Math143Hw10

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Exercise 1: Prove the following statements from lecture:

(a) Let $J \subseteq k[x_1, \dots, x_{n+1}]$ be ideals. If $\sqrt{J} \supseteq (x_1, \dots, x_{n+1})$, show that there exists an integer N such that $J \supseteq (x_1, \dots, x_{n+1})^N$, i.e. J contains all homogeneous polynomials of degree $\geqslant N$.

Proof. Since $\sqrt{J}$ is finitely generated, we know that $x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}} \in J$ for some powers $k_1, \dots, k_{n+1}$. Let $N = (n+1)\max(k_1, \dots, k_{n+1})$. By the pigeonhole principle, an arbitrary homogeneous polynomial in $(x_1, \dots, x_n)^N$ must be divisible by some $x_i^{\max(k_1, \dots, k_{n+1})}$. So $x_1^{k_1}, \dots, x_{n+1}^{k_{n+1}}$ generate $(x_1, \dots, x_n)^N$ and possible more, so $(x_1, \dots, x_{n+1})^N \subseteq J$. □

(b) Show that a projective algebraic set $X \subseteq \mathbb{P}^n$ is irreducible if and only if $\mathbb{I}(X)$ is prime. (You may do this directly or you might see how to reduce it to the affine case, which you can quote from lecture.)

Proof. If X is empty, then $\mathbb{I}(X) = k[x_1, \dots, x_{n+1}]$ which is prime. Otherwise, $\mathbb{I}(X) = I(C(X))$. We have that $\mathbb{I}(X)$ is prime iff $I(C(X))$ is prime, iff $C(X)$ is irreducible.

Next is to prove that $C(X)$ irreducible iff X irreducible. If X is reducible, then

$$X = A \cup B$$

and therefore,

$$\begin{aligned} C(X) &= \{(x_1, \dots, x_{n+1}) : [x_1, \dots, x_{n+1}] \in A \cup B\} \\ &= \{(x_1, \dots, x_{n+1}) : [x_1, \dots x_{n+1}] \in A\} \cup \{(x_1, \dots, x_{n+1}) : [x_1, \dots, x_{n+1}] \in B\} \\ &= C(A) \cup C(B) \end{aligned}$$

Since A, B are projective algebraic sets killed by some homogeneous polynomials, $C(A)$ are points killed by those same polynomials. Which shows that $C(X)$ is a union of algebraic sets and is reducible.

If $C(X)$ is reducible, we have:

$$C(X) = A \cup B$$

Because $C(X)$ is a union of lines through the origin, we know that each line is irreducible, so each line must be in either A or B. So A, B are cones to some sets:

$$C(X) = C(A') \cup C(B')$$

Since $C(A'), C(B')$ algebraic sets of lines through the origin, we know that it is the vanishing of some homogeneous polynomials. Call them $V_A(F_1, \dots, F_r)$, $V_B(G_1, \dots, G_s)$. These polynomials also kill the points in A', B' respectively. So A', B' are projective algebraic sets.

So $C(X)$ irreducible iff X irreducible, which finishes the proof. □

Exercise 2: Let $U_1, \dots, U_{n+1}$ be the affine charts on $\mathbb{P}^n$ and let $X \subseteq \mathbb{P}^n$ be any subset.

(a) Prove that if X is closed in the Zariski topology on $\mathbb{P}^n$, then $X \cap U_i$ is closed in the Zariski topology on each $U_i \cong \mathbb{A}^n$.

Proof. Since X is a projective algebraic set, $X = \mathbb{V}(F_1, \dots, F_r)$ for homogeneous poly F_i. Then
$$X \cap U_i = \{p = [x_1 : \cdots : 1 : \cdots : x_{n+1}] : F_i(p) = 0\}$$
which is the same as:
$$X \cap U_i = V(F_1(x_1, \dots, 1, \dots, x_{n+1}), \dots, F_r(x_1, \dots, 1, \dots, x_{n+1}))$$
which shows that $X \cap U_i$ is an algebraic subset of $U_i \cong \mathbb{A}^n$. □

(b) Prove that if $W \subseteq U_i$ is open in the Zariski topology on $U_i \cong \mathbb{A}^n$, then $W \subseteq \mathbb{P}^n$ is open in the Zariski topology on $\mathbb{P}^n$.

Proof. Since $W \subseteq U_i$ open in U_i, we know that $W^c_{U_i}$ is closed. We have:
$$U_i \cup U_i^c = \mathbb{P}^n$$
Then the complement of W in $\mathbb{P}^n$ is the union of the complement of W in U_i and the complement of W in U_i^c. Since $W \subseteq U_i$, then $W \cap U_i^c = \emptyset$. So
$$W = W^c_{U_i} \cup U_i^c = W^c_{U_i} \cup \mathbb{V}(x_i)$$
This is the union of two closed sets, which means that the complement of W in $\mathbb{P}^n$ is closed, so W in $\mathbb{P}^n$ is open. □

(c) Prove that if $X \cap U_i$ is closed in the Zariski topology on each $U_i \cong \mathbb{A}^n$, then X is closed in the Zariski topology on $\mathbb{P}^n$.

Proof. To show that $\mathbb{P}^n \backslash X = \bigcup_i U_i \backslash (U_i \cap X)$, we first have that $\bigcup_i U_i \backslash (U_i \cap X) \subseteq \mathbb{P}^n \backslash X$ because an element in $U_i \backslash (U_i \cap X)$ is not an element in X, but an element in $\mathbb{P}^n$. So an element in the union is not an element in X, but an element of $\mathbb{P}^n$.

For the other inclusion, $\mathbb{P}^n \backslash X \subseteq \bigcup_i U_i \backslash (U_i \cap X)$ if we have $X = \mathbb{P}^n$, then we are done as the empty set is a subset of all sets. Suppose that we have $[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n \backslash X$ and $X \neq \mathbb{P}^n$. Then not all x_i are 0. Say that $x_j \neq 0$. Then it lies in U_j. But because the point is not in X also, it lies in $U_j \backslash (U_j \cap X)$. So we have $\mathbb{P}^n \backslash X \subseteq \bigcup_i U_i \backslash (U_i \cap X)$.

Because each $U_i \cap X$ is closed on each U_i, we know that $U_i \backslash (U_i \cap X)$ is open in U_i. By the previous problem, we have that $U_i \backslash (U_i \cap X)$ is open in $\mathbb{P}^n$. Then $\bigcup_i U_i \backslash (U_i \cap X)$ is open in $\mathbb{P}^n$. So $\mathbb{P}^n \backslash X$ is open and therefore X is closed in $\mathbb{P}^n$. □

(d) Conclude that $X \subseteq \mathbb{P}^n$ is closed (resp. open) if and only if $X \cap U_i$ is closed (resp. open) for each i.

Proof. $X \subseteq \mathbb{P}^n$ closed $\rightarrow X \cap U_i$ closed on U_i for each i by part a, and the converse by part c.

$X \subseteq \mathbb{P}^n$ open implies that $X \cap U_i$ open on U_i for each i: If $X \subseteq \mathbb{P}^n$ open, $X \cap U_i$ is open on $\mathbb{P}^n$ because each U_i is open in $\mathbb{P}^n$.

$X \cap U_i$ open on each U_i implies that $X \subseteq \mathbb{P}^n$ is open by part b. □

Exercise 3: Practice with homogenization. Given an ideal $I \subseteq k[x_1, \dots, x_n]$, recall that we write
$$H(I) = (\{H(f) : f \in I\}) \subseteq k[x_1, \dots, x_{n+1}]$$
for the homogenization. Given a homogeneous ideal $J \subseteq k[x_1, \dots, x_{n+1}]$, let
$$J' = \{F(x_1, \dots, x_n, 1) : F \in J\} \subseteq k[x_1, \dots, x_n],$$
called the dehomogenization.

(a) (Optional) Check that J' is an ideal. You don't need to write this part up.

Proof. Let $f, g \in J'$. Then $f = F(x_1, \dots, x_n, 1), g = G(x_1, \dots, x_n, 1)$ where F, G homogeneous polynomials in J. Let $\deg F = n$, $\deg G = m$ where $m \leqslant n$. Then $F + x_{n+1}^{n-m} G$ is homogeneous of degree n. It also lies in J. Then

$$\begin{aligned} f + g &= F(x_1, \dots, x_n, 1) + G(x_1, \dots, x_n, 1) \\ &= F(x_1, \dots, x_n, 1) + 1^{n-m} G(x_1, \dots, x_n, 1) \end{aligned}$$

So $f + g \in J'$ because it is the evaluation of $F + x_{n+1}^{n-m} G$ for $x_{n+1} = 1$.

Suppose that $f \in J'$, $g \in k[x_1, \dots, x_n]$. Then $f = F(x_1, \dots, x_n, 1)$ for some $F \in J$. Since J is an ideal, we know that $F(x_1, \dots, x_n, x_{n+1}) g \in J$. We let

$$G = F(x_1, \dots, x_{n+1}) g = G_0 + G_1 + \cdots + G_d$$

and we can homogenize G by multiplying each form in G by various powers of x_{n+1}. This is possible because J homogeneous so each form lies in J:

$$G' = G_0 x_{n+1}^d + G_1 x_{n+1}^{d-1} + \cdots + G_{d-1} x_{n+1} + G_d$$

Then we have $G'(x_1, \dots, x_n, 1) = F(x_1, \dots, x_n, 1) g = fg \in J'$. □

(b) Prove that if $J \subseteq k[x_1, \dots, x_{n+1}]$ is a radical homogeneous ideal, then J' is radical.

Proof. Suppose that $F^d(x_1, \dots, x_n, 1) \in J'$. We want to show that $F(x_1, \dots, x_n, 1) \in J'$. Since $F^d(x_1, \dots, x_n, 1) \in J'$, we know that:

$$F^d(x_1, \dots, x_{n+1}) \in J$$

where F^d homogeneous by the definition of J'. Since J is radical, $F \in J$. Suppose that F is not homogeneous. Then

$$F = f_0 + f_1 + \cdots + f_k$$

and

$$F^d = (f_0 + f_1 + \cdots + f_k)^d$$

Take the lowest nonzero homogeneous form f_j where $j < k$. Then $f_j^d \neq 0$ and therefore, $F^d = f_j^d + \cdots + f_k^d$. So F^d is not homogeneous contradiction. So F is homogeneous and $F(x_1, \dots, x_n, 1) \in J'$ which concludes the proof. □

(c) Prove that if $I \subseteq k[x_1, \dots, x_n]$ is radical, then $H(I) \subseteq k[x_1, \dots, x_{n+1}]$ is radical.

Proof. We want to show that if

$$f^n = f_0 + f_1 + \cdots + f_d \in H(I)$$

then $f \in H(I)$ if I is radical. Since $H(I)$ homogeneous, we know that each $f_i \in H(I)$. Because

$$H(I) = (\{H(f) : f \in I\})$$

each f_i, homogeneous of degree i can be written as a sum of homogenized polynomials from I which generate $H(I)$, let's say g_{i_j} of degree i. Then

$$\begin{aligned} f_i(x_1, \dots, x_{n+1}) &= g_{i_1}(x_1, \dots, x_{n+1}) + \cdots + g_{i_j}(x_1, \dots, x_{n+1}) \\ f_i(x_1, \dots, x_n, 1) &= g_{i_1}(x_1, \dots, x_n, 1) + \cdots + g_{i_j}(x_1, \dots, x_n, 1) \end{aligned}$$

where each $g_{i_j}(x_1, \dots, x_n, 1) \in I$. So we know that $f^n(x_1, \dots, x_n, 1) \in I$. Then $f(x_1, \dots, x_n, 1) \in I$. But because

$$H(I) = (\{H(f) : f \in I\})$$

we have

$$H(f(x_1, \dots, x_n, 1)) = f(x_1, \dots, x_{n+1}) \in H(I)$$

which shows that $H(I)$ is radical. □

Exercise 4: Intersections of linear spaces. The vanishing of a linear equation on projective space is called a *hyperplane*. Let

$$\Lambda_1 = \mathbb{V}(a_{1,1}x_1 + \cdots + a_{1,n+1}x_{n+1}), \ldots, \Lambda_m = \mathbb{V}(a_{m,1}x_1 + \cdots + a_{m,n+1}x_{n+1})$$

be hyperplanes in $\mathbb{P}^n$ with $m \leqslant n$. Show that $\Lambda_1 \cap \cdots \cap \Lambda_m \neq \emptyset$.

Proof. Since we are looking for points in the intersection of $\mathbb{P}^n$, we want to find the points $[x_1 : \cdots : x_{n+1}]$ that satisfy the system:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n+1}x_{n+1} &= 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n+1}x_{n+1} &= 0 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n+1}x_{n+1} &= 0 \end{aligned}$$

From linear algebra, we are finding the kernel of T for the matrix:

$$T = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n+1} \end{bmatrix}$$

This maps an $n+1$ dimensional space onto one of m dimensions, where $n+1 > m$, so there is a non-trivial kernel, which means that there are nonzero solutions $[x_1 : \cdots : x_{n+1}]$, which is what we wanted. □

Exercise 5: Let $I \subseteq k[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal. Let $S_d \subseteq k[x_1, \ldots, x_{n+1}]/I$ be the set of degree d forms.

(a) Prove that S_d is a finite-dimensional vector space.

Proof. We first have that S_d is the image of the map

$$\varphi : k[x_1, \ldots, x_{n+1}] \to k[x_1, \ldots, x_{n+1}]/I$$

obtained from some restricted domain which is the set of homogeneous polynomials of degree d.

To show closure under addition: If $\overline{F}, \overline{G} \in S_d$, we have $F, G \in k[x_1, \ldots, x_{n+1}]$ of degree d. The sum of homogeneous polynomials of degree d is also of degree d. Then the image of $F + G$ is $\overline{F} + \overline{G} \in S_d$.

Closure under multiplication from k: Suppose that $\overline{F} \in S_d$. Then we have $F \in k[x_{11}, \ldots, x_{n+1}]$ of degree d. Then kF is also of degree d. So $k\overline{F} \in S_d$.

Finite dimensional. There are a finite number of generators for homogeneous polynomials of degree d in $k[x_1, \ldots, x_{n+1}]$. Notice that

$$H_d = \{x_1^{i_1} \cdots x_{n+1}^{i_{n+1}} : i_1, \ldots, i_{n+1} \geqslant 0, i_1 + \cdots + i_{n+1} = d\}$$

generate homogeneous polynomials of degree d in $k[x_1, \ldots, x_{n+1}]$. Then if we have a form $f \in k[x_1, \ldots, x_{n+1}]/I$ where $f = \varphi(F)$ for

$$F = \sum a_i x_1^{i_1} \cdots x_{n+1}^{i_{n+1}}$$

homogeneous of degree d, then

$$\varphi(F) = \sum a_i \varphi(x_1^{i_1} \cdots x_{n+1}^{i_{n+1}})$$

We find that
$$\{\varphi(x_1^{i_1}\cdots x_{n+1}^{i_{n+1}}) : x_1^{i_1}+\cdots+x_{n+1}^{i_{n+1}} \in H_d\}$$
span the image. We can reduce this to linearly independent basis by removing the linearly dependent terms. So S_d is a finite dimensional vector space. □

(b) (Extra Credit) Can you give an upper bound on the dimension of S_d in terms of n and d?

Proof. By the previous problem, our basis for S_d is a subset of
$$\mathcal{B}_d = \{\varphi(x_1^{i_1}\cdots x_{n+1}^{i_{n+1}}) : x_1^{i_1}+\cdots+x_{n+1}^{i_{n+1}} \in H_d\}$$
where
$$H_d = \{x_1^{i_1}\cdots x_{n+1}^{i_{n+1}} : i_1,\dots,i_{n+1} \geqslant 0, i_1+\cdots+i_{n+1} = d\}$$
We have that $|\mathcal{B}_d| \leqslant |H_d|$ because we have a surjective mapping from $H_d \to \mathcal{B}_d$ by taking φ of the elements of H_d. So we can always match distinct element of $\mathcal{B}_d$ with distinct elements of H_d. So $|H_d|$ is the upper bound. Using stars and bars, we get $\binom{d+n}{n}$ number of elements of H_d. So $\binom{d+n}{n}$ is the upper bound. □