Stat134Hw3

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Exercise 1: 46% of th electors of a town consider themselves as independent, whereas 30% consider themselves democrats and 24% republicans. In a recent election, 35% of the independents, 62% of the democrats and 58% of the republicans voted.

(a) What proportion of the total population actually voted?

Answer. If $P(R), P(D), P(I)$ represents the proportion of republican, democrat, and independents, and $P(V)$ is the proportion that voted, then

$$P(V) = P(V \cap R) + P(V \cap D) + P(I \cap D)$$

So

$$P(V) = P(V \mid R)P(R) + P(V \mid D)P(D) + P(I \mid D)/P(I)$$

therefore,

$$P(V) = (.58)(.24) + (.62)(.3) + (.35)(.46) = 0.4862$$

(b) A uniformly random voter X is picked. Given that X voted, what is the probability that X is independent? democrat? republican?

Answer. We want to find $P(I \mid V) = \frac{P(I \cap V)}{P(V)}$. Now that we have $P(V)$ and $P(I \cap V)$, we can compute:

$$P(I \mid V) = \frac{P(I \cap V)}{P(V)} = \frac{.35}{0.4862} = 0.71986836692719$$

Exercise 2: We choose a number from the set $\{1, 2, 3, \ldots, 100\}$ uniformly at random and denote this number by X. For each of the following choices decide whether the two events in question are independent or not.

(a) $A = \{X \text{ is even}\}$, $B = \{X \text{ is divisible by } 5\}$.

Answer. We have that $|A| = 50$, $|B| = 20$. So we check that $P(A)P(B) = P(A \cap B)$. Also, $|\{X : 10 \mid X\}| = 10$. The sample space Ω has cardinality 100. So we have

$$P(A) = \frac{50}{100} = \frac{1}{2}$$
$$P(B) = \frac{20}{100} = \frac{1}{5}$$
$$P(A \cap B) = \frac{10}{100} = \frac{1}{10}$$

Since $P(A)P(B) = P(A \cap B)$, the events are independent.

(b) $C = \{X \text{ has two digits}\}$, $D = \{X \text{ is divisible by } 3\}$.

Answer. Try to show the same thing as above.

$$\begin{aligned}|C| &= 90\\ |D| &= 33\\ |C \cap D| &= 33 - 3 = 30\end{aligned}$$

So

$$\begin{aligned}P(C) &= \frac{90}{100} = \frac{9}{10}\\ P(D) &= \frac{33}{100}\\ P(C \cap D) &= \frac{30}{100} = \frac{3}{10}\end{aligned}$$

Since $P(C)P(D) = \frac{297}{1000} \neq \frac{300}{1000} = P(C \cap D)$, the events are not independent.

(c) $E = \{X \text{ is a prime}\}$, $F = \{X \text{ has a digit } 5\}$. Note that 1 is not a prime number.

Answer. Try to show the same thing as above. We have $|F| = 10$. Also note that $|F \cap E| = 1$ since 5 is prime and any number with two or more digits that has a 5 at the end is divisible by 5. So

$$P(E \cap F) = \frac{1}{100} \text{ and } P(F) = \frac{10}{100}$$

So suppose that the events are independent for contradiction. Then

$$P(E \cap F) = P(E)P(F)$$

and

$$P(E) = \frac{P(E \cap F)}{P(F)} = \frac{1}{10} = \frac{10}{100}$$

But this is not true because there are more than 10 primes between 1 and 100:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

So the events are not independent.

Exercise 3: An urn contains 5 balls numbered from 1 to 5. We draw 3 of them at random without replacement.

(a) Let X be the largest number drawn. What is the probability mass function of X?

Answer. The possible values are $X = 3, 4, 5$ and we have:

$$\begin{aligned}P(X = 3) &= \frac{\binom{2}{2}}{10} = \frac{1}{10}\\ P(X = 4) &= \frac{\binom{3}{2}}{10} = \frac{3}{10}\\ P(X = 5) &= \frac{\binom{4}{2}}{10} = \frac{6}{10}\end{aligned}$$

which defines the probability mass function.

(b) Let Y be the smallest number drawn. What is the probability mass function of Y?

Answer. The values for the function are $X = 3, 2, 1$. Now compute their probabilities:

$$P(X = 3) = \frac{\binom{2}{2}}{10} = \frac{1}{10}$$
$$P(X = 2) = \frac{\binom{3}{2}}{10} = \frac{3}{10}$$
$$P(X = 1) = \frac{\binom{4}{2}}{10} = \frac{6}{10}$$

which defines the probability mass function.

Exercise 4: Let A and B be two disjoint events. Under what conditions are A and B independent?

Answer. We require that
$$P(A)P(B) = P(A \cap B)$$
Since A and B are disjoint, we have that $P(A \cap B) = 0$. So
$$P(A)P(B) = 0$$
This means that A, B are independent when $P(A)$ or $P(B)$ is 0.

Exercise 5: We flip a biased coin with probability of heads $\frac{1}{3}$. Let X denote the total number of heads after five flips. What is more probable: $X < 1.5$ or $X > 1.5$?

Answer. We have the probability mass function defined as

$$P(X = \alpha) = \left(\frac{1}{3}\right)^{\alpha} \left(1 - \frac{1}{3}\right)^{5-\alpha}$$

for $\alpha \in \{0, 1, 2, 3, 4, 5\}$. If $X < 1.5$, then $X = 0$ or 1. These events are disjoint, so we can add probability:

$$P(X = 0) = \left(1 - \frac{1}{3}\right)^5 = \frac{32}{243}$$
$$P(X = 1) = \left(\frac{1}{3}\right)\left(1 - \frac{1}{3}\right)^4 = \frac{16}{243}$$
$$P(X < 1.5) = P(X = 0) + P(X = 1) = \frac{48}{243} = 0.19753086419753$$

Since $P(X > 1.5) = 1 - 0.19753086419753 = 0.80246913580247 > P(X < 1.5)$, we conclude that $P(X > 1.5)$ is more probable.