Math55Hw12

Trustin Nguyen

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7.1: 12, 14, 18, 36, 44

Exercise 12: What is the probability that a five-card poker hand contains exactly one ace?

Let $D = \{1, 2, \ldots, 52\}$ be the set of 52 distinct cards. Let $1, 2, 3, 4$ be the four aces.

Experiment	Draw 5 cards without replacement from a deck of 52 cards
Sample Space	$S = \{H \subseteq D : \lvert H\rvert = 5\}$
Event	$E = \{H : 1 \in H, 2 \in H, 3 \in H, \text{ or } 4 \in H\}$

There are 4 ways to choose an ace, then $\binom{48}{4}$ ways to choose the remaining cards. Therefore,

$$|E| = 4\binom{48}{4}.$$

We have also have

$$|S| = \binom{52}{5}$$

Therefore,

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{4\binom{48}{4}}{\binom{52}{5}}$$

Exercise 14: What is the probability that a five-card poker hand contains cards of five different kinds?

Let

$$\begin{aligned} D_1 &= \{\text{cards that are numbered } 1\} \\ D_2 &= \{\text{cards that are numbered } 2\} \\ &\vdots \\ D_{13} &= \{\text{cards that are numbered } 13\} \end{aligned}$$

Observe that the deck of 52 cards is $D_1 \cup D_2 \cup \ldots \cup D_{13}$

Experiment	Draw 5 cards from a deck of 52 cards
Sample Space	$S = \{H \subseteq D : \lvert H\rvert = 5\}$
Event	$E = \{\{d_{i_1}, \ldots, d_{i_5}\} : i_k$ are distinct and $i_k \in \{1, \ldots, 13\}\}$

To count the $|E|$, notice that there are $\binom{13}{5}$ ways to choose which a subset of 5 numbers. Then there are $\binom{4}{1}$ ways to choose the suit for each one. So

$$\begin{aligned} |S| &= \binom{52}{5} \\ |E| &= 4\binom{13}{5} \end{aligned}$$

We have everything to calculate probability:

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{4\binom{13}{5}}{\binom{52}{5}}$$

Exercise 18: What is the probability that a five-card poker hand contains a straight flush, that is five cards of the same suit of consecutive kinds?

Let

$$\begin{aligned} H &= \{\text{set of hearts}\} \\ D &= \{\text{set of diamonds}\} \\ C &= \{\text{set of clubs}\} \\ S &= \{\text{set of spades}\} \end{aligned}$$

Experiment	Draw 5 cards from a deck of 52 cards.
Sample Space	$T = \{A \subseteq H \cup D \cup C \cup S : \lvert A\rvert = 5\}$
Event	$E = \{\{c_1, \ldots, c_5\} : c_1, \ldots, c_5$ consecutive and same suit$\}$

Observe that every flush is defined by its smallest numbered card. Therefore, if we have cards from $1, \ldots, 13$, we can have a straight flush described by $1, \ldots, 10$.

There are $\binom{10}{1}$ ways to pick the smallest card and $\binom{4}{1}$ ways to pick the suit. Therefore,

$$|T| = \binom{52}{5}$$
$$|E| = 4\binom{10}{1}$$

Now to calculate probability:

$$\mathbb{P}(E) = \frac{|E|}{|T|} = \frac{40}{\binom{52}{5}}$$

Exercise 36: Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

We start with calculating the probability for 2 dice:

Experiment	Roll two dice and record their numbers
Sample Space	$S = \{(d_1, d_2) : 1 \leq d_1, d_2 \leq 6\}$
Event	$E = \{D \subseteq S : d_1 + d_2 = 9\}$

Take note of all possible pairs that add to 8:

$$(2, 6) \quad (3, 5) \quad (4, 4)$$

We can reverse the order of each pair of numbers to get another pair that adds to 8 with the exception for $(4, 4)$. So

$$|S| = 6^2$$
$$|E| = 5$$
$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{5}{36}$$

Now for the probability for 3 dice:

Experiment	Roll three dice and record their numbers
Sample Space	$S = \{(d_1, d_2, d_3) : 1 \leq d_1, d_2, d_3 \leq 6\}$
Event	$E = \{D \subseteq S : d_1 + d_2 + d_3 = 8\}$

Let there be 8 1's. The sum of 1's that each dice holds will be its number that it rolled into. We will count E by giving each dice a 1 first. There are 5 1's left to distribute. So there are $\binom{5+2}{2} = 21$ ways to get an 8. So

$$|S| = 6^3$$
$$|E| = 21$$
$$\mathbb{P}(E) = \frac{21}{6^3}$$

Since the first probability is greater, it is more likely to get an 8 with two dice than three.

Exercise 44: Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, opens a losing door and gives you a chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

Experiment, Sample Space, Event:

Experiment	You pick a door, Monty picks a door, You stay
Sample Space	$S = \{\{\text{car, you, Monty}\} : \text{car, you, Monty} \in \{1,2,3,4\}$ and Monty $\neq$ you, car$\}$
Event	$E = \{C \in S : \text{you} = \text{car}\}$

We count S by first choosing Monty's door. We remove an element from the set $\{1,2,3,4\}$ and assign one of the three remaining numbers to you and car: $\binom{4}{1}\binom{3}{1}\binom{3}{1} = 36$.

We count S by first choosing Monty's door. We remove an element from the set $\{1,2,3,4\}$ and assign one of the three remaining numbers to you. We assign the same number to car: $\binom{4}{1}\binom{3}{1} = 12$. We calculate the probability of each $e \in E$: $1/4 \cdot 1/4 \cdot 1/3 = 1/48$

Therefore:

$$\mathbb{P}(E) = \sum_{e \in E} \frac{1}{48} = \frac{1}{4}$$

7.2: 8abc

Exercise 8abc: What is the probability of these events when we randomly select a permutation of $\{1,2,\ldots,n\}$ where $n \geq 4$?

a) 1 precedes 2

Experiment	Create permutations of $\{1,2,\ldots,n\}$
Sample Space	$S = \{(a_1, a_2, \ldots, a_n) : a_i \in \{1,2,\ldots,n\},$ $i \neq j \to a_i \neq a_j\}$
Event	$E = \{C \in S : a_i = 1, a_j = 2, i < j\}$

There is a bijection between the number of events where 1 precedes 2 and the number of events where 2 precedes 1. We establish this bijection by switching the positions of 1 and 2. So

$$\begin{aligned} E \cup \bar{E} &= S \\ |S| &= n! \\ |E| &= \frac{n!}{2} \\ \mathbb{P}(E) = \frac{|E|}{|S|} &= \frac{1}{2} \end{aligned}$$

b) 2 precedes 1

By the last item, it is $\frac{1}{2}$.

c) 1 immediately precedes 2

Experiment	Create permutations of $\{1, 2, \ldots, n\}$
Sample Space	$S = \{(a_1, a_2, \ldots, a_n) : a_i \in \{1, 2, \ldots, n\},$ $i \neq j \to a_i \neq a_j\}$
Event	$E = \{C \in S : a_i = 1, a_{i+1} = 2\}$

To count E, we stick 1 and 2 together as an element: 1-2. Now we count the number of permutations of $\{1-2, 3, 4, \ldots, n\} = (n-1)!$.

$$\begin{aligned} |S| &= n! \\ |E| &= (n-1)! \\ \mathbb{P}(E) = \frac{|E|}{|S|} &= \frac{(n-1)!}{n!} = \frac{1}{n} \end{aligned}$$