Math172Hw4

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September 21, 2023

Exercise 1: Find a closed expression for the Stirling number $S(n, n-1)$.

Proof. Notice that we have a recursive formula for the Stirling numbers which is given by:

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

Now plugging in $k = n-1$, we get:

$$S(n, n-1) = S(n-1, n-2) + k \cdot S(n-1, n-1)$$

But we have that $S(n-1, n-1)$ is the number of partition of $n-1$ objects into $n-1$ indistinguishable groups. Then there is only 1 way to do so:

$$S(n, n-1) = S(n-1, n-2) + k$$

We start with $S(1, 0) = 0$ and $S(2, 1) = 1$. By the recursive formula, we have

$$\begin{aligned} S(1,0) &= 0 \\ S(2,1) &= 1 \\ S(3,2) &= 3 \\ S(4,3) &= 6 \\ &\vdots \\ S(n, n-1) &= ? \end{aligned}$$

But we notice that we just add k for each step and these are just the triangle numbers. We have $\frac{n(n+1)}{2}$ partitions. ◻

Exercise 2: Recall that $p_k(n)$ is the number of partitions of n with k non-zero parts. Show that for any positive integers n, k we have

$$p_1(n) + p_2(n) + p_3(n) + \cdots + p_k(n) = p_k(n+k)$$

by considering the operation of removing the first column from a Young diagram of a partition of $n + k$ with k parts.

Proof. If we start with $n + k$ as a number to partition into k groups, we subtract out the first column. Now all that is left is to count the number of partitions of $n + k - k$ into parts of size $\leqslant k$. This is just the sum that is shown above:

$$p_1(n) + p_2(n) + p_3(n) + \cdots + p_k(n) = p_k(n+k)$$

which completes the proof. ◻

Exercise 3: Show that the number of strict partitions of n with k distinct parts is equal to $p_k\left(n - \frac{k(k-1)}{2}\right)$.

Proof. We first establish a strict partition with k parts where the i-th row starting from the top has $k - i - 1$ blocks. The number of blocks that we use up is $\frac{k(k+1)}{2}$. So now we count the number of ways to partition $n - \frac{k(k+1)}{2}$. This is just:

$$p_1(n - \frac{k(k+1)}{2}) + p_2(n - \frac{k(k+1)}{2}) + \cdots + p_k(n - \frac{k(k+1)}{2})$$

to which we get:

$$p_k\left(n - \frac{k(k+1)}{2} + k\right) = p_k\left(n + \frac{k(2-k-1)}{2}\right) = p_k\left(n + \frac{k(-k+1)}{2}\right) = p_k\left(n - \frac{k(k-1)}{2}\right)$$

which is what was wanted. □

Exercise 4: Show that for any permutation σ of $[n]$ we have $\sigma^{n!} = e$ where e is the identity permutation $123\ldots n$.

Proof. By a fact proved in class, we have that for any σ on $[n]$ and each $x_i \in [n]$, there is some $k_i \in [n]$ such that:

$$\sigma^{k_i}(x_i) = x_i$$

The we take the lcm over the $k_i's$:

$$\sigma^{\text{lcm}(k_1,\ldots,k_n)}$$

since each $k_i \mid \text{lcm}(k_1, \ldots, k_n)$, we have $k_i \cdot d = \text{lcm}(k_1, \ldots, k_n)$. Therefore, for the corresponding $x_i \in [n]$, we have:

$$\sigma^{k_i \cdot d}(x_i) = x_i$$

since the power k_i fixes x_i. We apply this d times and it will still fix x_i. Notice that our choice of i for k_i, x_i is arbitrary. So $\sigma^{n!}$ fixes all elements as $\text{lcm}(k_1, \ldots, k_n) \mid n!$. So we are done. □

Exercise 5: An inversion of a permutation σ of $[n]$ is a pair of integers (i, j) such that $i, j \in [n], i < j$ and $\sigma(i) > \sigma(j)$. Show that σ and σ^{-1} have the same number of inversions.

Proof. We will construct a bijection between the set of inversions on σ denoted $I(\sigma)$ and the set of inversions on σ^{-1} denoted $I(\sigma^{-1})$.

We will show that $\varphi : I(\sigma) \longrightarrow I(\sigma^{-1})$ is bijective, given by:

$$\varphi((g_1, g_2)) \mapsto (\sigma(g_2), \sigma(g_1))$$

We need to show that the image is indeed in $I(\sigma^{-1})$.

- Since we have $\sigma(g_2) < \sigma(g_1)$, it agrees with the first part of the definition of an inversion.
- Now we take $\sigma^{-1}\sigma(g_1) = g_1$ and $\sigma^{-1}\sigma(g_2) = g_2$, which we can do because permutations are bijection. But now $g_2 > g_1$. So indeed the image of φ is a subset of $I(\sigma^{-1})$.

(Surjectivity) Let $(i, j) \in I(\sigma^{-1})$. Then consider $(\sigma^{-1}(j), \sigma^{-1}(i))$. This is an inversion in $I(\sigma)$ because

- $\sigma^{-1}(j) < \sigma^{-1}(i)$
- $\sigma\sigma^{-1}(j) = j > i = \sigma\sigma^{-1}(i)$

and notice that in the second condition check, we have $(\sigma^{-1}(j), \sigma^{-1}(i)) \in I(\sigma)$ such that:

$$\varphi((\sigma^{-1}(j), \sigma^{-1}(i))) = (i, j)$$

(Injectivity) Suppose that $\varphi(i_1, j_1) = \varphi(i_2, j_2)$. Then that means that $(\sigma(j_1), \sigma(i_1)) = (\sigma(j_2), \sigma(i_2))$. Considering component wise:

$$\begin{aligned} \sigma(j_1) &= \sigma(j_2) & \qquad \sigma(i_1) &= \sigma(i_2) \\ \sigma^{-1}\sigma(j_1) &= \sigma^{-1}\sigma(j_2) & \qquad \sigma^{-1}\sigma(i_1) &= \sigma^{-1}\sigma(i_2) \\ j_1 &= j_2 & \qquad i_1 &= i_2 \end{aligned}$$

which shows that $(i_1, j_1) = (i_2, j_2)$ therefore proving injectivity. So we have a bijection and σ has the same number of inversions as σ^{-1}. $\square$