Math113Hw5

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Homework 5

Exercise 1: Calculate the size of the conjugacy class of $(1\,2\,3)$ as an element of S_4, S_5, S_6. Find the size of its centraliser in each case. Hence, to otherwise, calculate the size of the conjugacy class of $(1\,2\,3)$ in A_4, A_5, A_6.

Proof. Since conjugation does not change cycle type, the conjugacy class has order $2\binom{4}{3} = 8$ in S_4, $2\binom{5}{3} = 20$ in S_5 and $2\binom{6}{3} = 40$ in S_6. By orbit-stabilizer, the size of the centralizer for S_4 for $(1\,2\,3)$ is 3. The size of the centralizer for S_5 is 6, and the size of the centralizer for S_6 is $720/40 = 18$.
We can find an odd permutation that commutes with $(1\,2\,3)$ in both S_5 and S_6:

$$(4\,5)$$

so the size of $|C_{G_{A_n}}((1\,2\,3))| = \frac{1}{2}|D_{G_{S_n}}|((1\,2\,3))$. The conjugacy class of $(1\,2\,3)$ does not split in A_5, A_6.
Now for the A_4 case, using the orbit-stabilizer theorem,

$$|C_G((1\,2\,3))| = 3$$

and since the center cannot split, the conjugacy classes must split. So $\mathrm{ccl}((1\,2\,3))$ splits in A_4. □

Exercise 2: Show that D_{2n} has one conjugacy class of reflections if n is odd, and two conjugacy classes of reflections if n is even.

Exercise 3:

1. Let G be a finite group and let H be a subgroup of index $n \neq 1$ in G. Suppose that $|G|$ does not divide $n!$. Show that H contains a non-trivial normal subgroup of G.

   Proof. Let $G \circlearrowright G$ by left multiplication of the cosets of H where $|G : H| = n$:

   $$\varphi : G \to S_n$$

   The kernel is a subset of H. We have that the kernel is a subgroup of G also, so

   $$|G : \ker(\varphi)| \mid n!$$

   by the isomorphism theorem where the quotient group of G and the kernel is isomorphic to some subgroup of S_n. But

   $$|G| \nmid n!$$

   and by Lagrange,

   $$|G : \ker(\varphi)||\ker(\varphi)| \nmid n!$$

So we cannot have $\ker(\varphi) = 1$. Therefore, the kernel is non-trivial and must contain more elements than just e. $\square$

2. Show that if G is of order 28, and has a normal subgroup of order 4, then G is abelian.

Proof. Let $G \circlearrowleft H$ by conjugation. Observe that the normal subgroup H is made of a union of conjugacy classes. The identity element is its own conjugacy class. Then there are three elements left. The size of the conjugacy class divides order of G by orbit stabilizer so either we have H is made of conjugacy classes of size $1, 1, 1, 1$ or $1, 1, 2$. In the first case, if H is a subgroup of the center of G, then we take G/H and observe that since the group is cyclic and partitions G, generator $\langle gH \rangle$, every element can be written as

$$g^i h^j = h^j g^i$$

so G is abelian. In the case of $1, 1, 2$ size conjugacy classes in H. Let

$$|\mathrm{ccl}(h)| = 2$$

and

$$\mathrm{ccl}(h) = \{\sigma, \tau\}$$

for some $h \in H$. If $g\sigma g^{-1} = \tau$, for $g \notin H$ then

$$\begin{aligned} g\sigma g^{-1} &= \tau \\ g^2 \sigma g^{-2} &= \sigma \\ &\vdots \\ g^6 \sigma g^{-6} &= \sigma \\ g^7 \sigma g^{-7} &= \tau = e\sigma e^{-1} = \sigma \end{aligned}$$

contradiction. H is made of conjugacy classes of order 1 and G is therefore abelian. $\square$

Exercise 4: Let G be a non-abelian group of order p^3, where p is a prime number.

1. Show that the center of $Z(G)$ has order p.

Proof. Since the grouphas order p^a, the center is non-trivial and must divide p^3. Also, the center cannot have order p^3, since the group is non-abelian. If $|Z(G)| = p^2$, then $|G/Z(G)| = p$ and the quotient group is cyclic: $\langle gH \rangle$. So the cosets of the center partition the group, where every element in G can then be written as

$$g^i z^j = z^j g^i$$

with $z \in Z(G)$ which shows that G is abelian. So $|Z(G)| = p$. $\square$

2. Show that if $g \notin Z(G)$, then the centralizer $C(g)$ has order p^2.

Proof. Since

$$Z(G) = \bigcap_{g \in G} C_G(g)$$

if $g \notin Z(G)$, then $|Z(G)| < |C_G(g)|$ since $Z(G) \subseteq C_G(g)$. The centralizer is a group so its order must divide p^3. But the centralizer cannot have the size p^3 otherwise, g commutes with every element and actually belongs in the center. So we must have $|C_G(g)| = p^2$. $\square$

3. Find the number and sizes of the conjugacy classes in G.

Proof. We must have that the elements of the center have conjugacy classes of size 1 and that there are p of them. Since the conjugacy classes partition the group, we must have that the sum of the rest of the conjugacy classes's order must be ewual to $p^3 - p = (p^2 - 1)p$. Since the sizes of the centralizers for elements not in $Z(G)$ is p^2, by orbit stabilizer, the size of their conjugacy class is p. That means there are $p^2 - 1$ conjugacy classes with p elements:

	Size $= 1$	Size $= p$
Count	$\lvert\text{ccl}(g)\rvert = 1$	$\lvert\text{ccl}(g)\rvert = p$
Count	p	$p^2 - 1$

□

Exercise 5: Let G be a finite group acting on a set X. For $g \in G$. Let

$$\text{Fix}(g) = \{x \in x : gx = x\}$$

be the set of fixed points of g. Counting the set

$$\{(g, x) \in G \times X : gx = x\}$$

in two ways and using the orbit-stabilizer theorem, or otherwise, show that the number of orbits is given by

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

Show that if G acts transitively on X and $|X| > 1$, there is $g \in G$ with no fixed points.

Proof. Take an arbitrary stabilizer of our group action:

$$\text{Stab}(x) = \{e, g_1, g_2, \ldots\}$$

notice that for every element in the stabilizer of x, x belongs to the elements' fix set. So

$$\sum_{x \in X} |\text{Stab}(x)| = \sum_{g \in G} |\text{Fix}(g)|$$

Now take an arbitrary orbit:

$$\text{Orb}(x) = \{x_1, x_2, x_3, \ldots, x_n\}$$

and notice that

$$\text{Orb}(x_1) = \text{Orb}(x_2) = \cdots = \text{Orb}(x_n)$$

so

$$\frac{1}{\text{Orb}(x_1)} + \frac{1}{|\text{Orb}(x_2)|} + \cdots + \frac{1}{\text{Orb}(x_n)} = 1$$

by orbit stabilizer theorem,

$$\begin{aligned}
|G| &= |\text{Orb}(x)||\text{Stab}(x)| \\
\frac{|\text{Stab}(x)|}{|G|} &= \frac{1}{|\text{Orb}(x)|} \\
\sum_{x \in X} \frac{|\text{Stab}(x)|}{|G|} &= \sum_{x \in X} \frac{1}{|\text{Orb}(x)|} \\
\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| &= \sum_{x \in X} \frac{1}{|\text{Orb}(x)|}
\end{aligned}$$

But since the sum of the reciprocal of the orbits counts the number of unique orbits there are, we are done. □

Proof. If G acts transitively, we have

$$\sum_{g \in G} |\mathrm{Fix}(g)| = 1 \cdot |G|$$

We first single out $\mathrm{Fix}(e)$ which has a cardinality of $|X| > 1$ by definition. Suppose there are no non-zero $|\mathrm{Fix}(g)|$'s. Then the minimum value of $\sum_{g \in G} |\mathrm{Fix}(g)|$ is

$$|G| - 1 + |\mathrm{Fix}(e)| = |G| + 1$$

which is impossible. There must be an element that fixes nothing. $\square$