Math172Hw7

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October 22, 2023

Exercise 1: Show that the number of ways to split the multiplication $x_0x_1 \cdots x_n$ into n pairwise multiplications (ab) using brackets is equal to the Catalan number C_n. For example, for $n = 3$ we have $C_3 = 5$ ways:

$$(x_0(x_1(x_2x_3))), (x_0((x_1x_2)x_3)), ((x_0x_1)(x_2x_3)), ((x_0(x_1x_2))x_3), (((x_0x_1)x_2)x_3)$$

Proof. If B_n is the number of ways to split a multiplication of $x_0x_1 \ldots x_n$, then we will show that the relation:

$$\sum_{i \geqslant 1}^{n} B_{i-1}B_{n-i}$$

holds similarly to the Catalan numbers and that $C_0 = B_0$, $C_1 = B_1$. For B_0, B_1, we have

$$x_0, (x_0, x_1)$$

So $B_0 = B_1 = 1 = C_0 = C_1$. So the base case is done. Now consider the general case. Suppose that we have $x_0 \ldots x_n$. Then we will count this by:

- Then we count the number of ways to place the first parenthesis. The first outermost parenthesis will split the numbers into left and right portions. Suppose that we chose to place the first outermost parenthesis after x_i:

  $$(x_0x_1 \ldots x_{i-1})(x_i \ldots x_n)$$

  So for $i = 1$ to n, we have distinct ways of placing the outermost parenthesis.

- Now we apply induction on the LHS for $x_0 \ldots x_{i-1}$ to get B_{i-1} and for the RHS, we get also B_{n-i}. So therefore, for each $i = 1, n$, we have: $B_{i-1}B_{n-i}$ ways to split parenthesis:

  $$\sum_{i \geqslant 1}^{n} B_{i-1}B_{n-i}$$

which is what we wanted. □

Exercise 2: Show that Catalan numbers satisfy the following recurrence relation:

$$C_n = \frac{2(2n-1)}{n+1}C_{n-1}$$

Proof. We see that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

So multiplying them together:

$$\begin{aligned}
\frac{2(2n-1)}{n+1}C_{n-1} &= \frac{2(2n-1)}{n(n+1)}\binom{2n-2}{n-1} \\
&= \frac{2n(2n-1)(2n-2)(2n-3)\cdots(1)}{n^2(n+1)(n-1)!(n-1)!} \\
&= \frac{2n!}{(n+1)n!n!} \\
&= \frac{1}{n+1}\binom{2n}{n} \\
&= C_n
\end{aligned}$$

which concludes the proof. □

Exercise 3: This continuation of Problem 3 in Problem set 6. Let $F(x)$, $G(x)$ be formal power series.

- If $G(0) = 0$ then we have $\frac{d}{dx}(F(G(x))) = G'(x)F'(G(x))$.

  Proof. We will first show that if $F(x)$ is a power series, then

  $$\frac{d}{dx}(F(x))^n = nF'(x)(F(x))^{n-1}$$

  We start with a result from the last hw:

  $$\frac{d}{dx}F(x)G(x) = F'(x)G(x) + F(x)G'(x)$$

  So we will induct on the number n. If $n = 1$, then $\frac{d}{dx}F(x)^1 = 1F'(x)(F(x))^0 = F'(x)$, which is true.

  Inductive Step: Suppose that $\frac{d}{dx}(F(x))^n = nF'(x)(F(x))^{n-1}$. We will show this for $n+1$. Consider:

  $$\begin{aligned}
  \frac{d}{dx}(F(x))^{n+1} &= \frac{d}{dx}(F(x)^n F(x)) \\
  &= \frac{d}{dx}(F(x))^n F(x) + (F(x))^n F'(x) \\
  &= nF'(x)(F(x))^{n-1}F(x) + (F(x))^n F'(x) \\
  &= nF'(x)(F(x))^n + (F(x))^n F'(x) \\
  &= nF'(x)(F(x))^n + F'(x)(F(x))^n \\
  &= (n+1)F'(x)(F(x))^n
  \end{aligned}$$

  So now we can use this. We have that $F(x) = \sum_{i\geqslant 0} a_i x^i$, $G(x) = \sum_{i\geqslant 1} b_i x^i$. So $F(G(x)) = \sum_{i\geqslant 0} a_i G(x)^i$. And:

  $$\begin{aligned}
  \frac{d}{dx}F(G(x)) &= \frac{d}{dx}\sum_{i\geqslant 0} a_i G(x)^i \\
  &= \sum_{i\geqslant 1} a_i iG'(x)(G(x))^{i-1} \\
  &= G'(x)\sum_{i\geqslant 1} a_i iG(x)^{i-1} \\
  &= G'(x)F'(G(x))
  \end{aligned}$$

  which concludes the proof. □

- If $F'(x) = G'(x)$ then $F(x) - G(x) = const$, that is $F(x) - G(x)$ has only the constant term.

  Proof. Let $F(x) = \sum_{i \geqslant 0} a_i x^i$ and $G(x) = \sum_{j \geqslant 0} b_j x^j$. Then $F'(x) = \sum_{i \geqslant 1} i a_i x^{i-1}$ and $G'(x) = \sum_{j \geqslant 1} j b_j x^{j-1}$. Now:

  $$F'(x) - G'(x) = \sum_{i \geqslant 1} i(a_i - b_i)x^{i-1} = 0$$

  But we know that the coefficient of each x^i is 0. So

  $$i(a_i - b_i) = 0$$

  and $i \neq 0$, so $a_i - b_i = 0$, $a_i = b_i$ for $i \geqslant 1$. Then we have the rewrite:

  $$F(x) = a_0 + \sum_{i \geqslant 1} a_i x^i$$
  $$G(x) = b_0 + \sum_{i \geqslant 1} a_i x^i$$

  and therefore,

  $$F(x) - G(x) = a_0 - b_0 = \text{const}$$

  which concludes the proof. □

Exercise 4: Use generating functions to show that the number of partitions of n with not more than k parts of size k for each $k \geqslant 1$ is equal to the number of partitions of n without parts of sizes $2, 6, 12, \ldots, k(k+1), \ldots$ where k are integers.

Proof. We start by saying that m_i is the number of parts of the partition of size i. Then the number of partitions of n in the generating function is counted by having the total sum of im_i contribute 1 to the exponent of n:

$$\sum_{0 \leqslant m_1, m_2, \ldots} x^{m_1 + 2m_2 + \cdots}$$

and since each of the m_i are bounded by $0 \leqslant m_i \leqslant i$, we get:

$$\begin{aligned} \sum_{0 \leqslant m_1, m_2, \ldots} x^{m_1 + 2m_2 + \cdots} &= \left(\sum_{0 \leqslant m_1 \leqslant 1} x^{m_1} \right) \left(\sum_{0 \leqslant m_2 \leqslant 2} x^{m_2} \right) \cdots \\ &= (1+x)(1+x^2+x^4)(1+x^3+x^6+x^9)\cdots \\ &= \prod_{i \geqslant 1} (1 + x^i + \cdots + x^{i^2}) \end{aligned}$$

Then

$$(1 - x^i)(1 + x^i + \cdots + x^{i^2}) = 1 - x^{i^2 + i}$$

So we find:

$$\left(\prod_{i \geqslant 1} (1 + x^i + \cdots + x^{i^2}) \right) \left(\prod_{i \geqslant 1} (1 - x^i) \right) = (1 - x^2)(1 - x^6)\cdots = \prod_{i \geqslant 1} (1 - x^{i(i+1)})$$

In short:

$$\begin{aligned} \left(\prod_{i \geqslant 1} (1 + x^i + \cdots + x^{i^2}) \right) \left(\prod_{i \geqslant 1} (1 - x^i) \right) &= \prod_{i \geqslant 1} (1 - x^{i(i+1)}) \\ \prod_{i \geqslant 1} (1 + x^i + \cdots + x^{i^2}) &= \frac{\prod_{i \geqslant 1} (1 - x^{i(i+1)})}{\prod_{i \geqslant 1} (1 - x^i)} \\ &= \frac{1}{\prod_{\substack{i \geqslant 1 \\ i \neq k(k+1)}} (1 - x^i)} \end{aligned}$$

and just to verify that this is the number of partitions without $k(k+1)$ parts:

$$\sum_{\substack{0\leqslant m_1,m_2,\dots \\ m_i:i\neq k(k+1)}} x^{m_1+2m_2+\cdots}$$

which is just:

$$\begin{aligned}
\left(\sum_{m_1\geqslant 0} x^{m_1}\right)\left(\sum_{m_3\geqslant 0} x^{3m_3}\right)\cdots &= \prod_{m_i:i\neq k(k+1)}\left(\sum_{m_i\geqslant 0} x^{im_i}\right) \\
&= \prod_{\substack{i\geqslant 1 \\ i\neq k(k+1)}}\left(\frac{1}{1-x^i}\right)
\end{aligned}$$

which is the same as the expression above. □

Exercise 5: Let $p_{even}(n)$ denote the number of partitions of n with even number of parts and $p_{odd}(n)$ denote the number of partitions of n with odd number of parts.

- Show that the generating function of $p_{even}(n) - p_{odd}(n)$ is equal to $\prod_{i\geqslant 1}(1-x^{2i-1})$.

  Proof. We have that the number of partitions:

$$\sum_{n\geqslant 0} p(n)x^n = \sum_{n\geqslant 0}(p_{\text{even}} + p_{\text{odd}})x^n$$

  And we have

$$\sum_{n\geqslant 0} p(n)x^n = \sum_{0\leqslant m_1,m_2,\dots} x^{m_1+2m_2+\cdots}$$

  where m_i are the number of parts of size i. Then we denote $(-1)^{m_i}$ as the $-p_{\text{odd}}$ contribution to the partition:

$$\begin{aligned}
\sum_{0\leqslant m_1,m_2,\dots} &= \left(\sum_{m_1\geqslant 0} x^{m_1}\right)\left(\sum_{m_2\geqslant 0} x^{2m_2}\right)\cdots \\
&\to \left(\sum_{m_1\geqslant 0} (-1)^{m_1}x^{m_1}\right)\left(\sum_{m_2\geqslant 0} (-1)^{m_2}x^{2m_2}\right)\cdots \\
&= \left(\sum_{m_1\geqslant 0} (-x)^{m_1}\right)\left(\sum_{m_2\geqslant 0} (-x^2)^{m_2}\right)\cdots \\
&= \prod_{i\geqslant 1}\frac{1}{1+x^i}
\end{aligned}$$

  Now we just compare this to $\prod_{i\geqslant 1}(1-x^{2i-1})$. Observe that:

$$\prod_{i\geqslant 1}(1-x^{2i-1}) = (1-x)(1-x^3)(1-x^5)\cdots$$

  Now when we multiply both sides by $\prod_{i\geqslant 1}(1+x^i)$, we get:

$$\begin{aligned}
\prod_{i\geqslant 1}(1+x^i)\prod_{i\geqslant 1}(1-x^{2i-1}) &= (1-x)(1-x^3)(1-x^5)\cdots\prod_{i\geqslant 1}(1+x^i) \\
&= (1-x^2)(1-x^3)(1-x^5)\cdots\prod_{i\geqslant 2}(1+x^i) \\
&\vdots \\
&= 1
\end{aligned}$$

Therefore dividing both sides by $\prod_{i\geqslant 1}(1+x^i)$, we get:

$$\prod_{i\geqslant 1}(1+x^i)\prod_{i\geqslant 1}(1-x^{2i-1}) = 1$$

$$\prod_{i\geqslant 1}(1-x^{2i-1}) = \frac{1}{\prod_{i\geqslant 1}(1+x^i)}$$

and therefore, the formula we proved above was equivalent to $\prod_{i\geqslant 1}(1-x^{2i-1})$. $\square$

- For each n determine if $p_{even}(n) > p_{odd}(n), p_{even}(n) < p_{odd}(n)$ or $p_{even}(n) = p_{odd}(n)$.

  Proof. We can rewrite $\prod_{i\geqslant 1}(1-x^{2i-1})$ as

  $$\prod_{j\geqslant 1}(1-x)^j\prod_{i\geqslant 1}(1+x^2+\cdots+x^{2i})$$

  Now for some N sufficiently large, the coefficients of x^i no longer change in $\prod_{i\geqslant 1}(1-x^{2i-1})$. We take if i is even, we just need $N = i/2$ and if i is odd, we take $N = (i+1)/2$. This is because the product of the term with $(1-x^{2N-1})$ will no longer affect all coefficients less than i and it will subtract 1 from the current x^i coefficient. From this, for i even, we take:

  $$\prod_{j\geqslant 1}^{i}(1-x^j)\prod_{k\geqslant 1}^{i/2}(1+x^2+\cdots x^{2i})$$

  So the left product only has positive coefficients and the right product has negative coefficients for odd powers and positive ones for even ones. So all even n turns out to have $p_{\text{even}}(n) > p_{\text{odd}}(n)$ except for $n = 2$. This is because the coefficient of x^2 is 1 and we subtract 1 from it giving us $p_{\text{odd}}(2) = p_{\text{even}}(2)$. Now for odd n, we only have the left product to consider because the right product only contributes even summands. And the odd summands in the left product are negative, making the coefficients of odd terms negative. So $p_{\text{even}}(n) < p_{\text{odd}}(n)$ when n is odd. $\square$