TMA4212 Num.diff.
Spring 2025

Project 2

Norwegian University of Science and Technology
Department of Mathematical Sciences

Practical information

- *Deadline and hand-in:* Friday April 4 (before midnight). Hand in the project in ovsys.
- *Supervision:* There will be a few additional meeting hours, these will be announced at the wiki-page.
- *Report:* Submit the report as a Jupyter notebook or as a pdf-document (LaTeX) with the python code in a separate file. Write the report as a scientific report, not as a solution to an exercise. Write the report as a scientific report, not as a solution to an exercise. Meaning: Describe the problem you want to solve, describe the method you are using, write mathematical results as mathematical statements, and make sure there is a consistency between theoretical and numerical results etc. Use plots whenever appropriate, make sure they are readable, and explain clearly what you observe, and if they are as expected.

  Include the name of all the group members on the title page.

  The tex-report should not exceed 10 pages, all included (and it may be shorter).

  If you are using any kind of AI or other tools or references, it should be clearly stated how it is used, and what it is used for.
- *Grading:* Out of 20 points, the report counts for 5 points, Problem 1 for 8 points and Problem 2 for 7. Roughly.
- *Learning objectives:* When completed this project you should demonstrate that you are able to:
  - implement a FEM solver for the Poisson problem, and confirm theoretical results numerically.
  - understand how the use of a finite element space can be used in a different context, here, and optimal control problem,
  - communicate the results in a scientific manner.

Some advice:

- *Implementation:* Make a plan. Do not implement everything at once, split the work in small pieces, and make sure each of them works before you continue. If possible, use nontrivial test problems of which the numerical solution is exact to check that the implementation is correct, but please do not include such results in the report.

Make an implementation such that functions implemented for Problem 1 can be reused for Problem 2.

Use sparse solvers to solve linear systems of equations.

- *Writing:* Imagine you are writing to a fellow student, who do not know about this project. How you make him/her understand and be interested in what you have done and learned during the project?

  Do not include huge amounts of trivial calculations, this is a report, not an exam paper.

  Writing takes a lot of time, so start early. And accept that you may want to rewrite parts, that is a part of the writing process.

- *Time organisation:* Think about how much time you are willing to use on this project. If you are completely stuck at one point, maybe it is better to skip it and concentrate on writing a good report instead.

Introduction

In this exercise you are asked to solve two 1d problems, the Poisson equation and a problem from optimal control. In both cases, the exact solution is a function in $H_0^1(\Omega)$, and the numerical solution is found by searching for a solution in a finite dimensional subspace $V_h = \text{span}\{\varphi_0, \varphi_1, \dots, \varphi_M\}$. A function $v \in V_h$ can be written as

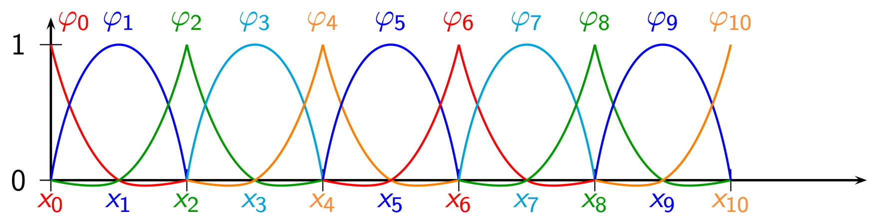
$$v(x) = \sum_{j=0}^{M} v_j \varphi_j(x),$$

that is, the function is completely defined by the vector $\boldsymbol{v} = (v_0, v_1, \dots, v_M) \in \mathbb{R}^{M+1}$.

In this project we will use the second degree Lagrange finite element space:

$$X_h^2 = \{v \in C^0(\Omega) \,:\, v|_K \in \mathbb{P}_2, \forall K \in \mathcal{T}_h.\}$$

where $\mathcal{T}_h$ is a partition of the domain. Below is an example with 5 elements and 11 basis functions, based on an equidistributed grid.



$\boxed{1}$ **Poisson equation.**

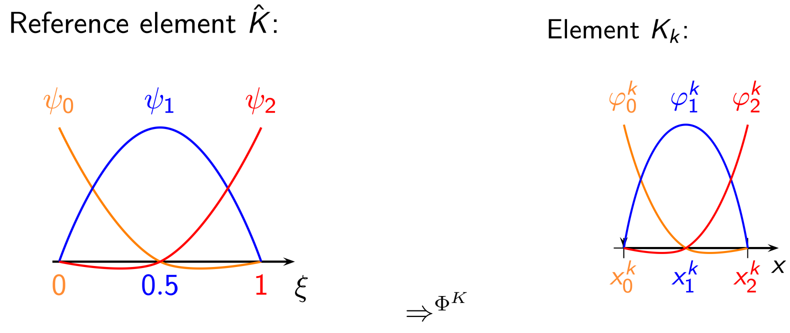


Figure 1: Shape functions on a reference element $\hat{K}$, and the corresponding basis functions on the physical element K_k.

It is already known that the variational form of Poisson equation

$$-\Delta u = f \text{ on } \Omega = (0,1), \quad u(0) = u(1) = 0$$

is:

$$\text{find } u \in V \text{ such that } a(u,v) = F(v), \quad \forall v \in V, \tag{1a}$$

with

$$V = H_0^1(\Omega), \quad a(u,v) = \int_0^1 u_x v_x \mathrm{d}x, \quad F(v) = \int_0^1 f v \mathrm{d}x. \tag{1b}$$

The Galerkin method is then

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v) = F(v), \quad \forall v \in V_h, \tag{2}$$

where V_h is some finite dimensional subspace of V.

Your task is to write and test a code for solving (2) with

$$V_h = X_h^2 \cap H_0^1(\Omega) \tag{3}$$

The code should work on a partition with variable element sizes.

a) To implement the code, follow the following steps:

- Choose a partition $\mathcal{T}_h$: That is, choose a sequence $0 = x_0 < x_2 < \cdots < x_{2M-2} < x_{2M} = 1$, the elements are $K_k = [x_{2k}, x_{2k+2}]$ for $k = 0, \ldots M-1$ and the stepsizes are $h_k = x_{2k+2} - x_{2k}$. Then $\mathcal{T}_h$ is the set of the elements K_k. In each elements there are three nodes: $x_{2k}, x_{2k+1} = x_{2k} + h_k/2$ and x_{2k+2}.
- Define a reference element $\hat{K} = [0,1]$, and define the shape functions $\Psi_i \in \mathbb{P}_2$ on $\hat{K}$. Choose those such that $\Psi_\alpha(\xi_\beta) = \delta_{\alpha,\beta}$ for $\xi_\beta \in \{0, 1/2, 1\}$, see Figure 1. Define a mapping $\Phi_K : \hat{K} \to K_k$, and a local to global mapping $i = \theta(k, \alpha)$.
- Use this to find the elemental matrix A^{K_k} and the elemental load vector b^{K_k}. Use some sufficiently accurate numerical methods to compute approximations to the integrals $\int_{K_k} f \varphi_i \mathrm{d}x$, e.g. Simpsons formula turns out to be very convenient in this case.

- Assemble the extended stiffness matrix and the extended load vector, and impose the boundaries.
  Hint: If you do not want to try other than the zero boundary conditions, the easiest is simply to remove the first and last row and column from the stiffness matrix, and the first and last element of the load vector, but take a few minutes to ensure yourself that this is the right thing to do.
- Test the code on some problem of your own choice, and compare with the exact solution. A good starting point can be $f = 1$, but you should also make more complicated examples.

b) Use Lemmata 4.3 and 4.4 in the note by Charles Curry to find an upper bound for the error $\|u - u_h\|_{L^2(\Omega)}$.
Given an equidistributed grid (constant stepsize). What is the expected order of the method.
Verify the results numerically.

$\boxed{2}$ **An optimal control problem**.

A physical object $\Omega = (0, 1)$ is to be heated or cooled to attain a desired temperature profile $y_d \in L^2(\Omega)$. The (relative) temperature is zero at the boundary. We control the distributed heat source u, and a cost $\alpha \in (0, \infty)$ is associated with applying heating or cooling. Our problem is to jointly minimise the discrepency from y_d and the total cost of heating/cooling,

$$\min_{y,u} J(y, u) = \frac{1}{2} \int_0^1 |y - y_d|^2 dx + \frac{\alpha}{2} \int_0^1 u^2 dx, \tag{4}$$

where y solves

$$-\Delta y = u \text{ and } y(0) = y(1) = 0, \tag{5}$$

in the weak sense. The above problem is a prototypical example of a PDE optimal control problem (OCP). The goal of an OCP is, in many industrial applications, to find the controlled parameters which optimises the outcome of some physical process, described by a dependent state variable. These problems are also found in image processing. In our case, the control variable is u, the desired outcome is modelled by $J(y, u)$, and $y = y(u)$ is the state variable.

We will approximate (4)–(5) with the following constrained FE minimisation problem,

$$\begin{aligned} &\min_{u_h, y_h \in V_h} \frac{1}{2} \|y_h - \bar{y}_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u_h\|^2_{L^2(\Omega)} \\ &\text{s.t. } a(y_h, v) = \langle u_h, v \rangle_{L^2(\Omega)} \text{ for all } v \in V_h, \end{aligned} \tag{6}$$

where $\bar{y}_d$ is the interpolation of y_d onto X_h^2, $V_h = X_h^2 \cap H_0^1(\Omega)$ and a is the bilinear form given in (1b).

a) To solve (6), we must interpret the problem as a real-valued minimisation problem on the unknown coeffients $\boldsymbol{u} = (u_1, ..., u_{2N-1})$, $\boldsymbol{y} = (y_1, ..., y_{2N-1})$, where N is the number of elements,

$$\min_{\boldsymbol{u}, \boldsymbol{y} \in R^{2N-1}} G(\boldsymbol{y}, \boldsymbol{u}) \text{ subject to } B\boldsymbol{y} = F\boldsymbol{u}. \tag{7}$$

Find G and the matrices $B, F \in \mathbf{R}^{(2N-1)\times(2N-1)}$.
Hint: What is $\langle u, v\rangle_{L^2(\Omega)}$ in matrix form when

1. $u, v \in V_h$?
2. $u \in V_h$ and $v \in X_h^2$?

b) We will use the method of Lagrange multipliers from TMA4105 Mathematics 2 to solve (7). Find the Lagrange function

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{y}, \boldsymbol{\lambda}) = G(\boldsymbol{y}, \boldsymbol{u}) - \boldsymbol{\lambda}^T(B\boldsymbol{y} - F\boldsymbol{u})$$

where $\boldsymbol{\lambda} \in R^{2N-1}$ is the vector of Lagrange multipliers. Set up the equation for critical points

$$\begin{cases} \nabla_{\boldsymbol{y}}\mathcal{L} = 0, \\ \nabla_{\boldsymbol{u}}\mathcal{L} = 0, \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L} = 0. \end{cases} \tag{8}$$

and solve these with respect to $\boldsymbol{y}$ and $\boldsymbol{u}$.

c) Solve (6) numerically for

$$y_d = \frac{1}{2}x(1-x), \tag{9}$$

$$y_d = 1, \tag{10}$$

$$y_d = \begin{cases} 1 & \text{for } x \in [1/4, 3/4], \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

For reference, typical values of α lie in the interval $(10^{-3}, 10^{-4})$.

(c1) Comment on the behavior of the optimal control u_h and the optimal state y_h for large costs $\alpha \in (10^{-2}, 1)$. Does this make sense with respect to the optimisation problem (6)?

(c2) What happens for very small costs $\alpha \in (10^{-6}, 10^{-8})$? Comment on the difference between $y_d \in H_0^1(\Omega)$ and $y_d \notin H_0^1(\Omega)$.