Math55Hw7

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5.1: 3, 4, 10, 18, 28, 32, 49, 54x, 62x, 64, 76x.

Exercise 3: Let $P(n)$ be the statement that $1^2+2^2+\ldots+n^2 = \dfrac{n(n+1)(n+2)}{6}$ for the positive integer n.

a) What is the statement $P(1)$?

$P(1)$ is the statement that $1^2 = \dfrac{(1)(2)(3)}{6}$

b) Show that $P(1)$ is true, completing the basis step of a proof that $P(n)$ is true for all positive integers n

$$\begin{aligned} 1^2 &= \frac{(1)(2)(3)}{6} \\ 1 &= 1 \end{aligned}$$

So $P(1)$ is true.

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all integers n?

The inductive hypothesis is that $1^2 + 2^2 + \ldots + k^2 = \dfrac{k(k+1)(k+2)}{6}$ for some arbitrary k.

d) What do you need to prove in an inductive step of a proof that $P(n)$ is true for all positive integers n?

You need to prove $P(k+1)$ using the inductive hypothesis.

e) Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.

Proof. Suppose k is arbitrary and $P(k)$ is true. Then

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(k+2)}{6}$$

Now observe:

$$\begin{aligned}
1^2 + 2^2 + \ldots + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + k + 1\right) \\
&= (k+1)\left(\frac{2k^2+7k+6}{6}\right) \\
&= (k+1)\left(\frac{(2k+3)(k+2)}{6}\right) \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6}
\end{aligned}$$

So $P(k+1) = \dfrac{(k+1)(k+2)(2(k+1)+1)}{6}$ as desired. □

f) Explain why these steps show that that this formula is true whenever n is a positive integer.

Since $P(1)$ and $P(k+1)$, then $P(2)$. Since $P(2)$, $P(3)$, and so on. This shows that $P(n)$ is true for all positive integers.

Exercise 4: Let $P(n)$ be the statement that $1^3 + 2^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$ for the positive integer n.

a) What is the statement $P(1)$?

The statement is that $1^3 = \left(\dfrac{1(1+1)}{2}\right)^2$

b) Show that $P(1)$ is true completing the basis step of $P(n)$ for all positive integers n.

$$\begin{aligned}
1^3 &= \left(\frac{1(2)}{2}\right)^2 \\
&= (1)^2
\end{aligned}$$

So $P(1)$ is true.

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?

The inductive hypothesis is that $1^3 + 2^3 + \ldots + k^3 = \left(\dfrac{k(k+1)}{2}\right)^2$ for some arbitrary k.

d) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?

You would need to prove that $P(k+1)$ is true from the inductive hypothesis.

e) Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.

Proof. Let k be arbitrary and $P(k)$ is true. Then

$$1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Now for $P(k+1)$:

$$\begin{aligned}
1^3 + 2^3 + \ldots + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \left(\frac{k^2(k+1)^2}{4}\right) + (k+1)^3 \\
&= (k+1)^2\left(\frac{k^2}{4} + (k+1)\right) \\
&= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) \\
&= (k+1)^2\left(\frac{(k+2)^2}{4}\right) \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}$$

Since $P(k+1) = \left(\frac{(k+1)(k+2)}{2}\right)^2$ as desired. □

f) Explain why these steps show that this formula is true whenever n is a positive integer.

Since $P(1)$ and $P(k+1)$, then $P(2)$. Since $P(2)$, $P(3)$, and so on. This shows that $P(n)$ is true for all positive integers.

Exercise 10:

a Find a formula for

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

$$\frac{1}{1\cdot 2} = \frac{1}{2},$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{3}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3} = \frac{2}{3},$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{12}{1\cdot 2\cdot 3\cdot 4} + \frac{4}{1\cdot 2\cdot 3\cdot 4} + \frac{2}{1\cdot 2\cdot 3\cdot 4} = \frac{3}{4}$$

Inductive hypothesis: $P(n) : \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

b) Prove the formula you conjectured in part (a).

Proof. Basis step: Observe that

$$\frac{1}{1\cdot 2} = \frac{1}{2}$$

So $P(1)$ holds.

Inductive Step: Suppose k is arbitrary and assume that $P(k)$ is true. Then

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Adding $1/(k+1)(k+2)$ to both sides,

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

since $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$, $P(k+1)$ is true as desired. □

Use mathematical induction to prove the inequalities in Exercises 18-30.

Exercise 18: Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

a) What is the statement $P(2)$?

The statement $P(2)$ is that $x! < 2^2$

b) Show that $P(2)$ is true completing the basis step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.

Since $2! = 2 < 2^2 = 4$, $P(2)$ holds.

c) What is the inductive hypothesis for a proof by mathematical induction that $P(n)$ for all integers n greater than 1?

The inductive hypothesis is that $k! < k^k$ for some arbitrary integer k greater than 1.

d) What do you need to prove in the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.

You would need to prove that $P(k+1)$ is true from the inductive hypothesis. That is, that $(k+1)! < (k+1)^{(k+1)}$.

e) Complete the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.

Proof. Suppose that k is arbitrary integer greater than 1 and that the inequality $k! < k^k$ holds. We wish to show that $(k+1)! < (k+1)^{(k+1)}$. Multiply both sides by $k+1$. Since $k+1 > 0$, we have:

$$\begin{aligned} k! &< k^k \\ (k+1)! &< k^k(k+1) \end{aligned}$$

But now observe that:

$$\begin{aligned} (k+1) &> k \\ (k+1)^k &> k^k \\ (k+1)^k(k+1) &> k^k(k+1) \\ (k+1)^{(k+1)} &> k^k(k+1) \end{aligned}$$

Therefore,

$$(k+1)! < k^k(k+1) < (k+1)^{(k+1)}$$

So the statement $(k+1)! < (k+1)^{(k+1)}$ is true, as desired. $\square$

f) Explain why these steps show that the inequality is true whenever n is an integer greater than 1.

Since we have show that $P(2)$ is true and the if $P(k)$ is true, $P(k+1)$ is true, we can conclude that the inequality holds for $P(3)$, $P(4)$, ..., and so on. So the inequality is true for all natural numbers greater than 1.

Exercise 28: Prove that $n^2 - 7n + 12$ is non-negative when n is an integer with $n \geq 3$.

Proof. We will prove that $n^2 - 7n + 12 \geq 0$ for $n \geq 3$ by induction.

Basis Step: We need to show that the statement holds for our base case of 3:

$$3^2 - 7(3) + 12 = 9 - 21 + 12 = 0 \geq 0$$

Inductive Step: Suppose that k is an arbitrary integer greater than or equal to 3. Suppose that $k^2 - 7k + 12 \geq 0$ is true. We wish to show that $(k+1)^2 - 7(k+1) + 12 \geq 0$. Note the expression $(k+1)^2 - 7(k+1) + 12$ expanded:

$$k^2 + 2k + 1 - 7k - 7 + 12 = k^2 - 5k + 6$$

Now consider the expression $2k - 6$ or $2(k-3)$. Observe that the expression is positive when

$$\begin{aligned} k - 3 &\geq 0 \\ k &\geq 3 \end{aligned}$$

Since k is indeed greater than or equal to 3, we have the inequalities:

$$\begin{aligned} k^2 - 7k + 12 &\geq 0 \\ 2k - 6 &\geq 0 \end{aligned}$$

Adding the inequalities, we get

$$\begin{aligned} k^2 - 7k + 12 + (2k - 6) &\geq 0 \\ (k+1)^2 - 7(k+1) + 12 &\geq 0 \end{aligned}$$

as desired. $\square$

Use mathematical induction in Exercises 31-37 to prove divisibility facts.

Exercise 32: Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Proof. Basis Step: We must show that 3 divides $n^3 + 2n$ for $n = 1$:

$$\begin{gathered} 1^3 + 2(1) = 3 \\ \text{Indeed, } 3|3 \end{gathered}$$

Inductive Step: Suppose k is an arbitrary positive integer and that 3 divides $k^3 + 2k$. Since 3 divides $k^3 + 2k$, let $k^3 + 2k = 3a$ for some $a \in \mathbb{Z}$. We must show that 3 also divides $(k+1)^3 + 2(k+1)$. Observe that:

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= k^3 + 3k^2 + 5k + 3 \\
&= k^3 + 2k + 3(k^2 + k + 1) \\
&= 3a + 3(k^2 + k + 1) \\
&= 3(k^2 + k + 1 + a)
\end{aligned}$$

So $(k+1)^3 + 2(k+1)$ is also divisible by 3, as desired. □

Exercise 49-51 present incorrect proofs using mathematical induction. You will need to identify an error in reasoning in eash exercise.

Exercise 49: What is wrong with this "proof" that all horses are the same color?

Proof. Let $P(n)$ be the proposition that all horses in a set of n horses have the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all horses in a set of k horses are the same color. Consider $k+1$ horses; number these as horses $1, 2, \ldots, k, k+1$. Now the first k of these horses must have the same color. The last k of this must also have the same color. Because the set of the first k horses and the last k horses overlap, all $k+1$ must be the same color. This shows that $P(k+1)$ is true and finishes the proof by induction. □

This is incorrect since the definition of $P(n)$ is incorrect. The proposition that is to be tested is that "n horses have the same color." If this definition was used, we cannot conclude that the last k horses all have the same color since we only know the color of the first k horses.

Exercise 54x: Use mathematical induction to prove that given a set of $n+1$ positive integers, none exceeding $2n$, there exists one integer in this set that divides another integer in the set.

Proof. We will proceed by induction:

Basis Step: We must show that given as a set of 2 positive integers, none exceeding 2, that there is an integer that divides another in the set.

There are 3 possible sets: {1,1} {1,2} {2,2}

By cases, we have verified the base case: 1|1 1|2 2|2

Inductive Step: Assume that k is an arbitrary positive integer. Suppose that a set of $k+1$ positive integers, none exceeding $2k$ has an integer that divides another in the set. We must show that a set of $k+2$ positive integers, none exceeding $2k+2$ has an integer that divides another in the set.

We can list all possible sets of $k+1$ elements less than $2k$ such that there is an element in the set that divides another:

$$
\begin{aligned}
K_1 &= \{k_{1_1}, k_{1_2}, \ldots, k_{1_{k+1}}\} \\
K_2 &= \{k_{2_1}, k_{2_2}, \ldots, k_{2_{k+1}}\} \\
&\vdots \\
K_n &= \{k_{n_1}, k_{n_2}, \ldots, k_{n_{k+1}}\}
\end{aligned}
$$

Notice that $2k+1, 2k+2 \notin K_1, K_2, \ldots, K_n$. Now construct all $K_{i_{2k+1}}$ and $K_{i_{2k+2}}$ which is defined by adding by adding the element $2k+1$ to the set K_i and $2k+2$ to the set K_i $(1 \leq i \leq n)$. Observe that all possible sets of $k+2$ elements can either be in the form $K_{i_{2k+1}}$ or $K_{i_{2k+2}}$ or have 2 or more elements that are in the form $2k+1$ or $2k+2$.

Case 1: They have two or more elements of the form $2k+1$ or $2k+2$. Since $2k+1|2k+1$ and $2k+2|2k+2$, we have shown that an element in the set divides another in the same set.

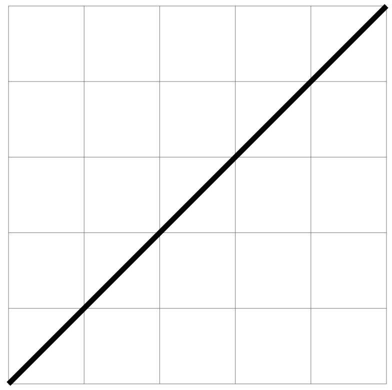
Case 2: Suppose the set is $K_{i_{2k+1}}$ or $K_{i_{2k+2}}$. Consider only one set without loss of generality. Then it follows that since K_i has an element that divides another in the set, and $K_i \subseteq K_{i_{2k+1}}$, the same holds for $K_{i_{2k+1}}$.

For both cases, we reached the conclusion that a set with $k+2$ positive integers no greater than $2k+2$ has an element that divides another element in the set, as desired. □

Exercise 62x: Show that n lines separate the plane into $\dfrac{n^2+n+2}{2}$ regions if no two are parallel and no three pass through the same point.

Proof. If no two lines are parallel, and no three intersect at a point, then each pair of lines has 1 intersection point.

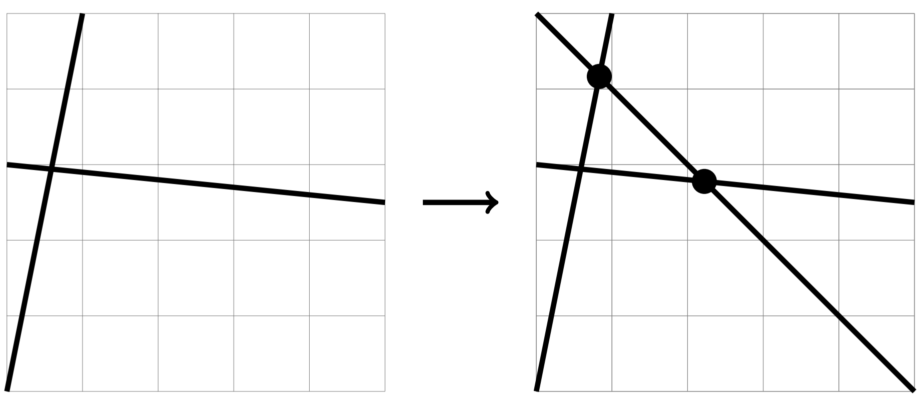
Basis Step: Observe that for 1 line we have 2 regions:



And $\dfrac{1^2+1+2}{2} = \dfrac{4}{2} = 2.$

Inductive Step: Suppose that k is an arbitrary positive integer and that k lines separate a plane into $\dfrac{k^2+k+2}{2}$ regions if no two are parallel and no three pass through the same point. We must show that for $k+1$ lines, there will be $\dfrac{(k+1)^2+(k+1)+2}{2}$ regions.

Observe that when an arbitrary line k' is added to the k lines, it must intersect each of the k lines once. So there will be k intersection points on our new line k'.



Since the line k' is divided into $k+1$ segments by the intersection points, k' must have run through $k+1$ regions. But by our basis step, a region divided by a single line yields two regions. So with $k+1$ regions divided, our plane with

k' has $k+1$ more regions than the plane without k':

$$\begin{aligned}\frac{k^2+k+2}{2}+(k+1) &= \frac{k^2+3k+4}{2}\\ &= \frac{(k+1)^2+k+3}{2}\\ &= \frac{(k+1)^2+(k+1)+2}{2}\end{aligned}$$

We have the expected formula for a plane with $k+1$ lines as desired. $\square$

Exercise 64: Use mathematical induction to prove Lemma 3 of Section 4.3 which states that if p is prime and $p|a_1a_2\ldots a_n$, where a_i is an integer for $i=1,2,\ldots,n$, the $p|a_i$ for some integer i.

Proof. We will proceed by mathematical induction.

Basis Step: We must show that $p|a_1$ implies that $p|a_i$ for some $i=1$. And indeed, $p|a_i$ for $i=1$.

Inductive Step: Suppose that k is some arbitrary positive integer and that $p|a_1a_2\ldots a_k$ means that $p|a_i$ for some $i=1,2,\ldots,k$. We must show that $p|a_1a_2\ldots a_{k+1}$ implies that $p|a_i$ for some $i=1,2,\ldots,k+1$. For $p|a_1a_2\ldots a_{k+1}$, we have two cases.

Case 1: $p|a_1a_2\ldots a_k$. Then $p|a_1a_2\ldots a_{k+1}$ and $p|a_i$ for $i=1,2,\ldots,k+1$.

Case 2: $p\nmid a_1a_2\ldots a_k$. Thus, $bp+r=a_1a_2\ldots a_k$ for some $b\in\mathbb{Z}_+$ and $0<r<p$. Observe:

$$\begin{aligned}bp+r &= a_1a_2\ldots a_k\\ a_{k+1}bp+a_{k+1}r &= a_1a_2\ldots a_{k+1}\end{aligned}$$

Now suppose that $p|a_1a_2\ldots a_{k+1}$. Then $cp=a_1a_2\ldots a_{k+1}$ for some $c\in\mathbb{Z}_+$. So

$$\begin{aligned}a_{k+1}bp+a_{k+1}r &= cp\\ a_{k+1}bp-cp &= -a_{k+1}r\\ p(a_{k+1}b-c) &= -a_{k+1}r\end{aligned}$$

So $p|-a_{k+1}r$. Since $p\nmid r$, $p|a_{k+1}$. We conclude that $p|a_i$ for some $i=1,2,\ldots,k+1$. Both cases give us our desired statement. $\square$

Exercise 76x: Suppose we want to prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\cdot\frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

a) Show that if we try to prove this inequality with mathematical induction, the basis step works, but the inductive step fails.

Basis Step: We need to verify that $\frac{1}{2} < \frac{1}{\sqrt{3}}$. Multiplying both sides by $2\sqrt{3}$, we get $\sqrt{3} < 2$ which is true.

Inductive Step: Suppose that k is an arbitrary positive integer and that $\frac{1}{2}\cdot\frac{3}{4}\cdots\cdot\frac{2k-1}{2k} < \frac{1}{\sqrt{3k}}$. We must show that $\frac{1}{2}\cdot\frac{3}{4}\cdots\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+3}}$.

Observe that:

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k}}\cdot\frac{2k+1}{2k+2}$$
$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k+1}{2k+2} < \frac{2k+1}{\sqrt{12k^3+24k^2+12k}}$$

But:

$$\frac{2k+1}{\sqrt{12k^3+24k^2+12k}} > \frac{2k+1}{\sqrt{12k^3+24k^2+15k+3}} = \frac{2k+1}{\sqrt{(3k+3)(2k+1)^2}} = \frac{1}{\sqrt{3k+3}}$$

So:

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k+1}{2k+2}\ ?\ \frac{1}{\sqrt{3k+3}} < \frac{1}{\sqrt{3k}}\cdot\frac{2k+1}{2k+2}$$

We cannot complete the proof since we still do not know if $\prod_{i=0}^{k}\frac{2j+1}{2j+2}$ is less than, greater than, or equal to $\frac{1}{\sqrt{3k+3}}$.

b) Show that mathematical induction can be used to prove the stronger inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

For all integers greater than 1, which, together with the verification of the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

Proof. Inductive Step: Suppose that k is an arbitrary positive integer and that $\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k-1}{2k} < \frac{1}{\sqrt{3k+1}}$. We must show that $\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}$.

Observe that:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2}$$
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k+1}{2k+2} < \frac{2k+1}{\sqrt{(3k+1)(2k+2)^2}}$$
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k+1}{2k+2} < \frac{2k+1}{\sqrt{12k^3+28k^2+20k+4}} \quad < \frac{2k+1}{\sqrt{12k^3+28k^2+19k+4}}$$
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k+1}{2k+2} < \frac{2k+1}{\sqrt{(3k+4)(2k+1)^2}}$$
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}$$

We have concluded the desired statement. □

5.2: 9, 38x

Exercise 9: Use strong induction to prove that $\sqrt{2}$ is irrational. [*Hint:* Let $P(n)$ be the statement that $\sqrt{2} \neq \frac{n}{b}$ for any positive integer b.

Proof. Basis Step: We must show that $\sqrt{2} \neq \frac{1}{b}$. Since $\sqrt{2} = 1.41\ldots$ we have $1 < \sqrt{2}$. Since $\frac{1}{b} < 1$, $\sqrt{2} \neq \frac{1}{b}$.

Inductive Step: Let b be positive and an arbitrary integer. Suppose that for $1, 2, \ldots, k$, $\sqrt{2} \neq \frac{k}{b}$. We must show that $\sqrt{2} \neq \frac{k+1}{b}$. We have three cases.

Case 1: Suppose that $k+1$ is odd. Suppose for contradiction that $\sqrt{2} = \frac{k+1}{b}$.

Then it follows that:

$$\frac{k+1}{b} = \sqrt{2}$$
$$(k+1)^2 = 2b^2$$

Therefore, $(k+1)^2$ is even. Contradiction. $\dfrac{k+1}{b} \neq \sqrt{2}$.

Case 2: $k+1$ is even and $b > k$. But that means that:

$$b \geq k+1$$
$$1 \geq \frac{k+1}{b}$$

We conclude that $\dfrac{k+1}{b} \neq \sqrt{2}$.

Case 3: $k+1$ is even and $b \leq k$. We know that $\dfrac{1}{b}, \dfrac{2}{b}, \ldots, \dfrac{k}{b} \neq \sqrt{2}$. So that implies that:

$$b, \frac{b}{2}, \ldots, \frac{b}{k} \neq \frac{1}{\sqrt{2}}$$
$$2b, \frac{2b}{2}, \ldots, \frac{2b}{k} \neq \sqrt{2}$$

Let use begin by listing out all rationals that we have not equal to $\sqrt{2}$ in an array.

$$\begin{array}{cccc} 1/1 & 2/1 & \ldots & k/1 \\ 1/2 & 2/2 & \ldots & k/2 \\ 1/3 & 2/3 & \ldots & k/3 \\ \vdots & \vdots & & \vdots \end{array}$$

We can impose these computations to create a new array that also has elements not equal to $\sqrt{2}$. First, we take the reciprocal of all elements and reorganize:

$$\begin{array}{cccc} 1/1 & 2/1 & 3/1 & \ldots \\ 1/2 & 2/2 & 3/2 & \ldots \\ \vdots & \vdots & \vdots & \\ 1/k & 2/k & 3/k & \ldots \end{array}$$

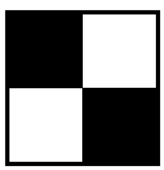
Then we multiply all elements by 2.

$$\begin{array}{cccc} {}^{2}/_{1} & {}^{4}/_{1} & {}^{6}/_{1} & \dots \\ {}^{2}/_{2} & {}^{4}/_{2} & {}^{6}/_{2} & \dots \\ \vdots & \vdots & \vdots & \\ {}^{2}/_{k} & {}^{4}/_{k} & {}^{6}/_{k} & \dots \end{array}$$

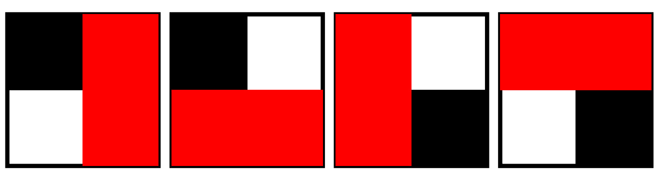
This is an array showing that all even numerator fractions are not equal to $\sqrt{2}$ such that our denominator b is less than or equal to k, as desired. $\square$

Exercise 38x Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

Proof. We will proceed by induction. Base case: The smallest, even celled checkerboard is a 2×2:



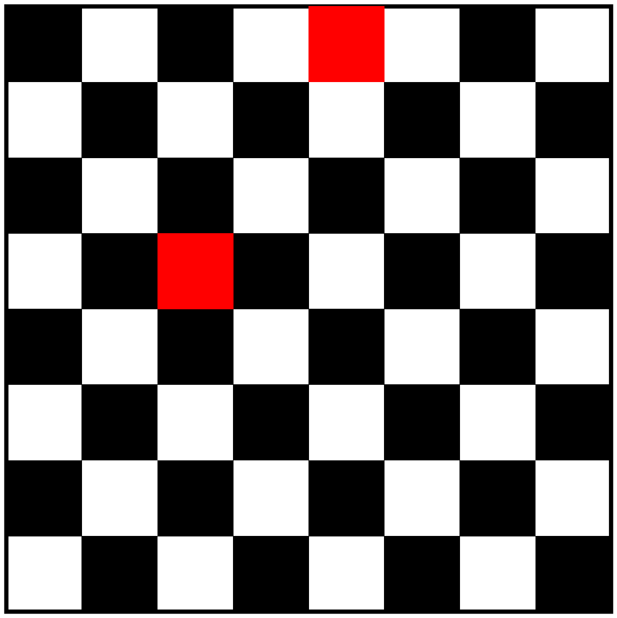
We have 4 cases of checkerboards with 2 squares removed, one white, one black:



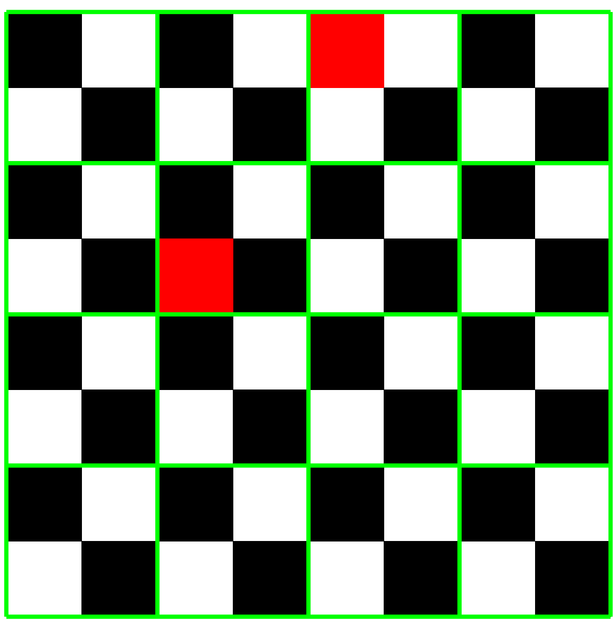
Each can be tiled with one domino.

Inductive Step: Suppose that we have a $2b \times 2b$ checkerboard with 1 black and white square removed which can be tiled with dominoes for $b = 1, 2, ...k$. We must show that a $2k + 2 \times 2k + 2$ checkerboard with one black and white square removed can be tiled with dominoes also.

For reference:



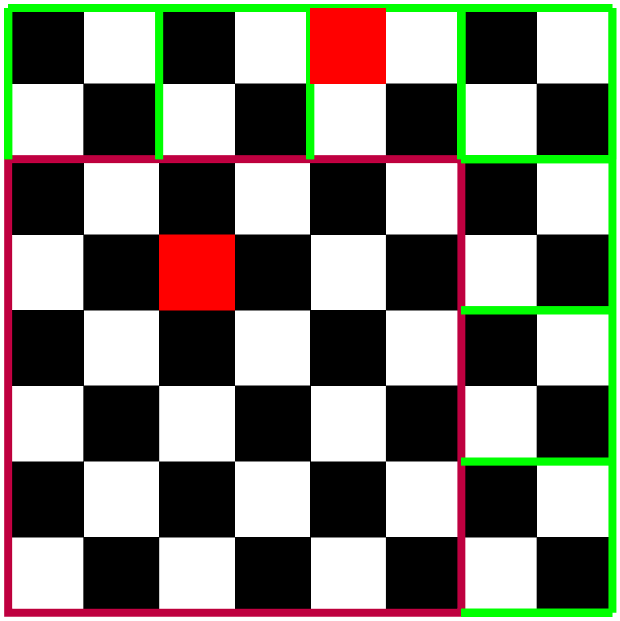
We can divide the boards into segments of 2×2 checkerboards:



Case 1: The red squares fall in the same 2×2 checkerboard. Then we can tile it. Observe that we can tile the remaining 2×2 checkerboards as desired.

Case 2: The red squares do not fall in the same 2×2 checkerboard. Then we can consider B, the largest $2k \times 2k$ checkerboard subset of our $2k + 2 \times 2k + 2$

checkerboard such that B contains red square r_1 but not r_2. Let A be the 2×2 checkerboard containing r_2.



Observe that B borders A. We can always place a domino that lies in both B and A. Let that domino be adjacent to r_2. Then it covers an opposite colored square than r_2 in A called r_2'. It also covers the same colored square as r_2 in B, called r_1'. Thus, r_1' and r_1 are opposite colored. Thus, we can tile A and B. We can also tile the remaining 2×2 checkerboards with 2 dominoes, so we are done. $\square$

5.3: 4ab, 8bd, 12, 14, 17x, 20, mt2-2016

Exercise 4: Find $f(2), f(3), f(4), f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and $n = 1, 2, \ldots$.

a) $f(n+1) = f(n) - f(n-1)$.
$f(2) = f(1) - f(0) = 1 - 1 = 0$
$f(3) = f(2) - f(1) = 0 - 1 = -1$
$f(4) = f(3) - f(2) = -1 - 0 = -1$
$f(5) = f(4) - f(3) = -1 - 1 = -2$

b) $f(n+1) = f(n)f(n-1)$.
$f(2) = f(1)f(0) = 1$
$f(3) = f(2)f(1) = 1$
$f(4) = f(3)f(2) = 1$
$f(4) = f(4)f(3) = 1$

Exercise 8: Give a recursive definition of the sequence $\{a_n\}, n = 1, 2, \ldots$ if

b) $a_n = 1 + (-1)^n$
$a_1 = 0, a_2 = 2$
$a_{n+1} = (-1/2)(a_n - a_{n-1}) + 1$

d) $a_n = n^2$
$a_1 = 1, a_2 = 4$
$a_{n+1} = (a_n - a_{n-1} + 2) + a_n = 2a_n - a_{n-1} + 2$

In Exercises 12-19, f_n is the nth Fibonacci number.

Exercise 12 Prove that $f_1^2 + f_2^2 + \ldots f_n^2 = f_n f_{n+1}$ when n is a positive integer.

Proof. We will proceed by induction.

Basis Step: We must show that for $f_1 = 0$, $f_2 = 1$, and $f_3 = 1$, that $f_1^2 + f_2^2 = f_2 f_3$. Plug in the numbers and verify: $f_1^2 + f_2^2 = 0^2 + 1^2 = 1^2 = f_2 f_3$.

Inductive Step: Suppose that k is arbitrary and that $f_1^2 + f_2^2 + \ldots + f_k^2 = f_k f_{k+1}$. We must show that $f_1^2 + f_2^2 + \ldots + f_{k+1}^2 = f_{k+1} f_{k+2}$. Observe the following algebraic manipulations:

$$\begin{aligned}
f_1^2 + f_2^2 + \ldots + f_k^2 &= f_k f_{k+1} \\
f_1^2 + f_2^2 + \ldots + f_{k+1}^2 &= f_k f_{k+1} + f_{k+1}^2 \\
f_1^2 + f_2^2 + \ldots + f_{k+1}^2 &= f_{k+1}(f_k + f_{k+1}) \\
f_1^2 + f_2^2 + \ldots + f_{k+1}^2 &= f_{k+1} f_{k+2}
\end{aligned}$$

We have verified the equality to be proved. □

Exercise 14: Show that $f_{n-1} f_{n+1} - f_n^2 = (-1)^n$ when n is a positive integer.

Proof. We will proceed by induction.

Basis Step: We should confirm that $f_0 f_2 - f_1^2 = (-1)^1$ for the base case $n = 1$, $f_0 = 0$, $f_1 = 1$, $f_2 = 1$. By computation: $f_0 f_2 - f_1^2 = 0 - 1 = (-1)^1$.

Inductive Step: Suppose that k is an arbitrary positive integer and that $f_{k-1} f_{k+1} - f_k^2 = (-1)^k$. We must prove that $f_k f_{k+2} - f_{k+1}^2 = (-1)^{k+1}$. From our first equation, add $f_k f_{k+1} - f_k f_{k+1} = 0$ to both sides:

$$\begin{aligned}
f_{k-1} f_{k+1} - f_k^2 &= (-1)^k \\
f_{k-1} f_{k+1} + (f_k f_{k+1}) - f_k^2 + (-f_k f_{k+1}) &= (-1)^k \\
f_{k+1}(f_k + f_{k+1}) - f_k(f_k + f_{k+1}) &= (-1)^k \\
f_{k+1}^2 - f_k f_{k+2} &= (-1)^k \\
f_k f_{k+2} - f_{k+1}^2 &= (-1)^{k+1}
\end{aligned}$$

We have concluded that $f_k f_{k+2} - f_{k+1}^2 = (-1)^{k+1}$ as desired. □

Exercise 17x: Determine the number of divisions used by the Euclidean Algorithm to find the greatest common divisor of the Fibonacci numbers f_n and f_{n+1}, where n is a nonnegative integer. Verify your answer using mathematical induction.

Proof. We will proceed with mathematical induction where the number of steps to compute $\gcd(f_n, f_{n+1})$ is $n-1$.

Basis Step: We need to show that to compute $\gcd(f_1, f_2)$ takes $1-1=0$ steps. Our Fibonacci numbers are $f_1 = 0, f_2 = 1$. So $\gcd(1,0) = 1$ which required 0 steps.

Inductive Step: Suppose that k is an arbitrary positive integer and that $\gcd(f_n, f_{n+1})$ takes $k-1$ steps of the Euclidean algorithm to compute. We must show that $\gcd(f_{n+1}, f_{n+2})$ takes k steps to compute.

Consider $\gcd(f_{n+1}, f_{n+2})$. By the Euclidean Algorithm, we have:

$$\begin{aligned}\gcd(f_{n+1}, f_{n+2}) &= \gcd(f_{n+2} \mod f_{n+1}, f_{n+1}) \\ &= \gcd(f_n, f_{n+1})\end{aligned}$$

So there are $k-1$ more steps after this to computing the gcd. Since we took one step of the Euclidean Algorithm (computing $f_{n+2} \mod f_{n+1}$), there are $k-1+1$ steps total or k, as desired. □

Exercise 20: Give a recursive definition of the functions max and min so that $\max(a_1, a_2, \ldots, a_n)$ and $\min(a_1, a_2, \ldots, a_n)$ are the maximum and minimum of the n numbers $a_1, a_2, \ldots, a_n$, respectively.

Maximum:

Basis Step: $\max(a_1) = a_1$

Recursive Step: $\max(a_1, a_2, \ldots, a_n) = \max(\max(a_1, a_2, \ldots, a_{n-1}), a_n)$

Minimum:

Basis Step: $\min(a_1) = a_1$

Recursive Step: $\min(a_1, a_2, \ldots, a_n) = \min(\min(a_1, a_2, \ldots, a_{n-1}), a_n)$

Midterm 1 (2016): (8 points) Consider the function recursively defined by:

$$f(1) = 1; \quad f(k+1) = \sqrt{1 + f(x)} \quad \text{when } k \geq 1$$

Given that $f(2) = \sqrt{2}$ is irrational, prove, using induction, that $f(n)$ is irrational for all integers $n \geq 2$.

Proof. We will proceed with induction.

Basis Step: We are already given that $f(2) = \sqrt{2}$ is irrational.

Inductive Step: Suppose that $k \geq 2$ is an arbitrary integer and that $f(k)$ is irrational. We must show that $f(k+1)$ is also irrational. Suppose for contradiction that $f(k+1)$ is rational. Then,

$$\begin{aligned} f(k+1) &= \sqrt{1 + f(k)} \\ f^2(k+1) &= 1 + f(k) \end{aligned}$$

Since $f(k+1)$ is rational, $f^2(k+1)$ is also rational and therefore, $1 + f(k)$ is rational. Contradiction. We know that 1 is rational and $f(k)$ is irrational. The sum of an irrational number and rational number is always irrational. $\square$