Math110Hw2

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Homework 2

Exercise 1: Prove or disprove (i.e, provide a counterexample): if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Proof. False. Consider $W = \mathbb{R}^3$ which is a vector space and

$$U_1 = \{(x, 0, 0) : x \in \mathbb{R}\}$$
$$U_2 = \{(x, y, 0) : x, y \in \mathbb{R}\}$$

Clearly, $U_1 + W = \mathbb{R}^3 = U_2 + W$ and U_1 is a subspace:

1. $0 \in U_1$
2. If $(x, 0, 0), (y, 0, 0) \in U_1$, then

$$(x, 0, 0) + (y, 0, 0) = (x + y, 0, 0) \in U_1$$

3. If $(x, 0, 0) \in U_1, \lambda \in \mathbb{R}$,

$$\lambda(x, 0, 0) = (\lambda x, 0, 0) \in U_1$$

The n same properties hold for U_2 also. Then $U_1 + W = U_2 + W$ but $U_1 \neq U_2$. $\square$

Exercise 2: Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let

$$U = \{(x, y, x + 2y, -x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

When $x = 0$ or $y = 0$ we get

$$(0, y, 2y, -y, 0) \qquad\qquad (x, 0, x, -x, 2x)$$

To ensure that no element can be written as two different sums, elements in W_1, W_2, W_3 cannot have that form. Observe that

$$\begin{aligned} W_1 &= \{(0,0,z,0,0) : z \in \mathbb{F}\} \\ W_2 &= \{(0,0,0,z,0) : z \in \mathbb{F}\} \\ W_3 &= \{(0,0,0,0,z) : z \in \mathbb{F}\} \end{aligned}$$

have no elements in the same form as that of U so their intersection is the 0 vector only. If we imagine the coverage of U, it is the two dimensional xy plane. The sum of W_1, W_2, W_3 will allow us to reach vectors in $\mathbb{F}^5$ by modifying the three last components.

Exercise 3: Let V be a vector space over $\mathbb{F}$. Suppose that $1+1 \neq 0$ in $\mathbb{F}$ and the list v_1, v_2, v_3, v_4 is linearly independent in V. Show that the list $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is also linearly independent in V.

Proof. Since v_1, v_2, v_3, v_4 are independent,

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$$

implies that

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$$

So we look to see what happens with a linear combination of the other vectors:

$$\begin{aligned} \lambda_5(v_1 - v_2) + \lambda_6(v_2 + v_1) + \lambda_7(v_3 - v_2) + \lambda_8(v_4 - v_1) &= 0 \\ (\lambda_5 - \lambda_8 + \lambda_6)v_1 + (\lambda_6 - \lambda_5 - \lambda_7)v_2 + \lambda_7 v_3 + \lambda_8 v_4 &= 0 \end{aligned}$$

We have the coefficients of $v_1, \ldots, v_4$ so we set them to 0:

$$\begin{aligned} \lambda_5 - \lambda_8 + \lambda_6 &= 0 & \lambda_5 + \lambda_6 &= 0 & \lambda_5 &= 0 \\ \lambda_6 - \lambda_5 - \lambda_7) &= 0 & \lambda_6 - \lambda_5 &= 0 & \lambda_6 &= 0 \\ \lambda_7 &= 0 & \lambda_7 &= 0 & \lambda_7 &= 0 \\ \lambda_8 &= 0 & \lambda_8 &= 0 & \lambda_8 &= 0 \end{aligned}$$

And we're done. □

Exercise 4: Does the statement of Problem 3 still hold if we replaced 'linearly independent' by 'a basis'?

The statement should still hold since v_1, v_2, v_3, v_4 being a basis of V implies that V is four dimensional. Since $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ are all linearly independent and there are four of them, they should span the same space as v_1, v_2, v_3, v_4. Therefore, these vectors are also a basis.

Exercise 5: Prove that the space $\mathbb{C}^{\mathbb{C}}$ is infinite-dimensional.

Proof. By definition, the space is defined as

$$\mathbb{C}^{\mathbb{C}} = \{f : \mathbb{C} \mapsto \mathbb{C}\}$$

We can make the observation that any field $\mathbb{F}^S$ for some set S will have dimension $|S|$. In this question, the dimension would be $|\mathbb{C}|$ which is a set that is infinitely uncountable. The idea is that if

$$\mathbb{F}^S = \{f : S \mapsto \mathbb{F}\}$$

Then we define function for $1 \leq i \leq |S|$,

$$f_{s_i}(x) = \begin{cases} 1 & \text{if } x = s_i \\ 0 & \text{if otherwise} \end{cases}$$

Observe that now, each $f(x) \in \mathbb{F}^S$ can be written as a linear combination of the functions

$$\lambda_1 f_{s_1} + \lambda_2 f_{s_2} + \dots \lambda_{|S|} f_{s_{|S|}} = f(x)$$

where

$$f(s_i) = \lambda_i$$

which concludes the proof. □

Exercise 6: Determine, with explanation, the dimension of

(a) $\mathbb{C}$ as a vector space over $\mathbb{C}$;

The dimension would be 1 by the previous problem where our function $f : \mathbb{C} \mapsto 1$ can be multiplied by a singular valued scalar in $\mathbb{C}$ to get any other function in $\mathbb{C}$.

(b) $\mathbb{C}$ as a vector space over $\mathbb{R}$;

Unlike the previous problem, the scalar must be from $\mathbb{R}$. Therefore, we need two functions to span all of $\mathbb{C}$ which are $f : \mathbb{C} \mapsto 1$ and $g : \mathbb{C} \mapsto i$. Taking a linear combination of these functions gives us another function with the form $h : \mathbb{C} \mapsto a + bi$ for some a, b.

(c) $\mathbb{C}^2$ as a vector space over $\mathbb{C}$;

By the previous problem, similar to (a), it is 2.

(d) $\mathbb{C}^5$ as a vector space over $\mathbb{R}$.

We encounter the same issue as in (b), and applying the same solution method gives us a dimension of 10.