Math55Hw9

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9.1: 1, 2, 3

Exercise 1: Show that graph $G = (V, E)$ is a tree if and only if for every distinct pair of vertices $x, y \in V$, there is a unique path between x and y in G.

Proof. ($\rightarrow$) We will prove the contrapositive. Suppose that there are two paths between an arbitrary x and y in V. We will show that G is not a tree:

$$x, e_1, \ldots, e_n, y \tag{1}$$
$$x, e'_1, \ldots, e'_m, y \tag{2}$$

We can reverse the sequence of path (2) and append it to path (1):

$$x, e_1, \ldots, e_n, y, e'_n, \ldots, e'_1, x \tag{3}$$

Observe that our new path (3) is a circuit as the first and last elements are equal. Notice that for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ there is at least one e_i and e'_j such that $e_i \neq e'_j$. Thus, we can remove all duplicate edges in path (3) and the resultant path will have at least 2 edges. We can also remove all vertices that are not incident to our remaining edges. Finally, if there are repeated vertices such as x_t, we can take the first and last occurrence of x_t and delete the content in between. It will be a cycle since it is simple. Thus, G is not a tree. Contradiction.

($\leftarrow$) Suppose that for every pair of vertices $x, y \in V$, there is a unique path between x and y in G. We must show that G is a tree. Since there exists a path between every pair of vertices, we know that G is connected. All that is left is to see if it is acyclic. Suppose for contradiction that G is cyclic. Then there exists a simple cycle with at least 3 distinct vertices:

$$x_0, e_1, x_1, \ldots, e_p, x_p \tag{4}$$

Where $x_0 = x_p$. So in path (4), there is some x_i where $x_i \neq x_0$. We can have the paths:

$$x_0, e_1, \ldots, x_i \tag{5}$$
$$x_i, \ldots, e_p, x_p \tag{6}$$

Paths (5) and (6) cannot be identical, since then, our original path (4) with at least 3 distinct vertices is not a cycle. If the paths are distinct, we have a contradiction since we have found two distinct paths from x_0 to x_i. Therefore, G is acyclic and a tree. $\square$

Exercise 2: Show that if a tree contains a vertex of degree k, then it must have at least k leaves.

Proof. Let $G = (V, E)$ be a tree with a vertex v_{0_0} where $\deg(v_{0_0}) = k$. We will show that it has at least k leaves. We know that v_{0_0} is adjacent to k other vertices $v_{0_1}, v_{1_1}, \ldots, v_{k-1_1}$. We can construct paths starting at v_0:

$$v_{0_0}, e_{0_1}, v_{0_1}, \ldots \tag{1}$$
$$v_{0_0}, e_{1_1}, v_{1_1}, \ldots \tag{2}$$
$$v_{0_0}, e_{2_1}, v_{2_1}, \ldots \tag{3}$$
$$\vdots \tag{4}$$
$$v_{0_0}, e_{k-1_1}, v_{k-1_1}, \ldots \tag{5}$$

For path (1), traverse some edge such that the vertex we arrive at is not any that we have seen before. Add that edge and vertex to our path. We can repeat this until we run out of unique edges to add or until all adjacent vertices are ones we have encountered before. In the first case, we have a leaf. In the second, a contradiction which implies that our tree has a cycle. But if we repeat this process for each path, we get k leaves, each being distinct, as desired. $\square$

Exercise 3: Show that if $T = (V, E)$ is a tree and $e \in E$, then the spanning subgraph $T' - \{e\}$ of T is disconnected.

Proof. Since T is a tree, the spanning tree of T is T. Consider the graph of $T - \{e\} = T'$. Since we know that T is a tree, by Exercise 1 Hw9.1, we know that there is only one unique path between every distinct pair of vertices. Then removing e means that there are two distinct vertices x and y such that there is no path between them. That means that T' is disconnected. $\square$

Exercise 4: Suppose G is a graph with n vertices and $n - 1$ edges. Show that G is connected if and only if G is acyclic.

Proof. ($\rightarrow$) We will show the contrapositive. Suppose that G is cyclic. Then there is a simple circuit:

$$x_0, e_1, \ldots, e_n, x_k$$

Where $x_0 = x_k$. Notice that a cycle with k vertices contains k edges. Since the circuit is connected, let us build a connected component G_1 that contains the cycle. For every new vertex we add to G_1, we must add at least 1 new edge.

Observe that the maximum number of vertices we can add to G_1 is $n-k-1$ since we only have $n-1$ edges. But that means that at least one vertex lies outside of G_1. Therefore, there must be other connected components: $G_2, G_3, \dots, G_i$ which all non-empty and disjoint. Since their union is G and we know that adding more vertices to any connected component makes it disconnected, G is disconnected.

($\leftarrow$) Suppose that $G = (V, E)$ is acyclic. We will prove that G is connected. Suppose for contradiction that G is disconnected. Then there are connected components $G_1, G_2, \dots, G_i$ where $i \geq 2$ such that the G_i are pairwise disjoint and their union is G. Observe that since G is acyclic, so is its connected components. Therefore, $G_1, \dots, G_i$ are all trees. We know that if a tree has k vertices, it has $k-1$ edges. Let $V_1, \dots, V_i$ be set of vertices in the connected components of $G_1, \dots, G_i$ respectively. Then $\sum_{j=1}^{i} |V_j| - 1 < |V| - 1 = |E|$. Contradiction, as desired. $\square$

9.2: 1, 2, 3, 4, 5

Exercise 1: Prove that if H is a connected component of G and v is a vertex in H, then the degree of v in H is equal to the degree of v in G.

Proof. Suppose H is a connected component of G and $v \in H$. Suppose for contradiction that the degree of v in H is not equal to the degree of v in G. But that must mean that there is a vertex in G adjacent to v that is not in H. So H is not a connected component, as desired. $\square$

Exercise 2: Prove the Claim in the proof of Theorem 2 above (which is the same as the claim from Lecture 17): If G is connected multigraph with at least 2 vertices and C is a cycle, If $H = G - E(C)$, for each connected component of H, H_i, there exists a (not necessarily unique) vertex $s_i \in H_i$ such that $s_i \in C$.

Proof. Suppose G is a connected multigraph with at least 2 vertices and C is a cycle in G. Suppose for contradiction that there is at least one H_i such that for any vertex $s_i \in C$, $s_i \notin H_i$. Call this one H_k. Observe that if we re-add $E(C)$ to H, H_k remains a connected component in G since there is no edge in C that is incident to any vertex in H_k. But G must contain another connected component since $H_k \neq G$. Therefore, G has at least 2 connected components and is disconnected. Contradiction, as desired. $\square$

Exercise 3: Prove that if a simple graph contains a circuit with no repeated edges, then it must contain a circuit with no repeated vertices (i.e., a simple circuit).

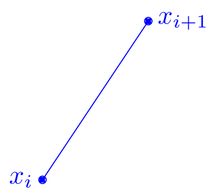
Proof. Suppose a simple graph contains a circuit with no repeated edges called C:

$$C = x_0, e_1, x_1, \ldots, e_n, x_n$$

where $x_0 = x_n$. If there are no repeated vertices, we are done. If there are repeated vertices $x_i = x_j$ such that $i < j$, we the elements that lie between then and name that a cycle. Suppose without loss of generality that x_i, x_j are the endpoints of the shortest circuit C'. Then:

$$C' = x_i, e_{i+1}, x_{i+1}, \ldots, x_j, x_j$$

If our circuit has one or two vertices we have:



or

Which are cycles. If there are three or more vertices, we can choose a vertex x_k such that $i < k < j$ and create two paths with distinct edges:

$$x_i, e_{i+1}, \ldots, x_k$$
$$x_k, e_{k+1}, \ldots, x_j$$

Since there are no repeated edges, there is no way to reorder one of the paths such that it equals the other. We have found two distinct simple paths from one vertex to another, so C' is a cycle, as desired. □

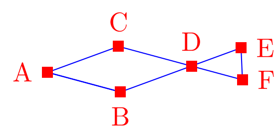
Exercise 4: Prove of Disprove: There exists a connected graph with 6 vertices and 5 edges which has an Euler circuit.

Proof. Disprove. We will show that for all connected graphs with 6 vertices and 5 edges, there does not exist an Euler circuit. Suppose G is a connected graph with 6 vertices and 5 edges. It must be acyclic also by Exercise 4 of Hw9.1. Therefore, G is a tree. Therefore, G has a leaf. Therefore, G has a vertex of odd degree. Therefore, there is no Euler circuit. □

Exercise 5: Prove or Disprove: There exists a connected graph with 6 vertices and 7 edges which has an Euler circuit.

Proof. The following graph has 6 vertices, 7 edges, and the Eulerian circuit:

$$A, \{A, C\}, C, \{C, D\}, D, \{D, F\}, F, \\ \{F, E\}, E, \{E, D\}, D, \{D, B\}, B, \{B, A\}, A$$



$\square$