Stat134Hw8

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Exercise 1: Professor Smith gives his lectures in T-shirts of three colors: gray, red, and blue. For each lecture (independently of others), he chooses gray with probability 0.5, red with probability 0.3, and blue with probability 0.2. There are 9 lectures in April.

(a) What is the probability that he wears T-shirts of each color three times in April?

Answer. We can represent this event as $(0.5x + 0.3y + 0.2z)^9$. Then the probability is given by the coefficient of $x^3y^3z^3$. This is

$$\begin{aligned}\binom{9}{3,3,3}(0.5)^3(0.3)^3(0.2)^3 &= \frac{9!}{3!3!3!}(0.125)(0.027)(0.008)\\ &= 7!(0.125)(0.027)(0.008) = 0.13608\end{aligned}$$

(b) What is the probability that he wears a gray T-shirt exactly 8 times in April?

Answer. This turns into a binomial dist, with gray being 0.5, and not gray as 0.5. So the probability is

$$\binom{9}{8}(0.5)^8 = 9 * (0.5)^8$$

Exercise 2: (X, Y) is a uniformly random point inside the triangle with vertices $(0, 0), (2, 0), (0, 1)$.

(a) Find the marginal probability density functions for X and Y.

Answer. The area under the triangle is 1. So for the pdf of X, we take the hypotenuse line as the function:

$$p_X(x) = \begin{cases} 1 - \frac{1}{2}x & \text{if } 0 \leqslant x \leqslant 2 \\ 0 & \text{if } otherwise \end{cases}$$

And do the same for Y:

$$p_Y(y) = \begin{cases} 2 - 2y & \text{if } 0 \leqslant y \leqslant 1 \\ 0 & \text{if } otherwise \end{cases}$$

(b) Are X and Y independent?

Answer. No since if $x = 0$, $y = 1$. This means that $P(X = 0, Y \neq 1) = 0 \neq P(X = 0) \cdot P(Y \neq 1)$

Exercise 3: For (X, Y) from Problem 2, compute $\mathbb{E}[XY]$.

Answer. By definition, we need to compute:

$$\int_0^1 \int_0^{2-2y} xyp_X(x)p_Y(y)\ dx\ dy$$

We have:

$$\begin{aligned}
\int_0^1 \int_0^{2-2y} xy(1 - x/2)(2 - 2y)\ dx\ dy &= \int_0^1 \int_0^{2-2y} xy(2 - x - 2y + xy)\ dx\ dy \\
&= \int_0^1 \int_0^{2-2y} 2xy - x^2y - 2xy^2 + x^2y^2\ dx\ dy \\
&= \int_0^1 (x^2y - \frac{x^3y}{3} - x^2y^2 + \frac{x^3y^2}{3})\Big|_0^{2-2y}\ dy \\
&= \int_0^1 (2-2y)^2y - (2-2y)^3\frac{y}{3} - (2-2y)^2y^2 + (2-2y)^3\frac{y^2}{3}\ dy \\
&= \frac{1}{9}
\end{aligned}$$

Exercise 4: Suppose that X and Y are jointly continuous with probability density function

$$f(x, y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x > 0, y > 0 \\ 0 & \text{if otherwise} \end{cases}$$

Are X and Y independent?

Answer. We first get the marginal distributions of X and Y by integrating along the other variables:

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{\infty} 6e^{-(2x+3y)} \, dy \\ &= \int_{0}^{\infty} 6e^{-(2x+3y)} \, dy \\ &= \left(-2e^{-(2x+3y)}\right) \Big|_0^{\infty} \\ &= 0 - (-2e^{-(2x)}) = 2e^{-2x} \end{aligned}$$

and for the other:

$$\begin{aligned} p_Y(y) &= \int_{-\infty}^{\infty} 6e^{-(2x+3y)} \, dx \\ &= \int_{0}^{\infty} 6e^{-(2x+3y)} \, dx \\ &= \left(-3e^{-(2x+3y)}\right) \Big|_0^{\infty} \\ &= 0 - (-3e^{-(3y)}) = 3e^{-3y} \end{aligned}$$

Now we check that the product gives us the pdf. For $x > 0, y > 0$:

$$p_X(x) \cdot p_Y(y) = 2e^{-2x} \cdot 3e^{-3y} = 6e^{-2x-3y} = f(x, y)$$

For when $x \leqslant 0$ or $y \leqslant 0$, we still get $p_X(x) \cdot p_Y(y) = 0 = f(x, y)$. So indeed, the variables are independent.

Exercise 5: We celebrate the solar eclipse on April 8 by establishing a remarkable fact about spheres, known already to Archimedes. We let a random vector (X, Y, Z) be uniformly distributed on the unit sphere in $\mathbb{R}^3$. Equivalently, denoting S to the sphere, $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, the distribution of (X, Y, Z) is given by setting

$$P((X, Y, Z) \in A) = \frac{Area(A)}{Area(S)}, \ A \subseteq S$$

(a) For each t, compute the area of the subset of S given by

$$\{(x, y, z) \in \mathbb{R}^3 : x \leqslant t, x^2 + y^2 + z^2 = 1\}.$$

Hint: You can use without a proof the following fact from multivariate calculus: If a surface in three-dimensional space is obtained by revolving the graph of function $y = f(x)$, $a \leqslant x \leqslant b$, around the x-axis, then its area is computed as

$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx\,.$$

Answer. We want to revolved a function $y = f(x)$ around the x-axis, so first, set $z = 0$:

$$x^2 + y^2 = 1$$

Now, we know that $-1 \leqslant x \leqslant t$, so we just plug in $f(x) = \sqrt{1 - x^2}$, $f'(x) = \frac{1}{2}(-2x)(1 - x^2)^{-1/2} = \frac{-x}{\sqrt{1-x^2}}$:

$$\begin{aligned} 2\pi \int_{-1}^t \sqrt{1 - x^2}\sqrt{1 + (f'(x))^2}\, dx &= 2\pi \int_{-1}^t \sqrt{1 - x^2}\sqrt{1 + \left(\frac{x^2}{1 - x^2}\right)}\, dx \\ &= 2\pi \int_{-1}^t \sqrt{1 - x^2}\sqrt{\frac{1}{1 - x^2}}\, dx \\ &= 2\pi \int_{-1}^t 1\, dx \\ &= 2\pi(t + 1) \end{aligned}$$

(b) Use (a) to compute the CDF of X and show that the marginal distribution of X is uniform, i.e. $X \sim \mathtt{Uniform}[-1, 1]$

Answer. We know that $P((X, Y, Z) \in A)$ is the area of A over S. So we compute the area of S by setting $t = 1$. We get

$$SurfaceArea(S) = 2\pi(2) = 4\pi$$

Now the CDF for $-1 \leqslant x \leqslant 1$:

$$\frac{2\pi(t + 1)}{4\pi} = \frac{t + 1}{2}$$

To get the pdf, take the derivative:

$$\frac{1}{2}$$

which shows that x is uniformly distributed between $-1, 1$.