THE POWER OF ABSTRACTION IN EDUCATION

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Give a person a fish

Give a person a fish

—and you feed the person for a day

Give a person a fish

—and you feed the person for a day

Teach the person how to fish

Give a person a fish

—and you feed the person for a day

Teach the person how to fish

—*and you feed the person for life*

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Facts are like fish

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—and you feed the person for a day

Teach the person how to fish

—*and you feed the person for life*

Facts are like fish

—Abstractions are like fishing equipment

<ins>Abstractions are like fishing equipment</ins>

When you deal with <ins>abstractions</ins>, not with the <ins>literal objects</ins>

—then you can often devise <ins>*operational names*</ins> for the objects

Abstractions are like fishing equipment

When you deal with abstractions, not with the literal objects

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These names can give you *computational control* over the objects

Abstractions are like fishing equipment

When you deal with abstractions, not with the literal objects

—then you can often devise *operational names* for the objects

These names can give you *computational control* over the objects

And—

You can create different names for different purposes!

What's in a name? (Shakespeare)

What's in a name? (Shakespeare)

You can devise *operational names* for the objects

—names that give you computational control

—different names for different purposes

What's in a name? (Shakespeare)

We discuss three abstraction-naming situations

1. The first is rather elementary (in required background)

What's in a name? (Shakespeare)

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   We discuss this one in detail

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1. The first is rather elementary (in required background)
   We discuss this one in detail
2. The second requires some computing/programming experience

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   I try to impress you by citing famous names ...

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EXAMPLES ARE SIMPLIFIED TO APPEAL TO EVEN BEGINNING STUDENTS

Situation #1

HOW TO THINK ABOUT *NUMBERS*

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HOW TO THINK ABOUT *NUMBERS*

The main message:

Try to think about numbers using a variety of "names".

Different "names" may suggest very different ways of thinking — and reasoning

Consider the following table.

$$\begin{array}{lcrcl}
1 & = & 1 & = & 1^2 \\
1+3 & = & 4 & = & 2^2 \\
1+3+5 & = & 9 & = & 3^2 \\
1+3+5+7 & = & 16 & = & 4^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Consider the following table.

$$\begin{array}{lcrcl}
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\end{array}$$

It *appears* that

each perfect square n^2 is the sum of the first n odd numbers.

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1+3+5+7 & = & 16 & = & 4^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It *appears* that

each perfect square n^2 *is the sum of the first* n *odd numbers.*

Let's *prove* that this is true for all positive integers n.

Each perfect square n^2 is the sum of the first n odd numbers.

We provide *three* proofs which use different “names” for the numbers.

Each perfect square n^2 is the sum of the first n odd numbers.

We provide *three* proofs which use different “names” for the numbers.

(We could provide more “names” and more proofs.)

Each perfect square n^2 is the sum of the first n odd numbers.

We provide *three* proofs which use different "names" for the numbers.

Some "names" for the numbers:

$Numeral$ $(Digit)$	$Token$ $(Number)$
1	●
2	● ●
3	● ● ●
4	● ● ● ●
5	● ● ● ● ●
6	● ● ● ● ● ●
7	● ● ● ● ● ● ●

Each perfect square n^2 is the sum of the first n odd numbers.

We provide *three* proofs which use different "names" for the numbers.

More "names" for the numbers:

$Numeral$ $(Digit)$	$Token$ $(Number)$	$Token$ $(Number + configuration)$
1	●	●
2	● ●	
3	● ● ●	● ● ●
4	● ● ● ●	
5	● ● ● ● ●	● ● ● ● ●
6	● ● ● ● ● ●	
7	● ● ● ● ● ● ●	● ● ● ● ● ● ●

A TEXT-ORIENTED PROOF

Each perfect square n^2 is the sum of the first n odd numbers.

A *text-oriented* proof: STEP 1

Write out the summation of interest:

$$S \; = \; 1 \; + \; 3 \; + \; 5 \; + \; 7 \; + \; 9$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *text-oriented* proof: STEP 2

Rewrite the summation — *in reverse order*.

$$
\begin{array}{rcccccccccc}
S & = & 1 & + & 3 & + & 5 & + & 7 & + & 9 \\
S & = & 9 & + & 7 & + & 5 & + & 3 & + & 1
\end{array}
$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *text-oriented* proof: STEP 3

Add the two versions of the summation.

$$\begin{array}{rcccccccccc} S & = & 1 & + & 3 & + & 5 & + & 7 & + & 9 \\ S & = & 9 & + & 7 & + & 5 & + & 3 & + & 1 \\ \hline 2S & = & 10 & + & 10 & + & 10 & + & 10 & + & 10 \end{array}$$

Each perfect square n^2 is the sum of the first n odd numbers.

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Note: All column-sums are identical.

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Note: All column-sums are identical.

We thereby find that

$$2S \quad = \quad 5 \times 10 \quad = \quad 50$$

so that

$$S \quad = \quad 25 \quad = \quad \boxed{5^2}$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *text-oriented* proof: STEP 3

Add the two versions of the summation.

$$\begin{array}{rcccccccccc} S & = & 1 & + & 3 & + & 5 & + & 7 & + & 9 \\ S & = & 9 & + & 7 & + & 5 & + & 3 & + & 1 \\ \hline 2S & = & 10 & + & 10 & + & 10 & + & 10 & + & 10 \end{array}$$

All column-sums are identical.

We thereby find that

$$2S \quad = \quad 5 \times 10 \quad = \quad 50$$

so that

$$S \quad = \quad 25 \quad = \quad \boxed{5^2}$$

Similar calculations work for any n.

A TOKEN-NUMBER-ORIENTED PROOF

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number-oriented* proof: STEP 1

Write out the summation of interest:

$$S = 1 + 3 + 5 + 7 + 9 = \begin{array}{ll} & \bullet \\ + & \bullet\ \bullet\ \bullet \\ + & \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\ + & \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\ + & \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet \end{array}$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number-oriented* proof: STEP 1

Write out the summation of interest:

$$S \;=\; \begin{array}{l} \quad \bullet \\ + \; \bullet \; \bullet \; \bullet \\ + \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \\ + \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \\ + \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \end{array}$$

The *token-based* form of the summation will expose the advantage of this way of "naming" numbers.

Each perfect square n^2 is the sum of the first n odd numbers.

A ***token-number-oriented*** proof: STEP 3

The *token-based* form of the summation provides the following value for S:

$$S \quad = \quad \begin{array}{ccccccccc} \bullet & & & & & & & & \\ \bullet & \bullet & \bullet & & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

S thus is close to *the area of the height-5, base-9 right triangle.*

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number-oriented* proof: STEP 3

The *token-based* form of the summation provides the following value for S:

$$S \quad = \quad \begin{array}{ccccccccc}
\bullet & & & & & & & & \\
\bullet & \bullet & \bullet & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}$$

S thus is close to *the area of the height-*5*, base-*9 *right triangle.*

Let us find S exactly by using this similarity.

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number-oriented* proof: STEP 4

$$S \;=\; \begin{array}{lllllllll} \bullet & & & & & & & & \\ \bullet & \bullet & \bullet & & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \qquad \approx \text{ AREA(height-}5\text{, base-}9\text{ right triangle)}$$

To convert this intuition into a visual computation of S:
We augment the sum-triangle with a "shadow" version:

$$2S \;=\; \begin{array}{cccccccccc} \bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \end{array}$$

The area of this (5×10) rectangle is just $2 \times S$

__Each perfect square n^2 is the sum of the first n odd numbers.__

A *__token-number-oriented__* proof: STEP 4

$$2S \quad = \quad \begin{array}{cccccccccc}
\bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ
\end{array}$$

The area of this rectangle is just $2 \times S$. Therefore:

$$S \quad = \quad \frac{1}{2} \times 5 \times 10 \;=\; \frac{1}{2} \times 50 \;=\; 25 \;=\; \boxed{5^2}$$

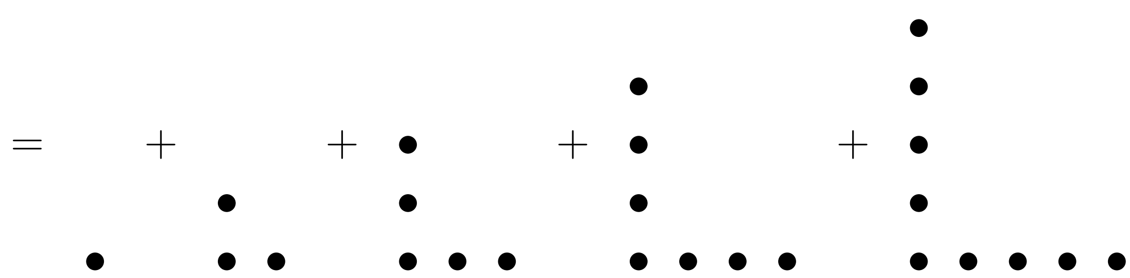
A TOKEN-NUMBER+CONFIGURATION-ORIENTED PROOF

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 1

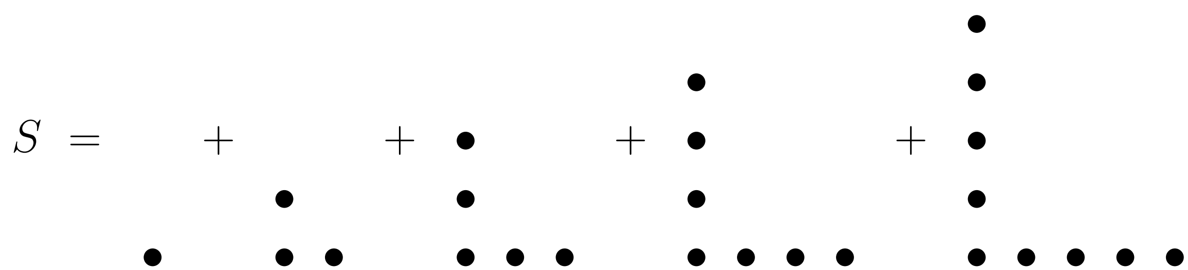
Write out the summaton of interest:

$$S = 1 + 3 + 5 + 7 + 9$$

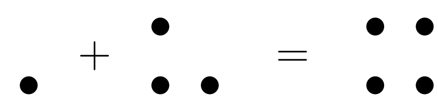


Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 2

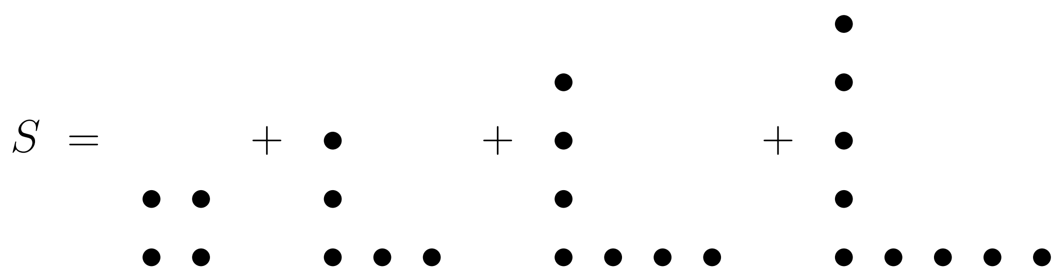


The first two summands sum by *nesting*:



Each perfect square n^2 is the sum of the first n odd numbers.

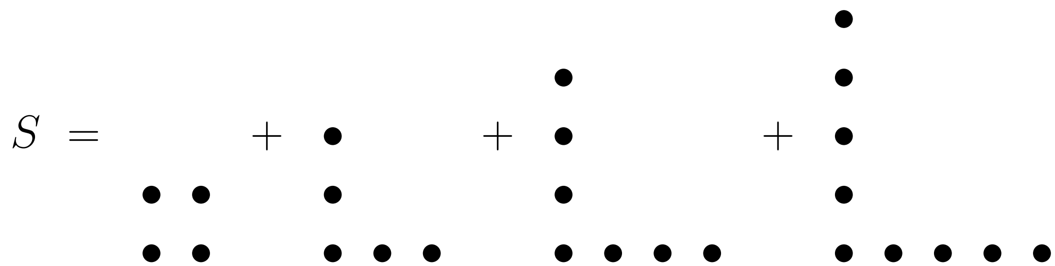
A *token-number+configuration-oriented* proof: STEP 3



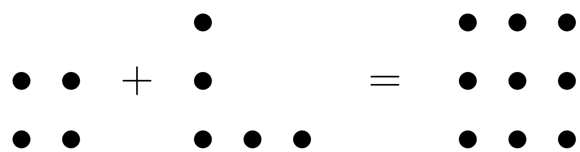
$$= 4 + 5 + 7 + 9$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 3

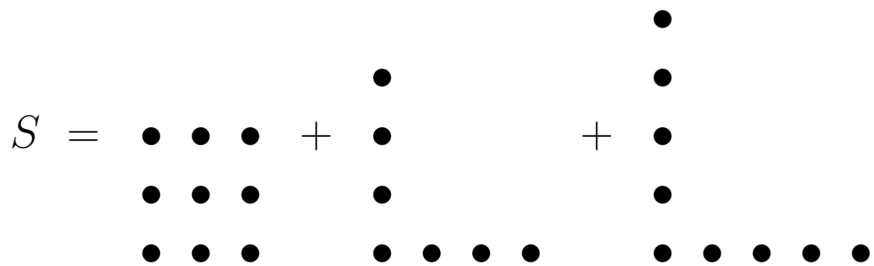


The first two summands sum by *nesting*:



Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 4



$$= 9 + 7 + 9$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 4

$$S = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} + \begin{array}{cccc} \bullet & & & \\ \bullet & & & \\ \bullet & & & \\ \bullet & \bullet & \bullet & \bullet \end{array} + \begin{array}{ccccc} \bullet & & & & \\ \bullet & & & & \\ \bullet & & & & \\ \bullet & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

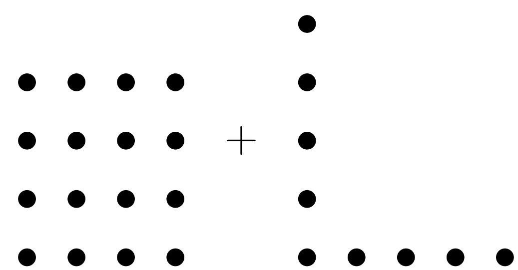
The first two summands sum by *nesting*:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} + \begin{array}{cccc} \bullet & & & \\ \bullet & & & \\ \bullet & & & \\ \bullet & \bullet & \bullet & \bullet \end{array} = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 5

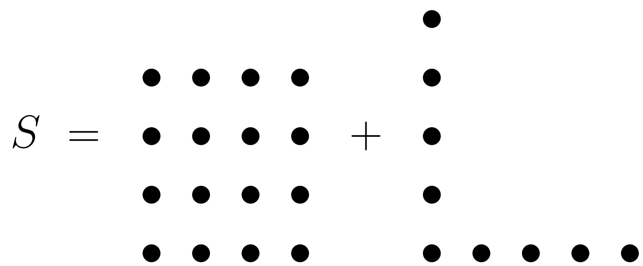
$S =$



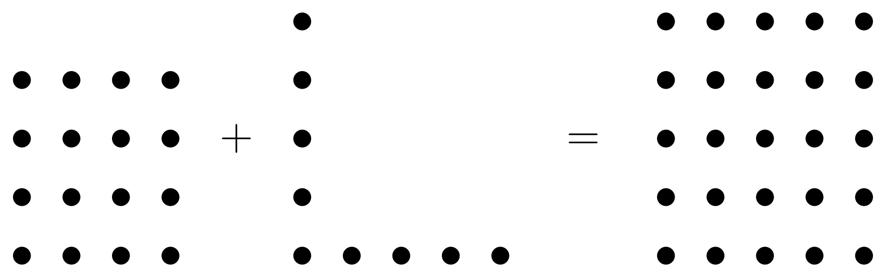
$$= 16 + 9$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 5



The final two summands sum by *nesting*:



Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 6

Finally:

$$\begin{aligned} S \;&=\; \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \\ &=\; 25 \\ &=\; \boxed{5^2} \end{aligned}$$

Each perfect square n^2 is the sum of the first n odd numbers.

A *token-number+configuration-oriented* proof: STEP 6

$$S = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

$$= 25$$

$$= \boxed{5^2}$$

As before, this illustration extends to arbitrary positive integers n

Situation #2

HOW TO THINK ABOUT *COMPUTATIONS*

Situation #2

HOW TO THINK ABOUT *COMPUTATIONS*

The main message:

Step back from details in algorithms and programs.

You may observe computationally advantageous nonobvious similarities.

Tasks in a computation cannot be executed in arbitrary orders.

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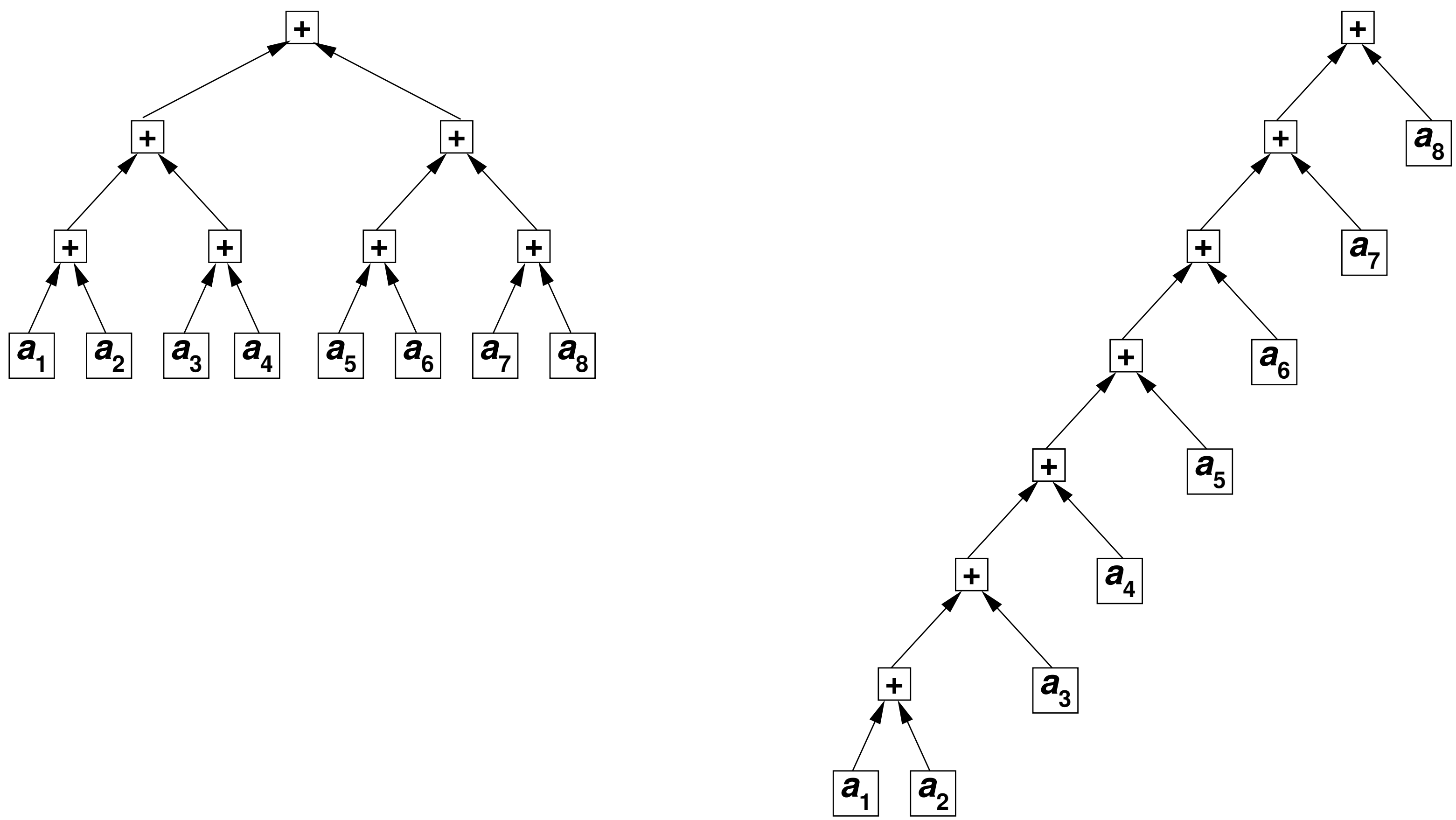
—Some tasks depend on inputs from other tasks

Two ways to sum 8 numbers — illustrating dependencies



Tasks in a computation cannot be executed in arbitrary orders.

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Tasks in a computation cannot be executed in arbitrary orders.

—Some tasks depend on inputs from other tasks

Dependencies constrain

- a $compiler$'s freedom to rearrange orders of task-execution

  —in order to optimize access to/consumption of resources

Tasks in a computation cannot be executed in arbitrary orders.

—Some tasks depend on inputs from other tasks

Dependencies constrain

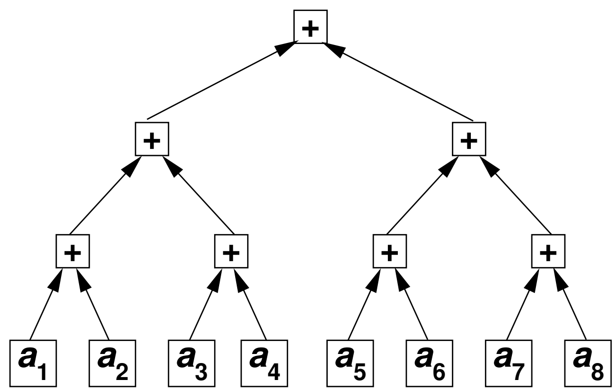
- a *compiler*'s freedom to rearrange orders of task-execution
- a *scheduler*'s freedom to have tasks execute concurrently

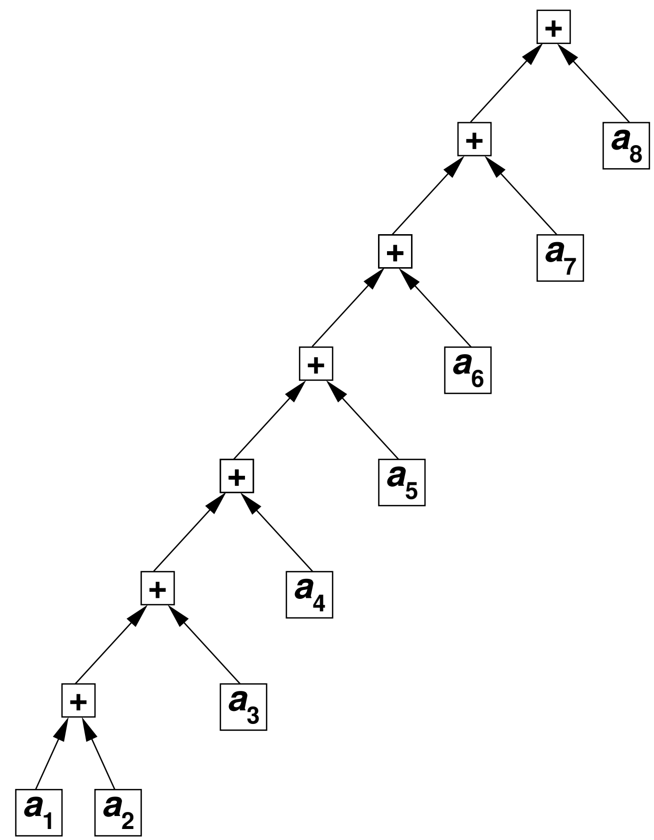
  —in order to optimize cumulative task-execution time

Dependencies constrain

- a *compiler*'s freedom to rearrange orders of task-execution
- a *scheduler*'s freedom to have tasks execute concurrently

The degree of constraint can vary considerably





Because of its importance to compilation and scheduling ...

understanding dependency structures is critical to

- the design of *efficient* algorithms

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  –all kinds of efficiency:

  - $\star$ operation speed

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  –all kinds of efficiency:

  - $\star$ operation speed
  - $\star$ utilization of memory/storage

Because of its importance to compilation and scheduling ...

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  –all kinds of efficiency:

  - $\star$ operation speed
  - $\star$ utilization of memory/storage
  - $\star$ conservation of power

Because of its importance to compilation and scheduling ...

understanding dependency structures is critical to

- the design of *efficient* algorithms
- the design of high-performance programs

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- the design of *efficient* algorithms
- the design of high-performance programs
- understanding that

  many algorithms, for many disparate tasks, share the same dependency structure

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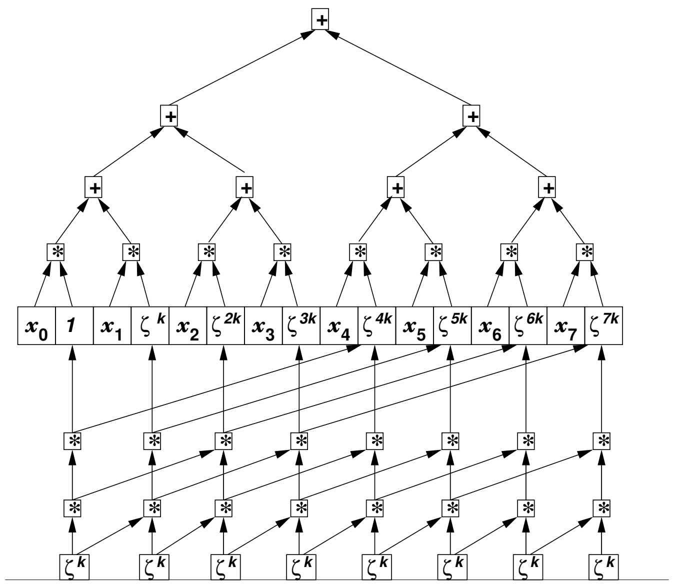
- the design of high-performance programs
- understanding that

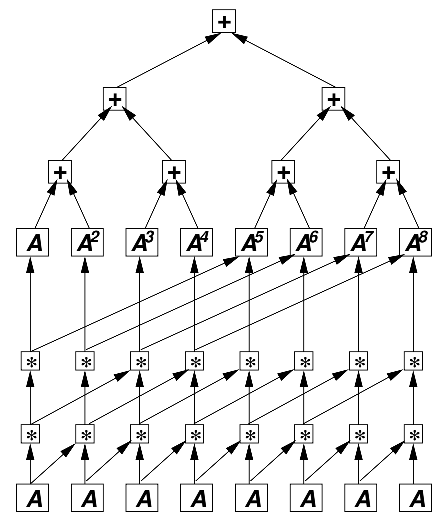
  many algorithms, for many disparate tasks, <u>share the same dependency structure</u>

Some examples:

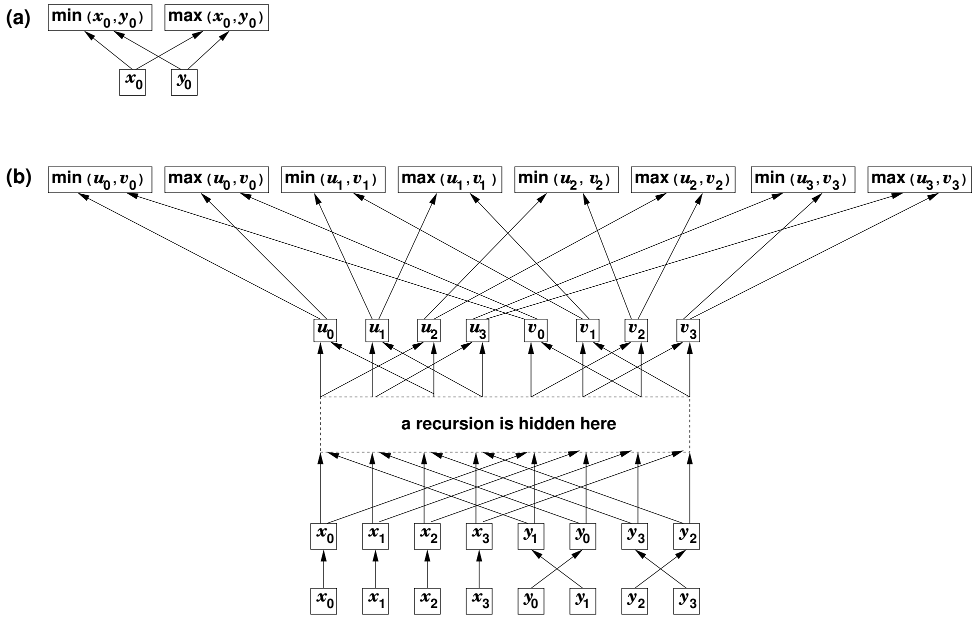
The Discrete Laplace Transform



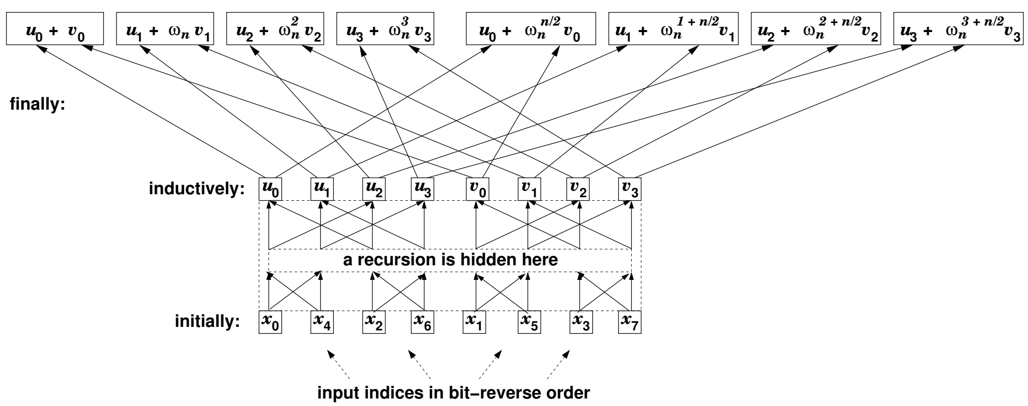
Finding all paths in a graph



A butterfly sorting algorithm



The Fast Fourier Transform



Situation #3

HOW TO THINK ABOUT *COMPUTING*

Situation #3

HOW TO THINK ABOUT *COMPUTING*

The main message:

The integer I have written behind this screen is actually . . .

Situation #3

HOW TO THINK ABOUT *COMPUTING*

The main message:

The integer I have written behind this screen is actually . . .

- just a number?

Situation $\#3$

HOW TO THINK ABOUT *COMPUTING*

The main message:

The integer I have written behind this screen is actually ...

- just a number?
- an encoding of a program that computes π to $10^{10^{10}}$ places?

Situation $\#3$

HOW TO THINK ABOUT *COMPUTING*

The main message:

The integer I have written behind this screen is actually . . .

- just a number?
- an encoding of a program that computes π to $10^{10^{10}}$ places?
- an encoding of the computation by that program?

Situation $\#3$

HOW TO THINK ABOUT *COMPUTING*

The main message:

The integer I have written behind this screen is actually ...

- just a number?
- an encoding of a program that computes π to $10^{10^{10}}$ places?
- an encoding of the computation by that program?
- an encoding of the elements of the set of all sets that can be written using $10^{10^{10}}$ characters?

Situation #3

HOW TO THINK ABOUT *COMPUTING*

The main message:

The integer I have written behind this screen is actually . . .

- just a number?
- an encoding of a program that computes π to $10^{10^{10}}$ places?
- an encoding of the computation by that program?
- an encoding of the elements of the set of all sets that can be written using $10^{10^{10}}$ characters?

ENCODINGS ARE POWERFUL — and mischievous — MECHANISMS

that span the chasm between NUMBER and PROGRAM and LANGUAGE

- Kurt Gödel:

  “This sentence is false”

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IS THIS SENTENCE TRUE?

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IS THIS SENTENCE TRUE?

- If it is true — then it is false!

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IS THIS SENTENCE TRUE?

- If it is true — then it is false!
- If it is false — then it is true!

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IS THIS SENTENCE TRUE?

- If it is true — then it is false!
- If it is false — then it is true!

This curious *self-referential* sentence teaches us:

$\boxed{\textit{Not all assertions are } \text{TRUE} \textit{ or } \text{FALSE.}}$

- Kurt Gödel:

  "This sentence is false"

IS THIS SENTENCE TRUE?

- If it is true — then it is false!
- If it is false — then it is true!

This curious <u>*self-referential*</u> sentence teaches us:

$\boxed{\textit{Not all assertions are } \text{TRUE} \textit{ or } \text{FALSE.}}$

$\boxed{\textit{Not all } \text{TRUE} \textit{ assertions are } \text{PROVABLE.}}$

- Kurt Gödel:

  "This sentence is false"

- Bertrand Russell:

  Let X be the set whose elements are
  all sets that *are not* elements of themselves

- Kurt Gödel:

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  <u>all sets that *are not* elements of themselves</u>

Note that sets *can* have sets as elements!

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  Let X be the set whose elements are
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IS X AN ELEMENT OF X?

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- If X is an element of X, then X is *not* an element of X

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IS X AN ELEMENT OF X?

- If X is an element of X, then X is *not* an element of X
- If X is *not* an element of X, then X is an element of X.

This $\textit{self-referential}$ paradox teaches us:

$$\boxed{\textit{Set } X \textit{ cannot exist.}}$$

- Kurt Gödel:

  "This sentence is false"

- Bertrand Russell:

  X *is the set of all sets that* are not *elements of themselves*

- Alan Turing:

  Let $\mathbf{P}$ be a program that takes programs as inputs ...

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  Let $\mathbf{P}$ be a program that takes programs as inputs ...

Clearly such programs exist – a compiler is an example.

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- Alan Turing:

  Let $\mathbf{P}$ be a program that takes programs as inputs ...
  AND
  <u>that enters an endless loop *precisely* when</u>
  <u>an input program Π *halts* on input Π.</u>

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DOES PROGRAM $\mathbf{P}$ HALT ON INPUT $\mathbf{P}$?

- If program $\mathbf{P}$ halts on input $\mathbf{P}$, then it *does not* halt on input $\mathbf{P}$.

- If program $\mathbf{P}$ does not halt on input $\mathbf{P}$, then it *does* halt on input $\mathbf{P}$.

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  Let $\mathbf{P}$ be a program that takes programs as inputs ...
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This *self-referential* paradox teaches us:

$\boxed{\textit{Program } \mathbf{P} \textit{ cannot exist.}}$

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- Alan Turing:

  $\mathbf{P}(\Pi)$ *fails to halt* just when $\Pi(\Pi)$ *halts*.

- Georg Cantor:

  Say that you present me with a program $\mathbf{P}$ that produces a
  <u>one-to-one correspondence between the integers and the real numbers</u>

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  Say that you present me with a program $\mathbf{P}$ that produces a
  <u>one-to-one correspondence between the integers and the real numbers</u>

In detail, I mean that the sequence

$$\mathbf{P}(1),\ \mathbf{P}(2),\ \mathbf{P}(3),\ \ldots$$

contains every real number *precisely once*.

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  Say that you present me with a program $\mathbf{P}$ that produces a one-to-one correspondence between the integers and the real numbers

I can write a program Π *such that* $\Pi(\mathbf{P})$

—i.e., the result produced by program Π on input $\mathbf{P}$

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  Say that you present me with a program $\mathbf{P}$ that produces a
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I can write a program Π *such that* $\Pi(\mathbf{P})$

is a real number that is not on your list!

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is a real number that is not on your list!

The lesson is:

$\boxed{\textit{There is no one-to-one correspondence between the integers and the real numbers}}$

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  Program $\mathbf{P}$ produces a one-to-one correspondence between the integers and the real numbers

$\approx\approx\approx\approx\approx$ MOST IMPORTANTLY $\approx\approx\approx\approx\approx$

$\boxed{\textit{ALL OF THE PROOFS}}$

— Gödel's, Russells', Turing's, and Cantor's —

$\boxed{\textit{ARE ENCODINGS OF ONE ANOTHER!}}$