Final

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Exercise 1: Let $T \in \mathcal{L}(\mathbb{R}^n), n \geq 2$. Prove that $\mathbb{R}^n$ has a 2-dimensional T-invariant subspace.

Proof. Suppose that $p_{\min}$, when fully factored, contains a quadratic term, say $(z^2 - r_n)$. Then we use the fact that $p(T)$ is the 0 operator and that each factor sends a specific vector to 0.

$$(T^2 - r_n)v = 0$$
$$T^2 v - r_n v = 0$$
$$T^2 v = r_n v$$

This tells us that $\text{Span}\{v\}$ is invariant under T^2 and that Tv, if added to the span, $\text{Span}\{v, Tv\}$ makes an invariant subspace under T. The idea is that v is not quite an eigenvector, but there is an intermediate subspace that v is sent to, and this, when looked at together with v makes a 2 dimensional invariant subspace. Suppose now that $p_{\min}$ contains no quadratic factors. Then T can be written as a diagonal matrix with respect to some basis, lets say $v_1, \dots, v_n$. So we now just take $\{v_1, v_2\}$ to be the invariant subspace because that is how we read off of diagonal matrices. The first vector is an eigenvector and the second can be written as a linear combination of itself with the first. $\square$

Exercise 2: Let $U_j, j \in \mathbb{N}$, be a family of finite-dimensional nested subspaces of a vector space V, i.e.,

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq U_k \subseteq \cdots$$

Prove that

(a) $U := \bigcup_{j=1}^{\infty} U_j$ is a subspace of V;

Proof. We will proceed by induction.
Base Case: For $n = 1$, since U_1 is a subspace of V, the U_1 is a subspace of V.
Inductive Step: Now suppose that $\bigcup_{j=1}^{n-1} U_j$ is a subspace of V. We will show that $\bigcup_{j=1}^{n} U_j$ is a subspace consequently. We check that 0 is in $\bigcup_{j=1}^{n}$ which is indeed true since 0 is in U_1. Suppose that $v, w \in \bigcup_{j=1}^{n}$. If both are in $U_1 \subseteq \ldots \subseteq U_{n-1}$ then we are done or if both are in U_n we are also done, as they are subspaces. Now if wlog v is in $U_1 \subseteq \ldots \subseteq U_{n-1}$ and $w \in U_n$, we are also done as we know that $w \in U_n$. So $v + w \in \bigcup_{j=1}^{n} U_j$. Now suppose $\lambda \in \mathbb{F}$ and that $v \in \bigcup_{j=1}^{n} U_j$. Then if $v \in \bigcup_{j=1}^{n-1} U_j$, then we are done. If $v \in U_n$, we are also done as U_n is a subspace. Therefore, $\bigcup_{j=1}^{n} U_j$ is a subspace of V. $\square$

(b) $\dim U \geq \dim U_k$ for all $k \in \mathbb{N}$.

Proof. We can use the fact that the dimension of a subspace of a vector space does not exceed the dimension of that vector space. So considering that $U_k \subseteq \bigcup_{j=1}^{n} U_j \subseteq U$, we can say that $\dim U_k \leq \dim \bigcup_{j=1}^{n} U_j \leq \dim U$. $\square$

Exercise 3: Let $\varphi_1, \varphi_2, \ldots, \varphi_m$ be linearly independent linear functionals on an n-dimensional vector space V over a field $\mathbb{F}$. Define the map $T : V \to \mathbb{F}^m$ by the formula

$$T(v) := (\varphi_1(v), \varphi_2(v), \ldots, \varphi_m(v)).$$

(a) Prove that T is a linear map.

Proof. Notice that $\mathbb{F}^m$ is a vector space, so this is a valid map. Now to prove linearity, we check that if $v, w \in V$,

$$\begin{aligned} T(v) + T(w) &= (\varphi_1(v), \ldots, \varphi_m(v)) + (\varphi_1(w), \ldots, \varphi_m(w)) \\ &= (\varphi_1(v) + \varphi_1(w), \ldots, \varphi_m(v) + \varphi_m(w)) \\ &= (\varphi_1(v + w), \ldots, \varphi_n(v + w)) \\ &= T(v + w) \end{aligned}$$

Where we used the fact that φ is linear. Now for multiplication, suppose that $\lambda \in \mathbb{F}$ and that $v \in V$:

$$\begin{aligned} T(\lambda v) &= (\varphi_1(\lambda v), \ldots, \varphi_n(\lambda v)) \\ &= \lambda(\varphi_1(v), \ldots, \varphi_n(v)) \\ &= \lambda T(v) \end{aligned}$$

Which concludes the proof. $\square$

(b) Determine, with proof, $\dim \ker T$ and $\dim \operatorname{Im}\{T\}$. When is T invertible?

Proof. Since $\varphi_1, \ldots, \varphi_m$ are linearly independent, we have some independent list $v_1, \ldots, v_m$ of V such that

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if otherwise} \end{cases}$$

So we find that for $(\varphi_1(v), \ldots, \varphi_m(v)) = 0$, we must have that

$$\varphi_1(v), \ldots, \varphi_m(v) = 0$$

So this means that for the basis $v_1, \ldots, v_m, \ldots, v_n$ which is an extended basis to V. So if

$$v = a_1 v_1 + \cdots + a_n v_n$$

we have that

$$\begin{aligned} \varphi_1(v) &= a_2 v_2 + \cdots + a_n v_n \\ &\vdots \\ \varphi_m(v) &= a_1 v_1 + \cdots + a_{m-1} v_{m-1} + a_{m+1} v_{m+1} + \cdots a_n v_n \end{aligned}$$

So if all the $\varphi_i(v) = 0$, then that means that $T(v)$ is exactly 0 whenever v is written as a linear combination of only the vectors in $\{v_1, \ldots, v_m\}$. This tells us that $\dim \ker T = m$ and by rank nullity, $\dim \operatorname{Im}\{T\} = n - m$. $\square$

Exercise 4: Let V be a finite-dimensional complex vector space, let $T \in \mathcal{L}(V)$ and let $\lambda \in \mathbb{C}$. Using the Jordan normal form of T, prove or disprove:

$$\dim \ker (T - \lambda I)^3 - \dim \ker (T - \lambda I)^2 \leq \dim \ker (T - \lambda I)^2 - \dim \ker (T - \lambda I)^1.$$

Proof. The expression is true. The LHS gives how many Jordan blocks there are that have size greater than 1×1 while the RHS gives the number of Jordan blocks that have size greater than 2×2. Clearly, the collection of Jordan blocks that have size greater than 2×2 is a subset of the collection of Jordan blocks that have a size greater than 1×1. Therefore,

$$\dim \ker (T - \lambda I)^3 - \dim \ker (T - \lambda I)^2 \leq \dim \ker (T - \lambda I)^2 - \dim \ker (T - \lambda I)^1.$$

□

Exercise 5: Let V be the real vector space of polynomials in x and y of (total) degree at most 2, and let $T \in \mathcal{L}(V)$ be defined as follows (you do not need to verity that $T \in \mathcal{L}(V)$; it is so):

$$(Tf)(x, y) := (y + 1)\frac{\partial}{\partial x} f(x, y) + (x + 1)\frac{\partial}{\partial y} f(x, y).$$

Find a basis of V that diagonalizes T and the resulting diagonal matrix representation $\mathcal{M}(T)$.

Proof. Note that a basis for V we could start with is $\{1, x, y, xy, x^2, y^2\}$. We know that $x + 1$ is in the final basis since:

$$(Tx)(x, y) := (y + 1)0 + (x + 1)(1) = x + 1$$

So the shortest dependence so far is $T(x + 1) = x + 1$ which implies that our $p_{\min}$ has factor $(z - 1)$. Also notice that this factor annihilates the basis vector $y + 1$ also. So our next basis is $\{x + 1, y + 1\}$. If we take the vector 1, we get $(T1) = 0$ so our next factor is x: $p_{\min} = z(z - 1)$. Now we can guess the last vectors. Try $(y + 1)^2$:

$$\begin{aligned} T(x + 1)^2 &= (2x + 2)(x + 1) \\ T(x + 1)^2 - 2(x + 1)^2 &= 0 \end{aligned}$$

Now we guess one for replacing xy which could be $(x + 1)(y + 1)$:

$$\begin{aligned} T(x + 1)(y + 1) &= (y + 1)(x + 1) + (x + 1)(y + 1) \\ Tv - 2v &= 0 \end{aligned}$$

So our minimal polynomial is $p_{min} = z(z - 1)(z - 2)$. Our basis is now $\{1, x + 1, y + 1, (x + 1)(y + 1), (x + 1)^2, (y + 1)^2\}$. □

Exercise 6: Find a function $f \in \operatorname{Span}\{1, \cos x, \sin x\}$ which minimizes the integral

$$\int_0^{2\pi} |x + 1 - f(x)|^2 \, \mathrm{d}x \, .$$

Proof. We observe that this can be minimized by projecting the vector $x + 1$ onto the space spanned by $\{1, \cos x, \sin x\}$. So we start by matching inner products:

$$\begin{aligned} \langle x + 1, 1 \rangle &= \langle a_0 + a_1 \cos x + a_2 \sin x \rangle \\ \langle x + 1, \cos x \rangle &= \langle a_0 + a_1 \cos x + a_2 \sin x \rangle \\ \langle x + 1, \sin x \rangle &= \langle a_0 + a_1 \cos x + a_2 \sin x \rangle \end{aligned}$$

After all that computation, we get the system of equations:

$$\begin{aligned} 2\pi^2 + 2\pi &= 2\pi a_0 \\ 0 &= a_1 \pi \\ -2\pi &= a_2 \pi \end{aligned}$$

We therefore, have solved for the values a_0, a_1, a_2 that minimize the function:

$$(\pi + 1) + 0 \cos x - 2 \sin x$$

□

Exercise 7: Let T be a self-adjoint operator and let S be a positive operator on a complex finite-dimensional inner product space. Prove that all eigenvalues of ST are real.

Exercise 8: Consider the complex inner product space

$$V = \mathrm{Span}\{1, \cos x, \sin x\}$$

with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$$

and the operator $T = I + D^2 : f(x) \mapsto f(x) + f''(x)$.

(a) Is T self-adjoint? Explain.

Proof. T is indeed self-adjoint. We will look at the the matrix representation of T by looking at its action on the basis vectors:

$$\begin{aligned} 1 &\mapsto 1 \\ \cos x &\mapsto \cos x + (-\cos x) \\ \sin x &\mapsto \sin x + (-\sin x) \end{aligned}$$

therefore, T is

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is self-adjoint as $T = \overline{T}^{\perp}$ □

(b) Determine the singular value decomposition of T.

Proof. We can find the singular values of T through the matrix representation of T^*T:

$$\mathcal{M}(T^*T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, our singular values are $\sigma_1 = 1$, $\sigma_2 = 0$, and $\sigma_3 = 0$. Now the goal is to represent the image of T as

$$Tv = \sigma_1\langle v, e_1\rangle f_1 + \sigma_2\langle v, e_2\rangle f_2 + \sigma_3\langle v, e_3\rangle f_3$$

for some $f_i \in \mathrm{Span}\{1, \cos x, \sin x\}$ and e_1, e_2, e_3 orthonormal basis vectors. We can orthonormalize the vectors in $\mathrm{Span}\{1, \cos x, \sin x\}$ which we have done before:

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\pi}, \frac{\sin x}{\pi}\right\}$$

Now we just take

$$f_1 = \frac{T(\frac{1}{\sqrt{2\pi}})}{1}$$

which is all that is needed because T sends the other basis vectors to 0. Our singular value decomposition is

$$Tv = 1\left\langle v, \frac{1}{\sqrt{2\pi}}\right\rangle \frac{1}{\sqrt{2\pi}}$$

□

Exercise 9: Decide if the following implications hold in the settings below. No need to justify your answers. You will receive 2pts for each correct answer, 1 pt for each black answer, opts for each incorrect answer. Please circle the best answer.

(a) $\{v_1, \dots, v_k\}^\perp = (\text{Span}\{v_1, \dots, v_k\})^\perp$ for any $v_1, \dots, v_k \in V$.

$\boxed{\text{ALWAYS TRUE}}$ TRUE ONLY IN FINITE DIMENSION FALSE

(b) $S, T \in \mathcal{L}(V), (\dim V < \infty)$ commute if and only if their matrix representations commute.

ALWAYS TRUE $\boxed{\text{TRUE ONLY IF USING SAME BASIS}}$ FALSE

(c) If V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$ is invertible, the $\dim V = \dim W$.

$\boxed{\text{TRUE OVER } \mathbb{C} \text{ and } \mathbb{R}}$ TRUE OVER $\mathbb{R}$ BUT NOT $\mathbb{C}$ FALSE

(d) Any normal operator on a finite-dimensional space is diagonalizable.

ALWAYS TRUE $\boxed{\text{TRUE OVER } \mathbb{C} \text{ BUT NOT } \mathbb{R}}$ FALSE

(e) If $T \in \mathcal{L}(V)$ is invertible, then T' is.

$\boxed{\text{ALWAYS TRUE}}$ TRUE ONLY IN FINITE DIMENSION FALSE