Stohastic models in Finance
mid-term.
May 2024

Exercise 1

a) In the context of a trinomial model for a risky asset, we need to examine the existence of an equivalent martingale measure (EMM) Q. An EMM is a probability measure under which the discounted price process of the risky asset becomes a martingale. This means that the expected discounted future price is equal to the current discounted price.

Now we will outline the trinomial model and derive the conditions for the existence of an EMM.

In a single time step, the price of a risky asset can move to the three states below.

$$S_{n+1} = H_{n+1} S_n \qquad H_{n+1} = \begin{cases} u, & p_1 = q_1 \\ m, & p_2 = q_2 \\ d, & p_3 = 1 - p_1 - p_2 = 1 - q_1 - q_2 = q_3 \end{cases}$$

$$E[H_{n+1}] = u \cdot p_1 + m \cdot p_2 + d \cdot (1 - p_1 - p_2)$$

$$E[S_{n+1}] = (u \cdot p_1 + m \cdot p_2 + d(1 - p_1 - p_2))^n \cdot S_0$$
$$= ((u-d) \cdot p_1 + (m-d) \cdot p_2 + d)^n \cdot S_0$$

$$S_n^* = (1+r)^{-n} \cdot S_n \to S_0^* = (1+r)^{0} \cdot S_0 = S_0$$

$$q_1 > 0 \quad , \quad q_2 > 0, \quad q_3 > 0$$

$$\frac{1}{1+r} E_Q[S_1] = S_0$$

$$E_Q[S_1] = (q_1 \cdot u + q_2 \cdot m + q_3 \cdot d) \cdot S_0 \iff$$

$$(1+r) \cdot S_0 = [q_1(u-d) + q_2(m-d) + d) \cdot S_0 \iff$$

$$1 = q_1 \frac{u-d}{1+r} + q_2 \frac{m-d}{1+r} + \frac{d}{1+r}$$

$$q_2 = \lambda \quad , \quad 0 < \lambda < 1 \quad , \quad 1 > 1 - q_1 - \lambda > 0$$

$$0 < q_1 < 1$$

$$1 + r = q_1 (u-d) + \lambda (m-d) + d \iff$$

$$0 < q_1 = \frac{1 + r - \lambda(m-d) - d}{u-d} < 1 \iff$$

$$0 < 1 + r - \lambda(m-d) - d < 1 \iff$$

$$-1 < \lambda(m-d) + d - (1+r) < 1 \iff$$

$$r < \lambda(m-d) + d < 2 + r \iff$$

$$r - d < \lambda(m-d) < 2 + r - d \iff$$

$$\frac{r-d}{m-d} < \lambda < \frac{2+r-d}{m-d}$$

$$\lambda \in \left(\max\left\{0, \frac{r-d}{m-d}\right\}, \min\left\{1, \frac{2+r-d}{m-d}\right\} \right)$$

$$q_3 = \frac{1 - 1 + r - \lambda(m-d) + d}{u-d} - \lambda. \iff$$

$$q_3 = \frac{u-d-(1+r-\lambda(m-d)+d) - \lambda(u-d)}{u-d}$$

$$= \frac{u-d-1-r+\lambda m-\lambda d-\lambda u+\lambda d-d}{u-d}$$

$$= \frac{u-2d-1-r+\lambda(m-u)}{u-d} \in (0,1)$$

$$0 < u-2d < (1+r)+\lambda(m-u) < u-d \iff$$

$$(1+r)-(u-2d) < \lambda(m-u) < u-d+(1+r)-(u-2d) \iff$$

$$\frac{1+r-(u-2d)}{m-u} > \lambda > \frac{u-d+(1+r)-(u-2d)}{m-u}$$

So these linear equations with three unknowns will have an infinate amount of solutions. So the trinomial model admits an infinate number of EMMs.

b) Arbitrage opportunities exist if there is a way to construct a portfolio that guarantees a positive payoff withe zero initial investment and no risk loss.
In our case iff there exist at least one set of positive risk-neutral probabilites i do not have an arbitrage.

so. in a) we've shown that it is possible to find positive risk-neutral probabilities when $1+r$ lies between d and u. Thus, given these conditions, there is no arbitrage opportunities.

Exercise 3

we need to show that:

$$S(t) \xrightarrow{d} \mu t + \sigma B(t)$$

for the binomial model we have

$$\frac{S_n}{S_0} = H_1 \dots H_n \quad , \quad H_i = \begin{cases} u, & p_u \\ d, & p_d = 1-p_u \end{cases}$$

$$\log\left(\frac{S_n}{S_0}\right) = \sum \log(H_i)$$

$$E\left[\log\left(\frac{S_n}{S_0}\right)\right] = n\left[\log u \cdot p_u + \log d(1-p_u)\right]$$

$$\sigma_n^2 = Var\left[\log\left(\frac{S_n}{S_0}\right)\right] = n\left[(\log u - \mu_n)^2 p_u + (\log d - \mu_n)^2 (1-p_u)\right]$$

set $u \cdot d = 1$

$$E\left[\log\left(\frac{S_n}{S_0}\right)\right] = n \underbrace{\log u \, (p_u - p_d)}_{\mu_1}$$

$$Var\left[\log\left(\frac{S_n}{S_0}\right)\right] = n \underbrace{\left[p_u(\log u - \mu_1) + (1-p_u)(\log u + \mu_1)^2\right]}_{\sigma_1^2}$$

and $\mu t = n\mu_1$ and $\sigma^2 t = n\sigma_1^2$

define $u = \frac{t}{n}$ and take

$\mu t = u\mu_1 \Rightarrow \mu t = \frac{t}{n}\mu_1 \Leftrightarrow \mu = \frac{\mu_1}{n}$

$\sigma^2 t = u\sigma_1^2 \Rightarrow \sigma^2 t = \frac{t}{n}\sigma_1^2 \Leftrightarrow \sigma^2 = \frac{\sigma_1^2}{n}$

The convergence in law

$\log\left(\frac{S_n}{S_0}\right) \xrightarrow{d} \mu t + \sigma B(t)$

is equivalent to $\dfrac{\log\left(\frac{S_n}{S_0}\right) - \mu t}{\sqrt{t}\,\sigma} \xrightarrow{d} N(0,1)$

The latter holds due to CLT

for the sequence $\log\left(\frac{S_n}{S_0}\right) = \sum_{k=1}^{n} H_k$

where $H_1, \dots H_n$ are i.i.d.