8 : z_9] & \text{if } z_8 \neq 0 \text{ over } U_8 \cap \Im\nu_{3,2} \\ [z_4 : z_7 : z_9 : z_{10}] & \text{if } z_{10} \neq 0 \text{ over } U_{10} \cap \Im\nu_{3,2} \end{cases}$$

Then the image is the set of points such that the mapping is well-defined:

$$I = \left\{ [z_1 : z_2 : \cdots : z_9 : z_{10}] : \mathrm{rank} \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_5 & z_6 & z_7 \\ z_3 & z_6 & z_8 & z_9 \\ z_4 & z_7 & z_9 & z_{10} \end{bmatrix} = 1 \right\} = \mathbb{V}(2 \times 2 \text{ minors})$$

We see that the map well-defined, because if $[z_1 : \cdots : z_{10}]$ satisfy all conditions, we have:

$$[z_1 : \cdots : z_{10}] \mapsto \begin{cases} [x_1^2 : x_1x_2 : x_1x_3 : x_1x_4] & \text{if } z_1 \neq 0 \\ [x_1x_2 : x_2^2 : x_2x_3 : x_2x_4] & \text{if } z_2 \neq 0 \\ [x_1x_3 : x_2x_3 : x_3^2 : x_3x_4] & \text{if } z_3 \neq 0 \\ [x_1x_4 : x_2x_4 : x_3x_4 : x_4^2] & \text{if } z_4 \neq 0 \end{cases}$$
$$= \begin{cases} [x_1 : x_2 : x_3 : x_4] & \text{if } z_1 \neq 0 \\ [x_1 : x_2 : x_3 : x_4] & \text{if } z_2 \neq 0 \\ [x_1 : x_2 : x_3 : x_4] & \text{if } z_3 \neq 0 \\ [x_1 : x_2 : x_3 : x_4] & \text{if } z_4 \neq 0 \end{cases}$$
$$= [x_1 : x_2 : x_3 : x_4]$$

This generalizes if it satisfies less than all 4 conditions.

We need to check composition of morphisms. Referring to the previous shown mapping of $[z_1 : \cdots : z_{10}] \mapsto$ stuff, we compose that with $\nu_{3,2}$ to get:

$$\begin{aligned} &\nu_{3,2}([x_1 : x_2 : x_3 : x_4]) = \\ &\quad [x_1 : x_2 : x_3 : x_4] \mapsto [x_1^2 : x_1x_2 : x_1x_3 : x_1x_4 : x_2^2 : x_2x_3 : x_2x_4 : x_3^2 : x_3x_4 : x_4^2] \\ &\quad = [z_1 : \cdots : z_{10}] \end{aligned}$$

That is the composition $\nu_{3,2}(\nu_{3,2}^{-1}([z_1 : \cdots : z_{10}]))$. For the other composition, we have

$$[x_1 : x_2 : x_3 : x_4] \mapsto [x_1^2 : x_1x_2 : x_1x_3 : x_1x_4 : x_2^2 : x_2x_3 : x_2x_4 : x_3^2 : x_3x_4 : x_4^2]$$

Then we see that it maps back to $[x_1 : x_2 : x_3 : x_4]$ under $\nu_{3,2}^{-1}$.

The rank is less than or equal to k because at least one of z_1, z_5, z_8, z_{10} are non-zero. Then wlog, say that $z_1 \neq 0$. For z_5, z_8, z_{10}, consider these cases similarly for each:

- $z_i = 0$. Then it follows that the entire row in the matrix is 0 because z_i is a factor of every element in that row. So we have a row linearly dependent to the first row.
- $z_i \neq 0$. Then it follows that dividing the entire row by z_i and multiplying by z_1 gives us the first row. So the rows are linearly dependent.

So we can have at most 1 linearly independent row, and we must have at least one linearly independent row. So $\mathrm{rank} = 1$. This shows the inclusion of $\Im \nu_{3,2}^{-1} \subseteq I$. □

(c) What is the preimage $\nu_{3,2}^{-1}(\mathbb{V}(z_1 + 4z_3 - 2z_7 + 5z_9))$?

Answer. The preimage is obtained by replacing the $z_1, \ldots, z_{10}$ with what they correspond to in the image of $[x_1 : x_2 : x_3 : x_4]$. So it is

$$\mathbb{V}(x_1^2 + 4x_1x_3 - 2x_2x_4 + 5x_3x_4)$$

Exercise 5: Let $X, Y \subseteq \mathbb{A}^2$ be affine plane curves and let $\overline{X}$ and $\overline{Y}$ be their projective closures. For each of the following statements, prove or give a counter example.

(a) If $\overline{X}$ and $\overline{Y}$ are projectively equivalent, then X and Y are isomorphic (as affine algebraic sets).

Answer. This is false. There is a projective equivalence between $\overline{X} = \mathbb{V}(x) \to \overline{Y} = \mathbb{V}(z)$ by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ 0 \end{bmatrix}$$

But we have $\overline{V(x)} = \mathbb{V}(x)$ and $\overline{V(1)} = \mathbb{V}(z)$. But there is no isomorphism between $V(x) \to V(1) = \emptyset$.

(b) If X and Y are isomorphic (as affine algebraic sets), then $\overline{X}$ and $\overline{Y}$ are projectively equivalent.

Proof. Don't know □