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# PREFACE

My purpose in writing this book was to provide a clear, accessible treatment of discrete mathematics for students majoring or minoring in computer science, mathematics, mathematics education, and engineering. The goal of the book is to lay the mathematical foundation for computer science courses such as data structures, algorithms, relational database theory, automata theory and formal languages, compiler design, and cryptography, and for mathematics courses such as linear and abstract algebra, combinatorics, probability, logic and set theory, and number theory. By combining discussion of theory and practice, I have tried to show that mathematics has engaging and important applications as well as being interesting and beautiful in its own right.

A good background in algebra is the only prerequisite; the course may be taken by students either before or after a course in calculus. Previous editions of the book have been used successfully by students at hundreds of institutions in North and South America, Europe, the Middle East, Asia, and Australia.

Recent curricular recommendations from the Institute for Electrical and Electronic Engineers Computer Society (IEEE-CS) and the Association for Computing Machinery (ACM) include discrete mathematics as the largest portion of "core knowledge" for computer science students and state that students should take at least a one-semester course in the subject as part of their first-year studies, with a two-semester course preferred when possible. This book includes all the topics recommended by those organizations and can be used effectively for either a one-semester or a two-semester course.

At one time, most of the topics in discrete mathematics were taught only to upper-level undergraduates. Discovering how to present these topics in ways that can be understood by first- and second-year students was the major and most interesting challenge of writing this book. The presentation was developed over a long period of experimentation during which my students were in many ways my teachers. Their questions, comments, and written work showed me what concepts and techniques caused them difficulty, and their reaction to my exposition showed me what worked to build their understanding and to encourage their interest. Many of the changes in this edition have resulted from continuing interaction with students.

## *Themes of a Discrete Mathematics Course*

Discrete mathematics describes processes that consist of a sequence of individual steps. This contrasts with calculus, which describes processes that change in a continuous fashion. Whereas the ideas of calculus were fundamental to the science and technology of the industrial revolution, the ideas of discrete mathematics underlie the science and technology of the computer age. The main themes of a first course in discrete mathematics are logic and proof, induction and recursion, combinatorics and discrete probability, algorithms and their analysis, discrete structures, and applications and modeling.

*Logic and Proof* Probably the most important goal of a first course in discrete mathematics is to help students develop the ability to think abstractly. This means learning to use logically valid forms of argument and avoid common logical errors, appreciating what it means to reason from definitions, knowing how to use both direct and indirect argument to derive new results from those already known to be true, and being able to work with symbolic representations as if they were concrete objects.

*Induction and Recursion* An exciting development of recent years has been the increased appreciation for the power and beauty of "recursive thinking." To think recursively means to address a problem by assuming that similar problems of a smaller nature have already been solved and figuring out how to put those solutions together to solve the larger problem. Such thinking is widely used in the analysis of algorithms, where recurrence relations that result from recursive thinking often give rise to formulas that are verified by mathematical induction.

*Combinatorics and Discrete Probability* Combinatorics is the mathematics of counting and arranging objects, and probability is the study of laws concerning the measurement of random or chance events. Discrete probability focuses on situations involving discrete sets of objects, such as finding the likelihood of obtaining a certain number of heads when an unbiased coin is tossed a certain number of times. Skill in using combinatorics and probability is needed in almost every discipline where mathematics is applied, from economics to biology, to computer science, to chemistry and physics, to business management.

*Algorithms and Their Analysis* The word *algorithm* was largely unknown in the middle of the twentieth century, yet now it is one of the first words encountered in the study of computer science. To solve a problem on a computer, it is necessary to find an algorithm or step-by-step sequence of instructions for the computer to follow. Designing an algorithm requires an understanding of the mathematics underlying the problem to be solved. Determining whether or not an algorithm is correct requires a sophisticated use of mathematical induction. Calculating the amount of time or memory space the algorithm will need in order to compare it to other algorithms that produce the same output requires knowledge of combinatorics, recurrence relations, functions, and $O$-, $\Omega$-, and $\Theta$-notations.

*Discrete Structures* Discrete mathematical structures are the abstract structures that describe, categorize, and reveal the underlying relationships among discrete mathematical objects. Those studied in this book are the sets of integers and rational numbers, general sets, Boolean algebras, functions, relations, graphs and trees, formal languages and regular expressions, and finite-state automata.

*Applications and Modeling* Mathematical topics are best understood when they are seen in a variety of contexts and used to solve problems in a broad range of applied situations. One of the profound lessons of mathematics is that the same mathematical model can be used to solve problems in situations that appear superficially to be totally dissimilar. A goal of this book is to show students the extraordinary practical utility of some very abstract mathematical ideas.

## Special Features of This Book

*Mathematical Reasoning* The feature that most distinguishes this book from other discrete mathematics texts is that it teaches—explicitly but in a way that is accessible to first- and second-year college and university students—the unspoken logic and reasoning that underlie mathematical thought. For many years I taught an intensively interactive transition-to-abstract-mathematics course to mathematics and computer science majors. This experience showed me that while it is possible to teach the majority of students to understand and construct straightforward mathematical arguments, the obstacles to doing so cannot be passed over lightly. To be successful, a text for such a course must address students' difficulties with logic and language directly and at some length. It must also include enough concrete examples and exercises to enable students to develop the mental

models needed to conceptualize more abstract problems. The treatment of logic and proof in this book blends common sense and rigor in a way that explains the essentials, yet avoids overloading students with formal detail.

*Spiral Approach to Concept Development* A number of concepts in this book appear in increasingly more sophisticated forms in successive chapters to help students develop the ability to deal effectively with increasing levels of abstraction. For example, by the time students encounter the relatively advanced mathematics of Fermat's little theorem and the Chinese remainder theorem in the Section 10.4, they have been introduced to the logic of mathematical discourse in Chapters 1 and 2, learned the basic methods of proof and the concepts of *mod* and *div* in Chapter 3, studied partitions of the integers in Chapter 5, considered *mod* and *div* as functions in Chapter 7, and become familiar with equivalence relations in Sections 10.2 and 10.3. This approach builds in useful review and develops mathematical maturity in natural stages.

*Support for the Student* Students at colleges and universities inevitably have to learn a great deal on their own. Though it is often frustrating, learning to learn through self-study is a crucial step toward eventual success in a professional career. This book has a number of features to facilitate students' transition to independent learning.

**Worked Examples**
The book contains over 500 worked examples, which are written using a problem-solution format and are keyed in type and in difficulty to the exercises. Many solutions for the proof problems are developed in two stages: first a discussion of how one might come to think of the proof or disproof and then a summary of the solution, which is enclosed in a box. This format allows students to read the problem and skip immediately to the summary, if they wish, only going back to the discussion if they have trouble understanding the summary. The format also saves time for students who are rereading the text in preparation for an examination.

**Exercises**
The book contains almost 2,500 exercises. The sets at the end of each section have been designed so that students with widely varying backgrounds and ability levels will find some exercises they can be sure to do successfully and also some exercises that will challenge them.

**Solutions for Exercises**
To provide adequate feedback for students between class sessions, Appendix B contains a large number of complete solutions to exercises. Students are strongly urged not to consult solutions until they have tried their best to answer the questions on their own. Once they have done so, however, comparing their answers with those given can lead to significantly improved understanding. In addition, many problems, including some of the most challenging, have partial solutions or hints so that students can determine whether they are on the right track and make adjustments if necessary. There are also plenty of exercises without solutions to help students learn to grapple with mathematical problems in a realistic environment.

**Figures and Tables**
Figures and tables are included in every case where it seemed that doing so would help readers to a better understanding. In most, a second color is used to add meaning.

**Reference Features**
Many students have written me to say that the book helped them succeed in their advanced courses. One even wrote that he had used the first edition so extensively that it had fallen apart and he actually went out and bought a copy of the second edition,

which he was continuing to use in a master's program. My rationale for screening statements of definitions and theorems, for putting titles on exercises, and for giving the meaning of symbols and a list of reference formulas in the endpapers is to make it easier for students to use this book for review in a current course and as a reference in later ones.

*Support for the Instructor* I have received a great deal of valuable feedback from instructors who have used previous editions of this book. Many aspects of the book have been improved through their suggestions.

**Exercises**
The large variety of exercises at all levels of difficulty allows instructors great freedom to tailor a course to the abilities of their students. Exercises with solutions in the back of the book have numbers in blue and those whose solutions are given in a separate Student Solutions Manual/Study Guide have numbers that are a multiple of three. There are exercises of every type that are represented in this book which have no answer in either location to enable instructors to assign whatever mixture they prefer of exercises with and without answers. The ample number of exercises of all kinds gives instructors a significant choice of problems to use for review assignments and exams. Instructors are invited to use the many exercises stated as questions rather than in "prove that" form to stimulate class discussion on the role of proof and counterexample in problem solving.

**Flexible Sections**
Most sections are divided into subsections so that an instructor who is pressed for time can choose to cover certain subsections only and either omit the rest or leave them for the students to study on their own. The division into subsections also makes it easier for instructors to break up sections if they wish to spend more then one day on them.

**Presentation of Proof Methods**
It is inevitable that the proofs and disproofs in this book will seem easy to instructors. Many students, however, find them difficult. In showing students how to discover and construct proof and disproofs, I have tried to describe the kinds of approaches that mathematicians use when confronting challenging problems in their own research.

**Instructor's Manual**
An instructor's manual is available to anyone teaching a course from this book. It contains suggestions about how to approach the material of each chapter, solutions for all exercises not fully solved in Appendix B, transparency masters, review sheets, ideas for projects and writing assignments, and additional exercises.

## Highlights of the Third Edition

The changes that have been made for this edition are based on suggestions from colleagues and other long-time users of the first and second editions, on continuing interactions with my students, and on developments within the evolving fields of computer science and mathematics.

**Improved Pedagogy**

- The number of exercises has been increased to almost 2,500. Approximately 980 new exercises have been added.
- Exercises have been added for topics where students seemed to need additional practice, and they have been modified, as needed, to address student difficulties.
- Additional full answers have been incorporated into Appendix B to give students more help for difficult topics.

- The exposition has been reexamined throughout and revised where needed.
- Careful work has been done to improve format and presentation.
- Discussion of historical background and recent results has been expanded and the number of photographs of mathematicians and computer scientists whose contributions are discussed in the book has been increased.

**Logic**

- The treatment of quantification has been significantly expanded, with a new section entirely devoted to multiple quantifiers.
- Exercises have been added using Tarski's World, an excellent pedagogical tool developed by Jon Barwise and John Etchemendy at Stanford University.
- Applications related to Internet searching are now included.
- Terms for various forms of argument have been simplified.

**Introduction to Proof**

- The directions for writing proofs have been expanded.
- The descriptions of methods of proof have been made clearer.
- Exercises have been revised and/or relocated to promote the development of student understanding.

**Induction and Recursion**

- The format for outlining proofs by mathematical induction has been improved.
- The subsections in the section on sequences have been reorganized.
- The sets of exercises for the sections on strong mathematical induction and the well-ordering principle and on recursive definitions have been significantly expanded.

**Number Theory**

- A subsection on open problems in number theory has been incorporated, and the discussion of recent mathematical discoveries in number theory has been expanded.
- A new section on modular arithmetic and cryptography has been added. It includes a discussion of RSA cryptography, Fermat's little theorem, and the Chinese remainder theorem.
- The discussion of testing for primality has been moved to later in Chapter 3 to make clear its dependence on indirect argument.

**Set Theory**

- The properties of the empty set are now introduced in the first section of Chapter 5.
- The second section of Chapter 5 is now entirely devoted to element proofs.
- Algebraic proofs of set properties and the use of counterexamples to disprove set properties have been moved to the third section of Chapter 5.
- The treatment of Boolean algebras has been expanded, and the relationship among logical equivalences, set properties, and Boolean algebras has been highlighted.

**Combinatorics and Discrete Probability**

- Exercises for the section on the binomial theorem has been significantly expanded.
- Two new sections have been added on probability, including expected value, conditional probability and independence, and Bayes' theorem.
- Combinatorial aspects of Internet protocol (IP) addresses are explained.

**Functions**

- Exercises about one-to-one and onto functions have been refined and improved.
- The set of exercises on cardinality with applications to computability has been significantly expanded.

**Efficiency of Algorithms**

- Sections 9.2 and 9.4 have been reworked to add $\Theta$- and $\Omega$-notations.
- Sections 9.3 and 9.5 have been revised correspondingly, with a clearer explanation of the meaning of order for an algorithm.
- The treatment of insertion sort and selection sort has been improved and expanded.

**Regular Expressions and Finite-State Automata**

- The previous disparate sections on formal languages and finite-state automata have been reassembled into a chapter of their own.
- A new section on regular expressions has been added, as well as discussion of the relationship between regular expressions and finite-state automata.

## *Website*

A website has been developed for this book that contains information and materials for both students and instructors. It includes

- descriptions and links to many sites on the Internet with accessible information about discrete mathematical topics,
- links to applets that illustrate or provide practice in the concepts of discrete mathematics,
- additional examples and exercises with solutions,
- review guides for the chapters of the book.

A special section for instructors contains

- transparency masters and PowerPoint slides,
- additional exercises for quizzes and exams.

## *Student Solutions Manual/Study Guide*

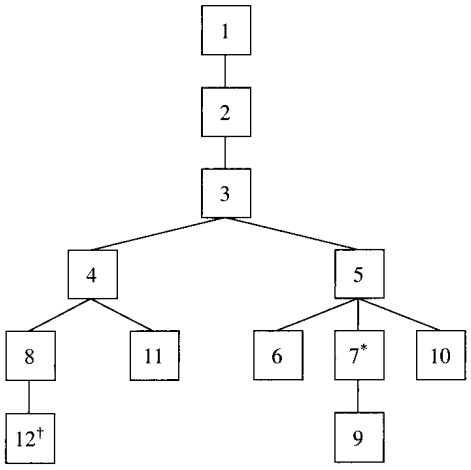
In writing this book, I strove to give sufficient help to students through the exposition in the text, the worked examples, and the exercise solutions, so that the book itself would provide all that a student would need to successfully master the material of the course. I believe that students who finish the study of this book with the ability to solve, on their own, all the exercises with full solutions in Appendix B will have developed an excellent command of the subject. Nonetheless, I have become aware that some students want the opportunity to obtain additional helpful materials. In response, I have developed a Student Solutions Manual/Study Guide, available separately from this book, which contains complete solutions to every exercise that is not completely answered in Appendix B and whose number is divisible by 3. The guide also includes alternative explanations for some of the concepts, and review questions for each chapter.

## *Organization*

This book may be used effectively for a one- or two-semester course. Each chapter contains core sections, sections covering optional mathematical material, and sections covering optional applications. Instructors have the flexibility to choose whatever mixture will best serve the needs of their students. The following table shows a division of the sections into categories.

| Chapter | Core Sections | Sections Containing Optional Mathematical Material | Sections Containing Optional Computer Science Applications |
|---|---|---|---|
| 1 | 1.1–1.3 | | 1.4, 1.5 |
| 2 | 2.1–2.4 | 2.2, 2.3 | 2.3 |
| 3 | 3.1–3.4, 3.6 | 3.5, 3.7 | 3.8 |
| 4 | 4.1–4.2 | 4.3–4.4 | 4.5 |
| 5 | 5.1 | 5.2–5.4 | 5.4 |
| 6 | 6.1–6.4 | 6.5–6.9 | 6.3 |
| 7 | 7.1–7.2 | 7.3–7.5 | 7.1, 7.2, 7.5 |
| 8 | 8.1, 8.2 | 8.3, 8.4 | 8.4 |
| 9 | 9.1, 9.2 | 9.4 | 9.3, 9.5 |
| 10 | 10.1–10.3 | 10.4, 10.5 | 10.4, 10.5 |
| 11 | 11.1, 11.5 | 11.2, 11.3, 11.4 | 11.1, 11.2, 11.5, 11.6 |
| 12 | 12.1, 12.2 | 12.3 | 12.1–12.3 |

The following tree diagram shows, approximately, how the chapters of this book depend on each other. Chapters on different branches of the tree are sufficiently independent that instructors need to make at most minor adjustments if they skip chapters but follow paths along branches of the tree.




*Instructors who wish to define a function as a binary relation can cover Section 10.1 before Section 7.1.

†Section 10.3 is needed for Section 12.3 but not for Sections 12.1 and 12.2.