Saad 9

The principle of preconditioning

$Ax = b$, A has slow convergence.

Re write the system by choosing $M \in \mathbb{R}^{n \times n}$

$M^{-1}Ax = M^{-1}b$ (left preconditioning) LPC

$AM^{-1}u = b$, $u = Mx$ or $x = M^{-1}u$ (right preconditioning) RPC

Apply RPC (to GMRES)

A, b, x_0 , $u_0 = Mx_0$, $r_0 = b - AM^{-1}u = b - Ax_0$

$\beta = \|r_0\|$, $v_1 = \frac{r_0}{\beta}$

Arnoldi

for $j = 1, \dots$

$w_j = AM^{-1}v_j$

for $i = 1, \dots, j$

$h_{ij} = \langle w_j, v_i \rangle$

$w_j = w_j - h_{ij} v_i$

$h_{j+1,j} = \|w_j\|$ if "0" stop

$v_{j+1} = \frac{w_j}{h_{j+1,j}}$

Solve

$\bar{H}_m y_m = \beta e_1$ in least square sense

$u_m = u_0 + V_m y_m$

$$x_m = M^{-1}u_0 + M^{-1}V_m y_m$$
$$= x_0 + M^{-1}V_m y_m$$

We never use u , We need to compute $M^{-1}v_j = z_j$

$\hookrightarrow$ $Mz_j = v_j$ for each iteration.

$r_m = b - Ax_m$ the same as the unconditioned.

Using LPC changes the residual.

we want M^{-1} to be close to A^{-1} s.t. $M^{-1}A \sim I_n$

Preconditioning the CG, Must be symmetric

A is SPD and $\tilde{A} = AM^{-1}$ also must be SPD
$\Rightarrow M$ is SPD, $M = L \cdot L^T$

$\tilde{A} = AM^{-1}$ or $\tilde{A} = M^{-1}A$?

$$M^{-1}Ax = M^{-1}b$$

$$\cancel{L^{-T}}\underbrace{L^{-1}AL^{-T}}_{\tilde{A}}\underbrace{L^{T}x}_{\tilde{x}} = \cancel{L^{-T}}\underbrace{L^{-1}b}_{\tilde{b}}$$

now $\tilde{A}$ is SPD

$$\tilde{A} = L^{-1}AL^{-T}, \quad \tilde{x} = L^{T}x, \quad \tilde{b} = L^{-1}b$$

$$\tilde{r} = \tilde{b} - \tilde{A}\tilde{x} = L^{-1}(b - Ax)$$

CG on $\tilde{A}\tilde{x} = \tilde{b}$

$$\tilde{r}_0 = L^{-1}(b - Ax_0), \quad \tilde{p}_0 = \tilde{r}_0$$

for $j = 0, 1, \dots$

$$\tilde{x}_{j+1} = \tilde{x}_j + \alpha_j \tilde{p}_j, \quad \alpha_j = \frac{(\tilde{r}_j, \tilde{r}_j)}{(\tilde{A}\tilde{p}_j, \tilde{p}_j)} = \frac{\|\tilde{r}_j\|^2}{\|\tilde{p}_j\|^2_{\tilde{A}}}$$

$$\tilde{r}_{j+1} = \tilde{r}_j - \alpha \tilde{A}\tilde{p}_j$$

$$\tilde{p}_{j+1} = \tilde{r}_{j+1} + \beta_j \tilde{p}_j, \quad \beta_j = \frac{(\tilde{r}_{j+1}, \tilde{r}_{j+1})}{(\tilde{r}_j, \tilde{r}_j)} = \frac{\|\tilde{r}_{j+1}\|^2}{\|\tilde{r}_j\|^2}$$

$$\alpha_j : \langle \tilde{r}_j, \tilde{r}_j \rangle = \tilde{r}_j^T \tilde{r}_j = \langle L^{-1}r, L^{-1}r \rangle = \langle r, L^{-T}L^{-1}r$$

$$= \langle r, M^{-1}r \rangle$$

$$\langle \hat{A}\tilde{p}_j, \tilde{p}_j \rangle = \langle L^{-1} A L^{-T} \tilde{p}_j, \tilde{p}_j \rangle$$ Let $p_j = L^{-T}\tilde{p}_j$

$$= \langle A p_j, L^{-T}\tilde{p}_j \rangle$$

$$= \langle A p_j, p_j \rangle$$

we multiply $\tilde{x}_j$ and $\tilde{p}_j$ by L^{-T} and $\tilde{r}_j$ by L

$$\Rightarrow \quad L^{-T}\hat{x}_{j+1} = L^{-T}\hat{x}_j + \alpha_j L^{-T}\tilde{p}_j = x_j + \alpha_j p_j$$

$$L\hat{r}_{j+1} = L\hat{r}_j - \alpha_j L L^{-1} A L^{-T} L^T p_j = r_j - \alpha_j A p_j$$

$$L^{-T}\tilde{p}_{j+1} = L^{-T}\hat{r}_{j+1} - \beta_j L^{-T}\tilde{p}_j$$

$$= L^{-T}L^{-1} r_{j+1} - \beta_j p_j$$

$$= M^{-1} r_{j+1} - \beta_j p_j$$

We have a new p_j and a new α_j

$r_0 = b - Ax_0$, $z_0 = M^{-1} r_0$, $p_0 = z_0$

for $j = 0, \ldots$

$$x_{j+1} = x_j + \alpha_j p_j, \qquad \alpha_j = \frac{\langle r_j, r_j \rangle}{\langle p_j, p_j \rangle_A}$$

$$r_{j+1} = r_j - \alpha_j A p_j$$

$$z_{j+1} = M^{-1} r_{j+1}$$

$$p_{j+1} = z_{j+1} + \beta_j p_j, \qquad \beta_j = \frac{\langle r_{j+1}, z_{j+1} \rangle}{\langle r_j, z_j \rangle}$$

Price: solve $M z_j = r_j$ pr. iteration, store z_j

How to choose M (M is a transformation, not "given" it's a matrix)

- $M \approx A$ s.t. $M^{-1}A \approx I_n$
- $\mathcal{K}(\hat{A}) \ll \mathcal{K}(A)$
- $Mz = v$ should be easy to solve
- M should have (almost) the same sparsity structure as A
- If A is SPD choose M SPD, (conjugate gradient)

In this course

1) Use one iteration of one of the classical methods Jacobi, Gauss-Seidel, SOR, ... and symmetric versions of those.

   $[J: \ M = D]$

2) Incomplete LU (ILU)

   $A \approx LU$, L, U keeping the sparsity structure.

3) Multigrid method (- Close to discretization of PDE's -)