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Problem 1

Proof. Suppose $G=(V,E)$ is a simple connected graph which is not a tree. Then G is cyclic with a cycle C that has at least 3 edges. Observe that for $x,y \in V$, if it uses an edge e of C, we can replace that edge with $C-e$ which would create a new path between x and y. More importantly deleting that edge in C does not make G disconnected. Since there is a cycle with at least 3 edges, we have

$$C = x_0, e_1, x_1, \ldots, e_n, x_n = x_0$$

Let $G_1, \ldots, G_n$ be the subgraphs of G obtained by deleting $e_1, \ldots, e_n$ respectively. $G_1, \ldots, G_n$ are connected so they contain a spanning tree: $T_1, \ldots, T_n$. But that spanning tree must also be in G since it shares the same set of vertices as G. Since $n \geq 3$, G has at least 3 spanning trees. $\square$

Problem 2

Proof. Suppose G has 10 connected components $H_1, \ldots, H_{10}$ each containing an Eulerian circuit. Since we want G to have an Eulerian circuit, we must make G connected. That is, for ~~each~~ $v_i \in H_i$, $v_j \in H_j$ $i \neq j$, there must be a path between v_i and v_j. If we add such an edge $\{v_i, v_j\}$, ~~then~~ observe that $H_i \cup H_j$ is now a connected component of $G + \{v_j, v_i\}$. We do this until $H_1 \cup H_2 \cup \ldots \cup H_{10}$ is a connected ~~component~~ component. So we need to add 9 edges. ~~Consider~~ Let this new graph be G'. Consider the subgraph of G', G'' ~~with only~~ ~~the~~ which contains the 9 added edges and their incident vertices. Since G'' is connected and has 10 vertices, $10-1$ edges, G'' is acyclic (by HW problem). Then it has at least 2 leaves: v_0, v_p. Since these have odd degree, we must ~~have~~ add an edge $\{v_0, v_p\}$ to give them even degree. Since G'' has 2 leaves (minimum case), the other 8 vertices must have degrees summing to $18-2 = 16$. So each vertex has degree 2 ~~which~~ which is even. So 10 edges are necessary to make G connected and have $\deg(v) \equiv 0 \pmod 2$ for all $v \in G$. $\square$