Math185Hw4

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February 21, 2024

Exercise 1: Show that two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ with positive radius of convergence sum to the same function if and only if $a_n = b_n$ for all n.

Proof. ($\rightarrow$) Suppose that

$$\sum_{n \geqslant 0} a_n z^n = \sum_{n \geqslant 0} b_n z^n$$

Then for $z = 0$, we get: $a_0 = b_0$. Taking the derivative, we get

$$\sum_{n \geqslant 1} n a_n z^{n-1} = \sum_{n \geqslant 1} n b_n z^{n-1}$$

Then for $z = 0$, we get: $a_1 = b_1$. So if we take the $k-$the derivative, we will see that $k! a_k = k! b_k$ and therefore $a_k = b_k$. We can take the derivative infinitely many times, so $a_n = b_n$ for all n.

($\leftarrow$) Suppose that $a_n = b_n$ for all n. Then $a_n - b_n = 0$ and therefore,

$$\sum_{n \geqslant 0} (a_n - b_n) z^n = 0$$

So we get $\sum_{n \geqslant 0} a_n z^n - \sum_{n \geqslant 0} b_n z^n = 0$, which is what we wanted. □

Exercise 2: Strengthen Q1 as follows: show that if $a(z) = \sum a_n z^n$ converges for small z and $a_n \neq 0$ for some $n > 0$, then for all sufficiently small $z \neq 0$ we have $a(z) \neq a_0$. In other words, the solution $z = 0$ to the equation $a(z) = a_0$ is *isolated*.

Hint: Write $a(z) = a_0 + z^k(a_k + \sum_{n>k} a_n z^{n-k})$ and exploit the continuity of the series.

Proof. If we write $a(z) = a_0 + z^k(a_k + \sum_{n>k} a_n z^{n-k})$, we see that $\sum_{n>k} a_n z^{n-k}$ has a radius of convergence at least as large as $a(z)$. Since it is continuous also, we have that $\lim_{z\to 0} \sum_{n>k} a_n z^{n-k} = 0$, and more precisely by epsilon-delta, for any $\varepsilon > 0$, there is a R such that for all $|z| < R$:

$$\left|\sum_{n>k} a_n z^{n-k}\right| < \varepsilon$$

then

$$-\varepsilon < \sum_{n>k} a_n z^{n-k} < \varepsilon$$

and so

$$-\varepsilon + a_k < a_k + \sum_{n>k} a_n z^{n-k} < \varepsilon + a_k$$

If we choose $\varepsilon < a_k$, then $0 < a_k + \sum_{n>k} a_n z^{n-k} = \delta$ so we have

$$a(z) = a_0 + z^k \delta > a_0$$

for some $|z| < R'$, R' sufficiently small, z non-zero. □

Exercise 3: Show that $\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} + \cdots$ for $|z| < 1$.

Proof. The derivative of arctan is $\frac{1}{1+z^2}$, which is equal to its Taylor Series:

$$\begin{aligned}
\frac{1}{1+z^2} &= \frac{1}{1-(-z^2)} \\
&= 1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + \cdots \\
(\arctan z)' &= 1 - z^2 + z^4 - z^6 + \cdots \\
\int (\arctan z)' \, dz &= \int 1 - z^2 + z^4 - z^6 + \cdots \, dz \\
&= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots
\end{aligned}$$

We know that this is the power series for $|z| < 1$ because the expansion of $\frac{1}{1+z^2}$ converges for $|z| < 1$, and when we take the integral, the radius of convergence is preserved. The last thing to check is that integrating does not introduce a constant. Since $\arctan 0 = 0$, the constant term is 0. $\square$

Exercise 4: Find the open region of convergence of

(a) $\sum_{n=0}^{\infty} \frac{(z+i)^n}{(n+1)(n+2)}$

Answer. Let $y = z + i$. Then we have:

$$\sum_{n=0}^{\infty} \frac{y^n}{(n+1)(n+2)}$$

By the ratio test, it converges when

$$\left| \frac{y(n+1)(n+2)}{(n+2)(n+3)} \right| < 1$$

or

$$|y| < \frac{n+3}{n+1} \to 1$$

So this is the circle of radius 1 centered at $-i$ as:

$$|z + i| < 1$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 3^n} \left(\frac{z+1}{z-1} \right)^n$

Answer. Let $y = \frac{z+1}{z-1}$. Then we have the series:

$$\sum_{n \geqslant 1} \frac{1}{n^2 \cdot 3^n} y^n$$

By the ratio test, this converges when

$$\left| \frac{y \cdot n^2 \cdot 3^n}{(n+1)^2 \cdot 3^{n+1}} \right| = \left| \frac{y \cdot n^2}{3(n+1)^2} \right| < 1$$

So

$$|y| < \frac{3(n+1)^2}{n^2} \to 3$$

Then the condition becomes:

$$\left| \frac{z+1}{z-1} \right| < 3$$

Exercise 5: Investigate the (a) absolute and (b) uniform convergence of the series of functions

$$\frac{z}{3} + \frac{z^2(3-z)}{3^2} + \frac{z^3(3-z)^2}{3^3} + \frac{z^4(3-z)^3}{3^4} + \cdots$$

Answer. (Part I) The series converges absolutely when

$$\sum_{n\geqslant 1} \left|\frac{z^{n+1}(3-z)^n}{3^{n+1}}\right| \text{ is finite}$$

By the ratio test, we then require:

$$\left|\frac{z^{n+2}(3-z)^{n+1}3^{n+1}}{z^{n+1}(3-z)^n 3^{n+2}}\right| = \left|\frac{z(3-z)}{3}\right| < 1$$

(Part II) Let the series of functions be $f_n(z) = \sum_{k=0}^{n} \frac{z^{k+1}(3-z)^k}{3^{k+1}}$. Then $\lim_{n\to\infty} f_n(z) = \sum_{k\geqslant 0} \frac{z^{k+1}(3-z)^k}{3^{k+1}}$. To compute this, let $C = \lim_{n\to\infty} f_n(z)$. Then:

$$\begin{aligned}
C &= \frac{z}{3} + \frac{z^2(3-z)}{3^2} + \frac{z^3(3-z)^2}{3^3} + \cdots \\
\frac{z(3-z)}{3}C &= \frac{z^2(3-z)}{3^2} + \frac{z^3(3-z)^2}{3^3} + \cdots \\
C - \frac{z(3-z)}{3}C &= \frac{z}{3} \\
3C - z(3-z)C &= z \\
z^2C - 3zC + 3C &= z \\
C(z^2 - 3z + 3) &= z \\
C &= \frac{z}{z^2 - 3z + 3}
\end{aligned}$$

Suppose that $f_n(z) \to C$ uniformly for contradiction. Since by the theorem, each $f_n(z)$ is continuous for $n \in \mathbb{N}$. But $C = \frac{z}{z^2-3z+3}$ is not continuous, which is a contradiction. It is not continuous because it is not defined when the denominator vanishes.

Exercise 6: If the power series $a(z)$ and $b(z)$ converge for $|z| < R$, we have seen that their product $a(z)b(z)$ also converges for $|z| < R$. Find an example in which the radius of convergence for $a(z)b(z)$ is *greater* than that of both $a(z)$ or $b(z)$.

Answer. Take $a(z) = (1-z)^\alpha = \sum_{k\geqslant 0} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} z^k$ and $b(z) = (1-z)^{1-\alpha} = \sum_{k\geqslant 0} \frac{(1-\alpha)(1-\alpha-1)\cdots(1-\alpha-k+1)}{k!} z^k$ for $|z| < 1$. Then $a(z)b(z) = 1 - z$, which has infinite radius of convergence. If we take $\alpha = .5$, then we know that $a(z)$ and $b(z)$ have the same radius of convergence $|z| < 1$.

Exercise 7: Find the series expansion of $f(z) = 1/(1 - z + z^2)$ by two different methods:

- By partial fraction expansion, and using the geometric series

  Proof. We first find the partial fraction decomposition by solving for the roots of $1 - z + z^2$:
$$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$
  Let:
$$z_1 = \frac{1 + i\sqrt{3}}{2}$$
$$z_2 = \frac{1 - i\sqrt{3}}{2}$$
  Now solve:
$$\frac{A}{z - z_1} + \frac{B}{z - z_2} = \frac{1}{1 - z + z^2}$$
  So we get:
$$Az - Az_2 + Bz - Bz_1 = 1$$
  Solve the system:
$$A + B = 0$$
$$-Az_2 - Bz_1 = 1$$
  expand with $A = -B$:
$$\begin{aligned} Bz_2 - Bz_1 &= B(z_2 - z_1) \\ &= -Bi\sqrt{3} \\ 1 &= -Bi\sqrt{3} \\ \frac{-1}{i\sqrt{3}} &= B \\ \frac{i\sqrt{3}}{3} &= B \end{aligned}$$
  So the decomposition is:
$$\frac{-i\sqrt{3}}{3} \frac{1}{z - z_1}$$
  □

- By setting up and solving a recursion for the coefficients.

  Answer. We note that $f(0) = 1$, so $a_0 = 1$. Next, we see that $f(z)(1 - z + z^2) = 1$. This means that for a general term, a_n, we have the relation that:
$$a_n z^n - z \cdot a_{n-1} z^{n-1} + z^2 \cdot a_{n-2} z^{n-2} = 0$$
  This tells us that
$$a_n - a_{n-1} + a_{n-2} = 0$$
  We also need to compute a_1. The relation limits to $a_n - a_{n-1} = 0$ for $n = 1$. So we get $a_1 = 1$. Now we figure out the rest of the terms using the recursion relation:
$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_n &= a_{n-1} - a_{n-2} \end{aligned}$$

Here is the list:

$$\begin{array}{ll}
n = 0 & a_0 = 1 \\
n = 1 & a_1 = 1 \\
n = 2 & a_2 = 0 \\
n = 3 & a_3 = -1 \\
n = 4 & a_4 = -1 \\
n = 5 & a_5 = 0 \\
n = 6 & a_6 = 1 \\
n = 7 & a_7 = 1 \\
n = 8 & a_8 = 0 \\
\vdots & \vdots
\end{array}$$

and we see that the pattern repeats. So the series is

$$1 + z + 0z^2 - z^3 - z^4 + 0z^5 + z^6 + z^7 + \cdots$$