

# Math250aHw7

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October 24, 2023

## Part A

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**Exercise 1:** Let  $\mathbb{Q} \subseteq \mathbb{R}$  be the topological subspace of rationals and  $\mathbb{Q} \rightarrow \mathbb{R}$  be the inclusion map as an epimorphism in the category of topological Hausdorff spaces and continuous maps. Show that dense subobjects can be defined by epimorphic monomorphisms.

*Answer.* We see that if we have a dense set  $D$  and  $T$  such that  $D \rightarrow T$  is a subobject, then it is a monomorphism by definition. Furthermore, it is epimorphic since for

$$D \longrightarrow T \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} T'$$

if  $D \rightarrow T \xrightarrow{f} T' = D \rightarrow T \xrightarrow{g} T'$ , we have that inclusion maps are unique since they are the kernel of some map out of  $T'$ . So  $f = g$  which shows that dense subobjects are defined by epimorphic monomorphisms.

## Part B

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**Exercise 1:** Show that if  $\mathcal{A}$  is the category of ordered sets and  $D : \mathcal{A} \rightarrow \mathcal{A}$  is a functor assigning a set to its dual, then the automorphism class group of  $\mathcal{A}$  has at least two elements.

*Answer.* We know that  $\text{id}_{\mathcal{A}}$  is a functor naturally equivalent to the identity functor. So  $\text{id}_{\mathcal{A}}$  belongs in  $I$ . We also have that  $D$  is an equivalence because if we take  $D^2$ , we get back our same set, as  $D$  just reverses the order.

We know that  $\text{id}, D$  are not of the same equivalence class because  $D^2 = \text{id}$  and  $\text{id} \neq D$ . Therefore, the automorphism group contains at least  $\text{id}, D$ .

**Exercise 2:** Let  $[\rightarrow]$  be the category with

- Objects  $L, R$
- Morphism  $L \rightarrow R$

and  $[\rightarrow \rightarrow]$  be the category:

- Objects  $L, M, R$
- Morphisms  $L \rightarrow M, M \rightarrow R, L \rightarrow R$

*Answer.* The morphisms in the image of the functor  $[\rightarrow] + [\rightarrow] \xrightarrow{\pi} [\rightarrow\rightarrow]$  does not form a category because if the objects are  $\pi(L_1) = L$ ,  $\pi(L_2) = \pi(R_1) = M$ , and  $\pi(R_2) = R$ , then we have the morphisms

- $\pi(L_1 \rightarrow R_1) = L \rightarrow M$
- $\pi(L_2) \rightarrow R_2 = M \rightarrow R$
- But there are no more morphisms because both  $[\rightarrow]$  have only one morphism.

So the composition  $L \rightarrow R$  does not exist in the image of  $\pi$ .

## Part C

**Exercise 1:** Let  $\mathcal{S}$  be the category of sets with morphisms as set functions. Prove that the automorphism class group of  $\mathcal{S}$  is trivial. Use the fact that if  $F : \mathcal{S} \rightarrow \mathcal{S}$  is an automorphism, then  $F(D)$  has one element, if  $D$  has one element. Now define for each  $A \in \mathcal{S}$ ,  $A \rightarrow F(A)$  where:

$$\begin{array}{ccc} D & \longrightarrow & F(D) \\ \downarrow x & & \downarrow F(x) \\ A & \longrightarrow & F(A) \end{array}$$

commutes for all  $x \in (D, A)$ .

*Proof.* So we can label the map  $\psi : D \rightarrow F(D)$  and  $\varphi : A \rightarrow F(A)$ . To make the map commute, we consider the mappings from the diagram:

$$\begin{array}{ccc} d & \longmapsto & d' \\ \downarrow & & \downarrow \\ a & & a' \end{array}$$

which means that

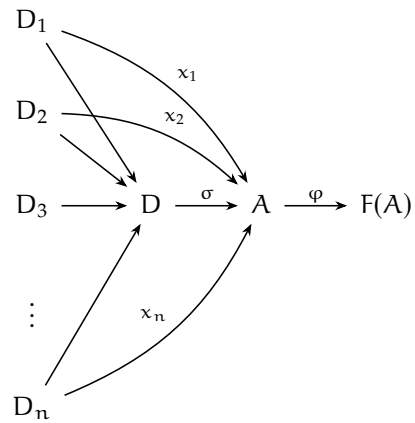
$$\varphi(a) = F(x)\psi(d)$$

sending all elements of  $A$  into one element of  $F(A)$ , making the diagram commute for any choice of  $x$ .

We conclude that there is a natural transformation from the identity functor:

$$\begin{array}{c} D \\ \downarrow x \\ A \end{array}$$

to our functor restricted to  $(D, A)$ . But now the action of this functor on  $(D, A)$  uniquely determines our functor, since:



we can decompose any  $n$  element set  $D$  into a disjoint union of 1 element sets. And since there is a mapping from  $D_1 \rightarrow F(A)$  where  $D_i$  have size 1, we know there is a mapping  $\varphi$  that gives us a natural transformation from the identity functor to any automorphism functor.

There is also a way to go backwards and find a natural transformation from  $F$  to the identity functor. So for  $F$  in the automorphism class group, it is isomorphic to the identity functor, so the group is trivial.  $\square$