Math185Hw7

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March 14, 2024

Exercise 1: Determine $\oint_C \frac{e^{3z}}{z-\pi i}\, dz$ if C is:

(a) the circle $|z-1| = 4$;

Answer. We have that

$$|i\pi - 1| = \sqrt{\pi^2+1} < 4$$

So by Cauchy's theorem, since e^{3z} is holomorphic in the region, the integral evaluates to $2ie^{3i\pi} = -2i$.

(b) the ellipse $|z-2| + |z+2| = 6$.

Answer. Plugging in $z = \pi i$, we get:

$$|\pi i - 2| + |\pi i + 2| = 2\sqrt{\pi^2+4} > 6$$

Since the function is holomorphic within the region, the integral evaluates to 0.

Exercise 2: Determine $\oint_C \frac{\cos \pi z}{z^2-1}$ dz around the rectangles with vertices at:

(a) $2 \pm i$, $-2 \pm i$;

Answer. There are two singular points $z = 1, z = -1$. By cauchy, the contribution of $z = 1$ and $z = -1$ are:

$$
\begin{aligned}
2\pi i \cdot \frac{\cos \pi \cdot z}{z+1} &= 2\pi i \cdot \frac{\cos \pi}{2} \\
&= 2\pi i \cdot \frac{-1}{2} \\
&= -\pi i \\
2\pi i \cdot \frac{\cos \pi \cdot z}{z-1} &= 2\pi i \cdot \frac{\cos -\pi}{-2} \\
&= 2\pi i \cdot \frac{1}{2} \\
&= \pi i
\end{aligned}
$$

So the integral is the sum of their contributions which is 0.

(b) $\pm i, 2 \pm i$

Answer. From the previous part, we can just take the contribution at the point $z = 1$ because minus that point, the function is holomorphic in the area bounded by C. So the integral is $-\pi i$.

Exercise 3: Evaluate $\int_0^{\pi} \frac{\cos\theta}{5+4\cos\theta}\, d\theta$.

Answer. Using the fact that $z = \cos\theta + i\sin\theta$, we have

$$\begin{aligned} z &= e^{i\theta} \\ dz &= ie^{i\theta} d\theta \\ \frac{i}{z} dz &= d\theta \end{aligned}$$

Now the bottom $\cos\theta = (z + \frac{1}{z})/2$. So the integral is the real part of for the half circle on the upper plane:

$$\int_C \frac{i}{5 + 2z + \dfrac{2}{z}}\, dz = i \int_C \frac{1}{2z^2 + 5z + 2}\, dz$$

Solving for the roots, we get:

$$z = \frac{-5 \pm \sqrt{25 - 16}}{4} = \frac{-5 \pm 3}{4} = -2, \frac{-1}{2}$$

We have that $-1/2$ is a singularity in the region. So the integral over the half circle in the top two quadrants is

$$i \int_C \frac{\frac{1}{z+2}}{(z + \frac{1}{2})}\, dz$$

By cauchy, we get:

$$-2\pi(\frac{2}{3}) = \frac{4}{3}\pi$$

Exercise 4: Apply Cauchy's formula to a first quadrant quarter-disk and take the radius $R \to \infty$ to show, for a fixed real number $a > 0$,

$$\int_0^\infty \frac{1}{x^4 + a^4}\, dx = \frac{\pi}{2\sqrt{2}a^3}, \text{ and } \int_0^\infty \frac{x}{x^4 + a^4}\, dx = \frac{\pi}{4a^2}$$

Answer. We have

$$\int_C \frac{1}{z^4 + a^4}\, dz = \int_{Q_R} \frac{1}{z^4 + a^4}\, dz + \int_0^R \frac{1}{z^4 + a^4}\, dz + \int_R^0 \frac{1}{z^4 + a^4}\, dz$$

The quarter circle integral goes to 0 because the integral is at most

$$\left|\frac{1}{z^4 + a^4} \cdot R \cdot \frac{\pi}{2}\right| = \frac{1}{R^4 + a^4} \cdot R\pi/2$$

which goes to 0 as R goes to ∞.

The integral

$$\int_R^0 \frac{1}{z^4 + a^4}\, dz$$

would be parametrized by

$$z = -ix$$
$$dz = -idx$$

So we have

$$i\int_R^0 \frac{1}{x^4 + a^4}\, dz$$

And:

$$\int_0^R \frac{1}{x^4 + a^4}\, dx - i\int_0^R \frac{1}{x^4 + a^4}\, dx = (1 - i)\int_0^R \frac{1}{x^4 + a^4}\, dx = \int_C \frac{1}{z^4 + a^4}\, dz$$

The roots of $z^4 + a^4 = 0$ are

$$ae^{i\pi/4}, ae^{3i\pi/4}, ae^{5\pi i/4}, ae^{7i\pi/4}$$

or

$$a(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}), a(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}), a(\frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}), a(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})$$

There is a singularity at $\sqrt[4]{a}e^{i\pi/4}$. So by cauchy, the integral evaluates to:

$$\int_C \frac{1}{z^4 + a^4}\, dz = 2\pi i(\frac{1}{(ae^{i\pi/4} - ae^{3i\pi/4})(ae^{i\pi/4} - ae^{5i\pi/4})(ae^{i\pi/4} - ae^{7i\pi/4})})$$

or

$$\frac{2\pi i}{a^3(\sqrt{2})(\sqrt{2} + i\sqrt{2})(i\sqrt{2})} = \frac{\pi}{a^3(\sqrt{2} + i\sqrt{2})} = (1 - i)\int_0^\infty \frac{1}{x^4 + a^4}\, dx$$

Now:

$$\frac{\pi}{a^3(\sqrt{2} + i\sqrt{2})} \cdot \frac{1}{1 - i} = \frac{\pi}{a^3(\sqrt{2} - i\sqrt{2} + i\sqrt{2} + \sqrt{2})} = \frac{\pi}{a^3 2\sqrt{2}}$$

So that is the first integral.

For the second, we have:

$$\int_C \frac{z}{z^4 + a^4}\, dz = \int_0^\infty \frac{z}{z^4 + a^4}\, dz + \int_{Q_R} \frac{z}{z^4 + a^4}\, dz + \int_R^0 \frac{z}{z^4 + a^4}\, dz$$

We see that the quarter circle arc integral vanishes as $R \to \infty$ because it is bounded by

$$\left|\frac{z}{z^4 + a^4}\right| \cdot R \cdot \frac{\pi}{4} = \frac{R^2}{R^4 + a^4}$$

which goes to 0. For the parametrization, we can use

$$z = -ix, \; dz = -idx$$

again. So we get:

$$\int_R^0 \frac{z}{z^4 + a^4} \; dz = -\int_R^0 \frac{x}{x^4 + a^4} \; dx$$

So in total,

$$\int_C \frac{z}{z^4 + a^4} \; dz = 2\int_0^R \frac{x}{x^4 + a^4} \; dx$$

The right hand side's denominator has the same roots:

$$ae^{i\pi/4}, ae^{3i\pi/4}, ae^{5\pi i/4}, ae^{7i\pi/4}$$

So:

$$\int_C \frac{z}{z^4 + a^4} \; dz = 2\pi i(\frac{ae^{i\pi/4}}{(ae^{i\pi/4} - ae^{3i\pi/4})(ae^{i\pi/4} - ae^{5i\pi/4})(ae^{i\pi/4} - ae^{7i\pi/4})})$$

which is

$$\frac{\pi e^{i\pi/4}}{a^2(\sqrt{2} + i\sqrt{2})} = \frac{\pi e^{i\pi/4}}{a^2 2e^{i\pi/4}} = \frac{\pi}{2a^2}$$

In total:

$$\int_C \frac{z}{z^4 + a^4} \; dz = \frac{\pi}{2z^2} = 2\int_0^\infty \frac{z}{z^4 + a^4} \; dz$$

and

$$\int_0^\infty \frac{z}{z^4 + a^4} \; dz = \frac{\pi}{4a^2}$$

Exercise 5: Apply Cauchy's formula to an upper half-disk and the function $\exp(iz)/(z^4+4)$ and take the radius $R \to \infty$ to find the value of

$$\int_0^\infty \frac{\cos x}{x^4+4}\, dx$$

Answer. Using the upper half disk, we can integrate

$$\int_C \frac{e^{iz}}{z^4+4}\, dz = \int_{H_R} \frac{e^{iz}}{z^4+4}\, dz + \int_{-\infty}^{\infty} \frac{e^{iz}}{z^4+4}\, dz$$

, Now the half circle arc contribution is 0 because the integral is bounded by

$$\left|\frac{e^{iz}}{z^4+4}\right| \cdot R\pi = \frac{e^{-y}}{R^4} \cdot \pi R$$

As $R \to \infty$, we are on the upper circle, so $e^{-y} \to 0$ and the whole quantity goes to 0. This means

$$\int_C \frac{e^{iz}}{z^4+4}\, dz = \int_{-\infty}^{\infty} \frac{e^{iz}}{z^4+4}\, dz$$

Now we can apply cauchy to the left side. The singularities are at $z = \sqrt{2}e^{i\pi/4}$ and $z = \sqrt{2}e^{3i\pi/4}$. So cauchy says that we get a total contribution of

$$\begin{aligned}\int_C \frac{e^{iz}}{z^4+4}\, dz = 2\pi i (&\frac{e^{i(1+i)}}{(\sqrt{2}e^{i\pi/4} - \sqrt{2}e^{3i\pi/4})(\sqrt{2}e^{i\pi/4} - \sqrt{2}e^{5i\pi/4})(\sqrt{2}e^{i\pi/4} - \sqrt{2}e^{7i\pi/4})})\\ &+ 2\pi i (\frac{e^{i(-1+i)}}{(\sqrt{2}e^{3i\pi/4} - \sqrt{2}e^{i\pi/4})(\sqrt{2}e^{3i\pi/4} - \sqrt{2}e^{5i\pi/4})(\sqrt{2}e^{3i\pi/4} - \sqrt{2}e^{7i\pi/4})})\end{aligned}$$

This is

$$\frac{\pi e^{i-1}}{2\sqrt{2}(\sqrt{2}+i\sqrt{2})} + 2\pi i \frac{e^{-i-1}}{2\sqrt{2}(-\sqrt{2})(i\sqrt{2})(-\sqrt{2}+i\sqrt{2})} = \frac{\pi e^{i-1}}{2\sqrt{2}(\sqrt{2}+i\sqrt{2})} + \frac{\pi e^{-i-1}}{2\sqrt{2}(\sqrt{2}-i\sqrt{2})}$$

We can factor out a $\pi e^{-1}/\sqrt{2}$:

$$\frac{\pi e^{-1}}{\sqrt{2}} \cdot \left(\frac{e^i}{\sqrt{2}+i\sqrt{2}} + \frac{e^{-i}}{\sqrt{2}-i\sqrt{2}}\right)$$

Normalize the denominator of each:

$$\frac{\pi e^{-1}}{2\sqrt{2}} \cdot \left(\frac{(\sqrt{2}-i\sqrt{2})e^i}{4} + \frac{(\sqrt{2}+i\sqrt{2})e^{-i}}{4}\right)$$

or

$$\frac{\pi}{2e} \cdot \left(\frac{e^i + e^{-i} - ie^i + ie^{-i}}{4}\right)$$

which is

$$\frac{\pi}{4e}(\cos 1 + \sin 1)$$

Exercise 6: Prove that $\int_0^\pi \log \sin \theta \ d\theta = -\pi \log 2$.

Exercise 7: By integrating the function $\exp(-z^2)$ around the circular sector of radius R, centered at 0, and bounded by the rays $\arg z = 0$ and $\arg z = \pi/8$, and letting $R \to \infty$, show that

$$\int_0^\infty e^{-t^2} \cos t^2 \, dt = \frac{1}{4}\sqrt{\pi}\sqrt{1+\sqrt{2}}$$

Explain why the contribution of the circular arc vanishes as $R \to \infty$. *Note*: $\int_0^\infty e^{-t^2} \, dt = \sqrt{\pi}/2$. Try to recall the 2-variable calculus trick that computes that one.

Answer. We see that if we use $z = e^{i\pi/8}x$, the integral of e^{-z^2} along the line of angle $\frac{\pi}{8}$ gives:

$$\int_0^{\pi/2} \int_0^\infty re^{i\pi/4} e^{e^{i\pi/4}r^2} \, dr \, d\theta$$

which will be the same as when we evaluate the integral

$$\int_0^\infty e^{-x^2} \, dx$$

Since the function e^{-z^2} is holomorphic, the integral along the circular sector goes to 0, so the integral of the circular arc vanishes for this one. Then we can conclude that the integral for the circular arc vanishes for $e^{-z^2} \cos z^2$ along the eighth circle arc because the magnitude of the integrand is smaller:

$$\left|e^{-z^2}\right| \geqslant \left|e^{-z^2} \cos z^2\right|$$

To integrate, we have

$$\begin{aligned} e^{-z^2} \cos z^2 &= \frac{e^{-z^2+iz^2} + e^{-z^2+iz^2}}{2} \\ &= \frac{e^{z^2(-1+i)} + e^{z^2(-1-i)}}{2} \end{aligned}$$

Now using the fact

$$\begin{aligned} z &= -e^{i\pi/8}x \\ dz &= -e^{i\pi/8}dx \end{aligned}$$

we have

$$\int_C e^{-z^2} \cos z^2 \, dz = \int_0^\infty e^{-x^2} \cos x^2 \, dx - \int_L e^{-z^2} \cos z^2 \, dz$$

where L represents the line going to infinity at angle $e^{i\pi/8}$. Since the function is holomorphic, we just have

$$\int_0^\infty e^{-z^2} \cos z^2 \, dz = \int_L e^{-z^2} \cos z^2 \, dz$$

Now we can expand:

$$\int_L e^{-z^2}\cos z^2\ dz = -\int_0^\infty e^{e^{-i\pi/4}x^2(-1+i)}e^{i\pi/8} + e^{-e^{i\pi/4}x^2(-1-i)}e^{i\pi/8}\ dx$$

$$\begin{aligned}
(\int_0^\infty e^{-e^{i\pi/4}x^2(-1+i)}e^{i\pi/8}\ dx)^2 &= \int_0^\infty\int_0^\infty e^{-e^{i\pi/4}r^2(-1+i)}e^{i\pi/4}\ dx\ dy \\
&= \int_0^{\pi/2}\int_0^\infty e^{-e^{i\pi/4}r^2(-1+i)}e^{i\pi/4}r\ dr\ d\theta \\
&= \int_0^{\pi/2}\left(\frac{e^{-\sqrt{2}r^2}}{(-1+i)\cdot 2}\right)\Bigg|_0^\infty d\theta \\
&= \left(\frac{1}{(-1+i)\cdot 2}\right)\Bigg|_0^{\pi/2} \\
&= \frac{\pi}{4(-1+i)}
\end{aligned}$$

Doing the same for the other integral yields

$$\frac{\pi}{4(-1-i)}$$

So the integral will evaluate to

$$\frac{1}{2}\left(\sqrt{\frac{-\pi}{4(-1+i)}}+\sqrt{\frac{-\pi}{4(-1-i)}}\right)$$

Let $C = \sqrt{\frac{-\pi}{4(-1+i)}}+\sqrt{\frac{-\pi}{4(-1-i)}}$. Then

$$\begin{aligned}
C^2 &= \frac{-\pi}{4(-1+i)}+\frac{-\pi}{4(-1-i)}+2\sqrt{\frac{\pi^2}{16(-1+i)(-1-i)}} \\
&= \frac{-\pi(-1-i)}{8}+\frac{-\pi(-1+i)}{8}+\frac{\pi}{2\sqrt{2}} \\
&= \frac{2\pi+2\sqrt{2}\pi}{8} \\
&= \frac{(1+\sqrt{2})\pi}{4}
\end{aligned}$$

Then $C = \frac{\sqrt{\pi}\sqrt{1+\sqrt{2}}}{2}$ and therefore, we take $\frac{1}{2}$ of that to get:

$$\frac{\sqrt{\pi}\sqrt{1+\sqrt{2}}}{4}$$

Exercise 8: Apply Cauchy's formula to the function $ze^{iz}/(z^4+4)$ on a large ($R \to \infty$) upper half-disk to show that

$$\int_0^\infty \frac{x \sin x}{x^4+4}\, dx = \frac{\pi}{4e} \sin 1$$

Answer. So we have:

$$\int_C \frac{ze^{iz}}{z^4+4}\, dz = \int_{H_R} \frac{ze^{iz}}{z^4+4}\, dz + \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^4+4}\, dz$$

We know that the half arc integral goes to 0 by Jordan's Theorem. Then by cauchy, we have that the lhs is equal to

$$\int_C \frac{ze^{iz}}{z^4+4}\, dz = 2\pi i\left(\frac{\sqrt{2}e^{i\pi/4}e^{i(i+1)}}{(\sqrt{2}e^{i\pi/4}-\sqrt{2}e^{3i\pi/4})(\sqrt{2}e^{i\pi/4}-\sqrt{2}e^{5i\pi/4})(\sqrt{2}e^{i\pi/4}-\sqrt{2}e^{7i\pi/4})}\right)$$
$$+ 2\pi i\left(\frac{\sqrt{2}e^{3i\pi/4}e^{i(i-1)}}{(\sqrt{2}e^{3i\pi/4}-\sqrt{2}e^{i\pi/4})(\sqrt{2}e^{3i\pi/4}-\sqrt{2}e^{5i\pi/4})(\sqrt{2}e^{3i\pi/4}-\sqrt{2}e^{7i\pi/4})}\right)$$

which is

$$\frac{\pi e^{i-1}e^{\pi i/4}}{2(\sqrt{2}+i\sqrt{2})} + \frac{\pi e^{-i-1}e^{3\pi i/4}}{2(\sqrt{2}-i\sqrt{2})} = \frac{\pi e^{-1}}{2}\left(\frac{e^{i}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}{\sqrt{2}+i\sqrt{2}} + \frac{e^{-i}\left(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}{\sqrt{2}-i\sqrt{2}}\right)$$

which simplifies to

$$\frac{\pi e^{-1}}{16}\left(2e^{i}-2e^{-i}\right)$$

which is

$$\frac{\pi}{4e}\sin 1$$

Exercise 9: Let z_0 be a fixed complex number and let the complex function f be defined and continuous in the disk $|z - z_0| < R$, and let C_r be the circle of radius $r < R$ centered at z_0. Show that

$$\lim_{r \to 0} \oint_{C_r} \frac{f(z)}{z - z_0} \, dz = 2\pi i \cdot f(z_0)$$

Note: We do not assume that f is holomorphic, or even real-differentiable.