Math55Hw14

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7.4: 4, 8, 10, 12

Exercise 4: A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is the expected number of heads that come up when it is flipped 10 times?

Let $X : S \to \mathbb{R}$ where S is the set of all bit-strings of length 10 where

$$X = \text{ number of 1's}$$

If we have a random variable X_i for the i-th element in the bit-string:

$$X_i = \begin{cases} 1 \text{ if the element is 1} \\ 0 \text{ if otherwise} \end{cases}$$

Then $X = X_1 + \cdots + X_{10}$. Therefore,

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10})$$

We compute the expectation for each X_i:

$$\mathbb{E}(X_i) = 1(0.6) = 0.6$$

So

$$\mathbb{E}(X) = 10(0.6) = 6$$

Exercise 8: What is the expected sum of the numbers that appear when three fair dice are rolled?

Let $X : S \to \mathbb{R}$ where $S = \{1, 2, 3, 4, 5, 6\}^3$ and X records the sum of the numbers in each ordered triple. If we have a random variable for each dice X_i for the i-th dice which outputs the number it rolled, then

$$X = X_1 + X_2 + X_3$$

So we can calculate the individual expectations $\mathbb{E}(X_i)$:

$$\mathbb{E}(X_i) = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$$

Therefore, $\mathbb{E}(X) = \frac{7}{2}(3) = \frac{21}{2}$.

Exercise 10: Suppose that we flip a fair coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?

Let $X : S \to \mathbb{R}$ where S is the sample space of the experiment described in the problem and X is the random variable for the number of times the coin is flipped. Notice that we only need to flip a coin twice:

$$X = \begin{cases} 2 \text{ if the first two flips are tails} \\ 6 \text{ if the first two are not both tails} \end{cases}$$

Now compute the expectation of X:

$$\mathbb{E}(X) = 2(.25) + 6(.75) = .5 + 4.5 = 5$$

Exercise 12: Suppose that we roll a fair die until a 6 comes up.

a) What is the probability that we roll the die n times?

The probability that we roll the die n times is the probability that we do not get a 6 on the first $n-1$ rolls but a 6 on the n-th roll. This is

$$p(n) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}$$

b) What is the expected number of times we roll the die?

If $X : S \to \mathbb{R}$ is the number of times the die is rolled for a given result in the experiment, by the formula for expected value, it is

$$\begin{aligned} \mathbb{E}(X) &= \sum_{n=1}^{\infty} n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1} \\ &= \frac{1}{6}\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1} \end{aligned}$$

Let $x = {}^5\!/_6$:

$$\begin{aligned} \frac{1}{6}\sum_{n=1}^{\infty} nx^{n-1} &= \frac{1}{6}(1 + 2x + 3x^2 + 4x^3 + \ldots) \\ S &= (1 + 2x + 3x^2 + 4x^3 + \ldots) \\ Sx &= (x + 2x^2 + 3x^3 + 4x^4 + \ldots) \\ S - Sx &= 1 + x + x^2 + \ldots \\ x(S - Sx) &= x + x^2 + \ldots \\ (1-x)(S - Sx) &= 1 \\ S &= \frac{1}{(1-x)^2} = \frac{1}{({}^1\!/_6)^2} = 36 \end{aligned}$$

Therefore, $\mathbb{E}(X) = \dfrac{36}{6} = 6$.