PHYSICS C161 Project: Black Hole Imaging

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1 Introduction

According to Eistein's general relativity, light bends in the presence of strong gravitational field. Whenever we take the picture of a black hole, what we see is actually a distorted image of what's behind the black hole. In our project we investigate this effect by using retracing method[1]. Using the fact that light paths are invertible, we suppose light rays comes out of an observer's eyes at certain angle and position in front of the black hole, and calculate the angle of the photon at a infinitely distance. We put our background picture at infinitely far, and what the photon hit is exactly what we can see.

This article is a writeup of the methods that we use to develop our black hole imaging code. We also put some results here. We first describe the analytical way to get the equations of motion for photons in a general Kerr space-time (easily set to Schwarzschild by setting spin parameter to zero) as a set of first order differential equations. Then we use 4th-order Runge Kutta method to calculate the trajectory of photons and build a connection between the initial angles and position of photon with the final angles. In this way, we can construct a picture of what we should see if there's a black hole in front of our background picture.

2 Calculation of Photon Trajectory

2.1 Equations of motions

There are several ways to analytically calculate the equation of motion for photons. The first one is to use the geodesic, which is a second-order differential equation that can be integrated. However, in our project we use constants of motion instead. In this way, we can obtain a set of first order differential equations. This should give us better precision and speed. In Kerr space-time, using Boyer-Lindquist coordinates we have

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2\mathrm{Mr}}{\Sigma}\right) & 0 & 0 & \frac{-2\,\mathrm{a}\sin^2\theta\mathrm{Mr}}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ \frac{-2\mathrm{a}\sin^2\theta\mathrm{Mr}}{\Sigma} & 0 & 0 & \left(\left(r^2+a^2\right)^2 - \Delta a^2\sin^2\theta\right)\frac{\sin^2\theta}{\Sigma} \end{pmatrix} \tag{1}$$

in which

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2Mr + a^2 \end{aligned} \tag{2}$$

The four momentum of the photon is

$$p^\mu = \left(\frac{\partial t}{\partial\lambda}, \frac{\partial x^i}{\partial\lambda}\right) \tag{3}$$

And it should satisfy $g_{\mu\nu}p^\mu p^\nu = 0$. For a general Kerr black hole, there are also two simple constants of motion that can be constructed by two killing vectors $\left(\frac{\partial}{\partial t}\right)^\mu$ and $\left(\frac{\partial}{\partial\phi}\right)^\mu$. We contract them with

[1] See [1] for detail descriptions.

p^μ respectively to get two constants of motion p_t and p_ϕ,

$$-\left(1-\frac{2\mathrm{Mr}}{\Sigma}\right)\left(\frac{dt}{d\lambda}\right)-\frac{2a\mathrm{sin}^2\theta Mr}{\Sigma}\left(\frac{d\phi}{d\lambda}\right)=-E \tag{4}$$

$$\left(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)+\left(\frac{-2a\mathrm{sin}^2\theta Mr}{\Sigma}\right)\left(\frac{dt}{d\lambda}\right)=L \tag{5}$$

By a combination of these two equations, we can deduce the time evolution equation of t and phi

$$\Sigma\left(\frac{dt}{d\lambda}\right)=-a^2E\sin^2\theta+La+\frac{\left(a^2+r^2\right)\left(E\left(r^2+a^2\right)-La\right)}{\Delta} \tag{6}$$

$$\Sigma\left(\frac{d\phi}{d\lambda}\right)\equiv-\left(aE-\frac{L}{\sin^2\theta}\right)+\frac{aT}{\Delta} \tag{7}$$

where $T\equiv E\left(r^2+a^2\right)-La$ is a quantity defined for convenience (and is not a constant). We can plug these two equation into $p_\mu p^\mu=0$. By a separation of variables, we obtain two equations and a constant. While the details are left out, we can obtain the last two equations

$$\Sigma\left(\frac{\partial\theta}{\partial\lambda}\right)=\pm\sqrt{\kappa-\cos^2\theta\left[a^2\left(E^2\right)+\frac{L^2}{\sin^2\theta}\right]} \tag{8}$$

$$\Sigma\left(\frac{\partial r}{\partial\lambda}\right)=\pm\sqrt{T^2-\Delta\left[(Ea-L)^2+\kappa\right]} \tag{9}$$

where $\kappa\equiv p_\theta^2+\cos^2\theta\left[a^2\left(\mu^2-p_t^2\right)+\frac{p_\phi^2}{\sin^2\theta}\right]$ is the Carter constant. But we have obtained it by separating variables. We also cross-checked our results with [2]. Now that we have obtained four ODEs, we can start to do numerical calculations. In practice, we start from the light emitted from an observer. For simplicity, we always set the observer on the equatorial plane, r_0 away from the black hole, and the direction of the light being labeled as θ_i and ϕ_i. Then we compute the three constants of motion and we can know how photons will travel. Note that there is a $\pm$ sign for $\frac{\partial r}{\partial\lambda}$ and $\frac{\partial\theta}{\partial\lambda}$. When we write the code, we also have to carefully switch signs. For details one can refer to the code.

2.2 Iterative algorithm: 4th order Runge Kutta Method

Now that we've derived a set of linear differential equations of first order, which can be generally written as

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f_1(x,y,z),\frac{\mathrm{d}z}{\mathrm{d}x}=f_2(x,y,z) \tag{10}$$

Here we only consider 2 unknown functions y and z of x for simplicity. We can pick a step-size $h>0$ and define

$$y_{n+1}=y_n+\frac{1}{6}(y_n^{(0)}+2y_n^{(1)}+2y_n^{(2)}+y_n^{(3)}) \tag{11a}$$

$$z_{n+1}=z_n+\frac{1}{6}(z_n^{(0)}+2z_n^{(1)}+2z_n^{(2)}+z_n^{(3)}) \tag{11b}$$

$$x_{n+1}=x_n+h \tag{11c}$$

where

$$\begin{cases} y_n^{(0)}=hf_1(x_n,y_n,z_n)\\ y_n^{(1)}=hf_1(x_n+h/2,y_n+hy_n^{(0)}/2,z_n+hz_n^{(0)}/2)\\ y_n^{(2)}=hf_1(x_n+h/2,y_n+hy_n^{(1)}/2,z_n+hz_n^{(1)}/2)\\ y_n^{(3)}=hf_1(x_n+h/2,y_n+hy_n^{(2)},z_n+hz_n^{(2)}) \end{cases} \tag{12}$$

and similarly

$$\begin{cases} z_n^{(0)}=hf_2(x_n,y_n,z_n)\\ z_n^{(1)}=hf_2(x_n+h/2,y_n+hy_n^{(0)}/2,z_n+hz_n^{(0)}/2)\\ z_n^{(2)}=hf_2(x_n+h/2,y_n+hy_n^{(1)}/2,z_n+hz_n^{(1)}/2)\\ z_n^{(3)}=hf_2(x_n+h/2,y_n+hy_n^{(2)},z_n+hz_n^{(2)}) \end{cases} \tag{13}$$

Here y_{n+1} is the approximation of $y(x_{n+1})$ and z_{n+1} is the approximation of $z(x_{n+1})$, by adding the weighted average of four increments where each increment is the product of the size of the interval h and an estimated slope specified by function f_1 or f_2. This method has the local truncation error on the order of $O(h^5)$, so this is called 4th order Runge Kutta (RK4) method. We use this method to numerically solve the above four coupled equations.

2.3 Image Processing

We will use numerical method to compute the trajectory to as large as we can. When the photon is very far from the black hole, we can assume it's moving in the final direction θ_f and ϕ_f. And using the RK4 method, we can obtain the map from the initial angles θ_i, ϕ_i to θ_f, ϕ_f. Thus, the remaining work is to convert this result into the image of our black hole.

What we need to do is to map the final angles θ_i, θ_f to a specific point on our background image. Our idea is extremely simple: we introduced an observer R_s pixels away from our imagelet light rays emitted from the observer with angles θ_f, ϕ_f. The mapped point $(x(\theta_f, \phi_f), y(\theta_f, \phi_f))$ is nothing but the intersection of the light ray and our background image. The whole process is illustrated in figure 1.

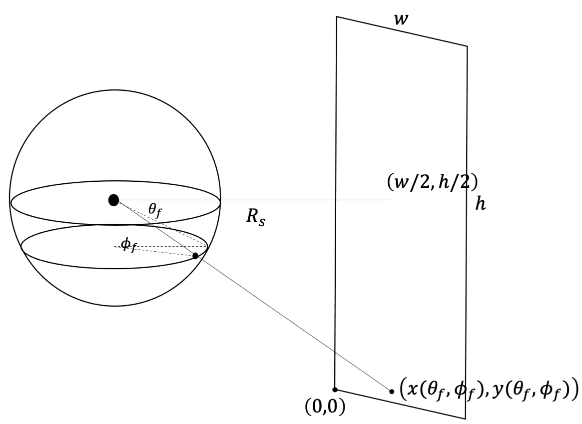

Figure 1: Illustration of the image-processing process.

With basic knowledge of geometry, one can read off the function $(x(\theta_f, \phi_f), y(\theta_f, \phi_f))$:

$$x = R_s \tan(\phi_f) + \frac{w}{2}, \quad y = \frac{h}{2} - \frac{R_s}{\cos(\phi_f)} \tan(\theta_f) \tag{14}$$

Where h and w is the height and width of our background picture pixels respectively.

Now we have the functions $(x(\theta_f, \phi_f), y(\theta_f, \phi_f))$, and we also obtained the functions $\theta_f(\theta_i, \phi_i), \phi_f(\theta_i, \phi_i)$ from the RK4 method. By simply combining those results, one can find the relation between the initial angles and the mapped point on our background image. To draw the picture, we scan through the mesh-grid of θ_i and ϕ_i, take the color of the mapped point as the color correspond to θ_i and ϕ_i, then plot everything down.

3 Result

3.1 Light Trajectory

3.1.1 Photon gaining angular momentum

First, we can graphically represent the trajectory of the photon (light) by transforming all the variables into Cartesian coordinates. When the light is emitted from the observer radially to the center of the black hole, it originally had no angular momentum. Because Kerr Black Hole is spinning, it brings angular momentum to the photon, which bents the light trajectory, as shown in figure 2.

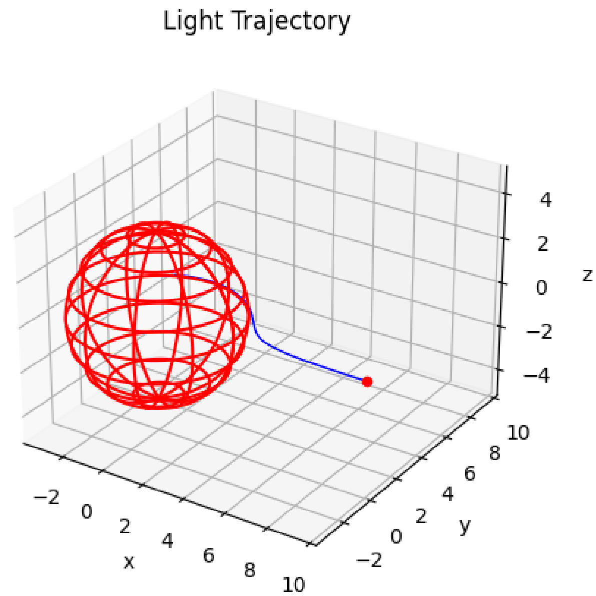

Figure 2: Graphical representation of the light rays (blue lines) and Kerr Black Hole (red sphere) with spin $a = 0.5M$. The red point in the figure corresponds to the position of the observer, which is on the equatorial plane with $r = 5M$. The angles at which the observer is looking, θ_i and ϕ_i, are both zero.

3.1.2 Falling into or passing by the Black Hole

By fixing the observer and changing the angles at which the observer is looking, we can see that the light trajectories have both the case of falling into the black hole and being bent by the black hole, respectively shown by figure 3 and 4. It's shown in both figures that the light trajectories of different angles (θ_i and ϕ_i) are not symmetric about the rz-plane ($\phi_i = 0$), which are caused by the difference of angular momentum on different directions. Even if the angle ϕ_i is approaching $\pi/2$, namely the light is emitted perpendicular to the r-direction, the light trajectory will bend towards the black hole due to the gravitational field.

3.2 Black Hole Image

3.2.1 Black Hole on a Four-color Background

To show the effects of a Kerr black hole to the background, we put it in front of a four-color background, as shown below in figure 5. Taking $R_s = 40$ pixels, varying different a, we can produce the black hole image as in figure 6 and 7. An interesting effect is that a black hole will look as if it has been shifted in the presence of spin. And there are many different colors around the black hole, showing chaotic behaviors.

3.2.2 What If We Put a Kerr Black Hole Inside the Campus?

We also produced a black hole image of resolution $256 * 256$, which takes approximately 18 hours. In theory if we have more time, we can reach even higher resolution. But such programs will be very difficult to run with our own computers. We use the picture of Berkeley as the background. We put the picture of Berkeley at sufficiently far and only use the angles θ_f, ϕ_f and R_s to calculate the color it corresponds to. It should be noted that the picture we see here is strongly distorted even without the black hole. That's because we plot it in terms of the angle. Of course we can try to map it into a picture that corresponds more to what we see in reality, but it is more of a photography problem, so we didn't do it here. To illustrate this effect, see (c) of 8. There is no black hole in this case, and the graph is plotted at $R_s = 200$. The boundary of the picture is always the most distorted, but the center will not be much affected. And this effect will enhance as R_s gets smaller, which will be very much the case for our plotting. As we can see, we see two towers as a lensing effect. And similarly we

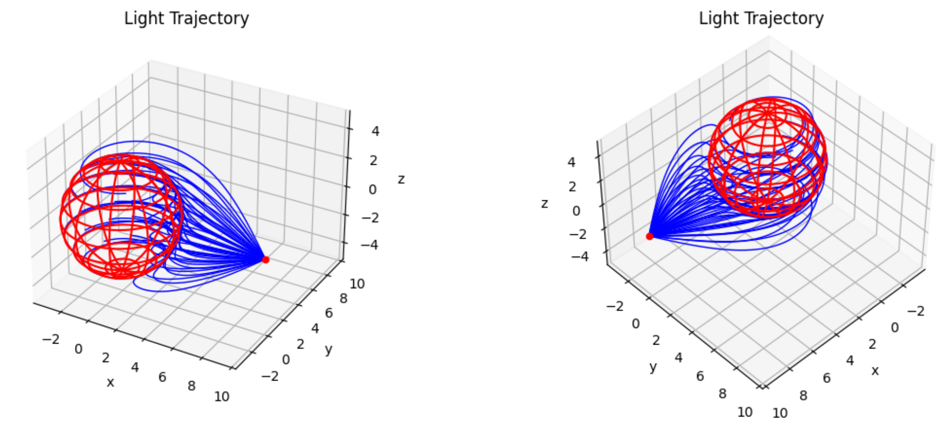

Figure 3: Graphical representation of the light rays (blue lines) and Kerr Black Hole (red sphere) with spin $a = 0.5M$ (looking from two perspectives). The red point in the figure corresponds to the position of the observer, which is on the equatorial plane with $r = 5M$. The angles at which the observer is looking, θ_i and ϕ_i, both range from $-\pi/6$ to $\pi/6$.

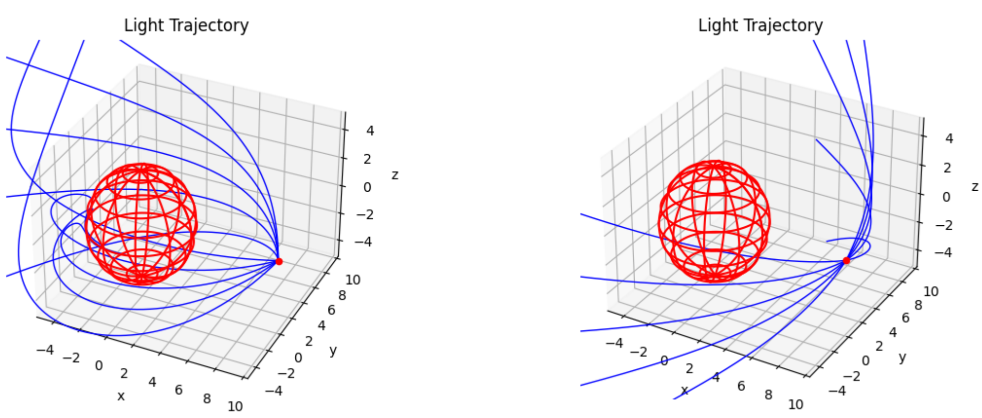

Figure 4: Graphical representation of the light rays (blue lines) and Kerr Black Hole (red sphere) with spin $a = 0.5M$. The red point in the figure corresponds to the position of the observer, which is on the equatorial plane with $r = 5M$. The angles at which the observer is looking, θ_i ranges from $-\pi/6$ to $\pi/6$, and (a) $\phi_i = \pm\pi/3$, (b) $\phi_i \approx \pm\pi/2$.

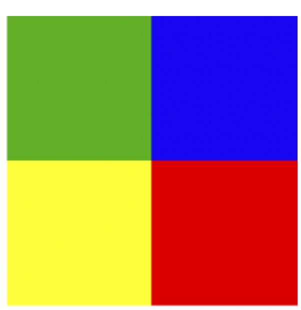

Figure 5: The four-color background we used.

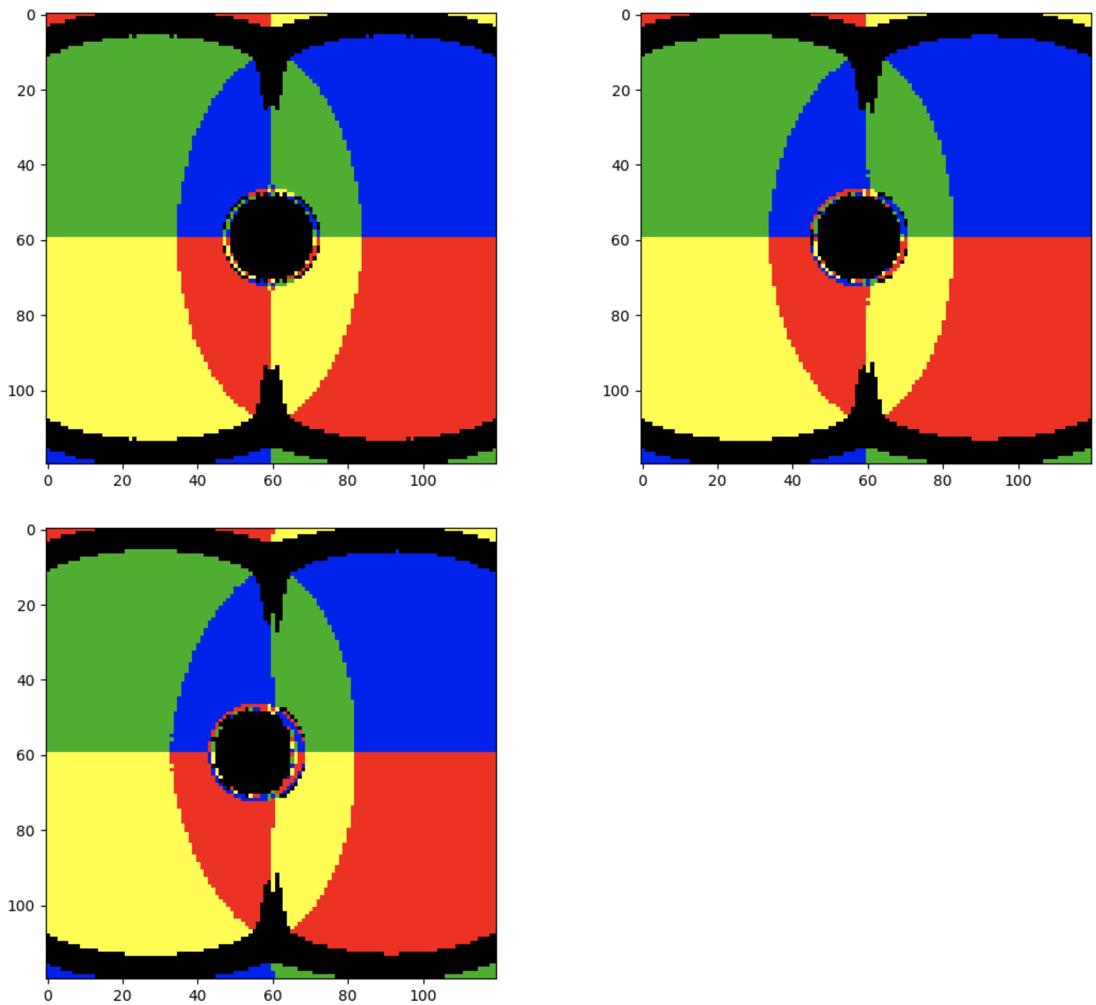

Figure 6: Kerr BH in front of the 4-color background, with (a)$a = 0$, (b)$a = 0.5M$, (c)$a = M$. The observer is located at $r = 50\frac{GM}{c^2}, \theta = \pi/2$ and $\phi = 0$.

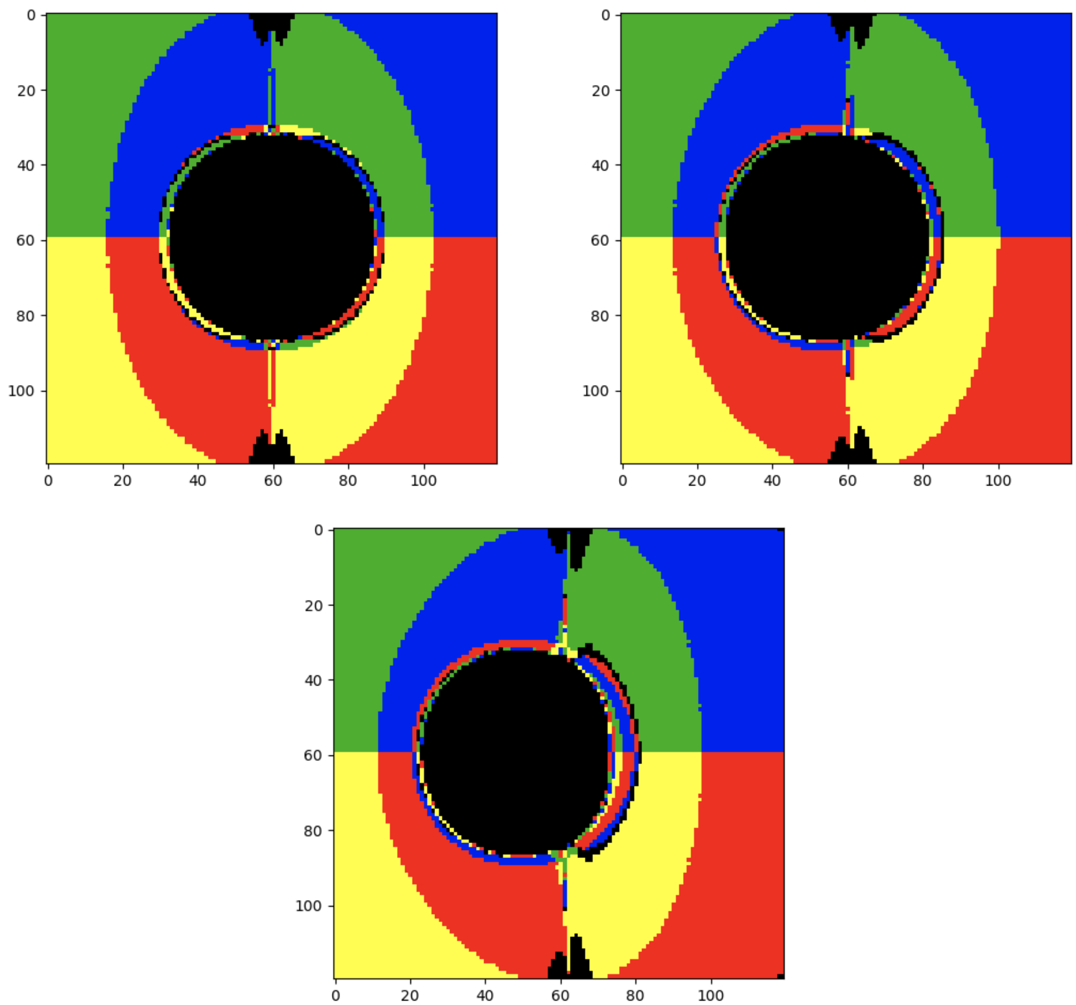

Figure 7: Kerr BH in front of the 4-color background, with (a)$a = 0$, (b)$a = 0.5M$, (c)$a = M$. The observer is located at $r = 20\frac{GM}{c^2}, \theta = \pi/2$ and $\phi = 0$.

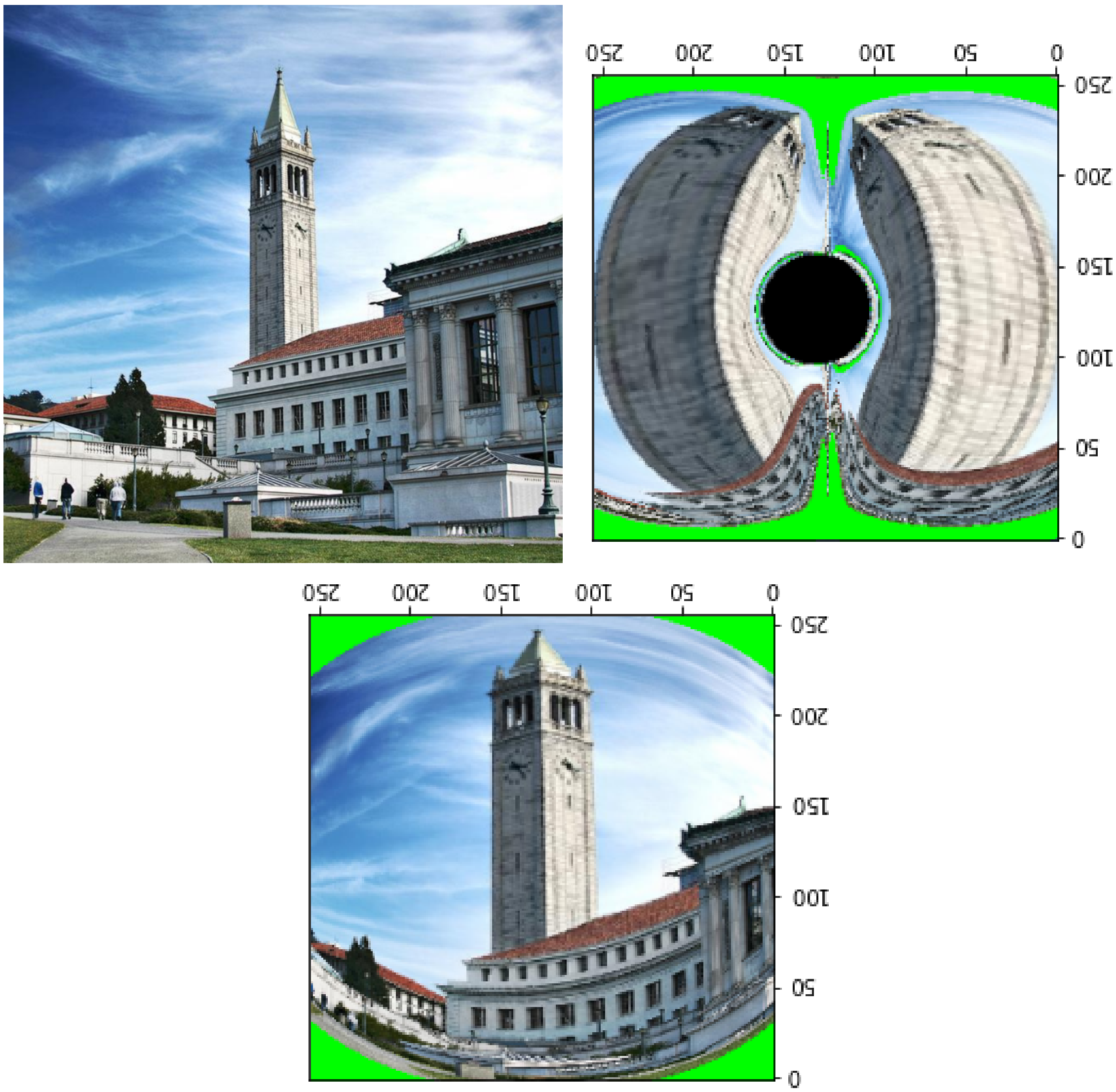

Figure 8: (a) Picture of the Sather Tower that we use. (b) Image of the Sather Tower in the presence of a Kerr black hole with spin $a = 0.5M$, $r_0 = 40$, $R_s = 35$. X axis is angle ϕ, y axis is angle θ, both range from $\pi/3$ ti $-\pi/3$. When the photon θ_f and ϕ_f is too large, they would not hit the background and we fill these grids with green color. (c) An illustration of the distortion effect induced by plotting the picture with θ and ϕ. Here $Rs = 200$.

can see a shift of the position of the black hole. These photons will go very near to the black hole and have large winding numbers. One can use our code to explore more interesting effects of black holes. The plotting can also be improved.

References

[1] Peng-Cheng Li, Minyong Guo, and Bin Chen. Shadow of a spinning black hole in an expanding universe. *Phys. Rev. D*, 101:084041, Apr 2020.

[2] James M. Bardeen, William H. Press, and Saul A. Teukolsky. Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation. *The Astrophysical Journal*, 178:347–370, 1972.