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Problem 1

Proof.
We can partition the numbers $2001, \ldots, 2120$ into three element sets where $S = \{2001, \ldots, 2120\}$. Let these three element sets be the pigeonholes:

$$P = \{\{n_1, n_2, n_3\} : n_1 \equiv 0 \pmod 3 \wedge n_2 = n_1 + 1 \wedge n_3 = n_1 + 2\}$$

for $n_1, n_2, n_3 \in S$. Let the 90 houses be pigeons. Since $|P| = \frac{2120 - 2000}{3} = 40$, by the generalized pigeonhole principle, at least one three element set of consecutive integers has at least $\lceil \frac{90}{40} \rceil$ pigeons $= 3$ pigeons; p_1, p_2, p_3. Then p_1, p_2, p_3 are sequential. $\square$

Problem 2

Proof. We start with $n=1$: (1)
with $n=2$: $(1,2), (2,1)$
with $n=3$: $(1,2,3), (1,3,2), (2,3,1), (2,1,3), (3,1,2), (3,2,1)$.

If f_n is the number of permutations of $1, \ldots, n$ written as $a_1, \ldots, a_n$ where $a_i \le a_{i+1} + 2$, then $f_1 = 1, f_2 = 2, f_3 = 6$. Observe that for $n=4$, there can only be two numbers that immediately ~~or~~ succeed 4: $3, 2$. Using this idea, we let

$$P = \{(a_1, \ldots, a_n) : a_i \le a_{i+1} + 2, \ a_i \in \{1, \ldots, n\}\}$$

for $1 \le i \le n-1$. Observe that if $p \in P$, then if $n = a_{j+1}$,

$$a_{j+1} \le a_{j+2} + 2$$
$$n \le a_{j+2} + 2$$

$$a_j \le n \le a_{j+2} + 2$$

So we can remove n from P and obtain a pretty cool permutation of $1, \ldots, n-1$. Observe that there are n spaces that we can place n back into the permutation, but n can only be put before $n-1$, $n-2$, or at the end of the sequence. So for every $1, \ldots, n-1$ pretty cool permutation, there are 3 ways to get a $1, \ldots, n$ pretty cool permutation. Therefore, $f_n = (f_{n-1})3$.

$f_4 = 18$, $f_5 = 18(3)$, $f_6 = 18(9)$, $f_7 = 18(3^3)$, $f_8 = 18(3^4)$, $f_9 = 18(3^5)$, $f_{10} = 18(3^6)$.