Math172Hw8

Trustin Nguyen

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Exercise 1: Compute all the numbers $p(1), p(2), \ldots, p(10)$ using the pentagonal numbers.

Answer. Using $p(n) = \sum_{i \neq 0} (-1)^{i-1} p(n - i(3i-1)/2)$ and $p(0) = p(1) = 1$, we have

$$\begin{aligned}
p(1) &= 1 \\
p(2) &= p(1) + p(0) = 2 \\
p(3) &= p(2) + p(1) = 3 \\
p(4) &= p(3) + p(2) = 5 \\
p(5) &= p(4) + p(3) - p(0) = 7 \\
p(6) &= p(5) + p(4) - p(1) = 11 \\
p(7) &= p(6) + p(5) - p(2) - p(0) = 15 \\
p(8) &= p(7) + p(6) - p(3) - p(1) = 22 \\
p(9) &= p(8) + p(7) - p(4) - p(2) = 30 \\
p(10) &= p(9) + p(8) - p(5) - p(3) = 42
\end{aligned}$$

is the answer

Exercise 2: Recall that the Bernoulli numbers B_n are defined by $\frac{x}{1-\exp(-x)} = \sum_{n \geqslant 0} B_n \frac{x^n}{n!}$ (during the lecture there was a sign error in this definition, here is the correct version). These numbers arise in the formula

$$1^k + 2^k + 3^k + \cdots + n^k = \sum_{i=0}^{k} B_{k-i} \binom{k}{i} \frac{n^{i+1}}{i+1}$$

- Compute the numbers B_0, B_1, B_2.

  Answer. For B_0 set $k = 0, n = 1$. Then we get:

  $$1^0 = \sum_{i=0}^{0} B_{0-i} \binom{0}{i} \frac{1}{i+1} = B_0$$

  Then for B_1, set $k = 1, n = 1$. So we get:

  $$1^1 = \sum_{i=0}^{1} B_{1-i} \binom{1}{i} \frac{1}{i+1} = B_1 + \frac{1}{2} B_0$$

  Then for B_2, set $k = 2, n = 1$. We then get:

  $$1^2 = \sum_{i=0}^{2} B_{2-i} \binom{2}{i} \frac{1}{i+1} = B_2 + B_1 + \frac{1}{3} B_0$$

So overall, $B_0 = 1$,

$$B_1 + \frac{1}{2}B_0 = 1$$

which means

$$B_1 + \frac{1}{2} = 1, B_1 = \frac{1}{2}$$

Then finally,

$$\begin{aligned} B_2 + B_1 + \frac{1}{3}B_0 &= 1 \\ B_1 + \frac{1}{2} + \frac{1}{3} &= 1 \\ B_1 + \frac{5}{6} &= 1 \\ B_1 &= \frac{1}{6} \end{aligned}$$

Overall,

B_0	B_1	B_2
1	$\frac{1}{2}$	$\frac{1}{6}$

- Show that all the numbers B_{2i+1} are equal to 0 with the exception of B_1.

  Proof. I couldn't solve it but here is some work I did:

  There is the decomposition:

  $$1 = \sum_{i=0}^{k} B_{k-i}\binom{k}{i}\frac{1}{i+1} = \sum_{i=0}^{k-1} B_{k-i}\binom{k}{i}\frac{1}{i+1} + \sum_{i=0}^{k-1} B_{k-i-1}\binom{k}{i}\frac{1}{i+2}$$

  We can complete the left summand on the RHS:

  $$1 = -\frac{B_0}{k+1} + \sum_{i=0}^{k} B_{k-i}\binom{k}{i}\frac{1}{i+1} + \sum_{i=0}^{k-1} B_{k-i-1}\binom{k}{i}\frac{1}{i+2}$$

  So

  $$\frac{1}{k+1} = \sum_{i=0}^{k-1} B_{k-i-1}\binom{k}{i}\frac{1}{i+2}$$

  Add B_k to both sides:

  $$B_k + \frac{1}{k+1} = \sum_{i=0}^{k-1} B_{k-i}\binom{k}{i+1}\frac{1}{i+1}$$

  □

Exercise 3: This is a continuation of Problem 3 in Problem sets 6,7. Define the following formal power series:

$$\ln(1+x) = \sum_{k\geqslant 1}(-1)^{k-1}\frac{x^k}{k}$$

- Show that $\frac{d}{dx}\exp(x) = \exp(x)$ and $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$.

  Proof. We have $\exp(x) = \sum_{i\geqslant 0}\frac{x^i}{i!}$. Then taking the derivative:

  $$\frac{d}{dx}\sum_{i\geqslant 0}\frac{x^i}{i!} = \sum_{i\geqslant 1}\frac{x^{i-1}}{(i-1)!} = \sum_{i\geqslant 0}\frac{x^i}{i!} = \exp(x)$$

And now for $\ln(1+x)$, use the definition and take the derivative:

$$\begin{aligned}
\ln(1+x) &= \sum_{k\geqslant 1}(-1)^{k-1}\frac{x^k}{k} \\
&\to \frac{d}{dx}\sum_{k\geqslant 1}(-1)^{k-1}\frac{x^k}{k} \\
&= \sum_{k\geqslant 1}(-1)^{k-1}x^{k-1} \\
&= \sum_{k\geqslant 0}(-1)^{k}x^{k} \\
&= \sum_{k\geqslant 0}(-x)^{k} \\
&= \frac{1}{1-(-x)} \\
&= \frac{1}{1+x}
\end{aligned}$$

which concludes the proof. □

- Show that $\ln(\exp(x)) = x$, where the left hand side is the result of plugging $\exp(x)-1$ instead of the variable t into $\ln(1+t)$. Note that in class we have mentioned, without proof, that the identity from part (2) implies that $\exp(\ln(1+x)) - 1 = x$, which is nontrivial to show by direct computation.

Proof. Consider the derivative of this term. Last time, it was proved that $\frac{d}{dx}F(G(x)) = G'(x)F(G(x))$ for two formal power series $F(x), G(x)$. Then

$$\begin{aligned}
\frac{d}{dx}\ln(\exp(x)) &= \frac{d}{dx}\ln\left(1+\sum_{k\geqslant 1}\frac{x^k}{k!}\right) \\
&= \exp(x)\frac{1}{1+\sum_{k\geqslant 1}\frac{x^k}{k!}} \\
&= \exp(x)\frac{1}{\sum_{k\geqslant 0}\frac{x^k}{k!}} \\
&= \exp(x)\frac{1}{\exp(x)} \\
&= 1
\end{aligned}$$

Taking the anti-derivative with respect to x, we get $x+c$. But we see that

$$\ln(\exp(x)) = \sum_{j\geqslant 1}(-1)^{j-1}\frac{\left(\sum_{k\geqslant 1}\frac{x^k}{k!}\right)^j}{j}$$

which x divides because for $\sum_{k\geqslant 1}\frac{x^k}{k!}$, x divides this. Therefore, $x \mid x+c$, $c=0$. So $\ln(\exp(x)) = x$. □

Exercise 4: Find the exponential generating function for the number of derangements of $[n]$ (we have found this number earlier using inclusion-exclusion).

Proof. The number of derangements of $[n]$ was given by

$$n!\sum_{i=0}^{n}(-1)^i\frac{1}{i!}$$

Then we get for the exponential generating function:

$$\sum_{n \geqslant 0} \left(\sum_{i=0}^{n} (-1)^i \frac{1}{i!} \right) x^n$$

But the coefficient of each power of x is given by the columns of:

$$\begin{aligned}
1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots \\
x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \cdots \\
+x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \cdots \\
+ x^3 - \frac{x^4}{1!} + \frac{x^5}{2!} - \cdots \\
\ddots \qquad\qquad
\end{aligned}$$

Then we can rewrite the generating function as:

$$\sum_{n \geqslant 0} x^n \exp(-1)$$

since each row in the sum represented $x^i \exp(-x)$. So we substituted $x = 1$. So that is the exponential generating function for the number of derangements. □