Math128aHw4

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Exercise Set 3.1

Exercise 2: For the given functions $f(x)$, let $x_0 = 1, x_1 = 1.25$, and $x_2 = 1.6$. Construct interpolation polynomials of degree at most one and at most two of approximate $f(1.4)$ and find the absolute error.

(d) $f(x) = e^{2x} - x$

Answer. Given x_0, x_1, x_2, we get that $y_0, y_1, y_2 = e^2 - 1, e^{2.5} - 1.25, e^{3.2} - 1.6$. Here is the degree 2 polynomial to approximate:

$$f(x) = (e^2 - 1)\frac{(x - 1.25)(x - 1.6)}{(1 - 1.25)(1 - 1.6)} + (e^{2.5} - 1.25)\frac{(x - 1)(x - 1.6)}{(1.25 - 1)(1.25 - 1.6)} + (e^{3.2} - 1.6)\frac{(x - 1)(x - 1.25)}{(1.6 - 1)(1.6 - 1.25)}$$

and here is the degree 1:

$$f(x) = (e^2 - 1)\frac{x - 1.6}{1 - 1.6} + (e^{3.2} - 1.6)\frac{x - 1}{1.6 - 1}$$

We get:

$$f(1.4) = (e^2 - 1)\frac{-.15 \cdot .2}{.15} + (e^{2.5} - 1.25)\frac{.4 \cdot -.2}{.25 \cdot -.35} + (e^{3.2} - 1.6)\frac{.4 \cdot .15}{.6 \cdot .35} = 15.2697633149$$

for the second degree approximation and

$$f(1.4) = (e^2 - 1)\frac{-0.2}{-.6} + (e^{3.2} - 1.6)\frac{.4}{.6} = 17.418038831$$

the true value is $f(1.4) = 15.0446467711$, so the absolute error for the first degree approximation is

$$|15.0446467711 - 15.2697633149| = 0.2251165438$$

and for the first degree approximation is

$$|15.0446467711 - 17.418038831| = 2.3733920599$$

Exercise 4: Use Theorem 3.3 to find an error bound for the approximations in Exercise 2.

(d) $f(x) = e^{2x} - x$

Answer. By the theorem, we know that there is some $\mathcal{E}$ in the interval $(1, 1.6)$ such that

$$f(x) = P(x) + \frac{f^{n+1}(\mathcal{E})}{(n + 1)!}(x - 1)(x - 1.25)(x - 1.6)$$

Considering the second degree polynomial interpolation, we get:

$$f(x) = P(x) + \frac{f^3(\mathcal{E})}{6}(x-1)(x-1.25)(x-1.6)$$

and

$$f^1(x) = 2e^{2x} - 1, f^2(x) = 4e^{2x}, f^3(x) = 8e^{2x}$$

The maximum of $f^3(x)$ on the interval is at $x = 1.6$, so we get:

$$f^3(1.6) = 8e^{3.2}$$

Plugging this in:

$$|\text{Error}| <= \left|\frac{8e^{3.2}}{6}(.4)(.15)(.2)\right| = 0.392520483154$$

Now for the first degree interpolation, the error will be given by:

$$\frac{f^2(\mathcal{E})}{2}(x-1)(x-1.6)$$

We have:

$$|f^2(\mathcal{E})| < |4e^{3.2}|$$

So plugging this in:

$$|\text{Error}| <= \left|\frac{4e^{3.2}}{2}(.4)(-.2)\right| = 3.92520483154$$

Exercise 6: Use appropriate Lagrange interpolating polynomials of degrees one, two and three to approximate each of the following:

(b) $f(0)$ if $f(-0.5) = 1.93750, f(-0.25) = 1.33203, f(0.25) = 0.800781, f(0.5) = 0.687500$

Answer. The first degree polynomial:

$$f(x) = 1.33203\frac{x-0.25}{(-0.25-0.25)} + 0.800781\frac{x+0.25}{(0.25+0.25)}$$

The second degree polynomial:

$$\begin{aligned} f(x) = {} & 1.93750\frac{(x+0.25)(x-0.25)}{(-0.5+0.25)(-0.5-0.25)} \\ & + 1.33203\frac{(x+0.5)(x-0.25)}{(-0.25+0.5)(-0.25-0.25)} \\ & + 0.800781\frac{(x+0.5)(x+0.25)}{(0.25+0.5)(0.25+0.25)} \end{aligned}$$

The third degree polynomial:

$$\begin{aligned} f(x) = {} & 1.93750\frac{(x+0.25)(x-0.25)(x-0.5)}{(-0.5+0.25)(-0.5-0.25)(-0.5-0.5)} \\ & + 1.33203\frac{(x+0.5)(x-0.25)(x-0.5)}{(-0.25+0.5)(-0.25-0.25)(-0.25-0.5)} \\ & + 0.800781\frac{(x+0.5)(x+0.25)(x-0.5)}{(0.25+0.5)(0.25+0.25)(0.25-0.5)} \\ & + 0.687500\frac{(x+0.5)(x+0.25)(x-0.25)}{(0.5+0.5)(0.5+0.25)(0.5-0.25)} \end{aligned}$$

Now we plug the values in:

$$f(0) = 1.33203\frac{0 - 0.25}{(-0.25 - 0.25)} + 0.800781\frac{0 + 0.25}{(0.25 + 0.25)} = 1.0664055$$

and

$$\begin{aligned} f(0) = 1.93750&\frac{(0 + 0.25)(0 - 0.25)}{(-0.5 + 0.25)(-0.5 - 0.25)} \\ &+ 1.33203\frac{(0 + 0.5)(0 - 0.25)}{(-0.25 + 0.5)(-0.25 - 0.25)} \\ &+ 0.800781\frac{(0 + 0.5)(0 + 0.25)}{(0.25 + 0.5)(0.25 + 0.25)} \\ &= 0.953123666667 \end{aligned}$$

and

$$\begin{aligned} f(0) = 1.93750&\frac{(0 + 0.25)(0 - 0.25)(0 - 0.5)}{(-0.5 + 0.25)(-0.5 - 0.25)(-0.5 - 0.5)} \\ &+ 1.33203\frac{(0 + 0.5)(0 - 0.25)(0 - 0.5)}{(-0.25 + 0.5)(-0.25 - 0.25)(-0.25 - 0.5)} \\ &+ 0.800781\frac{(0 + 0.5)(0 + 0.25)(0 - 0.5)}{(0.25 + 0.5)(0.25 + 0.25)(0.25 - 0.5)} \\ &+ 0.687500\frac{(0 + 0.5)(0 + 0.25)(0 - 0.25)}{(0.5 + 0.5)(0.5 + 0.25)(0.5 - 0.25)} \\ &= 0.984374 \end{aligned}$$

Exercise 8: The data for Exercise 6 were generated using the following functions. Use the error formula to find a bound for the error and compare the bound to the actual error for the cases $n = 1$ and $n = 2$.

(b) $f(x) = x^4 - x^3 + x^2 - x + 1$

Answer. We first find the second and third derivatives of the equation:

$$\begin{aligned} f^{(1)}(x) &= 4x^3 - 3x^2 + 2x - 1 \\ f^{(2)}(x) &= 12x^2 - 6x + 2 \\ f^{(3)}(x) &= 24x - 6 \end{aligned}$$

For the one degree polynomial interpolation, I used the x-values $-0.25, 0.25$, so the absolute value of the max of $f^{(2)}(x)$ on the interval is 4.25. Then the error bound is given by:

$$|\text{Error}| \leqslant \left|\frac{4.25}{2}(0 - 0.25)(0 + 0.25)\right| = 0.1328125$$

For the second degree polynomial interpolation, I used the x-values $-0.5, -0.25, 0.25$, so the absolute value of the max of $f^{(3)}(x)$ on the interval $(-0.5, 0.25)$ is -18. Then the error bound is given by:

$$|\text{Error}| \leqslant \left|\frac{18}{6}(0.5)(.25)(0.25)\right| = 0.09375$$

Error Comparison: We see that $f(0) = 1$, So the error computed for the first degree approximation and second degree approximation are $|1 - 1.0664055| = 0.0664055$ and $|1 - 0.953123666667| = 0.046876333333$. We see that the absolute error is indeed smaller than the error given by the error bound.

Exercise 21: Show that $\max_{x_j \leqslant x \leqslant x_{j+1}} |g(x)| = h^2/4$, where $g(x) = (x - jh)(x - (j+1)h)$.

Answer. Since $j \leqslant x \leqslant j+1$, we have $g(x) = h^2(x-j)(x-(j+1))$ achieving its max at the midpoint of the roots $j, j+1$. So the max is at $x = \frac{2j+1}{2}$, and $\frac{2j+1}{2} - j = \frac{1}{2}$. Therefore, $|g(\frac{2j+1}{2})| = \frac{h^2}{4}$. Which is the max of $|g(x)|$.

Exercise 22: Prove Taylor's Theorem 1.14 by following the procedure in the proof of Theorem 3.3. Hint: Let

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \cdot \frac{(t - x_0)^{n+1}}{(x - x_0)^{n+1}}$$

where P is the nth Taylor polynomial, and use Generalized Rolle's Theorem.

Answer. Let P be the nth Taylor expansion and $R_n(x) = \frac{f^{n+1}(x)}{(n+1)!}(x - x_0)$. If $x = x_0$, $\mathcal{E} = x_0$, and therefore, $R_n(\mathcal{E}) = \frac{f^{n+1}(\mathcal{E})}{(n+1)!}(0) = 0$, and $f(x_0) = x_0 = P(x_0)$. So

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \cdot \frac{(t - x_0)^{n+1}}{(x - x_0)^{n+1}}$$

is continuous. At $t = x_0$, we get $g(t) = 0$, and when $t = x$, we get $g(t) = 0$ also. Then this means that on the interval $[x_0, x]$, by generalized Rolle's theorem, there is an $\mathcal{E}$ such that:

$$g^{(n+1)}(\mathcal{E}) = 0 = f^{(n+1)}(\mathcal{E}) - P^{(n+1)}(\mathcal{E}) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \frac{(t - x_0)^{n+1}}{(x - x_0)^{n+1}}$$

and

$$\frac{d^{n+1}}{dt^{n+1}} \frac{(t - x_0)^{n+1}}{(x - x_0)^{n+1}} = \frac{(n+1)!}{(x - x_0)^{n+1}}$$

This means that

$$\begin{aligned} [f(x) - P(x)] \frac{(n+1)!}{(x - x_0)^{n+1}} &= f^{(n+1)}(\mathcal{E}) - P^{(n+1)}(\mathcal{E}) \\ f(x) - P(x) &= \frac{f^{(n+1)}(\mathcal{E})}{(n+1)!}(x - x_0)^{n+1} = R_n(\mathcal{E}) \end{aligned}$$

as P is a polynomial of degree n.

Discussion 2: If we decide to increase the degree of the interpolating polynomial by adding nodes, is there an easy way to use a previous interpolating polynomial to obtain a higher-degree interpolating polynomial, or do we need to start over?

Answer. Yes there is a way. Suppose that we have an interpolating polynomial for $x_0, x_1, \ldots, x_n$ called $f_n(x)$ which gives us the values $y_0, \ldots, y_n$ respectively. If we want to add another node x_{n+1} such that $f(x_i) = y_i$, we can do:

$$f(x) = f_n(x) \cdot \sum_{j=0}^{n} \frac{\prod_{i \neq j}^{n+1}(x - x_i)}{\prod_{i \neq j}^{n+1}(x_j - x_i)} + y_{n+1} \cdot \frac{(x - x_0) \cdots (x - x_n)}{(x_{n+1} - x_0) \cdots (x_{n+1} - x_n)}$$

To see that this works, in the term:

$$\sum_{j=0}^{n} \frac{\prod_{i \neq j}^{n+1}(x - x_i)}{\prod_{i \neq j}^{n+1}(x_j - x_i)}$$

each summand evaluates to 0 if $x = x_i$ for $i = 0, \dots, n+1, i \neq j$. But if $x = x_j$, then the numerator and denominator of the summand are equal and the summand evaluates to 1. Since x is only one of x_i for $i = 1, \dots, n$, We have that the whole sum is 1 if x is one of x_i for $i = 1, \dots, n$. Now if $x = x_{n+1}$, it evaluates to 0 because of the $x - x_{n+1}$ term in the numerator. The right term is just a Lagrange polynomial, so this shows that the interpolation for one extra term works.

Exercise Set 3.2

Exercise 6: Neville's method is used to approximate $f(0.5)$, giving the following table.

$$\begin{array}{llll} x_0 = 0 & P_0 = 0 & & \\ x_1 = 0.4 & P_1 = 2.8 & P_{0,1} = 3.5 & \\ x_2 = 0.7 & P_2 & P_{1,2} & P_{0,1,2} = \frac{27}{7} \end{array}$$

Determine $P_2 = f(0.7)$.

Answer. To get get P_2, we need to solve for $P_{1,2}$ first. We have the relation that:

$$P_{0,1,2}(x) = \frac{(x-0)P_{1,2}(x) + (x-0.7)P_{0,1}(x)}{(0.7-0)}$$

So

$$P_{0,1,2}(0.5) = \frac{0.5P_{1,2}(0.5) - 0.2(3.5)}{.7} = \frac{27}{7}$$

and therefore,

$$P_{1,2}(0.5) = ((27/7) * .7 + 0.2(3.5))/0.5 = 6.8$$

Now we can do the same thing but using $P_{1,2}, P_1$:

$$P_{1,2}(x) = \frac{(x-0.4)P_2(x) + (x-0.7)P_1(x)}{0.7-0.4}$$

then

$$P_{1,2}(0.5) = \frac{0.1P_2(0.5) - 0.5(2)}{0.3} = 6.8$$

so

$$P_2(0.5) = ((0.3) * 6.8 + 1)/0.1 = 30.4$$

Exercise 10: Neville's Algorithm is used to approximate $f(0)$ using $f(-2), f(-1), f(1)$, and $f(2)$. Suppose $f(-1)$ was overstated by 2 and $f(1)$ was understated by 3. Determine the error in the original calculation of the value of the interpolating polynomial to approximate $f(0)$.

Answer. Using Neville's algorithm, I got:

$$\begin{array}{cccccc} x_0 & -2 & f(-2) & & & \\ x_1 & -1 & f(-1) & f(-2)+2f(-1) & & \\ x_2 & 1 & f(1) & \frac{f(-1)-f(1)}{2} & \frac{-f(-2)-f(-1)-f(1)}{3} & \\ x_3 & 2 & f(2) & -f(2)-2f(1) & \frac{f(2)+3f(1)-f(-1)}{3} & \frac{1}{3}(f(-2)+2f(-1)+2f(1)+f(2)) \end{array}$$

So the rightmost column is the correct approximation. The original calculation would substitute $f(-1) \to f(-1) + 2, f(1) \to f(1) - 3$. This give:

$$\frac{1}{3}(f(-2) + 2f(-1) + 4 + 2f(1) - 6 + f(2)) = \frac{1}{3}(f(-2) + 2f(-1) + 2f(1) + f(2)) - \frac{2}{3}$$

So the error is $-2/3$.

Discussion 2: Can Neville's method be used to obtain the interpolation polynomial at a general point as opposed to a specific point?

Answer. Yes it can, by using Neville's method, but without fixing the approximation point.

Exercise Set 3.3

Exercise 8:

(a) Use Algorithm 3.2 to construct the interpolating polynomial of degree four for the unequally spaced points given in the following table:

x	f(x)
0.0	-6.00000
0.1	-5.89483
0.3	-5.65014
0.6	-5.17788
1.0	-4.28172

Answer. I got:

$$\begin{aligned} f(s) = s * ((((1135193048929907 * s)/18014398509481984 \\ + 3989974812726791/22517998136852480) * (s - 3/10) \\ + 5156621573339067/9007199254740992) * (s - 1/10) + 10517/10000) - 6 \end{aligned}$$

using matlab. Here is the code:

```
function f = NewtonForwardDifference(x, y)
    sz = size(x, 1);
    p = zeros(sz);
    p(:, 1) = y;
    for i = 2:size(x)
        for j = 2:i
            p(i, j) = (p(i, j-1) - p(i-1, j-1)) / (x(i) - x(i-j+1));
        end
    end
    f = zeros(sz, 1);
    for i = 1:sz
        f(i) = p(i, i);
    end

end

function f = forwardPoly(x, c)
    syms s;
    f = c(end);
    for i = 1:size(c)-1
        f = f * (s - x(end-i));
        f = f + c(end-i);
    end
end

x = [0.0; 0.1; 0.3; 0.6; 1.0];
y = [-6.00000; -5.89483; -5.65014; -5.17788; -4.28172];
F = NewtonForwardDifference(x, y)
f = forwardPoly(x, F)
f = matlabFunction(f)
```

(b) Add $f(1.1) = -3.99583$ to the table and construct the interpolating polynomial of degree five.

Answer. I got:

$$\begin{aligned} f(s) = s * (((((8162368155838225 * s)/576460752303423488 \\ + 2816380940991879 9/576460752303423488) * (s - 3/5) \\ + 3873095679539377/18014398509481984) * (s - 3/10) \\ + 5156621573339067/9007199254740992) * (s - 1/10) + 10517/10000) - 6 \end{aligned}$$

from the same matlab code above.

Exercise 13: The newton forward-difference formula is used to approximate f(0.3) given the following data.

x	0.0	0.2	0.4	0.6
f(x)	15.0	21.0	30.0	51.0

Suppose it is discovered that f(0.4) was understated by 10 and f(0.6) was overstated by 5. By what amount should the approximation to f(0.3) be changed?

Answer. Using matlab, I got an error of -5.9375. This means that we should add 5.9375 to our approximation to get the correct approximation. Here is my code:

```
function f = NewtonForwardDifference(x, y)
    sz = size(x, 1);
    p = zeros(sz);
    p(:, 1) = y;
    for i = 2:size(x)
        for j = 2:i
            p(i, j) = (p(i, j-1) - p(i-1, j-1)) / (x(i) - x(i-j+1));
        end
    end
    f = zeros(sz, 1);
    for i = 1:sz
        f(i) = p(i, i);
    end

end

function f = forwardPoly(x, c)
    syms s;
    f = c(end);
    for i = 1:size(c)-1
        f = f * (s - x(end-i));
        f = f + c(end-i);
    end
end

x = [0.0; 0.2; 0.4; 0.6];
y = [15.0; 21.0; 30.0; 51.0];
y_err = [15.0; 21.0; 20.0; 56.0];

c1 = NewtonForwardDifference(x, y);
f_true = matlabFunction(forwardPoly(x, c1));
a1 = f_true(0.3);

c2 = NewtonForwardDifference(x, y_err);
f_err = matlabFunction(forwardPoly(x, c2));
a2 = f_err(0.3);
```

```
err = a2 - a1
```

Exercise 18: The fastest time ever recorded in the Kentucky Derby was by a horse named Secretariat in 1973. He covered the $1\frac{1}{4}$ mile track in $1 : 59\frac{2}{5}$ (1 minute and 59.4 seconds). Times at the quarter-mile, half-mile, and mile poles were $0 : 25\frac{1}{5}, 0 : 49\frac{1}{5}$, and $1 : 36\frac{2}{5}$.

(a) Use interpolation to predict the time at the three-quarter mile pole and compare this to the actual time of $1 : 13$.

Answer. We are given the table:

x_0	.25	25.2
x_1	.5	49.2
x_2	1	96.4
x_3	1.25	119.4

We have that:

$$f[x_0, x_1] = \frac{49.2 - 25.2}{.25} = 96$$
$$f[x_1, x_2] = \frac{96.4 - 49.2}{.5} = 94.4$$
$$f[x_2, x_3] = \frac{119.4 - 96.4}{.25} = 92$$

and:

$$f[x_0, x_1, x_2] = \frac{94.4 - 96}{.75} = \frac{-1.6}{.75} = \frac{-6.4}{3}$$
$$f[x_1, x_2, x_3] = \frac{92 - 94.4}{.75} = \frac{-2.4}{.75} = \frac{-9.6}{3} = -3.2$$

and finally:

$$f[x_0, x_1, x_2, x_4] = \frac{-3.2 + \frac{6.4}{3}}{1} = \frac{-3.2}{3}$$

Putting this all together:

$$\begin{aligned} P_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\ &= 25.2 + 96(x - 0.25) - \frac{6.4}{3}(x - 0.25)(x - .5) - \frac{3.2}{3}(x - 0.25)(x - .5)(x - 1) \end{aligned}$$

Plugging in $x = .75$, I got $f(x) = 72.9666$ which is $1 : 12\frac{9}{10}$. This is very close to $1 : 13$.

(b) Use the derivative of the interpolating polynomial to estimate the speed of Secretariat at the end of the race.

Answer. Using matlab, I got $f'(119.4) = 0.0110$. This means that Secretariat was moving 0.0110 mi/s or 39.6 mph. Here is the code:

```
function f = NewtonForwardDifference(x, y)
    sz = size(x, 1);
    p = zeros(sz);
    p(:, 1) = y;
    for i = 2:size(x)
        for j = 2:i
            p(i, j) = (p(i, j-1) - p(i-1, j-1)) / (x(i) - x(i-j+1));
        end
```

```
    end
    f = zeros(sz, 1);
    for i = 1:sz
        f(i) = p(i, i);
    end

end

function f = forwardPoly(x, c)
    syms s;
    f = c(end);
    for i = 1:size(c)-1
        f = f * (s - x(end-i));
        f = f + c(end-i);
    end
end

x = [25.2; 49.2; 96.4; 119.4];
y = [0.25; 0.5; 1; 1.25];
c = NewtonForwardDifference(x, y)
f = forwardPoly(x, c)
f = diff(f);
f = matlabFunction(f)
s = double(f(119.4))
```

Exercise 20:

(a) Show that the cubic polynomials

$$P(x) = 3 - 2(x+1) + 0(x+1)(x) + (x+1)(x)(x-1)$$

and

$$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + (x+2)(x+1)(x)$$

both interpolate the data

x	-2	-1	0	1	2
f(x)	-1	3	1	-1	3

Answer. We can quickly evaluate P at $-1, 0, 1$:

$$\begin{aligned} P(0) &= 3 - 2 = 1 \\ P(-1) &= 3 \\ P(1) &= 3 - 4 = -1 \end{aligned}$$

Now for $P(-2), P(2)$:

$$\begin{aligned} P(-2) &= 3 + 2 + (-1)(-2)(-3) \\ &= 5 - 6 = -1 \\ P(2) &= 3 - 6 + 3(2)(1) \\ &= -3 + 6 = 3 \end{aligned}$$

We can quickly evaluate Q at $0, -1, -2$:

$$\begin{aligned} Q(-2) &= -1 \\ Q(-1) &= -1 + 4 = 3 \\ Q(0) &= -1 + 8 - 6 = 1 \end{aligned}$$

and for $Q(1), Q(2)$:

$$\begin{aligned}
Q(1) &= -1 + 12 - 3(3)(2) + (3)(2)(1) \\
&= 11 - 18 + 6 \\
&= -1 \\
Q(2) &= -1 + 16 - 3(4)(3) + (4)(3)(2) \\
&= 15 - 36 + 24 \\
&= 3
\end{aligned}$$

(b) Why does part (a) not violate the uniqueness property of interpolating polynomials?

Answer. It does not violate the uniqueness property because they are the same polynomial when expanded out.

Exercise 21: Given

$$\begin{aligned}
P_n(x) = f[x_0] &+ f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) \\
&+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots \\
&+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}),
\end{aligned}$$

use $P_n(x_2)$ to show that $a_2 = f[x_0, x_1, x_2]$.

Answer. Plugging in x_2, we get:

$$f(x_2) = P_n(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

Move a_2 to one side:

$$f(x_2) - f(x_0) - f[x_0, x_1](x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1)$$

and divide by $x_2 - x_1$:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} + \frac{f(x_1) - f(x_0)}{x_2 - x_1} - \frac{(f(x_1) - f(x_0))(x_2 - x_0)}{(x_1 - x_0)(x_2 - x_1)} = a_2(x_2 - x_0)$$

and so on:

$$\begin{aligned}
f[x_1, x_2] + \frac{(f(x_1) - f(x_0))(x_1 - x_0)}{(x_2 - x_1)(x_1 - x_0)} - \frac{(f(x_1) - f(x_0))(x_2 - x_0)}{(x_2 - x_1)(x_1 - x_0)} &= a_2(x_2 - x_0) \\
f[x_1, x_2] + \frac{(f(x_1) - f(x_0))(x_1 - x_2)}{(x_2 - x_1)(x_1 - x_0)} &= a_2(x_2 - x_0) \\
f[x_1, x_2] - \frac{f(x_1) - f(x_0)}{x_1 - x_0} &= a_2(x_2 - x_0) \\
f[x_1, x_2] - f[x_0, x_1] &= a_2(x_2 - x_0) \\
\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} &= a_2
\end{aligned}$$

which is the definition of $f[x_0, x_1, x_2]$.