Math55Hw11

Trustin Nguyen

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6.5: 32, 46, 50

Exercise 32: How many different strings can be made from the letters in $MISSISSIPPI$, using all the letters?

There are four boxes one for each letter. The M box has 1 element. The I box has 4 elements. The S box has 4 elements. The P box has 2 elements. There are 11 total letters. If we have a 11 tokens, each corresponding to a unique position, there are $\frac{11!}{1!4!4!2!}$ ways to put the tokens in the boxes.

Exercise 46: In how many ways can a dozen books be placed on four distinguishable shelves

a) if the books are indistinguishable copies of the same title?

There is a bijection to the number of bit strings of length 12 + 4 - 1, with exactly 3 1's. In this instance, the 1's represent separators that divide the zeroes into 4 groups. The count is $\binom{12+4-1}{4-1}$.

b) if no two books are the same and positions of the books on the shelves matter?

There is a bijection to the number of strings that can be made from the letters $ABCDEFGHIJKLIII$. Let the $I's$ be separators that serve the same role as in the previous problem and $A, B, C, D, E, F, G, H, I, J, K, L$ each represent a different book. By Exercise 32, this is just $\frac{12!}{3!}$.

Exercise 50: Prove Theorem 4 by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type i, $i = 1, 2, \ldots, k$, and the distribution of n objects into k boxes such that n_i objects are placed in box $i = 1, 2, \ldots, k$ and then applying Theorem 3.

Let $S = \{$strings of length n with n_i labeled objects of type i, $i = 1, 2, \ldots, k\}$

$T = \{$the distribution of n objects into k boxes such that n_i objects

are placed in box $i = 1, 2, \ldots, k$ $\}$

Define function $f : S \mapsto S'$ where f takes the string and gives each object a subscript corresponding to its location on the string from left to right. Observe that this is bijection. Since we assigned each object a position, we can define a bijective function $g : S' \mapsto S''$ that changes the position of each object of a string. Let g be the function that takes all objects of type 1 and places them in ascending subscript value at the beginning of the string, then objects of type 2 after type 1, ... all the way to type k. Let $h : S'' \mapsto S''$ which add dividers wherever there are two adjacent objects that are not of the same type. Now $j : S'' \mapsto T$ is bijective since each substring created by a divider represents a box and the subscripts of each object represents n_i elements chosen from $1, \ldots, n$ to be placed into box i.

8.1: 2a, 8, 12, 16x, 26, 32x

Exercise 2a:

a) Find a recurrence relation for the number of permutations of a set of n elements.

Let $f_i =$ the number of permutations of an i element set. We have

$$f_1 = 1$$
$$f_2 = 2$$
$$f_3 = 6$$

Suppose we wish to count f_n. Notice that each permutation begins with one of n elements. Now the next $n-1$ elements of the permutation are chosen from an $n-1$ element set. But this is just f_{n-1}. Since we broke it down into a series of choices, by the product rule, the recurrence is $f_n = nf_{n-1}$.

Exercise 8:

a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

Let $f_i =$ the number of bit strings of length i that contain three consecutive 0s. We have

$$f_1 = 0$$
$$f_2 = 0$$
$$f_3 = 1$$
$$f_4 = 3$$

We wish to count f_n. Notice that each bit string that contains three consecutive 0s can be partitioned into sets. Let

A be the set of bit strings that begin with 1

B be the set of bit strings that begin with 01

C be the set of bit strings that begin with 001

D be the set of bit strings that begin with 000

If n is a string of at least length 3, then the cardinality of A is just the strings of length $n-1$ that satisfy the property. We get $|A| = f_{n-1}$. By the same reasoning, $|B| = f_{n-2}, |C| = f_{n-3}$. Observe that for the set D, the remaining $n-3$ positions of the bit string can be whatever. So we get $|D| = 2^{n-3}$. Since the sets are disjoint and exhaustive, we add up $|A| + |B| + |C| + |D| = f_n = f_{n-1} + f_{n-2} + f_{n-3} + 2^{n-3}$.

b) What are the initial conditions?

The initial conditions are

$$
\begin{aligned}
f_1 &= 0 \\
f_2 &= 0 \\
f_3 &= 1 \\
f_4 &= 3
\end{aligned}
$$

c) How many bit strings of length 7 contain three consecutive 0s?

$f_5 = 3 + 1 + 0 + 2^2 = 8$

$f_6 = 8 + 3 + 1 + 2^3 = 20$

$f_7 = 20 + 8 + 3 + 2^4 = 47$

Exercise 12:

a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time?

The person can either take 1 stair on their first step leaving $n-1$ stairs left, two stairs on their first step leaving $n-2$ or three stairs $(n-3)$ on their first step. By that logic, if the number of ways to climb n stairs is f_n, then $f_n = f_{n-1} + f_{n-2} + f_{n-3}$

b) What are the initial conditions?

The initial conditions are

$$
\begin{aligned}
f_1 &= 1 \\
f_2 &= 2 \\
f_3 &= 4
\end{aligned}
$$

c) In how many ways can this person climb a flight of eight stairs.

$f_4 = 4 + 2 + 1 = 7$

$$
\begin{aligned}
f_5 &= 7 + 4 + 2 = 13 \\
f_6 &= 13 + 7 + 4 = 24 \\
f_7 &= 24 + 13 + 7 = 44 \\
f_8 &= 44 + 24 + 13 = 81
\end{aligned}
$$

Exercise 16x:

a) Find a recurrence relation for the number of ternary strings of length n that contain either two consecutive 0s or two consecutive 1s.

Let P be the property that the string contains either two consecutive 0s or two consecutive 1s. Let f_n be the number of strings of length n satisfying P. Let $\bar{P}$ be the negation of property P and $\bar{f}_n$ follow the same rule as f_n but satisfying $\bar{P}$.

We will count the complement. Let $\bar{f}_n(a)$ be the number of strings of length n that begin in an a which satisfies $\bar{P}$. Each string begins with either 0, 1, or 2.

If the string starts with 0, then we have $\bar{f}_{n-1}(1) + \bar{f}_{n-1}(2)$.

If the string starts with 1, then we have $\bar{f}_{n-1}(2) + \bar{f}_{n-1}(0)$.

If the string starts with 2, then we have $\bar{f}_{n-1}$.

Since this is exhaustive, we get:

$$\bar{f}_n = \bar{f}_{n-1}(0) + \bar{f}_{n-1}(1) + 2\bar{f}_{n-1}(2) + \bar{f}_{n-1}$$

But

$$\bar{f}_{n-1} = \bar{f}_{n-1}(0) + \bar{f}_{n-1}(1) + \bar{f}_{n-1}(2).$$

So the equation is

$$\bar{f}_n = 2\bar{f}_{n-1} + \bar{f}_{n-2}.$$

Since $3^n - f_n = \bar{f}_n$, we can rewrite our equation in terms of f_n:

$$
\begin{aligned}
3^n - f_n &= 2(3^{n-1} - f_{n-1}) + (3^{n-2} - f_{n-2}) \\
-f_n &= -2f_{n-1} + 3^{n-2} - f_{n-2} - 3^{n-1} \\
f_n &= 2f_{n-1} + f_{n-2} + 2(3^{n-2})
\end{aligned}
$$

This gives insight into another way to think of this: We can has strings that begin in 00, 11,

b) What are the initial conditions?

The initial conditions are

$$
\begin{aligned}
f_1 &= 0 \\
f_2 &= 2
\end{aligned}
$$

c) How many ternary strings of length six contain two consecutive 0s or two consecutive 1s?

$f_3 = 2(2) + 0 + 2(3^1) = 4 + 6 = 10$

$f_4 = 2(10) + 2 + 2(3^2) = 20 + 2 + 18 = 40$

$f_5 = 2(40) + 10 + 2(3^3) = 80 + 10 + 54 = 144$

$f_6 = 2(144) + 40 + 2(3^4) = 288 + 40 + 162 = 490$

Exercise 17:

a) Find a recurrence relation for the number of ternary strings of length n that do not contain consecutive symbols that are the same.

Let P be the property that the string has not consecutive symbols that are the same.

If we have a ternary string of length n, then it either starts with a 0, 1, or 2. Let their counts be $t_n(0), t_n(1), t_n(2)$ respectively. Observe that strings that start with 0 must have the remaining $n-1$ length string satisfy property P and start with either 1 or 2. So

$$t_n(0) = t_{n-1}(1) + t_{n-1}(2)$$
$$t_n(1) = t_{n-1}(2) + t_{n-1}(0)$$
$$t_n(2) = t_{n-1}(0) + t_{n-1}(1)$$

If t_n is the total number of strings that satisfy property P, then

$$t_n = 2t_{n-1}(0) + 2t_{n-1}(1) + 2t_{n-1}(2).$$

But

$$t_{n-1}(0) + t_{n-1}(1) + t_{n-1}(2) = t_{n-1}.$$

So our formula is

$$t_n = 2t_{n-1}$$

b) What are the initial conditions?

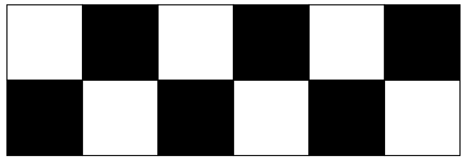
The initial condition is that $t_1 = 3$.

c) Wrong number :(

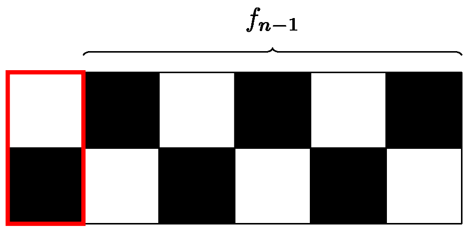
Exercise 26:

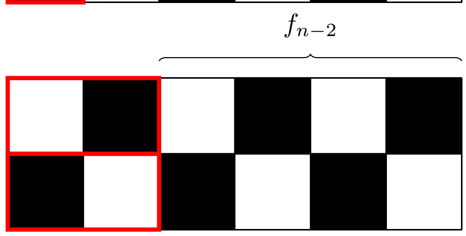
a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.

Suppose we had an $2 \times n$ checkerboard:



Observe that the first dominoes are fixed to two cases. Let f_n be the number of ways to tile a $2 \times n$ checkerboard with 1×2 dominoes:





By these two disjoint exhaustive cases, the $f_n = f_{n-1} + f_{n-2}$.

b) What are the initial conditions for the recurrence relations in part (a)?

The initial conditions are

$$f_1 = 1$$
$$f_2 = 2$$

c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

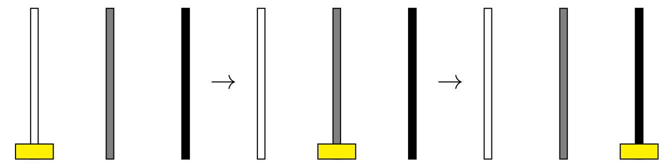
$f_3 = 3$	$f_4 = 5$	$f_5 = 8$
$f_6 = 13$	$f_7 = 21$	$f_8 = 34$
$f_8 = 55$	$f_9 = 89$	$f_{10} = 144$
$f_{11} = 233$	$f_{12} = 377$	$f_{13} = 610$
$f_{14} = 987$	$f_{15} = 1597$	$f_{16} = 2584$
$f_{17} = 4181$		

Exercise 32x: In the tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.

a) Find a recurrence relation for the number of moves required to solve the puzzle for n disks with this added restriction.

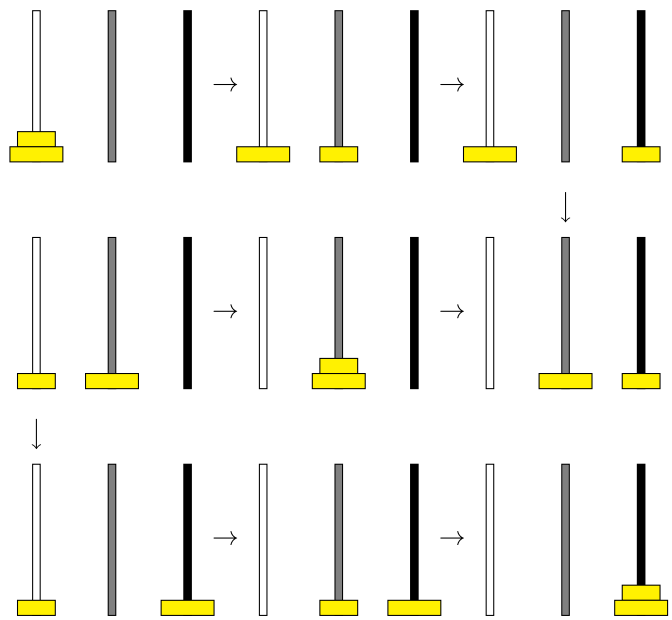
For $n = 1$, we need 2 moves to solve the puzzle:

$n = 1$



For $n = 2$, we need 8 moves:

$n = 2$



Observe that the process can be broken down into 5 key steps for any n:

(1) Solve the puzzle for $n - 1$ of the smallest disks.

(2) Move the bottom disk one pole to the right.

(3) Unsolve the puzzle for $n - 1$ of the smallest disks.

(4) Move the bottom disk one pole to the right.

(5) Resolve the puzzle for $n - 1$ of the smallest disks.

The formula we get if f_n is the number of moves needed to solve the Hanoi puzzle for n disks is $f_n = 3f_{n-1} + 2$.

b) Solve this recurrence relation to find a formula for the number of moves required to solve the puzzle for n disks.

Suppose $f_1 = a$. Let us list out f_2, f_3, f_4:

$$\begin{aligned}
f_1 &= a \\
f_2 &= 3a + 2 \\
f_3 &= 3(3a + 2) + 2 = 3^2 a + 3(2) + 2 \\
f_4 &= 3(3^2 a + 3(2) + 2) + 2 = 3^3 a + 3^2(2) + 3(2) + (2)
\end{aligned}$$

Observe that to obtain f_n, we multiply a by 3^{n-1} and compute the sum of increasing powers of 3 from $1, \ldots, k = n - 2$. Since $a = 2$, our formula is

$$f_n = 3^{n-1}(2) + 2 \sum_{i=0}^{n-2} 3^i$$

Or more simply:

$$f_n = 2 \sum_{k=0}^{n-1} 3^k$$

We can solve for $\sum_{k=0}^{n-1} 3^k$ by letting:

$$\begin{aligned}
S &= 1 + 3 + 3^2 + \cdots + 3^{n-1} \\
3S - S &= 3^n - 1 \\
2S &= 3^n - 1 \\
S &= \frac{3^n - 1}{2}
\end{aligned}$$

So

$$f_n = 3^n - 1$$

c) How many different arrangements are there of the n disks on three pegs so that no disk is on top of a smaller disk?

We break this down into a series of choices:

C_n We put the largest disk into any of the 3 pegs.

C_{n-1} We put the $(n-1)^{\text{th}}$ largest disk into any of the 3 pegs.

$\vdots$

C_1 We put the smallest disk into any of the 3 pegs.

There are three options for each choice, and running through all choices will produce all arrangements such that no disk is on top of a smaller disk. We get 3^n of such arrangements.

d) Show that every allowable arrangement of the n disks occurs in the solution of this variation of the puzzle.

We will proceed by induction. Let $P(n)$ be the statement that every allowable arrangement of the n disks occurs in the solution of this variation of the puzzle.

Proof. Basis Step: For 1 disk, it is easy to check that all 3 possible arrangements occur in the 2 move solution.

Inductive Step: Suppose for some arbitrary n that $P(n)$ is true. We wish to show that $P(n+1)$. We will break this into casework:

Case 1: The largest disk is in the first peg. Recall in (a) that we solved the puzzle for $n+1$ disks by first solving the puzzle for n disks, the largest excluded. By $P(n)$, all arrangements of the n disks are visited.

Case 2: The largest disk is in the second peg. Recall in (a) that we solved the puzzle for $n+1$ disks by secondly, unsolving the puzzle for n disks, the largest excluded. By $P(n)$, all arrangements of n disks are visited.

Case 3: The largest disk is in the last peg. Recall in (a) that we solved the puzzle for $n+1$ disks by thirdly, resolving the puzzle for n disks, the largest excluded. By $P(n)$, all arrangements of n disks are visited.

Since this is exhaustive of the location of the largest disk, all possible arrangements are used in the solution of the Hanoi puzzle for $n+1$ disks. □

Additional Exercise: Let T be a tree on n vertices. Give a recurrence counting the number of 3-colorings of T. What is is the number of 3-colorings when $n = 10$?

Proof. Observe that when we add a vertex v_0 to the tree T with $n-1$ vertices to create a new tree T' on n vertices. v_0 must only have degree 1, for if its degree is greater than 1, then let v_1, v_2 be adjacent to v_0 in T'. But T is connected so there is a path:

$$v_1, e_1, \dots, e_p, v_2$$

Observe that T' now has a cycle:

$$v_1, e_1, \dots, e_p, v_2, \{v_2, v_0\}, v_0, \{v_0, v_1\}, v_1$$

For each leaf l we add, it has to be a different color to its adjacent vertex. Since there is only one vertex adjacent to l, there are two options. We also need to choose which vertex that l is constructed adjacent to so there are $n-1$ options. So a recurrence formula is $f_n = 2(n-1)f_{n-1}$ if f_n is the number of 3-colorings for a tree with n vertices. The initial condition is $f_1 = 3$. □

Let us solve the recurrence where we suppose $f_1 = a$. List the numbers f_2, f_3, f_4:

$$\begin{aligned}
f_1 &= a \\
f_2 &= 2(1)a \\
f_3 &= 2(2)(2(1)(a)) = 2^2(2!(a)) \\
f_4 &= 2(3)(2^2(2!(a))) = 2^3(3!(a))
\end{aligned}$$

The pattern is $f(n) = 2^{n-1}(n-1)!(a) = 2^{n-1}3(n-1)!$, verifiable with induction, by plugging it back into the recursive definition and checking it holds for f_{n+1}. So

$$f(10) = 2^9(3)(9!) = 557383680.$$

6.2: 6, 10, 18, 28, 36, 42, 47x

Exercise 6: There are six professors teaching the introductory discrete mathematics class at a university. The same final exam is given by all six professors. If the lowest possible score on the final is 0 and the highest is 100, how many must there be to guarantee that there are two students with the same professor who earned the same final examination score?

Suppose the six professors are Professor A, B, C, D, E, F. We can make boxes specified by professor and score:

	A	B	C	D	E	F
0	A0	B0	C0	D0	E0	F0
1	A1	B1	C1	D1	E1	F1
2	A2	B2	C2	D2	E2	F2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We will have 600 of such boxes. By the pigeonhole principle, if we have 601 students taking the final exam, at least two students must be in the same box. But that would mean that they have the same professor and the same score.

Exercise 10: Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.

Proof. Suppose $|T| = n$ and that $T = \{t_1, \ldots, t_n\}$. Suppose that $|S| = m$ and that $S = \{s_1, \ldots, s_m\}$ where $m > n$. Let us make pigeonholes labeled $t_1, t_2, \ldots, t_n$. Let the pigeons be $f(s_1), f(s_2), \ldots, f(s_m)$. By the definition of the function, if $f(s_i) = t_j$, then place the $f(s_i)$ pigeon into the t_j pigeonhole. But since there are more pigeons than pigeonholes, there must be two pigeons that are in the same pigeonhole. Let the pigeons be $f(s_a), f(s_b)$ and the pigeonhole be t_k. Therefore, $f(s_a) = t_k$ and $f(s_b) = t_k$. We conclude that $f(s_a) = f(s_b)$ for $a \neq b$. $\square$

Exercise 18: How many numbers must be selected from the set

$$\{1, 3, 5, 7, 9, 11, 13, 15\}$$

to guarantee that at least one pair of these numbers add up to 16?

Proof. We start by finding all pairs of the numbers from the set

$$O = \{1, 3, 5, 7, 9, 11, 13, 15\}$$

such that they add up to 16:

$$\{1, 15\} \qquad \{3, 13\} \qquad \{5, 11\} \qquad \{7, 9\}$$

Let the smallest element in each of these sets be the label for a pigeonhole. Observe that if we select 5 distinct pigeons from the set O, then at least two pigeons j, k must be in the same pigeonhole i for $i = 1, 3, 5, 7$. But that means that $j, k \in \{i, 16-i\}$. Since $j \neq k$, then the two numbers must add up to 16. $\square$

Exercise 28: Show that in a group of 5 people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.

Proof. Let us represent this with a graph $G = (P, E)$ where $P = \{p_1, p_2, \ldots, p_5\}$. Suppose that if $\{p_i, p_j\} \in E$, then p_i, p_j are friends. We start by counting the number of possible groupings of three points which is $\binom{5}{3} = 10$. Let the sets of three edges that join any three points in a 3-cycle be pigeonholes called $g_1, g_2, \ldots, g_{10}$. Now we count the number of possible edges in the graph which is

$$\frac{n(n-1)}{2} = \frac{5(4)}{2} = 10$$

If we pick an edge $\square$

Exercise 36: Assuming that no one has more than 1,000,000 hairs on their head and that the population of New York City was 8,537,673 in 2016, show that there had to be at least nine people in New York City in 2016 with the same number of hairs on their heads.

Let there be 1,000,001 pigeonholes corresponding to the number of hairs from 0-1,000,000. Let the people be pigeons. By the generalized pigeonhole principle, there must be a pigeonhole with at least

$$\left\lceil \frac{8,537,673}{1,000,001} \right\rceil = 9$$

pigeons. So at least 9 people had the same number of hairs on their head.

Exercise 42: Prove that at a party where there are at least two people, there are two people who know the same number of other people.

Proof. Suppose $G = (V, E)$ is a graph where $|V|$ is the set of people in the party and if for $p_i, p_j \in V$, $\{p_i, p_j\} \in E$, then p_i knows p_j. Observe that now, we wish to prove that there are two vertices with the same degree. We know that $0 \leq \deg(p_n) \leq |V| - 1$. But observe that there cannot be a vertex with degree 0 and a vertex of degree $|V|-1$ in the same graph. So there are $|V|-1$ possible degrees and $|V|$ people. By the pigeonhole principle, two people must know the same number of people. ◻

Exercise 47x: Let x be an irrational number. Show that for some positive integer j exceeding the positive integer n, the absolute value of the difference between jx and the nearest integer to jx is less than $1/n$.

Proof. Suppose x is irrational. Let n be arbitrary. We will show that some

$$j_1 \neq j_2 \in J = \{n+1, 2n+2, 3n+3, \ldots, (n+1)(n+1)\}$$

has property that for some $1 \leq a \leq n$, and $b = \lfloor xj_1 \rfloor$, $c = \lfloor xj_2 \rfloor$,

$$\begin{aligned} b + \frac{a-1}{n} &< xj_1 < b + \frac{a}{n} \\ c + \frac{a-1}{n} &< xj_2 < c + \frac{a}{n} \end{aligned}$$

Let each interval:

$$(0, 1/n), (1/n, 2/n), \ldots, (n-1/n, n/n)$$

be a pigeonhole. Let each $|xj_i - \lfloor xj_i \rfloor|$ be a pigeon. By pigeonhole principle, at least two $|xj_i - \lfloor xj_i \rfloor|$ must lie in the same interval. Name their j_i components as $j_1 > j_2$. So

$$\begin{aligned} b + \frac{a-1}{n} &< xj_1 < b + \frac{a}{n} \\ -c + \frac{-a}{n} &< -xj_2 < -c + \frac{1-a}{n} \\ b - c + \frac{-1}{n} &< xj_1 - xj_2 < b - c + \frac{1}{n} \\ b - c + \frac{-1}{n} &< x(j_1 - j_2) < b - c + \frac{1}{n} \end{aligned}$$

Let $j = j_1 - j_2 > n$. So

$$\begin{aligned} \frac{-1}{n} &< xj - (b-c) < \frac{1}{n} \\ 0 &< |xj - (b-c)| < \frac{1}{n} \end{aligned}$$

Observe that $b - c$ is the closest integer to xj. If not, then there is a number closer to xj, and their difference would still be less than $1/n$ as desired. ◻