Math250aHw7

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Part A

Exercise 1: Let $\mathbb{Q} \subseteq \mathbb{R}$ be the topological subspace of rationals and $\mathbb{Q} \to \mathbb{R}$ be the inclusion map as an epimorphism in the category of topological Hausdorff spaces and continuous maps. Show that dense subobjects can be defined by epimorphic monomorphisms.

Answer. We see that if we have a dense set D and T such that $D \to T$ is a subobject, then it is a monomorphism by definition. Furthermore, it is epimorphic since for

$$\begin{tikzcd} D \arrow[r] & T \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & T' \end{tikzcd}$$

if $D \to T \xrightarrow{f} T' = D \to T \xrightarrow{g} T'$, we have that inclusion maps are unique since they are the kernel of some map out of T'. So $f = g$ which shows that dense subobjects are defined by epimorphic monomorphisms.

Part B

Exercise 1: Show that if $\mathcal{A}$ is the category of ordered sets and $D : \mathcal{A} \to \mathcal{A}$ is a functor assigning a set to its dual, then the automorphism class group of $\mathcal{A}$ has at least two elements.

Answer. We know that $\mathtt{id}_{\mathcal{A}}$ is a functor naturally equivalent to the identity functor. So $\mathtt{id}_{\mathcal{A}}$ belongs in I. We also have that D is an equivalence because if we take D^2, we get back our same set, as D just reverses the order.

We know that $\mathtt{id}, D$ are not of the same equivalence class because $D^2 = \mathtt{id}$ and $\mathtt{id} = \mathtt{id}$. Therefore, the automorphism group contains at least $\mathtt{id}, D$.

Exercise 2: Let $[\to]$ be the category with

- Objects L, R
- Morphism $L \to R$

and $[\to\to]$ be the category:

- Objects L, M, R
- Morphisms $L \to M$, $M \to R$, $L \to R$

Answer. The morphisms in the image of the functor $[\rightarrow] + [\rightarrow] \xrightarrow{\pi} [\rightarrow\rightarrow]$ does not form a category because if the objects are $\pi(L_1) = L$, $\pi(L_2) = \pi(R_1) = M$, and $\pi(R_2) = R$, then we have the morphisms

- $\pi(L_1 \to R_1) = L \to M$
- $\pi(L_2) \to R_2 = M \to R$
- But there are no more morphisms because both $[\rightarrow]$ have only one morphism.

So the composition $L \to R$ does not exist in the image of π.

Part C

Exercise 1: Let $\mathcal{S}$ be the category of sets with morphisms as set functions. Prove that the automorphism class group of $\mathcal{S}$ is trivial. Use the fact that if $F : \mathcal{S} \to \mathcal{S}$ is an automorphism, then $F(D)$ has one element, if D has one element. Now define for each $A \in \mathcal{S}$, $A \to F(A)$ where:

$$\begin{tikzcd} D \arrow[r] \arrow[d, "x"] & F(D) \arrow[d, "F(x)"] \\ A \arrow[r] & F(A) \end{tikzcd}$$

commutes for all $x \in (D, A)$.

Proof. So we can label the map $\psi : D \to F(D)$ and $\varphi : A \to F(A)$. To make the map commute, we consider the mappings from the diagram:

$$\begin{tikzcd} d \arrow[r, mapsto] \arrow[d, mapsto] & d' \arrow[d, mapsto] \\ a & a' \end{tikzcd}$$

which means that

$$\varphi(a) = F(x)\psi(d)$$

sending all elements of A into one element of $F(A)$, making the diagram commute for any choice of x.

We conclude that there is a natural transformation from the identity functor:

$$\begin{tikzcd} D \arrow[d, "x"] \\ A \end{tikzcd}$$

to our functor restricted to (D, A). But now the action of this functor on (D, A) uniquely determines our functor, since:

$$
\begin{tikzcd}
D_1 \arrow[ddr] \arrow[ddrr, bend left, "x_1"] & & & \\
D_2 \arrow[dr] \arrow[drr, bend left, "x_2"] & & & \\
D_3 \arrow[r] & D \arrow[r, "\sigma"] & A \arrow[r, "\varphi"] & F(A) \\
\vdots & & & \\
D_n \arrow[uur] \arrow[uurr, bend right, "x_n"'] & & &
\end{tikzcd}
$$

we can decompose any n element set D into a disjoint union of 1 element sets. And since there is a mapping from $D_1 \to F(A)$ where D_i have size 1, we know there is a mapping φ that gives us a natural transformation from the identity functor to any automorphism functor.

There is also a way to go backwards and find a natural transformation from F to the identity functor. So for F in the automorphism class group, it is isomorphic to the identity functor, so the group is trivial. $\square$