Math104Hw5

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Exercise 1: For any $x, y \in \mathbb{R}^2$, define $d_1(x, y) = \max(\{|x_1 - y_1|, |x_2 - y_2|\})$. Prove that d_1 is a metrics on $\mathbb{R}^2$.

Proof. We check three properties:

- We require that $d_1(x, x) = 0$. So we have $x = (x_1, y_1)$ and

  $$d(x, x) = \max(|x_1 - x_1|, |y_1 - y_1|) = \max(0, 0) = 0$$

  We need to show that if $a = (x_1, y_1) \neq (x_2, y_2) = b$, then $d(a, b) \neq 0$. So we have:

  $$d(a, b) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

  But we know that either $x_1 \neq x_2$ or $y_1 \neq y_2$. Wlog, if $x_1 \neq x_2$, then $x_1 - x_2 \neq 0$ and $|x_1 - x_2| \geqslant 0$. So $|x_1 - x_2| > 0$. So the maximum must be greater than 0 and therefore, $d(a, b) \neq 0$.

- We require that $d(a, b) = d(b, a)$. Indeed:

  $$d(a, b) = \max(|x_1 - x_2|, |y_1 - y_2|) = \max(|x_2 - x_1|, |y_2 - y_1|) = d(b, a)$$

  so second condition is verified.

- Lastly, we must prove that $d(a, c) \leqslant d(a, b) + d(b, c)$. So let $c = (x_3, y_3)$ in this case. We have:

  $$d(a, b) + d(b, c) = \max(|x_1 - x_2|, |y_1 - y_2|) + \max(|x_2 - x_3|, |y_2 - y_3|)$$

  But notice that

  $$\max(|a| + |c|, |b| + |d|) \leqslant \max(|a|, |b|) + \max(|c|, |d|)$$

  which will be proved at the end if you're interested. That aside, we get as a result:

  $$\max(|x_1-x_2|+|x_2-x_3|, |y_1-y_2|+|y_2-y_3|) \leqslant \max(|x_1-x_2|, |y_1-y_2|)+\max(|x_2-x_3|, |y_2-y_3|)$$

  and by the triangle inequality:

  $$\max(|x_1 - x_3|, |y_2 - y_3|) \leqslant \max(|x_1 - x_2| + |x_2 - x_3|, |y_1 - y_2| + |y_2 - y_3|)$$

  so therefore, we have a string of inequalities giving us:

  $$\max(|x_1 - x_3|, |y_2 - y_3|) \leqslant \max(|x_1 - x_2|, |y_1 - y_2|) + \max(|x_2 - x_3|, |y_2 - y_3|)$$

  which shows that the distance formula obeys triangle inequality.

  (Proof of Claim) We want to show:

  $$\max(|a| + |c|, |b| + |d|) \leqslant \max(|a|, |b|) + \max(|c|, |d|)$$

  We can see this by cases. If $|a| \geqslant |b|$ and $|c| \geqslant |d|$, then we get $|a| + |c|$ on the RHS and the same on the LHS. If we get $|c| \leqslant |d|$, then we get $|a| + |d|$ on the RHS and either $|a| + |c|$ or $|b| + |d|$ on the LHS. Either way, both are less than or equal to the RHS.

$\square$

Exercise 2: For any $x, y \in \mathbb{R}^2$, define $d_2(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Prove that d_2 is a metrics on $\mathbb{R}^2$.

Proof. We need to prove the three conditions:

- If we have $x = (x_1, y_1)$, then

$$d(x, x) = |x_1 - x_1| + |y_1 - y_1| = 0$$

as desired. If we have $y = (x_2, y_2) \neq x$, then we have:

$$d(x, y) = |x_1 - x_2| + |y_1 - y_2|$$

and since either $x_1 - x_2$ or $y_1 - y_2$ is non-zero, we know that one of $|x_1 - x_2|, |y_1 - y_2|$ is non-zero also. Since both absolute values are $\geqslant 0$, we know that:

$$d(x, y) > 0$$

which proves the first part.

- Now we need to prove symmetry:

$$d(x, y) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_1| = d(y, x)$$

so that is the second condition done.

- Now we need to show that:

$$d(x, z) \leqslant d(x, y) + d(y, z)$$

We have:

$$\begin{aligned} d(x, y) + d(y, z) &= |x_1 - x_2| + |y_1 - y_2| + |x_2 - x_3| + |y_2 - y_3| \\ &= |x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3| \\ &\geqslant |x_1 - x_3| + |y_1 - y_3| \text{ by triangle inequality} \\ &= d(x, z) \end{aligned}$$

so we have proven that this distance formula obeys triangle inequality.

$\square$

Exercise 3: In $\mathbb{R}$, find the interior, closure and boundary of the set $\{1/n^2 : n \in \mathbb{N}\}$.

Proof. (Interior) There are no points in the interior. We note that if $p \in \{1/n^2 : n \in \mathbb{N}\}$, then we cannot find an ε such that:

$$E = \{r \in \mathbb{R} : |p - r| < \varepsilon\} \subseteq \{1/n^2 : n \in \mathbb{N}\} = M$$

To prove this, we see that $s_n = \frac{1}{n^2}$ is a decreasing sequence. We must have $|E| > 1$. So we can pick one other point in E that is also in M. Call this $1/n^2$. If $p = 1/m^2 > 1/n^2$, then we can build a chain:

$$1/m^2 > 1/(m+1)^2 > \cdots \geqslant 1/n^2$$

But because the sequence is decreasing, we know that

$$\frac{1}{2}\left(\frac{1}{m^2} + \frac{1}{(m+1)^2}\right)$$

is in E but not M. If $p < 1/n^2$, then we build the chain backwards and say that

$$\frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right)$$

is in E but not M. So E is never a subset of M. So the interior is $\emptyset$.

(Closure) We note that the complement of $E \cup \{0\}$ is an open set. If our point p was greater than all points in the set, then we just need to choose a distance smaller than $\min(|p-1|, |p|)$, which accounts for the distances from p to 0 and 1. On the other hand, any other point in M^c satisfies

$$\frac{1}{(n+1)^2} < p < \frac{1}{n^2}$$

for some i and we can just take our radius to be $\varepsilon < \min(p - \frac{1}{(n+1)^2}, \frac{1}{n^2} - p)$. Because the sequence $1/n^2$ is decreasing, we know that none of the terms will be within this radius. So the closure is the set $\{1/n^2 : n \in \mathbb{N}\} \cup \{0\}$ because the complement is open.

(Boundary) If we look at the closure of M^c, we know that the rationals are dense in $\mathbb{R}$, so we can always find a rational sequence converging to every element of M. So this means that $\{1/n^2 : n \in \mathbb{N}\}$ must be a subset of the closure. Therefore the closure of M^c is just $\mathbb{R}$. The boundary is $M^c \cap (\{0\} \cup \{1/n^2 : n \in \mathbb{N}\})$ which is

$$\{1/n^2 : n \in \mathbb{N}\} \cup \{0\}$$

□

Exercise 4: In $\mathbb{R}$, find the interior, closure and boundary of the rational number set $\mathbb{Q}$.

Proof. (Interior) A proof of the fact that there is an irrational between any two rationals will be left after this proof. Anyways, the claim is that the interior is $\emptyset$. This is because for any point $p \in \mathbb{Q}$, we have $(p - \varepsilon, p + \varepsilon)$ must contain p and some other point. If it is irrational for all $\varepsilon > 0$, we are done. If there is a rational q, then using the fact that there is an irrational between p, q, we see that:

$$\{r \in \mathbb{R} : |p - r| < \varepsilon\} \not\subseteq \mathbb{Q}$$

since irrationals are not in $\mathbb{Q}$. So the interior is $\emptyset$.

(Closure) Since a closed set must contain the limit of all sequences on $\mathbb{Q}$, using the fact that the rationals are dense in $\mathbb{R}$, we can construct a sequence converging to an irrational, q. Pick an arbitrary rational say p_1. By the denseness of rationals, we can find a p_2 such that:

$$q < p_2 < p_1$$

if $p_1 > q$ or

$$p_1 < p_2 < q$$

We continue finding p_i and this creates a sequence converging to q. So the closure is the rationals union the irrationals which is all of $\mathbb{R}$.

The boundary is the closure of the complement intersect the closure. So the closure of the irrationals must contain all the limits on the irrationals. We will show that there exists a sequence converging to a rational p. Let q_1 be an irrational $q_1 < p$ wlog. Then we can find a rational p_1 between them:

$$q_1 < p_1 < p$$

But by what will be proved later, there is an irrational between p_1, p. So we can keep going:

$$q_1 < p_1 < q_2 < p_2 < q_3 < p_3 < \cdots < p$$

Now just take the irrational numbers in this and that forms a sequence converging to p. Therefore, the closure of $\mathbb{R} - \mathbb{Q}$ is $\mathbb{R} - \mathbb{Q} \cup \mathbb{Q} = \mathbb{R}$. So the boundary is $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$

(Proof of Claim) We want to show that between any two rationals, there is an irrational q:

$$\frac{a}{b} < q < \frac{c}{d}$$

We will look at the case when $\frac{a}{b} > 0$ first. We know that $1 < \sqrt{2} < \frac{3}{2}$. So we have:

$$\frac{a}{b} < \frac{a\sqrt{2}}{b}$$

If $\frac{a\sqrt{2}}{b} < \frac{c}{b}$, then we are done. Otherwise, take the ratio:

$$\frac{\frac{a\sqrt{2}}{b}}{\frac{c}{d}} = r_1$$

$$\frac{\frac{a\sqrt{2}}{b}}{\frac{a}{b}} = r_2$$

and find a rational number f such that:

$$r_1 < f < r_2$$

So we have:

$$\begin{aligned} \frac{c}{d} r_1 &= \frac{a\sqrt{2}}{b} \\ \frac{c}{d}\left(\frac{r_1}{f}\right) &= \frac{a\sqrt{2}}{bf} \\ \frac{c}{d} &> \frac{a\sqrt{2}}{bf} \end{aligned}$$

and

$$\begin{aligned} \frac{a}{b} r_2 &= \frac{a\sqrt{2}}{b} \\ \frac{a}{b}\left(\frac{r_2}{f}\right) &= \frac{a\sqrt{2}}{bf} \\ \frac{a}{b} &< \frac{a\sqrt{2}}{bf} \end{aligned}$$

Then

$$\frac{a}{b} < \frac{a\sqrt{2}}{fb} < \frac{c}{d}$$

as desired.

Now if $\frac{a}{b} \leqslant 0 < \frac{c}{d}$, then we just apply the same idea with some rational p between 0 and $\frac{c}{d}$. If $\frac{a}{b} < \frac{c}{d} < 0$, we multiply through by a negative to get $0 < \frac{-c}{d} < \frac{-a}{b}$ and find our irrational q such that $0 < \frac{-c}{d} < q < \frac{-a}{b}$, and multiply through by negative sign to get:

$$\frac{a}{b} < -q < \frac{c}{d} < 0$$

so we are done as we have shown for all cases. □

Exercise 5: Let E be a compact set of $\mathbb{R}$, prove that $\max(E)$ and $\min(E)$ exist.

Proof. We know that a set $E \subseteq \mathbb{R}$ is compact if and only if it is bounded and closed. Since it is bounded, by the completeness axiom, we know that it exists. □

Exercise 6: Let $S = \mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$ with the usual metric. Show that $(-\infty, 0)$ is a closed subset of S.

Proof. We just need to show that its complement is open. The complement is $(0, \infty)$. Suppose that $p \in (0, \infty)$. Then we just choose $r < p$ as:

$$\{s \in S : d(s, p) < r\}$$

we know that:

$$d(s, p) = |s - p| < p$$

which means:

$$-p < s - p < p$$

or

$$0 < s < 2p$$

so therefore, $s \in (0, \infty)$ which proves:

$$\{s \in S : d(s, p) < r\} \subseteq (0, \infty)$$

Since $(0, \infty)$ is equal to its interior, we know that it is open. Therefore, $(-\infty, 0)$ is closed. □