Math110Hw5

Trustin Nguyen

February 2023

Homework 5

Exercise 1: Let $V := \mathbb{C}^2$. Give an example of a map $T \in \mathcal{L}(V, V)$ such that $V = \operatorname{null} T \oplus \operatorname{range} T$, with both $\operatorname{null} T$ and $\operatorname{range} T$ non-zero, or prove that none exists.

Our linear map is $T \in \mathcal{L}(\mathbb{C}^2, \mathbb{C}^2)$, so by the Fundamental Theorem of Linear Maps, we have

$$\begin{gathered} 2 = \dim \operatorname{range} T + \dim \operatorname{null} T \\ \dim \operatorname{range} T = \dim \operatorname{null} T = 1 \end{gathered}$$

Since the dimensions of the range and null space are non-zero. Take the linear map

$$T := (a, b) \mapsto (0, b)$$

It is linear by checking

$$\begin{aligned} T(am, bm) + T(an, bn) &= (0, bm) + (0, bn) \\ &= (0, bm + bn) \\ &= (0, b(m + n)) \\ &= (m + n)T(a, b) \end{aligned}$$

It is clear that all elements in the range of T are of the form $(0, \lambda)$, so $(0, 1)$ spans $\operatorname{range} T$. Now all elements that are sent to $(0, 0)$ must have $b = 0$ and $a =$ anything. So they are of the form $(\lambda, 0)$. So $(1, 0)$ spans the null space of T. So we have found a linear map that satisfies $\mathbb{C}^2 = \operatorname{Span}\{(1, 0)\} \oplus \operatorname{Span}\{(0, 1)\}$.

Exercise 2: Give an example of a map $T \in \mathcal{L}(\mathbb{R}^6, \mathbb{R}^2)$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4, x_5, x_6) : x_1 = 2x_2,\ x_3 - x_5 = 0,\ x_1 + x_4 - x_5 = 0\}$$

or prove that none such exists.

Proof. We show that none such exists, first using the conditions

$$\begin{aligned} x_1 &= 2x_2, \\ x_3 - x_5 &= 0, \\ x_1 + x_4 - x_5 &= 0. \end{aligned}$$

we simplify the form of vectors in the null space to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ x_3 \\ x_3 - 2x_2 \\ x_3 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ -2x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_6 \end{bmatrix}$$

So the null space is spanned by

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

And the dimension of the null space is 3. Assume for contradiction that such a null space exists. By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned} \dim \mathbb{R}^6 = 6 &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ 3 &= \dim \operatorname{range} T \end{aligned}$$

But the range of T is a subspace of $\mathbb{R}^2$, so the dimension of it is at most 2. Contradiction. $\square$

Exercise 3: Let $T : f \mapsto f''' - 3f'' + f$. Write down its matrix representation

(a) as a map in $\mathcal{L}(\mathcal{P}_2, \mathcal{P}_2)$ using the standard monomial basis both for the domain and codomain;

		1	x	x^2
$T(1) = 1$	$\rightarrow$	1	0	0
$T(x) = x$	$\rightarrow$	0	1	0
$T(x^2) = x^2 - 6$	$\rightarrow$	-6	0	1

We get a matrix with the columns denoting the linear combination of basis vectors of W that give us each $T(v)$:

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using the standard monomial basis both for the domain and codomain;

		1	x	x^2	x^3
$T(1) = 1$	$\rightarrow$	1	0	0	0
$T(x) = x$	$\rightarrow$	0	1	0	0
$T(x^2) = x^2 - 6$	$\rightarrow$	-6	0	1	0
$T(x^3) = x^3 - 18x + 6$	$\rightarrow$	6	-18	0	1

We get a matrix with the columns denoting the linear combination of basis vectors of W that give us each $T(v)$:

$$\begin{bmatrix} 1 & 0 & -6 & 6 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) as a map in $\mathcal{L}(\mathcal{P}_2, \mathcal{P}_2)$ using a Newton basis $1, x, x(x-1)$ for the domain and the standard monomial basis for the codomain;

		1	x	x^2
$T(1) = 1$	$\rightarrow$	1	0	0
$T(x) = x$	$\rightarrow$	0	1	0
$T(x(x-1)) = x^2 - x - 6$	$\rightarrow$	-6	-1	1

We get a matrix with the columns denoting the linear combination of basis vectors of W that give us each $T(v)$:

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using a shifted monomial basis $1, x-1, (x-1)^2, (x-1)^3$ for the domain and a Newton basis $1, x-1, (x-1)x, (x-1)x(x+1)$

	1	$x-1$	$(x-1)x$	$(x-1)x(x+1)$
$T(1) = 1$	1	0	0	0
$T((x-1)) = x$	-1	1	0	0
$T((x-1)^2) = x^2 - 6$	-3	-1	1	0
$T((x-1)^3) = x^3 - 18x + 6$	6	-17	-3	1

We get a matrix with the columns denoting the linear combination of basis vectors of W that give us each $T(v)$:

$$\begin{bmatrix} 1 & -1 & -3 & 6 \\ 0 & 1 & -1 & -17 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 4: Suppose V and W are finite-dimensional vector spaces. Let $v \in V$ (fixed), and consider

$$E_v := \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that E_v is a subspace of $\mathcal{L}(V, W)$.

Proof. Note that E_v is a subset of the vector space of all linear maps. We check the three conditions.

1. The zero map is in E_v since $0v = 0$
2. The subspace is closed under their combination since if $T_1, T_2 \in E_v$,
$$(T_1 + T_2)v = T_1 v + T_2 v = 0$$
so $T_1 + T_2$ is in E_v also.
3. The subspace is closed under scalar multiplication. Suppose $\lambda \in \mathbb{R}$. Then observe that
$$(\lambda T)(v) = \lambda T(v) = \lambda \cdot 0 = 0$$
so (λT) is in E_v.

and we're done. □

(b) Suppose $v \neq 0$. What is $\dim E_v$?

Proof. Since $Tv = 0$, we consider the linear mapping to all other vectors other than the ones in $\operatorname{Span}(v)$:

$$T' : V \setminus \operatorname{Span}(v) \to W$$

which has dimension $\dim E_v = |W|(|V| - 1)$. $V \setminus \operatorname{Span}(v)$ has dimension $|V| - 1$ since we can construct a basis for V starting with v and remove that vector after to obtain a basis for our domain of T'. □