Math172Notes

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Contents

Chapter 1

Week 1

1.1 Pigeon-hole Principle

Theorem 1.1.1 **Pigeon-hole Principle**

Suppose there are n objects, which are put into m boxes where $n > m$. Then there is a box with at least two objects in it.

Proof. Assume for contradiction that we have placed n objects into m boxes but all boxes have at most 1 objects. This means that the total number of objects is less than or equal to m. But we know that the number of objects exceeds m. This is a contradiction. □

Example 1.1.1: Let m be an odd positive integer. Then $\exists k$ such that $2^k - 1$ is divisible by m.

Proof. We take a sequence of all numbers $2^k - 1$ where k is up to m:

$$1, 4-1, 8-1, \ldots, 2^m - 1$$

consider the remainders of these numbers from division by m. If one of the remainders are 0, then we are done. Otherwise, the remainders must be in the list:

$$1, \ldots, m-1$$

This means that there are $m-1$ possible values for the remainders. By the pigeonhole principle, two of the remainders from division by m in the first list must be equal. Lets say they are $2^a - 1$ and $2^b - 1$ which have the same remainder with division by m. Suppose that $a < b$. Let us take the difference which will be divisible by m:

$$2^b - 1 - 2^a + 1 = 2^b - 2^a$$

we have $2^a(2^{b-a} - 1)$ is divisible my m. But the below claim, we can say that

$$2^{a-1}(2^{b-a} - 1)$$

is divisible by m. We repeat this process until we have $2^{b-a} - 1$ is divisible by m.

Claim: Suppose that $2n$ is divisible by m. We claim that n is divisible by m.

Proof. We have that $2n = m \cdot k$. So $m \cdot k$ is even. But m is odd so k is even . This means that $k = 2l$. So we have that

$$2n = m \cdot 2l$$
$$n = m \cdot l$$

□

□

Theorem 1.1.2 **Generalized Pigeonhole Principle**

Suppose that we have three positive integers n, m, r. Suppose that we also have the inequality: $n > m \cdot r$. Then if we try to place n objects into m boxes, then there must be a box with at least $r + 1$ objects.

Proof. Assume that we have placed n objects into m boxes. For contradiction, say that each box has less than $r + 1$. Then the number of objects is less than or equal to $n \cdot m$ objects. This means that the total number of objects is $\leqslant n \cdot r$. This is a contradiction. □

Example 1.1.2: Suppose that we have 10 points in a unit square on a plane. Then the following statements are true about these points:

- There exists a triple which can be covered by a disc of radius $\frac{1}{2}$.

  Proof. If we split the square along the diagonals, then we have 10 points and 4 triangles. Using the pigeonhole principle with $r = 2$, we have that there is a triangle with at least 3 points in it. But now we take the midpoint of the side of the triangle shared with the side of the square. The disc of radius $\frac{1}{2}$ covers these three points. □

- There exists a couple of points such that the distance between these points is no greater than 0.48.

  Proof. Take the same square but split it into 9 smaller squares. By the pigeonhole principle with $r = 1$, we have that one of the squares has at least 2 points. So the maximal distance between these two points is $\frac{\sqrt{2}}{3}$. So $\sqrt{2} > 1.4$ and $\frac{1.4}{3} \approx .472 < .48$ □

1.2 Mathematical Induction

Definition 1.2.1 **Mathematical Induction**

Mathematical induction is used to prove a sequence of statements: $S_1, S_2, \ldots$ which iterates through the natural numbers. We have two steps:

- Prove that S_1 is true.
- Inductive Step: Assuming n statements are true, prove the S_{n+1} statement.

After these two steps, we have proven that all S_i for $i \geqslant 1$ is true.

Proof. Suppose that the initial step and inductive step are verified. Suppose for contradiction that S_n is the minimal statement with the smallest index. The case that $n = 1$ is impossible. Otherwise, we have $n > 1$. Since there is a statement before it, S_{n-1}, we know that this statement is true as $n - 1 < n$. But by inductive hypothesis, we have a contradiction. □

Example 1.2.1: We will prove that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. We will use induction on n.

- Base Case: $n = 1$. This is true: $1^2 = 1$ and $\frac{1(2)(3)}{6} = 1$.
- Inductive Case: Assume that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. We will show that $1^2 + 2^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$. Notice that we have:

$$\begin{aligned} 1^2 + 2^2 + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\left(\frac{2n^2+n}{6} + n + 1\right) \\ &= (n+1)\left(\frac{2n^2+7n+6}{6}\right) \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

□

Example 1.2.2: Suppose that m is an odd integer. We have that if $2^k n$ is divisible by m, then $m \mid n$.

Proof. Fix m, n. We will do induction on k:

- For $k = 0$, we have $m \mid n$ so $m \mid n$ which is true.
- Suppose that this is true up to p. Then we have:

$$m \mid 2^p n$$

Observe that for $p + 1$, we have:

$$2^{p+1} n = 2^p n \times 2$$

Since $m \mid 2^p n$, we have that $m \mid 2n$. So we have $m \mid n$.

□

Example 1.2.3: A set is an unordered collection of different objects. We have the following definitions:

- $a \in A$: a is in the set A
- $B \subseteq A$: Every element of b is an element of A.
- $[n]$ is the set of n elements: $\{1, 2, \ldots, n\}$.

Subsets of a Set

Theorem 1.2.1 If A is an n element set, then the number of subsets of A is 2^n.

Proof. If we have a set A, we can enumerate each element by a number. We proceed by induction:

- Initial Step: We have $A = \emptyset$. The subsets are only $\emptyset$
- Inductive case: Suppose that $|[n]| = 2^n$ is true. For $n + 1$, we have that we have subsets that do not have $n + 1$ and sets that do. So we have $2 * 2^n$ subsets or 2^{n+1} subsets.

$\square$

Chapter 2

Week 2

2.1 Strong Induction

Example 2.1.1: We have the Fibonacci numbers defined as for F_i:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_{n+1} = F_{n-1} + F_{n+1}$$

we will prove that

$$F_k = \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{\sqrt{5}\cdot 2^k}$$

Proof. We will prove this by induction:

- Base Cases: $F_0 = \frac{1-1}{\sqrt{5}} = 0$ and $F_1 = \frac{1+\sqrt{5}-(1-\sqrt{5})}{2\sqrt{5}} = 1$
- Inductive Case: We need more than just the previous statement because we have $F_n = F_{n-1} + F_{n-2}$. So we actually need to assume that the last two statements are true. So we compute F_n:

$$\begin{aligned} F_n &= \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{\sqrt{5}\cdot 2^{n-1}} + \frac{(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}}{\sqrt{5}\cdot 2^{n-2}} \\ &= \frac{(1+\sqrt{5})^{n-1} + 2\cdot(1+\sqrt{5})^{n-2}}{\sqrt{5}\cdot 2^{n-1}} \\ &= \frac{(1+\sqrt{5})^{n-2}(3+\sqrt{5})}{\sqrt{5}\cdot 2^{n-1}} \\ &= \frac{(1-\sqrt{5})^{n-2}(3-\sqrt{5})}{\sqrt{5}\cdot 2^{n-1}} \end{aligned}$$

  we use

$$(1+\sqrt{5})^2 = 2(3+\sqrt{5})$$
$$(1-\sqrt{5})^2 = 2(3-\sqrt{5})$$

□

Definition 2.1.1

Strong Induction

We have statements to prove: $S_1, S_2, S_3, \ldots$

- Initial Step: We prove a subset of the first statements: $S_1, \ldots, S_k$
- Induction Step: We assume for each $n > k$ that $S_1, \ldots, S_{n-1}$ are all true. Then we prove S_n.

2.2 Basic Counting Problems

The first objects that we are counting will be permutations:

Definition 2.2.1

Permutation

A permutation of A is an arrangement of elements of A in a linear order which uses elements of A exactly once.

Theorem 2.2.1

Permutations of a Set

There are $n!$ permutations of an n element set A.

Proof. There are n choices for the first element. Then there are $n-1$, so we continue and get $n!$. □

The more generalized version of a set is a multiset.

Definition 2.2.2

Multiset

A multiset is an unordered collection of elements where each element can appear with some multiplicity.

Theorem 2.2.2

Permutations of a Multiset

Suppose that A is a multiset with multiplicities $a_1, a_2, \ldots, a_k$. And $n = a_1 + \cdots + a_k$ is the size of the multiset. We have:

$$\frac{n!}{a_1!a_2!\cdots a_k!}$$

permutations of A.

Proof. Without loss of generality, suppose that:

$$A = \{\underbrace{1, 1, \ldots, 1}_{a_1 \text{ times}}, \underbrace{2, 2, \ldots, 2}_{a_2 \text{ times}}, \ldots, \underbrace{k, k, \ldots, k}_{a_k \text{ times}}\}$$

And we let B be the set $\{(1,1), (1,2), \ldots, (1, a_1), (2,1), \ldots, (2, a_2), \ldots\}$. There are $n!$ permutations of B. We have overcounted. So find how many permutations of B lead to a permutation of A which we call w. So we count the number of ways to permute a single element of multiplicity a_1 which is just $a_1!$. So there are $a_1!a_2!\cdots a_k!$ ways to add labels to a permutation of A. We have the number of permutations of A as X and the number of ways to add labels as $\prod_{i=1}^{k} a_i!$. Therefore:

$$X \cdot \prod_{i=1}^{k} a_i! = n!$$

So this concludes the proof. □

We consider strings to be a generalized version of the permutations on a multiset.

Definition 2.2.3 **Strings**

We generate strings by fixing some set A which we call the alphabet. Consider strings of length m with letters from A.

Theorem 2.2.3 **Number of strings**

The number of strings of length m in an n-letter alphabet is n^m.

Proof. Since we have a string of length m, we have m spots to fill with letters, each spot with n choices of letters. Since the choices in each position are independent from each other, we have $\underbrace{n \cdot n \cdots n}_{m \text{ times}}$ which is just n^m. □

2.3 Functions

If A, B are sets, then $f : A \to B$ is the assignment of elements in A to elements in B.

Definition 2.3.1 **Injective and Surjective**

A function $f : A \to B$ is called injective if $f(a_1) = f(a_2) \implies a_1 = a_2$. A function is called surjective if for every $b \in B$, there exists an $a \in A$ such that

$$f(a) = b$$

A function is called bijective if it is both injective and surjective.

Proposition: If A, B are two sets of sizes $|A| = n, |B| = m$. Assume that $f : A \to B$.

- If f is injective, then $|A| \leqslant |B|$
- If f is surjective, then $|A| \geqslant |B|$
- If f is bijective, then $|A| = |B|$

Definition 2.3.2 **Preimage of an element**

For $b \in B$, let

$$f^{-1}(b) = \{a \in A : f(a) = b\}$$

Then $f^{-1}(b)$ is a set that partitions A into parts.

Proof. Under the perspective of injectivity, we have that $|f^{-1}(b)| \leqslant 1$. This means that the collection of preimages splits A into b groups of size no greater than 1. So $|A| \leqslant |B|$. As for surjectivity, we note that each $|f^{-1}(b)| \geqslant 1$. Since these preimages splits A into

b groups of size greater than or equal to 1, we have $|A| \geqslant |B|$. Since a bijective is both injective and surjective, it follows that $|A| = |B|$ from the first two parts. □

Example 2.3.1: $[n]$ has 2^n subsets.

Proof. Construct a bijection from the subsets:

$$\{\text{subsets of } [n]\} \longrightarrow \{\text{strings of length } n \text{ in } \{0,1\}\}$$

Consider the string:

$$S_1 S_2 \dots S_n$$

defined as:

$$S_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

This means that the number of subsets of $[n]$ is equal to the number of strings of $0, 1$ strings of size n. We know that the number of strings is 2^n. □

Proposition: There are m^n functions from the set $[n] \to [m]$.

Proof. Each function is defined by a string: $f(1)f_2 \dots f(n)$. The string of length n with an alphabet of size m has size m^n. □

Proposition: There are $n!$ bijections from the set $[n] \to [n]$.

Proof. Each bijection f is determined by $f(1), f(2), \dots, f(n)$. Since f is a bijection, all $f(i)$ are distinct from $i \neq j$ so this string is a permutation of n elements. □

Proposition: Suppose that $m \leqslant n$ and count the number of injections from $[m] \to [n]$. This will be $\frac{n!}{(n-m)!}$.

Proof. Let us count sequences $f(1), \dots, f(m)$. Since f is injective, all $f(i)$ are different. There are n choices for the first, then $n-1$ options for $f(2)$, ..., there is $n-m+1$ option for $f(m)$. So there are $\frac{n!}{(n-m)!}$ options. □

Definition 2.3.3 **Number of Surjections**

Suppose that $k \leqslant n$. The number of surjections from $[n] \to [k]$ is denoted $k! \cdot S(n,k)$ where $S(n,k)$ is called the stirling number of the second kind.

Choice Problems: We want to count the number of subsets of a given set if we fix the size of our desired subset.

Definition 2.3.4 **Binomial Coefficients**

The number of k element subsets of $[n]$ is denoted $\binom{n}{k}$.

Theorem 2.3.1 **Binomial Coefficient Value**

The binomial coefficient value:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. Each subset $A \subseteq [n]$ is encoded by $S_1, \dots, S_n$ where:

$$S_i = \begin{cases} 0 & \text{if } i \notin A \\ 1 & \text{if } i \in A \end{cases}$$

Now we count the number of these strings with the restriction that there are exactly k 1's in this sequence. These strings are exactly permutations of the multiset with k instances of 1 and $n-k$ instances of 1. We have done this which is:

$$\frac{n!}{(n-k)!k!}$$

□

2.4 Counting with Binomial Coefficients

Example 2.4.1: Need to choose 5 days out of the month for meetings.

Answer. The solution is just 30 choose 5 or $\binom{30}{5}$

Example 2.4.2: Need to choose 5 days for meetings in 30 day month where we don't have a meeting on two consecutive days.

Answer. Let $a_1, a_2, \dots, a_5$ be the days that we have chosen for our meetings. Assume that $1 \leqslant a_1 < a_2 < \dots < a_5 \leqslant 30$. Sow we have $a_i - a_j \geqslant 2$ for $i > j$. Consider another sequence $b_1, b_2, \dots, b_5$ which is shifted: $(a_1, a_2 - 1, a_3 - 2, a_4 - 3, a_5 - 4)$ which means that $1 \leqslant b_1 < b_2 < b_3 < b_4 < b_5 \leqslant 26$. This is a simpler condition. So there are $\binom{26}{5}$ $b-$sequences.

Example 2.4.3: We need to choose days for 10 meetings in a 7 day week. Several meetings can be on the same day. Meetings are distinguishable.

Answer. Let $a_1, \dots, a_{10}$ be the days of the meetings. So we order then in strictly increasing order:

$$1 \leqslant a_1 \leqslant a_2 \leqslant \dots \leqslant a_{10} \leqslant 7$$

Consider the new sequence $b_1, \dots, b_{10}$ which is $(a_1, a_2 + 1, a_3 + 2, \dots, a_{10} + 9)$. So now we have a strictly increasing sequence:

$$1 \leqslant b_1 < b_2 < \dots < b_{10} \leqslant 16$$

So the answer is $\binom{16}{10}$.

Proposition: Number of multisets consisting from numbers in $[n]$ of size k:

$$\binom{n+k-1}{k}$$

Proof. Use the trick on $1 \leqslant a_1 \leqslant \dots \leqslant a_n \leqslant n$. □

So we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = number of ways to choose k element subsets of $[n]$.

Proposition: $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, $\binom{n}{k} = \binom{n}{n-k}$

Proof. $\binom{n}{0}$ is the number of ways to pick an empty set. There is only 1 way. $\binom{n}{n}$ is 1 because there is only 1 way to choose the whole set. We can build a correspondence between subsets of size k and $n-k$ by the complement set. $\square$

Pascal's Triangle

Theorem 2.4.1

$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Proof. What is $\binom{n+1}{k+1}$? This is the number of ways to choose subsets of size $k+1$ in the set $[n+1]$. We try to count this in a different way. We construct $A \subseteq [n+1]$ by first choosing if $n+1 \in A$ or $n+1 \notin A$. Then we choose the remaining elements from $[n]$.

- If $n+1 \in A$, then we pick k more elements from $[n]$ or $\binom{n}{k}$ options
- If $n+1 \notin A$, then we pick $k+1$ more elements from $[n]$ which is $\binom{n}{k+1}$ options.

In total, we have $\binom{n}{k} + \binom{n}{k+1}$ options. $\square$

Hockey Stick

Theorem 2.4.2

We have that:

$$\sum_{i=1}^{n-k} \binom{k+i}{k} = \binom{n+1}{k+1}$$

Proof. This is $\binom{n+1}{k+1}$ or how many ways to choose $k+1$ size subsets out of $[n+1]$. We find an alternative way to choose this:

- First choose x, the largest number of our subset. We have that $n+1 \geqslant x \geqslant k+1$.
- Then pick k elements which are numbers from 1 to $x-1$. We have $\binom{x-1}{k}$ options.

$\square$

Chapter 3

Week 3

3.1 Binomial Theorem

A binomial is the sum of two parts: $x + y$. We want to look at the powers of this binomial:

$$(x + y)^n$$

Theorem 3.1.1 **Binomial Theorem**

For $n \geqslant 0$,

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof. We consider the expansion: $(x + y)(x + y) \cdots (x + y)$:

$$x(x + y) \cdots + y(x + y)(x + y) \cdots$$

And we keep expanding:

$$x^2(x + y) + \ldots + xy(x + y) + \ldots + yx(x + y) + \ldots + y^2(x + y)$$

In general, each summand corresponds to our choice of x or y. So we are counting number of ways to choose k x's and $n - k$ y's. So that means that

$$x^k y^{n-k}$$

appears $\binom{n}{k}$ times. □

Example 3.1.1: We have that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

Proof. The first summand is the number of 0 element subsets of $[n]$. The second is the number of 2 element subsets of $[n]$,.... And the last is the number of n element subsets of $[n]$. Therefore, this is the number of subsets of $[n]$ which is 2^n. The other proof is to set $x = y = 1$ in the binomial theorem. □

Example 3.1.2: Another example is

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = \binom{n}{0} - \binom{n}{1} + \ldots = 0$$

Proof. We have $x = -1$ and $y = 1$, so by the binomial theorem we have 0. In the combinatorial proof, we can move all the negative terms to the other side:

$$\binom{n}{0} + \binom{n}{2} + \ldots = \binom{n}{1} + \binom{n}{3} + \ldots$$

So we wish to show that the number of even sized subsets of $[n]$ is equal to the number of subsets of odd size. Consider the bijection between the subsets containing n and those that do not contain n. Notice that the bijection is between even and odd element subsets. The bijections splits $[n]$ into pairs $(A, A\backslash\{n\})$. In each of the pairs, we have one even sized subset and one odd sized subset. □

Example 3.1.3: Another identity is:

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$$

In the algebraic proof, we have

$$\left(\frac{d}{dx}(x+1)^n\right)\bigg|_{x=1}$$

In the combinatorial proof, we have the following problem. Consider a group of n people, distinguishable. Count the number of ways to pick a team consisting of these people f arbitrary size and choosing one of the people of these to be the team captain. First choose the captain which is n options then pick an arbitrary subset of the remaining people. We get $n2^{n-1}$ options. Then we count the other side. First, choose subsets of size of team: $\binom{n}{i}$. Then for each team of size i, choose the team captain: $i\binom{n}{i}$. We sum over all the size. So we have the equivalence.

3.2 Multinomial Theorem

There is a more general theorem which comes from the question of what is $(x+y+z)^3$. We start expanding:

$$x(x+y+z)^2 + y(x+y+z)^2 + z(x+y+z)^2$$

Then we keep expanding:

$$x^2(x+y+z) + xy(x+y+z) + \ldots$$

Multinomial Theorem

Theorem 3.2.1

Suppose that $n \geqslant 0$ and $k \geqslant 1$. Then look at the expression $(x_1 + \ldots + x_k)^n$:

$$\sum_{\substack{a_1,\ldots,a_k \\ a_i > 0 \\ a_1+a_2+\ldots+a_k=n}} \binom{n}{a_1, a_2, \ldots, a_k} \cdot x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}$$

and

$$\binom{n}{a_1, \ldots, a_k} = \frac{n!}{a_1! \cdots a_k!}$$

which is the multinomial coefficient.

Proof. We need to find $(\sum_{i=1}^k x_i^n)$. Recall the distribution law:

$$\left(\sum_{i=0}^{k} a_i\right)\left(\sum_{j=0}^{m} b_j\right) = \sum_{i,j=0}^{i=n,j=m} a_i b_j$$

So we have

$$\left(\sum_{i=1}^{k} x_i\right)^n = \sum_{\substack{i_1,\ldots,i_n \\ i_j \in [1,\ldots,k]}} x_{i_1} x_{i_2} \ldots x_{i_n}$$

We want to count how many $x_{i_1} x_{i_2} \ldots x_{i_n}$ are equal to $x_1^{a_1} \ldots x_n^{a_n}$. This is the same as counting $(i_1, \ldots, i_n)$ which contain a_1 ones, a_2 twos, ... These are permutations of $\{1^{a_1}, \ldots, n^{a_n}\}$. There are

$$\frac{n!}{a_1! \ldots a_n!}$$

permutations. □

Proposition: $\binom{n}{a_1,\ldots,a_n}$ is the number of ways to split $[n]$ into disjoint subsets $A_1, \ldots, A_k$ where $|A_i| = a_i$.

Proof. Any way to split $[n]$ into this subset is equivalent to a permutation of the multiset by assigning $1, 2, \ldots, k$ to an element of n. □

Proposition: $\binom{n}{a_1,\ldots,a_k} = \binom{n}{a_1}\binom{n-a_1}{a_2}\binom{n-a_1-a_2}{a_3} \ldots \binom{n-a_1-a_2-\ldots-a_{k-1}}{a_k}$.

Proof. How to split $[n]$ into k groups of sizes $a_1, \ldots, a_k$. It can be counted in the following way:

- First choose the group A_1 which an a_1-element subset of $[n]$. There are $\binom{n}{a_1}$ options.
- Then choose A_2 which is an a_2-element subset of the set $[n] \backslash A_1$ which has size $n - a_1$. There are $\binom{n-a_1}{a_2}$ options.
- Repeat this process.
- At the end, you will choose the last group A_k in which you choose a_k elements in $[n] \backslash (A_1 \cup \ldots \cup A_{k-1})$ in which we have $\binom{n-a_1 \ldots a_{k-1}}{a_k}$ options.

□

Example 3.2.1: Take for example $(x + y + z)^3$. It will be:

$$x^3 + y^3 + z^3 + 3x^2y + 3y^2x + 3y^2z + 3z^2y + 3x^2z + 3z^2x + 6xyz$$

There is another way to generalize binomial theorem. Take $(1 + x)^a$ where a is an arbitrary number. There are examples like $\sqrt{1+x}$:

$$(1 + x)^a = \sum_{i \geqslant 0} \binom{a}{i} x^i$$

This will be postponed to the generating functions section.

There will be new types of counting problems: How many ways there are to split a number of objects into some number of groups.

Example 3.2.2: Splitting different objects into different groups. Suppose there are n balls which are numbered from $1, \ldots, n$. Say there are k boxes, which are numbered from $1, \ldots, k$. How many ways are there to place all the balls into some boxes? The number of ways would be the number of functions from $[n]$ to $[k]$.

Count the number of ways to place balls in a way where no box is left empty. This is equivalent to counting surjections from $[n]$ to $[k]$. This will be $k!S(n, k)$.

Example 3.2.3: Now we will be splitting identical objects into different groups. Suppose there are n identical balls and k boxes numbered from $1, \ldots, k$. The only important piece of data is how many balls are in each box. Let $a_1, \ldots, a_k$ be the sizes of these boxes. The only constraint is that $a_1 + \ldots + a_k = n$

Definition 3.2.1 **Weak Composition of n**

A sequence $(a_1, \ldots, a_k)$ such that a_i are integers $\geqslant 0$ and $a_1 + \cdots + a_k = n$ is called weak composition of n. The numbers a_i are called parts of the composition. If in addition $a_i \neq 0$, then the sequence $(a_1, \ldots, a_k)$ is called a composition of n.

Chapter 4

Week 4

4.1 Identical Objects, Different Boxes

Proposition: What is the number of weak compositions of n with k parts is equal to $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$.

Proof. (Stars and Bars) Want to count sequences $(a_1, \dots, a_k)$ where $a_i \geqslant 0$, $a_1 + \dots + a_k = n$. To think about these sequences, think about the configuration:

$$\underbrace{**\dots*}_{a_1 \text{ times}} \Big| \underbrace{**\dots*}_{a_2 \text{ times}} \Big| \dots \Big| \underbrace{**\dots*}_{a_k \text{ times}}$$

There should be in total n stars and $k-1$ bars. So weak compositions are in one to one correspondence with permutations of $\{*^n, |^{k-1}\}$ and there are $\binom{n+k-1}{n}$ permutations. □

Proposition: The number of compositions of n with k parts (each part must be $\geqslant 1$) is equal to $\binom{n-1}{k-1}$.

Proof. (First Proof) There is a bijection $(a_1, \dots, a_k)$, then consider the composition: $(a_1 + 1, \dots, a_k + 1)$. The first is a weak composition of n and the second is a composition of $n + k$. The compositions of n with k parts are in bijection with the number of weak compositions of $n - k$ with k parts. So substituting in $n - k = n$, we get $\binom{n-1}{k-1}$.

(Second Proof) Encode $(a_1, \dots, a_k)$ by stars and bars. Consider n stars placed in a row. Between each gap, you either do nothing or place a bar. So we have $n - 1$ gaps and $k - 1$ bars to place. Therefore, we have $\binom{n-1}{k-1}$ compositions. □

Proposition: The number of compositions of n is equal to 2^{n-1}.

Proof. We are counting all the possible ways to place bars between n stars. There are $n - 1$ gaps, so we just choose a subset of these gaps. There are 2^{n-1} subsets. □

For weak compositions because you can have groups of size 0. So there are infinitely many.

4.2 Different Objects, Identical Boxes

Set $[n]$. How many ways are there to split the set into k disjoint subsets.

Definition 4.2.1 **Partitions**

An ordered collection of non-empty subsets of $[n]$ such that each element of $[n]$ is exactly in one of the subsets is called a partition of the set $[n]$. These are called set partitions.

Proposition: The number of partitions of $[n]$ with k parts is equal to $S(n, k)$.

Proof. Recall that the number of surjections f from $[n]$ to $[k]$ is equal to $k!S(n, k)$. We will show that there for each partition of $[n]$ into k parts, we can assign to it $k!$ surjections. If $f : [n] \to [k]$ is a surjection, consider the partition of $[n]$ into subsets according to the value of $f(i)$, namely, if $f(i) = f(j)$, then i, j are in the same partition. Since f is a surjection, there are k such groups. Let us fix a partition of $[n]$ with k parts. To construct surjection we specify how values of f correspond to values of this partition. Since the values of f are numbers from 1 to k, we specify how to order these numbers. So there are $k!$ ways to get surjections from such a partition. There are no two equal parts of the partition. □

Theorem 4.2.1 **Stirling Numbers**

For any positive integers n, k,

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$

4.3 Stirling Numbers Continued

The stirling number formula looks similar to

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Example 4.3.1: We know that $S(n, 1) = 1 = S(n, n)$ since there is only way to split $[n]$ into 1 part or n parts for all $n \geqslant 1$.

Example 4.3.2: It is defined that $S(n, 0) = 0$ and $S(0, 0) = 1$.

Example 4.3.3: $S(1, 1) = 1, S(2, 1) = 1, S(2, 2) = 1, S(3, 1) = 1, S(3, 2) = 3, S(3, 3) = 1$

Proof. We will construct partitions of $[n]$ by first picking partitions of $[n - 1]$ and then adding n. In this construction, there are two cases.

- We can add n as its separate partition of $[n - 1]$
- We can either add n as a part to an existing partition of $[n - 1]$

For the first case, if we want k parts in the end, we start with a partition of $[n - 1]$ with $k - 1$ parts. There are $S(n - 1, k - 1)$ partitions of $[n - 1]$ into $k - 1$ parts. For the second case, we start with a partition of $[n - 1]$ with k parts and choose one of the k parts to add n to. There are k choices for each partition, so we have $k \cdot S(n - 1, k)$ partitions. The total would be

$$S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k)$$

proving the identity. □

Theorem 4.3.1 **Number of Mappings:** $[n] \to [m]$

For any positive integers n, m

$$m^n = \sum_{k=1}^{n} \binom{m}{k} S(n,k) \cdot k!$$

Proof. We have m^n is the number of mappings from $[n] \to [m]$. How to get another mapping $f : [n] \to [m]$? First pick $k = 1, \ldots, n$. Then pick a subset $I \subseteq [m]$ such that $|I| = k$. Finally, build f by choosing a surjection from $[n] \to I$. The idea is by choosing the image of f first and counting the number of surjections. There are $\binom{m}{k}$ ways to choose I. □

Example 4.3.4: We know that $S(n, 1) = 1$. Set $m = 2$. We have

$$2^n = \sum_{k=1}^{n} \binom{2}{k} k! \cdot S(n,k) = 2 \cdot S(n,1) + 2 \cdot S(n,2) = 2 + 2 \cdot S(n,2)$$

We get:

$$S(n,2) = 2^{n-1} - 1$$

Example 4.3.5: Now set $m = 3$:

$$3^n = \sum_{k=1}^{n} \binom{3}{k} k! \cdot S(n,k) = 3 \cdot S(n,1) + 6 \cdot S(n,2) + 6 \cdot S(n,3)$$

We get:

$$S(n,3) = \frac{3^n - 6(2^n - 1) - 3}{6} = \frac{3^{n-1} - 1}{2} - 2^{n-1} + 1$$

Definition 4.3.1 **Bell Number**

The number of partitions of $[n]$ is denoted $B(n)$ which is called Bell number:

$$B(n) = \sum_{k=1}^{n} S(n,k)$$

4.4 Identical Objects, Identical Boxes

Placing n identical objects into identical boxes is equivalent to picking a multiset $\{a_1, a_2, \ldots, a_k\}$ of how many objects of how many objects are in each box. We can assume that $a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots$.

Definition 4.4.1 **Integer Partitions**

A sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$ is called an integer partition of n.

The number $p(n)$ is the number of partitions of n and $p_k(n)$ is the number of partitions of n into k-parts.

Definition 4.4.2 **Young Diagram**

A Young diagram of a partition λ is an arrangement of square boxes in upper-left corner of a plane quadrant such that the first row has λ_1 boxes, second row has λ_2 boxes, ...

Definition 4.4.3 **Conjugate**

A conjugate of a partition λ is denoted by λ' and it is the partition which is obtained by reflecting the Young Diagram of λ with respect to the main diagonal.

Definition 4.4.4 **Self Conjugate**

A partition λ is called self conjugate if it is equal to its conjugate.

An alternative definition of λ' is

$$\lambda_i = \{j : \lambda_j \geqslant i\}$$

Example 4.4.1: λ'_1 =number of parts of λ,

λ'_2 =number of parts $\geqslant 2$

So we can get the number of parts = 1 by $\lambda'_1 - \lambda'_2$

Proposition: Number of partitions of n with parts $\leqslant k$ is equal to the number of partitions of n with $\leqslant k$ parts.

Proof. The number of partitions with $\leqslant k$ parts is the number of young diagrams with height less than or equal to k. The number of partitions of n parts that are all $\leqslant k$ is the conjugate of our partition, which has length $\leqslant k$. This induces a bijection. □

Definition 4.4.5 **Strict Partitions**

A partition λ is called strict if all parts are strictly decreasing $\lambda_1 > \lambda_2 > \ldots > \lambda_k$

Proposition: The number of conjugate partitions of n is equal to the number of strict partitions of n with only odd parts.

Proof. Take all hooks and construct a partition where μ is defined by its parts as μ_i = number of boxes in the i-th hook. We know that the parts are odd because the length will be k which will be equal to the height because it is self-conjugate. Then there are $2k - 1$ boxes in the young diagram. It is also strictly decreasing because the inner hook has length $\leqslant k - 1$. So $\mu_i > \mu_j$ if $j > i$. This is a bijection because you can get the young diagram from the strictly decreasing partition with only odd parts. □

4.5 Permutations Continued

A permutation of $[n]$ is a linear ordering of elements from $[n]$. If $\sigma : [n] \to [n]$ is a bijection, then we can interpret this σ as a permutation $\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(n)$.

Example 4.5.1: The sequence $2\,3\,1$ is the bijection

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 3 \\ 3 &\mapsto 1 \end{aligned}$$

Definition 4.5.1 **Composition of Permutations**

Let σ, ω be permutations of $[n]$. The product $\sigma\omega$ is a permutation defined by

$$[\sigma\omega](i) = \sigma(\omega(i))$$

Example 4.5.2: $2\,3\,1 \cdot 2\,1\,3 = 3\,2\,1$.

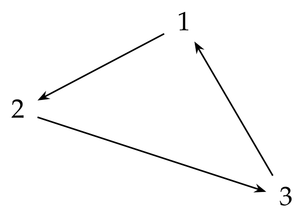
A remark of this operation is that it is associative:

$$(\sigma\omega)\tau = \sigma(\omega\tau)$$

This is not a commutative operation. So in general, $\sigma\omega \neq \omega\sigma$. There is an identity element: $e = 1\,2\,3\,\ldots\,n$, satisfying the property that e commutes with all permutations.

Finally, for any permutation σ, we can define σ^{-1}. If $\sigma(i) = j$, then $\sigma^{-1}(j) = i$. The operation satisfies $\sigma\sigma^{-1} = e = \sigma^{-1}\sigma$. Associativity, identity element, and existence of inverses make the set of permutations of $[n]$ into a group. The set of all permutations of $[n]$ is denoted S_n.

We will study behavior of σ^n. We have cycles:



Lemma: For any permutation σ of $[n]$ and any $x \in [n]$, there exists a number $j \in [n]$ such that $\sigma^j(x) = x$.

Proof. Consider the sequence $x, \sigma(x), \sigma^2(x), \ldots, \sigma^n(x)$. This is a sequence of $n+1$ numbers from $[n]$. By the pigeonhole principle, we have that two of the numbers must be equal:

$$\sigma^a(x) = \sigma^b(x) \text{ where } 0 \leqslant a < b \leqslant n$$

So multiplying from the left by σ^{-a}, we have $\sigma^{b-a}(x) = x$. □

Chapter 5

Week 5

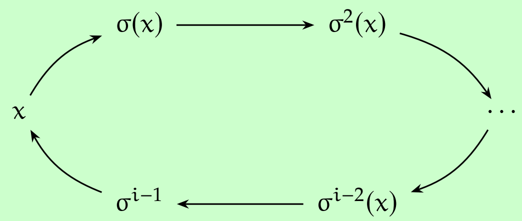
5.1 Cycles and Permutations

Definition 5.1.1 — **Cycles**

Let σ be a permutation of $[n]$ and $x \in [n]$. Let i be the minimum positive integer such that $\sigma^i(x) = x$. Then

$$x, \sigma(x), \sigma^2(x), \ldots, \sigma^{i-1}(x)$$

is called an i-cycle of σ. This cycle is denoted by $(x, \sigma(x), \ldots, \sigma^{i-1}(x))$. Cycles that differ by a rotation will be considered the same cycle.



Theorem 5.1.1 — **Disjoint Cycles**

Any permutation σ of $[n]$ can be decomposed into disjoint cycles (every element from $[n]$ will be in exactly one of the cycles).

Definition 5.1.2 — **Cycle Type**

A cycle type of a permutation σ of $[n]$ is the partition $(\lambda_1, \lambda_2, \ldots)$ of n such that $\lambda_1, \lambda_2, \ldots$ are exactly the lengths of cycles of σ.

Proposition: Let λ be a partition of n, and let m_i denote the number of parts of λ equal to i. Then the number of permutations with cycle type λ is equal to

$$\frac{n!}{\prod_{i \geqslant 1} m_i! i^{m_i}}$$

Proof. Construct σ as follows

- Start with permutation of $[n]$ denoted ω which we write as a linearly ordered sequence of numbers $\omega_1\omega_2\dots\omega_n$.
- We define by $(\omega_1\dots\omega_{\lambda_1})(\omega_{\lambda_1+1}\dots\omega_{\lambda_1+\lambda_2})\dots$. This gives σ with cycle type λ. However, different choices of ω might lead to the same σ.
- Count how many ω lead to the same σ. There are two ways to get the same ω. First, we can rotate each cycle to get the same σ. For each i-cycle, this gives i options:

  $$\prod_{i\geqslant 1} i^{m_i}$$

  We can also permute all i-cycles for a fixed i, so for each i, this gives $m_i!$ more options. So we get:

  $$\prod_{i\geqslant 1} m_i! i^{m_i}$$

  permutations leading to the same σ. Since there were $n!$ ways to start with a ω, we have

  $$\frac{n!}{\prod_{i\geqslant 1} m_i! i^{m_i}}$$

□

Example 5.1.1: There are $(n-1)!$ n-cycles of $[n]$.

Example 5.1.2: There are

$$\frac{(2n)!}{2^n n!}$$

ways to split $[2n]$ into pairs. This is the same as constructing permutations with only 2 cycles with has cycle type (2^n).

5.2 Stirling Numbers of the First Kind

Definition 5.2.1 **Unsigned Stirling Number of the First Kind**

The number of permutations of $[n]$ with exactly k-cycles is denoted by $c(n,k)$ and called unsigned Stirling number of the first kind. The number $(-1)^{n-k}c(n,k) = s(n,k)$ is called stirling number of the first kind. The stirling number

$$c(n,0) = 0\, n > 0, \qquad c(0,0) = 1$$

Recall that for stirling numbers of the second kind, we had:

$$s(n,k) = s(n-1,k-1) + ks(n-1,k)$$

Proposition: For any positive integers n, k we have

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

Proof. We have $c(n,k)$ is the number of permutations of $[n]$ with k cycles. We will count this relation in the following way:

- We will first take a permutation of $[n-1]$.
- Then add n to the permutation of $[n-1]$ in some way.

There are two ways to add n.

- We can add n as a 1-cycle to a permutation of $[n-1]$. In this case, there are $c(n-1, k-1)$ options because we have to start with $k-1$ cycles.
- We can add n to some cycle which already exists in a permutation of $[n-1]$. Since the number of permutations does not change, we have $c(n-1, k)$ options for the starting permutation. But we also have i different ways to add n into i-cycle. We should take this i for each cycle of the starting permutation. So the number of different ways will be the sum of all the lengths of all cycles which is equal to $n-1$. So there are $(n-1)c(n-1, k)$ options.

Taking these two cases combined, we get

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k)$$

□

We know that $c(n, 1) = (n-1)!$ and $c(n, n) = 1$. Using this recurrence relation, we can prove:

Identity on Polynomials

Theorem 5.2.1 For any $n \geqslant 0$,

$$\sum_{k=0}^{n} c(n, k)x^k = x(x+1)(x+2)\cdots(x+n-1)$$

Proof. Induction proof. □

Proposition: For $n \geqslant 0$,

$$\sum_{k=0}^{n} s(n, k)x^k = x(x-1)(x-2)\cdots(x-(n-1)) = (x)_n$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n} s(n, k)x^k &= \sum_{k=0}^{n} (-1)^{n-k} c(n, k)x^k \\
&= (-1)^n \sum_{k=0}^{n} c(n, k)(-x)^k \\
&= (-1)^n(-x)(-x+1)(-x+2)\cdots(-x+n-1) \\
&= (x)(x-1)(x-2)\cdots(x-(n-1))
\end{aligned}$$

□

Recall that

$$\sum_{k=0}^{n} S(n, k) \cdot k! \cdot \binom{m}{k} = m^n$$

We can rewrite:

$$\sum_{k=0}^{n} S(n, k) \cdot (m)_k = m^n$$

Theorem 5.2.2 If $f(x), g(x)$ are polynomials in x such that $f(a) = g(a)$ for infinitely many a's, then these polynomials are equal at every point: $f(x) = g(x)$.

Therefore, we can conclude

$$\sum_{k=0}^{n} S(n,k) \cdot (x)_k = x^n$$

The main idea is that signed stirling numbers of the first kind shows how to get from polynomials to falling factorials and the stirling number of the second kind shows how to get from falling factorials to polynomials.

Reminder: A polynomial in x of degree $\leqslant n$ is of the form:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = f(x)$$

we can rewrite

$$f(b) = \sum_{i=0}^{n} a_i b^i$$

Proof. Enough to prove that $(f - g)(a) = 0$ vanishes at infinitely many points, then $h(x) = f - g = 0$. More precisely, if $h(x_i) = 0$ for a sequence $(x_1, x_2, \ldots)$, them $h(x) = 0$.

Induction on $\deg h$:

- When $\deg h = 0$, then $h(x) = a_0$. But $a_0 = 0$ so $h(x) = 0$.
- Assume that we have proved it for $\deg h = n$. Take $h = n + 1$. We have $h(x_0) = 0$ for all x_i. Consider the substitution: $x = y + x_1$:

  $$\tilde{h}(y) = h(y + x_1)$$

  so $\tilde{h}$ has degree $n + 1$. We also have:

  $$\tilde{h}(x_i - x_1) = 0$$

  In particular, $\tilde{h}(0) = 0$. But $\tilde{h} = a_0 + a_1 y + \cdots + a_{n+1} y^{n+1}$ so $a_0 = 0$. We can factor: $\tilde{h} = y g(y)$. Where $g(y)$ is a polynomial of degree n. So $g(y) = 0 = \tilde{h}$.

□

5.3 Inclusion Exclusion Principle

There are 10 students who like apples. There are 15 students who like oranges. How many students like at least one of these fruits?

There are many cases:

- All students who like apples also like oranges: 15
- People who like apples hate oranges and vice versa. So we get 25.

Problem: 10 students like apples and 15 students like oranges, 5 students like both apples and oranges. How many students like at least one of the two fruits?

The answer is $10 + 15 - 5 = 20$.

Problem: 10 students like apples, 15 students like oranges, 20 students like bananas, 5 students like both apples and oranges, 7 students like both apples and bananas, 10 students like oranges and bananas, 4 students like all three fruits. How many students like at least one of these three fruits?

Solution: 10 + 15 + 20 counts students with one fruit once. Students who like two fruits are counted twice. Students who like all three are counted 3 times. So subtract out the redundant counts: $45-5-7-10 = 23$ counts the number of students who like one fruit one time, students who like two fruits are counted once, and students who like all three fruits are counted 0 times. Finally, $13+4 = 27$.

Inclusion-Exclusion Principle

Theorem 5.3.1 Let $(A_1, A_2, \dots, A_k)$ is a collection of k sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{j=1}^{k} (-1)^{j-1} \sum_{i_1,\dots,i_j} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}|$$

where $\sum_{i_1,\dots,i_j}$ is the sum over $\{i_1, \dots, i_j\} \subseteq [k]$ with j elements.

Example 5.3.1: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ and $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$.

Proof. Let $x \in A_1 \cup A_2 \cup \cdots \cup A_k$. We want to show that x will be counted once in

$$\sum_{j=1}^{k} (-1)^{j-1} \sum_{i_1,\dots,i_j} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}|$$

Let S denote the set of indices i such that $x \in A_i \iff i \in S$. We know $|S| \geqslant 1$. $x \in A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}$ iff $\{i_1, i_2, \dots, i_j\} \subseteq S$. This means that x is counted in $\sum_{i_1,\dots,i_j} |A_{i_1} \cap \cdots \cap A_{i_j}|$ exactly $\binom{n}{j}$ times where $n = |S|$. This is the number of ways to choose $\{i_1, \dots, i_j\} \subseteq S$. In total, x is counted

$$\sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j}$$

Recall that

$$\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} = 0 = 1 + \sum_{j=1}^{n} (-1)^{j} \binom{n}{j}$$

so we get:

$$\sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} = 1$$

which means x is counted once. □

Applications: Consider maps $[n] \to [k]$. A set A_i is the set of maps $f : [n] \to [k]$ which misses i, where $i \in [k]$. Then:

$$|A_i| = \text{Number of maps } [n] \to [k]/\{i\}$$

The number of such maps is $(k-1)^n$. Moreover, the intersection $A_{i_1} \cap \cdots \cap A_{i_j}$ corresponds to the number of maps $[n] \to [k] \backslash \{i_1, \dots, i_j\}$ which is $(k-j)^n$. By PIE, we have that:

$$|A_1 \cup \cdots \cup A_k| = \sum_{j=1}^{k} (-1)^{j-1} (k-j)^n \binom{k}{j}$$

These are maps from $[n] \to [k]$ which miss at least one element of $[k]$. This means that the number of surjections is:

$$k^n - |A_1 \cup \cdots \cup A_k|$$

So there is the following formula:

$$k!S(n,k) = k^n - \sum_{j=1}^{n} (-1)^{j-1}(k-j)^n \binom{k}{j} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

Now dividing by $k!$:

$$S(n,k) = \sum_{j=0}^{k} (-1)^j \frac{(k-j)^n}{j!(k-j)!}$$

Chapter 6

Week 6

6.1 PIE in permutations

We know that there are $n!$ permutations of $[n]$. Consider permutations σ of $[n]$ such that $\sigma(i) = i$ for a fixed $i \in [n]$. So there are $(n-1)!$ such permutations. Moreover, permutations σ of $[n]$ such that $\sigma(i_1) = i_1, \sigma(i_2) = i_2, \ldots, \sigma(i_j) = i_j$ where $\{i_1, \ldots, k_j\} \subseteq [n]$. Equivalently, we can look at the number of permutations of $[n] \backslash \{i_1, \ldots, i_j\}$ where there are $(n-j)!$ permutations.

Definition 6.1.1 **Derangement**

A permutation σ of $[n]$ is called derangement if it has no fixed points. That is, $\sigma(i) \neq i$ for any $i \in [n]$.

Theorem 6.1.1 **Number of Derangements**

The number of derangements of $[n]$ is equal to $n! \cdot \sum_{j=0}^{n} \frac{(-1)^j}{j!}$.

Proof. Let A_i denote the set of permutations which fix i. We know that $|A_i| = (n-1)!$ and we also know the sizes of the intersection of $|A_{i_1} \cap \cdots A_{i_j}|$ which is $(n-j)!$. The union of the sets:

$$|A_1 \cup \cdots \cup A_n| = \{\text{permutations that fix at least one point}\}$$

Now we apply inclusion exclusion. We have

$$
\begin{aligned}
|A_1 \cup \cdots A_n| &= \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1,\dots,i_j} |A_{i_1} \cap \cdots \cap A_{i_j}| \\
&= \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1,\dots,i_j} (n-j)! \\
&= \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)! \\
&= \sum_{j=1}^{n} (-1)^{j-1} \frac{n!}{j!} \\
&= -\sum_{j=1}^{n} (-1)^{j} \frac{n!}{j!}
\end{aligned}
$$

Number of derangements = $n! + \sum_{j=1}^{n} (-1)^j \frac{n!}{j!}$ □

Remark: If we take $n = \infty$ for $\sum_{j=0}^{n} \frac{(-1)^j}{j!}$ it will evaluate to e^{-1}. So for very large number of permutations, around 1/3 will have no fixed points.

6.2 Formal Power Series

Definition 6.2.1 **Formal Power Series**

A formal power series is a formal expression of the form:

$$\sum_{i \geqslant 0}^{\infty} a_i x^i$$

where x is a formal variable, and a_i are numbers which are coefficients of the power series.

Usually, $f(x)$ is used to denote a formal power series with variable x. When you plug $x = 0$, you get: $f(0) = a_0$. The k-th coefficient a_k is sometimes denoted $[x^k]f$.

Example 6.2.1: Every number is a formal power series :

$$a = a + 0 \cdot x + 0 \cdot x^2 + \cdots$$

Example 6.2.2: Every polynomial in x is a formal power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + 0 \cdot x^{n+1} + 0 \cdot x^{n+2} + \cdots$$

Example 6.2.3: We also have $\sum_{i=0}^{\infty} x^i$ is one where all coefficients are 1.

Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$. We define:

$$(f+g)(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

The product of this power series is:

$$(fg)(x) = \sum_{i \geqslant 0} \left(\sum_{j=0}^{i} a_i \cdot b_{i-j} \right) x^i$$

Properties:

- $(f+g)+h = f+(g+h)$
- $f+g = g+f$
- $f+0 = f$
- $\forall f$, you have $-f$ such that $f+(-f) = 0$
- $(fg)h = f(gh)$
- $fg = gf$
- $f \cdot 1 = f$
- $(f+g)h = fh + gh$

These properties says that the set of all formal power series form a ring denoted by $\mathbb{R}[[x]]$.

Example 6.2.4: $(1-x)(\sum_{i \geqslant 0} x^i) = 1 + \sum_{i \geqslant 1}(\sum_{j=0}^{i} a_j b_{i-j}) x^i = 1$. So we have:

$$\frac{1}{1-x} = \sum_{i \geqslant 0} x^i$$

Proposition: Let $F(x) = \sum_{i \geqslant 0} a_i x^i$. Then there exists $G(x)$ such that $F(x)G(x) = 1$ if and only if $F(0) \neq 0$. In other words,

$$\frac{1}{F(x)}$$

is a well-defined power series if and only if its constant term is non-zero.

Proof. If $F(0) = 0$, then $\forall G$, we have $F(0)G(0) = 0$ But this contradicts the fact that there is an inverse. Let $F(0) \neq 0$. Let us construct $G(x) = \sum_{i \geqslant 0} b_i x^i$ such that $F(x)G(x) = 1$. This means that $b_0 = a_0^{-1}$. For $k \geqslant 1$, $\sum_{i=0}^{k} a_i b_{k-i} = 0$ and equivalently, we can have $a_0 b_k = \sum_{i=1}^{k} a_i b_{k-i}$. Now:

$$b_k = \frac{-\sum_{i=1}^{k} a_i b_{k-i}}{a_0}$$

Notice that the next term depends on the previous terms. So we can construct our next terms. □

Example 6.2.5: $\frac{1}{1-x+x^2}, \frac{1}{1-x^2}, \frac{1}{2-2x+x^3+x^4}$ are formal power series

Example 6.2.6: $\frac{1}{x}$ is not a formal power series.

Another approach to the proof. Suppose that $F(0) \neq 0$. Then

$$F(x) = a + G(x)$$

where $G(0) = 0$. We want to have:

$$\frac{1}{F(x)} = \frac{1}{a + G(x)} = \frac{1}{a} + \frac{1}{1 + G(x)/a} = \frac{1}{a}\left(\sum_{j \geqslant 0}\left(\frac{-G(x)}{a}\right)^j\right)$$

Composition of Formal Power Series

Definition 6.2.2

Let $F(x) = \sum a_i x^i$ and $h(t) = \sum b_i t^i$ such that $h(0) = 0$. Then the formal power series $F(h(t))$ is defined as:

$$F(h(t)) = \sum_{i \geqslant 0} a_i h(t)^i$$

More precisely, the coefficient of t^k can be computed only using $\sum_{i=0}^{k} a_i (h(t))^i$ because $h(t)^n = at^n + \cdots$.

Example 6.2.7: $F(x) = \frac{1}{1-x} = 1 + x + x^2$

$h(t) = -t - t^2 - t^3 - \cdots$

Now computing:

$$\begin{aligned} F(h(t)) &= 1 + h(t) + h(t)^2 + \cdots \\ &= 1 + (-1)t + (-1+1)t^2 + (-1+2-1)t^3 + \cdots \\ &= 1 - t \end{aligned}$$

We also know that $F(h(t)) = \frac{1}{1+t+t^2+\cdots} = 1 - t$

Example 6.2.8: $F(x) = \frac{1}{1-x} = 1 + x^2 + x^3 + \cdots$

$h(t) = 2 + t$

$F(h(t)) = \sum_{i \geqslant 0}(2+t)^i = \sum_{i \geqslant 0} 2^i$

On the other hand

$$\frac{1}{1-(2+t)} = \frac{1}{-1-t}$$

is well-defined.

Theorem 6.2.1

Inverse Formal Power Series

Let $h(t)$ be a formal power series such that

$$h(t) = 0 + t + \sum_{i \geqslant 2} a_i t^i$$

then there exists a unique formal power series $g(x)$ such that

$$g(x) = 0 + x + \sum_{i \geqslant 2} b_i x^i$$

where $g(h(t)) = t$ and $h(g(x)) = x$.

6.3 Linear Recurrence Relations

Recall the Fibonacci numbers: $f_0, f_1, f_2, \dots$ and $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$. Take

$$\sum x^n f_n = \sum x^n f_{n-1} + \sum x^n f_{n-2}$$

and

$$\sum_{n \geqslant 2} x^n f_n = x \left(\sum_{n \geqslant 2} x^{n-1} f_{n-1} \right) + x^2 \sum_{n \geqslant 2} x^{n-2} f_{n-2}$$

which is the same as

$$\sum_{n \geqslant 0} x^n f_n - x = x \left(\sum_{n \geqslant 0} x^n f_n \right) + x^2 \sum_{n \geqslant 0} x^2 f_n$$

Definition 6.3.1

Generating Functions

A generating function of a sequence $(a_0, a_1, \dots)$ is a formal power series $\sum_{i \geqslant 0} a_i x^i$.

Let $F(x) = \sum_{n \geqslant 0} f_n x^n$. Then

$$\begin{aligned} F(x) - x &= xF(x) + x^2 F(x) \\ F(x) - xF(x) - x^2 F(x) &= x \\ F(x)(1 - x - x^2) &= x \\ F(x) &= \frac{x}{1 - x - x^2} \end{aligned}$$

We have $1 - x - x^2 = -(x^2 + x - 1) = -((x + \frac{1}{2})^2 - \frac{5}{4}) = -(x + \frac{1 - \sqrt{5}}{2})(x + \frac{1 + \sqrt{5}}{2})$. So the roots are:

$$a_1 = \frac{-1 + \sqrt{5}}{2}, a_2 = \frac{-1 - \sqrt{5}}{2}$$

we also observe that $a_1 a_2 = -1$. We can also rewrite:

$$x = \frac{a_2(x - a_1) - a_1(x - a_2)}{a_2 - a_1}$$

Plugging these two functions in:

$$F(x) = \frac{a_2(x - a_1) - a_1(x - a_2)}{-(a_2 - a_1)(a_1 - x)(a_2 - x)} = \frac{a_2(x - a_1)}{-(a_2 - a_1)(a_2 - x)(a_1 - x)} - \frac{a_1(x - a_2)}{-(a_2 - a_1)(a_1 - x)(a_2 - x)}$$

Some terms cancel out:

$$= \frac{a_2}{(a_2 - a_1)(a_2 - x)} - \frac{a_1}{(a_2 - a_1)(a_1 - x)}$$

Now the term $\frac{a}{a-x} = \frac{1}{1-\frac{x}{a}} = \sum(\frac{x}{a})^n$. So we get:

$$\frac{1}{a_2 - a_1}\left(\sum_{n\geqslant 0}\left(\frac{x}{a_2}\right)^2 - \sum_{n\geqslant 0}\left(\frac{x}{a_1}\right)^n\right)$$

So

$$F_n = \frac{1}{a_2 - a_1}\left(a_2^{-n} - a_1^{-n}\right) = \frac{(\sqrt{5}+1)^n - (1-\sqrt{5})^n}{\sqrt{5}\cdot 2^n}$$

General setting: Suppose you have a sequence $(a_0, a_1, a_2, \ldots)$ such that they satisfy a linear recurrence relation:

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$$

where $k \geqslant 1$ is an integer, $n \geqslant k$, and $c_1, \ldots, c_k$ are numbers.

Theorem 6.3.1

Let $F(x) = \sum_{n\geqslant 0} a_n x^n$. In the setting above,

$$F(x) = \frac{p(x)}{q(x)}$$

where $q(x) = 1 + c_1 x + c_2 x^2 + c_k x^k$ and $p(x)$ is some polynomial of degree $< k$ which is given in terms of c_i, $a_0, a_1, \ldots, a_{k-1}$.

Proof. Let $F_{\leqslant k}(x) = \sum_{n=0}^{k} a_n x^n$. We have the relation:

$$\sum_{n\geqslant k} x^n(a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k}) = 0$$

$$\sum_{n\geqslant k} a_n x^n + xc_1 \sum_{n\geqslant k} a_{n-1}x^{n-1} + x^2 c_2 \sum_{n\geqslant k} a_{n-2}x^{n-2} \cdots + x^k c_k \sum_{n\geqslant k} a_{n-k}x^{n-k} = 0$$

$$F(x) - F_{\leqslant k}(x) + xc_1(F(x) - F_{\leqslant k-2}(x)) + \cdots + x^k c_k F(x) = 0$$

$$F(x) + c_1 x F(x) + \cdots + c_k x^k F(x) = F_{\leqslant k-1}(x) + xc_1 F_{\leqslant k-2}(x) + x^2 c_2 F_{\leqslant k-3}(x) + \cdots = P(x)$$

So $P(x)$ is of degree $< k$ and is in terms of $a_0, a_1, \ldots, a_{k-1}$. On the left hand side:

$$F(x)(1 + c_1 x + c_2 x^2 + \cdots + c_k x^k) = Q(x)F(x)$$

so we are done. □

How to get the coefficients of the $P(x)/Q(x)$? Assume that $Q(x) = (1 - \frac{x}{z_1})(1 - \frac{x}{z_2})\cdots(1 - \frac{x}{z_k})$ where $z_1, \ldots, z_k$ are distinct and non-zero.

Theorem 6.3.2 **Partial Fraction Decomposition**

Assume that $\operatorname{def} P(x) < \deg Q(x) = k$. Then:

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{k} \frac{b_i}{1 - \frac{x}{z_i}} = \sum_{n\geqslant 0}\sum_{i=1}^{k} b_i z_i^{-n} x^n$$

where $b_i = \frac{P(z_i)}{\prod_{j\neq i}(1 - \frac{z_i}{z_j})}$

Proof. **Lemma**: If $P_1(x), P_2(x)$ are tow polynomials of degree $\leqslant k$, then $P_1(x) = P_2(x)$ if $P_1(z_i) = P_2(z_i)$ for $k+1$ different points. We need to prove that

$$P(x) = \sum_{i=1}^{k} b_i \prod_{j \neq i} (1 - \frac{x}{z_j})$$

Both polynomials have degree $< k$. By the lemma, it is enough to check equality for k different points: $x = z_i$:

$$P(z_i) = \sum_{s=1}^{k} b_s \prod_{i \neq j} (1 - \frac{z_i}{z_j}) = b_i \prod_{j \neq i} (1 - \frac{z_i}{z_j})$$

which finishes the proof. □

We get

$$a_n = \sum_{i=1}^{k} b_i z_i^{-n}$$

Chapter 7

Week 7

7.1 Binomial Theorem

We have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

Theorem 7.1.1

We say

$$\boxed{(1+x)^a} = \sum_{i\geqslant 0} \binom{a}{i} x^i \text{ where } \binom{a}{i} = \frac{a(a-1)(a-i+1)}{i!} = \frac{(a_i)}{i!}$$

When a is not an integer, what is $(1+x)^a$?

Theorem 7.1.2

We will show that $\boxed{(1+x)^a}$ behaves like $(1+x^a)$.

- $\boxed{(1+x)^a} \cdot \boxed{(1+x)^b} = \boxed{(1+x)^{ab}}$
- $\boxed{(1+x)^n} = (1+x)^n$ for all integers n
- $\boxed{(1+x)^a}\Big|_{x=0} = 1$

Proof. (Part I) For 1, we have:

$$\left(\sum_{i\geqslant 0} \binom{a}{i} x^i\right)\left(\sum_{i\geqslant 0} \binom{b}{i} x^i\right) = \sum_{i\geqslant 0} \binom{a+b}{i} x^i$$

This is equivalent to showing for any $k \geqslant 0$, that $\sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i} = \binom{a+b}{k}$. This is just looking at the coefficients of x^k. Recall:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

Then for any $n, m \geqslant 0$, integers,

$$(1+x)^n (1+x)^m = (1+x)^{n+m}$$

means that for any k, we have $\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}=\binom{n+m}{k}$. So we know that $\sum_{i=0}^{k}\binom{a}{i}\binom{b}{k-i}=\binom{a+b}{k}$ when $a=n, b=m$ positive integers. If you fix $b=m$, then we get two polynomials in the variable a which are equal when $a=n$ which means that they are equal for all a. So we know that $\sum_{i=0}^{k}\binom{a}{i}\binom{b}{k-i}=\binom{a+k}{i}$ holds when a is arbitrary and b is an integer. Now we fix a being arbitrary real and consider this for b. So the identity holds for real numbers. This identity is called the Chu-Vandermonde identity.

(Part II) If $n \geqslant 0$ integer, then $\boxed{(1+x)^n} = \sum_{i\geqslant 0}\binom{n}{i}x^i = (1+x)^n$. We also know that $\boxed{(1+x)^{-n}} \cdot \boxed{(1+x)^n} = \boxed{(1+x)^0} = 1$. So

$$\boxed{(1+x)^{-n}} = \frac{1}{(1+x)^n}$$

(Part III) This is trivial because $\boxed{(1+0)^a} = \binom{a}{0} = 1$ □

Example 7.1.1: Take $(1+x)^{-1} = \frac{1}{(1+x)} = \sum_{i\geqslant 0}(-x)^i$. We also have

$$\boxed{(1+x)^{-1}} = \sum_{i\geqslant 0}\binom{-1}{i}x^i$$

We have

$$\binom{-1}{i} = \frac{-1(-2)\cdots(-i)}{i!} = (-1)^i$$

Example 7.1.2: We have $(1+x)^{-2} = \sum_{i\geqslant 0}\binom{-2}{i}x^i$. Now:

$$\binom{-2}{i} = \frac{-2(-3)\cdots(-i-1)}{i!} = (-1)^i \cdot \frac{2\cdot 3\cdots(i+1)}{i!} = (-1)^i(i+1)$$

so

$$(1+x)^{-2} = \sum_{i\geqslant 0}(i+1)(-x)^i \text{ and } \frac{1}{(1-x)^2} = \sum_{i\geqslant 0}(i+1)x^i$$

i**Proposition**: For any k-positive integer, we have

$$\frac{1}{(1-x)^k} = \sum_{i\geqslant 0}\binom{k+i-1}{k-1}x^i$$

Proof. We know that

$$(1-x)^{-k} = \sum_{i\geqslant 0}\binom{-k}{i}(-x)^i$$

Now we compute:

$$\begin{aligned}\binom{-k}{i} &= \frac{(-k)(-k-1)(-k-2)\cdots(-k-i+1)}{i!}\\ &= (-1)^i\frac{k\cdot(k+1)\cdots(k+i-1)}{i!}\\ &= (-1)^i\frac{(k+i-1)!}{i!(k-1)!}\end{aligned}$$

So

$$(1-x)^{-k} = \sum_{i\geqslant 0} \binom{k+i-1}{k-1} x^i$$

which is what we wanted to show. □

Proposition: $\frac{1}{1-x}^k$ is the generating function for the number of weak compositions of n with fixed number of parts equal to k. This is equivalent to sequences $(a_1, \dots, a_n)$ where $a_1 + \cdots + a_k = n$ and $a_i \geqslant 0$

Proof. Fix k. Let $c_{n,k}$ denote the number of weak compositions of n with k parts. The generating function is

$$\sum_{n\geqslant 0} c_{n,k} x^n$$

We can view this as follows. Each $(a_1, \dots, a_k)$ contributes $x^{a_1+\cdots+a_k}$ to F

$$F = \sum_{a_1,\dots,a_k} x^{a_1+a_2+\cdots+a_k} = \left(\sum_{a_1\geqslant 0} x^{a_1}\right)\left(\sum_{a_1,\dots,a_k} x^{a_2+\cdots+a_k}\right) = \left(\sum_{a\geqslant 0} x^a\right)^k = \frac{1}{(1-x)^k}$$

□

Proposition: If f, g are two formal power series such that $f^2 = g^2$, then they differ by a sign.

Proof. We $f^2 = g^2$, we have $f^2 - g^2 = 0$. Or $(f-g)(f+g) = 0$. Suppose $f \neq \pm g$. Then $f + g = a_k x^k +$ higher powers of x, $a_k \neq 0$ and $f - g = b_m x^m +$ higher powers of x, $b_m \neq 0$. Then $(f+g)(f-g) = a_k b_m x^{m+k} +$ higher powers of x^{m+k}. So this is a contradiction. □

Proposition: Let f be a formal power series such that $f^2 = 1 + x$. Then $f = \pm \boxed{(1+x)^{\frac{1}{2}}} = \pm \sum_{i\geqslant 0} \binom{\frac{1}{2}}{i} x^i$

Proof. We have

$$\boxed{(1+x)^{\frac{1}{2}}}^2 = \boxed{(1+x)^1} = 1 + x = f^2$$

so

$$f = \pm \boxed{(1+x)^{\frac{1}{2}}}$$

□

Proposition: For any formal power series $f(x)$ such that $f(0) > 0$ for real numbers or $f(0) \neq 0$ for complex numbers, there is a g such that $g^2 = f$ where g is defined uniquely up to a sign.

Proof. We only need to show that g exists. Take:

$$f(x) = f(0) + h(x) \text{ where } h(0) = 0$$

Then we can write $f(x) = f(0)(1 + \frac{h(x)}{f(0)})$ and

$$g(x) = \sqrt{f(0)}\boxed{\left(1 + \frac{h(x)}{f(0)}\right)^{\frac{1}{2}}}$$

where $\boxed{\left(1 + \frac{h(x)}{f(0)}\right)^{\frac{1}{2}}}$ is the result of substitution $x = \frac{h(x)}{f(0)}$. □

Chapter 8

Week 8

8.1 Catalan Numbers

Definition 8.1.1

Catalan Number

A Catalan number C_n is the number of ways to split a regular $n+2$-gon into triangles by non-crossing diagonals.

Definition 8.1.2

Dyck Path

A dyck path is a path on a square grid $\{(x, y) : x, y \in \mathbb{Z}\}$ which starts at $(0, 0)$, and has steps of the form $(1, 1)$ or $(1, -1)$ and ends at some point of the form $(2n, 0)$ and never goes below x-axis. The number $2n$ is called the length.

Proposition: The number of Dyck paths connecting points $(0, 0)$ to $(2n, 0)$ is equal to the n-th Catalan number.

Proof. We will build a bijection between Dyck paths of length $2n$ and triangulations of an $n + 2$-gon. Consider an $n + 2$-gon and enumerate the vertices clockwise by numbers $-1, 0, \dots, n$. In any triangulation, there is a triangle with the edge $0, -1$. Let i be the third vertex of this triangle. The triangle will split the $n + 2$-gon into 2 parts. The first part will be an $i + 1$-gon formed by $(0, 1, \dots, i)$ and another $n - i + 2$-gon formed by $(i, \dots, n, -1)$. Now run the algorithm for each of the two parts. Fix the first part $(0, 1, \dots, i) \to (-1, \dots, i - 1)$ and for the second part: $(i, \dots, n, -1) \to (-1, \dots, n - i + 2)$. We obtain two dyck paths D_1, D_2 which correspond to the triangulation of these two paths. The first dyck path will be of length $2(i - 1)$ and the other length is $2(n - i)$. We construct the path by starting with $(0, 0) \to (1, 1)$, then insert D_1, then we have $(1, 1) \xrightarrow{D_1} (2i - 1, 1) \to (2i, 0) \xrightarrow{D_2}\to D(2n, 0)$.

To construct a triangulation out of a dyck path:

- Take a dyck path of length $2n$ and consider the leftmost point $(2i, 0)$ where the path touches the x-axis.
- This corresponds to the $n + 2$-gon initial triangulation where the last point of the triangle containing $(0, -1)$ is i, and we triangulate D_1, D_2 that comes from splitting the $n + 2$-gon.

since there is a way to go backwards, there is a bijection between dyck paths and catalan numbers. ◻

Other objects counted by Catalan numbers:

- Correct sequences of n opening an n closing brackets:

$$((())), (())(), ()(()), (()()), ()()()$$

- The number of ways to place brackets in multiplication of $x_0 x_1 \cdots x_n$

8.2 Catalan Numbers Continued

Proposition: For any $n \geqslant 1$, we have:

$$\begin{aligned} C_n &= C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0 \\ &= \sum_{i=1}^{n} C_{i-1} C_{n-i} \end{aligned}$$

Proof. We know that C_n is the number of triangulations of $n+2$-gon. We start with an $n+1$-gon. Now we label the vertices using $(-1, 0, 1, \ldots, n)$. In any triangulation, there is a triangle containing -1, 0. We have i is the third vertex of this triangle. Now all triangulations can be counted as follows:

- Determine $i \in [n]$
- Choose a triangulation for $(0, \ldots, i)$- $i+1$-gon. There are C_{i+1} options for this.
- Choose a triangulation for $(n-i+1)$-gon. So this will be C_{n-i} options.

◻

If you set $C_0 = 1$, then this relation tells:

$$\begin{aligned} C_1 &= C_0^2 \\ C_2 &= C_0 C_1 + C_1 C_0 \\ C_3 &= C_0 C_2 + C_1^2 + C_1 C_0 \\ &\vdots \end{aligned}$$

Theorem 8.2.1 **Closed Catalan Formula**

We have $C_n = \frac{1}{n+1}\binom{2n}{n}$

Proof. Let $F(x) = \sum_{n \geqslant 0} C_n x^n$. Then

$$\begin{aligned}
\sum_{n \geqslant 1} C_n x^n &= \sum_{n \geqslant 1} \left(\sum_{i=1}^{n} C_{i-1} C_{n-i} \right) x^{n-1} \\
F(x) - 1 &= xF(x)^2 \\
x^2 F(x)^2 - xF(x) + x &= 0 \\
\left(xF(x) - \frac{1}{2} \right)^2 + x - \frac{1}{4} &= 0 \\
(2xF(x) - 1)^2 &= 1 - 4x \\
2xF(x) - 1 &= \pm\sqrt{1 - 4x} = \pm \sum_{n \geqslant 0} \binom{\frac{1}{2}}{n} (-4x)^n \\
2xF(x) - 1 &= - \sum_{n \geqslant 0} \binom{\frac{1}{2}}{n} (-4x)^n \\
F(x) &= \frac{1 - \sum_{n \geqslant 0} \binom{\frac{1}{2}}{n} (-4x)^n}{2x} \\
C_n &= \binom{\frac{1}{2}}{n+1} (-4)^{n+1}/2 \\
\binom{\frac{1}{2}}{n+1} &= \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2) \cdots (\frac{1}{2} - n)}{(n+1)!} \\
&= \frac{\frac{1}{2}(-\frac{1}{2})(\frac{-3}{2}) \cdots (\frac{-2n+1}{2})}{(n+1)!} \\
&= \frac{1(-1)(-3) \cdots (-2n+1)}{2^{n+1}(n+1)!} \\
&= (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}(n+1)!} \\
&= (-1)^n \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{2^{n+1}(n+1)! \cdot 2 \cdot 4 \cdots 2n} \\
&= (-1)^n \frac{(2n)!}{2^{2n+1}(n+1)!n!} \\
&= (-1)^n \frac{1}{2^{n+1}(n+1)} \binom{2n}{n} \\
C_n &= -\frac{(-4)^{n+1}}{2} \binom{\frac{1}{2}}{n+1} \\
&= \frac{1}{n+1} \binom{2n}{n}
\end{aligned}$$

□

8.3 Partitions

Definition 8.3.1 **Partition**

A partition of n is a sequence $(\lambda_1, \ldots, \lambda_k)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ and $\lambda_1 + \cdots + \lambda_k = n$.

Proposition: The generating function of partitions:

$$\sum_{n \geqslant 0} p(n)x^n = \prod_{i \geqslant 1} \frac{1}{1-x^i}$$

Note that $\prod_{i\geqslant 1} \frac{1}{1-x^i}$ means that if we look at the coefficient of x^k in $\prod_{i=1}^N \frac{1}{1-x^i}$ it will be constant starting from some N. So the coefficient of x^k in $\prod_{i\geqslant 1} \frac{1}{1-x^i}$ is the coefficient of $\prod_{i=1}^N$ for large N.

Proof. A partition λ contributes $x^{\lambda_1+\lambda_2+\cdots+\lambda_k}$ to the generating function. For partition λ, we write $|\lambda| = \lambda_1 + \cdots + \lambda_k$. So

$$\sum_{n \geqslant 0} p(n)x^n = \sum_{\lambda} x^{|\lambda|}$$

Recall that a partition is $(\lambda_1, \ldots, \lambda_k)$ as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$. Alternatively, we have these multiplicities $m_i(\lambda)$ = number of parts of size i. Equivalently, we can write instead of $(\lambda_1, \ldots, \lambda_k)$ as a sequence $(1^{m_1}2^{m_2}3^{m_3}\ldots)$. Now the generating function:

$$\sum_{\lambda} x^{|\lambda|} = \sum_{m_1,m_2,\ldots\geqslant 0} x^{m_1+2m_2+3m_3+\cdots} = \left(\sum_{m_1\geqslant 0} x^{m_1}\right)\left(\sum_{m_2\geqslant 0} x^{2m_2}\right)\left(\sum_{m_3\geqslant 0} x^{3m_3}\right)\cdots$$

Now this gives us:

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots$$

which concludes the proof. □

Example 8.3.1: Let $a(n)$ denote the number of strict partitions of n. Then $\sum_{n\geqslant 0} a(n) = \prod_{i\geqslant 1}(1+x^i)$.

Proof. So we have instead where $m_i = 0, 1$:

$$\sum_{\lambda\text{-strict}} x^{|\lambda|} = \sum_{1\geqslant m_1,m_2,\ldots,\geqslant 0} x^{m_1+2m_2+\cdots} = (1+x)(1+x^2)(1+x^3)\cdots$$

□

The generating function for partitions does not actually make computation a lot easier. But there are some properties that can be derived:

Proposition: The number of partitions of n without parts equal to 1 is equal to $p(n)-p(n-1)$.

Proof. We have $\sum_{n\geqslant 0} a(n)x^n = \prod_{i\geqslant 2} \frac{1}{1-x^i} = (1-x)\prod_{i\geqslant 1} \frac{1}{1-x^i}$. This is $(1-x)(\sum_{n\geqslant 0} p(n)x^n) = 1 + (p(1)-p(0))x + (p(2)-p(1))x^2 + \cdots$. So you get $1 + \sum_{n\geqslant 1}(p(n)-p(n-1))x^n$. □

Proposition: The number of strict partitions of n is equal to the number of usual partitions of n with only odd parts.

Proof. Take $\sum q(n)x^n = \prod_{i\geqslant 1}(1+x^i)$, $q(n)$ is the number of strict partitions. We have:

$$\prod_{i\geqslant 1}(1+x^i) = \prod_{i\geqslant 1} \frac{1-x^{2i}}{1-x^i} = \frac{\prod_{i\geqslant 1}(1-x^{2i})}{\prod_{i\geqslant 1}(1-x^i)} = \prod_{i\text{-odd}} \frac{1}{1-x^i}$$

This corresponds to the number of partitions with only odd parts. □

Going back to partitions:

$$\sum_{n\geqslant 0} p(n)x^n = \prod_{i\geqslant 1} \frac{1}{1-x^i} = (1-x)(1-x^1)(1-x^2)(1-x^3)(1-x^4)\cdots$$

We have by ignoring x^5 and greater powers

$$\begin{aligned}(1-x-x^2+x^3)(1-x^3)(1-x^4) &= (1-x-x^2+x^3-x^3+x^4+\cdots)(1-x^4)\\ &= (1-x-x^2+x^4-x^4+x^5)\\ &= 1-x-x^2+x^5\end{aligned}$$

Euler's Pentagonal Number Theorem

Theorem 8.3.1 We have

$$\prod_{i\geqslant 1}(1-x^i) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2}$$

Consider the sequence of $k(3k-1)/2$:

$k=0$	$k=1$	$k=2$	$k=3$
0	1	5	12

You get these numbers by counting the number of dots in $k-1$ nested pentagons that shared the top point.

Proposition: $p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+\cdots$. This is:

$$\sum_{k\geqslant 0}(-1)^{k-1}p(n-\frac{3(3k-1)}{2})$$

where $p(i)$ for $i<0$ are 0.

Proof. We have:

$$\prod_{i\geqslant 1}(1-x^i)\cdot\prod_{i\geqslant 1}\frac{1}{(1-x)^i} = 1$$

So we have:

$$\left(\sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2}\right)\left(\sum_{n\geqslant 0}p(n)x^n\right) = 1$$

Now taking the coefficient of x^n for $n\geqslant 1$ we get:

$$\sum_{k=-\infty}^{\infty}(-1)^n p(n-\frac{k(3k-1)}{2}) = 0$$

So we take out $k=0$:

$$\sum_{k=-\inf}^{\inf}(-1)^{k-1}p(n-\frac{k(3k-1)}{2}) = p(n)$$

□

Chapter 9

Week 9

9.1 Partitions Continued

Proposition: $p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+\cdots = \sum_{k\geqslant 0}(-1)^{k-1}p(n-\frac{k(3k-1)}{2})$.

Proof. Recall that $\prod_{i\geqslant 1}(1+x^i) = \sum_{n\geqslant 0} q(n)x^n$ where $q(n)$ is the number of strict partitions of n. Let $q_{odd}(n)$ be the number of strict partitions of n with odd number of parts, $q_{even}(n)$ be the number of strict partitions of n with even number of parts. Then

$$\prod_{i\geqslant 1}(1-x^i) = \sum_{1\leqslant i_1<\cdots<i_k} (-1)^k x^{i_1+\cdots+i_k} = \sum_{\lambda-\text{strict}} (-1)^{l(\lambda)} x^{|\lambda|}$$

where $l(\lambda)$ = number of parts $\geqslant 1$ of partition λ, called the length of the partition. We see that

$$\prod_{i\geqslant 1}(1-x^i) = \sum_{n\geqslant 0}(q_{even}(n) - q_{odd}(n))x^n$$

Then Pentagonal theorem iff $q_{even}(n) = q_{odd}(n)$ if n is not pentagonal and $q_{even}(n) = q_{odd}(n)+1$ if $n = \frac{k(3k-1)}{2}$ with even k. And $q_{even}(n)+1 = q_{odd}(n)$ if $n = \frac{k(3k-1)}{2}$ with k odd.

We will construct a pairing between "almost" all strict partitions of n such that in each of the pairs, one partition will have an odd number of parts and the other an even number of parts. If λ is a strict partition, then let m denote the size of the minimal part of λ, l denote the total number of parts of λ. Let s be the minimal positive number such that $\lambda_{s+1} < \lambda_{s-1}$.

Now fix a strict partition λ. If $S < m$, then construct a new strict partition $\tilde{\lambda}$ by removing the S boxes of the right-most diagonal and placing these S boxes back as a new smallest part of the partition.

In the opposite case, we construct $\tilde{\lambda}$ by removing the smallest part and placing these m boxes as a new diagonal of the partition.

Properties:

- The operation changes the parity of the number of parts.
- $\tilde{\tilde{\lambda}} = \tilde{\lambda}$ because if $S < m$, $\tilde{\lambda}$ was constructed by taking S boxes from the diagonal to form a new smallest part. Then for the new partition $\tilde{m} = S$ and $\tilde{S} \geqslant S$. So $\tilde{S} \geqslant \tilde{m}$. So when we do the process, we reverse the operation and place the smallest part as a diagonal.

- If $S \geqslant m$, then $\tilde{\lambda}$: $\tilde{S} = m$ while $\tilde{m} > m$, so $\tilde{m} > \tilde{S}$ and to get $\tilde{\tilde{\lambda}}$, we move back the diagonal of boxes as a smallest part of the partition.
- Since $\tilde{\tilde{\lambda}} = \lambda$, there is no problem in defining the pairing of odd and even parts in a partition: $(\lambda, \tilde{\lambda})$.
- When does this operation work? The problem will appear when the length of the partition is equal to S. If $m > S$ the problem happens if $m = S + 1$.
- If $m \leqslant S$ and $S = l$, then the problem happens if $m = S$.
- If we are in the first case, $l = S = m - 1$, we have $S^2 + \frac{S(S+1)}{2} = \frac{S(3S+1)}{2} = \frac{-l(3(-l))-1}{2}$. So $l(\lambda) = l$ covers pentagonal number when $(-l) < 0$.
- In the other case, we have $l = S = m$ and $|\lambda| = S^2 + \frac{S(S-1)}{2} = \frac{l(3l-1)}{2}$ and $l(\lambda) = l$. This covers pentagonal numbers for $l \geqslant 0$.

$\square$

9.2 Exponential Generating Functions

Some numbers do not have a generating function such as the number of permutations of $[n]$. This is $n!$. We have

$$\sum n! x^n = ?$$

Exponential Generating Function

Definition 9.2.1 For a sequence of numbers $(a_0, a_1, \ldots)$ its exponential generating function is given by:

$$\sum_{n \geqslant 0} a_n \frac{x^n}{n!}$$

Example 9.2.1: The exponential generating function on the number of permutations is:

$$\sum_{n \geqslant 0} n! \frac{x^n}{n!} = \sum_{n \geqslant 0} x^n = \frac{1}{1-x}$$

Exponent

Definition 9.2.2 We have $\exp(x) = \sum_{n \geqslant 0} \frac{x^n}{n!}$. This is the Taylor expansion for e^x.

We have: $e^{x+y} = e^x e^y$.

Proposition: $\exp(x + y) = \exp(x)\exp(y)$

Proof. We have

$$\left(\sum_{n \geqslant 0} \frac{x^n}{n!}\right)\left(\sum_{m \geqslant 0} \frac{y^m}{m!}\right)$$

is

$$\sum_{n,m\geqslant 0}\frac{1}{n!m!}x^ny^m = \sum_{n,m\geqslant 0}\binom{n+m}{n}\frac{x^ny^m}{(n+m)!} = \sum_{S\geqslant 0}\sum_{n=0}^{S}\binom{S}{n}\frac{x^ny^{s-n}}{s!} = \sum_{s\geqslant 0}\frac{(x+y)^s}{s!} = \exp(x+y)$$

□

In calculus, $\exp(x)$ is related to $\frac{d}{dx}\exp(x) = \exp(x)$ which will be an exercise.

Example 9.2.2: Recall that $S(n,k)$ = Stirling numbers of the second kind. Fix $k \geqslant 1$:

$$\sum_{n\geqslant 0} S(n,k)\frac{x^n}{n!} = ?$$

Recall that $k!S(n,k)$ is the number of surjections from $[n] \to [k]$. These surjections can be counted as follows:

- Pick $(a_1, a_2, \ldots, a_k)$- composition of n. The number a_i will denote the number of elements that map to i.
- Then split $[n]$ into numbered groups of sizes $a_1, a_2, \ldots, a_k$.

So $k!S(n,k) = \sum_{a_1,a_2,\ldots,a_k}\binom{n}{a_1,a_2,\ldots,a_k}$:

$$\begin{aligned}
k!\sum_{n\geqslant 0} S(n,k)\frac{x^n}{n!} &= \sum_{n\geqslant 0}\sum_{a_1\ldots,a_k}\binom{n}{a_1,a_2,\ldots,a_k}\frac{x^n}{n!}\\
&= \sum_{a_1,a_2,\ldots,a_k\geqslant 1}\frac{(a_1+a_2+\cdots+a_k)!}{a_1!a_2!\cdots a_k!}\frac{x^{a_1+a_2+\cdots+a_k}}{n!}\\
&= \sum_{a_1,\ldots,a_k\geqslant 1}\frac{x^{a_1+a_2+\cdots+a_k}}{a_1!a_2!\cdots a_k!}\\
&= \left(\sum_{a_1\geqslant 1}\frac{x^{a_1}}{a_1!}\right)\left(\sum_{a_2\geqslant 1}\frac{x^{a_2}}{a_2!}\right)\cdots\left(\sum_{a_k\geqslant 1}\frac{x^{a_k}}{a_k!}\right)\\
&= (\exp(x)-1)^k
\end{aligned}$$

Therefore,

$$\sum_{n\geqslant 0} S(n,k)\frac{\lambda^n}{n!} = \frac{(\exp(x)-1)^k}{k!}$$

We also have:

$$\begin{aligned}
\frac{(\exp(x)-1)^k}{k!} &= \frac{\sum_{i=0}^{k}(-1)^{k-i}\exp(x)^i\cdot\binom{k}{i}}{k!}\\
&= \sum_{i=0}^{k}(-1)^{k-i}\frac{1}{i!(k-i)!}\exp(ix)
\end{aligned}$$

So now we need the coefficient of x^n which is $\frac{S(n,k)}{n!}$. We have:

$$S(n,k) = \sum_{i=0}^{k}(-1)^{k-i}\frac{1}{i!(k-i)!}i^n$$

Example 9.2.3: Suppose that:

$$b(n,k) = 1^k + 2^k + \cdots + n^k$$

Look at the generating function for fixed n:

$$\begin{aligned}
\sum_{k\geqslant 0} b(n,k)\frac{x^k}{n!} &= \sum_{i=1}^{n}\sum_{k\geqslant 0} i^n \frac{x^n}{n!} \\
&= \sum_{i=1}^{n}\sum_{k\geqslant 0} \frac{(ix)^n}{n!} \\
&= \sum_{i=1}^{n} \exp(ix) \\
&= \exp(x) + (\exp(x))^2 + (\exp(x))^3 + (\exp(x))^4 + \cdots + (\exp(x))^n \\
&= \frac{\exp(x)(1-\exp(x)^k)}{1-\exp(x)} \\
&= \frac{1-\exp(nx)}{\exp(-x)-1}
\end{aligned}$$

Bernoulli Numbers

Definition 9.2.3

We have:

$$\frac{x}{\exp(-x)-1} = \sum_{n\geqslant 0} B_n \frac{x^n}{n!}$$

where B_n are Bernoulli numbers.

So we have

$$\begin{aligned}
\sum_{k\geqslant 0} b(n,k)\frac{x^k}{n!} &= \frac{1-\exp(nx)}{x}\left(\sum_{n\geqslant 0} B_n \frac{x^n}{n!}\right) \\
&= -\sum_{i\geqslant 1} \frac{n^i x^{i-1}}{i!}\left(\sum_{n\geqslant 0} B_n \frac{x^n}{n!}\right) \\
&= \sum_{k\geqslant 0}\sum_{i=0}^{k}\left(-\frac{n^{i+1}}{(i+1)!}B_{k-i}\frac{1}{(k-i)!}\right)x^k
\end{aligned}$$

Now the coefficient:

$$b(n,k) = -\sum_{i=0}^{k} B_{k-i}\binom{k}{i}\frac{n^{i+1}}{i+1}$$

9.3 Graph Theory

Definition 9.3.1 **Graph**

A graph is denoted by the data, (V, E, r).

- V is the set of vertices of a graph.
- E is the set of edges
- $r : E \longrightarrow \{\{x, y\} : s, y \in V\}$.

Definition 9.3.2 **Walk, Trail, Eulerian, Closed paths**

A walk along the graph G is a pair of sequences $(v_0 v_1 v_2 \dots v_n)$ and $(e_1 e_2 \dots e_n)$ where v_i are vertices in V and e_i are edges that connect v_{i-1}, v_i. Also, n is called the length of this walk. A trail is a walk with all edges e_i being different. A trail is called Eulerian if this trail uses all edges of the graph. A walk or trail is closed if $v_0 = v_1$

Given a graph G, is there a closed Eulerian trail of the graph G.

Definition 9.3.3 **Connected**

A graph G is called connected if for any $u, v \in V$, there is a walk starting at u and ending at v.

Definition 9.3.4 **Subgraph**

A subgraph of $G = (V, E, r)$ is a graph $G' = (V', E', r')$ such that $V' \subseteq V$, $E' \subseteq E$ and for any $e \in E'$, $r(e) = \{x, y\}$ where $x, y \in V'$. And r' is the restriction of r.

Definition 9.3.5 **Connected Component**

A connected component of G is a maximal connected subgraph of $G' \subseteq G$. That is, G' is a connected subgraph such that adding any vertices or edges from G breaks connectivity of G'.

Definition 9.3.6 **Degree**

For a vertex v in G, the degree of v is the total number of times the vertex v appears as an endpoint of an edge $e \in E$.

Theorem 9.3.1 A connected graph G has a closed Eulerian trail iff all vertices of G have even degree.

Proof. $(\longrightarrow)$ Let $(v_0, v_1, v_2, \dots, v_n)$ and $(e_1, e_2, \dots, e_n)$ be the vertices and edges of the Eulerian trail. Pick a $v \in V$. If $\deg v = 0$, then we are done. Otherwise, v must appear in our trail. There are two cases:

- $v \neq v_0 = v_n$. Then v appears inside $(v_1, \dots, v_{n-1})$. Let a denote the number of times that v appears. Each time, $v = v_i$, we have two edges e_i, e_{i+1} which are connected to v_i. So in total, we have $2a$ edges in $(e_1, \dots, e_n)$. But $(e_1, \dots, e_n)$ is the set of all edges.
- If $v = v_0 = v_n$. Let a denote the number of times v appears in $(v_1, \dots, v_{n-1})$. Then we get $2a$ edges in this sequence $(e_2, \dots, e_{n-1})$ connected to v. In addition, e_1 and e_n are also connected to v. So the degree is $2n + 2$.

$(\longleftarrow)$ Let G be a graph with only even degrees and let v be a vertex of this graph such that $\deg v \in G$ is positive. Start a trail from the vertex v going arbitrarily without using any edge twice. When the walk enters $u \neq v$. For each of the previous visits, we have used a pair of different edges before we enter the vertex, it has even degree. So u has an odd number of unused edges. After entering it, we know that there is at least one unused edge that we can use to leave it. Since the graph is finite, the walk must end. And it must end at v.

- Given a graph with only even degrees and v of positive degree, this construction tells us how to construct a closed trail of positive length starting and ending at v.

- Fix a connected graph G with only even degrees. Let Σ be the set of all closed trails on G. This set is finite and must have at least one trail of positive length by (1). Look at possible lengths of trails in Σ. This is a finite subset of $\mathbb{N}$. Let N be the largest length of trails in Σ and let C be a trail in Σ of length N. We will show that C must be an Eulerian trail. This is the formal statement of increasing a trail until it becomes the largest.

  Assume for contradiction that C does not use all the edges of G. Let $\tilde{G}$ be the subgraph of G with the same set of vertices but only with edges not used in C. By assumption, $\tilde{G}$ has at least one edge. Moreover, we know that in $\tilde{G}$, every vertex has even degree by $(\rightarrow)$ part of the proof. We want to pick a vertex v of positive degree in $\tilde{G}$ which is visited by C. We know that such a vertex exists:

  - Case 1: If C has visited all vertices of G, then we can pick an arbitrary vertex of $\tilde{G}$ of positive degree.

  - Case 2: If C does not visit all the vertices, then let u, w be a pair of vertices such that u was visited by C but w was not. So since G is connected, there exists a walk from u to w. Let $v_0 = u, v_1, \ldots, v_n = w$ be the vertices. Let i be the least number such that v_i is not visited by C. We know $v_i \geqslant 1$. This number is well defined and means that v_{i-1} is visited by C. We have $e = (v_{i-1}, v_i)$ as an edge that exists in $\tilde{G}$ so the vertex v_{i-1} has positive degree in $\tilde{G}$.

  Now by the naive construction, we can take v which has positive degree in $\tilde{G}$ and construct a closed trail of positive length that starts v and ends at v. Call the new trail $\tilde{C}$. But now, we have a closed trail $C + \tilde{C}$ which is a closed trail which is obtained by taking $v \rightarrow v$ by C and $v \rightarrow v$ by $\tilde{C}$ which is a trail that is longer than C. So this is a contradiction with the assumption that C does not use all the edges.

$\square$

Chapter 10

Week 10

Proposition: Let G be a connected graph and u, v be a pair of different vertices of this graph. Then there is an Eulerian trail from $u \to v$ iff u, v are the only vertices of odd degree in G.

Proof. Let $\tilde{G}$ be the graph G with an additional edge e from u to v. So now there is a Eulerian trail $u \to v$ in G is equivalent to saying there is a closed Eulerian trail in $\tilde{G}$. This is equivalent to saying that all degrees in $\tilde{G}$ are even, so in G, we have two vertices that will have odd degree. □

Proposition: Let G be a graph. Then the number of vertices with odd degree in G is even.

Proof. Let e denote the number of edges. Let $d_1, d_2, \ldots, d_n$ be degrees in G. Then $d_1 + d_2 + \cdots + d_n = 2e$. So the number of vertices with odd degree is even. □

Definition 10.0.1 **Path, Cycles, Hamiltonian Paths/Cycles**

A path is a trail $(v_0, \ldots, v_n), (e_1, \ldots, e_n)$ such that all vertices v_i are different. A cycle is a closed trail such that all the vertices $v_1, \ldots, v_n$ are different. A Hamiltonian path or cycle is a path or cycle that uses all the vertices.

Question: Given a graph G, how can we find a Hamiltonian cycle. If G is not connected, there is no Hamiltonian cycle/path.

Definition 10.0.2 **Complete Graphs**

K_n is the graph with $V = [n]$ with exactly one edge between any pair of vertices.

For $n \geqslant 3$, K_n has a Hamiltonian cycle.

Given a graph G, how quickly can we determine/find a Hamiltonian cycle.

Definition 10.0.3 **Simple Graphs**

A graph G is simple if G has no loops and no repeated edges. For any $u, v \in V$, there is at most one edge between them.

Theorem 10.0.1

Let G be a simple graph $n \geqslant 3$ such that every vertex $v \in V$ has degree at least $n/2$ where n is the total number of vertices. Then there is a Hamiltonian cycle of the graph G.

Proof. We note that G must be connected. Suppose that we can split G into two parts. By the pigeonhole principle, one of the parts must have degree $\geqslant n/2$. If you pick a vertex in the other part, it has degree at most $\frac{n}{2} - 1$ neighbors which is a contradiction. Assume that there is no Hamiltonian cycle in G. Start adding edges to G without creating a Hamiltonian cycle while keeping the graph simple. Eventually, we get $G' \supseteq G$ such that adding any new edge to the graph will create a Hamiltonian cycle. So all degrees of G' are still $\geqslant n/2$. Let u, v be a pair of vertices not connected by an edge in G'. If we add an edge connecting u, v, we will have created a Hamiltonian cycle. We get a Hamiltonian path P from $u \to v$ inside G'. Let $(v_1, v_2, \ldots, v_n)$ be the vertices of P. Look at indices i such that v_i is a neighbor of u. We know there are at least $n/2$ indices i. We know that $i \in \{2, \ldots, n-1\}$. Consider the set of indices i such that v_{i-1} is connected to v. So among $\{3, \ldots, n\}$, there should be at least $n/2$ such i. In total, in $\{2, \ldots, n\}$ there are $n/2$ v_i which are connected to u and $n/2$ v_{i-1} connected to v. So there is an index i in $\{3, \ldots, n-1\}$ such that v_i is connected to u and v_{i-1} is connected to v. So there is a Hamiltonian cycle given by $u \to v_{i-1} \to v \to v_i \to u$. This is a contradiction because the graph has hamiltonian cycle. □

Recall that in the original definition, graphs were defined as $G(V, E, r)$. In simple graphs, each edge e can be interpreted as a two element subset of V. So the definition of graphs for simple graphs is $G(V, E)$.

Definition 10.0.4 **Simple Graph**

A simple graph G is a pair (V, E) where V is a set of vertices and $E \subseteq \{\{x, y\} : x \neq y, x, y \in V\}$.

Definition 10.0.5 **Isomorphism**

An isomorphism from $G = (V, E)$ to $G' = (V', E')$ is a bijective map $f : V \to V'$ such that for any pair of vertices $u, v \in V$, $\{u, v\} \in E \iff \{f(u), f(v)\} \in E'$.

To show that a graph G, G' are isomorphic, it is enough to construct an isomorphism f.

Q: How quickly can you check if G is isomorphic to G'

10.1 Trees

Definition 10.1.1 **Tree**

A tree is a simple graph $G = (V, E)$ which is minimally connected, or removing any edge from G results in G' which is not connected.

Proposition: For a connected graph G, the following statements are equivalent:

- G is minimally connected/G is a tree.
- G has no cycles.
- For any pair of vertices $u, v \in V$ there is a unique path from u to v.

Proof. $(2 \to 1)$ Let G be a graph with no cycles. Let e be an edge such that $G \backslash e$ is still connected. Then there was a cycle in G which is a contradiction.

($3 \to 2$) Suppose that G satisfies 3. Let C be a cycle in G. Then pick any two vertices in the cycle. Then there are two paths from $u \to v$. Then we have a contradiction.

($1 \to 3$) Suppose that G is minimally connected. Suppose that there are two different paths P, P' from u to v. Let e be the first edge of P' different from P. Then e does not appear in P'. Let $e = (w_1, w_2)$. After removing e, the graph is still connected. We can still go: $w_1 \to u \to v \to w_2$. So contradiction. ◻

Usually, it is said that G is connected if for any u, v there is a path between them.

Chapter 11

Week 11

Proposition: G has $n-1$ edges where n is the number of vertices iff G is a tree.

Leaf

Definition 11.0.1

A leaf is a vertex of degree one.

Lemma: Each tree with $\geqslant 2$ vertices has at least 2 leaves.

Proof. Let P be a longest path in a tree G where G has at least 2 vertices. $P = (v_0, v_1, \dots, v_n)$, and $E = (e_1, \dots, e_n)$. v_0 has degree at least 1. Assume we have another edge e connected to v_0:

$$u \xrightarrow{e} v_0 \xrightarrow{e_1} v_1$$

If u appears in P, then we have 2 paths connecting $u \to v_0$. This means that we have a path that is one step longer, which is impossible. So the degree of v_0 is 1 and similarly, the degree of v_n is 1. □

Proof. ($\rightarrow$) First, we will show that each tree $G = (V, E)$ has $|V| - 1$ edges. Induction on the number of vertices:

- When $|V| = 1$, there are 0 edges.
- Assume that this statement is true for $n - 1 \geqslant 0$. Let G be a tree on $n \geqslant 2$ vertices, by lemma, G has a leaf. Let $\tilde{G}$ the subgraph obtained by removing the vertex v with its edge. If $\tilde{G}$ has a cycle, then G also has a cycle. So $\tilde{G}$ is connected and is a tree with $n-1$ vertices and $n-2$ edges. So G has $n-1$ edges.

($\leftarrow$) In the other direction, let G be a connected graph with n vertices and $n-1$ edges. If G is not minimally connected, remove edges of G until there is a subgraph G' that is minimally connected. So G' is a tree with $n-1$ edges. So we removed 0 edges and G is a tree. □

Forest

Definition 11.0.2

A forest is a simple graph such that each connected component of this graph is a tree.

Proposition: A graph G is a forest if

$$|E| = |V| - k$$

where k is the number of connected components of G.

Proof. If G is a union of k trees, then

$$|E| = |V| - k$$

In the other direction. Let $n_1, \dots, n_k$ be the connected components and $m_1, \dots, m_k$ be the number of edges of each component. Each connected component is connected. Then $n_i \leqslant m_i + 1$. If $|E| = |V| - k$, we must have $n_i = m_i + 1$. □

Question: How many trees are there?

- How many trees up to isomorphism are there if $|V| = n$?
- How many trees are there on $[n]$?

Cayley's Theorem

Theorem 11.0.1 There are n^{n-2} trees on the vertex set $[n]$.

Prufer codes:

- Find the leaf with the minimal label
- Remove this leaf, and add to the code what the edge led to.
- Repeat this procedure for $n - 2$ times.
- Stop when there is a tree on two vertices.

Proof. Let G be a tree on $[n]$, let $A = (a_1, \dots, a_{n-2})$ its code. We claim that $\deg i - 1$ is equal to the multiplicity of i in A:

Fix i. During encoding,, we can either remove a neighbor of i or we can remove i itself. Other operations do not affect $\deg(i)$. When we remove a neighbor of i, we reduce the degree of i by 1 but i is added into the code. When i is removed, i is a leaf, $\deg(i) = 1$. If i is removed it means that we have first removed its neighbors for $\deg(i) - 1$ times and then removed i. If i is one of the two remaining vertices at the end of the procedure, i has one neighbor in the remaining two vertex tree, so it has degree 1 remaining. So the multiplicity of i in the code is still $\deg(i) - 1$.

In particular, if i does not appear in the code A, then it must be a leaf. We can reconstruct the first step of the encoding by taking the smallest number not in the code and connecting it to the first number of the code.

Reverse Algorithm: Let $A = (a_1, a_2, \dots, a_{n-2})$, let G be n disjoint vertices. Let $I = [n]$. Do the following operation $n - 2$ times at iteration i:

- Look at all numbers $j \in I$ which do not appear in $(a_i, a_{i+1}, \dots, a_{n-2})$.
- Pick smallest j, add the edge (j, a_i) and remove j from I.

In the end, we get some graph G, and I has 2 elements. Construction ends by connecting vertices in I by an edge.

Show that the resulting G is a tree. At each step, we add (j, a_i) where both $j, a_i \in I$. We remove j from I, so we cannot reuse (j, a_i), no loops, so G is simple. Moreover, each time we remove j from I, add an edge connecting j to the remaining I. This means that any vertex of G is connected to something in I. So after $n - 2$ steps, everything is connected to one of the two vertices in I, but in the final step, these two vertices are connected, so G is connected. Also, G has $n - 1$ edges, which means G is a tree. □

Consider the problem where we have a graph and each edge has a cost

Definition 11.0.3 **Spanning Tree**

A spanning tree of $G = (V, E)$ is a subgraph $T \subseteq G$ such that T is a tree and T has the same vertex set as G.

Example 11.0.1: The number of spanning trees of a complete graph K_n is n^{n-2}.

Proposition: A connected graph G always has a spanning tree.

Let $G = (V, E)$ be a connected graph. Let $w : E \to \mathbb{R}_{\geqslant 0}$ which is the weight function. If $T \subseteq G$, spanning tree, then we can define a weight of the tree by $w(T) = \sum_{i=1}^{n-1} w(t_i)$ where t_i are edges of the tree.

Problem: Given a graph G connected, and w, what is the spanning tree with the minimal weight?

If $G = (V, E)$ and $|V| = n$, consider the following algorithm:

- Pick edges of T one by one.
- Look at all edges such that they are not picked for the spanning tree, adding them to the spanning tree does not create cycles. Out of these edges, pick the one with minimal weight.

This is called a greedy algorithm. This gives a graph with no cycles which is a forest. There are $n - 1$ edges which makes it a tree.

Theorem 11.0.2 **Kruskal**

For this problem, the greedy algorithm provides the spanning tree with the minimal weight.

Lemma: Let F and F' be two forests with the same vertex set. Let F' has more edges than F. Then there is an edge of F', e, such that $F \cup e$ is a forest.

Proof. Let n be the number of vertices. Let k and k' be the number of connected components of F, F'. Then

$$n - k' > n - k \iff k > k'$$

Assume that adding any edge e of F' to F creates a cycle. So the endpoints of any edge in F' are connected in F. This is the same as saying that endpoints in F' are in the same component as that of F. This means that the number of connected components of $F' \geqslant F$ which is a contradiction. □

Proof. Let G be a connected graph, w is the weight function. Let T be the result of the greedy algorithm with $t_1, t_2, \ldots, t_{n-1}$ edges of T ordered by how they are picked.

First, we need to show that the algorithm can pick $n - 1$ edges/does not fail to pick an edge at some point $m < n - 1$. By the lemma, we have a forest at m-th step. Since G is connected, G has a spanning tree on $n - 1$ edges. Then we can add an edge to the forest which still keeps it a forest.

Assume that there is a spanning tree H where $w(H) < w(T)$. Let $h_1, \ldots, h_{n-1}$ be the edges of the tree. Order them so that $w(h_1) \leqslant w(h_2) \leqslant \ldots \leqslant w(h_{n-1})$. Since T was obtained using a greedy algorithm, we also know that $w(t_1) \leqslant w(t_2) \leqslant \cdots \leqslant w(t_{n-1})$. Let i be the minimal index where $\sum_{j=1}^{i} w(t_i) > \sum_{j=1}^{i} w(h_j)$, $i \geqslant 2$. This means that $\sum_{j=1}^{i-1} w(t_j) \leqslant \sum_{j=1}^{i-1} w(h_j)$. So $w(t_i) > w(h_i)$. We have that $(t_1, \dots, t_{i-1})$ forms a forest with $i-1$ edges, called F. And $(h_1, \dots, h_i)$ is a forest with i edges, called F'. By the lemma, we can add an edge from the larger forest to the smaller forest $h_j (j \leqslant i)$ to F. But $w(t_i) > w(h_i)$ and $w(h_{i-1}) \geqslant w(h_{i-2}) \geqslant \cdots \geqslant w(h_1)$. So $w(t_i) > w(h_j)$, but we could have chosen h_j instead of t_i. So the greedy algorithm should have chosen h_j. $\square$

Chapter 12

Week 12

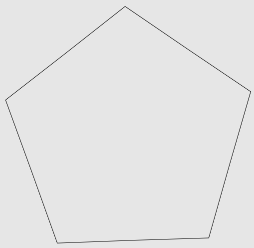
12.1 Graph Colorings

Definition 12.1.1

Chromatic Number

A chromatic number of G is denoted by $\chi(G)$ which is the minimal integer k such that we can color vertices of G in k colors without creating a monochromatic edge (Edge with both endpoints with the same color).

Example 12.1.1:



$= G \qquad \chi(G) = 3$

Tasks which cannot be done simultaneously → Graph vertices are tasks, edges are constraints, colorings are placings of tasks in a time slot.

Definition 12.1.2 ◆ **k-colorable**

A graph is called k-colorable if its vertices can be correctly colored using k-colors.

Definition 12.1.3 ◆ **Bipartite Graph**

A bipartite graph is a two colorable graph. Equivalently, $G = (V, E)$ is bipartite if $V = A \coprod B$ and all edges $e \in E$ are of the form $e = \{a, b\}$, where $a \in A, b \in B$.

Proposition: G is bipartite iff G has no cycles of odd length.

Proof. ($\longrightarrow$) Let G be bipartite and assume that G has an odd cycle C. Let $(v_1, v_2, \ldots, v_{2n+1})$ be the vertices of this cycle. Let blue be the color of v_1. Then v_2 is red. Then v_3 is blue, v_4 is red. We see that v_{2i} is red and v_{2i+1} is blue. Then v_{2n+1} is blue. But there is the edge $\{v_1, v_{2n+1}\}$ which is monochromatic, contradiction.

($\longleftarrow$) Let G be a graph without odd cycles. Without loss of generality, assume that G is connected. Pick a vertex v in G. Color it blue. Color each vertex w into blue if the shortest path from $v \longrightarrow w$ has even length and color it red otherwise. Assume that the coloring creates an edge $\{w, u\} \in E$, such that w, u have the same color. So we have the closed walk of odd length: $v \longrightarrow w \longrightarrow u \longrightarrow v$. Let C be the shortest closed walk of odd length in G. If C is not a cycle, then it splits into shorter closed walks C_1, C_2. One of them must be odd. So this is a contradiction, since C was not the shortest closed walk of odd length. So C is a cycle of odd length which is a contradiction. $\square$

Definition 12.1.4 ◆ $K_{a,b}$

$K_{a,b}$ is the bipartite graph with $a + b$ vertices $V = A \coprod B$ where $|A| = a$, $|B| = b$ and edges are all pairs of vertices $\{v, w\}, v \in A, w \in B$.

Proposition: If G is a bipartite graph with n vertices, then G has at most $n^2/4$ edges if n is even and $(n^2 - 1)/4$ edges if n is odd.

Proof. Let a, b be the sizes of color groups in the two coloring of G. Then G cannot have more edges than $K_{a,b}$ which is ab, which is $a(n - a)$. Maximal value of $a(n - a)$ for integer $a \in \mathbb{N}$. We have $a(n - a) = \frac{n^2}{4} - (a - \frac{n}{2})^2$. Minimize $(a - \frac{n}{2})^2$. If n-even, we can choose $a = \frac{n}{2}$. Then $a(n - a) = \frac{n^2}{4}$. If n is odd, then the possible values of $a - \frac{n}{2}$. Set $a = \frac{n \pm 1}{2}$. We have $b = \frac{n \mp 1}{2}$. So $ab = \frac{n^2-1}{4}$. $\square$

Theorem 12.1.1

If G is a graph with $2m$ vertices and $m^2 + 1$ edges, then G has m different triangles.

12.2 Matchings

Definition 12.2.1 ◆ **Matching**

A matching M in $G = (V, E)$ is a subset $M \subseteq E$ such that any pair of edges in M does not have common endpoints. A matching M is perfect if each vertex appears as an endpoint of an edge $e \in M$.

Definition 12.2.2

Let G be bipartite with X, Y color groups of vertices. Then a matching M is called a perfect matching of X into Y if every vertex of X is an endpoint of some edge in G.

Proposition: If X can be perfectly matched in to Y, then $|X| \leqslant |Y|$.

Proof. If M is a perfect matching of X into Y, then for any $x \in X$, there is an edge $\{x, f(x)\} \in M$ where $f(x) \in Y$. Since M is a matching, f is an injection. So $|Y| \geqslant |X|$. □

If X has 2 leaves with the same neighbor, there is no perfect matching of X into Y. Let $T \subseteq X$, the $N(T)$ is the set of neighbors of T.

Proposition: For any bipartite graph G with X, Y color classes, X can be perfectly matched into Y only if for any $T \subseteq X$ we have $|T| \leqslant |N(T)|$.

Proof. If M is a perfect matching of X into Y, then we have a perfect matching of T into $N(T)$. By the previous proposition, we have $|T| \leqslant |N(T)|$ □

Hall's Theorem

Theorem 12.2.1

For any bipartite G, X, Y color classes, X can be perfectly matched into Y iff for any $T \subseteq X$, we have $|T| \leqslant |N(T)|$.

Proof. ($\longrightarrow$) Was proved in Proposition

($\longleftarrow$) Induction on the number of vertices $|X|$.

- Base Case: $|X| = 1$. Then $|N(X)| \geqslant 1$. So there is at least one neighbor connected to X so there is a perfect matching of X into Y.
- Inductive Step: Let $n \geqslant 2$ and assume that the statement is true for $|X| < n$. Let $|X| = n$.
  - Case 1: Assume that there is $B \subseteq X$ where $0 < |B| < |X|$ where $|B| = |N_G(B)|$. Let G_1 be the subgraph which is $B \cup N(B)$. Let G_2 be the subgraph by removing $B \cup N_G(B)$ from G with adjacent edges. Check that $|T| \leqslant |N(T)|$ for G_1, G_2. Let $T \subseteq B$. Then $N_{G_1}(T) = N_G(T)$. Then

    $$|T| \leqslant |N_G(T)| = |N_{G_1}(T)|$$

    So we can use the inductive hypothesis to get some perfect matching in G_1. So we have M_1 a perfect matching of B into $N_G(B)$. For G_2, let $U \subseteq X - B$. Consider $N_G(U \cup B)$. Then $N_G(U \cup B) \subseteq N_G(B) \coprod N_{G_2}(U)$. The sets are disjoint, because otherwise, there is a vertex in $U \cup B$ such that it has a neighbor in $N_G(B)$ and $N_{G_2}(U)$. So that edge actually belongs to $N_G(B)$. And an element in the disjoint union is a neighbor of an element of either U or B. We also have the reverse containment because if $v \in N_G(U \cup B)$ because if $v \in N_G(B)$, then we are done. Otherwise, if $v \notin N_G(B)$, then $\exists$ edge v, u with $u \in U$ and $v, (v, u) \in G_2$ which means that $v \in N_{G_2}(U)$. Then

    $$|U| = |U \cup B| - |B| \leqslant |N_G(U \cup B)| - |N_G(B)| = |N_{G_2}(B)|$$

    Apply the inductive hypothesis to G_2. So we have M_2 a perfect matching of $X - B$ into $Y - N(B)$ in G_2. Then the union $M_1 \cup M_2$ give a perfect matching of X into Y.
  - Assume that for any $B \subseteq X$ where $B \neq X, \emptyset$, we have $|N| < |N_G(B)|$. Pick an arbitrary vertex $x \in X$. Since x has at least 1 neighbor, take $y \in N(\{x\})$. Let G' be the subgraph of G obtained by throwing away x, y and all attached edges. Let $T \subseteq X - \{x\}$. Then if T is empty, then indeed $0 \leqslant \ldots$. Otherwise,

$|T| < |N_G(T)|$. This is equivalent to $T \leqslant |N_G(T)| - 1$. If $v \in N_G(T)$, then either $v = y$ or $v \in N_{G'}(T)$. Then $|N_{G'}(T)| = |N_G(T)|$ or $|N_{G'}(T)| = |N_G(T)| - 1$. So we get:

$$|T| \leqslant |N_G(T)| - 1 \leqslant |N_{G'}(T)|$$

By the inductive hypothesis, we have M' perfect matching on G'. We union this to a perfect matching over G by adding in $\{x, y\}$ to M'.

$\square$

Chapter 13

Week 13

13.1 Hall's Theorem Examples

Example 13.1.1: Let G be a bipartite graph with X, Y color groups. All $x \in X$ have degree d_1 and $y \in Y$ have degree d_2, $d_1 \geqslant d_2 \geqslant 1$. Then there is a perfect matching of X into Y.

Proof. Check that $|S| \leqslant |N_G(S)|$. Let $S \subseteq X$. Let m denote the number of edges between S and $N(S)$. We know that $m = |S|d_1$ and $m \leqslant |N(S)|d_2$. So

$$|S|d_1 \leqslant |N(S)|d_2$$

and therefore

$$|S| \leqslant |N(S)|$$

□

Example 13.1.2: Fix $n \geqslant 1$. A Latin square is an $n \times n$ square of numbers for $[n]$ where every row/column appears once. For $k \leqslant n$, a Latin $k \times n$ rectangle is one with numbers from $[n]$ such that in each column, every number appears once, and each row has k distinct numbers. Claim: Every $k \times n$ Latin rectangle can be extended to a Latin square.

Proof. Show that $k \times n \to (k+1) \times n$. We want to add an extra column and draw edges from elements of $[n]$ to a row $a_{k+1,j}$ if x does not appear in $\begin{bmatrix} a_{1,j} & a_{2,j} & \cdots & a_{k,j} \end{bmatrix}$ Pick $x-$number where in original rectangle, there are k copies of k. So x has degree $n-k$. For any position, $a_{k+1,j}$ there are also $n-k$ options. So by the previous example, we are done. □

13.2 Extensions to Non-bipartite Graphs

For 2 colorable graphs, with maximum number of $|E|$ and fixed $|V|$ is achieved by $K_{a,b}$ where a, b are as closed as possible. Let $K_{a_1,a_2,\ldots,a_k}$ be the graph of groups of vertices of size $z_1, \ldots, a_k$. Then connect with edges all pairs (i, j) where i, j are different groups. Then let $n, k \in \mathbb{N}$, where $n = dk + r$. Consider $K_{d+1,\ldots,d+1,d,d,\ldots,d}$. Let $T(n, k)$ denote the number of edges.

Theorem 13.2.1 **Turan**

Let G be a simple graph with n vertices and more than $T(n,k)$ number of edges. Then G must contain a subgraph which is isomorphic to K_{k+1} and G is not k-colorable.

Lemma: $T(n,k) = \binom{k}{2} + (k-1)(n-k) + T(n-k,k)$ for any $n \geqslant k \geqslant 1$.

Proof. If $n = dk+r$, $n-k = (d-1)k+r$. So $T(n,k)$ is the number of edges in $T_{d+1,\dots,d+1,d,\dots,d}$ and $T(n-k,k)$ is number of edges in $K_{d,d,\dots,d,d-1,\dots,d-1}$. So $T(n,k)$ is the number of edges of the graph G obtained by adding a vertex to each group in $k_{(d)^k(d-1)^{k-r}}$ and adding connected new vertices to vertices in other groups. New edges between new vertices is $\binom{k}{2}$. Then a group of new edges which connect a new vertex to an old vertex. There are $n-k$ old vertices. Each old vertex is connected to $k-1$ new vertices. So there are $(n-k)(k-1)$ new edges added. □

Proof. Let G be the graph on n vertices which does not contain a graph on K_{k+1} with maximal possible number of edges. So adding an edge creates K_{k+1}. We want to show that G has at most $T(n,k)$ edges. Induction on n. For all $n \leqslant k$, the claim is trivial : $K_{1,1,\dots,0,\dots} = K_n$. So no K_{k+1} in G and no more than $T(n,k)$ edges. Suppose that Turan theorem was proved for $n-k$. Prove it for n:

Pick G as before. So $G \neq K_n$, there is a pair (v,w) not connected by an edge. In G, we have a complete graph on $k-1$ vertices with v,w connecting to all vertices in it. Then there is K_k in G. Let S be this subgraph. There are $\binom{k}{2}$ edges in S. There are at most $T(n-k,k)$ edges between vertices outside of S. There are at most $(k-1)(n-k)$ edges between S and $G-S$ because every vertex not in S cannot be connected to every vertex in S otherwise, there is a complete graph K_{k+1}. So $|E| \leqslant T(n,k)$ which completes the proof. □

Let $G = (V,E)$ be a graph. For $S \subseteq V$ let $G-S$ denote the graph obtained by removing all vertices in S and all edges connected to them. Let $c_{odd}(G-S)$ denote the number of connected components of $G-S$ with an odd number of vertices.

Theorem 13.2.2 **Tutte's Theorem**

For a simple graph $G = (V,E)$ there is a perfect matching in G iff for all subsets $S \subseteq V$, we have $|S| \geqslant |c_{odd}(G-S)|$.

Proof. ($\rightarrow$) Let $S \subseteq V$ assume that M is a perfect matching of G. Let $G_1,\dots,G_k$ be connected components of $G-S$. Consider the edges that are used in the perfect matching M. Let G_i be a component such that there is no edge in M which connects G_i and S. So G_i has an even number of vertices because M splits G_i into pairs. In total, there are $\leqslant |S|$ edges of M which connects S to some G_i. So $c_{odd}(G-S) \leqslant |S|$. □

13.3 Chromatic Number

Let $P(G,k)$ denote the number of ways to color vertices of G into k colors without creating a monochromatic edge.

Proposition: $P(G,k)$ is a polynomial in k

Proof. Let a_i denote the number of ways to properly color G using precisely i colors. Let n be the number of vertices in G. Then $a_i = 0$ for $i > n$. Then $P(G,k)$ can be counted by first choosing i colors to use $\binom{k}{i}$ options. Then color G using these colors, a_i options.

Then

$$P(G,k) = \sum_{i=0}^{n} a_i \binom{k}{i}$$

This is a polynomial in k. □

Chromatic Polynomial

Definition 13.3.1

$P(G,k)$ is called chromatic polynomial of G.

Proposition: $\chi(G)$ is the minimal non-negative integer $P(G,\chi(G)) \neq 0$.

Let $G = (V,E)$ and let $e \in E$. Let $G - e$ denote the graph G without edge e. Let G/e denote the graph obtained by contraction along edge e.

Deletion-Contraction Formula

Theorem 13.3.1

$P(G,k) = P(G-e,k) - P(G/e,k)$.

Proof. $P(G-e,k)$ counts all colorings of G using k colors where e is the only edge that is allowed to be monochromatic. Then $P(G-e,k) - P(G,k)$ is the number of proper colorings of $G - e$ where the endpoints of e are same color. These colorings are in bijection with the contracted graph. The bijection is obtained by taking the coloring of endpoints of e is the same as colors of merged vertex. □

Example 13.3.1: Square example.

13.4 Planar Graphs

Suppose there are three houses and three wells. Is it possible to connect them all without creating intersections? No

Planar Graph

Definition 13.4.1

A graph G is called planar if we can draw it on $\mathbb{R}^2$ in a way that edges intersect other edges only at endpoints. No vertex is inside an edge.

So the previous question is: Is $K_{3,3}$ planar or not?

Example 13.4.1: $K_{3,2}$, K_4, all trees are planar.

Faces

Definition 13.4.2

Let G be a planar graph. The edges of G split the plane into regions which are called faces of G.

Theorem 13.4.1 **Euler's Theorem**

Let $G \neq \emptyset$ be a planar connected graph drawn on a plane with V vertices, E edges, and F faces. Then $V + F - E = 2$. In particular, F does not depend on how the graph is drawn.

Proof. Induction on E.

- Base Case: $E = 0$. The graph contains only an isolated vertex, $V = 1, F = 1, E = 0$, so $1 + 1 - 0 = 2$.
- Inductive Step: Suppose that this is proved for $E - 1$ edges. Let G be a connected graph with E edges.
  - Case 1: If G has a leaf, v. Let G' be a graph obtained by removing v and its edge. So G' has $V' = V - 1$ vertices and $E' = E - 1$ edges. The number of faces is the same $F' = F$. G' is planar and connected. So we have that $V + F - E = V' + F' - E' = 2$.
  - Case 2: If G has no leaves, G is not a tree. So G has a cycle C. Let e be an edge of the cycle. Let $G' = G - e$. So G' has $V' = V, E' = E - 1, F' = F - 1$. C splits the plane into two regions. Then e has different faces on its sides. Then removing the edge merges the two faces. Then $V + F - E = V' + F' - E' = 2$.

So we are done.

□

Theorem 13.4.2

$K_{3,3}$ is not planar.

Proof. Suppose that $K_{3,3}$ is planar. Then $K_{3,3}$ has $V = 6, E = 9$. So $F = E - V + 2 = 5$. For each face, construct a closed walk W_i:

- Pick an edge on the boundary of some face.
- Orient this edge e in a way that the face is to the left
- Walk along G, such that at each vertex, take the left-most turn.
- At some point we return to the original edge. We get a closed walk.

Why 4 is true: Since G is finite, at some point, there is an edge repeated twice. Then we can retract steps until repeating e twice. For each of the 5 faces, we have W_i. An oriented edge of G can appear only in one of these walks. Look at the face to the left to recover the corresponding walk. Also, each oriented edge appears at most once in W_i. Since G is bipartite, all W_i have an even length. So $l(W_i) \geqslant 4$. There are 5 walks, so there are at least 20 different oriented edges. But the graph $K_{3,3}$ has 18 oriented edges, contradiction. □

Proposition: Let G be a planar connected graph with $|V| \geqslant 3$. Then $|E| \leqslant 3|V| - 6$.

Proof. For each face of G, construct a closed walk w_i where $i = 1, \ldots, F$.

- Pick any edge e on the boundary of the face and orient it so that the face is to your left
- Start walking, at each vertex, take the left-most turn.
- At some point, we try to go along e in the same direction as the start. So we stop.

If the length of the walk $w_i = 1$, then we have a loop. If the length of $w_i = 2$, then w_i has leaves as its endpoints, which is an isolated edge. So G only has two vertices, and $V = 2 \leqslant 3$. So each w_i has length $\geqslant 3$. Each oriented edge can appear at most once. So $3F \leqslant 2E$. Using Euler, $F = E - V + 2$, we get

$$3E - 3V + 6 \leqslant 2E \implies E \leqslant 3V - 6$$

□

Chapter 14

Week 14

14.1 More on Planar Graphs

Proposition: K_5 is not planar.

Proof. We have K_5 is simple connected, with 5 vertices, so by the previous proposition, we know that if K_5 is planar, then $E \leqslant 15 - 6 = 9$. But K_5 has $\binom{5}{2} = 10$ edges. □

Proposition: A planar graph G must have a vertex of degree $\leqslant 5$.

Proof. Assume that the graph has degree $\geqslant 6$. Suppose that G is a connected component of the planar graph. Then the sum of degrees is equal to 2E. So $6V \leqslant 2E$. Then $E \geqslant 3V$. But recall that $E \leqslant 3V - 6$ which is a contradiction. □

What are other examples of non-planar graphs?

- If a graph G contains K_5 or $K_{3,3}$ as a subgraph, then G is non planar.
- Another non-planar graph is with vertices on K_5 but each edge is drawn with an intermediate added vertex.

Definition 14.1.1 **Division of Graphs**

A graph G′ is called a division of a graph G if G′ is obtained from G by the following operations:

- Take an edge (u, v) and replace it by $(u, w), (w, v)$ where w is a new vertex.

Proposition: If G′ is a division of G, then G′ is planar iff G is planar.

If G contains a division of K_5 or $K_{3,3}$ as a subgraph, then G is non-planar.

Theorem 14.1.1 **Kuratowski**

A graph G is planar iff G does not contain a division of K_5 or $K_{3,3}$ as a subgraph.

Chapter 15

Week 15

15.1 Chromatic Number of Planar Graphs

We have K_4 is planar which means that there are planar graphs which require at least 4 colors.

Proposition: Every planar graph is 6 colorable.

Proof. Induction on the number of vertices. If $V \leqslant 6$, then we are done. For general V, suppose that it is proved for $V-1$ vertices. Let G be a planar graph on V vertices. Let v be a vertex of degree 5 or less. Let G' be the graph obtained by removing the vertex and its edges. Then G' is 6 colorable as it is still planar. Since v has $\leqslant 5$ neighbors, there is a color to choose for v in G. So G is 6-colorable. □

3 colors may not be enough because K_4 is planar.

Proposition: Every planar graph is 5 colorable.

Proof. By induction on the number of vertices. If $V \leqslant 5$, then the claim is true. Otherwise, if $V = n$, the claim is proved for $\leqslant n-1$ vertices. G has a vertex of degree $\leqslant 5$. Let v be this vertex. Let G' be the graph with this vertex removed. Then G' is 5 colorable. Let $C : V' \to [5]$ be the coloring of G'. If $\deg v \leqslant 4$, then we are done. If $\deg v = 5$, and some of the neighbors of v have the same color, then we are done. Let $v_1, \dots, v_5$ be the neighbors of v, which are ordered counter clockwise around v. Also, $C(v_i) = i$. We want to construct another coloring of G' such that v_i do not have different colors.

Let $G'_{1,3} \subseteq G'$ be the subgraph consisting only of vertices of G' with colors 1 or 3 in C. Similarly, define $G'_{2,4} \subseteq G'$ by keeping vertices of color $2, 4$. Note that $v_1, v_3 \in G'_{1,3}$.

Claim 1: If v_1, v_3 are not connected by a path in $G'_{1,3}$, then there is a coloring of G' such that $C(v_1) = C(v_3)$.

If v_1, v_3 are not connected in $G'_{1,3}$, then they are in different connected components. Let H be the connected component of v_1 in $G'_{1,3}$. Let $\tilde{C}$ be the coloring of G' obtained by

- Keeping the coloring of all vertices $\notin H$
- Changing 1 to 3 and 3 to 1 in H

Then $\tilde{C}$ is a proper coloring of G'. If e had endpoints both not in H, then the coloring is still valid. If e has both endpoints in H, then this is still a valid coloring. If one endpoint is in H and the other not, then $e \notin G'_{1,3}$. So swapping $1, 3$ does not make it monochromatic. So now, we have $\tilde{C}(v_1) = \tilde{C}(v_3)$.

So if v_1, v_3 are not connected by a path in $G'_{1,3}$, then we are done. Similarly, if v_2, v_4 are not connected by a path in $G'_{2,4}$, then we are done.

Claim 2: Either v_1, v_3 are not connected in $G'_{1,3}$ or v_2, v_4 are not connected in $G'_{2,4}$.

Suppose that this is not the cases. So P_1 connects v_1, v_3, P_2 connects v_2, v_4. Consider the cycle inside G which is $v \to v_1 \to v_3 \to v$. This cycle splits the plane into 2 regions. So v_2 and v_4 are in different sides of the cycle. Then P_2 intersects the cycle. So P_2 intersects P_1 are a common vertex. Recall that P_1, P_2 are in $G'_{1,3}$, $G'_{2,4}$, so they have a common vertex that is in both, which is a contradiction. □

Theorem 15.1.1 **Four Color Theorem**

Every planar graph is 4 colorable.

15.2 Polyhedral

Definition 15.2.1 **Polyhedral**

A convex polyhedron is a convex 3-d body in $\mathbb{R}^3$ such that the boundary is a finite collection of polygons, glued along the edges. Equivalently, a convex polyhedron is a convex subset of $\mathbb{R}^3$ defined by inequalities:

$$\begin{gathered} f_1(x, y, z) \leqslant a_1 \\ f_2(x, y, z) \leqslant a_2 \\ \vdots \end{gathered}$$

where $f_i(x, y, z) = \alpha x + \beta y + \gamma z$ where $a_i, \alpha, \beta, \gamma \in \mathbb{R}$.

Reminder: Convex: if $x, y \in P$, then $[x, y] \subseteq P$, or all $\lambda x + (1 - \lambda)y \in P$ for $\lambda \in [0, 1]$. We also assume that for a polyhedra, there is a $p \in P$ such that a small ball centered at P is a subset of P.

Let P be a convex polyhedron. Polygons on the boundary of P are faces. Sides of the polygons are edges of P. The corners of the polygons are vertices.

Definition 15.2.2 **Vertex**

Formally, a vertex is a point $p \in P$ such that there are no points $x, y \in P$ such that $p \in (x, y)$.

Definition 15.2.3 **1 - Skeleton**

A 1-skeleton of a polyhedron is a graph obtained by taking vertices and edges of the polyhedron.

Theorem 15.2.1

Let P be a convex polyhedron.

- The 1-skeleton of P is a planar graph
- $V + F - E = 2$ where V, F, E are the number of vertices, faces, and edges of a polyhedron.

Proof. We will show that the boundary of P without a point can be unwrapped in a way that we get a bijection (preserves lines and faces) with $\mathbb{R}^2$.

- Step 1: First draw the boundary of P on a sphere. Let $p \in P$ strictly inside P. Let S be a sufficiently large sphere centered at p which contains P. Consider all rays starting at p. Each ray intersects S at one point. Also, each ray intersects the boundary at a single point. Since P is convex, any line intersects P along one point, so the ray only passes through a boundary once. So we have a mapping of δP-the boundary of P to S along the corresponding ray starting from P. As a result, we draw a 1-skeleton of P on S without self-intersections. Faces of P correspond to regions on S bounded by the one skeleton.
- Step 2: We use stereographical projection to go from S to a plane. Pick a point x on S inside one of the faces. Let H be the plane tangent to S opposite of x. Send each point a on S to the intersection point of H and the line (x, a). The projection sends the 1-skeleton to a planar graph with the faces of P corresponding to faces of the planar graph.

□

Proposition: $3F \leqslant 2E \leqslant 6F - 12$. Also, $3V \leqslant 2E \leqslant 6V - 12$.

Proof. $3F \leqslant 2E$ since every face has at least 3 sides. Every edge appears in exactly 2 faces. Also $3V \leqslant 2E$ since each vertex has degree $\geqslant 3$. Every edge connects 2 vertices.

The other inequalities come from Euler's formula:

$$3(E + 2 - V) \leqslant 2E \iff E \leqslant 3V - 6$$

and similarly,

$$3(E + 2 - F) \leqslant 2E \iff E \leqslant 3F - 6$$

□

Corollary: Every polyhedron has a face with 5 or less sides.

Proof. If all faces have six sides or more, then

$$2E \geqslant 6F$$

But this contradicts the fact that

$$2E \leqslant 6F - 12$$

□

15.3 Platonic Solids

Platonic solids are convex polyhedron, regular faces, equal to each other. All vertices have the same degrees and all angles between the faces or the same.

Theorem 15.3.1 Up to scaling, there are five Platonic solids

Idea of proof:

- Let P be the platonic solid
- Let l be the number of sides of a face
- Let d be the degree of all vertices

Then $d = 3, 4, 5$, $l = 3, 4, 5$. So $2E = dV$ and $2E = lF$. By Euler's formula we have

$$\frac{2E}{d} + \frac{2E}{l} - E = 2$$

and

$$\frac{2}{d} + \frac{2}{e} - 1 > 0 \iff \frac{1}{d} + \frac{1}{l} > \frac{1}{2}$$

So this forbids the pairs (d, l) of $(4, 4), (4, 5), (5, 4), (5, 5)$. So we have:

- $(d, l) = (3, 3)$ Tetrahedron
- $(d, l) = (4, 3)$ Octahedron
- $(d, l) = (3, 4)$ Cube
- $(d, l) = (3, 5)$ Dodecahedron
- $(d, l) = (5, 3)$ Icosahedron

15.4 Other Surfaces

Definition 15.4.1 **Toroidal**

A graph is called toroidal if it can be drawn on a torus without self-intersections.

Example 15.4.1: Every planar graph is toroidal.

Example 15.4.2: K_5 and $K_{3,3}$ are toroidal.

Example 15.4.3: K_7 is toroidal. Every graph with $\leqslant 7$ vertices is toroidal.

Is K_8 toroidal?

For planar graphs, we have $V + F - E = 2$ and for toroidal: $V + F - E = ?$

Theorem 15.4.1

Let G be a toroidal graph, drawn on a torus in a way that each face is homeomorphic to a disk. Then $V + F - E = 0$. Homeomorphic to a disk means that one can deform a disk into a face without creating any holes, self-intersections, ...

For K_8, we have $V = 8$, $E = 28$. So it should have $F = 20$. Each face is a triangle. So $3F \leqslant 2E$. So $60 \leqslant 56$. If each face is not homeomorphic to a disc, we have $F \geqslant 20$ and the same proof.

$K_{4,5}$ is not toroidal. There is a family of minimal non-toroidal graphs similar to $K_{3,3}, K_5$ for planar graphs.

Chapter 16

Week 16

16.1 Review: Counting Problems

We have covered:

- Binomial and Multinomial Coefficients
- Placing objects into boxes
- Permutations
- Stirling Numbers
- Inclusion-Exclusion Principle

Stirling Numbers: $S(n,k), s(n,k)$.

Second Type: $S(n,k)$. Ways to place n objects, different into k boxes. This is the same as number of partitions of $[n]$ into k non-empty boxes. Then $k!S(n,k) =$ number of surjections $[n] \to [k]$.

First Type: $c(n,k)$ is the number of permutations of $[n]$ with exactly k cycles (unsigned). Then $(-1)^{n-k}c(n,k) = s(n,k)$ (signed).

Formula for Stirling number of 2nd kind using Inclusion-Exclusion:

Proof. $S(n,k) =$ number of surjections. $k^n - k!S(n,k)$ is the number of functions $[n] \to [k]$ not surjections. Then

$$S(n,k) = |A_1 \cup A_2 \cup \cdots \cup A_k|$$

where A_i is the number of maps where i is not in the image of your function. Then

$$\begin{aligned}|A_1 \cup \cdots \cup A_k| &= |A_1| + \cdots + |A_k| \\ &\quad - |A_1 \cap A_2| - |A_1 \cap A_3 - \cdots| \\ &\quad + \cdots\end{aligned}$$

Then $|A_{j_1} \cap \cdots \cap A_{j_l}| = |$ functions $[n] \to [k] \backslash \{j_1, \ldots, j_l\}|$. There are $(k-l)^n$ such functions. So

$$|A_1 \cap \cdots \cap A_k| = \binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n - \cdots$$

This is

$$\sum_{i\geqslant 1}\binom{k}{i}(k-i)^n(-1)^i$$

So

$$k^n - k!S(n,k) = \sum_{i\geqslant 1}\binom{k}{i}(k-i)^n(-1)^i$$

Rearranging:

$$\begin{aligned} S(n,k) &= \frac{1}{k!}(k^n = \sum \cdots) \\ &= \frac{1}{k!}\sum_{i\geqslant 0}(-1)^i\binom{k}{i}(k-i)^n \end{aligned}$$

Finally,

$$S(n,k) = \sum_{i=0}^{k}\frac{(k-i)^n}{i!(k-i)!}$$

□

$x^k, (x)_k = x(x-1)(x-2)\cdots(x-k+1)$.

$$x^n = \sum_{k\geqslant 0} S(n,k)(x)_k \qquad\qquad (x)_k = \sum_{k\geqslant 0} S(n,k)x^k$$

$S(n,k) = S(n-1,k-1) + kS(n-1,k)$. Also, $c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$.

Cycle type of permutation: Partition of n whose parts are the length of cycles. If λ is a partition of n, m_i = number of parts of size i, then the number of permutations given by λ is

$$\frac{n!}{\prod_{i\geqslant 1} m_i! i^{m_i}}$$

The i^{m_i} corresponds to rotating these cycles, and $m_i!$ corresponds to number of ways to swap cycle positions.

Objects and Boxes:

- n different objects and k different boxes. We have number of functions $[n] \to [k]$ which is k^n.
- Surjections: $k!S(n,k)$
- Injections: $\binom{k}{n}n!$
- n identical objects and k different boxes. This is the number of compositions $(a_1, a_2, \dots, a_k)$, $\sum a_i = n$
- Weak compositions: $(a_1, \dots, a_k)$, $a_i \geqslant 0$, and $a_1 + \cdots + a_k = n$. Stars and bars: $\binom{n+k-1}{k-1}$.
- Compositions: $(a_1, \dots, a_k)$, $a_i \geqslant 1$, $a_1 + \cdots + a_k = n$. This is $\binom{n-1}{k-1}$. Place $k-1$ bars between n stars.
- n different objects and k identical boxes. This number of partitions of $[n]$, with no empty parts is given by $S(n,k)$.
- n identical objects and k identical boxes, this is the number of partitions. $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$. $p(n)$ = number of partitions of n.

Generalized Binomial:

Example 16.1.1:

$$\begin{aligned}\sqrt{\frac{1_x}{1-x}} &= (1+x)\sqrt{\frac{1}{(1-x)(1+x)}}\\ &= (1+x)\sqrt{\frac{1}{1-x^2}}\\ &= (1+x)(1-x^2)^{\frac{-1}{2}}\\ &= (1+x)\sum_{n\geqslant 0}\binom{\frac{-1}{2}}{n}(-x^2)^n\\ \binom{a}{n} &= \frac{a(a-1)\cdots(a-n+1)}{n!}\end{aligned}$$

Catalan Numbers:

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 = \sum_{i=0}^{n-1} C_iC_{n-i-1}$$

Also:

$$\begin{aligned}C_n &= \frac{1}{n+1}\binom{2n}{n}\\ \sum_{n\geqslant 0} C_nx^n &= \frac{1-\sqrt{1-4x}}{2x}\end{aligned}$$

Generating Functions for Partitions:

$$\sum_{n\geqslant 0} p(n)x^n = \prod_{i\geqslant 1}\frac{1}{1-x^i}$$

This is because:

$$\sum_{\lambda} x^{|\lambda|} = \sum_{m_1,m_2,\ldots} x^{m_1+2m_2+\cdots}$$

For even parts only:

$$\prod_{i\geqslant 1}\frac{1}{1-x^{2i}}$$

For strict partitions:

$$\prod_{i\geqslant 1}(1+x^i)$$

For partitions of parts of sizes $3, 5, 10$:

$$\frac{1}{1-x^3}\frac{1}{1-x^5}\frac{1}{1-x^{10}}$$

Euler's pentagonal numbers:

$$\prod_{i\geqslant 1}(1-x^i) = \sum_{k\in\mathbb{Z}}(-1)^k x^{\frac{(3k-1)k}{2}}$$

So

$$p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k-1} p(n - \frac{k(3k-1)}{2})$$

Generating function: $(a_0, a_1, \ldots)$. Generating function is $\sum_{n \geqslant 0} a_n x^n$. Exponential generating functions: $\sum_{n \geqslant 0} a_n \frac{x^n}{n!}$.

$$\exp(x) = \sum_{n \geqslant 0} \frac{x^n}{n!}$$

Properties:

- $\exp(x) \exp(y) = \exp(x + y)$
- $dx \exp(x) = \exp(x)$.

Example 16.1.2: $t(n)$ is the number of permutations of $[n]$ such that $\sigma^3 = id$.

$$\begin{aligned}
\sum_{n \geqslant 0} \frac{t(n)}{n!} x^n &= \sum_{n \geqslant 0} \sum_{\substack{\sigma \text{ of } [n] \\ \text{3,1 cycles}}} \frac{x^n}{n!} \\
&= \sum_{n \geqslant 0} \sum_{k \leqslant \frac{n}{3}} \frac{n!}{3^k k! (n-3k)! \frac{x^n}{n!}} \\
&= \sum_{n \geqslant 0} \sum_{k \leqslant \frac{n}{3}} \frac{1}{3^k k! (n-3k)!} x^n \\
&= \sum_{k \geqslant 0} \sum_{n \geqslant 3k} \frac{x^n}{3^k k! (n-3k)!} \\
&= \sum_{k \geqslant 0} \sum_{n \geqslant 3k} \frac{x^{3k+(n-3k)}}{3^k k! (n-3k)!} \\
&= \sum_{k \geqslant 0} \frac{x^{3k}}{3^k k!} \cdot \sum_{m \geqslant 0} \frac{x^m}{m!} \\
&= \exp\left(\frac{x^3}{3}\right) \exp(x)
\end{aligned}$$

where $m = n - 3k$.