Visual Computing - Lecture notes week 5

- Author: Ruben Schenk
- Date: 02.11.2021
- Contact: ruben.schenk@inf.ethz.ch

6. Unitary Transform

6.1 Introduction

A digital image can be written as a **matrix**:

$$f = \begin{bmatrix} f(0,\,0) & f(1,\,0) & \cdots & f(N-1,\,0) \\ f(0,\,1) & f(1,\,1) & \cdots & f(N-1,\,1) \\ \vdots & \vdots & & \vdots \\ f(0,\,L-1) & f(1,\,L-1) & \cdots & f(N-1,\,L-1) \end{bmatrix}$$

The pixels $f(x,\,y)$ are sorted into the matrix in natural order. This results in $f(x,\,y) = f_{xy}$, where f_{xy} denotes an individual element in common matrix notation.

We might also write an image as a single **vector** of length $L \cdot N$. This makes the math easier:

$$\vec{f} = \begin{bmatrix} f(0,\,0) \\ f(1,\,0) \\ \vdots \\ f(N-1,\,0) \\ f(0,\,1) \\ \vdots \\ f(N-1,\,1) \\ \vdots \\ f(N-1,\,L-1) \end{bmatrix}$$

6.2 Linear Image Processing

Any linear image processing algorithm can be written as:

$$\vec{g} = H\vec{f}$$

We define a **linear operator** $O[.\,]$ as:

$$O[\alpha_1 \cdot \vec{f}_1 + \alpha_2 \cdot \vec{f}_2] = \alpha_1 \cdot O[\vec{f}_1] + \alpha_2 \cdot O[\vec{f}_2]$$

for all scalars $\alpha_1,\,\alpha_2$.

But how does one choose H

- so $\vec{g}$ separates the salient features from the rest of the image signal?
- so $\vec{g}$ looks better?
- in order for $\vec{g}$ to be sparse?

6.3 Unitary Transforms

For a **unitary transform** we might proceed as follows:

1. Sort samples $f(x,\ y)$ of a $M \times N$ image into a column vector of length MN
2. Compute the transform coefficients of $\vec{c} = A\vec{f}$, where A is a matrix of size $MN \times MN$

The transform A is said to be **unitary**, if and only if:

$$A^{-1} = A^{*T} = A^H$$

If A is real valued, i.e. if $A = A^*$, then the transform is said to be **orthonormal**.

6.3.1 Energy Conservation

For any unitary transform $\vec{c} = A\vec{f}$, we obtain

$$||\vec{c}||^2 = \vec{c}^H\vec{c} = \vec{f}^H A^H A\vec{f} = ||\vec{f}||^2$$

This means, that every unitary transform is simply a rotation of the coordinate system (and, possibly, sign flips). The vector lengths, i.e. the "energies", are conserved!

6.3.2 Image Collection

We might denote **image collections** in the following way:

1. f_i as one image
2. $F = [f_1\, f_2\, \cdots\, f_n]$ as an image collection
3. $R_{ff} = E[f_i \cdot\ f_i^H] = \frac{F.F^H}{n}$ as an image collection *auto-correlation function*

6.3.3 Energy Distribution

With unitary transforms, energy is conserved, but often will be *unevenly distributed* among the coefficients.

For the auto-correlation matrix, we have that:

$$R_{cc} = E[\vec{c}\vec{c}^H] = E[A\vec{f} \cdot \vec{f}^H A^H] = AR_{ff}A^H$$

This leads to the mean squared values (i.e. the "average energies") of the coefficients c_i being on the diagonal of R_{cc}:

$$E[c_i^2] = [R_{cc}]_{i,i} = [AR_{ff}A^H]_{i,i}$$

6.3.4 Eigenmatrix of Auto-Correlation Matrix

The **eigenmatrix** Θ of an auto-correlation matrix R_{ff} fulfills the following rules:

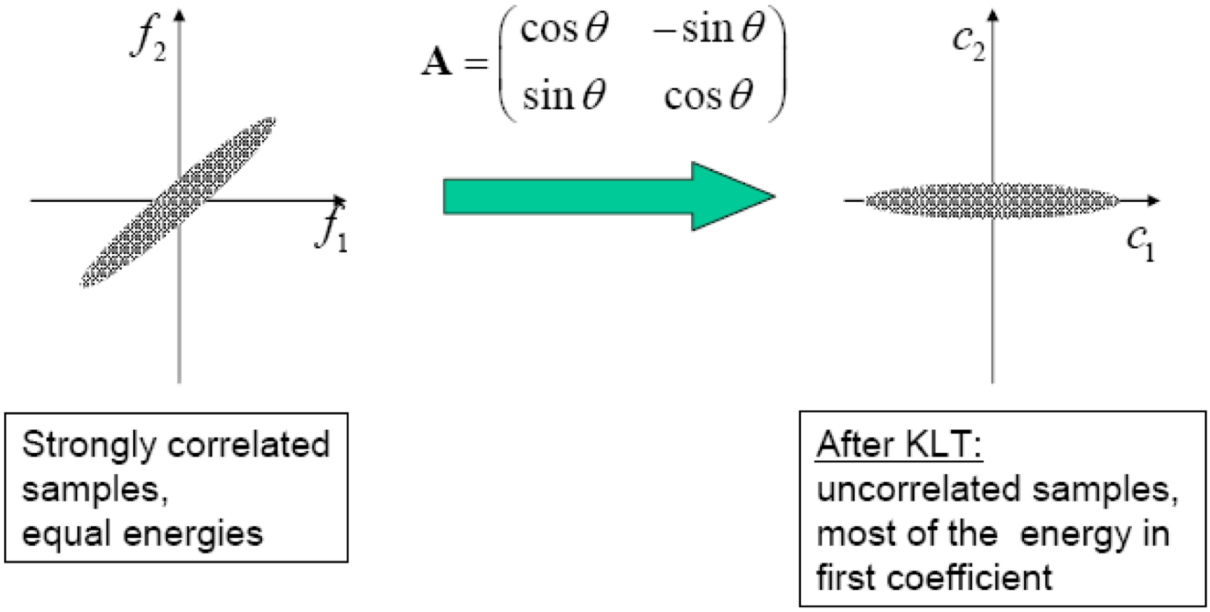
- Θ is unitary
- The columns of Θ form a set of eigenvectors of R_{ff}, i.e. $R_{ff}\Theta = \Theta\Lambda$, where Λ is a diagonal matrix of eigenvalues λ_i.
- R_{ff} is symmetric non-negative definite, hence $\lambda_i \geq 0$ for all i
- R_{ff} is a normal matrix, i.e. $R_{ff}^H R_{ff} = R_{ff}R_{ff}^H$, hence the unitary eigenmatrix exists

6.3.5 Karhunen-Loeve Transform (aka PCA)

The **Karhunen-Loeve transform** (*KL-transform*) describes the unitary transform with the matrix $A = \Theta^H$, where the columns of Θ are ordered according to decreasing eigenvalues.

- The transform coefficients are pairwise uncorrelated
- No other unitary transform packs as much energy into the first J coefficients, where J is arbitrary
- Mean squared approximation error by choosing only the first J coefficients is minimized

Example:



6.3.6 Basis Images and Eigenimages

For any unitary transform, the *inverse transform* $\vec{f} = A^H \vec{c}$ can be interpreted in terms of the superposition of basis images (columns of A^H) of size $M \times N$.
If the transform is a KL transform, the basis images, which are the eigenvectors of the auto-correlation matrix R_{ff}, are called **eigenimages.**
If energy concentration works well, only a limited number of eigenimages is needed to approximate a set of images with a small error. These eigenimages form an optimal linear subspace of dimensionality J.

To recognize complex patterns, e.g. faces, large portions of an image might have to be considered. With a transform of the form $\vec{c} = W\vec{f}$ we can reduce the dimensionality from $M \times N$ to J by representing the image by J coefficients.

6.3.7 Simple Recognition

For a **simple recognition** we can use the *simple euclidean distance (SSD)* between two images and let the best match "win":

$$\arg\min_i D_i = ||I_i - I||$$

However, this is computationally expensive, i.e. it requires the presented image to be correlated with every image in the database!

6.3.8 Eigenspace Matching

We consider the PCA (aka KLT aka closes rank-k approximation property of SVD) $\hat{I}_i \simeq Ep_i$. Then:

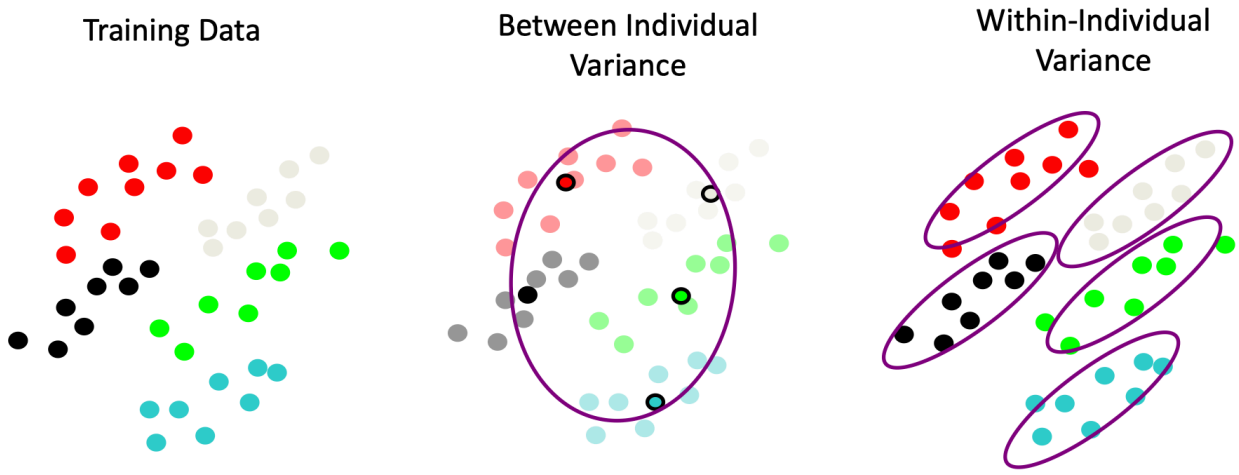
$$I_i - I = \hat{I}_i - \hat{I} \simeq E(p_i - p)$$
$$||I_i - I|| \simeq ||p_i - p||$$
$$\arg\min_i D_i = ||I_i - I|| \simeq ||p_i - p||$$

with $\hat{I} = I - \bar{I}$ and $p = E^T\hat{I}$. This is *much cheaper to compute!*

6.3.9 Fisherfaces

The key ideas of **Fisherfaces** are as follows:

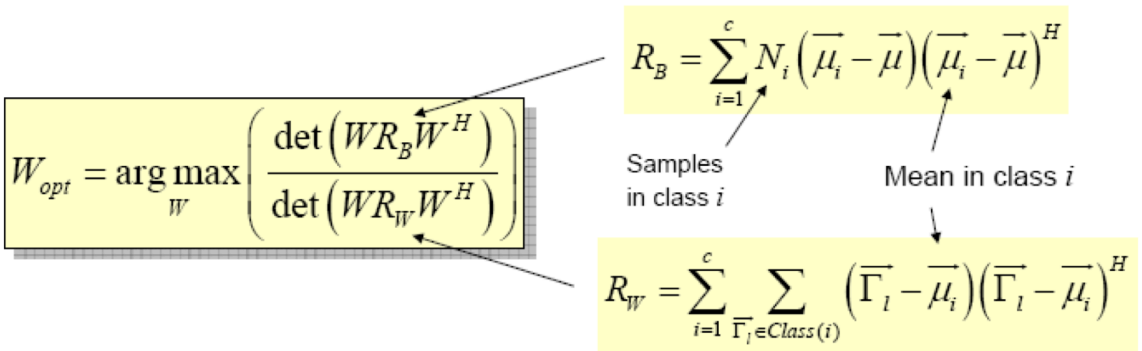
- Find directions where the ratio between/within individual variances are maximized
- Linearly project to the basis where the dimensions with good signal/noise ratios are maximized



The eigenimage method maximizes *scatter* within the linear subspace over the entire image set, regardless of the classification tasks:

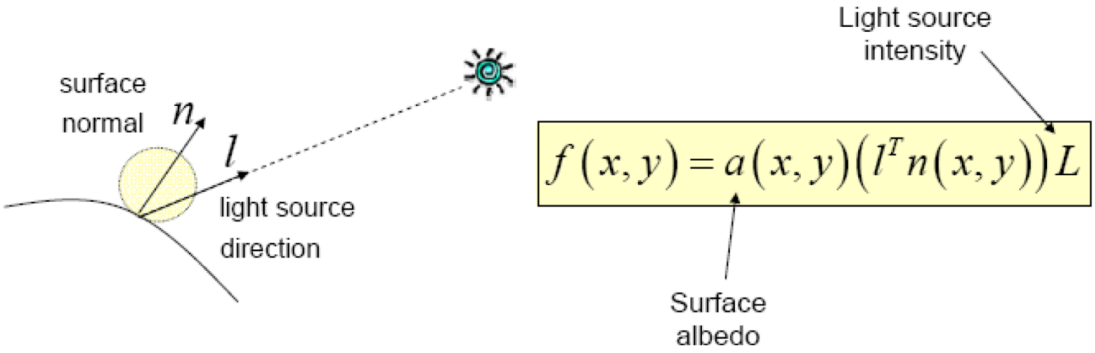
$$W_{opt} = \arg\max_W(\det(WRW^H))$$

The idea of the **Fisher linear discriminant analysis** is to maximize between-class scatter, while minimizing within-class scatter:



Fisher Images and Varying Illumination

All images of the same Lambertian surface with different illumination (without shadows) lie in a 3D linear subspace. There is a single point source at infinity such that:



7. Image Compression

7.1 JPEG Image Compression

7.1.1 Block-Based Discrete Cosine Transform (DCT)

We essentially do discrete **Cosine Transform** on our image we wish to compress:

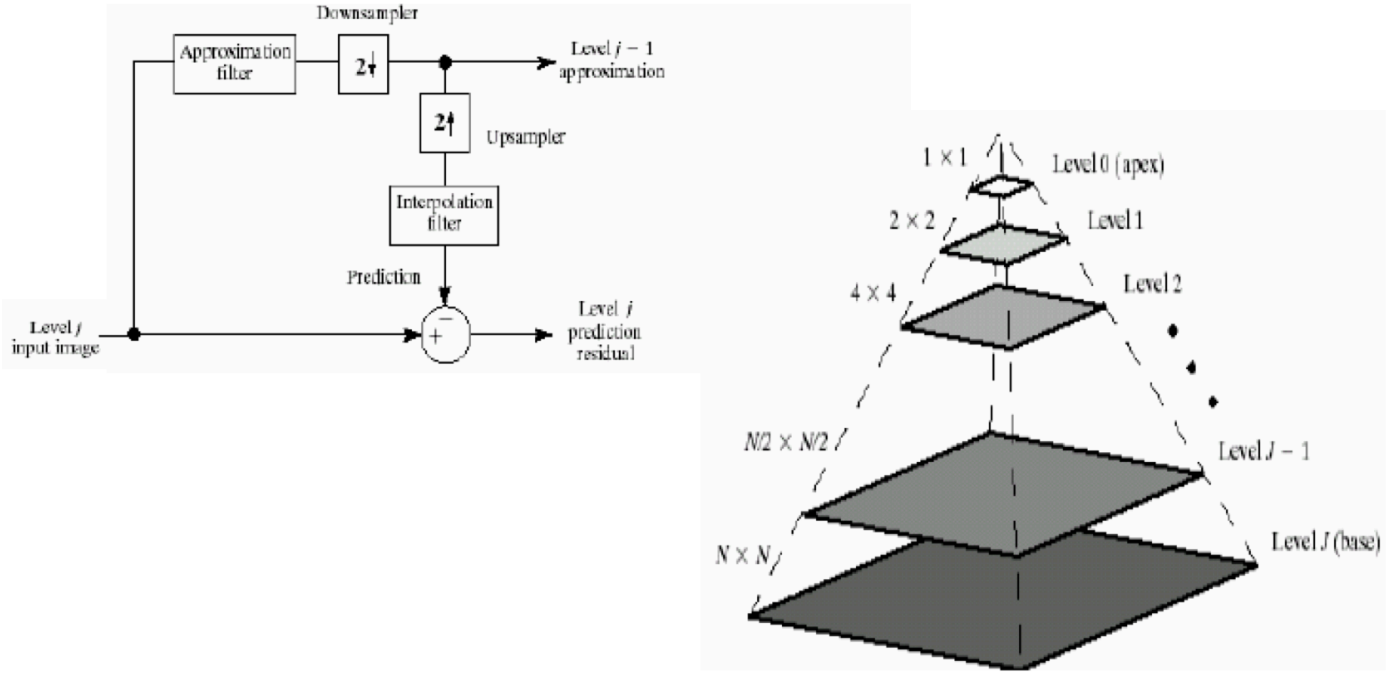
DCT is a variant of discrete Fourier transform with real numbers and a very fast implementation. We can choose the block size, its size will have an influence on the transform:

- Small blocks: faster computation and correlation between neighboring pixels
- Large blocks: better compression in smooth regions

7.1.2 Image Compression Using DCT

DCT enables *image compression* by concentrating most image information in the low frequencies. We loose unimportant image info, i.e. high frequencies, by cutting $B(u,\ v)$ in the bottom right corner. The decoder computes the inverse DCT (iDCT).

7.2 Image Pyramid



The application of **scaled representations,** such as *image pyramids*, is to look at coarse scaled and then refine with finer scaled. For example, a "good" edge at a finer scale has parents at some coarser scale.

7.2.1 Gaussian Pyramid

The different levels of the **Gaussian Pyramid** are smooth with Gaussians, since a Gaussian times a Gaussian is another Gaussian.

7.2.2 Laplacian Pyramid

For the synthesis of a **Laplacian Pyramid** we have that:

- It preserves the difference between unsampled Gaussian pyramid levels and Gaussian pyramid levels
- It's a band pass filter: Each level represents spatial frequencies unrepresented at other levels

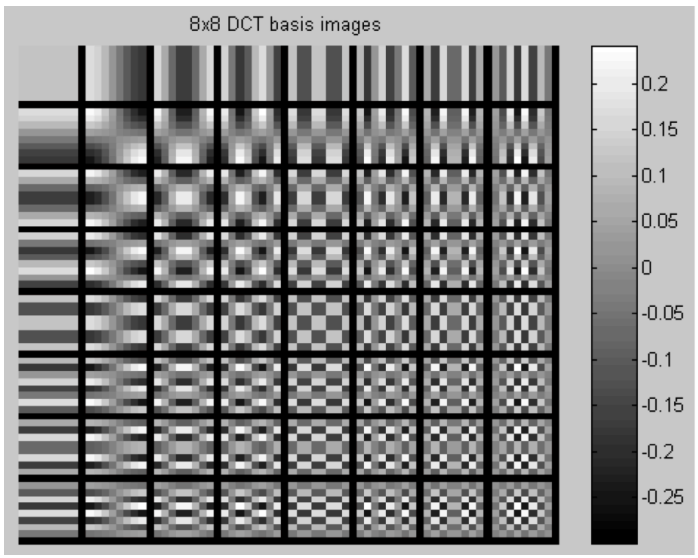
7.3 Discrete Wavelet Transform

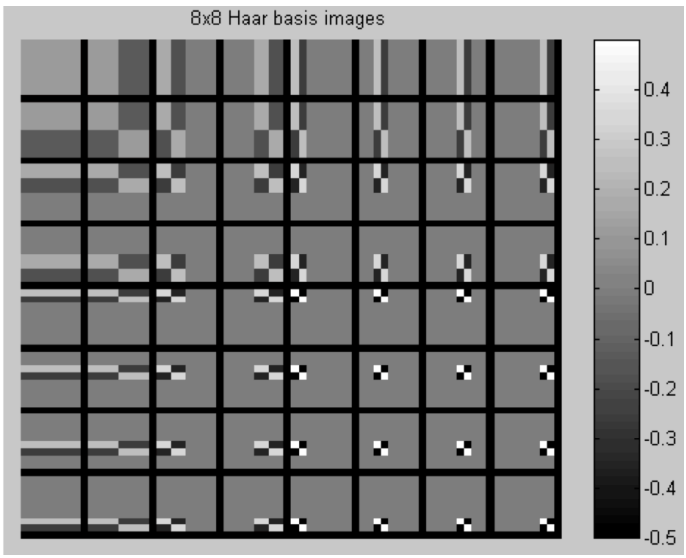
7.3.1 Haar Transform

The **Haar Transform** with the *Haar Basis* has the following properties:

- Real and orthogonal
- Transition at each scale p is localized according to q

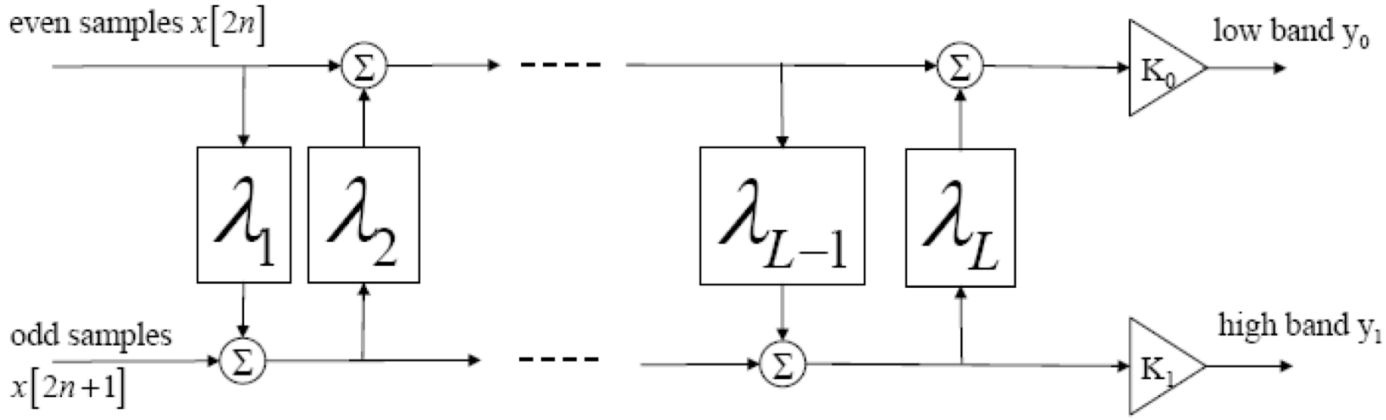
Comparison of DCT and Haar basis:





7.3.2 Lifting

Analysis filters:



Synthesis filters: