# Exercise 1

January 14, 2025

**Problem 1.**

Assume that $f, g\colon \mathbb{R}^d \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ are extended real valued functions and denote by $L_f(\alpha)$, $L_g(\alpha)$ for $\alpha \in \bar{\mathbb{R}}$ the level sets of $f$ and $g$, respectively.

a) Show that $L_f(\alpha) \subset L_f(\beta)$ whenever $-\infty \leq \alpha < \beta \leq +\infty$.

b) Assume that $f(x) \geq g(x)$ for all $x \in \mathbb{R}^d$. Show that $L_f(\alpha) \subset L_g(\alpha)$ for all $\alpha \in \bar{\mathbb{R}}$.

c) Define $h\colon \mathbb{R}^d \to \bar{\mathbb{R}}$ by $h(x) := \max\{f(x), g(x)\}$ for $x \in \mathbb{R}^d$. Show that $L_h(\alpha) = L_f(\alpha) \cap L_g(\alpha)$ for all $\alpha \in \bar{\mathbb{R}}$.

**Problem 2.**

For each of the following functions, check whether it is lower semi-continuous and/or coercive, and whether it admits a global minimum:

a) $\ell\colon \mathbb{R} \to \mathbb{R}$ defined as $\ell(x) = 5x^{10} + 8x^7 - 9x^2 + x + c$, where $c \in \mathbb{R}$ is a constant.

b) $m\colon \mathbb{R} \to \mathbb{R}$ defined as $m(x) = e^x - \frac{1}{1+x^2}$.

c) $p\colon \mathbb{R} \to \mathbb{R}$ defined as $p(x) = x^4 - 20x^3 + \sup_{k\in\mathbb{N}} \sin(k^2 x)$.

d) $q\colon \mathbb{R}^2 \to \mathbb{R}$ defined as $q(x) = x_1^2(1 + x_2^3) + x_1^2$.

**Problem 3.**

Find the gradient, Hessian, and local minimizers of the objective function $f$ of the optimization problem $\min_{(x,y)\in\mathbb{R}^2} f(x, y)$, where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined as:

a) $f(x, y) = \frac{x^2}{2} + x \cos y$.

b) $f(x, y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y$.

**Problem 4.**

Compute the gradient, Hessian and local minimisers of the Rosenbrock function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.

**Problem 5.**

For a matrix $A \in \mathbb{R}^{d\times d}$, we denote by

$$\|A\|_F := \left(\sum_{i,j=1}^{d} a_{ij}^2\right)^{\frac{1}{2}}$$

its *Frobenius norm.* Show that the optimization problem

$$\min_{\substack{A\in\mathbb{R}^{d\times d}\\ \det A>0}} \left(\|A\|_F + \frac{1}{\det A}\right)$$

admits a global minimum.