Math143Hw12

Trustin Nguyen

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Exercise 1: Segre embeddings.

(a) Let $\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the morphism given by

$$[x_1 : x_2] \times [y_1 : y_2] \mapsto [x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2]$$

Let $[z_1 : \cdots : z_4]$ be the coordinates on $\mathbb{P}^3$. Prove that $\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{V}(z_1z_4 - z_2z_3)$.

Proof. We have $\Im\sigma_{1,1} \subseteq \mathbb{V}(z_1z_4 - z_2z_3)$ because

$$\begin{gathered}[x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2] \in \Im\sigma_{1,1} \\ x_1y_1x_2y_2 - x_1y_2x_2y_1 = 0\end{gathered}$$

Now for the other containment, we need that $\mathbb{V}(z_1z_4 - z_2z_3) \subseteq \Im\sigma_{1,1}$. Suppose that $[z_1 : z_2 : z_3 : z_4] \in \mathbb{V}(z_1z_4 - z_2z_3)$. Then we have two cases:

- $z_1 = 0$. Then $z_2z_3 = 0$ and either z_2, z_3 is 0. If $z_2 = 0$, we have:

$$\begin{aligned}\sigma_{1,1}([0 : x_2] \times [y_1 : y_2]) &= [0 : 0 : y_1 : y_2] \\ \sigma_{1,1}([0 : 1] \times [z_3 : z_4]) &= [0 : 0 : z_3 : z_4]\end{aligned}$$

  If $z_3 = 0$, we have

$$\begin{aligned}\sigma_{1,1}([x_1 : x_2] \times [0 : y_2]) &= [0 : x_1y_2 : 0 : x_2y_2] \\ \sigma_{1,1}([z_2 : z_4] \times [0 : 1]) &= [0 : z_2 : 0 : z_4]\end{aligned}$$

  and if $z_2, z_3 = 0$,

$$\begin{aligned}\sigma_{1,1}([0 : x_2] \times [0 : y_2]) &= [0 : 0 : 0 : x_2y_2] \\ \sigma_{1,1}([0 : z_4] \times [0 : 1]) &= [0 : 0 : 0 : z_4]\end{aligned}$$

  so we have that in all cases, there is an element in the preimage that gets mapped to the element in $\mathbb{V}(z_1z_4 - z_2z_3)$.

- If $z_1 \neq 0$, we have

$$z_1z_4 - z_2z_3 = 0 \implies z_4 = \frac{z_2z_3}{z_1}$$

  Now if $[z_1 : z_2 : z_3 : z_4] \in \mathbb{V}(z_1z_4 - z_2z_3)$, then:

$$\begin{aligned}[z_1 : z_2 : z_3 : z_4] &= [1 : \frac{z_2}{z_1} : \frac{z_3}{z_1} : \frac{z_4}{z_1}] \\ &= [1 : \frac{z_2}{z_1} : \frac{z_3}{z_1} : \frac{z_2z_3}{z_1^2}] \\ &= \sigma_{1,1}([1 : \frac{z_3}{z_1}] \times [1 : \frac{z_2}{z_1}])\end{aligned}$$

  which completes the proof.

□

(b) Let $\sigma_{1,2} : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$ be the morphism given by

$$[x_1 : x_2] \times [y_1 : y_2 : y_3] \mapsto [x_1y_1 : x_1y_2 : x_1y_3 : x_2y_1 : x_2y_2 : x_2y_3]$$

Let $[z_1 : \cdots : z_6]$ be the coordinates on $\mathbb{P}^5$. Find a matrix M whose entries are polynomials in z_i and an integer k so that $\sigma_{1,2}(\mathbb{P}^1 \times \mathbb{P}^2) \subseteq \mathbb{P}^5$ is the set of points where $\mathrm{rank}\, M \leqslant k$. Prove that $\sigma_{1,2}(\mathbb{P}^1 \times \mathbb{P}^2) = \{[z_1 : \cdots : z_6] : \mathrm{rank}\, M \leqslant k\}$ for your chose M and k. (This implies that $\sigma_{1,2}(\mathbb{P}^1 \times \mathbb{P}^2)$ is the vanishing of the $(k+1) \times (k+1)$ minors of M.)

Proof. The matrix is

$$M = \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{bmatrix}$$

Let $k = 1$. Then what is to be proved is that:

$$\sigma_{1,2}(\mathbb{P}^1 \times \mathbb{P}^2) = \left\{ [z_1 : z_2 : \cdots : z_6] : \mathrm{rank}\left(\begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{bmatrix} \right) \leqslant 1 \right\} = J$$

($\mathfrak{I}\sigma_{1,2} \subseteq J$) If we have a point

$$[x_1y_1 : x_1y_2 : x_1y_3 : x_2y_1 : x_2y_2 : x_2y_3]$$

in the image, then we check that:

$$\begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \end{bmatrix}$$

has rank $\leqslant 1$. This is true because, either $x_1, x_2 \neq 0$, we can divide wlog the top row by x_1 and multiply by x_2 to get a rank of $\leqslant 1$.

($J \subseteq \mathfrak{I}\sigma_{1,2}$) The rank cannot be 0 because there is no origin in $\mathbb{P}^5$. Then if we have rank 1, one row is a scalar multiple of the other. So we have the set of points in J as

$$[z_1 : z_2 : z_3 : rz_1 : rz_2 : rz_3]$$

where z_1, z_2, z_3, r not all 0. But this is just in the image of $\sigma_{1,2}$ as

$$\sigma_{1,2}([1 : r] \times [z_1 : z_2 : z_3]) = [z_1 : z_2 : z_3 : rz_1 : rz_2 : rz_3]$$

which completes the proof. □

(c) (Optional) Do you see how to generalize this to $\sigma_{m,n}$?

Proof. In general, for the mapping:

$$\sigma_{m,n}([x_1 : x_2 : \cdots : x_m] \times [y_1 : y_2 : \cdots : y_n]) = [z_1 : z_2 : \cdots : z_{mn}]$$

we will have a matrix:

$$\begin{bmatrix} z_1 & z_2 & \cdots & z_n \\ z_{n+1} & z_{n+2} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{(m-1)n+1} & z_{(m-1)n+2} & \cdots & z_{mn} \end{bmatrix}$$

and the image will be given by when the rank is 1. □

Exercise 2: Function fields.

(a) Prove from the definition that the map $k(\mathbb{P}^1) \to k(x)$ defined by $F/G \mapsto F(x,1)/G(x,1)$ is an isomorphism of fields. (Check that it is one-to-one and onto.)

Proof. (Injectivity) Suppose that $F/G \mapsto \frac{F(x,1)}{G(x,1)} = 0$. Then $F(x,1) = 0$. We also know that $F \in \Gamma(\mathbb{P}^1) = k[x,y]$. So we have that

$$F(x,y) = a_0x^d + a_1x^{d-1}y + a_2x^{d-2}y^2 + \cdots + a_{d-1}xy^{d-1} + a_dxy^d$$

And therefore,

$$F(x,1) = a_0x^d + a_1x^{d-1} + \cdots + a_d = 0$$

and all $a_i = 0$. So when we rehomogenize, all the coefficients are still 0 and $F = 0$ so it is injective.

(Surjectivity) Suppose that

$$\frac{a_0 + a_1x + \cdots + a_dx^d}{b_0 + b_1x + \cdots + b_ex^e} \in k(x)$$

We can homogenize the denominator and numerator:

$$F'(x,y) = a_0y^d + a_1xy^{d-1} + \cdots + a_dx^d$$

and

$$G'(x,y) = b_0y^e + b_1xy^{e-1} + \cdots + b_ex^e$$

If the degree of G' is greater than that of F', we just multiply F' by y^{e-d}. So $y^{e-d}F(x,y)$ has the same degree as G'. Then

$$\varphi\left(\frac{y^{e-d}F'(x,y)}{G'(x,y)}\right) = \frac{F'(x,1)}{G'(x,1)} = \frac{a_0 + a_1x + \cdots + a_dx^d}{b_0 + b_1x + \cdots + b_ex^e}$$

On the other hand if the degree of G' is less than that of F', then we can multiply $G'(x,y)$ by y^{d-e} to get:

$$\varphi\left(\frac{F'(x,y)}{y^{d-e}G'(x,y)}\right) = \frac{F'(x,y)}{G'(x,y)} = \frac{a_0 + a_1x + \cdots + a_dx^d}{b_0 + b_1x + \cdots + b_ex^e}$$

So we have an element of the preimage. □

Optional: generalize this to an isomorphism $k(\mathbb{P}^n) \to k(x_1, \ldots, x_n)$.

(b) Suppose $\varphi : X \to Y$ is a dominant morphism of projective algebraic sets and $U \subseteq Y$ is a non-empty open subset. Prove that $\varphi^{-1}(U)$ is a non-empty open subset.

Proof. Suppose for contradiction that $U \cap \varphi(X) = \emptyset$. Since U is non-empty, we have U^c is not all of Y and it is a closed set. Furthermore, if $y \in \varphi(X)$, then $y \notin U$, therefore, $y \in U^c$. So $\varphi(X) \subseteq U^c$. Since U^c is closed, then the closure of $\overline{\varphi(X)} = U^c \neq Y$, contradiction. So there is an element of $U \cap \varphi(X)$. So there is an element $x \in X$ such that $\varphi(X) \in U$, and therefore, the preimage is non-empty. □

Exercise 3: Local rings.

(a) Suppose $\varphi : X \to Y$ is an isomorphism and $\varphi(P) = Q$. Prove that the pullback on function fields induces an isomorphism on local rings $\mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$.

Recall that we can view ideals $\mathfrak{m}_Q(Y) \subseteq \mathcal{O}_Q(Y)$ and $\mathfrak{m}_P(X) \subseteq \mathcal{O}_P(X)$ as abelian subgroups. Show that the isomorphism $\mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$ induces an isomorphism $\mathfrak{m}_Q(Y) \to \mathfrak{m}_P(X)$ (as abelian groups).

Proof. It was show that since $\varphi : X \to Y$ is an isomorphism, there is an isomorphism φ^* on $k(Y) \to k(X)$. Consider $\varphi^{*\prime} = \varphi^*_{|\mathcal{O}_Q(Y)}$ and $\psi^{*\prime} = \psi^*_{|\mathcal{O}_P(X)}$. And $\psi^* = (\varphi^*)^{-1}$. We need to show that $\varphi^{*\prime}\psi^{*\prime}$ is the identity on $\mathcal{O}_P(X)$ and $\psi^{*\prime}\varphi^{*\prime}$ is the identity on $\mathcal{O}_Q(Y)$.

(Part I) If $(U, \alpha) \in \mathcal{O}_P(X)$, then $\alpha(P)$ is defined, we have

$$\psi^{*\prime}(U, \alpha) = (U', \alpha \circ \psi)$$

We see that indeed the RHS is in $\mathcal{O}_Q(Y)$ because $(\alpha \circ \psi)(Q) = \alpha(P)$ which is defined. So $\alpha \circ \psi$ is defined at Q. Then with the last composition:

$$\varphi^{*\prime}(U', \alpha \circ \psi) = (U'', \alpha \circ \psi \circ \varphi)$$

We see that $\alpha \circ \psi \circ \varphi$ is defined again at P because $(\alpha \circ \psi \circ \varphi)(P) = \alpha(P)$ which is by definition, defined at P. Lastly, $(U'', \alpha \circ \psi \circ \varphi) = (U, \alpha)$ because $\alpha, \alpha \circ \psi \circ \varphi$ are defined in $U \cap U''$ by definition, and $\psi \circ \varphi$ is the identity on X. The same proof works for the composition $\psi^{*\prime}\varphi^{*\prime}$.

(Part II) We just need to show that the isomorphism on $\pi : \mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$ restricts to a mapping of non-units to non-units. This is because if a non-unit maps to a unit:

$$\pi(a) = b$$

Then b has an inverse, π is surjective, so

$$\pi(c) = b^{-1}$$

and therefore,

$$\pi(a)\pi(c) = 1 = \pi(ac)$$

Since π is injective, $ac = 1$, so a was a unit, contradiction. So non-units map to non-units.

Then this sends $\mathfrak{m}_Q(Y)$ to some subset of $\mathfrak{m}_P(X)$ in $\mathcal{O}_P(X)$. Since there is an isomorphism on the local rings, we also know that there is a mapping $\mathcal{O}_P(X) \to \mathcal{O}_Q(Y)$ that restricts to sending non-units to non-units. So it sends $\mathfrak{m}_P(X)$ to some subset of $\mathfrak{m}_Q(Y)$. But the mappings are injective, which shows an isomorphism of $\mathfrak{m}_P(X) \cong \mathfrak{m}_Q(Y)$. □

(b) (Extra credit - you may use this in the subsequent parts even if you do not solve it) Suppose X and Y are isomorphic. Prove that X is smooth if and only if Y is smooth. (Hint: Use the following alternate characterization of snoothness: X is smooth at P if and only if $\dim X = \dim_k \mathfrak{m}_P(X)/\mathfrak{m}_P(X)^2$. Here, $\mathfrak{m}_P(X)^2$ is the ideal generated by products ab with $a \in \mathfrak{m}_P(X)$ and $b \in \mathfrak{m}_P(X)$. We then view this as a subgroup of $\mathfrak{m}_P(X)$ and take the quotient as groups. This group has the structure of a vector space over k and is called the *Zariski tangent space*. You may assume without proof that $\dim X = \dim Y$.) Note that this implies that P is a smooth point of X if and only if Q is a smooth point of Y.

Proof. We want to show that X smooth $\implies$ Y smooth. Using the previous question, an isomorphism on X, Y induces an isomorphism on $\mathfrak{m}_P(X)$ and $\mathfrak{m}_Q(Y)$ for $P \in X$, $\varphi(P) = Q \in Y$. Let this isomorphism be π. We will prove that $\mathfrak{m}_P(X)^2 \cong \mathfrak{m}_Q(Y)^2$.

(Surjectivity) Suppose that we had

$$fg \in \mathfrak{m}_Q(Y)^2$$

for $f, g \in \mathfrak{m}_Q(Y)$. We have that $\pi(f') = f$, $\pi(g') = g$. Then $\pi(f'g') = fg$.

(Injectivity) We have that if $\pi(fg) = 0$, either $f = 0, g = 0$. Then π restricts to an isomorphism on $\mathfrak{m}_Q(Y)^2 \cong \mathfrak{m}_P(X)^2$.

As abelian groups, we have that there is a mapping obtained from π:

$$\pi' : \mathfrak{m}_P(X) \longrightarrow \frac{\mathfrak{m}_Q(Y)}{\mathfrak{m}_Q(Y)^2}$$
$$f \mapsto \pi(f) + \mathfrak{m}_Q(Y)^2$$

Clearly, $\mathfrak{m}_P(X)^2 \subseteq \ker \pi'$. Suppose that an element of $f \in \mathfrak{m}_P(X)$ is mapped to a product in $\mathfrak{m}_Q(Y)$. So

$$\pi(f) = f_1 f_2$$

Recall that π restricts to an isomorphism on $\mathfrak{m}_P(X)^2 \cong \mathfrak{m}_Q(Y)^2$. Then there is a backwards mapping showing that

$$\pi^{-1}(\pi(f)) = \pi^{-1}(f_1 f_2) = \pi^{-1}(f_1)\pi^{-1}(f_2) = f$$

So f is a product. So we get $\ker \pi' = \mathfrak{m}_P(X)^2$. We conclude that by the first isomorphism theorem,

$$\frac{\mathfrak{m}_P(X)}{\mathfrak{m}_P(X)^2} \cong \frac{\mathfrak{m}_Q(Y)}{\mathfrak{m}_Q(Y)^2}$$

Since X is smooth, $\dim X = \dim \frac{\mathfrak{m}_P(X)}{\mathfrak{m}_P(X)^2}$. It has a dimension over k, so we can find a basis $\{\overline{\lambda_1}, \ldots, \overline{\lambda_j}\}$. Where

$$\overline{\lambda_i} = \lambda_i + \mathfrak{m}_P(X)^2$$

Every element of $\frac{\mathfrak{m}_P(X)}{\mathfrak{m}_P(X)^2}$ can be uniquely expressed as

$$a_1\overline{\lambda_1} + \cdots + a_j\overline{\lambda_j}$$

We have shown an isomorphism of $\frac{\mathfrak{m}_P(X)}{\mathfrak{m}_P(X)^2} \cong \frac{\mathfrak{m}_Q(Y)}{\mathfrak{m}_Q(Y)^2}$. Let this mapping be given by φ. Let φ have an additional action on k as the identity such that:

$$\varphi(a_1\overline{\lambda_1} + \cdots + a_j\overline{\lambda_j}) = a_1\varphi(\overline{\lambda_1}) + \cdots + a_j\varphi(\overline{\lambda_j})$$

We need to show that $\varphi(\overline{\lambda_i})$ are linearly independent. If the RHS is 0, suppose there is a nontrivial relation, where some $a_i \neq 0$. Then

$$\varphi(a_1\overline{\lambda_1} + \cdots + a_j\overline{\lambda_j}) = 0$$

where

$$a_1\overline{\lambda_1} + \cdots + a_j\overline{\lambda_j} \neq 0$$

But $a_1\overline{\lambda_1} + \cdots + a_j\overline{\lambda_j} \in \frac{\mathfrak{m}_P(X)}{\mathfrak{m}_P(X)^2}$, and φ is an isomorphism. So the kernel of φ with its regular action (without acting as identity on k) is nontrivial which is a contradiction.

Because $\varphi(\overline{\lambda_i})$ are linearly independent, φ is injective. We also have an inverse map φ^{-1} which is also injective by the same reason above. So we actually get an isomorphism of vector spaces. We have the equality:

$$\dim X = \dim \frac{m_P(X)}{m_P(X)^2} = \dim \frac{m_Q(Y)}{m_Q(Y)^2}$$

and since $\dim X = \dim Y$, we have

$$\dim \frac{m_Q(Y)}{m_Q(Y)^2} = \dim Y$$

So that means that Y is smooth. The direction that Y is smooth $\implies$ X is smooth is symmetric, so we are done. □

(c) Prove that $V(y)$ and $V(y - x^3)$ are isomorphic affine varieties.

Proof. We need to show that there is a morphism and inverse morphism:

$$\varphi : V(y) \to V(y - x^3)$$
$$\psi : V(y - x^3) \to V(y)$$
$$\varphi \circ \psi = \mathrm{id}_{V(y-x^3)}$$
$$\psi \circ \varphi = \mathrm{id}_{V(y)}$$

Define:

$$\varphi(x, y) = (\varphi_1(x, y), \varphi_2(x, y))$$

where

$$\varphi_1(x, y) = x$$
$$\varphi_2(x, y) = y^3$$

φ is a polynomial map because $\varphi_1, \varphi_2 \in k[x, y]$. Similarly, define

$$\psi(x, y) = (x, 0)$$

Suppose that $P = (x, 0) \in V(y)$. Then

$$\psi(\varphi(P)) = \psi(x, 0) = (x, 0)$$

and if $Q = (x, x^3) \in V(y - x^3)$,

$$\varphi(\psi(Q)) = \varphi(x) = \varphi(x, 0) = (x, x^3)$$

which shows that they are isomorphic. □

(d) Prove that the projective closures of $V(y)$ and $V(y - x^3)$ are not isomorphic. Do you see why this happens geometrically?

Proof. Check that $V(y)$ is smooth:

$$f_x = 0$$
$$f_y = 1$$

So $f_x \neq f_y \neq 0$, and there are no singular points. In the last question, we proved an isomorphism, and since $V(y)$ is smooth, $V(y - x^3)$ is smooth also. By definition, the projective closures $\mathbb{V}(y)$ and $\mathbb{V}(z^2y - x^3)$ are smooth also. Check for singular points on $\mathbb{V}(z^2y - x^3)$:

$$F_x = 3x^2$$
$$F_y = z^2$$
$$F_z = 2yz$$

We see that $[0 : 1 : 0]$ makes all of them 0. And for $F = z^2y - x^3$, $F([0 : 1 : 0]) = 0$ also. So $\mathbb{V}(z^2y - x^3)$ is not smooth, contradiction. This happens because the projective closure of $y - x^3$ looks like $z^2 - x^3$ which is a cusp at $[0 : 1 : 0]$. □

Exercise 4: Let $F \in k[x, y, z]$ be a homogeneous polynomial of degree n.

(a) Show that $xF_x + yF_y + zF_z = nF$ where F_x, F_y, F_z denote the partial derivatives of F with respect to x, y, z respectively

Proof. We have that

$$F = \sum F_i$$

where each F_i are of the form $a_i x^{r_1} y^{r_2} z^{r_3}$ and $r_1 + r_2 + r_3 = n$. Then

$$F_x = \sum F_{ix}$$

and

$$xF_x + yF_y + zF_z = \sum xF_{ix} + yF_{iy} + zF_{iz}$$

We have:

$$\begin{aligned} xF_{ix} &= r_1 a_i x^{r_1} y^{r_2} z^{r_3} \\ yF_{iy} &= r_2 a_i x^{r_1} y^{r_2} z^{r_3} \\ zF_{iz} &= r_3 a_i x^{r_1} y^{r_2} z^{r_3} \end{aligned}$$

and therefore,

$$\begin{aligned} xF_{ix} + yF_{iy} + zF_{iz} &= a_i(r_1 + r_2 + r_3)x^{r_1} y^{r_2} z^{r_3} \\ &= a_i(n)x^{r_1} y^{r_2} z^{r_3} \\ &= nF_i \end{aligned}$$

Therefore

$$\begin{aligned} xF_x + yF_y + zF_z &= \sum nF_i \\ &= n \sum F_i \\ &= nF \end{aligned}$$

and we're done. $\square$

(b) Now suppose F has no repeated factors. Let $P \in U_i \subseteq \mathbb{P}^2$. Recall that we say $\mathbb{V}(F)$ is singular at P if $V(f)$ is singular at P where f is the dehomogenization of F with respect to the i-th coordinate. Show that a point $P \in \mathbb{P}^2$ is a singular point of $\mathbb{V}(F)$ if and only if $F(P) = F_x(P) = F_y(P) = F_z(P) = 0$.

Proof. The operation of taking the partial derivative and dehomogenization on a variable other than the derivative commutes. So that means $f_x(P) = 0 \iff F_x(P) = 0$.

($\rightarrow$) Since P is singular in $\mathbb{V}(F)$, we have $f_x(P) = f_y(P) = f(P) = 0$. And $f_x(P) = 0 \iff F_x(P) = 0$. Also, $f(P) = 0 \implies F(P) = 0$ because $F([x : y : 1]) = f(x, y)$. By the previous problem, we have

$$xF_x(P) + yF_y(P) + zF_z(P) = nF(P)$$

and therefore,

$$zF_z(P) = 0$$

so $F_z(P) = 0$. If the point does not lie in U_3, we can use another affine chart with the same result. So we are done here.

($\leftarrow$) $P \in \mathbb{P}^2$, it is in one of U_i wlog say U_1. Then $F(P) = F(1, P_2, P_3) = f(P_2, P_3) = 0$ or in other words, $P \in V(f)$.

Now we need to show $f_y(P) = f_z(P) = 0$. But that comes from the fact proved earlier that dehomogenization and partial differentiation commutes. So

$$F_y([1 : P_2 : P_3]) = (F(1, y, z))_y(P_2, P_3) = f_y(P_2, P_3) = 0$$

We conclude both $f_y(P) = f_z(P) = 0$. So P is singular on $V(f)$ and therefore on $\mathbb{V}(F)$. □

(c) Suppose that $P \in U_i$ is a smooth point of $\mathbb{V}(F)$. Recall that the projective tangent line at P is the projective closure of the tangent line of $V(f)$, where f is the dehomogenization of F with respect to the i-th coordinate. Prove that the projective tangent space at P is the vanishing of

$$xF_x(P) + yF_y(P) + zF_z(P)$$

Proof. Since P is smooth in $\mathbb{V}(F)$, then P is smooth in $V(f)$. Suppose wlog that $P = [P_1 : P_2 : 1] \in U_3$. Then we can dehomogenize F:

$$\begin{aligned} F &\longrightarrow F(x, y, 1) = F_d \\ F &= a_0 + a_1x + a_2y + a_3z + z_4x^2 + \cdots \\ F_d &= a_0 + a_3 + a_1x + a_2y + \cdots \end{aligned}$$

Now we want to find $T_{(P_1,P_2)}(F_d)$. Compute the translation, $x \mapsto x + P_1$, $y \mapsto y + P_2$. So we have that the tangent space is the vanishing of the degree 1 terms in

$$F_d(x + P_1, y + P_2) = c + F_{dx}(0, 0)x + F_{dy}(0, 0)y + \{\text{higher degree terms}\}$$

Then the tangent space is $\mathbb{V}(F_{dx}(0, 0)x + F_{dy}(0, 0)y)$. Undoing our change of variables, we have $\mathbb{V}(F_{dx}(0, 0)x + F_{dy}(0, 0)y - F_{dx}(0, 0)P_1z - F_{dy}(0, 0)P_2z)$. Since $F_x(P) = F_{dx}(0, 0)$, and so on, then this can be changed to

$$\mathbb{V}(F_x(P)x + F_y(P)y + (-F_x(P)P_1 - F_y(P)P_2)z)$$

Using the fact that

$$nF = xF_x + yF_y + zF_z$$

then

$$nF(P) = 0 = P_1F_x(P) + P_2F_y(P) + F_z(P)$$

or

$$-P_1F_x(P) - P_2F_y(P) = F_z(P)$$

So

$$\begin{aligned} &\mathbb{V}(F_x(P)x + F_y(P)y + (-F_x(P)P_1 - F_y(P)P_2)z) \\ &= \mathbb{V}(xF_x(P) + yF_y(P) + zF_z(P)) \end{aligned}$$

Since it does not matter what affine chart we started with, we will always get the same result. So we are done. □

Exercise 5: For each of the following projective plane curves, find their singular points and the multiplicities and tangent cone at each of the singular points.

(a) $x^2y^3 + x^2z^3 + y^2z^3$

Proof. By the previous problem, the singular points are when

$$x^2y^3 + x^2z^3 + y^2z^3 = 0$$

and

$$\begin{aligned} F_x(P) &= 0 \\ F_y(P) &= 0 \\ F_z(P) &= 0 \end{aligned}$$

We have

$$\begin{aligned} F_x &= 2xy^3 + 2xz^3 \\ F_y &= 3x^2y^2 + 2yz^3 \\ F_z &= 3x^2z^2 + 3y^2z^2 \end{aligned}$$

Now we simultaneously solve for the 0's:

$$\begin{aligned} 2xy^3 + 2xz^3 &= 0 & 3x^2y^2 + 2yz^3 &= 0 & 3x^2z^2 + 3y^2z^2 &= 0 \\ 2x(y^3 - z^3) &= 0 & y(3x^2y + 2z^3) &= 0 & 3z^2(x^2 + y^2) &= 0 \end{aligned}$$

From the first equation, we require $x = 0$, or $y^3 = z^3$. For the second, we require $y = 0$ or $3x^2y + 2z^3 = 0$. For the third, we require $3z^2 = 0$ or $(x^2 + y^2) = 0$. Go through the cases:

- $x = 0$. Then by the second equation, $y = 0$ or $2z^3 = 0$. In either case, the last equation is 0 also and $[0 : 0 : 1], [0 : 1 : 0] \in \mathbb{V}(F)$.
- $y^3 = z^3$. By the third equation, either $x = y = 0$ or $z = 0$. If $x = y = 0$, we have $x = y = z = 0$ which is impossible. If $z = 0, y = 0$, then $F_y(P) = 0$. So $[1 : 0 : 0]$ is another possible solution.

These are the singular points.

(Multiplicities) We see that each singular point is 0 in their affine chart. So we dehomogenize and find the lowest degree:

- $[1 : 0 : 0]$. We have
$$F(1, y, z) = y^3 + z^3 + y^2z^3$$
The lowest degree is 3 which is the multiplicity of $\mathbb{V}(F)$ at $[1 : 0 : 0]$.
- $[0 : 1 : 0]$. We have
$$F(x, 1, z) = x^2 + x^2z^3 + z^3$$
The lowest degree is 2 so the multiplicity is 2.
- $[0 : 0 : 1]$. We have
$$F(x, y, 1) = x^2y^3 + x^2 + y^2$$
and the lowest degree is 2.

(Tangent Cones) Take the lowest degree terms of the previous dehomogenizations:

- $\mathbb{TC}_{[1:0:0]}(\mathbb{V}(F)) = y^3 + z^3$.

– $\mathbb{T}C_{[0:1:0]}(\mathbb{V}(F)) = x^2$.

– $\mathbb{T}C_{[1:0:0]}(\mathbb{V}(F)) = x^2 + y^2$.

so we are done. □

(b) $y^2z - x(x-z)(x-\lambda z), \lambda \in k$

Proof. We require $F(P) = F_x(P) = F_y(P) = F_z(P) = 0$. So calculate the derivatives:

$$\begin{aligned} y^2z - x(x-z)(x-\lambda z) &= y^2z - x(x^2 - (\lambda+1)xz + \lambda z^2) \\ &= y^2z - x^3 + (\lambda+1)x^2z + \lambda xz^2 \end{aligned}$$

We have

$$\begin{aligned} F_x &= -3x^2 + 2(\lambda+1)xz + \lambda z^2 \\ F_y &= 2yz \\ F_z &= y^2 + (\lambda+1)x^2 + 2\lambda xz \end{aligned}$$

– Case 1: $F_y = 0 \implies y = 0$. Then

$$F(x,0,z) = 0 \implies x(x-z)(x-\lambda z) = 0$$

  * $x = 0$. This means that $F_x(0,0,z)$ implies $z = 0$ which is impossible.
  * $x = \lambda z$. Then
  $$F_z(\lambda z, 0, z) = (\lambda^3 + 3\lambda^2)z^2$$
  $z \neq 0$ so $\lambda = 0, -3$. Also,
  $$F_x(\lambda z, 0, z) = (-\lambda^2 + 3\lambda)z^2$$
  So $\lambda = 0, 3$. Then $\lambda = 0$, $x = 0 = z$, contradiction.

– Case 2: $F_y = 0 \implies z = 0$. Then

$$F_x(x,y,0) = -3x^2$$

So $x = 0$. Now plug this into F_z to get:

$$F_z(0,y,0) = y^2$$

So $y = 0$, which is impossible.

– $z = y = 0$. This is impossible since $F_x(x,0,0) = 0$ implies that $x = 0$.

None of the cases work out, so there are no singular points. □

(c) $x^n + y^n + z^n, n > 0$.

Proof. We need $x^n + y^n + z^n = 0$ and

$$\begin{aligned} F_x &= nx^{n-1} \\ F_y &= ny^{n-1} \\ F_z &= nz^{n-1} \end{aligned}$$

To be 0 at P. This is true when $P = [0:0:0] \notin \mathbb{P}^2$. So there are no singular points. □

Exercise 6: For each point $[a : b : c : d : e : f] \in \mathbb{P}^5$, we can associate the degree 2 plane curve

$$C = \mathbb{V}(ax^2 + bxy + cxz + dy^2 + eyz + fz^2) \subseteq \mathbb{P}^2$$

given $C \subseteq \mathbb{P}^2$, we write $[C] \in \mathbb{P}^5$ for the point $[a : b : c : d : e : f]$ corresponding to the coefficients. (Note: this is well-defined because if we rescale the coefficients a, b, c, d, e, f it does not change the vanishing set.) This problem is about relating degree 2 curves with certain properties to algebraic subsets of $\mathbb{P}^5$.

(a) Fix a point $P = [x_0 : y_0 : z_0] \in \mathbb{P}^2$. Prove that the set $\{[C] \in \mathbb{P}^5 : P \in C\}$ is a hyperplane in $\mathbb{P}^5$. (In fact, you should find it is the hyperplane $\nu_{2,2}(P)^*$.)

Proof. We have that the a, b, c, d, e, f that make

$$ax_0^2 + bx_0y_0 + cx_0z_0 + dy_0^2 + ey_0z_0 + fz_0^2 = 0$$

can be seen as variables of the hyperplane with coefficients $x_0^2, x_0y_0, \ldots, z_0^2$ because they are fixed:

$$x_0^2a + x_0y_0b + x_0z_0c + y_0^2d + y_0z_0e + z_0^2f = 0$$

So the set

$$\{[C] \in \mathbb{P}^5 : P \in C\} = \mathbb{V}(x_0^2a + x_0y_0b + x_0z_0c + y_0^2d + y_0z_0e + z_0^2f)$$

as desired. $\square$

(b) Prove that there exists a curve $C = \mathbb{V}(ax^2 + bxy + cxz + dy^2 + eyz + fz^2)$ through any 5 points $P_1, \ldots, P_5 \in \mathbb{P}^2$.

Proof. We can use a matrix argument:

$$\begin{bmatrix} x_1^2 & x_1y_1 & x_1z_1 & y_1^2 & y_1z_1 & z_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_5^2 & x_5y_5 & x_5z_5 & y_5^2 & y_5z_5 & z_5^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has non-trivial 0's because this is a mapping from a 6 dimensional vector space to one of 5 dimensions. So there exists a, b, c, d, e, f not all 0 such that the curve C passes through $P_1, \ldots, P_5$. $\square$

(c) Prove that the set

$$\{[a : b : c : d : e : f] \in \mathbb{P}^5 : \mathrm{mult}_P(ax^2 + bxy + cxz + dy^2 + eyz + fz^2) \geqslant 2\}$$

is isomorphic to $\mathbb{P}^2$. (Hint: you might want to perform a change of coordinates to reduce to the case $P = [0 : 0 : 1]$.)

Proof. Perform a change of coordinates so that $P \to [0 : 0 : 1]$. This corresponds to some rotation.

Then we get some new vanishing $\mathbb{V}(ax^{2\prime} + bx'y' + cx'z' + dy^{2\prime} + ey'z' + fz^{2\prime})$. So dehomogenize:

$$\mathbb{V}(ax^{2\prime} + bx'y' + cx' + dy^{2\prime} + ey' + f)$$

So the multiplicity is $\geqslant 2$ when

$$c = e = f = 0$$

and this is isomorphic to $\mathbb{P}^2$ because we just have

$$\{[a : b : d] : a, b, d \in k \text{ not all } 0\}$$

$\square$

(d) Prove that the set $\{[C] \in \mathbb{P}^5 : C \text{ is a line}\}$ is projectively equivalent to $\nu_{2,2}(\mathbb{P}^2) \subseteq \mathbb{P}^5$.

Proof. We have that

$$C = \mathbb{V}(ax^2 + bxy + cxz + dy^2 + eyz + fz^2)$$

is the vanishing of a degree 2 polynomial in $\mathbb{P}^2$, so it splits into two lines. Then we require that it splits into a product of two lines that are the same, so

$$C = \mathbb{V}((ux + vy + wz)^2)$$

Then

$$(ux + vy + wz)^2 = u^2x + v^2y^2 + w^2z^2 + 2uvxy + 2vwyz + 2uwxz$$

So we get

$$[a : b : c : d : e : f] = [u^2 : 2uv : 2uw : v^2 : 2vw : w^2]$$

and therefore, we take the set of all such points for u, v, w varied over k:

$$\{[C] \in \mathbb{P}^5 : C \text{ is a line}\} = \{[u^2 : 2uv : 2uw : v^2 : 2vw : w^2] : u, v, w \in k \text{ not all } 0\}$$

Now $\nu_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ is

$$[x : y : z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2]$$

So there is a change of coordinates given by the invertible matrix:

$$\begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ xz \\ y^2 \\ yz \\ z^2 \end{bmatrix} = \begin{bmatrix} u^2 \\ 2uv \\ 2uw \\ v^2 \\ 2vw \\ w^2 \end{bmatrix}$$

So we have $\{[C] : C \text{ is a line}\}$ is projectively equivalent to $\nu_{2,2}(\mathbb{P}^2)$. $\square$

Exercise 7: Suppose k is algebraically closed. Let $F \in k[x, y, z]$ be an irreducible homogeneous polynomial of degree 2. Prove that $\mathbb{V}(F)$ is projectively equivalent to $\mathbb{V}(yz - x^2)$. In other words, all irreducible conics are projectively equivalent.

(Hint: Let P be a point in $\mathbb{V}(F)$. There is a change of coordinates that takes P to $[0 : 1 : 0]$. In these coordinates, if we write $F = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$, then what do you know about d? Can b and e both vanish? Find a change of coordinates so that $F = a'x^2 + c'xz + yz + f'z^2 = a'x^2 + (c'x + y + f'z)z$. Can a' vanish?)

Proof. If we do a change of coordinates so that $P \to [0 : 1 : 0]$ and write

$$F = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$$

then $[0 : 1 : 0]$ must vanish on this so

$$0 = d$$

and so

$$F = ax^2 + bxy + cxz + eyz + fz^2$$

Suppose for contradiction that $b = e = 0$. Then we have

$$F = ax^2 + cxz + fz^2$$

Notice that when we dehomogenize, the polynomial becomes reducible:

$$F(x, 1) = ax^2 + cx + f = gh$$

So when we rehomogenize, the polynomial becomes reducible, which is a contradiction. Then we have

$$F = ax^2 + cxz + fz^2 + bxy + eyz$$

Wlog, suppose that $e \neq 0$. Then we have the change of coordinates

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y \\ bx + ez &\mapsto z \end{aligned}$$

Then the inverse of this change of coordinates is given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & e \end{bmatrix}$, and since $e \neq 0$, we have that this is invertible. If instead, $e = 0, b \neq 0$, then we can modify the map to $bx + ez \mapsto x$. So under this change of coordinates, we have F goes to

$$\begin{aligned} ax^2 + cxz + fz^2 + y(bx + ez) &\mapsto a'x^2 + c'xz + f'z^2 + yz \\ &= a'x^2 + (c'x + y + f'z)z \end{aligned}$$

We have that a' cannot vanish, otherwise, we have that F was not irreducible as we get $(c'x + y + f'z)z$. Now we have a change of coordinates $c'x + y + f'z \mapsto a'y$. The inverse of this is given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ c' & a' & f' \\ 0 & 0 & 1 \end{bmatrix}$ which is invertible. Under this mapping, we have

$$F \mapsto a'x^2 + a'yz$$

Rescaling, we see that $\mathbb{V}(F)$ is projectively equivalent to $\mathbb{V}(x^2 + yz)$ because there is a change of coordinates by composition. So all irreducible conics are projectively equivalent. $\square$

Exercise 8: (Extra Credit) Recall that given a point $P = [a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$, we write

$$P^* = \mathbb{V}(a_1 x_1 + \cdots + a_{n+1} x_{n+1}) \subseteq \mathbb{P}^n$$

for the corresponding hyperplane in $\mathbb{P}^n$. Let $P \in \mathbb{P}^m$ and $Q \in \mathbb{P}^n$. Prove that

$$\sigma_{m,n}^{-1}(\sigma_{m,n}(P \times Q)^*) = P^* \times \mathbb{P}^n \cup \mathbb{P}^m \times Q^* \subseteq \mathbb{P}^m \times \mathbb{P}^n$$

Proof. □