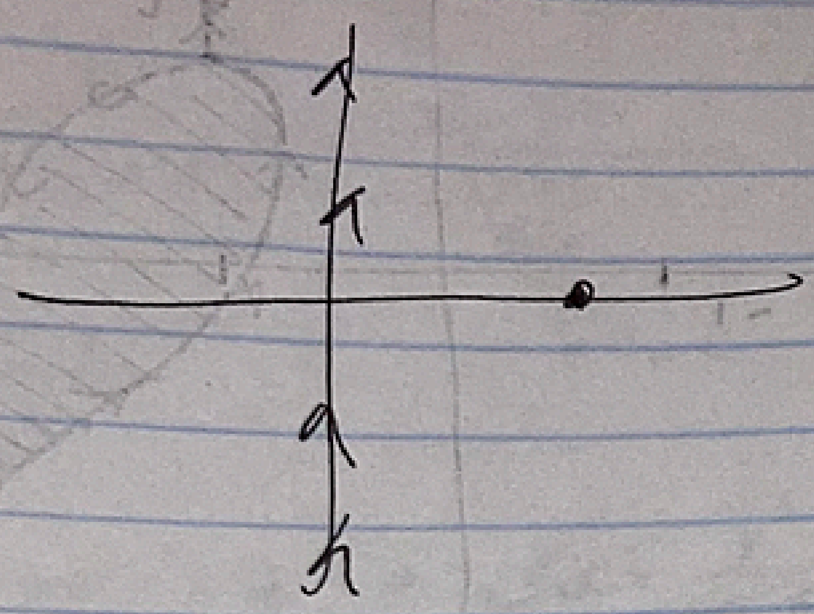
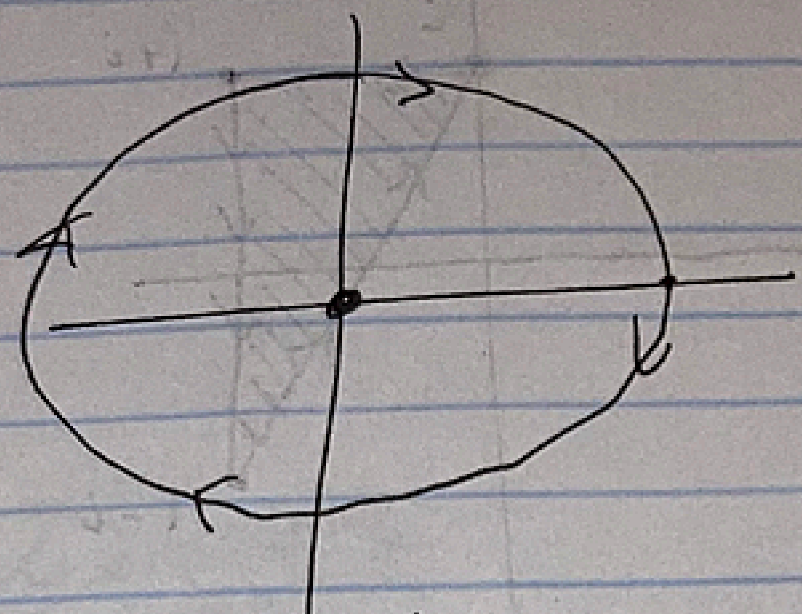
① $[z:1:0:\infty i] = [w:0:1:e^{i\theta}]$





$$z = \frac{w-1}{w-e^{i\theta}} \Big/ \frac{0-1}{0-e^{i\theta}} = \frac{w-1}{w-e^{i\theta}} \Big/ \frac{1}{e^{i\theta}}$$

$$= \frac{we^{i\theta}-e^{i\theta}}{w-e^{i\theta}} = \frac{w-1}{we^{-i\theta}-1}$$

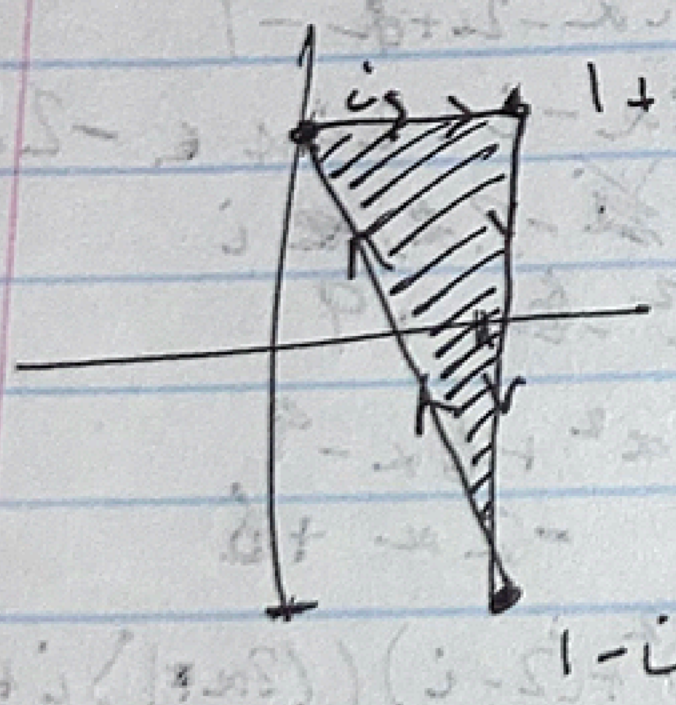
$$zwe^{-i\theta} - z = w-1$$

$$-z = w - zwe^{-i\theta} - 1$$

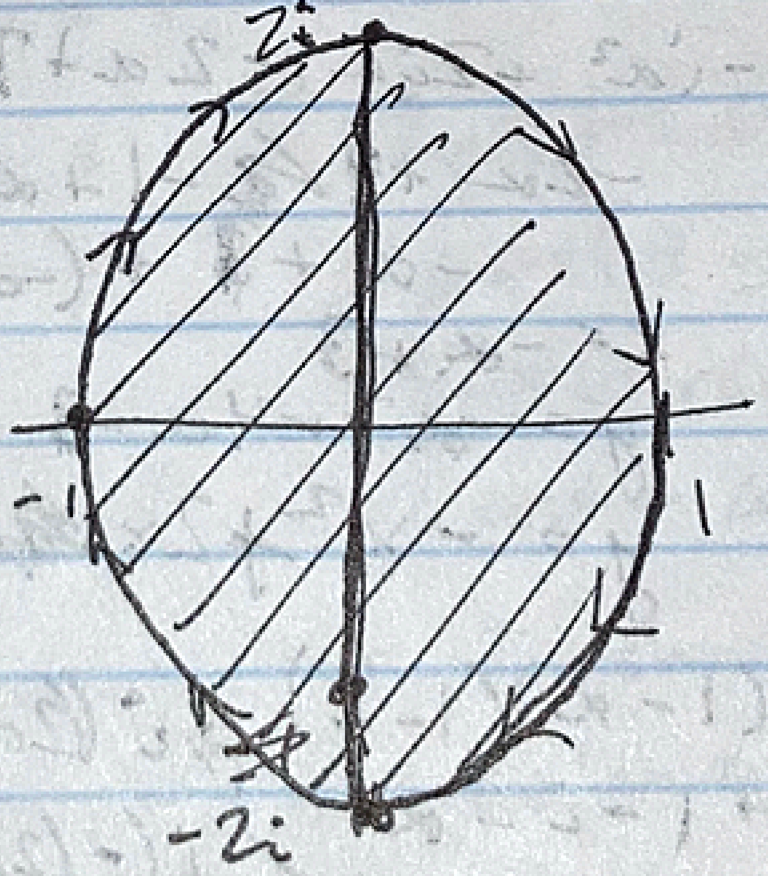
$$-z = w(1-ze^{-i\theta}) - 1$$

$$\boxed{w = \frac{-z+1}{1-ze^{-i\theta}}} \quad \theta \in [0, 2\pi)$$

④



a)



~~line $i + \alpha i \to (i+\alpha i)^2$~~

~~$= (i(1+\alpha))^2$~~

~~$= -(1+2\alpha+\alpha^2)$~~

line $i+\alpha \to (i+\alpha)^2$

$= -1+\alpha^2+2i\alpha$

$x = -1+\alpha^2$

$y = 2\alpha$

$x+1 = \left(\frac{y}{2}\right)^2$

$x = \frac{y^2}{4} - 1$

line $1-i+\alpha i$

$= 1+(\alpha-1)i \to (1+(\alpha-1)i)^2$

$= 1+2(\alpha-1)i-(\alpha-1)^2$

$1-(\alpha^2-2\alpha+1) + 2(\alpha-1)i$

$-\alpha^2+2\alpha + 2(\alpha-1)i$

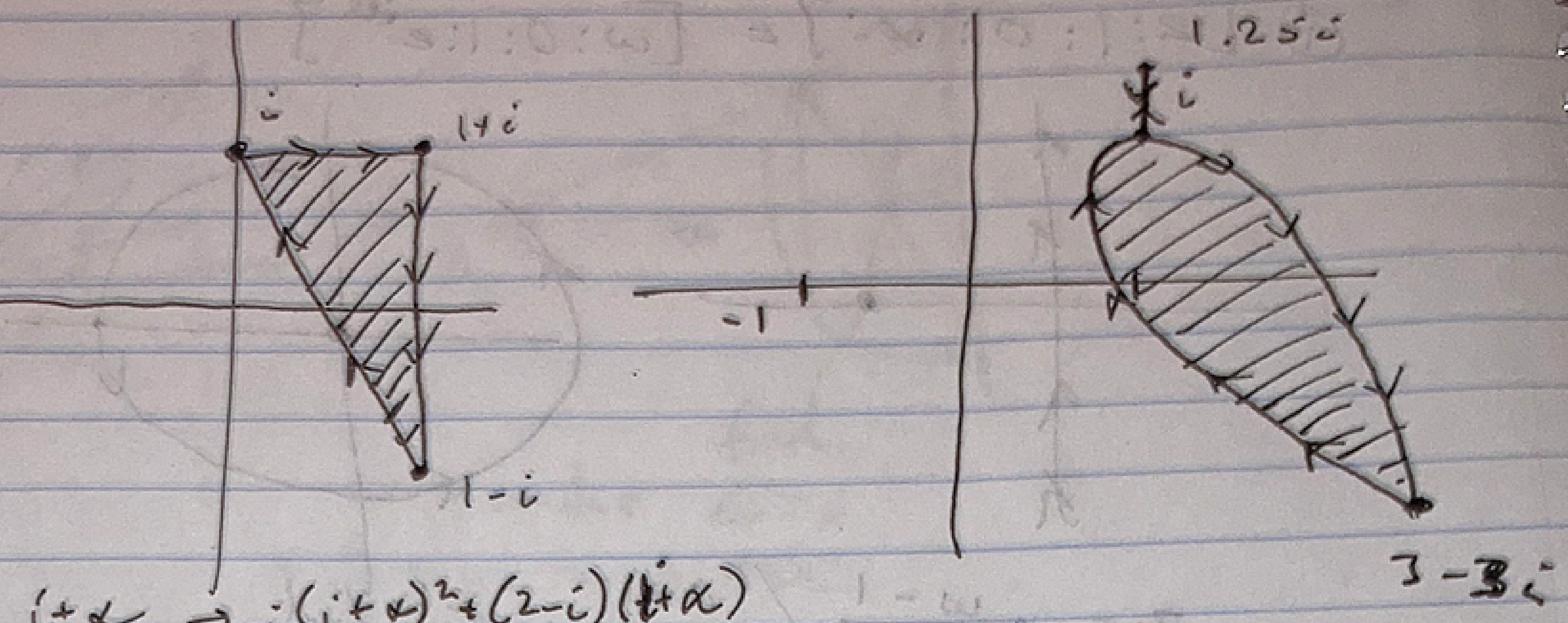
$x = -\alpha^2+2\alpha$

$y = 2(\alpha-1)$ $\quad y^2 = 4(\alpha^2-2\alpha+1$

$y^2 = 4(-x+1)$

$\left(1-\frac{y^2}{4}\right) = x$

b)



line $i+\alpha \to i(i+\alpha)^2+(2-i)(\cancel{1}i+\alpha)$

$= i(-1+2i\alpha+\alpha^2)+(2i+2\alpha+1-\alpha i)$

$= \cancel{-2\alpha+2\alpha}+1 \cancel{-}i+i\alpha^2+2i-\alpha i$

$= 1+(-1-\alpha+2+\alpha^2)i = 1+(1-\alpha+\alpha^2)i$

$x=1$ for $\alpha\in[0,1]$

$y=1-\alpha+\alpha^2$

line $1+(\alpha-1)i \to i(1+(\alpha-1)i)^2+(2-i)(1+(\alpha-1)i)$

$i(-\alpha^2+2\alpha+2(\alpha-1)i)+2-i+(2\alpha-2-i\alpha+i)i$

$-i\alpha^2+2i\alpha-2\alpha+2+2-i+2i\alpha-2i+\alpha-1$

$-2\alpha+4\cancel{2}-1+\alpha-i\alpha^2+2i\alpha-i+2i\alpha-2i$

$-\alpha+\cancel{1}+(-\alpha^2+4\alpha-1\cancel{2}-2\cancel{\alpha})\cancel{\alpha}i$

$x=-\alpha+3$ $\qquad x^2=\alpha^2-6\alpha+9$

for $\alpha\in[0,2]$ $\quad y=-\alpha^2+4\alpha-3$ $\qquad -x^2=-\alpha^2+6\alpha-9$

$y=-x^2+2x\cancel{+}$ $\qquad -2\alpha+6$

line $\alpha i+(1-\alpha)(1-i) \to i((2\alpha-1)i+1-\alpha)^2+(2-i)((2\alpha-1)i+1-\alpha)$

$=2\alpha i+1-i-\alpha$ $\qquad i(-(2\alpha-1)^2+(1-\alpha)^2+2(2\alpha-1)(1-\alpha)i)$

$=(2\alpha-1)i+1-\alpha$ $\qquad +(4\alpha-2)i+2-2\alpha+(2\alpha-1)-(1-\alpha)i$

$\alpha\in[0,1]$ $\quad x=-2(2\alpha-1)(1-\alpha)+2\cancel{-2\alpha+2\alpha}-1$

$y=-(2\alpha-1)^2+(1-\alpha)^2+(4\alpha-2)-(1-\alpha)$

$x=-2(2\alpha-2\alpha^2-1+\alpha)+1$

$y=-(4\alpha^2-4\alpha+1)+(1-2\alpha+\alpha^2)+5\alpha-3$

$x=4\alpha^2-6\alpha+3$ $\qquad y=-3\alpha^2+7\alpha-3$

$7x+6y=10\alpha^2+3$ $\qquad 3x+4y=10\alpha-3$

$70x+60y-(3x+4y+3)^2=30 \quad 30=70x+60y-(3x+4y+3)^2$

(5) $z \to \sqrt{z^2-1}$

$z = \alpha + \beta i$

Fix α;

$z \mapsto \sqrt{(\alpha+\beta i)^2 - 1} = \sqrt{\alpha^2 + 2\alpha\beta i - \beta^2 - 1}$

$= \sqrt{\alpha^2 - \beta^2 - 1 + 2\alpha\beta i}$

$z = re^{i\theta}$ $\quad \sqrt{(re^{i\theta}-1)(re^{i\theta}+1)}$

$0 < \theta < \pi$

$r > 0$ $\quad = \sqrt{re^{i\theta}-1}\,\sqrt{re^{i\theta}+1}$

(6) $\cancel{\varphi(re^{2\theta})}$

$\varphi(re^{i\theta}) = \frac{1}{2\pi}(1-r^2)\int_0^{2\pi} \frac{u(e^{i\alpha})\,d\alpha}{1+r^2-2r\cos(\theta-\alpha)}$

$= \frac{1}{2\pi}(1-r^2)\int_0^{2\pi} \frac{\cos 2e^{i\alpha}}{1+r^2-2r\cos(\theta-\alpha)}\,d\alpha$

$= \frac{1}{2\pi}(1-r^2)\int_0^{2\pi} \frac{2\cos^2 e^{i\alpha} - 1}{1+r^2-2r\cos(\theta-\alpha)}\,d\alpha$

$\zeta = e^{i\alpha}$

$z = re^{i\theta}$

$= \frac{1}{2\pi i}\oint_C \cos 2\zeta \left(\frac{1}{1-z\zeta^{-1}} + \frac{\bar{z}\zeta}{1-\bar{z}\zeta}\right)\frac{d\zeta}{\zeta}$

$= \frac{1}{2\pi i}\oint_C \frac{\cos 2\zeta\, d\zeta}{\zeta - z} + \frac{\cos 2\zeta\, \bar{z}}{1-\bar{z}\zeta}\,d\zeta$

$= \frac{1}{2\pi i}\cos 2z - \oint_C \frac{\cos 2\zeta}{\zeta - \frac{1}{\bar{z}}}\,d\zeta$

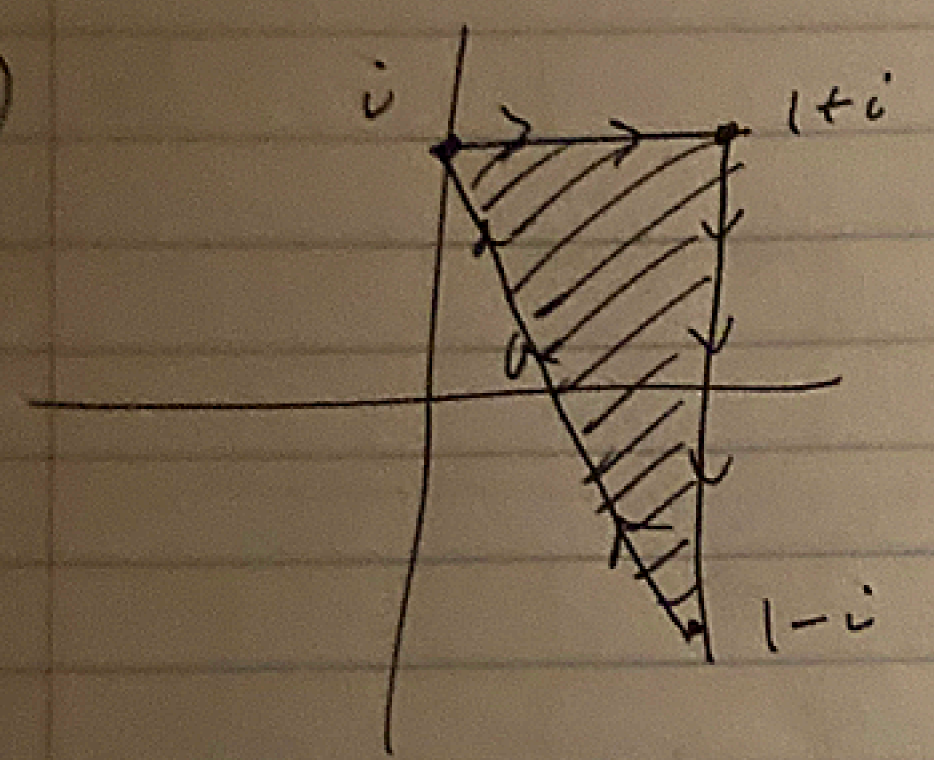
$= \frac{1}{2\pi i}\left(\cos 2z - \cos\frac{2}{\bar{z}}\right)$

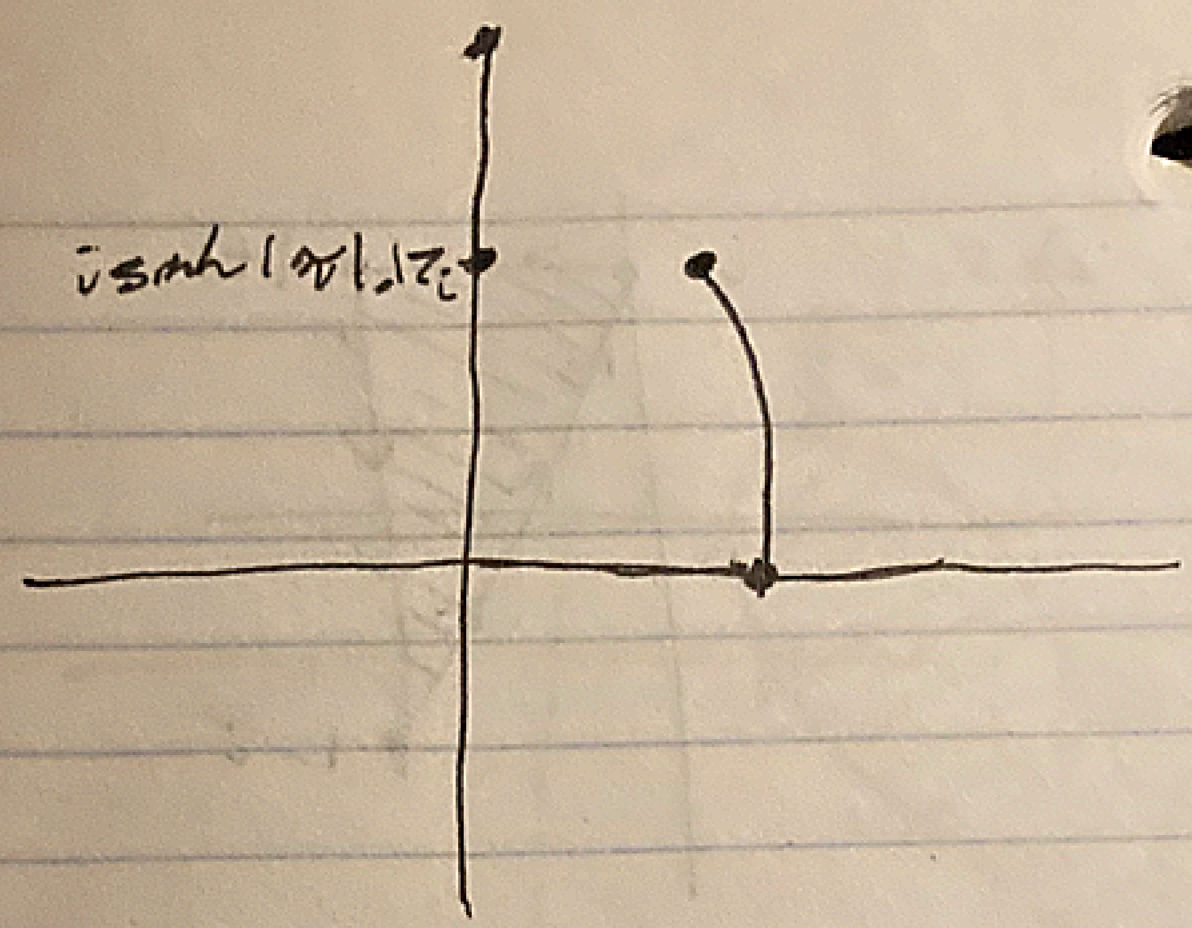
$\cancel{= \frac{1}{2\pi i}\left(\cos 2re^{i\theta} - \cos\frac{2}{r}e^{-i\theta}\right)}$

$\cancel{e^{i2re^{i\theta}} \quad e^{-i2re^{i\theta}}}$

$= \frac{1}{2\pi i}\left(\cos 2z - \cos\frac{2z}{\|z\|}\right)$

d)





line $i+\alpha \to i+\alpha+\frac{1}{i+\alpha} = i+\alpha+\frac{\alpha-i}{\alpha^2+1}$

$= \frac{(\alpha^2+1)(i+\alpha)+\alpha-i}{\alpha^2+1}$

$= \frac{\alpha^2 i+\alpha^3+i+\alpha+\alpha-i}{\alpha^2+1}$

$= \frac{\alpha^3+2\alpha+(\alpha^2)i}{\alpha^2+1}$

$x = \frac{\alpha^3+2\alpha}{\alpha^2+1} \qquad y = \frac{\alpha^2}{\alpha^2+1}$

$x^2 = \frac{\alpha^2(\alpha^2+2)^2}{(\alpha^2+1)^2} = y \cdot \frac{(\alpha^2+2)^2}{(\alpha^2+1)}$

$= y \cdot \frac{\alpha^4+4\alpha^2+4}{\alpha^2+1} =$

$x = \alpha y + \frac{2\alpha}{\alpha^2+1}$

$\cosh^2\alpha - \sinh^2\alpha$

$= \left(\frac{e^\alpha+e^{-\alpha}}{2}\right)^2 - \left(\frac{e^\alpha-e^{-\alpha}}{2}\right)^2$

$= 1$

$= \cos(i+\alpha)$

$= \cos i \cos\alpha - \sin i \sin\alpha$

$= \cosh 1 \cos\alpha - \sinh 1 \cdot i \sin\alpha$

$x = \cosh 1 \cos\alpha$

$y = -\sinh 1 \sin\alpha$

$\frac{x^2}{\cosh^2 1} + \frac{y^2}{\sinh^2 1} = 1$

$\alpha = 0,\ x = \cosh 1,\ y = 0$

$\alpha = 1,\ x = .83,\ y = -.98$

line $1+(\alpha-1)i \to \cos(1+(\alpha-1)i) = \cos 1 \cos(\alpha-1)i - \sin 1 \sin(\alpha-1)i$

$= \cos 1 \cosh(\alpha-1) - i \sin 1 \sinh(\alpha-1)$

$x = \cos 1 \cosh(\alpha-1) \qquad y = -\sin 1 \sinh(\alpha-1)$

$\frac{x^2}{\cos^2 1} - \frac{y^2}{\sin^2 1} = 1$

when $\alpha = 0$, $x = .83$ $y = .98$

when $\alpha = 2$, $x = .83$, $y = -.98$

line $\alpha i + (1-\alpha)(1-i) = (2\alpha-1)i + 1-\alpha$

$z \to \cos(z) = \cos(1-\alpha + (2\alpha-1)i)$

$= \cos(1-\alpha)\cosh(2\alpha-1) - \sin(1-\alpha)\sin((2\alpha-1)i)$

$= \cos(1-\alpha)\cosh(2\alpha-1) - i\sin(1-\alpha)\sinh(2\alpha-1)$

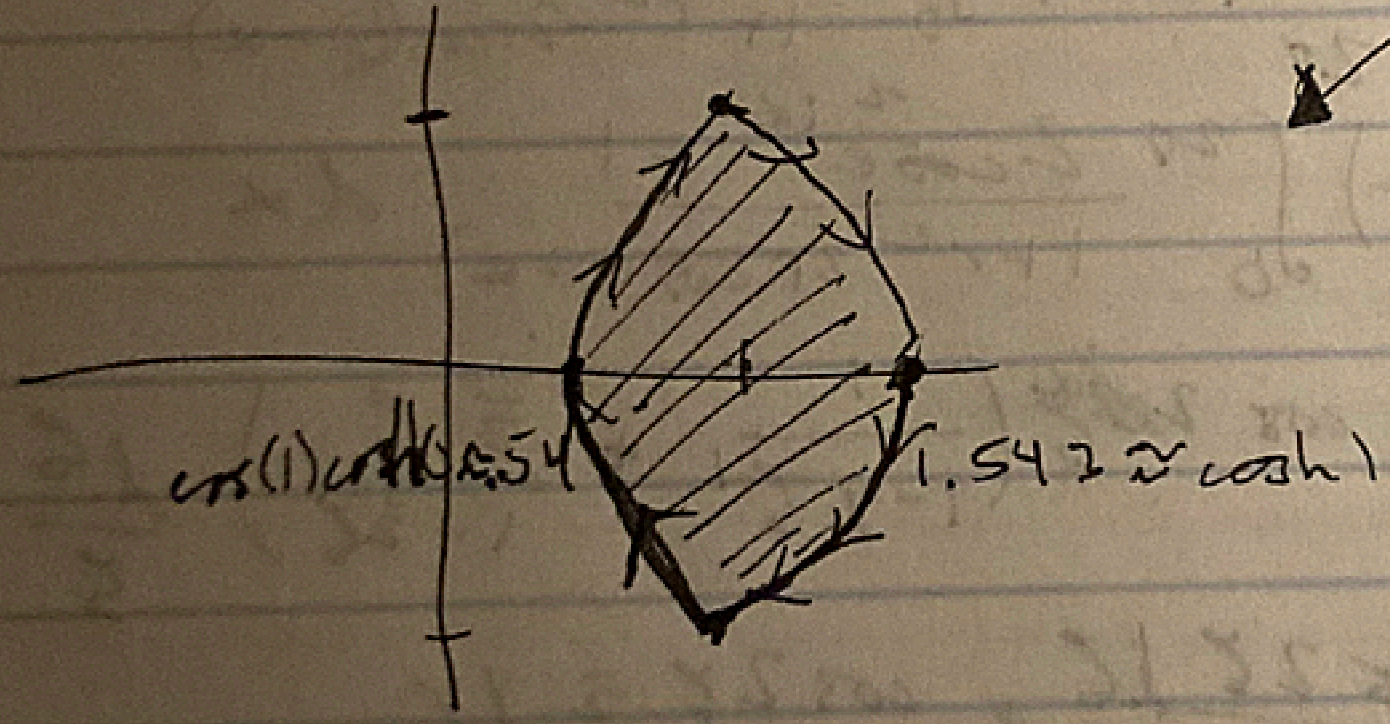
$x = \cos(1-\alpha)\cosh(2\alpha-1)$

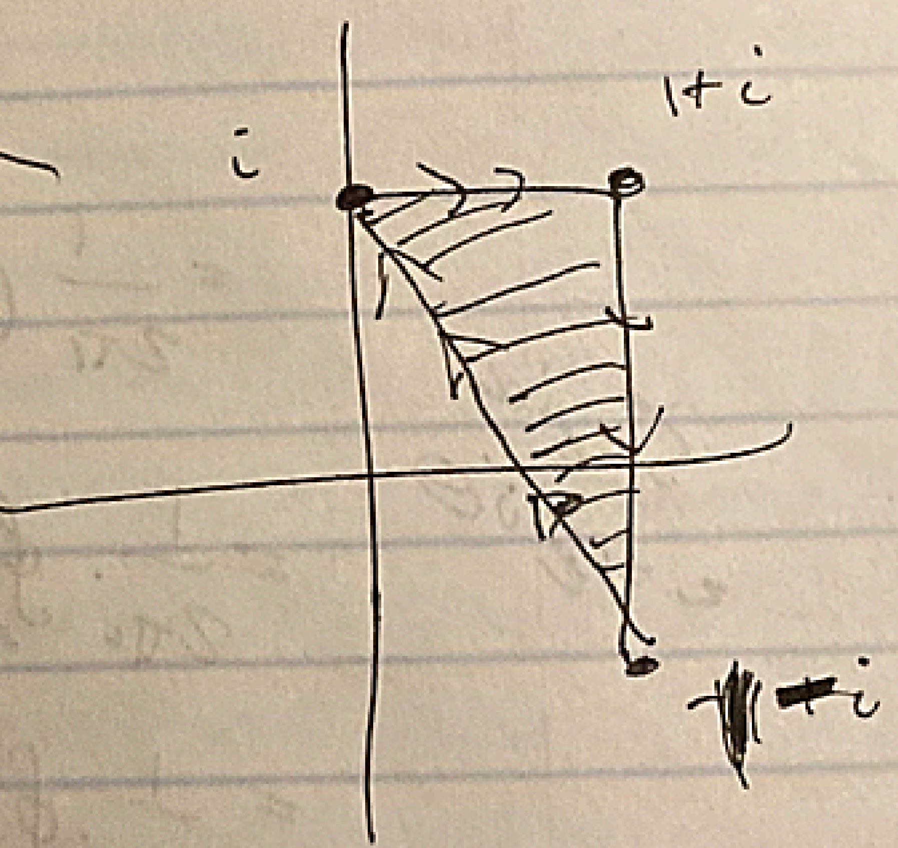
$y = -\sin(1-\alpha)\sinh(2\alpha-1)$

$$\frac{x^2}{\cosh^2(2\alpha-1)} + \frac{y^2}{\sinh^2(2\alpha-1)} = 1$$

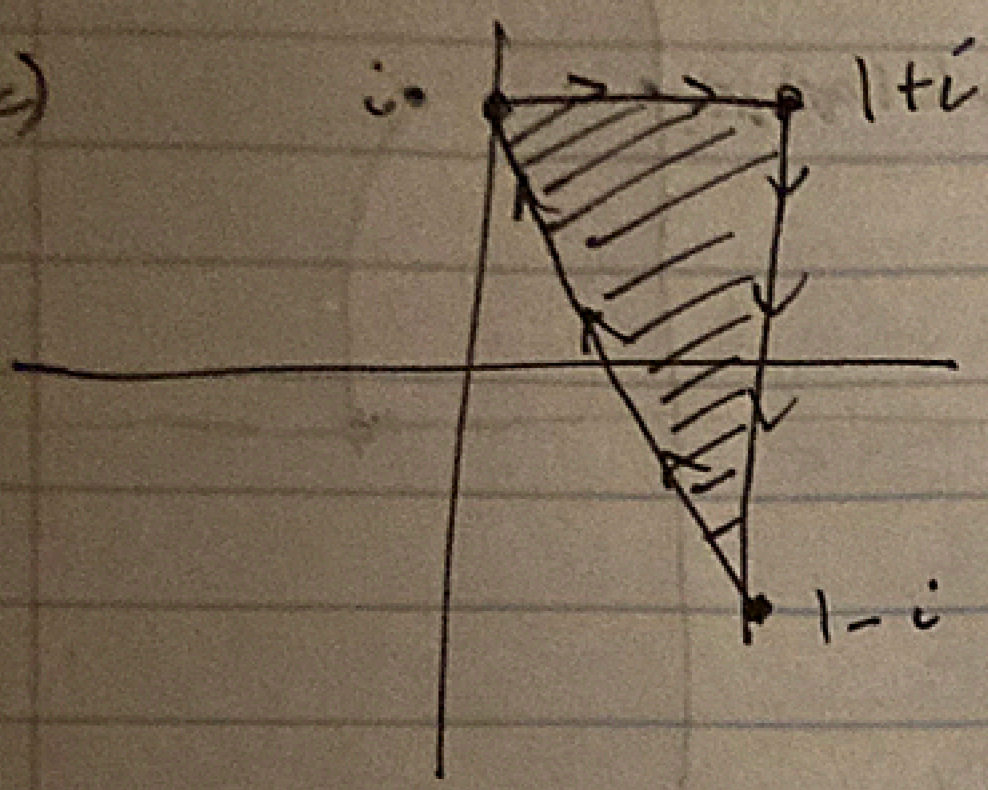
$x = (\cos 1 \cos\alpha + \sin 1 \sin\alpha)(\cosh 2\alpha \cosh 1 - \sinh 2\alpha \sinh 1)$

$$\frac{x^2}{\cos^2(1-\alpha)} - \frac{y^2}{\sin^2(1-\alpha)} = 1$$





c)



line: $i+\alpha \to \cancel{i\alpha + \frac{1}{i+\alpha}} = \cancel{i+\alpha + \frac{\alpha - i}{\alpha^2+1}}$

$$2\cos(i+\alpha) = e^{i\alpha - 1} + e^{1-i\alpha}$$

$$= e^{-1}(\cos\alpha + i\sin\alpha) + e(\cos -\alpha + i\sin -\alpha)$$

$$x = e^{-1}\cos\alpha + e\cos\alpha = (e^{-1}+e)\cos\alpha$$

$$y = e^{-1}\sin\alpha - e\sin\alpha = (e^{-1}-e)\sin\alpha$$

line: $1+(\alpha-1)i \to$

(5)

$z = e^{i\theta}$ $\quad z \to ze^{-\alpha} + z^{-1}e^{\alpha}$

$$z \to e^{i\theta-\alpha} + e^{\alpha - i\theta} = 2\cosh(i\theta - \alpha)$$

$$= 2\cos(\theta + i\alpha)$$

$$= 2\cos\theta\cos i\alpha - 2\sin\theta\sin i\alpha$$

$$= 2\cosh\alpha\cos\theta - 2i\underline{\sinh\alpha}\sin\theta$$

$$x = 2\cosh\alpha\cos\theta$$

$$y = 2\sinh\alpha\sin\theta$$

we have $\frac{x^2}{4\cosh^2\alpha} + \frac{y^2}{4\sinh^2\alpha} = 1$

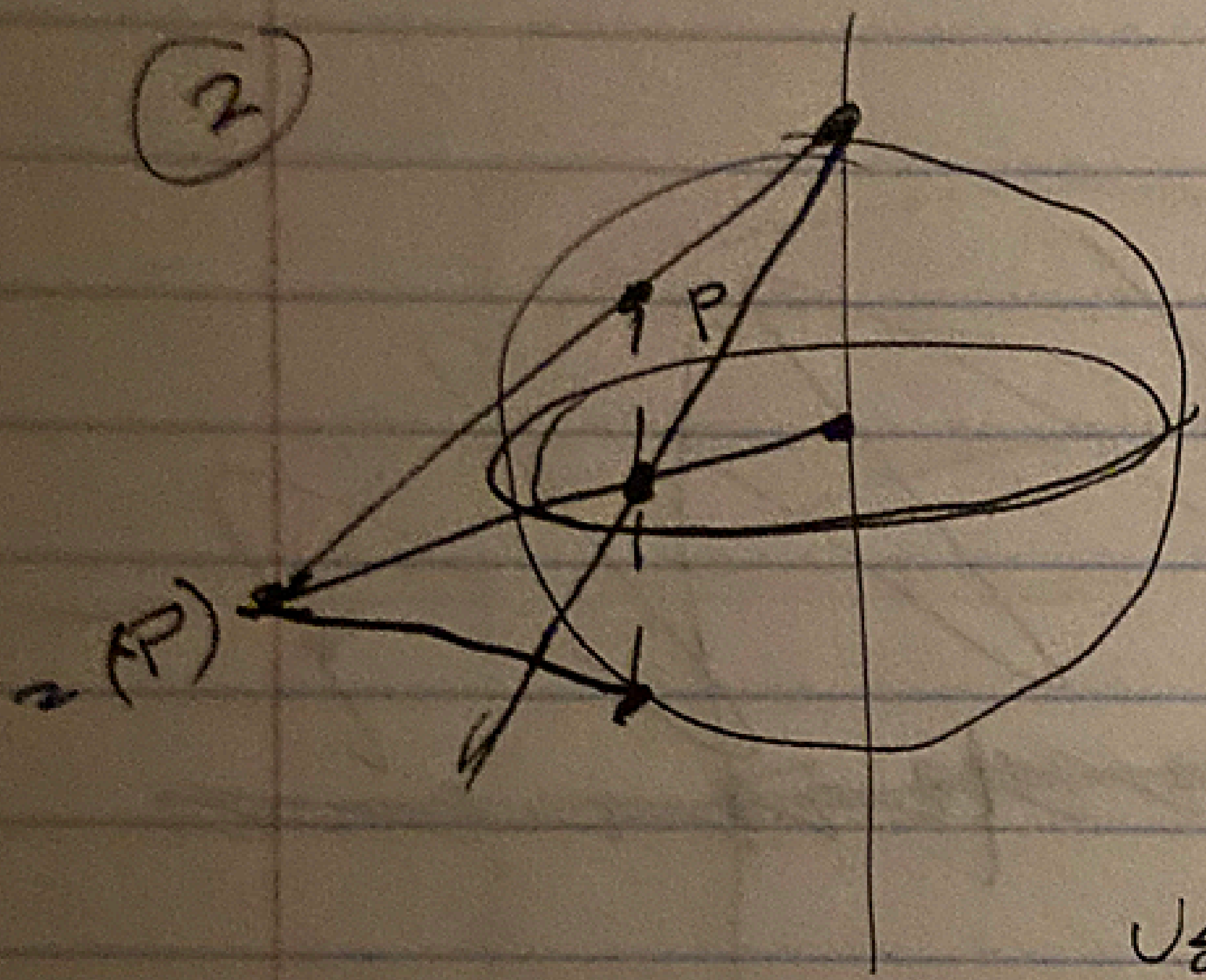
which is equation of ellipse

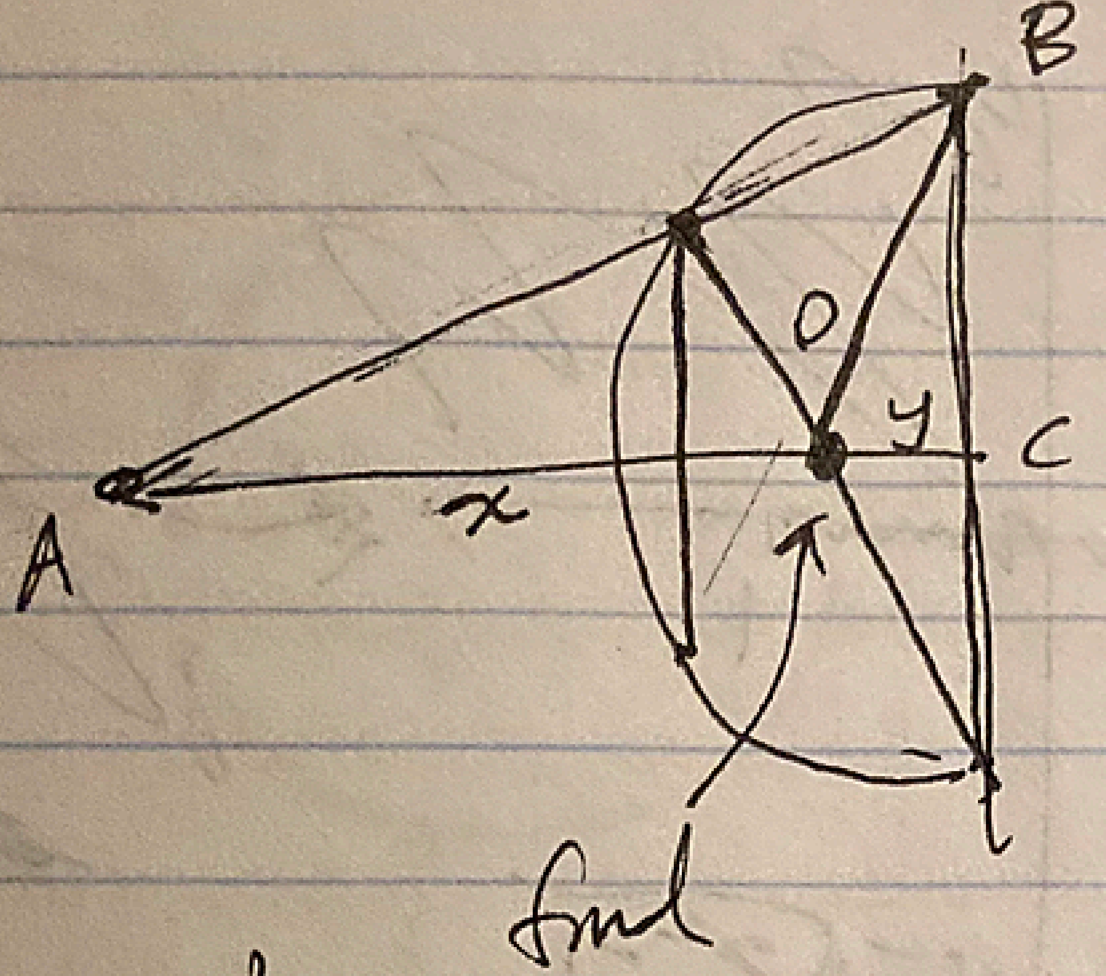
Interior maps to ext because

$$0 \to 0 + \frac{1}{0}e^{\alpha} = \infty$$

$z \to ze^{-\alpha} + z^{-1}e^{\alpha}$ is holomorphic, so interior of curves map to interiors of curve in codomain.

(2)





Using similar Δ's,

$\Delta ABC \sim \Delta BDC$

so $\dfrac{AC}{BC} = \dfrac{BC}{DC} \to \dfrac{AC}{1} = \dfrac{1}{DC}$

$\hookrightarrow |z| = \dfrac{1}{|m(z)|}$

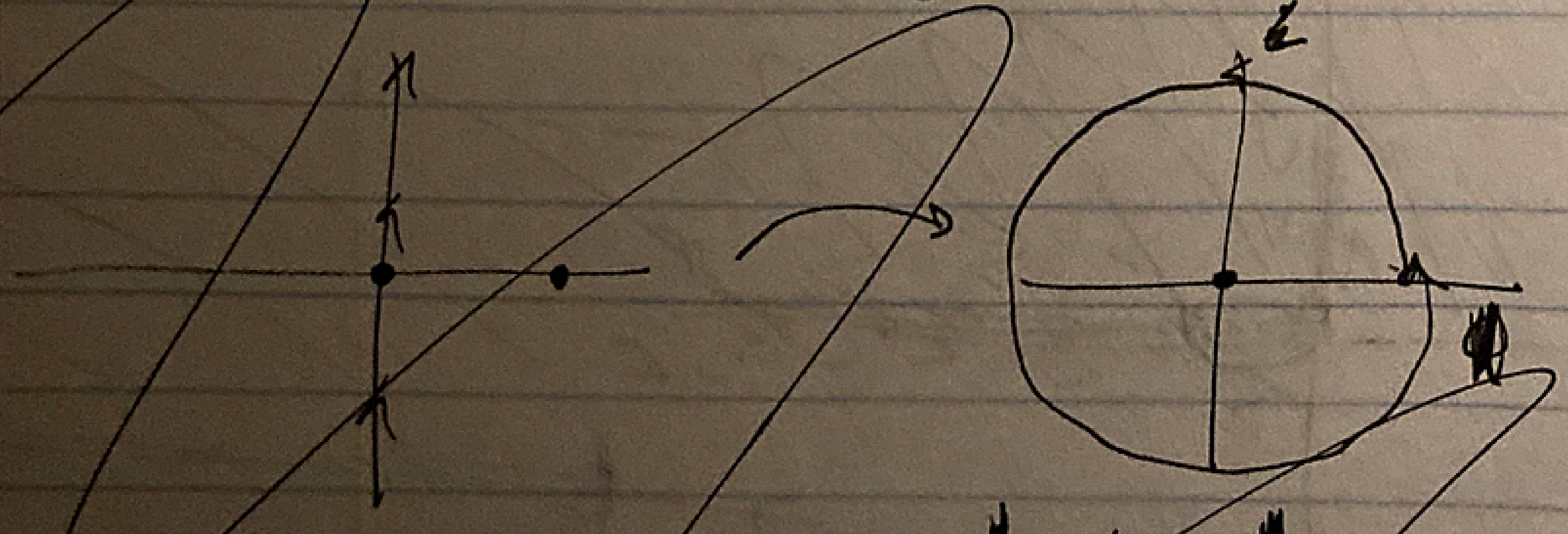
$\dfrac{1}{|z|} = |m(z)|$

a) Denote the map as $m(z)$

then the map sends $z \mapsto \dfrac{1}{z} = \dfrac{z}{\|z\|}$

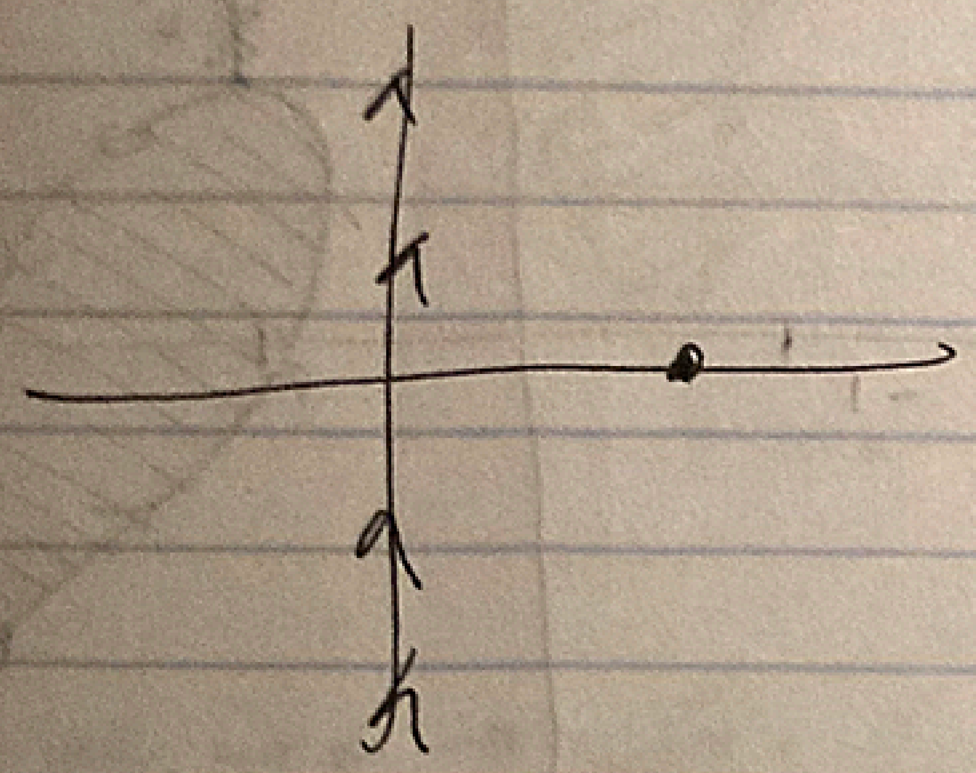
b) Geometrically, this is a ~~sliding~~ ~~~~ inverts the scaling of z by its norm.

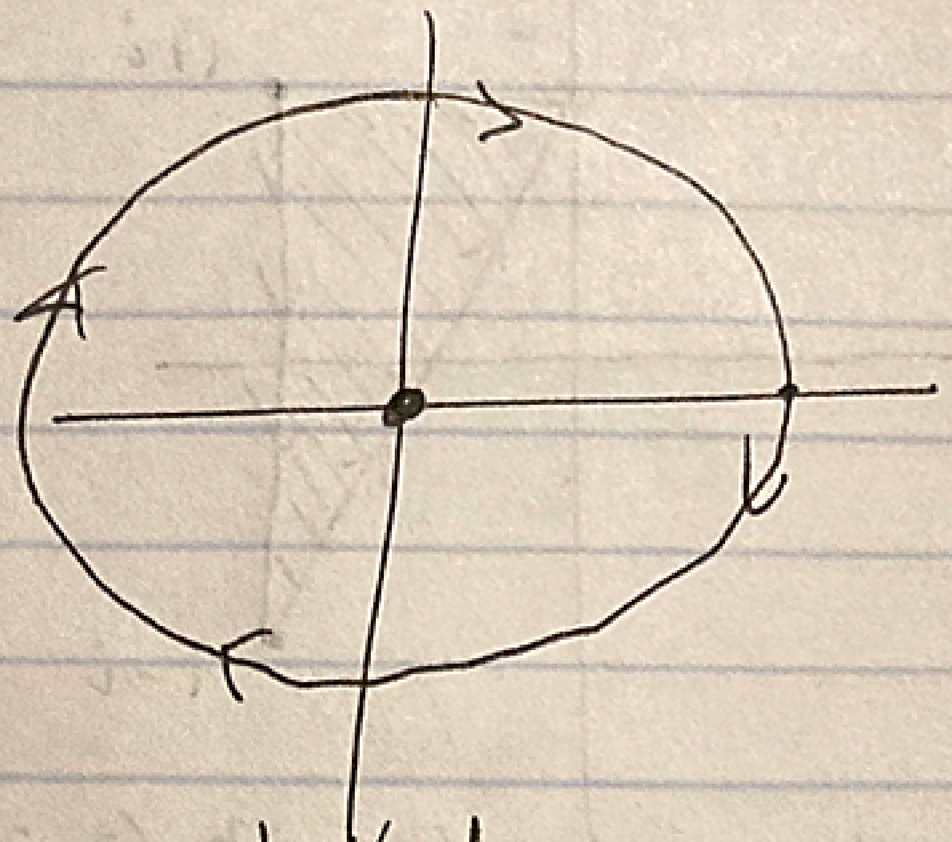
(1) $[z : 1 : 0 : \infty i] = [w : 0 : \; : e^{i\theta}]$



$\dfrac{z-0}{z-\infty i} \Big/ \dfrac{1-0}{1-\infty i} = \dfrac{w - \;}{w - e^{i\theta}} \Big/ \dfrac{0 - \;}{0 - e^{i\theta}}$

① $[z:1:0:\infty i] = [w:0:1:e^{i\theta}]$





$$z = \frac{w-1}{w-e^{i\theta}} \Big/ \frac{0-1}{0-e^{i\theta}} = \frac{w-1}{w-e^{i\theta}} \Big/ \frac{1}{e^{i\theta}}$$

$$= \frac{we^{i\theta}-e^{i\theta}}{w-e^{i\theta}} = \frac{w-1}{we^{-i\theta}-1}$$

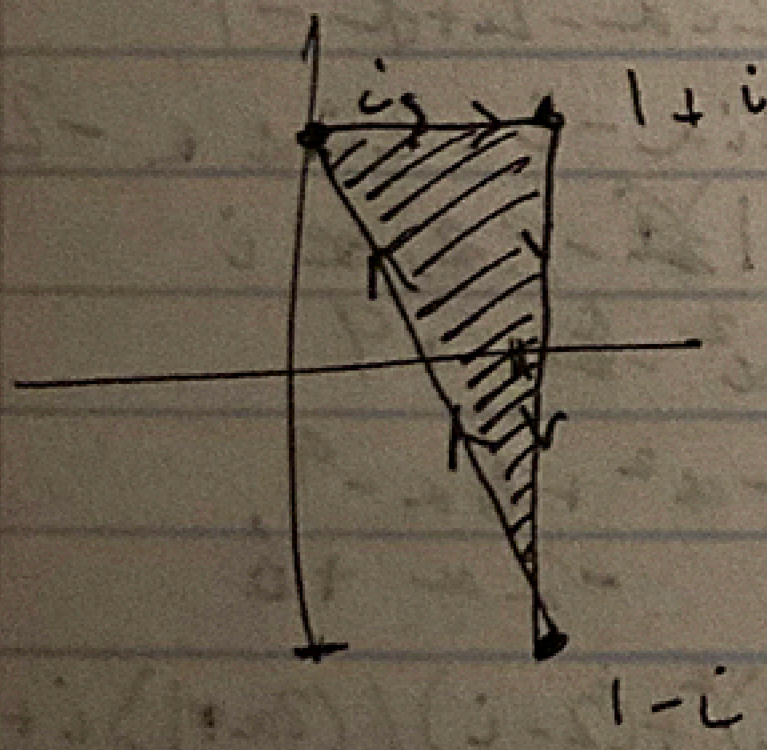
$$zwe^{-i\theta} - z = w - 1$$

$$-z = w - zwe^{-i\theta} - 1$$

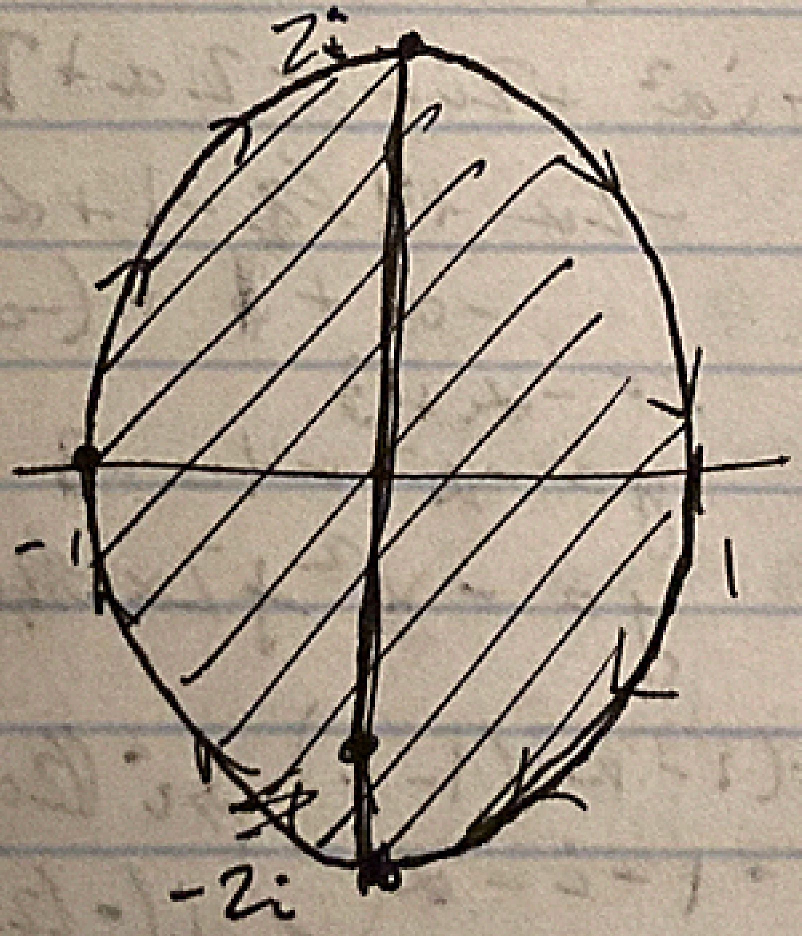
$$-z = w(1-ze^{-i\theta}) - 1$$

$$\boxed{w = \frac{-z+1}{1-ze^{-i\theta}}} \qquad \theta \in [0, 2\pi)$$

④



a)



~~line $i+\alpha i \to (i+\alpha i)^2 = (i(1+\alpha))^2 = -(1+2\alpha+\alpha^2)$~~

line $i+\alpha \to (i+\alpha)^2$

$= -1+\alpha^2+2i\alpha$

$x = -1+\alpha^2$

$y = 2\alpha$

$x+1 = \left(\frac{y}{2}\right)^2$

$x = \frac{y^2}{4} - 1$

line $1-i+\alpha i$

$= 1+(\alpha-1)i \to (1+(\alpha-1)i)^2$

$= 1+2(\alpha-1)i-(\alpha-1)^2$

$1-(\alpha^2-2\alpha+1)+2(\alpha-1)i$

$-\alpha^2+2\alpha+2(\alpha-1)i$

$x = -\alpha^2+2\alpha$

$y = 2(\alpha-1)i \qquad y^2 = 4(\alpha^2-2\alpha+1)$

$y^2 = 4(-x+1) \qquad \left(1-\frac{y^2}{4}\right) = x$