Math250aHw2

Trustin Nguyen

September 11, 2023

Exercise 1: **Direct sums**:

- Prove in detail that the conditions given in Proposition 3.2 for a sequence to split are equivalent. Show that a sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

splits if and only if there exists a submodule N of M such that M is equal to the direct sum $\mathfrak{I}f \oplus N$, and that if this is the case, then N is isomorphic to M''. Complete all the details of the proof of Proposition 3.2.

Proof. First, suppose that there is a homomorphism $\varphi : M'' \to M$ such that $g \circ \varphi = \mathrm{id}$. Then we have the diagram:

$$\begin{tikzcd} M \arrow[r, shift left, "g"] & M'' \arrow[l, shift left, "\varphi"] \arrow[r] & 0 \end{tikzcd}$$

Suppose that $x \in M$. Then we have that $x - \varphi(g(x))$ is in the kernel of g. So x is written as the sum of an element of the kernel above, and an element of the image of φ. To see that it is direct, suppose that $m \in \ker g, \mathfrak{I}\varphi$. Then $\varphi(m_0) = m$. Therefore, $g(\varphi(m_0)) = m_0 = g(m) = 0$. Then $m_0 = 0$ which implies that $\varphi(m_0) = m = 0$. So we have that $M = \ker g \oplus \mathfrak{I}\varphi$.

Now consider the mapping f which is isomorphic to $\ker g$ because the mapping is a homomorphism that is bijective. Then we have an inverse mapping from $M \to M'$ which sends all elements of $\mathfrak{I}\varphi$ to 0 and acts like the inverse of f for elements in $\ker g$. Then this means that there is a homomorphism $\psi : M \to M'$ such that $\psi \circ f = \mathrm{id}$. So we have proved that one of the conditions implies the other. Now suppose that $x \in M$. Then

$$x - f(\psi(x))$$

is in the kernel of ψ and $f(\psi(x))$ is in the image of f. So therefore:

$$M = \mathfrak{I}f \oplus \ker \psi$$

By the fact that $M' \cong \mathfrak{I}f = \ker g$ and $M'' \cong \mathfrak{I}\varphi$, we have

$$M \cong M' \oplus M''$$

Now suppose that we started with statement 2 which was that there was a ψ where $\psi \circ f = \mathrm{id}$. Then we get that $M \cong \mathfrak{I}f \oplus \ker \psi$ which we already showed. To prove that statement 2 implies statement 1, consider the mapping of g restricted to $\ker \psi$. Since $\ker \psi \cap \mathfrak{I}f = 0$, then the mapping is injective. Since it is also surjective, we have an isomorphism and therefore, an inverse. So we have shown the details of the proof. $\square$

Here is the proof of the next biconditional:

Proof. ($\rightarrow$) We have already proved the first part as $M \cong \ker g \oplus \Im\varphi$. We know that $\Im\varphi \cong M''$.

($\leftarrow$) Suppose that

$$M \cong \Im f \oplus N \text{ or } M \cong \ker g \oplus N$$

where N is a submodule of M. Then let $m \in M$ where $m = g_0 + n$ and $g_0 \in \ker g, n \in N$. Notice that we have:

$$g(m) = g(g_0) + g(n) = g(n)$$

Since $\ker g \cap N = 0$, the mapping is injective and furthermore, g is surjective since our sequence is direct. Therefore, we have an isomorphism with $N \cong M''$. This also means that there is a homomorphism $\varphi : M'' \to M$ such that $g \circ \varphi = \mathrm{id}$. We have therefore shown that the sequence splits. □

- Let E and $E_i (i = 1, \ldots, m)$ be modules over a ring. Let $\varphi_i : E_i \to E$ and $\psi_i : E \to E_i$ be homomorphisms having the following properties:

$$\psi_i \circ \varphi_i = id, \qquad \psi_i \circ \varphi_j = 0 \qquad \text{if } i \neq j,$$

$$\sum_{i=1}^{m} \varphi_i \circ \psi_i = id$$

Show that the map $x \mapsto (\psi_1 x, \ldots, \psi_m x)$ is an isomorphism of E onto the direct product of the $E_i (i = 1, \ldots, m)$, and that the map

$$(x_1, \ldots, x_m) \mapsto \varphi_1 x_1 + \cdots + \varphi_m x_m$$

is an isomorphism of this direct product onto E.

Proof. Suppose that we have the conditions listed above. Clearly,

$$x \mapsto (\psi_1 x, \ldots, \psi_m x)$$

is a homomorphism as each ψ_i are homomorphisms. We will show injectivity. Suppose that $x \mapsto (0, 0, \ldots, 0)$. Then we must have

$$x = \sum_{i=1}^{m} (\varphi_i \circ \psi_i)(x)$$

But computing this sum using $\psi_1 x = 0, \ldots, \psi_m x = 0$, we get that $x = 0$. Now to prove surjectivity, suppose we desired a mapping

$$x \mapsto (y_1, \ldots, y_m)$$

Then consider $x = \varphi_1 y_1 + \ldots + \varphi_m y_m$. Then we have:

$$\begin{gathered} \psi_1 x = \psi_1 \varphi_1 y_1 = y_1 \\ \vdots \\ \psi_m x = \psi_m \varphi_m y_m = y_m \end{gathered}$$

So we are done. Now we just need to show that the other mapping is an inverse, because the inverse of an isomorphism is an isomorphism. We have

$$\begin{aligned} x &\mapsto (\psi_1 x, \ldots, \psi_m x) \\ (\psi_1 x, \ldots, \psi_m x) &\mapsto \varphi_1 \psi_1 x + \ldots + \varphi_m \psi_m x \\ &= \sum_{i=1}^{m} (\varphi_i \circ \psi_i)(x) = x \end{aligned}$$

So the mapping

$$(x_1, \ldots, x_m) \mapsto \varphi_1 x_1 + \ldots + \varphi_m x_m$$

was indeed an inverse and therefore also an isomorphism. $\square$

Conversely, if E is equal to the direct product (or direct sum) of submodules $E_i (i = 1, \ldots, m)$, if we let φ_i be the inclusion of E_i in E, and ψ_i the projection of E on E_i, then these maps satisfy the above-mentioned properties.

Proof. Suppose that we had an isomorphism. So there are inverse from $\varphi : E \to \bigoplus E_i$ and $\psi : \bigoplus E_i \to E$ such that $\varphi \circ \psi = \text{id}$. Now define φ as

$$\varphi(x) = (\varphi_1 x, \ldots, \varphi_n x)$$

and ψ as

$$\begin{aligned} \psi &: E_i \to E \\ \psi &:= e_i \mapsto \psi(e_i) \end{aligned}$$

Then action of φ:

$$\varphi(e_1) = (\varphi_1 e_1, 0, \ldots, 0)$$

Then we have the action of ψ :

$$(\varphi_1 e_1, 0, \ldots, 0) \mapsto \psi_1 \varphi_1 e_1 = e_1$$

Since φ and ψ are inverses, we conclude that $\psi_1 \varphi_1 = \text{id}$. We can generalize this for any i. In the previous proof, we also concluded that $\varphi \circ \psi = \sum_{i=1}^{m} \varphi_i \circ \psi_i$. Since it is an isomorphism, we have that $\sum_{i=1}^{m} \varphi_i \circ \psi_i = \text{id}$. Now with the first condition we proved and this condition, we have

$$\psi_1 \sum_{i=1}^{m} \varphi_i \circ \psi_i = \text{id}$$

so we can prove that for each $i \neq j$, $\psi_i \circ \varphi_j = 0$. $\square$

Exercise 2: Let R be a principal ideal domain, and let M be a finitely generated R-module. Show that $\text{Hom}_R(M, R)$ is a free R-module.

Proof. We will show that $\text{Hom}_R(M, R) = \bigoplus \text{Hom}_R((f_i), R)$ for a certain set of generators f_i of R. Let $f_1, \ldots, f_m$ be the set of generators of M. Let $f_1, \ldots, f_n$ be the maximal set of independent generators of M, which we can see, is non-empty. Then we have:

$$a_1 f_1 + \ldots + a_n f_n = 0$$

implies that all $a_i f_i$ are 0. Since the set is maximal, suppose that $f_r \notin f_1, \ldots, f_n$. Then

$$a f_r + a_1 f_1 + \ldots + a_n f_n = 0$$

means that at least two terms are nonzero. We note that $a f_r \neq 0$, otherwise, f_r belongs in the set of independent elements. So we can therefore write $a f_r$ as a combination of the other generators. Suppose that we had some homomorphism in $\text{Hom}_R((f_r), R)$ determined by:

$$\varphi := \varphi(f_r) \mapsto m$$

To show surjectivity, we must show that we can add homomorphisms from $\text{Hom}_R((f_1), R), \ldots, \text{Hom}_R(($ to obtain φ shown above. Notice that we can simply map:

$$\psi(a f_r) = a m$$

We have

$$\psi(af_r) = a\psi(f_r) = am$$

Therefore,

$$\psi(f_r) \mapsto m$$

Since this is a module homomorphism, and that af_r can be written as a combination of the other generators, then the homomorphism action on the generators $f_1, \ldots, f_n$ also determine the homomorphism action on the generators f_r for $f_r \notin f_1, \ldots, f_n$.

So $\mathrm{Hom}_R(M, R) = \mathrm{Hom}_R((f_1, R)) + \ldots + \mathrm{Hom}_R((f_n, R))$. The sum is direct because the intersection of any two $\mathrm{Hom}_R((f_i), R)$ and $\mathrm{Hom}_R((f_j), R)$ is $\{0\}$ since none of the generators are multiples of each other. So we are done. □

Exercise 3: Let R be a principal ideal domain, and let M be a finitely generated R-module. If $0 \neq r \in R$, show that $\mathrm{Hom}_R(R/(r), M) \cong M_r$, the set of elements of M annihilated by r.

Proof. Suppose that we have a module homomorphism $\varphi \in \mathrm{Hom}_R(R/(r), M)$. Then we consider what the multiplicative identity in R gets mapped to:

$$\varphi(1) = m \in M$$

Note that because this is a module homomorphism, we have:

$$r\varphi(1) = \varphi(r) = 0 = rm$$

Then it must be that $m \in M_r$. So every homomorphism is determined uniquely by what the identity in R maps to in M_r. So we have a bijection between these sets. Let φ_{m_1} be the notation for the homomorphism $\varphi \in \mathrm{Hom}_R(R/(r), M)$ which sends the identity in $R/(r)$ to $m_1 \in M_r$. Consider the mappings:

$$\pi : \mathrm{Hom}_R(R/(r), M) \to M_r$$
$$\pi := \varphi_{m_1} \mapsto m_1$$

To prove this is a homomorphism:

$$\begin{aligned}\pi(\varphi_{m_1}\varphi_{m_2}) &= \pi(\varphi_{m_1 m_2}) \\ &= m_1 m_2 \\ &= \pi(\varphi_{m_1})\pi(\varphi_{m_2})\end{aligned}$$

and for the additive part:

$$\begin{aligned}\pi(\varphi_{m_1} + \varphi_{m_2}) &= \pi(\varphi_{m_1+m_2}) \\ &= m_1 + m_2 \\ &= \pi(\varphi_{m_1}) + \pi(\varphi_{m_2})\end{aligned}$$

And as a module homomorphism:

$$\begin{aligned}\pi(a\varphi_{m_1}) &= \pi(\varphi_{am_1}) \\ &= am_1 \\ &= a\pi(\varphi_{m_1})\end{aligned}$$

Since it is a module homomorphism and a bijection, it is an isomorphism. □

Exercise 4: Let R be an integral domain (Lang: entire ring), and let

$$Q := \{(r, s) \in R^2 : s \neq 0\}/\sim$$

where $\sim$ is the equivalence relation $(r, s) \sim (u, v)$ if $rv = su$. Show that the map $R \to Q : r \mapsto (r, 1)$ is a monomorphism (that is, a homomorphism of rings that is one-to-one as a map of sets), and that Q is a field. The field Q is called the quotient field of R.

Proof. We first show that Q is a ring. Define addition to be such that:

$$(r, s) + (t, v) = (rv + st, sv)$$

and multiplication to be:

$$(r, s)(t, v) = (rt, sv)$$

Then the additive identity is $(0, m)$, because $(0, m) + (r, s) = (r, s) + (0, m) = (rm, sm) = (r, s)$. The inverse of an element (r, s) is then $(-r, s)$, and the addition commutes from the fact that R is commutative. The multiplicative identity is $(1, 1)$ since $(r, s)(1, 1) = (r, s)$. We have both closure under addition and multiplication, so we have a ring. Now define the mapping in the question $R \to Q : r \mapsto (r, 1)$ to be φ. This is a homomorphism because:

$$\varphi(rs) = (rs, 1) = (r, 1)(s, 1) = \varphi(r)\varphi(s)$$

and

$$\varphi(r + s) = (r + s, 1) = (r, 1) + (s, 1) = \varphi(r) + \varphi(s)$$

and to check injectivity, suppose $\varphi(r_1) = \varphi(r_2)$. Then $(r_1, 1) = (r_2, 1)$. By the equivalence relation, it follows that $r_1 \cdot 1 = r_2 \cdot 1$ so $r_1 = r_2$. So φ is a monomorphism. Q is a field because the multiplicative inverse an element $(r, s) \in Q$ is (s, r) as:

$$(r, s)(s, r) = (1, 1)$$

This completes the proof. □

Exercise 5: Let R be a principal ideal domain, let M be a finitely generated torsion R-module, and let Q be the quotient field of R. Show that $\mathrm{Hom}_R(M, Q) = 0$.

Proof. Suppose that $\varphi \in \mathrm{Hom}_R(M, Q)$. Then for an arbitrary $m \in M$, we have:

$$\varphi(m) = q$$

for some $q \in Q$. Because φ is a module homomorphism, we have for some $r \neq 0$, $rm = 0$:

$$r\varphi(m) = \varphi(rm) = 0 = rq$$

But Q is an integral domain, therefore, $q = 0$ which means that the only homomorphism of $\mathrm{Hom}_R(M, Q)$ is the 0 map. So $\mathrm{Hom}_R(M, Q) = 0$. □

Exercise 6: Let R be a principal ideal domain, let M be a finitely generated R-module, and let Q be the quotient field of R. Show that $\mathrm{Hom}_R(M, Q) \cong \mathrm{Hom}_R(M/M_{\mathrm{tors}}, Q)$. (Hint: Start with the result of the previous problem, which is a special case).

Proof. By a theorem in the book, M is isomorphic to the direct sum of its torsion submodule and free module:

$$M \cong M_{\mathrm{tors}} \oplus M/M_{\mathrm{tors}}$$

So now let $\varphi \in \mathrm{Hom}_R(M, Q)$ and let $x \in M$ such that $x = x_{\mathrm{tors}} + f$. So we have:

$$\varphi(x) = \varphi(x_{\mathrm{tors}}) + \varphi(f)$$

But $\varphi(x_{\mathrm{tors}}) = 0$ by the previous part. So the homomorphisms of $\mathrm{Hom}_R(M, Q)$ are exactly the ones in $\mathrm{Hom}_R(M/M_{\mathrm{tors}}, Q)$. □

Exercise 7: Let $\mathbb{R}$ be a field of real numbers, and let A be a 4×4 matrix with real entries and minimal polynomial $(x^2 + a)^2$, where a is a positive number. Use the structure theorem for modules of $\mathbb{R}[x]$ to produce the $\mathbb{R}-$canonical form for A. (The answer should be a 4×4 matrix with real entries similar to A).

Proof. We can define $A : V \to V$ where V is a vector space over $\mathbb{R}$. Consider the module structure of the modules A with minimal polynomial $(x^2 + a)^2$ on $\mathbb{R}[x]$ by the action of x as A:

$$x \cdot v = Av$$

So we have a module over a PID. Then M is a torsion module that can be decomposed with respect to some basis. We have

$$M = M(p)$$

for $p = (x^2 + a)$ and so:

$$M = \mathbb{R}[x]/(p) \oplus \mathbb{R}[x]/(p^2)$$

The first summand has elements of degree 1 and 0 and the second has degree $\leqslant 3$. So maybe:

$$1, x, (x^2 + a), x(x^2 + a)$$

So since action of A is determined by multiplication by x, we look at the basis elements times x to get:

$$\begin{bmatrix} 0 & -a & 0 & 3a^2 \\ 1 & 0 & 0 & a \\ 0 & 1 & 0 & -2a \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is the canonical form? $\square$