Stat134Hw5

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Exercise 1: Let X be uniform random variable on the interval $[1, 3]$. Compute $\mathbb{E}[X^3 - X]$.

Answer. We have that

$$X = \begin{cases} \frac{1}{2} & \text{if } 1 \leqslant x \leqslant 3 \\ 0 & \text{if } otherwise \end{cases}$$

Then $\mathbb{E}[X^3 - X] = \mathbb{E}[X^3] - \mathbb{E}[X] = \mathbb{E}[X^3] - 2$. Also,

$$\begin{aligned} \mathbb{E}[X^3] &= \int_{-\infty}^{\infty} x^3 p(x)\ dx \\ &= \int_1^3 x^3 \frac{1}{2}\ dx \\ &= \left(\frac{x^4}{8}\right)\Big|_1^3 \\ &= \frac{81}{8} - \frac{1}{8} = 10 \end{aligned}$$

which tells us that $\mathbb{E}[X^3 - X] = 10 - 2 = 8$.

Exercise 2: Let X be a continuous random variable with triangular density

$$f(x) = \begin{cases} c \cdot (1-x) & \text{if } 0 \leqslant x \leqslant 1 \\ c \cdot (1+x) & \text{if } -1 \leqslant x < 0 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Here c is a positive constant, which makes f(x) a probability density function. Find c and then compute the expectation and variance of X.

Answer. We must have that

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

which can be broken up as:

$$\int_{-\infty}^{-1} 0\ dx + \int_{-1}^{0} c \cdot (1+x)\ dx + \int_{0}^{1} c \cdot (1-x)\ dx + \int_{1}^{\infty} 0\ dx$$

or

$$\int_{-1}^{0} c \cdot (1+x)\ dx + \int_{0}^{1} c \cdot (1-x)\ dx$$

Now evaluate:

$$\begin{aligned} \int_{-1}^{0} c \cdot (1+x)\ dx + \int_{0}^{1} c \cdot (1-x)\ dx &= c\left(\int_{-1}^{0} 1 + x\ dx + \int_{0}^{1} 1 - x\ dx\right) \\ &= 2c \int_{0}^{1} 1 - x\ dx \\ &= 2c\left(x - \frac{x^2}{2}\right)\Big|_0^1 \\ &= 2c\left(1 - \frac{1}{2}\right) \\ &= 2c/2 = c \end{aligned}$$

So $c = 1$ since the integral must be 1.

(Expectation) The expectation is

$$\int_{-1}^{1} x \cdot f(x)\ dx = \int_{-1}^{0} x(1+x)\ dx + \int_{0}^{1} x(1-x)\ dx$$

which is

$$\begin{aligned} \int_{-1}^{0} x + x^2\ dx + \int_{0}^{1} x - x^2\ dx &= \int_{-1}^{0} x\ dx + \int_{-1}^{0} x^2\ dx + \int_{0}^{1} x\ dx + \int_{0}^{1} -x^2\ dx \\ &= \int_{-1}^{0} x\ dx + \int_{0}^{1} x\ dx \\ &= 0 \end{aligned}$$

(Variance) The variance is given by $\mathbb{E}[X^2] - \mathbb{E}[X]^2$. To calculate the value of $\mathbb{E}[X^2]$, we have the integral:

$$\int_{-1}^{0} x^2(1+x)\ dx + \int_{0}^{1} x^2(1-x)\ dx = \int_{-1}^{0} x^2\ dx + \int_{-1}^{0} x^3\ dx + \int_{0}^{1} x^2\ dx + \int_{0}^{1} -x^3\ dx$$

which is

$$2\int_{-1}^{0} x^2\ dx - 2\int_{0}^{1} x^3\ dx$$

So calculate:

$$
\begin{aligned}
2\left(\int_{-1}^{0} x^2 \, dx - \int_{0}^{1} x^3 \, dx\right) &= 2\left(\left(\frac{x^3}{3}\right)\Big|_{-1}^{0} - \left(\frac{x^4}{4}\right)\Big|_{0}^{1}\right) \\
&= 2\left(\frac{1}{3} - \frac{1}{4}\right) \\
&= 2(1/12) = 1/6
\end{aligned}
$$

So the variance is $\frac{1}{6} - 0 = \frac{1}{6}$.

Exercise 3: Let X be Normal random variable $\mathcal{N}(2, 9)$ of mean 2 and variance 9. Using the table from Appendix E of the textbook (copied on the next page; this is for standard normal $\mathcal{N}(0, 1)$) estimate $P(X < 5)$ and $P(|X| > 8)$.

Answer. If $Z \sim \mathcal{N}(0, 1)$, then $X = 2 + Z \cdot 3$. So we want to find

$$\mathbb{P}(X = 3Z + 2 < 5) = \mathbb{P}(X < 1)$$

Using the table, we get the value 0.8413

For $\mathbb{P}(|X| > 8)$, we can split it as:

$$\mathbb{P}(X > 8) + \mathbb{P}(X < -8)$$

Now plug in $X = 3Z + 2$:

$$\mathbb{P}(3Z + 2 > 8) + \mathbb{P}(3Z + 2 < -8) = \mathbb{P}(Z > 2) + \mathbb{P}(Z < \frac{-10}{3})$$

We know that $\mathbb{P}(Z > 2) = 1 - \mathbb{P}(Z \leqslant 2)$, where $\mathbb{P}(Z \leqslant 2) = 0.9772$. Also, $\mathbb{P}(Z < \frac{-10}{3}) = 1 - \mathbb{P}(Z < \frac{10}{3})$. Using the table, $\mathbb{P}(X < \frac{10}{3}) = 0.9996$. So

$$1 - \mathbb{P}(Z \leqslant 2) + 1 - \mathbb{P}(Z < \frac{10}{3}) = 0.0228 + 0.0004 = 0.0232$$

Exercise 4: Let X be Normal random variable $\mathcal{N}(0,2)$ of mean 0 and variance 2. Compute $\mathbb{E}X^3$ and $\mathbb{E}X^4$.

Answer. (Part I : $\mathbb{E}[X^3]$) The pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}} = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{8}}$$

We have that $\mathbb{E}[X^3] = 0$ because x^3 is odd and $p(x)$ is even. So $x^3p(x)$ is odd.

(Part II : $\mathbb{E}[X^4]$) Integrate by parts using:

$$\int U'V\,dx + \int UV'\,dx = \int (UV)'\,dx$$

Let $U = x^3$, $V = f(x)$. Then

$$\int_{-\infty}^{\infty} 3x^2 f(x)\,dx + \int_{-\infty}^{\infty} x^3 f'(x)\,dx = \int_{-\infty}^{\infty} F'(x)\,dx$$

We have that $\int_{-\infty}^{\infty} F'(x)\,dx = 0$ because $F(x) = x^3p(x)$ is odd. Notice that the left equation is one for the variance:

$$3\int_{-\infty}^{\infty} x^2 f(x)\,dx = 3\mathbb{E}[X^2] = 3\sigma^2$$

So we are down to:

$$3\sigma^2 = -\int_{-\infty}^{\infty} x^3 f'(x)\,dx$$

Now

$$f'(x) = -\frac{x}{4} \cdot f(x)$$

So

$$-\int_{-\infty}^{\infty} x^3 f'(x)\,dx = \int_{-\infty}^{\infty} \frac{x^4}{4} f(x)\,dx = \frac{1}{4}\int_{-\infty}^{\infty} x^4 f(x)\,dx = \frac{1}{4}\mathbb{E}[X^4]$$

So the answer is

$$\frac{1}{4}\mathbb{E}[X^4] = 3\sigma^2$$

and therefore

$$\mathbb{E}[X^4] = 4 \cdot 3 \cdot 2^2 = 48$$

Exercise 5:

(a) Find all $\alpha > 0$ for which there is a finite constant c_α so that the function $f_\alpha(x) = c_\alpha \frac{1}{1+|x|^\alpha}$ is a probability density function.

Answer. We want to have

$$\int_{-\infty}^{\infty} c_\alpha \frac{1}{1+|x|^\alpha}\,dx = 1$$

which means that we want the integral to converge. Using the fact that

$$\int_0^\infty \frac{1}{x^p}\,dx$$

converges when $p > 1$ and diverges when $p \leqslant 1$, we have

$$\int_{-\infty}^{\infty} \frac{1}{|x|^p}\,dx = 2\int_0^\infty \frac{1}{x^p}\,dx$$

So

$$\int_{-\infty}^{\infty} \frac{1}{|x|^p}\,dx$$

converges under the same conditions when $p > 1$. Since $\frac{1}{1+|x|^\alpha} < \frac{1}{|x|^\alpha}$, we know that it converges for $\alpha > 1$.

(b) Suppose that $\alpha > 0$ is a number for which f_α is a PDF. Suppose that X is a random variable with PDF f_α. For which $\alpha > 0$ will $\mathbb{E}X$ exist?

Answer. We now want for the integral:

$$\int_{-\infty}^{\infty} c_\alpha \frac{x}{1+|x|^\alpha}\,dx$$

to be finite. Now we want

$$\int_0^\infty c_\alpha \frac{x}{1+x^\alpha}\,dx$$

to converge because the integrand is symmetric. Again, $\frac{x}{1+x^\alpha} < \frac{x}{x^\alpha} = \frac{1}{x^{\alpha-1}}$. So it converges for $\alpha > 2$.

(c) Suppose that X has a PDF given by f_α and X has a finite expectation. What is $\mathbb{E}X$?

Answer. The expectation is 0 because $\frac{x}{1+|x|^\alpha}$ is an odd function. So the negative part and positive part integrate to values that cancel when $\alpha > 2$.