Math250aHw6

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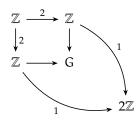
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Exercise 1: Let $G = \langle a, b : a^2 = b^2 \rangle$ be the coproduct in the category of groups of the following diagram

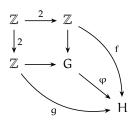
$$\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \\
\downarrow_{2} & & \downarrow \\
\mathbb{Z} & \longrightarrow & G
\end{array}$$

Prove (by exhibiting certain surjections out of the coproduct) that G is nonabelian and infinite.

Proof. We know that it is infinite because there is a surjective mapping to $2\mathbb{Z}$:



Given by a multiplication of $z \in \mathbb{Z}$ by 2, then by 1. Now consider the more general diagram:



We have that $f(2\mathbb{Z}) = g(2\mathbb{Z})$ and $\varphi(a) = f(1)$, $\varphi(b) = g(1)$. So

$$\varphi(a^n) = f(n), \varphi(b^n) = g(n)$$

and by the condition that $f(2\mathbb{Z}) = g(2\mathbb{Z})$, we let $\mathfrak{a},\mathfrak{b}$ be sent to two cycles that don't commute such as $(1\,2),(2\,3)$. These generate D_6 . So for $H=D_6$, there is a surjection which says that G is nonabelian. \square

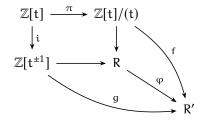
Exercise 2: Let $R = \mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[t]} (\mathbb{Z}[t]/t)$ be the coproduct in the category of rings of the following diagram

$$\begin{array}{ccc} \mathbb{Z}[t] & \longrightarrow & \mathbb{Z}[t]/t \\ \downarrow & & \downarrow \\ \mathbb{Z}[t^{\pm 1}] & \longrightarrow & R \end{array}$$

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Show that R = 0 is the zero ring by showing that it admits no maps to any nonzero ring.

Proof. We observe that with the natural mappings, we must have in the diagram:



with f(1) = ?, and $t \mapsto 0 \mapsto 0$ under f. As for the morphism under g, we have g(1) = ? and $t \mapsto t \mapsto 0$ under g. But by the fact that g is a morphism, we must have $g(t \cdot t^{-1}) = g(1) = 1$. But $g(t \cdot t^{-1}) = g(0) = 0$. Therefore, g(1) = 0. So f(1) = 0 and there are no rings other than the zero ring that accept these maps.

This means that the coproduct exists: For any two mappings $\mathbb{Z}[t]/(t) \to R'$ and $\mathbb{Z}[t^{\pm 1}] \to R'$, there exists a unique map $R = 0 \to R'$ that makes the diagram commute. Furthermore, we see that shown above, that if $1 \neq 0$, there are no maps $R \to R'$ making the diagram commute.

Let $F,G:C\to D$ be two functors. Recall that a natural transformation $\nu:F\to G$ is a collection of morphisms $\nu x:F(X)\to G(X)$ such that for any morphism $g:X\to Y$ the following square commutes

$$F(x) \xrightarrow{\quad vx \quad} G(x)$$

$$\downarrow^{F(g)} \qquad \downarrow^{G(g)}$$

$$F(Y) \xrightarrow{\quad vY \quad} G(Y)$$

Exercise 3: Let G be a group. Let BG be the category with a single object * and morphisms Hom(*,*) = G. Show that $Hom(id_{BG},id_{BG}) = Z(G)$. (In other words, under composition natural transformations from the identity functor to itself form a group, isomorphism to the center of G.)

Proof. Using the definition of natural transformations, we want to find the elements of $Hom(id_{BG}, id_{BG})$ by the morphisms $g_0 : id_{BG}(*) \rightarrow id_{BG}(*)$ making the diagram

$$id_{BG}(*) \xrightarrow{g_0} id_{BG}(*)$$

$$\downarrow id_{BG}(g) \qquad \downarrow id_{BG}(g)$$

$$id_{BG}(*) \xrightarrow{g_0} id_{BG}(*)$$

commute. We can simplify this down to

$$\begin{array}{ccc}
* & \xrightarrow{g_0} & * \\
\downarrow g & & \downarrow g \\
* & \xrightarrow{g_0} & *
\end{array}$$

and find that $g_0g = gg_0$. So the morphisms that send the identity functor to itself making the diagram commute is isomorphic to the elements of G that commute with all $g \in \text{Hom}(*,*)$.

Exercise 4: Let X, Y_i be vector spaces. The set $Hom(X, Y_i)$ is naturally a vector space. Construct a natural map

$$\bigoplus_i \text{Hom}(X,Y_i) \to \text{Hom}(X,\bigoplus_i Y_i)$$

Here $\bigoplus \text{Hom}(X, Y_i)$ is the coproduct in the category of vector spaces. Give an example where this map is not surjective.

Proof. We have mappings from $X \to Y_i$ in each column:

Where the basis vectors of X map to vectors of Y_i . Then we can map this information to:

$$x_1 \mapsto (y_{1_1}, y_{2_1}, \dots, y_{i_1})$$

 $x_2 \mapsto (y_{1_2}, y_{2_2}, \dots, y_{i_2})$
 \vdots
 $x_n \mapsto (y_{1_n}, y_{2_n}, \dots, y_{i_n})$

which is an element in $Hom(X, \bigoplus_i Y_i)$.

Exercise 5: Show that the functor $Ab \rightarrow Group$ from Abelian groups to all groups does not admit a right adjoint.

Proof.