Math250aHw10

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December 9, 2023

Exercise 4: Let $\varphi : A \to B$ be a commutative ring homomorphism. Let E be an $A-$ module and F a B-module. Let F_A be the A-module obtained from F via the operation of A on F through φ, that is for $y \in F_A$ and $a \in A$ this operation is given by

$$(a, y) \mapsto (\varphi(a)y)$$

Show that there is a natural isomorphism

$$\mathrm{Hom}_B(B \otimes_A E, F) \cong \mathrm{Hom}_A(E, F_A).$$

Proof. Consider the diagram:

$$\begin{tikzcd}
(B,(E,F)) \arrow[r] & (E,F_A) \\
(B,(E',F)) \arrow[r] \arrow[u] & (E',F_A) \arrow[u]
\end{tikzcd}$$

We know that

$$\mathrm{Hom}_B(B \otimes_A E, F) \cong (B,(E,F))$$

So if $\pi \in (B,(E,F))$, we can send $\pi \mapsto \pi(1_B)$ for the top morphism. If $\psi : E \to E'$, then we have that the left morphism sends $f \in (B,(E',F)) \mapsto f(-) \circ \psi \in (B,(E,F))$. For the bottom morphism, we can take $f(-) \circ \psi \mapsto f(1_B) \circ \psi$. So we have

$$\begin{tikzcd}
f(-) \circ \psi \arrow[r, "ev_{1_B}"] & f(1_B) \circ \psi \\
f(-) \arrow[r, "ev_{1_B}"'] \arrow[u, "- \circ \psi"] & f(1_B) \arrow[u, "- \circ \psi"']
\end{tikzcd}$$

This is invertible because . . . idk.

As for naturality in the other part, we have:

$$\begin{tikzcd}
(B,(E,F)) \arrow[r] \arrow[d] & (E,F_A) \arrow[d] \\
(B,(E,F')) \arrow[r] & (E,F'_A)
\end{tikzcd}$$

If we have a morphism $\psi : F \to F'$ and $f \in (E, F)$, we have the natural transformations:

$$\begin{tikzcd}
f(-) \arrow[r, "ev_{1_B}"] \arrow[d, "\psi \circ -"] & f(1_B) \arrow[d, "\psi \circ -"] \\
\psi \circ f(-) \arrow[r, "ev_{1_B}"] & \psi \circ f(1_B)
\end{tikzcd}$$

$\square$

Exercise 6: Let M, N be flat. Show that $M \otimes N$ is flat.

Proof. Let $E' \to E \to E''$ be an exact sequence. By definition, we have that

$$E' \otimes M \to E \otimes M \to E'' \otimes M$$

is exact also. Now we tensor again by N which should also preserve flatness:

$$(E' \otimes M) \otimes N \to (E \otimes M) \otimes N \to (E'' \otimes M) \otimes N$$

is exact. By the associative property of tensor products, we have that for any exact sequence

$$E' \to E \to E''$$

tensoring with $M \otimes N$ preserves exactness, so $N \otimes M$ is flat. □

Exercise 8: Prove Proposition 3.2

Proof. (Part I) Let S be a multiplicative subset of R. Then $S^{-1}R$ is flat over R.

We need to show that if we have an exact sequence

$$0 \to E' \to E$$

then we also have the exact sequence

$$0 \to R[S^{-1}] \otimes E' \to R[S^{-1}] \otimes E$$

Then we just need to show that

$$0 \to E[S^{-1}] \to E'[S^{-1}]$$

is exact. Suppose that we have some elements $\frac{e_1}{s_1}, \frac{e_2}{s_2}$ such that

$$\varphi(\frac{e_1}{s_1}) = \varphi(\frac{e_2}{s_2})$$

Then we have

$$\varphi(\frac{e_1}{s_1} - \frac{e_2}{s_2}) = 0$$

Since this is a module morphism we can multiply by anything in $r \in S^{-1}$ and we see that

$$\varphi(e_1 s_2 - e_2 s_1) = 0$$

(Part II) A module M is flat over R if and only if the localization M_p is flat over R_p for each prime ideal p of R.

($\to$) Suppose that M is flat over R. Then we have $M_p \cong M \otimes R_p$. Since M is flat and R_p is flat, we have that $M_p \cong M \otimes R_p$ is flat over R. Then it is also flat over R_p. So this is true for any p prime.

($\leftarrow$) Idk < Help needed.

(Part III) ($\to$) Consider an injective mapping

$$\varphi := r \to ar$$

with the corresponding exact sequence:

$$0 \longrightarrow R \longrightarrow R$$

Since F is flat:

$$0 \longrightarrow R \otimes F \longrightarrow R \otimes F$$

is exact and the morphism is given by

$$\varphi \otimes 1 := r \otimes f \mapsto ar \otimes f$$

which is injective, so $af \neq 0$ for any a.

($\leftarrow$) Torsion free modules over a PID are free, since any module over a PID can be written as:

$$E = E_{tor} \oplus F$$

for F free. Free modules are flat modules. $\square$

Exercise 9: We continue to assume that rings are commutative. Let M be an A-module. We say that M is **faithfully flat** if M is flat, and if the functor

$$T_M : E \mapsto M \otimes_A E.$$

is faithful, that is $E \neq 0$ implies $M \otimes_A E \neq 0$. Prove that the following conditions are equivalent.

- M is faithfully flat.
- M is flat, and if $u : F \to E$ is a homomorphism of A-modules, $u \neq 0$, then $T_M(u) : M \otimes_A F \to M \otimes_A E$ is also $\neq 0$.
- M is flat, and for all maximal ideals m of A, we have $mM \neq M$.
- A sequence of A-modules $N' \to N \to M''$ is exact if and only if the sequence tensored with M is exact.

Proof. ($1 \to 2$) Consider the diagram:

$$\begin{tikzcd}
F \arrow[r, "u"] \arrow[d, "T_M(F)"'] & E \arrow[d, "T_M(E)"] \\
M \otimes_A F \arrow[r, "T_M(u)"'] & M \otimes_A (E)
\end{tikzcd}$$

Since $E \neq 0$, and supposing $u \neq 0$, we can construct the exact sequence $K \to F \to E$:

$$\begin{tikzcd}
K \arrow[r, "k"] & F \arrow[r, "u"] \arrow[d, "T_M(F)"] & E \arrow[d, "T_M(E)"] \\
M \otimes_A K \arrow[r] & M \otimes_A (F) \arrow[r, "T_M(u)"] & M \otimes_A (E)
\end{tikzcd}$$

where k is the kernel of u. Since M is flat, we have an exact sequence on the bottom also. Since $M \otimes_A (E)$ is not 0, $M \otimes_A K \to M \otimes_A (F)$ is not surjective. Then the image is not all of $M \otimes_A F$ and therefore, the kernel of $T_M(u)$ is not all of $M \otimes_A F$. So $T_M(u)$ is not the 0 map.

($2 \to 3$) Consider the exact sequence:

$$0 \longrightarrow m \longrightarrow R \longrightarrow R/m \longrightarrow 0$$

Since M is flat, tensoring, we get:

$$0 \longrightarrow m \otimes M \longrightarrow R \otimes M \longrightarrow R/m \otimes M \longrightarrow 0$$

is exact or

$$0 \longrightarrow \mathfrak{m}M \longrightarrow M \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

But because $R \to R/\mathfrak{m}$ is not the zero map, $R \otimes M \to R/\mathfrak{m} \otimes M$ is not the zero map also. Then $M/\mathfrak{m}M$ is not the zero module. Therefore, $\mathfrak{m}M \to M$ is not surjective, and so $\mathfrak{m}M$ is not equal to M for any $\mathfrak{m}$.

($3 \to 4$) If M is flat, by definition, we already have the $\to$ direction. Suppose that

$$N' \otimes M \longrightarrow N \otimes M \longrightarrow M'' \otimes M$$

is exact. Then we tensor everything with $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. To get:

$$N' \otimes M/\mathfrak{m}M \longrightarrow N \otimes M/\mathfrak{m}M \longrightarrow M'' \otimes M/\mathfrak{m}M$$

where $R/\mathfrak{m} \otimes M$ is a vector space and therefore a free module. Furthermore, it is a one dimensional vector space because we only have one copy of $R/\mathfrak{m}$. Then for any module T, we just have $T \otimes M/\mathfrak{m}M \cong T$. So we have:

$$N' \longrightarrow N \longrightarrow M''$$

is exact.

($4 \to 1$) First, assume that tensoring M with any short exact sequence on three modules gives a short exact sequence. Then by definition, M is flat. So now we will show that the functor

$$T_M : E \mapsto M \otimes_A E$$

is faithful. We will prove the contrapositive. Suppose that $M \otimes E = 0$. Then we have that the sequence

$$0 \otimes M \longrightarrow E \otimes M \longrightarrow 0 \otimes M$$

is exact. But by assuming 4, we can say that detensoring by M gives an exact sequence:

$$0 \longrightarrow E \longrightarrow 0$$

and we therefore conclude that $E = 0$. So we have proved equality of the four conditions. $\square$