Math55Hw8

Trustin Nguyen

October 2022

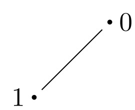
8.1: 1, 2, 3

Exercise 1: Let n be a positive integer. Prove that the hypercube on n bits (as defined in class) is connected.

Proof. We will proceed by induction. Define a hypercube as a graph $G(V, E)$ such that:

$$V = \{0,1\}^n = \{(a_1, a_2, \ldots, a_n : a_i \in \{0,1\}\}$$
$$x = (x_1 \ldots x_n) \text{ is adjacent to } y = (y_1 \ldots y_n)$$
$$\text{whenever they differ in exactly one entry}$$

Basis Step: We must prove that $G(V, E)$ is connected for $V = \{0,1\}^1 = \{a_1 : a_i \in \{0,1\}\}$ and $x = (x_1)$ is adjacent to $y = (y_1)$ when they differ in exactly one entry. We can prove this with a picture:



Observe that the graph is connected as we can create a path from 1 to 0: $1, \{1, 0\}, 0$ which also means we have a path from 0 to 1. Thus, for every $v \in V$, there is a path between them.

Inductive Step: Suppose that k is an arbitrary integer greater than zero. Let G(V,E) be a connected graph such that $V = \{0,1\} = \{a_1 \ldots a_k : a_i \in \{0,1\}\}$ and $x = x_1 \ldots x_k$ and $y = y_1 \ldots y_k$ are adjacent when they differ by exactly one entry. We must show that the graph of $V = \{0,1\}^{k+1}$ with $x = x_1 \ldots x_{k+1}$ and $y = y_1 \ldots y_{k+1}$ are adjacent when they differ by one entry, is connected. Let

$$S = \{a_1 \ldots a_k 0 : a_i \in \{0,1\}\}$$
$$\text{and}$$
$$T = \{a_1 \ldots a_k 1 : a_i \in \{0,1\}\}$$

Observe that the graphs $G(S,E)$ and $G(T,E)$ are connected. We will show that the graph $G(S \cup T, E)$ is connected. Consider the fact that for $x_1 \ldots x_k = 0$, $x_1 \ldots x_k 0$ and $x_1 \ldots x_k 1$ differ in exactly one entry. By definition also, $x_1 \ldots x_k 0 \in S$ and $x_1 \ldots x_k 1 \in T$. Thus, $x_1 \ldots x_k 0$ and $x_1 \ldots x_k 1$ in $S \cup T$ are adjacent. But then we are done, since if we start with some element of S (without loss of generality), $y_1 \ldots y_k 0$, we can find a path to all elements ending in 0 since S is connected but we can also find a path to all elements in T by constructing the path:

$$y_1 \ldots y_k 0, \to, x_1 \ldots x_k 0, \{x_1 \ldots x_k 0, x_1 \ldots x_k 1\}, x_1 \ldots x_k 1, \to, z_1 \ldots z_k 1$$

Where $z_1 \ldots z_k 1 \in T$. Since $S \cup T = \{0,1\}^{k+1}$, we have proved a connected graph for the $k+1$ case from k, as desired. □

Exercise 2: Suppose that G is a connected graph. Show that between every pair of distinct vertices of G, there is a simple path connecting x to y in G.

Proof. By definition, there exists a path between every pair of distinct vertices of G. We must now show that if it is a non-simple path, there exists a simple path. Suppose that our path from x to y contains a repeated edge called $\{e_1, e_2\}$:

$$x, \ldots, e_1, \{e_1, e_2\}, e_2, \ldots, e_2, \{e_2, e_1\}, e_1, \ldots, y$$

We can take the leftmost $\{e_1, e_2\}$ and rightmost $\{e_1, e_2\}$ and remove the sets and all elements in between. Observe that we have a new path without the repeat:

$$x, \ldots, e_1, e_1, \ldots, y$$

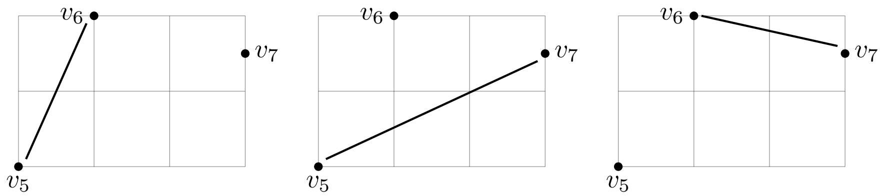
We can repeat the process until all repeated edges are removed. We will have a simple path as desired. □

Exercise 3: The *degree sequence* of a graph is a list of the degrees of its vertices in nonincreasing order. Prove that if a simple graph has a degree sequence $2,2,2,1,1,1,1$ then it must be disconnected.

Proof. Let $G = (V,E)$ be the graph with degree sequence $2,2,2,1,1,1,1$. Without loss of generality, suppose $v_1, v_2, v_3, v_4 \in V_1$ have degree 1 and $v_5, v_6, v_7 \in V_2$ have degree 2. Suppose for contradiction that $G = (V,E)$ is connected. It follows that $\{v_i, v_j\} \notin E$ if $v_i, v_j \in V_1$. Since if not, then all paths consisting of v_i or v_j are a subsequence of the sequence:

$$v_i, \{v_i, v_j\}, v_j, \{v_i, v_j\}, v_i, \ldots$$

This implies that G is not connected as v_5, v_6, v_7 are not in the path. So each v_1, v_2, v_3, v_4 are adjacent to a $v_k \in V_2$. Call the set of these edges E_1. Observe that the graph $G = (V_2, E - E_1)$ must be disconnected. There can only be one edge in $G = (V_2, E - E_1)$ so our possible graphs are:



All three possibilities are disconnected: let v_5, v_6 be the vertices with no path between them and v_7 be adjacent to v_5 w.l.o.g. in $G(V_2, E - E_1)$. Now consider the graph of $G = (V, E)$. Let us try to construct a path between v_5, v_6:

$$v_5, \{v_5, ?\}, \dots, \{v_6, ?\}, v_6$$

Observe that the vertex between v_5, v_6 in the path must have degree at least 2. Thus, we have:

$$v_5, \{v_5, v_7\}, v_7, \dots, \{v_6, ?\}, v_6$$

But v_7 is the only other vertex with degree ≥ 2 and we have already proven that the graph of V_2 is disconnected. Therefore, $G = (V, E)$ is disconnected. □

8.2: 1, 2, 3, 4, 5

Defn. If $G = (V, E)$ is a graph, a subset $S \subset V$ is a *clique* if $xy \in S$ for every pair of distinct vertices $x, y \in S$. S is called an *independent set* if $xy \notin E$ for every pair of distinct vertices $x, y \in S$.

Exercise 1: Prove that if $G = (\{1, \dots, 6\}, E)$ is a graph, then it contains either 3 vertices that form a clique or 3 vertices that form an independent set. (hint: see lecture 15 part III)

Proof. Suppose that $G(\{1, \dots, 6\}, E)$ is a graph. We will show that there must be a clique of 3 vertices or independent set of 3 vertices. Let x be an arbitrary vertex.

Case 1: deg $(x) \leq 2$. Then we can choose three vertices not adjacent to x called a, b, c. It follows that these must form a clique. If not, then without loss of generality, let the edge $\{a, b\} \notin E$. Then we have $\{x, a\}, \{x, b\}, \{a, b\} \notin E$ and a, b, x would form an independent set.

Case 2: deg $(x) \geq 3$. Let a, b, c be the three vertices adjacent to x. We can have a, b, c form an independent set, but if not, without loss of generality, suppose that $\{a, b\} \in E$. Then $\{x, a\}, \{x, b\}, \{a, b\} \in E$, so a, b, x form a clique.

Since in either case we must have either a clique or independent set, as desired. □

Exercise 2: Prove that if $G = (V, E)$ is k-colorable, then there are disjoint sets $V_1, V_2, \dots, V_k \subseteq V$ such that each V_i is an independent set.

Proof. Suppose that $G = (V, E)$ is k-colorable. This means that we have a function $f : V \to \{1, 2, \ldots, k\}$ such that if $\{x, y\} \in E$, then $f(x) \neq f(y)$. Let $V_1, V_2, \ldots, V_k$ be the sets:

$$\begin{aligned} V_1 &= \{x \in V : f(x) = 1\} \\ V_2 &= \{x \in V : f(x) = 2\} \\ &\vdots \\ V_k &= \{x \in V : f(x) = k\} \end{aligned}$$

We will prove that for every V_i, $i \in \{1, 2, \ldots, k\}$, V_i is disjoint with V_j, $j \in \{1, 2, \ldots, k\} - \{i\}$. Let V_i be arbitrary. Let $v_i \in V_i$ be an arbitrary element. It follows that $f(v_i) = i \notin \{1, 2, \ldots, k\} - \{i\}$. Thus, $f(v_i) = i \neq j$ and $v_i \notin V_j$. We have proven $V_1, V_2, \ldots, V_k$ are disjoint.

We will now show that each V_i is an independent set. Again, suppose V_i is arbitrary. Suppose for contradiction that V_i is not an independent set. So there exist two vertices $v_{i_1}, v_{i_2} \in V_i$ such that $\{v_{i_1}, v_{i_2}\} \in E$. But by our k-coloring, that implies that $f(v_{i_1}) \neq f(v_{i_2})$ and $i \neq i$ which is absurd.

Lastly, we know that each $V_i \subseteq V$ by the way we defined $V_1, V_2, \ldots, V_k$. If $x \in V_i$ is an arbitrary element, then $x \in V$, which completes the proof. $\square$

Defn. If $G = (V, E)$ is a graph, a subgraph H of G is called a *connected component* of G if H is connected and *maximal*, i.e., it is not possible to add any vertices or edges to H while keeping it connected (formally: if H' is a subgraph of G such that $H \neq H'$, and H is a subgraph of H', then H' must be disconnected.).

Exercise 3: Prove that every nonempty graph contains at least one connected component. (hint: choose an arbitrary vertex and put it in H. If H is a maximal subgraph we are done, otherwise choose a vertex to add to H, etc.)

Proof. Let $G = (V, E)$ be a nonempty graph. Define the subgraph of G called $H = (V', E')$. Define $E' = \{\{x, y\} : \{x, y\} \in E \text{ and } x, y \in V'\}$. We can start by taking an arbitrary element $v_1 \in V$ and adding it to V'. If H is maximal, we are done. If not, there is some $v_2 \in V$ and $v_2 \notin V'$ such that $\{v_1, v_2\} \in E$. Then we can add v_2 to V'. We can repeat this process until:

Case 1: We run out of vertices. If there are no more vertices to add, then $V' = V$. So we know that $H = (V, E')$ is connected. But now, $E' = \{\{x, y\} : \{x, y\} \in E \text{ and } x, y \in V\} = E$. Thus, our resultant graph $H = (V, E)$ is maximal and connected.

Case 2: We do not run out of vertices. Then there is a vertex in V called v_i such that for every $v_j \in V'$, $\{v_j, v_i\} \notin E$. Observe that v_i in H has degree 0.

Since H has at least one vertex, H must be disconnected since there is no path to v_i. So $H = (V' - \{v_i\}, E')$ is maximal and connected. □

Exercise 4: Prove that every graph $G = (V, E)$ is a disjoint union of its connected components, i.e., there are subgraphs $G_1, \ldots, G_k$ for some k with disjoint sets of vertices V_i such that each G_i is a connected component of G and $\cup_{i=1}^{k} V_i = V$. (hint: induction)

Proof. We will proceed by induction. Let $P(n)$ be the statement that for $|V|= n$, $G = (V, E)$ is a disjoint union of its connected components.

Basis Step: We must show that $P(1)$ is true. Observe that for a graph with 1 vertex v_1, we have one connected component $G_1 = (\{v_1\}, E)$. Observe that $P(1)$ holds since $G = G_1$.

Inductive Step: Suppose that $k \in \mathbb{Z}_+$ is arbitrary and that $P(k)$ is true. We must show that $P(k+1)$ is true. Let $G(V, E)$ be the graph with k vertices. Let $G(V \cup \{v_{k+1}\}, E')$ be the graph with $k+1$ vertices where $v_{k+1} \notin V$. We say that there exists disjoint connected components of $G = (V, E) = G_1 \cup G_2 \cup \ldots \cup G_k$. There are two possibilities:

Case 1: v_{k+1} has degree 0. We can place vertex v_{k+1} in an empty subgraph G_{k+1}. Observe that G_{k+1} is a connected component. Also notice that adding v_{k+1} to $G_1, G_2, \ldots,$ or G_k makes one of the subgraphs disconnected. Then $V_1 \cup V_2 \cup \ldots \cup V_{k+1} = V \cup \{v_{k+1}\}$ and G is a disjoint union of connected components.

Case 2: v_{k+1} has degree $n \geq 1$. Then it follows that we can add v_{k+1} to, without loss of generality, $G_1, G_2, \ldots, G_n$ such that the subgraphs will remain connected. Observe that $V_1, V_2, \ldots, V_n$ are not disjoint, so take their union and call it V'_k.

G'_k is connected: Suppose two arbitrary vertices a, b are in different graphs of $G_1, G_2, \ldots, G_n$. Let a', b' be vertices in the same graph as a, b respectively such that $\{a', v_{k+1}\}, \{b', v_{k+1}\} \in E$. We can construct a path between a, b:

$$a, \ldots, a', \{a', v_{k+1}\}, v_{k+1}, \{b', v_{k+1}\}, b', \ldots, b$$

G'_k is maximal: Observe that $G_{n+1}, \ldots, G_k$ are maximal. It follows that we cannot add vertices or edges to the subgraphs such that they remain connected. But that means that any vertex from $G_{n+1}, \ldots, G_k$ we add to G'_k will make G'_k disconnected. Thus, G'_k is maximal.

$V_{n+1}, \ldots, V_k, V'_k$ are disjoint. Observe that $V_1, \ldots, V_k$ are disjoint.

We will prove that for three disjoint sets, the union of two of them will be disjoint to the third. Let A, B, C be disjoint sets. We call A, B disjoint if for any $y \in A$, $y \notin B$. Consider $A \cup B$. Let $x \in A \cup B$ be arbitrary. Observe that $x \in A$ or $x \in B$. Then in either case, $x \notin C$. So $A \cup B$ is disjoint to C.

Since $V_1, \ldots, V_k$ are disjoint, $V_1 \cup \ldots \cup V_n, V_{n+1}, \ldots, V_k$ are disjoint. Since $v_{k+1} \notin V_{n+1}, \ldots, V_k$, we have $V_1 \cup \ldots \cup V_n \cup \{v_{k+1}\}, V_{n+1}, \ldots, V_k$ are disjoint. Thus, $V_{n+1}, \ldots, V_k, V'_k$ are disjoint. We conclude that G is a disjoint union of connected components $G_{n+1}, \ldots, G_k, G'_k$ as desired. □

Exercise 5: The complement of a simple graph $G = (V, E)$ is the graph

$$\bar{G} = (V, \{\{u, v\} : \{u, v\} \text{ is not an element of } G\}),$$

i.e., the graph whose edges are the non-edges of G. Show that for every graph G, if G is not connected, then its complement $\bar{G}$ must be connected. (hint: consider the connected components of G)

Proof. Suppose that G is not connected. We know that every graph is made up of disjoint connected components. Let G be the disjoint union of connected components $G_1, \ldots, G_n$ where $n \geq 2$. We will show that $\bar{G}$ is connected. Observe that for each element $v_i \in G_i$, there will be an edge $\{v_i, v_j\} \in \bar{G}$ for all $v_j \in G_j$ where $G_j \neq G_i$. We have two cases where we need to show a path exists in $\bar{G}$:

Case 1: Two vertices are in the same disjoint connected component of G. Observe that for two arbitrary vertices $v_{i_1}, v_{i_2} \in G_i$, we can construct a path between them in the graph $\bar{G}$:

$$v_{i_1}, \{v_{i_1}, v_j\}, v_j, \{v_{i_2}, v_j\}, v_{i_2}$$

Case 2: Two vertices are not in the same disjoint connected component of G. Observe for two arbitrary vertices v_i, v_j not both in the same subgraph, $G_1, G_2, \ldots, G_n$, we can construct a path between them:

Subcase 1: $n = 2$. We have two disjoint subgraphs G_1, G_2 and by definition of $\bar{G}$, $\{v_1, v_2\}$ is an edge in $\bar{G}$ ($v_1 \in G_1$ and $v_2 \in G_2$). Thus, our path in $\bar{G}$ is:

$$v_1, \{v_1, v_2\}, v_2$$

Subcase 2: $n > 2$. We have three or more disjoint subgraphs. Then we can consider an arbitrary subgraph G_h where $G_h \neq G_i, G_j$. Let $v_h \in G_h$ be arbitrary. By definition of $\bar{G}$, $\{v_h, v_i\}, \{v_h, v_j\}$ are edges in $\bar{G}$. Thus, our path looks like:

$$v_i, \{v_h, v_i\}, v_h, \{v_h, v_j\}, v_j$$

Therefore, we know that all vertices in our graph $\bar{G}$ are connected by some path, as desired. □