Math250aHw6

Trustin Nguyen

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Exercise 1: Let $G = \langle a, b : a^2 = b^2 \rangle$ be the coproduct in the category of groups of the following diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & G \end{tikzcd}$$

Prove (by exhibiting certain surjections out of the coproduct) that G is nonabelian and infinite.

Proof. We know that it is infinite because there is a surjective mapping to $2\mathbb{Z}$:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"] & \mathbb{Z} \arrow[d] \arrow[ddr, bend left, "1"] & \\ \mathbb{Z} \arrow[r] \arrow[drr, bend right, "1"'] & G & \\ & & 2\mathbb{Z} \end{tikzcd}$$

Given by a multiplication of $z \in \mathbb{Z}$ by 2, then by 1. Now consider the more general diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"] & \mathbb{Z} \arrow[d] \arrow[ddr, bend left, "f"] & \\ \mathbb{Z} \arrow[r] \arrow[drr, bend right, "g"'] & G \arrow[dr, "\varphi"] & \\ & & H \end{tikzcd}$$

We have that $f(2\mathbb{Z}) = g(2\mathbb{Z})$ and $\varphi(a) = f(1), \varphi(b) = g(1)$. So

$$\varphi(a^n) = f(n), \varphi(b^n) = g(n)$$

and by the condition that $f(2\mathbb{Z}) = g(2\mathbb{Z})$, we let a, b be sent to two cycles that don't commute such as $(1\,2), (2\,3)$. These generate D_6. So for $H = D_6$, there is a surjection which says that G is nonabelian. $\square$

Exercise 2: Let $R = \mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[t]} (\mathbb{Z}[t]/t)$ be the coproduct in the category of rings of the following diagram

$$\begin{tikzcd} \mathbb{Z}[t] \arrow[r] \arrow[d] & \mathbb{Z}[t]/t \arrow[d] \\ \mathbb{Z}[t^{\pm 1}] \arrow[r] & R \end{tikzcd}$$

Show that $R = 0$ is the zero ring by showing that it admits no maps to any nonzero ring.

Proof. We observe that with the natural mappings, we must have in the diagram:

$$\begin{tikzcd}
\mathbb{Z}[t] \arrow[r, "\pi"] \arrow[d, "i"] & \mathbb{Z}[t]/(t) \arrow[d] \arrow[ddr, bend left, "f"] & \\
\mathbb{Z}[t^{\pm 1}] \arrow[r] \arrow[drr, bend right, "g"'] & R \arrow[dr, "\varphi"] & \\
 & & R'
\end{tikzcd}$$

with $f(1) =?$, and $t \mapsto 0 \mapsto 0$ under f. As for the morphism under g, we have $g(1) =?$ and $t \mapsto t \mapsto 0$ under g. But by the fact that g is a morphism, we must have $g(t \cdot t^{-1}) = g(1) = 1$. But $g(t \cdot t^{-1}) = g(0) = 0$. Therefore, $g(1) = 0$. So $f(1) = 0$ and there are no rings other than the zero ring that accept these maps.

This means that the coproduct exists: For any two mappings $\mathbb{Z}[t]/(t) \to R'$ and $\mathbb{Z}[t^{\pm 1}] \to R'$, there exists a unique map $R = 0 \to R'$ that makes the diagram commute. Furthermore, we see that shown above, that if $1 \neq 0$, there are no maps $R \to R'$ making the diagram commute. $\square$

Let $F, G : C \to D$ be two functors. Recall that a *natural transformation* $\nu : F \to G$ is a collection of morphisms $\nu x : F(X) \to G(X)$ such that for any morphism $g : X \to Y$ the following square commutes

$$\begin{tikzcd}
F(x) \arrow[r, "\nu x"] \arrow[d, "F(g)"] & G(x) \arrow[d, "G(g)"] \\
F(Y) \arrow[r, "\nu Y"] & G(Y)
\end{tikzcd}$$

Exercise 3: Let G be a group. Let BG be the category with a single object $*$ and morphisms $\mathrm{Hom}(*, *) = G$. Show that $\mathrm{Hom}(\mathrm{id}_{BG}, \mathrm{id}_{BG}) = Z(G)$. (In other words, under composition natural transformations from the identity functor to itself form a group, isomorphism to the center of G.)

Proof. Using the definition of natural transformations, we want to find the elements of $\mathrm{Hom}(\mathrm{id}_{BG}, \mathrm{id}_{BG})$ by the morphisms $g_0 : \mathrm{id}_{BG}(*) \to \mathrm{id}_{BG}(*)$ making the diagram

$$\begin{tikzcd}
\mathrm{id}_{BG}(*) \arrow[r, "g_0"] \arrow[d, "\mathrm{id}_{BG}(g)"] & \mathrm{id}_{BG}(*) \arrow[d, "\mathrm{id}_{BG}(g)"] \\
\mathrm{id}_{BG}(*) \arrow[r, "g_0"] & \mathrm{id}_{BG}(*)
\end{tikzcd}$$

commute. We can simplify this down to

$$\begin{tikzcd}
* \arrow[r, "g_0"] \arrow[d, "g"] & * \arrow[d, "g"] \\
* \arrow[r, "g_0"] & *
\end{tikzcd}$$

and find that $g_0 g = g g_0$. So the morphisms that send the identity functor to itself making the diagram commute is isomorphic to the elements of G that commute with all $g \in \mathrm{Hom}(*, *)$. $\square$

Exercise 4: Let X, Y_i be vector spaces. The set $\mathrm{Hom}(X, Y_i)$ is naturally a vector space. Construct a natural map

$$\bigoplus_i \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \bigoplus_i Y_i)$$

Here $\bigoplus \mathrm{Hom}(X, Y_i)$ is the coproduct in the category of vector spaces. Give an example where this map is not surjective.

Proof. We have mappings from $X \to Y_i$ in each column:

$$\begin{array}{ccc} x_1 \mapsto y_{1_1} & x_1 \mapsto y_{2_1} & x_1 \mapsto y_{i_1} \\ x_2 \mapsto y_{1_2} & x_2 \mapsto y_{2_2} & x_2 \mapsto y_{i_2} \\ \vdots & \vdots & \vdots \\ x_n \mapsto y_{1_n} & x_n \mapsto y_{2_n} & x_n \mapsto y_{i_n} \end{array}$$

Where the basis vectors of X map to vectors of Y_i. Then we can map this information to:

$$\begin{aligned} x_1 &\mapsto (y_{1_1}, y_{2_1}, \ldots, y_{i_1}) \\ x_2 &\mapsto (y_{1_2}, y_{2_2}, \ldots, y_{i_2}) \\ &\vdots \\ x_n &\mapsto (y_{1_n}, y_{2_n}, \ldots, y_{i_n}) \end{aligned}$$

which is an element in $\mathrm{Hom}(X, \bigoplus_i Y_i)$. ☐

Exercise 5: Show that the functor $\mathrm{Ab} \to$ Group from Abelian groups to all groups does not admit a right adjoint.

Proof. ☐