Linear Algebra Notes

Trustin Nguyen

May 14, 2023

Spectral Theorem, Positive/Negative Operators

Need more info on the min polynomial of T over $\mathbb{C}$. In $\mathbb{C}$, this factors into linear factors $(z - z_j)$ we already know each z_j must be real since T is self-adjoint. Over $\mathbb{R}$, we need to rule out quadratic factors: $z^2 + az + b$ or in other words, we require that $a^2 - 4b < 0$. Plug in T and consider

$$\begin{aligned}
\langle Tv, v\rangle = \langle (T^2 + aT + bI)v, v\rangle &= \langle T^2 v, v\rangle + a\langle Tv, v\rangle + b\langle v, v\rangle \\
&= \langle Tv, Tv\rangle + a\langle Tv, v\rangle + b\langle v, v\rangle \\
&= \|Tv\|^2 + a\langle Tv, v\rangle + b\|v\|^2 \\
\geq \|Tv\|^2 - |a|\|Tv\|\|v\| + b\|v\|^2 &= (\|Tv\| - \frac{|a|}{2}\|v\|)^2 - \frac{|a|^2}{4}\|v\|^2 + b\|v\|^2 \\
&= (\|Tv\| - \|v\|\frac{|a|}{2})^2 + (b - \frac{a^2}{4})\|v\|^2 \geq 0
\end{aligned}$$

This expression can be 0 only if $\|v\| = 0$ which means that $v = 0$. So $T^2 + aT + bI$ is an invertible operator. So if $p_{\min}(z)$ contains $q(z)$ as a factor, we would have

$$p_{\min}(T) = q(T) \cdot h(T) = 0 \qquad \text{for some polynomial } h(z)$$

So $h(T) = 0$ and $q(T)$ does not belong in the minimal polynomial. So if $T = T^*$, its minimal polynomial has only linear factors $z - z_j$, where each $z_j \in \mathbb{R}$. So $\mathcal{M}(T)$ is upper triangular in some orthonormal basis. But then $\mathcal{M}(T^*) = \overline{\mathcal{M}(T)^T} = \mathcal{M}(T)$. That means that $\mathcal{M}(T)$ is diagonal.

Complex Spectral Theorem

Theorem

Complex Spectral Theorem

Over $\mathbb{C}$, the following are equivalent

(a) $T \in \mathcal{L}(V)$ is normal

(b) There is a diagonal matrix representation for T with respect to some orthonormal basis.

Main points of the proof: $(a) \to (b)$ we start by Schur's theorem with an upper-triangular form for T with respect to some orthonormal basis:

$$\begin{aligned}\mathcal{M}(T) &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \\ &= \|Te_1\|^2 = \left|a_{11}^2\right| \\ &= \|T^* e_1\|^2 = \sum_{j=1}^{n} |a_{1,j}|^2\end{aligned}$$

Since the norms are equal, then for $a_{1,j}$ for $j \neq 1$, they must be 0. By applying this same observation, to each Te_j and $T^* e_j$, we conclude all off-diagonal entries must be zero. $(b) \to (a)$ is clear since any 2 diagonal matrices commute.

Nonnegative and Positive Operators

Definition

Positive and Nonnegative Operators

Definition: Let V be a finite-dimensional inner product space. An operator $T \in \mathcal{L}(V)$ is called nonnegative if $T = T^*$ and

$$\langle Tv, v \rangle \geq 0 \qquad \forall v \in V$$

T is called positive if $T = T^*$ and

$$\langle Tv, v \rangle > 0 \qquad \forall v \in V \backslash \{0\}$$

Characterization of nonnegative and positive operators. Let $T \in \mathcal{L}(V)$, then the following are equivalent

(a) T is positive / T is nonnegative

(a) $T = T^*$ and all its eigenvalues are positive / $T = T^*$ and all its eigenvalues are nonnegative.

(c) With respect to some orthonormal basis, $\mathcal{M}(T)$ is diagonal with all diagonal terms positive / With respect to some orthonormal basis, $\mathcal{M}(T)$ is diagonal with all diagonal terms nonnegative

(d) T has a positive square root / T has a nonnegative square root.

(e) T has a self-adjoint square root

T is invertible No requirements

(f) $T = R^* R$ for some R invertible / $T = R^* R$ for some R

Proof. $(a) \to (b) \to (c)$ is straightforward.
$(c) \to (a)$: Any vector $v \in V$ can be written as $v = a_1 e_1 + \ldots + a_n e_n$ where (e_j) are orthonormal and $Te_j = \lambda_j e_j$.

$$Tv = a_1 \lambda_1 e_1 + \ldots + a_n \lambda_n e_n$$

So

$$\begin{aligned}\langle Tv, v \rangle &= \langle a_1 \lambda_1 j e_1 + a_2 \lambda_2 e_2 + \ldots + a_n \lambda_n e_n, a_1 e_1 + a_2 e_2 + \ldots + a_n e_n \rangle \\ &= \lambda_1 |a_1|^2 + \lambda_2 |a_2|^2 + \cdots + \lambda_n |a_n|^2 \geq 0\end{aligned}$$

Moreover, in case $\lambda_j > 0$ for all $j = 1, \ldots, n$, this expression is positive unless every $a_j = 0$. This would imply that $v = 0$. $\square$

What about square roots? We say that R is a square root of T if $R = R^*, T = T^*, R^2 = T$. If T is positive, it has a positive square root and if T is nonnegative, it has a nonnegative square root. Actually, the positive/nonngeative square root is necessasrily unique.

$$Te_j = \lambda_j e_j$$
$$Re_j = \sqrt{\lambda_j} e_j$$

the eigenvector of R is the same as with T, and the eigenvalues are determined by T. So the square root is unique.

(e) corresonds to dropping the positivity / nonnegativity condition on the square root and (f) corresponds to dropping the self-adjointness condition on the square root.

$$R^* R = \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix}$$

SVD

Theorem

Suppose that $T \in \mathcal{L}(V, W)$. Then

(a) T^*T is nonnegative

(b) $\ker T^*T = \ker T$

(c) $\mathrm{Im}\{T^*T\} = \mathrm{Im}\{T^*\}$

(d) $\dim \mathrm{Im}\{T^*T\} = \dim \mathrm{Im}\{T^*\} = \dim \mathrm{Im}\{T\}$

Proof. (a) $(T^*T)^* = T^*T^{**} = T^*T \to \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \geq 0$
(b) $\ker T \subseteq \ker T^*T$. Suppose $v \in \ker T^*T$ so $T^*Tv = 0$ so $\|Tv\| = 0 \to Tv = 0 \to v \in \ker T$
(c) $\mathrm{Im}\{T^*T\} = [\ker (T^*T)^*]^\perp = (\ker T^*T)^\perp = (\ker T)^\perp = \mathrm{Im}\{T^*\}$
(d) $\dim \mathrm{Im}\{T^*T\} = \dim \mathrm{Im}\{T^*\} = \dim \mathrm{Im}\{T\}$

$\square$

Definition

Singular Value Decomposition

The singular values of T are defined as the square roots of the eigenvalues of T^*T, usually ordered from largest to smallest and called

$$s_1 \geq s_2 \geq \ldots \geq s_n \geq 0$$

Theorem

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Then there exist orthonormal vectors $e_1, \ldots, e_m$ in V and $f_1, \ldots, f_m$ in W such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_m\langle v, e_m\rangle f_m$$

Proof. Recall the eigenvalues of T^*T are $s_1^2, \ldots, s_m^2, \underbrace{s_{m+1}^2, \ldots, s_n^2}_{=0}$. Since T^*T is self-adjoint, there is an orthonormal basis $e_1, e_2, \ldots, e_n$ such that $T^*Te_j s_j^2 e_j$. Define $f_j := \frac{1}{s_j} Te_j$ for $j = 1, \ldots, m$

$$\begin{aligned}
\langle f_j, f_k \rangle &= \frac{1}{s_j s_k} \langle Te_j, Te_k \rangle \\
&= \frac{1}{s_j s_k} \langle \underbrace{T^*Te_j}_{s_j^2 e_j}, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

An arbitrary $v \in V$ can be written as $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$. the action:

$$\begin{aligned}
Tv &= \langle v, e_1 \rangle Te_1 + \cdots + \langle v, e_n \rangle Te_n \\
&= s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m.
\end{aligned}$$

□

Isometries

Special class of operators/linear maps where $S \in \mathcal{L}(V, W)$ iff

$$\begin{aligned}
\|Sv\| &= \|v\| \qquad \forall v \in V \\
\langle Sv, Sv \rangle &= \langle v, v \rangle \\
\langle S^*Sv, v \rangle &= \langle v, v \rangle \\
\langle (S^*S - I)v, v \rangle &= 0 \\
S^*S &= I
\end{aligned}$$

Jordan Normal Form

$$\begin{bmatrix}
\begin{matrix} \lambda_1 & 1 & \ldots & 0 \\ 0 & \lambda_1 & 1 & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_1 \end{matrix} & & & \\
& \begin{matrix} \lambda_2 & 1 & \ldots & 0 \\ 0 & \lambda_2 & 1 & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_2 \end{matrix} & & \\
& & \ddots & \\
& & & \begin{matrix} \lambda_n & 1 & \ldots & 0 \\ 0 & \lambda_n & 1 & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_n \end{matrix}
\end{bmatrix}$$

Disclaimer: to guarantee this decomposition, we need to work over $\mathbb{R} = \mathbb{C}$.

1. If $\mathbb{F} = \mathbb{C}$ and $\dim B \geq 1$, then T always has an eigen value. Call it $\lambda_1 \in \mathbb{C}$. Consider the chain

$$\ker{(T - \lambda_1 I)} \subseteq \ker{(T - \lambda_1 I)^2} \subseteq \ker{(T - \lambda_1 I)^3}$$

Since V is finite-dimensional, there exists a k such that

$$\ker{(T-\lambda_1 I)^k} = \ker{(T-\lambda_1 I)^{k+j}}$$

Now consider

$$\ker{(T-\lambda_1 I)^k} \cap \mathrm{Im}\big\{(T-\lambda_1 I)^k\big\}$$

If $v \in \ker{(T-\lambda_1 I)^k}$, then $v = (T-\lambda_1 I)^k u$ and that $(T-\lambda_1 I)^k v = 0$, or in other words, $(T-\lambda_1 I)^{2k} u = 0$. But then $(T-\lambda_1 I)^k u = 0$. So $v = \ker{(T-\lambda_1 I)^k} \oplus \mathrm{Im}\big\{(T-\lambda_1 I)^k\big\}$. Both of these spaces are T invariant.

This reduces the problem to the case of a single eigenvalue. In fact, wlog, $\lambda_1 = 0$, by shifting. Recall $T^k = 0$ on our subspace but $T^{k-1} \neq 0$. So there exists $v \in$ our subspace such that $T^{k-1}v \neq 0$. Then thereexists a $u \in$ the same subspace such that $\langle T^{k-1}v, u\rangle \neq 0$. Here our subspace is $\ker{(T-\lambda_1 I)^k}$. Call $T' = T - \lambda_1 I$. Consider the matrix $(\langle T'^{j-1}v, T^{*k-j}\rangle u)^{i-1,\dots,k}_{j=1,\dots,k}$. This is a matrix of size $k \times k$.

$$\begin{aligned} &\langle T^{j-1}v, T^{*k-i}u\rangle \\ = \langle T^{k+j-i-1}v, u\rangle = &\begin{cases} \neq 0 & \text{if } i-j \\ 0 & \text{if } j > i \end{cases} \end{aligned}$$

This matrix is invertible because it's triangular with nonzeros on the main diagonal. Take the vectors

$$v, Tv, \dots, T^{k-1}v$$

Since any dependence among these vecotrs would give rise to the same dependence among the columns of the above matrix, they must be independent. So

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since every $T-$invariant space is subject to this process, we can split the entire space into subspaces spanned by these Jordan chains.

Instructions for Jordan Normal Form

Examples

1. Find the JNF and Jordan basis for $D : \mathcal{L}(V)$ where $V = \mathcal{P}_3(\mathbb{R})$.

$$\mathrm{JNF}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} f_1(x) &= 1 \\ f_2(x) &= x \\ f_3(x) &= \frac{x^2}{2} \\ f_4(x) &= \frac{x^3}{3!} \end{aligned}$$

2.

$$\begin{bmatrix} 1 & 1 & 0 & & & & & & & & \\ 0 & 1 & 1 & & & & & & & & \\ 0 & 0 & 1 & & & & & & & & \\ & & & 1 & 1 & & & & & & \\ & & & 0 & 1 & & & & & & \\ & & & & & 0 & 1 & 0 & & & \\ & & & & & 0 & 0 & 1 & & & \\ & & & & & 0 & 0 & 0 & & & \\ & & & & & & & & 0 & 1 & 0 \\ & & & & & & & & 0 & 0 & 1 \\ & & & & & & & & 0 & 0 & 0 \end{bmatrix}$$

$\dim V = 11$j	$\dim \ker T = 2$
$\dim \ker T^2 = 4$	$\dim \ker T^3 = 6$
$\dim \ker T^4 = 6 = \dim \ker (T^6)$	

We also have

(a) $\dim \ker T - I = 2$

(b) $\dim \ker T - I^2 = 4$

(c) $\dim \ker T - I^3 = 5$

(d) $\dim \ker T - I^4 = 5$

3. Given the following info:

(a) T has 3 eigenvalues: $i, -i, 0$

(b) $\dim \ker (T) = 3$

(c) $\dim \ker T^2 = 5$

(d) $\dim \ker T^3 = 6$

(e) $\dim \ker T - iI = 2$

(f) $\dim \ker T + iI^2 = 4$

(g) $\dim \ker T + iI^3 = 6$

(h) $\dim \ker T - iI = 4$

(i) $\dim \ker T - iI = 6$

We have

$$
\begin{bmatrix}
0 \\
& 0 & 1 \\
& 0 & 0 \\
& & & 0 & 1 & 0 \\
& & & 0 & 0 & 1 \\
& & & 0 & 0 & 0 \\
& & & & & & -i & 1 & 0 \\
& & & & & & 0 & -i & 1 \\
& & & & & & 0 & 0 & -i \\
& & & & & & & & & 0 & 1 & 0 \\
& & & & & & & & & 0 & -i & 1 \\
& & & & & & & & & 0 & 0 & -i \\
& & & & & & & & & & & & -i \\
& & & & & & & & & & & & & -i \\
& & & & & & & & & & & & & & -i & 1 \\
& & & & & & & & & & & & & & 0 & -i \\
& & & & & & & & & & & & & & & & -i & 1 \\
& & & & & & & & & & & & & & & & 0 & -i
\end{bmatrix}
$$

4. $\mathcal{P}_2^{x,y}(\mathbb{R})$ are bivariate polynomials of degree ≤ 2.

Let the operator be $T : \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and V has a basis $\{1, x, y, x^2, xy, y^2\}$. The action of T reduces the degree of the polynomials. The eigenvalue is 0 because we need to apply T enough times to kill all the basis vectors. We need to apply T 3 times to send all vectors to 0. So the $\dim \ker (T)^3 = 6$, as it kills the whole space. What are

(a) $\dim \ker T^3 = 6$

(b) $\dim \ker T^2 = 5$

(c) $\dim \ker T = 3$

Take

$$
\begin{aligned}
x^2 &\mapsto 2x \mapsto 2 \\
xy &\mapsto x + y \mapsto 2 \\
y^2 &\mapsto 2y \mapsto 2 \\
x &\mapsto 1 \mapsto 0 \\
y &\mapsto 1 \mapsto 0 \\
1 &\mapsto 0 \mapsto 0
\end{aligned}
$$

We can find the dimension of the null space by looking at the dimension of the range:

(a) $\dim \operatorname{Im}\{T\} = 3 \to \dim \ker T = 3$. In fact, $1, x - y, (x - y)^2 \in \ker T$

(b) $\dim \operatorname{Im}\{T^2\} = 1 \to \dim \ker T^2 = 5$. So we have $1, x-y, (x-y)^2, x, x^2 - y^2 \in \ker T^2$.

So we can find the Jordan Normal Form:

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
& & & 0 & 1 \\
& & & 0 & 0 \\
& & & & & 0
\end{bmatrix}
$$

Now for the Jordan Basis, The first column goes to 2, then $2x$, then, x^2. Take the next block to have $2(x-y), x^2-y^2$. Last basis vector is $(x-y)^2$.