Math55Hw13

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7.2: 12, 16, 24, 26

Exercise 12: Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$

Proof. By the sum rule,

$$\begin{aligned} p(E \cup F) &= p(E) + p(F) - p(E \cap F) \leq 1 \\ &= 0.8 + 0.6 - p(E \cap F) \leq 1 \\ &= -p(E \cap F) \leq -.4 \\ &= p(E \cap F) \geq .4 \end{aligned}$$

By conditional probability,

$$\begin{aligned} p(E|F) &= \frac{p(E \cap F)}{p(F)} \leq 1 \\ &= p(E \cap F) \leq 0.6 \end{aligned}$$

Therefore, we can use the intersection to construct a sum rule bound:

$$\begin{gathered} -p(E \cap F) \geq -0.6 \\ p(F) + p(E) - p(E \cap F) \geq -0.6 + 0.6 + 0.8 \\ p(E \cup F) \geq 0.8 \end{gathered}$$

□

Exercise 16: Show that if E and F are independent events, then $\bar{E}$ and $\bar{F}$ are also independent events.

Proof. If we know that E and F are independent, then

$$p(E)p(F) = p(E \cap F)$$

We start with finding the value of $p(\bar{E}|\bar{F})$:

$$\begin{aligned}\frac{\left|\bar{E}\cap\bar{F}\right|}{\left|\bar{F}\right|} &= \frac{1-|E\cup F|}{1-|F|} \\ &= \frac{1-|E|-|F|+|E\cup F|}{1-|F|} \\ &= \frac{1-|E|-|F|+|E|\,|F|}{1-|F|} \\ &= 1-|E| \\ &= \left|\bar{E}\right|\end{aligned}$$

Since $p(\bar{E}|\bar{F}) = p(\bar{E})$, $\bar{E}$ and $\bar{F}$ are independent. □

Exercise 24: What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

Experiment	Flip a coin five times and record the result as a bit-string of H and T of length 5
Sample Space	$S = \{H, T\}^5$
Event	$E = A \cap B$
	$A = \{e_1 \in S : T \text{ is the first element of the string}\}$
	$B = \{e_2 \in S : \text{Last four elements are heads}\}$

The conditional probability that we want to find is

$$p(B|A) = \frac{|A\cap B|}{|A|} = \frac{|E|}{|A|}$$

Since there is only one element in $E : THHHH$, we get $|E| = 1$. To count A, note that the first element is T. The number of bit strings of length 5 that began with a T, upon removal of the T will be in bijection with the number of bit strings of length 4. So $|A| = 2^4$. Plugging in these cardinalities gives us:

$$p(B|A) = \frac{1}{2^4}$$

Exercise 26: Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

Proof. We must see if $p(E|F) = p(E)$.

Experiment	Generate bit strings of length 3
Sample Space	$S = \{0, 1\}^3$
Event	$E = \{e_1 \in \{0,1\}^3 : e_1 \text{ contains an odd number of 1's}\}$
	$F = \{e_2 \in \{0,1\}^3 : e_2 \text{ starts with a } 1\}$

We start by calculating $|E \cap F|$. There are exactly two bit strings of length 3 that begin with 1 and contain an odd number of 1's: 100 and 111. So $|E \cap F| = 2$. We next find $|F|$. The number of bit-strings of length 3 that start with 1 is in bijection upon removal of the 1, with the number of bit strings of length 2. We get $2^2 = 4 = |F|$. We finally calculate $|E| = 4$ since there are four of such strings: $100, 010, 001, 111$.

$$\begin{aligned} p(E|F) &= \frac{|E \cap F|}{|F|} = \frac{2}{4} = \frac{1}{2} \\ p(E) &= \frac{|E|}{|S|} = \frac{4}{8} = \frac{1}{2} \end{aligned}$$

Since $p(E|F) = p(E)$, the events are independent. $\square$

7.3: 2, 4, 10ac, 15

Exercise 2: Suppose that E and F are two events in the sample space and $p(E) = 2/3, p(F) = 3/4,$ and $p(F|E) = 5/8$. Find $p(E|F)$.

We can use Bayes' Theorem:

$$\begin{aligned} p(F|E)p(E) &= p(E|F)p(F) \\ \left(\frac{5}{8}\right)\frac{2}{3} &= p(E|F)\left(\frac{3}{4}\right) \\ p(E|F) &= \frac{4}{3}\left(\frac{5}{4}\right) \\ p(E|F) &= \frac{5}{3} \end{aligned}$$

Exercise 4: Suppose that Ann selects a ball by first picking one of the two boxes at random and then selecting a ball from this box. The first box contains three orange balls and four black balls, and the second box contains five orange balls and six black balls. What is the probability that Ann picked a ball from the second box if she has selected an orange ball?

- Let O set of events where Ann selects an orange ball.
- Let B be the set of events where Ann selects a black ball.
- Let T_1 be the set of events where Ann selects a ball from the first box.
- Let T_2 be the set of events where Ann selects a ball from the second box.

We wish to calculate $p(T_2|O) = \frac{|T_2 \cap O|}{|O|}$. We have $|T_2 \cap O| = 5$ and $|O| = 8$. Therefore, the answer is $\frac{5}{8}$.

Exercise 10ac: Suppose that 4% of the patients tested in a clinic are infected with avian influenza. Furthermore, suppose that when a blood test for avian influenza is given, 97% of the patients infected with avian influenza tested positive and that 2% of the patients not infected with avian influenza tested positive. What is the probability that

a) a patient testing positive for avian influenza with this test is infected with it?

If $p(P)$ is the probability of testing positive for avian influenza and $p(I)$ is the probability of being infected with avian influenza, then we need to find $p(I|P)$. By the Law of Total Probability,

$$\begin{aligned} p(P) &= p(P|I)p(I) + p(P|\bar{I})p(\bar{I}) \\ p(P) &= .97(.04) + .02(.96) \end{aligned}$$

Using that fact that $p(P|I)p(I) = p(I|P)p(P)$, we can find $p(I|P)$:

$$\begin{aligned} p(P|I)p(I) &= p(I|P)p(P) \\ .97(.04) &= p(I|P)p(P) \\ p(I|P) &= \frac{.97(.04)}{.97(.04) + .02(.96)} \end{aligned}$$

c) a patient testing negative for avian influenza with this test is infected with it?

If $p(N)$ is the probability of testing negative and $p(I)$ is the probability of testing negative, then we want to find $p(I|N)$:

$$\begin{aligned} p(I) &= p(I|N)p(N) + p(I|\bar{N})p(\bar{N}) \\ p(I|N) &= \frac{p(I) - p(I|P)p(P)}{p(N)} \end{aligned}$$

Since $p(N) = 1 - p(P)$, we can use **a)** to calculate the answer.

Exercise 15: In this exercise we will use Bayes' theorem to solve the Monty Hall puzzle (Example 10 in Section 7.1). Recall that in this puzzle you are asked to select one of three doors to open. There is a large prize behind one of the three doors and the other two doors are losers. After you select a door, Monty hall opens one of the two doors you did not select that he knows is a losing door, selecting a random if both are losing doors. Monty asks you whether you would like to switch doors. Suppose that the three doors in the puzzle are labeled 1, 2, and 3. Let W be the random variable whose value is the number of the winning door; assume that $p(W = k) = 1/3$ for $k = 1, 2, 3$. Let M demote the random variable whose value is the number of the door that Monty opens. Suppose you choose door i.

a) What is the probability that you will win the prize if the game ends without Monty asking you whether you want to change doors?

Experiment	You select one of three doors.
	One of the doors has a car prize behind it.
Sample Space	$S = \{1, 2, 3\}^2$
Event	$W = \{e \in S :$ the elements of the string are equal$\}$

There are three tertiary strings of length 2 where the elements are the same: 11, 22, and 33. There are 3^2 tertiary strings of length 2 in total. So

$$p(W) = \frac{|W|}{|S|} = \frac{3}{3^2} = \frac{1}{3}$$

b) Find $p(M = j|W = k)$ for $j = 1, 2, 3$ and $k = 1, 2, 3$.

We have

$$p(M = j|W = k) = \frac{|M = j \cap W = k|}{|W = k|}$$

again, we create an experiment:

Experiment	You choose a door, Monty chooses a losing door.
Sample Space	$S = \{(s_1, s_2, s_3) : s_i \in \{1, 2, 3\} \wedge s_3 \neq s_1, s_2\}$
Event	$(M = j \cap W = k) = \{e_1 \in S : s_3 = j, s_2 = k\}$

Since we know that s_1, s_3 are fixed, we have two options for s_2:

- $s_2 = s_1$
- $s_2 \neq s_1, s_2$

We now count $|W = k| = 3$. Therefore the probability is $^2/_3 = \dfrac{2}{3}$.

c) Use Bayes' theorem to find $p(W = j|M = k)$ where i and j and k are distinct values.

By Bayes' Theorem,

$$\begin{aligned} p(W = j|M = k)p(M = k) &= p(M = j|W = k)p(W = k) \\ p(W = j|M = k)\frac{4}{12} &= \frac{2}{3}\left(\frac{1}{3}\right) \\ p(W = j|M = k) &= \frac{2}{3} \end{aligned}$$

d) Explain why the answer to part (c) tells you whether you should change doors when Monty gives you the chance to do so.

From **a)**, we know that the probability of winning if you stay is $^1/_3$. But by **c)**, after Monty chooses a door, the probability of winning goes up to $^2/_3$.

This says that you should switch the $^1/_3$ probability for $^2/_3$ by switching doors.

2016-mt2 question 2ab: Nikhil is trying to solve a certain mathematical problem. There is a 2/3 chance that he will pass out while writing. If he passes out, there is a 3/4 chance that he will make an arithmetic mistake. If he doesn't pass out, there is only a 3/16 chance.

a) What is the probability that Nikhil will make a mistake?

   If $p(M)$ is the probability of making a mistake and $p(P)$ is the probability of passing out, we know $p(P) = {}^2/_3$, $p(M|P) = {}^3/_4$, $p(M|\bar{P}) = {}^3/_{16}$. Therefore,

$$\begin{aligned} p(M) &= p(M|P)p(P) + p(M|\bar{P})p(\bar{P}) \\ &= \frac{3}{4}\left(\frac{2}{3}\right) + \frac{3}{16}\left(\frac{1}{3}\right) \\ &= \frac{1}{2} + \frac{1}{16} = \frac{9}{16} \end{aligned}$$

b) Given that he does make a mistake, what is the probability that he passed out?

   We want to find

$$\begin{aligned} p(P|M) &= \frac{p(M|P)p(P)}{p(M)} \\ &= \frac{{}^3/_4({}^2/_3)}{{}^9/_{16}} \\ &= \frac{8}{9} \end{aligned}$$

7.2: 15

Exercise 15: Show that if $E_1, E_2, \ldots, E_n$ are events from a finite sample space, then

$$\begin{aligned} &p(E_1 \cup E_2 \cup \cdots \cup E_n) \\ &\qquad \leq p(E_1) + p(E_2) + \cdot + p(E_n) \end{aligned}$$

This is known as **Boole s inequality**.

Proof. Let $|E_i| = e_i$. Since the sample space is finite, call it $|S| = s$. Then

$$\sum_{i=1}^{n} p(E_i) = \frac{\sum_{i=1}^{n} e_i}{s}$$

Observe that $|E_1 \cup \cdots \cup E_n| \leq \sum_{i=1}^{n} e_i$ and therefore:

$$\frac{|E_1 \cup \cdots \cup E_n|}{s} \leq \frac{\sum_{i=1}^{n} e_i}{s}$$

So we have proven Boole's inequality, as desired. $\square$

7.4: 4, 8, 18, 24, 26x, Mt2prac 6, Finalprac 7a

Exercise 4: A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is the expected number of heads that come up when it is flipped 10 times.

Let $X(s)$ be the number of heads that are obtained after flipping a coin 10 times. If

$$X_i = \begin{cases} 1 \text{ when the } i\text{-th position is } H \\ 0 \text{ when the } i\text{-th position is } T \end{cases}$$

then

$$\sum_{i=1}^{10} X_i = X(s)$$

For each X_i, their expectation is:

$$\mathbb{E}X_i = .6(1) + .4(0)$$

Therefore,

$$\mathbb{E}X(s) = .6 \cdot 10 = 6$$

Exercise 8: What is the expected sum of the numbers that appear when three fair dice are rolled?

Let $X(s)$ be the sum of the numbers of the dice. Let X_i be the number rolled on the i-th dice. Then

$$X(s) = X_1 + X_2 + X_3$$

We can now calculate the expected number rolled on each dice:

$$\sum_{r \in \text{range } X(s)} r\mathbb{P}(X = r) = \frac{1}{6}(1 + 2 + \cdots 6) = \frac{21}{6}$$

Therefore,

$$\mathbb{E}X(s) = 3\left(\frac{21}{6}\right) = \frac{21}{2} = 10.5$$

Exercise 18: Suppose that X and Y are random variables and that X and Y are nonnegative for all points in a sample space S. Let Z be the random variable defined by $Z(s) = \max(X(s), Y(s)$ for all elements $s \in S$. Show that $E(Z) \leq E(X) + E(Y)$.

Since $X(s), Y(s) \geq 0$,

$$\max(X(s), Y(s)) \leq X(s) + Y(s)$$

We can easily derive this by breaking it into cases at each $s \in S$ whether $X(s) > Y(s)$ or $Y(s) > X(s)$. Since expectation is linear,

$$\mathbb{E}(Z) = \mathbb{E}\max(X(s), Y(s)) \leq \mathbb{E}(X) + \mathbb{E}(Y)$$

Exercise 24: Let A be an event. Then I_A, the **indicator random variable** of A, equals 1 if A occurs and equals 0 otherwise. Show that the expectation of the indicator random variable of A equals the probability of A, that is, $E(I_A) = p(A)$.

Proof. Let $A = \{a_1, \ldots, a_n\}$ where each element has a $p(a_i)$ for $1 \leq i \leq n$. We can break the indicator function down into

$$I_A = I_{A_1} + \cdots I_{A_n}$$

where $A_i = \{a_i : a_i = i\text{-th largest element in } A\}$ and $|A_i| = 1$. Observe that

$$\mathbb{E}I_{A_i} = p(a_i)$$

Therefore,

$$\mathbb{E}I_A = \sum_{i=1}^{n} p(a_i) = p(A)$$

as desired. □

Exercise 26x: Let $X(s)$ be a random variable, where $X(s)$ is a nonnegative integer for all $s \in S$, and let A_k be the event that $X(s) \geq k$. Show that $\mathbb{E}(X) = \sum_{k=1}^{\infty} p(A_k)$.

Proof. Let p(X = s) = p(s). We expand the right-hand side:

$$\begin{aligned}\sum_{k=1}^{\infty} p(A_k) &= \sum_{s=1}^{\infty} p(s) + \sum_{s=2}^{\infty} p(s) + \cdots + \sum_{s=\infty}^{\infty} p(s) \\ &= \sum_{s=1}^{\infty} sp(s) = \sum_{s=1}^{\infty} sp(X = s)\end{aligned}$$

as desired. □

Mt2prac 8: Suppose everyone in a class of 260 students is born on a uniformly random day of the year chosen from $\{1, \ldots, 365\}$. Let us call an (unordered) pair of students *magical* if they have the same birthday. What is the expected number of magical pairs in the class? (*hint:* linearity of expectation)

Let $X(c)$ be the random variable that assigns classroom to an integer denoting the number of magical pairs it has. Let

$$S = \{c = \{\text{pairs of students in the classroom}\}\}$$

For each pair, assign a random variable:

$$X(c)_i = \begin{cases} 1 \text{ if the students have the same birthday} \\ 0 \text{ if the students have different birthdays} \end{cases}$$

for $1 \leq i \leq \binom{260}{2}$. Observe that

$$X(c) = \sum_{i=1}^{\binom{260}{2}} X(c)_i$$

Each $\mathbb{E}(X(c)_i)$ is therefore $1/365$. Therefore,

$$\mathbb{E}(X(s)) = \frac{\binom{260}{2}}{365}$$

Finalprac 7a: Suppose I randomly throw 10 balls into 4 bins (where each ball is thrown independently and uniformly into one of the bins). (a) What is the expected number of empty bins?

Let there be 4 bins labelled 1, 2, 3, 4. We assign each bin a random variable:

$$X_i = \begin{cases} 1 \text{ if empty} \\ 0 \text{ if otherwise} \end{cases}$$

We find the probability that bin i is empty. There are $\binom{13}{3}$ ways to throw the balls by stars and bars. Then by stars and bars, there are $\binom{12}{2}$ ways after we remove bin 1. Observe that $\mathbb{E}(X_i) = \frac{\binom{12}{2}}{\binom{13}{3}} = \frac{3}{13}$. Therefore, $\mathbb{E}(X_i) = \frac{3}{13}$. So the expected number of empty bins is $\frac{12}{13}$.