Math143Hw4

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September 21, 2023

Exercise 1: Let R be an integral domain and let

$$\mathtt{Frac}(R) = \{a/b : a, b \in R, b \neq 0\}/\sim$$

where $a/b \sim c/d$ if $ad = bc \in R$.

(a) Prove that the map $\iota : R \longrightarrow \mathtt{Frac}(R)$ that sends $a \mapsto a/1$ is injective.

Proof. Suppose that we have $\iota(r_1) = \iota(r_2)$. Then $r_1/1 = r_2/1$. But by the equivalence relation, we have that $1 \cdot r_1 = 1 \cdot r_2$ which means that $r_1 = r_2$. So the mapping is injective. □

(b) Prove that if K is a field and $\varphi : R \longrightarrow K$ is any homomorphism, then there exists a map $\beta : \mathtt{Frac}(R) \longrightarrow K$ such that $\varphi = \beta \circ \iota$.

Proof. For any homomorphism φ, we can say that

$$\varphi(r_i) = k_i$$

for $r_i \in R, k_i \in K$. Then define the mapping:

$$\beta(r_i/1) = k_i = \varphi(r_i)$$

We will show that β is a homomorphism:

$$\begin{aligned}\beta(r_i/1)\beta(r_j/1) &= \varphi(r_i)\varphi(r_j)\\ &= \varphi(r_i r_j)\\ &= \beta(r_i r_j/1)\end{aligned}$$

and for addition:

$$\begin{aligned}\beta(r_i/1) + \beta(r_j/1) &= \varphi(r_i) + \varphi(r_j)\\ &= \varphi(r_i + r_j)\\ &= \beta((r_i + r_j)/1)\end{aligned}$$

since

$$\beta(\iota(r_i)) = \beta(r_i/1) = \varphi(r_i)$$

we are done. □

Exercise 2:

(a) Suppose $\psi : k \longrightarrow R$ is a ring homomorphism with k a field. Prove that ψ is either injective or the zero map.

Proof. Suppose that ψ is not injective. We will prove that it is the zero map. Then we have a nontrivial kernel with an element $g \in K$. Then (g) generates the entire field k. But $g \mapsto 0$. So we say that for $k \in K$ $ag = k$ for some $a \in K$ and $\psi(ag) = 0$. So we are done. □

(b) Does there exist a surjective map $k[x_1, \ldots, x_n] \to k(y)$? Give an example explain why none exists. (You may quote results from class.)

Exercise 3: Practice with quotients:

(a) Let k be any field and $f \in k[x]$ a polynomial of degree $n > 0$. Show that images of $1, x, \ldots, x^{n-1}$ in $k[x]/(f)$ form a basis for $k[x]/(f)$ as a vector space over k.

Proof. Let $f = a_d x^d + a_{d-1}x^{d-1} + \cdots + ax + 1$. We know that every polynomial in $k[x]/(f)$ is of degree $< d$, since otherwise, we can use the substitution:

$$x^d = -a_d^{-1}(a_{d-1}x^{d-1} + \cdots + ax + 1)$$

Since any polynomial of degree $< d$ can be written as a linear combination of $1, x, \ldots, x^{n-1}$, then $1, x, \ldots, x^{n-1}$ generate $k[x]/(f)$. Now suppose that

$$K = k_0 + k_1 x + \cdots + k_{n-1}x^{n-1} = 0$$

Then $K \in (f)$ but K has degree $< d$ so it must be $0 \in (f)$. But the polynomial K as a degree $\leqslant n-1$ has at most $n-1$ roots. Therefore, it cannot have degree $\geqslant 0$. So all coefficients $k_i = 0$ which shows linear independence. □

(b) Let $I \subseteq k[x, y]$ be the ideal generated by monomials of degree d

$$I = (\{x^i y^j : i + j = d\})$$

What is the dimension of $k[x, y]/I$?

Proof. We can create an array of such x, y pairs:

$$\begin{array}{ccccc}
(0,0) & (1,0) & (2,0) & \cdots & (d,0) \\
(0,1) & (1,1) & (2,1) & \cdots & (d,1) \\
(0,2) & (1,2) & (2,2) & \cdots & (d,2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(0,d) & (1,d) & (2,d) & \cdots & (d,d)
\end{array}$$

If we draw a line through the diagonal that goes through $(d, 0)$ and $(0, d)$, we notice that all polynomials denoted by those points will lie in I. So the basis is given by everything above that diagonal to which there are $d(d-1)/2$ elements. □

(c) (Optional, Extra Credit) Let $I \subseteq k[x_1, \ldots, x_n]$ be the ideal generated by monomials of degree d

$$I = (\{x_1^{i_1} \cdots x_n^{i_n} : i_1 + \cdots + i_n = d\}).$$

What is the dimension of $k[x_1, \ldots, x_n]/I$?

Proof. We can generalize the method used at the top:

$$\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
n=0 & n=1 & n=2 & n=3 & n=4 & n=5 & n=6
\end{array}$$

Note that the n denotes the diagonal of the triangle. So on n variables, we wish to compute the cumulative sum of the column, which is given by the values of the next column by the hockey stick theorem. So for degree d, we need to look at the $d-1$-th row of that column. Therefore, we have a dimension of $\binom{n+d-1}{d-1}$. □

Exercise 4: Suppose $P_1, \ldots, P_m$ are distinct points in $\mathbb{A}^n$. Prove that, for each j, there exists a polynomial f such that $f(P_i) = 0$ if $i \neq j$ and $f(P_j) = 1$.

Proof. We can find a polynomial that kills one point. Let us do this for P_1. Notice that we have

$$P_1 = (q_1, q_2, \ldots, q_n)$$

an n-tuple of q_i's. Then taking the polynomial:

$$f_1(x_1, \ldots, x_n) = (x_1 - q_1) + \cdots + (x_n - q_n)$$

will send this point to 0. We can do this because the q_i are in our underlying field. Since all P_i are distinct, we can take for some $i \neq 1$, P_i where $P_i - P_1 \neq (0, 0, \ldots, 0)$. Therefore,

$$f_1(P_i) = r \neq 0$$

So we can take an inverse:

$$r^{-1} f_1(P_i) = 1$$

So we have found a polynomial f_1 that sends $P_1 \mapsto 0$ and $P_2 \mapsto 1$. In general, we can do this process for any two pairs of points. Suppose we are given points $P_1, \ldots, P_m$. Then we can find polynomials f_{ij} such that:

$$f_{ij}(P_k) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k = j \end{cases}$$

Consider the product:

$$f_i = \prod_{s \neq i \geqslant 1}^{m} f_{is}$$

Then $f_i(P_s) = 0$ for when $i \neq s$ since f_{is} is a factor of f_i and f_{is} sends P_s to 0. Now when $i = s$, we have that:

$$f_i(P_s) = f_{i1}(P_s) \cdot f_{i2}(P_s) \cdots f_{im}(P_s)$$

But all f_{ij} are equal to 1, since $s = i$. So $f_i(P_s) = 1$. So we have found such a polynomial. We can repeat this process for the other points. □

Exercise 5: Suppose $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^r$ are algebraic sets and $\varphi : X \to Y$ and $\psi : Y \to Z$ are polynomial maps.

(a) Show that the composition $\psi \circ \varphi : X \to Z$ is a polynomial map.

Proof. If φ is a polynomial map, that means that there exist φ_i such that

$$\varphi(p) = (\varphi_1(p), \varphi_2(p), \ldots, \varphi_r(p))$$

And the same for ψ:

$$\psi(p) = (\psi_1(p), \psi_2(p), \ldots, \psi_s(p))$$

where $Z \subseteq \mathbb{A}^s$. So now we look at the composition $\psi \circ \varphi$:

$$(\psi \circ \varphi)(p) = (\psi_1((\varphi_1(p), \ldots, \varphi_r(p))), \ldots, \psi_s((\varphi_1(p), \ldots, \varphi_r(p))))$$

Since we have $\psi_i \in k[x_1, \ldots, x_r]$, we observe that $\psi_i(\varphi_1(p), \ldots, \varphi_r(p))$ is just the substitution of

$$\begin{aligned} x_1 &= \varphi_1(p) \\ x_2 &= \varphi_2(p) \\ &\vdots \\ x_r &= \varphi_r(p) \end{aligned}$$

Indeed this gives us a polynomial in $k[x_1, \ldots, x_n]$ since each of the $\varphi_i \in k[x_1, \ldots, x_n]$. So we know there are polynomials $\pi \in k[x_1, \ldots, x_n]$ such that

$$(\psi \circ \varphi)(p) = (\pi_1(p), \ldots, \pi_s(p))$$

□

(b) Show that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Proof. Suppose that $\psi : Y \to Z$ and $\varphi : X \to Y$. We just need to look at the action of $(\psi \circ \varphi)^*$ on f and $\varphi^* \circ \psi^*$ on $f \in \Gamma(Z)$:

$$\begin{aligned} (\psi \circ \varphi)^*(f) &= f \circ (\psi \circ \varphi) \\ &= f \circ \psi \circ \varphi \end{aligned} \qquad \begin{aligned} (\varphi^* \circ \psi^*)(f) &= \varphi^*(f \circ \psi) \\ &= (f \circ \psi) \circ \varphi \end{aligned}$$

and indeed they are equal. □