Math104Hw3

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Exercise 1: Use the limit Theorem 9.2 – 9.7 to prove that $\lim(\frac{n^2}{n^2+1}) = 1$. Justify all steps.

Proof. By the theorems:

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}}$$

we have $\lim(\frac{1}{n^2}) = 0$. The limit of the sum of the series $t_n = 1$ and $s_n = \frac{1}{n^2}$ is just the sum of the limits:

$$\lim_{n\to\infty} \frac{1}{1}$$

Now since the numerator as a sequence and denominator as a sequence don't converge to 0, the limit of the division is the division of their limits which is just 1. □

Exercise 2: Assume that $\lim(s_n)$ and $\lim(t_n)$ exist, and $t_n \geqslant s_n$ for all n. Prove that $\lim(t_n) \geqslant \lim(s_n)$.

Proof. We have that $t_n - s_n \geqslant 0$ for all n. Therefore, our sequence $(t_n - s_n)$ is bounded by 0 on the lower end. The limit of this sequence is therefore greater than or equal to 0. But the limit of the sequence $t_n - s_n$ is the sum of the limits. So we have $\lim(t_n - s_n) = \lim(t_n) - \lim(s_n) \geqslant 0$. So we get

$$\lim(t_n) \geqslant \lim(s_n)$$

as desired. □

Exercise 3: Find the limit of the following sequences if they exist, otherwise write DNE. No proof is required.

- n^n;

  Answer. The limit of the sequence is ∞. We can show that it diverges to ∞ by showing that $\forall M > 0, \exists N$ such that $\forall n > N$, we have

  $$n^n > M$$

  Now pick $N = \max(M, 1)$. Then we have $n \geqslant M$ and $n \geqslant 1$. Therefore,

  $$n^n \geqslant n$$

  but $n > N \geqslant M$ so $n > M$. Therefore, we have found an N such that for all $n > N$,

  $$n^n > M$$

- $(-n)^n$;

  Answer. The limit does not exist. If we take the subsequence of this sequence where n is even and the subsequence where n is odd, we see that there are two limits. But if S is the set of subsequential limits, then there is a limit iff $|S| = 1$. In this case, we have the limits $\infty, -\infty$.

- $(1.1)^n$.

  Answer. The limit of the sequence is ∞. We can rewrite this as

  $$\frac{11^n}{10^n}$$

  Divide the numerator and denominator by 11^n:

  $$\frac{1}{\frac{10^n}{11^n}}$$

  The denominator converges to 0, since $\frac{10}{11} < 1$. Therefore, we have that the number goes to ∞.

Exercise 4: Let $s_n = \cos \frac{n\pi}{3}$. Use the definition to find $limsup(s_n)$ and $liminf(s_n)$, then explain why s_n has no limit.

Proof. We have by definition:

$$liminf(s_n) = \lim_{N\to\infty} inf\{s_n : n > N\}$$
$$limsup(s_n) = \lim_{N\to\infty} sup\{s_n : n > N\}$$

We know that $-1 \leqslant \cos x \leqslant 1$. We will show that $-1, 1 \in \{s_n : n > N\}$ for any N. Clearly, $3N > N$. Now let $n = 3N$. Then we have $\cos N\pi \in \{s_n : n > N\}$. If N is even, then we therefore have $1 \in \{s_n : n > N\}$. The we can also let $n = 3(N + 1)$, which gives us $\cos (N + 1)\pi = -1 \in \{s_n : n > N\}$ also. For the case when N is odd, we also have that $1, -1 \in \{s_n : n > N\}$.

Now we prove that 1 is the supremum and -1 is the infimum of $\{s_n : n > N\}$ for any N. This is immediate because any upper bound less than 1 is not an upper bound because $1 \in \{s_n : n > N\}$. Same for the infimum argument.

So now we just have:

$$liminf(s_n) = \lim_{N\to\infty} -1 = -1$$
$$limsup(s_n) = \lim_{N\to\infty} 1 = 1$$

so since $liminf(s_n) \neq limsup(s_n)$, the limit does not exist. □

Exercise 5: Define $s_1 = 1$ and $s_{n+1} = \frac{s_n+1}{4}$ for all $n \in \mathbb{N}$. Prove that:

- for all n we have $1 \geqslant s_n \geqslant 1/3$

  Proof. We will show this by induction:

  – Base Case: For s_1, we have $1 \geqslant s_1 \geqslant 1/3$.

- Inductive Case: Suppose that $1 \geqslant s_n \geqslant 1/3$ Now we have

$$\frac{4}{3} \leqslant s_n + 1 \leqslant 2$$

and therefore,

$$\frac{1}{3} \leqslant \frac{s_n + 1}{4} = s_{n+1} \leqslant 1/2 \leqslant 1$$

so we have as desired

$\square$

- (s_n) is decreasing;

Proof. We will check this by taking the difference:

$$s_n - s_{n+1} = s_n - \frac{s_n + 1}{4} = \frac{3s_n + 1}{4}$$

But since $s_n > 0$, we have

$$s_n - s_{n+1} > 0$$

Therefore, (s_n) is decreasing. $\square$

- $\lim(s_n)$ exists and find $\lim(s_n)$.

Proof. (Part I) The limit exists because lower bounded decreasing sequences are convergent. We can pick the infimum of the set

$$\{s : s \in (s_n)\}$$

and say that $inf(s_n) + \varepsilon$ is not a lower bound, meaning we can find an S_N such that

$$S_N < inf(s_n) + \varepsilon$$

but since (s_n) is decreasing, we have that for all $n > N$,

$$-\varepsilon < s_n - inf(s_n) < \varepsilon$$

so it has a limit.

Now we know that there is a limit, so any subsequence converges to this same limit. So we have $\lim(s_{n+1}) = \lim(s_n)$. We can use the limit theorems to get the following simplifications:

$$\begin{aligned}
\lim(s_{n+1}) &= \lim(\frac{s_n + 1}{4}) \\
\lim(s_n) &= \lim(\frac{s_n}{4}) + \lim(\frac{1}{4}) \\
\lim(\frac{3s_n}{4}) &= \frac{1}{4} \\
\frac{3}{4}\lim(s_n) &= \frac{1}{4} \\
\lim(s_n) &= \frac{1}{3}
\end{aligned}$$

which is our limit. $\square$

Exercise 6: Directly use the definition of the Cauchy sequence to show that:

- $a_n = 1/n$ is a Cauchy sequence;

Proof. We will show that $\forall \varepsilon > 0$, there is an N such that $\forall n, m > N$ we have

$$|a_n - a_m| < \varepsilon$$

By triangle inequality:

$$|a_n - a_m| \leqslant ||a_n| + |a_m||$$

it suffices to show that we can find an N such that $\forall n, m > N$:

$$a_n + a_m = \frac{1}{n} + \frac{1}{m} < \varepsilon$$

So we want:

$$\frac{1}{n} < \frac{\varepsilon}{2}$$

or in other words,

$$n > \frac{2}{\varepsilon} \implies N = \frac{2}{\varepsilon}$$

So we can check:

$$\begin{aligned} n &> \frac{2}{\varepsilon} & m &> \frac{2}{\varepsilon} \\ \frac{1}{n} &< \frac{\varepsilon}{2} & \frac{1}{m} &< \frac{\varepsilon}{2} \\ \frac{1}{n} + \frac{1}{m} &< \varepsilon \end{aligned}$$

so we have as desired. $\square$

- $b_n = (-1)^n$ is not a Cauchy sequence.

  Proof. So we want to show that there is an $\varepsilon > 0$ such that $\forall N$, we have there are are $n, m > N$ such that

  $$|b_n - b_m| \geqslant \varepsilon$$

  Take $\varepsilon = 1$ and let N be arbitrary. Clearly, $2N > N$ and we have:

  $$b_{2N} = (-1)^{2N} = 1$$

  and we also have

  $$b_{2N+1} = (-1)^{2N+1} = -1$$

  Now take $n, m = 2N, 2N + 1$. Then

  $$|b_n - b_m| = |1 - (-1)| = 2 \geqslant 1 = \varepsilon$$

  so this is not a Cauchy sequence. $\square$