CSED601
Dependable Computing
Lecture 12

Jong Kim

Dept. of CSE

POSTECH

Review of Previous Lecture

- Reliability Evaluation Techniques
  - Combinatorial Model
    - Series/Parallel Model
    - Reliability Block Diagram
    - Non-series/Non-parallel model
    - M-out-of-N model
  - Markov modeling

Markov Process

- A stochastic process is a function whose values are random variables
- The classification of a random process depends on different quantities
  - state space
  - index (time) parameter
  - statistical dependencies among the random variables $X(t)$ for different values of the index parameter t.

Markov Process

- State Space
  - the set of possible values (states) that X(t) might take on.
  - if there are finite states => discrete-state process or chain
  - if there is a continuous interval => continuous process
- Index (Time) Parameter
  - if the times at which changes may take place are finite or countable, then we say we have a discrete-(time) parameter process.
  - if the changes may occur anywhere within a finite or infinite interval on the time axis, then we say we have a continuous-parameter process.

Markov Process

- States must be
  - mutually exclusive
  - collectively exhaustive
- Let $P_i(t)$ = Probability of being in state S_i at time t.

  $\Sigma P_i(t) = 1$
- Markov Properties
  - future state prob. depends only on current state
    - independent of time in state
    - path to state

Markov Process

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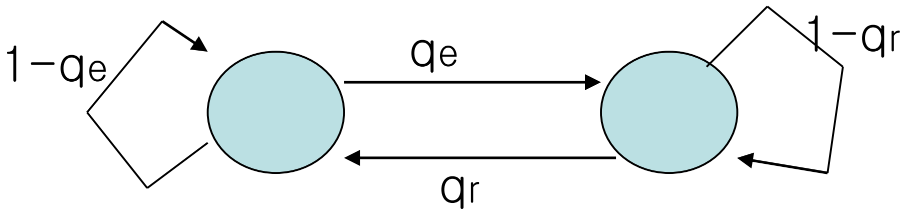
Discrete parameter Markov chain

- Concept
  - A Markov process whose state space is discrete is called as a Markov chain
  - State transition occurs at discrete time intervals
  - Let $P(X_n=j) = P_j(n)$ and $P_{jk}(m,n) = P(X_n=k|X_m=j)$ where $P_{jk}(m,n)$ represents the probability that the precedence makes a transition from state j at step m to state k at step n.
  - $P_{jk}(n) = P(X_{m+n}=k|X_m=j)$: n-step transition probability
  - $P_{jk}(1) = P_{jk}$: one-step transition probability and used to denote the transition probability matrix.
  - $0 <= P_{jk} <= 1$ and $\Sigma P_{jk} = 1$

Discrete parameter Markov chain

- Example



  - P = [Pij] = [$1 - q_e \quad q_e$]
    [$q_r \quad 1 - q_r$]
  - How to find n step transition probability matrix from P

    P(n) = P^n : multiply P matrix with itself n-1 times

Discrete parameter Markov chain

- State Probability
  - Let initial state $P(0) = [p_0(0), p_1(0), \ldots.,]$
  - $P_j(n) = P(X_n=j) = \Sigma\, P(X_0=i)P(X_n=j|X_0=i)$
    $= \Sigma\, P_i(0)P_{ij}(n)$
  - Pn = P(0) P(n) = P(0) P^n
- Example
  - P = [1 – a, a] P^n(a+b) = [(b+a(1-a-b)^n), (a-a(1-a-b)^n)]
    [b, 1 – b] [(b-b(1-a-b)^n), (a+b(1-a-b)^n)]

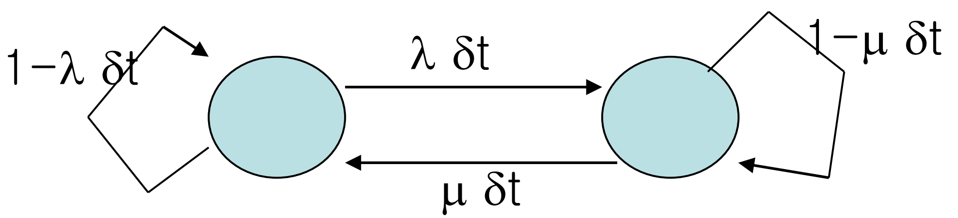
Types of State

- Types of State
  - Transient State : (non-recurrent state)
    - A state j is called a transient state if there is a positive probability that it would not come back to state j.
  - Recurrent State
    - A state j is called recurrent if starting from state j it eventually returns to state j with probability 1.
  - Absorbing State
    - A state j is called an absorbing state if there is a positive probability that there is a transition to state j from other states, but there is no transition to other states from state j.
- Irreducible Markov chain
  - Iff each state can be reached from each other state.

Continuous parameter Markov chain

- Concept
  - The transition from one state to another can be at any instance of time.
  - Transitions are given by rate s.
  - The time spent in a state is exponentially distributed.
  - M/M/1 (arrival / service / # of server)

    Markovian arrival/ Markovian service/ one server



Continuous parameter Markov chain

- Equation development
  - λ, μ : rates
  - $\lambda\ \delta t$, $\mu\ \delta t$: probabilities
  - $P = [\ 1 - \lambda\ \delta t,\ \lambda\ \delta t]$
    $[\ \mu\ \delta t,\ 1 - \mu\ \delta t]$
  - $P(t + \delta t) = [p_0(t+\delta t), p_1(t+\delta t)]$
    $= [p_0(t), p_1(t)]\ X\ P$

  ➔ $p_0(t+\delta t) = p_0(t)(1 - \lambda\ \delta t) + p_1(t)\ \mu\ \delta t$
  $p_1(t+\delta t) = p_0(t)(\lambda\ \delta t) + p_1(t)\ (1 - \mu\ \delta t)$

  ➔ $(p_0(t+\delta t) - p_0(t)) / \delta t = - \lambda\ p_0(t) + \mu\ p_1(t)$
  $(p_1(t+\delta t) - p_1(t)) / \delta t = \lambda\ p_0(t) - \mu\ p_1(t)$

  ➔ $dp_0(t) / dt = - \lambda\ p_0(t) + \mu\ p_1(t)$ ➔ $P'(t) = P(t)\ X\ [\ -\lambda, \lambda] = P(t)\ X\ T$
  $dp_1(t) / dt = \lambda\ p_0(t) - \mu\ p_1(t)$ $[\mu, -\mu]$

  Chapman-Kolmogorv Equation

Continuous parameter Markov chain

- How to solve?
  - Using Laplace transform
    - $dp_0(t) / dt = -\lambda p_0(t) + \mu p_1(t)$
    - $dp_1(t) / dt = \lambda p_0(t) - \mu p_1(t)$

➔

- $s p_0(s) - p_0(0) = -\lambda p_0(s) + \mu p_1(s)$
- $s p_1(s) - p_1(0) = \lambda p_0(s) - \mu p_1(s)$

➔

- $[p_0(0), p_1(0)] = [p_0(s), p_1(s)] \text{ X } \begin{bmatrix} s+\lambda, & -\lambda \\ -\mu, & s+\mu \end{bmatrix}$

➔ $P(0) = P(s) \text{ X } [sI - T]$ ➔ $P(s) = P(0) \text{ X } [sI - T]$ ^(-1)

➔ Take Inverse Laplace transform to find p(t) : transient probability

Continuous parameter Markov chain

- Steady-state analysis
  - All states are in steady-state.
  - What does it mean?
    - In-transition is equal to out-transition.
    - Change rate per time is zero → $dPi(t) / dt = 0$
  - Hence, $P \times T = 0$
    - Solve n-variable 1st order n-eqautions

Continuous parameter Markov chain

- Example: Reliability of TMR without repair
  - State diagram : Draw your one here
  - T?
- Example: Availability of single system with repair
  - State diagram
  - T?
  - Result?