Math113Hw2

Trustin Nguyen

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Homework 2

Exercise 1:

1. Write the following permutations as products of disjoint cycles and hence calculate their orders and signs:

   (a) $(1\,2)(1\,2\,3\,4)(1\,2)$:

   We see that $1 \mapsto 2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 1 \mapsto 2$, $2 \mapsto 1 \mapsto 2 \mapsto 1$. So the cycle is $(1\,3\,4\,2)$.

   (b) $(1\,2\,3)(2\,3\,5)(3\,4\,5)(4\,5)$

   There is $1 \mapsto 2$, $2 \mapsto 3 \mapsto 1$, $4 \mapsto 5 \mapsto 3 \mapsto 5$, $5 \mapsto 4 \mapsto 5 \mapsto 2 \mapsto 3$. Therefore, we have the cycle: $(1\,2)(3\,4\,5)$.

2. What is the largest possible order of an element of S_5? What about S_9? Justify your answers.

   The largest possible order of S_5 is 6 since we have a cycle of lenth 2 and another of length 3. We want the closest pair of nonequal primes which add up to 5, which will yield the largest lcm.

   As for S_9 we notice that the largest lcm cannot have 3 prime factors since $2, 3, 5$ the samallest ones add up to a number greater than 9. Then by the previous rule we made, we chose 3, 5, which add up to 8. The largest order is 15.

3. Show that any element of S_{10} of order 14 is odd.

   We observe that order 2 cycles and order 7 cycles (disjoint) must make up the element. since

   $$2x + 7y < 10$$

   for positive x, y is only satisfied when $x = y = 1$, the element has one 7 order cycle and one 2 order cycle. Noting that signs are a homomorphism,

   $$\mathrm{sgn}(c_1c_2) = \mathrm{sgn}(c_1)\mathrm{sgn}(c_2)$$

   where c_1 can be broken down into a product of 2 - 1 transpositions and c_1 is a product of $7 - 1$ transpositions.

   $$\mathrm{sgn}(c_1c_2) = (-1)^1(-1)^6 = (-1)^7 - 1$$

   which says that it is odd.

Exercise 2:

1. Show that for any divisor d of 24, there is a subgroup of S_4 of order d.

   Proof. The divisors 1 and 24 are trivial, 12 is the order of A_4. For $2, 3, 4$ we take $\langle(1\,2)\rangle, \langle(1\,2\,3)\rangle, \langle(1\,2\,3\,4)\rangle$. For $6, 8$, D_6 and D_8 are subgroups. $\square$

2. Find two non-isomorphic subgroups of S_4 of order 4.

Proof. Let the groups be

$$G = \{e, (1\,2\,3\,4), (1\,3)(2\,4), (1\,4\,3\,2)\} H = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$$

Notice that G is cyclic and H is isomorphic to $C_2 \times C_2$. these are not isomorphic. □

3. Find a subgroup of A_4 of order 4. Is there a subgroup of order 6? Justify your answer.

Proof. The subgroup of A of order 4 is the group $C_2 \times C_2$ since all elements are written as a product of an even number of transpositions. Notice that A_4 contains all three cycles and the disjoint transpositions that make up the $C_2 \times C_2$ groups. These are the minimal subgroups of A_4, the ones with the least order other than the trivial subgroup. But it we have a group of order 6, it must be either dihedral or cyclic. It cannot be dihedral because we have no C_2 group. It is not cyclic because then it need to have 6 elements and our group only has 4. □

Idea behind this proof: I first attempted to look at how the 6 group was generated which led to a lot of case work such as how the elements of order 3 mulitplied with that of order 4. This was messy and did not look at the group structure of the group. This proof is much better because it looks at what the A_4 group looks like: it contains 3-cycles and product of two transpositions. And this helps to see that there is no group of order 6 simply because these minimal subgroups cannot generate it and that if the group of order 6 was not generated by these groups, it would be too big.

Exercise 3: Let G be a group which contains elements of order 6 and 10. Show that G contains at least 30 elements.

Proof. By Lagrange's Theorem, we have

$$|K||G : K| = |G|$$

So for a cyclic subgroup of G, K, the order of that element divides the order of G, therefore, $6 \bigm| |G|$, $10 \bigm| |G|$, $\operatorname{lcm}(6, 10) \bigm| |G|$. And we have the least common multiple as 30, since G is non-empty, there is an element of order 30 which is the group operation of the one of order 6 and the other of order 10. □

Exercise 4: We say that a finite group G is generated by a set T if any element of G can be written as a finite product (with repetitions) of powers of elements of T. Show that S_n is generated by each of the following sets:

1. The set $\{(j\,k) : 1 \leq j \leq k \leq n\}$;

Proof. We know that every permutation can be written as a product of cycles, each of which can be written as a product of transpositions. This is the set of all transpositions in S_n, so we can generate every permutation with a product of the elements in the set. □

2. The set $\{(j\,j + 1) : 1 \leq j \leq n\}$;

Proof. This seems like a set that generates the previous set. Suppose $(j\,k)$ is a transposition where $j = k - a$ for some $a \geq 0$. For $a = 0$,

$$(j\,j + 1)(j\,j + 1)$$

Gives us the identity map for j. For $a = 1$, this follows by definition. For $a > 1$,

$$\begin{gathered}(j\,j + 1)(j + 1\,j + 2) \cdots (j + a\,j + a - 1) \cdots (j + 1\,j + 2)(j\,j + 1)\\ (j\,j + a)(j + 1)\\ (j\,k)\end{gathered}$$

So this set generates the previous set, and therefore generates G. $\square$

3. The set $\{(1\,k) : 1 < k \leq n\}$;

Proof. This set generates the set in 1, since for $1 < k < n, k < j \leq n$,

$$(1\,k)(1\,j)(1\,k) = (1)(k\,j) = (k\,j)$$

which is what we wanted. $\square$

4. The set $\{(1\,2), (1\,2\,\dots\,n)\}$;

Proof. We can create a construction of the identity:

$$\underbrace{(1\,2\,\dots\,n)\dots(1\,2\,\dots\,n)}_{n\text{ times}} = e$$

$$\underbrace{(1\,2\,\dots\,n)\cdots(1\,2\,\dots\,n)}_{n-1\text{ times}} = (n\,\dots 2\,1)$$

So the invers of both elements exist. Now a special property:

$$(1\,2\,\dots\,n)(j\,j+1)(n\,\dots\,2\,1) = (j+1\,j+2)$$

By induction using $(1\,2)$, we have $\{(j\,j+1) : 1 \leq j \leq n\}$ is a subset of the group generated by $\{(1\,2), (1\,2\,\dots\,n)\}$ which completes the proof by number 2. $\square$

Idea behind this proof: Trying to match up this set to the previous subsets that we know. The fact that there were inverses is cool because the idea is that by using the base $(1\,2)$, we can generate the next element by taking $c_1(1\,2)c_1^{-1}$ which is cool and this happens to match up with one of our previous generating sets.

Exercise 5: Let G be a finite group in which every element other than the identity has order 2. Show that $|G|$ is a power of 2.

Proof. Since G is a finite group with elements of order 2 only, if $G = \{e\}$, $|G| = 2^0$. If G contains exactly one transposition, we have $|G| = 2^1$. If G has more than 1 transposition, we prove that they are disjoint. Suppose they are not. Then for $b \neq c$,

$$(a\,b)(a\,c) = (a\,c\,b)$$

which is in G but has order 3. Contradiction. Therefore, we take the set of all transposiions and for each transposition, a unique element in G has property:

$$g = \tau_1^{k_1}\tau_2^{k_2}\cdots\tau_n^{k_n}$$

Since all $k_i = 0$ or 1, we have $|G| = 2^n$. $\square$