Stat134Hw7

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Exercise 1: Suppose that $\mathcal{L}$ is a continuous random variable with the Laplace distribution of density

$$f(x) = \frac{1}{2}e^{-|x|},\ x \in \mathbb{R}$$

Find the moment generating function of $\mathcal{L}$ and use it to compute the variance of $\mathcal{L}$.

Answer. By definition:

$$M_{\mathcal{L}}(t) = \mathbb{E}e^{t\mathcal{L}} = \int_{-\infty}^{\infty} \frac{1}{2}e^{-|x|}e^{tx}\ dx$$

Then we have:

$$\int_{-\infty}^{0} \frac{1}{2}e^{-|x|+tx}\ dx + \int_{0}^{\infty} \frac{1}{2}e^{-|x|+tx}\ dx$$

On the negative values, $|x| = -x$ and for positive, $|x| = x$. So:

$$\int_{-\infty}^{0} \frac{1}{2}e^{x+tx}\ dx + \int_{0}^{\infty} \frac{1}{2}e^{-x+tx}\ dx$$

So we have:

$$\frac{1}{2}\left(\left(\frac{e^{x(t+1)}}{t+1}\right)\Big|_{-\infty}^{0} + \left(\frac{e^{x(t-1)}}{t-1}\right)\Big|_{0}^{\infty}\right)$$

When we evaluate the limit $\lim\limits_{x\to-\infty} e^{x(t+1)}$, we require that the exponent is negative, so that gives the restriction $-1 < t$. For the other summand, we require $t < 1$ for the same reason. So after evaluation:

$$\frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right) = \frac{1}{2}\left(\frac{t-1}{t^2-1} - \frac{t+1}{t^2-1}\right)$$

which is

$$\frac{1}{2}\left(\frac{-2}{t^2-1}\right) = -\frac{1}{t^2-1}$$

for $-1 < t < 1$.

Exercise 2: In this problem you are given MGF $M_X(t) = \mathbb{E}e^{tX}$ of a discrete random variable X and your task is to find the probability mass function of X:

(a) $M_X(t) = \frac{4}{7}\left(1 + \frac{1}{2}e^{-t} + \frac{1}{4}e^{-2t}\right)$.

Answer. Multiply out:

$$M_X(t) = \frac{4}{7}e^{(0)t} + \frac{2}{7}e^{(-1)t} + \frac{1}{7}e^{(-2)t}$$

This is the formula for the expectation, so we can recover the pmf:

$$\begin{aligned} p_X(0) &= \frac{4}{7} \\ p_X(-1) &= \frac{2}{7} \\ p_X(-2) &= \frac{1}{7} \end{aligned}$$

Since the MGF uniquely determines the distribution, this defines $p_X(x)$ fully.

(b) $M_X(t) = \frac{1}{9}\left(1 + e^t + e^{2t}\right)^2$.

Answer. Expand:

$$\begin{aligned} M_X(t) &= \frac{1}{9}(1 + e^t + e^{2t} + e^t + e^{2t} + e^{3t} + e^{2t} + e^{3t} + e^{4t}) \\ &= \frac{1}{9}(1 + 2e^t + 3e^{2t} + 2e^{3t} + e^{4t}) \\ &= \frac{1}{9}e^{0t} + \frac{2}{9}e^{1t} + \frac{1}{3}e^{2t} + \frac{2}{9}e^{3t} + \frac{1}{9}e^{4t} \end{aligned}$$

So again we can uniquely recover the pmf:

$$\begin{aligned} p_X(0) &= \frac{1}{9} \\ p_X(1) &= \frac{2}{9} \\ p_X(2) &= \frac{1}{3} \\ p_X(3) &= \frac{2}{9} \\ p_X(4) &= \frac{1}{9} \end{aligned}$$

(c) $M_X(t) = \frac{1}{2-e^{-t}}$

Answer. We see that:

$$M_X(t) = \frac{1}{1 - \frac{e^{-t}}{2}} = \sum_{n\geqslant 0}(e^{-t}/2)^n = \sum_{n\geqslant 0}\frac{1}{2^n}e^{-tn}$$

So we see that:

$$p_X(x) = \begin{cases} \frac{1}{2^x} & \text{if } x \in \mathbb{Z}_{\geqslant 0} \\ 0 & \text{if } otherwise \end{cases}$$

Exercise 3: Find the marginal distribution of random variable Y, if the joint pmf of (X, Y) is:

$X \setminus Y$	1	2	3
0	1/21	2/21	1/7
1	4/21	5/21	2/7

which are values of $P(X = k, Y = m)$.

Answer. The marginal distribution is would be fixing each Y value and varying over probabilities of X:

$$P_Y(1) = \frac{1}{21} + \frac{4}{21}$$
$$P_Y(2) = \frac{2}{21} + \frac{5}{21}$$
$$P_Y(3) = \frac{1}{7} + \frac{2}{7}$$

so

$$P_Y(1) = \frac{5}{21}$$
$$P_Y(2) = \frac{1}{3}$$
$$P_Y(3) = \frac{3}{7}$$

Exercise 4: Recall that the addition mod 2 is defined by the rules:

$$0+0 \equiv 0 \pmod 2,\ 0+1 \equiv 1 \pmod 2,\ 1+0 \equiv 1 \pmod 2,\ 1+1 \equiv 0 \pmod 2$$

let X and Y be two independent Bernoulli random variables with $P(X=0) = P(X=1) = P(Y=0) = P(Y=1) = \frac{1}{2}$, and let $Z = X + Y \pmod 2$.

(a) Compute the joint probability mass function of X, Y, Z.

Answer. To calculate the joint pmf, we first calculate the pmf of Z:

$$P_Z(0) = P(X=0, Y=0) + P(X=1, Y=1) = \frac{1}{2}$$
$$P_Z(1) = P(X=1, Y=0) + P(X=0, Y=1) = \frac{1}{2}$$

We immediately see that:

$$P(X=0, Y=1, Z=0) = 0$$
$$P(X=1, Y=0, Z=0) = 0$$
$$P(X=0, Y=0, Z=1) = 0$$
$$P(X=1, Y=1, Z=1) = 0$$

What is left are the four cases:

$$P(X=0, Y=1, Z=1) = ?$$
$$P(X=1, Y=0, Z=1) = ?$$
$$P(X=0, Y=0, Z=0) = ?$$
$$P(X=1, Y=1, Z=0) = ?$$

But we know that the probability of $Z = 0$ or 1 is 1 given that X, Y are a certain value. So Z is completely dependent on X, Y, we can simplify:

$$P(X=0, Y=1, Z=1) = P(X=0, Y=1) = 1/4$$
$$P(X=1, Y=0, Z=1) = P(X=1, Y=0) = 1/4$$
$$P(X=0, Y=0, Z=0) = P(X=0, Y=0) = 1/4$$
$$P(X=1, Y=1, Z=0) = P(X=1, Y=1) = 1/4$$

(b) Show that (Y, Z) are independent, but (X, Y, Z) are not independent.

Answer. We see that (X, Y, Z) is not independent because

$$P(X=0, Y=1, Z=0) = 0 \neq P(X=0)P(Y=1)P(Z=0) = 1/8$$

To get (Y, Z), we fix instances of Y, Z and add up the variation among X:

$$P(Y=0, Z=0) = P(X=0, Y=0, Z=0) + P(X=1, Y=0, Z=0)$$
$$P(Y=0, Z=1) = P(X=0, Y=0, Z=1) + P(X=1, Y=0, Z=1)$$
$$P(Y=1, Z=0) = P(X=0, Y=1, Z=0) + P(X=1, Y=1, Z=0)$$
$$P(Y=1, Z=1) = P(X=0, Y=1, Z=1) + P(X=1, Y=1, Z=1)$$

We see that all these sums are $1/4$ and they obey they product rule for independence:

$$P(Y=a)P(Z=b) = P(Y=a, Z=b)$$

Exercise 5: Sisters Anna and Mary bought a box with four individually wrapped chocolate truffles. On Monday and Tuesday, Anna was coming home very hungry. Each day she was flipping a fair coin. If it comes heads, Anna eats one chocolate truffle and replaces it with a fake plastic truffle in the same wrapping. On Wednesday Mary took two random truffles out of the four in the box. Let X be the total number of truffles Anna ate and let Y be the number of true (rather than fake plastic ones) truffles which Mary took.

(a) Find the joint probability mass function of X and Y.

Answer. Make a table:

$X \backslash Y$	0	1	2	sum
0	1/4	0	0	1/4
1	?	?	0	2/4
2	?	?	?	1/4

(b) Find $\mathbb{E}Y$.