Math 124 - Programming for Mathematical Applications

UC Berkeley, Spring 2023

Project 5 - Neural Networks for Character Recognition

Due Friday, April 28

In [1]:
```
using PyPlot, Random, LinearAlgebra, Optim   # Packages needed
```

Description

In this project, you will implement an artificial neural network for *Optical Character Recognition* (OCR). We will use a so-called *Multilayer Perceptron* (MLP) with a single hidden layer.

Preliminaries

First we define the characters that we will use to train the network. Each character will be an image of size 7-by-6, represented as a vector of 42 values, and we will use the 4 characters "MATH". In the network, these characters will be encoded using two output variables $y = (y_1, y_2)$, where $y = (0, 0)$ represents "M", $y = (0, 1)$ represents "A", and so on. The code below defines these images and returns the following variables:

- `training` , size 42-by-4 array containing the 42 pixel-values for each of the 4 characters
- `target` , size 2-by-4 array containing the desired output for each of the 4 characters
- `mapstr` , the string "MATH" which is the characters that each `target` output corresponds to

In [2]:
```
charstr = """
          000000  000000  000000  00..00
          000000  000000  000000  00..00
          0.00.0  00..00  ..00..  00..00
          0.00.0  00..00  ..00..  000000
          0....0  000000  ..00..  00..00
          0....0  00..00  ..00..  00..00
          0... 0  00..00  ..00..  00..00
          """

training = reshape(collect(charstr), :, 7)
training = Int.(training[[1:6;9:14;17:22;25:30],:] .== '0')
training = reshape(training', 7*6, 4)
target = [0 0; 0 1; 1 0; 1 1]'
mapstr = "MATH";
```

We also define the plotting function below, which takes an array `images` with 42 rows and one column for each image, and shows the images in a grid:

In [3]:
```
function plot_chars(images)
    gray()
    n_images = size(images,2)
    for j = 1:n_images
        subplot(ceil(Int, n_images/4), 4, j)
        im = 1 .- reshape(images[:,j], 7, 6)
        imshow(im); axis("off");
    end
end
plot_chars(training)
```

Problem 1 - Generating noisy test characters

To test our trained OCR code, we need noisy or perturbed characters. Here we will artificially produce these by modifying the true character images in `training`.

Write a function `make_testdata(training)` which returns an array `testdata` of size 42-by-20. The first 4 images (columns) of `testdata` are identical copies of the `training` array. Generate the next 4 images by randomly choosing 2 pixels in each training image, and flip their values (that is, 0 becomes 1, 1 becomes 0). For the next 4 images you choose 2x2=4 random pixels to flip, then 2x3=6 pixels, and finally 2x4=8 pixels. This gives a total of 16 new perturbed images, and 20 columns total including the original images.

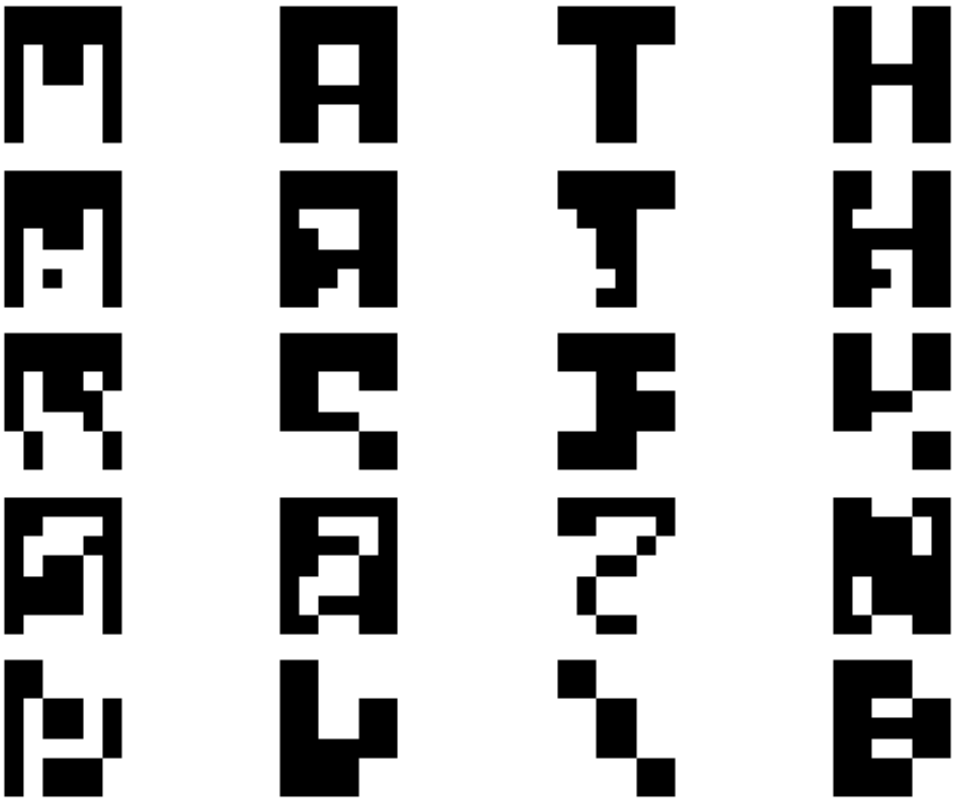
Plot your test data using the commands below. This should show a 5-by-4 array of successively worse letters MATH in each row.

In [4]:

```
function make_testdata(training)

    function pixel_mod(dim1, dim2, images)
        images2 = zeros(Int64, size(images, 1), size(images, 2))
        range1 = 7 - dim1 + 1
        range2 = 6 - dim2 + 1
        row1 = rand(collect(1:range1))
        col1 = rand(collect(1:range2))
        row2, col2 = 0, 0

        while true
            row2 = rand(collect(1:range1))
            col2 = rand(collect(1:range2))
            if row2 != row1 || col2 != col1
                break
            end
        end

        for k = 1:size(images, 2)
            A = reshape(images[:, k], 7, 6)
            for i = 1:dim1
                for j = 1:dim2

                    A[row1 + i - 1, col1 + j - 1] = 1 - A[row1 + i
                    A[row2 + i - 1, col2 + j - 1] = 1 - A[row2 + i
                end
            end
            images2[:,k] = reshape(A, 1, 42)
        end

        return images2

    end

    return training2 = [training pixel_mod(1, 1, training) pixel_m

end
```

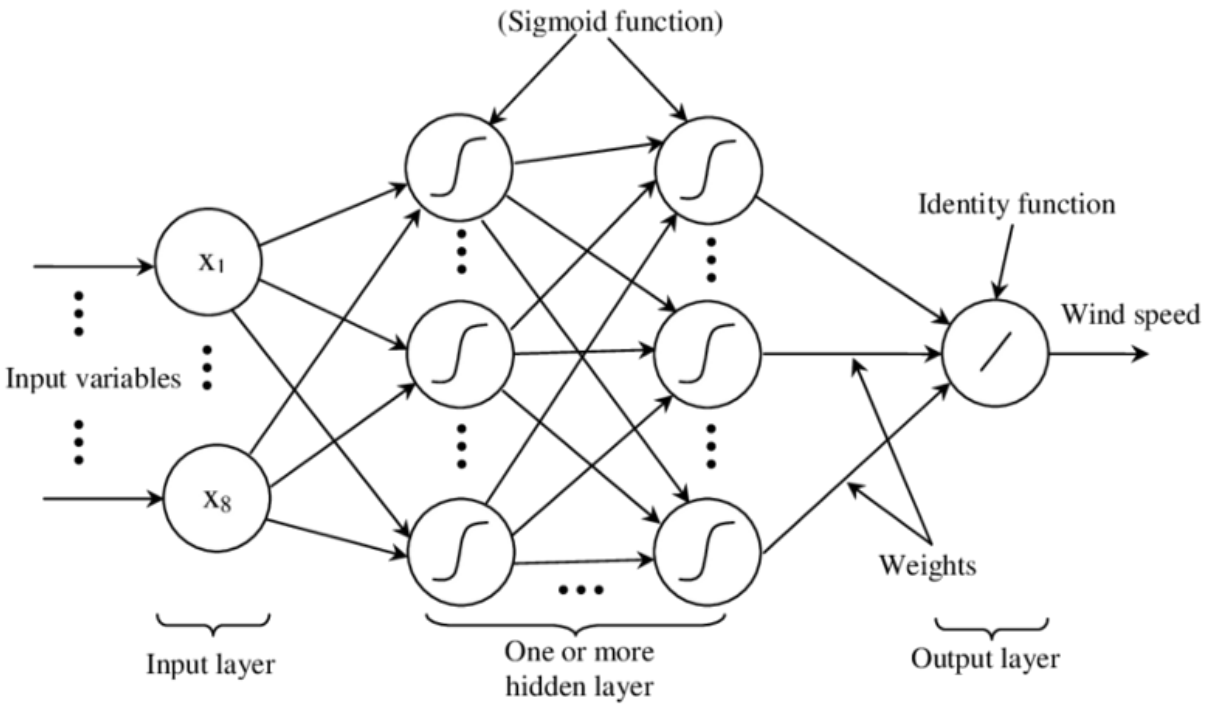
Out[4]: make_testdata (generic function with 1 method)

In [5]:
```
testdata = make_testdata(training);
plot_chars(testdata)
```



Machine learning using a multilayer perceptron

Our network is illustrated below. The input $x = (x_1, \ldots, x_{42})$ is in our case an image represented as a vector of length 42, and the output $y = (y_1, y_2)$ is a vector of length 2, with the predicted character encoded as described above.



(by V. Palaniappan, https://medium.com/engineer-quant/multilayer-perceptron-4453615c4337 (https://medium.com/engineer-quant/multilayer-perceptron-4453615c4337))

Here, the *sigmoid function* is defined by

$$\sigma(x) = \frac{1}{1 + e^{0.5-x}}$$

The *weights* (that is, the arrows between the layers) can be represented in a linear form as a matrix multiplication. The goal of the training is to determine these weights, that is, the entries of the matrices.

We will use a hidden layer with 10 neurons. This means that the first layer can be written as a matrix-vector product Vx for some weight matrix V of size 10-by-42. Next we apply the sigmoid function (element-wise) to get $r = \sigma(Vx)$, apply another weight matrix W of size 2-by-10, and finally apply the sigmoid function again to predict the output y. In total this gives $10 \cdot 42 + 2 \cdot 10 = 440$ weights that need to be determined during the training. The entire network can be written in a compact form as

$$y(x) = \sigma(W\sigma(Vx))$$

We will train our network by solving the following optimization problem: Find $V \in \mathbb{R}^{10\times 42}$, $W \in \mathbb{R}^{2\times 10}$ that minimize the following so-called *misfit function*

$$E(V, W) = \frac{1}{2}\sum_{j=1}^{4} \|y(x_j) - t_j\|_2^2$$

Here, x_j is the jth training vector (from the `training` array), t_j is the desired target output for the input x_j (from the `target` array), and $y(x)$ is the network described above.

Stochastic Gradient Descent training

Finding the true gradient of E is somewhat involved, and instead we will use the so-called Stochastic Gradient Descent method. At each iteration, we pick:

- A random number $j \in \{1, 2, 3, 4\}$ to decide which training vector x_j and target output t_j to use
- A random number $k \in \{1, 2\}$ to decide which component of the vector $y(x_j) - t_j$ to consider

We then only compute the gradient direction for the part of $E(V, W)$ that contribute to output k for training vector j. The gradient calculations get simplified with these assumptions, and they can be computed with the following sequence of computations:

$$
\begin{aligned}
r &= \sigma(V x_j) && \text{column-vector (matrix-vector product)} \\
y &= \sigma(W_k \cdot r) && \text{scalar (dot-product, } W_k \text{ is } k\text{th row of } W) \\
q &= (y - t_{kj}) y (1 - y) && \text{scalar } (t_{kj} \text{ is component } k \text{ of } t_j) \\
u &= W_k^T r (1 - r) && \text{column-vector (all products are element-wise)} \\
\nabla_{W_k} E(V, W) &= q r^T && \text{row-vector (scalar times row-vector)} \\
\nabla_V E(V, W) &= q u x_j^T && \text{matrix (scalar times outer product)}
\end{aligned}
$$

Using these gradients, we update V and the kth row of W using standard gradient descent iterations for a given stepsize α.

Problem 2 - Training using SGD

Implement a function

```
function train_sgd(; maxiter=10000, rate=1)
```

which initializes V, W to normal-distributed random numbers and performs `maxiter` stochastic gradient descent iterations as described above. Use the step-size α set to `rate` (the so-called *learning rate*). The function returns V, W, and can be tested using the code below.

In [6]:
```
maxiter=10000
rate=1
function train_sgd(; maxiter=10000, rate=1)

    α = rate
    V = reshape(randn(420), 10, 42)
    W = reshape(randn(20), 2, 10)

    for i = 1:α:maxiter

        j = rand(collect(1:4))
        k = rand([1, 2])

        σ(x) = 1 / (1 + exp(0.5 - x))
        r = σ.(V * training[:,j])
        y = σ.(W[k,:] ⋅ r)
        q = (y - target[k,j])*y*(1 - y)
        u = W[k,:] .* r .* (1 .- r)

        ΔW = q .* transpose(r)
        ΔV = q .* u[:,:] * transpose(training[:,j][:,:])
        W[k,:] -= α * transpose(ΔW)
        V -= α * ΔV

    end

    return V, W

end
```

Out[6]:
```
train_sgd (generic function with 1 method)
```

In [7]:
```
V,W = train_sgd()
```

Out[7]:
```
([-0.10825595586026325 0.0331458569540207 … -2.048336177478499 -1.097966823695898; 0.05614314715294488 -0.10207318114433704 … -0.6696065741167333 -1.0773834045219517; … ; 0.6626871883600354 0.009559912284699055 … 0.18650223741770477 1.0283653526404901; 0.7374185610854761 -1.1658514061187704 … 0.6262096445010441 1.2309072812497177], [4.716185175211911 -1.627875174615892 … -1.144231978254841 -2.0399458752900785; -0.859484338806269 1.6736657349852249 … -0.7219365928937113 0.9720185138070203])
```

Problem 3 - Predict output for noisy characters

We can now apply the trained network V, W to the noisy characters in `testdata`, and see how well they match. We do this by simply computing $y(x_j)$ for each image x_j in `testdata`, rounding the outputs y_1, y_2 to integers, and mapping the four possibilities to the letters in `mapstr`.

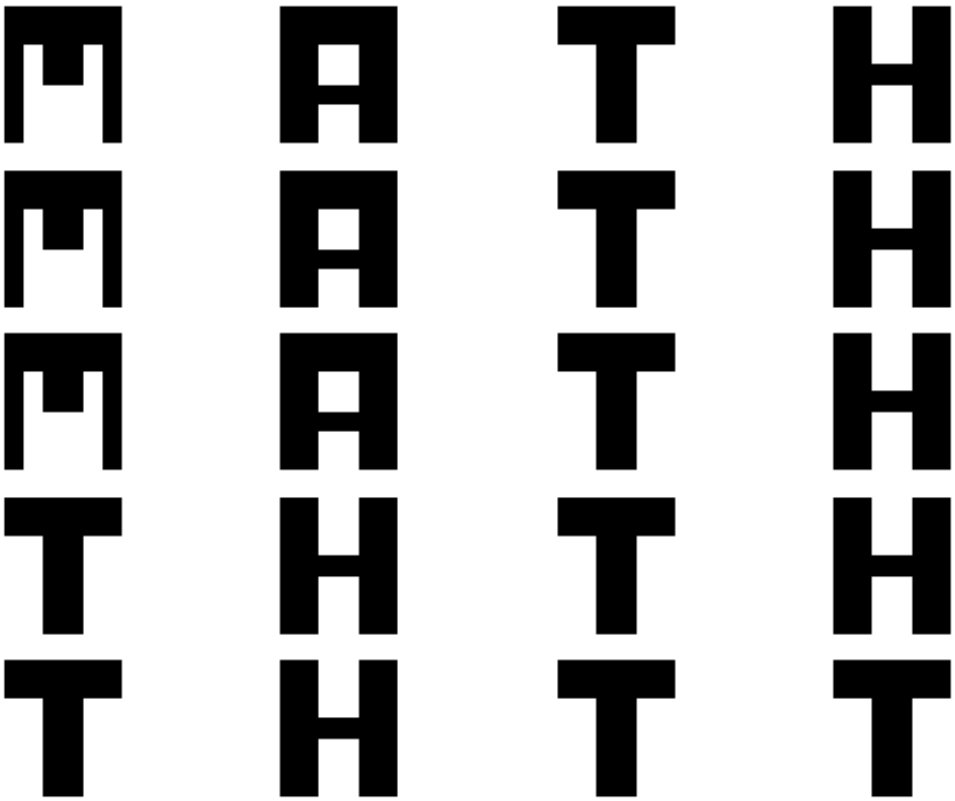
Write a function `predict(testdata, V, W)` which performs these operations, and returns a 5-by-4 character array with the predicted characters. The ideal output would be 5 rows of `'M', 'A', 'T', 'H'`, but we do not of course expect to be perfect for the highly noisy cases.

In [8]:

```
function predict(testdata, V, W)

    σ(x) = 1 / (1 + exp(0.5 - x))
    min = -Inf
    newdata = zeros(Int64, size(testdata, 1), size(testdata,2))

    for i = 1:size(testdata,2)

        y = σ.(W*σ.(V*testdata[:,i]))

        for j = 1:2
            if abs(y[j] - floor(y[j])) ≤ abs(y[j] - ceil(y[j]))
                y[j] = floor(y[j])
            else
                y[j] = ceil(y[j])
            end
        end

        if y[1] == 0
            if y[2] == 0
                newdata[:,i] = training[:,1]
            else
                newdata[:,i] = training[:,2]
            end
        elseif y[2] == 0
            newdata[:,i] = training[:,3]
        else
            newdata[:,i] = training[:,4]
        end
    end

    return plot_chars(newdata)

end
```

Out[8]: predict (generic function with 1 method)

In [9]:
```
plot_chars(testdata)
predict(testdata, V, W)
```



Problem 4 - True gradient descent using the `Optim` package

Since the `Optim` package can perform automatic differentiation, we can try to solve the optimization problem using true gradient descent. Write a function `train_optim()` which does this and returns V, W like before. Use the solver `GradientDescent()`, and remember to set `autodiff=:forward`.

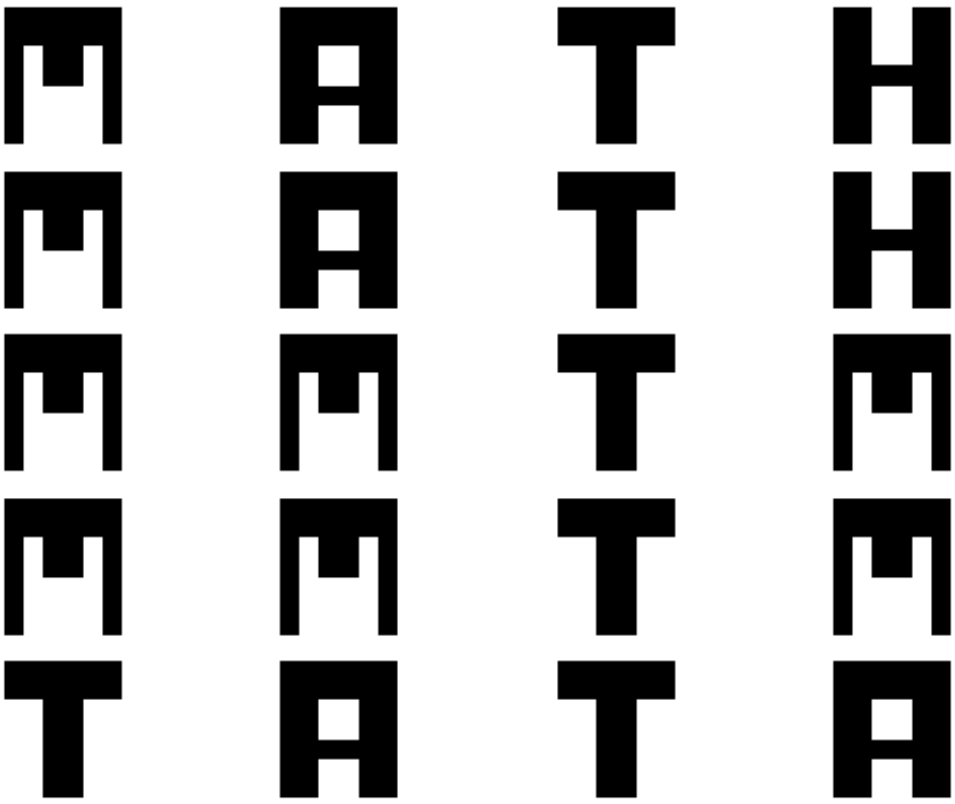
One tricky thing with this is that `optimize` expects a single input vector x, but we have two matrices V, W. This is a quite common problem, and you can get around it by converting the 420+20=440 numbers in V, W to a vector. In the objective function, you also have to convert back to matrices in order to evaluate the misfit function $E(V, W)$.

Run your function, then predict the output using the resulting network V, W with the code below.

In [10]:
```
using Optim
function train_optim()
    X = randn(440)
    σ(x) = 1 / (1 + exp(0.5 - x))
    E(x) = 0.5 * sum([norm(σ.(reshape(x[421:end], 2, 10) * σ.(resh
    res = optimize(E, X, GradientDescent(); autodiff=:forward)
    return reshape(res.minimizer[1:420], 10, 42), reshape(res.mini
end
```

Out[10]: train_optim (generic function with 1 method)

In [11]:
```
plot_chars(testdata)
V,W = train_optim()
predict(testdata, V, W)
```



In [ ]:
```

```