Extra Problems on Jordan Normal Form

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Exercise 1: Let V be a complex n-dimensional space and let $T \in \mathcal{L}(V)$ be such that $\ker T^{n-3} \neq \ker T^{n-2}$. How many distinct eigenvalues can T have?

Proof. Notice that the kernel of T^{n-2} is non-trivial, as it is a superset of $\ker T^{n-3}$. This means that there is a Jordan Block for the eigenvalue of 0 and additionally, there exists at least one block of size $n-2 \times n-2$. This tells us that for n larger than or equal to 4, there is exactly one Jordan Block of size $n-2 \times n-2$ for the eigenvalue 0 and this means that the maximum number of distinct eigenvalues is 3 as 0 is an eigenvalue and that we can add two more eigenvalues corresponding to linearly independent vectors. This corresponds to a maximum of n eigenvectors. $\square$

Exercise 2: Let V be a complex finite-dimensional vector space and let $T \in \mathcal{L}(V)$ have eigenvalues $-1, 0, 1$. Given the dimensions of the corresponding null spaces below, determine the Jordan normal form of T.

λ	$\dim\ker(T-\lambda I)$	$\dim\ker(T-\lambda I)^2$	$\dim\ker(T-\lambda I)^3$	$\dim\ker(T-\lambda I)^4$	$\dim\ker(T-\lambda I)^5$
-1	3	5	6	6	6
0	2	4	6	7	7
1	3	4	5	5	5

Proof. We see that for the eigenvalue of -1, there are a total of 3 Jordan blocks, since each Jordan block contributes a dimension of 1 to the null space of $T - \lambda I$. For the next column, there is are 2 Jordan blocks with dimension greater than 1, because the Jordan blocks with size 1×1 stabilize at $\ker(T-\lambda I)$ while the blocks with greater dimension do not stabilize. Now for the next column, there is one block that has dimension greater than 2×2 since $6 - 5 = 1$. Repeating the same process, we deduce that there are 2 Jordan blocks for eigenvalue 0, one of which has dimension 3, and the other dimension 4. For eigenvalue 1, there are three Jordan blocks, two having dimenision 1×1 and one having dimension 3×3. We put this in a matrix:

$$\mathrm{JNF}(T) = \left[\begin{array}{cccccccccccccccccc}
-1 \\
& -1 & 1 \\
& 0 & -1 \\
& & & -1 & 1 & 0 \\
& & & 0 & -1 & 1 \\
& & & 0 & 0 & -1 \\
& & & & & & 0 & 1 & 0 \\
& & & & & & 0 & 0 & 1 \\
& & & & & & 0 & 0 & 0 \\
& & & & & & & & & 0 & 1 & 0 & 0 \\
& & & & & & & & & 0 & 0 & 1 & 0 \\
& & & & & & & & & 0 & 0 & 0 & 1 \\
& & & & & & & & & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & 1 & 1 & 0 \\
& & & & & & & & & & & & & & & 0 & 1 & 1 \\
& & & & & & & & & & & & & & & 0 & 0 & 1
\end{array}\right]$$

□

Exercise 3: Let $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{C}))$ be the operator

$$T : f(x) \mapsto f(x-1) + x^3 f'''(x)/3$$

Find the Jordan normal form and a Jordan basis for T.

Proof. We first look at the action of T on our basis vectors $\{1, x, x^2, x^3\}$:

$$\begin{aligned}
T &: 1 \mapsto 1 \\
T &: x \mapsto x - 1 \\
T &: x^2 \mapsto (x-1)^2 \\
T &: x^3 \mapsto (x-1)^3 + 2x^3
\end{aligned}$$

It can be seen that our matrix representation with respect to this basis is

$$T = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

So the Jordan normal form to aim for is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

□

Exercise 4: Let V be a complex (finite-dimensional) vector space and let $T \in \mathcal{L}(V)$. Prove that there exist operators D and N in $\mathcal{L}(v)$ such that $T = D + N$, D is diagonalizable, N is nilpotent, and $DN = ND$.

Proof. Clearly, there exists a Jordan normal form for T, so we have that

$$\text{JNF}(T) = \begin{bmatrix} \lambda_1 & 1 & 0 & & & & & & & \\ 0 & \ddots & 1 & & & & & & & \\ 0 & 0 & \lambda_1 & & & & & & & \\ & & & \lambda_2 & 1 & 0 & & & & \\ & & & 0 & \ddots & 1 & & & & \\ & & & 0 & 0 & \lambda_2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \lambda_n & 1 & 0 \\ & & & & & & & & \lambda_n & 1 \\ & & & & & & & & 0 & \lambda_n \\ & & & & & & & & & \\ & & & & & & & & & \end{bmatrix}$$

which has a diagonal component and 1's on the top. Observe that N is upper triangular with $0'$s on the diagonal. This tells us that the minimal polynomial has only the root 0 where z is the only factor. Therefore, we have that $P(N) = N^k = 0$ for some k. So therefore, N is nilpotent. □

Exercise 5: Suppose the Jordan form of an operator $T \in \mathcal{L}(V)$ consists of Jordan blocks of size 3×3, 4×4, $1 \times q$, 5×5, 2×2, corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_2, \lambda_1$, repectively. Assuming that $\lambda_i \neq \lambda_j$ for $i \neq j$, find the minimal polynomial of T.

Proof. To find the minimal polynomial, we note that we will have factors $(z-\lambda_1), (z-\lambda_2), (z-\lambda_3)$. We choose the multiplicity corresponding the the maximal block size. Therefore, our $p_{\min}$ is

$$p_{\min} = (z - \lambda_1)^3 (z - \lambda_2)^5 (z - \lambda_3)$$

□