Math 124 - Programming for Mathematical Applications

UC Berkeley, Spring 2023

Homework 6

Due Wednesday, March 1

In [1]:
```
using LinearAlgebra, PyPlot
```

Problem 1 - Hilbert matrices

A *Hilbert matrix* H of size n-by-n has entries

$$H_{ij} = \frac{1}{i + j - 1}$$

Problem 1(a)

Create a 2D array with a Hilbert matrix H of size $n = 6$.

In [2]:
```
H = [1/(i + j - 1) for i = 1:6, j = 1:6]
```

Out[2]:
```
6×6 Matrix{Float64}:
 1.0       0.5       0.333333  0.25      0.2       0.166667
 0.5       0.333333  0.25      0.2       0.166667  0.142857
 0.333333  0.25      0.2       0.166667  0.142857  0.125
 0.25      0.2       0.166667  0.142857  0.125     0.111111
 0.2       0.166667  0.142857  0.125     0.111111  0.1
 0.166667  0.142857  0.125     0.111111  0.1       0.0909091
```

Problem 1(b)

Convert H to Julia's `LinearAlgebra.Symmetric` matrix.

In [3]:
```
Symmetric(H)
```

Out[3]:
```
6×6 Symmetric{Float64, Matrix{Float64}}:
 1.0       0.5       0.333333  0.25      0.2       0.166667
 0.5       0.333333  0.25      0.2       0.166667  0.142857
 0.333333  0.25      0.2       0.166667  0.142857  0.125
 0.25      0.2       0.166667  0.142857  0.125     0.111111
 0.2       0.166667  0.142857  0.125     0.111111  0.1
 0.166667  0.142857  0.125     0.111111  0.1       0.0909091
```

Problem 1(c)

Create the matrix $G = H^2$.

In [4]:
```
G = H^2
```

Out[4]:
```
6×6 Matrix{Float64}:
 1.49139   0.857143  0.616071  0.484788  0.401091  0.342691
 0.857143  0.511797  0.375     0.298611  0.249074  0.214078
 0.616071  0.375     0.277422  0.222222  0.186111  0.160438
 0.484788  0.298611  0.222222  0.178657  0.15      0.129545
 0.401091  0.249074  0.186111  0.15      0.126157  0.109091
 0.342691  0.214078  0.160438  0.129545  0.109091  0.0944211
```

Problem 1(d)

Consider the linear system $G\boldsymbol{x} = \boldsymbol{b}$, where

$$b_i = \sum_{j=1}^{n} G_{ij}$$

What is the exact solution $\boldsymbol{x}$?

The exact solution x is the vector with all 1's, since b_i is the sum of row entries in the $i - th$ row. When we take the product $G * I$, we get exactly that.

Problem 1(e)

Solve numerically for $\boldsymbol{x}$.

In [5]:
```
x = G \ sum(G, dims = 2)
```

Out[5]:
```
6×1 Matrix{Float64}:
 1.0000003101050903
 0.9999911806804064
 1.0000594947495467
 0.9998456070994495
 1.0001700969032143
 0.9999330808965133
```

Problem 1(f)

Compute $\|\boldsymbol{x} - \mathbf{1}\|_2$, where $\mathbf{1}$ is a vector with all entries = 1.

In [6]:
```
norm(x - ones(6))
```

Out[6]:
```
0.0002467099357988225
```

Problem 1(g)

This is an example of a highly *ill-conditioned* matrix, which means operations such as solving linear systems can be very inaccurate. Compute the so-called *condition number* of G, defined by:

$$\kappa(G) = \frac{\lambda_{\max}}{\lambda_{\min}}$$

that is, the ratio of the largest and the smallest eigenvalues of G.

In [7]:
```
maximum(eigvals(G))/minimum(eigvals(G))
```

Out[7]:
```
2.251832632804391e14
```

Problem 2 - The Strassen algorithm

The Strassen algorithm is a method for matrix-matrix multiplication which performs asymptotically fewer operations than the standard method for large matrices (but it is still slower in practice for most matrices). Consider the matrix-matrix product $C = AB$, where A, B, C are n-by-n matrices and n is a power of 2. Partition the matrices as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where all submatrices are of size $n/2$-by-$n/2$. Now evaluate the following 7 (smaller) matrix-matrix products recursively:

$$\begin{aligned}
M_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22})B_{11} \\
M_3 &= A_{11}(B_{12} - B_{22}) \\
M_4 &= A_{22}(B_{21} - B_{11}) \\
M_5 &= (A_{11} + A_{12})B_{22} \\
M_6 &= (A_{21} - A_{11})(B_{11} + B_{12}) \\
M_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}$$

and finally form C from the following submatrices:

$$\begin{aligned}
C_{11} &= M_1 + M_4 - M_5 + M_7 \\
C_{12} &= M_3 + M_5 \\
C_{21} &= M_2 + M_4 \\
C_{22} &= M_1 - M_2 + M_3 + M_6
\end{aligned}$$

Implement this algorithm as a function `strassen(A,B)`, which assumes the size of `A` and `B` are powers of 2. The base case is when the matrix sizes are 1-by-1, and the multiplication is a scalar multiplication. Note: this means that your implementation cannot perform matrix multiplication at any point, only scalar multiplication.

In [8]:
```
function strassen(A,B)
    n = size(A, 1)
    if n == 1
        A * B
    else
        A11 = A[1:n ÷ 2, 1:n ÷ 2]
        A12 = A[1:n ÷ 2, n ÷ 2 + 1: n]
        A21 = A[n ÷ 2 + 1:n, 1:n ÷ 2]
        A22 = A[n ÷ 2 + 1:n, n ÷ 2 + 1:n]
        B11 = B[1:n ÷ 2, 1:n ÷ 2]
        B12 = B[1:n ÷ 2, n ÷ 2 + 1: n]
        B21 = B[n ÷ 2 + 1:n, 1:n ÷ 2]
        B22 = B[n ÷ 2 + 1:n, n ÷ 2 + 1:n]

        M1 = strassen(A11 + A22, B11 + B22)
        M2 = strassen(A21 + A22, B11)
        M3 = strassen(A11, B12 - B22)
        M4 = strassen(A22, B21 - B11)
        M5 = strassen(A11 + A12, B22)
        M6 = strassen(A21 - A11, B11 + B12)
        M7 = strassen(A12 - A22, B21 + B22)

        C11 = M1 + M4 - M5 + M7
        C12 = M3 + M5
        C21 = M2 + M4
        C22 = M1 - M2 + M3 + M6
        C1112 = [C11 C12]
        C2122 = [C21 C22]
        C = [C1112 ; C2122]
    end
end
```

Out[8]:
```
strassen (generic function with 1 method)
```

Test your function using the commands below.

In [9]:
```
A = randn(256,256)
B = randn(256,256)
C = strassen(A,B)
D = A * B
maximum(abs.(C-D))     # Should be very small
```

Out[9]:
```
5.858424856342026e-12
```

Problem 3 - Polynomial data fitting

Generalize the example on linear regression from the lecture notebook, to fit a polynomial of degree $p \geq 1$ to the data (the linear regression example corresponds to $p = 1$).

Problem 3(a)

Write a function with the syntax `pol = polyfit(x, y, p)` which computes a polynomial `pol` of degree `p` that is a least-squares fit of the data `x,y` .

In [10]:
```
function polyfit(x, y, p)
    A = [x[i]^(j - 1) for i = 1:length(x), j = 1:p + 1]
    return A \ y
end
```

Out[10]:
```
polyfit (generic function with 1 method)
```

Problem 3(b)

Write a function with the syntax `yy = polyval(pol, xx)` which evaluates the polynomial `pol` at all the x-values in `xx` .

In [11]:
```
function polyval(pol, xx)
    return sum([pol[i]a^(i - 1) for a in xx, i = 1:length(pol)], c
end
```

Out[11]:
```
polyval (generic function with 1 method)
```

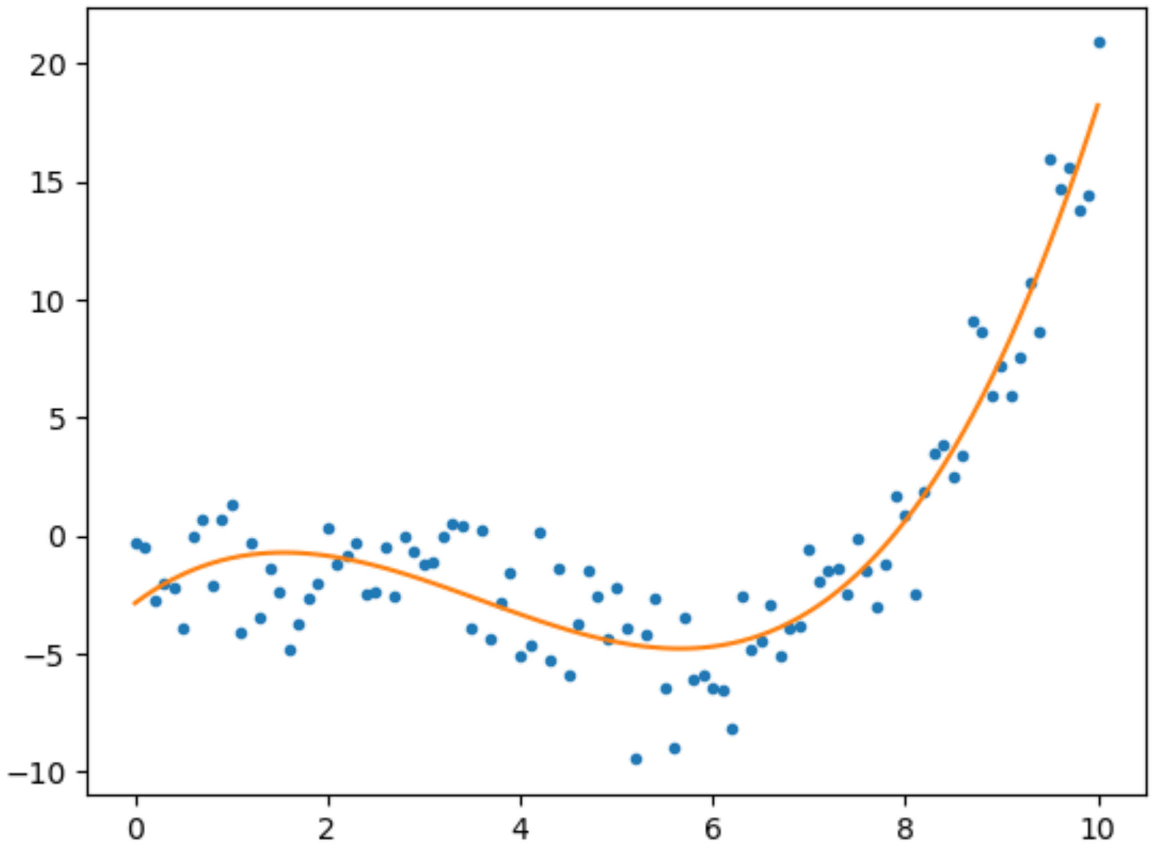
Problem 3(c)

Demonstrate your functions by fitting a cubic polynomial to the following data, and plotting in the same way as in the lecture notebook:

In [12]:

```
using PyPlot
x = 0:0.1:10
noise = 2randn(size(x))
y = @. 0.1x^3 - x^2 + 2x - 2 + noise; # Example data: cubic with r

pol = polyfit(collect(x), y, 3)
yy = polyval(pol, collect(x))
plot(x,y, ".")
plot(x,yy)
```



Out[12]:

```
1-element Vector{PyCall.PyObject}:
 PyObject <matplotlib.lines.Line2D object at 0x7f49036350a0>
```

Problem 4 - Strings and File Processing

From Think Julia:

> Give me a word with three consecutive double letters. I'll give you a couple of words that almost qualify, but don't. For example, the word committee, c-o-m-m-i-t-t-e-e. It would be great except for the i that sneaks in there. Or Mississippi: M-i-s-s-i-s-s-i-p-p-i. If you could take out those i's it would work. But there is a word that has three consecutive pairs of letters and to the best of my knowledge this may be the only word. Of course there are probably 500 more but I can only think of one. What is the word?

Write a program to find these words. First download the file https://github.com/BenLauwens/ThinkJulia.jl/blob/master/data/words.txt (https://github.com/BenLauwens/ThinkJulia.jl/blob/master/data/words.txt) to your computer, and upload it to the datahub in the same directory that you keep your notebook. Then read each line of the file, and if the you find the pattern described above, print the word.

In [15]:
```
words = readlines("words.txt")

for w in words
    three_consec(w)
end
```

```
bookkeeper
bookkeepers
bookkeeping
bookkeepings
```

In [14]:

```
function three_consec(f)
    index = 1
    convec = []
    function find_consec(f, index)
        g = f[index:end]

        if length(g) ≤ 1
        elseif g[1] == g[2] && (length(convec) == 0 || g[1:2] != c
            push!(convec, g[1:2])
            index += 2
            find_consec(f, index)
        else
            index += 1
            find_consec(f, index)
        end
        convec
    end
    find_consec(f,index)

    k = false
    found = []
    if length(convec) ≥ 3
        for i = 1:length(convec) - 2
            trio = []
            trio = string(convec[i], convec[i + 1], convec[i + 2])
            push!(found, findfirst(trio,f))
        end
    end
    for a in found
        if a != nothing
            k = true
        end
    end
    if k
        println(f)
    end
end
```

Out[14]: three_consec (generic function with 1 method)