(5) $X_1 = X_2$ $\mu = 1$, $\sigma^2 = 1$ Let $S_n = X_1 + X_2$

$Cov(X_1, X_2) = 1$

$$\frac{Var(S_n)}{n^2} = \frac{Var X_1 + Var X_2 - 2 Cov(X_1, X_2)}{4}$$

$$= \frac{2 - 2}{4} = 0$$

Take $X_1 = X_2$, with $\mu = 1$, $\sigma^2 = 10$

Then $Cov(X_1, X_2) = \mathbb{E} X_1 X_2 - \mathbb{E} X_1 \mathbb{E} X_2$

$$= \mathbb{E} X_1 X_2 - 1 = \mathbb{E} X_1^2 - 1 = 8$$

$$\mathbb{E} X_1^2 = Var X_1 - (\mathbb{E} X_1)^2$$

$$= 10 - 1 = 9$$

$$Var\left(\frac{S_n}{n}\right) = \frac{Var(X_1 + X_2)}{\quad}$$

(6) Take $X_1 = X_2$ then

$$\frac{Var(X_1 + X_2)}{2^2} = \frac{Var\, 2X_1}{2^2} = Var X_1$$

If $Var X_1 = 10$,

we have $\omega \nleq \frac{1}{2}$

$$1 \le 1$$

Inductive Case: Suppose $Var\left(\frac{S_{n-1}}{n-1}\right) \le \frac{1}{n-1}$.

(5) $Var(S_n) = \mathbb{E} S_n^2 - (\mathbb{E} S_n)^2$

$$= \mathbb{E}(X_1 + \dots + X_n)^2 - (n)^2$$

$$= \sum_i \mathbb{E} X_i^2 + 2 \sum_{1 \le i < j \le n} \mathbb{E} X_i X_j - n^2$$

$Var X_i = \mathbb{E} X_i^2 - (\mathbb{E} X_i)^2$ $\qquad$ $\mathbb{E} X_i X_j - \mathbb{E} X_i \cdot \mathbb{E} X_j = Cov X_i X_j$

$1 = \mathbb{E} X_i^2 - 1$ $\qquad$ $\mathbb{E} X_i X_j = Cov X_i X_j + 1$

$2 = \mathbb{E} X_i^2$ $\qquad$ $\mathbb{E} X_i X_j < 1$

$$Var S_n = 2n + 2 \sum_{1 \le i < j \le n} \mathbb{E} X_i X_j - n^2$$

$$\le 2n + 2 \cdot \binom{n}{2} - n^2$$

$$= 2n + 2 \cdot \frac{n(n-1)}{2} - n^2$$

$$= 2n + n(n-1) - n^2 = 2n + n^2 - n - n^2$$

$$= n$$

$Var S_n \le n$ $\qquad$ so $Var\left(\frac{S_n}{n}\right) = \frac{Var S_n}{n^2} \le \frac{n}{n^2} = \frac{1}{n}$ ✓

(4) a) $Var(A+B) = Var(A) + Var(B) - 2Cov(A,B)$

$Cov(A,B) = \mathbb{E}(AB) - \mathbb{E}A\,\mathbb{E}B$

$= \mathbb{E}(2X^2 + 3XY + 2X + 3Y) - \mathbb{E}(X+1)\mathbb{E}(2X+3Y)$

$= 2 + \mathbb{E}(3XY) + 2 - (\mathbb{E}X + 1)(2)$

$= 4 + 3\mathbb{E}XY - 4 = 3\mathbb{E}XY = \boxed{3}$

$Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y$

$1 = \mathbb{E}XY$

b) $\cancel{Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var X}\sqrt{Var Y}} = \frac{3}{\sqrt{2}\sqrt{3}} = \frac{3}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}}$

$Corr(A,B) = \frac{Cov(A,B)}{\sqrt{Var A}\sqrt{Var B}} = \frac{3}{\sqrt{2}\sqrt{20}} = \frac{3}{\sqrt{40}} = \frac{3\sqrt{40}}{40} = \boxed{\frac{3\sqrt{10}}{20}}$

$Var A = \mathbb{E}A^2 - (\mathbb{E}A)^2 = \mathbb{E}(X^2 + 2X + 1) - (\mathbb{E}(X+1))^2$

$= 6 - 4 = 2$

$Var B = \mathbb{E}(B^2) - (\mathbb{E}B)^2 = \mathbb{E}(4X^2 + 12XY + 9Y) - (\mathbb{E}(2X+3Y))^2$

$12 + 12 - 4 = 20$

$\mathbb{E}X^2 = Var X + (\mathbb{E}X)^2$

$= 2 + 1 = 3$

(5) $Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2} Var(S_n)$

$\mathbb{E}X_iX_j = Cov(X_i, X_j) + 1$

$0 \leq \mathbb{E}X_iX_j < 1$

1) $n=1$ $Var S_1 = Var X_1 = 1 \to \frac{1}{n^2} \leq \frac{1}{n}$ ✓

2) Suppose true for $S_1, \ldots, S_{n-1}$ that $Var\left(\frac{S_i}{n}\right) \leq \frac{1}{n}$

Then $Var S_n = Var(X_1 + \ldots + X_n)$

$= Var(X_1 + \ldots + X_{n-1}) + Var(X_n) - 2Cov(X_1 + \ldots + X_{n-1}, X_n)$

$= Var(X_1 + \ldots + X_{n-1}) + 1 - 2(\mathbb{E}(X_1 + \ldots + X_{n-1})\mathbb{E}(X_n)) - \mathbb{E}(X_1 + \ldots + X_{n-1})\mathbb{E}(X_n))$

$\leq Var(X_1 + \ldots + X_{n-1}) + 1 - 2(0 - (n-1))$

$\leq Var(S_{n-1}) + 1 + 2n - 2 \leq n + 1 + 2n - 2 = 3n - 1$

$\frac{Var S_n}{n^2} \leq \frac{3n-1}{n^2}$

$\frac{Var S_{n-1}}{n^2} \leq \frac{1}{n} \to Var S_{n-1} \leq n$

③ $p_{X,Y,Z} = e^{-x}$ PDF of $X+Y+Z =$

$$\cancel{p_{X+Y+Z}(w=k) \int_{-\infty}^{\infty} p_X(u)\, p_Y(v)\, p_Z(k-u-v)\,du}$$

$$\cancel{p_{X+Y}(w=k) = \int_{-\infty}^{\infty} p_X(u)\, p_Y(k-u)\,du}$$

$$\cancel{= \int_{-\infty}^{\infty} e^{-u} \cdot e^{u-k}\,du}$$

$$\cancel{= \int_{-\infty}^{\infty} e^{-k}\,du =}$$

$$p_{X+Y}(w=k) = \int_0 p_X(u)\, p_Y(k-u)\,du$$

$$= \int_0^k \cancel{k} \cdot e^{-u} \cdot e^{-k+u}\,du$$

$$= \int_0^k e^{-k}\,du = \left(ue^{-k}\right)\Big|_0^k$$

$$= ke^{-k}$$

$$p_{X+Y+Z}(w=k) = \int_0 p_{X+Y}(u)\, p_Z(k-u)\,du$$

$$= \int_0^k ue^{-u} \cdot e^{-k+u}\,du$$

$$= \int_0^k ue^{-k}\,du = \left(\frac{u^2}{2}e^{-k}\right)\Big|_0^k$$

$$= \frac{k^2}{2}e^{-k}$$

$$\cancel{p_{X+Y+Z}(w) = \begin{cases} \frac{w^2}{2}e^{-w}, & \text{if } w \geq 0 \\ 0, & \text{if } w < 0 \end{cases}}$$

$$\boxed{p_{X+Y+Z}(w) = \begin{cases} \frac{w^2}{2}e^{-w} & \text{if } w \geq 0 \\ 0 & \text{if } w < 0 \end{cases}}$$

① $P(X+Y=n) = \sum_{m=0,1} P_X(k-m)\, P_Y(m)$

$P(X=k)\,P_Y(0) + P(X=k-1)\,P_Y(Y=1)$

$\cancel{\binom{n}{k} p^n (1-p) \cdot (1-p) + \binom{n}{n-1} p^{n} (1-p)(1-p)}$

$\cancel{\left(\binom{n}{n-1} p^{n-1} (1-p)\right)(p+1)}$

$\binom{n}{k} p^k (1-p)^0 \cdot (1-p) + \binom{n}{k-1} p^k (1-p)$

$\left(\binom{n}{k} + \binom{n}{k-1}\right) p^k (1-p)$

$= \binom{n+1}{k} p^k (1-p)$

$X+Y \sim \text{Binom}(n+1, p)$

② $P(X+Y=k) = \sum_{m=0,1} P_Y(m)\, P_X(k-m)$

for $n+1 > k > 0$

$= P_Y(Y=1)\,P_X(X=k-1) + P_Y(Y=0)\,P_X(X=k)$

$= p \cdot \frac{1}{n} + (1-p) \cdot \frac{1}{n} = \frac{1}{n}$

for $n+1 > k > 0$

If $k = n+1$, $P_X(X=n+1) = 0 \to P(X+Y=k) = p \cdot \frac{1}{n}$

If $k = 0$, $P_Y(Y=1)\,P_X(X=k-1) = 0 \to P(X+Y=k) = (1-p) \cdot \frac{1}{n}$

we have

$$p_{X+Y}(z) = \begin{cases} \frac{1}{n}, & \text{if } z \in \{2, \dots, n\} \\ p \cdot 1/n, & \text{if } z = n+1 \\ (1-p) \cdot 1/n, & \text{if } z = 1 \\ 0 & \text{otherwise} \end{cases}$$