Math172Hw6

Trustin Nguyen

October 21, 2023

Exercise 1: Find an explicit formula for a_n if $a_0 = 1$ and for any $n \geqslant 1$ we have $a_n = 3a_{n-1} + 2^{n-1}$.

Proof. We start with $a_0 = 1$ and list out the numbers in the process:

- $a_0 = 1$
- $a_1 = 3 + 1$
- $a_2 = 3^2 + 1 \cdot 3 + 2^1$
- $a_3 = 3^3 + 1 \cdot 3^2 + 2^1 \cdot 3 + 2^2$
- $\vdots$

It is visible that

$$2a_n + 3^{n+1} - 3^n = 3a_n + 2^n$$

This is because which we will prove inductively below.

$$a_n = 3^n + \sum_{k=0}^{n-1} 3^k \cdot 2^{(n-1)-k}$$

which evaluates to the same thing for both sides:

$$3^n + 2\sum_{k=0}^{n-1} 3^k \cdot 2^{(n-1)-k} + 3^{n+1} = 3^{n+1} + \sum_{k=0}^{n} 3^k \cdot 2^{n-k}$$

for the RHS and

$$\begin{aligned} 3^{n+1} + \sum_{k=0}^{n-1} 3^{k+1} 2^{(n-1)-k} + 2^n &= 3^{n+1} + 3^n + \sum_{k=0}^{n-2} 3^{k+1} 2^{(n-1)-k} + 2^n \\ &= 3^{n+1} + 3^n + \sum_{k=0}^{n-1} 3^k 2^{n-k} \\ &= 3^{n+1} + \sum_{k=0}^{n-1} 3^k 2^{n-k} \end{aligned}$$

Notice that the evaluation of the RHS shows that the form of a_n holds inductively. We have shown the inductive case for this equivalence, which also holds for $n = 0$ because $a_0 = 1$, which indeed fits the description of a_n we have given above. Now we just solve for a_n:

$$2a_n + 3^{n+1} - 3^n = 3a_n + 2^n$$

which shows that:

$$a_n = 3^{n+1} - 3^n - 2^n$$

□

Exercise 2: Find an explicit formula for a_n if $a_0 = 1$, $a_1 = 4$ and for any $n \geqslant 2$ we have $a_n = 8a_{n-1} - 16a_{n-2}$.

Proof. We have

$$\begin{aligned}
\sum_{n\geqslant 2} a_n x^n &= \sum_{n\geqslant 2} 8a_{n-1}x^n + \sum_{n\geqslant 2} -16a_{n-2}x^{n-2} \\
\sum_{n\geqslant 0} a_n x^n - 4x - 1 &= 8x \sum_{n\geqslant 2} a_{n-1}x^{n-1} - 16x^2 \sum_{n\geqslant 2} a_{n-2}x^{n-2} \\
\sum_{n\geqslant 0} a_n x^n - 4x - 1 &= 8x \sum_{n\geqslant 0} a_n x^n - 8x - 16x^2 \sum_{n\geqslant 0} a_n x^n
\end{aligned}$$

if we let $F(x) = \sum_{n\geqslant 0} a_n x^n$, then we have:

$$\begin{aligned}
F(x) - 4x - 1 &= 8xF(x) - 8x - 16x^2F(x) \\
F(x) - 8xF(x) + 16x^2F(x) &= -4x + 1 \\
F(x)(1 - 8x + 16x^2) &= -4x + 1 \\
F(x) &= \frac{-4x + 1}{1 - 8x + 16x^2}
\end{aligned}$$

Now we calculate the roots of the denominator:

$$Q(x) = (4x - 1)(4x - 1)$$

or we just simplify:

$$F(x) = \frac{-1}{4x - 1} = \frac{1}{1 - 4x}$$

But we know that:

$$\frac{1}{1 - y} = 1 + y + y^2 + \cdots$$

So

$$\frac{1}{1 - 4x} = 1 + 4x + (4x)^2 + \cdots$$

Now a_n is the coefficient of x^n in $F(x)$. Therefore, $a_n = 4^n$. □

Exercise 3: Let $F(x) = \sum_{k\geqslant 0} a_k x^k$ be a formal power series. The *formal derivative* of $F(x)$ is defined as the formal power series $\sum_{k\geqslant 0} k a_k x^{k-1}$ and is denoted by $F'(x)$ of $\frac{d}{dx}F(x)$.

- Show that for any formal power series $F(x), G(x)$ we have $\frac{d}{dx}(F(x) + G(x)) = F'(x) + G'(x)$ and $\frac{d}{dx}(F(x)G(x)) = F'(x)G(x) + F(x)G'(x)$.

  Proof. We have $F(x) = \sum_{k\geqslant 0} a_k x^k$ and $G(x) = \sum_{j\geqslant 0} b_j x^j$. Then:

$$F(x) + G(x) = \sum_{k\geqslant 0} (a_k + b_k)x^k$$

  and therefore,

$$\begin{aligned}
\frac{d}{dx}(F(x) + G(x)) = \sum_{k\geqslant 0} k(a_k + b_k)x^{k-1} &= \sum_{k\geqslant 0} k a_k x^{k-1} + k b_k x^{k-1} \\
&= \sum_{k\geqslant 0} k a_k x^{k-1} + \sum_{j\geqslant 0} j b_j x^{j-1} \\
&= F'(x) + G'(x)
\end{aligned}$$

so we are done with the additive one. Now for the multiplicative one:

$$F(x)G(x) = \sum_{i \geqslant 0} \sum_{j \geqslant 0}^{i} a_j b_{i-j} x^i$$

So the derivative:

$$\frac{d}{dx} F(x)G(x) = \sum_{i \geqslant 0} \sum_{j \geqslant 0}^{i} (i) a_j b_{i-j} x^{i-1}$$

One the other hand,

$$F'(x)G(x) = \sum_{i \geqslant 0} \sum_{j \geqslant 0}^{i} (i - j + 1) a_j b_{i-j+1} x^i$$

and

$$\begin{aligned} F(x)G'(x) &= \sum_{i \geqslant 0} \sum_{j \geqslant 0}^{i-1} (j+1) a_{j+1} b_{i-j} x^i \\ &= \sum_{i \geqslant 0} \sum_{j \geqslant 1}^{i} (j) a_j b_{i-j+1} x^i \end{aligned}$$

so

$$\begin{aligned} F(x)G'(x) + F'(x)G(x) &= \sum_{i \geqslant 0} \sum_{j \geqslant 0}^{i} (i - j + 1) a_j b_{i-j+1} x^i + \sum_{i \geqslant 0} \sum_{j \geqslant 1}^{i} (j) a_j b_{i-j+1} x^i \\ &= \sum_{i \geqslant 0} \sum_{j \geqslant 0}^{i} (i + 1) a_j b_{i-j+1} x^i \\ &= \sum_{i \geqslant 0} \sum_{j \geqslant 0}^{i} (i) a_j b_{i-j} x^{i-1} = \frac{d}{dx} F(x)G(x) \end{aligned}$$

so we are done. □

- Show that $\frac{d}{dx}(1+x)^a = a(1+x)^{a-1}$ where a is an arbitrary number (we have defined these expressions using binomial theorem).

Proof. We know that

$$(1+x)^a = \sum_{k \geqslant 0} \binom{a}{k} x^k$$

So taking the derivative:

$$\begin{aligned} \frac{d}{dx}(1+x)^a = \sum_{k \geqslant 0} k \binom{a}{k} x^{k-1} &= \sum_{k \geqslant 1} \frac{a(a-1)(a-2)\cdots(a-k+1)}{(k-1)!} x^{k-1} \\ &= \sum_{k \geqslant 1} a \frac{(a-1)\cdots(a-k+1)}{(k-1)!} x^{k-1} \\ &= a \sum_{k \geqslant 1} \frac{(a-1)!}{(a-k)!(k-1)!} x^{k-1} \\ &= a \sum_{k \geqslant 1} \binom{a-1}{k-1} x^{k-1} = a \sum_{k \geqslant 0} \binom{a-1}{k} x^k = a(1+x)^{a-1} \end{aligned}$$

so we are done. □

Exercise 4: Find a closed expression for the generating function for the number of functions $[n] \to [k]$ with fixed k, that is find expression for

$$\sum_{n \geqslant 0} k^n x^n$$

which does not involve infinite sums.

Proof. We have

$$S = \sum_{n \geqslant 0} k^n x^n$$

and

$$xkS = \sum_{n \geqslant 0} k^{n+1} x^{n+1}$$

Then

$$S - xkS = 1$$

and

$$S(1 - xk) = 1$$

so

$$S = \frac{1}{1 - xk}$$

which is the expression. □

Exercise 5: Let $\kappa(n, k, j)$ denote the number of weak compositions of n into k parts that each part is less than j.

- Compute the generating function $\sum_{n \geqslant 0} \kappa(n, k, j) x^n$ (more precisely, express it as a rational function in x without using summations with the number of summands depending on k or j).

  Proof. Notice that the number of weak compositions such that no part has less than j things plus the number of compositions where at least one part has j things is equal to the number of compositions total give by $\frac{1}{(1-x)^k}$. So we have

  $$\kappa(n, k, j) + \kappa(n - j, k, n + 1) = \kappa(n, k, n + 1)$$

  or

  $$\kappa(n, k, j) = \kappa(n, k, n + 1) - \kappa(n - j, k, n + 1)$$

  so this is given by:

  $$\frac{1}{(1 - x)^k} - \frac{x^j}{(1 - x)^k}$$

  But our choice of which part received at least a count of j was not arbitrary. So we apply principle of inclusion exclusion:

  $$\frac{1}{(1 - x)^k} - \frac{kx^j}{1 - x^k} + \frac{\binom{k}{2} x^{2j}}{1 - x^k} - \cdots$$

  □

- Prove that

  $$\kappa(n, k, j) = \sum_{r + sj = n} (-1)^s \binom{k + r - 1}{r} \binom{k}{s}.$$

Proof. By the previous part, we first know that

$$\frac{1}{(1-x)^k} = \sum_{r \geqslant 0} \binom{k+r-1}{r} x^r$$

The second term is just $\binom{k}{1}$ times this but with the sign flipped:

$$\frac{kx^j}{(1-x)^k} = \sum_{r \geqslant j} (-1) \binom{k+r-1}{r} \binom{k}{1}$$

And in general:

$$\frac{\binom{k}{s} x^{sj}}{(1-x)^k} = \sum_{r+sj=n} (-1)^s \binom{k+r-1}{r} \binom{k}{s}$$

and we keep going, summing up these terms over s until $n = sj + r$ where $r < j$ which gives us the formula seen up top. $\square$