Math113Hw7

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Exercise 1: let I be an ideal of a ring R and $P_1, \ldots, P_n$ be prime ideals. Show that if $I \subseteq \bigcup P_i$, then $I \subseteq P_j$ for some j.

Proof. We will proceed by induction on the number of prime ideals.

1. Base Case: If I is a subset of P_1, it is indeed a subset of P_1.
2. Inductive Case: Suppose that I is a subset of $P_1 \cup \ldots \cup P_n$ implies that I is a subset of some P_i. Suppose that $I \subseteq P_1 \cup \ldots \cup P_{n+1}$ where none of the prime ideals are subsets of each other. Consider the $p_i \in P_i$. We choose p_1, then p_2 such that it is not in the previous prime ideals and so on. Now the element $p_n + p_{n-1} \cdots p_1 \in I$ has two cases:
   (a) $p_n + p_{n-1} \cdots p_1 \in P_1 \cup \cdots \cup P_{n-1}$. By the inductive hyposthesis, this belongs to some P_i where i is between 1 and $n-1$. So $p_i \in P_i$ which is a contradiction by the p_n we chose.
   (b) $p_n + p_{n-1} \cdots p_1 \in P_n$. Then $p_{n-1} \cdots p_1 \in P_n$ and since this is a prime ideal, all of the p_k are in P_n so $I \subseteq P_n$.

$\square$

Exercise 2:

1. Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

   Proof. This is a Euclidean domain because there is a Euclidean function $\varphi : \mathbb{Z}[\sqrt{-2}]/\{0\} \to \mathbb{Z}_{\geq 0}$ defined by

$$\varphi : a + b\sqrt{-2} \mapsto (a + b\sqrt{-2})(a - b\sqrt{-2})$$

   which is the product of a complex number with its conjugate. We check the conditions:

(a) $\varphi(ab) \geq \varphi(b)$: We have $a + b\sqrt{-2} \mapsto a^2 + 2b^2$. Since this is non-zero, elements in the range are atleast 1 which guaantees that

$$\varphi(ab) = \varphi(a)\varphi(b) \geq (1)\varphi(b)$$

(b) If we were to have a division algorithm, for some $a, b, q, r \in \mathbb{Z}[\sqrt{-2}]$,

$$a = bq + br$$

where $0 \leq \varphi(br) < \varphi(b)$. Therefore,

$$\left|\frac{a}{b} - 1\right| < 1$$

So for anything in $\mathbb{C}$, there is a $q \in \mathbb{Z}[\sqrt{-2}]$ which satisfies the equation. Notice that for an arbitrary real componenet $a \in \mathbb{C}$, it lies between two consecutive integers $a_1 < a < a_1$. We have that

$$(a_1 - a) + (a - a_0) = 1$$

So we can choose an a_0 or a_1 whichever leads to a difference $\leq .5$. We can repeaat the same idea for the complex componenet $c \in \mathbb{C}$ where we can choose a $b \in \mathbb{Z}$ where $\left|b\sqrt{2} - c\right| \leq \frac{\sqrt{2}}{s}$. We can conclude that we can find an element $q \in \mathbb{Z}[\sqrt{-2}]$ such that

$$|x - 1| \leq \left|.5 + \frac{\sqrt{2}}{2}i\right| = .25 + .5 < 1$$

$\square$

2. Show that the norm doesn't make $\mathbb{Z}[\sqrt{-3}]$ into a euclidean domain. Is there another φ for which $\mathbb{Z}[\sqrt{-3}]$ is a Euclidean domain? Justify your answer.

Proof. If we take the norm, then the homomorphism fails the 2nd condition which is that there is a division algorithm. If we weant to find a $a + b\sqrt{-3} \in \mathbb{Z}[\sqrt{-3}]$ such that

$$\left|a + b\sqrt{-3} - z\right| < 1$$

for $z \in \mathbb{C}$, then we have that the difference of the real components will have a max value of .5 while the max difference in the complex componenets will be $\frac{\sqrt{3}}{3}$. This means that

$$\left|a + b\sqrt{-3} - z\right| \leq \left|\frac{1}{2} + \frac{\sqrt{3}}{2}i\right| = \frac{1}{4} + \frac{3}{4} = 1$$

So if we choose a z such that the difference is 1, it will not work. An example is $z = \frac{1}{2} + \frac{\sqrt{-3}}{2}$. There is also no homomorphism that makes $\mathbb{Z}[\sqrt{-3}]$ a Euclidea domain. If it is a euclidea domain, it is a UFD, but we have

$$(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4 = 2 * 2$$

But this shows that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD since $1 + \sqrt{-3}$ and 2 are not associates. □

Exercise 3:

1. Show that $\mathbb{Z}[\sqrt{-7}]$ is not a principal ideal domain.

   Proof. Take the ideal $(2, 1 + \sqrt{-7})$. Notice that 2 is irreducible and that $1 + \sqrt{-7}$ is too. If this was generated by a single element, we have that $a + b\sqrt{-7}$ divides $1 + \sqrt{-7}, 2$. But this is impossible, since they are both irreducibles and that 2 does not divide $1 + \sqrt{-7}$. Also, 1 is not in the ideal, so we are not generating the whole group. □

2. Exhibit an element of $\mathbb{Z}[\sqrt{-7}]$ which is a product of 2 irreducibles and also a product of 3 irreducbles.

   An element would be 8 which would be $(1 - \sqrt{-7})(1 + \sqrt{-7})$ and $2 * 2 * 2$.

Exercise 4: Find all integer solutions to $x^2 + 2 = y^3$.

Proof. Notice all prossible $x^2 = 2$ is some element of $\mathbb{Z}[\sqrt{-2}]$. as proved in exercise 2, $\mathbb{Z}[\sqrt{-2}]$ is a euclidean domain and therefore a unique factorization domain. so $x^2 + 2$ is written as a product of two numbers which is $x + \sqrt{-2}$, $x - \sqrt{-2}$. since irreducibles of conjugates do not divide the other conjugate in a UFD, $x + \sqrt{-2}$ is a cube of an element. So

$$\begin{aligned}
x + \sqrt{-2} &= (a + b\sqrt{-2})^3 \\
&= (a^2 - 2b^2 + 2ab + \sqrt{-2})(a + b\sqrt{-2}) \\
&= a^3 - 2ab^2 + 2a^{2b\sqrt{-2}} + a^2b\sqrt{-2} - 2b^3\sqrt{-2} - 4ab^2 \\
&= a^3 - 6ab^2 + (3a^2b - 2b^3)\sqrt{-2}
\end{aligned}$$

therefore,

$$\begin{aligned}
3a^2b - 2b^3 &= 1 \\
b(3a^2 - 2b^2) &= 1 \\
b &= \pm 1 \\
3a^2 - 2 &= 1 \\
a &= \pm 1
\end{aligned}$$

Cases:

1. $a = 1, b = \pm 1$: $x = a^3 - 6ab^2 = -5$

2. $a = -1 b = \pm 1$: $x = a^3 - 6ab^2 = 5$

So the solutions are $x = 5, -5$ and $y = 3$. $\square$

Exercise 5: Consider the subring $\mathbb{Z}[\sqrt{2}]$ of $\mathbf{R}$. Show that it is a Euclidean domain and find the units.