Stat134Hw6

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Exercise 1: Let X be Normal random variable $\mathcal{N}(3,4)$ of mean 3 and variance 4. What is $\mathbb{E}(X^2)$?

Answer. We have that variance is $\mathbb{E}(X^2) - (\mathbb{E}X)^2$. Also, $(\mathbb{E}(X))^2 = 9$ and the variance is 4. So $3 + 9 = \mathbb{E}(X^2)$ and $\mathbb{E}(X^2) = 12$.

Exercise 2: Suppose 10 percent of households earn over 80,000 dollars a year, and 0.25 percent of households earn over 450,000. A random sample of 400 households has been chosen. In this sample, let X be the number of households that earn over 80,000, and let Y be the number of households that earn over 450,000. Use either the Normal or Poisson approximation, whichever is appropriate in either case, to find the simplest estimates you can for the probabilities $P(X \geqslant 48)$ and $P(Y \geqslant 2)$.

Answer. We will use a Normal distribution since $np(1-p) = 40 * .9 = 36$ is sufficiently large. We want:

$$P(X \geqslant 48) \implies P(X - 40 \geqslant 8) \implies P(\frac{X-40}{\sqrt{36}} = \frac{X-40}{6} \geqslant 4/3) = 1 - P(\frac{X-40}{6} \leqslant 4/3) = 1 - .9082$$

So the answer is .0918.

For the other one, we have $np(1-p) = 400 * 0.0025 * .9975 = 1 * .9975$ which is small. We will want to use the Poisson distribution. So:

$$P(Y \geqslant 2) = 1 - P(Y < 2) = 1 - P(Y = 1) - P(Y = 0)$$

this is

$$1 - \frac{1^0}{0!e} - \frac{1^1}{1!e}$$

Exercise 3: Let $X \sim \text{Exp}(2)$ be exponential random variable of rate 2 (its density is $2e^{-2x}$, $x > 0$). Compute the conditional probabilities

$$P(X > 2 \mid X > 1) \text{ and } P(X > 1 \mid X > 2)$$

Answer. The conditional probability:

$$P(X > 2 \mid X > 1) = \frac{P(X > 2 \cap X > 1)}{P(X > 1)} = \frac{P(X > 2)}{P(X > 1)}$$

with pmf $p(x) = 2e^{-2x}$ for $x \geqslant 0$. So for $x > 1$, integrate:

$$\int_a^\infty 2e^{-2x}\, dx = -e^{-2x}\Big|_a^\infty = e^{-2a}$$

So $P(X > 2) = e^{-4}$ and $P(X > 1) = e^{-2}$. So the answer is e^{-2}

For the other one, we have

$$P(X > 1 \,\big|\, X > 2) = \frac{P(X > 1 \cap X > 2)}{P(X > 2)} = \frac{P(X > 2)}{P(X > 2)} = 1$$

Exercise 4: Recall that PMF of Poisson(λ) is $P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, 2, \ldots$.

(a) Let $X \sim \mathrm{Poisson}(\lambda)$ and consider the sequence $p(0), p(1), p(2), \ldots$ with $p(k) = P(X = k)$. Show that $p(k)$ first increases and then decreases. At what k does the change happen?

Answer. Consider the ratio of the term after over the term before:

$$\frac{\lambda^{k+1}k!}{\lambda^k(k+1)!} = \frac{\lambda}{k+1}$$

We see that it is increasing when

$$\frac{\lambda}{k+1} \geqslant 1 \implies \lambda \geqslant k+1$$

and is decreasing or the same otherwise. So as k increases, we will reach a point where the ratio of the terms gets smaller: $\frac{\lambda}{k+1} > \frac{\lambda}{k+2}$. This change happens when $k = \lambda - 1$.

(b) Let $X \sim \mathrm{Bin}(n, p)$ and consider the sequence $p(0), p(1), \ldots, p(n)$ with $p(k) = P(X = k)$. Show that $p(k)$ first increases and then decreases. At what k does the change happen?

Answer. The probability is $\binom{n}{k}p^k(1-p)^{n-k}$. Now a ratio of the next term with the previous:

$$\frac{\binom{n}{k+1}p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{(n-k)!k!p}{(n-k-1)!(k+1)!(1-p)} = \frac{(n-k)\cdot p}{(k+1)(1-p)}$$

The change occurs when

$$\frac{(n-k)p}{(k+1)(1-p)} = 1$$

or

$$np - kp = -kp - p + k + 1$$

So

$$np + p = k + 1 \implies k = np + p - 1$$

We see that if k is sufficiently far from n, then $k+1$ results in an increase, if $k + 1 < np + p - 1$. And the opposite is true for when k is close to n.

Exercise 5: Let $X \sim \mathrm{Poisson}(1)$ be Poisson random variable of mean 1. Compute

$$\mathbb{E}[(\ln(2))^X]$$

Answer. By definition, the expectation is

$$e^{-1}\sum_{k\geqslant 0}\frac{1}{k!}\ln(2)^k = e^{-1}\frac{\ln(2)^k}{k!}$$

and $\frac{\ln(2)^k}{k!} = e^{\ln(2)} = 2$. So the answer is $2e^{-1}$.

Exercise 6: Let X_n be iid random variables, such that $X_n = e$ with probability $1/2$ and $X_n = \frac{1}{e}$ with probability $1/2$. Compute (in any form you can)

$$\lim_{n\to\infty} (X_1 X_2 \cdots X_n)^{1/\sqrt{n}}$$

Answer. Let $Y = \log\Big((X_1 X_2 \cdots X_n)^{1/\sqrt{n}}\Big)$. We first compute the limit of this as $n \to \infty$. Now we can rewrite Y as:

$$\frac{1}{\sqrt{n}} \sum_{i\geqslant 1}^{n} \log(X_i)$$

Taking the limit as $n \to \infty$. We compute the expectation of $\log(X_i)$. We have

$$P(\log(X_i) = k) = \begin{cases} \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{2} & \text{if } k = -1 \end{cases}$$

The expectation is 0. The variance is defined by $\mathbb{E}[\log(X_i)^2]$. So we have:

$$\sum_k P(\log(X_i) = k)k^2 = 1$$

By the central limit theorem, we have that it is normally distributed:

$$\lim_{n\to\infty} \frac{Y - 0n}{1\sqrt{n}} \sim \mathcal{N}(0, 1)$$

Now to recover the random variable $\lim\limits_{n\to\infty} (X_1 X_2 \cdots X_n)^{1/\sqrt{n}}$, we take e^Y. So the limit is the moment generating function of $\mathcal{N}(0, 1)$ at $t = 1$. This was show to be $e^{t^2/2}$. So taking $t = 1$, we get that the limit is $e^{1/2}$.