Math104Hw12

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Exercise 1: Let $f(x) = |x| + |x-2|, x \in \mathbb{R}$. Find points where f is not differentiable.

Proof. Notice that $f(x)$ is not differentiable at 0 or 2. This is because $f(x)$ on the domain $x \in \{0\}$ has the property:

$$f(x) = |x-2|$$

and $f(x)$ restricted to the domain $x \in \{2\}$ is just $|x|$.

Furthermore, we know that $|x|$ is differentiable everywhere else besides 0 and $|x-2|$ is differentiable everywhere else but 2. The sum of two differentiable functions is differentiable, so $f(x)$ is differentiable on $\mathbb{R} \backslash \{0, 2\}$. □

Exercise 2: Let $f(x) = x^2 \sin 1/x$ when $x \neq 0$, $f(0) = 0$. For any $a \neq 0$, find $f'(a)$.

Proof. Let $g(x) = x^2$ and $h(x) = \sin \frac{1}{x}$. Then use the product rule:

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$

Then use chain rule on $h(x)$:

$$h'(x) = -\frac{1}{x^2} \cos \frac{1}{x}$$

So we have:

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

and then we evaluate at a:

$$f'(a) = 2a \sin \frac{1}{a} - \cos \frac{1}{a}$$

so we are done. □

Exercise 3: In Q2, use the definition to find $f'(0)$ and show that f' is not continuous at 0.

Proof. We need to see if

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

with

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \frac{x^2 \sin 1/x}{x} = x \sin \frac{1}{x}$$

Then the derivative is 0 because that is the left side and right side limit and $-1 \leqslant \sin \frac{1}{x} \leqslant 1$. But it is not continuous because we require:

$$\lim_{x \to 0} x \sin \frac{1}{x} \neq 0 \sin \frac{1}{0}$$

where $\frac{1}{0}$ is undefined. □

Exercise 4: Prove that $|\cos x - \cos y| \leqslant |x - y|$ for any x, y.

Proof. We first make a change of variables $y = x + a$:

$$|\cos x - \cos(x + a)| \leqslant |a|$$

is what we want to prove. In other words, the function $f_a(x) = \cos x - \cos x + a$ is bounded by a:

$$-a \leqslant f_a(x) \leqslant a$$

Take the derivative:

$$f'_a(x) = \sin(x + a) - \sin x$$

We see that

$$f'_a(x) = 0$$

when

$$\sin(x + a) = \sin x$$

or $(2n + 1)\pi = x + (x + a) = 2x + a$ for $n \in \mathbb{Z}$. So we know that the maximum and minimum values are at $x = \frac{(2n+1)\pi - a}{2}$. So we plug this back into $f_a(x)$:

$$\begin{aligned} f_a\left(\frac{(2n+1)\pi - a}{2}\right) &= \cos\left(\frac{(2n+1)\pi}{2} - \frac{a}{2}\right) - \cos\left(\frac{(2n-1)\pi}{2} + \frac{a}{2}\right) \\ &= \sin\left(\frac{-a}{2}\right) - \sin\left(\frac{a}{2}\right) \\ &= -2\sin\left(\frac{a}{2}\right) \\ &= -4\sin\left(\frac{a}{4}\right)\cos\left(\frac{a}{4}\right) \end{aligned}$$

With $|\sin x| \leqslant |x|$, we have

$$|f_a(x)| \leqslant 4\left|\sin\left(\frac{a}{4}\right)\right|\left|\cos\left(\frac{a}{4}\right)\right| \leqslant 4\left|\frac{a}{4}\right| \leqslant |a|$$

which concludes the proof. □

Exercise 5: Assume that f is differentiable on $\mathbb{R}$ such that $f(0) = 0$, $f(1) = 1$, $f(2) = 1$. Show that $\exists x \in (0, 2)$ such that $f'(x) = \frac{1}{10}$.

Proof. By the MVT, we know that there exists an $a_1 \in (0, 1)$ such that

$$f'(a_1) = \frac{f(1) - f(0)}{1 - 0} = 1$$

We also know that there is an $a_2 \in (1, 2)$ such that

$$f'(a_2) = \frac{f(2) - f(1)}{2 - 1} = 0$$

Since $f'(a_2) < \frac{1}{10} < f'(a_1)$ by the IVT for derivatives, there is some $a_1 < x < a_2$ such that $f'(x) = \frac{1}{10}$. □

Exercise 6: Show that $\frac{x}{\sin x}$ is strictly increasing on $(0, \pi/2)$.

Proof. We take the derivative, which is possible because $x, \sin x$ are differentiable. Since $\sin x \neq 0$ for $x \in (0, \pi/2)$, we have:

$$\left(\frac{x}{\sin x}\right)' = \frac{\sin x - x\cos x}{\sin^2 x}$$

Now, $\sin^2 x > 0$. We require that $\sin x - x\cos x \geqslant 0$. Consider the derivative:

$$\cos x - (\cos x - x\sin x) = x\sin x$$

This is positive on $(0, \pi/2)$, then since $\sin x - x\cos x$ is 0 at $x = 0$, we have that $\sin x - x\cos x > 0$ for $x \in (0, \pi/2)$. So the derivative of $\frac{x}{\sin x}$ is greater than 0, which shows that it is strictly increasing on $(0, \pi/2)$. □