Math128aHw7

Trustin Nguyen

October 23, 2024

Exercise Set 4.3

Exercise 2: Approximate the following integrals using the Trapezoidal rule.

(b) $\int_{-0.5}^{0} x \ln x + 1 \, dx$

Answer. I got $\int_{-0.5}^{0} x \ln x + 1 \, dx = 0.061301828313232886724648551535211$. Here is my code:

```
function y = Trapezoid(f, a, b, n)
y = 0;
h = (b - a) / n;
fX = f(a);
for i = 1:n
    newFX = f(a + i * h);
    y = y + h * (newFX + fX) / 2;
    fX = newFX;
end
end

syms s;

f(s) = s * log(s + 1);
res = vpa(double(Trapezoid(f, -0.5, 0, 2)))
```

Exercise 4: Find a bound for the error in Exercise 2 using the error formula and compare this to the actual error.

(b) $\int_{-0.5}^{0} x \ln x + 1 \, dx$

Answer. The error formula is given by

$$\left| \frac{h^3}{12} f''(\mathcal{E}) \right|$$

We have that $h = 0.25$.

$$f'(x) = \ln x + 1 + \frac{x}{x+1}$$

Then the second derivative:

$$f''(x) = \frac{1}{x+1} + \frac{x+1-x}{(x+1)^2} = \frac{1}{x+1} + \frac{1}{(x+1)^2} = (x+1)^{-1} + (x+1)^{-2}$$

Now we need to determine the maximum value of this function on the interval $[-0.5, 0]$. Clearly, it achieves its max when the denominator is small, so the max is at $x = -0.5$. Plugging this in, we get:

$$f''(-0.5) = \frac{1}{0.5} + \frac{1}{0.5^2} = 2 + 4 = 6$$

Putting this all together, the error is at most:

$$\left|\frac{0.25^3}{12} \cdot 6\right| = \frac{0.015625}{2} = 0.0078125$$

The actual value is 0.05256980729. Then the error is

$$0.06130182831323288672464855153521 1 - 0.05256980729 = 0.0087320210232329$$

which is indeed less than the error bound we calculated.

Exercise 6: Repeat Exercise 2 using Simpson's rule.

(b) $\int_{-0.5}^{0} x \ln x + 1 \; dx$

Answer. I got $\int_{-0.5}^{0} x \ln x + 1 \; dx = 0.052854638560979466666012172026967$. Here is my code:

```
function y = Simpson(f, a, b)
x1 = (a + b) / 2;
h = x1 - a;
y = (h / 3) * (f(a) + 4 * f(x1) + f(b));
end

syms s;

f(s) = s * log(s + 1);
res = vpa(double(Simpson(f, -0.5, 0)))
```

Exercise 8: Repeat Exercise 4 using Simpson's rule and the results of Exercise 6.

Answer. The error formula given is

$$\frac{h^5}{90} f^{(4)}(\mathcal{E})$$

Then calculating the fourth derivative:

$$f^{(2)}(x) = (x + 1)^{-1} + (x + 1)^{-2}$$

and therefore:

$$f^{(3)}(x) = -(x + 1)^{-2} - 2(x + 1)^{-3}$$

so

$$f^{(4)}(x) = 2(x + 1)^{-3} + 6(x + 1)^{-4}$$

Then we see that the max on the interval is still at $x = -0.5$. Evaluating:

$$f^{(4)}(-0.5) = 2(.5)^{-3} + 6(0.5)^{-4} = 2 \cdot 8 + 6 \cdot 16 = 16 + 96 = 112$$

Now $h = 0.25$. So putting this all together:

$$\frac{0.25^4}{90} \cdot 112 = \frac{112}{256 \cdot 90} = \frac{112}{23040} = 0.0048611111111111$$

The actual value is 0.05256980729. The value we got is 0.052854638560979466666012172026967. Then the actual error is

$$0.052854638560979466666012172026967 - 0.05256980729 = 0.00028483127097947$$

which we see is bounded by the error bound.

Exercise 19: Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)\ dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

Answer. If $f(x)$ is linear:

$$f(x) = ax + b$$

and

$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} ax + b\ dx = \left(\frac{a}{2}x^2 + bx\right)\Big|_{-1}^{1}$$

Evaluating:

$$\frac{a}{2} + b - (\frac{a}{2} - b) = 2b$$

On the LHS:

$$-\frac{a\sqrt{3}}{3} + b + (\frac{a\sqrt{3}}{3} + b) = 2b$$

So it works for linear functions.

Now consider $f(x) = x^2$. We have

$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} x^2\ dx = \left(\frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{2}{3}$$

It is clear that the LHS also evaluates to 2/3. So it works for all quadratics, going off of the linearity of integration.

Now for $f(x) = x^3$, it is an odd function so it integrates to 0.

Let $f(x) = x^i$ for i of even degree. Then

$$\int_{-1}^{1} f(x)\ dx = \left(\frac{x^{i+1}}{i+1}\right)\Big|_{-1}^{1} = \frac{2}{i+1}$$

And

$$f\left(\frac{-\sqrt{3}}{3}\right) = f\left(\frac{\sqrt{3}}{3}\right)$$

Since i is even, $i = 2j$. So

$$f\left(\frac{\sqrt{3}}{3}\right) = \frac{3^j}{3^i} = \frac{1}{3^j}$$

So the LHS evaluates to

$$\frac{2}{3^j}$$

We see that this equation first fails when $i = 4$. So therefore, the degree of precision is 3.

Exercise 22: The quadrature formula $\int_0^2 f(x)\,dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than or equal to two. Determine c_0, c_1, and c_2

Answer. For constant polynomial:

$$\int_0^2 p_0\,dx = p_0 x\Big|_0^2 = 2p_0$$

Then it follows that:

$$c_0 + c_1 + c_2 = 2$$

For linear polynomials:

$$\int_0^2 x\,dx = \left(\frac{x^2}{2}\right)\Big|_0^2 = 2$$

So then:

$$c_1 + 2c_2 = 2$$

Finally, for quadratic polynomials:

$$\int_0^2 3x^2\,dx = \left(x^3\right)\Big|_0^2 = 8$$

and therefore,

$$3c_1 + 12c_2 = 8$$

From the last two equations, $6c_2 = 2$ or $c_2 = \frac{1}{3}$. So $c_1 = \frac{4}{3}$. So $c_0 = \frac{1}{3}$.

Then the approximation is:

$$\frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2)$$

Exercise Set 4.4

Exercise 2: Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

(b) $\int_{-0.5}^{0.5} x \ln x + 1 \, dx$, $n = 6$.

Answer. I got 0.093630139742868559449284759921284. Here is my code:

```
function y = CompositeTrapezoidal(f, a, b, n)
h = (b - a) / n;
xI0 = f(a) + f(b);
xI1 = 0;
for i = 1:n-1
    x = a + i * h;
    xI1 = xI1 + f(x);
end
y = h * (xI0 + 2 * xI1) / 2;
end

syms s

f(s) = s * log(s + 1);
res = vpa(double(CompositeTrapezoidal(f, -0.5, 0.5, 6)))
```

Exercise 4: Use the Composite Simpson's rule to approximate the integrals in Exercise 2.

(b) $\int_{-0.5}^{0.5} x \ln x + 1 \, dx$, $n = 6$

Answer. I got 0.088092210960428865562654721088862. Here is my code:

```
function y = CompositeSimpson(f, a, b, n)
h = (b - a) / n;
xI0 = f(a) + f(b);
xI1 = 0;
xI2 = 0;
for i=1:n-1
    x = a + i*h;
    if mod(i, 2) == 1
        xI1 = xI1 + f(x);
    else
        xI2 = xI2 + f(x);
    end
end
y = h * (xI0 + 2 * xI2 + 4 * xI1) / 3;
end

syms s

f(s) = s * log(s + 1);
res = vpa(double(CompositeSimpson(f, -0.5, 0.5, 6)))
```

Exercise 11: Determine the values of n and h required to approximate

$$\int_0^2 e^{2x} \sin 3x \, dx$$

to within 10^{-4}. Use

(b) Composite Simpson's rule.

Answer. The error term is given by

$$-\frac{(b-a)}{180}h^4 f^{(4)}(\mu)$$

for $\mu \in (a,b)$. The fourth derivative is

$$-e^{2x}(119\sin(3x) + 120\cos(3x))$$

also, $b - a = 2$. So we want:

$$-\frac{1}{90}h^4(-e^{2x}(119\sin 3x + 120\cos 3x)) < 10^{-4}$$

or

$$h^4 < \frac{9 \cdot 10^{-3}}{e^{2x}(119\sin 3x + 120\cos 3x)}$$

We see that

$$e^{2x}(119\sin 3x + 120\cos 3x) < 120e^{2x}\sqrt{2}$$

which is at most $120e^4\sqrt{2}$ on the interval. So this means that:

$$h < \sqrt[4]{\frac{1}{9000 \cdot 120e^4\sqrt{2}}} < \sqrt[4]{\frac{1}{160000e^4}}$$

We can get $n = (b-a)/h$

Exercise 16: In multivariable calculus and statistics courses, it is shown that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)(x/\sigma)^2}\, dx = 1$$

for any positive σ. The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)(x/\sigma)^2}$$

is the *normal density function* with *mean* $\mu = 0$ and *standard deviation* σ. The probability that a randomly chosen value described by this distribution lies in $[a,b]$ is given by $\int_a^b f(x)\, dx$. Approximate to within 10^{-5} the probability that a randomly chosen value described by this distribution will lie in

(a) $[-\sigma, \sigma]$

Answer. I got 0.682698220175433

(b) $[-2\sigma, 2\sigma]$

Answer. I got 0.954463324707435

(c) $[-3\sigma, 3\sigma]$

Answer. I got 0.997195309084966. Here is my code:

```
function y = CompositeSimpson(f, a, b, n)
h = (b - a) / n;
xI0 = f(a) + f(b);
xI1 = 0;
xI2 = 0;
```

```
for i=1:n-1
    x = a + i*h;
    if mod(i, 2) == 1
        xI1 = xI1 + f(x);
    else
        xI2 = xI2 + f(x);
    end
end
y = h * (xI0 + 2 * xI2 + 4 * xI1) / 3;
end

syms s

sigma = 1;
mean = 0;
f(s) = exp(-.5 * (s/sigma)^2) / (sigma * (2 * pi)^(.5));

res = [0;0;0];
n = 10;
for i = 1:3
    res(i) = vpa(double(CompositeSimpson(f, -i*sigma, i*sigma, n)))
end
```

Exercise Set 4.5

Exercise 2: Use Romberg integration to compute $R_{3,3}$ for the following integrals.

(b) $\int_{-0.75}^{0.75} x \ln x + 1 \, dx$

Answer. I got 0.32795861011519389371926536114188. Here is the code:

```
function R = Romberg(f, a, b, n)
h = b - a;
R = zeros(2, n);
R(1, 1) = h/2 * (f(a) + f(b));
for i = 2:n
    s = 0;
    for k = 1:2^(i - 2)
        s = s + f(a + (k - 0.5) * h);
    end
    R(2, 1) = (R(1, 1) + h * s) / 2;

    for j = 2:i
        numerator = R(2, j - 1) - R(1, j - 1);
        denominator = 4^(j - 1) - 1;
        R(2, j) = R(2, j - 1) + numerator / denominator;
    end

    h = h / 2;
    R(1, :) = R(2, :);
end
end

syms s

f(s) = s * log(s + 1);
R = vpa(Romberg(f, -0.75, 0.75, 3))
```

Exercise 9: Romberg integration is used to approximate

$$\int_2^3 f(x) \, dx \, .$$

If $f(2) = 0.51342$, $f(3) = 0.36788$, $R_{31} = 0.43687$, and $R_{33} = 0.43662$, find $f(2.5)$.

Answer. Using the extrapolation formula:

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1} - 1}(R_{k,j-1} - R_{k-1,j-1})$$

we can solve for $R_{2,1}$. We have the equations:

$$R_{2,2} = \frac{4R_{2,1} - R_{1,1}}{3}$$
$$R_{3,2} = \frac{4R_{3,1} - R_{2,1}}{3}$$
$$R_{3,3} = \frac{16R_{3,2} - R_{2,2}}{15}$$

If follows that we want to plug it all in and simplify:

$$R_{3,3} = \frac{16\left(\frac{4R_{3,1} - R_{2,1}}{3}\right) - \left(\frac{4R_{2,1} - R_{1,1}}{3}\right)}{15}$$

Which is

$$R_{3,3} = \frac{64R_{3,1} - 16R_{2,1} - 4R_{2,1} + R_{1,1}}{45} = \frac{64R_{3,1} - 20R_{2,1} + R_{1,1}}{45}$$

Now we solve for $R_{2,1}$:

$$45R_{3,3} = 64R_{3,1} - 20R_{2,1} + R_{1,1}$$

or

$$R_{2,1} = -\frac{45R_{3,3} - 64R_{3,1} - R_{1,1}}{20}$$

We already have $R_{3,1} = 0.43687$ and $R_{3,3} = 0.43662$. Now $R_{1,1}$ is the trapezoid with one node:

$$R_{1,1} = \frac{f(2) + f(3)}{2} = \frac{0.51342 + 0.36788}{2} = 0.44065$$

Plugging this all in:

$$R_{2,1} = -\frac{45(0.43687) - 64(0.43662) - 0.44065}{20} = 0.436259$$

Recall that:

$$R_{2,1} = \frac{b-a}{4}(f(2) + f(3) + 2f(2.5)) = \frac{1}{4}(0.51342 + 0.36788 + f(2.5)) = 0.436259$$

So

$$1.745036 = 0.51342 + 0.36788 + 2f(2.5)$$

and

$$2f(2.5) = 1.745036 - 0.51342 - 0.36788 = 0.863736$$

therefore,

$$f(2.5) = 0.863736/2 = 0.431868$$

Exercise 14: In Exercise 24 of Section 1.1, a Maclaurin series was integrated to approximate $\mathrm{erf}(1)$, where $\mathrm{erf}(x)$ is the normal distribution error function defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\,.$$

Approximate $\mathrm{erf}(1)$ to within 10^{-7}.

Answer. I got: 0.842700792949714894142232424201 21. Here is my code:

```
function R = Romberg(f, a, b, n)
h = b - a;
R = zeros(2, n);
R(1, 1) = h/2 * (f(a) + f(b));
for i = 2:n
    s = 0;
    for k = 1:2^(i - 2)
        s = s + f(a + (k - 0.5) * h);
    end
    R(2, 1) = (R(1, 1) + h * s) / 2;
```

```
        for j = 2:i
            numerator = R(2, j - 1) - R(1, j - 1);
            denominator = 4^(j - 1) - 1;
            R(2, j) = R(2, j - 1) + numerator / denominator;
        end

        h = h / 2;
        R(1, :) = R(2, :);
    end
end

syms s;

f(s) = 2 * exp(-(s)^2) / (sqrt(pi));

res = vpa(double(Romberg(f, 0, 1, 10)))
```

Exercise 18: Show that, for any k,

$$\sum_{i=1}^{2^{k-1}-1} f\left(a + \frac{i}{2}h_{k-1}\right) = \sum_{i=1}^{2^{k-2}} f\left(a + \left(i - \frac{1}{2}\right)h_{k-1}\right) + \sum_{i=1}^{2^{k-2}-1} f(a + ih_{k-1}).$$

Answer. Notice that we can break the LHS into a sum over i even and i odd:

$$\sum_{i=1}^{2^{k-1}-1} f\left(a + \frac{i}{2}h_{k-1}\right) = \sum_{i=\text{even}}^{2^{k-1}-2} f\left(a + \frac{i}{2}h_{k-1}\right) + \sum_{i=\text{odd}}^{2^{k-1}-1} f\left(a + \frac{i}{2}h_{k-1}\right)$$

For the even one, we can take $i = 2j$:

$$\sum_{i=\text{even}}^{2^{k-1}-2} f\left(a + \frac{i}{2}h_{k-1}\right) = \sum_{j=1}^{2^{k-2}-1} f(a + jh_{k-1})$$

Now for $i = 2j - 1$:

$$\sum_{i=\text{odd}}^{2^{k-1}-1} = f\left(a + \frac{i}{2}h_{k-1}\right) = \sum_{j=1}^{2^{k-2}} f\left(a + \frac{2j-1}{2}h_{k-1}\right) = \sum_{j=1}^{2^{k-2}} f\left(a + \left(j - \frac{1}{2}\right)h_{k-1}\right)$$

so we are done.

Exercise 19: Use the result of Exercise 18 to show that for all k,

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f\left(a + \left(i - \frac{1}{2}\right)h_{k-1}\right)\right]$$

Answer. Recall that:

$$R_{k,1} = \frac{h_{k-1}}{2}\left(f(a) + f(b) + \sum_{i=1}^{2^{k-1}-1} f\left(a + \frac{i}{2}h_{k-1}\right)\right)$$

By the previous problem, this is:

$$R_{k,1} = \frac{h_{k-1}}{2}\left(f(a) + f(b) + \sum_{i=1}^{2^{k-2}-1} f(a + ih_{k-1}) + \sum_{i=1}^{2k-2} f\left(a + \left(i - \frac{1}{2}\right)h_{k-1}\right)\right)$$

Pulling the $h_{k-1}/2$ inside and simplifying, this is:

$$\frac{1}{2}\left(R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f\left(a + \left(i - \frac{1}{2}\right)h_{k-1}\right)\right)$$

so we are done.