Math128aHw12

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Exercise Set 6.1

Exercise 6: Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

d.

$$\begin{aligned} x_1 + x_2 + x_4 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \end{aligned}$$

Answer. Write it as a matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & 5 & 5 & -1 & 9 \\ -1 & 2 & 3 & -1 & 4 \\ 0 & 5 & 8 & -1 & 9 \end{bmatrix}$$

which becomes:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & 5 & 5 & -1 & 9 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & 0 & 3 & 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 5 & -1 & 9 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

So we get $x_1 = -1, x_2 = 2, x_3 = 0, x_4 = 1$.

Exercise 9: Given the linear system

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

a. Find value(s) of α for which the system has no solutions.

Answer. Multiply both equations by:

$$6\alpha x_1 - 18\alpha^2 x_2 = 9\alpha$$
$$-6\alpha x_1 + 2x_2 = -3$$

So we get the matrix:

$$\begin{bmatrix} 6\alpha & -18\alpha^2 & 9\alpha \\ 0 & 2-18\alpha^2 & 9\alpha - 3 \end{bmatrix}$$

There are no solutions when $2 - 18\alpha^2 = 0$ or $\alpha = \pm\frac{1}{3}$. But $9\alpha - 3$ should also be nonzero, so no solutions when $\alpha = -\frac{1}{3}$.

b. Find value(s) of α for which the system has an infinite number of solutions.

Answer. From the previous problem, there are infinite solutions when $\alpha = \pm\frac{1}{3}$, or when both $2 - 18\alpha^2 = 0$ and $9\alpha - 3 = 0$.

c. Assuming a unique solution exists for a given α, find the solution.

Answer. Reduce the matrix from part a:

$$\begin{bmatrix} 1 & -3\alpha & 3/2 \\ 0 & 1 & \dfrac{9\alpha - 3}{2 - 18\alpha^2} \end{bmatrix}$$

So we get:

$$x_1 = \frac{3}{2} - 3\alpha\left(\frac{9\alpha - 3}{2 - 18\alpha^2}\right)$$
$$x_2 = \frac{9\alpha - 3}{2 - 18\alpha^2}$$

Exercise 12: A Fredholm integral equation of the second kind is an equation of the form

$$u(x) = f(x) + \int_a^b K(x,t)u(t)\,dt\,,$$

where a and b and the functions f and K are given. To approximate the function u on the interval $[a, b]$, a partition $x_0 = a < x_1 < \cdots < x_{m-1} < x_m = b$ is selected, and the equations

$$u(x_i) = f(x_i) + \int_a^b K(x_i, t)u(t)\,dt$$

are solved for $u(x_0), u(x_1), \ldots, u(x_m)$. The integrals are approximated using quadrature formulas based on the nodes $x_0, \ldots, x_m$. In our problem, $a = 0, b = 1, f(x) = x^2$, and $K(x,t) = e^{|x-t|}$.

a. Show that the linear system

$$u(0) = f(0) + \frac{1}{2}[K(0,0)u(0) + K(0,1)u(1)]$$
$$u(1) = f(1) + \frac{1}{2}[K(1,0)u(0) + K(1,1)u(1)]$$

must be solved when the Trapezoidal rule is used.

Answer. In the trapezoidal rule, we take 1/2 of the sum of the endpoints. So for the first node $x_0 = a = 0$, we have:

$$\int_a^b K(x_0, t)u(t)\ dt \approx \frac{1}{2}[K(x_0, 0)u(0) + K(x_0, 1)u(1)]$$

and for $x_1 = b = 0$:

$$\int_a^b K(x_1, t)u(t)\ dt \approx \frac{1}{2}[K(x_1, 0)u(0) + K(x_1, 1)u(1)]$$

putting this with $u(x)$, we get the equations above.

b. Set up and solve the linear system that results when the Composite Trapezoidal rule is used with $n = 4$.

c. Repeat part (b) using the composite Simpson's rule.

Exercise 13: Show that the operations

a. $(\lambda E_i) \to (E_i)$.

b. $(E_i + \lambda E_j) \to (E_i)$

c. $(E_i) \iff (E_j)$

do not change the solution set of a linear system.

a. $(\lambda E_i) \to (E_i)$.

Answer. If E_i represents the coefficients at row i, we have the relation:

$$a_{i,0}x_{i,0} + \cdots + a_{i,n}x_{i,n} = b_i$$

So if $x_{i,0}, \dots, x_{i,n}$ is our solution, we see that it still remains the solution in:

$$\lambda(a_{i,0}x_{i,0} + \cdots + a_{i,n}x_{i,n}) = \lambda b_i$$

for any λ.

We also see that if y is not a solution, then y is not a soluiton under the new coefficients also.

b. $(E_i + \lambda E_j) \to (E_i)$

Answer. Recall

$$a_{i,0}x_{i,0} + \cdots + a_{i,n}x_{i,n} = b_i$$

and let the product of the coefficients with $x_{i,0}, \dots, x_{i,n}$ be the linear transformation. Then we have:

$$T_i(x) + \lambda T_j(x)$$

where T_i represents a dot product with x. Then by the properties of a dot product:

$$T_i(x) + \lambda T_j(x) = (T_i + \lambda T_j)(x) = b_i + \lambda b_j$$

and therefore we see that x remains a solution under the new coefficients.

It is also clear that if y is a vector that is not in the solution set, it is also not a solution under the new coefficients.

c. $(E_i) \iff (E_j)$

Answer. If x is a solution to $E_1, \dots, E_n$, permuting the E_i does not change the fact that x remains a solution. And if y does not satisfy one of the E_i, permuting the E_i does not change the fact that y does not satisfy one of the E_i.

do not change the solution set of a linear system.

Exercise Set 6.2

Exercise 2: Find the row interchanges that are required to solve the following linear systems using Algorithm 6.1.

a.

$$\begin{aligned} 13x_1 + 17x_2 + x_3 &= 5 \\ x_2 + 19x_3 &= 1 \\ 12x_2 - x_3 &= 0 \end{aligned}$$

Answer. I got $x_1 = 0.3743$, $x_2 = 0.0046$, $x_3 = 0.0553$. Here is my code:

```
function out = GEBackward(A)
n = size(A, 1);
out.flag = 0;

for i = 1:n-1
    p = -1;
    for j = i:n
        if A(j, i) ~= 0
            p = j;
            break
        end
    end
    if p == -1
        % no solution exists
        out.flag = 1;
        return;
    end

    if p ~= i
        % interchange
        tmp = A(i, :);
        A(i, :) = A(p, :);
        A(p, :) = tmp;
    end

    for j = i+1:n
        m = A(j, i) / A(i, i);
        A(j, :) = A(j, :) - m * A(i, :);
    end
end

if A(n, n) == 0
    out.flag = 1;
    return;
end

out.x = zeros(n, 1);
out.x(end) = A(n, n + 1) / A(n, n);
for i = n-1:-1:1
    out.x(i) = (A(i, n + 1) - dot(A(i, i+1:n), out.x(i+1:n))) / A(i, i);
end

end
```

```
A = [
    13, 17,  1, 5;
    0,   1, 18, 1;
    0,  12, -1, 0;
    ];

out = GEBackward(A);
out.flag
out.x
```

Exercise 4: Repeat Exercise 2 using Algorithm 6.2.

Answer. Here is the code:

```
function out = GEPP(A)
n = size(A, 1);
NROW = 1:n;
out.flag = 0;
out.x = zeros(n, 1);

for i = 1:n-1
    p = -1;
    m_p = -inf();
    for j = i:n
        if abs(A(NROW(j), i)) > m_p
            p = j;
            m_p = abs(A(NROW(j), i));
        end
    end

    if A(NROW(p), i) == 0
        out.flag = 1;
        return
    end

    if NROW(i) ~= NROW(p)
        NCOPY = NROW(i);
        NROW(i) = NROW(p);
        NROW(p) = NCOPY;
    end

    for j = i+1:n
        m = A(NROW(j), i) / A(NROW(i), i);
        A(NROW(j), :) = A(NROW(j), :) - m * A(NROW(i), :);
    end

end

if A(NROW(n), n) == 0
    out.flag = 1;
    return
end

out.x(n) = A(NROW(n), n + 1) / A(NROW(n), n);

for i = n-1:-1:1
```

```
    num = A(NROW(i), n + 1) - dot(A(NROW(i), i+1:n), out.x(i+1:n));
    dom = A(NROW(i), i);
    out.x(j) = num / dom;
end

end

A = [
    13, 17,  1, 5;
    0,   1, 18, 1;
    0,  12, -1, 0;
    ];

out = GEPP(A);
out.flag
out.x
```

Exercise 6: Repeat Exercise 2 using Algorithm 6.3.

Exercise Set 6.3

Exercise 6: Determine which of the following matrices are nonsingular and compute the inverse of those matrices:

a.
$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix}$$

Answer. Row reduce
$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 6 & 2 \end{bmatrix}$$
and
$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -10 \end{bmatrix}$$
so it is invertible.

Exercise 9:

a. The product is as shown because of the linearity of matrix products. We can take multiply matrices block wise to get AB.

b. No because the dimensions do not match up for the matrix multiplication.

c. The vertical split of A must match up with the size of the horizontal split of B.