Math143Hw8

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October 26, 2023

Exercise 1: Check the following statements from class:

(a) If $\varphi : X \to Y$ is an isomorphism that sends P to Q, then $\varphi^* : \mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$ is an isomorphism.

Proof. Since φ is an isomorphism, we find that there is a ψ such that $\psi\varphi = \mathrm{id}$. So we have

$$\psi\varphi(P) = \psi(Q) = P$$

Furthermore, we can say there exists a $\psi^* : \Gamma(X) \to \Gamma(Y)$ such that $\psi^*\varphi^* = \mathrm{id}$ and $\varphi^*\psi^* = \mathrm{id}$. Then proved in lecture was that $\varphi^* : \mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$ is well defined if we have the mapping:

$$\varphi^*(\frac{g}{h}) = \frac{\varphi^*(g)}{\varphi^*(h)}$$

and also,

$$\varphi^*(h)(P) = h(\varphi(P)) = h(Q) \neq 0$$

Now if we define ψ^* as:

$$\psi^*(\frac{g}{h}) = \frac{\psi^*(g)}{\psi^*(h)}$$

for g, h in $\Gamma(X)$ and $\frac{g}{h} \in \mathcal{O}_P(X)$. Then using the fact that $\psi(Q) = P$, we get:

$$\psi^*(h)(Q) = h(\psi(Q)) = h(P) \neq 0$$

This tells us that $\frac{\psi^*(g)}{\psi^*(h)} \in \mathcal{O}_Q(Y)$ and the denominator is not the zero polynomial. Now we see that:

$$\varphi^*\psi^*(\frac{g}{h}) = \frac{\varphi^*\psi^*(g)}{\varphi^*\psi^*(h)} = \frac{g}{h}$$

since $\varphi^*\psi^* = \mathrm{id}_{\Gamma(X)}$. Now because $\psi^*\varphi^* = \mathrm{id}_{\Gamma(Y)}$, we have:

$$\psi^*\varphi^*(\frac{g}{h}) = \frac{\psi^*\varphi^*(g)}{\psi^*\varphi^*(h)} = \frac{g}{h}$$

So both compositions are identities. So φ^* is an isomorphism. □

(b) Let $P = (0, 0)$. Prove directly from the definition that $I_P(x, y) = 1$.

Proof. By definition, we have:

$$I_P(x, y) = \dim_k \left(\frac{\mathcal{O}_P(\mathbb{A}^2)}{(x, y)} \right)$$

Let $f(x, y), g(x, y)$ have non-zero constant term. We will show that $\frac{k_1}{f(x,y)}$ is a generator of $\mathcal{O}_P(\mathbb{A}^2)/(x, y)$. Suppose that $\frac{k_2}{g(x,y)} \in \mathcal{O}_P(\mathbb{A}^2)/(x, y)$. Then observe that we desire a $k_0 \in k$ such that:

$$\left(\frac{k_1}{f(x, y)} \right) \cdot k_0 = \frac{k_2}{g(x, y)}$$

we get:

$$
\begin{aligned}
k_1k_0g(x,y) &= k_2f(x,y)\\
k_1k_0g_0 &= k_2f_0 \mod (x,y)\\
k_0 &= k_2f_0k_1^{-1}g_0^{-1} \in k
\end{aligned}
$$

So we have found a k_0. Then $\frac{k_1}{f(x,y)}$ is a generator of $\mathcal{O}_P(\mathbb{A}^2)/(x,y)$. So it is one-dimensional. □

(c) Suppose f and g have no repeated factors, P is a smooth point of $V(f)$ and $V(g)$ and the tangent lines to $V(f)$ and $V(g)$ at P are distinct. Prove that $I_P(f,g) = 1$.

Proof. If $P \neq 0$, then first compute the pullback and everything will be preserved, such as the multiplicity of P in f, g, the distinct tangent lines, and that P is smooth. Because the tangent lines are distinct, we know that $I_P(f,g) = \text{mult}_P(f)\,\text{mult}_P(g)$. Because P is smooth, by the formula for a tangent line:

$$
\begin{aligned}
f_x(p)(x-x_0) + f_y(p)(y-y_0) &= 0\\
g_x(p)(x-x_0) + g_y(p)(y-y_0) &= 0
\end{aligned}
$$

we know that there is exactly one tangent line for f, called T_f, and one for g, called T_g. So $V(T_f), V(T_g)$ are the tangent cones of f, g. And we see that both T_f, T_g are homogeneous of degree 1, therefore, $\text{mult}_P(f) = 1$ and $\text{mult}_P(g) = 1$. So the product is equal to $I_P(f,g) = 1$. □

Exercise 2: Let $P = (0,0)$ and $k = \mathbb{C}$. Compute the following intersection numbers using the properties from class. There may be many possible routes to do so!

(a) $I_P(x^2-y, y^2-x^3)$

Answer. We have

$$
\begin{aligned}
I_P(x^2-y, y^2-x^3) &= I_P(y^2-y, x^2-y)\\
&= I_P(y, x^2-y) + I_P(y-1, x^2-y)\\
&= I_P(y, x^2)\\
&= 2I_P(x,y)\\
&= 2
\end{aligned}
$$

(b) $I_P(x-y^2, x+y^2)$

Answer. We have

$$
\begin{aligned}
I_P(x-y^2, x+y^2) &= I_P(2x, x+y^2)\\
&= I_P(x, x+y^2)\\
&= I_P(x, y^2)\\
&= 2I_P(x,y)\\
&= 2
\end{aligned}
$$

(c) $I_P(x^3+xy, 3x^2y+xy^2)$

Answer. We have

$$
\begin{aligned}
I_P(x^3+xy, 3x^2y+xy^2) &= I_P(xy, x^3+xy) + I_P(x^3+xy, 3x+y)\\
&= I_P(xy, x^3) + I_P(x^3-3x^2, 3x+y)\\
&= I_P(x, x^2) + I_P(x^3-3x^2, 3x+y)\\
&= \infty
\end{aligned}
$$

(d) $I_P(x+y+y^2x, x+y+x^2-y^2+y^3)$

Answer. We have

$$\begin{aligned}
I_P(x+y+y^2x, x+y+x^2-y^2+y^3) &= I_P(x^2-y^2+y^3-y^2x, x+y+y^2x)\\
&= I_P(y^2(y-x)+(-1)(x-y)(x+y), x+y+y^2x)\\
&= I_P((y^2-1)(y-x)(x+y), x+y+y^2x)\\
&= I_P(y^2-1, x+y+y^2x) + I_P(y-x, x+y+y^2x) + I_P(x+y, x+\\
&= 0 + I_P(y-x, x+y+y^2x) + I_P(x+y, x+y+y^2x)\\
&= I_P(y-x, 2y+y^2x) + I_P(x+y, y^2x)\\
&= I_P(y, y-x) + I_P(y-x, 2+yx) + I_P(x+y, y^3)\\
&= I_P(y,x) + 0 + 3I_P(x+y, y)\\
&= I_P(x,y) + 3I_P(x,y)\\
&= 4I_P(x,y)\\
&= 4
\end{aligned}$$

Exercise 3: Let $g, h \in k[x,y]$ and let $P=(0,0)$.

(a) Prove that $I_P(y, g+h) \geqslant \min(\{I_P(y,g), I_P(y,h)\})$

Proof. If $g(x,0)=0$, then $y \mid g$ and so we can say that $I_P(y, g+h) = I_P(y,h)$ because $g \in (y)$. The vanishing of g and y have a common component so $I_P(y,g)=\infty$, and therefore, $I_P(y,h) \geqslant I_P(y,h)$ which is true. If both $g(x,0)=0=h(x,0)$, then the above equation turns to:

$$I_P(y) \geqslant \min(I_P(y), I_P(y))$$

which is true. So suppose that $y \nmid g, h$. Then we can write $I_P(y,g) = I_P(y,g')$ and the same for $h \to h'$ for g', h' polynomials in terms of x. And then $(y, g+h) = (y, g'+h')$. So we have:

$$\begin{aligned}
g'(x) &= g_0 + g_1 + \cdots\\
h'(x) &= h_0 + h_1 + \cdots
\end{aligned}$$

Let g_i be the first non-zero homogeneous form in g' and h_k be the first non-zero homogeneous form in h'. Then we have:

$$\begin{aligned}
g' &= x^i(g_0 + g_1 + \cdots)\\
h' &= x^k(h_0 + h_1 + \cdots)
\end{aligned}$$

by factoring out the x's and note that $g_0, h_0 \neq 0$. Then

$$\begin{aligned}
I_P(y,g') &= I_P(x^i, y) + I_P(g_0+g_1+\cdots, y) \quad & I_P(y,h') &= I_P(x^k, y) + kI_P(h_0+h_1+\cdots, y)\\
&= i + 0 & &= k+0
\end{aligned}$$

We note that $I_P(g_0+g_1+\cdots, y) = 0$ because the y vanishes on $(0,0)$ but the $g_0+g_1+\cdots$ does not because $g_0 \neq 0$. So the RHS turns out to be

$$\min(i,k)$$

While

$$I_P(y, g+h) = I_P(y, g'+h')$$

and we note that $g'+h'$ has multiplicity greater than or equal to g' and h'. So in the same process above,

$$g'+h' = (g'+h')_0 + (g'+h')_1 + \cdots$$

and the first non zero homogeneous form is at least $\min(i,k)$. So repeating the process gives us $I_P(y, g+h) \geqslant \min(i,k)$, so we are done. $\square$

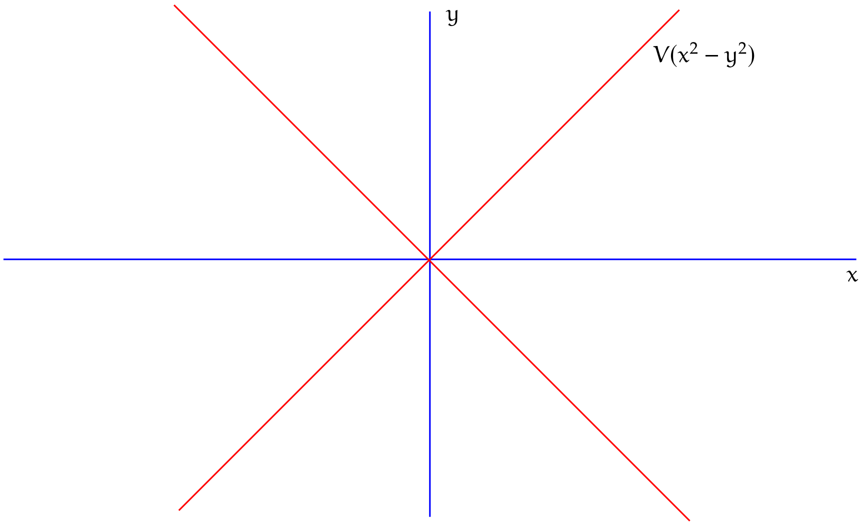
(b) It turns out part (a) is true whenever we replace y with a polynomial f such that $f_1 \neq 0$ (but you do not need to prove this). However, it can be false when $f_1 = 0$. Find an example of polynomials f, g and h so that $I_P(f, g+h) < \min(\{I_P(f,g), I_P(f,h)\})$.

Proof. The counterexample is at the end if explanation isn't needed.

To construct an example, we see that:

$$\begin{aligned} I_P(f, g+h) &\geqslant \operatorname{mult}_P(f)\operatorname{mult}_P(g+h) \\ &\geqslant \operatorname{mult}_P(f)\min(\operatorname{mult}_P(g), \operatorname{mult}_P(h)) \\ &= \min(\operatorname{mult}_P(f)\operatorname{mult}_P(g), \operatorname{mult}_P(f)\operatorname{mult}_P(h)) \end{aligned}$$

So we want $I_P(f,g) \neq \operatorname{mult}_P(f)\operatorname{mult}_P(g)$ and $I_P(f,h) \neq \operatorname{mult}_P(f)\operatorname{mult}_P(h)$. Otherwise, we get $I_P(f, g+h) \geqslant \min(I_P(f,g), I_P(f,h))$. So this means that the f, g share some tangent line and f, h share some tangent line. But we want $g+h$ to not share a tangent line with f to minimize $I_P(f, g+h)$. Considering:



We see that $g = x+y$, $h = x-y$ satisfy the requirements and $g+h = 2x$ has no tangent lines in common with $x^2 - y^2$. So to verify:

$$\begin{aligned} I_P(f,g) &= I_P(x^2-y^2, x+y) \\ &= \infty \end{aligned} \qquad \begin{aligned} I_P(f,h) &= I_P(x^2-y^2, x-y) \\ &= \infty \end{aligned} \qquad \begin{aligned} I_P(f,g+h) &= I_P(x^2-y^2, 2x) \\ &= I_P(x-y, x) + I_P(x+y, x) \\ &= 1+1 = 2 \end{aligned}$$

And indeed, $I_P(f, g+h) < \infty$. Let $P = (0,0)$. We have:

$$I_P(x^2-y^2, 2x) < \min(I_P(x^2-y^2, x-y), I_P(x^2-y^2, x+y))$$

$\square$

Exercise 4: Nodes: Let $f \in k[x,y]$ be a polynomial with no repeated factors. We say that f has a node at P if P has multiplicity 2 in $V(f)$ and the tangent cone of f is two distinct lines. Prove that P is a node of $V(f)$ if and only if $f_{xy}(P) \neq f_{xx}(P)f_{yy}(P)$. Here $f_{xy} = \frac{d}{dx}\left(\frac{d}{dy}f\right)$ and $f_{xx} = \frac{d^2}{dx^2}f$ and $f_{yy} = \frac{d^2}{dy^2}f$ are second derivatives.

Proof. Suppose that P has multiplicity 2 in $V(f)$. Then it also has multiplicity 2 in the pullback where if $P = (p_1, p_2)$:

$$\varphi(x,y) = (x+p_1, y+p_2)$$

and

$$\varphi^* f(x, y) = f(x + p_1, y + p_2)$$

So we will consider $\varphi^* f$ and $(0, 0) \in V(\varphi^* f)$. Since we know that it has multiplicity 2 also, we have that $V(\varphi^* f_2)$ is the tangent cone, which decomposes into $V(L_1 L_2)$ where L_1, L_2 are distinct lines. Then

$$\begin{aligned} L_1 &: y = a_1 x \\ L_2 &: y = a_2 x \end{aligned}$$

And

$$L_1 L_2 = (y - a_1 x)(y - a_2 x) = y^2 - (a_1 + a_2)xy + a_1 a_2 x^2$$

Notice that since

$$f = f_2 + f_3 + \cdots + f_m$$

Then $f_{xy}(P), f_{xx}(P), f_{yy}(P)$ are determined only by $(f_2)_{xx}(P), (f_2)_{yy}(P), (f_2)_{xy}(P)$, as the homogeneous terms of higher degree will still contain a x or y variable and evaluate to 0 upon plugging in $P = (0, 0)$. Then:

$$\begin{aligned} (f_2)_{xx} &= 2a_1 a_2 \\ (f_2)_{xy} &= -(a_1 + a_2) \\ (f_2)_{yy} &= 2 \end{aligned}$$

Then

$$f_{xy}^2(P) = a_1^2 + 2a_1 a_2 + a_2^2 = 4a_1 a_2 = f_{xx}(P) f_{yy}(P)$$

implies that

$$a_1^2 - 2a_1 a_2 + a_2^2 = 0$$

which means that $(a_1 - a_2)^2 = 0$ or $a_1 = a_2$ which is a contradiction. So $f_{xy}^2(P) \neq f_{xx}(P) f_{yy}(P)$. This argument also works in reverse because we made no non-arbitrary assumptions about f_2. So its iff. □

Exercise 5: Cusps: Let $f \in k[x, y]$ be a polynomial with no repeated factors and suppose P is a point of multiplicity 2 in $V(f)$. Furthermore, suppose now that the tangent cone of $V(f)$ is a single line $V(L)$.

(a) Show that $I_P(f, L) \geqslant 3$. If equality holds, we say $V(f)$ has a cusp at P.

Proof. We know that intersection multiplicity is preserved with a change of coordinates, so we can look at $I_{(0,0)}(\varphi^* f, \varphi^* L)$ where φ^* is some translation. By the fact that:

$$I_P(f, g) \geqslant \operatorname{mult}_P(f) \operatorname{mult}_P(g)$$

we have

$$I_{(0,0)}(\varphi^* f, \varphi^* L) \geqslant \operatorname{mult}_{(0,0)}(\varphi^* f) \operatorname{mult}_{(0,0)}(\varphi^* L) = 2\cdot$$

But because $(0, 0)$ as a point in f has multiplicity 2 with tangent line L in common, equality does not hold, so in fact,

$$I_{(0,0)}(\varphi^* f, \varphi^* L) \geqslant 3$$

the translation is an isomorphism, and is invertible, so we know that $I_P(f, L) \geqslant 3$. □

(b) Suppose $P = (0, 0)$ and $L = y$. Show that P is a cusp if and only if $f_{xxx}(P) \neq 0$, where f_{xxx} is the third partial derivative of f with respect to x.

Proof. We have that

$$f = f_2 + f_3 + \cdots$$

Since $P = (0,0)$, with multiplicity 2, we know that $V(f_2) = V(y)$, as the tangent cone. Then $V(f_2) = V(A) \cup V(B) = V(y)$, but $V(y)$ is irreducible, so we know that $A = y, B = y$, as neither of the vanishings can be empty. Now looking at the f_3 term,

$$f_3 = a_3x^3 + a_2x^2y + a_1xy^2 + y^3$$

Calculating $(f_3)_{xxx}(P)$, we get:

$$(f_3)_{xxx}(P) = 6a_3$$

In the context of f, we have:

$$f_{xxx}(P) = (f_2)_{xxx}(P) + (f_3)_{xxx}(P) + (f_4)_{xxx}(P) + \cdots$$

And because all terms of $(f_i)_{xxx}$ for $i > 3$ contain either an x or a y, we know that evaluation at P returns 0, so:

$$f_{xxx}(P) = (f_3)_{xxx}(P) = 6a_3$$

Suppose that $f_{xxx}(P) \neq 0$. We will prove a cusp, with a chain of iffs. Then

$$f_{xxx}(P) \neq 0 \iff 6a_3 \neq 0 \iff a_3 \neq 0$$

Then a_3x^3 is a summand of f and $y \nmid f$. We know that

$$I_P(f, L) = I_P(y^2 + f_3 + \cdots, y) = I_P(f_3 + \cdots, y)$$

reducing the right polynomial $f_3 + f_4 + \cdots$ into a polynomial in terms of just x, we get:

$$\begin{aligned} I_P(f, L) &= I_P(a_3x^3 + a_4x^4 + \cdots + a_nx^n, y) \\ &= I_P(a_3 + a_4x + \cdots + a_nx^{n-3}, y) + I_P(x^3, y) \\ &= 3I_P(x, y) + I_P(a_3 + a_4x + \cdots + a_nx^{n-3}, y) \\ &= 3 + I_P(a_3 + a_4x + \cdots + a_nx^{n-3}, y) \end{aligned}$$

But we know that $a_3 \neq 0$. So $(0,0) \notin V(a_3 + a_4x + \cdots + a_nx^{n-3})$. This means that the intersection multiplicity for the right summand is 0. So we have

$$I_P(f, L) = 3$$

The chain of iffs means that this is a biconditional. □

(c) Show that if P is a cusp, then $V(f)$ has only one irreducible component passing through P.

Proof. If P is a cusp at $V(f)$, then we can do some change of coordinates by translation or rotations to get that $(0,0)$ has the same multiplicity as P and that it is in $V(f')$ where the tangent line at P is $V(y)$. If $V(f) = V(g) \cup V(h)$ where both algebraic sets are proper subsets of $V(f)$, then suppose that they both contain P. We must have:

$$I_P(f, g) = I_P(g, y) + I_P(h, y)$$

We have three possibilities:

– The tangent cone of $V(g)$ is y^i for $i \geqslant 2$ (Sharing common tangent line with y). Then:

$$I_P(g, y) = I_P(x^k + x^{k+1} + \cdots)$$

where $k \geqslant 3$. And we know that there is at least one x^i term because $y \nmid g$. Then $I_P(g, y) \geqslant 3$, which is a contradiction because $I_P(h, y) \geqslant 1$ and the total $I_P = 3$

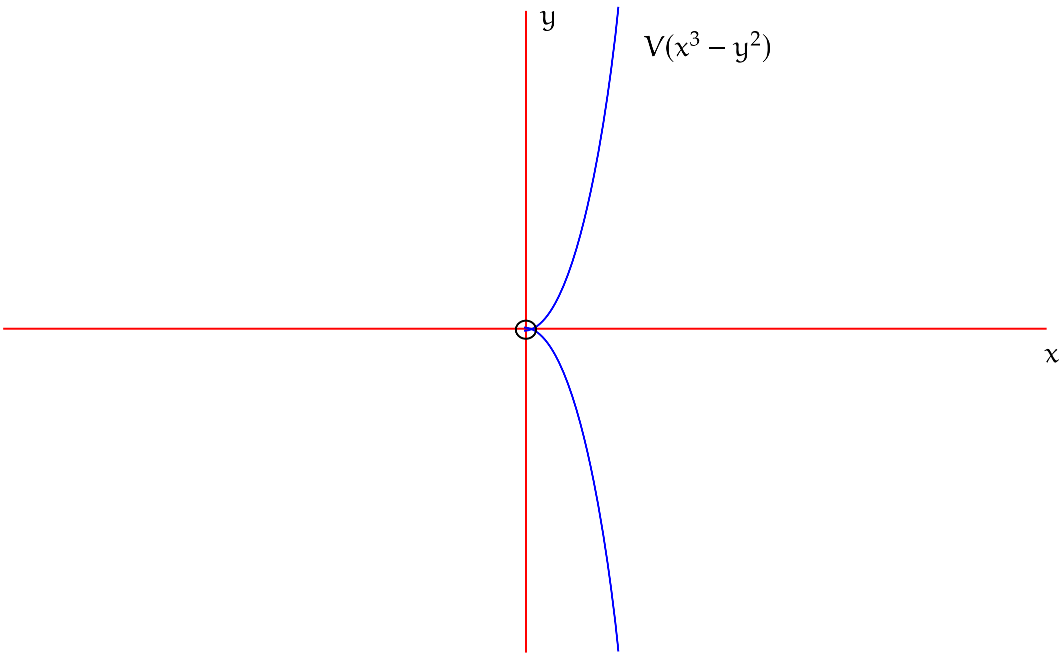
- The tangent cone of $V(g)$ shares no common factor with y. Then $I_P(g, y) = \text{mult}_P(g)\,\text{mult}_P(y) = \text{mult}_P(g)$. Then $I_P(f, y) = \text{mult}_P(g) + \text{mult}_P(h)$. Wlog, assume $\text{mult}_P(g) = 1$. Then $g = k_0 x + \cdots$ and $h = k_2 x^2 + k_3 xy + k_4 y^2 + \cdots$. But this is a contradiction because $V(g) \cup V(h) = V(gh) = V(f)$. And gh and f now have different tangent cones, where the tangent cone of gh is $a_3 x^3 + a_2 x^2 y + a_1 x y^2 + a_0 y^3$ where $a_3 \neq 0$. So it does not contain a $V(y)$ line.

- If the tangent cone of $V(g), V(h)$ share at least one common factor, then we know that $I_P(g, y) = (x^2 + \cdots)$ because the tangent cone is divisible by y and terms that contain only x are those that are homogeneous of degree at least 2. Then $I_P(g, y) + I_P(h, y) \geqslant 4 \neq 3$. So contradiction.

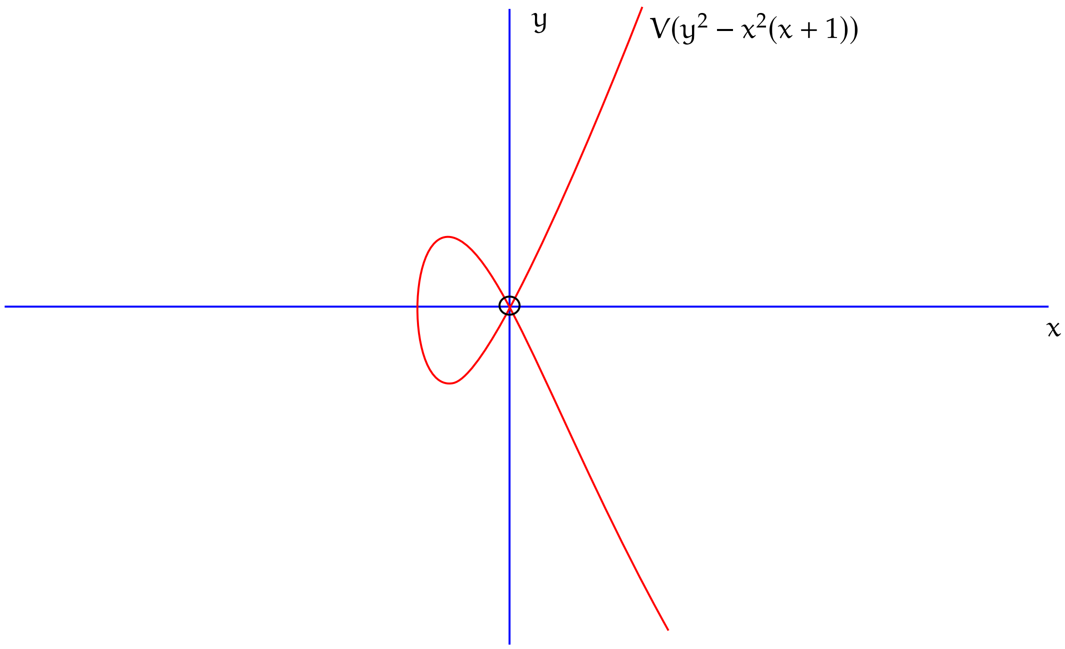
So there is only one irreducible component passing through P. □

Optional: You may wish to look back at Homework 5 problems 1 and 2. One had a node and one had a cusp - do you see which is which and prove it?

Answer. The cusp is $(0, 0)$ in $V(x^3 - y^2)$ because $f_{xxx}((0, 0)) = 6 \neq 0$. This one is:



The node is $(0, 0)$ in $V(y^2 - x^3 + x^2)$ because $f_{xx}(P) = 2$, $f_{yy}(P) = 2$, and $f_{xy}(P) = 0$. We have $f_{xy}^2(P) = 0 \neq 4 = f_{xx}(P) f_{yy}(P)$. This one is:



y
$V(y^2 - x^2(x+1))$
x