Math143Hw6

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Exercise 1: Practice with topology

(a) Let X be an algebraic set and let $\mathcal{U} = \{X \backslash Z : Z \subseteq X \text{ is an algebraic subset}\}$. Check that $\mathcal{U}$ is a topology on X. This is called the *Zariski topology on* X.

Proof. We just need to check that the set of complements of elements of $\mathcal{U}$ define a topology based on closed sets. Let Z_i be algebraic sets. We know that

$$\bigcap_{i=1}^{\infty} Z_i$$

is also an algebraic set. We also have that

$$\bigcup_{i=1}^{n} \mathbb{Z}_i$$

form an algebraic set. So if C denotes the set of algebraic set, then C is a topology and therefore, the set of open sets, $\mathcal{U}$ is also a topology. □

(b) Let X and Y be algebraic sets and let $\varphi : X \to Y$ be a morphism. Consider X and Y with their Zariski topology. Prove that φ is a continuous map.

Proof. We know that φ is a continuous map if the preimage of a closed set is a closed set. Since for polynomial maps between algebraic sets, we know that the preimage of an algebraic set is an algebraic set, this says that the map is continuous. □

(c) Let $Z \subseteq \mathbb{A}^n \cong \mathbb{C}^n$ be an algebraic subset. Prove that $Z \subseteq \mathbb{C}^n$ is closed in the classical topology on $\mathbb{C}^n$.

Proof. We know that $Z = V(f_1, \dots, f_r)$. So Z is the collection of points that vanish at each f_i. Now consider the polynomial $f_1 \cdots f_r$. We will show that $\{0\}$ is closed in $\mathbb{C}^n$ later. If we have that $\{0\}$ is closed, we know that its preimage is a closed set in the classical topology, which is just Z.

($\{0\}$ is closed). Consider $\mathbb{C}^n \backslash \{(0, \dots, 0)\}$. Then for a point $p \in \mathbb{C}^n$, we can choose $\varepsilon < \|p\|$ which gives us:

$$\{r \in \mathbb{C}^n : \|p - r\| < \varepsilon\}$$

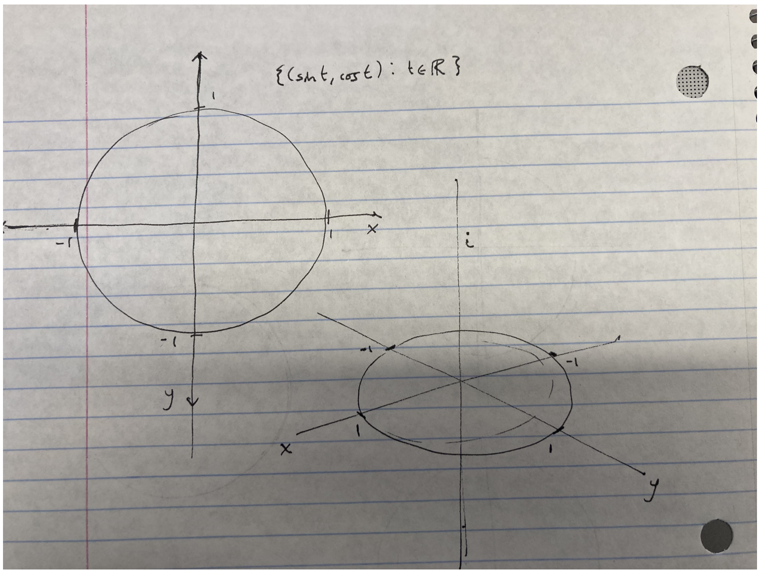
does not include 0, and therefore, is an open set. So $\{0\}$ is closed.

This generalizes to any field with a measure of distance, so in general, we have that if Z is an algebraic subset of $\mathbb{A}^n$, then Z is closed in the classical topology on $\mathbb{A}^n$. This also means that open sets in the Zariski topology are also open sets in the classical topology, and therefore, $\mathcal{U}$ as a Zariski topology is coarser than $\mathcal{V}$ as a classical topology because $\mathcal{U} \subseteq \mathcal{V}$ where $\mathcal{U}, \mathcal{V}$ are the sets of open sets on each topology. □

Exercise 2: Find the Zariski closures of the following subsets of $\mathbb{A}^n$. Draw a picture of the set (or at least its real points) and its closure. Justify your answer.

(a) $\{(\sin t, \cos t) : t \in \mathbb{R}\} \subseteq \mathbb{A}^2_{\mathbb{C}}$

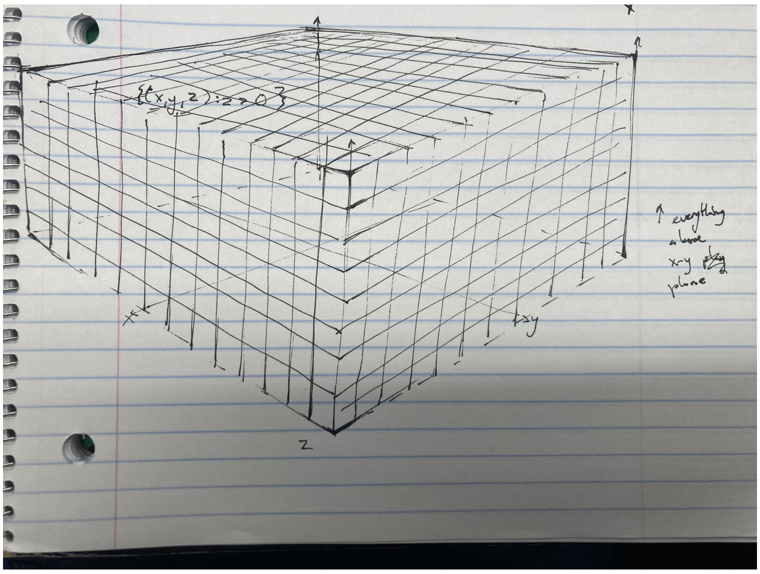
Proof. We know that $\{(\sin t, \cos t : t \in \mathbb{R})\} \subseteq V(x^2 + y^2 - 1)$. But $V(x^2 + y^2 - 1) \not\subseteq \{(\sin t, \cos t) : t \in \mathbb{R}\}$ as $(i, \sqrt{2}) \in V(x^2 + y^2 - 1)$. So I don't know. Here's the picture anyway:



□

(b) $\{(x, y, z) : z > 0\} \subseteq \mathbb{A}^3_{\mathbb{R}}$

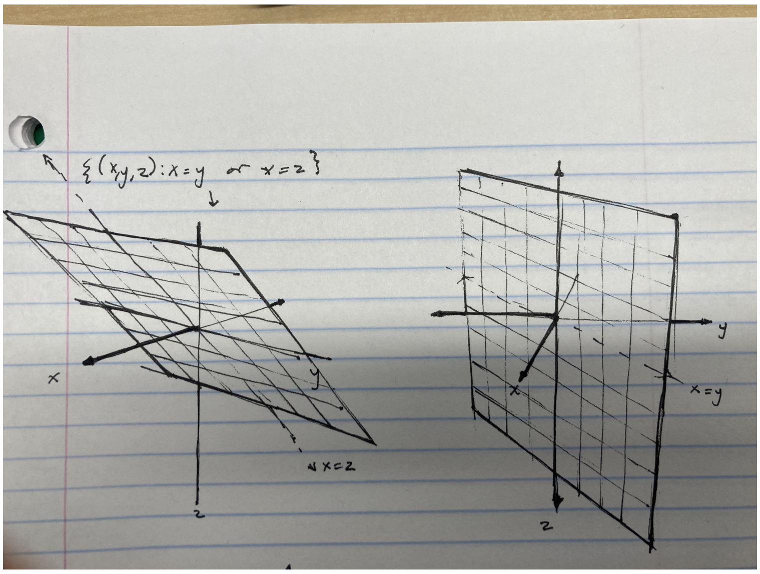
Proof. Suppose that $f \in I(\{(x, y, z) : z > 0\})$. If we look at $f(x, y, z)$ as a polynomial on one variable z, we conclude that $f = 0$ because $f \in V(\mathbb{A}^1)$. So now we look at f over $k[x, y, z]$. Since $f = 0$ over $k[x, y]$, then $f = 0$ over $k[x, y, z]$. So now we take $V(I(\{(x, y, z) : z > 0\})) = V(\{0\}) = \mathbb{A}^3_{\mathbb{R}}$. So the closure is the entire space.



□

(c) $\{(x, y, z) : x = y \text{ or } x = z\} \subseteq \mathbb{A}^3_{\mathbb{C}}$

Proof. This set is an algebraic set because it is equivalent to $V(((x-z)(x-y)))$. If $(x,x,z) \in \{(x,y,z) : x = y \text{ or } x = z\}$, then we have for $f \in ((x-y)(x-z))$, $f((x,x,z)) = g(x,y,z) \cdot (x-x)(x-z) = 0$. The same case applies if $(x,y,x) \in \{...\}$ or in the case where $x = y$ and $x = z$. And points that lie in $V(((x-z)(x-y)))$ are killed by $f(x,y,z) = (x-z)(x-y)$ at the very least. So $0 = (x-z)(x-y)$. We are over an integral domain, $\mathbb{C}$. So either one has to be 0. So $x = y$ or $x = z$ which shows the other inclusion. Since it is an algebraic set, it is equal to its closure.



□

Exercise 3: Let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be the morphism given by $(x,y) \mapsto (xy, y)$.

(a) Find the pullback map $\varphi : \Gamma(\mathbb{A}^2) \to \Gamma(\mathbb{A}^2)$.

Proof. The pullback is

$$\varphi^* : k[x,y] \to k[u,v]$$

where

$$x \mapsto uv$$
$$y \mapsto v$$

□

(b) What is the image of φ? Is φ dominant?

Proof. We have that

$$\mathfrak{I}\varphi = \{(xy,y) : x,y \in \mathbb{A}\}$$

To see if φ is dominant or not, we need to look at injectivity of φ^*. Suppose that

$$\varphi^*(f) = 0$$

for some $f \in k[x,y]$. Then we need to show that $f = 0 \in k[x,y]$. Suppose that $f \neq 0$. Then we have that the degree of $f \geqslant 1$. So now we make the substitution on all x in $f(x,y)$ as $x = uy$. We note that for a given term $cx^{k_1}y^{k_2}$ in f, we have:

$$cu^{k_1}y^{k_1+k_2}$$

and we find its inverse:

$$cu^{k_1}y^{k_1+k_2} + g(uy,y) = 0$$

The degree of u tells us that x has k_1 factors in $g(x, y)$. And that immediately tells us the degree of y. Finally, the coefficient and the fact that k is an integral domain says that the inverse is just the negative version of the polynomial. So this means that substitution of $x = uy$ preserves additive inverses and therefore, $f(uy, y)$ is non-zero. Therefore, this contradicts the fact that $f(uv, v) = 0$. So $f = 0$ and φ is dominant. □

(c) Although φ is continuous, prove that the image is neither closed nor open in the Zariski topology.

Proof. (Not Closed) We have that since φ is dominant,

$$I(\varphi(\mathbb{A}^2)) = I(\mathbb{A}^2) = 0$$

and now the vanishing:

$$V(I(\varphi(\mathbb{A}^2))) = \mathbb{A}^2$$

but $\varphi(\mathbb{A}^2) \neq \mathbb{A}^2$ because the mapping is not surjective because we do not have points $(p_1, 0)$ for non-zero $p_1 \in k$. This is because if $(a, b) \in \mathbb{A}^2$, then we require:

$$\varphi(x, y) = (a, b)$$

or

$$xy = a, y = b$$

and therefore,

$$x = \frac{a}{b}$$

which does not work when $y = b = 0$. So this tells us that $\varphi(\mathbb{A}^2)$ is not closed because it is not equal to its closure.

(Not Open) By the previous part, we see that the elements not in $\varphi(\mathbb{A}^2)$ are those of the form $(p_1, 0)$ for $p_1 \neq 0 \in k$. So now we just need to show that the complement is not closed. Let

$$Y = \varphi(\mathbb{A}^2)^c = \{(p, 0) : p \in k, p \neq 0\}$$

Now the closure is just $\mathbb{A}^1$ because $Y \subseteq \mathbb{A}^1$ and is an infinite subset. But we know that the only algebraic subsets of $\mathbb{A}^1$ are either finite or $\mathbb{A}^1$ itself. So we have

$$V(I(Y)) = \mathbb{A}^1 \neq Y$$

So the complement is not closed and therefore, $\varphi(\mathbb{A}^2)$ is not open. □

Exercise 4: A subset $A \subseteq X$ is called *dense* if the closure of A is all of X.

(a) Suppose $f \in \Gamma(X)$ is a polynomial function and $A \subseteq X$ is a dense subset. Prove that if $f(P) = 0$ for all $P \in A$, then $f = 0 \in \Gamma(X)$.

Proof. This means that $f \in I(A)$. Now we have:

$$V(I(A)) = X$$

which comes from the fact that A is dense and

$$I(A) \subseteq I(V(I(A))) = I(X)$$

But since $f \in I(A)$, we have $f \in I(X)$ and therefore, $f = 0 \in \Gamma(X)$. □

(b) Prove that $\varphi : X \to Y$ is dominant if and only if $\varphi(X)$ is dense in Y. (Recall that we defined φ to be dominant if $I(\varphi(X)) = I(Y)$).

Proof. ($\rightarrow$) If φ is dominant, we get

$$I(\varphi(X)) = I(Y)$$

and applying V to both sides:

$$V(I(\varphi(X))) = V(I(Y)) = Y \text{ since } Y \text{ is algebraic}$$

so this means that the closure of $\varphi(X)$ is Y. So $\varphi(X)$ is dense in Y.

($\leftarrow$) Suppose that $\varphi(X)$ is dense in Y. Then that means

$$V(I(\varphi(X))) = Y$$

We can also see that

$$I(\varphi(X)) \subseteq I(V(I(\varphi(X)))) = I(Y)$$

So we have proven one inclusion. Now we also have $\varphi(X) \subseteq Y$. By inclusion reversing,

$$I(\varphi(X)) \supseteq I(Y)$$

So $I(\varphi(X)) = I(Y)$ which tells us that $\varphi : X \to Y$ is dominant. □

(c) Suppose X is an algebraic set. Is the intersection of dense subsets always dense? Prove or give a counter example. (Do not assume that the subsets are open.)

Proof. Let $X = \mathbb{A}^1$ where $k = \mathbb{R}$. Now take the subsets $X_1 = \{x \in \mathbb{A}^1 : x > 0\}$ and $X_2 = \{x \in \mathbb{A}^1 : x < 0\}$. We notice that

$$V(I(X_1)) = X \text{ and } V(I(X_2)) = X$$

because the algebraic sets of $\mathbb{A}^1$ are either finite or the entire field. But we observe that $X_1 \cap X_2 = \emptyset$ which is an algebraic set of $\mathbb{R}$. These are the vanishing of the irreducible quadratics of $\mathbb{R}[x]$. So the closure is $\emptyset \neq \mathbb{A}^1$. □

(d) Prove that if X is an irreducible algebraic set and $U \subseteq X$ is a non-empty open subset, then U is dense.

Proof. Since U is open, we have $X - U$ is closed. Then:

$$X = U \cup X - U$$

So now we just apply the vanishing and ideal operators:

$$I(X) = I(U \cup X - U) = I(U) \cap I(X - U)$$

and

$$V(I(X)) = V(I(U) \cap I(X - U)) = V(I(U)) \cup V(I(X - U))$$

Now since $X - U$ is algebraic, we know that its closure is itself, and the same goes for X:

$$X = V(I(U)) \cup (X - U)$$

But since X is irreducible, it must be that either $X - U$ or $V(I(U))$ is a non-proper algebraic set. Since U is non-empty, $X - U$ is not all of X, so it must be that $V(I(U)) = X$. This means that U is dense in X. □

Exercise 5: Let $X \subseteq \mathbb{A}^n$ be an algebraic set, and let $f \in \Gamma(X)$ be a polynomial function. The *graph* of f is the set

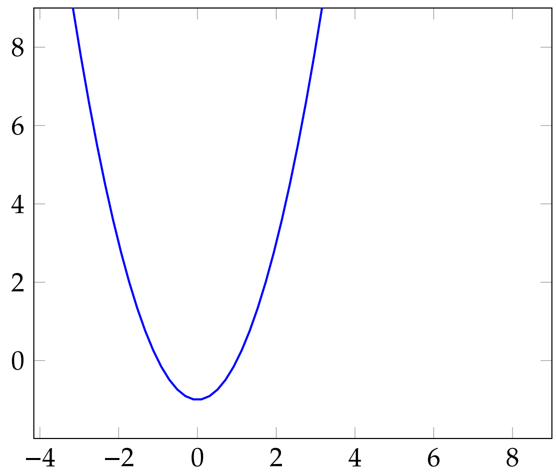
$$G(f) = \{(a_1, \dots, a_n, a_{n+1}) \in \mathbb{A}^{n+1} : (a_1, \dots, a_n) \in X \text{ and } a_{n+1} = f(a_1, \dots, a_n)\}.$$

(a) Let $X = \mathbb{A}^1$. Apply the definition above to describe the graph $x^2 - 1 \in \Gamma(\mathbb{A}^1) = k[x]$ and draw a picture.

Answer. By the definition, we have

$$G(x^2 - 1) = \{(a_1, a_1^2 - 1) \in \mathbb{A}^2\}$$

Graphing this:



(b) Prove that $G(f)$ is an algebraic set.

Proof. We will show that $G(f) = V((x^2 - 1 - y))$. Certainly, if $(a, a^2 - 1) \in G(f)$, and $g \in (x^2 - 1 - y)$,

$$g(x, y) = h(x, y)(x^2 - 1 - y)$$

so

$$g(a, a^2 - 1) = h(a, a^2 - 1)(a^2 - 1 - (a^2 - 1)) = 0$$

which means $(a, a^2 - 1) \in V((x^2 - 1 - y))$.

Now if $p = (x_0, y_0) \in V((x^2 - 1 - y))$, then we require:

$$x_0^2 - 1 - y_0 = 0$$

or

$$y_0 = x_0^2 - 1$$

therefore, $p = (x_0, x_0^2 - 1) \in G(f)$. So $G(f) = V(x^2 - 1 - y)$. □

(c) Prove that $G(f)$ is isomorphic to X.

Proof. We have a natural mapping from $\varphi : \mathbb{A}^1 \to G(f)$ given by

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to G(f) \\ \varphi(a) &= (\varphi_1(a), \varphi_2(a)) \\ \varphi_1(a) &= a \\ \varphi_2(a) &= a^2 - 1 \end{aligned}$$

This is a polynomial map because $\varphi_1, \varphi_2 \in k[x]$. Now we also have an inverse given by:

$$\begin{aligned} \varphi^{-1} : G(f) &\to \mathbb{A}^1 \\ \varphi^{-1}(a, b) &= a \end{aligned}$$

which is a polynomial map. Now we just have to check that $\varphi\varphi^{-1} = \mathtt{id}_{G(f)}$ and $\varphi^{-1}\varphi = \mathtt{id}_{\mathbb{A}^1}$.

($\varphi\varphi^{-1} = \mathtt{id}_{G(f)}$) We have:

$$\begin{aligned}\varphi\varphi^{-1}(a, a^2 - 1) &= \varphi(a)\\ &= (\varphi_1(a), \varphi_2(a))\\ &= (a, a^2 - 1)\end{aligned}$$

($\varphi^{-1}\varphi = \mathtt{id}_{\mathbb{A}^1}$) We have:

$$\begin{aligned}\varphi^{-1}\varphi(a) &= \varphi^{-1}((\varphi_1(a), \varphi_2(a)))\\ &= \varphi^{-1}((a, a^2 - 1))\\ &= a\end{aligned}$$

which concludes the proof. $\square$