Athens University of Economics and Business

Stohastic Finance Final Assignment

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July 5, 2024

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Chapter 1

Introduction

This project explores the theory , computation and application of time series models in financial asset analysis, focusing on the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model and its variants. Time series models are pivotal in capturing the dynamics of financial markets, where volatility is a key factor. By analyzing data from Yahoo's stock close prices, this study aims to understand and forecast financial time series behavior. We will delve into the theoretical underpinnings of stationarity and heteroskedasticity, implement computational techniques using R, and discuss the practical implications of these models in risk management and financial forecasting. The project seeks to provide a comprehensive understanding of GARCH models' capabilities and limitations in financial time series analysis.

Chapter 2

Stationarity

A time series (Y_t) is called (weak) stationary if

- The mean function $\mathbb{E}[Y_t]$ is constant over time
- The variance function $Var(Y_t)$ is constant over time
- The autocovariance function $cov(Y_t, Y_{t+h})$ depends only on lag h.

There are several way to check stationarity in time series graphically or using tests such the Dickey-Fuller test. For the latter, we take $Y_t = a + b \cdot Y_{t-1} + \epsilon_t$

- H_0 : The series is not stationary(b=1)
- H_1 : The series is stationary (b¡1)

The corresponding test statistic is $t = \frac{\hat{b}}{s.e(\hat{b})}$. Due to H_0 t doesn't follow the t-student distribution. In order to reject or not the Null, we use some 'corrected' critical values. We reject H_0 if t is smaller than the critical value.

2.1 Heteroskedasticity models

Modelling financial time series from real life data, we can meet concrete behaviors such some heteroskedastic effects, meaning that the volatility of the considered process is generally not constant. The presence of heteroskedasticity is ignored in some financial models such as the Black-Scholes model, which is widely used to determine the fair pricing of European-style options. It needs assumptions about the distribution and stationarity of the underlying process which are unrealistic in general.

In order to address the heteroskedasticity effects we construct time series models of heteroscedasticity:

1. ARCH(p): Autoregressive Conditional Heteroscedasticity models were introduced by Robert Engle (1982). The Arch(p) model can be written by

$$R(t) = \sigma(t) \cdot \epsilon(t),\, \epsilon(t) \overset{i.i.d}{\sim} \mathcal{N}(0,1) \tag{2.1}$$

$$\sigma^2(t) = \omega + b_1 \cdot R^2(t) + \ldots + a_p \cdot R^2(t-p) \tag{2.2}$$

   where $\omega > 0$, $ai \geq 0$ in order to be well defined the conditional variance. The conditional variance depends on lagged squared value of R.

2. Garch(p,q): Tim Bollerslev (1986) extended the ARCH model to allow conditional to have an additional autoregressive structure. The Garch(p,q) model can be determined by

$$R(t) = \sigma(t) \cdot \epsilon(t),\, \epsilon(t) \overset{i.i.d}{\sim} \mathcal{N}(0,1) \tag{2.3}$$

$$\sigma^2(t) = \omega + b_1 \cdot R^2(t) + \ldots + a_p \cdot R^2(t-p) + b_1 \cdot \sigma^2(t-1) + \ldots + b_q \cdot \sigma^2(t-q) \tag{2.4}$$

   where $\omega > 0$, $a_i \geq 0$, $b_i \geq 0$ in order to be well defined the conditional variance. The conditional variance depends on lagged R^2 and on lagged conditional variances.

3. Garch(p,q) with errors which follow the t-student distribution with unknown degrees of freedom

4. EGarch(p,q): Nelson (1991) defined the following model:

$$\begin{aligned} R(t) &= \sigma(t) \cdot \epsilon(t),\, \epsilon(t) \overset{i.i.d}{\sim} \mathcal{N}(0,1) \\ \log\left(\sigma^2(t)\right) &= \omega + \sum_{i=1}^{q} a_i \cdot \log\left(\sigma^2(t-i)\right) + \sum_{i=1}^{p} \left[b_i \cdot \epsilon(t-i) + \gamma_i \cdot \left[|\epsilon(t-i)| - \mathbb{E}\left[|\epsilon(t-i)|\right]\right]\right] \end{aligned}$$

The GARCH model extends the ARCH model by incorporating past conditional variances, allowing for a more flexible and realistic modeling of financial time series data. In a GARCH(p,q) model, the current variance is a function of past squared returns and past variances, capturing volatility clustering more effectively. The standard GARCH assumes normally distributed errors, which can be limiting. To address heavy tails often observed in financial data, the GARCH model with Student-t errors is used. This extension allows for modeling of extreme values and provides a better fit for financial returns. Both models are crucial for understanding and forecasting volatility, making them indispensable in risk management and financial modeling.

Chapter 3

Data and analysis

To apply the above theory we will analyze data from Yahoo's stock close using R's quantmod package. Data is daily and dates from '2015-01-05' to '2023-03-25'. This process will be denoted by $S(t)$.

Stationarity is crucial in time series analysis as it implies that statistical properties such as mean, variance, and autocovariance remain constant over time. Non-stationary data can lead to misleading results in modeling and forecasting. To address non-stationarity, we use the first-order difference of logarithms, $R(t) = \log(S(t)/S(t-1))$, which stabilizes the mean and reduces trends and seasonal effects. This transformation helps in achieving a stationary series suitable for further analysis with GARCH models.

First, we will check stationarity of the available data graphically and using the unit root test.

In the following graph, we notice that the time series doesn't move near a constant mean and consequently seems to be nonstationary.

Fig. 1: Plot of $S(t)$

The same inference can be derived by the unit root test, because the corresponding p-value is bigger that 0.05 and thus we can't reject H_0. Therefore, our data is not stationary.

```
    Augmented Dickey-Fuller Test


Dickey-Fuller = -2.8898, Lag order = 12, p-value = 0.2016
alternative hypothesis: stationary
```

In order to address this problem, we use the first order difference between the corresponding logarithms, i.e

$$R(t) := \log\left(\frac{S(t)}{S(t-1)}\right). \tag{3.1}$$

In the following graph we notice that this time series seems to be stationary.

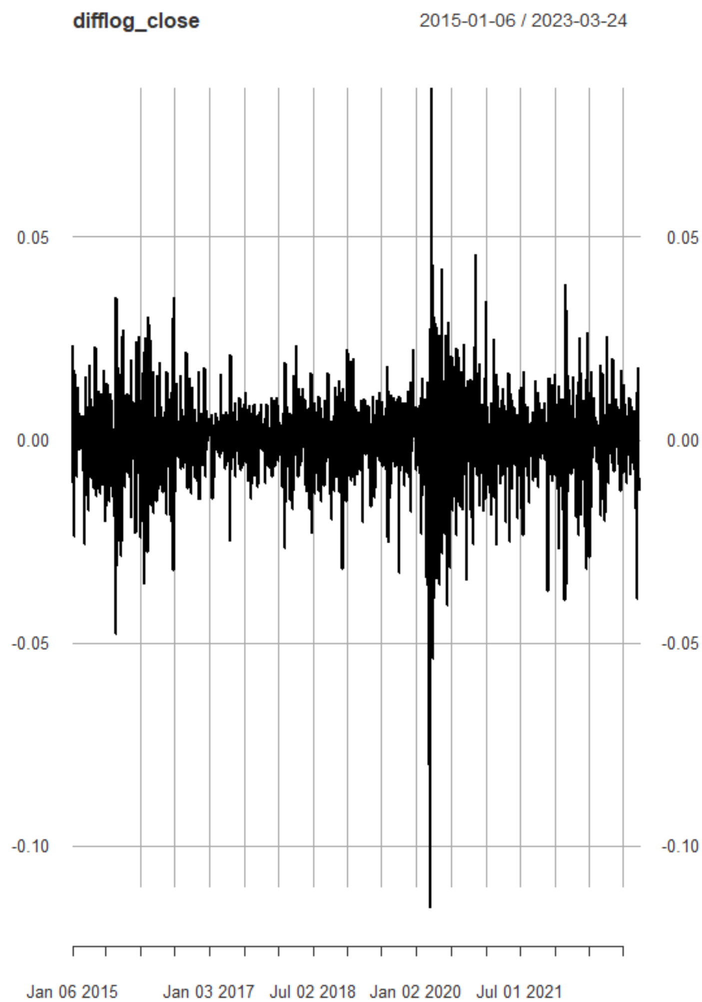


Fig. 2: Plot of $R(t)$

Indeed, the Dickey-Fuller test gives us the same result. The observed significance level is smaller than 0.05 and reject the Null. Therefore, timeseries is stationary.

```
    data:   difflog_close
Dickey-Fuller = -13.789, Lag order = 12, p-value = 0.01
alternative hypothesis: stationary
```

This timeseries has the property of a weak stationary process but seems to randomly fluctuate around zero, meaning there is little autocorrelation. This is confirmed by a plot of the sample autocorrelation function.The ACF plot of $R(t)$ in Fig. 3 shows the autocorrelation of the differenced log returns over various lags. The plot indicates that the autocorrelations are not significant, implying that the differenced series is approximately white noise. This confirms that the transformation to $R(t)$ has effectively removed the autocorrelation present in the original series, making it suitable for further analysis using GARCH models.

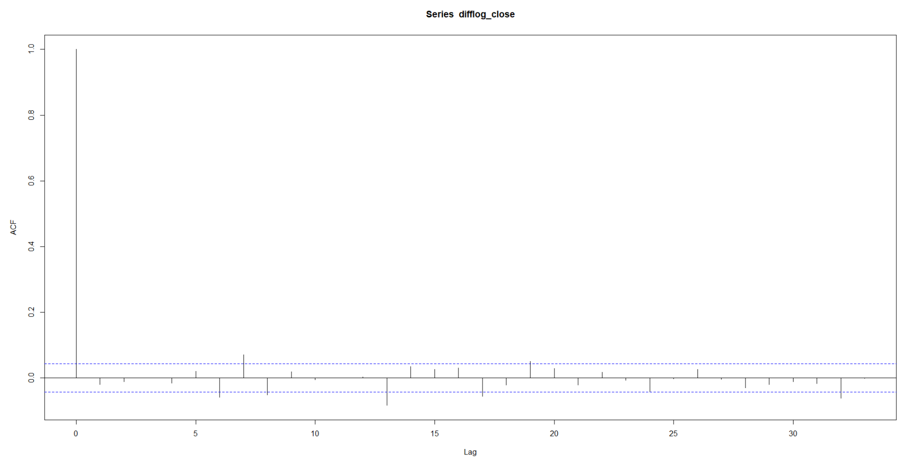


Fig. 3: ACF plot of $R(t)$

Using a histogram and normal quantile plot of $R(t)$, we conclude the non-normality of $R(t)$.

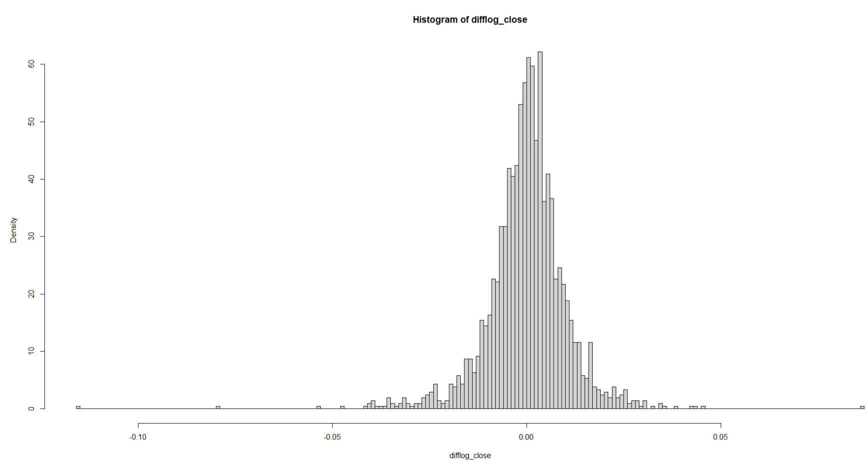
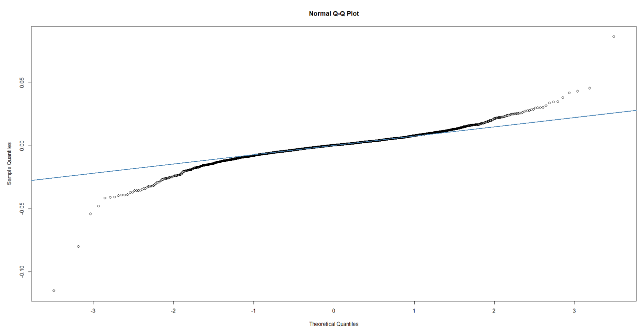
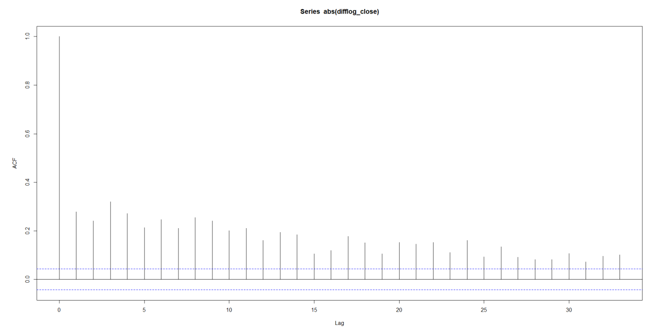


Fig. 4: Histogram of $R(t)$



(a) Normal quantile plot of $R(t)$



(b) Autocorrelation plot of $|R(t)|$

In addition, as shown in Fig. 2, there is visual evidence that the series of returns exhibits conditional heteroskedasticity since we observe volatility clustering. This is confirmed by the following graphs:

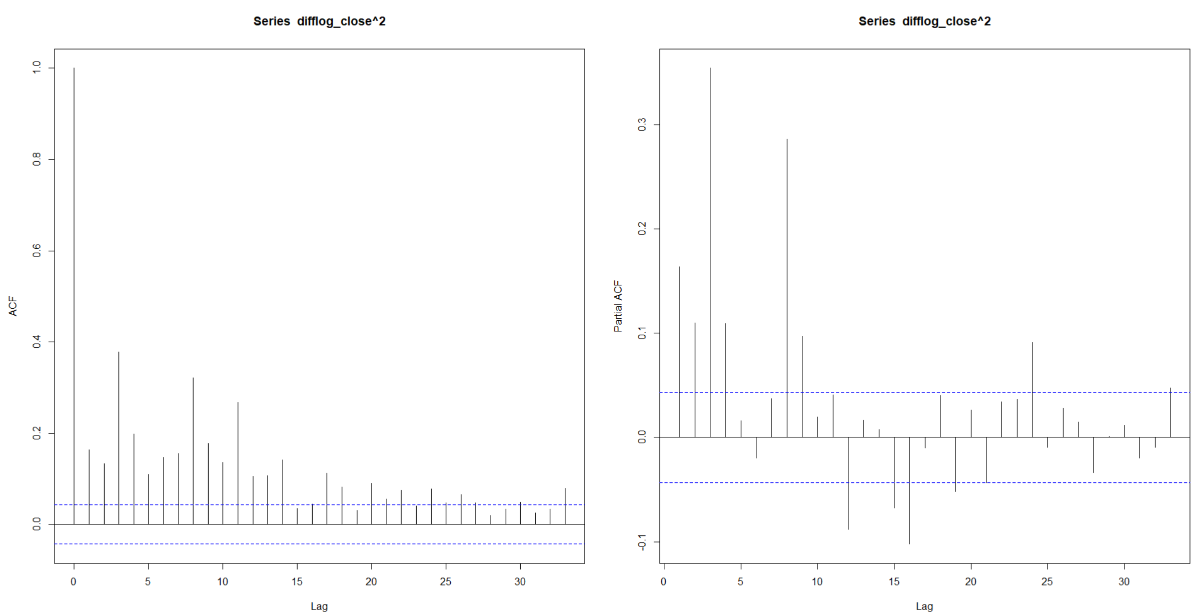


Fig. 6: ACF and PACF plots of $R^2(t)$

The ACF and PACF plots of $R^2(t)$ confirm the presence of heteroskedasticity, as they show significant autocorrelation in the squared returns over various lags. This indicates that the volatility of the time series is not constant. (Fig. 6) do the same here,fit them in a page ,scale if needed..put the 2 first plots next to each other first,and then fit everything in a page

Chapter 4

The standard Garch Model

Consider the standard Garch(1,1) model

$$\begin{aligned} R(t) &= \sigma(t)\cdot\epsilon(t),\, \epsilon(t)\sim\mathcal{N}(0,1) && (4.1)\\ \sigma^2(t) &= \omega + a_1\cdot R^2(t) + b_1\cdot\sigma^2(t-1) && (4.2)\end{aligned}$$

We'll estimate this model for $R(t)$ using the package "rugarch" in R and we obtain:

```
    omega        alpha1        beta1
5.884036e-06  1.570747e-01  7.849571e-01
```

Then, the model can by determined by

$$\begin{aligned} \hat{R}(t) &= 0 && (4.3)\\ \hat{\sigma^2}(t) &= 5.884036\cdot 10^{-6} + 0.1570747\cdot R^2(t) + 0.7849571\cdot\sigma^2(t-1) && (4.4)\end{aligned}$$

we take $\hat{a}_1 + \hat{b}_1 = 0.9420318 < 1$, and notice that our model is stationary. Hence, the unconditional variance is constant and equal to

$$\bar{\sigma}^2 = \frac{5.884036\cdot 10^{-6}}{1-0.9420318} = 0.0001015046. \quad (4.5)$$

Then, for the conditional volatility $\mathbb{E}_{t-1}\left[R^2(t+n)\right]$ holds

$$\begin{aligned} \mathbb{E}_{t-1}\left[R^2(t+n)\right] &= 0.0001015046 + 0.9420318^n\cdot(0.000170213 - 0.0001015046)\\ &= 0.0001015046 + 0.9420318^n\cdot 6.87084 * 10^{-5}\end{aligned}$$

The plot in Fig. 7 shows the estimated conditional standard deviations ($\pm$ 2 standard deviations) of $R(t)$, highlighting periods of high and low volatility (volatility clustering). The GARCH model adjusts these based on recent market movements, predicting higher volatility following a spike. The $\pm$ 2 standard deviations range indicates where about 95% of returns are expected to lie, assuming normality. This range is essential for risk management, helping to assess potential extreme movements in stock prices. The parameters α_1 and β_1 influence the width of these bands, with higher values indicating more significant and persistent impacts on future volatility.

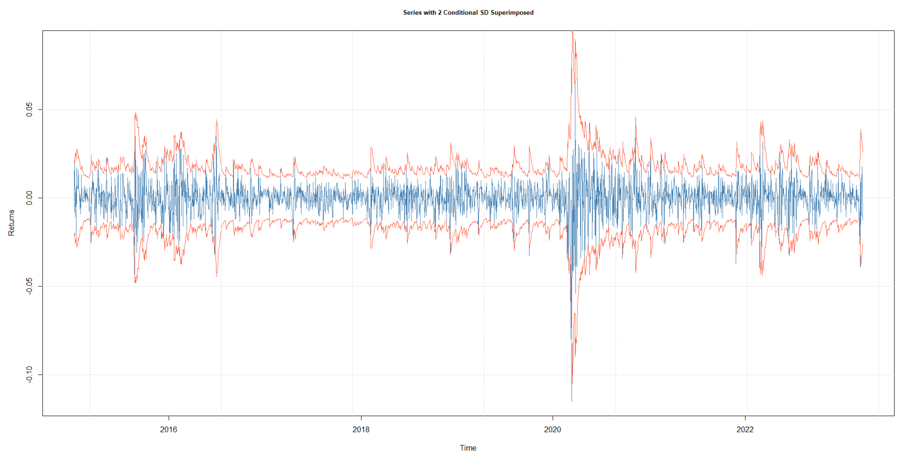


Fig. 7: estimated of +- 2 conditional standard deviations of $R(t)$

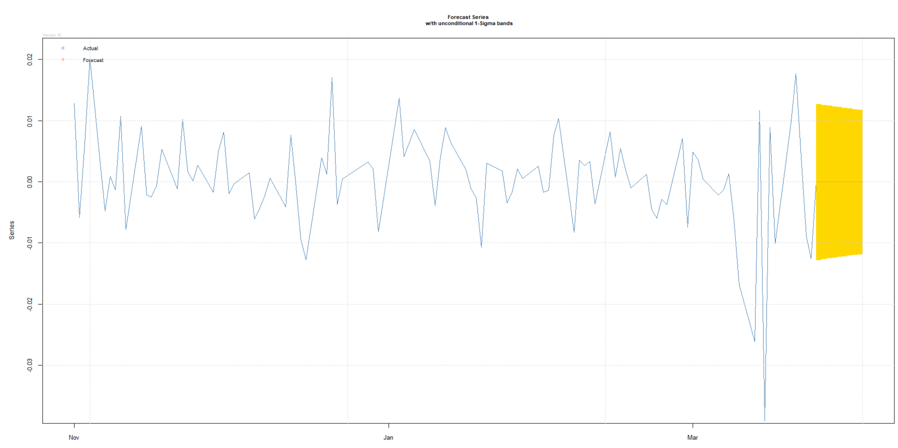


Fig. 8: 10 predicted values of $R(t)$

The predicted values align closely with observed values, demonstrating the model's accuracy in forecasting returns. This indicates the model's effectiveness in capturing the underlying patterns and trends in the time series data. (Fig. 8) The predicted unconditional sigma values are stable and consistent, indicating reliable long-term volatility predictions. This stability suggests that the model can be trusted for ongoing risk management and financial decision-making. (Fig. 9)

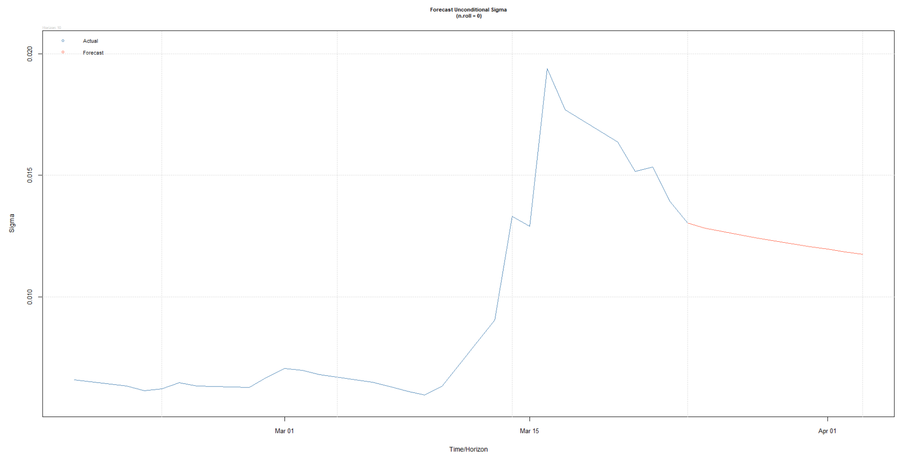


Fig. 9: 10 predicted values of uncoditional sigma

4.1 The t-student Garch model

As we said earlier, while the standard GARCH model assumes normally distributed errors, financial return series often exhibit heavy tails that the normal distribution cannot capture. To address this, we use Student-t errors in the GARCH model. This extension allows for a better fit to financial data by modeling extreme values more accurately, providing a more robust understanding of volatility in financial time series.

Consider the standard Garch(1,1) model

$$R(t) = \sigma(t) \cdot \epsilon(t),\, \epsilon(t) \sim \mathcal{N}(0,1) \tag{4.6}$$

$$\sigma^2(t) = \omega + a_1 \cdot R^2(t) + b_1 \cdot \sigma^2(t-1) \tag{4.7}$$

We'll estimate this model for $R(t)$ using the package "rugarch" in R and we obtain:

```
       omega       alpha1        beta1        shape
4.410999e-06 1.547134e-01 8.090703e-01 5.309958e+00
```

Then, the model can by determined by

$$\hat{R}(t) = 0 \tag{4.8}$$

$$\hat{\sigma^2}(t) = 4.410999 \cdot 10^{-6} + 0.1547134 \cdot R^2(t) + 0.8090703 \cdot \sigma^2(t-1) \tag{4.9}$$

we take $\hat{a}_1 + \hat{b}_1 = 0.9637837 < 1$, and notice that our model is stationary. Hence, the unconditional variance is constant and equal to

$$\bar{\sigma}^2 = \frac{4.410999 \cdot 10^{-6}}{1 - 0.9637837} = 0.0001217959. \tag{4.10}$$

Then, for the conditional volatility $\mathbb{E}_{t-1}\left[R^2(t+n)\right]$ holds

$$\begin{aligned}\mathbb{E}_{t-1}\left[R^2(t+n)\right] &= 0.0001217959 + 0.9637837^n \cdot (0.0001859365 - 0.0001217959) \\ &= 0.0001015046 + 0.9637837^n \cdot 6.414056 * 10^{-5}\end{aligned}$$

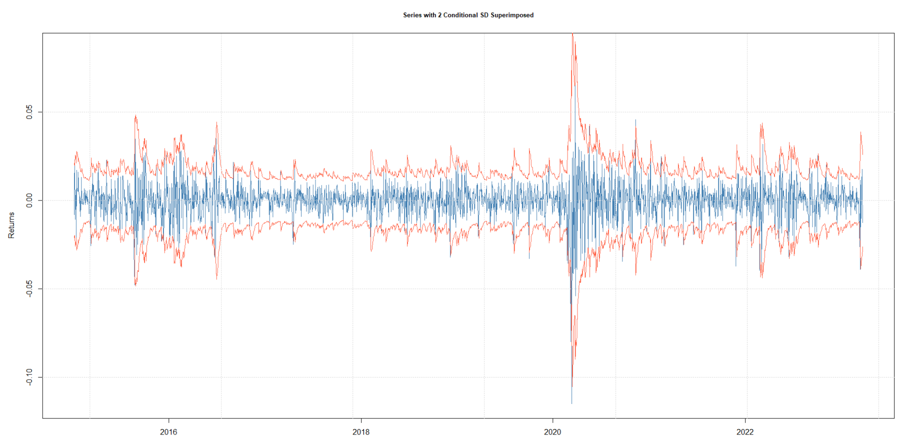
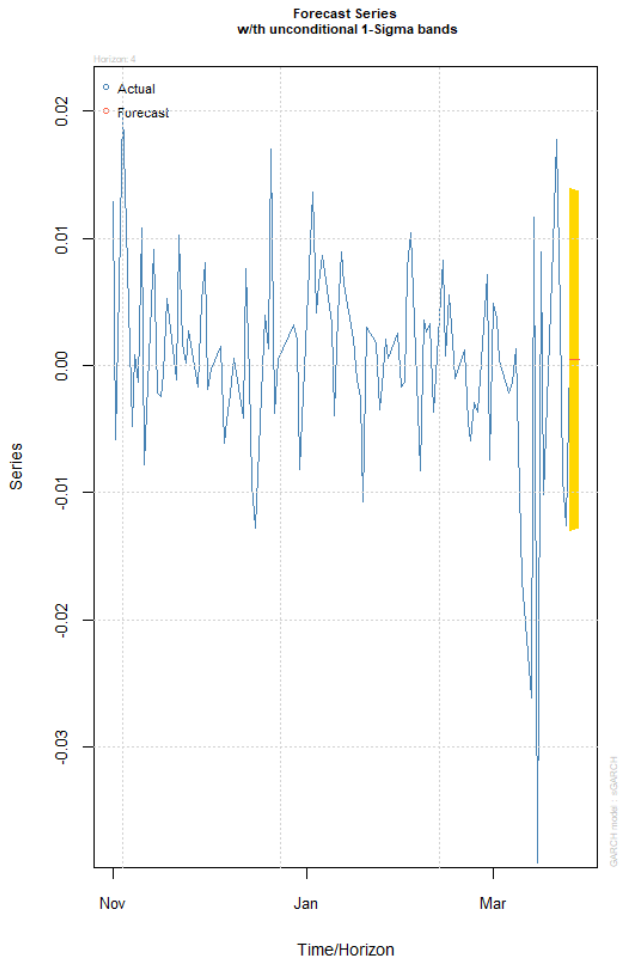
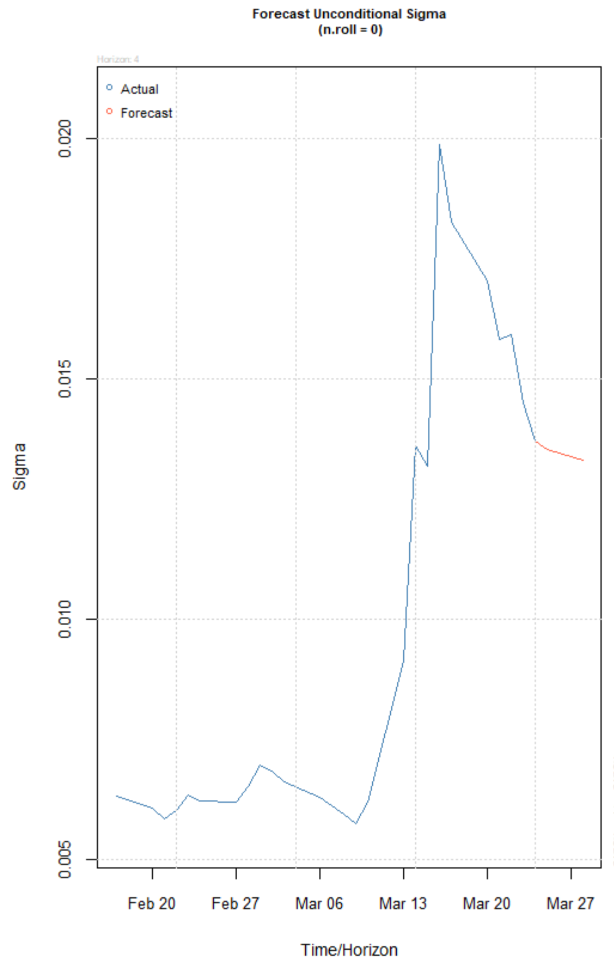


Fig. 10: estimated of +- 2 conditional standard deviations of $R(t)$

The model captures extreme values more effectively, providing better predictions of high volatility periods. This improvement highlights the advantage of using the T-Student distribution for modeling financial data with heavy tails. (Fig. 10) The T-Student GARCH model improves fit, capturing heavy tails and providing more accurate volatility forecasts. This enhancement makes the model more robust for financial time series analysis and risk assessment. (Fig. 11)



(a) 10 predicted values of $R(t)$



(b) 10 predicted values of unconditional sigma

Fig. 11: Predicted values

Chapter 5

The EGarch(1,1) Model

Consider the standard EGarch(1,1) model

$$\begin{aligned} R(t) &= \sigma(t)\cdot\epsilon(t),\ \epsilon(t) \overset{i.i.d}{\sim} \mathcal{N}(0,1) \\ \log\left(\sigma^2(t)\right) &= \omega + b_1\cdot\log\left(\sigma^2(t-1)\right) + a_1\cdot\epsilon(t-1) + \gamma_1\cdot\left[|\epsilon(t-1)| - \mathbb{E}\left[|\epsilon(t-1)|\right]\right] \end{aligned}$$

Set $g(t-i) =: a_1\cdot\epsilon(t-i) + \gamma_1\cdot\left[|\epsilon(t-i)| - \mathbb{E}\left[|\epsilon(t-i)|\right]\right]$ and $\log\left(\sigma^2(t)\right)$ can be rewritten

$$\begin{aligned} \log\left(\sigma^2(t)\right) &= \omega + b_1\cdot\log\left(\sigma^2(t-1)\right) + g(t-1) \\ &= \omega + b_1\cdot\left[\omega + b_1\cdot\log\left(\sigma^2(t-2)\right) + g(t-2)\right] + g(t-1) \\ &= \omega + b_1\cdot\omega + b_1^2\cdot\log\left(\sigma^2(t-2)\right) + b_1\cdot g(t-2) + g(t-1) \\ &= \omega\cdot[1+b_1]\, a_1^2\cdot\log\left(\sigma^2(t-2)\right) + b_1\cdot g(t-2) + g(t-1) \\ &= \ldots \\ &= \omega\cdot\sum_{j=0}^{n} b_1^j + b_1^n\cdot\log\left(\sigma^2(t-n)\right) + \sum_{j=1}^{n} b_1^{j-1}\cdot g(t-j) \end{aligned}$$

In order to prove when the above process is weakly stationary, we examine the limiting behavior of the sequence $S_n := \sum_{j=1}^n a_1^{j-1}\cdot g(t-j)$. The random variables $Y_j = a_1^{j-1}\cdot g(t-j)$ form a sequence of independent random variables. Then, we have:

$$\begin{aligned} \mathbb{E}[Y_k] &= 0 \\ \mathbb{V}[Y_k] &= a_1^{2j-2}\cdot s^2,\ s^2 := \mathbb{V}[g(t-j] \end{aligned}$$

Therefore, $s_n^2 = \sum_{j=1}^n (a_1^{j-1})^2\cdot s^2 = \frac{s^2\cdot(1-a^n)}{1-a_1^2}$.

If $|b_1| < 1$, we take $\lim s_n^2 = \frac{s^2}{1-b_1^2}$ and $\mathbb{E}\left[\log\left(\sigma^2(t)\right)\right] = \frac{\omega}{1-b_1}$, and

$$\mathbb{V}\left[\log\left(\sigma^2(t)\right)\right] = b_1^{2n} \cdot \mathbb{V}\left[\log\left(\sigma^2(t-n)\right)\right] + s_n^2$$

Define $v_1^2 := \mathbb{V}\left[\log\left(\sigma^2(t)\right)\right]$ we take

$$v^2 = b_1^{2n} v^2 + s_n^2 \to \frac{s^2}{1-b_1^2}$$

$$\begin{aligned} s^2 &= a_1^2 \cdot \mathbb{V}[\epsilon] + \gamma_1^2 \mathbb{V}[|\epsilon|] + 2 \cdot a_1 \cdot \gamma_1 Cov(\epsilon, |\epsilon|) \\ &= a_1^2 + b_1^2 \cdot (1 - \frac{2}{\pi}) + 2 \cdot a_1 \cdot \gamma_1 \cdot 0 \\ &= a_1^2 + b_1^2 \cdot (1 - \frac{2}{\pi}) \end{aligned}$$

Using an Exponential Garch(1,1) we take:

omega	alpha1	beta1	gamma1
−0.3419884	−0.1562591	0.9634306	0.1392084

Then, the model can by determined by

$$\hat{R}(t) = 0 \tag{5.1}$$

$$\log\left(\sigma^2(t)\right) = -0.3419884 - 0.1562591\epsilon(t-1) + 0.9634306 \cdot \log\left(\sigma^2(t-1)\right) \tag{5.2}$$

$$+ \; 0.1392084 \cdot \left[|\epsilon(t-1)| - \mathbb{E}\left[|\epsilon(t-1)|\right]\right] \tag{5.3}$$

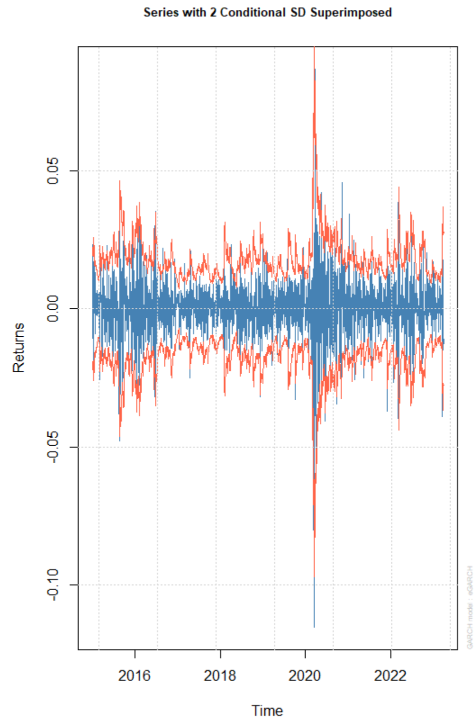
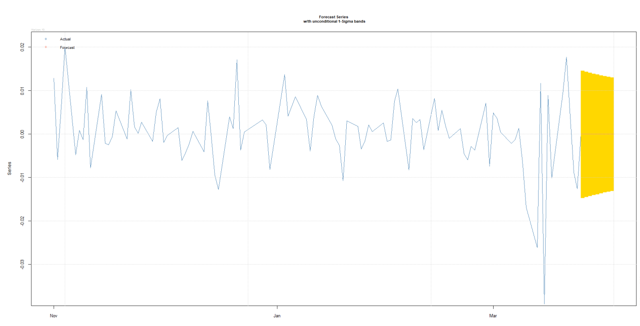
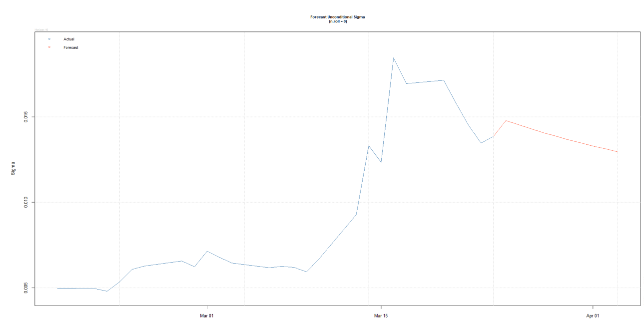


Fig. 12: Estimated of +- 2 conditional standard deviations of $R(t)$

We take $\hat{b_1} = 0.9634306$ and notice that our model is stationary.



(a) 10 predicted values of $R(t)$



(b) 10 predicted values of unconditional sigma

Fig. 13: Predicted values

Chapter 6

Conclusions

This project demonstrates the critical role of GARCH models in financial asset analysis, specifically in capturing and forecasting market volatility. By analyzing Yahoo's stock close prices, the study highlights the effectiveness of these models in addressing the complexities of financial time series.

The GARCH(1,1) model successfully captures volatility clustering, indicating periods of high and low market volatility. Its parameters show a robust fit, making it a valuable tool for risk management. The t-student GARCH model further improves predictive accuracy by accommodating heavy tails in financial data, enhancing its reliability in forecasting extreme market movements.

The EGARCH model offers a unique advantage by modeling the logarithm of the conditional variance, effectively handling data asymmetries. This approach not only maintains stationarity but also provides consistent long-term volatility predictions.

Overall, the GARCH models provide essential insights into market dynamics, supporting informed decision-making in financial risk management. The study underscores the strengths of each model, offering a nuanced understanding of their applications and limitations in financial data analysis.

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