Math172Hw12

Trustin Nguyen

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Exercise 1:

- Let G be a graph obtained from K_6 by removing two edges. Is it possible that G is planar?

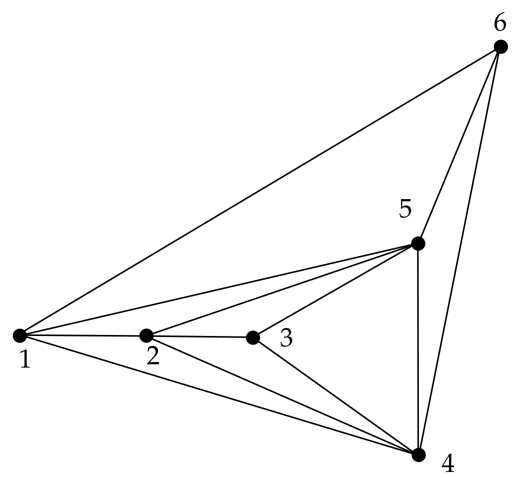
  Proof. We can use the fact that if we have a planar connected graph, then $E \leqslant 3V - 6$, for connected graphs with at least 3 vertices. Then we have $|E| = \binom{6}{2} - 2$ and $|V| = 6$. Then $|E| = 15 - 2 = 13$. But

  $$13 = |E| > 3|V| - 6 = 12$$

  which shows that K_6 with two edges removed is not planar. □

- Let G be a graph obtained from K_6 by removing three edges. Is it possible that G is planar?

  Answer. Yes it is possible. We have the graph:



  We have that the vertex 1 is missing the edge to 3. 6 is missing an edge to 3 and 2. Adding each of these edges in gives every vertex degree 5, which means that this is K_6 with 3 edges removed.

Exercise 2: Let G be a convex octagon, and let S be a set of 10 points inside G in general position (no three on the same line). Assume, after drawing some non-intersecting straight segments between points in S and vertices of G, we have split the interior of G into triangles, with the vertices of these triangles being either a vertex of G or a point in S and every point of S being a vertex of some triangle. How many triangles are formed? (You can assume without proof that the resulting graph on 18 vertices, with edges being the drawn line-segments and the sides of G, is connected).

Proof. We have by Euler's formula that $V + F - E = 2$. Then we know that $|V| = 18$. Notice that each face of the triangle uses 3 oriented edges, except for the face that is on the exterior of the octagon, which uses 8 oriented edges. There are $2|E|$ oriented edges total, so we have:

$$3(|F| - 1) + 8 = 2|E|$$

and

$$18 + |F| - |E| = 2$$

So we must solve the system:

$$\begin{aligned} 3|F| - 2|E| &= -5 \\ |F| - |E| &= -16 \end{aligned}$$

We have:

$$\begin{aligned} |E| &= 43 \\ |F| &= 27 \end{aligned}$$

There are 27 faces total, minus the exterior face, we have 26 faces in the octagon. So there are 26 triangles. □

Exercise 3: In class we prove that every planar graph should have a vertex of degree at most 5. In this problem we construct an example showing that this bound is strict.

- Let G be a simple planar connected graph with all vertices having degree at least 5. Show that G has at least 12 vertices.

  Proof. Since every vertex has degree $\geqslant 5$, we know that $2|E| \geqslant 5|V|$. Since G is simple planar connected, we also know

  $$|E| \leqslant 3|V| - 6$$

  Suppose for contradiction that G has less than 12 vertices. Then

  $$2|E| \geqslant 5 \cdot 11 = 55$$

  so

  $$|E| \geqslant 28$$

  But using the planar bound above, we get:

  $$|E| \leqslant 33 - 6 = 27$$

  which is a contradiction. □

- Show that if G from part (1) has exactly 12 vertices, then it has 20 faces, all faces are triangles and all vertices have degree exactly 5.

  Proof. Using the planar bound above, we know that

  $$|E| \leqslant 3(12) - 6 = 30$$

  Then we have that all degrees are at least 5, so the sum of all degrees is at least $5|V| \leqslant 2|E|$ o r

  $$60 \leqslant 2|E| \implies |E| \geqslant 30$$

  Therefore, $|E| = 30$. Then using Euler's formula that:

  $$|V| + |F| - |E| = 2$$

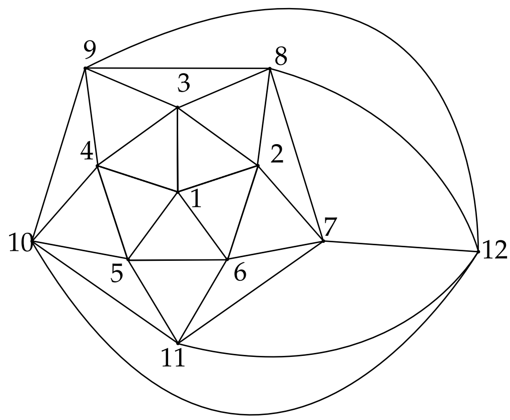
  we get

  $$12 + |F| - 30 = 2$$

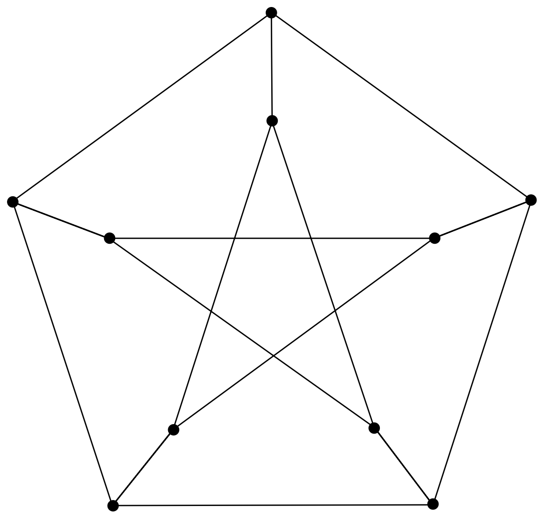
  Then $|F| = 20$. All vertices have degree 5 because there are 30 edges, so total degree is 60. Each vertex has degree at least 5, but if a vertex has degree more than 5, the total degree will exceed 60. So each vertex has degree exactly 5. □

- Construct G as in part (2).

  Answer.

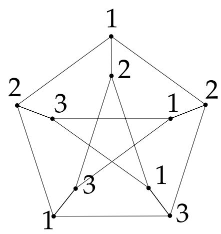


Exercise 4: Consider the following graph G with 10 vertices and 15 edges:



- Find the chromatic number $\chi(G)$ of this graph.

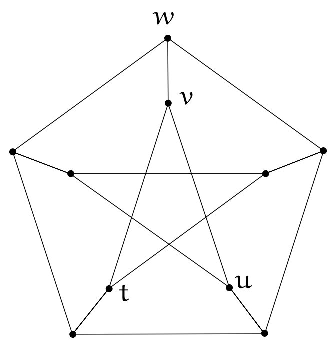
  Proof. The chromatic number is greater than 2, because there is an odd cycle. We also see from the coloring:



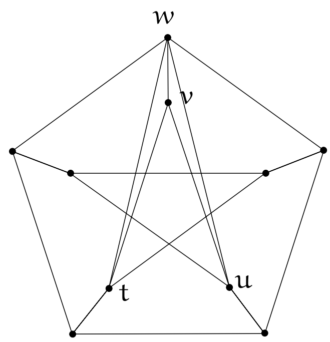
  That the graph is 3 colorable. So $\chi(G) = 3$. □

- Is G planar or not?

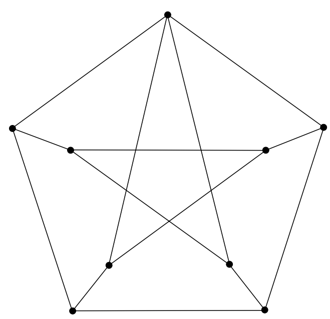
  Proof. Suppose for contradiction that G is planar. Then we have some planar drawing of G with no self-intersections. Consider the vertices t, u, v, w shown below:



Consider the plane containing the vertices u, v, w. We know that such a plane exists in some planar drawing. This is because if we travel from $u \to v$, it is the edge immediately oriented to the right. Since these lie in the same plane, then there are no edges in the interior, so we can draw an edge u, w, which still makes it planar. Similarly, we can draw an edge t, w by the same reasoning. So we get:



Now we can remove the edges $(t, v), (v, w), (u, v)$ to get a planar graph:



Repeating this on the other corners of the star, we get that K_5 is planar, which is a contradiction. □