Math55Hw15

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7.4: 12, 16, 31, 32, 34x, 37x

Exercise 12: Suppose that we roll a fair die until a 6 comes up.

a) What is the probability that we roll the die n times?

The probability of rolling the die n times is the probability of getting anything but a 6 in the first $n-1$ rolls, then a 6 in the n-th roll which is

$$p(n) = \frac{1}{6}\left(\frac{5}{6}\right)^{(n-1)}$$

b) What is the expected number of times we roll the die?

Let $X : S \mapsto \mathbb{R}$ be the random variable that gives the number of rolls for an output.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{n=1}^{\infty} np(X = n) \\
&= \sum_{n=1}^{\infty} n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{(n-1)} \\
&= \frac{1}{6}\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{(n-1)}
\end{aligned}$$

Use generating functions:

$$\begin{aligned}
S &= 1 + 2x + 3x^2 + 4x^3 + \dots \\
xS &= 0 + x + 2x^2 + 3x^3 + \dots \\
S - xS &= 1 + x + x^2 + x^3 + \dots \\
x(S - xS) &= 0 + x + x^2 + \dots \\
(S - xS) - x(S - xS) &= 1 \\
(S - xS)(1 - x) &= 1 \\
S &= \frac{1}{(1-x)^2}
\end{aligned}$$

We can now solve for $\mathbb{E}(X)$:

$$\begin{aligned}
\mathbb{E}(X) &= \frac{1}{6} \cdot \frac{1}{(1 - {}^5\!/_6)^2} \\
&= \frac{1}{6} \cdot \frac{1}{{}^1\!/_{36}} \\
&= \frac{1}{6} \cdot 36 \\
&= 6
\end{aligned}$$

Exercise 16: Let X and Y be the random variables that count the number of heads and the number of tails that come up when two fair coins are flipped. Show that X and Y are not independent.

We have a sample space with possibilities that $X = 1$, $X = 2$, $Y = 1$, and $Y = 2$. We need to see if these events are independent. Upon observation, if $X = 1$, then $Y = 1$ since if only one head is obtained from two coin flips, the other must be tails. That means that $X = 1$ and $Y = 2$ event sets are disjoint. So $p(X = 1|Y = 2) = 0 \neq p(X = 1)$. The random variables are dependent.

Exercise 31: Let $A(X) = \mathbb{E}(|X - \mathbb{E}(X)|)$, the expected value of the absolute value of the deviation of X, where X is a random variable. Prove or disprove that $A(X + Y) = A(X) + A(Y)$ for all random variables X and Y.

Proof. We know that the statement is true for the variance:

$$\mathbb{E}((X - \mathbb{E}(X))^2)$$

when X and Y are independent. Therefore,

$$\mathbb{V}(X + Y) = \mathbb{V}(X) + \mathbb{V}(Y)$$

But observe what happens when we get the standard deviation from the variance:

$$\begin{aligned}
A(X + Y) = \sqrt{\mathbb{V}(X + Y)} = &\sqrt{\mathbb{V}(X) + \mathbb{V}(Y)} \neq \\
&\sqrt{\mathbb{V}(X)} + \sqrt{\mathbb{V}(Y)} = A(X) + A(Y)
\end{aligned}$$

So we have found a counter example. $\square$

Exercise 32: Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances when the random variables are not independent.

Let there be two coin flips and have X be the number of heads while Y be the number of tails of a string of length 2. Calculate their variances:

$$\begin{aligned}\mathbb{E}((X-\mathbb{E}(X))^2) &= \mathbb{E}((X-1)^2)\\ &= \mathbb{E}(X^2)+2\mathbb{E}(X)+1\\ &= \mathbb{E}(X^2)+3\end{aligned}$$

We let $X = X_1 + X_2$, where

$$X_i = \begin{cases}1 \text{ if H}\\ 0 \text{ if T}\end{cases}$$
$$\mathbb{E}(X_i) = 1\cdot\frac{1}{2} = \frac{1}{2}$$

Note that $X_i^2 = X_i$. Substituting, we get

$$\begin{aligned}\mathbb{E}((X-\mathbb{E}(X))^2) = \mathbb{E}(X^2)+3\\ \mathbb{E}(X_1^2+2X_1X_2+X_2^2)+3\\ 2\mathbb{E}(X_1X_2)+4\end{aligned}$$

We now see if X_1 and X_2 are independent:

$$\begin{aligned}p(X_1=1\cap X_2=1) = .25 = p(X_1=1)p(X_2=1)\\ p(X_1=0\cap X_2=1) = .25 = p(X_1=0)p(X_2=1)\\ p(X_1=1\cap X_2=0) = .25 = p(X_1=1)p(X_2=0)\\ p(X_1=0\cap X_2=0) = .25 = p(X_1=0)p(X_2=0)\end{aligned}$$

So we get

$$\begin{aligned}\mathbb{E}((X-\mathbb{E}(X))^2) &= 2\mathbb{E}(X_1X_2)+4\\ &= 2\mathbb{E}(X_1)\mathbb{E}(X_2)+4\\ &= \frac{1}{2}+4 = \frac{9}{2}\end{aligned}$$

The same should be for $\mathbb{E}((Y-\mathbb{E}(Y))^2)$. So the sum of variances is 9. We define a new random variable $Z = X + Y$ where

$$Z = 2$$

Since the number of heads plus tails is just the number of coin flips. We compute the variance of that:

$$\begin{aligned}\mathbb{E}((Z-\mathbb{E}(Z))^2) &= \mathbb{E}((2-2)^2)\\ &= 0\end{aligned}$$

Since $0 \neq 9$, we have shown that variance of the sum of two random variables is not always equal to the sum of their variances.

Exercise 34x: Prove the general case of Theorem 7. That is, show that if $X_1, X_2, \ldots, X_n$ are pairwise independent random variables on the sample space X, where n is a positive integer, then $V(X_1+X_2+\cdots+X_n) = V(X_1)+V(X_2)+\cdots+V(X_n)$.

Proof. We first start with two variables: X_1, X_2:

$$\begin{aligned}\mathbb{E}((X_1+X_2-\mathbb{E}(X_1+X_2))^2) &= \mathbb{E}((X_1-\mathbb{E}(X_1)+X_2-\mathbb{E}(X_2))^2)\\ &= \mathbb{E}((X_1-\mathbb{E}(X_1))^2+2(X_1-\mathbb{E}(X_1))(Y_1-\mathbb{E}(Y_1))+(Y_1-\mathbb{E}(Y_1))^2)\\ &= \mathbb{V}(X)+\mathbb{V}(Y)+2\mathbb{E}(X_1-\mathbb{E}(X_1))(Y_1-\mathbb{E}(Y_1))\end{aligned}$$

By the fact that

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

We get

$$\mathbb{E}((X_1+X_2-\mathbb{E}(X_1+X_2))^2) = \mathbb{V}(X)+\mathbb{V}(Y)$$

In general, we let $W(X_i) = X_i - \mathbb{E}(X_i)$ so that

$$\begin{aligned}\mathbb{E}((X_1+\ldots+X_n-\mathbb{E}(X_1+\ldots+X_n))^2) &= \mathbb{E}\left(\sum_{i=1}^{n} W(X_i)\right)^2\\ &= \sum_{i=1}^{n}\mathbb{V}(X_i) + 2\sum_{1\leq i<j\leq n}\mathbb{E}(W(X_i)W(X_j))\end{aligned}$$

But observe that

$$\begin{aligned}\mathbb{E}(W(X_i)W(X_j)) &= \mathbb{E}(X_i-\mathbb{E}(X_i))(X_j-\mathbb{E}(X_j))\\ &= \mathbb{E}(X_iX_j - X_i\mathbb{E}(X_j) - X_j\mathbb{E}(X_i) + \mathbb{E}(X_i)\mathbb{E}(X_j))\\ &= 2\mathbb{E}(X_i)\mathbb{E}(X_j) - 2\mathbb{E}(X_i)\mathbb{E}(X_j) = 0\end{aligned}$$

So we get

$$\mathbb{E}((X_1+\ldots+X_n-\mathbb{E}(X_1+\ldots+X_n))^2) = \sum_{i=1}^{n}\mathbb{V}(X_i)$$

□

Exercise 37x: Let X be a random variable on a sample space S such that $X(s) \leq 0$ for all $s \in S$. Show that $p(X(x) \geq a) \leq \mathbb{E}(X)/a$ for every positive real number a. This inequality is called Markov's inequality.

Proof. Use the fact that

$$\begin{aligned}\mathbb{E}(X) = \sum_{n=1}^{\infty} np(X = n) &\geq \sum_{n=a}^{\infty} ap(X = n)\\ &= a\sum_{n=a}^{\infty} p(X = n)\\ &= ap(X \geq a)\end{aligned}$$

□

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Exercise 28: What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?

Let X be the number of times that a 6 appears when a die is rolled 10 times. Let

$$X_i = \begin{cases} 1 \text{ if a 6 is rolled on } i\text{-th dice} \\ 0 \text{ if otherwise} \end{cases}$$

Then

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10})\\ &= \frac{1}{6}(10) = \frac{10}{6} = \frac{5}{3}\end{aligned}$$

Therefore,

$$V(X) = \mathbb{E}(X^2) - \mathbb{E}(X))^2 = \left(\frac{5}{3}\right)^2 - \frac{5}{3} = \frac{1-}{3}$$

Exercise 36: Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.6 is tossed n times deviates from the mean by more than $\sqrt{n}$.

We want to find $\mathbb{P}(|X - \mathbb{E}(X))| \geq \sqrt{n}) \leq \dfrac{V(X)}{n}$. Observe that this is a binomial distribution with n flips and bias 0.6. Therefore, the variance is $n(.6)(.4) = .24n$. Now plug it into the formula:

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{.24}{n}$$

Exercise 38: Suppose that the number of cans of soda pop filled in a day at a bottling plant is a random variable with an expected value of 10,000 and a variance of 1000.

a) Use Markov's inequality (Exercise 37) to obtain an upper bound on the probability that the plant will fill more than 11,000 cans on a particular day.

Markov's Inequality is

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}$$

We let $t = 110000$:

$$\mathbb{P}(X \geq 11000) \leq \frac{10000}{11000} = \frac{10}{11}$$

b) Use Chebyshev's inequality to obtain a lower bound on the probability that the plant will fill between 9000 and 11,000 cans on a particular day.

Since 9000 and 11000 both differ from the mean by 1000, by Chebyshev's inequality:

$$\begin{aligned} \mathbb{P}(|X - \mathbb{E}(X)| \geq 1000) &\leq \frac{1000}{1000^2} = \frac{1}{1000} \\ -\mathbb{P}(|X - \mathbb{E}(X)| \geq 1000) &\geq -\frac{1}{1000} \\ 1 - \mathbb{P}(|X - \mathbb{E}(X)| \geq 1000) &\geq \frac{999}{1000} \end{aligned}$$

We have the lower bound of .999.

Exercise 43: What is the variance of the number of **fixed elements**, that is, elements left in the same position, of a randomly selected permutation of n elements?

Define X to be the number of fixed elements in an element s from the sample space S. Let

$$X_i = \begin{cases} 1 \text{ if the } i\text{-th element is unchanged after the permutation} \\ 0 \text{ if otherwise} \end{cases}$$

Observe that $X = \sum_{i=1}^{n} X_i$. Therefore, $\mathbb{E}(X) = \mathbb{E}(\sum_{i=1}^{n} X_i)$. Calculate $\mathbb{E}(X_i)$:

$$\mathbb{E}(X_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

Therefore, $\mathbb{E}(X) = 1$. We next find $\mathbb{E}(X^2)$:

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(X_1^2 + \cdots + X_n^2 + 2\sum_{1 \le i < j \le n} X_i X_j) \\
&= 1 + 2\binom{n}{2}\left(\frac{(n-2)!}{n!}\right) \\
&= 1 + 2\left(\frac{n!}{2!(n-2)!}\right)\left(\frac{(n-2)!}{n!}\right) \\
&= 1 + 2\left(\frac{1}{2}\right) \\
&= 2
\end{aligned}$$

To calculate variance, we take

$$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2 - 1^2 = 1$$

Mt2prac-2016 7: True or False: If X and Y are random variables, then $V(X+Y) \ge V(X) + V(Y)$, where V is the variance.

By the formula,

$$\begin{aligned}
V(X+Y) &= \mathbb{E}((X + Y - \mathbb{E}(X+Y))^2) \\
&= \mathbb{E}((X - \mathbb{E}(X) + Y - \mathbb{E}(Y))^2) \\
&= V(X) + V(Y) + 2\mathbb{E}(X - \mathbb{E}(X))(Y - \mathbb{E}(Y)) \\
&= V(X) + V(Y) + 2(\mathbb{E}(XY) - 2\mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y)) \\
&= V(X) + V(Y) + 2(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y))
\end{aligned}$$

The equation is not true when $\mathbb{E}(X)\mathbb{E}(Y) > \mathbb{E}(XY)$. Suppose we flip a coin. Let

$$X = \begin{cases} 1 \text{ if heads} \\ -1 \text{ if tails} \end{cases}$$

$$Y = \begin{cases} 1 \text{ if tails} \\ -1 \text{ if heads} \end{cases}$$

Observe that

$$XY = \begin{cases} -1 \text{ if heads} \\ -1 \text{ if tails} \end{cases}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(X), \mathbb{E}(Y) &= 0 \\
\mathbb{E}(XY) &= -1
\end{aligned}$$