Math110Hw7

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Exercise 1: Let $p \in \mathcal{P}(\mathbb{C})$. Define $q : \mathbb{C} \to \mathbb{C}$ by the formula

$$q(z) := p(z)\overline{p(\bar{z})}$$

Prove that $q \in \mathcal{P}(\mathbb{R})$. If $\deg p = n$, then what is $\deg q$? Explain.

Proof. The degree of q is $2 \deg p = 2n$. The degree of $\overline{p(\bar{z})}$ is also n. Since polynomial multiplication adds the exponents when two terms are multiplied together, the largest sum will be the largest exponent with non-zero coefficient in p plus the largest exponent with non-zero coefficient in $\bar{p}$ □

Proof. Let $p(z)$ be the polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

Then the polynomial $\overline{p(\bar{z})}$ is defined as

$$\overline{p(\bar{z})} = \bar{a_0} + \bar{a_1} z + \ldots + \bar{a_n} z^n$$

due to the linearity of the bar operator (a proof will be given after this proof). Here is a table of the expanded multiplication terms of $\overline{p(\bar{z})}$ and $p(z)$:

	a_0	a_1	$\ldots$	a_n
$\bar{a_0}$	$\bar{a_0}a_0$	$\bar{a_0}a_1$	$\ldots$	$\bar{a_0}a_n$
$\bar{a_1}$	$\bar{a_1}a_0$	$\bar{a_1}a_1$	$\ldots$	$\bar{a_1}a_n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\bar{a_n}$	$\bar{a_n}a_0$	$\bar{a_n}a_1$	$\ldots$	$\bar{a_n}a_n$

Let i be the row number and j the column. When $i = j$, we get for $a_i = a + bi$ for $a, b \in \mathbb{R}$,

$$\begin{aligned} \bar{a_i}a_j &= (a - bi)(a + bi) \\ &= a^2 - bi^2 \\ &= a^2 + b^2 \in \mathbb{R} \end{aligned}$$

When $i \neq j$, let $a_i = a + bi, a_j = c + di$, then

$$\begin{aligned}\bar{a_i}a_j + \bar{a_j}a_i &= (a - bi)(c + di) + (c - di)(a + bi)\\ &= ac + (ad - bc)i + bd + ac + (cb - ad)i + bd\\ &= 2ac + 2bd \in \mathbb{R}\end{aligned}$$

In total, the entries of the diagonal in the table are real numbers, and for the ones not in the diagonal, which lie at row i, column j, we add it to the entry at row j column i to get a real number. We can add them because both entries are coefficients to the term x^{i+j}. So all terms of the polynomial will be real. □

Proof. The conjugation operator is linear. Let $T : \mathbb{C} \to \mathbb{C}$ by $a + bi \mapsto a - bi$. To show that it is linear, suppose that $a + bi, c + di \in \mathbb{C}$. Then

1. $$\begin{aligned}T(a + bi + c + di) &= T(a + c + (b + d)i)\\ &= a + c - (b + d)i\\ &= a - bi + c - di\\ &= T(a + bi) + T(c + di)\end{aligned}$$

2. Let $\lambda \in \mathbb{C}$
$$\begin{aligned}T(\lambda(a + bi)) &= T(a\lambda + b\lambda i)\\ &= a\lambda - b\lambda i\\ &= \lambda(a - bi)\\ &= \lambda T(a + bi)\end{aligned}$$

which shows that T or conjugation is linear. □

Exercise 2: Let $p \in \mathcal{P}_n(\mathbb{R})$ have n distinct real zeros and let p' be a non-zero polynomial. Prove that the zeros of p and of p' *interlace*, i.e., between any two consecutive zeros of one of them lies exactly one zero of the other. Explain why, as a consequence, p' has $n - 1$ distinct real zeros.

Proof. We will use induction. For the degree of $p = 1$, $p = 2$ we have

$$p(z) = z - a_0 \implies p'(z) = 1$$

Indeed, $p(z)$ has 1 root and $p'(z)$ has 0 roots which interleave with $p(z)$.

$$p(z) = (z - a_0)(z - a_1) \implies p'(z) = 2z - a_0 - a_1$$

And the zero $\dfrac{a_0 + a_1}{2}$ interleaves with a_0, a_1.

Inductive case: suppose that for $\deg p = m$, that the zeros of p' interleave with that of p. For a polynomial of degree $m+1$, we have such that $a_{m+1} > a_m > \ldots > a_0$

$$p(z) = (z - a_0)(z - a_1) \cdots (z - a_{m+1})$$

Take the derivative by product rule

$$p'(z) = (z - a_0)(z - a_1) \cdots (z - a_m)'(z - a_{m+1}) + (z - a_0)(z - a_1) \cdots (z - a_m)$$

Define for convenience

$$\begin{aligned}
y_1(z) &= (z - a_0)(z - a_1) \cdots (z - a_m)' \\
y_1(z) &= (z - a_0')(z - a_1') \cdots (z - a_{m-1}') \\
y_2(z) &= (z - a_0)(z - a_1) \cdots (z - a_m) \\
p'(z) &= y_1(z)(z - a_{m+1}) + y_2(z)
\end{aligned}$$

where $a_0 < a_0' < a_1 < a_1' < \ldots < a_{m-1}' < a_m$. Consider a_i' between any two consecutive zeros of $p(z)$ called a_i, a_{i+1}. Since the zeros of $y_1(z), y_2(z)$ interleave, we have that

$$\begin{aligned}
p'(a_i) &= y_1(a_i)(a_i - a_{m-1}), \\
p'(a_i') &= y_2(a_i'), \\
p'(a_{i+1}) &= y_1(a_{i+1})(a_{i+1} - a_{m-1})
\end{aligned}$$

On the interval (a_i, a_{i+1}), since there are $m - i$ roots greater than a_i for $y_2(z)$, the sign of $y_2(z)$ is $(-1)^{m-i}$. But for $y_1(z)$, since the zeros of the derivative interleave with that of $y_2(z)$, the sign of $y_1(z)$ from (a_i, a_i') will be opposite of that from (a_i', a_{i+1}). Without loss of generality, suppose $y_1(z)$ has a different sign than $y_2(z)$ on the interval (a_i', a_{i+1}). By the fact that

$$\begin{aligned}
p'(a_i') &= y_2(a_i'), \\
p'(a_{i+1}) &= y_1(a_{i+1})(a_{i+1} - a_{m-1})
\end{aligned}$$

the value of p' goes from positive to negative or vice versa in the interval, therefore, there must be a zero somewhere in the interval. In fact, since $y_2(z)$ is either only strictly increasing or decreasing on the interval, as $y_1(z)$ is either positive or negative on the interval, there is exactly one zero. So the converse holds. So interleaving holds for $p(z)$ with degree $m+1$.

This shows that $p'(z)$ has $n-1$ distinct real zeros if $p(z)$ has n real zeros is because by interleaving, we can place a zero of $p'(z)$ between every two consecutive zeros. Each bin is determined by the least zero in the pair, so if $a_0 < a_1 < a_2 < \cdots < a_n$, then $a_0 < a_1 < \cdots < a_{n-1}$ are the zeros that determine the bins that we place the zeros of $p'(z)$ in. $\square$

Exercise 3: Suppose $p \in \mathcal{P}_d(\mathbb{C}), d \geq 2$, satisfies the condition $p(z_0) = p'(z_0) = \cdots = p^{(k)}(z_0) = 0$ for some $z_0 \in \mathbb{C}$ and some $k < d$. What is the multiplicity of the factor $z - z_0$ in the factorization of $p(z)$ over $\mathbb{C}$? Justify your answer.

Proof. We will prove that the multiplicity of $z - z_0$ is at least $k+1$. Suppose for contradiction that it is $j \leq k$. Then

$$p(z) = (z - z_0)^j q(z)$$

By taking the derivative, we get

$$\begin{aligned} p'(z) &= j(z - z_0)^{j-1} q(z) + (z - z_0)^j q'(z) \\ &= (z - z_0)^{j-1}(jq(z) + (z - z_0)q'(z)) \end{aligned}$$

Since $(z - z_0)$ is not a factor of $q(z)$, we cannot factor out any more $(z - z_0)$ terms. Notice that now, $p'(z)$ has $z - z_0$ with multiplicity $j - 1$ and that this process is repeatable with

$$p'(z) = (z - z_0)^{j-1} r(z)$$

where

$$r(z) = jq(z) + (z - z_0)q'(z)$$

So when we take the derivative of p k times, we have that the multiplicity of $z - z_0$ is $j - k \leq 0$ which is impossible. The polynomial $p^{(k)}$ must have a $z - z_0$ as a factor since $p^{(k)}(z_0) = 0$. So the multiplicity of $z - z_0$ is at least $k+1$. $\square$

Exercise 4: [Lagrange interpolation.] Prove *using duality*: given distinct real *data sites* x_j and arbitrary real *data* $y_j, j = 0, \ldots, n$, there is a unique polynomial $p \in \mathcal{P}_n(\mathbb{R})$ such that $p(x_j) = y_j$, for all $j = 0, \ldots, n$.

Proof. We can construct a polynomial which forces p to have $p(x_j) = y_j$ by the following

$$\begin{aligned} p(x) = y_0 &\left(\frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} \right) \\ + y_1 &\left(\frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} \right) \\ &\vdots \\ + y_n &\left(\frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_1 - x_1) \cdots (x_1 - x_{n-1})} \right) \end{aligned}$$

Each row represents the action of a linear functional and the overall polynomial gives rise to a linear mapping:

$$\begin{aligned} T &: \mathbb{R}^n \to \mathcal{P}_n(\mathbb{R}) \\ T(y_0, \ldots, y_n) &:= y_0 v_0 + y_1 v_1 + \ldots + y_n v_n \end{aligned}$$

where $\{v_0, \ldots, v_n\}$ is the basis for $\mathcal{P}_n(\mathbb{R})$ defined as:

$$\begin{aligned}
v_0 &= (x - x_1)(x - x_2)\cdots(x - x_n) \\
v_1 &= (x - x_0)(x - x_2)\cdots(x - x_n) \\
&\vdots \\
v_n &= (x - x_0)(x - x_1)\cdots(x - x_{n-1})
\end{aligned}$$

Check that they are linearly independent:

$$\begin{aligned}
a_0v_0 + a_1v_1 + \ldots + a_nv_n &= 0 \\
(a_0v_0 + a_1v_1 + \ldots + a_nv_n)(x_0) &= 0 \\
a_1v_1(x_0) &= 0 \\
a_1 &= 0
\end{aligned}$$

And repeat that for all x_i to get that $a_0 = a_1 = \ldots = a_n = 0$ so the v_i form a basis for $\mathcal{P}_n(\mathbb{R})$. Now we can define

$$\phi_i(v_j) = \begin{cases} v_j \mapsto 1 & \text{if } i = j \\ v_j \mapsto 0 & \text{if otherwise} \end{cases}$$

which lives in $\mathcal{P}'_n(\mathbb{R})$. And now we have a dual map $T' \in \mathcal{L}(\mathcal{P}'_n(\mathbb{R}), \mathbb{R}^{n'})$.

$$T'(\phi_i) = \phi_i(T) = \psi_i \in \mathbb{R}^{n'}$$

which lives in $\mathbb{R}^{n'}$. Notice that the dual map is surjective, as we map ϕ_i to all basis vectors in $\mathbb{R}^{n'}$. So the original map T must be injective, as the dimension of the domain and codomain are equal. This completes the proof that there is a unique polynomial. □

Exercise 5: Let $p \in \mathcal{P}_n(\mathbb{C})$ for some n and suppose there exist distinct real numbers $x_0, x_1, \ldots, x_n$ such that $p(x_j) \in \mathbb{R}$ for all $j = 0, \ldots, n$. Prove that all coefficients of p are real.

Proof. Suppose there was a polynomial that has real values at $x_0, x_1, \ldots, x_n$. Then denote that as $p_r(x)$. By Lagrange interpolation, there is a unique polynomial $p_l(x)$ that evaluates at $x_0, x_1, \ldots, x_n$ such that

$$p_r(x_i) = p_l(x_i)$$

for all i. Then

$$p_r(x) - p_l(x)$$

has roots at $x_1, x_2, \ldots, x_n$.

$$p_r(x) - p_l(x) = \lambda(x - x_1)(x - x_2)\cdots(x - x_n)$$

□