# Math110Hw4

Trustin Nguyen

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## Homework 4

**Exercise 1**: Let $V = \mathcal{P}_2(\mathbb{R}), W = \mathbb{R}$. Are the maps

$$T : f \mapsto f(2), S : f \mapsto \int_0^1 f(x)\,\mathrm{d}x$$

in $\mathcal{L}(V, W)$? Are they linearly independent?

*Proof.* We see what happens with $a_1T + a_2S = 0$. Let $f$ be an arbitrary function. Then we have

$$a_1Tf_1 + a_2Sf_1 = 0a_1Tf_2 + a_2Sf_2 = 0$$

Let $f_1 = 3x^2$

$$a_1Tf_1 + a_2Sf_1 = 12a_1 + a_2 = 0$$

Where $a_2 = -12a_1$. But for $f_2 = 2x$,

$$a_1Tf_2 + a_2Sf_2 = 4a_1 + a_2$$

Where $a_2 = -4a_1$. So by the two equations,

$$a_2 = -12a_1a_2 = -4a_1 \qquad -4a_1 = -12a_1$$

So $a_1 = 0$ and that means $a_2 = 0$. The linear maps are linearly independent. □

**Exercise 2**: Suppose $V$ is a nonzero finite-dimensional vector space and $W$ is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

*Proof.* It was proved in class that there was a bijection from

$$M : \mathcal{L}(V, W) \to \mathbb{F}^{\dim W \times \dim V}$$

(Injective) Suppose that $M(R) = M(S)$. Then $M(R - S) = 0$. We look at an element in the null-space $T$. Then $Tv_0 = 0$ for any basis vector of $V$ and therefore, any $v \in V$. So $R - S = 0$ and $R = S$.

(Surjective) Suppose that there is a $W$ in the image of the linear transformation in $\mathcal{L}(V, W)$. Let $\{w_1, \ldots, w_{\dim W}\}$ be the basis vectors of W.

$$A = \begin{bmatrix} a_{1,1} & \ldots & a_{1,v} \\ \vdots & \ddots & \vdots \\ a_{w,1} & \ldots & a_{w,v} \end{bmatrix}$$

If $Tv_i$ were set to equal the linear combination of $\{w_1, \ldots, w_{\dim W}\}$ with coefficients "$a$" of the $i - th$ column, then that is a defined linear mapping. Since there is a bijection, the dimensions of $\mathcal{L}(V, W)$ and $\mathbb{F}^{\dim W \times \dim V}$ are equal. Since $W$ is infinite dimensional, then $\mathcal{L}(V, W)$ is infinite dimensional. $\square$

**Exercise 3**: Suppose $V$ is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\text{range}S \subset \text{null}T.$$

Prove that $(ST)^2 = 0$.

*Proof.* We start from $ST^2 = STST$. Let $v \in V$. Then

$$STSTv = STS(Tv)$$

Notice that $S(Tv) \in \text{range}S$. Then $S(Tv) \in \text{null}T$. Therefore, $TS(Tv) = 0$. So the equation can be simplified down to

$$S(0) = 0$$

since linear maps send 0 to 0. $\square$

**Exercise 4**: Suppose $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ is defined by the formula $(Tf)(x) = 2xf''(x) - f'$. Check $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ and find a basis for the null space and a basis for the range of $T$.

*Proof.* (Linearity) To check that $T$ is linear, we have two properties:

1. $T(f_1 + f_2) = T(f_1) + T(f_2)$:
$$\begin{aligned} T(f_1 + f_2) &= 2x(f_1 + f_2)''(x) - (f_1 + f_2)' \\ &= 2xf_1''(x) + 2xf_2''(x) + f_1' + f_2' \\ &= T(f_1) + T(f_2) \end{aligned}$$

2. $T(\lambda f) = \lambda T(f)$
$$\begin{aligned} T(\lambda f) &= 2x(\lambda f)''(x) - (\lambda f)' \\ &= 2x\lambda f''(x) - \lambda f' \\ &= \lambda T(f) \end{aligned}$$

(Null Space) To find the basis for the null space, we must find an $f$ such that :

$$\begin{gathered}(Tf)(x) = 2xf''(x) - f' = 0 \\ f' = 2xf''(x)\end{gathered}$$

So we are looking at a function $f(x) = ax^3 + bx^2 + cx + d$ such that the equation is satisfied

$$\begin{aligned}f(x) &= ax^3 + bx^2 + cx + d \\ f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 6ax + 2b \\ f'(x) &= 2xf''(x) \\ 3ax^2 + 2bx + c &= 12ax^2 + 4bx \\ 9ax^2 + 2bx - c &= 0\end{aligned}$$

Therefore, $a = 0$, $b = 0$, $c = 0$, $d =$ anything. So $\{1\}$ is a basis of the null space.

(Range or $T$) Elements in the range of $T$ have the form

$$(Tf)(x) = 2xf''(x) - f'$$

So we repeat the process by breaking down the form of $f$:

$$\begin{aligned}f(x) &= ax^3 + bx^2 + cx + d \\ f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 6ax + 2b \\ 2xf''(x) - f' &= 9ax^2 + 2bx - c\end{aligned}$$

Notice that $a, b, c$ can be anything. So the basis is $\{x^2, x, 1\}$. □

**Exercise 5**: Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T$ is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $TS$ is the identity map on $W$.

*Proof.* ($\rightarrow$) Suppose that $T$ is surjective. Then for every $w \in W$, there is a $v \in V$ such that $Tv_0 = w_0$. We can take a function $S$ such that

$$Sw_0 = v_0$$

We must check that there is only one $v_0 \in V$ that $S$ maps $w_0 \in W$ to, which is not true, since the function $T$ might not be injective. We solve the problem by picking the least $v$ in $\hat{V_0} = \{v \in V : Tv = w_0\}$. Now we take

$$TSw_0 = Tv_0$$

By definition of the set $\hat{V_0}$, this is $w_0$, so $TS$ is the identity on $W$.

($\leftarrow$) Suppose that there is an $S \in \mathcal{L}(W, V)$ such that $TS$ is the identity map on $W$. That means for every $w \in W$,

$$\begin{aligned} TSw &= w \\ T(Sw) &= w \end{aligned}$$

So this implies that every $w \in W$ is an image of an element in $v$ under $T$. So $T$ is surjective. $\square$