Math104Hw8

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Exercise 1: Use the $\varepsilon - \delta$ definition to show that $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

Proof. We want to show that $\forall \varepsilon > 0$, there $\exists \delta > 0$ such that if

$$|x - y| < \delta$$

then

$$|f(x) - f(y)| < \varepsilon$$

Then we have

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x - y||x + y| \\ &\leqslant 2|x - y| \end{aligned}$$

which we want to be

$$2|x - y| < \varepsilon$$

So

$$|x - y| < \frac{\varepsilon}{2}$$

and we can choose $\delta = \frac{\varepsilon}{2}$. Therefore, on the interval $[0, 1]$, we have that $|f(x) - f(y)| \leqslant 2|x - y| < \frac{2\varepsilon}{2} = \varepsilon$. Since δ does not depend on our choice of x, we are done. $\square$

Exercise 2: Show that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$.

Proof. Suppose for contradiction that f is uniformly continuous on $[0, \infty)$. Then that means that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y$ where

$$|x - y| < \delta$$

we have

$$|f(x) - f(y)| < \varepsilon$$

Take $\varepsilon = 1$. Then there is a δ such that

$$|x^2 - y^2| < 1$$

Take $y = x + \frac{\delta}{2}$. Then

$$|x - y| = |x - (x + \frac{\delta}{2})| = \frac{\delta}{2} < \delta$$

Then because it is uniformly continuous, we know that

$$|x^2 - y^2| = |x^2 - (x^2 + \delta x + \frac{\delta^2}{4})| = |-\delta x - \frac{\delta^2}{4}| = \delta x + \frac{\delta^2}{4} < 1$$

But if we take

$$x = \frac{1}{\delta}$$

we have a contradiction because

$$\delta x + \frac{\delta^2}{4} = 1 + \frac{\delta^2}{4} \not< 1$$

Since $x, y \in [0, \infty)$, f is not uniformly continuous on $[0, \infty)$ □

Exercise 3: Assume that f is uniformly continuous on a bounded set S, prove that $f(S)$ is bounded.

Proof. Pick any sequence $(s_n) \subseteq S$. Since the sequence is bounded, we know that it has a convergent subsequence. Then the subsequence is cauchy because it is convergent. Say that (s_{n_k}) converges to s. Then we know that $(f(s_{n_k}))$ is cauchy also by theorem 19.4. That means that for any $\varepsilon > 0$, there is an N such that $n_k, n_j > N$ for some N,

$$|f(s_{n_k}) - f(s_{n_j})| < \varepsilon$$

This means that f is bounded by ε for elements of our sequence. Since it converges, we know that f has an extension at the supremum and infimum of S. And therefore, for other sequences converging to elements of s_{n_k}, we know that their image is also bounded. Therefore, $f(S)$ is bounded. □

Exercise 4: Let $f(x) = \frac{1}{(x-1)(x-3)^2}$, determine $\lim\limits_{x\to 1} f(x), \lim\limits_{x\to 2} f(x), \lim\limits_{x\to 3} f(x)$.

Proof. Suppose we have a sequence $(s_n) \in (-\infty, 1)$ converging to 1. Then

$$\lim_{x\to 1^-} f(x) = \lim_{n\to\infty} \frac{1}{(s_n - 1)(s_n - 3)^2}$$

We know that $s_n - 1$ as a sequence converges to 0 and $(s_n - 3)$ converges to 4. Then the denominator converges to 0 and therefore, the fraction diverges to $-\infty$. Now if we take the interval $(1, 3)$, we see that

$$\lim_{x\to 1^+} f(x) = \lim_{n\to\infty} \frac{1}{(t_n - 1)(t_n - 3)^2}$$

for $(t_n) \in (1, 3)$ converging to 1, by similar reasoning, we get that it diverges to ∞. So the limit $\lim\limits_{x\to 1} f(x)$ does not exist. Now for $\lim\limits_{x\to 2}$, we see that the limit is $\frac{1}{2}$. To prove this, we need to show that for all $\varepsilon > 0$, there is a $\delta > 0$ such that if

$$|x - 2| < \delta$$

we have

$$|f(x) - \frac{1}{2}| < \varepsilon$$

So we have:

$$|\frac{1}{(x-1)(x-3)^2} - \frac{1}{2}| < \varepsilon$$

means that

$$|\frac{2 - (x-1)(x-3)^2}{2(x-1)(x-3)^2}| = |\frac{x^3 - 7x^2 + 15x - 11}{2(x-1)(x-3)^2}| < \varepsilon$$

But notice that $(x-2)^3 = x^3 - 2x^2 + 4x - 8 > x^3 - 7x^2 + 15x - 11$ since

$$5x^2 - 11x + 3 > 0$$

(Moved everything to the RHS) for when $x \geqslant \frac{19}{10}$. So we say that $\delta \leqslant \frac{19}{10}$. Then take

$$|\frac{x^3 - 7x^2 + 15x - 11}{2(x-1)(x-3)^2}| < |\frac{(x-2)^3}{2(x-1)(x-3)^2}| < \varepsilon$$

Now notice that

$$|x - 1| \leqslant |x - 2| + 1$$

and

$$|x - 3| \leqslant |x - 2| + 1$$

by triangle inequality. So

$$|\frac{1}{(x-1)(x-3)^2} - \frac{1}{2}| < |\frac{(x-2)^3}{2(|x-2|+1)^2}| < |\frac{(x-2)^3}{2(x-2)^2}| = |\frac{x-2}{2}| < \varepsilon$$

if we have $\delta < 2\varepsilon$. So we just require $\delta < \min(\frac{19}{10}, 2\varepsilon)$. This show that the limit as $x \longrightarrow 2$ is $\frac{1}{2}$ by epsilon delta property. As for the last one, we have:

$$\lim_{x \to 3^-} \frac{1}{(x-1)(x-3)^2} = -\infty$$

while

$$\lim_{x \to 3^+} \frac{1}{(x-1)(x-3)^2} = \infty$$

because the denominator shrinks. But because the limits are not equal, the limit does not exist. □

Exercise 5: Let $f(x) = \frac{x^2+1}{x-1}$ when $x > 0$, and $f(x) = -\cos x$ when $x < 0$. Does f admit an extension $\tilde{f}$ on $\mathbb{R}$, which is continuous at $x = 0$? Explain why.

Proof. It admits the extension, $\tilde{f}(0) = -1$. This is because we have the limit of $\frac{x^2+1}{x-1}$ as $x \longrightarrow 0^+$ is -1 while the limit as $-\cos x$ as $x \longrightarrow 0^-$ is -1. Then since the limits from the left equals the limit from the right, we can take $\tilde{f}(0)$ to be -1. Since all three limits match up, $\tilde{f}$ is continuous at 0. □

Exercise 6: Let f, g be two continuous functions on $\mathbb{R}$ so that $\lim_{x \to 0^-} f(x) = 1$, $g(0) = 10$, $\lim_{x \to 1} g(x) = 2$. Decide the value of $\lim_{x \to 0} g \circ f(x)$ if it exists.

Proof. Since f is continuous at 0, we know that

$$\lim_{x \to 0} f(x) = f(0)$$

Since the limit exists, and f is continuous on all of $\mathbb{R}$, we know that f is defined everywhere on $\mathbb{R}$, so

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = f(0) = 1$$

It was proved in class that if g is continuous at $f(a)$ and $\lim_{x \to a} f(x)$ is finite, then

$$\lim_{x \to a} g \circ f(x) = g(f(a))$$

Since this is the case, we have

$$\lim_{x \to 0} g(f(x)) = g(f(0)) = g(1) = 2$$

□