Math104Hw1

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Exercise 1: Prove that $\forall a, b, c \in \mathbb{R}$, $|a + b + c| \leqslant |a| + |b| + |c|$. (Hint: Use the triangle inequality twice).

Proof. We have by triangle inequality:

$$|(a + b) + c| \leqslant |a + b| + |c|$$

and by triangle inequality again,

$$|a + b| + |c| \leqslant |a| + |b| + |c|$$

□

Exercise 2: $\forall a, b \in \mathbb{R}$ and $c > 0$, prove that

$$|a - b| \leqslant c \iff b - c \leqslant a \leqslant b + c$$

Proof. ($\rightarrow$) We note that there are two cases.

- $a - b \geqslant 0$. Then $|a - b| = a - b$ and so we get

  $$a - b \leqslant c$$

  So we get

  $$a \leqslant b + c$$

- $a - b < 0$. Then we have that $|a - b| = b - a$ and so

  $$b - a \leqslant c$$

  or in other words,

  $$b - c \leqslant a$$

If we consider both cases, we have that:

$$b - c \leqslant a \leqslant b + c$$

which is possible since $c > 0$. □

Exercise 3:

- Construct a set $S_1 \subseteq \mathbb{R}$ so that $\sup S_1$ exists and $\sup S_1 \notin S_1$:

  Proof. Take the set $S_1 = \{a \in \mathbb{R} : 0 < a < 1\}$. We see that an upper bound is 1 since $1 > s$ if $s \in S_1$. We will prove that an upper bound less than 1 does not exist. Suppose that s_u is an upper bound such that $s_u < 1$. Then we have two cases:

- $s_u \leqslant 0$. Then we notice that clearly, $.5 \in S_1$ but $s_u < .5$ so this is not an upper bound.
- $s_u > 0$. Then $s_u \in S_1$. But then there is always a rational number between any two real numbers, which was proved in class. This means that there is a $q \in \mathbb{Q}$ such that $s_u < q < 1$. Then $q \in S_1$ but $q > s_u$. Therefore, s_u is not an upper bound. Therefore, 1 is the supremum and $1 \notin S_1$.

So our construction S_1 satisfies the qualities. □

- Construct a set $S_2 \subseteq \mathbb{R}$ so that S_2 is bounded above but not bounded below.

Proof. Take the set $S = \{a \in \mathbb{R} : a < 0\}$. This is bounded above by 0 because for an arbitrary $s \in S$, then we have $s < 0$ and therefore, $s < 0$. So 0 is an upper bound. Now we show that it is not bounded below. Suppose it is bounded below by $s_b < 0$. Then we note that $s_b - 1 \in \mathbb{R}$ and $s_b - 1 < s_b$. So s_b is not a lower bound. Contradiction. So our construction S satisfies the requirements. □

Exercise 4: Assume that $S, T \subseteq \mathbb{R}$ be two non-empty bounded sets. Prove that:

- $S \cup T$ is a bounded set

Proof. (Lower Bound) Since S, T are bounded, we have s_b and t_b are lower bounds of S, T respectively. Then we take $l = \min(s_b, t_b)$. We claim that this is the lower bound of $S \cup T$. Suppose that $x \in S \cup T$. Then $x \in S$ or $x \in T$. In either case, we see that $l < x$ because l is a lower bound of S and T.

(Upper Bound) The upper bound is the same case, but for completion, say that s_u, t_u are upper bounds of S, T respectively. Then we take $u = \max(s_u, t_u)$. We see that for an arbitrary $x \in S \cup T$, $x \in S$ or $x \in T$. Since u is the upper bound of both S and T, we have that $u > x$. □

- $\inf(S \cup T) = \min(\inf S, \inf T)$

Proof. Suppose that $i_t = \inf T$ and $i_s = \inf S$. Let $i = \min(i_s, i_t)$. Wlog, suppose that $i = i_t$. Then that means that $i_t \leqslant i_s$. Suppose that $x \in S \cup T$. Then $x \in S$ or $x \in T$. We have three cases:

- $x \in S$. So
$$i = i_t \leqslant i_s \leqslant x$$
so we are done
- $x \in T$. So
$$i = i_t \leqslant x$$
so we are done
- $x \in S \wedge x \in T$. We reduce this to the previous two cases.

We can conclude that $i \leqslant x$ for all $x \in S \cup T$. So $i = \inf S \cup T$. □

Exercise 5: Let $S = \{r \in \mathbb{Q} : \sqrt{2} \geqslant r \geqslant 0\}$. Show that $\sup S = \sqrt{2}$.

Proof. Clearly, by definition of our set S, $\sqrt{2}$ is an upper bound. Suppose for contradiction there exists an upper bound $s_u < \sqrt{2}$. Then we are reduced to two cases:

- $s_u < 0$. Then we see that $0 \in S$ but $0 > s_u$. This means that s_u is not an upper bound.

- $s_u \geqslant 0$. Then we note that $s_u \in S$. But by the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that there is a $q \in \mathbb{Q}$ such that
$$s_u < q < \sqrt{2}$$
But since $q < \sqrt{2}$, $q \in S$ but $q > s_u$. Therefore, s_u is not an upper bound.

Since s_u is not an upper bound in either case, we conclude that there is not upper bound less than $\sqrt{2}$. So $\sup(S) = \sqrt{2}$. □

Exercise 6: Decide whether the following sequences converge or diverge, find the limit if it converges. No proof required.

- $s_n = \sqrt{n}, n \in \mathbb{N}$;

  Answer. Diverges. We need to show that there is an $\varepsilon > 0$ such that for any N there is an $n > N$ where it is not true that
$$|\sqrt{n} - L| < \varepsilon$$
  Let $\varepsilon = \frac{1}{2}$. We have two cases:

  – $\sqrt{n} - L \geqslant 0$. Then we have $\sqrt{n} - L < \frac{1}{2}$ or $\sqrt{n} < \varepsilon + L$ or $n < (\frac{1}{2} + L)^2$. But this means that n has an upper bound. This means that there is no N such that all $n > N$ works.

  – $\sqrt{n} - L < 0$. Then we have $L - \sqrt{n} < \frac{1}{2}$ or $n > (L - \frac{1}{2})^2$. But it does not matter because first case failed. We note that the first case is possible because we can choose $n \geqslant L^2$

  So there is an upper bound on our interval of convergence, and therefore, the series diverges.

- $a_n = \sin n\pi, n \in \mathbb{N}$;

  Answer. Diverges. We need to show that there is an $\varepsilon > 0$ such that for any N there is an $n > N$ where it is not true that
$$|a_n - L| < \varepsilon$$
  We choose ε to be $\frac{1}{10}$. Suppose that for contradiction we have an N such that for all $n > N$,
$$|\sin n\pi - L| < \frac{1}{10}$$
  So
$$\frac{-1}{10} < \sin n\pi - L < \frac{1}{10} \implies \frac{-1}{10} + L < \sin n\pi < \frac{1}{10} + L$$
  We notice that $-1 \leqslant \sin n\pi \leqslant 1$, so we have conditions on L:
$$\begin{aligned} \frac{1}{10} + L &> 1 \\ L &> \frac{9}{10} \\ \frac{-1}{10} + L &< -1 \\ L &< \frac{-9}{10} \end{aligned}$$
  But that is a contradiction. So the limit does not exist.

- $b_n = \cos n\pi, n \in \mathbb{N}$;

Answer. Diverges. We need to show that there is an $\varepsilon > 0$ such that for any N there is an $n > N$ where it is not true that

$$|b_n - L| < \varepsilon$$

We choose ε to be $\frac{1}{10}$. Suppose that for contradiction we have an N such that for all $n > N$,

$$|\cos n\pi - L| < \frac{1}{10}$$

So

$$\frac{-1}{10} < \cos n\pi - L < \frac{1}{10} \implies \frac{-1}{10} + L < \cos n\pi < \frac{1}{10} + L$$

We notice that $-1 \leqslant \cos n\pi \leqslant 1$, so we have conditions on L:

$$\begin{aligned} \frac{1}{10} + L &> 1 \\ L &> \frac{9}{10} \\ \frac{-1}{10} + L &< -1 \\ L &< \frac{-9}{10} \end{aligned}$$

But that is a contradiction. So the limit does not exist.

- $c(n) = \frac{10^{20}}{n}, n \in \mathbb{N}$.

  Answer. This converges. As $n \to \infty$, $10^2/n \to 0$. We want that $\forall \varepsilon > 0$, there should be an N such that $\forall n > N$,

  $$\left| \frac{10^{20}}{n} - 0 \right| < \varepsilon$$

  So we need

  $$\frac{10^{20}}{n} < \varepsilon$$

  or

  $$n > \frac{10^{20}}{\varepsilon}$$

  The limit is 0.