Math104FinalPractice

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Exercise 1:

- Suppose that $A \subseteq \mathbb{R}$ is a finite set. Prove that A cannot be dense in $\mathbb{R}$.

  Proof. Suppose that A if finite, for contradiction. Then there exists a supremum of the set S. So for all $a \in A$, we have $a \leqslant S$. Then we have

  $$A \cap (S, \infty) = \emptyset$$

  demonstrates and empty intersection with an open set. So A is not dense. $\square$

- Suppose that $A \subseteq \mathbb{R}$ satisfies the property:

  $$A \text{ is bounded above, and } \sup A \notin A.$$

  First, give an example of a set A which satisfies $(*)$. Then, prove that any set A which satisfies $(*)$ must have infinitely many elements.

  Proof. One such set is $A = (0, 1)$. We have $\sup A = 1 \notin (0, 1)$. Suppose that we have a set that is bounded above and that $\sup A \notin A$. Then for all $a \in A$, we have that $a \leqslant \sup A$. Also, if we have M such that $a \leqslant M$ for all $a \in A$, then $\sup A \leqslant M$. Notice that our set must be non-empty. Then there exists an element such that $a_0 < \sup A$. Since a_0 is not the supremum, then there exists an element not in A larger. So $a_0 < a_1 < \sup A$. But we can continue this process forever. So the size of the set is infinite. $\square$

Exercise 2:

- Using the $\varepsilon - \delta$ definition of continuity, prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 6x - 5$ is continuous on $\mathbb{R}$.

  Proof. We need to show that $\forall \varepsilon > 0$, we have that $\exists \delta > 0$ such that if

  $$|x - y| < \delta$$

  then

  $$|f(x) - f(y)| < \varepsilon$$

  So we want to find when

  $$\begin{aligned} |6x - 5 - (6y - 5)| &< \varepsilon \\ |6x - 6y| &< \varepsilon \\ 6|x - y| &< \varepsilon \\ |x - y| &< \frac{\varepsilon}{6} \end{aligned}$$

  So there is a $\delta = \frac{\varepsilon}{6}$. And we are done. $\square$

- Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Must the set $g([0,1]) \subseteq \mathbb{R}$ be bounded? Must it be closed? Justify your answers.

  Proof. We know that $g([0,1])$ is bounded. Suppose wlog that $g([0,1])$ is unbounded above. Then we have that there is a sequence $g(x_n)$ such that

  $$g(x_1) < g(x_2) < \cdots$$

  diverges to ∞. Or we have $(a_n) < (g(x_n))$, where $a_n = n$. □

Exercise 3:

- Prove that the series $\sum_{n=1}^{\infty} \frac{2+\sin n}{n^3+5}$ converges.

  Proof. By comparison test, we have

  $$\frac{2+\sin n}{n^3+5} \leqslant \frac{3}{n^3}$$

  we also see that

  $$\frac{3}{n^3} \leqslant \frac{1}{n^2}$$

  when $n \geqslant 3$. Since $\sum \frac{1}{n^2}$ converges, we have that $\sum_{n=1}^{\infty} \frac{2+\sin n}{n^3+5}$. □

- Let $(a_n)_{n=1}^{\infty}$ be a sequence of strictly positive real numbers which satisfies $\lim_{n\to\infty} a_n = 2$. Determine if the series

  $$\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot a_n}$$

  converges (always, sometimes, or never). Justify your answer.

  Proof. The sequence always converges. Because we have that $\lim_{n\to\infty} a_n = 2$, this means that there is an N such that $\forall n > N$, we have

  $$|a_n - 2| < 1$$

  This means that

  $$1 < a_n < 3$$

  So we have that

  $$\left|\frac{1}{n^2 \cdot a_n}\right| \leqslant \frac{1}{n^2}$$

  and that the RHS converges. Then so does the LHS by the comparison test. Since it converges when we sum for terms above N and for terms below N, we have that the entire sum converges. □

Exercise 4: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$.

- Prove that

  $$\lim_{h\to 0^+} \frac{1}{h} \int_0^h f(x)\, dx = f(0)$$

  Proof. Since f is continuous, then it is integrable, and we have

  $$\lim_{h\to 0^+} \frac{1}{h}(F(h) - F(0)) = \lim_{h\to 0^+} F'(h) = f(0)$$

  So this is the answer. □

- Prove that if

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^{1+h} f(x)\, dx \text{ exists (i.e. is some real number)},$$

then $\int_0^1 f(x)\, dx = 0$.

Proof. □

Exercise 5:

- Consider the power series $\sum_{n=0}^{\infty}(\frac{1}{5})^n x^n$. Find all points $x \in \mathbb{R}$ for which this series converges.

Proof. This converges when

$$\limsup \left|\left(\frac{1}{5}\right)^n x^n\right|^{\frac{1}{n}} < 1$$

So we require

$$|x| < 5$$

Now, checking the endpoints, we have

$$\sum_{n=0}^{\infty} 1^n$$

and

$$\sum_{n=0}^{\infty} (-1)^n$$

which both diverge. So the radius of convergence is $(-5, 5)$. □

- Let $f(x) = \sum_{n=0}^{\infty}(\frac{1}{5})^n x^n$ for all x values for which the right-hand side is well-defined. Is f differentiable at $x = 0$? If so, explain why and find $f'(0)$.

Proof. f is differentiable at $x = 0$ because the sum converges at $x = 0$. We need to verify that by the definition:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

converges. Then this is

$$\lim_{x\to 0} \frac{\sum_{n=0}^{\infty}(\frac{1}{5}x)^n - 1}{x} = \lim_{x\to 0} \frac{\sum_{n=1}^{\infty}(\frac{1}{5}x)^n}{x} = \lim_{x\to 0} \sum_{n=1}^{\infty}(\frac{1}{5})^n x^{n-1} = \frac{1}{5}$$

So this is the limit and the derivative. □

Exercise 6: For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be the function which satisfies

$$f_n(x) = \begin{cases} n & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{if } x \in \{0\} \cup [\frac{1}{n}, 1]. \end{cases}$$

- Find the function $f : [0,1] \to \mathbb{R}$ so that $f_n \to f$ pointwise in $[0,1]$, and justify.

Proof. The function converges to 0. We need to show that for all $x \in [0,1]$, we have for all $\varepsilon > 0$:

$$|f_n(x)| < \varepsilon$$

If we pick $n = \varepsilon/2$, we see that the function has a value of either $\varepsilon/2$ or 0 which is less than ε. □

- Determine if $f_n \to f$ uniformly on $[0, 1]$, and explain.

  Proof. It converges uniformly because n does not depend on x. □

Exercise 7:

- Show that if f is continuous with $f \geqslant 0$ and $\int_0^1 f(x)\, dx = 0$, then $f \equiv 0$.
- Give an example which shows the statement in the previous part may not be true if we do not assume f is continuous, and justify.

Exercise 8: Suppose that $(x_n)_{n=1}^{\infty}$ is convergent sequence in $\mathbb{R}$.

- Show that the set $\{x_n : n \in \mathbb{N}\}$ is compact. The set should be $\{x_n : n \in \mathbb{N}\} \cup \{\lim_{n\to\infty} x_n\}$.

  Proof. We have that the set is bounded because it is a convergent sequence. This means that there is an N such that $\forall n > N$, we have

  $$|x_n - L| < 1$$

  Then we have that

  $$-1 + L < x_n < 1 + L$$

  So we see that (x_n) is bounded above by $\max(1 + L, x_0, x_1, \ldots, x_N)$ and below by $\min(-1 + L, x_0, x_1, \ldots, x_N)$. To show that it is closed, we need to prove that all the limits of any sequence in $M = \{x_n : n \in \mathbb{N}\} \cup \{\lim_{n\to\infty} x_n\}$ lies within the set. Suppose for contradiction that we had some other convergent series (y_n) where $y_n \in M$. Then we have

  $$\lim_{n\to\infty} y_n = L' \neq L$$

  Since there is infinitely many y_n, we must have some $x_m = y_n$ for any $m > N$. Consider the difference $|L' - L|$. First, we consider distances between y_i by the cauchy criterion, there is a B such that for all $a, b > B$, we have

  $$|y_a - y_b| < |L' - L|/3$$

  Now we also have a B' such that for all $b' > B'$,

  $$|y_{b'} - L'| < |L' - L|/3$$

  and finally, there is a B'' such that for some $b'' > B''$, we have

  $$|y_{b''} - L| < |L' - L|/3$$

  Now take $B''' = \max(B, B', B'')$, and we see that this situation is impossible, for some $y_{b'''}$ where $b''' > B'''$. □

- Show that the set $\{x_n : n \in \mathbb{N}\}$ is not connected. The set should be assumed to have at least two distinct elements.

  Proof. Suppose for contradiction that it was connected. Then we have that $J = \{x_n : n \in \mathbb{N}\}$ is a union of two open sets A_1, A_2 such that

  - $(A_1 \cap J) \cup (A_2 \cap J) = J$,
  - $A_1 \cap J, A_2 \cap J \neq \emptyset$,
  - $(A_1 \cap J) \cap (A_2 \cap J) = \emptyset$.

  □

Exercise 9: Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable (i.e. f' and f'' exist and are continuous on all of $\mathbb{R}$) with $f(0) = 0$ and $f'(0) < 0$.

- Prove that there exists $\varepsilon > 0$ such that

$$\frac{f(x)}{x} < 0$$

for all $x \in (-\varepsilon, \varepsilon)\backslash\{0\}$.

Proof. We see that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

Since $\frac{f(x)}{x}$ converges to some negative value, we have for $\varepsilon = -\frac{f'(0)}{2}$ there is a δ such that for

$$|x| < \delta$$

we have

$$\left|\frac{f(x)}{x} - f'(0)\right| < \varepsilon$$

so this shows that $\frac{f(x)}{x}$ is less than 0 in an interval. □

- Suppose $f(1) = 1$ (recall also from above that $f(0) = 0$ and $f'(0) < 0$). Show that for some $x \in [0, 1]$ we must have $f''(x) > 0$.

Proof. By the MVT, we can see that there is an x_0 in $(0, 1)$ such that $f'(x_0) = 1$. Now applying the mean value theorem again, there exists an x_0' between $(0, x_0)$ such that

$$f''(x_0') = \frac{f'(x_0) - f'(0)}{x_0 - 0}$$

Then

$$f''(x_0') = \frac{1 - f'(0)}{x_0}$$

and because the numerator and denominator are positive, we have $f''(x_0') > 0$. □