Exercise 2

January 21, 2025

Problem 1.

Define the function $f\colon \mathbb{R}^3 \to \mathbb{R}$,

$$f(x,y,z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9.$$

a) Assess whether this function is (strictly) convex.

b) Find all local and global minimisers of f.

Problem 2.

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a convex function.

a) Show that the set of minimisers of f is a convex set.

b) Assume that f is strictly convex. Show that the problem $\min_{x\in\mathbb{R}^d} f(x)$ has at most one global solution. In addition, find a strictly convex function f that has no global minimiser at all.

Problem 3.

Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \log\bigl(e^x + e^y\bigr)$$

is convex.

Problem 4.

Assume that $f\colon \mathbb{R}^d \to \mathbb{R}$ is a function satisfying $f(x) > 0$ for all $x \in \mathbb{R}^d$ and define $g\colon \mathbb{R}^d \to \mathbb{R}$, $g(x) := \log(f(x))$. Assume that g is convex. Show that f is then convex as well.

Problem 5.

a) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ (see Exercise 1, Problem 3b),

$$f(x,y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y.$$

Perform one step of the gradient descent method with backtracking (Armijo) line search starting from the point $x_0 = (0,0)$. Start with an initial step length $\alpha = 1$ and use the parameters $c = 0.1$ (sufficient decrease parameter) and $\rho = 0.1$ (contraction factor).

b) Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = x^4y^2 + x^4 - 2x^3y - 2x^2y - x^2 + 2x + 2.$$

Perform one step of the gradient descent method with backtracking (Armijo) line search starting from the point $x_0 = (0, 0)$. Start with an initial step length $\alpha = \frac{1}{2}$ and use the parameters $c = \frac{1}{2}$ (sufficient decrease parameter) and $\rho = 0.1$ (contraction factor).

Problem 6.

Assume that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is generated by the gradient descent method with backtracking (Armijo) line search for the minimisation of a continuously differentiable function $f \colon \mathbb{R}^d \to \mathbb{R}$, and that $\nabla f(x_k) \neq 0$ for all k. Moreover, assume that $\bar{x}$ is an accumulation point[1] of the sequence $\{x_k\}_{k\in\mathbb{N}}$. Show that $\bar{x}$ is not a local maximum of f.

Remark: It is in theory possible, if highly unlikely, that one of the iterates in the gradient descent method turns out to be a critical point, in which case the iteration terminates. If this is not the case, then this result shows that the gradient descent method at least will not converge to a maximum of f.

Problem 7.

We say that a function $f \colon \mathbb{R}^d \to \mathbb{R}$ is *quasi-convex*, if for every $\alpha \in \mathbb{R}$, the level set $L_f(\alpha)$ is convex.

a) Show that every convex function is quasi-convex.

b) Find a quasi-convex function that is not convex.

c) Show that a function $f \colon \mathbb{R}^d \to \mathbb{R}$ is quasi-convex, if and only if for all $x, y \in \mathbb{R}^d$ and $0 < \lambda < 1$ we have that

$$f\big(\lambda x + (1 - \lambda)y\big) \leq \max\big\{f(x), f(y)\big\}.$$

d) Show by means of an example that a local minimum of a quasi-convex function need not be a global minimum.

Hint: Consider a function on $\mathbb{R}$ that is locally constant.

[1] Recall that $\bar{x}$ is an accumulation point of the sequence $\{x_k\}_{k\in\mathbb{N}}$, if there exists a subsequence $\{x_{k'}\}_{k'}$ with $\bar{x} = \lim_{k'} x_{k'}$.