Math172Hw11

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Exercise 1: Let T be a tree with n labeled vertices. How many ways there are to color T into k colors so that no two vertices of the same color are connected by an edge?

Proof. We will color the tree with a process:

- Choose a leaf to color called x_1. Let $N_1 = \{x_1\}$ and N_2 be all the neighbors of vertices in N_1.
- Color all vertices of N_i. Let N_{i+1} be the set of neighbors of N_i not in the previous N_j for $j \leqslant i$. This means that we do not recolor a vertex we have already colored.
- Repeat until all vertices are colored.

This process will use all vertices because trees are connected, and on a finite graph, we have a finite path from our starting leaf x_1 to any vertex x_i. So x_i is in one of the N_j.

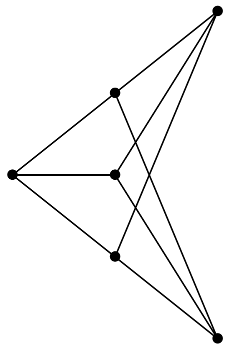
We see that on the first step, there are k ways to color the leaf. For the second step, notice that the subgraph of T on N_i contains no edges. This is because if there was an edge $\{v_1, v_2\}$, where $v_1, v_2 \in N_i$, then there is a closed walk $x_1 \to v_1 \to v_2 \to x_1$. Since v_1, v_2 have neighbors in N_{i-1}, we turn this into a trail of length $\geqslant 3$ by removing duplicate edges and vertices in this walk. Therefore, there is a cycle, contradiction.

This also shows that vertices in N_i cannot contain edges connecting to the same vertex in N_{i+1}, by extending the previous argument. Then by these two properties, we conclude that each vertex in N_i can be colored with one of the $k - 1$ colors different from its neighbor in N_{i-1}.

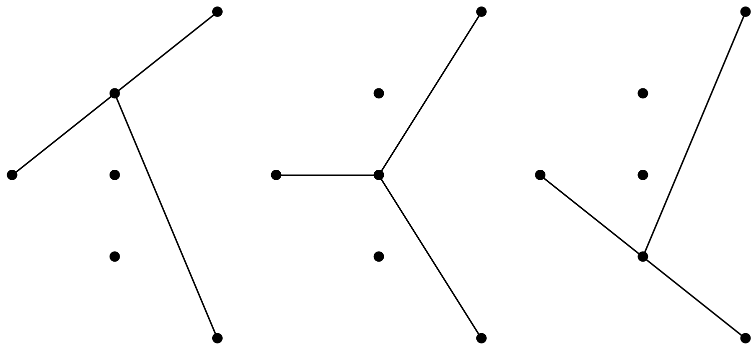
The process above gives all valid colorings because the order of our coloring does not influence the validity of a coloring. So there are $k(k-1)^{n-2}$ ways to color T. □

Exercise 2: How many ways there are to color $K_{3,3}$ into k colors so that no two vertices of the same color are connected by an edge (all vertices are labeled)?

Proof. We know that we can divide this into groups A, B such that A has three vertices, B has three vertices, and that between every vertex $a \in A$, $b \in B$ there is an edge $\{a, b\}$. Now draw out the graph:



We can first color the three "outside vertices" then color the ones along the middle column as shown below:



We have

- k ways to color all three outside vertices the same color.
- $k(k-1)\binom{3}{2}$ ways to color with two outside vertices having the same color.
- $k(k-1)(k-2)$ ways to color with all outside vertices having different colors.

Notice that we can choose the coloring of the vertices along the middle column independently because there is no edge connecting any of them. In the first case, we have $(k-1)^3$ ways to color the middle vertices if all outside vertices are the same.

In the second case, we have $(k-2)^3$ ways to color the middle vertices if exactly 2 outside vertices are same color.

In the third case, we have $(k-3)^3$ ways to color the middle vertices given that all outside vertices are different color. So putting it all together, we have:

$$k(k-1)^3 + k(k-1)\binom{3}{2}(k-2)^3 + k(k-1)(k-2)(k-3)^3$$

If we simplify, we get:

$$k(k-1)^3 + k(k-1)(k-2)(k^3 - 3k - 15)$$

□

Exercise 3: Let G be the following graph with $2n$ vertices and $2n$ edges: take a cycle with n vertices and attach a leaf to each of the vertices of this cycle. How many ways there are to color G into k colors so that no two vertices of the same color are connected by an edge (all vertices are labeled)?

Proof. Pick one vertex of the cycle on n vertices to color first. Then we start coloring in a circle. There are $k-1$ options for the next vertex, and so on. For the last one, there are $k-2$ options. So we have $k(k-1)^{n-2}(k-2)$ ways to color a cycle on n vertices.

Then add in the leaves. Since each leaf only has degree 1, it has one neighbor. Then we have $k-1$ options to color the leaf. There are n leaves, so we have $(k-1)^n$ ways to color the leaves. Putting this together, there are

$$k(k-1)^{n-2}(k-2)(k-1)^n = k(k-1)^{2n-2}(k-2)$$

ways to color this graph. □

Exercise 4: Is there a bipartite graph with 9 vertices which have the following multi-set of degrees: $3, 3, 3, 3, 3, 5, 6, 6, 6$?

Proof. Suppose that the graph G is bipartite. Consider the subgraph G' with the vertex of minimal degree removed. Then it has

$$\frac{5(3) + 5 + 3(6) - 6}{2} = 16$$

edges. Remove another vertex of minimal degree in the graph. Then we will have removed 3 edges, so there are 13 edges left on 7 vertices in G''. But the maximum number of edges in a bipartite graph was $\frac{7^2-1}{4} = 12$. So this subgraph of G'' is not two colorable. Therefore, G is not two-colorable/bipartite. □

Exercise 5: Fix integers n, k such that $n \geqslant 2$ and $0 \leqslant k < n/2$. Let X be a set of all k-element subsets of $[n]$, Y be the set of all $k+1$-element subsets of $[n]$ and connect $A \in X$ to $B \in Y$ if $A \subseteq B$. Show that there is a perfect matching of X into Y in the obtained bipartite graph.

Proof. We must show that for any subset $S_X \subseteq X$, the set of the neighbors of S_x, $N(S_X)$ has size greater than or equal to S_X. Notice that every element in the set Y is connected to exactly $k+1$ elements in X. Then we have that for a given vertex in X, called $v = \{e_1, \dots, e_k\}$, it has k elements of $[n]$. It is connected to $n-k$ vertices in Y, because there are $n-k$ ways to add an element to $v \in X$ to get an element $v' \in Y$ such that $v \subseteq v'$.

So given a set $S_X \subseteq X$, there are $|S_X|(n-k)$ edges connecting its vertices to vertices in Y. Each vertex in the preimage can use up at most $k+1$ edges. So S_X has at least $\frac{|S_X|(n-k)}{k+1}$ neighbors. We have from

$$0 \leqslant k < n/2$$

that

$$0 \leqslant 2k < n \implies 0 \leqslant k < n-k$$

Therefore, $k+1 \leqslant n-k$ and $1 \leqslant \frac{n-k}{k+1}$. So $|S_X| \leqslant \frac{|S_X|(n-k)}{k+1} \leqslant N(S_X)$ as desired. □