Math172Hw10

Trustin Nguyen

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Exercise 1: How many different labeled trees are there on $[n]$ that have no vertices of degree more than 2?

Proof. Because trees are connected, we know that each vertex has degree at least 1. We also know that a tree has $n-1$ edges, and therefore, the total degree is $2n-2$. Let k be the number of vertices with degree 2 and j be the number of vertices with degree 1. Then $k+j=n$ and $2k+j=2n-2$. Solve the system:

$$\begin{aligned} k+j &= n \\ 2k+j &= 2n-2 \end{aligned}$$

we get:

$$k = n-2,\ j = 2$$

So we construct our tree by the following algorithm:

(a) Choose an element of $I = [n]$ called k_1 as a leaf. Remove n_1 from I

(b) Pick an element of I to connect to k_1, called k_2, remove it from I, and repeat this process $n-2$ more times. We will get the path $k_1, k_2, \ldots, k_{n-1}$.

(c) Take the last element of I to append to the path. We will have $k_1, k_2, \ldots, k_n$ as the final path.

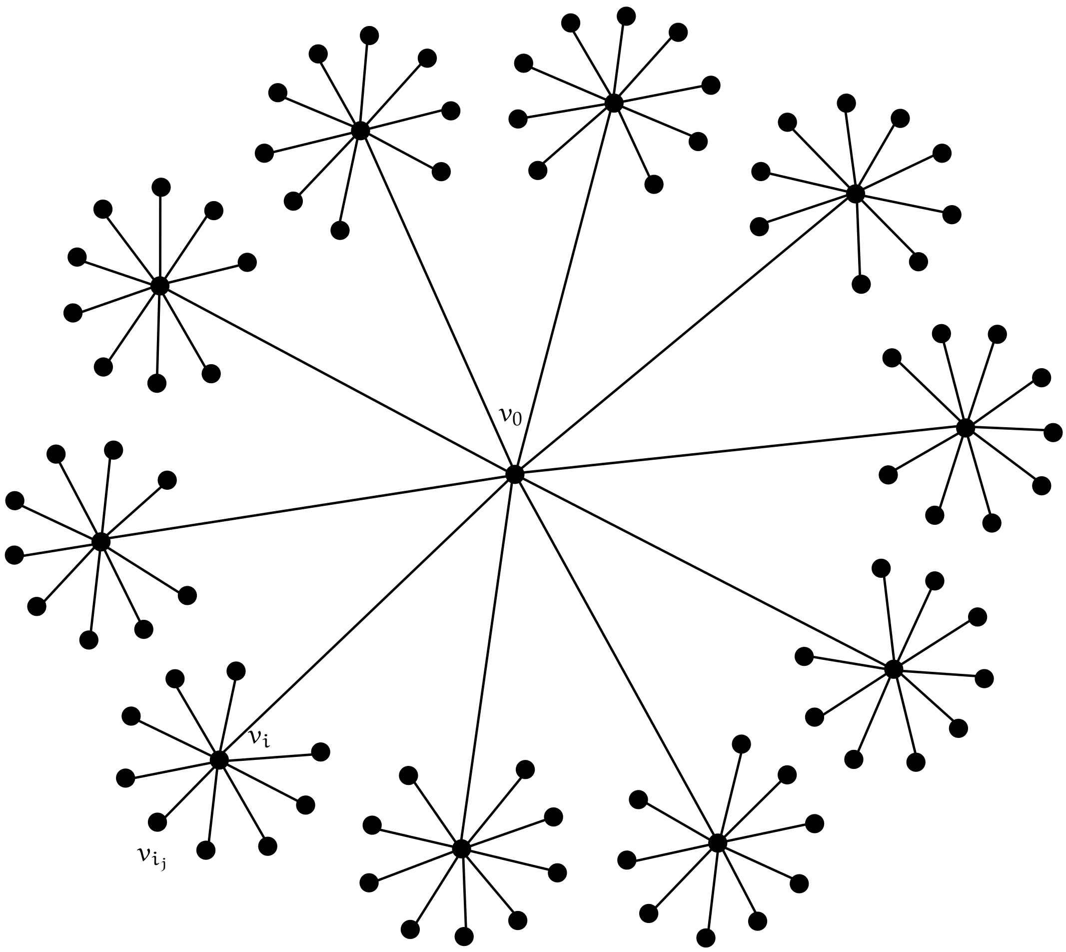
Such a path completely describes the tree, because each step is forced. If we first choose a leaf and the vertex connected to it, subsequent connections cannot reference a vertex more than twice, otherwise, that vertex will have degree $\geqslant 3$. Step 2 cannot terminate early, because our graph must be connected, so this means that our last or first vertex was not a leaf.

So there are $n!$ such constructions. But there is a bijection between a path and the path with the ordering reversed. So there are $n!/2$ such trees. For $n=1$, there is 1 such tree. □

Exercise 2: Let T be a tree with 101 vertices so that the largest degree in T is ten. Is it true that T must contain a path of length 5?

Proof. No it does not have to have a path of length 5. Let v_0 be the vertex with degree 10. Then v_0 is connected to $v_1', \ldots, v_{10}'$ such that the shortest path between $v_0 \to v_i'$ has length 1. Now let each v_i' have degree 10, where v_i' is connected to $v_0, v_{i_1}, \ldots, v_{i_9}$. Notice that this is a graph on 101 vertices because we have v_0 and for each v_i, there are 9 other vertices connected to it. So we have $|\{v_0\}| + |\{v_i\}| + |\{v_{i_j}\}| = 1 + 10 + 90 = 101$:

Example Graph



Let P be the longest path in our graph. Then its endpoints must be leaves, otherwise, there is a vertex and edge that we can append to the graph which shows that one of the endpoints was not a leaf, as it was actually a neighbor of two vertices. The only leaves in our graphs are of the form v_{i_j} for some i, j.

- If our path is between $v_{i_{j_1}}$ and $v_{i_{j_2}}$, then we must have the path $v_{i_{j_1}}, v_i, \ldots, v_i, v_{i_{j_2}}$. But notice that it must be the path $v_{i_{j_1}}, v_i, v_{i_{j_2}}$, otherwise, we will have created a cycle: $v_i, \ldots, v_i$. This path is of length less than 5

- If our path is between $v_{i_{k_1}}, v_{j_{k_2}}$ for $i \neq j$, then our path is of the form $v_{i_{k_1}}, v_i, \ldots, v_j, v_{j_{k_2}}$. Since no vertices in the path $v_i, \ldots, v_j$ are leaves, then v_0 connects them, so we have $v_i, v_0, \ldots, v_0, v_j$. Again, there are no vertices between the sub path $v_0, \ldots, v_0$ otherwise, we had a cycle. So our path is of the form $v_{i_{k_1}}, v_i, v_0, v_j, v_{j_{k_2}}$. This uses 4 edges which is less than 5.

Then the longest path has length $4 < 5$, which shows that there might not be a path of length 5. □

Exercise 3: Let G be a connected simple graph. Which of the following statements are true:

- If e is an edge of G, then there must be a spanning tree of G that contains e.

  Proof. True. We know that there exists a spanning tree $T = (V, E')$ of $G = (V, E)$ because G is connected. Suppose that $e \in G$ but $e \notin T$. Then we first note that

e connects two vertices v_1, v_2, and our spanning tree is connected, so there is a path from $v_1 \to v_2$, not using e. Suppose that that path is $(v_1, v_2', v_3', \ldots, v_n', v_2)$. Suppose we remove v_n', v_2 as an edge and add in e. Then our new graph is still connected, because any path that uses $\{v_n', v_2\}$, we can replace with $v_n', \ldots, v_1, e$. Suppose that removing the edge and adding in e created a cycle. Then we have a cycle, $c_1, c_2, \ldots, c_n, e$. But since traveling along e is the same as traveling along $v_1, v_2', v_3', \ldots, v_n', v_2$, we actually have that:

$$c_1, c_2, \ldots, c_n, v_1, v_2', v_3', \ldots, v_n', v_2$$

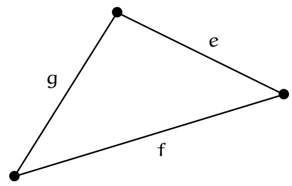
is a cycle. So T has a cycle, contradiction. Therefore, we have a way of getting e into a new spanning tree. □

- If e and f are edges of G, then there must be a spanning tree of G that contains e and f.

  Proof. This is true also. By the last proof we know that if e connects $v_1 \to v_2$ and f connects $v_3 \to v_4$, then we can take out the necessary edges and put in e, f to create a spanning subgraph. This process works even if f is in a path going from $v_1 \to v_2$ or if e is in the path $v_3 \to v_4$, there are no conflicts. By similar reasoning as above, we also get a tree also. □

- If e, f, g are edges of G, then there must be a spanning tree of G that contains e, f, g.

  Answer. This is false. Consider the graph:



  There is no spanning tree that contains all three edges, otherwise, we get the cycle, e, f, g.