Math104Hw2

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Exercise 1: State what it means mathematically that a sequence (s_n) converges to $s \in \mathbb{R}$.

Answer. For a sequence (s_n) to converge to s, we must have that $\forall \varepsilon > 0$, there exists an N such that $\forall n > N$, we have that:

$$|s_n - s| < \varepsilon$$

Exercise 2: Assume that $\lim(s_n) = 0$. Prove that $\lim(\sqrt[3]{s_n}) = 0$.

Proof. Since $\lim(s_n) = 0$, we have that for all ε, there is an N such that $\forall n > N$,

$$|s_n| < \varepsilon$$

or

$$-\varepsilon < s_n < \varepsilon$$

If we take the cube root of both sides, we have

$$\sqrt[3]{-\varepsilon} < \sqrt[3]{s_n} < \sqrt[3]{\varepsilon}$$

But now we are done as we can replace $\sqrt[3]{\varepsilon} = \delta$ and say that there is an N such that for all $n > N$,

$$|\sqrt[3]{s_n}| < \delta$$

We would just choose N based on the epsilon for our previous sequence, where $\varepsilon = \delta^3$. □

Exercise 3: Use the definition of convergence to prove $\lim(\frac{3n}{n+1}) = 3$; no theorem in Ross 9 is allowed.

Proof. We need to show that $\forall \varepsilon > 0$, we have an N such that $\forall n > N$,

$$\left| \frac{3n}{n+1} - 3 \right| < \varepsilon$$

So we can collect the terms in the absolute value:

$$\left| \frac{-3}{n+1} \right| < \varepsilon$$

Therefore, we require that:

$$\frac{3}{n+1} < \varepsilon \text{ or } n > \frac{3}{\varepsilon} - 1$$

Now let $N = \frac{3}{\varepsilon} - 1$. Then we plug in $n > \frac{3}{\varepsilon} - 1$ into

$$\left|\frac{3n}{n+1} - 3\right| = \left|\frac{-3}{n+1}\right|$$

We get that

$$\begin{aligned} n &> \frac{3}{\varepsilon} - 1 \\ n+1 &> \frac{3}{\varepsilon} \\ \varepsilon &> \frac{3}{n+1} \end{aligned}$$

But $|\frac{-3}{n+1}| = \frac{3}{n+1}$. So we can conclude that for any ε, we have found an N such that $\forall n > N$,

$$\left|\frac{3n}{n+1} - 3\right| < \varepsilon$$

So we are done. □

Exercise 4: Use the definition of convergence to prove $\lim(\frac{n-1}{n^2+1}) = 0$; no theorem in Ross 9 is allowed.

Proof. We want that $\forall \varepsilon > 0$, there exists an N such that $\forall n > N$, we have

$$\left|\frac{n-1}{n^2+1}\right| < \varepsilon$$

We can choose an intermediate function:

$$\left|\frac{n-1}{n^2+1}\right| < \left|\frac{n-1}{n^2-1}\right| < \varepsilon$$

So we require that $N \geqslant 1$. But now we see that

$$\frac{n-1}{n^2-1} = \frac{1}{n+1} < \varepsilon$$

and so we take

$$\frac{1}{\varepsilon} - 1 < n$$

Using the original constraints, let $N = \max(1, \frac{1}{\varepsilon} - 1)$. Now if $n > \max(1, \frac{1}{\varepsilon} - 1)$, we get

$$\begin{gathered} n - 1 > 0 \\ \frac{1}{n+1} < \varepsilon \end{gathered}$$

Therefore,

$$\left|\frac{n-1}{n^2-1}\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} < \varepsilon$$

For $n > 1$, we have

$$\left|\frac{n-1}{n^2+1}\right| = \frac{n-1}{n^2+1} < \frac{n-1}{n^2-1} = \left|\frac{n-1}{n^2-1}\right|$$

So we have that $\forall \varepsilon$, there is an N such that $\forall n > N$,

$$\left|\frac{n-1}{n^2+1}\right| < \left|\frac{n-1}{n^2-1}\right| < \varepsilon$$

which finishes the proof. □

Exercise 5: Use the definition of convergence to prove that the sequence (s_n) diverges where $s_n = \sqrt{n}$.

Proof. We wish to show that the sequence does not converge. So we want that $\exists \varepsilon > 0$ we have that for any N, there is an $n > N$ such that

$$|\sqrt{n} - L| \geqslant \varepsilon$$

where L is an arbitrary number in $\mathbb{R}$. Suppose for contradiction, there was an N such that for all $n > N$,

$$|\sqrt{n} - L| < \varepsilon$$

Take $\varepsilon = 1$. Take an arbitrary $n_0 > N$ and square it. We have $N < n_0 \leqslant n_0^2$. So plugging in $n = n_0^2$, we get:

$$-1 + L < n_0 < 1 + L$$

But now consider $N < n_0^2 < (n_0 + 5)^2$. So

$$\begin{aligned} -1 + L < n_0 + 5 < 1 + L \\ -1 - L < -n_0 < 1 - L \end{aligned}$$

The sum of the inequalities gives us:

$$-2 < 5 < 2$$

which is a contradiction. So for any N, there is an $n > N$ such that

$$|\sqrt{n} - L| \geqslant \varepsilon$$

This shows non-convergence. $\square$

Exercise 6: Assume that $\lim(s_n) = s$ and $s_n \geqslant 0$. Prove that $s \geqslant 0$.

Proof. Since the limit exists, we have that $\forall \varepsilon > 0$, there is an N such that $\forall n > N$, we have

$$|s_n - s| < \varepsilon$$

This also means

$$-\varepsilon + s < s_n < \varepsilon + s$$

But since $s_n \geqslant 0$, we have

$$0 \leqslant s_n < \varepsilon + s \implies 0 \leqslant \varepsilon + s$$

Suppose for contradiction that $s < 0$. We want to find an ε small enough such that its sum with s lies below 0. Take $\varepsilon = |\frac{s}{2}|$. So

$$\left|\frac{s}{2}\right| + s = s - \frac{s}{2} = \frac{s}{2} < 0$$

But that is impossible because we started with: $\forall \varepsilon > 0, 0 \leqslant \varepsilon + s$. Therefore, $s \geqslant 0$. $\square$