Math143Notes

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Contents

Chapter 1

Week 1

1.1 Algebra and Geometry

Algebra equations:

Example 1.1.1: This is a line

$$y = 2x + 1$$

Example 1.1.2: This is a circle:

$$x^2 + y^2 = 1$$

Example 1.1.3: And this:

$$y^2 = x^3$$

How does algebra of equations relate to geometry of solution? Relate systems of polynomial equations to the geometry of solutions.

Foundations of Geometry: Let k be a field, and typically, $k = \mathbb{R}$ or $\mathbb{C}$. A few examples are $\mathbb{C}, \mathbb{R}, \mathbb{F}_p, \mathbb{Q}$

Definition 1.1.1 **Affine Space**

Affine space over k is $\mathbb{A}^n = k^n = \{(a_1, a_2, \ldots, a_n) : a_i \in k\}$

Definition 1.1.2 **Polynomials**

A polynomial in $X_1, \ldots, X_n$ over k is a finite sum:

$$f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where the coefficient lies in $\mathbb{A}$. We define the degree to be the max of the sum of the degrees:

$$\deg(f) = \max(\{\alpha_1 + \ldots + \alpha_n : c_\alpha \neq 0\})$$

Definition 1.1.3

Polynomial space

$k[X_1, \ldots, X_n]$ is the ring of polynomials in $X_1, \ldots, X_n$.

Given $f \in k[X_1, \ldots, X_n]$, we get a function $\mathbb{A}^n \to k$ by:

$$(a_1, \ldots, a_n) = p \mapsto f(a_1, \ldots, a_n) = f(p)$$

Definition 1.1.4

Hypersurfaces

Write $V(f) = \{p \in \mathbb{A}^n : f(p) = 0\}$. This is called a hypersurface. ~~The degree of V)(f) is deg(f)~~. For example, the hypersurface of a circle of radius 1 would be $V(x^2 + y^2 - 1)$

Example 1.1.4: $V(x^2+y^2-z^2-1)$: Start in 3 dimensional space but 2d set of solutions.

Example 1.1.5: $x^3 + y^3 + z^2 + 1 - (x + y + z + 1)^3$

Definition 1.1.5

Affine Algebraic Set

Given $S \subseteq k[X_1, \ldots, X_n]$ define

$$V(S) = \{p \in \mathbb{A}^n : f(p) = 0 \, \forall f \in S\} = \bigcap_{f \in S} V(f)$$

Example 1.1.6: $f_1 = y - x^2, f_2 = y - 2$. Intersection: $V(f_1, f_2) = \{(\sqrt{2}, 2), (-\sqrt{2}, 2)\}$

Warning: An intersection could be empty over $\mathbb{R}$, but non-empty over $\mathbb{C}$.

Example 1.1.7: $f_1 = y - x^2, f_2 = y + 1$ has solutions in $\mathbb{C}$ but not $\mathbb{R}$

Definition 1.1.6

Algebraically Closed

A field k is algebraically closed if every polynomial in $k[X]$ has a root in k. Equivalently, every $f \in k[X]$ factors into linear factors $f = (x - r_1) \cdots (x - r_d)$.

Example 1.1.8: $\mathbb{C}$ is algebraically closed: Fundamental Theorem of Algebra.

Example 1.1.9: $\mathbb{R}$ is not algebraically closed because $x^2 + 1$ has no real roots.

Example 1.1.10: Is $\emptyset$ an algebraic set? In $\mathbb{R}$, the nonzero polynomial has solution set $\emptyset$

Example 1.1.11: $\mathbb{A}^n = V(0)$

Example 1.1.12: A non algebraic set of $\mathbb{A}$: $\mathbb{R}_+ \subseteq \mathbb{A}$. This is because $\mathbb{V}(f)$ is a finite set. So every algebraic set of A is finite or all of $\mathbb{A}$ or $\emptyset$.

Example 1.1.13: Every finite subset in $\mathbb{A}$ is algebraic because you can put them as $(x - r)$ factors in a polynomial

Intersections and Unions:

Intersection of algebraic sets is an algebraic set:

$$\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} S_i)$$

What about unions?

$$V(f) \cup V(g) = V(fg)$$

$$V(x, y) \cup V(z) = (V(x) \cap V(y)) \cup V(z) = (V(x) \cup V(z)) \cap (V(y) \cup V(z))$$

This s $V(xz, yz)$ We have:

$$\bigcup_{i \in I} V(S_i) \text{ is algebraic if } |I| \text{ is finite}$$

Chapter 2

Week 2

2.1 Ideals

Definition 2.1.1 **Ideal**

Let R be a commutative ring. An ideal $I \subseteq R$ is a subset which is closed under addition and satisfies $r : a \in I$ for all $a \in I, r \in R$.

Let $S \subseteq k[X_1, \ldots, X_n]$ and let I be the ideal generated by S. So I is the set of all finite sums of the form $\sum h_i s_i$ where $h_i \in k[X_1, \ldots, X_n]$ and $s_i \in S$.

Proposition: $V(S) = V(I)$.

Proof. We have that $V(I) \subseteq V(S)$. Suppose that $p \in V(S)$. Then $f(p) = 0$ for all $f \in I$. This means that $V(S) \subseteq V(I)$. □

So this means that every algebraic set is $V(I)$ for some ideal I.

Definition 2.1.2 **Ideal of X**

Given a subset $X \subseteq \mathbb{A}^n$, we have that:

$$I(X) = \{f \in k[X_1, \ldots, X_n] : f(p) = 0 \forall p \in X\}$$

Lemma: $I(X) \subseteq k[X_1, \ldots, X_n]$ is an ideal.

Proof. If $f, g \in I(X)$, then

$$(f + g)(p) = f(p) + g(p) = 0 \forall p \in X$$

which means that $f + g \in I(X)$.

Now if $h \in k[X_1, \ldots, X_n]$, then

$$(hf)(p) = h(p) \cdot f(p) = 0 \forall p \in X$$

which means that $hf \in I(X)$. □

Example 2.1.1: $X = \{(1,2)\} \subseteq \mathbb{A}^2$: $I(X) = (x-1, y-2)$. Notice that $X = V(I(X))$, and that this is always true when X is an algebraic set.

Example 2.1.2: We say that $X = \{(a,0) : a \in \mathbb{Z}\} \subseteq \mathbb{A}^2$ and that $I(X) = (y)$. In other words, $V(I(X)) = \{(x,y) : y = 0\}$ and this is the smallest algebraic set containing X.

Example 2.1.3:

- $I(\emptyset) = k[X_1, \ldots, X_n]$ because this equates to saying that if $x \in \emptyset$, then $f(x) = 0$. So since the first part is false, then the second part is automatically true for all $p \in k[X_1, \ldots, X_n]$.
- $I(\mathbb{A}^n) = (0)$ because the only polynomial that vanishes at all points in $\mathbb{A}^n$ is the 0 polynomial.

So we have this relation between V and I:

$$\{\text{ideals in } k[X_1, \ldots, X_n]\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{algebraic sets in } \mathbb{A}^n\}$$

Basic Properties:

$$\begin{aligned} I \subseteq J &\implies V(I) \supseteq V(J) \\ X \subseteq Y &\implies I(X) \supseteq I(Y) \end{aligned}$$

Lemma: If X is an algebraic set, then $V(I(X)) = X$.

Proof. We have that $X \subseteq V(I(X))$. Since $X = V(S)$ is algebraic, if $p \notin X$, then $\exists f \in S \subseteq I(X)$ such that $f(p) \neq 0$, so $p \notin V(I(X))$ □

Question: If J is an ideal, then is $I(V(J)) = J$?

Example 2.1.4: Not necessarily:

$$\begin{aligned} J &= (x^2) \subseteq k[x] \\ V(J) &= \{0\} \\ I(V(J)) &= (x) \neq J \end{aligned}$$

The Nullstellensatz says that this issue with powers is the only part that goes wrong. This tells us how ideals and algebraic sets are related.

Definition 2.1.3 **Radical**

An ideal I is radical if $f^r \in I \implies f \in I$.

Lemma: $I(X)$ is radical

Proof. If $f^r(p) = 0$, then for any p in X, $f(p) = 0$ for all $p \in X$ which means that $f \in I(X)$. □

Definition 2.1.4 **Radical of an Ideal**

Let $I \subseteq R$ be an ideal. The radical of I is:

$$\sqrt{I} = \{f \in R : f^n \in I \text{for some} n\}$$

Proposition: The radical of I is an ideal.

Example 2.1.5: (x^2) is not radical since $x \notin I$. Observe that taking the radical of an ideal enlarges it: $\sqrt{(x^2)} = (x)$ and could be used to solve our problem with $I(V(J)) = J$ when only $\supseteq$ holds in general.

Theorem 2.1.1 **Nullstellensatz**

If k is algebraically closed and $J \subseteq k[X_1, \ldots, X_n]$ is any ideal, then $I(V(J)) = \sqrt{J}$.

We will prove this later, since more algebra is needed.

2.2 Hilbert Basis Theorem

We will show that it is always possible to define an algebraic set with a finite number of polynomials.

Definition 2.2.1 **Noetherian**

A ring is Noetherian if every ideal is finitely generated.

Example 2.2.1: Fields are Noetherian, since the only ideals are $(0), (1)$

Example 2.2.2: $\mathbb{Z}$ is Noetherian because it is a PID

Theorem 2.2.1 **Noetherian and Polynomial Rings**

$k[X_1, \ldots, X_n]$ is Noetherian.

Proof. We know this because k is a field, so we have the Euclidean Domain on the polynomial ring. Now this means that every ideal can be reduced to a principle ideal. □

The Geometric Interpretation is that every algebraic set is the intersection of a finite number of hypersurfaces. The idea behind this is that $V(S) = V(I)$, where I is some ideal, and this becomes the problem of showing that every ideal is finitely generated.

Theorem 2.2.2 **Hilbert Basis Theorem**

If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.

Proof. Let $I \subseteq R[x]$. We should find a finite set of generators for I. We write each $f \in R[x]$ as $f = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x + a_0$. Call a_d the leading coefficient: $LC(F) = a_d$, and let:

$$J = \{LC(F) : F \in I\} \subseteq R$$

We claim that $J \subseteq R$ is an ideal.

- If $a = LC(f), b = LC(g)$, and wlog $\deg(f) \leqslant \deg(g)$, then we have:

$$a + b = LC(fx^n + g) \qquad n = \deg(g) - \deg(f)$$

- If $a = LC(f)$ and $r \in R$, then $ra = LC(rf)$.

So we know that J is finitely generated:

$$J = \langle LC(F_1), \dots, LC(F_r) \rangle$$

Let N be an integer $> \deg(F_i)$. For each $m \leqslant N$, let:

$$J_m = \{LC(F) : F \in I \text{ and } \deg(F) \leqslant m\}$$

which is also an ideal of R. Since R is Noetherian, J_m is finitely generated, so there are $F_{mj} \in I$ of degree $\leqslant m$ such that $J_m = \langle LC(F_{mj}) \rangle$. Let

$$I' = \langle F_i, F_{m,j} \rangle_{m \leqslant N} \subseteq R[x]$$

Claim: $I = I'$.

- Certainly, $I' \subseteq I$. Suppose that they are not equal and let $G \in I$ be an element of lowest degree that is not in I'
- If $\deg(G) > N$ then there is a $Q_i \in R[x]$ such that $\sum Q_i F_i$ and G have the same leading term and same degree. Since $LC(F_i)$ generates J, there is a $q_i \in R$ such that $LC(G) = \sum q_i LC(F_i)$. Now set $Q_i = q_i x^{\deg(G) - \deg(F_i)}$. Then

$$\begin{aligned} LC(\sum Q_i F_i) &= \sum LC(Q_i F_i) \\ &= \sum LC(Q_i) LC(F_i) \\ &= \sum q_i LC(F_i) \\ &= LC(G) \end{aligned}$$

  So $G - \sum Q_i F_i$ has a lower degree than G. But G was minimal degree among the elements of I not in I', therefore, we get that $G \in I'$.

- If $\deg(G) = m \leqslant N$ then $\exists Q_j \in R[x]$ such that $\sum Q_j F_{m,j}$ and g have the same leading term. Then $G - \sum Q_j F_{mj} \in I' \implies G \in I'$.

$\square$

Corollary: $k[X_1, \dots, X_n]$ is Noetherian. We note that the infinite polynomial ring is not Noetherian however. So we can use this to prove the minuteness of algebraic sets.

Chapter 3

Week 3

3.1 The Nullstellensatz

Recall:

$$\{\text{ideals}\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{algebraic sets}\}$$

These are inclusion reversing. Nullstellensatz: $I(V(J)) = \sqrt{J}$ if k is algebraically closed. We consider the fact that the smallest algebraic set should correspond to the largest ideal. So the weak Nullstellensatz is that we have a bijection between $\emptyset$ and the entire ring $k[X_1, \dots, X_n]$.

Example 3.1.1: If we have $k = \mathbb{R}$, $I = (x^2+1)$, then the vanishing of I is $\emptyset$ but $I \neq \mathbb{R}[x]$. Other examples are (x^2+3) and (x^2+y^2+1).

Theorem 3.1.1 **Weak Nullstellensatz 1**

Assume k is algebraically closed. If $V(I) = \emptyset$, then $I = k[X_1, \dots, X_n]$. Equivalently, if $I \subset k[X_1, \dots, X_n]$, then $V(I) \neq \emptyset$.

Nullstellensatz: If G vanishes on $V(J)$, then $\exists$ an equation:

$$G^N = A_1F_1 + \ldots + A_rF_r \text{ for } A_i \in k[X_1, \dots, X_n]$$

where F_r are generators of (J).

We need to prove that $\sqrt{J} \subseteq I(V(J))$ and $I(V(J)) \subseteq \sqrt{J}$.

Proof. ($\sqrt{J} \subseteq I(V(J))$) For the first inclusion, we have $F \in \sqrt{J}$ means that $F^n \in J$ and that for any $p \in V(J)$, we have $F^n(p) = 0$. Therefore, $F(p) = 0$. So $F \in I(V(P))$.

($I(V(I)) \subseteq \sqrt{I}$) Suppose that $I = \langle F_1, \dots, F_r \rangle$ and $G \in I(V(I))$. Let $J = \langle F_1, \dots, F_r, x_{n+1}G - 1 \rangle \subseteq k[X_1, \dots, X_n, X_{n+1}]$. Claim: $V(J) = \emptyset$. If $p \in \mathbb{A}^{n+1}$, and $F_i(p) = 0$, then $(x_{n+1}G - 1)(p) = x_{n+1}G(p) - 1 = -1 \neq 0$. So there is no point where all the polynomials vanish. If the point is where the first r vanish, the last $x_{n+1}G - 1$ does not. Now the weak

Nullstellensatz says that $J = k[X_1, \dots, X_n]$ which contains the element 1. This means:

$$1 = \sum A_i(X_1, \dots, X_n, X_{n+1})F_i + B(X_1, \dots, X_{n+1})(X_{n+1}G - 1)$$

Let $Y = \frac{1}{x_{n+1}}$. If we sub in $x_{n+1} = \frac{1}{Y}$. We get:

$$1 = \sum A_i(X_1, \dots, X_n, \frac{1}{Y})F_i + B(X_1, \dots, \frac{1}{Y})(\frac{1}{Y}G - 1)$$

we have denominators in Y. There $\exists N > 0$ such that if we multiply the expression by Y^N, we can clear out the denominators.

$$Y^N = \sum C_i(X_1, \dots, X_n, Y)F_i + D(X_1, \dots, X_n, Y)(G - Y)$$

Substitute $Y = G$. We have:

$$G^N = \sum G_i(X_1, \dots, X_n, G)F_i + D(X_1, \dots, X_n)(G - G)$$

Therefore, $G^N \in I$ therefore, $G \in \sqrt{I}$. $\square$

To prove WN1, we fix the bijection one step larger:

$$\begin{tikzcd} \{\text{radical ideals}\} \arrow[r, bend left, "V"] & \{\text{algebraic sets}\} \arrow[l, bend left, "I"] \end{tikzcd}$$

$$k[X_1, \dots, X_n] \xleftrightarrow{\text{WN1}} \emptyset$$

The next smallest algebraic sets are single points:

$$\text{largest proper ideals} \longrightarrow (a_1, \dots, a_n) \in \mathbb{A}^n$$

Definition 3.1.1 **Maximal**

An ideal $I \subseteq R$ is called maximal if $I \neq R$ and if $I \subseteq J \subset R$, then $I = J$.

Example 3.1.2: $(p) \subseteq \mathbb{Z}$, $(x) \subseteq k[x]$. If we take the quotient, we get fields: $\mathbb{Z}/(p) = \mathbb{F}_p$ and $k[X]/(x) = k$

An ideal I is maximal $\iff$ R/I is a field. Last time, we had that I is prime $\iff$ R/I is a domain. Maximal ideals are prime ideals.

Key Example: Say $p = (a_1, \dots, a_n) \in \mathbb{A}^n$. Then $I(p) = (x_1 - a_1, \dots, x_n - a_n)$ is claimed to be maximal. Why? Consider the map:

$$k[X_1, \dots, X_n] \to k$$

by the evaluation map at p. The map is surjective and the kernel is the $I(p)$. This means that $k[x_1, \dots, x_n]/I(p) = k$. Note: $I(p)$ is maximal then $I(p)$ is prime. So p is irreducible.

Now we need to prove that every maximal ideal is the ideal of some point.

Example 3.1.3: What is an ideal in $\mathbb{R}[x]$ that is maximal but not $I(P)$ for any $P \in \mathbb{R}$? We have $(x^2+1) \subseteq \mathbb{R}[x]$ is maximal because $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$. But (x^2+1) is not $I(p)$ for any $p \in \mathbb{R}$.

Theorem 3.1.2 **Weak Nullstellensatz 2**

If k is algebraically closed, then every maximal ideal in $k[X_1, \ldots, X_n]$ is $I(p)$ for some $p \in \mathbb{A}^n$.

Now we will show that WN2 $\implies$ WN1.

Lemma: In a Noetherian ring, every ideal is contained in a maximal ideal.

Proof. Suppose for contradiction that I is not contained in any maximal ideal. Then we some I_1 such that $I \subset I_1 \subset \ldots$. Where none of the I_n are maximal. So we get an infinite chain that is ascending. This is a contradiction. □

Now suppose that $I \subset k[X_1, \ldots, X_n]$. By the lemma, I is contained in some maximal ideal. Assuming WN2, that ideal is the ideal of a point. So $I \subseteq I(p)$. But now by reverse inclusion, $V(I) \supseteq V(I(p)) \ni p$. So $V(I) \neq \emptyset$. So we proved WN1 by contrapositive.

3.2 Nullstellensatz Day 2

Last class, we've shown:

$$\text{WN2} \implies \text{WN1} \implies \text{Nullstellensatz}$$

Weak Nullstellensatz 2: If k is algebraically closed, then every maximal ideal in $k[X_1, \ldots, X_n]$ is $I(p)$ for some $p \in \mathbb{A}^n$. There is a bijection between maximal ideals and points.

Suppose that $\mathfrak{m} \subseteq k[X_1, \ldots, X_n]$ is a maximal ideal. Then we have the quotient:

$$k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_n]/\mathfrak{m} = L$$

for L, a field that contains k. In general, an inclusion $k \subseteq L$ of fields is called a field extension.

Example 3.2.1: If $k = \mathbb{R}$, consider $\mathbb{R} \subseteq \mathbb{R}[X] \to \mathbb{R}[X]/(X^2+1) \cong \mathbb{R} \oplus \mathbb{R}x \cong \mathbb{C}$.

Example 3.2.2: Let $k = \mathbb{Q}$. Consider $\mathbb{Q} \subseteq \mathbb{Q}[X] \to \mathbb{Q}[X]/(x^2-2) \cong \mathbb{Q} \oplus \mathbb{Q}x \cong \mathbb{Q}[\sqrt{2}]$

Example 3.2.3: $\mathbb{R} \subseteq \mathbb{C}$ is a field extension and $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{2}]$ is also a field extension.

Note that the larger field is a vector space over the smaller field.

Definition 3.2.1 **Field of Rational Functions**

The field of rational functions over k is

$$k(x) = \{\frac{f(x)}{g(x)} : f(x), g(x) \in k[x] \wedge g(x) \neq 0\}$$

An example is $\mathbb{R}(x)$ which contains:

$$\frac{x^2 + 1}{3x^2 + 7x + 5}$$

Definition 3.2.2 **Field of Fractions**

If R is an integral domain, then

$$\text{Frac}(R) = \{\frac{a}{b} : a, b \in R \wedge b \neq 0\}/(\frac{a}{b} \sim \frac{c}{d} \text{ when } ad = bc \in R)$$

An example would be how $\mathbb{Q} = \text{Frac}(\mathbb{Z})$ and $k(x) = \text{Frac}(k[x])$. We have:

$$k(X_1, \ldots, X_n) = \text{Frac}(k[X_1, \ldots, X_n])$$

where an element would look like:

$$\frac{x_1^2 + x_2}{3x_1x_3}$$

Definition 3.2.3 **Finite Extensions**

Suppose $k \subseteq L$ is a field extension. We say L is a finite extension of k if L is a finite dimensional vector space over k.

Example 3.2.4: $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ is a field extension with a basis $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$.

More generally, if we have:

$$(a \cdot 1 + b\sqrt{2} + c\sqrt{3} + d\sqrt{6})(a' \cdot 1j + b'\sqrt{2} + c'\sqrt{3} + d'\sqrt{6})$$

is again a linear combination of $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$.

If $k \subseteq k(x)$ finite? Now because with the denominator as 1, we have $1, x, x^2, x^3, \ldots$.

Definition 3.2.4 **Algebraic over k**

An element α of L is called algebraic over k if there exists a polynomial $f \in k[x]$ such that $f(\alpha) = 0$ where $f \neq 0$.

Example 3.2.5: $\sqrt{2}$ is algebraic over $\mathbb{Q}$ since $x^2 - 2 = 0$. π is not algebraic over $\mathbb{Q}$. If we had any field k, and $L = k(x)$, then x is not algebraic over k

Definition 3.2.5 **Algebraic Extension**

If every element of L is algebraic over k, then L is called an algebraic extension of k.

Lemma: If $k \subseteq L$ is a finite extension, then $k \subseteq L$ is an algebraic extension.

Proof. Suppose that L has dimension n as a vector space over k. Now take an $\alpha \in L$, We need to show that α satisfies a polynomial in $k[x]$. Consider $1, \alpha, \alpha^2, \ldots, \alpha^n \in L$. This is a list of $n + 1$ elements. So we have that one of the elements can be written as a linear combination of the others. So we have a polynomial that kills α. □

Counterexample for the converse: If we let $k = \mathbb{F}_p$, then for each n, take $\mathbb{F}_{p^n}$ which is finite but their union is not a finite extension of $\mathbb{F}_p$. You can also take $k = \mathbb{Q}$ but $\mathbb{Q}[\sqrt{2}, \sqrt{3}, \ldots]$ which is not a finite extension.

Is $\mathbb{Q}(\pi)$ a finite extension over $\mathbb{Q}$? No since π is not algebraic, so it is not finite by the converse of what was shown for the first lemma.

Lemma. If $\varphi : k[X_1, \ldots, X_n] \to L$ and $\varphi(x_i) = a_i \in L$ is algebraic over k, then L is a finite extension of k.

Proof. To say that φ is surjective is to say every element in L can be written as a polynomial in the a_i with coefficients in k. By assumption, each a_i is algebraic over k, so there is an n_i where we can write $a_i^{n_i} = \sum_{j \leqslant n_i} c_{ij} a_i^j$

Claim: Every polynomial in $a_1, \ldots, a_n$ is equal to a linear combination of $\{a_1^{e_1} \ldots a_n^{e_n} : e_i < n_i\}$. But this is a finite set. So we are done. □

Theorem 3.2.1 **Weak Nullstellensatz 3**

Let k be a field, and let m be a maximal ideal in $k[x_1, \ldots, x_n]$. Then $k[x_1, \ldots, x_n]/m = L$ is a finite extension of k.

Proof. (WN3 $\implies$ WN2) If k is algebraically closed and L is a finite extension of k, then since L is algebraic over k, we must have $L = k$. Every element of L is a solution of k. But k is algebraically closed, so that element is in k. □

Chapter 4

Week 4

4.1 Nullstellensatz Day 3

Recall the statements of weak nullstellensatz 2 and nullstellensatz 3.

WN2: If k is algebraically closed, then every maximal ideal in $k[x_1, \ldots, x_n]$ is $I(p)$ for some $p \in \mathbb{A}^n$.

WN3: Suppose $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ is a maximal ideal. Then $L = k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite extension of k.

We will see why weak nullstellensatz 3 implies weak nullstellensatz 2.

Proof. Suppose that $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ is a maximal ideal. Consider the quotient $k[x_1, \ldots, x_n]/\mathfrak{m} = L$. By weak nullstellensatz 3, L is a finite extension of k. We know that L is an algebraic extension. We assume that k is algebraically closed, so any element of L is in k. So we see $L = k$. Now let $a_i = \varphi(x_i)$. Then $(x_1 - a_1, \ldots, x_n - a_n)$. This is in the kernel of φ. But this ideal is of a point which is maximal. So it is equal to the kernel of φ which is maximal. So every maximal ideal is the ideal of a point. ☐

Now we will prove Weak Nullstellensatz 3:

Proof. Let $\varphi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/\mathfrak{m} = L$. First suppose that $a_i = \varphi(a_i) \in L$ is algebraic over k. Then L is a finite extension of k by a lemma from last class. Suppose for contradiction that some $a_i = \varphi(x_i)$ is not algebraic over k. Assume that a_1 is not algebraic over k. In this case, we have the following inclusions:

$$\begin{tikzcd}
k \subseteq & k(\alpha_1) \subseteq & & L \\
& \text{field of rational functions} \arrow[u] & & & \frac{f_1(a_1)}{g_1(a_1)} = \frac{f_2(a_1)}{g_2(a_1)} \arrow[ul, "\text{if}"']
\end{tikzcd}$$

Then clearing denominators, we get:

$$f_1(a_1)g_2(a_1) - g_1(a_1)g_2(a_1) = 0$$

So these two rational functions are the same rational function. This shows an injective mapping from the field of rational functions on a_1 to L. If L is a finite extension of $k(a_1)$, then set $k' = k$ and we have $k' \subseteq k'(a) \subseteq L$ where L is finite over $k'(a)$. If L is not a finite extension of $k(a_1)$, then there is some a_i which is not algebraic over $k(a_1)$. Then we get

$k \subseteq k(a_1) \subseteq k(a_1, a_2) \subseteq L$. If L is finite over $k(a_1, a_2)$, set $k' = k(a_1)$. Then we have the situation:

$$k' \subseteq k'(a) \subseteq L$$

where L is a finite extension. Continuing in this way, we can assume we have $k' \subseteq k'(a) \subseteq L$ with L finite extension of $k'(a)$. Since L is finite, we can choose a basis for L as a vector space over $k'(a)$. Call that basis $e_0 = 1, \dots, e_n$. Now let $c_{ijk} \in k'(a)$ be elements such that

$$e_i e_j = \sum c_{ijk} e_k$$

There are finitely many c_{ijk} because we have finitely many $e_i e_j$ multiplication pairings. Let $t \in k'(a)$ be a common denominator for all c_{ijk}s. Now, let $d_{ij} \in k'(a)$ be elements such that

$$a_i = \varphi(x_i) = \sum d_{ij} \cdot e_j$$

We only have a finite number of d_{ij}s that are required to do this. So we can find a common denominator of all d_{ij}. Let $s \in k'[a]$ be this common denominator. Now suppose $F(a_1, \dots, a_n)$ is any polynomial. Then, there exists N, M so that

$$s^N t^M \cdot F(a_1, \dots, a_n) \qquad \text{has no denominators}$$

So any element in the image of φ which is of the form $\alpha = F(a_1, \dots, a_n)$ such that $s^N t^M \alpha$ has no denominators. Let $u \in k'[a]$ be an irreducible element that is not a factor in s, t. Then $\frac{1}{u} \in k'(a) \subseteq L$. We claim that $\frac{1}{u} \notin \Im\varphi$. This is because u is not a factor in s or t, so there is no M, N such that $s^N t^M \cdot \frac{1}{u} \in k'[\alpha]$. This is a contradiction since φ was the surjective quotient map. $\square$

Corollary: If $I \subseteq k[x_1, \dots, x_n]$ is a radical ideal, then $I(V(I)) = I$. We also have the following bijections:

$$\begin{tikzcd} \{\text{radical ideals}\} \arrow[rr, bend left, "V"] & & \{\text{algebraic sets}\} \arrow[ll, bend left, "I"] \end{tikzcd}$$

$$k[x_1, \dots, x_n] \xleftrightarrow{\text{WN1}} \emptyset$$

$$\text{maximal ideals} \longleftrightarrow \text{points}$$

$$\text{radical principal ideals} \longleftrightarrow \text{hypersurfaces}$$

If $I = (f)$, then $V(I) = V(f)$. If $X = V(f)$, then $I(X) = \sqrt{(f)} = (f_1, \dots, f_r)$ for $f = f_1^{e_1} \cdots f_r^{e_r}$. Note that $X = V(f_1) \cup \cdots \cup V(f_r)$ is a decomposition of the algebraic set into irreducible components.

We have that WN3 says that if m is maximal, then $k[x_1, \dots, x_n]/m$ is a finite extension.

Question: If I is an ideal and $k[x_1, \dots, x_n]/I$ is finite extension, is I maximal?

Answer. Counterexample. Take $I = (x^2)$ and have $k[x]/(x^2)$ is a finite k-vector space. But (x^2) is not maximal.

Finite Extension Results

Theorem 4.1.1 Let $I \subseteq k[x_1, \ldots, x_n]$. Assume k is algebraically closed. Then $X = V(I)$ is a finite set if and only if $k[x_1, \ldots, x_n]/I$ is a finite dimensional k-vector space. Moreover, $|V(I)| \leqslant \dim_k(k[x_1, \ldots, x_n])/I$.

Proof. $(\longrightarrow)$ Assume that $k[x_1, \ldots, x_n]/I$ is finite dimensional. Suppose $p_1, \ldots, p_r$ are distinct points in $V(I)$. We want to show that $r \leqslant \dim_k(k[x_1, \ldots, x_n]/I)$.

Claim 1: $\exists F_1, \ldots, F_r$ such that

$$F_i(P_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Claim 2: With F_i as above,

$$F_1, \ldots, F_r \in k[x_1, \ldots, x_n]/I$$

are linearly independent.

We will prove claim 2. Suppose that there is a relation $\sum \lambda_i \overline{F}_i = 0 \in k[x_1, \ldots, x_n]/I$ for $\lambda_i \in k$. This means that

$$\sum \lambda_i F_i \in I$$

Now $\lambda_j = \sum \lambda_i F_i(P_j) = 0$. This means that $F_1, \ldots, F_r$ are 0, so $\overline{F}_i$ are independent. So we are done.

$(\longleftarrow)$ Now assume that $V(I)$ is a finite set. We need to show that the quotient is finite dimensional. Let $P_i = \{a_{i_1}, \ldots, a_{i_n}\}$. For each $j \in \{1, \ldots, n\}$, define $f_j = (s_j - a_{1_j})(x_j - a_{2_j}) \cdots (x_j - a_{i_j})$. So $f_j(p_i) = 0$ for all i, j. So $f_j \in I(V(I)) = \sqrt{I}$. Therefore, there is an $f_j^{n_j} \in I$. Then, in $k[x_1, \ldots, x_n]/I$, $\overline{x}_j^{n_j r}$ is a k-linear combination of lower degree terms. So

$$\{\overline{x}_1^{e_1} \cdots \overline{x}_n^{e_n} : e_j < n_j r\}$$

spans $k[x_1, \ldots, x_n]/I$. □

Example 4.1.1: If $I = (0) \subseteq k[x]$, then $V(I) = \mathbb{A}^1$ which is infinite. We have $k[x]/(0) = k[x]$ which is an infinite dimensional vector space.

Example 4.1.2: If $f \in k[x]$ is a polynomial of degree d, then $\dim_k(k[x]/(f)) = d$. A basis of the quotient is given by the images $1, x, \ldots, x^{d-1}$.

4.2 Polynomial Functions and Polynomial Maps

Let $X \subseteq \mathbb{A}^n$ be an algebraic set and let $\mathcal{F}(x, k)$ be the set of all functions $X \longrightarrow k$. It has the structure of a ring. Suppose that $f, g \in \mathcal{F}(X, k)$. Then

$$(f + g)(p) = f(p) + g(p)$$
$$(fg)(p) = f(p)g(p)$$

The additive identity is the $0(p) = 0$ map. The multiplicative identity is the constant function $1(p) = 1$.

Definition 4.2.1 **Polynomial Function**

A function $f : X \to k$ is a polynomial function is $\exists F \in k[x_1, \dots, x_n]$ such that $f(p) = F(p)$. If $P(a_1, \dots, a_n) \in \mathbb{A}^n$, then $f(p) = F(a_1, \dots, a_n)$.

- Polynomial functions form a subring of $\mathcal{F}(X, k)$

Definition 4.2.2 **Subring of Polynomial Functions**

The subring of polynomial functions is called the coordinate ring of X which is denoted $\Gamma(x) \subseteq \mathcal{F}(X, k)$.

Example 4.2.1: $\Gamma(\mathbb{A}^n) = k[x_1, \dots, x_n]$. We just consider the functions on $\mathcal{F}(\mathbb{A}^n, k)$. All functions from $\mathbb{A}^n$ to k can be expressed as an element in $k[x_1, \dots, x_n]$. Therefore, $\Gamma(\mathbb{A}^n) = k[x_1, \dots, x_n]$.

Example 4.2.2: $X = V(y - x^2)$

There is a natural map $k[x_1, \dots, x_n] \to \Gamma(x)$.

If $F, G \in k[x_1, \dots, x_n]$, then F and G define the same polynomial function on $X \iff F - G \in I(X)$.

Definition 4.2.3 **Coordinate Ring**

The coordinate ring of X is $\Gamma(X) = \frac{k[x_1,\dots,x_n]}{I(x)}$.

If x is irreducible, then $\Gamma(X)$ is an integral domain.

Definition 4.2.4 **Varieties**

A variety is an irreducible algebraic set.

Definition 4.2.5 **Polynomial Maps**

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets. A map $X \to Y$ is called a polynomial map or morphism if $\exists T_1, \dots, T_m \in k[x_1, \dots, x_n]$ such that $\varphi(p) = (T_1(p), \dots, T_m(p))$.

Example 4.2.3: Define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ by

$$t \mapsto (t, t^2)$$

The image is $V(y - x^2) = 0$. Note: A polynomial map from $X \to \mathbb{A}^1$ is the same as a polynomial function. A polynomial map $X \to \mathbb{A}^m$ is determined by m polynomial functions.

HW: A composition of polynomial maps is again a polynomial map:

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

How are polynomial maps between algebraic sets related to coordinate rings. Suppose $\varphi : X \to Y$ is a polynomial map. We can define a map on coordinate rings $\varphi^* : \Gamma(Y) \to \Gamma(X)$ as follows:

$$p \mapsto g(\varphi(p)) \in \Gamma(X)$$

We now have

$$\varphi^*(g) = g \circ \varphi$$
$$(\varphi^* g)(p) = (g \circ \varphi)(p) = g(\varphi(p))$$

φ^* is called the pullback map. φ^* is a polynomial map because composition of polynomial maps are polynomial maps. We also use the fact that polynomial maps to $\mathbb{A}^1$ are polynomial functions and vice versa.

Example 4.2.4: $\varphi : \mathbb{A}^3 \to \mathbb{A}^2$ with

$$(x, y, z) \mapsto (x^2 y, x - z)$$

The pullback map:

$$\varphi^* : \Gamma(\mathbb{A}^2) \to \Gamma(\mathbb{A}^3)$$

Let elements of $\mathbb{A}^2$ be defined as $k[u, v]$ and $k[x, y, z]$ for $\mathbb{A}^3$. We have $\varphi^*(u) = x^2 y$. We also have $v \in \Gamma(\mathbb{A}^2)$ which is the projection of the v-coordinate on k. So $\varphi^*(v) = x - z$.

Example 4.2.5: If $\varphi : X \circlearrowleft \mathbb{A}^n$. Then $\varphi^* : \Gamma(\mathbb{A}^n) = k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/I(x) = \Gamma(x)$.

Example 4.2.6: $i : X \subseteq Y \subseteq \mathbb{A}^n$, then $I(Y) \subseteq I(X)$. Describe the pullback map from $\Gamma(Y) \to \Gamma(X)$: $k[x_1, \ldots, x_n]/I(Y) \to k[x_1, \ldots, x_n]/I(X)$. It is a quotient mapping: $k[x_1, \ldots, x_n]/I(Y)/I(X)/I(Y)$.

Proposition: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets. There is a one-to-one correspondence between {polynomial maps $X \to Y$} and {homomorphisms $\Gamma(Y) \to \Gamma(X)$}. We get this by sending $\varphi \mapsto \varphi^*$.

Proof. Given a map $\alpha : \Gamma(Y) \to \Gamma(X)$, we want to construct a polynomial map $\varphi : X \to Y$ such that $\alpha = \varphi^*$. Suppose that

$$\Gamma(Y) = k[y_1, \ldots, y_m]/I(Y) \text{ and } \Gamma(X) = k[x_1, \ldots, x_n]/I(X)$$

First, construct a map $X \to \mathbb{A}^m$. Then show that the image is contained in Y. Let $\psi_i = \alpha(\overline{y}_i) \in \Gamma(X)$. So $\psi_i : X \to k$ is a polynomial function. Build a map $\psi : X \to \mathbb{A}^m$:

$$p \mapsto (\psi_1(p), \ldots, \psi_m(p))$$

Claim 1: If $p \in X$, then $\psi(p) \in Y$.

Suppose $f \in I(Y)$. Then $f(\psi(p)) = f((\psi_1(p), \ldots, \psi_m(p))) = f(\psi_1, \ldots, \psi_m)(p)$:

$$f(\alpha(\overline{y}_1), \ldots, \alpha(\overline{y}_m))(p)$$

We have $f(\psi(p)) = \alpha(f(\overline{y}_1, \ldots, \overline{y}_m))(p)$. The f part is occurring in $\Gamma(Y)$. Since $f \in I(Y)$, $f = 0$. So f is 0 whenever f is in the ideal of Y. So $\psi(p) \in V(I(Y)) = Y$. So we have $\psi : X \to Y$.

Claim 2: $\alpha = \psi^*$. For $f \in \Gamma(Y)$, we have

$$(\psi^* f)(p) = f(\psi(p))$$

This should describe a function in $\Gamma(X)$. This is

$$\alpha(f(\overline{y}_1, \ldots, \overline{y}_m))(p) = f(\alpha(\overline{y}_1)(p), \ldots, \alpha(\overline{y}_m)(p))$$

Uniqueness: If $\varphi^* = \psi^*$, then $\varphi = \psi$. Suppose for contradiction that $\varphi^* = \psi^*$ but $\varphi \neq \psi$. Then there is some $p \in x$ such that $\varphi(p) \neq \psi(p) \in Y \subseteq \mathbb{A}^m$. Let $\varphi(p) = (a_1, \ldots, a_m)$, $\psi(p) = (b_1, \ldots, b_n)$. So there $\exists j$ such that $a_j \neq b_j$. We have $f = x_j - a_j$ is a polynomial function $\overline{f} \in \Gamma(Y)$ such that

$$\varphi^* \overline{f}(p) = (\varphi(p)) = 0$$
$$\psi^* \overline{f}(p) = \overline{f}(\psi(p)) \neq 0$$

□

4.3 Last Class Continued

Last Class: Let X be an algebraic set.

- **Coordinate Ring**: $\Gamma(X) = \{\text{polynomial functions } X \to k\} = k[x_1, \ldots, x_n]/I(X)$
- **Polynomial maps/Morphisms**: $\varphi : X \to Y$
- **Pullback**: $\varphi^* : \Gamma(Y) \to \Gamma(X)$.

There is a bijection between

$$\{\text{polynomial maps} X \to Y\} f \longleftarrow \{\text{homomorphisms } \Gamma(Y) \to \Gamma(X)\}$$

This is between φ and its pullback.

Example 4.3.1: Suppose $r \leqslant n$. What polynomial map of what algebraic sets corresponds to the inclusion of rings $k[x_1, \ldots, x_r] \circlearrowleft k[x_1, \ldots, x_n]$? We want $k[x_1, \ldots, x_r] = \Gamma(Y)$ and $k[x_1, \ldots, x_n] = \Gamma(X)$. We have $\varphi : X = \mathbb{A}^n \to Y = \mathbb{A}^r$ This is a projection map. Suppose that you do not know the map. We consider $\varphi(p) = (b_1, \ldots, b_r)$. We have $b_i = X_i(\varphi(p)) = (\varphi^* X_i)(p) = X_i(p) = a_i$ where $p = (a_1, \ldots, a_n)$.

Example 4.3.2: Let $X = V(u - v^3) \subseteq \mathbb{A}^2$ Consider $\Gamma(\mathbb{A}^3) \to \Gamma(X)$:

$$k[x, y, z] \to \frac{k[u, v]}{(u - v^3)}$$

with mapping $x \mapsto u, y \mapsto 2u, z \mapsto 3u$. What is the polynomial map corresponding the this pullback?

Definition 4.3.1

A polynomial map $\varphi : X \to Y$ is an isomorphism if $\exists \psi : Y \to X$ a polynomial map such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$.

Lemma: $\varphi : X \to Y$ is an isomorphism if and only if $\varphi^* : \Gamma(Y) \to \Gamma(X)$ is an isomorphism of rings.

Proof. Suppose that φ^* is an isomorphism. Then it has an inverse $(\varphi^*)^{-1}$ The inverse:

$$(\varphi^*)^{-1} : \Gamma(X) \to \Gamma(Y)$$

which corresponds to a mpa $\psi : Y \to X$. Now consider the composition $\psi \circ \varphi$ which we want to show is the identity on X. This is true if and only if $(\psi \circ \varphi)^* = \mathrm{id}_X^* = \mathrm{id}_{\Gamma(X)}$. We have $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ which is indeed the identity. $\square$

Example 4.3.3: $\varphi : \mathbb{A}^1 \to V(y - x^2) \subseteq \mathbb{A}^2$ where $t \mapsto (t, t^2)$. The inverse map is the projection map of $(t, t^2) \to t$. You can also look at the map on the coordinate rings. Consider $\varphi^* : \frac{k[x,y]}{(y-x^2)} \to k[t]$.

Coordinate Changes

Definition 4.3.2

A coordinate change is a type of polynomial map. When $T : \mathbb{A}^n \to \mathbb{A}^n$ given by

$$p \to (T_1(p), \dots, T_m(p))$$

is a bijection and all T_i are degree 1 polynomials, then T is called a coordinate change. In this case, $T = T'' \circ T'$ where T' is a k -linear map and T'' is a translation. This is always an isomorphism because T' has an inverse and so does T''.

Example 4.3.4: $T : \mathbb{A}^2 \to \mathbb{A}^2$ where

$$(x, y) \mapsto (2x + 1, x + y + 2)$$

This is

$$(x, y) \mapsto (2x, x + y) \to (2x + 1, x + y + 2)$$

If $T : \mathbb{A}^n \to \mathbb{A}^n$, is an isomorphism, then for any algebraic set $X \subseteq \mathbb{A}^n$, then $T\big|_X : X \to \Gamma(X)$ is also an isomorphism. The inverse is given by $T^{-1}\big|_{T(X)}$.

Returning to an example form Lecture 1:

$$V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2 \qquad V(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$$

The coordinate change is:

$$(x, y) \mapsto (x, iy)$$

Chapter 5

Week 5

5.1 Algebraic Subset

Definition 5.1.1 **Algebraic Subset**

Let $X \subseteq \mathbb{A}^n$ be an algebraic set. Given $\bar{f} \in \Gamma(X)$, we write $V(\bar{f}) = \{p \in X : \bar{f}(p) = 0\} \subseteq X$. We claim that $V(\bar{f})$ is an algebraic set in X.

Proof. We have $k[x_1, \ldots, x_n] \to \Gamma(X)$ is surjective and suppose $f \in k[x_1, \ldots, x_n]$ is sent to $\bar{f} \in \Gamma(X)$ under the quotient map. Then $V(\bar{f}) = V(f) \cap X$. Every algebraic set $Z \subseteq X$ is of this form because if $Z = V(f_1, \ldots f_r)$ for $f_i \in k[x_1, \ldots, x_n]$, then $Z = V(\bar{f}_1, \ldots, \bar{f}_r)$ □

.

5.2 Images and Preimages

Lemma: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets and suppose $\varphi : X \to Y$ is a polynomial map. If $Z \subseteq Y$ is an algebraic set, then $\varphi^{-1}(Z) \subseteq X$ is an algebraic set.

Proof. $Z = V(f_1, \ldots, f_r)$ for $f_i \in k[y_1, \ldots, y_m]$. Each defines a polynomial function $\bar{f}_i \in \Gamma(Y) = k[y_1, \ldots, y_m/I(Y)]$. □

Example 5.2.1: Suppose $\varphi : \mathbb{A}^2 \to \mathbb{A}^3$ where

$$(u, v) \mapsto (-u, v, u^2 + v^2)$$

where $\varphi : X \to Y$ by $X = V(u - v)$ and $Y = V(x^2 + y^2 - z)$.

Question: If $A \subseteq X$ is an algebraic set and $\varphi : X \to Y$ is a polynomial map, is $\varphi(A) \subseteq Y$ an algebraic set?

Lemma: let $\varphi : X \to Y$ be a morphism. Suppose $Z \subseteq Y$ is an algebraic set. If $\varphi^{-1}(Z)$ is irreducible, then Z is irreducible.

Proof. Suppose for contradiction that $\varphi^{-1}(Z)$ is irreducible by $Z = A \cup B$ with $A, B \subset Z$ algebraic subsets. Then $\varphi^{-1}(Z) = \varphi^{-1}(A) \cup \varphi^{-1}(B)$. Since $\varphi^{-1}(Z)$ is irreducible, $\varphi^{-1}(Z) = \varphi^{-1}(A)$ or $\varphi^{-1}(B)$. Without loss of generality, assume $\varphi^{-1}(Z) = \varphi^{-1}(A)$. But then $Z = \varphi(\varphi^{-1}(Z)) = \varphi(\varphi^{-1}(Z)) = A$. $\square$

Question: IF $\varphi(A)$ is irreducible, is A irreducible?

5.3 Injectivity and Surjectivity

Lemma: Suppose that $\varphi : X \to Y$ is surjective. Then $\varphi^* : \Gamma(Y) \to \Gamma(X)$ is injective.

Proof. Let $f \in \Gamma(Y)$ and suppose that $\varphi^* f = 0$. We want to show that $f = 0$.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend left, "\varphi^* f = 0"] & Y \arrow[r, "f"] & k \end{tikzcd}$$

Suppose that $q \in Y$ is any point. Because φ is surjective, there is a $p \in X$ such that $q = \varphi(p)$. Then $f(q) = f(\varphi(p)) = \varphi^* f(p) = 0$. Then $f(q) = 0$ for all $q \in Y$ so $f = 0$. $\square$

Example 5.3.1: Projection map $\mathbb{A}^n \to \mathbb{A}^r$ corresponds to $k[x_1, \ldots, x_r] \to k[x_1, \ldots, x_n]$

Question: If φ^* is injective, is φ surjective. No.

Definition 5.3.1 **Dominant**

A morphism $\varphi : X \to Y$ is dominant if $I(\varphi(X)) = I(Y)$.

Applying the vanishing:

$$V(I(\varphi(X))) = V(I(Y)) = Y$$

and $V(I(\varphi(X)))$ is the closure of X, the smallest algebraic set that contains X.

The example $V(xy - 1)$ to the projection on $\mathbb{A}$ by y is dominant but not surjective.

Proposition: Let $\varphi : X \to Y$ be a morphism. Then φ^* is injective iff φ is dominant.

Proof. ($\longrightarrow$) Assume that φ^* is injective. Since $\varphi(X) \subseteq Y$, we have $I(\varphi(X)) \supseteq I(Y)$. Let $Y \subseteq \mathbb{A}^m$ and $X \subseteq \mathbb{A}^n$. Let $f \in I(\varphi(X)) \subseteq k[y_1, \ldots, y_m]$. Then $\Gamma(\overline{f})(p)$ for $p \in X$. We get $f(\varphi(p)) = 0$ since $f \in I(\varphi(X))$ so f vanishes on points in $\mathfrak{I}\varphi$. So $\varphi^*(\overline{f}) = 0$ meaning $\overline{f} = 0$. So $f \in I(Y)$.

($\longleftarrow$) Now suppose that φ is dominant. Suppose $\overline{f} \in \Gamma(Y)$ and $\varphi^*(\overline{f}) = 0$. For all $p \in X$, we have $0 = (\varphi^* \overline{f}(p)) = \overline{f}(\varphi(p))$. Suppose $f \in k[y_1, \ldots, y_m]$ has image $\overline{f} \in \Gamma(Y)$. Then $\overline{f}(\varphi(p)) = f(\varphi(p)) \implies f \in I(\varphi(p))$. So $f \in I(Y)$. So $\overline{f} = 0$. $\square$

Chapter 6

Week 6

6.1 More on Morphisms

Last class, we showed that $\varphi : X \to Y$ is dominant iff φ^* is injective.

Proposition: Let $\varphi : X \to Y$ be a morphism of algebraic sets.

$$\varphi^* : \Gamma(Y) \to \Gamma(X)$$

is surjective iff $\varphi(X)$ is an algebraic set and $X \to \varphi(X)$ is an isomorphism.

Proof. ($\leftarrow$) Suppose that φ is an isomorphism onto $\varphi(X) = Y' \subseteq Y$ where Y' is an algebraic set.

$$\begin{tikzcd} X \arrow[d, "\sim"] \arrow[drr, "\varphi"] & & \\ Y' \subseteq & & Y \end{tikzcd} \qquad \begin{tikzcd} \Gamma(X) \arrow[d, "\sim"] & & \\ \Gamma(Y') & & \Gamma(Y) \arrow[ll] \arrow[ull, "\varphi'"'] \end{tikzcd}$$

($\rightarrow$) Suppose $\varphi^* : \Gamma(Y) \to \Gamma(X)$ is surjective. Let $Y' \subseteq Y$ be defined by $Y' = V(\ker \varphi^*)$. Now we claim that $\varphi(X) \subseteq Y'$. Suppose that $\varphi(p) \in \varphi(X)$. If $\overline{f} \in \ker \varphi^*$,

$$\overline{f}(\varphi(p)) = (\varphi^* \overline{f})(p) = 0$$

which because $\overline{f} \in \ker \varphi^*$. So we have that $\varphi(p) \in V(\ker \varphi^*)$.

Claim 2: $X \cong Y'$. Consider the maps:

$$\begin{tikzcd} \Gamma(Y) \arrow[r, "\varphi^*"] \arrow[d] & \Gamma(X) \\ \Gamma(Y') = \Gamma(Y)/\ker \varphi^* \arrow[ur, "\cong"] & \end{tikzcd}$$

Since we have $\Gamma(Y') \cong \Gamma(X)$ means that $Y' \cong X$. □

Example 6.1.1: Consider the inclusion $\mathbb{A}^2 \to \mathbb{A}^3$ which sends $(u, v) \mapsto (u, v, 0)$. The pullback $k[x, y, z] = \Gamma(\mathbb{A}^3) \to \Gamma(\mathbb{A}^2) = k[u, v]$. We have $f(x, y, z) \mapsto \overline{f}(u, v, 0)$. So $k[x, y, z] \to k[x, y, z]/(z) = k[u, v]$. We have that $\ker \varphi^* = (z)$ and the image of φ is

$V(\ker \varphi)$

6.2 Classical Topology

Recall open and closed intervals on $\mathbb{R}$. We have

$$(a, b) = \{r \in \mathbb{R} : a < r < b\}$$

and closed intervals:

$$[a, b] = \{r \in \mathbb{R} : a \leqslant r \leqslant b\}$$

More generally, given a subset $U \subseteq \mathbb{R}^n$, we say that U is open in the classical topology if $\forall x \in U, \exists \varepsilon > 0$ such that

$$B_\varepsilon(x) = \{y : \|y - x\| < \varepsilon\} \subseteq U$$

We will check that open intervals are open with this definition. If $x \in (a, b)$. Then $a < x < b$ so $\exists \varepsilon > 0$ such that $\varepsilon < b - x, x - a$. Then $B_\varepsilon(x) \subseteq (a, b)$. For $\mathbb{R}^1$, $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

Definition 6.2.1 **Closed Sets**

We say $Z \subseteq \mathbb{R}^n$ is closed (in the classical topology) if the complement $Z^c \subseteq \mathbb{R}^n$ is open.

We can check that the closed intervals are closed. We have that $[a, b]^c = (-\infty, a) \cup (b, \infty)$.

Properties: If U_i are open, then

$$\bigcup_{i \in I} U_i \text{ is an open set}$$

and

$$\bigcap_{i \in I}^{n} U_i \text{ is an open set}$$

Proof. Suppose that $x \in \bigcup_{i \in I} U_i$. Then $x \in U_i$ for some i. Because U_i is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_i \subseteq \bigcup_{i \in I} U_i$.

If $x \in \bigcap_{i \in I}^{n} U_i$. Then $x \in U_i \forall i$. So for each i, $\exists \varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subseteq U_i$. Set $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_n)$, then $B_\varepsilon(x) \subseteq U_i \forall i$. So $B_\varepsilon(x) \subseteq \bigcap_{i \in I} U_i$. □

Note that in general, an infinite intersection of open sets need not be open. One example is $\bigcap_{i=1}^{\infty}(1 - \frac{1}{i}, 1 + \frac{1}{i}) = \{1\}$.

These open sets define the classical topology on $\mathbb{R}^n$. We can similarly define open sets in $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 6.2.2 **Topology**

A topology on a space, X is a collection $\mathcal{U}$ of subsets of X that we call "open sets" satisfying

(a) If $U_i = \mathcal{U}$, then $\bigcup_{i \in I} U_i \in \mathcal{U}$

(b) If $U_i \in \mathcal{U}$, then $\bigcap_{i=1}^{n} U_1 \in \mathcal{U}$

(c) $\emptyset, X \in \mathcal{U}$.

Definition 6.2.3

If $\mathcal{U}$ is a topology on X, we call a subset $Z \subseteq X$ closed if $Z^c \in \mathcal{U}$.

A topology can also be defined by its closed sets. If C is the collection of closed sets in a topology, then the collection of open sets in $\mathcal{U} = \{Z^c : Z \in C\}$.

Properties of Closed Sets:

(a) If $C_i \in C$, then $\bigcup_{i=1}^n C_i \in C$. This is because $(\bigcup_{i=1}^n C_i)^c = \bigcap_{i=1}^n C_i^c \in \mathcal{U}$.

(b) If $C_i \in C$, then $\bigcap_{i\in I} C_i \in C$. We have $(\bigcap_{i\in I} C_i)^c = \bigcup_{i\in I} C_i^c \in \mathcal{U}$.

- $\emptyset, X \in C$.

A finite union of algebraic sets is an algebraic set. An arbitrary intersection of algebraic sets is an algebraic set. We also have $\mathbb{A}^n, \emptyset$ are algebraic sets. This says that the set of all algebraic sets satisfies the rules to be the closed sets of a topology.

Definition 6.2.4 **Zariski Topology**

We define the Zariski topology to be the topology on $\mathbb{A}^n$ where the closed set $C = \{\text{algebraic sets in } \mathbb{A}^n\}$. Equivalently, the Zariski topology in $\mathbb{A}^n$ is the topology where the collection of open sets is $\mathcal{U} = \{Z^c : Z \subseteq \mathbb{A}^n \text{ is an algebraic set}\}$.

Definition 6.2.5

We call an algebraic set $Z \subseteq X$ a closed set and a subset $U \subseteq X$ is called an open set if U^c is a closed set

Example 6.2.1: $\mathbb{A}^1 \backslash \{0\}$ is an open set in $\mathbb{A}^1$.

Example 6.2.2: $\mathbb{A}^2 \backslash V(x^2 + y^2 + 1)$ is an open set in $\mathbb{A}^2$.

Example 6.2.3: $\{(x, y) : x \neq 0, y = 0\}$ is not open in $\mathbb{A}^2$ but $\{(x, y) : s \neq 0, y = 0\} \subseteq V(y)$ is an open set.

Definition 6.2.6 **Closure**

Given a subset A of a set X with a topology $\mathcal{U}$, the closure of A is the smallest closed set containing A.

Example 6.2.4: In $\mathbb{R}$ with the Euclidean topology, the closure of the open interval $(-1, 2)$ is the closed interval $[-1, 2]$. But in $\mathbb{R}$ with the Zariski topology, the closure of $(-1, 2)$ is all of $\mathbb{R}$.

6.3 More on Closure and Continuity

Recall: Suppose X is a set with a topology. Given a subset $A \subseteq X$, the closure of A is the smallest closed set containing A. This is often denoted as $\overline{A}$.

$$\overline{A} = \bigcap_{Z \supseteq A, Z \text{ is closed}} Z$$

Lemma: Let X be an algebraic set with the Zariski topology. If $A \subseteq X$ is any subset, the closure of A is $V(I(A))$.

Proof. We have $V(I(A)) \supseteq A$ is a closed set. Now we need to show that it is contained in every closed set that contains A. Suppose that $V(S) \supseteq A$ is another closed set that contains A. So $S \subseteq I(A)$ and $V(S) \supseteq V(I(A))$. □

Warning: Open sets in the Zariski topology are very big.

Hw: If X is an irreducible algebraic set, and $U \subseteq X$ is open, then $\overline{U} = X$.

Lemma: Suppose X is an irreducible algebraic set (aka a variety). If $U_1, U_2 \subseteq X$ are non-empty open subsets, then $U_1 \cap U_2 \neq \emptyset$.

A topology is "Hausdorff" if $\forall x, y \in X$ with $x \neq y$, $\exists U \ni x$ and $Vk \ni y$ open sets such that $U \cap V = \emptyset$. This means that for every point, you can find spheres containing them that do not intersect. The lemma says that the Zariski topology is not Hausdorff.

Proof. Suppose for contradiction that $U_1 \cap U_2 = \emptyset$. Then $(U_1 \cap U_2)^c = X$ and $U_1^c \cup U_2^c$ So X is a union of two closed sets, which are algebraic sets. So X is U_i^c which means $U_i = \emptyset$. □

The Zariski topology is very coarse where there are not many open subsets. Suppose X has two topologies $\mathcal{U}$ and $\mathcal{V}$. We say $\mathcal{U}$ is coarser than $\mathcal{V}$ if $\mathcal{U} \subseteq \mathcal{V}$. We say $\mathcal{V}$ is finer than $\mathcal{U}$. The coarsest topology on X is $\{\emptyset, X\}$. The finest topology on X is {all subsets of X}.

Definition 6.3.1 **Continuity**

Suppose that X and Y are topological spaces. Then we say a map $\varphi : X \to Y$ is continuous if for any open set $U \subseteq Y$, $\varphi^{-1}(U) \subseteq X$ is an open subset in X. Equivalently, if for every closed set $Z \subseteq Y$, the preimage $\varphi^{-1}(Z) \subseteq X$ is a closed subset of X. These are equivalent because $\varphi^{-1}(Z^c) = (\varphi^{-1}(Z))^c$

This definition generalizes continuous functions from $f : \mathbb{R} \to \mathbb{R}$ if we use the classical topology. The definition of continuity:

$$f : \mathbb{R} \to \mathbb{R} \text{ is continuous if } \lim_{x \to y} f(x) = f(y)$$

A function if continuous if $\forall \varepsilon > 0, \exists \delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. If $U \subseteq \mathbb{R}$ is open, why does this mean $f^{-1}(U)$ is open? Suppose $x \in f^{-1}(U)$. We must show that $\exists \delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$. Since U is open and $f(x) \in U$, $\exists \varepsilon$ such that $B_\varepsilon(f(x)) \subseteq U$. So $\exists \delta$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))) \subseteq f^{-1}(U)$.

Now for the other direction. If $f^{-1}(U)$ is open for all U open, why is f continuous by the calculus definition? We have

$$\forall \varepsilon > 0, B_\varepsilon(f(x)) \text{ is open}$$

so $f^{-1}(B_\varepsilon(f(x)))$ is open. Now this means that $x \in f^{-1}(B_\varepsilon(f(x)))$. Sin the ball is open, $\exists \delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$.

We have already shown that morphisms are continuous in the Zariski topology. In fact, the Zariski topology is the coarsest topology such that

- A point $\in \mathbb{A}^1$ is closed
- Polynomial maps are continuous.

6.4 Rational Maps

Let X be a variety (irreducible algebraic set). The $\Gamma(X)$ is an integral domain.

Definition 6.4.1 **Field of Rational Functions**

The field of rational functions on X is

$$k(X) = \mathrm{Frac}(\Gamma(X)) = \left\{ \frac{f}{g} : f, g \in \Gamma(X), g \neq 0 \right\} / \frac{f_1}{g_1} \sim \frac{f_2}{g_2} \iff f_2 g_1 = f_1 g_2$$

Example 6.4.1: $X = \mathbb{A}^1, \Gamma(X) = k[x], k(X) = k(x)$.

$k(\mathbb{A}^n) = \mathrm{Frac}(k[x_1, \dots, x_n]) = k(x_1, \dots, x_n)$.

Example 6.4.2: $X = V(xy - z^2) \subseteq \mathbb{A}^3$. The rational function $\frac{x}{z}$ is the same as $\frac{z}{y}$ because $\frac{x}{z} \sim \frac{z}{y}$ as $z^2 = xy$.

Definition 6.4.2 **Defined at p**

We say that a rational function $f \in k(X)$ is defined at $p \in X$ if $\exists a, b \in \Gamma(X)$ such that $f = \frac{1}{b}$ and $b(p) \neq 0$.

Example 6.4.3: It is not always clear where a rational function is defined. Is $\frac{x}{z}$ defined at $(0, 1, 0)$? Yes because there is a representative where the point is defined. In fact, it is defined at (x, y, z) whenever $z \neq 0$ or $y \neq 0$. The pole set of $\frac{x}{z}$ is $V(z, y)$.

Definition 6.4.3 **Pole**

Let $f \in k(X)$ be a rational function. If f is not defined at $p \in X$, then we say p is a pole of f.

Example 6.4.4: $\mathbb{A}^2 \to \mathbb{A}^1$ as a projection from the origin onto $x = 1$. It sends $(x, y) \mapsto \frac{y}{x}$. This is not defined at $V(x)$.

Proposition: The pole set of a rational function $f \in k(X)$ is an algebraic subset of X.

Proof. Let $J_f = \{g \in \Gamma(X) : fg \in \Gamma(X)\}$ you can verify that it is an ideal. The claim is that $V(J_f)$ = pole set of f. A point p is not a pole of f iff $\exists a, b \in \Gamma(X)$ such that $f = \frac{a}{b}$ where $b(p) \neq 0$. That means that there is $b \in J_f$ so that $b(p) \neq 0$. This means that $p \notin V(J_f)$. $\square$

Chapter 7

Week 7

Plan for today:

- Local ring at a point
- Local rings (general definition)
- Pullbacks on local rings

Last Class: Let X be a variety (irreducible algebraic set). Then

$$k(X) = \mathrm{Frac}(\Gamma(X)) \text{ is the field of rational functions on } X$$

7.1 Local Rings

We say $f \in k(X)$ is defined at P if $\exists a, b \in \Gamma(X)$ where $f = \frac{a}{b}$ and $b(P) \neq 0$.

Definition 7.1.1 **Local Ring at a Point**

The local ring of X at P is denoted $\mathcal{O}_p(X)$ is the subring $\mathcal{O}_p(X) = \{f \in k(X) : f \text{ is defined at } P\} \subseteq k(X)$.

Caution: $\mathcal{O}_p(X) \neq k(P)$

Example 7.1.1: $X = \mathbb{A}^1$. Consider $P = V(x)$. Then $k(P) = \mathrm{Frac}(\Gamma(P)) = \mathrm{Frac}(k) = k$

Meanwhile, $k(X) = k(\mathbb{A}^1) = k(x)$.

$$\mathcal{O}_p(X) = \{\frac{f}{g} : f, g \in k[x] \text{ and constant term of } g \neq 0\}$$

Although $x \in k(X)$ vanishes at $p = 0$, the function x is not the zero function in $k(X)$.

We have $k \subseteq \Gamma(X) \subseteq \mathcal{O}_p(X) \subseteq k(X)$.

Proposition: Let k be algebraically closed and let X be a variety. Then $\Gamma(X) = \bigcap_{p \in X} \mathcal{O}_p(X)$.

Proof. We have the containment of $\Gamma(X) \subseteq \bigcap_{p\in X} \mathcal{O}_p(X)$. Suppose that $f \in \bigcap_{p\in X} \mathcal{O}_p(X)$. Then f is defined every where so if $J_f = \{g \in \Gamma(X) : g \cdot f \in \Gamma(X)\}$, then $V(J_f) = \emptyset$. Recall $\pi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/I(X) = \Gamma(X) \supseteq J_f$. We have $V(\pi^{-1}(J_f)) = V(J_f) = \emptyset$. Now apply WN1 to say that $\pi^{-1}(J_f) = k[x_1, \ldots, x_n]$. So $J_f = \Gamma(X)$. Since $1 \in J_f$, we have $1 \cdot f \in \Gamma(X)$. □

If $f \in \mathcal{O}_p(X)$ then there is an evaluation of f at p.

- So choose $a, b \in \Gamma(X)$ such that $b(P) \neq 0$ and $f = \frac{a}{b}$. The $f(P) = \frac{a(P)}{b(P)}$.
- If $a', b' \in \Gamma(X)$ satisfy $b'(P) \neq 0$ and $f = \frac{a'}{b'}$, then we have

$$\frac{a'}{b'} = \frac{a}{b} \iff a'b = b'a \in \Gamma(X)$$

This means that:

$$a'(P)b(P) = b'(P)a$$

so

$$\frac{a'(P)}{b'(P)} = \frac{a(P)}{b(P)}$$

which makes evaluation well-defined.

Evaluation at p gives a map $\mathcal{O}_p(X) \to k$ which is a surjective map. This means that the kernel is a maximal ideal which we denote as $\mathfrak{m}_p(X) = \{f \in \mathcal{O}_p(X) : f(P) = 0\}$. Claim: This is the set of {non-units in $\mathcal{O}_p(X)$}. In other words, if $g \notin \mathfrak{m}_p(X)$, then g is a unit of $\mathcal{O}_p(X)$. If $g \notin \mathfrak{m}_p(X)$, the $g = \frac{a}{b}$ where $b(P) \neq 0$, $a(P) \neq 0$. So $\frac{b}{a} \in \mathcal{O}_p(X)$.

Definition 7.1.2 **Local Ring**

We call a ring R a local ring if one of the following equivalent conditions is satisfied:

- {non-units in R} $\subseteq$ R is an ideal
- R has a unique maximal ideal.

Proof. ($1 \to 2$) Let $\mathfrak{m}$ = {non-units in R} is an ideal. Every proper ideal $I \subset R$ is contained in $\mathfrak{m}$. If $I \subset \mathfrak{m}$, then I has a unit making $I = R$. So $\mathfrak{m}$ is the unique maximal ideal since any other maximal ideal must be in $\mathfrak{m}$.

($2 \to 1$) Suppose that R has a unique maximal ideal. The claim is that $\mathfrak{m}$ is the set of non-units. If $a \in R$ that is not a unit, then the ideal generated by this element is not R. So $(a) \subseteq \mathfrak{m}$. So $\mathfrak{m}$ is the set of all non-units. □

Example 7.1.2: Let $R = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \text{ is odd}\} \subseteq \mathbb{Q}$. An element $c \in \mathbb{R}$ is a unit iff $c = \frac{2a}{b} \in (\frac{2}{1})$.

Example 7.1.3: Non-example. $k[x]$ is not a local ring: $(x), (x+1)$ are non-units but $x - x + 1 = 1$ is a unit. The ideals $(x) \neq (x+1)$ are maximal.

Example 7.1.4: Let $R = \{\frac{a}{b} \in k(x) : a, b \in k[x], b_0 \neq 0\} = \mathcal{O}_0(\mathbb{A}^1)$. This is a local ring with a unique maximal ideal $(\frac{x}{1})$.

Proposition: $\mathcal{O}_p(X)$ is Noetherian.

Proof. Suppose that $I \subseteq \mathcal{O}_p(X)$ is an ideal. Consider $J = I \cap \Gamma(X)$. J is an ideal in $\Gamma(X)$ and $\Gamma(X)$ is Noetherian. It follows that $J = (f_1, \dots, f_r)$. Consider $f = \frac{a}{b} \in I \subseteq \mathcal{O}_p(X)$, $a, b \in \Gamma(X)$ and $b(P) \neq 0$. Then $a = fb \in \mathcal{O}_p(X)$ and $a \in \Gamma(X)$, $a \in I$, so $a \in J$. This means that

$$a = a_1 f_1 + \cdots + a_r f_r \text{ for } a_i \in \Gamma(X)$$

So we divide through by b:

$$f = \frac{a}{b} = \left(\frac{a_1}{b}\right) f_1 + \cdots + \left(\frac{a_r}{b}\right) f_r \in (f_1, \dots, f_r)$$

so $I = (f_1, \dots, f_r)$. $\square$

7.2 Local rings and Pullbacks

Suppose $\varphi : X \to Y$ is a morphism of varieties. Then

$$\begin{tikzcd} \Gamma(Y) \arrow[r, "\varphi^*"] \arrow[d, hook] & \Gamma(X) \arrow[d, hook] \\ k(Y) \arrow[r, "?"] & k(X) \end{tikzcd}$$

If it extends, then $\frac{f}{g} \mapsto \frac{\varphi^* f}{\varphi^* g}$. If $g \in \ker \varphi^*$, then this does not work.

Let $p \in X$ and let $Q = \varphi(p) \in Y$. Suppose $f \in \mathcal{O}_Q(Y)$. Then $\exists g, h \in \Gamma(Y)$ so that $f = \frac{g}{h}$ and $h(Q) \neq 0$. Then

$$(\varphi^* h)(p) = h(\varphi(p)) = h(Q) \neq 0$$

So $\frac{\varphi^* g}{\varphi^* h}$ is defined at P. This induces a well-defined map

$$\mathcal{O}_Q(Y) \xrightarrow{\varphi^*} \mathcal{O}_p(X)$$

If $f = \frac{g'}{h'}$, where $h'(Q) \neq 0$, then

$$\frac{g}{h} = \frac{g'}{h'} \iff h'g = g'h$$

which means

$$\varphi^* h' \varphi^* g = \varphi^* g' \varphi^* h$$

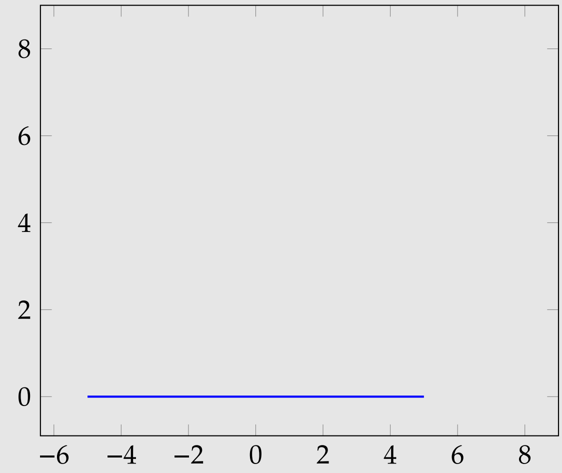
Since $\varphi^* h$, $\varphi^* h' \neq 0$,

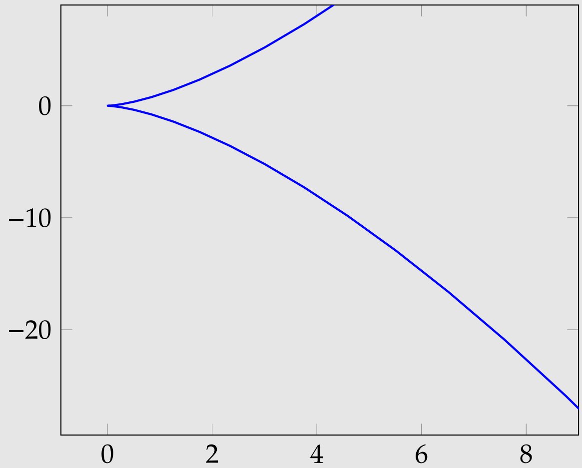
$$\frac{\varphi^* g}{\varphi^* h} = \frac{\varphi^* g'}{\varphi^* h'}$$

7.3 Tangent Spaces

Recall: It is not always possible to pullback rational functions along $\varphi : X \to Y$. If $p \in X$, $Q = \varphi(p) \in Y$, then $\mathcal{O}_Q(Y) \to \mathcal{O}_p(X)$

Example 7.3.1:





We have $\mathcal{O}_1(\mathbb{A}^1) = \{\frac{a}{b} : a, b \in k[t], b(1) \neq 0\}$ and $\mathcal{O}_{(1,1)}(Y) = \{\frac{g}{b} : g, h \in \frac{k[x,y]}{(y^2-x^3}, h(1,1) \neq 0\}$. Since x does not vanish at $(1,1)$, we have $\frac{y}{x} \longrightarrow t$.

(Onto) We have $\frac{g(t)}{h(t)} \leftarrow \frac{g(\frac{y}{x})}{h(\frac{y}{x})}$.

(Injective) If $0 = \frac{\varphi^* g}{\varphi^* h}$, then $\varphi^* g = 0$. This means $g(t^2, t^3) = 0$ and g vanishes on Y. Also possible to show that φ is dominant which implies that φ^* is injective.

On the other hand the map: $\mathcal{O}_0(\mathbb{A}^n) = \{\frac{a}{b} : a, b \in k[t], b \neq 0\} \leftarrow \mathcal{O}_{(0,0)}(Y) = \{\frac{f}{g} : f, g \in \frac{k[x,y]}{(x^3)-y^2}, g(0,0) \neq 0\}$ is not an isomorphism.

Claim: $t \in \mathcal{O}_0(\mathbb{A}^1)$ is not in the image. Suppose that $\frac{f(x,y)}{g(x,y)}$ where $\varphi^* \frac{f(x,y)}{g(x,y)} = t$. So $f(t^2, t^3) = g(t^2, t^3)t$. So $t \mid f(t^2, t^3)$ and $t^2 \mid f(t^2, t^3)$. So $t \mid g(t^2, t^3)$ and $g(0,0) = 0$.

Example 7.3.2: For $f = x^2 + y^2 - 1 = 0$, $V(f)$.

$$f_x = \frac{d}{dx} = 3x$$
$$f_y = \frac{d}{dy} = 2y$$

and $f_x(p) = -\sqrt{2}$, $f_y(p) = \sqrt{2}$. Tangent line to $V(f)$ at $p = (\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is

$$f_x(p)(x - x_0) + f_y(p)(y - y_0) = 0 \implies y = x + \sqrt{2}$$

Definition 7.3.1 **Singular vs Smooth Points**

If $f_x(p) = 0$ and $f_y(p) = 0$, then p is a singular point of $V(f)$. Otherwise, p is a smooth point.

Remark: This works when f has no repeated factors or (f) is radical.

Definition 7.3.2 **Smooth vs Singular Sets**

We say that $v(f)$ is smooth if $V(f)$ is smooth at every point $p \in V(f)$. Otherwise, we call $V(f)$ singular.

Example 7.3.3: $f(x, y) = y^2 - x^3 + x$.

Example 7.3.4: $f = (y - x^2)(y + 1) = y^2 + y - x^2y - x^2$ is smooth over reals but not over the complex numbers.

$$f_x = -2xy - 2x$$
$$f_y = 2y + 1 - x^2$$

$0 = 2x(y + 1)$ and $x^2 = 2y + 1$. Cases:

- $x = 0$: Then $2y + 1 = 0$, $y = \frac{-1}{2}$.
- $y = -1$: Then $x^2 = -1$. These points are singular on $V(f)$. So $V(f)$ is singular.

Example 7.3.5: $V(y^2 - x^3)$.

Definition 7.3.3 **Forms/Homogeneous Polynomials**

Given a polynomial $f \in k[x, y]$, We can always write

$$f = f_0 + f_1 + f_2 + \cdots + f_d$$

where f_i is a linear combination of monomials with degree i.

Claim: If $f_1 \neq 0$, then $V(f_1)$ is the tangent line to $V(f)$ at $(0, 0)$. If $f_1 = 0$, then $V(f)$ is singular at $(0, 0)$.

In either case, we define the tangent space to $V(f)$ at $(0, 0)$ to be $V(f_1)$ denoted $T_{(0,0)}V(f) = V(f_1)$.

Chapter 8

Week 8

8.1 Tangent Cones

Given $0 \neq f \in k[x, y]$, we defined f_i homogeneous parts of degree i.

Claim: If $f_1 \neq 0$, then $V(f_1)$ is the tangent line to $V(f)$ at $(0, 0)$. If $f_1 = 0$, then $V(f)$ is singular at $(0, 0)$.

In either case, we define the tangent space to $V(f)$ at $(0, 0)$ to be $V(f_1)$ denoted $T_{(0,0)}V(f) = V(f_1)$.

Proof. (a) If $f_1 \neq 0$ and $f = ax + by + \cdots$, then $f_x = a + \cdots$ and $f_y = b + \cdots$ so $f_x(0, 0) = a$, $f_y(0, 0) = b$. The tangent line is $f_x(p)(x - x_0) + f_y(p)(y - y_0) = 0$. So we have $ax + by = 0$.

(b) If $f_1 = 0$, we have $f = cx^2+dxy+ey^2+\cdots$. Now $f_x = 2cx+dy+\cdots$, $f_y = dx+2ey+\cdots$. Plug in: $f_x(0, 0) = 0$, $f_y(0, 0) = 0$. So $V(f)$ is singular at $(0, 0)$.

□

If $f_1 = 0$, the next best thing is to look at f_2.

Example 8.1.1: $f = y^2 - x^2 - x^3$.

Definition 8.1.1 **Tangent Cone**

Suppose that $f_0 = 0$, so $(0, 0) \in V(f)$. Let m be the minimal number so that $f_m \neq 0$. Then $V(f_m)$ is called the tangent cone to $V(f)$ at $(0, 0)$. Denoted $TC_{(0,0)}V(f)$. Here, m is called the multiplicity of $(0, 0)$.

If $(0, 0) \in V(f)$ has multiplicity 1, then the tangent cone is a line, and it is smooth at $(0, 0)$. If it is greater than 1, it is a union of lines contained in tangent space.

Proposition: Suppose $F \in k[x, y]$ is homogeneous of degree m. Assuming k is algebraically closed, F factors into m linear factors.

If $G(x, y)$ is dehomogenized: $G(x, 1)$, the form cannot be recovered unless the degree is known.

Proof. Say $F(x, y)$ is homogeneous. Write $F(x, y) = y^r G(x, y)$ where $y \nmid G$. Consider $G(x, y) = a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_1 x y^{d-1} + a_0 y^d j$. And $G(x, 1)k = a_d x^d + \cdots + a_1 x + a_0$. Because k is algebraically closed, $G(x, 1)$ factors to linear factors. $G(x, 1) = a_d \prod_{i=1}^{d}(x - \lambda_i)$. So $G(x, y) = a_d \prod_{i=1}(x - \lambda_i y)$. ◻

Let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be the map such that

$$\varphi((a, b)) = (x + a, y + b)$$

We also have:

$$\begin{aligned} \varphi^* : \Gamma(\mathbb{A}^2) &\to \gamma \\ f(x, y) &\mapsto f(x + a, y + b) \end{aligned}$$

If $(a, b) \in V(f)$, then $(0, 0) \in V(\varphi^* f) = \varphi^{-1}(V(f))$. Define the multiplicity of $(a, b) \in V(f)$ to be the multiplicity of $(0, 0) \in V(\varphi^* f)$. Define the tangent cone to X at (a, b) to be φ(tangent cone of $V(\varphi^* f)$ at $(0, 0)$).

Suppose X is an algebraic set in $\mathbb{A}^n$. The tangent space to X at $(0, \ldots, 0)$ is $T_{(0,\ldots,0)} X = V(\{f_1 : f \in I(X)\})$. The tangent cone to X at $(0, \ldots, 0)$ to be

$$TC_{(0,\ldots,0)} X = V(\{f_m : f \in I(X)\})$$

Extend this to arbitrary points in $\mathbb{A}^n$ using translation.

For any algebraic set X, we say X is smooth iff $\dim T_p X = \dim X$ for all $p \in X$.

Relationship with the local ring: Let I, J be ideals in a ring R. Then

$$IJ = \langle ab : a \in I, b \in J \rangle$$

And for powers of ideals:

$$I^n = \langle a_1 \cdots a_n : a_i \in I \rangle$$

Suppose $P \in X$. Consider $m_p = m_p(X) \subseteq \mathcal{O}_p(X)$.

Definition 8.1.2 **Zariski Tangent Space**

The Zariski tangent space of X at P is the dual vector space of m_p / mp^2 which is the space of linear maps $m_p / mp^2 \to k$.

Example 8.1.2: Suppose $f = y - 3x + x^3$.

Chapter 9

Week 9

Let X be an algebraic set in $\mathbb{A}^n$ and suppose $(0,\ldots,0) \in x$. Then $T_{(0,\ldots,0)}X = V(\{f_1 : f \in I(X)\})$.

Suppose $p \in X$ any point. Recall that $\mathfrak{m}_p(X) \subseteq \mathcal{O}_{p(X)}$ to be the ideal of functions that vanish at p. The more abstract definition of the tangent space is the definition of the Zariski Tangent Space.

Example 9.0.1: Suppose that $f = y - 3x + x^3$ and $p = (0,0) \in V(f) = X$. Then $\mathfrak{m}_p(X) = (\frac{x}{1}, \frac{y}{1})$. We also know that $y = 3x - x^3$ so $\mathfrak{m}_p(X) = (\frac{x}{1})$. So $\mathfrak{m}_p(X)^2 = (\frac{x^2}{1})$.

Claim: $\mathfrak{m}_p(X)/\mathfrak{m}_p(X)^2$ is a one dimensional vector space spanned by $\frac{x}{1}$.

Proof. If $\frac{g}{h} \in \mathfrak{m}_p(X)$. Then $\frac{g}{h} = \frac{g_1+\cdots}{h_0+h_1+\cdots}$.

$$\frac{g_1}{h_0} + g_1\left(\frac{1}{h} - \frac{1}{h_0}\right) + \frac{g-g_1}{h}$$

We have $g_1 \in \mathfrak{m}_p(X)$ and

$$\frac{1}{h} - \frac{1}{h_0} = \frac{h_0 - h}{hh_0} \in \mathfrak{m}_p(X)$$

So the quotient is a one-dimensional vector space.

In $\mathfrak{m}_p/\mathfrak{m}_p^2$, $\overline{y} = 3\overline{x}$. Then we have linear maps $\mathfrak{m}_p/\mathfrak{m}_p^2 \to k$ as $\overline{x} \mapsto a$, $\overline{y} \mapsto 3a$. This determines a vector $(a, 3a)$ which lies in the tangent space. □

Consider the case where $X = \mathbb{A}^n = V(0)$, $p = (0,\ldots,0)$. What is $T_p(\mathbb{A}^n)$. In this case, $T_p(\mathbb{A}^n) = \mathbb{A}^n$. We also have:

$$\mathcal{O}_p(\mathbb{A}^n) = \{\frac{g}{h} : g, h \in k[x_1,\ldots,x_n], h(p) \neq 0\}$$

and

$$\mathfrak{m}_p(\mathbb{A}^n) = (\frac{x_1}{1},\ldots,\frac{x_n}{1}), \mathfrak{m}_;(\mathbb{A}^n)^2 = \left(\frac{x_1^2}{1},\ldots,\frac{x_ix_j}{1}\ldots,\frac{x_n^2}{1}\right)$$

Claim: $\mathfrak{m}_p(\mathbb{A}^n)/\mathfrak{m}_p(\mathbb{A}^n)^2$ has basis $x_1/1,\ldots,x_n/1$. Each of these define a linear form on $\mathbb{A}^n \to k$.

We say that:

$$x_i(a_1, \dots, a_n) \mapsto a_i$$

So m_p/m_p^2 is the dual to $T_p(\mathbb{A}^n)$. And $(m_p/m_p^2)^\nu$ is $((T_p(\mathbb{A}^n))^\nu)^\nu \cong T_p\mathbb{A}^n$.

In general, suppose $X = V(I)$. Then $T_p(X) = V(\{f_1 : f \in I(X)\})$. Notice that $\{f_1 : f \in I(X)\}$ forms a vector space. Choose a basis $L_1, \dots, L_r$ for this vector space, so $T_p(X) = V(L_1, \dots, L_r)$

Now $m_p(X)$ is $(\frac{x_1}{1}, \dots, \frac{x_n}{1})$, but in $m_p(X) \subseteq \mathcal{O}_p(X)$, we have $L_i(\frac{x_1}{1}, \dots, \frac{x_n}{1})$+(higher degree terms) $\in I(X)$. In m_p/m_p^2, we have $L_i(\frac{x_1}{1}, \dots, \frac{x_n}{1}) = 0$. In fact, we have a short exact sequence of vector spaces:

$$0 \to \mathrm{Span}\,\{L_1, \dots, L_r\} \to \mathrm{Span}\,\{X_1, \dots, X_n\} \to m_p/m_p^2 \to 0$$

Take duals:

$$0 \leftarrow \mathrm{Span}\,\{L_1, \dots, L_r\}^\nu \leftarrow \mathrm{Span}\,\{X_1, \dots, X_n\}^\nu \leftarrow (m_p/m_p^2)^\nu \leftarrow 0$$

So we have $\mathrm{Span}\,\{x_1, \dots, x_n\}^\nu = ((T_p(\mathbb{A}^n))^\nu)^\nu = \mathbb{A}^n$ and in the dual map, i

$$(a_1, \dots, a_n) \to \begin{bmatrix} L_1(a_1, \dots, a_n) \\ L_2(a_1, \dots, a_n) \\ \vdots \\ L_r(a_1, \dots, a_n) \end{bmatrix}$$

So the dual of m_p/m_p^2 map to k is the kernel of the mapping to the tangent space.

9.1 Intersection Multiplicity

Definition 9.1.1 **Intersection Multiplicity**

Given $f, g \in k[x, y]$, we define

$$I_p(f, g) = \dim_k \left(\frac{\mathcal{O}_p(\mathbb{A}^2)}{(\frac{f}{1}), \frac{g}{1}} \right)$$

Properties of $I_p(f, g)$:

- If $V(f)$ and $V(g)$ have a common component passing through p, then $I_p(f, g) = \inf$. Otherwise, $I_p(f, g) < \inf$.

Example 9.1.1: Consider $V(xy), V(x(y - x^3))$. Then $\frac{\mathcal{O}_{(0,0)}(\mathbb{A}^2)}{(\frac{xy - x^4}{1}, \frac{xy}{1})}$. We claim that $1, \overline{y}, \overline{y}^2, \dots$ are linearly independent. Suppose there is dependence in the powers of $\overline{y}$. Then there is a polynomial in y that lies in $(\frac{xy-x^4}{1}, \frac{xy}{1})$.

On the other hand, if $V(f), V(g)$ have no common component, then $V(f, g)$ is finite. This means $\frac{k[x,y]}{(g,f)}$ is finite dimensional. Additionally:

$$\frac{k[x,y]}{(f,g)} = \bigoplus_{p_i \in V(f,g)} \frac{\mathcal{O}_{p_i}\mathbb{A}^2}{(f,g)}$$

- $I_p(f,g) = 0$ iff $p \notin V(f) \cap V(g)$.

  Proof. If $p \notin V(f) \cap V(g) = V(f,g)$ means that there is an $h \in (f,g) \subseteq k[x,y]$ such that $h(p) \neq 0$. Then h is a unit in $(f,g) \subseteq \mathcal{O}_p(\mathbb{A}^2)$. So $(f,g) = \mathcal{O}_p(\mathbb{A}^2)$, and the quotient is 0. □

  We also have that $I_p(f,g)$ depends only on the components of $V(f), V(g)$ that pass through p.

  Proof. If $p \notin V(f_2)$, then $f_2(p) \neq 0$. So f_2 is a unit in $\mathcal{O}_p(\mathbb{A}^2)$. So $(f,g) = (f_1 f_2, g) = (f_1, g)$. □

- If $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ is a change of coordinates (i.e. Translation) with $\varphi(p) = q$, then $I_q(f,g) = I_p(\varphi^* f, \varphi^* g)$.
- $I_p(f,g) = I_p(g,f)$
- Let $\mathrm{mult}_{(0,0)(f)}$ = smallest m such that $f_m \neq 0$. Then $I_p(f,g) \geqslant \mathrm{mult}_p(f)\,\mathrm{mult}_p(g)$. Equality holds iff the tangent cones of $V(f)$ and $V(g)$ have no lines in common.

Example 9.1.2: Consider $I_P(y^2-x^2-x^3, y^2-x^3)$. $TC_P(V(f)) = V(y-x) \cup V(y+x)$ while $TC_P(V(g)) = V(y^2) = V(y)$. $I_P(f,g) = \mathrm{multi}_P(f)\,\mathrm{multi}_P(g) = 2 \cdot 2 = 4$

- If $f = \prod f_i^{r_i}, g = \prod g_j^{s_j}$, then $I_P(f,g) = \sum_{i,j} r_i s_j I_P(f_i, g_j)$.

Example 9.1.3: $I_P(x^2, y^3)$:

$$\begin{aligned} I_P(x^2,y^3) &= \dim_k\left(\frac{\mathcal{O}_{(0,0)}(\mathbb{A}^2)}{(x^2,y^3)}\right) \\ &= \dim_k(Span\left\{1, x, y, xy, y^2, xy^2\right\}) \\ &= 6 \end{aligned}$$

Alternatively:

$$\begin{aligned} \dim_k\left(\frac{\mathcal{O}_{(0,0)}(\mathbb{A}^2)}{(x^2,y^3)}\right) &= 2 \cdot 3 \cdot I_P(x,y) \\ &= 6 \end{aligned}$$

Example 9.1.4: Another way to think about it: $I_P(x^2 - \varepsilon, y(y^2 - \varepsilon))$. Then $V(x^2 - \varepsilon) = V(x - \sqrt{\varepsilon}) \cup V(x + \sqrt{\varepsilon})$ with the same for $V(y(y^2 - \varepsilon))$. So there are 3 y-axis lines meeting 2 x-axis lines at $(0,0)$, which counts for multiplicity 6.

- For any $a \in k[x, y]$, we have $I_P(f, g) = I_P(f, g + af)$. Using the definition:

$$\frac{\mathcal{O}_P(\mathbb{A}^2)}{(f, g)} = \frac{\mathcal{O}_P(\mathbb{A}^2)}{(f, g + af)}$$

Example 9.1.5: $I_P(y, y - x^2) \to I_P(y, y - x^2 + (3y^2 + x)y)$.

9.2 Computing $I_P(f, g)$

Example 9.2.1: $P = (0, 0)$, $I_P(y^3 - y^2x^3, y - x^2)$. Since $V(f) = V(y^2) \cup V(y - x^3)$.

- Step 1: Translate the point to the origin. Compute the pullback of f, g.
- Step 2: Check if f and g have a common factor that vanishes at P.
- Step 3: Check if $P \in V(f) \cap V(g)$. If not $I_P(f, g) = 0$.
- Step 4: Find the tangent cones $TC_P(V(f)), TC_P(V(g))$. If $TC_P(V(f)), TC_P(V(g))$ have no lines in common, then $I_P(f, g) = \mathrm{mult}_P(f)\,\mathrm{mult}_P(g)$.
- Step 5: Choose a common line in TC. Do a change of coordinates so that the common line is $V(y)$. Consider $f(x, 0), g(x, 0)$. We have $f = 0, g = -x^2$.
- Step 6: Case 1 (one of $f(x, 0), g(x, 0) = 0$): $y^r \mid f(x, y)$.

$$\begin{aligned} I_P(f, g) &= I_P(y^r, g) + I_P(h, g) \\ &= I_P(y^2, y - x^2) + I_P(y - x^3, y - x^2) \\ &= 2I_P(y, y - x^2) + I_P(y - x^3, y - x^2) \\ &= 4 + I_P(y - x^3, y - x^2) \end{aligned}$$

  And repeat the process on the other part. In general for $I_P(y^r, g)$, write $g(x, 0) = x^m(a_0 + a_1x + \cdots)$. So $g(x, y) = x^mA + yB$. Then compute $I_P(y^r, g) = rm + I_P(y, Ax) = rm(I_P(y, A) + I_P(y, x)) = rm(0 + 1)$.
- Step 6: Case 2 (neither $f(x, 0), g(x, 0) = 0$): Consider $h = f - x^{r-s}g$ where $f(x, 0) = x^r + \cdots, g(x, 0) = x^s + \cdots$. In our example, we get $(y - x^3) - x(y - x^2) = y - xy$. So $I_P(y - x^3, y - x^2) = I_P(y - xy, y - x^2)$. Now go back to the beginning.

$$\begin{aligned} I_P(y - x^3, y - x^2) &= I_P(y - xy, y - x^2) \\ &= I_P(y(1 - x), y - x^2) \\ &= I_P(y, y - x^2) + I_P(1 - x, y - x^2) \\ &= I_P(y, -x^2) + 0 \\ &= 2I_P(y, -x) = 2 \end{aligned}$$

- Add them all up. We have

$$I_P(f, g) = I_P(y^2, g) + I_P(h, g) = 4 + 2 = 6$$

Chapter 10

Week 10

10.1 Projective Space

Does the total intersection number
$$\sum_{p\in\mathbb{A}^2} I_p(f,g)$$
satisfy some nice properties? Does it depend only on the degree of f and g.

Intersection "runs off to ∞".

Example:

Vertical lines only meet once. Other point meets at infinity.

Definition 10.1.1 **Projective Space**

Projective space $\mathbb{P}^n$ is an enlargement of $\mathbb{A}^n$ made by taking $\mathbb{A}^n\cup$ "points at ∞" or "points on the horizon".

For every line through $(0,0)$, we have a corresponding point that meets on $\{(x,1)\}$ except $V(y)$. We call the set of all lines through the origin $\mathbb{P}^1$. This looks like a circle for $\mathbb{P}^1_{\mathbb{R}}$. $\mathbb{P}^1_{\mathbb{C}}$ over the complex numbers looks like a sphere with a point at infinity.

Now for $\mathbb{A}^2 \subseteq \mathbb{P}^2$. Consider $\{(x,y,1)\} \subseteq \mathbb{A}^3$. For each point on the plane $(x,y,1)$, we can associate with it a line through $(0,0,0)$. The lines in $\mathbb{A}^3$ that do not meet the plane are lines in $V(z)$. So
$$\mathbb{P}^2 = \{\text{all lines in } \mathbb{A}^3 \text{ through } (0,0,0)\} = \mathbb{A}^2 \cup \{\text{lines in } V(z)\} = \mathbb{A}^2 \cup \mathbb{P}^1$$

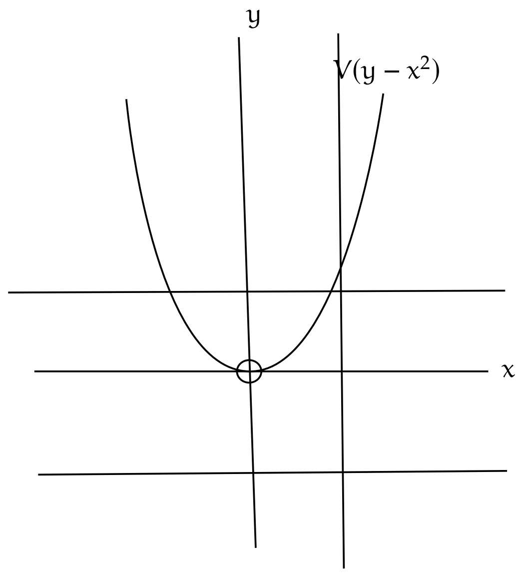
In general, define $\mathbb{P}^n$ to be $\{$lines through the origin in $\mathbb{A}^{n+1}\}$.

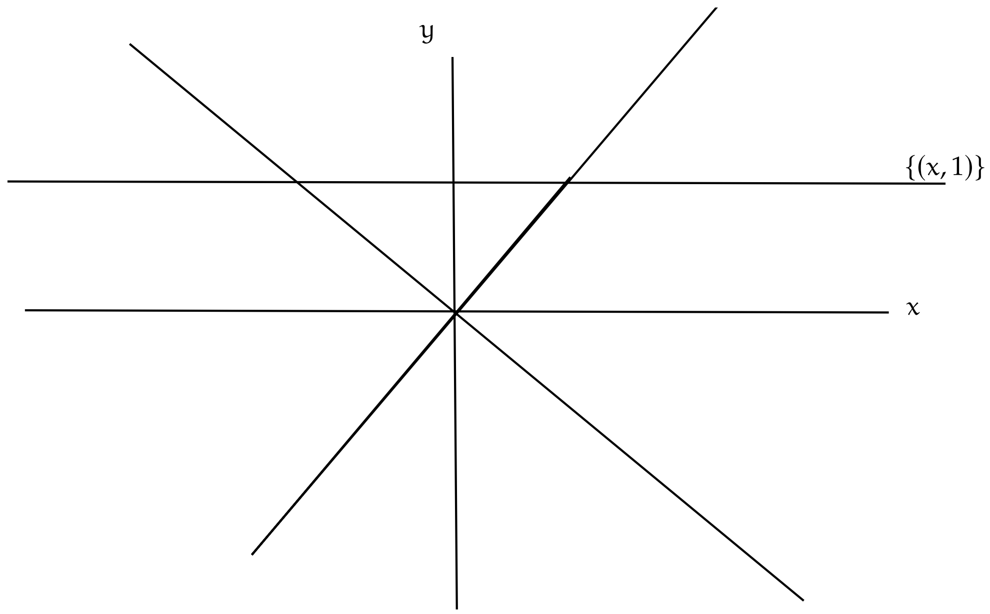
What is $\mathbb{P}^0$? A single point.

Any point $(x_1,\ldots,x_{n+1}) \in \mathbb{A}^n \neq (0,\ldots,0)$ determines a line through the origin determines the line $\{(\lambda x_1,\ldots,\lambda x_{n+1}) : \lambda \in k\}$.

Two points $(x_1,\ldots,x_{n+1})$ and $(y_1,\ldots,y_{n+1})$ determine the same line iff $(x_1,\ldots,x_{n+1}) = (\lambda y_1,\ldots,\lambda y_{n+1})$. Define two such points to be equivalent.

Alternate Definition: $\mathbb{P}^n_k = \mathbb{A}^{n+1}\backslash(0,\ldots,0)/(x \sim \lambda x \lambda \neq 0)$





to each point $(x, 1)$ we can associate a line through $(0, 0)$ and that point

We write points in $\mathbb{P}^n$ as $[x_1 : \cdots : x_{n+1}]$ where the values of x_i is not well defined. However, if $x_i \neq 0$, then x_i/x_j is well-defined.

If $x_{n+1} \neq 0$, then we can rescale:

$$U_{n+1} \cong \{[x_1 : \cdots : x_{n+1}] : s_{n+1} \neq 0\} = \{[x_1/x_{n+1} : \cdots : x_n/x_{n+1} : 1]\} = \mathbb{A}^n$$

So:

$$\mathbb{P}^n = U_{n+1} \cup \{[x_1 : \cdots : x_n : 0]\} = U_{n+1} \cup \mathbb{P}^{n-1}$$

So we see that as $x_{n+1} \to \infty$, the points of U_{n+1} tend to infinity.

Iterating this procedure,

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cdots \mathbb{A}^1 \cup \text{pt}.$$

Why just use the last coordinate? For any i, we define $U_i = \{[x_1 : \cdots : x_{n+1}] : x_i \neq 0\}$. For each point $p \in U_i$, we can write $p = [x_1 : \cdots : x_{i-1} : 1 : \cdots x_{n+1}]$ with no restrictions on the other x_j. These $U_i \cong \mathbb{A}^n \subseteq \mathbb{P}^n$ are called affine charts. So $\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i$.

Example 10.1.1: Affine charts on $\mathbb{P}^1$. We have $U_2 = \{[x_1 : x_2] : x_2 \neq 0\}$. And $U_1 = \{[x_1 : x_2] : x_1 \neq 0\}$.

Example 10.1.2: In $\mathbb{P}^2$, we have 3 affine charts.

$$\begin{aligned} U_1 &= \{[1 : x_2 : x_3]\} \ni [1 : 2 : 0] \\ U_2 &= \{[x_1 : 1 : x_3]\} \ni [\frac{1}{2} : 1 : 0] \\ U_3 &= \{[x_1 : x_2 : 1]\} \not\ni [1 : 2 : 0] \end{aligned}$$

10.2 Projective Algebraic Sets

Recall: $\mathbb{P}^n = \{\text{lines through } (0, \ldots, 0) \text{ in } \mathbb{A}^{n+1}\} = \mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\}/\text{scalar}$. $[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n$. We defined

$$U_i = \{[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n : x_i \neq 0\} = \mathbb{A}^n$$

We focus on $U_{n+1} \subseteq \mathbb{P}^n$ where we call the complement the hyperplane at ∞ or H_∞.

Example 10.2.1: Consider $L = V(y - mx - b) \subseteq \mathbb{A}^2$. Identify $\mathbb{A}^2 \cong U_3 \hookrightarrow \mathbb{P}^2$ where $(x, y) \to [x : y : 1]$. Then

$$L = \{(x, y) \in \mathbb{A}^2 : y = mx + b\} = \{[x : y : 1] : y = mx + b\}$$

If we choose another representative:

$$[\lambda x : \lambda y : \lambda] \to \lambda y = m\lambda x + b?$$

Instead:

$$\{[x : y : z] : y = mx + bz\} \cap U_3$$

Notice that

$$y = mx + bz$$

is homogeneous. Let

$$L' = \{x : y : z \in \mathbb{P}^2 : y = mx + bz\}$$

Then $L = L' \cap U_3$. What is $L' \cap H_\infty$? This is

$$\{[x : y : 0] : y = mx\} = \{[1 : m : 0]\}$$

Example 10.2.2: Parallel Lines: Consider $V(y-1), V(y) \subseteq \mathbb{A}^2$. Corresponding lines in $\mathbb{P}^2$:

$$\{[x : y : z] : y = z\} \text{ and } \{[x : y : z] : y = 0\}$$

These meet in $[1 : 0 : 0]$

Recall: $F \in k[x_1, \ldots, x_{n+1}]$ is homogeneous of degree d if F is a linear combination of monomials of degree d. If F is homogeneous of degree d, then $F(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d F(x, \ldots, x_{n+1})$. This means that homogeneous polynomials have well defined vanishings in $\mathbb{P}$.

Projective Hypersurface

Definition 10.2.1

For $F \in k[x_1, \ldots, x_n]$ homogeneous, let

$$\mathbb{V}(F) = \{[x_1 : x_2 : \cdots : x_{n+1}] \in \mathbb{P}^n : F(x_1, \ldots, x_{n+1}) = 0\}$$

Given any set $S \subseteq k[x_1, \ldots, x_{n+1}]$ of homogeneous polynomials, the projective algebraic set is

$$\mathbb{V}(S) = \bigcap_{F \in S} \mathbb{V}(F)$$

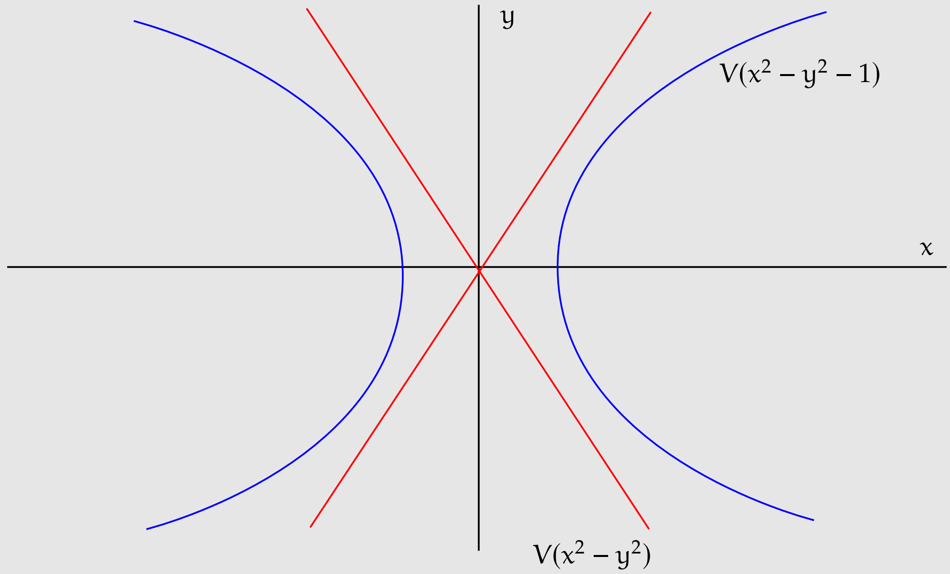
Example 10.2.3: Consider $x^2y - y^3 \in k[x, y]$. We have:

$$V(x^2y - y^3) \subseteq \mathbb{A}^2$$

Instead,

$$\mathbb{V}(x^2y - y^3) = \{[-1 : 1], [1 : 1], [1 : 0]\}$$

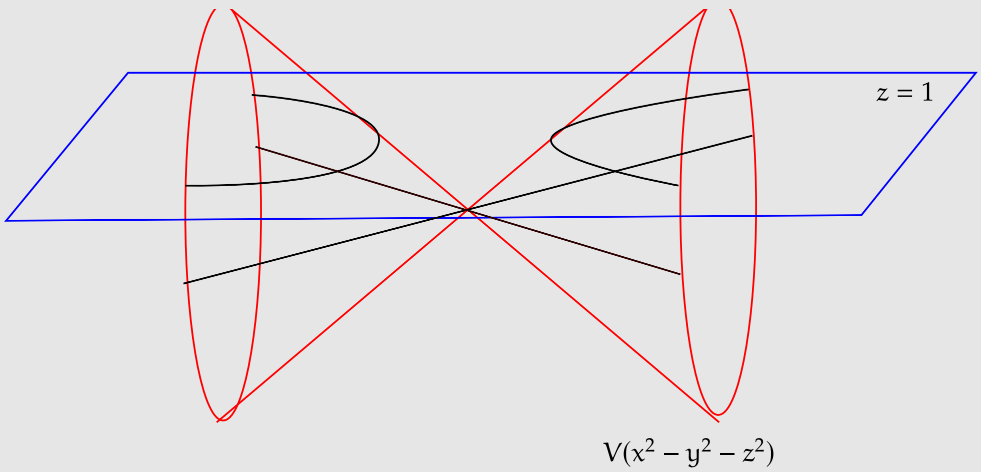
Example 10.2.4: Consider $V(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$ which is a hyperbola. Corresponding algebraic subset of $\mathbb{P}^2$ is



$$\mathbb{V}(x^2 - y^2 - z^2) = \{[x : y : z] : x^2 - y^2 - z^2 = 0\}$$

What is $\mathbb{V}(x^2-y^2-z^2) \cap H_\infty$? $= \{[x : y : z] : x^2-y^2 = 0, z = 0\} = \{[1 : 1 : 0], [1 : -1 : 0]\}$. The way that it meets the point at infinity is based on the higher degree terms.

We can also take $V(x^2-y^2-z^2) \subseteq \mathbb{A}^3$ which is a cone. Then the usual $V(x^2-y^2-1) = V(x^2 - y^2 - z^2) \cap V(z-1)$ and $\mathbb{V}(x^2 - y^2 - z^2) \cap H_\infty$ is the cone intersected at $z = 0$



We have $V(x^2 - y^2 - 1) = \mathbb{V}(x^2 - y^2 - z^2) \cap U_3$. But what does $\mathbb{V}(x^2 - y^2 - z^2) \cap U_1$ look like? It is a circle. This is

$$\{[x : y : z] : x^2 = y^2 + z^2 \wedge x \neq 0\} = \{[1 : y : z] : y^2 + z^2 = 1\}$$

Affine Cone

Definition 10.2.2

Given a projective algebraic set, $X \subseteq \mathbb{P}^n$, we define the affine cone over X to be

$$C(X) = \{(x_1, \ldots, x_{n+1}) \subseteq \mathbb{A}^{n+1} : [x_1 : \cdots : x_{n+1} \in X \text{ or } (x_1, \ldots, x_{n+1}) = (0, \ldots, 0)]\}$$

Note: If $X = \mathbb{V}(F_1, \ldots, F_r)$ with $F_i \in k[x_1, \ldots, x_n]$ homogeneous, then $C(X) =$

$$V(F_1, \ldots, F_r) \subseteq \mathbb{A}^{n+1}.$$

More generally, an algebraic set $Y \subseteq \mathbb{A}^{n+1}$ is called a cone if $\forall (x_1, \ldots, x_{n+1}) \in Y$, we have $(\lambda x_1, \ldots, \lambda x_{n+1}) \in Y$

We have

$$\begin{tikzcd}
\{\text{homogeneous poly in } k[x_1, \ldots, x_{n+1}]\} \arrow[r, "V"] \arrow[d, "\mathbb{V}"] & \{\text{cones in } \mathbb{A}^{n+1}\} \arrow[dl, bend left, "\text{projectivize}"] \\
\{\text{algebraic sets in } \mathbb{P}^n\} \arrow[ur, "C"] &
\end{tikzcd}$$

Homogeneous Ideals

Definition 10.2.3

Given $X \subseteq \mathbb{P}^n$ a projective algebraic set, define $\mathbb{I}(X) \subseteq k[x_1, \ldots, x_{n+1}]$ to be the ideal generated by $\{\text{homogeneous } F \in k[x_1, \ldots, x_n] : F(x_1, \ldots, x_{n+1}) = 0 \text{ for } [x_1 : \cdots : x_{n+1}] \in X\}$

Example 10.2.5: What is $\mathbb{I}(\{[1:2],[3:4]\})$? It is $y - 2x$ and $3y - 4x$ so $((y - 2x)(3y - 4x)) = \mathbb{I}(\{[1:2],[3:4]\})$.

Definition 10.2.4

An ideal $I \subseteq k[x_1, x_{n+1}]$ is called homogeneous if it satisfies either of the following equivalent conditions:

- I is generated by homogeneous polynomials
- $\forall f \in I$, if $f = f_0 + f_1 + \cdots + f_d$ where each f_i is homogeneous, of degree i, then $f_i \in I$.

Proof. ($2 \to 1$) If $I = (f^{(1)}, \ldots, f^{(s)})$ and 2 holds, then $f_j^{(i)} \in I$, so $I = \{f_j^{(i)}\}$.

($1 \to 2$) Suppose that $I = (\{F^{(\alpha)}\})$ is generated by homogeneous polynomials with degree $F^{(\alpha)}$ equal to d_α. Given $f \in I$ with $f = f_m + f_d$. First show that $f_m \in I$. Write $f = \sum A^{(\alpha)} F^{(\alpha)}$ and consider the degree m terms.

$$f_m = \sum A^{(\alpha)}_{m-d_\alpha} F^{(\alpha)} \in I$$

Now $f - f_m \in I$. Now write $f - f_m = f_{m+1} + \cdots + f_d$ and repeat. □

Chapter 11

Week 11

11.1 Projective Algebraic Sets

Recall: An ideal in $k[x_1, \dots, x_{n+1}]$ is called homogeneous if it is generated by homogeneous polynomials $I = (\{F^{(\alpha)}\})$. If $f \in I$ and we write

$$f = f_0 + f_1 + \cdots + f_d$$

with f_i homogeneous of degree i, then each $f_i \in I$.

It makes sense to take $\mathbb{V}(I)$ when I is homogeneous:

$$\mathbb{V}(I) = \mathbb{V}(\{F^{(\alpha)}\})$$

Diagram:

$$\begin{tikzcd}
\{\text{homogeneous ideals in } k[x_1, \dots, x_{n+1}]\} \arrow[r, "\mathbb{V}"] \arrow[dr, bend right, "V"'] & \{\text{projective algebraic sets in } \mathbb{P}^n\} \arrow[l, bend right, "\mathbb{I}"'] \arrow[d, bend left, "C"] \\
& \{\text{cone in } \mathbb{A}^{n+1}\} \arrow[ul, "I"] \arrow[u, bend left, "\text{projectivize}"]
\end{tikzcd}$$

Given $X \subseteq \mathbb{P}^n$ a projective algebraic set, $\mathbb{V}(\mathbb{I}(X)) = X$.

Proof. Suppose $X = \mathbb{V}(F_1, \dots, F_r)$. Then

$$\mathbb{I}(X) = (\{\text{homogeneous } F : F(P) = 0 \forall P \in X\}) \supseteq (F_1, \dots, F_r)$$

so

$$X \subseteq \mathbb{V}(\mathbb{I}(X)) \subseteq \mathbb{V}(F_1, \dots, F_r) = X$$

□

What about $\mathbb{I}(\mathbb{V}(J))$ for J homogeneous? We have $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$.

Proposition: Assume k is an infinite field. If $X \subseteq \mathbb{P}^n$ is nonempty, then

$$\mathbb{I}(X) = I(C(X))$$

Proof. ($I(C(X)) \subseteq \mathbb{I}(X)$) Suppose that $f \in k[x_1, \ldots, x_{n+1}]$ such that

$$f(\lambda a_1, \ldots, \lambda a_{n+1}) = 0$$

for all $[a_1 : \cdots : a_{n+1}] \in X$. If

$$f = f_1 + f_1 + \cdots + f_d$$

then $f_i(a_1, \ldots, a_{n+1}) = 0$. So $f_i \in \mathbb{I}(X)$. So $f \in \mathbb{I}(X)$.

($\mathbb{I}(X) \subseteq I(C(X))$) Since $\mathbb{I}(X)$ is generated by homogeneous polynomials, it is enough to show that if $F \subseteq \mathbb{I}(X)$ is homogeneous, then $F \in I(C(X))$. Suppose $F(x_1, \ldots, x_{n+1}) = 0$ for all $[x_1 : \cdots : x_{n+1}] \in X$. This means that

$$F(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d F(x_1, \ldots, x_{n+1}) = 0$$

for all $\lambda \in k \backslash \{0\}$. Since X is non-empty, so $\deg F > 0$, so $F(0, \ldots, 0) = 0$. So $F \in I(C(X))$. □

Theorem 11.1.1 **Projective Nullstellensatz**

Let $J \subseteq k[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal. Let k be algebraically closed.

- $\mathbb{V}(J) = \emptyset$ if and only if $\exists N$ such that $J \supseteq (x_1, \ldots, x_{n+1})^N$. J contains all homogeneous polynomials of degree $\geqslant N$.
- If $\mathbb{V}(J) \neq \emptyset$, then $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$.

Example 11.1.1: Nullstellensatz pt1: $\mathbb{V}(x^3, x^2y, xy^2, y^3) \subseteq \mathbb{V}(x^3, y^3) = \mathbb{V}(x, y) = \emptyset \subseteq \mathbb{P}^1$, because there is no $[0 : 0] \in \mathbb{P}^1$.

Proof. (Part I) $\mathbb{V}(J) = \emptyset$ iff $V(J) \subseteq \{(0, \ldots, 0)\}$ iff $I(V(J)) \supseteq I(\{0, \ldots, 0\}) = (x_1, \ldots, x_{n+1})$. By usual Nullstellensatz, $\sqrt{J} \supseteq I(\{0, \ldots, 0\}) = (x_1, \ldots, x_{n+1})$.

(Part II) $\mathbb{I}(\mathbb{V}(J)) = I(C(\mathbb{V}(J))) = I(V(J)) = \sqrt{J}$ □

So we have a bijection:

$$\begin{tikzcd} \{\text{radical homogeneous ideals in } k[x_1, \ldots, k_{n+1}] \text{ besides } (x_1, \ldots, x_{n+1})\} \arrow[r, bend left, "\mathbb{V}"] & \{\text{projective algebraic sets} \arrow[l, bend left, "\mathbb{I}"] \end{tikzcd}$$

Sometimes, $(x_1, \ldots, x_{n+1})$ is called irrelevant ideal.

11.2 Projective Zariski Topology

Definition 11.2.1 **Irreducible Projective Algebraic Sets**

A projective algebraic set $X \subseteq \mathbb{P}^n$ is irreducible if it is not a union of two smaller projective algebraic sets. If $X = X_1 \cup X_2$, then $X = X_i$. An irreducible projective algebraic set is called a projective variety.

Definition 11.2.2 ◆ **Zariski Topology**

The Zariski Topology on $\mathbb{P}^n$ to be the topology whose closed sets are the projective algebraic sets.

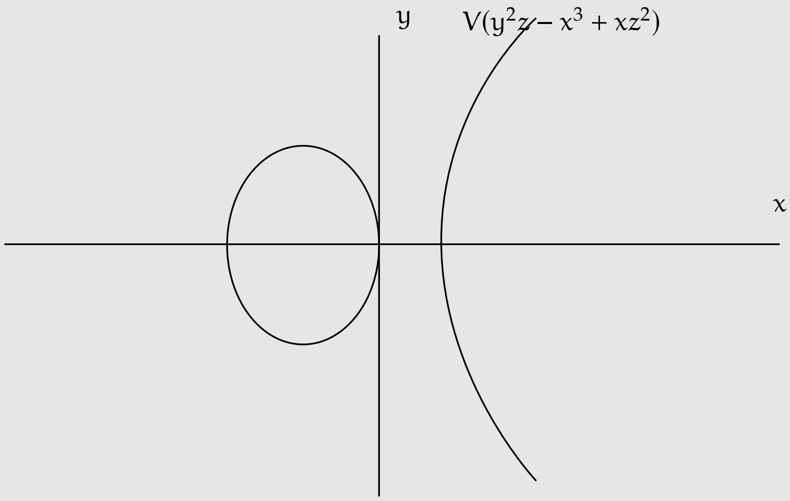
Example 11.2.1: $U_i = \mathbb{P}^n \setminus \mathbb{V}(x_i)$ is open. Hw: $X \subseteq \mathbb{P}^n$ is closed iff $X \cap U_i$ is closed.

Definition 11.2.3 ◆ **Zariski Topology on** X

Given $X \subseteq \mathbb{P}^n$ a projective algebraic set, we define the Zariski Topology on X to be the topology whose closed sets are the projective algebraic subsets of X.

Projective Closure: Given an algebraic set $X \subseteq \mathbb{A}^n \cong U_{n+1} \subseteq \mathbb{P}^n$, define the projective closure $\overline{X} \subseteq \mathbb{P}^n$ to be the smallest projective algebraic set that contains X. It is the closure of X in the Zariski Topology on $\mathbb{P}^n$.

Example 11.2.2: Suppose $V(y^2 - x^3 + x) \subseteq \mathbb{A}^2 \cong U_3 \hookrightarrow \mathbb{P}^2$.



The projective closure is $\mathbb{V}(y^2 z - x^3 + xz^2)$. This meets $\mathbb{V}(z)$ in $\{[0:1:0]\}$.

Given $f \in k[x_1, \ldots, x_n]$, let $H(f) \in k[x_1, \ldots, x_{n+1}]$ be the homogeneous polynomial of the same degree.

$$H(y^2 - x^3 + x) = y^2 z - x^3 + xz^2$$

Note: $\mathbb{V}(H(f)) \cap U_{n+1} = V(f)$.

Given an ideal $I \subseteq k[x_1, \ldots, x_n]$, let $H(I) = (\{H(f) : f \in I\}) \subseteq k[x_1, \ldots, x_{n+1}]$.

Lemma: Suppose $X \subseteq \mathbb{A}^n \cong U_{n+1} \hookrightarrow \mathbb{P}^n$ is an algebraic set. The projective closure of X is $\overline{X} = \mathbb{V}(H(I(X)))$.

Proof. We have $\mathbb{V}(H(I(X))) \cap U_{n+1} = V(I(X)) = X$. So $X \subseteq \mathbb{V}(H(I(X)))$. Now show that if $Y \subseteq \mathbb{P}^n \supseteq X$, then $Y \supseteq \mathbb{V}(H(I(X)))$. We have $\mathbb{I}(Y) \subseteq \mathbb{I}(X)$. Now $\mathbb{I}(X) = H(I(X))$. Clearly, $H(I(X)) \subseteq \mathbb{I}(X)$. Suppose $F \in \mathbb{I}(X)$ homogeneous. Then $F(x_1, \ldots, x_n, 1) \in I(X)$. We can recover the homogeneous form up to powers of x_{n+1}. So $F(x_1, \ldots, x_{n+1}) = x_{n+1}^{\alpha} H(F(x_1, \ldots, x_n, 1)) \in H(I(X))$. So $\mathbb{I}(X) = H(I(X))$ and applying $\mathbb{V}$, we have

$$\mathbb{V}(\mathbb{I}(X)) = \mathbb{V}(H(I(X))) \subseteq \mathbb{V}(\mathbb{I}(Y)) = Y$$

□

Here is an overview of the following interactions between affine space and projective space:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{radical}\\ \text{homogeneous ideals}\\ \text{in } k[x_1, \ldots, x_{n+1}]\end{array}\right\} \arrow[rr, bend left, "\mathbb{V}"] \arrow[d, bend left, "\text{set } x_{n+1}=1"] & & \left\{\begin{array}{c}\text{projective algebraic}\\ \text{sets in } \mathbb{P}^n\end{array}\right\} \arrow[ll, bend left, "\mathbb{I}"] \arrow[d, bend left, "\cap U_{n+1}"] \\
\left\{\begin{array}{c}\text{ideals in}\\ k[x_1, \ldots, x_n]\end{array}\right\} \arrow[u, bend left, "H"] \arrow[rr, bend left, "V"] & & \left\{\text{algebraic sets in } \mathbb{A}^n\right\} \arrow[ll, bend left, "I"] \arrow[u, bend left, "\text{projective closure } \overline{(\cdot)}"]
\end{tikzcd}$$

Note that taking the projective closure of an algebraic set and taking its intersection with U_{n+1} gives back the algebraic set: If $X \subseteq \mathbb{A}^n$, then $X^{-} \cap U_{n+1} = X$.

Example 11.2.3: Consider $\mathbb{V}(x_{n+1}) \subseteq \mathbb{P}^n$. Then $\mathbb{V}(x_{n+1}) \cap U_{n+1} = \emptyset$. Then $\overline{\mathbb{V}(x_{n+1}) \cap U_{n+1}} = \emptyset \neq \mathbb{V}(n+1)$.

Example 11.2.4: Take the ideal $(x_{n+1}) \subseteq k[x_1, \ldots, x_{n+1}]$. Set x_{n+1} to be 1, we get: $(1) = k[x_1, \ldots, x_n]$. Then $H((1)) = (1) \neq (x_{n+1})$.

Example 11.2.5: $V(y^2 - x^3 + x) \subseteq \mathbb{A}^2$. If we homogenize to higher degree, $\mathbb{V}(y^2z^2 - x^3z + xz^3)$, then we get: $\mathbb{V}(z) \cup \mathbb{V}(y^2z - x^3 + xz^2)$.

Proposition: If $I = (f) \subseteq k[x_1, \ldots, x_n]$, then $H(I)$ is $(H(f))$.

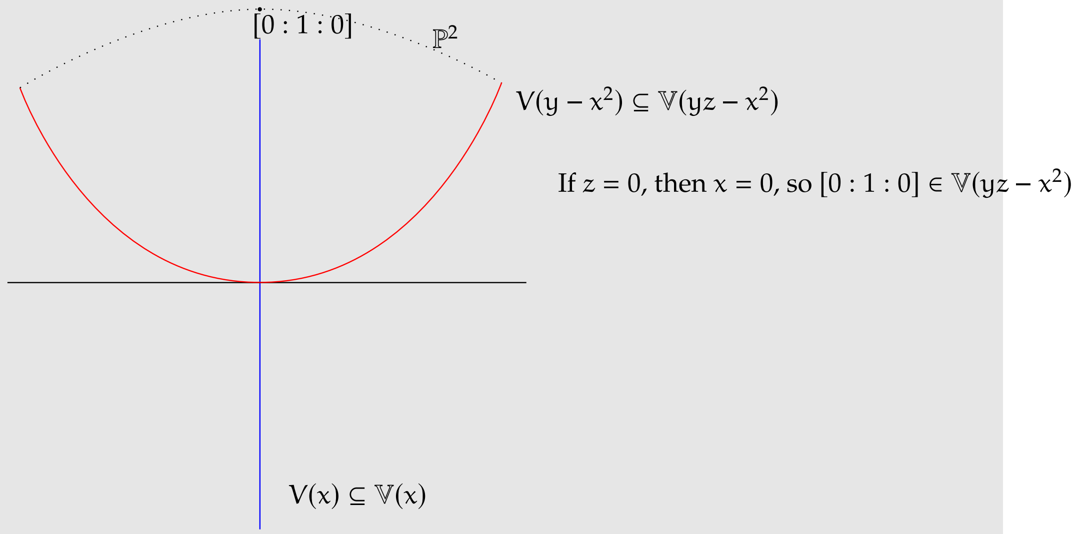
Proof. We have $(H(f)) \subseteq H(I)$. To show the other containment, consider

$$\begin{aligned}\{H(g) : g \in (f)\} &= \{H(af) : a \in k[x_1, \ldots, x_n]\} \\ &= \{H(a)H(f) : a \in k[x_1, \ldots, x_n]\} \subseteq (H(f))\end{aligned}$$

So $H(I) \subseteq (H(f))$. □

Warning: In general, if $I = (f_1, \ldots, f_r)$, then $H(I) \neq (H(f_1), \ldots, H(f_r))$.

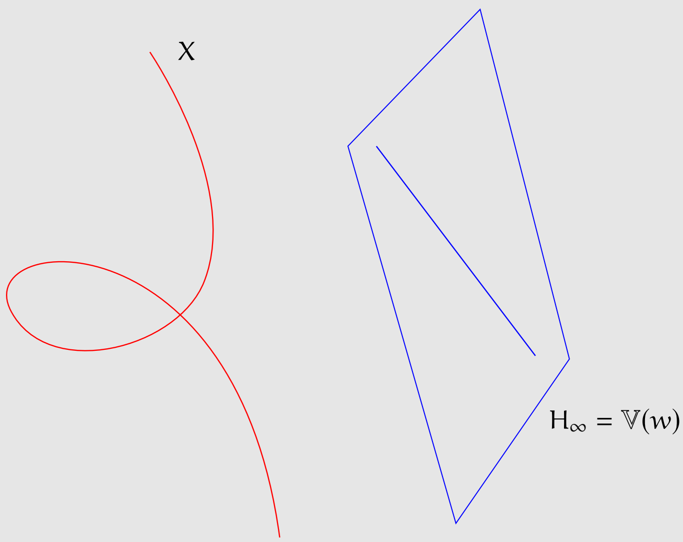
Example 11.2.6: Let $I = (y - x^2, x) \ni y$, so $y \in H(I)$. However, $(H(y - x^2), H(x)) = (yz - x^2, x) \not\ni y$. What went wrong?



Example 11.2.7: Let $X = \{(t, t^2, t^3) : t \in k\} \subseteq \mathbb{A}^3$. What is $\overline{X} \subseteq \mathbb{P}^3$?
Let $x = t, y = t^2, z = t^3$. Take:

$$I(X) = (y - x^2, z - x^3)$$

If we homogenize generators, we have $H(y - x^2) = yw - x^2$ and $H(z - x^3) = zw^2 - x^3$. Consider $\mathbb{V}(yw - x^2) \cap \mathbb{V}(zw^2 - x^3) = \mathbb{V}(w, x) \cup X \subseteq \mathbb{P}^3$.



We claim $\overline{X} = Y = \mathbb{V}(wy - x^2, xz - y^2, zw - xy)$.
Check if $w \neq 0$, then set $w = 1$ to get: $y = x^2, xz = y^2, z = xy = x^3$. So

$$Y \cap U_4 = \{[x : x^2 : x^3 : 1]\} = X \subseteq U_4$$

Now check $V \cap \mathbb{V}(w) = \mathbb{V}(w, x, y) = \{[0 : 0 : 1 : 0]\}$. Suffices to show that any projective algebraic set that contains X contains $[0 : 0 : 1 : 0]$.

Consider $X \cap U_3 = \{[t : t^2 : t^3 : 1] : t^3 \neq 0\}$. Then this is the same as:

$$\{t^{-2} : t^{-1} : 1 : t^{-3}\} = \{[s^2 : s : 1 : s^3] : s \neq 0\}$$

Suppose $F(s^2, s, 1, s^3) = 0$ for all $s \neq 0$. Then $F(s^2, s, 1, s^3) \in k[s]$ has infinitely many roots, so it is the zero polynomial. So $F(0, 0, 1, 0) = 0$. So if $\mathbb{V}(F) \subseteq X$, then $\mathbb{V}(F) \ni \{[0 : 0 : 1 : 0]\}$ so any algebraic set containing X contains $\{[0 : 0 : 1 : 0]\}$.

11.3 Homogeneous Coordinate Rings

Definition 11.3.1 **Homogeneous Coordinate Ring**

Given a projective algebraic set $X \subseteq \mathbb{P}^n$, we define the homogeneous coordinate ring to be

$$\Gamma_h(X) = \frac{k[x_1, \ldots, x_{n+1}]}{I(X)} = \frac{k[x_1, \ldots, x_{n+1}]}{I(C(X))} = \Gamma(C(X))$$

Warning: Elements of $\Gamma_k(X)$ are not functions on X. For example, take $X = \mathbb{P}^1$, $\Gamma_h(\mathbb{P}^1) = k[x, y]$, $f = x + y \in k[x, y]$ is not a function on $\mathbb{P}^1$, because $f([1 : 2]) = 3 \neq 6 = f([2 : 4])$

Definition 11.3.2 **Forms**

Suppose $I \subseteq k[x_1, \ldots, x_{n+1}]$ is a homogeneous ideal and let $\Gamma = \frac{k[x_1, \ldots, x_{n+1}]}{I}$. We say that $0 \neq f \in \Gamma$ is a form of degree d if $\exists F \in k[x_1, \ldots, x_{n+1}]$ that is homogeneous of degree d such that $\overline{F} = f \in \Gamma$.

Check that the degree is well-defined: Suppose F, G both satisfy that $\overline{F}, \overline{G} = f$. If this holds, then $F - G \in I$. If $\deg F \neq \deg G$, then since I is homogeneous, then $F, G \in I$. So this means that $\overline{G}, \overline{F} = 0$.

Proposition: Every $f \in \Gamma$ can be written uniquely as $f = f_0 + \cdots + f_d$ where f_i is a form of degree i.

Proof. Suppose $f = g_0 + \cdots + g_d$ is another representation of f with g_i's a form of degree i. Then $\exists F_i, G_i \in k[x_1, \ldots, x_{n+1}]$ that are homogeneous of degree i so that $\overline{F_i} = f_i, \overline{G_i} = g_i$. We have that

$$\sum \overline{F_i} = f = \sum \overline{G_i}$$

Then $\sum F_i - \sum G_i = \sum (F_i - G_i) \in I$. Since I is homogeneous, each $F_i - G_i \in I$ which means that $f_i = g_i$. □

11.4 Morphisms of Projective Algebraic Sets

Example 11.4.1: $\mathbb{P}^1 \to \mathbb{P}^2$ by:

$$[s, t] \mapsto [s^2 : st : t^2]$$
$$[\lambda s, \lambda t] \mapsto [\lambda^2 s^2 : \lambda^2 st : \lambda^2 t^2]$$

This morphism is well-defined because a scaling of the input gives a scaling of the output.

Definition 11.4.1 **Morphism**

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective algebraic sets. A map $\varphi : X \to Y$ is called a morphism if for every point $P \in X$, there exists an open set $U \subseteq X$ where $P \in U$, and there are homogeneous polynomials $F_1, \dots, F_{m+1}$ of the same degree such that $\varphi|_U$ agrees with the map

$$U \to \mathbb{P}^m$$
$$Q \mapsto [F_1(Q) : \cdots : F_{m+1}(Q)]$$

In the previous example, for each point $P \in \mathbb{P}^1$, take the open set $U = \mathbb{P}^1$. Then the polynomials that define the map are $F_1 = s^2$, $F_2 = st$, $F_3 = t^2$.

Let $Y = \mathbb{V}(zx - y^2)$. Then consider $\varphi : Y \to \mathbb{P}^2$ defined by

$$[x : y : z] \mapsto \begin{cases} [x : y] & \text{if } x \neq 0 (U_1 \cap Y) \\ [y : z] & \text{if } z \neq 0 (U_3 \cap Y) \end{cases}$$

This is well defined because if $[x : y : z] \in U_1 \cap U_3 \cap Y$, $x, z \neq 0$. So $y \neq 0$. So

$$[x : y] = [xy : y^2] = [xy : xz] = [y : z]$$

Chapter 12

Week 12

12.1 Morphisms Continued

Definition 12.1.1 **Isomorphism of Projective Algebraic Sets**

If X and Y are projective algebraic sets and $\varphi : X \to Y$ is a morphism we call φ an isomorphism if $\exists \psi : Y \to X$ such that $\varphi \circ \psi = id_Y, \psi \circ \varphi = id_X$ for ψ a morphism.

Example 12.1.1: $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ where:

$$\varphi([s : t]) \mapsto [s^2 : st : t^2]$$

So $\varphi : \mathbb{P}^1 \to \mathbb{V}(xz - y^2) = Y$. The inverse map $Y \to \mathbb{P}^1$ is defined by

$$[x : y : z] \mapsto \begin{cases} [x : y] & x \neq 0 \text{ on } U_1 \cap Y \\ [y : z] & z \neq 0 \text{ on } U_3 \cap Y \end{cases}$$

Why is $(U_1 \cap Y) \cup (U_3 \cap Y)$? If $x = z = 0$, then $y = 0$, so the complement of the union is empty. If $[x : y : z] \in U_1 \cap U_3 \cap Y$, then $xz \neq 0 \implies y \neq 0$

$$[x : y] = [xy : y^2] = [xy : xz] = [y : z]$$

So if an element lies in both, then the preimage is equal. It is the inverse:

$$[s : t] \mapsto [s^2 : st : t^2] \mapsto \begin{cases} [s^2 : st] = [s : t] & s \neq 0 \\ [st : t^2] = [s : t] & t \neq 0 \end{cases}$$

The other composition is also the identity on Y.

Warning: X and Y projective algebraic sets, isomorphic, do not necessarily have isomorphic $\Gamma_h(X) \cong \Gamma_h(Y)$.

In fact, $\Gamma_h(X) = \Gamma(C(X))$. So the coordinate rings are isomorphic iff the cones of $C(X) \cong C(Y)$.

We have:

$$C(\mathbb{P}^1) = \mathbb{A}^2$$
$$C(Y) = V(xz - y^2)$$

But the coordinate rings are not isomorphic:

$$\Gamma_h(\mathbb{P}^1) = k[x, y]$$
$$\Gamma_h(Y) = \frac{k[x, y, z]}{(xz - y^2)}$$

Lemma: If $\varphi : X \to Y$ is a morphism of projective algebraic sets, then φ is continuous in the Zariski topology.

Proof. Q Suppose that $Z \subseteq Y$ is a closed set, $Z = \mathbb{V}(G_1, \dots, G_r)$. We must show that $\varphi^{-1}(Z) \subseteq X$ is closed. We can write $X = \bigcup_\alpha U_\alpha$ with U_α open such that φ_{U_α} is given by $Q \mapsto [F_1^\alpha(Q), \dots, F_m^\alpha(Q)]$. Then

$$\varphi^{-1}(Z) \cap U_\alpha = \varphi_{U_\alpha}^{-1}(Z) = \mathbb{V}(G_1(F_1^\alpha, \dots, F_{m+1}^\alpha), \dots, G_r(F_1^\alpha, \dots, F_{m+1}^\alpha)) \cap U_\alpha = Z_\alpha \subseteq \mathbb{P}^n$$

Claim: $U_\alpha \backslash (\varphi^{-1}(Z) \cap U_\alpha) \subseteq X$ is an open subset. Indeed, the complement is $[U_\alpha \backslash (Z_\alpha \cap U_\alpha)]^c = (Z_\alpha \cap X) \cup U_\alpha^c$ has closed components. We have $X \backslash \varphi^{-1}(Z) = \bigcup_\alpha U_\alpha \backslash (\varphi^{-1}(Z) \cap U_\alpha)$ is a union of open sets, hence it is open. So $\varphi^{-1}(Z)$ is closed. □

Lemma: Let X be a projective algebraic set. If $X = \bigcup_{\alpha \in A} U_\alpha$ is a union of open sets, then $\exists$ a finite subset $A' \subseteq A$ so that $X = \bigcup_{\alpha \in A'} U_\alpha$. In other words, X is compact in the Zariski Topology.

Proof. Suppose for contradiction that no finite subset $A' \subseteq A$ exists. Then we can build an infinite ascending chain of open sets:

$$W_1 \subset W_1 \subset \cdots \subset X$$

Take complements in X:

$$X \supset Z_1 \supset \cdots \supset Z_1$$

Infinite descending chain of closed sets. Now apply $\mathbb{I}$:

$$\mathbb{I}(A) \subset \mathbb{I}(Z_1) \subset \cdots \subset \mathbb{I}(X) \subset k[X_1, \dots, X_{n+1}]$$

So this is an infinite ascending chain in a Noetherian ring, which is a contradiction. This also works for algebraic sets in $\mathbb{A}^n$. □

12.2 Projective Change of Coordinates

Suppose $T : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ invertible linear transformation with $T(0, \dots, 0) = (0, \dots, 0)$. Then T sends line through the origin to lines through the origin. So it induces a morphism from $\mathbb{P}^n \to \mathbb{P}^n$. Since T is invertible, there is T^{-1} which induces an isomorphism from $\mathbb{P}^n \to \mathbb{P}^n$.

We call this isomorphism $\mathbb{P}^n \to \mathbb{P}^n$ a projective change of coordinates. Note that λT defines the same map.

Definition 12.2.1 — GL_{n+1} **and** PGL_{n+1}

The group of invertible linear transformations is called GL_{n+1}. The quotient group of $GL_{n+1}/k\{I\}$ is called PGL_{n+1}, the projective general linear group.

In fact, PGL_{n+1} is the group of all automorphisms of $\mathbb{P}^n$.

Definition 12.2.2 — **Projective Equivalence**

$X, Y \subseteq \mathbb{P}^n$ are projectively equivalent if $\exists$ a projective change of coordinates $\mathbb{P}^n \to \mathbb{P}^n$ that restricts to an isomorphism $X \to Y$.

Example 12.2.1: We have $\mathbb{V}(x) \subseteq \mathbb{P}^2$ and $\mathbb{V}(y) \subseteq \mathbb{P}^2$ are projectively equivalent. The projective change of coordinates inducing the equivalence is $[x : y : z] \mapsto [y : x : z]$ with the corresponding matrix as:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: If $T : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ is an invertible linear transformation inducing change of coordinates $\mathbb{P}^n \to \mathbb{P}^n$ by $p \mapsto [T_1(p) : T_2(p) : \cdots : T_{n+1}(p)]$, then

$$T^{-1}(\mathbb{V}(F_1, \ldots, F_r)) = \mathbb{V}(F_1(T_1, \ldots, T_{n+1}), \ldots, F_r(T_1, \ldots, T_{n+1}))$$

When X, Y are projectively equivalent, $C(X) \cong C(Y)$ so there is an isomorphism $\Gamma_h(X) \cong \Gamma_h(Y)$.

12.3 Examples of Morphisms

Rational Normal Curves: A morphism $\mathbb{P}^1 \to \mathbb{P}^2$ with $[s : t] \mapsto [s^2 : st : t^2]$. Image in $\mathbb{P}^2$ is $\mathbb{V}(x_1x_3 - x_2^2)$.

Twisted Cubic Example: Consider $\mathbb{P}^1 \to \mathbb{P}^3$ with

$$[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$$

Image: $\mathbb{V}(x_1x_3 - x_2^2, x_2x_4 - x_3^2, x_1x_4 - x_2x_3)$. Consider the matrix:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$$

Veronese embedding $\nu_{1,d}$: Consider $\mathbb{P}^1 \to \mathbb{P}^d$.

$$[s : t] \mapsto [s^d : s^{d-1}t : \cdots : st^{d-1} : t^d]$$

Claim: The image is:

$$Y = \{[x_1 : \cdots : x_{d+1}] \in \mathbb{P}^d : \operatorname{rank} \begin{bmatrix} x_1 & x_2 & \cdots & x_d \\ x_2 & x_3 & \cdots & x_{d+1} \end{bmatrix} \leqslant 1\} = \mathbb{V}(2 \times 2 \text{ minors}) = \mathbb{V}(\{x_1x_j - x_{j-1}xx_{i+1}\})$$

Proof. We have that the image of the map is in Y because:

$$\begin{bmatrix} s^d & s^{d-1}t & \dots & st^{d-1} \\ s^{d-1}t & s^{d-2}t & \dots & t^d \end{bmatrix}$$

has linearly dependent rows if $t = 0$ and if $t \neq 0$, the top row is $\frac{s}{t}$ times the bottom one.

For $Y \subseteq$ Image, we have $\nu_{1,d}([0:1])$ corresponds to first row is all zero. If 1st row is non-zero the second row is a multiple of it. $x_2 = ux_1, x_3 = ux_2, \dots, x_4 = ux_3, \dots$. So $[x_1 : x_2 : \dots : x_{d+1}] = [x_1 : ux_1 : \dots : xu^d x_1]$. This is $[1 : u : u^2 : \dots : u^d]$. This is the image of $\nu_{1,d}([1:u])$. □

Veronese embedding $\nu_{2,2}\colon \mathbb{P}^2 \to \mathbb{P}^5$:

$$[x:y:z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2]$$

$\nu_{2,2}$ is an isomorphism onto its image.

Proof. The inverse morphism is:

$$[x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto \begin{cases} [x_1 : x_2 : x_3] & \text{if } x_1 \neq 0 \\ [x_2 : x_4 : x_5] & \text{if } x_4 \neq 0 \\ [x_3 : x_5 : x_6] & \text{if } x_6 \neq 0 \end{cases}$$

For the first case, since $x_1 \neq 0$, then $x \neq 0$ and we can rescale by x. If $x_4 \neq 0$, then $y \neq 0$ and you can rescale by y. Check that $\nu_{2,2}(\mathbb{P}^2) \subseteq U_1 \cup U_4 \cup U_6$. This is because at least one of x, y, z is non-zero. □

The image can be described as

$$\{[x_1 : x_2 : \dots : x_6] \in \mathbb{P}^5 : \operatorname{rank} \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \leqslant 1\} = \mathbb{V}(\{2 \times 2 \text{ minors of } M\})$$

If we restrict $\nu_{2,2}$ to $\mathbb{P}^1 = \mathbb{V}(x) \subseteq \mathbb{P}^2$, it sends

$$[0 : y : z] \mapsto [0 : 0 : 0 : y^2 : yz : x^2]$$

So it resembles the rational normal curve in $\mathbb{V}(x_1, x_2, x_3) \cong \mathbb{P}^2 \subseteq \mathbb{P}^5$.

What is the preimage $\nu_{2,2}^{-1}(\mathbb{V}(x_2 + 2x_4 - x_6))$? $\mathbb{V}(xy + 2y^2 - z^2)$

What about $\nu_{2,2}^1(\mathbb{V}(a_1x_1 + a_2x_2 + \dots + a_6x_6))$ for $[x : y : z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2]$? $\mathbb{V}(a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2)$.

As we vary over a_i, we get all degree 2 equations in $\mathbb{P}^2$. The set

$$\{\text{hyperplanes } \mathbb{V}(a_1x_1 + \dots + a_6x_6) \subseteq \mathbb{P}^5\}$$

is a copy of $\mathbb{P}^5$:

$$\{(a_1, \dots, a_6) : (a-1), \dots, a_6 \neq 0\}/(a_1, \dots, a_6) \sim (\lambda a_1, \dots, \lambda a_6)$$

which is $\{[a_1 : \dots : a_6] \in \mathbb{P}^5\}$. This copy of $\mathbb{P}^5$ is the dual projective space. Each $[a_1, \dots, a_6] \iff \mathbb{V}(a_1x^2 + a_2xy + \dots + a_6z^2)$. This $\mathbb{P}^5$ is the moduli space of plane conics.

The Veronese embedding $\nu_{2,d}$:

$$\nu_{2,d} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^{N-1}$$

where

$$[x : y : z] \mapsto [x^d : x^{d-1}y : \cdots : z^d]$$

where N is the number of degree d monomials in x, y, z.

Veronese Embedding $\nu_{n,d}$:

$$\begin{aligned} \nu_{n,d} : \mathbb{P}^n &\to \mathbb{P}^{N-1} \\ [x_1 : \cdots : x_{n+1}] &\mapsto [\cdots] \end{aligned}$$

Chapter 13

Week 13

13.1 Segre Embedding

The Segre embedding $\sigma_{1,1}$:

$$\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$[x_1 : x_2] \times [y_1 : y_2] \mapsto [x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2]$$

Well Defined:

$$[\lambda x_1 : \lambda x_2] \times [\mu y_1 : \mu y_2] \mapsto [\lambda\mu x_1y_1 : \lambda\mu x_2y_1; \lambda\mu x_2y_1 : \lambda\mu x_2y_2]$$

What equations define the image? $\mathbb{V}(z_1z_4 - z_2z_3) \supseteq \sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$. Prove:

$$\mathbb{V}(z_1z_4 - z_2z_3) = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^3 : \mathrm{rank} \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \leqslant 1\}$$

This is a determinantal variety. The surface $\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$ contains a lot of lines. For each $[a_1 : a_2] \in \mathbb{P}^1$,

$$\sigma_{1,1}([a_1 : a_2] \times \mathbb{P}^1) = \{[a_1y_1 : a_1y_2 : a_2y_1 : a_2y_2]\} = \mathbb{V}(a_2z_1 - a_1z_3, a_2z_2 - z_1z_4)$$

If we look in the chart $z_4 \neq 0$ which is U_4, then $\mathbb{V}(z_1z_4 - z_2z_3) \cap U_4 = V(z_1 - z_2z_3) \subseteq \mathbb{A}^3$. Then a projective change of coordinates on $\mathbb{P}^3$. Replace $z_1 = w_1 + w_2$, $z_4 = w_1 - w_2$, $z_2 = w_3 + w_4$, $z_3 = w_3 - w_4$. We have

$$(w_1 + w_2)(w_1 - w_2) - (w_3 + w_4)(w_3 - w_4) = w_1^2 - w_2^2 - (w_3^2 - w_4^2) = w_1^2 + w_4^2 - w_2^2 - w_3^2$$

Now look in the chart where $w_4 \neq 0$: $V(w_1^2 + 1 - w_2^2 - w_3^2)$.

Recall that the image of the twisted cubic $\nu_{1,3}$ was

$$\nu_{1,3}(\mathbb{P}^1) = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^3 : \mathrm{rank} \begin{bmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{bmatrix} \leqslant 1\} = \mathbb{V}(z_1z_3 - z_2^2, z_1z_4 - z_2z_3, z_2z_4 - z_3^2)$$

Notice that

$$\mathbb{V}(z_1z_3 - z_2^2, z_1z_4 - z_2z_3, z_2z_4 - z_3^2) \subseteq \mathbb{V}(z_1z_4 - z_2z_4) = \sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$$

What is $\sigma_{1,1}^{-1}(\mathbb{V}(z_2^2 - z_1z_3)) = \mathbb{V}((x_1y_2)^2 - (x_1y_1)(x_2y_1))$. This is homogeneous of bidegree $(2, 2)$.

$$\begin{aligned} \mathbb{V}((x_1y_2^2) - (x_1y_1)(x_2y_1)) &= \mathbb{V}(x_1(x_1y_2^2 - y_1^2x_2)) \\ &= \mathbb{V}(x_1) \cup \mathbb{V}(x_1y_2^2 - y_1^2x_2) \end{aligned}$$

The segre embedding $\sigma_{m,n}$

$$\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$
$$[x_1 : \cdots : x_{m+1}] \times [y_1 : \cdots : y_{n+1}] \mapsto [x_1y_1 : x_1y_2 : \cdots : x_iy_j : \cdots : x_{m+1}y_{n+1}]$$

13.2 Rational Functions

Let $X \subseteq \mathbb{P}^n$ be a projective variety.

Definition 13.2.1 **Homogeneous Function Field**

The homogeneous function field of X is

$$k_h(X) = \operatorname{Frac}\Gamma(X)$$

Most elements of $k_h(X)$ do not determine functions on an open subset of X. However, if we take ratios of forms of the same degree in $\Gamma(X)$, then we get a function. If $F, G \in k[x_1, \ldots, x_{n+1}]$ are homogeneous of degree d, then $\overline{F}, \overline{G} \in \Gamma(X)$ are forms of degree d. Furthermore, $\frac{\overline{F}(\lambda a_1,\ldots,\lambda a_{n+1})}{\overline{G}(\lambda a_1,\ldots,\lambda a_{n+1})} = \frac{\lambda^d(\overline{F}(a_1,\ldots,a_{n+1}))}{\lambda^d\overline{G}(a_1,\ldots,a_{n+1})}$. So $\overline{F}/\overline{G}$ defined a function on $X \backslash \mathbb{V}(G)$.

Example 13.2.1: $\frac{x_1}{x_2}$ is a rational function on $\mathbb{P}^2$ that is defined on U_2.

Definition 13.2.2 **Field of Rational Functions**

The field of rational functions on $X \subseteq \mathbb{P}^n$ is

$$k(X) = \left\{ z \in k_h(X) : z = \frac{\overline{F}}{\overline{G}} \text{ for } F, G \text{ homogeneous of the same degree}, \overline{G} \neq 0 \right\}$$

Note: $k \subseteq k(X) \subseteq k_h(X)$ by $\lambda \mapsto \frac{\lambda}{1}$ but typically $\Gamma(X) \not\subseteq k(X)$ because $f \mapsto \frac{f}{1}$.

Example 13.2.2: What is $k(\mathbb{P}^1)$? There is a map from $k(\mathbb{P}^1) \longrightarrow k(\mathbb{A}^1) = k(X)$. Think of $\mathbb{A}^1$ as U_2.

$$k(\mathbb{P}^1) \longrightarrow k(\mathbb{A}^1)$$
$$\frac{F(x,y)}{G(x,y)} \mapsto \frac{F(x,1)}{G(x,1)}$$

13.3 Local Ring

Definition 13.3.1

Let $X \subseteq \mathbb{P}^n$ be a projective variety. Let $P \in X$, $\alpha \in k(X)$. Then we say that α is defined at P if $\exists \overline{F}, \overline{G} \in \Gamma(X)$ of the same degree with $\overline{G}(P) \neq 0$ and $\alpha = \frac{\overline{F}}{\overline{G}}$.

Definition 13.3.2 **Local Ring**

The local ring of X at P is $\mathcal{O}_P(X) = \{\alpha \in k(X) : \alpha \text{ is defined at } P\}$.

If $P \in U_i$ any affine chart $U_i \subseteq \mathbb{P}^n$, then $\mathcal{O}_P(X) = \mathcal{O}_P(X \cap U_i)$. The map $\mathcal{O}_P(X) \to \mathcal{O}_P(X \cap U_i)$:

$$\frac{F(x_1, \ldots, x_{n+1})}{G(x_1, \ldots, x_{n+1})} \mapsto \frac{F(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1})}{G(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1})}$$

Example 13.3.1: Let $P = [0 : 0 : 1] \in \mathbb{P}^2$.

$$\begin{aligned}\mathcal{O}_P(\mathbb{P}^2) &= \left\{\frac{F}{G} : F, G \in k[x, y, z] \text{homogeneous, same degree, } G(0, 0, 1) \neq 0\right\} \\ &= \left\{\frac{F}{H + z^d} : \ F, H \text{ homogeneous of degree } d\right\}\end{aligned}$$

Consider $U_3 \subseteq \mathbb{P}^2$. Our point $P \in U_3 \cong \mathbb{A}^2$ is the origin $(0, 0)$. Then

$$\begin{aligned}\mathcal{O}_P(\mathbb{P}^2) &\to \mathcal{O}_{(0,0)}(\mathbb{A}^2) \\ \frac{F}{H + z^d} &\mapsto \frac{F(x, y, 1)}{H(x, y, 1) + 1}\end{aligned}$$

(Injective) $\frac{F}{H+z^d}$ is in the kernel iff $F(x, y, 1) = 0$ iff $F \in (z - 1)$. But F is homogeneous, so this happens if $F = 0$.

(Surjective) Given any $f/g \in \mathcal{O}_{(0,0)}(\mathbb{A}^2)$, consider $H(f)$ and $H(g)$. If they are of the same degree, then f/g is the image of $H(f)/H(g)$. If they are not of the same degree, we can multiply one of them with powers of z until they are of the same degree.

Alternative description of $k(X)$: Let $X \subseteq \mathbb{P}^n$ be a projective variety.

$S = \{(U, \alpha) : U \subseteq X$ open $\alpha : U \to k$ st $\exists F, G \in k[x_1, \ldots, x_{n+1}]$ homogeneous of same degree, $\alpha(P) = F(P),$

Where $(U, \alpha) \sim (U', \alpha')$ if $\alpha(P) = \alpha'(P) \forall P \in U \cap U'$.

We can make S into a ring by

$$[(U, \alpha)] + [(U', \alpha')] = [(U \cap U', \alpha |_{U \cap U'} + \alpha' |_{U \cap U'})]$$

and multiplication by:

$$[(U, \alpha)][(U', \alpha')] = [(U \cap U'), \alpha |_{U \cap U'} \alpha' |_{U \cap U'}]$$

The inverse is defined as: If (U, α) where $\alpha \neq 0$, then $\alpha = \frac{F}{G}$, $F \neq 0$ homogeneous polynomials F, G. Then $[(U \backslash \mathbb{V}(F), \frac{G}{F})][(U, \frac{F}{G})] = [(U \backslash \mathbb{V}(F), 1)]$. And $[(U \backslash \mathbb{V}(F), 1)] = [(X, 1)]$.

Example 13.3.2: $X = \mathbb{P}^2, (U_1, \frac{x_2}{x_1}) + (U_2, \frac{x_1}{x_3})$ is $(U_1 \cap U_3, \frac{x_2 x_3 + x_1^2}{x_1 x_3})$. The inverse of $[(U_1, \frac{x_2}{x_1})]$ is $[(U_1 \cap U_2, \frac{x_1}{x_2})] = [(U_2, \frac{x_1}{x_2})]$.

There is a map $k(X) \to S$ where $\alpha \to (U, \alpha)$ where U is the set where α is defined. This map is surjective.

Proposition: If $\alpha, \alpha' \in k(X)$, and $(U, \alpha) \sim (U', \alpha')$, then $\alpha = \alpha'$.

Proof. Suppose $\alpha = \frac{\overline{F}}{\overline{G}}$, $\alpha' = \frac{\overline{F'}}{a\overline{G'}}$. Since $(U, \alpha) \sim (U', \alpha')$, This means that $\frac{\overline{F}}{\overline{G}}(P) = \frac{\overline{F'}}{\overline{G'}}(P)$ for all $P \in U \cap U'$. Clearing denominators: $(\overline{FG'}) - \overline{F'G}(P) = 0$. So $(FG' - F'G)(P) = 0$ because they differ by a polynomial that vanishes on X.

$\mathbb{V}(FG' - F'G) \supseteq U \cap U'$ means that $\mathbb{V}(FG' - F'G = X$. $FG' - F'G \in \mathbb{I}(X)$ So $\overline{FG'} - \overline{F'G} = 0$ □

Let X, Y be irreducible algebraic sets. Suppose that $\varphi : X \to Y$ is dominant, $\overline{\varphi(X)} = Y$. Then there is a well-defined pull-back map $\varphi^* : k(Y) \to k(X)$. If $\varphi : X \to Y$ is dominant, then for any $U \subseteq Y$ open, we have $\varphi^{-1}(U) \subseteq X$ is non-empty.

So we can define the pullback by

$$(U, \alpha) \mapsto (\varphi^{-1}(U), \alpha \circ \varphi)$$

Algebraically, suppose $\varphi : X \to Y$ is given on some open subset $W \subseteq X$ by

$$P \to [f_1(P) : \cdots : f_{m+1}(P)]$$

for f_i homogeneous of the same degree. Then we can define the pullback $\varphi^* : k(Y) \to k(X)$:

$$(U, \alpha) \mapsto (\varphi^{-1}(U) \cap W, \alpha \circ \varphi)$$

Note: If $\alpha = \frac{F}{G}$, then $(\alpha \circ \varphi)(P) = F(f_1(P), \ldots, f_{m+1}(P))/G(f_1(P), \ldots, f_{m+1}(P))$.

Lemma: If $\varphi : X \to Y$ is an isomorphism, then $\varphi^* : k(Y) \to k(X)$ is also an isomorphism.

Proof. Let $\psi : Y \to X$ be the inverse morphism. Check that $\psi^* : k(X) \to k(Y)$ is the inverse. If we have $(U, \alpha) \in k(Y)$, we have

$$(U, \alpha) \xrightarrow{\varphi^*} (U', \alpha \circ \varphi) \xrightarrow{\psi^*} (U'', \alpha \circ \varphi \circ \psi) \sim (U, \alpha)$$

Above, $U' \subseteq \varphi^{-1}(U)$ on which φ is described by polynomial and $U'' \subseteq \psi^{-1}(U')$ where ψ is described by polynomials. □

Example 13.3.3: $\mathbb{P}^1 \to \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2$ by

$$[s : t] \mapsto [s^2 : st : t^2]$$

$k(\mathbb{P}^1) = k(\frac{s}{t}), k(\mathbb{V}(xz - y^2)) \cong k(V(\frac{x}{z} - (\frac{y}{z})^2))$. With

$$\frac{F(x, y, z)}{G(x, y, z)} \mapsto \frac{F(\frac{x}{z}, \frac{y}{z}, 1)}{G(\frac{x}{z}, \frac{y}{z}, 1)}$$

Then we have $\operatorname{Frac} \frac{k[\frac{x}{z}, \frac{y}{z}]}{(\frac{x}{z} - (\frac{y}{z})^2)} = \operatorname{Frac} k[\frac{y}{z}] = k(\frac{y}{z})$. The pullback sends $\frac{y}{z} \mapsto \frac{st}{t^2} = \frac{s}{t}$.

Definition 13.3.3 **Local Ring**

Given $X \subseteq \mathbb{P}^n$, the local ring of X at $P \in X$ is the subring $\mathcal{O}_P(X) \subseteq k(X)$ of functions defined at P. Or just $\{(U, \alpha) : P \in U\}$.

If $P \in U_i$ an affine chart, then $\mathcal{O}_P(X) = \mathcal{O}_P(X \cap U_i)$.

Example 13.3.4: Let $P = [1:0:1] \in \mathbb{P}^2$. What is the tangent line to $C = \mathbb{V}(yz^2 - x^3 + x^2z)$ at P? We have $P \in U_3$, and $C \cap U_3 = V(y - x^3 + x^2) \subseteq \mathbb{A}^2$.Take f_x, f_y:

$$\begin{aligned} f_x &= -3x^2 + 2x & f_x(P) &= -1 \\ f_y &= 1 & f_y(P) &= 1 \end{aligned}$$

Tangent line:

$$T_P(C \cap U_3) = V(-(x-1) + y) = V(y - x + 1) \subseteq U_3$$

And

$$\mathbb{T}_P(C) = \mathbb{V}(y - x + z) \subseteq \mathbb{P}^2$$

If $P \in U_1$, then we have $C \cap U_1 = V(yz^2 - 1 + z)$. Then

$$\begin{aligned} f_y &= z^2 & f_y(P) &= 1 \\ f_z &= 2yz + 1 & f_z &= 1 \end{aligned}$$

Then

$$T_P(C \cap U_1) = V(y + z - 1) \subseteq U_1$$

so

$$\mathbb{T}_P(C \cap U_1) = \mathbb{V}(y + z - x)$$

Chapter 14

Week 14

Definition 14.0.1 **Projective Tangent Space**

Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set and let $P \in X$, where $P \in U_i$. The projective tangent space to X at P to be the projective closure of $T_P(X \cap U_i)$. This is denoted as $\mathbb{T}_P(X)$.

The projective tangent cone to X at P is the projective closure of $TC_P(X \cap U_i)$. It is denoted $\mathbb{T}C_P(X)$.

Example 14.0.1: What is the projective tangent cone to $\mathbb{V}(x^3yz + 2x^2yz^2 + y^2x^3) \subseteq \mathbb{P}^2$ at $P = [1 : 0 : 0]$? Claim: $\mathbb{T}C_P = \mathbb{V}(yz + y^2)$.

$$\mathbb{V}(yz + 2yz^2 + y^2x^3) \cap U_1 = V(yz + 2yz^2 + y^2) \subseteq U_1$$
$$TC_P = V(yz + y^2)$$

Definition 14.0.2 **Singular**

Given $P \in U_i$, we say X is singular at P if $X \cap U_i$ is singular at P.

Definition 14.0.3 **Multiplicity**

The multiplicity of a homogeneous polynomial $F \in k[x, y, z]$ at $P \in U_i$ is the multiplicity of f at P where f is the dehomogenization of F with respect to the i-th coordinate.

Example 14.0.2: What is the multiplicity of $xz^2 + y^2z$ at $[1 : 1 : 0]$?

In U_2, dehomogenize $f = z^2x + z$. Want to find $\mathrm{mult}_{(1,0)}(f)$. Let $x' = x - 1$ so that $x' = 0$ iff $x = 1$. So $x = x' + 1$. We have:

$$f = z^2(x' + 1) + z$$

So $TC_P = V(z) \subseteq U_2$. Take the closure $\mathbb{T}C_{[1:1:0]} = \mathbb{V}(z)$.

Intersection Multiplicity

Definition 14.0.4 Let $F, G \in k[x, y, z]$ be homogeneous polynomials and let $P \in U_i$. Let f, g be the dehomogenizations of F, G with respect to the i-th coordinate. Then,

$$I_P(F, G) = I_P(f, g)$$

Proof. Independent of Affine Chart: Suppose that P is in U_1 and U_2. In U_1, we have that $I_P(F, G)$ is $\dim_k(\frac{\mathcal{O}_P(\mathbb{A}^2)}{(F(1,y,z),G(1,y,z))})$. Suppose $\deg F = m$, $\deg G = n$. We claim that this equals $\dim_k(\frac{\mathcal{O}_P(\mathbb{P}^2)}{(\frac{F}{x^m},\frac{G}{x^n})})$. Indeed, there is an isomorphism $\mathcal{O}_P(\mathbb{P}^2) \to \mathcal{O}_P(\mathbb{A}^2)$:

$$\frac{a}{b} \mapsto \frac{a(1, y, z)}{b(1, y, z)}$$

This gives an isomorphism of the above two quotients by $(\frac{F}{x^m}, \frac{G}{x^n}) \mapsto F(1, y, z), G(1, y, z)$. We have the same argument switching the roles of x, y we get that

$$I_P(F, G) = \dim_k\left(\frac{\mathcal{O}_:(\mathbb{P}^2)}{\left(\frac{F}{y^m}, \frac{G}{y^n}\right)}\right)$$

Since $P \in U_1 \cap U_2$, it follows that $\frac{x}{y} \in \mathcal{O}_P(\mathbb{P}^2)$ is a unit. Then

$$\left(\frac{F}{x^m}, \frac{G}{x^n}\right) = \left(\frac{x^m F}{x^m y^m}, \frac{x^n G}{x^n y^n}\right) = \left(\frac{F}{y^m}, \frac{G}{y^n}\right)$$

□

Bezout's Theorem

Theorem 14.0.1 Let k be algebraically closed and suppose that $F, G \in k[x, y, z]$ homogeneous polynomials of degree m, n. If $\mathbb{V}(F, G)$ is a finite set, then

$$\sum_{P \in \mathbb{P}^2} I_P(F, G) = mn$$

Note: $\mathbb{V}(F, G)$ is finite iff F, G have no common factors.

Proof. (Setup) Since $\mathbb{V}(F, G)$ is a finite set of points, $\exists$ a change of coordinates so that none of the points lie on $\mathbb{V}(z)$. Let $f = F(x, y, 1)$ and $g = G(x, y, 1)$ be the dehomogenizations. Then $\mathbb{V}(F, G) \cap U_3 = V(f, g) = \mathbb{V}(F, G)$. It follows that

$$\sum_{P \in \mathbb{P}^2} I_P(F, G) = \sum_{P \in \mathbb{A}^2 \cong U_3} I_P(f, g)$$

Lemma: $\sum_{P \in \mathbb{A}^2} I_P(f, g) = \dim_k\left(\frac{k[x,y]}{(f,g)}\right)$:

Proposition 6 in 2.9 says that there is an isomorphism of rings

$$k[x, y]/(f, g) \cong \bigoplus_{P_i \in V(f,g)} \frac{\mathcal{O}_{P_i}(\mathbb{A}^2)}{\left(\frac{f}{1}, \frac{g}{1}\right)}$$

Let $\Gamma = \frac{k[x,y,z]}{(F,G)}$. Because we quotient by a homogeneous ideal, Γ has the added structure of forms of degree d. Let Γ_d be the vector space of forms of degree d.

Claim: When $d \geqslant m + n$, we have

(a) $\dim_k(\Gamma_d) = mn$

(b) $\dim_k(\Gamma_d) = \dim_k \frac{k[x,y]}{(f,g)}$

Let $R = k[x, y, z]$. Let R_d be the vector space of homogeneous polynomials of degree d. Let $\pi : R \to \Gamma$ be the quotient map. So $\ker \pi = (F, G) = \{AF + BG : A, B \in k[x, y, z]\}$. Let $\varphi : R \times R \to R$ defined by $(A, B) \mapsto AF + BG$. We have that $\Im\varphi = \ker \pi$. What is $\ker \varphi$. If $AF + BG = 0$, $AF = -BG$, so $F \bigm| BG$. But F, G have no common factor, so $F \bigm| B$. Similarly, $G \bigm| A$. Moreover, if $C = \frac{A}{G}$, then $\frac{B}{F} = \frac{-A}{G} = -C$. Hence

$$\ker \varphi = \{(A, B) : A = CG, B = -CF\}$$

In other words, $\ker \varphi = \Im\psi : R \to R \times R$ by $C \mapsto (CG, -CF)$. $\square$

Chapter 15

Week 15

Last Lecture: Recall Bezout's Theorem: Let $F, G \in k[x, y, z]$ be homogeneous of degrees m and n. Suppose that $\mathbb{V}(F, G)$ is finite. Then

$$\sum_{p \in \mathbb{P}^2} I_P(F, G) = mn$$

We also defined $I_P(F, G)$ to be $I_P(f, g)$ when $P \in U_i$ and f, g are dehomogenized of F, G on i-th coordinate.

Want to find $\sum_{p \in \mathbb{A}^2_{\mathbb{C}}} I_P(ax + by + c, y - x^2)$. This is equal to $\sum_{P \in \mathbb{P}^2} I_P(ax + by + cz, yz - x^2) - \sum_{p \in \mathbb{V}(z)} I_P(ax + by + cz, yz - x^2)$. We have

$$= \begin{cases} \infty & \text{if } a = b = c = 0 \\ 0 & \text{if } a = b = 0, c \neq 0 \\ 1 & \text{if } b = 0, a \neq 0 \\ 2 & \text{if } b \neq 0 \end{cases}$$

We have $\mathbb{V}(yz - x^2) \cap \mathbb{V}(z) = \mathbb{V}(x, z) = \{[0 : 1 : 0]\}$. We have that $[0 : 1 : 0] \in \mathbb{V}(ax + by + cz)$ iff $b = 0$. We compute that:

$$I_{[0:1:0]}(ax + cz, yz - x^2) = I_P(ax + cz, z - x^2) = \begin{cases} 1 & \text{if } b = 0, a \neq 0 \\ 2 & \text{if } b = 0, a = 0 \end{cases}$$

Proof. Following notation from before, let $R = k[x, y, z]$ and $\Gamma = \frac{R}{(F,G)}$. Let R_d be the vector space of homogeneous polynomials of degree d. Let Γ_d be the vector space of forms of degree d in Γ. Last class, it was needed to be shown that for d large enough,

- $\dim \Gamma_d = mn$
- $\dim \Gamma_d = \dim_k \left(\frac{k[x,y]}{(f,g)} \right)$. This holds when $\mathbb{V}(F, G) \cap \mathbb{V}(z) = \emptyset$.

(Part a) Consider the following sequence of maps:

$$R \xrightarrow{\psi} R \times R \xrightarrow{\varphi} R \xrightarrow{\pi} \Gamma \longrightarrow 0$$

$$(A, B) \xrightarrow{mapsto} AF + BG$$

$$C \xrightarrow{mapsto} (CG, -CF)$$

We have $\ker \varphi = \Im\psi$, $\Im\varphi = \ker \pi$ and $\ker \psi = 0$. For forms in degree d:

$$0 \longrightarrow R_{d-m-n} \xrightarrow{\psi} R_{d-m} \times R_{d-n} \xrightarrow{\varphi} R_d \xrightarrow{\pi} \Gamma_d \longrightarrow 0$$

Recall that for any linear map of vector spaces, $V \to W$, we know that $\dim(\ker T) + \dim(\Im T) = \dim V$. Putting this all together:

We have:

$$\begin{aligned}
\dim \Gamma_d &= \dim \Im\pi \\
&= \dim R_d - \dim \ker \pi \\
&= \dim R_d - \dim \Im\varphi \\
&= \dim R_d - (\dim R_{d-m} \times R_{d-n} - \dim \ker \varphi) \\
&= \dim R_d - (\dim R_{d-m} \times R_{d-n} - \dim \Im\psi) \\
&= \dim R_d - (\dim R_{d-m} \times R_{d-n} - (\dim R_{d-m-n} - \dim \ker \psi)) \\
&= \binom{d+2}{2} - \binom{d-m+2}{2} - \binom{d-n+2}{2} + \binom{d-m-n+2}{2} \\
&= mn
\end{aligned}$$

(Part b) Show that for $d \geqslant m + n$, a basis for Γ_d dehomogenizes to form a basis for $k[x, y]/(f, g)$.

Lemma: For $d \geqslant m + n$, the map $\alpha : \Gamma_d \to \Gamma_{d+1}$ defined by

$$\alpha(\overline{H}) = \overline{z}\overline{H}$$

is an isomorphism of vector spaces.

(Proof of Lemma) α is a linear map between vector spaces of the same dimension. So it is enough to show that $\ker \alpha = 0$. If $\alpha(\overline{H}) = 0$, then $\overline{H} = 0$. So $\overline{z}\overline{H} = 0$ and $zH \in (F, G)$. So $zH = AF + BG$ for homogeneous polynomials $A, B \in k[x, y, z]$. Plug in $z = 0$:

$$0 = A(x, y, 0)F(x, y, 0) + B(x, y, 0)G(x, y, 0) = A_0F_0 + B_0G_0$$

Recall that $\mathbb{V}(F, G) \cap \mathbb{V}(z) = \emptyset$. Then F_0, G_0 have no common factor. Then $-A_0F_0 = B_0G_0$. So $F_0 \mid B_0$ and $G_0 \mid A_0$. If $C = \frac{B_0}{F_0} = \frac{-A_0}{G_0}$, we have

$$B_0 = CF_0, A_0 = -CG_0$$

Let $A_1 = A + CG$ and $B_1 = CF$. So when we set $z = 0$, $A_1 = B_1 = 0$. So $A_1 = zA'$, $B_1 = zB'$. Consider $A_1F + B_1G = AF + CGF + BG - CFG = AF + BG = zH$. So $zA'F = zB'G = zH$, we have $k[x, y, z]$ is a domain, we have $H = A'F + B'G \in (F, G)$.

Final Step: Let $A_1, \ldots, A_{mn} \in k[x, y, z]$ be homogeneous of degree d so that $\overline{A_1}, \ldots, \overline{A_{mn}}$ is a basis for Γ_d. Show that

$$\{\overline{A_i(x, y, 1)}\}$$

forms a basis for $\frac{k[x,y]}{(f,g)}$. By the lemma, $\cdot z : \Gamma_d \to \Gamma_{d+1}$ is an isomorphism. We have

$$\{\overline{z^r A_i}\}$$

is a basis for Γ_{d+r} for all $r \geqslant 0$. Let $a_i = \overline{A_i(x, y, 1)}$.

- a_i span $\frac{k[x,y]}{(f,g)}$. Suppose that $\overline{h} \in k[x,y]/(f,g)$ with $h \in k[x,y]$. We can homogenize to $H(h)$. For some $N >> 0$, $z^N H(h)$ has degree $d+r$, $r \geqslant 0$. So we have $z^N(H(h)) = \sum \lambda_i z^r A_i + BF + CG$. Set $z = 1$, we have $h = \sum \lambda_i A_i(x,y,1) + B(x,y,1)f + C(x,y,1)f$ which shows that $\overline{h} \in \mathrm{Span}\{a_i\}$.
- Linear Independent.

□

Theorem 15.0.1 **Bezout's Theorem in $\mathbb{P}^n$**

Let $F_1, \ldots, F_n \in k[x_1, \ldots, x_{n+1}]$ be homogeneous of degrees $d_1, \ldots, d_n$. If $\mathbb{V}(F_1, \ldots, F_n)$ is finite, then

$$\sum_{P \in \mathbb{P}^n} I_P(F_1, \ldots, F_n) = d_1 d_2 \cdots d_n$$

15.1 Circles of Apollonius

Question: Given 3 fixed circles, how many circles are tangent to all 3?

Definition 15.1.1

Given 2 smooth curves C, C', we say that C and C' are tangent at P if $P \in C$, $P \in C'$ and $T_P(C) = T_P(C')$. Equivalently, $I_P(C, C') \geqslant 2$. If the tangent lines are distinct, then $I_P(C, C') = \mathrm{mult}_P(C)\,\mathrm{mult}_P(C') = 1$.

Typically, two circles meet in 2 points or not at all. Suppose we have a circle $C : (x - x_0)^2 + (y - y_0)^2 - r^2 = 0$. The projective closure:

$$\overline{C} : (x - x_0 z)^2 + (y - y_0 z)^2 - r^2 z^2 = 0$$

And $\overline{C} \cap \mathbb{V}(z) = \mathbb{V}(z, x^2 + y^2) = \{[1 : i : 0], [1 : -i : 0]\}$.

C, C' are tangent iff they meet in a single point (mult 2). There are also concentric circles, which do not meet in $\mathbb{A}^2_{\mathbb{C}}$.

Moduli Spaces: We saw that the moduli space of conics:

$$[a : b : c : d : e : f] \iff \mathbb{V}(ax^2 + bxy + cxz + dy^2 + eyz + fz^2) \subseteq \mathbb{P}^2$$

Definition 15.1.2

Define the moduli space of complex projective circles as $\mathbb{V}(b, a - d) \subseteq \mathbb{P}^5$.

This is isomorphic to $\mathbb{P}^3$:

$$[a : c : e : f] \iff \mathbb{V}(a(x^2 + y^2) + cxz + eyz + fz^2) \subseteq \mathbb{P}^2$$

As $a \longrightarrow 0$, we get a line. If $a \neq 0$, we can rescale to make $a = 1$. Now compare it with

$$(x - x_0 z)^2 + (y - y_0 z)^2 - r^2 z^2$$

We have

$$\frac{c}{a} = -2x_0$$
$$\frac{e}{a} = -2y_0$$
$$\frac{f}{a} = x_0^2 + y_0^2 - r^2$$

and

$$\begin{aligned}
x_0 &= \frac{-c}{2a} \\
y_0 &= \frac{-e}{2a} \\
r^2 &= x_0^2 + y_0^2 - \frac{f}{a} \\
&= \frac{1}{4}\left(\left(\frac{c}{a}\right)^2 + \left(\frac{e}{a}\right)^2\right) - \frac{f}{a}
\end{aligned}$$

After a projective change of coordinates,

$$\begin{aligned}
x &\mapsto x - x_0 z \\
y &\mapsto y - y_0 z \\
z &\mapsto r_0 z
\end{aligned}$$

We can bring the first fixed circle to the unit circle.

$$V(x^2 + y^2 - 1) \to \mathbb{V}(x^2 + y^2 - z^2) \iff [1 : 0 : 0 : -1]$$

When is a circle with center (α, β) and radius r tangent to the unit circle? Using the Pythagorean theorem,

$$\alpha^2 + \beta^2 = (1 + r)^2 = r^2 + 2r + 1$$

so

$$\alpha^2 + \beta^2 - r^2 = 2r + 1$$

If the circle is internally tangent, we have

$$\alpha^2 + \beta^2 = (1 - r)^2 = r^2 - 2r + 1$$

so

$$\alpha^2 + \beta^2 - r^2 = -2r + 1$$

Recall that the LHS is $\frac{f}{a}$, so we need either $\frac{f}{a} - 1 - 2r = 0$ or $\frac{f}{a} - 1 + 2r = 0$. Take the product

$$(\frac{f}{a} - 1 - 2r)(\frac{f}{a} - 1 + 2r) = \left(\frac{f}{a} - 1\right)^2 - 4r^2$$

Plug in

$$\frac{1}{4}\left(\left(\frac{c}{a}\right)^2 + \left(\frac{e}{a}\right)^2\right) - \frac{f}{a} = r^2$$

We get

$$\left(\frac{f}{a} - 1\right)^2 - \left(\left(\frac{c}{a}\right)^2 + \left(\frac{e}{a}\right)^2 - \frac{4f}{a}\right)$$

So

$$\left(\frac{f}{a} + 1\right)^2 - \left(\left(\frac{c}{a}\right)^2 + \left(\frac{e}{a}\right)^2\right) = 0$$

Take the projective closure: $(f + a)^2 - (c^2 + e^2) = 0$ in $\mathbb{P}^3$. We have the intersection of three cones in $\mathbb{P}^3$, and if this intersection is finite, then the number of intersections is $d_1 \cdot d_2 \cdot d_3 = 8$.