Math110Hw11

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Exercise 1: Let $T \in \mathcal{L}(V, W)$. Prove

1. T is injective if and only if T^* is sujective;

   Proof. ($\rightarrow$) Using the proof that was shown in class, we have that $\ker(T) = (\text{Im}\{T^*\})^\perp$. This tells us that if T is injective of that $\ker T = \{0\}$, then that means that $\text{Im}\{T^*\}^\perp = \{0\}$. Additionally, the set $\{0\}$ is a subspace so therefore, $\text{Im}\{T^*\} = V$. So T^* is surjective.

   ($\leftarrow$) For the other direction, we can argue backwards since it was a string of equalities. $\square$

2. T^* is injective if and only if T is surjective.

   Proof. ($\rightarrow$) Suppose that T^* is injective. That means that $\ker T^* = \{0\}$ and using the fact that $\ker T^* = (\text{Im}\{T\})^\perp$, we can conclude that $(\text{Im}\{T\})^\perp = \{0\}$. Since $\{0\}$ is a subspace, we can conclude that the orthogonal complement of $(\text{Im}\{T\})^\perp$ is W. By the fact that $\text{Im}\{T\}^\perp$ and $\text{Im}\{T\}$ form a direct sum to W. Therefore, T is sujective.

   ($\leftarrow$) As before, we can argue backwards because we have used a string of if and only ifs/ equalities. $\square$

Exercise 2: Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Proof. ($\rightarrow$) Suppose that ST is self-adjoint. Then we have that

$$ST = (ST)^* = T^*S^* = TS$$

which is what we wanted.

($\leftarrow$) Suppose now that $ST = TS$. Then

$$ST = S^*T^* = (TS)^* = (ST)^*$$

which is what we wanted. $\square$

Exercise 3: Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that there is a subspace U of V such that $P_U = P$ if and only if P is self-adjoint.

Proof. ($\rightarrow$) Since we know that $P_U = P$ we can consider the matrix representation of this projection with respect to basis vectors of U and that of V. Let $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ be a orthogonal basis. We can do this by taking a basis of U and orthogonalizing with with respect to som inner product, then for the basis vectors corresponding to v, we do the same but by using Gram-schmidt on the basis for U we now have. Notice that the projection matrix maps the u basis vectors to itself, because u is orthogonal to all other vectors except for itself within the subspace U. For the v basis vectors, the projection maps these to 0 because these are orthogonal to the subspace U. So our matrix ends up looking like

$$P_U = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is certainly diagonal. Therefore, when we take the conjugate transpose, we get the T^* matrix representation which is definately the same as T, as it is a symmetric matrix. So $T = T^*$.

($\leftarrow$) For the other direction, we use the fact that $P^2 = P$ and that $P^* = P$. By computation, observe that for arbitrary vectors v_1, v_2,

$$\langle Pv_1, P^* v_2 \rangle = \left\langle v_1, P^{2*} v_2 \right\rangle = \langle v_1, P^* v_2 \rangle$$

Which tells us that P is an orthogonal projection onto the range of P^*. We can easily verify the range to be a subspace of V because of the linearity of P^*. $\square$

Exercise 4: Let $n \in \mathbb{N}$ be fixed. Consider the real space

$$V := \operatorname{Span}\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

with inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x \,.$$

Show that the differentiation operator $D \in \mathcal{L}(V)$ is *anti-Hermitian*, i.e., satisties $D^* = -D$.

Proof. We have shown before that this is a basis. Furthermore, in class, it was shown that the matrix representation of any linear operator $T \in \mathcal{L}(V)$ has its entries as the conjugate transpose of the operator T^*. We can use this. Consider the matrix representation of D. We can infer what it looks like based on its action on the basis. Take an arbitrary $\cos kx$ and observe that this is the $2k-th$ entry of the basis. The derivative of this is $-k \sin kx$ and note that $\sin kn$ is the $2k+1-th$ element of the basis. This tells us that $-k$ lies in the $2k+1$ row and $2k-th$ column. Also note that no other entries lie in this specific column. We do the same for $\sin kx$ with derivative $k \cos kx$ which tells us that k lies in the $2k-th$ row and $2k+1-th$ column. All other entries in this column are also 0. We also note that there is only one nonzero entry per row because if we have two, take a, b corresponding to $\cos kx$ wlog, we get that

$$\int a \cos kx \,\mathrm{d}x = \frac{a}{k} \sin kx$$
$$\int b \cos kx \,\mathrm{d}x = \frac{b}{k} \sin kx$$

which tells us that the basis vectors are linearly dependent, which is impossible. We have shown that the non-zero entries, when transposed and multiplied by negative 1 will remain the same. Now the rest is 0 as we have just proved. So the operator is *anti-Hermitian.* (*) The conjugate requirement is left out because we are in a real space. □

Exercise 5: Let T be a normal operator on V. Evaluate $\|T(v-w)\|$ given that

$$Tv = iv, \qquad Tw = (3+i)w, \qquad \|v\| = \|w\| = 1.$$

Proof. We first evaluate as much as we can:

$$\|T(v-w)\| = \|Tv - Tw)\| = \sqrt{\langle Tv - Tw, Tv - Tw \rangle}$$
$$\langle Tv, Tv \rangle - \langle Tw, Tv \rangle - \langle Tv, Tw \rangle + \langle Tw, Tw \rangle$$
$$\langle iv, iv \rangle - \langle Tw, Tv \rangle - \langle Tv, Tw \rangle + \langle (3+i)w, (3+i)w \rangle$$
$$1 - \langle Tw, Tv \rangle - \langle Tv, Tw \rangle + 10$$
$$11 - \langle Tw, Tv \rangle - \langle Tv, Tw \rangle$$

but now we rewrite the cross terms and observe what happens between w, v:

$$11 - (3+i)(-i) \langle w, v \rangle - (i)(3-i) \langle Tv, Tw \rangle$$

But notice that v, w are eigenvectors of T which is a normal operator, so as they correspond to different eigenvalues, the vectors are orthogonal. The cross terms become 0. Therefore, the norm is $\sqrt{11}$. □

Exercise 6: Suppose T is normal. Prove that, for any $\lambda \in \mathbb{F}$ and any $k \in \mathbb{N}$,

$$\ker (T - \lambda I)^k = \ker (T - \lambda I)$$

Proof. □