Math110Hw10

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Homework 10

Exercise 1: find a polynomial $p \in \mathcal{P}_2(\mathbb{R})$ such that

$$q\prime(1) = \int_0^1 p(t)q(t)\ \mathrm{d}t \qquad \text{for all } q \in \mathcal{P}_2(\mathbb{R})$$

Proof. From class, it was proved that we can represent any linear function as an inner product with a fixed second entry. So to represent $q\prime(1)$, we have

$$\varphi_1 := q\prime(1) = \langle \cdot, p_{\varphi_1}(t) \rangle$$

where $p_{\varphi_1}(t)$ is the function:

$$p_{\varphi_1}(t) = \varphi_1(e_1)e_1 + \ldots + \varphi_1(e_n)e_n$$

for an orthonormal basis $e_1, \ldots, e_n$. Starting with the basis $\{1, x, x^2\}$, use Gram-Schmidt and orthogonalize:

$$\begin{aligned} v_2 &= x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} \\ &= x - \frac{\int_0^1 x\ \mathrm{d}x}{\int_0^1 1\ \mathrm{d}x} \\ &= x - \frac{1}{2} \end{aligned}$$

So our basis vectors are $\{1, x - \frac{1}{2}\}$. Now to orthogonalize x^2 to this:

$$\begin{aligned} v_3 &= x^2 - \frac{\langle x^2, x - \frac{1}{2} \rangle}{x - \frac{1}{2}, x - \frac{1}{2}} \left(x - \frac{1}{2} \right) - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \\ &= x^2 - \frac{\int_0^1 x^3 - \frac{x^2}{2}\ \mathrm{d}x}{\int_0^1 x^2 - x + \frac{1}{4}\ \mathrm{d}x} \left(x - \frac{1}{2} \right) - \frac{\int_0^1 x^2\ \mathrm{d}x}{\int_0^1 1\ \mathrm{d}x} \\ &= x^2 - \frac{\frac{1}{12}}{\frac{1}{12}} \left(x - \frac{1}{2} \right) - \frac{1}{3} \\ &= x^2 - x + \frac{1}{6} \end{aligned}$$

to normalize:

$$\begin{aligned} \int_0^1 \left(x^2 - x + \frac{1}{6} \right) \mathrm{d}x &= \int_0^1 x^4 + x^2 + \frac{1}{36} - 2x^3 + \frac{1}{3}x^2 - \frac{1}{3}x\ \mathrm{d}x \\ &= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} + \frac{1}{9} - \frac{1}{6} \\ &= \frac{1}{5} + \frac{1}{36} - \frac{18}{36} + \frac{5}{36} + \frac{6}{36} \\ &= \frac{1}{5} - \frac{7}{36} = \frac{1}{180} \end{aligned}$$

So the orthonormal basis is $\{1, \sqrt{12}x - \frac{\sqrt{12}}{s}, 6x^2\sqrt{5} - 6x\sqrt{5} + \sqrt{5}\}$. So all that is left is to take φ_1 of each basis vector:

$$\varphi_1(e_1) = 0, \varphi_1(e_2) = \sqrt{12}, \varphi_1(e_3) = 12\sqrt{5} - 6\sqrt{5} = 6\sqrt{5}$$

Therefore, the p that we want is

$$\begin{aligned} p &= 12x - 6 + 180x^2 - 180x + 30 \\ &= 180x^2 - 168x + 24 \end{aligned}$$

□

Exercise 2: Let V be the vector space $\mathbb{R}^3$ equipped with the standard inner product. Prove or disprove: any linear operator $P \in \mathcal{L}(V)$ such that $P^2 = P$ is an orthogonal projector.

Proof. Take the linear operator T such that

$$T : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$T : \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$T : \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

we can take a vector say $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and observe that under the transformation, we get $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$, and if we do it again, we get $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ but notice that it is not orthogonal because if we take the original vector minus the projected vector, we get $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, and then the dot product with the projection:

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = 3$$

□

Exercise 3: Suppose that $e_1, \ldots, e_n$ is a list of vectors in V of length 1 (i.e., $\|e_k\| = 1$ for all $k = 1, \ldots, n$) such that

$$\|v\| = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2 \qquad \text{for all } v \in V$$

Prove that $e_1, \ldots, e_n$ is an orthonormal basis of V.

Proof. Let v be one of the vectors say e_1. Then

$$\begin{aligned} \|e_1\| &= |\langle e_1, e_1 \rangle|^2 + \cdots + |\langle e_1, e_n \rangle|^2 \\ 1 &= 1 + |\langle e_1, e_2 \rangle|^2 + \cdots + |\langle e_1, e_n \rangle|^2 \end{aligned}$$

since the perfect squares are greater than or equal to o, they must be o. So e_1 is orthogonal to the other vectors. We can repeat this for all the e_i. So $e_1, \ldots, e_n$ are orthonormal. Now let v

be arbitrary in V. If we compute the projection

$$P_{\{e_1,\ldots,e_n\}}(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$$

we find that the norm of this projection squared is

$$\left\|P_{\{e_1,\ldots,e_n\}}(v)\right\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 = \|v\|$$

but by the fact that

$$\left\|P_{\{e_1,\ldots,e_n\}}(v)\right\|^2 + \left\|v - P_{\{e_1,\ldots,e_n\}}(v)\right\|^2 = \|v\|^2$$

since the projection and v minus the projection are orthogonal, we must have that

$$\begin{aligned}\left\|v - P_{\{e_1,\ldots,e_n\}}(v)\right\|^2 &= 0\\ \left\langle v - P_{\{e_1,\ldots,e_n\}}(v), v - P_{\{e_1,\ldots,e_n\}}(v)\right\rangle &= 0\end{aligned}$$

telling us that v is equal to its projection. So v is in the span of $e_1, \ldots, e_n$. Since the list is linearly independent, spanning, and orthonormal, it is an orthonormal basis. □

Exercise 4: Let $V = C[-\pi, \pi]$ with the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}\ \mathrm{d}t\,.$$

Determine the orthogonal projection of the function $h(x) = e^{2ix}$ on the subspace

1. $\mathrm{Span}\{1, \cos x, \sin x\}$
2. $\mathrm{Span}\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$
3. $\mathrm{Span}\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\}$ for $n > 2$

Exercise 5: Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(-1) = 0$, $p\prime(-1) = 0$, and the following is minimized:

$$\int_0^1 |1 - 5x - p(x)|^2\ \mathrm{d}x\,.$$

Proof. Suppose that $p(x) = ax^3 + bx^2 + cx + d$. We will minimize the integral first. Let

$$\int_0^1 f(x)g(x)\ \mathrm{d}x$$

be the inner product. If we consider the stuff inside, the integral is finding the norm of this, so we minimize the norm by taking it as a projection. Observe that

$$p(x) + 5x - 1$$

is $p(x) - (5x + 1)$ if we take $5x + 1$ to be the projection, we can find the projection by solving the system of equations with the formula:

$$\langle P_{5x+1}(p(x)), u \in \mathrm{Span}\{5x + 1\}\rangle = \langle p(x), u \in \mathrm{Span}\{5x + 1\}\rangle$$

so here are the calculations:

$$\begin{aligned}\int_0^1 p(x)(-5x + 1)\ \mathrm{d}x &= \langle -5x + 1, -5x + 1\rangle\\ 52 &= -9a - 11b - 8c - 29d\end{aligned}$$

Now we solve for the other variables using the imposed conditions:

$$\begin{aligned}p(-1) &= -a + b - c + d = 0\\ p\prime(-1) &= 3a - 2b + c = 0\end{aligned}$$

In the system of equations, a is free, so we can take $a = 1$ to get:

$$p(x) = x^3 + \left(\frac{73}{56} - \frac{13}{14}\right) x^2 + \left(-\frac{11}{28} - \frac{13}{7}\right) x + \left(-\frac{39}{56} - \frac{13}{14}\right)$$

$\square$