Math143Hw9

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Exercise 1: Homogeneous polynomials. Assume k is an infinite field.

(a) Let $f \in k[x_1, \dots, x_{n+1}]$ and write $f = f_0 + f_1 + \dots + f_d$ where f_i is homogeneous of degree i. Prove that if $f(\lambda a_1, \dots, \lambda a_{n+1}) = 0$ for all non-zero scalars $\lambda \in k^\times$, then $f_i(a_1, \dots, a_{n+1}) = 0$ for all $i = 0, \dots, d$.

Proof. Let $p = (a_1, \dots, a_{n+1})$. We have

$$f(p) = f_0(p) + \lambda f_1(p) + \lambda^2 f_2(p) + \dots + \lambda^d f_d(p) = 0$$

Now each of the $f_i(p) \in k$. So we can think of this as a polynomial in $k[\lambda]$, where λ is a variable:

$$k_0 + k_1\lambda + k_2\lambda^2 + \dots + k_d\lambda^d = 0$$

But we know that the vanishing of a polynomial that is not a line must intersect the line at most $\deg f$ times by some problem from Homework 1. But it is 0 at infinitely many $\lambda \in k$. So we know all the coefficients must be 0. But since $0 = k_i = f_i(p)$, we have as desired. □

(b) Conclude that if $Y \subseteq \mathbb{A}^{n+1}$ is a cone, then $I(Y) \subseteq k[x_1, \dots, x_{n+1}]$ is homogeneous ideal.

Proof. If $Y \subseteq \mathbb{A}^{n+1}$ is a cone and $f \in I(Y)$, then we have for a $p \in Y$, $\lambda p \in Y$ also. Then

$$f = f_0 + f_1 + \dots + f_d$$

and since $f \in I(Y)$, $f(\lambda p) = 0$ for any $\lambda \in k$. Therefore, by the previous problem, $f_i(p) = 0$, then that means that each of the $f_i \in I(Y)$. So we have shown that if we have a polynomial of any degree, it can be written as a sum of homogeneous polynomials of smaller degree each in the ideal. So homogeneous polynomials generate the ideal. □

Exercise 2: The projective plane $\mathbb{P}^2$. Let U_1, U_2, U_3 be the affine charts on $\mathbb{P}^2$.

(a) Let L_i be the complement of U_i. Find the intersections $L_1 \cap L_2, L_1 \cap L_3, L_2 \cap L_3$. Let $k = \mathbb{R}$ and draw a picture of three lines that meet in this way; label the lines and intersections points.

Proof. We have

$$\begin{aligned} L_1 &= \{[0 : y : z] \in \mathbb{P}^2\} \\ L_2 &= \{[x : 0 : z] \in \mathbb{P}^2\} \\ L_3 &= \{[x : y : 0] \in \mathbb{P}^2\} \end{aligned}$$

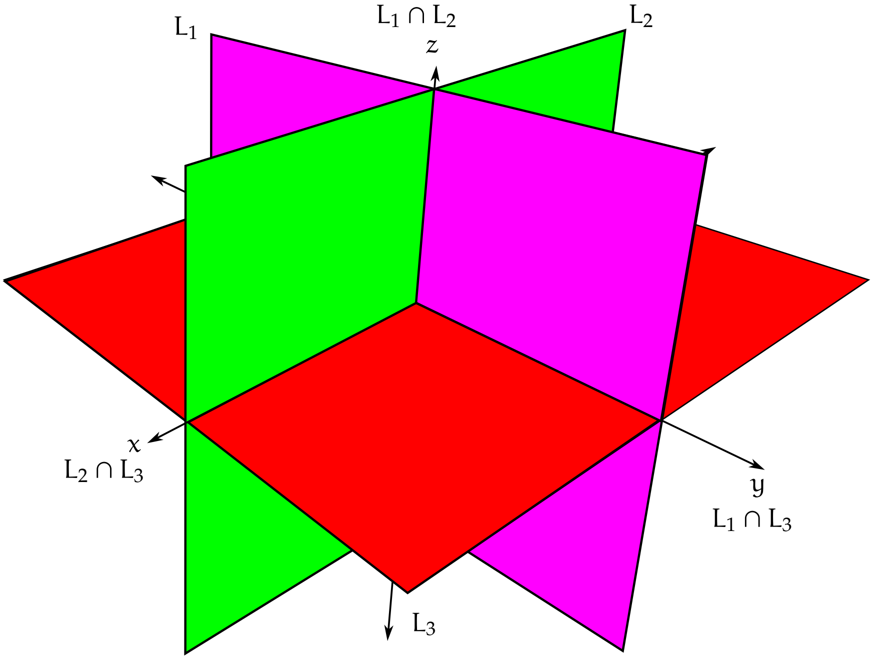
Then we see that:

$$\begin{aligned} L_1 \cap L_2 &= \{[0 : 0 : z] \in \mathbb{P}^2\} \\ L_2 \cap L_3 &= \{[x : 0 : 0] \in \mathbb{P}^2\} \\ L_3 \cap L_1 &= \{[0 : y : 0] \in \mathbb{P}^2\} \end{aligned}$$

Or

$$\begin{aligned} L_1 \cap L_2 &= \{[0:0:1]\} \\ L_2 \cap L_3 &= \{[1:0:0]\} \\ L_3 \cap L_1 &= \{[0:1:0]\} \end{aligned}$$

As for the projective lines that intersect like this, we can take:



□

(b) Which points in $\mathbb{P}^2$ belong to all three affine charts?

Answer. These are the points where none of the coordinates are 0:

$$\{[x:y:z] \in \mathbb{P}^2 : x, y, z \neq 0\}$$

(c) Which points in $\mathbb{P}^2$ belong to only one affine chart?

Proof. Suppose that a point is in U_1, but not U_2, U_3 wlog. Then it must be in the complement of U_2, U_3, or in $L_2 \cap L_3$. Since it is also in U_1, we have that these are the points in

$$U_1 \cap L_2 \cap L_3 = \{[x:0:0] \in \mathbb{P}^2\} = L_2 \cap L_3$$

since $[0:0:0] \notin \mathbb{P}^2$. Then we just take the union of these sets:

$$(L_1 \cap L_2) \cup (L_2 \cap L_3) \cup (L_3 \cap L_1)$$

which is:

$$\{[1:0:0], [0:1:0], [0:0:1]\}$$

which is the answer. □

Exercise 3: Consider $V(x - y^2) \subseteq \mathbb{A}^2$. Let $k = \mathbb{R}$ or $\mathbb{C}$.

(a) What is the corresponding projective algebraic set in $\mathbb{P}^2$? Write your answer as $\mathbb{V}(F)$ for $F \in k[x, y, z]$.

Proof. We have:

$$V(x - y^2) = \{(x, y) \in k^2 : x = y^2\}$$

and we have the mapping $(x, y) \mapsto [x : y : 1]$. And so the set is:

$$\mathbb{V}(F) = \{[\lambda x : \lambda y : \lambda] : \lambda x = y^2\} = \{[x : y : z] : zx = y^2\}$$

So it is $\mathbb{V}(y^2 - xz)$. □

(b) What is the intersection of the set in part (a) with the line at infinity? (Recall the line at infinity is $\mathbb{V}(z)$)

Proof. We have

$$\mathbb{V}(z) = \{[x : y : z] : z = 0\}$$

and

$$\mathbb{V}(y^2 - xz) = \{[x : y : z] : y^2 = xz\}$$

Then the intersection is

$$\mathbb{V}(z) \cap \mathbb{V}(F) = \{[x : y : z] : y^2 = 0\} = \{[x : 0 : 0]\} = \{[1 : 0 : 0]\}$$

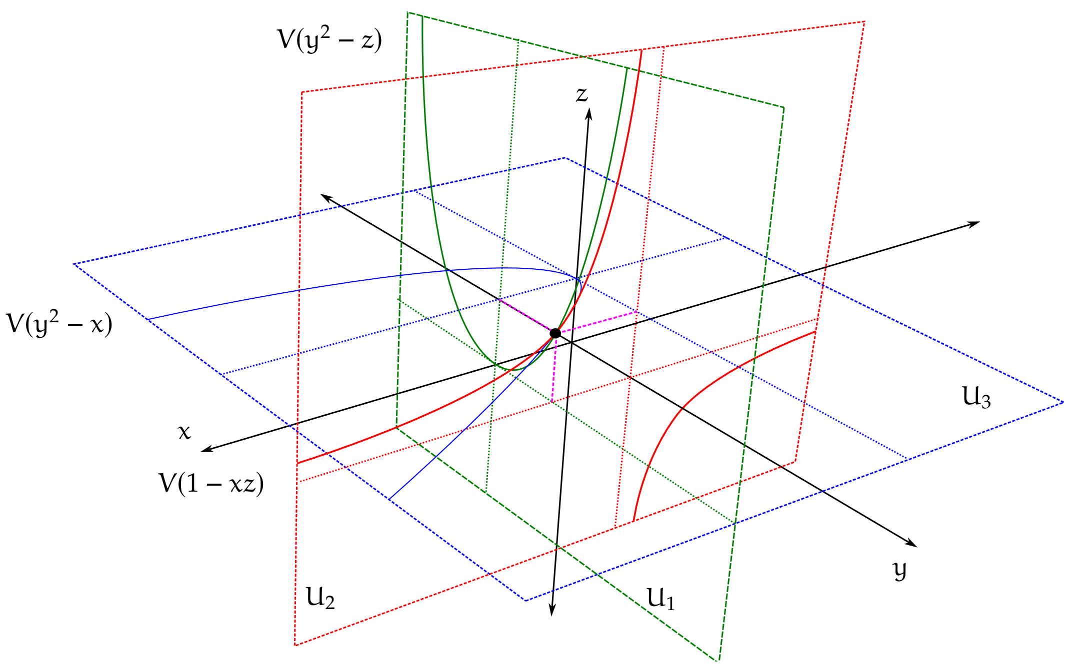
which is the intersection at infinity. □

(c) Describe the intersection of the set in part (a) with each of the affine charts U_1, U_2, U_3. Draw a picture for each. Also draw $\mathbb{V}(z) \cap U_i$ in each.

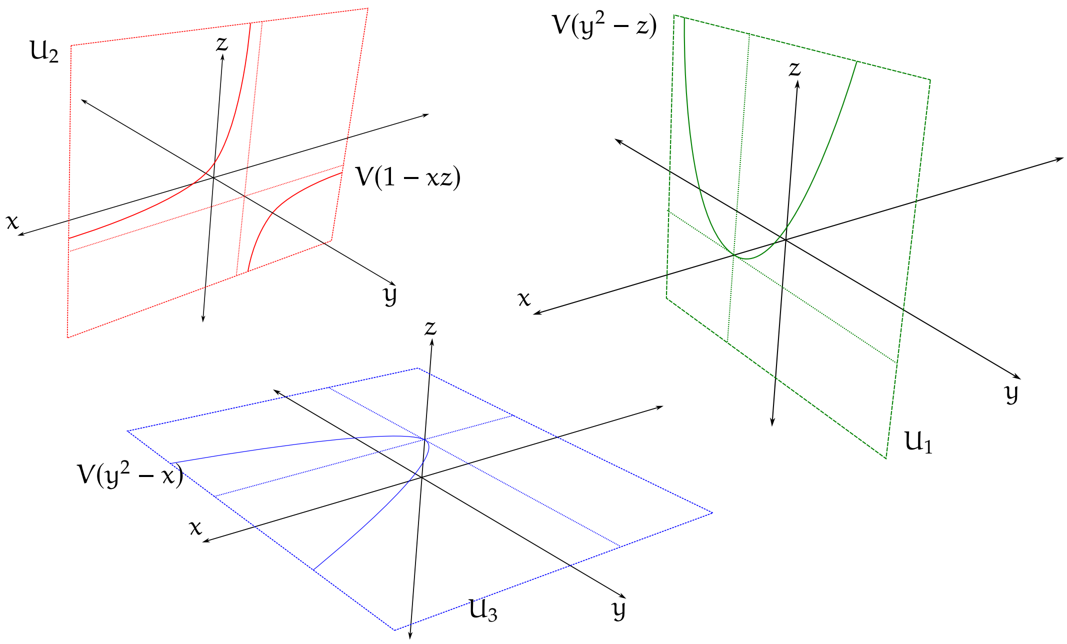
Proof. For each of the intersections, we just plug in $x = 1, y = 1$, or $z = 1$. So:

$$\begin{aligned}
\mathbb{V}(y^2 - xz) \cap U_1 &= V(y^2 - z) \\
\mathbb{V}(y^2 - xz) \cap U_2 &= V(1 - xz) \\
\mathbb{V}(y^2 - xz) \cap U_3 &= V(y^2 - x)
\end{aligned}$$

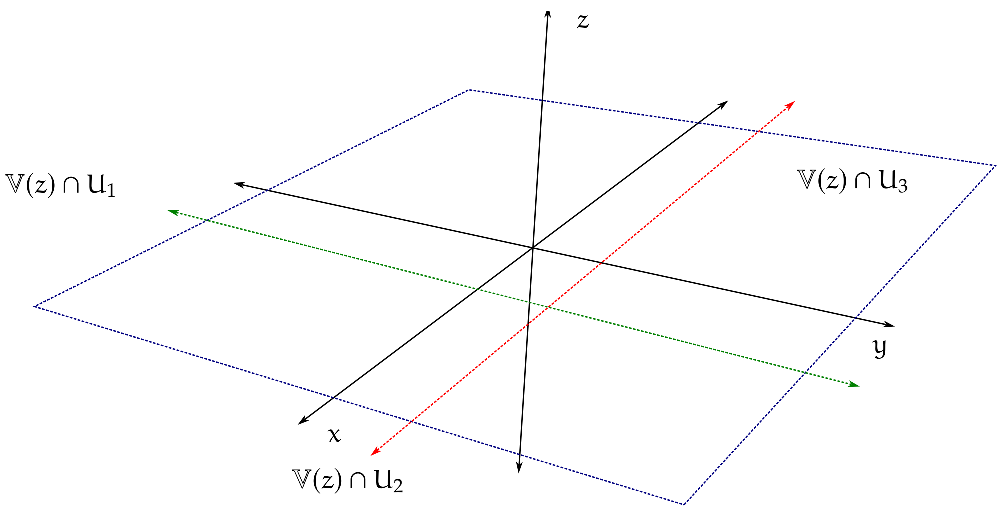
The first and third intersections are parabolas. The second intersection looks like a rotated hyperbola. Figures below:



and each individually:



And for $\mathbb{V}(z) \cap U_i$:



where $\mathbb{V}(z) \cap U_1$ is the line $V(x-1, z)$, $\mathbb{V}(z) \cap U_2$ is the line $V(y-1, z)$ and $\mathbb{V}(z) \cap U_3$ is the plane $V(z)$ $\square$

Exercise 4: Homogeneous ideals. Let I be a homogeneous ideal in $k[x_1, \ldots, x_{n+1}]$.

(a) Show that I is prime if and only if the following condition is satisfied. For every pair of homogeneous polynomials F and G, if $FG \in I$, then $F \in I$ or $G \in I$. (In other words, it is enough to check the usual condition just for homogeneous polynomials.)

Proof. ($\rightarrow$) If I is prime, then if we have $FG \in I$, by definition, either $F \in I$ or $G \in I$.

($\leftarrow$) Now suppose that if $FG \in I$ for homogeneous polynomials F, G, then $F \in I$ or $G \in I$. Let f be a polynomial of degree d and g be one of degree h. Then

$$g = g_0 + g_1 + g_2 + \cdots + g_h$$
$$f = f_0 + f_1 + f_2 + \cdots + f_d$$

and their product:

$$fg = \sum_{i=0}^{d} \sum_{j=0}^{h} f_i g_j$$

Now each homogeneous form in the sum above must lie in I because I is a homogeneous ideal. Then we take homogeneous form with the lowest degree. This is

$$f_0 g_0 \in I$$

Then either f_0 or g_0 is in I. Suppose wlog that $f_0 \in I$. Then we can say that the modified sum:

$$(f - f_0)g = \sum_{i=1}^{d} \sum_{j=0}^{h} f_i g_j$$

is in I also by subtracting off each factor that contains f_0. We continue this process. In each step of the process, we claim that the homogeneous form of the lowest degree will be exactly one term of the form $f_i g_j$. We see this because if we have a modified product of the form:

$$\sum_{i=k_1}^{d} \sum_{j=k_2}^{h} f_i g_j$$

Then the lowest degree will be $k_1 k_2$. But that corresponds to exactly the $f_{k_1} g_{k_2}$ term. Repeating this process, one of the big summations will eventually disappear. But that will mean that we have concluded that $f_0, \ldots, f_d \in I$ or $g_0, \ldots, g_h \in I$, to which we conclude that either $f \in I$ or $g \in I$. This finishes the proof. $\square$

(b) Show that the radical of I is also a homogeneous ideal.

Proof. Suppose I is homogeneous. We need to show that $\sqrt{I}$ is homogeneous also. Let

$$f = f_0 + f_1 + f_2 + \cdots + f_d$$

Then we have:

$$f^m = \sum_{i_1=0}^{d} \sum_{i_2=0}^{d} \cdots \sum_{i_m=0}^{d} f_{i_1} f_{i_2} \cdots f_{i_m}$$

Suppose that $f^m \in I$. Then we know that $f \in \sqrt{I}$. We want to prove that as a result, each $f_i \in \sqrt{I}$, making $\sqrt{I}$ homogeneous.

Consider the lowest degree homogeneous form. This is f_0^m which is in I because I is homogeneous. Then we know that $f_0 \in \sqrt{I}$. This means that we can eliminate all terms with f_0 and the result will still be in $\sqrt{I}$. So:

$$f' = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_m=1}^{d} f_{i_1} f_{i_2} \cdots f_{i_m} \in \sqrt{I}$$

Then we take the next homogeneous form of lowest degree. This will be f_1^m which might not lie in I. But we do know that $f' \in \sqrt{I}$, so $f'^{m'} \in I$ for some m' (*see after proof). Then we know that $f_1^{mm'} \in I$. So $f_1 \in \sqrt{I}$. This process repeats until we find that all $f_i \in \sqrt{I}$, so $\sqrt{I}$ is homogeneous also.

(*) Quick proof of this if needed. This is because if $f = \sum a_i \in \sqrt{I}$, we know that $\sqrt{I}$ is generated by the base powers of polynomials in I. So we have $a_1^{i_1}, a_2^{i_2}, \ldots, a_n^{i_n} \in I$. Then we have $(a_1 + \cdots + a_n)^{\max(i_1,\ldots,i_n)} \in I$ and therefore, $f^{m'} \in I$ for some power m'. $\square$

Exercise 5: (Spooky Halloween trick?) Let $f \in \mathbb{R}[x,y]$ be an irreducible degree 2 polynomial, and let $F \in \mathbb{R}[x,y,z] \subseteq \mathbb{C}[x,y,z]$ be its homogenization of degree 2. Prove that $V(f) \subseteq \mathbb{A}^2_{\mathbb{R}}$ is a circle (its center could be anywhere) if and only if $\mathbb{V}(F) \subseteq \mathbb{P}^2_{\mathbb{C}}$ meets the line at infinity in $\{[1:i:0],[1:-i:0]\}$ and $V(f) \subseteq \mathbb{A}^2_{\mathbb{R}}$. (These points are classically called the "circular points at infinity.")

Proof. ($\longrightarrow$) Suppose that $V(f)$ is a circle of radius r centered at (a,b). Then we know that f is of the form:

$$(x-a)^2 + (y-b)^2 = r^2$$

or $f = (x-a)^2 + (y-b)^2 - r^2$. The we expand:

$$\begin{aligned} f &= (x-a)^2 + (y-b)^2 - r^2 \\ &= x^2 - 2ax + a^2 + y^2 - 2by + b^2 - r^2 \\ F &= x^2 - 2axz + a^2z^2 + y^2 - 2byz + b^2 - r^2z^2 \end{aligned}$$

Then we find its intersection with the hyperplane at infinity, or $\mathbb{V}(z)$. Then we take F and plug in 0 for z to get:

$$F(x,y,0) = x^2 + y^2$$

Take the vanishing:

$$\begin{aligned} 0 &= x^2 + y^2 \\ &= (x+iy)(x-iy) \end{aligned}$$

Now fix $x = 1$, since we are in projective space, only scaling matters. Then $y = \pm i$. So $\mathbb{V}(F)$ meets the line at infinity at

$$\{[1:i:0],[1:-i:0]\}$$

($\longleftarrow$) We first start by taking the ideal of

$$\{[1:i:0],[1:-i:0]\}$$

which is

$$F(x,y,0) \in ((x-iy)(x+iy)) = (x^2+y^2)$$

Then we see that

$$F(x,y,z) = a_1x^2 + a_1y^2 + a_2xz + a_3yz + a_4z^2$$

and dehomogenize by plugging in $z = 1$:

$$F(x,y,1) = a_1x^2 + a_1y^2 + a_2x + a_3y + a_4 = f(x,y)$$

Now we want to see the vanishing of f:

$$\{(x,y) : x^2 + y^2 + a_1x + a_2y + a_3 = 0\}$$

Solving by perfect squares, we get:

$$\left(x - \frac{z_1}{2}\right)^2 + \left(y - \frac{z_2}{2}\right)^2 + r = 0$$

Observe that $r \leqslant 0$, otherwise, we see that $(i\sqrt{r} + \frac{z_1}{2}, \frac{z_2}{2})$ is a solution that lies in $\mathbb{C}^2$which is not what we want. Then we have

$$\left(x - \frac{z_1}{2}\right)^2 + \left(y - \frac{z_2}{2}\right)^2 = (\sqrt{-r})^2$$

So the vanishing of f is a circle. $\square$