Math143Hw5

Trustin Nguyen

September 28, 2023

Exercise 1: Let $k = \mathbb{C}$. Let $Y = V(x^3 - y^2) \subseteq \mathbb{A}^2$. Consider the morphism $\psi : \mathbb{A}^1 \to Y$ given by $\psi(t) = (t^2, t^3)$.

(a) Show that ψ is a bijection but not an isomorphism.

Proof. (Surjectivity) Suppose that $(a, b) \in Y$. Then we must have that

$$a^3 - b^2 = 0$$

or in other words:

$$a^3 = b^2 \implies a = (\sqrt[3]{b})^2$$

Now take $t = \sqrt[3]{b}$. So we have:

$$\psi(\sqrt[3]{b}) = ((\sqrt[3]{b})^2, b) = (a, b)$$

so it is surjective.

(Injectivity) Suppose that $\psi(t_1) = \psi(t_2)$. Then $(t_1^2, t_1^3) = (t_2^2, t_2^3)$. So $t_1^2 = t_1^3$ and $t_2^2 = t_2^3$ which means $t_1 = t_2$.

(Not Isomorphism) For there to be an isomorphism, we require that:

$$\psi^{-1}(\psi(t)) = t$$

of for there to be a polynomial map $Y \to \mathbb{A}^1$. Such that:

$$\psi^{-1}((t^2, t^3)) = t$$

The right hand side can be viewed as a polynomial with respect to t with degree 1. And ψ^{-1} will be a polynomial on two variables such that evaluation at t^2, t^3 gives a polynomial in 1 variable with degree 1. There does not exist such a polynomial because any with degree $\geqslant 1$ will contain t^k for $k \geqslant 2$. And ψ^{-1} cannot be the constant map. So this is not an isomorphism. □

(b) Show that Y is not isomorphic to $\mathbb{A}^1$ (Note: in (a) you might have shown that ψ is not an isomorphism. Now you should show there does not exist any isomorphism.)

Proof. We can show that there does not exist an isomorphism in the pullback map ψ^*. We have

$$\psi^* : \frac{k[x, y]}{(x^3 - y^2)} \to k[t]$$
$$\psi^*(x) = t^{k_1}$$
$$\psi^*(y) = t^{k_2}$$

We first require injectivity or $0 \mapsto 0$. So

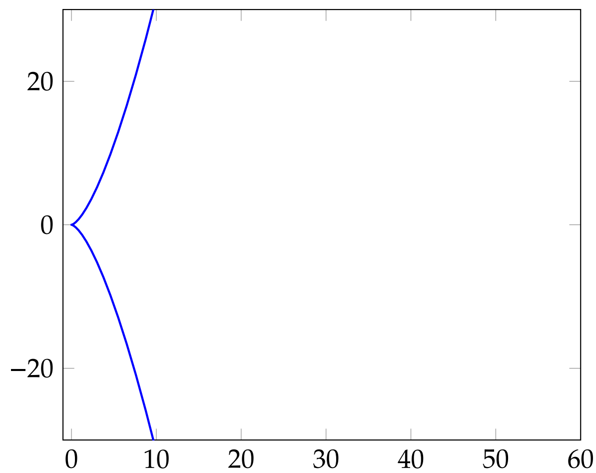
$$\psi^*(x^3) - \psi^*(y^2) = t^{3k_1} - t^{2k_2} = 0$$

So

$$3k_1 = 2k_2$$

This means that $2 \mid k_1$ and $3 \mid k_2$ for positive k_1, k_2. But this does not yield a surjection, as there is no linear combination of products of t^{k_1}, t^{k_2} that would hit t in the image. □

(c) Draw a picture of Y. (This is of course just a picture of the real points.) What do you notice about it?



There is a point at $(0,0)$ where the graph is non-differentiable.

(d) Find $\psi^* : \Gamma(Y) \to \Gamma(\mathbb{A}^1)$. Let $f = 3x^2 + y + 5$ and let $\bar{f} \in \Gamma(Y)$ be the corresponding polynomial function. What is $\psi^*\bar{f} \in \Gamma(\mathbb{A}^1) = k[t]$?

Answer. We have by the mapping $\psi : \mathbb{A}^1 \to Y$ given by:

$$\psi(t) = (t^2, t^3)$$

So now we take $f_1(x,y) = x + (x^3 - y^2)$, $f_2(x,y) = y + (x^3 - y^2)$, and $f_3(x,y) = 1 + (x^3 - y^2)$ which span $\Gamma(Y)$. We observe that:

$$\begin{aligned}(\psi^* f_1)(t) &= (f_1 \circ \psi)(t) \\ &= t^2\end{aligned} \qquad \begin{aligned}(\psi^* f_2)(t) &= (f_2 \circ \psi)(t) \\ &= t^3\end{aligned} \qquad \begin{aligned}(\psi^* f_3)(t) &= (f_3 \circ \psi)(t) \\ &= 1\end{aligned}$$

This uniquely determines ψ^*.

If $f = 3x^2 + y + 5$, we have $\bar{f} = 3x^2 + y + 5 + (x^3 - y^2)$:

$$\psi^*\bar{f} = (\bar{f} \circ \psi)(t) = 3t^4 + t^3 + 5$$

Exercise 2: Let $k = \mathbb{C}$. Consider the morphism $\psi : \mathbb{A}^1 \to \mathbb{A}^2$ given by $\psi(t) = (t^2 - 1, t(t^2 - 1))$.

(a) Find $\psi^* : \Gamma(\mathbb{A}^2) \to \Gamma(\mathbb{A}^1)$. What is $\psi^*(y)$.

Answer. We have $\psi(t) = (t^2 - 1, t(t^2 - 1))$. So the image is an algebraic set, to which we want to find the ideal of. Let $x = t^2 - 1$ and $y = t(t^2 - 1)$. Notice that $y^2 - (x+1)x^2 = 0$ so the image of the map is the algebraic set $V(y^2 - x^3 - x^2)$.

We clearly have that everything in $\mathfrak{I}\psi \subseteq V(y^2 - x^3 - x^2)$. To prove the other containment, suppose x, y satisfy:

$$y^2 - x^3 - x^2 = 0$$

or

$$\begin{aligned} y^2 &= x^2(x+1) \\ y &= \pm x\sqrt{x+1} \end{aligned}$$

We take

$$x = t^2 - 1 \text{ or } t = \pm\sqrt{x+1}$$

So we have $\psi(\sqrt{x+1}) = (x, x\sqrt{x+1})$ and $\psi(-\sqrt{x+1}) = (x, -x\sqrt{x+1})$. So in both situations in whether y is positive or negative, we have found a t such that $\psi(t) = (x, y)$. So $\mathfrak{I}\psi = V(y^2 - x^3 - x^2)$. Since $(y^3 - x^3 - x^2)$ is prime, it is a radical ideal, so we have:

$$\begin{aligned} \psi^* &: k[x, y]/(y^2 - x^3 - x^2) \to k[t] \\ \psi^* &:= 1 \mapsto 1 \\ &:= x \mapsto t^2 - 1 \\ &:= y \mapsto t(t^2 - 1) \end{aligned}$$

So we have $\psi^*(y) = t(t^2 - 1)$.

(b) Find $\psi^{-1}(V(y)) \subseteq \mathbb{A}^1$.

Answer. We have $V(y)$ is just when $y = 0$. So we take:

$$t(t^2 - 1) = 0$$

and find that $t = 0, t = -1, t = 1$. So our algebraic set is $\{-1, 0, 1\}$.

(c) Let $Y = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$. Show that ψ is one-to-one and onto Y, except that $\psi(1) = \psi(-1)$.

Proof. We have shown a surjection in part (a). Now to prove injectivity, suppose that $\psi(t_1) = \psi(t_2)$. Then we have:

$$(t_1^2 - 1, t_1(t_1^2 - 1)) = (t_2^2 - 1, t_2(t_2^2 - 1))$$

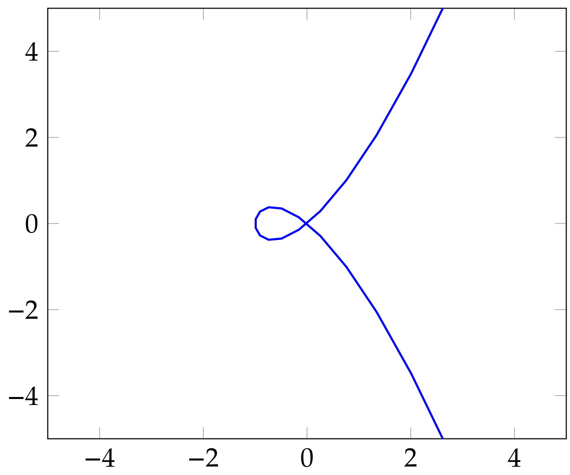
So

$$\begin{aligned} t_1^2 &= t_2^2 & \qquad t_1(t_1^2 - 1) &= t_2(t_2^2 - 1) \\ t_1 &= \pm t_2 & t_1^3 - t_1 &= t_2^3 - t_2 \\ &\Longrightarrow & t_1 &= t_2 \end{aligned}$$

which shows injectivity for $t_1 \neq \pm 1$. When $t_1 = \pm 1$, we have $\psi(1) = (0, 0), \psi(-1) = (0, 0)$. □

(d) Draw a picture of Y. Also draw $V(y)$. (Again, this is just a picture of the real points.) Use this picture and part (c) to explain your answer to (b).

Answer. Here is the picture:



To look at the vanishing of y, we restrict our focus to only the y-levels of the graph. And taking the pre-image of that, we trace along the curve until we hit the first $y = 0$ level, at which $t = -1$. Since we know that $\psi(-1) = \psi(1)$, we know that $y = 0$ when $t = 1$ also. Finally, the curve at $t = 0$ also has a y-level of 0. So that is our pre-image.

Exercise 3: Suppose $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are algebraic sets.

(a) Prove that $X \times Y \subseteq \mathbb{A}^{n+m}$ is an algebraic set.

Proof. Since $X \subseteq \mathbb{A}^n$ is an algebraic set, we have that $X = V(f_1) \cup \cdots \cup V(f_i)$ for $f_i \in k[x_1, \ldots, x_n]$ and $Y = V(g_1) \cup \cdots \cup V(g_j)$ for $g_j \in k[y_1, \ldots, y_m]$. Then now we take

$$W = (V(f_1) \cap Y) \cup \cdots \cup (V(f_i) \cap Y)$$

Suppose that $(p_1, p_2) \in X \times Y$. Then

$$f_i((p_1, ?)) = 0 \text{ for some } f_i \in I(X)$$

and

$$g_j((?, p_2)) = 0 \text{ for some } g_j \in I(Y)$$

That means that $(p_1, p_2) \in V(f_i) \cap V(g_j)$, so $(p_1, p_2) \in W$. Now suppose that $p \in W$. Then $p \in V(f_k) \cap Y$ for some k wlog. So

$$f_k(p) = 0 \wedge g_l(p) = 0$$

Therefore, if $p = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$, we know that $f_k(a_1, \ldots, a_n) = 0$ and $g_l(a_{n+1}, \ldots, a_{n+m}) = 0$. So $p \in X \times Y$. We know that the finite intersection and union of algebraic sets is algebraic. Therefore, $X \times Y = W$ which is algebraic. □

(b) Prove that the projection maps $X \times Y \to X$ and $X \times Y \to Y$ are morphisms.

Proof. We just need to show this for one of them wlog. It will be show that $X \times Y \to X$ is a morphism. We see that this map acts as the identity on each component $p_1, \ldots, p_n$ of a point $p \in X \times Y$, where $p = (p_1, \ldots, p_n, p_{n+1}, \ldots, p_{n+m})$. So for $\varphi : X \times Y \to X$, define $\varphi_i \in k[x_1, \ldots, x_{n+m}]$ to be:

$$\varphi(p) = (\varphi_1(p), \ldots, \varphi_n(p))$$
$$\varphi_i(p) = p_i$$

Each φ_i is a polynomial function to $\mathbb{A}^1$ and therefore a morphism. So we have found the φ_i polynomial maps that turn this into a morphism. □

(c) Prove that if $X \times Y$ is irreducible then X and Y are irreducible.

Proof. We know that if the pre-image of a mapping is irreducible, then the image is irreducible. So if $X \times Y$ is irreducible, by the projection map $X \times Y \to X$, X is the image and is therefore irreducible. The same goes for Y as we can create a projection map onto Y. □

(d) (extra credit) Prove that if X and Y are irreducible, then $X \times Y$ is irreducible. (Hint: Suppose that $X \times Y = A \cup B$ and consider the sets $X_A := \{p \in X : p \times Y \subseteq A\}$ and $X_B := \{p \in X : p \times Y \subseteq B\}$.)

Proof. Suppose for contradiction that $X \times Y$ is reducible. Then $X \times Y = A \cup B$ where A, B are algebraic sets. Then we know that $X_A \cup X_B = X$ and X_A, X_B are proper subsets of X. Since A, B are algebraic sets, we can consider $I(A)$ and $I(B) \in \Gamma(X \times Y)$. This means that $X_A = V(I(X)) \cap V(I(A))$, so it is an algebraic set and the same for X_B. So that is a contradiction. □

Exercise 4: Let $V \subseteq \mathbb{A}^n$ be a non-empty variety (i.e. irreducible algebraic set). Show that the following are equivalent:

(i) V is a point

(ii) $\Gamma(V) = k$

(iii) $\dim_k \Gamma(V) < \infty$

You may assume k is algebraically closed if you wish, but it is true over any field.

Proof. First is $(i) \to (ii)$. Now assuming k is algebraically closed, since V is a point, $I(V)$ is maximal by Nullstellensatz. This also means that $k[x_1, \dots, x_n]/I(V) = k$ because the ideal is generated by linear factors in n variables.

We have $(ii) \to (iii)$ because the dimension of $\Gamma(V)$ would be $1 < \infty$ as 1 generates k.

Finally, for $(iii) \to (i)$ we have that $\Gamma(V)$ is a finite extension of k. So this is an algebraic extension, and therefore, the roots of $f \in I(V)$ are in k. Since it is irreducible, We cannot write V as a union of algebraic sets. Suppose that $V = V(f_1, \dots, f_n)$. Then we will show that each f_i are linear factors. We know that if V is irreducible, $I(V)$ is prime. So if we have a f_i that has more than one linear factor, we can claim a smaller subset of those factors must be in V and by induction, we have that each f_i are linear. And therefore, V is a point as maximal ideals are in bijection with points.

Since we have proved $(i) \to (ii) \to (iii) \to (i)$, we have shown equivalence. □

Exercise 5: Assume that k is algebraically closed. Prove that the algebraic subsets of X are in bijection with the radical ideals in $\Gamma(X)$.

Proof. We know that there is a bijection between the radical ideals of $\Gamma(X)$ and the radical ideals of $k[x_1, \dots, x_n]$ by:

$$k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]/I(X) = \Gamma(X)$$

by the homomorphism $J \subseteq k[x_1, \dots, x_n] \mapsto J/I(X)$ which was proved in question 1 of homework 3. By the Nullstellensatz, we now also know that $V(I(V(Y)))$ for some $Y \subseteq X$ is Y and $I(V(J)) = J$ for $J \subseteq k[x_1, \dots, x_n]$. So this is the bijection between radical ideals of $k[x_1, \dots, x_n]$ and the algebraic sets of X. □