Math104Notes

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Contents

Chapter 1

Week 1

Notations: For sets, we have a set S is a collection of elements. Either $x \in S$ or $x \notin S$. If $S' \subseteq S$ if S' has elements that all are in S. Intersection and union:

$$\begin{aligned} S_1 \cap S_2 &= \{s : s \in S_1 \text{ and } s \in S_2\} \\ S_1 \cup S_2 &= \{s : s \in S_1 \text{ or } s \in S_2\} \\ S_1 - S_2 &= \{s : s \in S_1 \text{ and } s \notin S_2\} \end{aligned}$$

We also have other common types of sets:

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, \ldots\} \\ \mathbb{Z} &= \{\ldots, -1, 0, 1, \ldots\} \\ \mathbb{Q} &= \{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\} \\ \mathbb{R} &= \text{Real numbers} \end{aligned}$$

1.1 Absolute Values (Ross 3)

Definition 1.1.1 **Absolute Values**

$\forall a \in \mathbb{R}$:

$$|a| = \begin{cases} a \text{ if } a \geqslant 0 \\ a \text{ if } a < 0 \end{cases}$$

You could also say that the absolute value is the distance between a and 0.

Theorem 1.1.1 **Absolute values**

The following are facts about absolute values:

- $|a| \geqslant 0$

  Proof. We follow by the function of the definition of absolute values. Use case work. □

- $|ab| = |a||b|$

  Proof. □

- $|a+b| \leqslant |a| + |b|$

  Proof. For two numbers we have:

$$-|a| \leqslant a \leqslant |a|$$
$$-|b| \leqslant b \leqslant |b|$$

  We have that
$$-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$$
  We then have

$$a + b \leqslant |a| + |b|$$
$$-(a + b) \leqslant |a| + |b|$$

  By case work, $|a+b| \leqslant |a|+|b|$. This is called the triangle inequality because it says that the sum of two sides of a triangle is greater than or equal to the third. □

Example 1.1.1: Show that $\sqrt{2}$ is irrational.

Proof. Suppose for contradiction that $\sqrt{2}$ is a rational number. So we have p, q which are integers such that:
$$\sqrt{2} = \frac{p}{q}$$
where $q \neq 0$ and p, q are relatively prime.

$$\frac{p}{q} = \sqrt{2}$$
$$\frac{p^2}{q^2} = 2$$
$$p^2 = 2q^2$$

So p^2 must be even. So there exists an m such that $p = 2m$. Now we have

$$(2m)^2 = 2q^2$$
$$4m^2 = 2q^2$$
$$2m^2 = q^2$$

This also means that q^2 is even. But that is a contradiction as p, q are relatively prime. □

1.2 Lower and Upper Bounds/ Completeness Axiom

Definition 1.2.1 **Maximum/Minimum of $S \subseteq \mathbb{R}$**

If $\exists x_0 \in S$ such that $\forall x \in S$, $x_0 \geqslant x$, we call x_0 the maximum of S. $x_0 = \max(S)$

If $\exists x_0 \in S$ such that $x_0 \leqslant x$ for all $x \in S$, then x_0 is the minimum of S.

The maximum or minimum may not exist. It is unique if it exists also.

Example 1.2.1: We have the set $\{1,2,3,4\}$. The max is 4 and the min is 1

Example 1.2.2: We have the set $(0,10) = \{x \in \mathbb{R} : x > 0 \wedge x < 10\}$. There is no maximum nor minimum.

Example 1.2.3: We have the set $[0,\infty) = \{x \in \mathbb{R} : x \geqslant 0\}$. The maximum does not exist. The minimum is 0.

Definition 1.2.2 **Upper/Lower Bound**

If $\exists M \in \mathbb{R}$ where $\forall x \in S$, $M \geqslant x$, then M is the upper bound of a set S. Note that unlike the maximum, M does not have to be in S.

If $\exists m \in \mathbb{R}$ where $\forall x \in S$, $m \leqslant x$, then m is the lower bound of S.

We say that S is bounded above if the upper bound exists. If S has a lower bound, we say that S is bounded below. If S is bounded above and below, S is said to be bounded.

Example 1.2.4: $S = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$. We have the upper bound as $M \geqslant 1$. The lower bound is $m \leqslant 0$.

Example 1.2.5: $S = (-\infty, 1]$. The upper bound is $M \geqslant 1$. There is no lower bound.

Remark: The upper and lower bound may not exist. They also need not be unique. If the maximum of S exists, then the maximum is an upper bound. The same can be said for the minimum of S.

Definition 1.2.3 **Sup/Inf**

If $\exists x_0 \in \mathbb{R}$ such that:

- x_0 is an upper bound of S.
- x_0 is the smallest upper bound,

then we say that x_0 is the supremum of S. $\text{Sup}(S) = x_0$.

If $\exists x_0' \in \mathbb{R}$ such that:

- x_0' is a lower bound of S
- x_0' is the largest lower bound of S,

then we call x_0 the infimum of S or $\text{Inf}(S) = x_0'$.

Note that the infimum and supremum may not exist but if they do, they are unique.

If $\max(S)$ exists, then it is equal to $\sup(S)$. If $\sup(S)$ exists, then $\max(S)$ may not exist.

Example 1.2.6: $S = (0, 2)$. We have $\inf(S) = 0$ and $\sup(S) = 2$. Notice that the maximum and minimum does not exist.

Example 1.2.7: $S = \{x : x^2 < 2\}$. This is just $(-\sqrt{2}, \sqrt{2})$. This goes back to the previous example.

Question: Can we always find a smallest upper bound if an upper bound exists?

Completeness Axiom: If S is bounded above, then $\sup(S)$ exists. Corollary: If S is bounded below, then $\inf(S)$ will exist.

Example 1.2.8: Consider the set $A = \{x \in \mathbb{Q} : x \geqslant 0 \wedge x^2 \leqslant 2\} = (o, \sqrt{2}) \cap \mathbb{Q}$. A is bounded, but A has no supremum in $\mathbb{Q}$.

Theorem 1.2.1 **Archimedian Property**

If $a, b > 0$, then $\exists n \in \mathbb{N}$, we have $na > b$.

Proof. Assume for any a, $na \leqslant b$ for all $n \in \mathbb{N}$. Let $S = \{na : n \in \mathbb{N}\}$. Then S is bounded above. So $\sup(S) - a$ is not an upper bound of S. So $\exists n_0 \in \mathbb{N}$ such that $na > \sup(S) - a$: $(n_0 + 1)a > \sup(S)$. But $n_0 + 1 \in \mathbb{N}$, so it must be in S. This is a contradiction. $\square$

Theorem 1.2.2 **$\mathbb{Q}$ is dense**

If $a, b \in \mathbb{R}$ and $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

The idea is that we want to find $\frac{m}{n}$ that is between a and b. This is equivalent to $an < m < bn$. We want n such that $nb - na > 1$. But this is just the archimedian property: $n(a - b) > 1$

Chapter 2

Week 2

We will prove the denseness of the rationals using the archimedean property:

Proof. Let a, b be in $\mathbb{R}$. Choose a $k \in \mathbb{N}$ such that $|nb|, |na| < k$. Consider the set of $\{n \in \mathbb{Z} : na < j < k\}$. This set is finite, so we can find the minimum of this set. We have that $m - 1 \leqslant na$ which means that $m \leqslant na + 1 < nb$. So $na < m < nb$. □

2.1 Sequence and Convergence

Definition 2.1.1

Sequence

A sequence is a function from $\{n \in \mathbb{Z} : n \geqslant m\}$ where we usually choose $m = 0, 1$. The value is $S(n) \in \mathbb{R}$. We write the sequence $(S_n)_{n=m}^{\infty}$. If $m = 1$, we just write (S_n).

Remark: Sometimes, we use notations $(a_n), (b_n), (C_n)$

Example 2.1.1: We can have $S_n = \frac{1}{n^2}, n \in \mathbb{N}$ which is the sequence:

$$(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots)$$

Example 2.1.2: $a_n = (-1)^n$ where $n \in \mathbb{Z}_{\geqslant 0}$ which is the sequence:

$$(1, -1, 1, -1, \ldots)$$

Example 2.1.3:

- $a_n = \cos \frac{n\pi}{3}, n \in \mathbb{N}$.
- $a_1 = \cos \frac{\pi}{3} = \frac{1}{2}$
- $a_2 = \cos \frac{2\pi}{3} = \frac{-1}{2}$
- $a_3 = \cos \pi = -1$

- $(\frac{1}{2}, -\frac{1}{2}, -1, \dots)$

Example 2.1.4: $a_n = \sqrt[n]{n}, n \in \mathbb{N}$ with the sequence:

$$(1, \sqrt{2}, \sqrt[3]{3}, \dots)$$

We will now consider limits of a sequence:

Limits of a Sequence

Definition 2.1.2

Consider $(S_n)_{n\in\mathbb{N}}$ or (S_n) is said to converge to some real number $s \in \mathbb{R}$ provided that $\forall \varepsilon \geqslant 0, \exists N \in \mathbb{N}$ such that $\forall n > N$, we have $|S_n - S| < \mathcal{E}$.

Remark: We usually use ε for very small positive numbers.

- Choice of N depended on our choice of ε.

Today, we will cover rigorous proofs on convergence.

Proposition: If (S_n) has limits, then the limits are unique.

Proof. Assume $\lim S_n = S$, and that $\lim S_n = T$. By definition $\forall \mathcal{E} > 0, \exists N_1$ such that $\forall n > N$,

$$|S_n - S| < \mathcal{E}$$

But we have a second limit so $\exists N_2$ such that $\forall n > N_2$:

$$|S_n - T| < \mathcal{E}$$

Recall that $|a + b| \leqslant |a| + |b|$. Take $a = S_n - S, b = T - S_n$. Then $a + b = T - S$ and

$$\begin{aligned} |a + b| = |T - S| &\leqslant |S_n - S| + |T - S_n| \\ &< 2\mathcal{E} \end{aligned}$$

So $|T - S| = 0$. So $T = S$. □

How to prove $\lim(S_n) = S$ when S_n is a function of n. So for $\forall \mathcal{E} > 0$, we find N such that $\forall n > N$, we have $|S_n - S| < \mathcal{E}$. So we reduce $|S_n - S| < \mathcal{E}$ to the inequality $n >?$.

Example 2.1.5: Prove that $\lim(\frac{1}{n^2}) = 0$

Proof. $\forall \mathcal{E} > 0$, we want to find N such that $\forall n > N$, $|\frac{1}{n^2} - 0| = \frac{1}{n^2} < \mathcal{E}$. So we get:

$$n > \frac{1}{\sqrt{\mathcal{E}}}$$

So we choose $N = \frac{1}{\sqrt{\mathcal{E}}}$. So

$$|\frac{1}{n^2}| = \frac{1}{n^2}$$

and since $n > \frac{1}{\sqrt{\varepsilon}}, \frac{1}{n^2} < \frac{1}{(\frac{1}{\sqrt{\varepsilon}})^2} = \varepsilon$. Then $\lim(\frac{1}{n^2}) = 0$ □

Example 2.1.6: Prove $\lim(\frac{3n+1}{7n-4}) = \frac{3}{7}$. Idea to find the limit:

$$\frac{3n+1}{7n-4} = \frac{3+\frac{1}{n}}{7-\frac{4}{n}} = \frac{3}{7}$$

Proof. $\forall \varepsilon > 0$, we want to find N such that $\forall n > N$, we have $|\frac{3n+1}{7n-4} - \frac{3}{7}| < \varepsilon$

$$\frac{3n+1}{7n-4} - \frac{3}{7} = \frac{7(3n+1) - 3(7n-4)}{(7n-4)7} = \frac{19}{7(7n-4)}$$

So we want to find N such that:

$$|\frac{19}{7(7n-4)}| < \varepsilon$$

or

$$\varepsilon < \frac{19}{7(7n-4)} < \varepsilon$$

Since $7n - 4 > 0$, $\frac{19}{7(7n-4)} > 0$. Solve:

$$\frac{19}{7(7n-4)} < \varepsilon \implies n > \frac{19}{49\varepsilon} + \frac{4}{7}$$

So now we take $N = \frac{19}{49\varepsilon} + \frac{4}{7}$. Then verify. □

Example 2.1.7: Prove that $\frac{4n^3+3n}{n^3-6} = 4$.

Proof. $\forall \varepsilon > 0$, find N such that $\forall n > N$, we have:

$$|\frac{4n^3+3n}{n^3-6} - 4| < \varepsilon$$

We have:

$$\frac{4n^3+3n}{n^3-6} - 4 = \frac{3n+24}{n^3-6}$$

When $n \geqslant 2$, $n^3 - 6 > 0$, $\frac{3n+24}{n^3-6} > 0$. We want to find function f such that:

$$\frac{3n+24}{n^3-6} < f$$

and $f < \varepsilon$. Since $3n + 24 \leqslant 27n$ and $n^3 - 6 \geqslant \frac{1}{2}n^3$. When $n \geqslant 3$. So when $n \geqslant 3$, $\frac{3n+24}{n^3-6} \leqslant \frac{27n}{\frac{1}{2}n^3} = \frac{54}{n^2}$. So for $\frac{54}{n^2} < \varepsilon$, we need: $n > \sqrt{\frac{54}{\varepsilon}}$. So we need $n > \max(3, \sqrt{\frac{54}{\varepsilon}})$ □

2.2 Divergence

Example 2.2.1: Prove $a_n = (-1)^n$ diverges. Idea assume $\lim a_n = a$. We know that

$$\text{dist}(-1, a) + \text{dist}(1, a) \geqslant \text{dist}(-1, 1) = 2$$

One of $\text{dist}(-1, a)$ or $\text{dist}(1, a)$ must be greater than or equal to 1.

Proof. Assume $\lim(a_n) = a$. Let $\varepsilon = 1$. By definition, there is an N such that for all $n > N$,

$$|(-1)^n - a| < 1$$

We choose an odd and an even $n > N$ so

$$|1 - a| < 1 \qquad \text{and} \qquad |-1 - a| < 1$$

By triangle inequality:

$$|1 - a| + |a + 1| \geqslant |2| = 2$$

But their sum should be less than 2. So that is a contradiction. □

Example 2.2.2: Assume $s_n > 0$ and $\lim s_n = s > 0$. Prove

$$\lim \sqrt{s_n} = \sqrt{s}.$$

Idea: For $\forall \varepsilon > 0$, we want to find N such that $\forall n > N$, $|\sqrt{s_n} - \sqrt{s}| < \varepsilon$. Recall that $(a + b)(a - b) = a^2 - b^2$. So $a - b = \frac{a^2 - b^2}{a+b}$. We have

$$|\sqrt{s_n} - \sqrt{s}| = |\frac{s_n - s}{\sqrt{s_n} + s}|$$

Since $\sqrt{s_n} + \sqrt{s} > \sqrt{s}$, we have

$$\begin{aligned} \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}} &= \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \\ &< \frac{|s_n - s|}{\sqrt{s}} \end{aligned}$$

So we solve for ε:

$$\frac{|s_n - s|}{\sqrt{s}} < \varepsilon \iff |s_n - s| < \varepsilon\sqrt{s} = \varepsilon$$

Proof. We need to show that $\forall \varepsilon > 0$, we choose $\varepsilon = \varepsilon\sqrt{s}$. For ε', there is an N such that $\forall n > N$, $|S_n - s| < \varepsilon' = \varepsilon\sqrt{s}$. We have

$$|\sqrt{s_n} - \sqrt{s}| = s_n - s\sqrt{s_n} + \sqrt{s} < \frac{|s_n - s|}{\sqrt{s}} < \frac{\varepsilon\sqrt{s}}{\sqrt{s}} = \varepsilon$$

Why can't we choose $\varepsilon = (\sqrt{s_n} + \sqrt{s})\varepsilon$? Because we don't want ε to depend on n. □

2.3 Limit Theorems

Bounded Sequences

Definition 2.3.1

A sequence (s_n) is bounded if $\{s_n : n \in \mathbb{N}\}$ is bounded. Equivalently, we can require $\exists M > 0$ such that for all n,
$$|s_n| \leqslant M$$

Convergent Sequences are Bounded

Theorem 2.3.1

If $\lim S_n = s$, then (S_n) is bounded.

Proof. We choose $\varepsilon = 1$. So there is N such that $\forall n > N$,
$$|s_n - s| < 1$$

By the triangle inequality, $|s_n - s| + |s| > |s_n|$. Then $|s_n| < |s| + |s_n - s| < |s| + 1$. Take $m = \max(|s| + 1, |s_1|, |s_2|, \ldots, |s_N|)$. So $\forall n > N$, $|s_n| < |s| + 1 \leqslant m$. And $\forall n \leqslant N$, $|s_n| \leqslant m$. So $\forall n$, $|s_n| \leqslant m$. □

The idea is that for ε we know that the infinite number lies within some interval $(s - \varepsilon, s + \varepsilon)$ and we know that finitely many lie outside the interval. So taking the max over $s + \varepsilon$ and all elements s_n for $n \leqslant N$, to which there are a finite number will be the max of the entire sequence.

Theorem About Limits

Theorem 2.3.2

Assume that $\lim s_n = s$ and $\lim t_n = t$. We have

- $\forall r \in \mathbb{R}$, $\lim(rs_n) = rs$
- $\lim(s_n + t_n) = s + t$
- $\lim s_n t_n = st$
- $s_n \neq 0, \lim \frac{t_n}{s_n} = \frac{t}{s}$

Chapter 3

Week 3

3.1 Examples

Last Lecture: We had that a sequence (S_n) converses to S if $\forall \varepsilon > 0$, there $\exists N$ such that $|S_n - S| < \varepsilon$. If $\lim S_n = S$, $\lim(T_n) = T$, then

- $\lim(kS_n) = kS$
- $\lim(S_n + T_n) = S + T$
- $\lim S_n T_n = ST$
- $\lim(\frac{T_n}{S_n}) = \frac{T}{S}$ The condition is that we need S_n large enough that is non-zero. At the start of the sequence, our terms may not be defined, but for large enough n, we have a non-zero denominator, so there is a limit.

Last time, it was shown that

- $\lim \frac{1}{n^p} = 0$ for $p > 0$.
- We also have $\lim a^n = 0$ if $|a| < 1$.
- $\lim n^{\frac{1}{n}} = 1$
- $\lim a^{\frac{1}{n}} = 1$ for $a > 0$

Proof. Use Binomial Theorem. We have:

$$(1 + x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \ldots + x^n$$

□

Proof. We have $|a| = \frac{1}{b+1}$ or $b = \frac{1}{|a|} - 1 > 0$. Then

$$(1 + b)^n \geqslant 1 + nb$$

so we have $|a|^n = (\frac{1}{b+1})^n < \frac{1}{nb}$. So for all $\varepsilon > 0$, we let $N = \frac{1}{b\varepsilon}$. We get this by solving:

$$\frac{1}{nb} < \varepsilon \implies n > \frac{1}{b\varepsilon}$$

So we have that $\forall \varepsilon > 0$, if we let $N = \frac{1}{b\varepsilon}$, then for all $n > N$,

$$|a^n| = |a|^n < \frac{1}{nb} < \varepsilon$$

□

For the proof of (c):

Proof. We have $S_n = n^{\frac{1}{n}} - 1 \geqslant 0$. We also have:

$$(1+S_n)^n = n$$

by moving the 1 to the other side. So we have:

$$(1+S_n)^n \geqslant 1 + nS_n + \frac{1}{2}n(n-1)S_n^2 > \frac{1}{2}n(n-1)S_n^2$$

So we have

$$n > \frac{1}{2}n(n-1)S_n^2, S_n^2 < \frac{2}{n-1}$$

If we take the square root of both sides:

$$S_n < \sqrt{\frac{2}{n-1}} < \varepsilon$$

So if we solve:

$$\sqrt{\frac{2}{n-1}} < \varepsilon \implies \frac{2}{n-1} < \varepsilon^2 \implies n > \frac{2}{\varepsilon^2} + 1$$

Let $N = \frac{2}{\varepsilon^2} + 1$. Then $|S_n - 0| = S_n < \sqrt{\frac{2}{n+1}} < \varepsilon$. So the limit of $n^{\frac{1}{n}} = \lim S_n + 1 = 1$. □

Not enough time, so assume (d) is true.

Example 3.1.1: Let $S_n = \frac{n^3+6n^2+7}{4n^2+3n-4}$. Show that $\lim(S_n) = \frac{1}{4}$. We can change the form:

$$S_n = \frac{1 + \frac{6}{n} + \frac{7}{n^3}}{4 + \frac{3}{n^2} - \frac{4}{n^3}}$$

So $\lim S_n = \frac{\lim()}{\lim()} = \frac{1}{4}$.

Divergence

Definition 3.1.1 A sequence (S_n) diverges to $+\infty/-\infty$ if $\forall M > 0$, $\exists N$ such that $\forall n > N$, we have $S_n > M$. Similarly, $\lim S_n = -\infty$ if $\forall M < 0$, $\exists N$ such that $\forall n > N$, we have $S_nj < M$.

Example 3.1.2: Let $S_n = \sqrt{n} + 7$, $\lim(S_n) = +\infty$.

Proof. $\forall M > 0$, choose N such that for $n > N$, we have that $S_n > M$. So we want:

$$\sqrt{n} + 7 > M$$

or

$$n > (M-7)^2$$

Choose $N = (M-7)^2$. □

Remark: Theorem 9.2 - 9.6 may not hold for $\lim = -\infty/+\infty$. No definition for $\infty - \infty$.

Theorem 3.1.1 **9.9**

Assume that $\lim(S_n) = \infty$, $\lim(T_n) = \infty$. Then $\lim(S_n T_n) = \infty$.

Theorem 3.1.2 **9.10**

Assume that $S_n > 0$ for all n. Then

$$\lim(S_n) = \infty \iff \lim(\frac{1}{S_n}) = 0$$

3.2 More on Convergence and Divergence

Last Lecture, we have shown that

- $\lim \frac{1}{n^p} = 0$ if $p > 0$
- $\lim a^n = 0$ if $|a| < 1$
- $\lim n^{\frac{1}{n}} = 1$
- $\lim a^{\frac{1}{n}} = 1$ if $a > 0$
- $\lim S_n = \infty$ if $\forall M > 0, \exists N$ such that $\forall n > N, S_n > M$
- $\lim S_n = -\infty$ if $\forall M < 0, \exists N, \forall n > N$, such that $S_n < M$.

Definition 3.2.1 **Increasing, Decreasing, Monotone**

(S_n) is increasing if $\forall n, S_n \leqslant S_{n+1}$. (S_n) is decreasing if $\forall n, S_n \geqslant S_{n+1}$. (S_n) is a monotone sequence if it is increasing or decreasing.

Example 3.2.1: Increasing: $a_n = 1 - \frac{1}{n^2}, b_n = n^3, c_n = (1 + \frac{1}{n})^n$

Example 3.2.2: Decreasing: $d_n = \frac{1}{n}$

Example 3.2.3: Monotone: $s_n = (-1)^n$

Theorem 3.2.1 **Convergence Theorem**

Any bounded monotone sequence converges.

Proof. Assume (S_n) is a bounded increasing sequence. Let $S = \{S_n : n \in \mathbb{N}\}$ is bounded. So by the completeness axiom, $\mathrm{Sup}(S)$ exists. Let $u = \mathrm{Sup}(S)$. Then for every $\varepsilon > 0$, since $u - \varepsilon$ is not an upper bound. Then there is an $N \in \mathbb{N}$ such that $S_N > u - \varepsilon$. Since (S_n) is increasing, for all $n > N$, we have:

$$S_n \geqslant S_N > u - \varepsilon \implies |u - S_n| < \varepsilon$$

for all $n > N$. So $\lim(S_n) = u$. $\square$

Divergence Theorem

Theorem 3.2.2 If (S_n) is unbounded increasing, then (S_n) diverges to ∞. If (S_n) is unbounded and decreasing, then $\lim(S_n) = -\infty$.

Proof. Since (S_n) is unbounded and increasing, the it is bounded below by S_1. Then (S_n) has no upper bound. Then we have $\forall M$, M is not an upper bound of (S_n). So we have $N \in \mathbb{N}$ such that $S_n > M$. So we have for any $n > N$, we have

$$S_n > S_N > M$$

So this is the definition of divergence to ∞. □

If (S_n) is monotone. Then $\lim(S_n)$ exists as $\mathbb{R}$ or $-\infty/\infty$. What we will do next is check that (S_n) converges or not by itself. Sometimes, (S_n) is not defined explicitly as a function.

Given (S_n) for any $N \in \mathbb{N}$, we can define $U_N = \inf(\{S_n : n > N\})$ and $V_N = \sup(\{S_n : n > N\})$. An example is:

$$U_3 = \inf(\{S_4, S_5, \ldots\})$$

and

$$V_{10} = \sup(\{S_{11}, S_{12}, \ldots\})$$

So we have that $\forall N$, U_N is a lower bound of a smaller set: $\{S_n : n > N + 1\}$. We have $U_{N+1} \geqslant U_N$. So U_N is increasing. Similarly, V_N is decreasing. Define

$$\liminf(S_n) = \lim_{N\to\infty} U_N$$

and

$$\limsup(S_n) = \lim_{N\to\infty} V_N$$

Since $U_N \leqslant V_N$, $\limsup(S_n) \geqslant \liminf(S_n)$.

10.7

Theorem 3.2.3 If $\lim(S_n)$ is defined in $\mathbb{R}/-\infty/\infty$. Then $\liminf(S_n) = \limsup(S_n) = \lim(S_n)$. We also have that if $\liminf(S_n) = \limsup(S_n)$, then $\lim(S_n)$ exists and they are all equal.

Proof. (Part I) Only prove the case in $\mathbb{R}$. For the first part, assume $\lim(S_n) = S \in \mathbb{R}$. Then this means that $\forall \varepsilon > 0$, $\forall n > N$, we have

$$|S_n - S| < \varepsilon$$

In particular, $\forall n > N$, we have that:

$$S - \varepsilon < S_n < S + \varepsilon$$

So for $m > N$, we have $U_m = \inf(\{S_n : n > m\})$. So $U_m \geqslant S - \varepsilon$ for all $m > N$. So

$$\lim_{m\to\infty} = U_m \geqslant S - \varepsilon$$

so

$$\lim_{m\to\infty} \geqslant S$$

By a similar argument, we have

$$\lim_{m\to\infty} \leqslant S$$

But $S \geqslant \lim_{m\to\infty} V_m \geqslant \lim_{m\to\infty} U_m \geqslant S$. So they are all equal.

(Part II) Assume $\liminf(S_n) = \limsup(S_n) = S$. Take

$$U_n = \inf(\{S_n : n > N\})$$

Then for all $\varepsilon > 0$, there is an N_0 such that $\forall n > N_0$, $|U_n - S| < \varepsilon$. We also have $\forall n > N_0 + 2$, and $S_n \in \{S_{N_0+2}, S_{N_0+3}, \ldots\}$. So $S_n \geqslant U_{N_0+1}$. We have

$$|U_{N_0+1} - S| < \varepsilon$$

So $U_{N_0+1} < S - \varepsilon$ so $U_{N_0+1} \geqslant S - \varepsilon$ and $S_n > S - \varepsilon$. Similarly, if we used $\limsup(V_N) = S_n$, we get

$$S_n < S + \varepsilon$$

$\square$

Chapter 4

Week 4

4.1 Cauchy Sequence

Last Lecture, we have that monotone sequences were either

- Increasing: $S_n \leqslant S_{n+1}$
- Decreasing: $S_n \geqslant S_{n+1}$

Our definitions were of limsup and liminf:

$$\liminf = \lim_{N \to \infty} \inf\{S_n : n > N\}$$
$$\limsup = \lim_{N \to \infty} \sup\{S_n : n > N\}$$

We also found that $\lim(S_n)$ exists if and only if

Definition 4.1.1 **Cauchy Sequence**

(S_n) is a Cauchy Sequence if $\forall \varepsilon > 0$, there is an N such that $\forall n, m > N$,

$$|S_n - S_m| < \varepsilon$$

Lemma: Any Cauchy sequence is bounded.

Theorem 4.1.1 **10**

A sequence (S_n) is a convergent if and only if it is a cauchy sequence.

Proof. $(\longrightarrow)$ Assume that $\lim(S_n) = S$. Then $\forall \varepsilon > 0$, there is an N such that $\forall n > N$, $|S_n - S| < \frac{\varepsilon}{2}$. Therefore, we have that for all $n, m > N$,

$$|S_n - S| < \frac{\varepsilon}{2}$$
$$|S_m - S| < \frac{\varepsilon}{2}$$

So $|S_n - S_m| \leqslant |S_n - S| + |S_m - S| < \varepsilon$

$(\longleftarrow)$ We want to show that $\limsup = \liminf$ when (S_n) is Cauchy. By definition, of Cauchy, we have $\forall \varepsilon > 0$, there is an N such that $\forall n > N$, $|S_n - S_m| < \varepsilon$. Take

$V_N = \sup\{S_n : n > N\}$. Then $\forall m > N$, we have $\forall n > N$, $|S_n - S_m| < \varepsilon$. In other words:

$$S_n < S_m + \varepsilon$$

which means that $S_m + \varepsilon$ is an upper bound of $\{S_n : n > N\}$. So $V_N \leqslant S_n + \varepsilon$. This gives us $S_m \geqslant V_n - \varepsilon$. We find that $V_n - \varepsilon$ is a lower bound of $\{S_m : m > N\}$. Now let $U_N = \inf\{S_m : m > N\}$. We have $U_N \geqslant V_N - \varepsilon$. So we have $\liminf(S_n) \geqslant U_N \geqslant V_N - \varepsilon \geqslant \limsup(S_n) - \varepsilon$. If a_n is decreasing and bounded, the $\lim\limits_{n\to\infty} a_n = a$. We have $a = \inf\{a_n : n > N\}$.

We take the first and last inequality:

$$\liminf S_n \geqslant \limsup(S_n) - \varepsilon$$

So

$$\liminf(S_n) \geqslant \limsup(S_n)$$

By definition, $\liminf \leqslant \limsup$. So they must be equal. □

4.2 Subsequences

Why subsequences? Let $a_n = (-1)^n$. Choose $b_n = a_{2n} = (a_2, a_4, \ldots)$.

Definition 4.2.1 **Subsequences**

A subsequence of (S_n) is a sequence of the form (t_k) where $\forall k$ there is $n_k \in \mathbb{N}$ such that $n_1 < n_2 < n_3 < n_4 < \ldots < n_k < n_{k+1} <$ and take $t_k = S_{n_k}$. Sometimes, we just write (S_{n_k}) for a subsequence.

Example 4.2.1: $S_n = n$, $(1, 2, 3, 4, 5, \ldots)$. We can take $n_k = 2k, \forall k \in \mathbb{N}$ (S_{n_k}) is $(2, 4, 6, 8, \ldots)$. $(2, 1, 3, 4, 5, 6, \ldots)$ is not a subsequence since $n_1 > n_2$. If we have $(1, 1, 2, 3, 4, 5, \ldots)$ is not a subsequence because $n_1 = n_2$. We actually need $n_2 > n_1$. Finite sequences are not subsequences because sequences have infinitely many elements.

Theorem 4.2.1 **11.2**

Let (S_n) be a sequence. Then

- $\forall t \in \mathbb{R}$, there $\exists$ a subsequence (S_{n_k}) converging to t if and only if $\forall \varepsilon > 0$, $\{n \in \mathbb{N} : |S_n - t| < \varepsilon\}$ has infinitely many elements.
- If (S_n) is not bounded above, then it has a subsequence that diverges to ∞.
- If (S_n) is not bounded below, then it has a subsequence that diverges to $-\infty$.

Moreover, the subsequence can be chosen to be monotone if it satisfies one to the three conditions above.

Proof. (Part I) ($\rightarrow$) Assume $\lim\limits_{k\to\infty}(S_{n_k}) = t$, then $\forall \varepsilon > 0$, there is an N such that $\forall k > N$, $|S_{n_k} - t| < \varepsilon$. So the set $\{n \in \mathbb{N} : |S_n - t| < \varepsilon\}$ contains all of n_k.

($\leftarrow$) Let $\varepsilon_k = \frac{1}{k}$ for all $k \in \mathbb{N}$. Construct n_k inductively such that $n_k > n_{k-1}$ and $|S_{n_k} - t| < \varepsilon_k$. Choose n_1 to be any element in $\{n \in \mathbb{N} : |S_n - t| < \varepsilon_1 = 1\}$. Choose n_2 to be an element in the set $\{n \in \mathbb{N} : |S_n - t| < \varepsilon_2 = \frac{1}{2}, n > n_1\}$. There are infinitely many n in the set satisfying the first condition. Since there are finitely many numbers less than or equal to n_1, we know that the set is non empty. So we continue and choose an n_k to be an element in $\{n \in \mathbb{N} : |S_n - t| < \varepsilon_k, n > n_{k-1}\}$. □

4.3 Subsequences Continued

Last Lecture, we had that (S_n) is Cauchy if $\forall \varepsilon > 0$, there is an N such that for all $m, n > N$,

$$|S_n - S_m| < \varepsilon$$

We also had a theorem that says that a sequence converges if and only if it was Cauchy. We defined a subsequence (S_{n_k}) which must satisfy $(S_n) : n_1 < n_2 < n_3 < \ldots < n_k < n_{k+1} < \ldots$. So the index must be strictly increasing. Another theorem we have is that there exists a subsequence that converse to t if an d only if

$$\forall > 0, \{n \in \mathbb{N} : |S_n - t| < \varepsilon\}$$

is an infinite set.

If (S_n) is not bounded at above, then there exists a subsequence that diverges to ∞. If a (S_n) is not bounded below, then there exists a subsequence that diverges to $-\infty$.

We will see that if the original sequence converges, then any subsequence will converge to the same limit:

Subsequence Convergence

Theorem 4.3.1 If $\lim S_n = S$, then any subsequence converges to S.

Proof. Choose any subsequence say (S_{n_k}). Then $n_1 \geqslant 1$. $n_2 > n_1 \geqslant 1$. This also means that $n_2 \geqslant 2$. By induction, $n_k \geqslant k$. So we use the definition for convergence:

$$\forall \varepsilon > 0, \exists N \text{ such that } \forall n > N$$

we have

$$|S_n - S| < \varepsilon$$

Then for any $k > N$, $n_k \geqslant k > N$, we have

$$|S_{n_k} - S| < \varepsilon$$

Then $\lim_{k \to \infty} (S_{n_k}) = S$. □

Monotone Subsequence

Theorem 4.3.2 Every sequence has a monotone subsequence.

Proof. Given (S_n), $\forall n \in \mathbb{N}$, we call S_n dominant if $\forall m > n$, we have $S_n > S_m$. We have two cases:

- If (S_n) contains infinitely many dominant elements. Then we take the collection of dominant terms which will form a descending subsequence. This is monotone.
- If we have finitely many dominant terms, we choose n_1 is after all dominant terms. So S_{n_1} is not dominant. So we can find $n_2 > n_1$ such that $S_{n_2} \geqslant S_{n_1}$. Then S_{n_2} is not dominant. We continue, which gives us an increasing subsequence.

□

Theorem 4.3.3 **Bolzono-Weierstrass**

Every bounded sequence has a convergent subsequence.

Proof. Using the previous proof, we know there is a monotone sequence that is bounded. We know that bounded monotone sequences converge. □

Remark: The limit of a subsequence depends on the choice ot the subsequence:

$$(-1)^n$$

You can either choose all even n or all odd n as a subsequence.

Definition 4.3.1 **Subsequential Limit**

A subsequential limit of (S_n) is the limit of some subsequence of (S_n).

Theorem 4.3.4 **11.7**

For any (S_n), there exists a monotone subsequence whose limit is $\limsup S_n$ and there exists a monotone subsequence whose limit is $\limsup S_n$.

Theorem 4.3.5 **11.8**

Let S be the set of all subsequential limits of (S_n).

- S is non-empty
- $\sup S = \limsup S_n$
- $\inf S = \liminf S_n$
- $\lim S_n$ exists if and only if S has exactly one element.

Example 4.3.1: $S_n = (-1)^n n^2$. We have subsequences:

$$S_{2k} \to \infty$$
$$S_{2k+1} \to -\infty$$

How do we know that there are no more subsequential limits? We know that $n_k \geqslant k$. So $|S_{n_k}| = |n_k|^2 \geqslant k^2$. So we know that it cannot converge. So

$$S = \{-\infty, \infty\}$$

Example 4.3.2: $a_n = \sin \frac{n\pi}{3}$. We have that a_n could be $-\frac{\sqrt{3}}{2}$ or 0 or $\frac{\sqrt{3}}{2}$. The function is periodic and the image is restricted to those three values. So these are our subsequential limits:

$$S = \left\{ \frac{-\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2} \right\}$$

Example 4.3.3: We will use the fact that $\mathbb{Q}$ can be listed as a sequence (r_n). Start with

$$\begin{array}{ccccccc} -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ -1 & \frac{-1}{2} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \frac{-2}{3} & \frac{-1}{3} & 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

We will take the sequence $0, 1, \frac{1}{2}, 0, \frac{-1}{2}, -1, -2, -1, \ldots$. We claim that $S = \mathbb{R} \cup \{-\infty, \infty\}$.

11.9

Theorem 4.3.6 Let S be the set of subsequential limits of (S_n). Assume that (t_n) is a sequence in $S \cap \mathbb{R}$ and $\lim t_n = t$. Then $t \in S$.

Example 4.3.4: Suppose we have (S_n) and $S = \{\frac{1}{k} : k \in \mathbb{N}\}$. Then S converges to 0. So the theorem says that you can find a subsequence with a limit of 0.

4.4 More on limsup and liminf

Last Lecture: Any subsequence of a convergent sequence converges. Any sequence has a monotone subsequence. We also found that any bounded sequence has a convergent subsequence. There also exists a monotone subsequence whose limit is $\limsup(S_n)$. Let s be the set of subsequence limits. Then

- S is non-empty.
- $\text{Sup}(S) = \limsup(S_n)$,
- An $\lim(S_n)$ exists $\iff$ S has one element

The last theorem was that if $t_n \in S \cap \mathbb{R}$ and $\lim(t_n) = t$, then $t \in S$. We have that S cannot be $(0, 1)$.

Today, we will talk about lim sup and lim inf. Recall the definitions:

- $\limsup(S_n) = \lim_{N\to\infty} \sup\{S_n : n > N\}$
- $\liminf(S_n) = \lim_{N\to\infty} \inf\{S_n : n > N\}$

Remark: $\limsup(S_n)$ is not necessarily an upper bound of (S_n). An example is $\frac{1}{n} = (S_n)$. We know that $\limsup(S_n) = \lim(S_n)$ but 0 is not an upper bound.

12.1

Theorem 4.4.1 Consider $(s_n), (t_n)$. Suppose that

$$\lim(s_n) = s > 0$$

Then
$$\limsup(s_n t_n) = s \cdot \limsup(t_n)$$
Convention: $s \cdot \infty = \infty$ and $s \cdot -\infty = -\infty$.

In this case, notice that we only assume s_n converges. We don't say that t_n converges.

Proof. Only prove for one case where $\limsup(t_n) = \beta \in \mathbb{R}$. So there is a subsequence of (t_{n_k}) such that
$$\lim(t_{n_k}) = \beta$$
Then consider (S_{n_k}) converges to s. So we have the product theorem:
$$\lim(t_{n_k} s_{n_k}) = \lim(t_{n_k}) \lim(s_{n_k}) = \beta \cdot s$$
So βs is a subsequential limit of $(t_n s_n)$. Since $\limsup(t_n s_n)$ is the sup of all subsequential limits of $(t_n s_n)$,
$$\limsup(s_n t_n) \geqslant \beta \cdot s$$
Now we want the other direction. To get that
$$\limsup(s_n t_n) \leqslant \beta \cdot s$$
we consider $\frac{1}{s_n}$. This is well defined because $\lim(s_n) = s > 0$, we have that there is an N such that $\forall n > N$,
$$|s_n - s| < \frac{s}{2}$$
so
$$s_n > \frac{s}{2} > 0$$
By ignoring the first few elements, we can assume $s_n > 0 \forall n$. Take $s'_n = \frac{1}{s_n}$ and $t'_n = s_n t_n$ and $\lim(s'_n) = \frac{1}{s}$. So now we use the inequality proved above:
$$\lim(t'_n s'_n) \geqslant \limsup(t'_n) \cdot \frac{1}{s}$$
and
$$\limsup(t_n) \geqslant \frac{1}{s} \cdot \limsup(s_n t_n)$$
We have proven both directions of inequality, so they must be equal. □

12.2

Theorem 4.4.2

Assume that $\forall n$, $s_n \neq 0$, then
$$\liminf(\left|\frac{s_{n+1}}{s_n}\right|) \leqslant \liminf(|s_n|^{\frac{1}{n}}) \leqslant \limsup(|s_n|^{\frac{1}{n}}) \leqslant \limsup(\left|\frac{s_{n+1}}{s}\right|)$$

Why are we comparing $|\frac{s_{n+1}}{s_n}|$ and $|s_n|^{\frac{1}{n}}$.

Model case: Assume $\frac{s_{n+1}}{s_n} = 10$ and $s_1 = 10$, then by induction, we get $s_n = 10^n$. So $(s_n)^{\frac{1}{n}} = 10$.

Proof. Only prove $\limsup(|s_n|^{\frac{1}{n}}) \leqslant \limsup(|\frac{s_{n+1}}{s_n}|)$. Let $\limsup(|\frac{s_{n+1}}{s_n}|) = L \geqslant 0$. If $L = \infty$ we are done. Assume $L < \infty$. So this means that $\forall \varepsilon > 0$, we have that there $\exists N > N_0 - 2$ such that
$$|\sup\left\{\left|\frac{s_{n+1}}{s}\right| : n > N\right\} - L| < \varepsilon$$

Choose $N = N_0 - 1 > N_0 - 2$. So

$$\left|\sup\left\{\left|\frac{s_{n+1}}{s}\right| : n > N_0 - 1\right\} - L\right| < \varepsilon$$

So

$$\sup\left\{\left|\frac{s_{n+1}}{s}\right| : n > N_0 - 1\right\} < L + \varepsilon$$

We have that $\forall n > N_0 - 1$,

$$\left|\frac{s_{n+1}}{s}\right| < L + \varepsilon$$

Consider a rewite of $|S_n|$ given by

$$|S_n| = \left|\frac{s_n}{s_{n-1}}\right|\left|\frac{s_{n-1}}{s_{n-2}}\right| \dots \left|\frac{s_{N_0+1}}{s_{N_0}}\right| \cdot |S_{N_0}|$$

We have $n - N_0$ fractions and each fraction satisfies the inequality. So we can replace:

$$|s_n| < (L + \varepsilon)^{n - N_0} \cdot |s_{N_0}|$$

Let $a = (L + \varepsilon)^{-N_0} \cdot |s_{N_0}|$. Then we have

$$|s_n| < (L + \varepsilon)^n \cdot a$$

Taking the n-th root:

$$|s_n|^{\frac{1}{n}} < (L + \varepsilon) \cdot a^{\frac{1}{n}}$$

Take limsup of both sides:

$$\limsup(|s_n|^{\frac{1}{n}}) \leqslant \limsup(L + \varepsilon \cdot a^{\frac{1}{n}}) = L + \varepsilon$$

We can remove the ε, so we are done:

$$\limsup(|s_n|^{\frac{1}{n}}) \leqslant L = \limsup(\left|\frac{s_{n+1}}{s_n}\right|)$$

$\square$

Example 4.4.1: Assume $L = \limsup(S_n) \neq \infty \forall \alpha > L$. Then the set

$$\{n : s_n > \alpha\}$$

is finite.

Example 4.4.2: The set

$$\{n : S_n > \limsup(s_n)\}$$

can be infinite. $s_n = \frac{1}{n}, \limsup(s_n) = 0$.

Chapter 5

Week 5

5.1 Metrics

Last Lecture, we have that

- $\limsup(S_n) = \lim_{N\to\infty} \sup\{s_n : n > N\}$
- $\liminf(S_n) = \lim_{N\to\infty} \inf\{s_n : n > N\}$

We had theorems:

- If $\lim(S_n) = S > 0$, then $\limsup(S_n T_n) = S \limsup(T_n)$
- We also have $\liminf(|\frac{s_{n+1}}{s_n}|) \leqslant \liminf(|s_n|^{\frac{1}{n}}) \leqslant \limsup(|s_n|^{\frac{1}{n}}) \leqslant \limsup(|\frac{s_{n+1}}{s_n}|)$

Definition 5.1.1 **Metric Space**

Let S be a set and d is a function for all pairs of (x, y) where $x, y \in S$ such that

- $d(x, x) = 0$ and $d(x, y) > 0 \forall x \neq y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leqslant d(x, y) + d(y, z)$

This d is called the distance function or a metric on S. (S, d) is called a metric space.

Example 5.1.1: $S = \mathbb{R}$

- $(\mathbb{R}, \text{dist})$ where $\text{dist}(x, y) = |x - y|$
- $(\mathbb{R}, d')$ where $d'(x, x) = 0$ and $d'(x, y) = 1$ for any $x \neq y$.

Example 5.1.2:

- $\mathbb{R}^2, x \in \mathbb{R}^2$ where $x = (x_1, x_2)$ Then $\forall x, y \in \mathbb{R}^2$, we have distance as

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

- Let $k \in \mathbb{N}, \mathbb{R}^2$ then we have $x \in \mathbb{R}^k$ where $x = (x_1, x_2, \dots, x_n)$

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

Convergence and Cauchy on a Metric Space

Definition 5.1.2

Let $s_n \in S$ and say that (s_n) converges to $s \in S$ if $\lim_{n \to \infty} d(s_n, s) = 0$. We say that (s_n) is cauchy if $\forall \varepsilon > 0$, there is a N such that $\forall m, n > N$, we have

$$d(s_n, s_m) < \varepsilon$$

Remark:

- On $(\mathbb{R}, \text{dist})$, the definition is the same as usual.

Complete Metric Spaces

Definition 5.1.3

We call (S, d) complete if any cauchy sequence converges.

Remark:

- $(\mathbb{R}, \text{dist})$ is complete
- Let $S = (0, 1)$. Consider $((0, 1), \text{dist})$ is not complete. We can find a sequence that gets closer to 0, but since the limit is not contained in $(0, 1)$ it doesn't converge.
- $([0, 1], \text{dist})$ is complete
- Construct a non-bounded metric on $\mathbb{R}$.

Notation: We write a sequence $(x^{(n)})$ instead of (x_n) for $\mathbb{R}^R$.

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)})$$

Lemma: Consider $(x^{(n)}) \in \mathbb{R}^k$.

- It converges iff for each $j = 1, 2, \dots, k$, $(x_j^{(n)})$ converges in $\mathbb{R}$.
- $(x^{(n)})$ is cauchy iff for each $j = 1, 2, \dots, k$, $(x_j^{(n)})$ is cauchy.

$\mathbb{R}^k$

Theorem 5.1.1

$\mathbb{R}^k$ is complete

Proof. Assume $(x^{(n)})$ is cauchy. This means that for each j, we have (x_j^n) is cauchy. Then for each (x_j^n), it converges. So $(x^{(n)})$ converges by the lemma. □

Bounded

Definition 5.1.4

We define a subset S of $\mathbb{R}^k$ is bounded if there is an $M > 0$ such that $\forall x \in S$, we have $\max(|x_j| : j = 1, 2, \dots, k)$ is always $\leqslant M$.

Theorem 5.1.2 **Bolzano-Weierstrass**

Any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. Assume $(x^{(n)}) \subseteq \mathbb{R}^k$ is bounded. Then $(x_j^{(n)})$ is also bounded. So there exists a subsequence $(x^{(n_k)})$ where $(x_1^{n_k})$ converges. We can choose a subsequence of $(x^{(n_k)})$ saying $(x^{(n_{k_l})})$ such that $(x_2^{(n_{k_l})})$ also converges. Repeating k times, we get a convergent subsequence. □

5.2 Topology on a Metric Space

Last Lecture: Recall that we went over metric spaces where (S, d) is a vector space with a distance formula such that:

- $d(x, x) = 0$
- $d(x, y) > 0$
- $d(x, y) = d(y, x)$
- $d(x, z) \leqslant d(x, y) + d(y, z)$

One example of a metric space was $(\mathbb{R}^k, d(x, y))$ where

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_k - y_k)^2}$$

Convergence was defined by: $\lim(d(S_n, S)) = 0$ if S_n converges to S.

Cauchy: If $\forall \varepsilon > 0$, there is an N such that $m, n > 0$, we have $d(S_n, S_m) < \varepsilon$. A complete metric space is one where all cauchy sequences converge.

We showed that $\mathbb{R}^k$ is complete, and that every bounded sequence on $\mathbb{R}^k$ has a convergent subsequence.

Definition 5.2.1 **Interior**

Let (S, d) be a metric space and $E \subseteq S$. We say $s_0 \in E$ is interior to E if $\exists r > 0$ such that

$$\{s \in S : d(s, s_0) < r\} \subseteq E$$

Write E^0 for interior. Call E open if $E = E^0$

Consider a closed curve which we call E. If r is within the bounds of this curve, we can draw a circle with radius r such that that subset still lies in E. If s_0 is on the boundary, no matter the radius of the circle we choose, there will be points not in E.

Proposition:

- S is an open set
- $\emptyset$ is an open set
- Union of any open sets is an open set
- Intersection of finitely many open sets is an open set

Proof. (III) Let U_α be an open set where $\alpha \in A$. Take $x \in \bigcup_{a \in A} U_\alpha$. By definition, $\exists \alpha_0 \in A$ such that $x \in U_{\alpha_0}$. So we can find $r > 0$ such that

$$\{s \in S : d(s, x) < r\} \subseteq U_{\alpha_0} \subseteq \bigcup_{a \in A} U_\alpha$$

Since this holds for the union, x is an interior point of the union. Since x is any point, the union is open.

(IV) Assume $U_1, U_2, \dots, U_k$ are open. Then we take an arbitrary $x \in \bigcap_{i=1}^k U_i$. So for each set, we have $r_1 > 0$ such that

$$\{S : d(x, s) < r_1\} \subseteq U_1$$

Since we have finitely many r_i, take $r = \min(r_1, r_2, \dots, r_k)$. Now:

$$\{s \in S : d(x, s) < r\} \subseteq \bigcap_{i=1}^{k} \{s \in S : d(s, s) < r_i\} \subseteq \text{each } U_i = \bigcap_{i=1}^{k} U_i$$

□

Example 5.2.1: In the metric space $(\mathbb{R}, \text{dist})$, we have (a, b) as open. $[a, b]$ is not open.

$$\bigcap_{\varepsilon > 0} (a - \varepsilon, b + \varepsilon) = [a, b]$$

Definition 5.2.2 **Closed Sets**

On (S, d), we call E closed if $S - E$ is open.

Here are some properties:

- $S, \emptyset$ are closed sets.
- The intersection of any closed sets is closed
- The union of finitely many closed sets is closed.

Definition 5.2.3 **Closure of E**

Closure $\overline{E}$ of E is the intersection of all closed sets containing E.

Remark: $\overline{E}$ is the smallest closed set containing E.

Example 5.2.2: (a, b) is open and $[a, b]$ is closed. If we consider $[a, b)$, then it is not open nor closed. All have interior (a, b). Closure is $[a, b]$.

Definition 5.2.4

Boundary

The boundary of E is $\overline{E} = E^0$.

Example 5.2.3: On $\mathbb{R}^k$,

$$\{x : d(x, 0) < r\}$$

is an open set. On $\mathbb{R}^2$, it is a filled circle. The closed set is

$$\{x : d(x, 0) \leqslant r\}$$

is closed. The boundary is the circle perimeter.

Proposition: Assume $E \subseteq (S, d)$.

- Then E is closed iff $E = \overline{E}$.
- E is closed iff it contains the limit of any convergent sequence of points in E.
- An element is in $\overline{E}$ iff it is the limit of some sequence of points in E
- A point is a boundary point iff it is in $\overline{E} \cap \overline{(S - E)}$

5.3 More on Closed and Open Sets

Last Lecture: Metric Spaces (S, d), $E \subseteq S$. A point $s_0 \in E$ is an interior point if $\exists r > 0$ such that:

$$\{s \in S : d(s, s_0) < r\} \subseteq E$$

A point $x \in S$ is in the closure of E if $\exists s_n \in E$ such that $\lim(S_n) = x$. $E^0 \subseteq E \subseteq E^- \subseteq S$. E is open if $E = E^0$ and E is closed if $E = E^-$. The boundary of E is $E^- - E^0$. E is open iff $S - E$ is closed.

Example 5.3.1: $(\mathbb{R}, d), E = [1, 2), E^0 = (1, 2), E^- = [1, 2]$. The boundary is $\{1, 2\}$.

Example 5.3.2: $(S = [1, 2), d), E = [\frac{3}{2}, 2)$. First find the closure: $E^- = [\frac{3}{2}, 2)$. For example, $s_n = 2 - \frac{1}{n}$ does not converge because the limit is not in S. So 2 is not a point of the closure. E is closed, and the boundary is $\{\frac{3}{2}\}$.

Theorem 5.3.1

Let (F_n) be a sequence of decreasing closed bounded sets in R^k with $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$. Then

$$F = \bigcap_{n=1}^{\infty} F_n$$

is closed, bounded and non-empty.

Remark: Not true for open sets:

$$U_n = (0, \frac{1}{n}) \subseteq \mathbb{R}$$

where

$$(0,1) \supseteq (0, \frac{1}{2}) \supseteq (0, \frac{1}{3}) \supseteq \cdots$$

Their intersection:

$$\bigcap_{n=1}^{\infty} U_n = \emptyset$$

since $\forall x > 0, \exists N$ such that $\frac{1}{N} < x$, so $x \notin (0, \frac{1}{n})$.

Informal proof for $k = 1, F_n = [a_n, b_n]$. Since

$$F_n \supseteq F_{n+1} \forall n$$

$[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$. So a_n is an increasing sequence and b_n is a decreasing sequence. We also have $a_n \leqslant b_n \leqslant b_1$. So a_n is bounded monotone sequence. So $\lim(a_n) = a$ exists and $\lim(b_n) = b$ exists. Since $b_n \geqslant a_n$, we have $b \geqslant a$. We shall show $F = \bigcap_{n=1}^{\infty} = [a, b]$. First show that $[a, b] \subseteq F$. We have $a = \sup a_n \geqslant a_n$. And $b = \inf b_n \leqslant b_n$. This means that $[\sup a_n, \inf b_n] \subseteq F_i$ for all i. So it is in F. Now we wan to show that $\forall x > b = \inf b_n$, $x \notin F$. So x is not a lower bound of $\{b_n, n \in N\}$. So there is an n such that $b_n < x$. So $[a_n, b_n]$ does not contain x. So x is not in the intersection $F = \bigcap F_i$. We can prove the same for $\sup a_n$. So we have proved that $F \subseteq [a, b]$. Therefore, $F = [a, b]$.

Open Cover

Definition 5.3.1 (S, d) is a metric space. A family U of open sets is an open cover of a set E if E if $E \subseteq \bigcup_{v \in U} V$.

Example 5.3.3: $(\mathbb{R}, d), E = [0, 1]$. Take $U = \{(-1, \frac{1}{2}), (0, 1), (\frac{1}{2}, 2), (-1, 2)\}$. We have $\bigcup_{v \in U} V = (-1, 2) \supseteq E$. So U is an open cover of E.

Subcover

Definition 5.3.2 Define a subcover U' of U is a subset of U such that U' is an open cover.

Example 5.3.4: $(\mathbb{R}, d), E = [0, 1]$. Take $U = \{(-1, \frac{1}{2}), (0, 1), (\frac{1}{2}, 2), (-1, 2)\}$. We have $\bigcup_{v \in U} V = (-1, 2) \supseteq E$. So U is an open cover of E. Now take $U' = \{(-1, 2)\}$ is a subcover.

Finite Open Covers

Definition 5.3.3 An open cover/sub cover is finite if it contains finitely many sets.

Compact

Definition 5.3.4

A set $E \subseteq S$ is compact if any open cover of E has a finite subcover.

Heine-Borel

Theorem 5.3.2

In $\mathbb{R}^k$, $E \subseteq \mathbb{R}^k$ is compact iff E is closed and bounded.

Example 5.3.5: Bounded open $(0,1)$ is not compact. Take $U = \{(\frac{1}{n},1), n \in \mathbb{N}\}$is an open cover of $(0,1)$.

Take any finite subset of U. Let N be the max of n: $(\frac{/1}{n},1) \in U'$. Then the union is also $(\frac{1}{n},1) \not\supseteq (0,1)$.

Chapter 6

Week 6

Last Lecture: Theorem 13.14 If $(s_n) \subseteq \mathbb{R}^k$ is bounded closed nonempty decreasing set $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ then $\bigcap_{n=1}^{\infty} F_n$ is nonempty, closed, bounded

And theorem 13.12 If $E \subseteq \mathbb{R}^k$ is compact iff it is closed and bounded.

6.1 Series

Definition 6.1.1 **Series**

(a_k) is a sequence. We fix $m \in \mathbb{N}$ where $\forall n \geqslant m$, we have

$$S_n = \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

Example 6.1.1: if we take $m = 1$, $n = 3$, then $S_n = a_1 + a_2 + a_3$.

If $\lim(S_n)$ exists, then $\sum_{k=m}^{\infty} a_k = \lim(S_n)$ called an infinite series.

Remark:

- If (s_n) converges/diverges/diverges to $\infty/-\infty$, we say that $\sum_{k=m}^{\infty} a_k$ converges/diverges/diverges to $\infty/-\infty$.
- The existence of $\sum_{k=m}^{\infty} a_k$ does not depend on the choice of m.

Remark: If $a_k \geqslant 0$, S_n is increasing since

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n$$

so $\lim(s_n)$ exists if s_n is bounded above. For any (b_n) if $\sum |b_n| < \infty$, then we call $\sum b_n$ absolutely convergent.

Example 6.1.2: Suppose $a_k = r^k$ where $r \neq 1$. We have:

$$\sum_{k=0}^{n} a_k = (1 + r + r^2 + \cdots + r^n)$$

take:

$$\begin{aligned}(1-r)(1+r^2+\cdots+r^n) &= (1+r+r^2+\cdots+r^n) - (r+r^@+\cdots+r^n+r^{n+1}) \\ &= 1 - r^{n+1}\end{aligned}$$

so

$$\sum_{k=0}^{n} a_k = (1 + r + \cdots + r^n) = \frac{1-r^{n+1}}{1-r} = S_n$$

If $|r| < 1$, then $r^n \to 0$, $\lim S_n = \frac{1}{1-r}$.

Definition 6.1.2

Cauchy Criterion

We say $\sum a_n$ satisfies the cauchy criterion if (S_n) is a cauchy sequence: $\forall \varepsilon > 0, \varepsilon N$ such that $m, n > N$:

$$|S_m - S_n| < \varepsilon$$

The same criterion: $\forall \varepsilon > 0, \exists N$ such that $\forall n \geqslant m > N$, we have:

$$|\sum_{k=m}^{n} a_k| < \varepsilon$$

Theorem 6.1.1

14.4

A series $\sum a_n$ converges iff it satisfies the cauchy sequence.

Corollary: If $\sum a_n$ converges, then $\lim a_n = 0$.

Proof. Suppose it satisfies the cauchy criterion. $\forall \varepsilon > 0, \exists N$ such that $\forall n \geqslant m > N$,

$$|\sum_{k=m}^{n}| < \varepsilon$$

So $\forall n > N$, take $m = n$. So:

$$|a_n| < \varepsilon$$

or $\lim a_n = 0$. □

Remark: $\lim a_n = 0$ does not imply that $\sum a_n$ converges. We have $a_n = \frac{1}{n} \to 0$ but $\sum \frac{1}{n}$ diverges.

Theorem 6.1.2

Comparison Test

Assume $a_n \geqslant 0$ and $\forall n$

- If $\sum a_n$ converges and $|b_n| \leqslant a_n$, then $\sum b_n$ converges
- If $\sum a_n$ diverges to ∞ and $b_n \geqslant a_n$ then $\sum b_n = \infty$.

Proof. We have $\forall n \geqslant m$:

$$|\sum_{k=m}^{n} b_k| \leqslant \sum_{k=m}^{n} |b_k| \leqslant \sum_{k=m}^{n} a_k$$

but because each a_k is positive, we have:

$$\sum_{k=m}^{n} a_k = |\sum_{k=m}^{n} a_k|$$

and since $\sum a_k$ satisfies the Cauchy criterion, $\sum b_k$ also satisfies it. □

Corollary: Absolutely converges implies converges: If $\sum |b_n|$ converges, then $\sum b_n$ converges.

6.2 Summation Series Continued

Last Lecture: We had series:

$$S_n = \sum_{k=m}^{n} a_k$$

and

$$\sum_{k \geqslant 1}^{\infty} a_k = \lim(S_n) \text{ if it exists}$$

The sum

$$\sum a_k$$

satisfies Cauchy criterion if (S_n) is a Cauchy series or $\forall \varepsilon > 0$, there is an N such that $\forall n \geqslant m$,

$$|\sum_{k=m}^{n} a_k| < \varepsilon$$

Theorem 14.4 says that $\sum a_k$ converges if and only if it satisfies the Cauchy criterion

Corollary: If $\sum a_k$ converges, then $\lim(a_k) = 0$.

Comparison test: Assume $a_k \geqslant 0$:

- If $\sum a_k$ converges and $|b_k| \leqslant a_k$ then $\sum b_k$ converges
- If $\sum a_k$ diverges and $b_k \geqslant a_k$, then $\sum b_k$ diverges also.

Another test is the ratio test:

Theorem 6.2.1 **Ratio Test**

Assume $a_n \neq 0$. Then $\sum a_n$

- Converges absolutely if
$$\limsup(|\frac{a_{n+1}}{a_n}|) < 1$$
- Diverges if
$$\liminf(|\frac{a_{n+1}}{a_n}|) > 1$$

- No information if
$$\limsup(|\frac{a_{n+1}}{a_n}| \geqslant 1 \geqslant \liminf(|\frac{a_{n+1}}{a_n}|))$$

There is also the root test:

Theorem 6.2.2 **Root Test**

Let $\alpha = \limsup(|a_n|^{\frac{1}{n}})$ then $\sum a_k$

- Converges absolutely if $\alpha < 1$
- Diverges if $\alpha > 1$
- No information if $\alpha = 1$

Remark: Root test $\implies$ ratio test by theorem 12.2:

$$\liminf(|\frac{a_{n+1}}{a_n}|) \leqslant \liminf(|a_n|^{\frac{1}{n}}) \leqslant \limsup(|a_n|^{\frac{1}{n}}) \leqslant \limsup(|\frac{a_{n+1}}{a_n}|)$$

Proof.

- $\alpha < 1$ choose $\varepsilon > 0$ where $\alpha + \varepsilon < 1$. Then there is an N such that
$$|\sup\{|a_n^{\frac{1}{n}} : n > N|\} - \alpha| < \varepsilon$$
so
$$\sup\{|a_n^{\frac{1}{n}} : n > N|\} < \alpha + \varepsilon$$
So $(|a_n|) < (\alpha + \varepsilon)^n$. So
$$\sum_{n \geqslant N+1}^{\infty} |a_n| < \sum_{n \geqslant N+1}^{\infty} (\alpha + \varepsilon)^n$$
So the right hand side converges and so does the left hand side.
- This is similar. We would get $\sum(\alpha - \varepsilon)^n$ diverges so $\sum \alpha_n$ diverges.
- Take $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. We have
$$\lim(a_n^{\frac{1}{n}}) = 1$$
and
$$\lim(b_n^{\frac{1}{n}}) = 1$$
But $\sum a_n$ diverges and $\sum b_n$ converges.

□

Theorem 6.2.3 **15.1**

$\sum \frac{1}{n^p}$ converges iff $p > 1$.

Proof. Use integral test. ($\leftarrow$) Choose $p > 1$. $y = \frac{1}{x^p}$. The area under the curve for two points $x = n, x = n-1$, it has height $\frac{1}{n^p}$ so $A = \frac{1}{n^p}$. Now consider the area under the curve:

$$\int_{n-1}^{n} \frac{1}{x^p}\, dx \geqslant \frac{1}{n^p}$$

Now:

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^p} &= 1 + \sum_{n=2}^{N} \frac{1}{n^p} \\ &\leqslant 1 + \sum_{n=2}^{N} \int_{n-1}^{n} \frac{1}{x^p}\, dx \\ &= 1 + \int_{1}^{N} \frac{1}{x^p}\, dx \\ &= 1 + \frac{1}{p-1}\left(1 - \frac{1}{N^{p-1}}\right) \to 1 + \frac{1}{p-1}\end{aligned}$$

So it converges.

($\rightarrow$) It is enough to show that $\sum \frac{1}{n}$ diverges since $\frac{1}{n^p} \geqslant \frac{1}{n}$ when $p < 1$. We also consider the area. If we take the rectangle of height $\frac{1}{n}$ from points $n, n+1$, it is larger than that of $\int_n^{n+1} \frac{1}{x}\, dx \leqslant \frac{1}{n}$. So we have

$$\begin{aligned}\int_{1}^{N} \frac{1}{x}\, dx &\leqslant \sum_{n=1}^{N} \frac{1}{n} \\ \ln(n) &= \end{aligned}$$

Since $\ln(n)$ diverges, we have $\sum \frac{1}{n}$ diverges. □

Example 6.2.1: Take $\sum \frac{n}{n^2+3}$. No information from root or ratio test. We can use the comparison test. We have

$$\frac{n}{n^2+3} \geqslant \frac{n}{n^2+3n^2} = \frac{1}{4n}$$

Since $\frac{1}{4n}$ diverges, $\sum \frac{n}{n^2+3}$ diverges

Example 6.2.2: We have $a_n = 2^{(-1)^n - n}$ when n is even, $a_n = 2^{1-n}$ and when odd, $a_n = 2^{-1-n}$. No information from ratio test. But for root test:

$$|a_n|^{\frac{1}{n}} = 2^{((-1)^n - n)(\frac{1}{n})} = 2^{\frac{(-1)^n}{n} - 1}$$

and this goes to $\frac{1}{2}$. So the limsup of this is $\frac{1}{2} < 1$.

Chapter 7

Week 7

Last Lecture: We had a ratio test:

$$\limsup(|a_{n+1}/a_n|) < 1$$

then $\sum a_n$ converges absolutely. We also had that if

$$\liminf(|a_{n+1}/a|) > 1$$

then $\sum a_k$ diverges.

There was a root test also where if we let $\alpha = \limsup(|a_n|^n)$, and $\alpha < 1$, then a_n diverges. If $\alpha = 1$, then there is no information and if $\alpha > 1$, then it diverges.

Theorem 15.1 says that $\sum \frac{1}{n^p}$ converges iff $p > 1$.

Theorem 7.0.1 **Alternating Series**

If $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots \geqslant 0$ and $\lim(a_k) = 0$, then the alternating series:

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

converges.

Remark: $\sum(-1)^{n+1}\frac{1}{n}$ converges but not absolutely.

Proof. Let $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$. We want to show that S_n converges. The idea is that $(S_{2n}) = (S_2, S_4, S_6, \ldots)$ and $(S_{2n+1}) = (S_3, S_5, S_7, \ldots)$. We want to show $(S_{2n}) \to s \in \mathbb{R}$ and $(S_{2n+1}) \to s$.. Since $S_{2n+2} - S_{2n} = (-1)^{2n+1}a_{2n+1} + (-1)^{2n+2}a_{2n+2} = a_{2n+1} - a_{2n-2} \geqslant 0$. So (S_{2n}) is increasing. By a similar argument, S_{2n+1} is decreasing. Since $S_{2n+1} - S_{2n} = a_{2n+1} \geqslant 0$. Since $S_{n+1} \geqslant S_n$.

We claim: $\forall m, n \in \mathbb{N}$,

$$S_{2m} \leqslant S_{2n+1}$$

If $m \leqslant n$, then $S_{2m} \leqslant S_{2n} \leqslant S_{2n+1}$. If $m > n$, then $S_{2n+1} \geqslant S_{2m+1} \geqslant S_{2m}$. So (S_{2n}) is bounded above and (S_{2n+1}) is bounded below. So $\lim(S_{2n}) = s, \lim(S_{2n+1}) = t$. We have $t - s = \lim(S_{2n+1}) - \lim(S_{2n}) = \lim((S_{2n+1}) - \lim(S_{2n})) = \lim(a_{2n+1}) = 0$. □

7.1 Continuous Functions

We consider a function f such that $\operatorname{dom} f \subseteq \mathbb{R}$ and $f(x) \in \mathbb{R} \forall x \in \operatorname{dom} f$.

Natural domain: $f(x) = \frac{1}{x}$, domain is $(-\infty, 0) \cup (0, \infty)$.

$f(x) = \sqrt{x}$ dom is $[0, \infty]$.

Continuous

Definition 7.1.1

$x_0 \in \operatorname{dom} f$, f is continuous at x_0 if $\forall$ sequences $(x_n) \subseteq \operatorname{dom} f$, such that $\lim x_n = x_0$, we have $\lim f(x_n) = f(x_0)$.

We say that f is continuous on a set $S \subseteq \operatorname{dom} f$ if f is continuous at $\forall x_0 \in S$. We say f is continuous if it is continuous at $\forall x_0 \in \operatorname{dom} f$.

17.2

Theorem 7.1.1

$\varepsilon - \delta$ definition of continuous. f is continuous at $x_0 \iff \forall \varepsilon > 0, \varepsilon\delta > 0$, such that $\forall x \in \operatorname{dom} f$ such that

$$|x - x_0| < \delta$$

, we have

$$|f(x) - f(x_0)| < \varepsilon$$

Proof. $(\leftarrow)$ $\forall x_n \in \operatorname{dom} f$ such that $\lim x_n = x_0$. We have $\forall \varepsilon > 0$, there is an $\delta > 0$ such that $\forall x$ such that $|x - x_0| < \delta$, we have:

$$|f(x) - f(x_0)| < \varepsilon$$

So ε, $\exists N$ such that $\forall n > N$, we have $|x_n - x_0| < \delta$. This implies that

$$|f(x_n) - f(x_0)| < \varepsilon$$

So we have $\lim f(x_n) = f(x_0)$

$(\rightarrow)$ Assume $\varepsilon - \delta$ definition fails. Remark on how to write a statement fails. We have

$$\text{assumption} \implies \text{conclusion}$$

If we say it fails, we say that $\varepsilon \rightarrow \forall$ and claim that the conclusion does not hold. So if $\varepsilon - \delta$ fails, we have $\exists \varepsilon > 0$ such that $\forall \delta > 0$, where $\exists x \in \operatorname{dom} f$ such that

$$|x - x_0| < \delta$$

we have

$$|f(x) - f(x_0)| \geqslant \varepsilon$$

$\forall n \in \mathbb{N}$, take $\delta = \frac{1}{n}$. So we have $\exists x_n \in \operatorname{dom} f$ such that

$$|x_n - x_0| < \frac{1}{n}$$

and

$$|f(x_n) - f(x_0)| \geqslant \varepsilon$$

So $\lim x_n = x_0$, but $f(x_n)$ does not converge to $f(x_0)$ which is a contradiction. Therefore, the two definitions are equivalent. □

Example 7.1.1: Suppose $f(x) = 2x^2 + 1$ and prove that f is continuous on $\mathbb{R}$ by the continuous definition and definition of $\varepsilon - \delta$.

Proof. Let x_n be a sequence such that $\lim x_n = x_0$. Then $\lim f(x_n) = \lim(x_n^2+1) = \lim x_n^2 + \lim 1 = x_0^2 + 1 = f(x_0)$. □

Example 7.1.2: $f(x) = x^2 \sin\frac{1}{x}$ $x \neq 0$. We have $f(0) = 0$ and prove that f is continuous at 0.

Proof. $\forall x \to 0$, $|f(x_n) - 0| = |x_n^2 \sin\frac{1}{x_n}| \leqslant x_n^2 \to 0$ So $\lim f(x_n) = 0$. □

Example 7.1.3: $f(x) = \frac{1}{x}\sin\frac{1}{x^2}$ for $x \neq 0$. $f(0) = 0$. Prove that f is not continuous at 0.

Proof. Choose $x_n = \sqrt{\frac{1}{2n\pi+\frac{\pi}{2}}}$. We have $\lim x_n = 0$ but $f(x_n) = \frac{1}{x_n} \to \infty$. □

7.2 Continuous Functions Continued

Given a function f, we can define a new function kf where $k \in \mathbb{R}$:

$$(kf)(x) = k \cdot f(x)$$

We also have $|f| = |f|(x) = |f(x)|$.

17.3

Theorem 7.2.1 If f is continuous at x_0, then $|f|$ and kf are continuous for any $k \in \mathbb{R}$ at x_0.

Proof. For any $x_n \in \operatorname{dom} f$, converging to x_0, since f is continuous at x_0, we have $\lim f(x_n) = f(x_0)$. Then $\lim kf(x_n) = \lim d \cdot f(x_n) = k \cdot \lim f(x_n) = k \cdot f(x_0)$.

Now consider $||f(x_n)| - |f(x_0)|| \leqslant |f(x_n) - f(x_0)|$. So $\lim f(x_n) = f(x_0)$. □

Given functions f, g define

- $(f+g)(x) = f(x) + g(x)$
- $(fg)(x) = f(x)g(x)$
- $(f/g)(x) = f(x)/g(x)$
- $g \circ f(x) = g(f(x))$
- $\max(f,g)(x) = \max(f(x), g(x))$
- $\min(f,g)(x) = \min(f(x), g(x))$

Example 7.2.1: $f(x) = \frac{1}{x}$, $g(x) = \sin x$. We have $|f| = |\frac{1}{x}|$, $kf = \frac{k}{x}$, $f+g = \frac{1}{x} + \sin x$, $fg = \frac{\sin x}{x}$, $f/g = \frac{1}{x \sin x}$, $g \circ f = \sin\frac{1}{x}$, and $\max(f,g) = \frac{1}{x}$ when $\frac{1}{x} \geqslant \sin x$ and $\sin x$ when $\sin x \geqslant \frac{1}{x}$

Remark: The domain of the first 3 are $\operatorname{dom} f \cap \operatorname{dom} g$. The fourth one has domain $\{x \in \operatorname{dom} f : f(x) \in \operatorname{dom} g\}$.

17.4

Theorem 7.2.2

If f, g are continuous, then $f+g, fg, f/g$ are continuous at x_0.

Proof. Consider any $x_n \in \operatorname{dom} f \cap \operatorname{dom} g$ such that $\lim x_n = x_0$. Since f, g are continuous, we have

$$\lim f(x_n) = f(x_0), \lim g(x_n) = g(x_0)$$

Then

$$\lim f(x_n) + g(x_n) = \lim f(x_n) + \lim g(x_n) = f(x_0) + g(x_0)$$

we also have

$$\lim f(x_n)g(x_n) = \lim f(x_n) \lim g(x_n) = f(x_0)g(x_0)$$

Since $g(x_0)$ is non-zero, $g(x_n)$ is not 0 when n is large enough. So

$$\lim \frac{f}{g}(x_n) = \lim \frac{f(x_n)}{g(x_n)} = \frac{f(x_0)}{g(x_0)}$$

□

Theorem 7.2.3

If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. If $x_n \in \{x \in \operatorname{dom} f : f(x) \in \operatorname{dom} g\}$, such that $\lim x_n = x$, since f is continuous at x_0, $\lim f(x_n) = f(x_0)$. Since g is continuous at $f(x_0)$, we have $\lim g(f(x_n)) = g(f(x_0))$. So

$$\lim g \circ f(x_n) = g(f(x_0)).$$

□

Example 7.2.2: If f, g are continuous at x_0, then

$$\max(f, g)$$

is continuous at x_0.

Proof. Use the formula that $\forall a, b \in \mathbb{R}$, we have:

$$\max(a, b) = \frac{a+b}{2} + \frac{|a-b|}{2}$$

so

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2}$$

which is still continuous at x_0. □

Example 7.2.3: We know that $\sin x, \frac{1}{x}, x^2$ are continuous. So

$$x^2 \sin \frac{1}{x} \text{ is continuous } x \neq 0$$

18.1

Theorem 7.2.4

Let f be a continuous function on a closed interval $[a, b]$. Then f is bounded and it can achieve its maximum and minimum. That is $\exists M > 0$ such that $\forall x \in [a, b]$ $|f(x)| \leqslant M$ and $\exists x_0, y_0 \in [a, b]$ such that $f(x_0) \leqslant f(x) \leqslant f(y_0) \forall x \in [a, b]$ is.

Remark: The theorem does not hold on open intervals. One example is

$$f(x) = \frac{1}{x}$$

in $(0, 1)$.

Proof. (Bounded) Assume that $f(x)$ is unbounded. Then $\forall n \in \mathbb{N}$, we can find $x_n \in [a, b]$ such that $f(x_n) > n$. By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence $(x_{n_k}) \to x_0$. Then $\lim f(x_{n_k}) = f(x_0)$. But by our choice of x_{n_k}, $f(x_{n_k}) > n_k$ but $n_k \to \infty$, which is a contradiction.

(Maximum) Since $f(x)$ is bounded, it has a supremum M. So for any $n \in \mathbb{N}$, $M - \frac{1}{n}$ is not an upper bound. So there exists $x_n \in [a, b]$ such that $f(x_n) > M - \frac{1}{n}$. But $M \geqslant f(x_n) > M - \frac{1}{n}$. So $\lim f(x_n) = M$. Take convergent subsequence $(x_{n_k}) \to y_0$. By continuity, we have $\lim f(x_{n_k}) = f(y_0) = M$. ◻

7.3 Intermediate Value Theorem

Last Lecture, we had that f is continuous at a point if $\forall x_n \to x_0$, we have $f(x_n \to f(x_0))$. Theorem: if f, g are continuous, then $kf, |f|, f + g, f \cdot g, f/g, g \circ f, \max(f, g)$ are all continuous.

Another theorem: if f is continuous on $[a, b]$, then f is bounded and you can get the max/min values on $[a, b]$.

Theorem 7.3.1 **Intermediate Value Theorem**

If f is continuous on an interval I, then for any $a, b \in I$, and assume $f(a) < y < f(b)$. Then there exists some $x \in (a, b)$ such that $f(x) = y$.

Proof. Assume $f(a) < y < f(b)$. Take $S = \{x \in (a, b) : f(x) < y\}$. So s is non-empty since $a \in S$ and S is bounded. So the supremum exists. Let $x_0 = \sup S$. For $n \in \mathbb{N}$, since $x = \frac{-1}{n}$ is not an upper bound of S, so there is an x_n such that $x_0 \geqslant x_n \geqslant x_0 - \frac{1}{n}$. So $\lim x_n = x_0$. Since f is continuous, $\lim f(x_n) = f(x_0)$. Since $x_n \in s$, $f(x_n < y)$. So $\lim f(x_n) = f(x_0) \leqslant y$.

Now take $t_n = \min(b : x_0 + \frac{1}{n})$. Then $x_0 + \frac{1}{n} \geqslant t_n \geqslant x_0$. So $\lim t_n = s_0$ and $\lim f(t_n) = f(x_0)$. Since $t_n \notin S$, $f(t_n) \geqslant y$. So $f(x_0) = y$. ◻

Corollary 13.8: If f is continuous on an interval I, then $f(I) = \{f(x) : x \in I\}$ is an interval or a single point.

We just take the supremum and infimum of the set. By the intermediate value theorem, we can see that any values between $f(a), f(b)$ are still in the interval.

Example 7.3.1: Assume f is continuous and $f : [0, 1] \to [0, 1]$. Then f has a fixed point x_0 such that $f(x_0) = x_0$.

Proof. Consider $g(x) = f(x) - x$. We have:

$$g(0) = f(0) - 0 \geqslant 0$$

and

$$g(1) = f(1) - 1 \leqslant 0$$

If $g(0) = 0$ or $g(1) = 0$, then $f(0) = 0$ or $f(1) = 1$. Otherwise, $g(0) > 0$ and $g(1) < 0$.

We can use IVT to say that there is some $0 < x < 1$ where $g(x) = 0$. Then this means that x is a fixed point. □

Remark: In the intermediate value theorem, assume $f(a) \leqslant y \leqslant f(b)$ or $f(b) \leqslant y \leqslant f(a)$, then $\exists x \in [a, b]$ such that $f(x) = y$. We just changed the interval to a closed one.

Definition 7.3.1 **Strictly Increasing**

f is strictly increasing if $\forall x_1 < x_2$, $f(x_1) < f(x_2)$.

Fact: If f is strictly increasing, let $s = \{f(x) : x \in \operatorname{dom} f\}$. So for any $x \in \operatorname{dom} f$, $y = f(x)$. Now $f^{-1}(y) = x$. This is well defined and strictly increasing.

Example 7.3.2: We have $f(x) = |x|$ on $\mathbb{R}$ is not strictly increasing. So f^{-1} is not well defined. This is because there are two choices for what $f^{-1}(1) =?$. But $f(x) = |x|$ on $[0, \infty]$ has an inverse. It is $f^{-1}(y) = y$.

Theorem 7.3.2 **18.5**

Let f be a strictly increasing function on an interval such that $g(J)$ is an interval. Then g is continuous on J.

Proof. For any $x_0 \in J$, we want to show that g is continuous at x_0. Assume $g(x_0)$ is not an endpoint. So there is ε_0 such that $(g(x_0) - \varepsilon_0, g(x_0) + \varepsilon_0) \subseteq g(J)$. Check $\varepsilon - \delta$. So for all $\varepsilon > 0$, we can take $\varepsilon < \varepsilon_0$. So there is $x_1, x_2 \in J$ such that

$$g(x_1) = g(x_0) - \varepsilon$$

and

$$g(x_2) = g(x_0) + \varepsilon$$

Take $\delta = \min(x_1 - x_0, x_0 - x_2)$. Then $\forall x$ such that $d(x, x_0) < \delta$, then $x_1 < x < x_2$ so $g(x) - g(x_0) < \varepsilon$. □

Theorem 7.3.3 **18.4**

If f is strictly increasing and continuous, then f^{-1} is also strictly increasing and continuous.

Chapter 8

Week 8

8.1 Uniform Continuity

Idea: Difference between $f(x) = \frac{1}{x^2}$ and $y(x) = x^2$ on $(0, 1)$. Recall that f is continuous if $\forall x_0 \in$ dom and $\varepsilon > 0$, $\exists \delta > 0$ such that $\forall x \in$ dom such that

$$|x - x_0| < \delta$$

we have

$$|f(x) - f(x_0)| < \varepsilon$$

Remark: δ depends on ε and x in general.

Example 8.1.1: $f(x) = \frac{1}{x}$ and $(0, 1)$. We must have

- δ depends on ε if ε is very small, δ should be small also
- δ depends on $x_0 \in (0, 1)$ we claim $\delta < x_0$ otherwise, $\delta \geqslant x_0$,

  $$|x - x_0| < \delta$$

  contains $(0, x_0)$, but $\frac{1}{x^2}$ is unbounded on $(0, x_0)$ so contradiction.

Definition 8.1.1 **Uniformly Continuous**

f is uniformly continuous on S if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in S$, and

$$|x - y| < \delta$$

we have

$$|f(x) - f(y)| < \varepsilon$$

Remark: Uniform Continuity = Continuity + δ independent of x_0.

Example 8.1.2: $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$ since $\delta < x_0$.

Example 8.1.3: $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ for $a > 0$.

Proof. $\forall \varepsilon > 0, x \geqslant a, y \geqslant a$.

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{y^2 - x^2}{x^2y^2}\right| \\ &= \frac{|y-x|(x+y)}{x^2y^2} \\ &= |y-x|\left(\frac{x}{x^2y^2} + \frac{y}{x^2y^2}\right) \\ &= |y-x|\left(\frac{1}{x^2y} + \frac{1}{xy^2}\right) \\ &\leqslant |y-x|\left(\frac{1}{a^3} + \frac{1}{a^3}\right) \\ &= \frac{2}{a^3}|y-x| \end{aligned}$$

take $\delta = \frac{\varepsilon a^3}{2}$. So if $|x-y| < \frac{\varepsilon a^3}{2}$, then $|f(x)-f(y)| \leqslant \frac{2}{a^3}|y-x|$, and $\frac{2}{a^3}|y-x| < \varepsilon$. □

Example 8.1.4: $f(x) = x^2$ on $(0, 1)$ is uniformly continuous.

Proof. $\forall \varepsilon > 0, x, y \in \operatorname{dom} f$, we check

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y||x+y| < 2|x-y|$$

because $x, y < 1$. Then take

$$|x - y| < \frac{\varepsilon}{2}$$

take $\delta = \frac{\varepsilon}{2}$. Then $\forall x, y \in \operatorname{dom} f$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| < 2|x-y| < \varepsilon$$

which shows that x^2 is uniformly continuous on $(0, 1)$. □

So we know that a uniformly continuous function is continuous. But what about the converse?

19.2

Theorem 8.1.1

If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Remark: This holds on compact domains. So it works on continuous functions over bounded and closed sets.

Proof. Assume that f is not uniformly continuous on $[a, b]$. Then there is an $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x, y \in [a, b]$ such that

$$|x - y| < \delta$$

but

$$|f(x) - f(y)| \geqslant \varepsilon$$

Take $\delta = \frac{1}{n}$, $\exists x_n, y_n \in [a, b]$. Then $|x_n - y_n| < \frac{1}{n}$ but

$$|f(x_n) - f(y_n)| \geqslant \varepsilon$$

Since x_n is bounded, it has a convergent subsequence (x_{n_k}) where $x_0 = \lim x_{n_k}$. Since $[a, b]$ is closed, for any convergent sequence, $x_0 \in [a, b]$. So since f is continuous, $\lim f(x_{n_k}) = f(x_0)$. Since

$$|y_{n_k} - x_{n_k}| < \frac{1}{n_k} \to 0$$

$\lim y_{n_k} = x_0$. So $\lim f(y_{n_k}) = f(x_0)$. Then $\lim f(x_{n_k}) - f(y_{n_k}) = 0$. But

$$|f(x_{n_k}) - f(y_{n_k})| \geqslant \varepsilon$$

which is a contradiction. □

Remark: $f(x) = \frac{1}{x^2}$ on $(0, 1)$. Take $x_n = \frac{1}{n} \to 0$ But $f(x_n) = x^2 \to \infty$.

19.4

Theorem 8.1.2 If f is uniformly continuous function on S and $(S_n) \subseteq S$ is a Cauchy sequence and $f(S_n)$ is also cauchy.

Proof. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in S$, and $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon$$

For this delta, there exists N such that $\forall n, m > N$, we have

$$|s_n - s_m| < \delta$$

So we take $x = s_m, y = s_n$. Then

$$|f(s_n) - f(s_m)| < \varepsilon$$

So by definition, $(f(S_n))$ is a cauchy sequence. □

Last Lecture: f is a uniform continuous function on a set S if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in S$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon$$

Example 8.1.5: $f(s) = \frac{1}{x^2}$ is not uniform continuous on $(0, 1)$.

19.2

Theorem 8.1.3 If f is continuous on $[a, b]$, then f is uniform continuous on $[a, b]$.

19.4

Theorem 8.1.4 If (S_n) is a Cauchy sequence and f is a uniform sequence, then $(f(S_n))$ is cauchy.

Extension of Functions

Definition 8.1.2 We say $\tilde{f}$ is an extension of f if $\operatorname{dom} f \subseteq \operatorname{dom} \tilde{f}$ and $\forall x \in \operatorname{dom} f$, $\tilde{f}(x) = f(x)$.

Example 8.1.6: $f(x) = x \sin \frac{1}{x}, x \in (0, \frac{1}{\pi}]$. We can define $\tilde{f}(x) = x \sin \frac{1}{x}, x \in (0, \frac{1}{\pi})$ and $\tilde{f} = 0, x = 0$. $\tilde{f}$ is an extension of f.

$\tilde{f}$ is continuous at $x = 0$ since $\forall (x_n) \subseteq [0, \frac{1}{\pi}]$ which converges to 0, $|f(x_n)| \leqslant |x_n| \to 0$. So $\lim \tilde{f}(x_n) = 0$

Example 8.1.7: $f(x) = \frac{1}{x}, x \in (0, \infty)$. Define

$$\tilde{f}(x) = \begin{cases} \dfrac{1}{x} & x > 0 \\ c & x = 0 \end{cases}$$

for any c, $\tilde{f}$ is not continuous at $x = 0$.

19.5

Theorem 8.1.5

f is uniform continuous on (a, b) iff f can be extended to a continuous extension on $[a, b]$.

Proof. $(\leftarrow)$ $\tilde{f}$ is an extension on $[a, b]$. So $\tilde{f}$ is uniformly continuous on $[a, b]$. So $\tilde{f}$ is uniformly continuous on (a, b). Since $\tilde{f} = f$ on (a, b), f is also uniformly continuous on (a, b).

$(\leftarrow)$ We want to define $\tilde{f}(a), \tilde{f}(b)$ such that $\tilde{f}$ is continuous on $[a, b]$. So only consider $\tilde{f}(a)$.

Claim 1: If $(S_n) \subseteq (a, b)$ converges to a, then $\lim f(S_n)$ exists.

Claim 2: If $(S_n), (T_n)$ are two sequences in (a, b) converging to a, then $\lim f(S_n) = \lim f(T_n)$.

If claim 1 holds, we can define $\tilde{f}(a) = \lim f(S_n)$ where $S_n \subseteq (a, b)$ converges to a. Now $\tilde{f}$ is continuous at a by definition.

(Proof of claim 1) Since (S_n) is a cauchy sequence, $(f(S_n))$ is a cauchy sequence. So it converges and the limit exists.

(Proof of claim 2) Take $(U_n) = s_1, t_1, s_2, t_2, \ldots$. Then s_n, t_n are subsequences of U_n. So $\lim U_n = a$. By claim 1, $\lim f(U_n)$ exists. Since $f(t_n), f(s_n)$ are subsequences of $f(U_n)$, they converge to the same value. □

8.2 Limits of Functions

Recall that $S_n = \frac{1}{n}$ means $\lim_{n\to\infty} S_n = 0$. We want to define $\lim_{x\to 2} x^2 = 4$.

Limit

Definition 8.2.1

Let $S \subseteq \mathbb{R}$ and a is a real number or ∞, $-\infty$ which is a limit of a sequence in S. And L is a umber of $\infty/-\infty$. We write $\lim_{x\to a} f(x) = L$ if f is a function on S and for any sequence, $x_n \subseteq S$ which has limit a, we have $\lim f(x_n) = L$

Remark: f is a continuous function at a iff a is in dom f and $\lim_{x\to a} f(x) = f(a)$. Another remark is that $\lim_{x\to a} f(x)$ is unique if it exists. Last remark is that a is not necessarily in S.

Example 8.2.1: $f(x) = x \sin \frac{1}{x}$. We can choose $x \in (0, \infty)$ but $\lim_{x \to 0} f(x) = 0$.

More definitions:

Definition 8.2.2 **Notation**

- If $a \in \mathbb{R}$, we write $\lim_{x \to a} f(x) = L$, when we take $S = J \setminus \{a\}$ where J is an open interval containing a.
- We write $\lim_{x \to a+} f(x) = L$ if we take $S = (a, b)$ and $\lim_{x \to a-} f(x) = L$ if we take $S = (c, a)$.
- We write $\lim_{x \to \infty} f(x) = L$ if $S = (c, \infty)$ and $\lim_{x \to -\infty} f(x) = L$ if $S = (-\infty, b)$.

Example 8.2.2: We define $f(x) = -1$ if $x < 0$, 0 if $x = 0$ and 1 if $x > 1$.

$\lim_{x \to 0-} = -1$, $\lim_{x \to 0+} = 1$, but $\lim_{x \to 0}$ does not exist.

Last Lecture: We had theorem 19.5 which says that f is uniformly continuous on $[a, b]$ iff there exists $\tilde{f}$ an extension which is continuous on $[a, b]$. Definition: $\lim_{x \to a^s} f(x) = L$ if f is a function on S and $\forall (x_n) \subseteq S$ with limit a we have $\lim f(x_n) = L$.

Remark: $f(x)$ is continuous at a iff $a \in \text{dom}(f)$ and $\lim_{x \to a^{\text{dom}}} f(x) = f(a)$.

Special limits:

$$\lim_{x \to a}$$

where $S = J \setminus \{a\}$. J is an open interval containing a.

We have:

$$\lim_{x \to a^-}$$

where $S = (b, a)$, $b < a$.

$$\lim_{x \to a^+}$$

where $S = (a, c)$, $c > a$.

$$\lim_{x \to \infty}$$

where $S = (c, \infty)$.

$$\lim_{x \to -\infty}$$

where $S = (-\infty, b)$.

Example 8.2.3: $f(x) = \frac{1}{x-2}$. We have

$$\lim_{x \to 2^+} f(x) = +\infty$$

and

$$\lim_{x\to 2^-} f(x) = -\infty$$

and

$$\lim_{x\to 2} f(x) DNE$$

Now for ∞:

$$\lim_{x\to\infty} f(x) = 0$$

and

$$\lim_{x\to-\infty} f(x) = 0$$

20.5

Theorem 8.2.1 Assume $L = \lim_{x\to a} f(x)$ is finite. g is defined on $f(S) \cup \{L\}$ and is continuous at L. Then $\lim_{x\to a^s} g \circ f(x) = f(L)$.

Proof. $\forall (x_n) \subseteq S$ with limit a, we want to show that

$$\lim g \circ f(x_n) = g(L)$$

Since $\lim_{x\to a} f(x) = L$, we have

$$\lim f(x_n) = L$$

Since g is continuous at L,

$$\lim g(f(x_n)) = g(L)$$

So this means

$$\lim_{x\to a} g(f(x)) = g(L)$$

□

Example 8.2.4: Assume $\lim_{x\to a^s} f(x) = L \in \mathbb{R}$. By taking $g(x) = |x|$. Then

$$\lim_{x\to a^s} |f(x)| = |L|$$

similarly, we have

$$\lim_{x\to a^s} e^{f(x)} = e^L$$

We can also take

$$\lim_{x\to a^s} \sin f(x) = \sin L$$

20.6

Theorem 8.2.2 Assume a and L are finite. Then

$$\lim_{x\to a^s} f(x) = L \iff$$

$\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in S$ and

$$|x - a| < \delta$$

we have

$$|f(x) - L| < \varepsilon$$

Example 8.2.5: $f(x) = 1$ if $x \neq 0$ and 0 if $x = 0$. We have $\lim_{x \to 0} f(x) = 1$. So $\forall \varepsilon > 0$, there $\exists \delta > 0$, lets say 1 such that $x \in J \backslash \{0\}$ and

$$|x - 0| < 1$$

then

$$|f(x) - 1| = 0 < \varepsilon$$

Corollary: $\lim_{x \to a} f(x) = L$ iff $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < |x - a| < \delta$$

we have

$$|f(x) - L| < \varepsilon$$

The difference is that $x \neq a$ because we require $0 < |x - a|$. So the set of x is equal to $(a - \delta, a) \cup (a, a + \delta)$.

Corollary: We have

$$\lim_{x \to a^-} f(x) = L$$

iff $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$a - \delta < x < a$$

we have

$$|f(x) - L| < \varepsilon$$

(Case where L is infinite)

- $x_0 \in \mathbb{R}$. Then $\lim_{x \to x_0} f(x) = \infty$ iff $\forall M > 0, \exists \delta > 0$ such that $\forall x \in S$ and

$$|x - x_0| < \delta$$

  we have

$$f(x) > M$$

- $y_0 \in \mathbb{R}$, $\lim_{x \to -\infty} f(x) = y_0$ iff $\forall \varepsilon > 0\ \exists M < 0$ such that $\forall x < M$ we have

$$|f(x) - y_0| < \varepsilon$$

20.10

Theorem 8.2.3

Let $a \in \mathbb{R}$. Then $\lim_{x \to a} f(x)$ exists iff

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^+} f(x)$$

exist and are equal. In this case, all limits are equal.

Proof. Only consider the case where limits are finite.

($\rightarrow$) $\lim\limits_{x\to a} f(x) = L \in \mathbb{R}$. Then $\forall \varepsilon > 0, \exists \delta > 0$ such that if

$$|x - a| < \delta$$

we have

$$|f(x) - L| < \varepsilon$$

Then

$$a - \delta < x < a$$

implies that

$$0 < |x - a| < \delta$$

Therefore,

$$|f(x) - L| < \varepsilon$$

By the corollary above, the LHS limit is equal to L. Similarly, the RHS limit is also equal to L.

($\leftarrow$) We assume the limit exists for RHS and LHS. We choose the smallest δ such that for $a - \delta < x < a + \delta$, we get

$$|f(x) - L| < \varepsilon$$

$\square$

Chapter 9

Week 9

Last Lecture: If $\lim_{x \to a} f(x) = L$ is finite and $g(x)$ is defined on $f(S) \cup \{L\}$ and g is continuous at L, then $\lim_{x \to a} g \circ f(x) = g(L)$. If a, l are finite, then $\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in S$ and $|x - a| < \delta$, we have

$$|f(x) - f(a)| < \varepsilon$$

We also have $\lim_{x \to a} f(x)$ exists iff

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+}$$

exists.

9.1 Continuity on Metric Spaces

Recall that $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if for any sequence $x_n \to x_0$, we have

$$\lim f(x_n) = f(x_0)$$

which holds iff $\forall \varepsilon > 0, \exists \delta > 0$ such that for any x such that

$$|x - x_0| < \delta$$

we have

$$|f(x) - f(x_0)| < \varepsilon$$

We define f is uniformly continuous on $E \subseteq \mathbb{R}$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in E$, we have

$$|f(x) - f(y)| < \varepsilon$$

Recall the metric space has a distance (S, d). We say that $(x_n) \to x_0$ if $d(x_n, x_0) \to 0$.

Definition 9.1.1

Take two metric spaces (S, d) and (S^*, d^*). We have

$$f : S \to S^*$$

which is continuous at $s_0 \in S$ if $\forall \varepsilon > 0$, we have $\delta > 0$ such that $d(s, s_0) < \delta$ means that:

$$d^*(f(s), f(s_0)) < \varepsilon$$

and for continuity, we have that $\forall (S_n) \subseteq S$ which converges to S_0, we have $\lim f(S_n) = S_0$.

Example 9.1.1: $S = \mathbb{R}^2 = (x_1, x_2)$ and $S^* = \mathbb{R}^2 = (y_1, y_2)$.

$$f(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$$

and

$$g(x_1, x_2) = (x_1 x_2, x_1^2 + x_2^2)$$

We define f is uniformly continuous on E if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall s, t \in E$ and $d(s,t) < \delta$, we have

$$d^*(f(s), f(t)) < \varepsilon$$

If we take $S^* = \mathbb{R}^k$, we can take $d^*(x,y)$ with the usual metric:

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_k - y_k)^2}$$

Paths and Curves

Definition 9.1.2 Define

$$\gamma : \mathbb{R} \to \mathbb{R}^k$$

is a path if γ is continuous. The image $\gamma(\mathbb{R})$ is called a curve. If $\gamma : [a,b] \to \mathbb{R}^k$ is continuous, we still call it a path.

Example 9.1.2: $\gamma : \mathbb{R} \to \mathbb{R}$ given by:

$$\gamma(t) = (\cos t, \sin t)$$

so γ is a path and the circle is a curve.

Example 9.1.3: If $f : \mathbb{R} \to \mathbb{R}$ is continuous, we consider $\gamma(t) = (t, f(t))$.

Proposition: $\gamma : [a,b] \to \mathbb{R}^k$ and write $\gamma(t) = (f_1(t), f_2(t), \ldots, f_k(t))$. Then γ is continuous iff f_j is continuous for each j.

Recall a set $U \subseteq S$ is open if $\forall s \in U$, there is an r such that

$$\{x : d(s,x) < r\} \subseteq U$$

21.3

Theorem 9.1.1 We have

$$f : S \to S^*$$

is continuous iff $\forall$ open sets $U \subseteq S$, we have $f^{-1}(U)$ is open.

Remark: If we take $f(x) = x^2$ and $f^{-1}([0,1]) = [-1,1]$ and $f^{-1}([4,9]) = [-3,-2] \cup [2,3]$. So f is continuous if for any closed set $E \subseteq S$, we have $f^{-1}(E)$ is closed.

Proof. ($\rightarrow$) If f is continuous and U is open in S^*, we want to show $\forall S_0 \in f^{-1}(U)$, S_0 is an interior point. Since U is open, there exists $\varepsilon > 0$ such that $\{s^* \in S^* : d^*(s^*, f(s_0)) < \varepsilon\} \subseteq U$. Since f is continuous, $\exists \delta > 0$ such that $\forall d(s, s_0) < \delta$, we have $d^*(f(s), f(s_0)) < \varepsilon$. So take $f(s) = s^*$ and say that we have found an ε that make the set open. Then we have that there is a delta that makes $f^{-1}(S)$ open.

($\leftarrow$). $\square$

Last Lecture: Suppose we had two metric spaces $(S, d), (S^*, d^*)$. $f : S \rightarrow S^*$ is continuous at $s \in S$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$|d(s, s_0)| < \delta$$

implies that:

$$d^*(f(s), f(s_0)) < \varepsilon$$

We also say that f is uniformly continuous on $E \subseteq S$ if $\forall \varepsilon > 0$, $\delta > 0$ such that $x, y \in E$ and $d(x, y) < \delta$ means that:

$$d^*(f(x), f(y)) < \varepsilon$$

Proposition: If $\delta : [a, b] \rightarrow \mathbb{R}^k$, $\delta(t) = (f_1(t), f_2(t), \ldots, f_k(t))$, Then δ is continuous iff f_i are continuous. Lastly, there is a theorem which says that $f : S \rightarrow S^*$ is continuous iff $f^{-1}(U)$ where U is open, is open.

Recall: $E \subseteq S$ is a compact set of S if any open cover of E has a finite subcover.

Theorem 9.1.2 **13.12**

$E \subseteq \mathbb{R}^k$ is compact iff it is closed and bounded.

Theorem 9.1.3 **21.4**

Assume $f : S \rightarrow S^*$ is continuous, E is a compact subset of S. Then f(E) is compact as well. f is uniformly continuous on E.

Proof. Only prove for the case $S = \mathbb{R}^k$ and $S^* = \mathbb{R}^k$ so that we can use theorem 13.12. The theorem does in general hold for two metric spaces in general. Then E is bounded and closed.

(Part I) We want to show that f(E) is closed and bounded. Suppose that f(E) is not bounded. This means that there is $x_n \in E$ such that $d^*(0,, f(x_n)) \rightarrow +\infty$. Since $(x_n) \subseteq E$ is bounded, we can choose a convergent subsequence s_{n_k} which converges to x_0. Since E is closed, $x_0 \in E$. Since f is continuous, $\lim f(x_n) = x_0$. Then $\lim d^*(f(x_n), 0) = d^*(f(x_0), 0)$. The LHS is infinite but the RHS is finite. So contradiction.

To prove that f(E) is closed, assume that f(E) is not closed. So we can find $y_0 \in f(E)^- - f(E)$. So $\exists x_n \in E$ such that $\lim f(x_n) = y_0$. Since $y_0 \in f(E)^-$. So there is a subsequence $x_{n_k} \rightarrow x_0$. Since f is continuous, $\lim f(x_{n_k}) = f(x_0) = y_0 \in f(E)$. This is a contradiction.

(Part II) Recall from Theorem 19.2 where if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[a, b]$, then it is uniformly continuous on $[a, b]$ $\square$

Corollary: Assume $f : S \rightarrow \mathbb{R}$ is continuous and $E \subseteq S$. Then

- f(E) is bounded
- f assumes a maximum and minimum value on E.

Definition 9.1.3

Dense Sets

(S, d) is a metric space. We call $D \subseteq S$ is dense if every nonempty open set intersects D. We call $E \subseteq S$ nowhere dense if E^- has no interior.

Example 9.1.4: $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 9.1.5: $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

Remark:

- D is dense in S iff the closure $D^- = S$.
- E is nowhere dense iff $S - E^-$ is dense.

Theorem 9.1.4

Baire Category Theorem

Any complete metric space (S, d) is a not a union of a sequence of nowhere dense subsets of S. If $S = \bigcup_{n=1}^{\infty} E_n$, then at least one of E_n is not nowhere dense.

9.2 Metric Spaces and Connectedness

Idea: On $\mathbb{R}$, $(-1, 1)$ is connected. Consider $(-1, 0) \cup (0, 1)$ which should be disconnected. In $(-1, 0) \cup (0, 1)$, we can separate it into two parts $(-1, 0)$ and $(0, 1)$. On the other hand, we can separate $(-1, 1)$ into $(-1, 0]$ and $(0, 1)$. But notice that $(-1, 0]$ has a boundary point.

Definition 9.2.1

Connected and Disconnected Sets

Define $E \subseteq S$ is disconnected if it satisfies one of the following two conditions:

- There exists open sets $U_1, U_2 \subseteq S$ such that:

$$(E \cap U_1) \cap (E \cap U_2) = \emptyset$$

  and

- $(E \cap U_1) \cup (E \cap U_2) = E$ where $E \cap U_1 \neq \emptyset$ and $E \cap U_2 \neq \emptyset$.

So this says that we separate E into disjoint sets. The second condition says that their union is all of E. If there are disjoint nonempty subsets $A, B \subseteq E$ such that:

- $E = A \cup B$ and
- $A^- \cap B = \emptyset$, $B^- \cap A = \emptyset$

This says that the sets don't contain boundary points.

Remark: To prove that these two definitions are the same, prove that $a \to b$ by $A = E \cap U_1$, $B = E \cap U_2$. And for $b \to a$, take $U_1 = S - B^-$ and $U_2 = S - A^-$.

Last Lecture: If $f : S \to S^*$, $E \subseteq S$ is compact, then

- $f(E)$ is compact

- f is uniform continuous on E.

We call $E \subseteq S$ is dense if E intersects all open sets in S or the closure of E is S.

We call E nowhere dense if E^- has no interior. Any complete metric space cannot be a union of nowhere dense subsets.

$E \subseteq S$ is disconnected if one of the following conditions hold:

- $\exists U_1, U_2$ open sets of S such that $(E\cap U_1)\cap(E\cap U_2)$ is empty and $(E\cap U_1)\cup(E\cap U_2) = E$, $E \cap U_1$, $E \cap U_2$ non-empty
- $\exists$ disjoint non-empty subsets A, B such that $A^- \cap B = \emptyset$ and $B^- \cap A = \emptyset$ and $A \cup B = E$

We say E is connected if it is not disconnected.

Example 9.2.1: $(-1, 0) \cup (0, 1)$ is disconnected in $\mathbb{R}$. Take $U_1 = (-1, 0)$, $U_2 = (0, 1)$. So it is disconnected.

Example 9.2.2: Take $(-1, 1)$. Prove it is connected

Proof. Assume $(-1, 1)$ is disconnected. So there exists open sets U_1, U_2 satisfying that there is $a \in E\cap U_1, b \in E\cap U_2$. Assume $a < b$. Let $t = \sup([a, b)\cap U_1)$. Since U_1, u_2 are open, the neighborhood of $a \subseteq U_1$. The neighborhood of $b \subseteq U_2$ Then the neighborhood of $b \cap U_1 = \emptyset$. So $a < t < b$. So $t \in U_1$ or $t \in U_2$. If $t \in U_1$, then there is an r such that $(t - r, t + r) \subseteq U_1 \cap [a, b)$. But $\sup(U_1 \cap [a, b)) \geqslant t + r$ which is a contradiction.

If $t \in U_2$, there is an r such that $(t-r, t+r) \subseteq U_2\cap[a, b)$. So $(t-r, t+r)\cap U_1 \neq \emptyset$. We find that $\sup(U_1 \cap [a, b))$ is less than $t - r$ or greater than $t + r$. Contradiction. □

Remark: $E \subseteq \mathbb{R}$ is connected iff E is an interval. For the $\rightarrow$ direction, assume that E is not an interval. Then there is an $a < b < c$ such that $a, c \in E$ but $b \notin E$. Take $U_1 = (-\infty, C), U_2 = (C, \infty)$. U_1 and U_2 separate E.

Theorem 9.2.1 22.2

If $f : S \rightarrow S^*$ is continuous, $E \subseteq S$ is connected, then $f(E)$ is connected also.

Proof. Assume that $f(E)$ is disconnected. Then there are open sets V_1, V_2 separating $f(E)$. Since f is continuous. Then $U_1 = f^{-1}(V_1), U_2 = f^{-1}(V_2)$ are open. We can show U_1, U_2 separate E. □

Corollary: If $f : S \rightarrow \mathbb{R}$ is continuous and $E \subseteq S$ is connected, then $f(E)$ is an interval. So the IVT holds.

Definition 9.2.2 Path Connected

$E \subseteq S$ is path connected if $\forall x, y \in E$, there is a continuous function $\gamma : [a, b] \rightarrow E$ such that $\gamma(a) = x, \gamma(b) = y$.

Theorem 9.2.2 22.5

If $E \subseteq S$ is path connected, then E is connected.

Proof. Assume that E is disconnected. Then there are open sets U_1, U_2 that separate E. So we can choose $x \in U_1 \cap E, y \in U_2 \cap E$. So there is $\gamma : [a, b] \rightarrow E$ such that $\gamma(a) = x$,

$\gamma(b) = y$. Then U_1, U_2 separate $\gamma([a,b])$. Then $\gamma([a,b])$ is disconnected which is a contradiction. □

Example 9.2.3: Consider $\mathbb{R}^2$ and

$$E = \begin{cases} y = \sin\dfrac{1}{x} & \text{if} x > 0 \\ -1 \leqslant y \leqslant 1 & \text{if} x = 0 \end{cases}$$

This is a case where E is connected but not path connected.

Chapter 10

Week 10

10.1 Power Series

Definition 10.1.1 **Power Series**

$(a_n)_{n>0}^{\infty}$ is a sequence of real numbers. We all

$$a_0 + \sum_{n=1}^{\infty} a_n x^n$$

a power series. We write

$$\sum_{n=0}^{\infty} a_n x^n$$

with convention $0^0 = 1$.

Recall the root test: Let $\alpha = \limsup |S_n|^{\frac{1}{n}}$. If $\alpha < 1$, $\sum S_n$ converges. If $\alpha > 1$, then $\sum S_n$ diverges.

Theorem 10.1.1 **23.1**

For $\sum_{n=0}^{\infty} a_n x^n$, let $\beta = \limsup |a_n|^{\frac{1}{n}} \wedge \mathrm{let} m R = \frac{1}{\beta}$. If $\beta = 0$, we let $R = \infty$ and if $\beta = \infty$, let $R = 0$. Then

- $\sum_{n=0}^{\infty} a_n x^n$ converges when $|x| < R$
- $\sum_{n=0}^{\infty} a_n x^n$ diverges when $|x| > R$.

Proof. $\limsup |a_n x^n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} |x| = |x| \limsup |a_n|^{\frac{1}{n}} = |x|\beta$.

If $|x| < R$, $|x|\beta < 1$ and if $|x| > R$, then $|x|\beta > 1$. □

Recall that for $\sum S_n$, if $\lim |\frac{S_{n+1}}{S_n}|$, then $\lim |S_n|^{\frac{1}{n}} = \lim |\frac{S_{n+1}}{S_n}|$.

Example 10.1.1: We have $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $a_n = \frac{1}{n!}$. This case, we can use the ratio test:

$$\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0$$

So $\beta = 0$. So $R = \infty$, which means that when $|x| < \infty$ the series converges or just for any $x \in \mathbb{R}$.

Example 10.1.2: $\sum_{n=0}^{\infty} x_n, a_n = 1$. Then $\beta = 1$ and $R = 1$. So it converges when $|x| < 1$.

Example 10.1.3: Consider $\sum_{n=0}^{\infty} \frac{1}{n} x^n$. Recall that $\lim n^{\frac{1}{n}} = 1$. So

$$\limsup |a_n|^{\frac{1}{n}} = \limsup \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$$

So $\beta = 1$ and $R = 1$. So it converges for when $|x| < 1$. We also need to check for when $x = 1, -1$ separately. For $x = 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$$

When $x = -1$,

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

which converges. So the power series converges on $x \in [-1, 1)$.

Example 10.1.4: Consider $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$. Because $\limsup n^{2/n} = 1$ so $\beta = 1$. $R = 1$. If $x = -1$, then $\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n$ converges. And for $x = 1$, we get $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges. So the power series converges on $x \in [-1, 1]$.

Example 10.1.5: Consider the power series $\sum_{n=0}^{\infty} n! x^n$, and by the ratio test, we get $\beta = \infty$, $R = 0$. So the power series converges when $x = 0$.

Remark: For $\sum_{n=0}^{\infty} a_n x^n$, one of the following is true:

- Converges for any $x \in \mathbb{R}$
- Only converges on $x = 0$
- Converges for x in bounded interval centered at 0: $(-R, R), [-R, R), (-R, R], (-R, R)$.

We consider $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ where x_0 is a fixed point. We can take $y = x - x_0$ as a change of variables. Then we have:

$$\sum_{n=0}^{\infty} a_n y^n$$

Example 10.1.6: Consider $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$. Take $y = x-1$, $a_n = \frac{(-1)^{n+1}}{n}$. So we have if $y = 1$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

which converges. If $y = -1$, we get:

$$\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty}\frac{-1}{n}$$

which diverges to $-\infty$. So the power series converges on $y \in (-1,1]$. So $x = y+1$. So $x \in (0,2]$. Converges on an interval centered at x_0.

Convergence of Function Sequences

Definition 10.1.2

Let (f_n) be a sequence of functions that converges pointwise to f on S if $\forall x \in S$,

$$\lim f_n(x) = f(x)$$

$f_n = \sum_{k=0}^{n} a_n x^k$ converges pointwise to $\sum_{k=0}^{\infty} x^k$ on S which is the area of convergence. $\sum_{k=0}^{\infty} a_k x^k$ converges to some x_0 iff $\sum_{k=0}^{\infty} a_k x_0^k$ converges iff $\sum_{k=0}^{n} a_k x_0^k$ converges as $n \to \infty$ which is $(\sum_{k=0}^{n} a_k x_0^k) = f_n(x_0)$.

Last Lecture: Defined power series:

$$\sum_{i=0}^{\infty} a_i x^i$$

and proved that $\beta = \lim\sup|a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\beta}$. Then it converges when $|x| < R$ and diverges when $|x| > R$. No information when $x = \pm R$ if $R \neq 0, \infty$.

We can also use the ratio test if $\lim|\frac{a_{n+1}}{a_n}|$ exists. Then we let $\beta = \lim|\frac{a_{n+1}}{a_n}|$ and check.

A sequence of function (f_n) converges pointwise to f on S if for any $x \in S$, the $\lim f_n(x)$ converges to $f(x)$. For this, we write $f_n \to f$ pointwise. Remark: $f_n \to f$ pointwise in S iff $\forall x \in S, \forall \varepsilon > 0, \exists N$ such that $\forall n > N$, we have

$$|f_n(x) - f(x)| < S$$

Example 10.1.7: Let $f_n(x) = x^n, x \in [0,1]$. Recall that $\forall 0 < x < 1, x^n \to 0$. So $f_n \to f$ pointwise on $[0,1]$. So

$$f = \begin{cases} 1, x = 1 \\ 0, x \in [0,1) \end{cases}$$

Definition 10.1.3

Uniform Convergence

A sequence (f_n) converges uniformly to f on S if $\forall S$, $\exists N$ such that $\forall n > N$ and $x \in S$, we have
$$|f_n(x) - f(x)| < \varepsilon$$

Remark: Uniform Convergence $\longrightarrow$ pointwise convergence.

Example 10.1.8: $f(x) = (1 - |x|)^n$ where $x \in (-1, 1)$. When $x \neq 0$, then $1 > 1 - |x| > 0$. So $f_n(x) \longrightarrow 0$. When $x = 1$, $f_n(x) \longrightarrow 1$. So it converges pointwise on the interval to f where:
$$f = \begin{cases} 0, \text{if } 0 < |x| < 1 \\ 1, \text{if } x = 0 \end{cases}$$

Claim: $f_n \longrightarrow f$ is not uniformly convergent on S.

Proof. Assume that $f_n \longrightarrow f$ uniformly on $(-1, 1)$. Take $\varepsilon = \frac{1}{2}$. So then by definition, $\exists N$ such that $\forall n > N$, we have:
$$|f_n(x) - f(x)| < \frac{1}{2}$$

Choose $n = N + 1$, and take $x \in (0, 1)$. Then
$$(1 - x)^n < \frac{1}{2} \implies 1 - x < \frac{1}{\sqrt[n]{2}}$$

which means that
$$x > 1 - \frac{1}{\sqrt[n]{2}}$$

which is a contradiction because we can choose a smaller x than this number but greater than 0. □

Example 10.1.9: Let $f_n(x) = \frac{\sin nx}{n}, x \in \mathbb{R}$. Since $|f_n(x)| \leqslant \frac{1}{n}$. So $f_n \longrightarrow 0$ pointwise on $\mathbb{R}$.

Claim is that $f_n \longrightarrow 0$ uniformly on $\mathbb{R}$.

Proof. $\forall \varepsilon > 0$, we want to find N such that $\forall n > N$ and $x \in \mathbb{R}$, we have
$$|f_n(x) - f(x)| < \varepsilon$$

Since $|f_n(x)| \leqslant \frac{1}{n}$. We let $\frac{1}{n} < \varepsilon$, we get $n > \frac{1}{\varepsilon}$. So we can take $N = \frac{1}{\varepsilon}$. Then we take $\forall n > N, x \in \mathbb{R}$,
$$|f_n(x) - 0| = |\frac{\sin nx}{n}| \leqslant \frac{1}{n} < \varepsilon$$
□

Theorem 10.1.2

24.3

The uniform limit of continuous functions is continuous. More specifically, if $f_n \longrightarrow f$ uniformly on S, and each f_n is continuous at x_0, then f is continuous at x_0.

Proof. For any $\varepsilon > 0$, there is a N such that $\forall n > N$, and $\forall x \in S$, we have $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. Take $n = N + 1$. Since f_n is continuous at x_0, $\exists \delta > 0$ such that $\forall x \in S$, and $|x - x_0| < \delta$, we have

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$$

Then we have $\forall x \in S$, and $|x - x_0| < \delta$, we want to show that:

$$|f(x) - f(x_0)| < \varepsilon$$

We have

$$|f(x) - f(x_0)| \leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$

□

Remark: The uniform limit of uniformly continuous functions is also uniformly continuous.

Example 10.1.10: Let $f_n(x) = x^n$ where $x \in [0,1]$. We have show that $f_n \to f$ pointwise on $[0,1]$ where

$$f = \begin{cases} 0, \text{ if } x \in [0,1) \\ 1, \text{ if } x = 1 \end{cases}$$

Then $f_n \to f$ is not uniformly continuous since f is not continuous

Remark: $f_n \to f$ uniformly on S iff $\lim\limits_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0$.

Last Lecture: Defined $f_n \to f$ pointwise converges on S if $\forall x \in S$, $\lim f_n(x) = f(x)$. We write down the definition: $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon$$

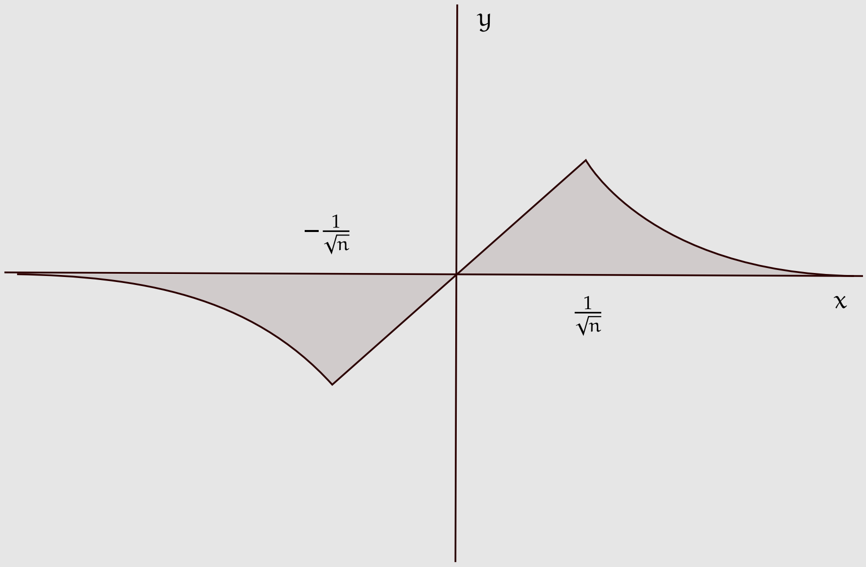
We defined $f_n \to f$ uniform convergence if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, $x \in S$,

$$|f_n(x) - f(x)| < \varepsilon$$

Theorem: If $f_n \to f$ uniformly on S and all f_n are continuous, then f is continuous. Remark that $f_n \to f$ uniformly convergent iff $\lim \sup\{f_n(x) - f(x) : x \in S\} = 0$

Proof. We have $\forall \varepsilon > 0$, $\forall x \in S$, $|f_n(x) - f(x)| \leqslant \varepsilon$ iff $\sup\{|f_n(x) - f(x)| : s \in S\} \leqslant \varepsilon$ □

Example 10.1.11: Consider $f_n = \frac{x}{1+nx^2}$, $x \in \mathbb{R}$. For $f_n(0) = 0$, and if $x \neq 0$, $1+nx^2 \to \infty$ when $n \to \infty$. So $f_n(x) \to 0$. So $f_n \to 0$ on $x \in \mathbb{R}$. To check $\sup\{|f_n(x)| : x \in \mathbb{R}\}$. We need to find supremum of f_n and $\inf f_n$. We find the derivative. The derivative is $\frac{1-nx^2}{(1+nx^2)^2}$. We have $f'_n(x) = 0$ when $x = -\frac{1}{\sqrt{n}}$ or $\frac{1}{\sqrt{n}}$. If $x \to \infty$, then $\lim\limits_{x\to\infty} \frac{1}{1+nx^2} = 0$ and $\lim\limits_{x\to\infty} \frac{x}{1+nx^2} = 0$.



We have $\max(f_n(x)) = f_n(\frac{1}{\sqrt{n}}) = \frac{1}{2\sqrt{n}}$. And $\min(f_n(x)) = f_n(-\frac{1}{\sqrt{n}}) = \frac{-1}{2\sqrt{n}}$. So $\sup\{|f_n(x)| : s \in \mathbb{R}\} = \frac{1}{2\sqrt{n}} \to 0$. So $f_n \to 0$ uniformly converges.

Example 10.1.12: Consider $f_n(x) = n^2x^n(1-x)$. It converges when $x \in [0,1]$. We have $\lim f_n(1) = 0$. If $0 \leqslant x < 1$, since $(n^2x^n(1-x))^{\frac{1}{n}} = (n^{\frac{2}{n}})x(1-x)^{\frac{1}{n}} \to x < 1$. So we know $\sum n^2x^n(1-x)$ converges. Then $n^2x^n(1-x) \to 0$. So $f_n \to 0$ pointwise. Check $\sup\{|f_n(x)| : x \in [0,1]\}$. Find the maximum and minimum values of f_n. Recall critical value theorem: If f is differentiable on $[a,b]$, then maximum or minimum values are at one of the critical points x such that $f'(x) = 0$ or a, b. Find $f_n'(x) = nx^{n-1} - (n+1)x^n = x^{n-1}(n-(n+1)x)$. So $f_n'(x) = 0$. So $x = 0$ or $\frac{n}{n+1}$. So $f_n(0) = f_n(1) = 0$. We have $f(\frac{n}{n+1}) = \frac{n^2}{n+1}(\frac{n}{n+1})^n$. We consider what happens when $n \to \infty$. We have

$$
\begin{aligned}
(\frac{n}{n+1})^n &= \frac{1}{(\frac{n+1}{n})^n} \\
&= \frac{1}{(1+\frac{1}{n})^n} \\
&= \frac{1}{e} \\
\frac{n^2}{n+1} &\to \infty
\end{aligned}
$$

So $\frac{n^2}{n+1}(\frac{n}{n+1})^n$ goes to infinity, which is not 0, so it is not uniformly convergent.

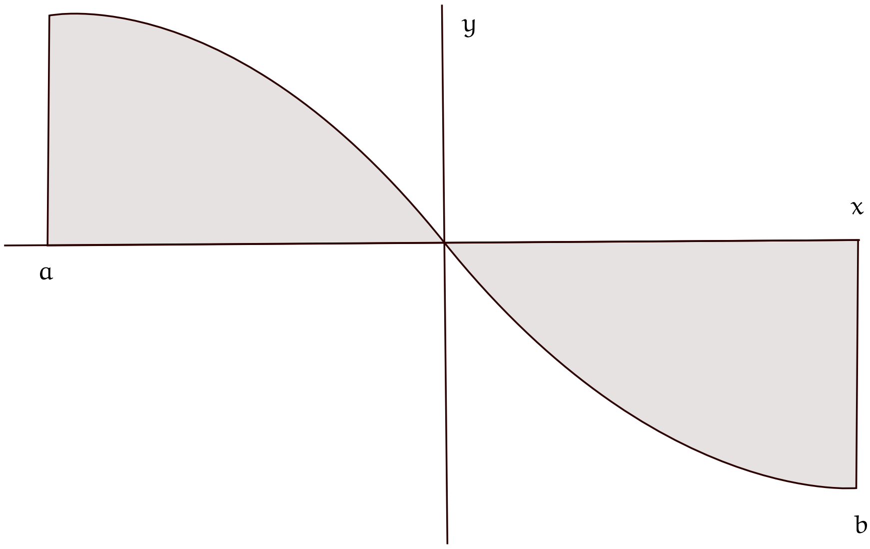
25.2

Theorem 10.1.3

If $f_n \to f$ uniformly on $[a,b]$ and f_n are continuous on $[a,b]$, then $\lim \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx = \int_a^b \lim f_n(x)\,dx$

Recall:

- The integral $\int_a^b f(x)\ dx$ = signed area between $f(x)$ and x coordinate on $[a, b]$



- Any continuous function is integrable $g(x)$
- If $g(x) \leqslant h(x)\ \forall x \in [a, b]$, then $\int_a^b g(x)\ dx \leqslant \int_a^b h(x)\ dx$
- $|\int_a^b g(x)\ dx| \leqslant \int_a^b |g(x)|\ dx$

Proof. By theorem 24.3, So f is continuous. Then $f_n - f$ is integrable. Then $\forall \varepsilon > 0$, there is N such that $\forall n > N$, $x \in [a, b]$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b - a}$$

Then $\forall n > N$, we have

$$\begin{aligned} |\int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx| &= |\int_a^b f_n(x) - f(x)\ dx| \\ &\leqslant \int_a^b |f_n(x) - f(x)|\ dx \\ &< \int_a^b \frac{\varepsilon}{b - a}\ dx \\ &= \varepsilon \end{aligned}$$

So $\lim \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx$. □

Chapter 11

Week 11

Last Lecture: $f_n \to f$ pointwise on S if $\forall x \in S$, $\lim f_n(x) = f(x)$. Then $\forall x \in S$, and $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon$$

Definition: $f_n \to f$ uniformly if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ and $x \in S$ we have

$$|f_n(x) - f(x)| < \varepsilon$$

Remark: $f_n \to f$ uniformly iff $\lim \sup\{|f_n(x) - f(x)| : x \in S\} = 0$. Theorem: If $f_n \to fk$ uniformly on $[a, b]$ and f_n is continuous, then

$$\lim \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$$

11.1 Series of Functions

Definition 11.1.1 **Cauchy Series**

(f_n) is uniformly cauchy on S if $\forall \varepsilon > 0$, $\exists N$ such that $\forall x \in S$ and $m, n > N$, we have

$$|f_n(x) - f_m(x)| < \varepsilon$$

Theorem 11.1.1 **Uniform Convergence**

If (f_n) uniformly cauchy on S, then $\exists f$ on S such that $f_n \to f$ uniformly on S.

Definition 11.1.2 **Series of Functions**

$\sum_{k=0}^{\infty} g_k$ is the limit if

$$\sum_{k=0}^{n} g_k$$

exist. If $\sum_{k=0}^{n} g_k$ on uniformly on S, we say that the series is uniformly convergent.

Example 11.1.1: Power Series:

$$\sum_{k=0}^{\infty} a_k x^k$$

is a series of functions. $g_k = a_k x^k$.

Example 11.1.2: $\sum_{k=0}^{\infty} \frac{x^n}{1+x^n}$ is a series of functions but not a power series.

Cauchy Criterion

Definition 11.1.3 The series converges uniformly on S if $\forall \varepsilon > 0, \exists N$ such that $\forall n \geqslant m > N$ and $x \in S$,

$$|\sum_{k=m}^{n} g_k(x)| < \varepsilon$$

25.6

Theorem 11.1.2 If $\sum_{k=0}^{\infty} g_k$ satisfies Cauchy criterion uniformly on S, then $\sum g_k$ converges uniformly on S.

25.7

Theorem 11.1.3 If real numbers $M_k \geqslant 0$ and $\sum M_k < \infty$, if $|g_k(x)| \leqslant M_k$ for any $x \in S$, then $\sum g_k$ converges uniformly on S.

Proof. Prove Cauchy criterion uniformly on S. $\forall \varepsilon > 0$, $\sum M_k$ converges. So $\exists N$ such that $\forall n \geqslant m > N$ we have

$$\sum_{k=m}^{n} M_k < \varepsilon$$

For any $x \in S$, $|\sum_{k=m}^{n} g_k(x)| \leqslant \sum_{k=m}^{n} |g_k(x)| \leqslant \sum_{k=m}^{n} M_k < \varepsilon$. □

25.5

Theorem 11.1.4 If g_k is continuous and $\sum_{k=0}^{\infty} g_k$ converges uniformly on S, then the limit of the series is continuous.

Remark: If $\sum_{k=0}^{\infty} g_k$ converges uniformly on S, then (g_k) converges to 0 on S.

Example 11.1.3: Consider the power series $\sum_{k=1}^{\infty} 2^{-k} x^k$. It converges pointwise to a continuous function on $(-2, 2)$ but not uniformly.

Proof. Take $a_k = 2^{-k}$, $\lim \sqrt[k]{a_k} = \frac{1}{2} = \beta$. Then $R = \frac{1}{\beta} = 2$. If $x = 2$, $\sum_{k=1}^{\infty} 1$ diverges and if $x = -2$, $\sum_{-1}^{\infty}(-1)^k$ also diverges. Then radius of convergence is $(-2, 2)$.

(The limit is continuous on $(-2, 2)$). Take any $0 < a < 2$, then for any $x \in [-a, a]$, we have

$$|2^{-n} x^n| \leqslant 2^{-n} a^n = \left(\frac{a}{2}\right)^n, \frac{a}{2} < 1$$

so $\sum_{k=0}^{\infty} g_k$ converges uniformly on $[-a, a]$. Then $\sum_{k=1}^{\infty} 2^{-n}x^n$ is continuous on $[-a, a]$. Since a is any number between $0, 2$, the limit is continuous on $(-2, 2)$.

(Not uniform) Assume that the convergence is uniform. Then $(g_k) = x^n 2^{-n}$ goes to 0 uniformly on $(-2, 2)$. But $\sup\{|x^n 2^{-n}| : x \in (-2, 2)\} = 1$. So contradiction.

□

26.1

Theorem 11.1.5

If we consider $\sum_{n=0}^{\infty} a_n x^n$ which has radius of convergence $r > 0$, then the power series converges uniformly $[-R_1, R_1]$ where $0 < R_1 < R$

Corollary: $\sum a_n x^n$ converges to a continuous function on $(-R, R)$. Remark: $\sum a_n x^n$ is not necessarily uniformly convergent on $(-R, R)$.

11.2 Derivative and Integral of Power Series

Last Lecture: Recall that $f_n \to f$ uniformly on S if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ and $x \in S$

$$|f_n(x) - f(x)| < \varepsilon$$

Theorem: If (f_n) are uniformly cauchy on S, then $\exists f$ such that $f_n \to f$ uniformly on S.

Theorem for Series: If $\sum g_k$ satisfies the cauchy criterion on S, then $\sum g_k$ converges uniformly on S.

Theorem: If $M_k \geqslant 0$ and $\sum M_k < \infty$. If $|g_k(x)| \leqslant M_k$ $\forall x \in S$, $\forall k \in \mathbb{N}$, then $\sum g_k$ converges uniformly on S.

Theorem for power series: $\sum a_n x^n$ has $R > 0$. Then $\forall R_1 < R$, the power series $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$.

Today: Derivative and integral on Power Series.

Recall: $(a_n x^n)' = n a_n x^{n-1}$ and $\int a_n x^n \, dx = \frac{a_n}{a_{n+1}} x^{n+1}$.

Lemma: If $\sum a_n x^n$ has R then $\sum n a_n x^n$ and $\sum \frac{a_n}{n+1} x^{n+1}$ have radius of convergence R.

Proof. We find $\beta = \limsup |a_n|^{\frac{1}{n}}$, $R = \frac{1}{\beta}$. Take $\limsup |n a_n|^{\frac{1}{n}} = \limsup n^{\frac{1}{n}} |a_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} = \beta$. For the anti-derivative, we get:

$$\begin{aligned} \limsup \left|\frac{a_n}{n+1}\right|^{\frac{1}{n}} &= \limsup \frac{|a_n|^{\frac{1}{n}}}{(n+1)^{\frac{1}{n}}} \\ &= \limsup |a_n|^{\frac{1}{n}} = \beta \end{aligned}$$

□

26.4

Theorem 11.2.1

If $\sum a_n x^n$ has a radius $R > 0$, then $\forall |x| < R$, we have $\int_0^x \sum_{n=0}^{\infty} a_n t^n \, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

Proof. If $x > 0$ and $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} \frac{a_{n+1}}{n+1} x^{n+1}$ uniformly converge on $[0, x]$. Use the theorem by 25.2 we have:

$$\begin{aligned}
\int_0^x \sum_{n=0}^{\infty} a_n t^n \, dt &= \int_0^x \lim_{k \to \infty} \sum_{n=0}^{k} a_n t^n \, dt \\
&= \lim_{k \to \infty} \int_0^x \sum_{n=0}^{k} a_n t^n \, dt \\
&= \lim_{k \to \infty} \sum_{n=0}^{k} \int_0^x a_n t^n \, dt \\
&= \lim_{k \to \infty} \sum_{n=0}^{k} \frac{a_{n+1}}{n+1} x^{n+1} \\
&= \sum_{k=0}^{\infty} \frac{a_{n+1}}{n+1} x^{n+1}
\end{aligned}$$

□

26.5

Theorem 11.2.2

Assume that $f(x_n) = \sum_{n=0}^{\infty} a_n x^n$ has $R > 0$. Then $f(x)$ is differentiable on $(-R, R)$ and $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

Proof. Take $g(x) = \sum_{n=0}^{\infty} n a_n x^{n+1}$. $\forall |x| < R$. Using the last theorem:

$$\begin{aligned}
\int_0^x f(t) \, dt &= \sum_{n=1}^{\infty} a_n x^n \\
&= f(x) - a_0
\end{aligned}$$

But

$$f(x) = \int_0^x g(t) \, dt + a_0$$

and $g(t)$ is continuous on $(-R, R)$. By the fundamental theorem of calculus, $f'(x) = g(x)$. □

Example 11.2.1: Recall that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. Then $R = 1$. So differentiate:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

We can also integrate:

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

for $|x| < 1$

Abel's Theorem

Theorem 11.2.3

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has $0 < R < \infty$, and if f converges at $x = R$, then f is continuous at $x = R$. If f converges at $x = -R$, then f is continuous at $x = -R$.

Example 11.2.2: $-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ for $|x| < 1$. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ converges at $x = -1$. By Abel's theorem, it is continuous at $x = -1$. This means that $\forall x_n \to -1^+$, $\lim \frac{x_n^{n+1}}{n+1} = \frac{(-1)^{n+1}}{n+1} = -\ln(2)$. Then we get:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Chapter 12

Week 12

Last Lecture: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has $R > 0$. Then $\forall |x| < R$,

$$\int_0^x f(t)\, dt = \sum_{n=0} L^{\infty} \frac{a_n}{n+1} x^{n+1}$$

and

$$\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Abel's theorem: If $\sum a_n x^n$ has $\infty > R > 0$, if it converges at R or $-R$, then it is continuous at R or $-R$.

12.1 Derivatives

Definition 12.1.1 **Derivative**

Let f be a function defined on an open interval containing x_0. Then f is differentiable or f has derivative at x_0 if $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ converges.

Example 12.1.1: $g(x) = x^2$. We have $\lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{x^2-x_0^2}{x-x_0}$:

$$\lim_{x \to x_0} \frac{(x-x_0)(x+x_0)}{x-x_0} = 2x_0$$

So $g'(x) = 2x$.

In general, $(x^n)' = nx^{n-1}$.

Theorem 12.1.1 **28.2**

If f is differentiable at x_0, then f is continuous at x_0.

Proof. $\forall x \in \text{dom}(f) - \{x_0\}$. Then $f(x) = (x - x_0)\frac{f(x)-f(x_0)}{x-x_0} + f(x_0)$. If $x \to x_0$, $(x - x_0) \to 0$, $\frac{f(x)f(x_0)}{x-x_0} \to f'(x_0)$, and $f(x_0) \to f(x_0)$. Therefore, $f(x) \to 0 \cdot f'(x_0) + f(x_0) = f(x_0)$. Then f is continuous at x_0. □

28.3

Theorem 12.1.2

If f and g are differentiable at x_0, then

- $\forall c \in \mathbb{R}$, $(cf)'(x_0) = ic \cdot f'(x_0)$.
- $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- $(fg)'(x) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
- If $g(x_0) \neq 0$, then $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$

Proof. (III) Consider $\frac{(fg)(x)-(fg)(x_0)}{x-x_0}$. We have:

$$\begin{aligned}\frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} &= f(x)\frac{g(x) - g(x_0)}{x - x_0} + g(x_0)\frac{f(x) - f(x_0)}{x - x_0} \\ &\to f(x_0)g'(x_0) + g(x_0)g'(x_0)\end{aligned}$$

(IV) Consider $\frac{f}{g}(x) - \frac{f}{g}(x_0)$

$$\begin{aligned}\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)} &= \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)} \\ &= \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}\end{aligned}$$

Then

$$\frac{\frac{f}{g}(x) - \frac{f}{g}(x_0)}{x - x_0} = \frac{1}{g(x)g(x_0)}\left(\frac{f(x) - f(x_0)}{x - x_0}g(x_0) - f(x_0)\frac{g(x) - g(x_0)}{x - x_0}\right)$$

Take $x \to x_0$:

$$\frac{1}{g^2(x_0)}\left(f'(x_0)g(x_0) - f(x_0)g'(x_0)\right) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$

□

Chain Rule

Theorem 12.1.3

If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.

Proof. Show $\forall$ sequence $x_n \to x_0$,

$$\lim_{n\to\infty} \frac{(g \circ f)(x_n) - (g \circ f)(x_0)}{x_n - x_0} = g'(f(x_0))f'(x_0)$$

Consider two cases:

- Case 1: Assume that $\exists$ an open interval J containing x_0 such that $\forall x \in J - \{x_0\}$, $f(x) \neq f(x_0)$. So $f(x_n) \neq f(x_0)$ when n is large. Then

$$\begin{aligned}\frac{(g \circ f)(x_n) - (g \circ f)(x_{0n})}{x_n - x_0} &= \frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)}\left(\frac{f(x_n) - f(x_0)}{x_0 - x_0}\right) \\ &= g'(f(x_0))f'(x_0)\end{aligned}$$

  So when $n \to \infty$, it goes to $g'(f(x_0))f'(x_0)$

- If the assumption in case 1 fails, there $\exists z_n \to x_0$, $z_n \neq x_0$ and $f(z_n) = f(x_0)$. Then $\lim_{n\to\infty} \frac{f(z_n)-f(x_0)}{z_n - x_0} = 0 = g'(f(x_0))f'(x_0)$. We need to show $(g \circ f)$ is differentiable at x_0 and derivative is 0. Since $g'(f(x_0))$ exists, then

$$\frac{g(y) - g(f(x_0))}{y - f(x_0)}$$

is bounded for any y near $f(x_0)$. That is, $\exists$ an open interval I containing $f(x_0)$ such that $\forall y \in I - \{f(x_0)\}$, we have:

$$\left|\frac{g(y) - g(f(x_0))}{y - f(x_0)}\right| \leqslant c$$

then we have

$$\left|\frac{(g\circ f)(x_n) - (g\circ f)(x_0)}{x_n - x_0}\right| \leqslant c \cdot \left|\frac{f(x_n) - f(x_0)}{x_n - x_0}\right|$$

This is because if $f(x_n) \neq f(x_n)$, then LHS is $\frac{(g\circ f)(x_n)-(g\circ f)(x_0)}{f(x_n)-f(x_n)} \frac{f(x_n)-f(x_0)}{x_n - x_0}$. If $f(x_n) = f(x_0)$, then $0 \leqslant c \cdot |\ldots|$. Using this inequality, we know that as we take the limit, as $n \to \infty$, we have that the RHS is equal to 0. So the LHS goes to 0. So we have $g'(f(x_0))f'(x_0) = 0 = (g \circ f)'(x_0)$ as desired.

□

12.2 Mean Value Theorem

Last Lecture: f is differentiable at a if the derivative:

$$f(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

converges.

If f is differentiable at a, then it is continuous at a. If f, g are differentiable on $\mathbb{R}$ and $c \in \mathbb{R}$, then

- $(f + g)'(x) = f' + g'$
- $(cf)' = cf'$
- $(fg)' = f'g + fg'$
- $(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$ when $g(x) \neq 0$
- $(g \circ f)'(x) = g'(f(x))f'(x)$

29.1

Theorem 12.2.1 If f is defined on an open interval containing x_0 and assumes its maximum and minimum at x_0 and is differentiable, then $f'(x_0) = 0$.

Proof. Let f be defined on the interval (a, b), achieving a maximum at x_0. For $x < x_0$, we have

$$\frac{f(x) - f(x_0)}{x - x_0}$$

So $f(x) < f(x_0)$, $f(x) - f(x_0) \leqslant 0$. So $x - x_0 < 0$. Then $f'(x_0) \geqslant 0$.

Now for $x > x_0$, consider again $\frac{f(x)-f(x_0)}{x-x_0}$. We have $f(x) - f(x_0) \leqslant 0$ and $x - x_0 > 0$. So $f'(x_0) \leqslant 0$. Then $f'(x_0) = 0$ putting the two together. □

Recall that any continuous function on $[a, b]$ achieves its maximum and minimum on $[a, b]$.

Theorem 12.2.2 **Rolle's Theorem**

Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that it is differentiable on (a, b). Suppose that $f(a) = f(b)$. Then there is $x \in (a, b)$ where $f'(x) = 0$.

Proof. So there is $x_0, y_0 \in [a, b]$ such that $f(x_0)$ is the minimum of f and $f(y_0)$ is the max of f on $[a, b]$. Then we have cases:

- x_0, y_0 are endpoint of $[a, b]$. Then f is constant since $f(a) = f(b)$. Then $f'(x) = 0$ for all $x \in (a, b)$.
- Otherwise, one of $x_0, y_0 \in (a, b)$ and apply the previous theorem.

$\square$

Theorem 12.2.3 **Mean Value Theorem**

Let f be continuous on $[a, b]$, and differentiable on (a, b). Then there exists $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

Proof. Consider the line on $[a, b]$ going through $f(a), f(b)$. Define

$$L = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Then consider $f - L$. Since $(f - L)(a) = (f - L)(b) = 0$, we know that there is some $x \in (a, b)$ such that $(f - L)'(x) = 0$. Then $f'(x) - L'(x) = 0$. So $f'(x) = L'(x)$. We have $L'(x) = \frac{f(b)-f(a)}{b-a}$. $\square$

Corollary: If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for $x \in (a, b)$, then f is the constant function.

Proof. Suppose that there is some $f(x_0) \neq f(y_0)$ then by mean value theorem, we have that $f'(z_0) = \frac{f(x_0)-f(y_0)}{x_0-y_0}$ which is not 0. $\square$

Corollary: If f, g are two differentiable functions on $(a, b) \to \mathbb{R}$ and $f' = g'$ on (a, b), then there is some constant c such that $f(x) = g(x) + c$.

Definition 12.2.1 **Increasing, Strictly Increasing, Decreasing, Strictly Decreasing**

Suppose f is defined on an interval I. We say that f is strictly increasing if on the interval I, for all $x_1, x_2 \in I$, $x_1 < x_2$, then $f(x_1)k < f(x_2)$. Similar definitions for other kinds of increasing/decreasing.

Corollary: If $f : (a, b) \to \mathbb{R}$ is differentiable, then

- f is strictly increasing if $f'(x) > 0$ for all $x \in (a, b)$
- f is strictly decreasing if $f'(x) < 0$ for all $x \in (a, b)$
- f is increasing if $f'(x) \geqslant 0$ for all $x \in (a, b)$
- f is decreasing if $f'(x) \leqslant 0$ for all $x \in (a, b)$

Proof. If f is not strictly increasing, then $\exists x_1 < x_2$ such that $f(x_1) \geqslant f(x_2)$. Then by mean value theorem we have $f'(x) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \leqslant 0$, which contradicts that $f' > 0$. $\square$

Theorem 12.2.4 **Intermediate Value Theorem for Derivatives**

Let $f : (a, b) \longrightarrow \mathbb{R}$ be differentiable. If $a < x_1 < x_2 < b$ and c is some value between $f'(x_1), f'(x_2)$, then there is some $x_1 < x < x_2$ such that $f'(x) = c$.

Proof. Consider $g(x) = f(x) - cx$. This is differentiable on (a, b). We can write $g'(x_1) < 0 < g'(x_2)$. Then g must assume its minimum in $[x_1, x_2]$. New x_1, we have

$$0 > g'(x_1) = \lim_{x \to x_1^-} \frac{g(x) - g(x_1)}{x - x_1}$$

Then $g(x) - g(x_1) < 0$. So $g(x) < g(x_1)$.

Near x_2, we have $g'(x_2) > 0$. Then we have

$$0 < g'(x_2) = \lim_{x \to x_2^-} \frac{g(x) - g(x_2)}{x - x_2}$$

We have $x - x_2 < 0$. So $g(x) - g(x_2) < 0$ and $g(x) < g(x_2)$. So the minimum of g is in (x_1, x_2). So g achieves its minimum at $x_0 \in (x_1, x_2)$, we have $g'(x) = 0 = f'(x) - c$. So $f'(x) = c$. □

Example 12.2.1: Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$x \mapsto \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f is differentiable on $\mathbb{R}$ but f' is not continuous at $x = 0$.

Chapter 13

Week 13

Last Lecture: If f is differentiable at x_0 and gets max/min at x_0 then $f(x_0) = 0$. Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

Intermediate Value Theorem for derivatives: If f is differentiable on (a, b) and $a < x_1 < x_2 < b$ and c is between $f'(x_1)$ and $f'(x_2)$, then there is an $x \in (x_1, x_2)$ such that $f'(x) = c$.

13.1 Derivative of Inverse Functions

29.9

Theorem 13.1.1 Let f be one-to-one continuous on I, $J = f(I)$. If f is differentiable at $x_0 \in I$, $f'(x_0) \neq 0$, then $g = f^{-1}$ is differentiable at $f(x_0) = y$, $g'(y_0) = \frac{1}{f'(x_0)}$.

Proof. $f(x) = y$, $g(y) = x$. Since f is continuous and I is connected, $J = f(I)$ is connected. So J is an interval. $g : J \to I$, $F : I \to J$. So

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \neq 0$$

Then

$$\begin{aligned}\lim_{x\to x_0} \frac{x - x_0}{f(x) - f(x_0)} &= \frac{1}{f'(x_0)} \neq 0 \\ &= \lim_{x\to x_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}\end{aligned}$$

We want to show that $\lim\limits_{y\to y_0} \iff \lim\limits_{x\to x_0}$. We want to show that f, g continuous. By definition, f is continuous and one-to-one. This means that f is strictly increasing or decreasing by theorem 18.6. This means that $g = f^{-1}$ is continuous by theorem 18.4. □

Example 13.1.1: $f(x) = x^n, x \in [0, \infty)$. Then $f^{-1}(y) = g(y) = y^{\frac{1}{n}}, x \in [0, \infty)$. So $f'(x) = nx^{n-1}$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{n(y^{\frac{1}{n}})^{n-1}}$. So this is $\frac{1}{n} y^{\frac{1}{n}-1}$.

Theorem 13.1.2 **Generalized MVT**

Let f, g be continuous on $[a, b]$ and differentiable at (a, b). Then there exists $x \in (a, b)$ such that $f(x)(g(b) - g(a)) = g(x)(f(b) - f(a))$.

Remark: If $g(x) = x$, it is MVT.

Proof. If $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. Take

$$h(b) = f(b)g(b) - f(b)g(a) - g(b)f(b) + g(b)f(a)$$
$$h(a) = g(b)f(a) - f(b)g(a) = h(b)$$

By Rolle's theorem, $\exists x \in (a, b)$ such that $h'(x) = 0$. □

Theorem 13.1.3 **L'Hopital's Rule**

Let s signify $a, a^+, a^-, -\infty, \infty$ where $a \in \mathbb{R}$ with f, g differentiable. Then limit

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L$$

exists. If $\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$ or if $\lim_{x \to s} |g(x)| = \infty$, then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = L$$

Proof. Prove the case $s = a^+$ and L is finite,

$$\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$$

Since $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$ exists. $g'(x)$ cannot be 0 near a^+. By IVT for derivative $g'(x)$ is always > 0 or < 0 near a^+ for $x \in (a, b)$. Assume $g'(x) < 0$ for all $x \in (a, b)$. Then g is strictly decreasing on (a, b). So it is one-to-one on (a, b). At most one $x \in (a, b)$ such that $g(x) = 0$. By choosing b smaller, we assume $g(x)$ never vanishes on (a, b). Then $\forall \varepsilon > 0, \exists \delta > 0$ such that $a + \delta < b$ and $\forall a < x < a + \delta$, we have

$$|\frac{f'(x)}{g'(x)} - L| < \varepsilon$$

So $\forall a < y < x < a + \delta$, by theorem 30.1, we have

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)}$$

by generalized MVT for $z \in (x, y)$. So

$$\left|\frac{f'(z)}{g'(z)} - L\right| < \varepsilon$$

So

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - L\right| < \varepsilon$$

Let $y \to a^+$. Then

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon$$

for $x \in (a, a + \delta)$. So $\lim_{x \to s} \frac{f(x)}{g(x)} = L$. □

Example 13.1.2: Consider

$$\lim_{x\to 0} \frac{\sin x}{x}$$

The derivative of $\sin x$ is $\cos x$ and $\frac{d}{dx}x = 1$. We have

$$\lim_{x\to 0} \frac{\cos x}{1} = 1$$

Example 13.1.3: Consider

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}}$$

We have $\frac{d}{dx}\frac{2x}{3e^{3x}}, \frac{d}{dx}\frac{2}{9e^{3x}} \longrightarrow 0$. Then

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} = 0$$

Last Lecture: Let f be one-to-one on I, f is differentiable at $x_0 \in I$, $f'(x_0) \neq 0$ then $g = f^{-1}$ is differentiable at $y_0 = f(x_0)$, $g'(y_0) = \frac{1}{f'(x_0)}$.

Let S signify $a, a^-, a^+, \infty, -\infty$, f, g differentiable. If

$$\lim_{x\to s} \frac{f'(x)}{g'(x)} = L$$

exists, if $\lim\limits_{x\to s} f(x) = \lim\limits_{x\to s} g(x) = 0$ or $\lim\limits_{x\to s}|g(x)| = \infty$, then $\lim\limits_{x\to s} \frac{f(x)}{g(x)} = L$.

Example 13.1.4: $\lim\limits_{x\to 0^+} x\ln x$. We can consider instead $x \ln x = \frac{\ln x}{\frac{1}{x}}$. Then the derivative is $\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim\limits_{x\to 0^+} -x = 0$

Example 13.1.5: $\lim\limits_{x\to 0^+} x^x$. Use $x = e^{\ln x}$. So

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} (e^{\ln x})^x \\ &= \lim_{x\to 0^+} e^{x\ln x} \\ &= e^0 \\ &= 1\end{aligned}$$

Example 13.1.6: $\lim\limits_{x\to 0} h(x)$ where $h(x) = \frac{1}{e^x - 1} - \frac{1}{x}$. Then

$$h(x) = \frac{x - e^x 1}{x(e^x - 1)}$$

Find the derivative:

$$\lim_{x\to 0}\frac{1-e^x}{e^x-1+xe^x}\to\lim_{x\to 0}\frac{-e^x}{e^x+e^x+xe^x}$$
$$=\frac{-1}{2}$$

Given $\sum_{k=0}^{\infty} a_k x^k$ with $R>0$. So $\sum_{k=0}^{\infty} a_k x^k = f(x)$ on $(-R,R)$. It is differentiable. So we have $\sum_{k=1}^{\infty} k a_k x^{k-1} = f'(x)$ on $(-R,R)$. We have

$$f'(0)=a_1$$

Then $\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = f''(x)$. Then

$$f''(0)=2a_2$$

Taking the derivative again: $\sum_{k=3}^{\infty} k(k-1)(k-2)a_k x^{k-3} = f'''(x)$. Then

$$f'''(0)=6a_3$$

If we continue, we get $f^{(n)}(0)=n!\cdot a_n$. Then $a_k=\frac{f^{(k)}(0)}{k!}$. Then

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

for $x\in(-R,R)$. We want to see if this equation is true for other functions like $\sin x, \cos x, \ldots$.

Remark: Given $\sum_{k=0}^{\infty} a_k(x-c)^k$, where c is constant, with $c\in\mathbb{R}$, $R>0$, we have

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

for $x\in(c-R,c+R)$.

Definition 13.1.1 **Taylor Series**

Let f be a function defined on an open interval containing c, and f has a derivative of all orders. Then we call $\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$ the Taylor Series. We call $R_n(x) = f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k$ the remainder.

Remark: $\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$ converges to $f(x)$ iff $\lim_{n\to\infty} R_n(x)=0$.

Theorem 13.1.4 **31.3**

Let f be defined on (a,b) such that $a<c<b$ and $f^{(n)}$ exists. Then $\forall x\neq c$ in (a,b) $\exists y$ between c and x such that $R_n(x)=\frac{f^{(n)}(y)}{n!}(x-c)^n$.

Proof. Fix $x\neq c$ and $n\geqslant 1$. Let M be the unique solution of the equation:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k+M\frac{(x-c)^n}{n!}$$

We need to show that $f^{(n)}(y) = M$ for some y between x, c. Let $g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}(t - c)^k + \frac{M(t-c)^n}{n!} - f(t)$. Then

$$g(c) = f(c) - f(c) = 0$$

Consider $g'(c) = f'(c) - f'(c) = 0$. So $g^{(k)}(c) = 0$ for $k < n$. Then $g(x) = 0$. By Rolle's Theorem, $\exists x_1$ between, x, c such that $g'(x_1) = 0$. By Rolle's Theorem for g', there exists x_2 between x_1, c such that $g''(x_2) = 0$, and so on. Then there is x_n such that $g^{(n)}(x_n) = 0$ for x_n between x_{n-1} and c. Take $y = x_n$. Then $g^{(n)}(y) = 0$. So

$$g^{(n)}(y) = M - f^{(n)}(y) = 0$$

□

Corollary: If $|f^{(n)}(x)| \leqslant c$, then $R_n(x) \to 0$.

Proof. We have $R_n(x) = \frac{f^{(n)}(y)}{n!}(x - c)^n$. Then it is $\leqslant \frac{c}{n!}(x - c)^n \to 0$. □

Last Lecture: Recall that the Taylor Series for f about c is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

Remainder is $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k$. We have

$$f(x) = \text{Taylor Series}$$

iff $\lim\limits_{n\to\infty} R_n(x) \to 0$.

Theorem: $\forall x \neq c$ in (a, b), $\exists y$ between x, c such that

$$R_n(x) = \frac{f^{(n)}(y)}{k!}(x - c)^n$$

Corollary: If $\exists c > 0$ such that $\forall n \in \mathbb{N}$, $|f^{(n)}| < c$, then $R_n(x) \to 0$. So if the n-th derivative is always bounded, then $f(x)$ is equal to the Taylor series.

Example 13.1.7: Consider $f(x) = e^x$. We have $f^{(n)} = e^x$. We have $f^{(n)}(0) = 1$. So we want to show

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Then $\forall x \in \mathbb{R}$, $\exists M > 0$ such that $|x| < M$. On $(-M, M)$ $f^{(n)}$ is bounded by e^M.

Example 13.1.8: $f(x) = \sin x$. We have

$$f^{(n)}(x) = \begin{cases} \cos x & \text{if } n = 1 \\ -\sin x & \text{if } n = 2 \\ -\cos x & \text{if } n = 3 \\ \sin x & \text{if } n = 4 \end{cases}$$

We have

$$f^{(n)}(0) = \begin{cases} 1 & \text{if } n = 1, 4, 9, \ldots \\ -1 & \text{if } n = 3, 7, 11, \ldots \\ 0 & \text{if } n = \text{ anything else} \end{cases}$$

So for any odd number, $n = 2k+1$. $f^{(n)}(0) = (-1)^k$. So the Taylor series is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Since $|f^{(n)}(x)| \leqslant 1$, so by the corollary,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

31.5

Theorem 13.1.5

Let f be defined on (a,b) where $a < c < b$. If $f^{(n)}$ exists, and is continuous on (a,b), then $\forall x \in (a,b)$, $R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt$.

Proof. For $n = 1$, only need to show that $R_1(x) \int_c^x f'(t)\, dt$. By definition, $R_1(x) = f(x) - f(c) = \int_c^x f'(t)\, dt$. In general, we will use induction on n and integration by parts. □

Corollary: $\forall x \neq c$ in (a,b) there exists y between c and x such that $R_n(x) = (x-c)\frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$. This is called the cauchy form of the remainder.

Proof. Use theorem 31.5 and IVT for integrals (33.9) covered later. □

Binomial Series Theorem

Theorem 13.1.6

$\forall \alpha \in \mathbb{R}$, $|x| < 1$, then

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

Proof. Let $f(x) = (1+x)^\alpha$. Then $f'(x) = \alpha(1+x)^{\alpha-1}$. And $f'(0k = \alpha)$. Then $f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$. So $f''(0) = \alpha(\alpha-1)$.

In the right hand side, it is the Taylor series of f about 0. Only need to show that $\lim_{n\to\infty} R_n(x) = 0$ $\forall |x| < 1$. Let $a_k = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$. Consider the ratio test:

$$\lim \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{a-k}{k+1}\right| = 1$$

So $\beta = 1$, $R = \frac{1}{\beta} = 1$. Similarly, $\sum k a_k x^{k-1}$ has radius convergence of 1. So $\forall |x| < 1$, we have $\lim k a_k x^{k-1} = 0$. By previous corollary, $R_n(x) = (x-c)\frac{(x-y_n)^{n-1}}{(n-1)!} f^{(n)}(y_n)$. This is equal to $(x-c)\frac{(x-y_n)^{n-1}}{(n-1)!}\alpha(\alpha-1)\cdots(\alpha-k+1)(1+y_n^{\alpha-1})$. Keep simplifying:

$$\ldots = x\frac{(x-y_n)^{n-1}}{(1+y_n)^{n-1}}(1+y_n)^{\alpha-1} n a_n$$

Let $y_n = z_n x$, $z_n \in (0,1)$. So

$$\left|\frac{x-y_n}{1+y_n}\right| = \left|\frac{x-z_n x}{1+z_n x}\right| = |x|\left|\frac{1-z_n}{1+z_n}x\right| < |x|$$

So

$$
\begin{aligned}
|R_n(x)| &= x\frac{(x-y_n)^{n-1}}{(1+y_n)^{n-1}}(1+y_n)^{\alpha-1}na_n \\
&< |x||x|^{n-1}na_n(1+y_n^{\alpha-1})
\end{aligned}
$$

Since $(1+y_n)^{\alpha-1}$ is bounded and the other stuff converges to 0, $|R_n(x)| \to 0$. □

Example 13.1.9: $f(x) \neq$ Taylor Series. Define

$$
f(x) = \begin{cases} e^{\frac{-1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 \end{cases}
$$

. Claim: $\forall n \in \mathbb{N}$, $f^{(n)}(0) = 0$. Taylor Series = 0.

We have

$$
\begin{aligned}
\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}}
\end{aligned}
$$

Use L'Hopital to get the limit is 0. Then $\forall x > 0$, $f^{(n)}(x) = e^{\frac{-1}{x}} \cdot$ polynomial of $\frac{1}{x}$. We see that $f(x) \neq 0$ which is its Taylor Series.

Chapter 14

Week 14

Last Lecture: Taylor Series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

And

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^k(c)}{k!}(x-c)^k$$

Theorem: $R_n(x) = \int_c^x \frac{(x-c)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt$. We had a corollary:

$$R_n(x) = (x-c)\frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$$

And that $\forall \alpha \in \mathbb{R}, |x| < 1$ we have

$$(1+x)^{\alpha} = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

14.1 Riemann Integral

Idea: If f is a continuous function on $[a, b]$ then $\int_a^b f(x)\, dx$ is the area. We use rectangles to estimate.

Definition 14.1.1 **Darboux Integral**

Let f be bounded on $[a, b]$. Then $\forall S \subseteq [a, b]$, define $M(f, S) = \sup\{f(x) : x \in S\}$. Define $m(f, S) = \inf\{f(x) : x \in S\}$. A partition of $[a, b]$ is

$$P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$$

The upper Darboux sum is

$$U(f, p) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

and the lower Darboux sum

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

Then the upper Darboux integral:

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

The lower Darboux integral:

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$$

We say that f is integral on $[a, b]$ if $U(f) = L(f)$. We write this as $\int_a^b f(x)\,dx$

Example 14.1.1: (Non-Integrable) Define

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

on $[a, b]$. We have

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we have

$$U(f, P) = \sum_{k=1}^{n} (t_k - t_{k-1}) = b - a$$

Now for

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

is

$$L(f, P) = \sum_{k=1}^{n} 0 = 0$$

But $U(f) \neq L(f)$.

Lemma: If P, Q are partitions, and $P \subseteq Q$, then $L(f, P) \leqslant L(f, Q) \leqslant U(f, Q) \leqslant U(f, P)$.

Lemma: If P, Q are partitions, then we have $L(f, P) \leqslant U(f, Q)$.

Proof. Take $P \cup Q$ as a partition of $[a, b]$. Then

$$L(f, P) \leqslant L(f, P \cup Q) \leqslant U(f, P \cup Q) \leqslant U(f, Q)$$

□

32.4

Theorem 14.1.1

We have $L(f) \leqslant U(f)$ by the previous lemma.

Theorem 14.1.2

32.5

Assume that f is bounded in $[a, b]$. Then f is integrable iff $\forall \varepsilon > 0$, there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$.

Proof. ($\longleftarrow$) Suppose that $U(f, P) < L(f, P) + \varepsilon$. Then

$$U(f) \leqslant U(f, P) < L(f, P) + \varepsilon \leqslant L(f) + \varepsilon$$

then

$$U(f) \leqslant L(f) \text{ and } L(f) \leqslant U(f)$$

so $U(f) = L(f)$.

($\longrightarrow$) $\forall \varepsilon > 0$, $\exists P_1, P_2$ such that

$$L(f, P_1) > L(f) - \frac{\varepsilon}{2}$$

and similarly,

$$U(f, P_2) < U(f) + \frac{\varepsilon}{2}$$

Take $P = P_1 \cup P_2$. Then

$$U(f, P) - L(f, P) \leqslant U(f, P_2) - L(f, P_1) < U(f) - L(f) + \varepsilon = \varepsilon$$

□

Definition 14.1.2

Mesh

The mesh of $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is

$$\mathrm{mesh}(P) = \max(t_k - t_{k-1} : k = 1, 2, \ldots, n)$$

Theorem 14.1.3

32.7

Assume f is bounded on $[a, b]$. Then f is integrable iff $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for any partition such that

$$\mathrm{mesh}(P) < \delta$$

we have

$$U(f, P) - L(f, P) < \varepsilon$$

Chapter 15

Week 15

15.1 Properties of Integrals

Last Lecture: A partition on $[a, b]$ with $p = \{a = t_0 < t_1 < \cdots < t_n = b\}$. The Darboux sum:

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

and the lower Darboux sum

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

The upper Darboux integral:

$$U(f) = \inf\{U(f, p) : p \text{ is a partition}\}$$

and the lower Darboux integral:

$$L(f) = \sup\{L(f, p) : p \text{ is a partition}\}$$

f is integrable on $[a, b]$ if $U(f) = L(f)$ and we use:

$$\int_a^b f(x)\, dx$$

We had that $mesh(P) = \max(t_k - t_{k-1} : k = 1, 2, \ldots, n)$. Theorem: f is integrable iff $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall P$ with mesh $< \delta$, we had

$$U(f, P) - L(f, P) < \varepsilon$$

Definition 15.1.1 **Riemann Sum**

The Riemann sum of f with P is the sum of the form

$$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

where $x_k \in [t_{k-1}, t_k]$. This is not unique such as $\sup S = U(f, P)$ and $\inf S = L(f, P)$.

f is Riemann integrable if $\exists r \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall \operatorname{mesh}(P) < \delta$, any Riemann sum S of f with P we have

$$|S - r| < \varepsilon$$

We call r is the Riemann integral.

32.9

Theorem 15.1.1

Riemann Integrable iff integrable.

33.1

Theorem 15.1.2

Any monotonic function f on the closed interval $[a, b]$ is integrable

Proof. f is increasing. Then $\forall x \in [a, b]$ we have $f(a) \leqslant f(x) \leqslant f(b)$. So $\forall \varepsilon > 0$ take $\delta = \frac{\varepsilon}{f(b)-f(a)}$. Then $\forall \operatorname{mesh}(P) < \delta$, we want to show that

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]))(t_k - t_{k-1})$$

Using the fact that f is increasing:

$$\begin{aligned}\sum_{k=1}^{n} = (f(t_k) - f(t_{k-1}))(t_k - t_{k-1}) < \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))\delta \\ = (f(b) - f(a))\delta = \varepsilon\end{aligned}$$

□

33.2

Theorem 15.1.3

Any continuous function on $[a, b]$ is integrable.

Proof. We have that f is uniformly continuous. So $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in [a, b]$ and

$$|x - y| < \delta$$

We have

$$|f(x) - f(y)| < \frac{\varepsilon}{b - a}$$

$\forall \operatorname{mesh}(P) < \delta$, we have that since f is continuous on $[t_{k-1}, t_k]$, f achieves its maximum and minimum on the closed interval. We have

$$M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b - a}$$

Then consider:

$$\begin{aligned}U(f, P) - L(f, P) &< \sum_{k=1}^{n} \frac{\varepsilon}{b - a}(t_k - t_{k-1}) \\ &= \frac{\varepsilon}{b - a}(b - a) = \varepsilon\end{aligned}$$

□

33.3

Theorem 15.1.4

Assume f, g are integrable on $[a, b]$ and $c \in \mathbb{R}$. Then

- $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$

- $\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$.

33.4

Theorem 15.1.5

- If f, g are integrable on $[a, b]$ and $f(x) \geqslant g(x)$, then

$$\int_a^b f(x)\, dx \geqslant \int_a^b g(x)\, dx$$

- If g is continuous and $\geqslant 0$ on $[a, b]$ and

$$\int_a^b g(x)\, dx = 0$$

then $g = 0$ on $[a, b]$.

Proof. (Part I) Take $h = f - g \geqslant 0$ on $[a, b]$. Therefore, for any partition P, $L(h, P) \geqslant 0$. So $L(h) \geqslant 0$. So by the previous theorem, h is integrable. So

$$\int_a^b h(x)\, dx = L(h) \geqslant 0$$

and

$$\int_a^b f(x)\, dx \geqslant \int_a^b g(x)\, dx$$

so we are done.

(Part II) Suppose that $g(t) > 0$ for some $t \in [a, b]$. So $\forall \varepsilon = \frac{g(t)}{2}$, $\exists \delta$ such that $\forall x \in [a, b]$ an

$$|x - t| < \delta$$

we have

$$|g(x) - g(t)| < \frac{g(t)}{2}$$

Then $g(x) > \frac{g(t)}{2}$. So $\exists (c, d) \subseteq [a, b]$ such that

$$g(x) > \frac{g(t)}{2} \text{ on } (c, d)$$

So

$$\int_a^b g(x)\, dx \geqslant \int_c^d g(x)\, dx > \int_c^d \frac{g(t)}{2}\, dt = \frac{g(t)}{2}(d - c) > 0$$

which is a contradiction. □

Last Lecture: Riemann sums

$$S = \sum_{k=1}^{n} f(x_k(t_k - t_{k-1}))$$

where $x_k \in [t_{k-1}, t_k]$. A function is Riemann integrable iff f is integrable.

Any monotone function is integrable. Any continuous function is integrable.

Integration is linear operation. If $f \geqslant g$, then integral of f is greater than that of g.

33.5

Theorem 15.1.6

If f is integrable on $[a, b]$ then $|f|$ is integrable and

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$$

Proof. (Part I) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall\, \mathrm{mesh}(P) < \delta$ we have

$$U(f, P) - L(f, P) < \varepsilon$$

Claim: $M(|f|, S) - m(|f|, S) \leqslant M(f, S) - m(f, S)$.

If f is greater than or equal to 0, they are the same. If some part of f is less than 0, $m(|f|, S)$ is greater than $m(f, S)$, positive. So the LHS is less than RHS.

Therefore,

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{k=1}^{n} (M(|f|, [t_{k-1}, t_k]) - m(|f|, [t_{k-1}, t_k]))(t_k - t_{k-1}) \\ &\leqslant \sum_{k=1}^{n} (M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_{rk}]))(t_k - t_{k-1}) \\ &= U(f, P) - L(f, P) < \varepsilon \end{aligned}$$

So $|f|$ is integrable.

(Part II) We have $-|f| \leqslant f \leqslant |f|$. We have

$$-\int_a^b |f(x)|\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b |f(x)|\,dx$$

So

$$\left|\int_a^b |f(x)|\,dx\right| \leqslant \int_a^b |f(x)|\,dx$$

□

33.6

Theorem 15.1.7

If f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$. Also, $\int_a^b f(x)\,dx = \int_a^c f(x)j\,dx + \int_c^b f(x)\,dx$.

32.5

Theorem 15.1.8

A function is integrable iff $\forall \varepsilon > 0, \exists P$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. Since f is integrable on $[a, c]$, $[c, b]$, there exists a P_1 on $[a, c]$ and P_2 on $[c, b]$ such that

$$U_a^c(f, P_1) - L_a^c(f, P_1) < \frac{\varepsilon}{2}$$
$$U_c^b(f, P_2) - L_c^b(f, P_2) < \frac{\varepsilon}{2}$$

Take $P = P_1 \cup P_2$ on $[a, b]$ and $U_a^b(f, P) = U_a^c(f, P_1) + +U_c^b(f, P_2)$, $L_a^b(f, P) = L_a^c(f, P_1) + L_c^b(f, P_2)$. By the previous constraints,

$$U_a^b(f, P) - L_a^b(f, P) < \varepsilon$$

So f is integrable on $[a, b]$. We have

$$\int_a^b f(x)\,dx \leqslant U_a^b(f, P) = U_a^c(f, P_1) + U_c^b(f, P_2)$$
$$< L_a^c(f, P_1 + \frac{\varepsilon}{2}) + L_c^b(f, P_2) + \frac{\varepsilon}{2}$$
$$\leqslant \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon$$

Take ε arbitrarily small:

$$\int_a^b f(x)\,dx \leqslant \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

We have

$$\int_a^b f(x)\,dx \geqslant L_z^b(f, P) = L_a^c(f, P_1) + L_c^b(f, P_2)$$
$$> U_a^c(f, P_1) - \frac{\varepsilon}{2}$$
$$\geqslant \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \varepsilon$$

take ε arbitrarily small:

$$\int_a^b f(x)\,dx \geqslant \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

so they are equal. □

33.7

Definition 15.1.2

f on $[a, b]$ is piecewise monotonic if $\exists P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ such that f is monotonic on each (t_{k-1}, t_k). f is piecewise continuous if $\exists P$ such that f is uniformly continuous on each (t_{k-1}, t_k).

33.8

Theorem 15.1.9

If f is piecewise continuous on $[a, b]$ or it is bounded and piecewise monotonic on $[a, b]$ then f is integrable on $[a, b]$.

Intermediate Value Theorem of Integrals

Theorem 15.1.10

If f is continuous on $[a, b]$, then $\exists x \in (a, b)$ such that $f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Proof. Let $M = \max(f(x), x \in [a, b])$. Let $m = \min(f(x), x \in [a, b])$. If $M = m$, then f is constant. Then for any x, $f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$. If $M \neq m$, then $M \geqslant f \geqslant m$. Take the integral:

$$\int_a^b M\,dx \geqslant \int_a^b f(x)\,dx \geqslant \int_a^b f(x)\,dx$$

and

$$M \geqslant \frac{1}{b-a}\int_a^b f(x)\,dx \geqslant m$$

By IVT, there exists $x \in (a, b)$ such that $f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$. □

Last Lecture: We had that f is integrable then $|f|$ is integrable and that

$$\left|\int_a^b f(x)\ dx\right| \leqslant \int_a^b |f(x)|\ dx$$

Also, if f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and that

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

Suppose that f is piecewise continuous or piecewise monotonic. Then f is integrable.

Theorem 33.9 was the IVT for integrals. If f is continuous on $[a, b]$, there there is $x \in (a, b)$ such that

$$f(x) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

15.2 Fundamental Theorem of Calculus

Theorem 15.2.1

If g is continuous on $[a, b]$, differentiable on (a, b) and g' integrable on $[a, b]$, then

$$\int_a^b g'(x)\ dx = g(b) - g(a)$$

Proof. $\forall \varepsilon > 0$, there is a partition $\{a = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = b\}$. This a partition such that

$$U(g', P) - L(g', P) < \varepsilon$$

For each $[t_{k-1}, t_k]$, by MVT, there exists $x_k \in (t_{k-1}, t_k)$ such that

$$g'(x_k) = \frac{g(t_k - g_{t_{k-1}}}{t_k - t_{k-1}}$$

We have

$$\begin{aligned} g(b) - g(a) &= \sum_{k=1}^{n}(g(t_k) - g(t_{k-1})) \\ &= \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) \end{aligned}$$

So we have $U(g', P) \geqslant g(b) - g(a) \geqslant L(g', P)$. We also have $U(g', P) \geqslant \int_a^b g'(x)\ dx \geqslant L(g', P)$. So

$$\left|\int_a^b g'(x)\ dx - (g(b) - g(a))\right| < \varepsilon$$

Let $\varepsilon \to 0$. We have $\int_a^b g'(x)\ dx = g(b) - g(a)$. □

Integration by Parts

Theorem 15.2.2

If U, V are continuous on $[a, b]$, differentiable on (a, b) and U, V are integrable on $[a, b]$. Then we have

$$\int_a^b U'V + \int_a^b UV' = U(b)V(b) - U(a)V(a)$$

Proof. Let $g = uv, g' = uv + vu$. Then

$$\int_a^b g'(x)\, dx = g(b) - g(a) = u(b)v(b) - u(a)v(a)$$

□

Example 15.2.1: Find $\int_0^\pi x \cos x\, dx$. Let $u = x, v = \sin x, v' = \cos x$. Then

$$\int_0^\pi uv' = u(\pi)v(\pi) - u(0)v(0) - \int_a^b u'v'$$

This is

$$-\int_0^\pi \sin x\, dx = -(-\cos x))\Big|_0^\pi = \cos\pi - \cos 0 = -2$$

Fundamental Theorem of Calc II

Theorem 15.2.3

If f is integrable on $[a, b]$, let

$$F(x) = \int_a^x f(t)\, dt\,,\ \forall x \in [a, b]$$

Then F is continuous. If f is is continuous at $x_0 \in (a, b)$, then F derivative at x_0: $F'(x_0) = f(x_0)$.

Proof. Choose $B > 0$ such that $|f| \leqslant B$. Then $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{B}$. Then $\forall x, y \in [a, b]$ and $|x - y| < \delta$, we have

$$|F(x) - F(y)| = |\int_a^x f(t)\, dt - \int_a^y f(t)\, dt| = |\int_y^x f(t)\, dt|$$

This is less than or equal to $\int_y^x |f(t)|\, dt, x > y$. Since $|f| \leqslant B$, we have

$$\int_y^x |f(t)|\, dt \leqslant \int_y^x B\, dt = (x - y)B < \varepsilon$$

(Part II) Suppose that f is continuous at x_0. Then $\forall \varepsilon > 0$, we have $\exists \delta > 0$ such that $\forall t \in (a, b), |t - x| < \delta$, we have

$$|f(t) - f(x_0)| < \varepsilon$$

So $\forall x \in [a, b]$ and $|x - x_0| < \delta$. We have

$$\begin{aligned}
\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| &= \left|\frac{1}{x - x_0}(\int_a^x f(t)\, dt - \int_a^{x_0} f(t)\, dt) - f(x_0)\right| \\
&= \left|\frac{1}{x - x_0}\int_{x_0}^x f(t)\, dt - \frac{1}{x - x_0}\int_{x_0}^x f(x_0)\, dt\right| \\
&= \left|\frac{1}{x - x_0}\int_{x_0}^x f(t) - f(x_0)\, dt\right| \\
&\leqslant \frac{1}{x - x_0}\int_{x_0}^x |f(t) - f(x_0)|\, dt \\
&< \frac{1}{x - x_0}\int_{x_0}^x \varepsilon\, dt \\
&= \varepsilon
\end{aligned}$$

So $F'(x_0) = f(x_0)$. □

Change of Variables

Theorem 15.2.4

If u is differentiable on an open interval J and $u(J) \subseteq I$, f is continuous on I. Then $f \circ u$ is continuous on I. Also,

$$\int_a^b f \circ u(x) u'(x)\, dx = \int_{u(a)}^{u(b)} f(t)\, dt$$

for $a, b \in J$.

Proof. Fix $c \in I$, let $F(u) = \int_c^a f(t)\, dt$. Then $F'(u) = f(u)$ for $u \in J$. Let $g = F \circ u$. Then $g' = F'(u(x))u'(x)$. Then

$$\int_a^b f(u(x))u'(x)\, dx = \int_a^b g'(x)\, dx$$

and by the First fundamental theorem

$$\begin{aligned}\int_a^b g'(x)\, dx &= g(b) - g(a)\\ &= F(u(b)) - F(u(a))\\ &= \int_{u(a)}^{u(b)} f(t)\, dt\end{aligned}$$

□