Math110Hw8

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Exercise 1: Let V be the complex vector space of bivariate polynomials of total degree at most 2, and let T be the linear operator $T : p \mapsto \frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}$. Determine, with proof, (a) the minimal polynomial, (b) all eigenvalues, and (c) the corresponding eigenvectors of T.

Proof. Choose an arbitrary element of the vector space:

$$x^2 + y^2$$

and consider:

$$f = x^2 + y^2, Tf = 2x - 2y, T^2 f = 0$$

The shortest dependence is

$$0f + 0Tf + 1T^2 f = 0$$

so the polynomial that sends f, Tf to 0. Now consider the remaining elements in the range of $p_1(v) = v^2$. After double differentiation, we only get constants, so the polynomial that sends the remaining stuff to 0 is $p_2(v) = v$. We can take $p_{\min} = p_1(v)p_2(v) = v^3$ which is the 0 map with respect to T. □

Exercise 2: Let $S, T \in \mathcal{L}(V)$ and suppose S is invertible. (a) Prove that, for any polynomial $p \in \mathcal{P}(\mathbb{F})$,

$$p(STS^{-1}) = Sp(T)S^{-1}$$

(b) How are the subspaces of V invariant under T related to the subspaces invariant under STS^{-1}?

(a) *Proof.* Consider the general polynomial of degree n:

$$p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Now consider an arbitrary term after plugging in STS^{-1}:

$$a_i x^i \mapsto a_i(STS^{-1})^i = a_i(ST^i S^{-1})$$

So we can write:

$$\begin{aligned} p(STS^{-1}) &= a_0SS^{-1} + a_1STS^{-1} + a_2ST^2S^{-1} + \cdots + a_nST^nS^{-1} \\ &= Sp(T)S^{-1} \end{aligned}$$

as desired. $\square$

(b) ~~Not interested~~

Exercise 3: Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). (a) Prove that $TS = ST$. (b) Give an example of such operators T and S on $\mathbb{R}^2$, neither of which is a multiple of the identity operator.

(a) If T has distinct eigenvalues, then we can conclude that there are $\dim V$ linearly independent eigenvectors. This forms a basis for T. We can consider the action of ST and TS with respect to this basis.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues for T for the eigenbasis for V: $\{v_1, v_2, \ldots, v_n\}$ and $\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be the eigenvalues for S for the same eigenbasis, corresponding eigenvectors.

Now for an arbitrary basis vector v_i we have

$$\begin{aligned} STv_i &= S(\lambda_i v_i) = \sigma_i\lambda_i v_i \\ TSv_i &= T\sigma_i v_i = \lambda_i\sigma_i v_i \\ STv_i &= \sigma_i\lambda_i v_i = \lambda_i\sigma_i v_i = TSv_i \end{aligned}$$

So the maps ST, TS are the same.

(b) Consider the operators

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad S = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Now compute TS, ST.

$$\begin{aligned} TS &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 0 \\ 0 & 4 \end{bmatrix} \\ ST &= \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 21 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

So this is an example where $ST = TS$.

Exercise 4: Let V be a finite-dimensional real vector space and let $T \in \mathcal{L}(V)$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\lambda) := \dim \operatorname{range}(T - \lambda I).$$

Formulate and prove a condition on T equivalent to f being a continuous function on $\mathbb{R}$.

Proof. We know that V has finitely many eigenvalues since it is finite dimensional. Observe that the map $T - \lambda I$ maps all eigenvectors of λ to 0. If there are no eigenvectors for a λ, then there are no vectors in the null space, except for the trivial one:

$$\begin{aligned}(T - \lambda I)v &= 0 \\ Tv - \lambda v &= 0 \\ Tv &= \lambda v\end{aligned}$$

but T has no eigenvectors, so $v = 0$. Therefore, the dimension of the range is equal to the dimension of V by the rank-nullity theorem. If there is an eigenvector, for the λ, continuity of f is broken because the function steps down. The limit as f approaches λ from the right is not equal to the actual value of λ:

$$\begin{aligned}\lim_{x \to \lambda +} f(x) &= \dim V \\ f(\lambda) &< \dim V\end{aligned}$$

To fix this, we must not have any eigenvalues. This is possible if all the zeroes to the minimal polynomial are not in $\mathbb{R}$. Since this means that $p(T)$ can be factored into linear degree 1 polyomials with complex roots, we conclude that T is upper triangular with complex values on the diagonal. $\square$

Exercise 5: Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$ satisfies the condition: For any $\phi \in V', \lim_{n\to\infty} \phi(T^n v) = 0$. What does this imply about the eigenvalues of T?

Proof. A basis of linear functionals map the corresponding basis vectors from V to 1 and the other basis vectors to 0. So we must have that

$$(a_1\phi_1 + a_2\phi_2 + \ldots + a_{\dim V}\phi_{\dim V})(T^n v) = 0$$

for any a_n in our field. If the basis for V is $\{v_1, \ldots, v_{\dim V}\}$,

$$T^n v = b_1 v_1 + b_2 v_2 + \ldots + b_{\dim V} v_{\dim V}$$

we get that any linear functional in V must have the condition that

$$\begin{aligned}(a_1\phi_1 + a_2\phi_2 + \ldots + a_{\dim V}\phi_{\dim V})(T^n v) &= 0 \\ (a_1 b_1 + a_2 b_2 + \ldots + a_{\dim V} b_{\dim V}) &= 0\end{aligned}$$

This is true only when $b_1 = \ldots = b_{\dim V} = 0$, so $T^n v = 0$. Now if T has an eigenvalue λ with eigenvector v, we know that

$$\lim_{k\to\infty} T^k v = \lim_{k\to\infty} \lambda^k v = 0$$

So as k goes to infinity, it must be that since v is not 0, the value λ^k must go to 0, so $-1 < \lambda < 1$. $\square$