Stat134Notes

Trustin Nguyen

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Contents

Chapter 1

Week 1

Last Lecture: Formalize random experiments by probability space: $(\Omega, \mathcal{F}, \mathbb{P})$.

- Ω : sample space (all possible outcomes).
- $\mathcal{F}$: collection of subsets in Ω called events
- P: probability measure (distribution) $P : \mathcal{F} \to \mathbb{R}_{\geqslant 0}$.

Conditions:

(i) $0 \leqslant P(A) \leqslant 1$

(ii) $P(\emptyset) = 0$ and $P(\Omega) = 1$

(iii) A_n disjoint events, then $P(\cup_{n-1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.

Corollary 1: If A and B are two disjoint events, then

$$P(A \cup B) = P(A) + P(B)$$

Proof. Take $A_1 = A$ and $A_2 = B$, $A_3 = A_4 = \cdots = \emptyset$. By (iii), we have $P(A \cup B) = P(A) + P(B)$. □

Corollary 2: Let A^c be complement of A:

$$A^c = \Omega \backslash A$$

Then $P(A^c) = 1 - P(A)$.

Proof. Take $B = A^c$ and use corollary 2:

$$P(A) + P(A^c) = P(\Omega) = 1$$

□

Example 1.0.1: Loaded die. The bottom part is heavier than top. Suppose that 6 is twice as likely than any other outcome. Then $P(6) = \frac{2}{7}$ and $P(1) = P(2) = \cdots = P(5) = \frac{1}{7}$.

- Probability of even outcome:

$$\begin{aligned} P(\text{even outcome}) &= P(\{2,4,6\}) \\ &= P(\{2\}) + P(\{4\}) + P(\{6\}) \\ &= \frac{4}{7} \end{aligned}$$

- Probability of odd outcome:

$$P(\text{odd outcome}) = 1 - P(\text{even outcome}) = \frac{3}{7}$$

There are three methods for creating experiments with random outcomes: ball and urn, dice, coin.

Example 1.0.2: There are 3 ways to retrieve n balls from an urn.

- Sampling with replacement: Take a ball, each n balls have equal probability of being selected, then record the number. Return the ball to the urn. Repeat k times.

  Outcome: sequence $(s_1, \ldots, s_k)$.

  $$\Omega = \{(s_1, \ldots, s_k) : 1 \leqslant s_i \leqslant n\}$$

  Then $|\Omega| = n^k$.
- Sampling without replacement:
  - Order matters: Same urn but we do not return the ball once selected. We still remember the order of which the balls were selected. Repeat k times.

    Outcome: sequence $(s_1, \ldots, s_k)$ with the restriction that s_i distinct.

    $$|\Omega| = \binom{n}{k} k!$$
  - Order does not matter: Same urn, take out balls and do not record their order. It only matters which balls are outside the urn at the end of the experiment.

    Outcome: sets $\{s_1, \ldots, s_k\}$ of an n element set.

    $$|\Omega| = \binom{n}{k}$$
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Example 1.0.3: Urn with $\{1,2,3,4,5\}$, sample 3 balls with replacement. Then

$$\Omega = \{1,2,3,4,5\}^3$$

and

$$|\Omega| = 125$$

so

$$P(w) = \frac{1}{125}$$

An example is that $P(153) = P(224)$.

Example 1.0.4: Same urn with $\{1, 2, 3, 4, 5\}$ and take 3 balls without replacement. Then

$$\Omega = \{(s_1, s_2, s_3) : s_i \text{ distinct}\}$$

and

$$|\Omega| = \binom{5}{3} 3!$$

so

$$P(w) = \frac{1}{\binom{5}{3} 3!}$$

In this example, $P(153) = P(w)$ while $P(224) = 0$.

Example 1.0.5: $P(w) = \frac{1}{\binom{n}{k}}$.

Example 1.0.6: Same urn with $\{1, 2, 3, 4, 5\}$ and take 3 balls without replacement and ignoring the order.

$$P(\{153\}) = \frac{1}{\binom{5}{3}}$$