Hw 1.8 ④ $x^3+y^3=z^3$

Proof. ~~Since $z^3 < 10$~~ Suppose x, y, z are positive integers and less than 10. Now consider the equation $x^3+y^3=z^3$. We wish to show that for any x, y, z, $x^3+y^3 \neq z^3$. The possible values for z are

$z = 1, 2, 3, 4, 5, 6, 7, 8, 9$

therefore,

$z^3 = 1, 8, 27, 64, 125, 196, 343, 512, 729$.

Notice that each ~~digit~~ value of z^3 has a unique one's digit. The same must go for x^3 and y^3. So for a set z^3 and $y^3 < z^3$, there is only one x^3 value such that $\frac{x^3+y^3}{10}$ has the same remainder as $\frac{z^3}{10}$.

Case 1: $z^3 = 729$:

$1+8 \neq 729$, $27+512 \neq 729$, $64+125 \neq 729$, $196+343 \neq 729$

Therefore, $z^3 \neq 729$

Case 2: $z^3 = 512$

$8+64 \neq 512$, $27+125 \neq 512$, $196+196 \neq 512$, $1+1 \neq 512$

Therefore, $z^3 \neq 512$

Case 3: $z^3 = 343$

$8+125 \neq 343$, $27+196 \neq 343$,

Therefore, $z^3 \neq 343$

Case 4: $z^3 = 196$

$1+125 \neq 196$, $8+8 \neq 196$

Therefore, $z^3 \neq 196$

Case 5: $z^3 = 125$

$1+64 \neq 125$, $8+27 \neq 125$

Therefore, $z^3 \neq 125$

Case 6: $z^3 = 64$: ~~No two numbers share the property~~ $27+27 \neq 64$

Case 7: $z^3 = 27$: No two numbers share the property

Case 8: $z^3 = 8$: No two numbers share the property

Case 9: $z^3 = 1$: No cubes less than 1.

As desired.

(12) $\neg$ perfect square $(2 \cdot 10^{500} + 15) \vee \neg$ perfect square $(2 \cdot 10^{500} + 16)$.
Suppose for contradiction, $2 \cdot 10^{500} + 15$ and $2 \cdot 10^{500} + 16$ are both perfect squares. Then

$$2 \cdot 10^{500} + 15 = n^2$$
$$2 \cdot 10^{500} + 16 = (n+a)^2$$

For integers n, a where $a \geq 1$. Observe that

$$(2 \cdot 10^{500} + 16) - (2 \cdot 10^{500} + 15) = (n+a)^2 - n^2$$
$$1 = 2an + a^2$$

Notice that

$$2an > 1$$

If ~~a,n≠2~~ $a, n \neq 0$. Since $a \geq 1$ and $n = \sqrt{2 \cdot 10^{500} + 15} \neq 0$, we cannot have the case that

$$1 = 2an + a^2$$

Contradiction as desired.

- This is nonconstructive since it is not known if $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ are perfect squares or not.

(16) $\forall a, b (a, b \in \mathbb{Q} \rightarrow a^b \in \mathbb{Q})$
Proof by counterexample. We will prove that there exists $a, b \in \mathbb{Q}$ such that $a^b \notin \mathbb{Q}$. Let $a = 2$, $b = \frac{1}{2}$. Then $a^b = 2^{1/2} = \sqrt{2}$. So $a^b = \sqrt{2}$ which is irrational as desired.

(17) Show that if there are two or more inputs that make $P(x)$ true, then the inputs are not unique.

a) $\exists x \forall y (P(y) \leftrightarrow x = y)$
Suppose $P(x_1)$ and $P(x_2)$. Since $\exists x \forall y$, we can choose an x_3 such that $x_3 = x_1$ and $x_3 = x_2$. Therefore $x_1 = x_2$ as desired.

b) $\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$
Suppose $P(x_1)$ and $P(x_2)$. Since $P(x) \wedge P(y) \rightarrow x = y$, $x_1 = x_2$ as desired.

c) $\exists x (P(x) \wedge \forall y (P(y) \rightarrow x = y))$
~~Since $\exists x P(x)$ choose an x_1 such that $P(x_1)$ and~~
Suppose $P(x_1)$ and $P(x_2)$. There also exists an x_3 such that $x_3 = x_1, x_2$ and $P(x_3)$. The solutions are not unique as desired.

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(36) $\sqrt[3]{2} \notin \mathbb{Q}$

Suppose $\sqrt[3]{2}$ is rational for contradiction. Then

$$\sqrt[3]{2} = \frac{p}{q}$$

for some $p, q \neq 0$ and relatively prime integers. It follows that

$$2 = \frac{p^3}{q^3} \rightarrow 2q^3 = p^3$$

So p^3 is even. Now to prove that if p^3 is even, p is even, suppose p is odd. So

$$p = 2a+1$$

Then

$$p^3 = (2a+1)^3 = 8a^3 + 3(4a^2) + 3(2a) + 1 = 2(4a^3 + 6a^2 + 3a) + 1$$

Since p^3 is odd, we have proven the contrapositive.

Since p^3 is even (original problem), p is even:

$$p = 2b \qquad b \in \mathbb{Z}$$

Now

$$2q^3 = (2b)^3 = 8b^3$$

$$q^3 = 4b^3$$

So q^3 is also even and thus q is even. This contradicts the fact that p and q are relatively prime as desired.

2.1 (1a) $x^2 = 1$ members: $1, -1$

$x = \pm 1$

$\{1, -1\}$

d) $\{x \mid x \in \mathbb{Z} \wedge x^2 = 2\}$

$x = \pm\sqrt{2} \notin \mathbb{Z}$

$\{\ \} \rightarrow$ no members

(19) $(A \subseteq B \wedge B \subseteq C) \rightarrow A \subseteq C$

Proof. Suppose $x \in A$. Since $A \subseteq B$, $x \in B$. Now since $B \subseteq C$, $x \in C$. Since $x \in A \rightarrow x \in B$ and $x \in B \rightarrow x \in C$, $x \in A \Rightarrow x \in C$ so $A \subseteq C$.

(24) $P(A) = P(B)$

$P(A) \subseteq P(B) \wedge P(B) \subseteq P(A)$ $\qquad S \in P(A)$

$S \in P(B)$

$S \subseteq A \wedge S \subseteq B \leftrightarrow$ $\qquad x \in S$

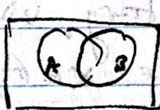
$x \in A \qquad x \in B$

$A \subseteq B$

Proof: Let A and B be two sets and $P(A)=P(B)$. So $P(A)\subseteq P(B)$ and $P(B)\subseteq P(A)$. We will show that $A\subseteq B$ and $B\subseteq A$.
Suppose $s\in P(A)$.
Since $P(A)\subseteq P(B)$, $s\in P(B)$ also. With $s\in P(A)$ and $s\in P(B)$, $s\subseteq A$ implies that $s\subseteq B$.
Now let $x\in s$. So $x\in A$ also. Since $s\subseteq B$, $x\in B$.
We now have from $x\in A$ implies $x\in B$ that $A\subseteq B$.
Notice that the second containment $B\subseteq A$ is completely analogous with the roles of A and B reversed.
With $A\subseteq B$ and $B\subseteq A$, $A=B$ as desired.

Hw 2.2 (2) a) $A\cap B$ c) $A\cup B$
b) $A\cap\overline{B}$ d) $\overline{A\cap B}$

(15) a) $\overline{A\cup B}\subseteq\overline{A}\cap\overline{B} \wedge \overline{A}\cap\overline{B}\subseteq\overline{A\cup B}$



Proof. Let A and B be sets. Let $x\in\overline{A\cup B}$ be arbitrary. So $x\notin A\cup B$. In other words, $x\notin A$ and $x\notin B$. From $x\notin A$, $x\in\overline{A}$ and from $x\notin B$, $x\in\overline{B}$. Since $x\in\overline{A}$ and $x\in\overline{B}$, $x\in\overline{A}\cap\overline{B}$.
Therefore, $\overline{A\cup B}\subseteq\overline{A}\cap\overline{B}$.
For $\overline{A}\cap\overline{B}\subseteq\overline{A\cup B}$, let $x\in\overline{A}\cap\overline{B}$ be arbitrary. So $x\in\overline{A}$ and $x\in\overline{B}$. It follows that $x\notin A$ and $x\notin B$. We can conclude that $x\notin A\cup B$ or $x\in\overline{A\cup B}$. Therefore, $\overline{A}\cap\overline{B}\subseteq\overline{A\cup B}$ as desired.

b)

A	B	$A\cup B$	$\overline{A\cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A}\cap\overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Proof. The membership table shows that for every combination of the membership of an element in sets A and B, $\overline{A\cup B}$ and $\overline{A}\cap\overline{B}$ have the same columns and therefore same memberships. The sets are equivalent.

(20) c) $(A-B)-C\subseteq A-C$
Proof. Suppose A, B, C are sets. Let $x\in(A-B)-C$ be arbitrary. Then $x\in A$ and $x\notin B$ and $x\notin C$. From $x\in A$ and $x\notin C$, we have $x\in A-C$. Since $x\in(A-B)-C$ implies $x\in A-C$, the containment is proven.

(2) $A \subseteq B$

a) $A \cup B = B$

Suppose A and B are sets such that $A \subseteq B$. ~~Let $a \in A$~~
We need to prove that $A \cup B \subseteq B$ and $B \subseteq A \cup B$.

For $A \cup B \subseteq B$, suppose $x \in A \cup B$ is arbitrary. Then by definition, $x \in A$ or $x \in B$.

Case 1: $x \in A$. It follows from $A \subseteq B$ that $x \in B$. Thus, ~~$A \subseteq B$~~, $A \cup B \subseteq B$

Case 2: $x \in B$. We assumed $x \in A \cup B$, so $A \cup B \subseteq B$.

Case 3: $x \in A$ and $x \in B$. ~~From $x \in B$~~, Since no contradiction arises from case 1 and 2, refer to them to conclude $A \cup B \subseteq B$.

Now for $B \subseteq A \cup B$, suppose $x \in B$ is arbitrary. By definition of $A \cup B$, $x \in A \cup B$ is true. Thus, $B \subseteq A \cup B$.

From $A \cup B \subseteq B$ and $B \subseteq A \cup B$, we can conclude $A \cup B = B$ if $A \subseteq B$.

b) $A \cap B = A$

Suppose A and B are sets such that $A \subseteq B$. We need to prove that $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

For $A \cap B \subseteq A$, suppose $x \in A \cap B$ is arbitrary. Then $x \in A$ and $x \in B$. Therefore, $A \cap B \subseteq A$.

For $A \subseteq A \cap B$, suppose $x \in A$. From $A \subseteq B$, it follows that $x \in B$ also. Therefore, $x \in A \cap B$ as desired.

From $A \cap B \subseteq A$ and $A \subseteq A \cap B$, we have $A \cap B = A$ if $A \subseteq B$.

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HW2.3 (43) $P(A\times B) = P(A)\times P(B)$
Counterexample. Let $A=\{1\}, B=\{1\}$. Then $A\times B=\{(1,1)\}$ and $P(A\times B)$ is subsequently $\{\emptyset, \{(1,1)\}\}$.

For $P(A)\times P(B)$, $P(A)=\{\emptyset,\{1\}\}$ and $P(B)=\{\emptyset,\{1\}\}$. That product is $\{(\emptyset,\emptyset),(\emptyset,\{1\}),(\{1\},\emptyset),(\{1\},\{1\})\}$. Notice that ~~these~~ the sets $P(A\times B)$ and $P(A)\times P(B)$ are not equal.

(44) $A\times B = A\times C$

$(x,y)\in A\times B$ $\quad x\in A \wedge y\in B$

$(x,y)\in A\times C$ $\quad x\in A \wedge y\in C$

Proof. Let A,B,C be sets such that $A\times B = A\times C$. Then $A\times B\subseteq A\times C$ and $A\times C\subseteq A\times B$. We wish to show that $B\subseteq C$ and $C\subseteq B$.

For the first containment $B\subseteq C$, suppose $(x,y)\in A\times B$ is arbitrary. By definition, $x\in A$ and $y\in B$. Also $(x,y)\in A\times C$ so $x\in A$ and $y\in C$. Since $y\in B$ and $y\in C$, $B\subseteq C$.

For the second containment, $C\subseteq B$, suppose $(x,y)\in A\times C$ is arbitrary. From $A\times C\subseteq A\times B$, $x\in A\times B$ also. So $x\in A$ and $x\in C$. Also, $x\in A$ and $x\in B$. Since $x\in C$ and $x\in B$, $C\subseteq B$.

Since $B\subseteq C$ and $C\subseteq B$, $B=C$ as desired.

(36) b) $A\times(B\cap C) = (A\times B)\cap(A\times C)$

Proof. Suppose A,B,C are sets. To prove equality, it is necessary to show that $A\times(B\cap C)\subseteq(A\times B)\cap(A\times C)$ and $(A\times B)\cap(A\times C)\subseteq A\times(B\cap C)$.

For the first containment, suppose $(x,y)\in A\times(B\cap C)$ is arbitrary. Then $x\in A$ and $y\in B$ and $y\in C$. It follows from $x\in A$ and $y\in B$ that $(x,y)\in A\times B$, but since $y\in C$ also, it must be true that $(x,y)\in A\times C$. Therefore, $(x,y)\in(A\times B)\cap(A\times C)$.

For the second containment, suppose $(x,y)\in(A\times B)\cap(A\times C)$. By definition $(x,y)\in A\times B$ and $(x,y)\in A\times C$. Therefore, $x\in A$ and $y\in B$ from the first statement and $x\in A$ and $y\in C$ from the second. Since $y\in B$ and $y\in C$, $y\in B\cap C$. Therefore, $(x,y)\in A\times(B\cap C)$ as desired.

Since both containments hold the equality is proven.

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(6) b) Domain: $\mathbb{Z}_+$ Range: ~~0-9~~ Integers ~~0-9~~ 1-9
$\{x \mid x \in \mathbb{Z} \wedge 0 < x \leq 9\}$

d) Domain: $\mathbb{Z}_+$. Range: $\mathbb{Z}_+$

(14) a) $f(m,n) = 2m - n$

Yes. Let $y \in \mathbb{Z}$ be arbitrary. Let $m = 0$ and $n = -y$. We have $f(0,-y) = y$. So there is an m and n such that $2m - n = y$.

b) $f(m,n) = m^2 - n^2$

Proof by Contradiction

No. Let $f(m,n) = 2$. $2 = m^2 - n^2$ or $(m+n)(m-n) = 2$

we have two cases:

① $m+n=1$, $m-n=2$ ② $m+n=2$, $m-n=1$

Now solve for m and n for each

① $m+n=1$, $m-n=2$ $2m=3$, $m=3/2$

② $m+n=2$, $m-n=1$ $2m=3$, $m=3/2$

Since m is not an integer, contradiction as desired.

(20) a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ d) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 7$

(28) $f: \mathbb{R} \to \mathbb{R}$

Choose a $b \in \mathbb{R}$ such that $b < 0$. For there to be a one to one correspondence, $b = e^x$ for some $x \in \mathbb{R}$. Therefore, $e^x < 0$ which is impossible. Therefore, the function is not invertible.

$f: \mathbb{R} \to \mathbb{R}_+$

Let $b \in \mathbb{R}_+$ be arbitrary. For $b = e^x$, $x = \ln(b)$ so the function is onto. ~~Since the [illegible] has only one [illegible] value,~~ Now suppose we have two inputs $x = a$ and $x = b$. Then $f(a) = e^a$ and $f(b) = e^b$. If $e^a = e^b$, then $a = b$. Since it is a bijection, the inverse can be defined as

$f^{-1}: \mathbb{R}_+ \to \mathbb{R}$

$f^{-1}(x) = \ln(x)$

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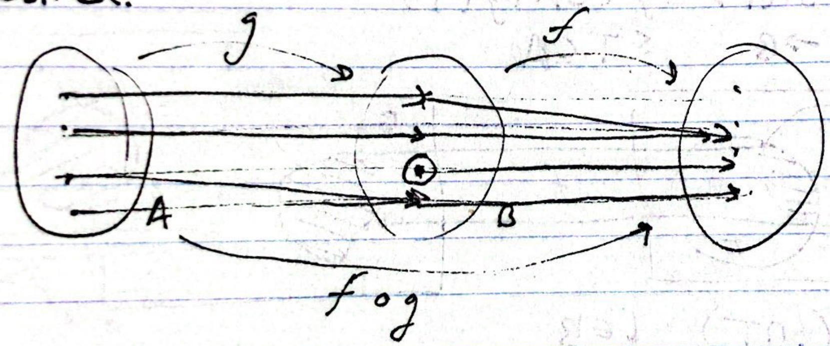
(34) $g: A \to B$ $f: B \to C$

a) Suppose $f(g(x))$ is onto. Then by definition, for some arbitrary $c \in C$, we can find an $a \in A$ such that $f(g(a)) = c$. Notice that $g(a) = b$ for some $b \in B$. Therefore $f(b) = c$. We have found at least one $b \in B$ for every $c \in C$.

b) Suppose $f(g(x))$ is one to one. Now suppose for contradiction that $g(x)$ is not one to one. So for some ~~$c \in C$, there are~~ $b \in B$, there are $a_1, a_2 \in A$ where $a_1 \neq a_2$ such that $g(a_1) = b$ and $g(a_2) = b$. It follows that $f(g(a_1)) = f(g(a_2))$. But $a_1 \neq a_2$. Therefore $f(g(x))$ is not one to one. Contradiction, as desired.

c)



$f(g(a_1)) = c$ $f(g(a_2)) = c$

$a_1 = a_2$

g is onto $\to$ f is one to one

f is one to one $\to$ g is onto

Proof. Suppose $f \circ g$ is a bijection. We wish to prove that g is onto implies f is one to one and that f is one to one implies g is onto.

g is onto $\to$ f is one to one:

Since $f \circ g$ is a bijection and from Proof (b), g is one to one. Suppose g is onto. Then g is a bijection. Suppose for contradiction that f is not one to one. So there is a $a_1, a_2 \in A$, $a_1 \neq a_2$, and $g(a_1), g(a_2) \in B$, $g(a_1) \neq g(a_2)$ such that for a $c \in C$, $g(a_1) = c$ and $g(a_2) = c$. But now $f(g(a_1)) = f(g(a_2))$ so $f \circ g$ is not a bijection. Contradiction as desired.

f is one to one $\to$ g is onto:

Since $f \circ g$ is a bijection and from Proof (a), f is onto. Suppose f is one to one. Then f is a bijection. Suppose for contradiction that g is not onto. Then there is $b \in B$ such that ~~If~~ we cannot find an $a \in A$ where $g(a) = b$. But $f(b) = c$ for some $c \in C$ as f is a bijection. Therefore, we cannot find an $a \in A$ such that $f(g(a)) = c$ so f is not a bijection. Contradiction. Since the implication is shown both ways, ~~it~~ we can conclude that g is onto if and only if f is one to one if $f \circ g$ is a bijection.

36 Yes, the proof was in problem 34b

(42b) $f(S \cap T) \subseteq f(S) \cap f(T)$

Proof. Suppose $a \in f(S \cap T)$ is arbitrary. It is necessary to show that $a \in f(S) \cap f(T)$ also. Define:

$$f(M) = \{x : g(y) = x \wedge y \in M\}$$

Since $a \in f(S \cap T)$, there is a $y \in S \cap T$ such that $g(y) = a$. Notice that $y \in S$ and $y \in T$. According to the definition $f(S) = \{x : g(y) = x \wedge y \in S\}$, it follows that $a \in f(S)$. For the same reasoning, $a \in f(T)$. Therefore, $a \in f(S) \cap f(T)$ as desired.

(46b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Proof. Suppose $a \in f^{-1}(S \cap T)$ is arbitrary. It is necessary to show that $a \in f^{-1}(S) \cap f^{-1}(T)$ also. Define:

$$f^{-1}(M) = \{x : g(y) = x \wedge y \in M\}$$

Since $a \in f^{-1}(S \cap T)$, there is a $y \in S \cap T$ such that $g(y) = a$. Notice that $y \in S$ and $y \in T$. According to the definition $f^{-1}(S) = \{x : g(y) = x \wedge y = S\}$, it follows that $a \in f^{-1}(S)$. For the same reasoning, $a \in f^{-1}(T)$. Therefore, $a \in f^{-1}(S) \cap f^{-1}(T)$ as desired.

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