Math172Ex10

Trustin Nguyen

November 7, 2023

Exercise 4: Fix $n \geqslant 3$ and let A be the graph obtained from K_n by removing one edge. Find the number of spanning trees of A (we count them as labeled trees, so different trees will be counted separately even if they are isomorphic).

Proof. We know that the number of spanning trees on a complete graph is n^{n-2}, because that is just the number of trees on $[n]$ which was proved in class. Now the number of spanning trees is a subset of these n^{n-2} strings. Here is the method for the code:

- Find the leaf with the minimal label
- Remove this leaf, and add to the code what the edge led to.
- Repeat this procedure for $n - 2$ times.
- Stop when there is a tree on two vertices.

Without loss of generality, remove the edge connecting $1, n$ in K_n. Then there are two cases:

- Converting a prufer code to a tree ends in adding the edge $\{1, n\}$ at the end. Because there is a bijection between prufer codes and trees, consider the process of constructing a prufer code.

  First show that one of the remaining vertices is always n. Every tree has at least 2 leaves. Then since n is the largest number in our graph, there will always be a smaller leaf n' that we will remove instead of n. So n will be one of the remaining vertices at the end.

  Suppose the last two vertices are $1, n$. We know that whatever leaf we removed before must be connected to either 1 or 5. Suppose that it is connected to 5, and had the label m. But that is a contradiction, because we must have the graph:

$$\begin{tikzcd} n \arrow[r, dash] \arrow[d, dash] & 1 \\ m & \end{tikzcd}$$

  But then since 1 is the smallest leaf, we will have removed $\{1, n\}$ as an edge, so $1, n$ are not the last vertices remaining. Then we know that graphs that contain the edge $\{1, n\}$ contribute to prufer codes that end in 1.

- The edge $\{1, n\}$ is removed before the last step. Since the vertex n is never a leaf that is removed until the last step, we know that it is the leaf 1 that is removed that is connected to the vertex n. Then that means that in our prufer code, this corresponds to adding n to our code. We must also have no 1's appearing after.

  If $\{1, n\}$ was removed in the first step, then the prufer code contains no 1's and starts with vertex n. Otherwise, 1 was not a leaf. Then we must have removed several edges connecting to 1, which would turn 1 into a leaf, which is the smallest one. So then an instance of $1, n$ in the prufer code, followed by 0 instances of 1 indicates that we have removed $\{1, n\}$ as an edge.

Notice that the set of prufer codes that account for these cases are disjoint. For the first case, we require that 1 appears as the last element of the code. So this is just n^{n-3}, because we set the last element in our $n-2$ element string to be 1 and have n options for the rest of $n-3$ places in our string.

In the next case, suppose we had a string of length $n-3$. Then we can account for an instance of $1, 5$ in our code with no 1's after by taking the last instance of 1 in our code and replacing it with $1, 5$. If there are no 1's, then we append a 5 to the start of the prufer code. There are n^{n-3} ways to have such strings in this case.

So now we take the number of prufer codes total corresponding to the number of spanning trees on K_n

$$n^{n-2}$$

and remove the number of prufer codes that use the edge $\{1, n\}$:

$$n^{n-3} + n^{n-3}$$

to get the number of spanning trees on K_n with one edge removed:

$$n^{n-2} - 2n^{n-3}$$

this is not defined for $n = 1, n = 2$, so the number of spanning trees for those is 1 which completes the proof. □