Math185Hw8

Trustin Nguyen

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Exercise 1: Prove that if a holomorphic function f has an isolated singularity at 0, then the principal part of its Laurent expansion converges everywhere on $\mathbb{C}\backslash\{0\}$.

Answer. It was shown in class that a holomorphic function converges to its Laurent expansion on the punctured disk around its singularity. So since it converges to its Laurent series, then the principal part must converge for $\mathbb{C}\backslash\{0\}$.

Exercise 2: Find the Laurent series of the function $f(z) = (z^2 - 1)\sin\frac{1}{z^2}$ which converges in the region $0 < |z| < \infty$.

Answer. The Taylor series for $\sin z$ centered at $z = 0$ is

$$\sum_{n\geqslant 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Now plugging in $\frac{1}{z^2}$, we get:

$$\sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)!z^{4n+2}}$$

Finally, multiply by $(z^2 - 1)$:

$$\begin{aligned}
(z^2 - 1)\sin\frac{1}{z^2} &= (z^2 - 1)\sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)!} z^{4n+2} \\
&= \sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)!z^{4n}} - \sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)!z^{4n+2}} \\
&= \sum_{n\geqslant 0} \frac{(-1)^{\lfloor (n+1)/2 \rfloor}}{(2\lfloor n/2 \rfloor + 1)!} z^{-2n}
\end{aligned}$$

which is the Laurent series for $f(z)$.

Exercise 3: Find the residues of the following functions at each of their isolated singularities:

(a) $\frac{z^p}{1-z^q}$ for $(p, q \in \mathbb{Z}_{>0})$

Answer. The singularities are at $e^{ik\pi/q}$. The for each singularity, define $h(z) = -(z - e^{ik\pi/q})f(z)$ where $f(z) = \frac{z^p}{1-z^q}$. So to evaluate $(e^{ik\pi/q} - e^{ik\pi/q})/(z^q - 1)$, use L'Hopital to get:

$$\lim_{z\to e^{ik\pi/q}} \frac{z - e^{ik\pi/q}}{z^q - 1} = \frac{1}{qz^{q-1}} = \frac{1}{qe^{ik\pi(q-1)/q}}$$

So for a singularity $e^{ik\pi/q}$, the residue is

$$-\frac{e^{ikp\pi/q}}{qe^{-ik\pi/q}} = \frac{e^{ik(p+1)\pi/q}}{q}$$

(b) $\frac{z^5}{(z^2-1)^2}$

Answer. We define the holomorphic function $h(z) = (z+1)^2 \frac{z^5}{(z^2-1)^2} = \frac{z^5}{(z-1)^2}$. Then the residue at -1 is $\mathrm{Res}_{z=-1} f(z) = \frac{1}{(2-1)!} h^{2-1}(-1)$. So

$$h'(z) = \frac{(z-1)^2 \cdot 5z^4 - z^5 \cdot 2(z-1)}{(z-1)^4}$$

and

$$\begin{aligned} h'(-1) &= \frac{4 \cdot 5 - (-1) \cdot 2(-2)}{(-2)^4} \\ &= \frac{20-4}{16} \\ &= 1 \end{aligned}$$

So the residue at $z = -1$ is 1. Now for the residue at $z = 1$, we define $h(z) = (z-1)^2 \frac{z^5}{(z^2-1)^2} = \frac{z^5}{(z+1)^2}$. The residue at $z = 1$ is $\mathrm{Res}_{z=1} \frac{1}{(2-1)!} h^{2-1}(1)$. And

$$h'(z) = \frac{(z+1)^2 \cdot 5z^4 - z^5 \cdot 2(z+1)}{(z+1)^4}$$

Now evaluating at $z = 1$:

$$\begin{aligned} h'(1) &= \frac{2^2 \cdot 5 - 2 \cdot 2}{2^4} \\ &= \frac{16}{16} \\ &= 1 \end{aligned}$$

So the residue at $z = 1$ is also 1.

(c) $\frac{\cos z}{1+z+z^2}$

Answer. It has singularities at $z = e^{2i\pi/3}, e^{4i\pi/3}$. So let $h(z) = (z - e^{2i\pi/3}) f(z)$ where $f(z) = \frac{\cos z}{1+z+z^2}$. Now we calculate $h(e^{2i\pi/3})$ which is

$$\frac{\cos z}{(z - e^{4\pi i/3})} = \frac{\cos e^{2i\pi/3}}{i\sqrt{3}}$$

which is the contribution at $e^{2i\pi/3}$. Now for the contribution at $e^{4i\pi/3}$, use $h(z) = (z - e^{4i\pi/3}) h(z)$ and evaluate at $z = e^{4i\pi/3}$:

$$\frac{\cos z}{(z - e^{2\pi i/3})} = \frac{\cos e^{4i\pi/3}}{-i\sqrt{3}}$$

which is the contribution at $e^{4i\pi/3}$.

Exercise 4: Give a formula for the residue at 0 of the function $\sin z + z^{-1}$.

Answer. We have

$$\sin z = \sum_{n \geqslant 0} \frac{z^{2n+1}(-1)^n}{(2n+1)!}$$

So

$$\sin z + z^{-1} = \sum_{n \geqslant 0} \frac{(z+z^{-1})^{2n+1}(-1)^n}{(2n+1)!}$$

We can use binomial expansion:

$$(z+z^{-1})^{2n+1} = \sum_{m \geqslant 0} \binom{2n+1}{m} z^{-m} z^{2n+1-m} = \sum_{m \geqslant 0} \binom{2n+1}{m} z^{2n-2m+1}$$

and the residue is obtained precisely when $m = n+1$. So we get the z^{-1} term is

$$\binom{2n+1}{n+1} z^{-1}$$

Plugging this back into our $\sin z + z^{-1}$ formula, we get the formula:

$$\sum_{n \geqslant 0} \frac{\binom{2n+1}{n+1}(-1)^n}{(2n+1)!} = \sum_{n \geqslant 0} \frac{1}{(n+1)!(n)!}(-1)^n$$

as the coefficient of z^{-1}, and it converges by alternating series test.

Exercise 5: Determine $\int_C \frac{\exp(iz)}{z^3}\, dz$ around the circle C $|z| = 1$.

Answer. Using Cauchy's formula for derivatives, we have:

$$(\exp(iz))'' = \frac{2!}{2\pi i} \int_C \frac{\exp(iz)}{z^3}\, dz$$

So

$$\pi i \cdot (\exp(iz))'' = -i\pi \exp(iz)\Big|_{z=0} = -i\pi$$

Exercise 6: Determine $\int_C \frac{\exp(tz)}{(z^2+1)^2}\, dz$ when $t > 0$ and C is the circle $|z| = 2$.

Answer. We have two singularities, one at $z = i$, the other, $z = -i$. So the integral is the sum of the contributions:

$$\int_C \frac{\frac{\exp(tz)}{(z-i)^2}}{(z+i)^2}\, dz + \int_C \frac{\frac{\exp(tz)}{(z+i)^2}}{(z-i)^2}\, dz$$

Now for each, we can use Cauchy's integral formula for derivatives:

$$\left(\frac{\exp(tz)}{(z-i)^2}\right)'\Bigg|_{z=-i} = \frac{1}{2\pi i} \int_C \frac{\frac{\exp(tz)}{(z-i)^2}}{(z+i)^2}\, dz$$

$$\left(\frac{\exp(tz)}{(z+i)^2}\right)'\Bigg|_{z=i} = \frac{1}{2\pi i} \int_C \frac{\frac{\exp(tz)}{(z-i)^2}}{(z+i)^2}\, dz$$

So compute both derivatives:

$$\left(\frac{\exp(tz)}{(z-i)^2}\right)' = \frac{(z-i)^2 t\exp(tz) - \exp(tz)2(z-i)}{(z-i)^4}$$
$$\left(\frac{\exp(tz)}{(z+i)^2}\right)' = \frac{(z+i)^2 t\exp(tz) - \exp(tz)2(z+i)}{(z+i)^4}$$

Evaluate both at $z = -i$, $z = i$:

$$\begin{aligned}\left(\frac{\exp(tz)}{(z-i)^2}\right)'\Bigg|_{z=-i} &= \frac{-4\cdot t\exp(-it) - \exp(-it)(-4i)}{(-2i)^4}\\ &= \frac{(-4t+4i)\exp(-it)}{16}\\ \left(\frac{\exp(tz)}{(z+i)^2}\right)'\Bigg|_{z=i} &= \frac{-4t\exp(it) - \exp(it)(4i)}{(2i)^4}\\ &= \frac{(-4t-4i)\exp(it)}{16}\end{aligned}$$

Multiplying both by $2\pi i$:

$$\begin{aligned}&\to \frac{\pi i\cdot(-t+i)\exp(-it)}{2}\\ &= \frac{(-\pi i t-\pi)\exp(-it)}{2}\end{aligned}$$

and

$$\begin{aligned}&\to \frac{\pi i(-t-i)\exp(it)}{2}\\ &= \frac{(-\pi i t+\pi)\exp(it)}{2}\end{aligned}$$

So the answer is

$$\frac{(-\pi i t+\pi)\exp(it)}{2} + \frac{(-\pi i t-\pi)\exp(-it)}{2}$$

Exercise 7: Show that, for any circle enclosing the point $z = -1$,

$$\int_C \frac{ze^{tz}}{(z+1)^3}\ dz = (t - t^2/2)e^{-t}$$

Answer. Use Cauchy's formula for derivatives:

$$(ze^{tz})''\Big|_{z=-1} = \frac{2!}{2\pi i}\int_C \frac{ze^{tz}}{(z+1)^3}\ dz$$

So we get:

$$\begin{aligned}(ze^{tz})' &= e^{tz} + tze^{tz}\\ (ze^{tz})'' &= te^{tz} + t(e^{tz} + tze^{tz})\\ &= te^{tz} + te^{tz} + t^2ze^{tz}\\ &= 2te^{tz} + t^2ze^{tz}\end{aligned}$$

and evaluate at $z = -1$ to get:

$$2te^{-t} - t^2e^{-t} = 2te^{-t} - t^2e^{-t} = (2t - t^2)e^{-t}$$

Then we divide by 2:

$$(t - \frac{t^2}{2})e^{-t}$$

Exercise 8: By choosing two different annuli, both centered at 0, in which the function below is holomorphic, find two different Laurent expansions for it in powers of z. Describe their regions of convergence.

$$f(z) = \frac{1}{z^2(1-z)}$$

Answer. We can have one annuli as $0 < |z| < 1$ and the other $1 < |z|$. Then on one annuli, we have that $1 - z = \sum_{n \geqslant 0} z^n$. So the Laurent series is

$$\frac{1}{z^2} \sum_{n \geqslant 0} z^n$$

which is

$$\sum_{n \geqslant -2} z^n$$

And for the other, we have that $|\frac{1}{z}| < 1$. This means we can expand:

$$\frac{1/z}{z^2\left(\frac{1}{z} - 1\right)} = -\frac{1}{z^3(1 - 1/z)}$$

This turns into

$$\frac{-1}{z^3} \cdot \sum_{n \geqslant 0} \left(\frac{1}{z}\right)^n = -\sum_{n \geqslant 3} \left(\frac{1}{z}\right)^n$$

Exercise 9: Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in Laurent series convergent for:

(a) $|z| < 1$;

Answer. We can use partial fraction decomposition to get:

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

Then since $|z| < 1$ and $|\frac{z}{2}| < 1$, we have the sum of two geometric series:

$$-\sum_{n \geqslant 0} z^n + \sum_{n \geqslant 0} \left(\frac{z}{2}\right)^n$$

(b) $1 < |z| < 2$;

Answer. Using the same decomposition:

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

Since $1 < |z|$, $\left|\frac{1}{z}\right| < 1$, so we can expand $\frac{1/z}{1-1/z}$ instead:

$$\frac{1}{z} \cdot \sum_{n \geqslant 0} \left(\frac{1}{z}\right)^n + \sum_{n \geqslant 0} \left(\frac{z}{2}\right)^n$$

(c) $|z| > 2$;

Answer. Using the same decomposition:

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

We instead can expand $-\frac{2/z}{1-2/z}$ instead for the second fraction:

$$\frac{1}{z}\sum_{n\geqslant 0}\left(\frac{1}{z}\right)^n - \sum_{n\geqslant 0}\left(\frac{2}{z}\right)^{n+1}$$

(d) $|z-1| > 1$;

Answer. Let $w = z-1$. Then the $f(z) = -\frac{w+1}{w(w-1)}$. This is

$$-\frac{w+1}{w^2-w} = -\frac{w+1}{w^2(1-1/w)}$$

Since $|\frac{1}{w}| < 1$, we can expand:

$$-\frac{w+1}{w^2}\cdot\sum_{n\geqslant 0}\left(\frac{1}{w}\right)^n$$

and re-substitute $w = z-1$:

$$-\frac{z}{(z-1)^2}\cdot\sum_{n\geqslant 0}\left(\frac{1}{z-1}\right)^n = -z\cdot\sum_{n\geqslant -2}\left(\frac{1}{z-1}\right)^n$$

(e) $0 < |z-2| < 1$.

Answer. Using $w = z-2$, we have:

$$f(z) = \frac{z}{(z-1)(2-z)} = \frac{w+2}{(w+1)(-w)} = -\frac{w+2}{w(w+1)}$$

So we can expand $(w+1)^{-1}$ as

$$-\frac{w+2}{w}\cdot\sum_{n\geqslant 0}(-w)^n$$

Now re-substitute in $w = z-2$:

$$-\frac{z}{z-2}\cdot\sum_{n\geqslant 0}(2-z)^n = z\cdot\sum_{n\geqslant -1}(2-z)^n$$