Math143Hw2

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Exercise 1: Suppose $I, J \subseteq k[x_1, \ldots, x_n]$ are ideals.

(a) Prove that $I + J := \{f + g : f \in I, g \in J\}$ is an ideal.

Proof. We will prove that it is an additive group. Closure under addition for $f_1, f_2 \in I$ and $g_1, g_2 \in J$:

$$(f_1 + g_1) + (f_2 + g_2) = (f_1 + f_2) + (g_1 + g_2)$$

We note that $f_1 + f_2 \in I$ and $g_1 + g_2 \in J$. Therefore, by definition of $I + J$, $(f_1 + g_1) + (f_2 + g_2) \in I + J$. The sum of two elements in $I + J$ is again in $I + J$. There is the additive identity because $0 \in I, 0 \in J$ implies that $0 + 0 = 0 \in I + J$. Associativity is inherited from the ring $k[X_1, \ldots, X_n]$.

Now we show a closure under multiplication of elements in $k[X_1, \ldots, X_n]$. Let $a \in k[X_1, \ldots, X_n]$. Then for $f + g \in I + J$, $f \in I$ and $g \in J$, we have:

$$a(f + g) = af + ag$$

but $af \in I$ and $ag \in J$, so $af + ag \in I + J$. We are done. □

(b) Prove that $V(I + J) = V(I) \cap V(J)$.

Proof. We have one inclusion because $I \subseteq I + J$ and $J \subseteq I + J$. Suppose that $p \in V(I + J)$. Then we have that for all polynomials of the form $f + g$,

$$(f + g)(p) = 0$$

Then we must also have the case where $f = 0$ or $g = 0$. Therefore, p must be solution to all polynomials of the form:

$$(f + 0)(p) = f(p) = 0 \text{ and } (0 + g)(p) = g(p) = 0$$

So we have $V(I + J) \subseteq V(I) \cap V(J)$. Now suppose that $p \in V(I) \cap V(J)$. The we must have both:

$$f(p) = 0 \text{ and } g(p) = 0$$

for all $f \in I$ and $g \in J$. We take the sum which is also 0:

$$f(p) + g(p) = (f + g)(p) = 0$$

Therefore $p \in V(I + J)$. So there is the other inclusion $V(I + J) \supseteq V(I) \cap V(J)$. □

Exercise 2: This problem will practice decomposing an algebraic set into irreducible components, similar to the example in lecture.

(a) Show that $V(y - x^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$ is irreducible.

Proof. We will prove that $y - x^2$ is prime. Let $y - x^2 = fg$. Then $\deg(f) = \deg(g) = 1$.

If this is not the case, then one of our polynomials will have degree $\geqslant 2$. Then the other polynomial will have degree 0, otherwise, we will get nonzero cross terms. If one of our factors has degree 0, we can multiply by a unit to get 1, therefore showing that $y - x^2$ divides either f or g.

Otherwise, we have wlog:

$$f = a_1x + b_1y + c, g = a_2x + b_2y$$

So fg is

$$a_1a_2x^2 + b_1b_2y^2 + a_1b_2xy + a_2b_1xy + a_2cx + b_2cy$$

We must have $a_1, a_2 \neq 0$ but $a_1b_2 = 0$ and $b_2c = 0$. So this case is impossible. We conclude that $(y - x^2)$ is a prime ideal in $\mathbb{C}[x, y]$. By what is proved in question 3, and the Nullstellensatz, we have:

$$I(V(y - x^2)) = \sqrt{(y - x^2)}$$

and since prime ideals are radical ideals,

$$I(V(y - x^2)) = (y - x^2)$$

Therefore, since $I(V(y - x^2))$ is prime, $V(y - x^2)$ is irreducible. □

(b) Decompose $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$ into irreducible components.

Answer. We want to see when

$$y^4 - x^2 = 0$$

So we have the factoring

$$(y^2 - x)(y^2 + x) = 0 \implies x = y^2, x = -y^2$$

Now we have irreducibles $(y^2 - x)$ and $(y^2 + x)$. Next, plug in the conditions into the second polynomial:

$$x^2 - x^3 + x^2 - x^3 = 2x^2 - 2x^3 = 0 = 2x^2(x - 1) = 0$$
$$x^2 + x^3 - x^2 - x^3 = 0$$

So we require that $x = 1$ or $x = 0$. Therefore, we have our decomposition:

$$V(y^2 + x) \cup (1, 1) \cup (0, 0) \cup (1, -1)$$

Exercise 3: Let I be an ideal in a ring R.

(a) If $a^n \in I, b^m \in I$, show that $(a + b)^{n+m} \in I$.

Proof. Suppose that $a^n \in I, b^m \in I$. Then we consider:

$$(a + b)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} a^i b^{n+m-i}$$

We have two cases. If $i \geqslant n$ then the summands at or after the first n summands will have the factor a^n, so it will be in I. Now if $i < n$, then we have that $n + m - i > m$. Therefore, the first $n - 1$ summands will have the factor b^m. Since each summand is an element of I and I is an ideal, then $(a + b)^{n+m} \in I$. □

(b) Show that $\sqrt{I}$ is an ideal.

Proof. Suppose that $a \in \sqrt{I}, b \in \sqrt{I}$. Then $a^n \in I$ and $b^m \in I$. Since $(a+b)^{n+m} \in I$ also, $a + b \in \sqrt{I}$. Now let $r \in R$. Then $r^n \in R$. Since I is an ideal, we have that $r^n a^n \in I$. Therefore, $(ra)^n \in I$ and we have $ra \in I$. □

(c) Show that $\sqrt{I}$ is a radical ideal.

Proof. ($\sqrt{\sqrt{I}} \subseteq I$) Suppose that $a \in \sqrt{\sqrt{I}}$. Then that means that $a^k \in \sqrt{I}$ for some $k > 0$. But since $a^k \in \sqrt{I}$, we must have $a^{kn} \in I$. Therefore, $a \in \sqrt{I}$. So we have $\sqrt{\sqrt{I}} \subseteq \sqrt{I}$.

($\sqrt{I} \subseteq \sqrt{\sqrt{I}}$) Suppose that $a \in \sqrt{I}$. Then we have $a^k \in I$ for some $k > 0$. But then that means that $a^k \in \sqrt{I}$ since $(a^k)^1 \in I$. Since $a^k \in \sqrt{I}$, we have $a \in \sqrt{\sqrt{I}}$.

Therefore, $\sqrt{I}$ is a radical ideal. □

(d) Prove that any prime ideal is a radical ideal.

Proof. Suppose that I is prime and that $a \in I$. We immediately have that $a \in \sqrt{I}$. Now suppose that $a \in \sqrt{I}$. Then $a^k \in I$ for some $k > 0$. If $k = 1$, then we are done. If $k > 1$, then we have that $a^k = a^{k-1} \cdot a \in I$. Therefore, either $a \in I$ or a^{k-1}. So by induction, we continue the decomposition until we can conclude that $a \in I$. Therefore, $I = \sqrt{I}$. □

Exercise 4: Prove or give a counter example to the following statements.

(a) If X and Y are algebraic sets, then $I(X \cup Y) = I(X) \cap I(Y)$.

Proof. ($I(X \cup Y) \subseteq I(X) \cap I(Y)$) Suppose that $p \in X \cup Y$ as an arbitrary point, and let $f \in I(X \cup Y)$ be arbitrary also. Then we know that $f(p) = 0$. Therefore, $f \in I(X)$ and $f \in I(Y)$. So $f \in I(X) \cap I(Y)$

($I(X) \cap I(Y) \subseteq I(X \cup Y)$) Suppose that $f \in I(X)$ and $f \in I(Y)$. Then for any $x \in X$, $y \in Y$, we have $f(x) = 0, f(y) = 0$. So if $p \in X \cup Y$, then $p \in X \vee Y$ and $f(p) = 0$. Therefore, $f \in I(X \cup Y)$. □

(b) If X and Y are algebraic sets, then $I(X \cap Y) = I(X) + I(Y)$.

Proof. ($I(x) + I(Y) \subseteq I(X \cap Y)$) We have that if $f \in I(x) + I(y)$, then $f = g + h$ where $g(x) = 0, h(y) = 0$. Since $g, h \in I(X \cap Y)$, then $f \in I \cap Y$.

I don't know how to prove the other inclusion, nor can I find a counter example. □

Exercise 5: In class we proved that if R is Noetherian, then every ascending chain of ideals

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

in R is finite. Prove the converse is true. Namely, if every ascending chain of ideals

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

in R is finite, then R is Noetherian.

Proof. Suppose for contradiction that there was an ideal I_0 that was infinitely generated by $f_1, f_2, \ldots$. Then we pick one of the generators and consider the ideal generated say $I_1 = (f_1)$. Then we define each ideal I_n to be

$$I_n = (f_1, f_2, \cdots, f_n)$$

Clearly,

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

But this is an infinite chain. So contradiction. $\square$