Math110Hw6

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Homework 6

Exercise 1: Give an example of $V, W, S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$ such that

(a) $TS = I$ but S is not invertible.

An example would be for $V = \mathbb{R}$, $W = \mathcal{P}_{\leq 1}(\mathbb{R})$. We would have for $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(W, V)$:

$$\begin{aligned} S(v) &= vx \\ T(p(x)) &= \frac{d}{dx}p(x) \end{aligned}$$

where x is a variable in the polynomial. We observe that linear map results in an element of the codomain, and that it is linear:

$$\begin{aligned} S(\lambda_1 v_1 + \lambda_2 v_2) &= x\lambda_1 v_1 + x\lambda_2 v_2 \\ &= \lambda_1 S(v_1) + \lambda_2 S(v_2) \\ T(\lambda_1 p(x) + \lambda_2 q(x)) &= \lambda_1 p'(x) + \lambda_2 q'(x) \\ &= \lambda_1 T(p(x)) + \lambda_2 T(q(x)) \end{aligned}$$

We also have that

$$TS(v) = T(vx) = v$$

So TS is the identity. But we know that a linear map is invertible if and only if it is bijective. It cannot be the case that S is bijective because the dimension of the domain is not equal to that of the range. V has dimension 1 while W has dimension 2 based on their bases:

$$\begin{aligned} V &= \operatorname{Span}(1) \\ W &= \operatorname{Span}(1, x) \end{aligned}$$

(b) $TS = I$ but T is not invertible.

We can use the same examples from the previous section with $V = \mathbb{R}$, $W = \mathcal{P}_{\leq 1}(\mathbb{R})$. We would have for $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(W, V)$.

$$\begin{aligned} S(v) &= vx \\ T(p(x)) &= \frac{d}{dx} p(x) \end{aligned}$$

and

$$TS(v) = T(vx) = v$$

but for the same reason as the last, T is invertible if and only if it is bijective. That cannot be the case since the dimensions of V and W are different:

$$\begin{aligned} V &= \text{Span}\,(1) \\ W &= \text{Span}\,(1, x) \end{aligned}$$

Exercise 2: Let V be a vector space over $\mathbb{F}$. Give a constructive proof that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic, i.e., construct an explicit isomorphism between these spaces.

Proof. It was proved in class that $\mathcal{L}(\mathbb{F}, V) \cong \mathbb{F}^{\dim V \times \dim \mathbb{F}}$. Consider the mapping from domain V with basis vector $v_1, v_2, \ldots, v_n$

$$\begin{aligned} T &: V \to \mathbb{F}^{n \times 1} \\ T(v \in V) &= T(a_1 v_1 + \ldots + a_n v_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \end{aligned}$$

We will show that it is linear

$$\begin{aligned} T\left(\lambda_1 \sum_{i=1}^{n} a_i v_i + \lambda_2 \sum_{i=1}^{n} b_i v_i\right) &= \begin{bmatrix} \lambda_1 a_1 + \lambda_2 a_1 \\ \lambda_1 a_2 + \lambda_2 a_2 \\ \vdots \\ \lambda_1 a_n + \lambda_2 a_n \end{bmatrix} \\ &= (\lambda_1 + \lambda_2) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\ &= \lambda_1 T\left(\sum_{i=1}^{n} a_i v_i\right) + \lambda_2 T\left(\sum_{i=1}^{n} b_i v_i\right) \end{aligned}$$

Since the dimension of the domain and codomain are equal, we just need to prove surjectivity to prove a bijection. If we have an element of $\mathbb{F}^{\dim V \times \dim \mathbb{F}}$,

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

we have an element in the domain that gets mapped to it, which is

$$a_1 v_1 + \ldots + a_n v_n$$

So the map is surjective and since the dimension of the domain and codomain are equal, it is injective and therefore bijective as well. Since the mapping is linear, this is an isomorphism

$$V \cong \mathbb{F}^{\dim V \times \dim \mathbb{F}}$$

But since we know there is an isomorphism given by $T : V \mapsto \mathbb{F}^{n \times 1}$

$$\mathcal{L}(\mathbb{F}, V) \cong \mathbb{F}^{\dim V \times \dim \mathbb{F}}$$

By the matrix linear map defined in class,

$$M : \mathcal{L}(\mathbb{F}, V) \to \mathbb{F}^{\dim V \times \dim \mathbb{F}}$$

we can take the composition

$$M^{-1} \circ T : V \to \mathcal{L}(\mathbb{F}, V)$$

which is also linear as the composition of linear maps is linear, and the composition of bijective maps is bijective. This is an isomorphism. $\square$

Exercise 3: Which of the following maps on $\mathcal{P}(\mathbb{R})$ are linear functional?

(a) $p(x) \mapsto \int_{-1}^{x} p(t)\, dt$

This is not a linear functional, since linear functionals are linear mappings from the vector space to $\mathbb{R}$. Since $\int_{-1}^{x} p(t)dt$ is another polynomial and a real number, the mapping is not a linear functional.

(b) $p(x) \mapsto \int_{0}^{2} t^2 p(2t^3 - 1)\, dt$

We can check if it is linear:

$$\begin{aligned}
\phi(\lambda_1 p(x) + \lambda_2 q(x)) &= \int_0^2 t^2(\lambda_1 p(2t^3 - 1) + \lambda_2 q(2t^3 - 1))dt \\
&= \int_0^2 t^2 \lambda_1 p(2t^3 - 1) + t^2 \lambda_2 q(2t^3 - 1)dt \\
&= \lambda_1 \int_0^2 p(2t^3 - 1)dt + \lambda_2 \int_0^2 q(2t^3 - 1)dt \\
&= \lambda_1 \phi(p(x)) + \lambda_2 \phi(q(x))
\end{aligned}$$

Observe that the domain is $\mathcal{P}(\mathbb{R})$ and the range is as subset of $\mathbb{R}$. So the mapping is in $\mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ and is therefore a linear functional.

(c) $p(x) \mapsto p''(x)$

Check if it is linear:

$$\begin{aligned} \phi(\lambda_1 p(x) + \lambda_2 q(x)) &= \lambda_2 p''(\pi) + \lambda_2 q''(\pi) \\ &= \lambda_1 \phi(p(x)) + \lambda_2 \phi(q(x)) \end{aligned}$$

Also, the result is the evaluation of a polynomial, which therefore lives in $\mathbb{R}$, so the mapping lives in $\mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ and is therefore a linear functional.

(d) $p(x) \mapsto 2p(1)x^2$

Take $p(x) = x$ which is a polynomial. This gets mapped to $2(1)x^2 \in \mathcal{P}(\mathbb{R})$. Therefore, the mapping is not in $\mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ and is therefore not a linear functional.

Exercise 4: Let $V = \mathcal{P}_2(\mathbb{R})$ and suppose $\phi_j(p) = p(j-1)$, $j = 1, 2, 3$. Prove that (ϕ_1, ϕ_2, ϕ_3) is a basis for $\mathcal{P}_2(\mathbb{R})'$ and find a basis (p_1, p_2, p_3) of $\mathcal{P}_2(\mathbb{R})$ whose dual is (ϕ_1, ϕ_2, ϕ_3).

Proof. We have that

$$\phi_1(p) = p(0),\ \phi_2(p) = p(1),\ \phi_3(p) = p(3)$$

Since the dimension of a space is the same as its dual, we either need to show linear independence or spanning. We will go with linear independence. To prove linear independence, find the dual basis to ϕ_1, ϕ_2, ϕ_3:

$$\left\{ \frac{1}{2}(x-1)(x-2), -x(x-2), \frac{1}{2}x(x-1) \right\}$$

We write the polynomials as a linear combination and solve the system for $\lambda_1, \lambda_2, \lambda_3$:

$$-\lambda_1 x^2 + \lambda_1 2 - \frac{1}{2}\lambda_2 x^2 + \lambda_2 \frac{3}{2}x + \lambda_2 + \frac{1}{2}\lambda_3 x^2 - \frac{1}{2}\lambda_3 x = 0$$

So since the terms of the polynomial must be 0 for the polynomial to be 0, we have

$$\begin{aligned} -\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 &= 0 \\ \frac{3}{2}\lambda_2 - \frac{1}{2}\lambda_3 &= 0 \\ 2\lambda_1 + \lambda_2 &= 0 \end{aligned}$$

which gives $\lambda_1, \lambda_2, \lambda_3 = 0$. Since

$$\begin{aligned}
\phi_1(\frac{1}{2}(x-1)(x-2)) &= 1 \\
\phi_1(-x(x-2)) &= 0 \\
\phi_1(\frac{1}{2}x(x-1)) &= 0
\end{aligned}$$

$$\begin{aligned}
\phi_2(\frac{1}{2}(x-1)(x-2)) &= 0 \\
\phi_2(-x(x-2)) &= 1 \\
\phi_2(\frac{1}{2}x(x-1)) &= 0
\end{aligned}$$

$$\begin{aligned}
\phi_3(\frac{1}{2}(x-1)(x-2)) &= 0 \\
\phi_3(-x(x-2)) &= 0 \\
\phi_3(\frac{1}{2}x(x-1)) &= 1
\end{aligned}$$

ϕ_1, ϕ_2, ϕ_3 is a dual basis. □

Exercise 5: Let V be a finite dimensional vector space and let U be its proper subspace (i.e. $U \neq V$). Prove that there exists $\phi \in V'$ such that $\phi(u) = 0$ for all $u \in U$ but $\phi \neq 0$.

Proof. Since $U \neq V$ and the dimension of a subspace does not exceed the dimension of the vector space, $\dim(U) < \dim(V)$. By a theorem proved in class,

$$\dim(U^\circ) = \dim(V) - \dim(U)$$

So the dimension of U° is non-zero, and therefore, contains a linear functional other than the 0 one that annihilates everthing in U. □