Math128aHw1

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Section 1.1

Exercise 2: Show that the following equations have at least one solution in the given intervals.

(c) $-3\tan(2x) + x = 0$, $[0, 1]$

Answer. Observe that plugging in 0, we get:

$$-3\tan(0) + 0 = 0 + 0 = 0$$

as desired.

(d) $\ln(x) - x^2 + \frac{5}{2}x - 1$, $[\frac{1}{2}, 1]$

Answer. We plug in $\frac{1}{2}$ to get:

$$\ln\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 + \frac{5}{4} - 1 = \ln\left(\frac{1}{2}\right) + 2 - 1 = \ln\left(\frac{1}{2}\right) - 1 < 0$$

Now plug in 1:

$$\ln(1) - 1 + \frac{5}{2} - 1 = 0 - 2 + \frac{5}{2} > 0$$

Because $f(x) = \ln x - x^2 + \frac{5}{2}x - 1$ is a continuous function on the interval $[\frac{1}{2}, 1]$, we know that by the intermediate value theorem, there is an $x \in [\frac{1}{2}, 1]$ such that $f(x) = 0$

Exercise 4: Find intervals containing solutions to the following equations.

(d) $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$.

Answer. Taking the derivative, we get $3x^2 + 4.001x + 4.002$. The determinant

$$b^2 - 4ac = 4.001^2 - 4 * 4.002 * 3 < 0$$

This means the graph is strictly increasing, there is one solution. If we plug in -4 and 0, we get:

$$-4^3 + 4.001 * (-4)^2 + 4.002(-4) + 1.101 < 0$$

So an interval with the solution is $[-4, 0]$.

Exercise 6: Find $\max_{a \leqslant x \leqslant b}(|f(x)|)$ for the following functions and intervals.

(a) $f(x) = 2x/(x^2 + 1)$, $[0, 2]$.

Answer. Find the derivative:

$$f'(x) = \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = \frac{-2(x^2-1)}{x^4+2x^2+1}$$

The function is negative when $-2(x^2-1) < 0$ or when $x > 1, x < -1$. This means that $f(x)$ is increasing on $[0, 1]$, decreasing on $[1, 2]$. So the max is at $x = 1$, $f(1) = 2/2 = 1$

Exercise 14: Let $f(x) = 2x\cos 2x - (x-2)^2$ and $x_0 = 0$.

(a) Find the third Taylor polynomial $P_3(x)$ and use it to approximate $f(0.4)$.

Answer. $f(0) = 0 - 4 = -4$,
$f'(x) = 2\cos 2x - 4x\sin 2x - 2(x-2)$, $f'(0) = 6$,
$f''(x) = -8\sin 2x - 8x\cos 2x - 2$, $f''(0) = -2$
S $f'''(x) = -24\cos 2x + 16x\sin 2x$, $f'''(0) = -24$
So the polynomial is $P_3(x) = -4x^3 - x^2 + 6x - 4$

(b) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$. Compute the actual error.

Answer. The error is given my the next term:

$$\left|\frac{f^4(\varepsilon(x))}{4!}x^4\right|$$

The next derivative is:

$$f^{(4)} = 64\sin 2x + 32x\cos 2x$$

So the error is $(\frac{8}{3}\sin 2\varepsilon(x) + \frac{4}{3}\varepsilon(x)\cos 2\varepsilon(x))x^4$. Since $0 \leqslant \varepsilon(x) \leqslant 0.4$, we just compute the max of the function on that interval. Taking the derivative, we get:

$$\frac{20}{3}\cos 2x - \frac{8}{3}x\sin 2x$$

It is easy to see that this function is positive on the interval because $\cos 2x > .5$ on the interval and $\sin 2x < 1$ on the interval. So $x\sin 2x < .4$, and it follows:

$$.5 * \frac{20}{3} - \frac{8}{3}.4 > 0$$

So the max error is $(.4)^4(\frac{8}{3}\sin .8 + \frac{4}{3} * .4\cos .8)$. The actual error can be obtained by plugging in the numbers:

$$\left|.8\cos .8 - (1.8)^2 - (-4(.4)^3 - (.4)^2 + 6(.4) - 4)\right|$$

(c) Find the fourth Taylor polynomial $P_4(x)$ and use it to approximate $f(0.4)$.

Answer. The fourth derivative was calculated in the previous part:

$$f^{(4)}(x) = 64\sin 2x + 32x\cos 2x$$

so $f^4(0) = 0$ and $P_4(x) = -4x^3 - x^2 + 6x - 4$. Then the approximation of $f(0.4)$ is $-4(.4)^3 - (0.4)^2 + 6(0.4) - 4$.

(d) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_4(0.4)|$. Compute the actual error.

Answer. This is the error as part b because $P_3(x) = P_4(x)$. The upper bound would be a little different. Calculate the derivative of

$$f^{(4)}(x) = 64 \sin 2x + 32x \cos 2x$$

which is

$$f^{(5)}(x) = 160 \cos 2x - 64x \sin 2x$$

So we have to find the max of

$$|160 \cos 2\varepsilon(x) - 64\varepsilon(x) \sin 2\varepsilon(x)|(0.4)^5/5!$$

in the interval $[0, 0.4]$. Taking the derivative:

$$-384 \sin 2x - 64x \cos 2x$$

This is clearly < 0 on the interval, so the max is at $\varepsilon(x) = 0$. This gives the max error of $160 * (0.4)^5/5! = 4 * (0.4)^5/3$

Exercise 26: Prove the Generalized Rolle's Theorem, Theorem 1.10, by verifying the following.

(a) Use Rolle's Theorem to show that $f'(z_i) = 0$ for $n - 1$ numbers in $[a, b]$ with $a \leqslant z_1 < z_2 < \cdots < z_{n-1} \leqslant b$.

Answer. Suppose that $f \in C[a, b]$ is $n - 1$ times differentiable on (a, b) and that $f(x) = 0$ at n distinct points $a < z_1 < z_2 < \cdots < z_{n-1} < b$. Then we can apply Rolle's theorem to each interval $(z_1, z_2), (z_2, z_3), \ldots, (z_{n-2}, z_{n-1})$. Since $f(z_i) = f(z_{i+1})$, by Rolle's theorem there are $n - 1$ distinct x such that $f'(x) = 0$.

(b) Use Rolle's Theorem to show that $f^{(2)}(w_i) = 0$ for $n - 2$ numbers in $[a, b]$ with $z_1 < w_1 < z_2 < w_2 < \cdots < w_{n-2} < z_{n-1} < b$.

Answer. We have that $f'(z_i) = 0$. We apply Rolle's Theorem to the intervals $(z_1, z_2), (z_2, z_3), \ldots, (z_{n-2}, z_{n-1})$. Since $f'(z_i) = f'(z_{i+1})$, by the theorem, there exists $z_i < w_i < z_{i+1}$ such that $f''(w_i) = 0$. Repeated for all intervals, we have at least $n - 2$ unique roots for f'' on the interval (a, b)

(c) Continue the arguments in parts (a) and (b) to show that for each $j = 1, 2, \ldots, n - 1$, there are $n - j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0.

Answer. Base case: $j = 1$. Done in part a.

Inductive case: Suppose that this is true for $j = i$. We will prove this holds for $i + 1 < n - 1$.

Since there are $n - i$ values in the interval $a < w_1 < w_2 < \cdots < w_{n-i} < b$ such that $f^{(i)}(w_k) = 0$, we apply Rolle's Theorem to the intervals $(w_1, w_2), (w_2, w_3), \ldots, (w_{n-i-1}, w_{n-i})$ to get $n - i - 1 = n - (i + 1)$ values $a < w_1 < q_1 < w_2 < q_2 < \cdots < q_{n-i-2} < w_{n-i-1} < q_{n-i-1} < w_{n-i}$ where $f^{i+1}(q_k) = 0$. So we are done.

(d) Show that part (c) implies the conclusion of the theorem.

Answer. Part c tells us that when $j = n - 1$, then there is 1 distinct numbers in $[a, b]$ where $f^{(n-1)}(q) = 0$. This is the statement for Rolle's Theorem for when f is differentiable $n - 1$ times.

Section 1.2

Exercise 2: Compute the absolute error and relative error in approximations of p by p^*.

(c) $p = 8!, p^* = 39900$

Answer. We have $p = 40320$. The absolute error is $p - p^* = 420$. The absolute error is $|p - p^*|/|p| = 0.010416666666667$.

Exercise 4: Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for each value of p.

(b) e

Answer. We have an interval of the form $[e-\delta, e+\delta]$ and such that $|e-p^*|/e \leqslant 10^{-4}$. If $e \geqslant p^*$,

$$e - p^* \leqslant e * 10^{-4}$$

which means the lower bound is $e(1 - 10^{-4})$. If $e \leqslant p^*$, then

$$p^* - e \leqslant e * 10^{-4}$$

which means the upper bound is $e(1 + 10^{-4})$. So the interval is $[e(1 - 10^{-4}), e(1 + 10^{-4})]$.

Exercise 12: The number e can be defined by $e = \sum_{n=0}^{\infty}(1/n!)$, where $n! = n(n-1)\cdots 2\cdot 1$ for $n \neq 0$ and $0! = 1$. Compute the absolute error and relative error in approximations of e:

(a) $\sum_{n=0}^{5} \frac{1}{n!}$

Answer. The absolute error is $\sum_{n=0}^{\infty}(1/n!) - \sum_{n=0}^{5}(1/n!) = \exp(1) - (1 + 1 + (1/2) + (1/6) + (1/24)) = \exp(1) - 2.70833333333333 = 0.00994849512574$. The relative error is $0.00994849512574/\exp(1) = 0.00365984682735$

(b) $\sum_{n=0}^{10} \frac{1}{n!}$

Answer. The absolute error is $\sum_{n=0}^{\infty}(1/n!) - \sum_{n=0}^{10}(1/n!)$ which is

$$\exp(1)-(1+1+(1/2)+(1/6)+(1/24)+(1/120)+(1/720)+(1/5040)+(1/40320)+(1/362880)+(1/3628800))$$

where $(1 + 1 + (1/2) + (1/6) + (1/24) + (1/120) + (1/720) + (1/5040) + (1/40320) + (1/362880) + (1/3628800)) = 2.7182818011464$. Then the absolute error is $\exp(1) - 2.7182818011464 = 2.73126450345e-8$. The relative error is then $2.73126450345e-8/\exp(1) = 9.1042150115e-9$

Exercise 22: The Taylor polynomial for $f(x) = e^x$ is $\sum_{i=0}^{n}(x^i/i!)$. Use the Taylor polynomial of degree nine and three-digit chopping arithmetic to find an approximation to e^{-5} by each of the following methods.

(a) $e^{-5} \approx \sum_{i=0}^{9} \frac{(-5)^i}{i!} = \sum_{i=0}^{9} \frac{(-1)^i 5^i}{i!}$

Answer. We have that

$$\sum_{i=0}^{9} \frac{(-1)^i 5^i}{i!} = -1.82710537919 = -.182710537919 \times 10^1 \rightarrow -0.182 \times 10^1$$

(b) $e^{-5} = \frac{1}{e^5} \approx \frac{1}{\sum_{i=0}^{9} \frac{5^i}{i!}}$

Answer. We have

$$\frac{1}{\sum_{i=0}^{9} \frac{5^i}{i!}} = 0.00695945286365 = 0.657452463635 \times 10^{-2} \rightarrow 0.657 \times 10^{-2}$$

(c) An approximate value of e^{-5} correct to three digits is 6.74×10^{-3}. Which formula, (a) or (b), gives the most accuracy, and why?

Answer. For e^{-5},

$$e^{-5} = 0.00673794699909 = 0.673794699909 \times 10^{-2} \rightarrow 0.673 \times 10^{-2}$$

Method b is more accurate

Section 1.3

Exercise 8: Suppose that $0 < q < p$ and that $\alpha_n = \alpha + O(n^{-p})$.

(a) Show that $\alpha_n = \alpha + O(n^{-q})$.

Answer. Since $\alpha_n = \alpha + O(n^{-p})$, we know that the convergence of α_n is bounded by that of n^{-p}:

$$|\alpha_n - \alpha| \leqslant k|n^{-p}|$$

Since $q < p$, we know that:

$$|n^q| < |n^p| \implies |n^{-p}| < |n^{-q}|$$

putting this in our previous equation, we get:

$$|\alpha_n - \alpha| \leqslant k|n^{-p}| < k|n^{-q}|$$

and therefore, $\alpha_n = \alpha + O(n^{-q})$.

(b) Make a table listing $1/n, 1/n^2, 1/n^3$, and $1/n^4$ for $n = 5, 10, 100$, and 1000 and discuss varying rates of convergence of these sequences as n becomes large.

Answer. Table:

	$1/n$	$1/n^2$	$1/n^3$	$1/n^4$
$n = 5$	1/5	1/25	1/125	1/625
$n = 10$	1/10	1/100	1/1000	1/1000
$n = 100$	1/100	1/10000	1/1000000	1/1000000
$n = 1000$	1/1000	1/10000000	1/1000000000	1/1000000000000

As n becomes large, the sequences with higher exponent in the denominator converge faster.

Exercise 15:

(a) How many multiplications and additions are required to determine a sum of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{i} a_i b_j?$$

Answer. The number of multiplications is $1 + 2 + 3 + 4 + \cdots + n$. This gives us $n(n+1)/2$ multiplications. The number of additions is $(n(n+1)/2) - 1$ because there are $n(n+1)/2$ terms after multiplication.

(b) Modify the sum in part (a) to an equivalent form that reduces the number of computations.

Answer. We can pull the a_i out:

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{i} b_j$$

This gives us n multiplications and $n + (0 + 1 + 2 + \cdots n - 1)$ additions, or just $1 + 2 + \cdots n = (n+1)n/2$ additions.

Exercise 2: Construct an algorithm that has as input an integer $n \geqslant 1$, numbers $x_0, x_1, \ldots, x_n$, and a number x and that produces as output the product $(x - x_0)(x - x_1) \cdots (x - x_n)$.

Answer. Algorithm:

```
def f(roots, x):
    return prod([(x - r) for r in roots])
```

Section 2.1

Exercise 6: Use the Bisection method to find solutions, accurate to within 10^{-5} for the following problems.

(d) $x + 1 - 2\sin \pi x = 0$ for $0 \leqslant x \leqslant 0.5$ and $0.5 \leqslant x \leqslant 1$.

Answer. Here is my bisection.m file:

```
function p = bisection(f, a, b, t)

while 1
    p = (a + b) / 2;
    if abs(f(p)) < t, break; end
    if f(a) * f(p) > 0
        a = p;
    else
        b = p;
    end
end

end
```

And my function:

```
function y = myfunc(x)
y = x - tan(x);
end
```

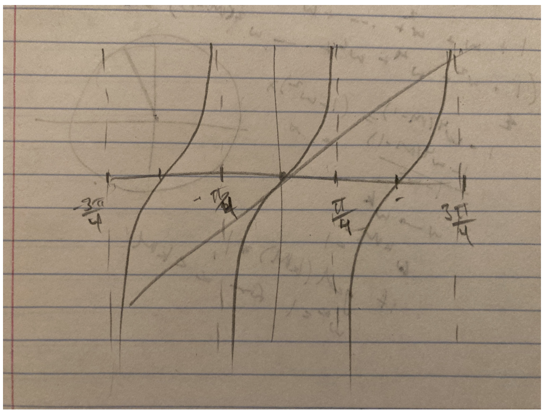
And the script that is ran:

```
a = vpa(bisection(@myfunc, 0, 0.5, 0.00001));
```

The output given is $x = 0.2060356140136718 75$.

Exercise 8:

(a) Sketch the graphs of $y = x$ and $y = \tan x$.



(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = \tan x$.

Answer. Here is my bisection.m file:

```
function p = bisection(f, a, b, t)

while 1
    p = (a + b) / 2;
    if abs(f(p)) < t, break; end
    if f(a) * f(p) > 0
        a = p;
    else
        b = p;
    end
end

end
```

And my function:

```
function y = myfunc(x)
y = x - tan(x);
end
```

And the script that is ran:

```
a = vpa(bisection(@myfunc, 0, 0.5, 0.00001));
```

The output given is $x = 0.015625$.

Exercise 20: Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1+1/n$. Show that $|f(p_n)| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

Answer. We have that $f(p_n) = 1/n^{10} = n^{-10}$. Since $n > 1$, $n^{10} > n^3$ and therefore, $n^{-3} > n^{-10}$. So $|f(p_n)| = n^{-10} < 10^{-3}$. Now we want to see when

$$|1/n| < 10^{-3}$$

This is true when

$$1/n < 10^{-3} \text{ or } -1/n > 10^{-3}$$

This gives:

$$1/10^{-3} < n \text{ or } -1/10^{-3} > n$$

so:

$$n > 1000 \text{ or } n < -1000$$