Full proof of Nash Theorem in high dimensions

By using Brouwer’s fixed point theorem

Reference:

- A Tutorial on the Proof of the Existence of Nash Equilibria
    - Albert Xin Jiang, Kevin Leyton-Brown , November 09, 2007
    - Department of Computer Science, University of British Columbia

(NashReport.pdf (ubc.ca))

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Proof of Sperner's lemma

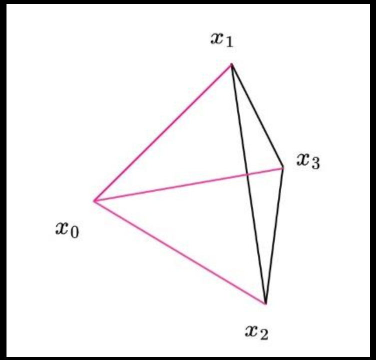
By using mathematical induction

Some Definition

- Convex combination:
  - For a vector: $x = (x_0, x_1, \dots, x_n)$ and a nonnegative scalars $\lambda_0, \dots, \lambda_n, \sum_{i=0}^{n} \lambda_i$ = 1
    - The vector: $\sum_{i=0}^{n} \lambda_i x_i$ is the convex combination of x.

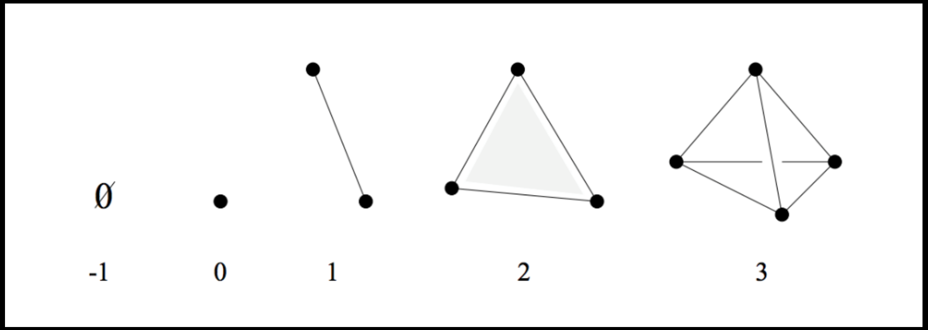
- Affine independence:

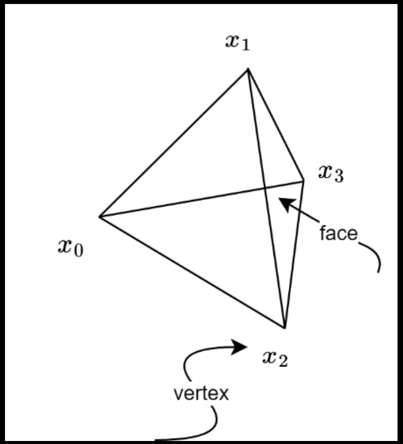
  The set X: $\{x_0, x_1, \dots, x_n\}$ is Affine independent if the set $\{x_1 - x_0, x_2 - x_0, \dots, x_n - x_0\}$ is linearly independent.



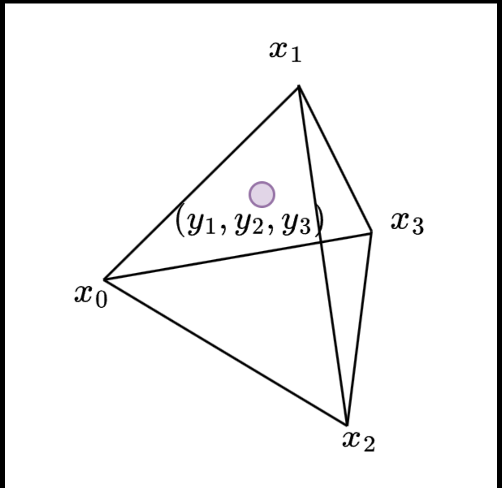
Sperner's lemma

Simplex





- $n-$ Simplex ("Triangle" in n-dimensions):
  - set of all convex n-simplex combinations of the affinely independent set of vectors.
  - $(x_0, x_1, .., x_n) = \{\sum_{i=0}^{n} \lambda_i x_i : \forall i \in \{0, .., n\}, \lambda_i \geq 0 \; and \; \sum_{i=0}^{n} \lambda_i = 1\}$
    - $n-$ Standard Simplex:
      - $\{\sum_{i=1}^{n} y_i = 1 \mid \forall i \, , y_i > 0 \}$ (The triangle that constructed by $\{O, e_1, e_2, \dots e_n\}$)

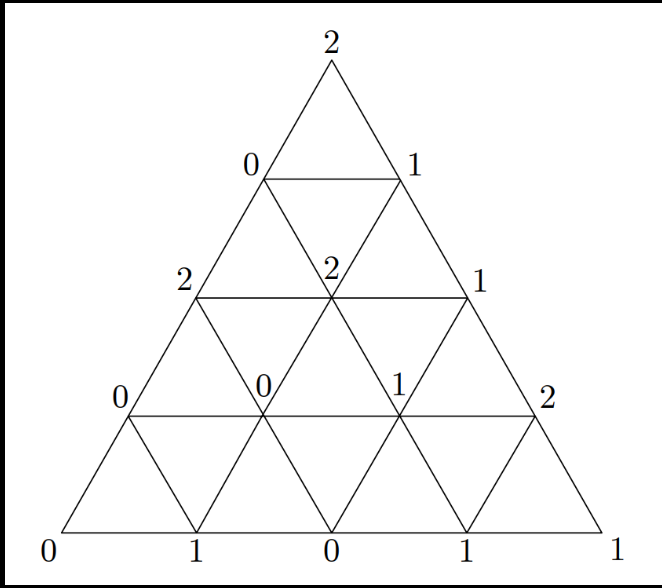


- A point y in the Simplex T with the vertices $\{x_0, x_1, \dots, x_n\}$:
  - $y = \sum_{i=0}^{n} \lambda_i x_i$ (convex combination of vertices.

Simplicial subdivision and proper labeling

- A simplex (T) is divided to a set of small simplexes ($\{T_i\}$) :
  - $\bigcup_{T_i \in T} T_i = T$
  - When 2 T_i overlap, the intersection must be an entire **face** of both subsimplexes.

  => an $(n - 1)$-face of an n-subsimplex in a simplicially subdivided:
    - either on an $(n - 1)$-face of T_n
    - or the **intersection** of two n-subsimplexes.



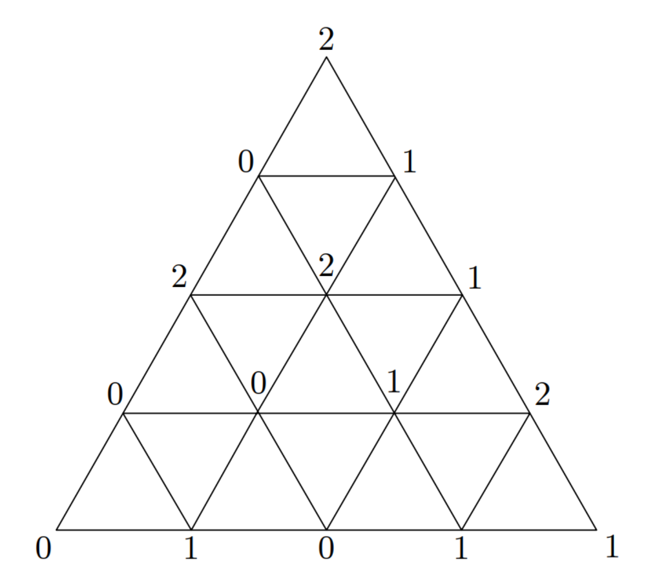
- $\chi(y) := \{i : \lambda_i > 0\}$
  - The indices of vertices that "constructs" it.

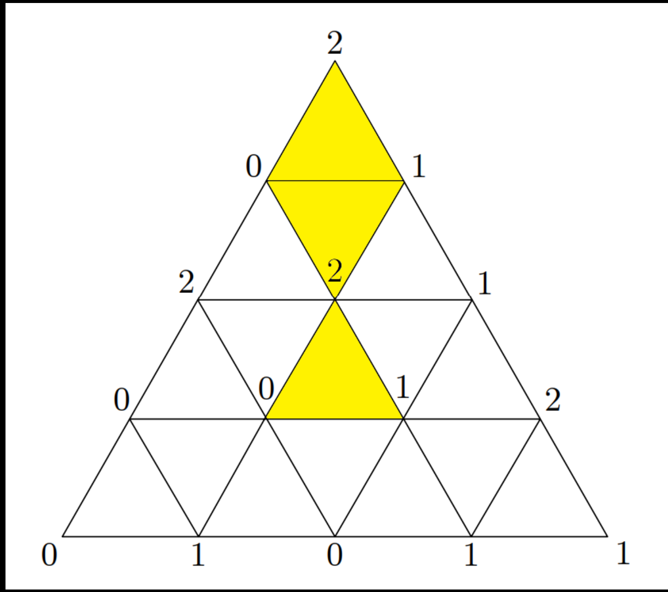
- L: $\mathrm{V} \rightarrow \{0, \dots, \mathrm{n}\}$
  - V: the set of all distinct vertices of all the subsimplexes)
  - Proper labeling : $L(y) \in \chi(y)$

  => the opposition $(n - 1)$-face of one vertex won't contain its index.

Completely labeling

- A subsimplex is completely labeled if L assumes completely labeled subsimplex all the values 0, . . . , n on its set of vertices

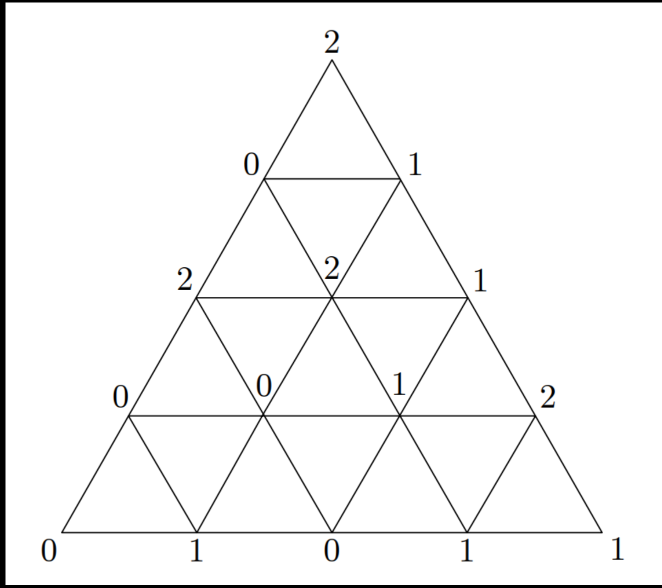


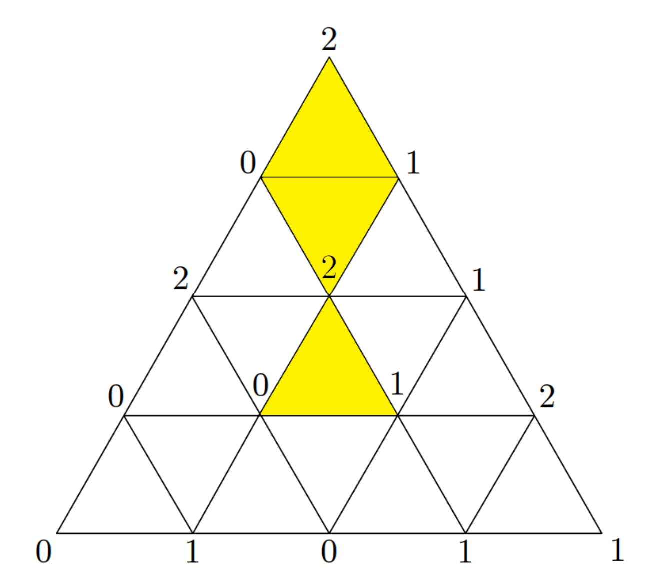


Sperner's lemma

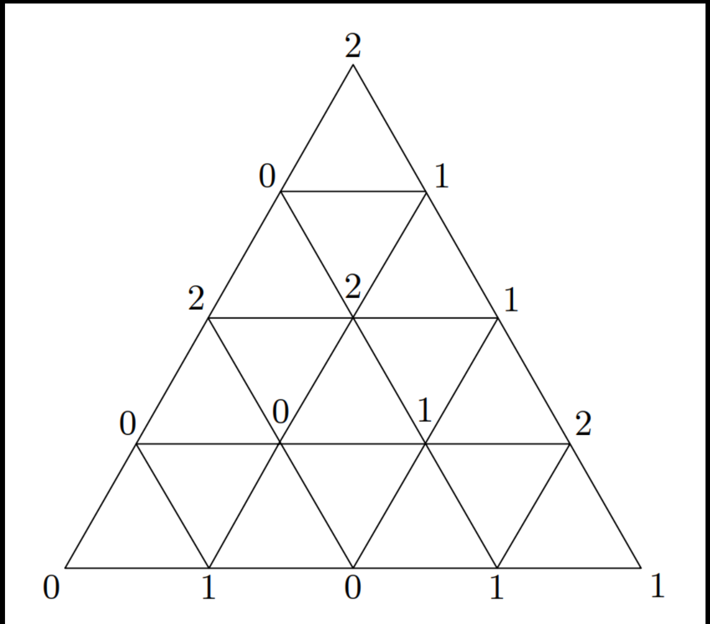
- Let $T_n = (x_0, .., x_n)$ be simplicially subdivided
- L: a **proper labeling** function

Then there are an ***odd*** number of completely labeled subsimplexes in the subdivision





Proof (by mathematical induction)

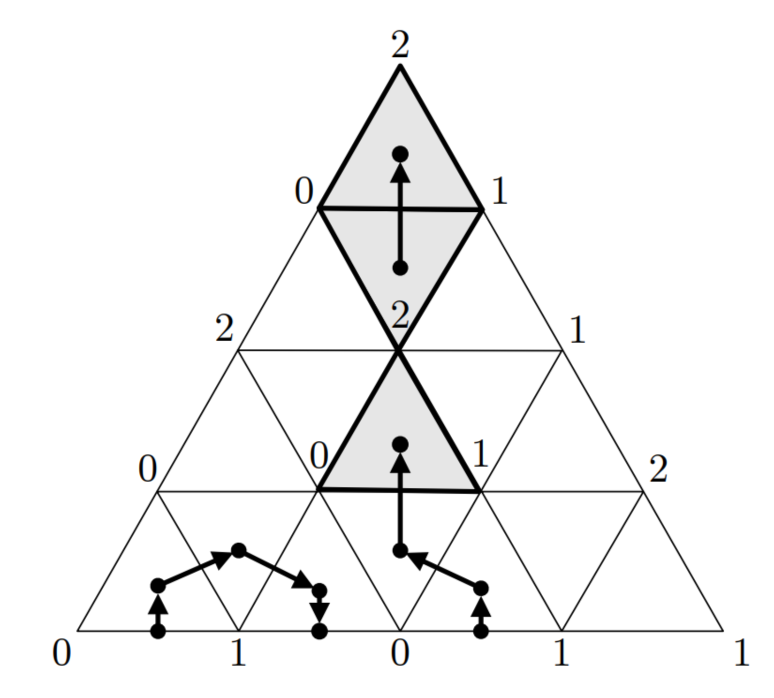


- T_0 : only possible simplicial subdivision is $\{x_0\}$
  - $L(x_0) = 0$: 1 completely labeled subsimplex
  - Satisfied

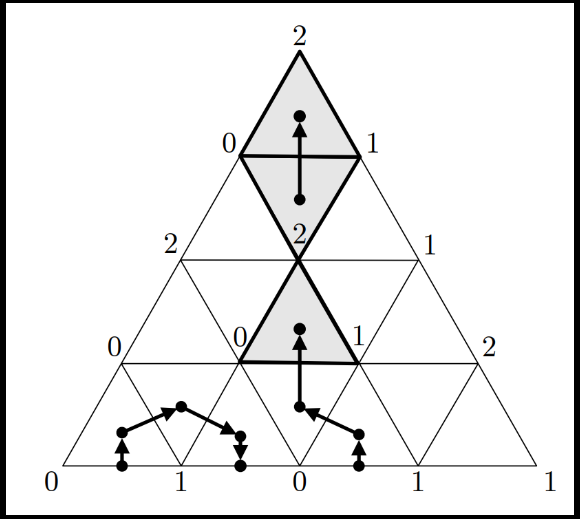
- Assume that T_{n-1} satisfied this lemma. Then for T_n:
  - T_n's face is an $(n-1)$-simplex
    - By the assumption, it has odd number of completely labeled subsimplexes in the subdivision.

- Walk:
  - Begin from b:
    - an $(n-1)$ - subsimplex with label $\{0,1,\ldots,n-1\}$ on the face of T_n
  - $\exists!\, d \in n-$ subsimplex, d has b as one of its face.
    - $ver(d) = ver(b) \cap$ an extra point z .
    - If $z.label$ is n :
      - d has the labels $(0,1,\ldots,n)$
      - a completely labeled subsimplex!
    - Else:
      - d.labels contains $(0,\ldots,n-1)$
        - Only one of the labels (j) is repeated , and the label n is missing.

      - $\exists!\, e \neq b \ \in (n-1)$-subsimplex that is a face of d and bears the labels $(0,\ldots,n\ -\ 1)$.
        - because each $(n\ -\ 1)$-face of d is defined by all but one of d's vertices.
      - b': the unique other n-subsimplex having e as a face
        - If b' is intersection : walk into it.
        - Else :b' is an $(n-1)$-face on T_n , stop.



Proof (by mathematical induction)



- The walk ends :
  - At a completely labeled n-subsimplex: (1)
    - Find a completely labeled subsimplexes
  - At a $(n-1)$-subsimplex with labels $(0,\ldots,n-1)$ on the face T_{n-1} (2):
    - Since the labels are from proper labeling method, start point and end point are at same $(n-1)$-face indeed.
    - These walks can be backward ! => once an (s,t) show up, (t,s) is a walk, too.
      - paired ! (Even number)
- The assumption : T_{n-1} has an odd number of completely labeled subsimplexes in the subdivision.
  - i.e. has an odd number of the subsimplexes with label $(0,\ldots,n-1)$
  - T_n has an odd(all) – even(case (2)) = odd (case (1)) number of completely labeled n-subsimplex.

Proof of Brouwer’s fixed point theorem

by Sperner’s lemma

Brouwer's fixed point theorem

Let $f : \triangle m \rightarrow \triangle m$ be continuous. ($\triangle m$: a standard simplex)

$$\{\sum_{i=1}^{n} y_i = 1 \mid \forall i , y_i > 0 \}$$

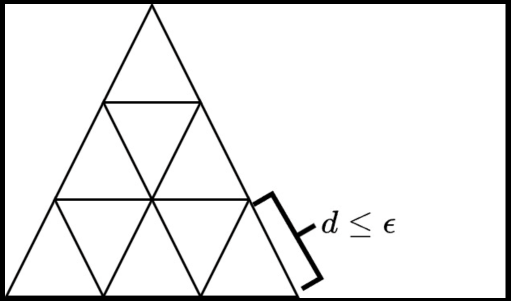
f has a fixed point.

That is, there exists some $z \in \triangle m$ such that $f(z) = z$

constructing a proper labeling

$f : \triangle m \rightarrow \triangle m$ be continuous. ($\triangle m$: a standard simplex)

- constructing a proper labeling of $\triangle m$
  - Let $\epsilon > 0$, simplicially subdivide $\triangle m$:
    - Distance of any two points in the same m-subsimplex $\leq \epsilon$
  - A proper labeling function $L(v) \in \chi(v) \cap f_i(v) \leq v_i$
    - Can be any label i such that $v_i > 0$ and f weakly decreases the i^{th}component of v.



Brouwer's fixed point theorem

To show that L is well-defined (by contradiction)

- $L(v) \in \chi(v) \cap f_i(v) \leq v_i$
- Assume $f_i(v) > v_i$ for $all\ i \in \chi(v)$:
  - Δm standard simplex: $\sum_{j\in\chi(v)} v_j = \sum_{i=1}^{m} v_i = 1$
  - $f_j(v) > v_j$ for $all\ j \in \chi(v)$:

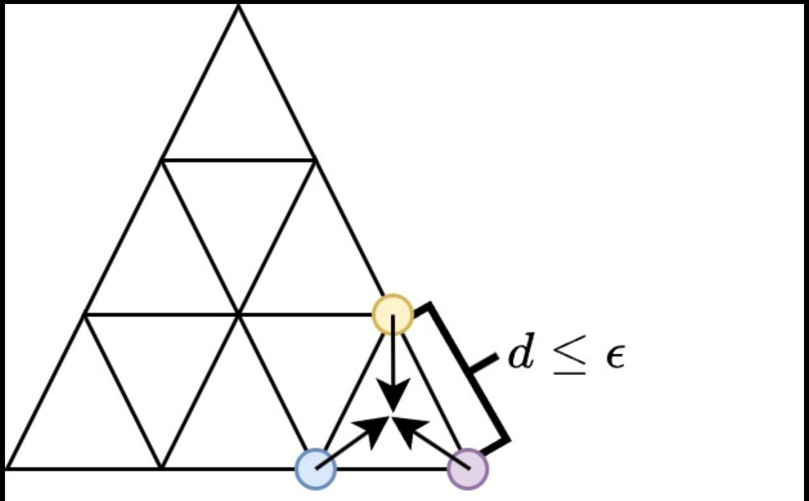
$$\sum_{j\in\chi(v)} f_j(v) > \sum_{j\in\chi(v)} v_j = \sum\nolimits_{i=1}^{m} v_i = 1$$

$\Rightarrow \sum_{j\in\chi(v)} f_j(v) > 1$

  - However, the results of f are in standard simplex, $\sum_{j\in\chi(v)} f_j(v)$=1
  - Contradiction => L is well-defined
- L is a legal proper labeling function

Brouwer's fixed point theorem

Proof (by Sperner's lemma)



- Since L is a legal proper labeling function :
  - By Sperner's lemma, there is at least one completely labeled subsimplex
    $$p_0 \cdots p_m \text{ such that } f_i(p_i) \leq p_i \text{ for each } i$$
  - Let $\epsilon \to 0$:
    - consider the sequence of centroids of completely labeled subsimplexes:
      - Since $\triangle$m is compact, there is a convergent subsequence.
      - A limit z: $p_i \to z$ for all i
  - f is continuous $\Rightarrow f_i(z) \leq z_i$ for all $i \Rightarrow f(z) \leq z$
    - Otherwise, $1 = \sum_i f_i(z_i) < \sum_i z_i = 1$ contradiction

Note

- The situation to prove the existence of NE is on simplotope (a Cartesian product of simplexes), which is the Homeomorphism of simplex.

- Brouwer’s fixed point theorem can extend to simplotope.
  - (The detail proof is in reference)

Proof the existence of NE

by Brouwer’s fixed point theorem

A (finite, n-person) normal-form game

- N: n players, indexed by i
- $A = (A_1, .., A_n), A_i$: a finite set of actions available to player i
  - $a = (a_1, a_2, .., a_n) \in A$ is a pure strategy profile
- $u = (u_1, \dots, u_n), u_i$: utility function for player i
- Mixed strategy:
  - For player i , S_i: $\Pi(A_i)$: set of all probability distributions over A_i
  - Mixed strategy profile: $s \in S = S_1 \times S_2 \times \cdots \times S_n$ (Cartesian product for all players)
    - $s_t(a_j)$: The probability that taking an action a_j under profile s_t,

Utility for a player under mixed strategy

- For a mixed stragtegy profile $s \in S$, :
  - $u_i(s) = \sum_{a \in A}[\, u_i(a) \prod_{j=1}^{n} s_j(a_j)]$
    - Utility under fixed strategy x the probability of the profile happens.

Best Response(BR) and NE

- BR_i to the profile s_{-i} :
  - A mixed strategy $s^* \in S_i$ $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$

- Nash equilibrium:
  - A strategy profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium if,

  for all agents i, s_i is a best response to s_{-i} .

  - i.e. No one can gain more expected utility by a unilateral change of strategy if the strategies of the others remain unchanged

Existence of NE

- Given a strategy profile $s \in S$, for all $i \in N \; and \; a_i \in A_i$:
  - $\phi_{i,a_i}(s) = \max\{0, u_i(a_i, s_{-i}) - u_i(s)\}$
  - $f : S \rightarrow S \; by \; f(s) = s'$ (finding new strategy profile s') where

$$s_i'(a_i) = \frac{s_i(a_i)+\phi_{i,a_i}(s)}{\sum_{b_i \in A_i} s_i(b_i)+\phi_{i,b_i}(s)} = \frac{s_i(a_i)+\phi_{i,a_i}(s)}{1+\sum_{b_i \in A_i} \phi_{i,b_i}(s)}$$

f: continuous (since each ϕ is continuous) S: convex and compact
$\Rightarrow f$ has at least one fixed point ($s = s'$)

- (Brouwer's fixed point theorem for simplotope)

- At least exist one NE for a finite, n-person normal-form game

Thanks