Math104Hw7

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October 21, 2023

Exercise 1: Show that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$

Proof. Let $p \in [0, \infty)$. We need to show that f is continuous at p. That means that $\forall \varepsilon > 0$, there is a $\delta > 0$ such that if $|x - p| < \delta$, then

$$|f(x) - f(p)| < \varepsilon$$

So we want to show that there is a δ such that

$$|\sqrt{x} - \sqrt{p}| < \varepsilon$$

We first notice that

$$|\sqrt{x} - \sqrt{p}||\sqrt{x} + \sqrt{p}| = |x - p|$$

Furthermore, we claim:

$$|\sqrt{x} - \sqrt{p}| \leqslant |\sqrt{x} + \sqrt{p}|$$

since

$$|\sqrt{x} - \sqrt{p}| \leqslant |\sqrt{x}| + |\sqrt{p}| = \sqrt{x} + \sqrt{p} = |\sqrt{x} + \sqrt{p}|$$

So if we take $|x - p| < \varepsilon^2$, we see that

$$|\sqrt{x} - \sqrt{p}|^2 \leqslant |\sqrt{x} - \sqrt{p}||\sqrt{x} + \sqrt{p}| < \varepsilon^2$$

So

$$|\sqrt{x} - \sqrt{p}| < \varepsilon$$

so we have found $\delta = \varepsilon^2$. This does not work for when $p = 0$, since $[0, \infty)$ is not open in $\mathbb{R}$. To fix this, let x_n be any sequence in $[0, \infty)$ converging to 0. Since $f(x_n)$ is bounded, decreasing, it converges. ◻

Exercise 2: Use the $\varepsilon - \delta$ property to show that $f(x) = x^2$ is continuous at $x_0 = 3$.

Proof. We will show that for all $\varepsilon > 0$, there is a $\delta > 0$ such that if

$$|x - x_0| < \delta$$

then

$$|f(x) - f(x_0)| < \varepsilon$$

Then writing it out properly, we get:

$$|x^2 - 9| < \varepsilon$$

But this is

$$|x - 3||x + 3| < \varepsilon$$

Now if

$$|x - 3| < 1$$

then we have that:
$$|x| < 4$$
and therefore,
$$|x+3| \leqslant |x| + 3 < 7$$
Then we require
$$|x-3||x+3| < \varepsilon$$
or
$$|x-3| < \frac{\varepsilon}{7}$$
So we take $\delta = \min(1, \frac{\varepsilon}{7})$. □

Exercise 3: Let f, g to be two continuous functions on $[a, b]$ and $f(a) \geqslant g(a), f(b) \leqslant g(b)$. Prove that $\exists x_0 \in [a, b]$, such that $f(x_0) = g(x_0)$.

Proof. Let $h(x) = f(x) - g(x)$ which is continuous because f, g are continuous. Then we know that $h(a) \geqslant 0$, $h(b) \leqslant 0$. We know that $[a, b]$ is non-empty, so by the intermediate value theorem, there exists an x_0 such that $0 \leqslant h(x_0) \leqslant 0$ or $h(x_0) = 0$. This means that $f(x_0) - g(x_0) = 0$ or $f(x_0) = g(x_0)$. □

Exercise 4: Prove $x = \cos x$ for some $x \in (0, \pi/2)$.

Proof. Let $f = x - \cos x$. Then $f(0) = -1$, $f(\pi/2) = \pi/2$. By the intermediate value theorem, we know that there is an x_0 in $[a, b]$ such that $f(x_0) = 0$ since $-1 < 0 < \pi/2$. Then we must have:
$$f(x_0) = 0 = x_0 - \cos x_0$$
Then
$$x_0 = \cos x_0$$
which shows that $x = \cos x$ for $x_0 \in [a, b]$. □

Exercise 5: Let E be a non-closed set in $\mathbb{R}$ and $s \in E^- - E$. Prove that $\frac{1}{x-s}$ is continuous on E but $f(E)$ is not bounded.

Proof. Since $s \in E^- - E$, we know that for any $p \in (0, 1)$, $(x_n) \subseteq E$:
$$\lim \frac{1}{x_n - s} = \frac{\lim 1}{\lim x_n - s} = \frac{1}{p - s} = f(p)$$
Which is possible because the denominator does not converge to 0 for large values of n. So $f(x) = \frac{1}{x-s}$ is continuous in $(0, 1)$.

(Not Bounded) Since $s \in E^- - E$, we know that there is a sequence $(x_n) \subseteq E$ converging to s. So then $x_n - s$ is a sequence converging to 0. By definition, we have that $\forall \varepsilon > 0$, there is an N such that $\forall n > N$,
$$|x_n - s| < \varepsilon$$
or
$$\frac{1}{|x_n - s|} > \frac{1}{\varepsilon}$$
because $(x_n) \subseteq E$, $s \notin E$, so $x_n - s \neq 0$ for any n. Suppose that f is bounded above by M and below by L. Let $M' = \max(|M|, |L|)$. Then we can find an $\varepsilon = \frac{1}{M'}$ such that:
$$\left|\frac{1}{x_n - s}\right| > M'$$

This means that:

$$\frac{1}{x_n - s} > M' \text{ or } \frac{1}{x_n - s} < -M'$$

Then it cannot be that:

$$\frac{1}{x_n - s} > M'$$

because $M' > |M| > M$ so

$$\frac{1}{x_n - s} > M$$

contradiction. So if $\frac{1}{x_n - s} < -M'$. We have $M' > |L|$. But that means:

$$-M' < L < M'$$

Which also leads to a contradiction because that implies:

$$\frac{1}{x_n - s} < -M' < L$$

so it cannot be that we have both an upper and lower bound M, L. So $f(E)$ is not bounded. □

Exercise 6: Assume $f(x)$ is a continuous function on $[0, 1]$, prove that $\exists x \in (0, 1)$ such that $f(x) < 1/x$.

Proof. Suppose for contradiction that $f(x) \geqslant \frac{1}{x}$ for all $x \in (0, 1)$. Let x_n be a sequence in $(0, 1)$ converging to 0. Then we will show that $\frac{1}{x_n}$ diverges. Let M be any number greater than 0. Then since x_n converges to 0, we know that $\forall \varepsilon > 0$, there is an N such that $\forall n > N$,

$$|x_n| < \varepsilon$$

which means that

$$0 < x_n < \frac{1}{M}$$

for all $n > N$. So then since $x_n > 0$, we have:

$$x_n < \frac{1}{M} \implies \frac{1}{x_n} > M$$

so for any $M > 0$, there is an n such that for all $n > N$, we have:

$$\frac{1}{x_n} > M$$

which means that $\frac{1}{x_n}$ diverges to ∞. But then we have $f(x_n) \geqslant \frac{1}{x_n}$ implies that $f(x_n)$ diverges to:

$$f(x_n) \geqslant \frac{1}{x_n} > M$$

But that contradicts the continuity of $f(x)$ on $[0, 1]$, because $\lim f(x_n) = \infty \neq f(0)$ which is finite. Therefore, we must have $f(x) < \frac{1}{x}$ for some x. □