Visual Computing - Notes Week 3

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3. Convolution and Filtering

3.1 Linear Shift-Invariant Filtering

Linear shift-invariant filtering is about modifying pixels base don its *neighborhood*. Linear means that it should be a *linear combination* of neighbors. Shift-invariant means that we do the *same thing for each pixel*. This approach is useful for:

- Low-level image processing operations
- Smoothing and noise reduction
- Sharpening
- Detecting or enhancing features

3.1.1 Linear Filtering

L is a **linear** operation if:

$$L[\alpha I_1 + \beta I_2] = \alpha L[I_1] + \beta L[I_2]$$

Linear operations can be written as:

$$I'(x,\, y) = \sum_{(i,\, j) \in \mathcal{N}(x,\, y)} K(x,\, y;\, i,\, j) I(i,\, j)$$

Where I is the input image, I' is the output of the operation, and K is the **kernel** of the operation. $\mathcal{N}(m,\, n)$ denotes the *neighborhood* of $(m,\, n)$.

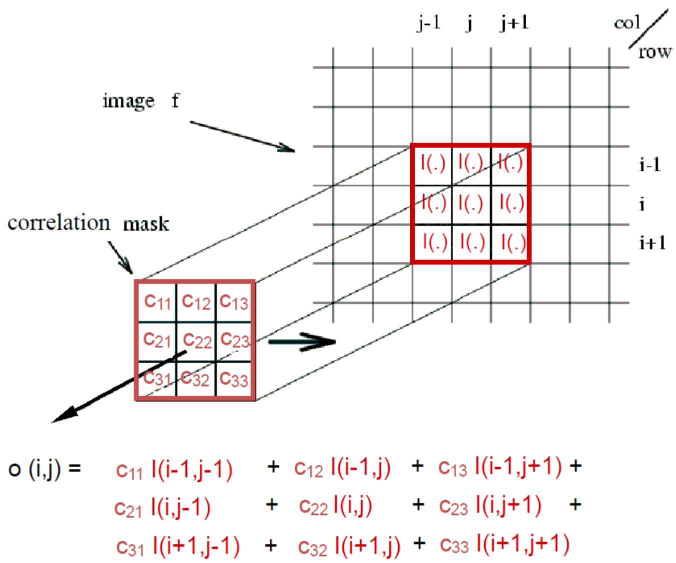
> Operations are **shift-invariant** if K does *not* depend on $(x,\, y)$, i.e. we when using the same weights everywhere!

3.2 Correlation

In this approach, we take a **correlation mask** and apply it to an image.

> Correlation is as if we had a template (the mask), and search for it in our image.

This would look as follows:



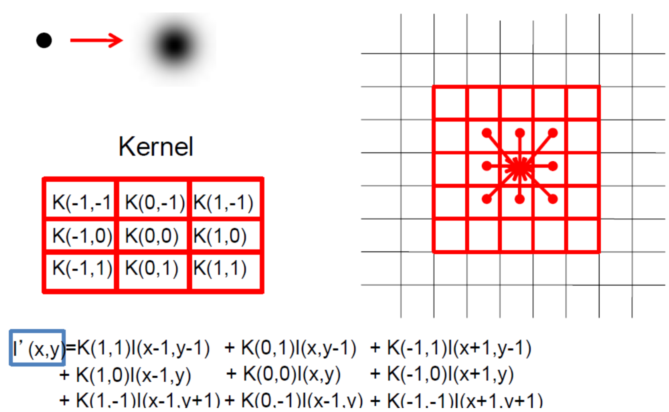
The linear operation of correlation looks as follows:

$$I' = K \circ I I'(x, y) = \sum_{(i, j) \in \mathcal{N}(x, y)} K(i, j) I(x + i, y + j)$$

This represents the linear weights as an image.

3.3 Convolution

Compared to correlation, where we looked at the neighborhood of a pixel and applied what we learned from the neighborhood to the single pixel, in convolution we look at a single pixel and apply what we can learn from it to its neighborhood.

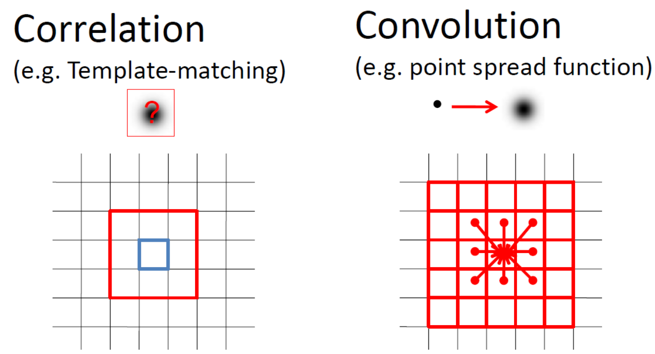


The linear operation of convolution is given by:

$$I' = K * I I'(x, y) = \sum_{(i, j) \in \mathcal{N}(x, y)} K(i, j) I(x - i, y - j)$$

This too represents the linear weights as an image, it is actually the same as correlation, but with a reversed kernel.

3.3.1 Correlation vs Convolution



$I' = \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(i,j) I(x+i, y+j)$ $\quad$ $I' = \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(i,j) I(x-i, y-j)$

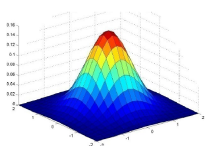
3.4 Separable Kernels

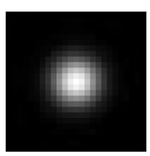
Separable filters can be written as $K(m, n) = f(m)g(n)$. For a rectangular neighborhood with size $(2M+1) \times (2N+1)$, $I'(m, n) = f * (g * I(\mathcal{N}(m, n)))$. We can rewrite this to:

$$I''(m, n) = \sum_{j=-N}^{N} g(j) I(m, n-j) I'(m, n) = \sum_{i=-M}^{M} f(i) I''(m-i, n)$$

3.5 Gaussian Kernel

The idea of the **Gaussian kernel** is that we weight the contributions of neighboring pixels by their nearness:





0.003	0.013	0.022	0.013	0.003
0.013	0.059	0.097	0.059	0.013
0.022	0.097	0.159	0.097	0.022
0.013	0.059	0.097	0.059	0.013
0.003	0.013	0.022	0.013	0.003

5 x 5, σ = 1

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

3.5.1 Gaussian Smoothing Kernels

The amount of smoothing when using Gaussian kernels depends on σ and on the window size.

The top 5 reasons to use Gaussian smoothing are:

1. Rotationally symmetric
2. Has a single lobe (neighbor's influence decreases monotonically)
3. Still one lobe in frequency domain (no corruption from high frequencies)
4. Simple relationship to σ
5. Easy to implement efficiently

3.6 Filter Examples

Differential filters

- *Prewitt operator:*

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

- *Sobel operator:*

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

High-pass filters

- *Laplacian operator:*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- *High-pass filter:*

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

4. Image Features

4.1 Template Matching

Template matching describes the problem of locating an object, described by a template $t(x, y)$, in the image $s(x, y)$. This is done by searching for the best match by minimizing mean-squared error:

$$\begin{aligned} E(p, q) &= \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} [s(x, y) - t(x-p, y-q)]^2 \\ &= \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} |s(x, y)|^2 + \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} |t(x, y)|^2 - 2 \cdot \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} s(x, y) \cdot t(x-p, y-q) \end{aligned}$$

Equivalently, we can *maximize* the **area correlation**:

$$r(p, q) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} s(x, y) \cdot t(x-p, y-q) = s(p, q) * t(-p, -q)$$

The area correlation is equivalent to the convolution of image $s(x, y)$ with impulse response $t(-x, -y)$.

4.2 Edge Detection

One idea, in a continuous-space, is to detect the local gradient:

$$|\mathrm{grad}(f(x, y))| = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$$

We mostly use the following **edge detection filters**:

Prewitt $\begin{pmatrix} -1 & 0 & 1 \\ -1 & [0] & 1 \\ -1 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -1 & -1 & -1 \\ 0 & [0] & 0 \\ 1 & 1 & 1 \end{pmatrix}$

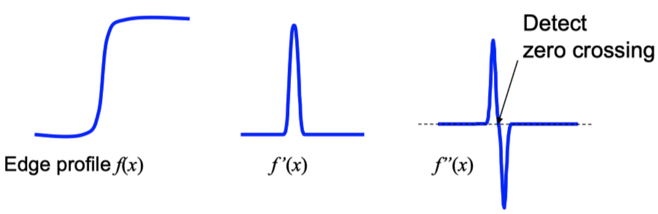
Sobel $\begin{pmatrix} -1 & 0 & 1 \\ -2 & [0] & 2 \\ -1 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -1 & -2 & -1 \\ 0 & [0] & 0 \\ 1 & 2 & 1 \end{pmatrix}$

Roberts $\begin{pmatrix} [0] & 1 \\ -1 & 0 \end{pmatrix}$ $\begin{pmatrix} [1] & 0 \\ 0 & -1 \end{pmatrix}$

4.2.1 Laplacian Operator

The idea of a **Laplacian operator** is to detect discontinuities by considering the second derivative and searching for *zero-crossings* (those mark edge locations):

$$\nabla^2 f(x, y) = \frac{\partial^2 fx, y()}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2}$$



We can do a *discrete-space approximation* by convolution with a 3×3 impulse response:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

However, the Laplacian operator is sensitive to very fine detail and noise, so we might want to blur the image first.

Laplacian of Gaussian Blurring the image with Gaussian and Laplacian operator can be combined into convolution with **Laplacian of Gaussian operator** (LoG):

$$\mathrm{LoG}(x, y) = -\frac{1}{\pi\sigma^4}[1 - \frac{x^2 + y^2}{2\sigma^2}]exp(-\frac{x^2 + y^2}{2\sigma^2})$$

4.2.2 Canny Edge Detector

The **Canny edge detector** works with the following steps:

1. Smooth the image with a Gaussian filter
2. Compute the gradient magnitude and angle (Sobel, Prewitt, etc.):

$$M(x,\, y) = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} \quad \text{and} \quad \alpha(x,\, y) = \tan^{-1}(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x})$$

3. Apply nonmaxima suppression to gradient magnitude image
4. Double thresholding to detect strong and weak edge pixels
5. Reject weak edge pixels not connected with strong edge pixels

Canny nonmaxima suppression The **Canny nonmaxima suppression** works as follows:

1. Quantize the edge normal to one of the four directions: horizontal, -45deg, vertical, +45deg
2. If $M(x,\, y)$ is smaller than either of its neighbors in edge normal direction, then suppress is, else keep it

Canny Thresholding When using a Canny edge detector, we do double thresholding of the gradient magnitude:

- Strong edge: $M(x,\, y) \geq \Theta_{high}$
- Weak edge: $\Theta_{high} > M(x,\, y) \geq \Theta_{low}$

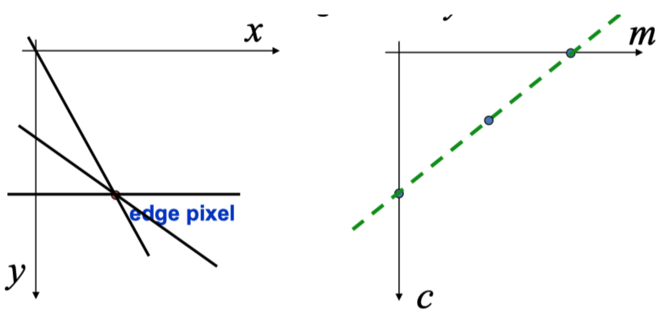
A typical setting for the thresholds would be $\frac{\Theta_{high}}{\Theta_{low}} \in [2,\, 3]$.

4.2.3 Hough Transform

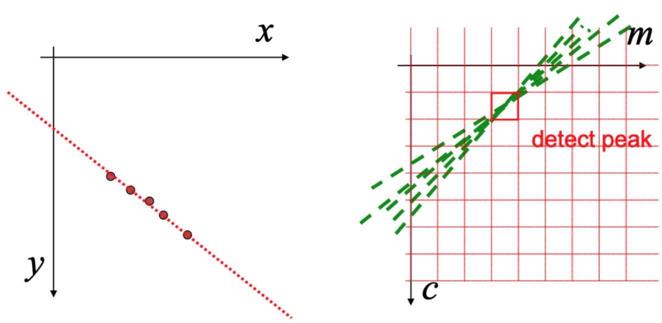
The **Hough transform** solves the problem of fitting a straight line (or curve) to a set of edge pixels. The Hough transform is a *generalized template matching technique.*

It works the following way:

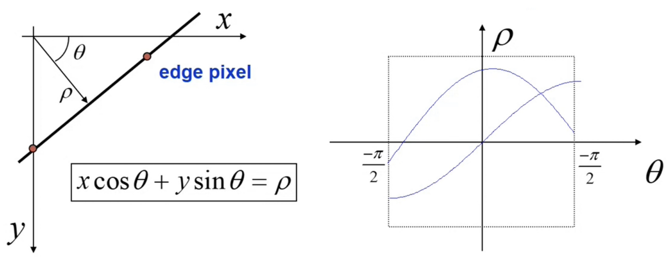
1. For an edge pixel in the $(x,\, y)$ plane we can draw the different lines that cross the edge pixel (all lines have the form $y = mx + c$). We can draw the m and c values in a $(m,\, c)$ plane and see that all those lines are linearly dependent:



2. If we have multiple edge pixels, we can do the same procedure for each of those, giving us a line in the $(m,\, c)$ plane for each edge pixel.
3. We then subdivide the $(m,\, c)$ plane into discrete "bins" and initialize the bin count of each bin to 0. Each time a bin is crossed by one of the lines of the different edge pixels, we increase its count by one.
4. We then simply have to detect the peaks in the $(m,\, c)$ plane to get our fitted straight line:



We might encounter an infinite-slope problem, which can be avoided with an alternative parameterization:



4.3 Detecting Corner Points

Many applications benefit from features localized in (x, y). If edges are well localized but only in one direction, we might want to **detect corners**.

The desirable properties of a corner detector are:

- accurate localization
- invariance against shift, rotation, scale, and brightness change
- robust against noise, high repeatability

something something what patterns can be localized most accurately?

- Local displacement sensitivity

$$S(\Delta x, \Delta y) = \sum_{(x,y)\in window} \left[f(x,y) - f(x+\Delta x, y+\Delta y) \right]^2$$

- Linear approximation for small $\Delta x, \Delta y$

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

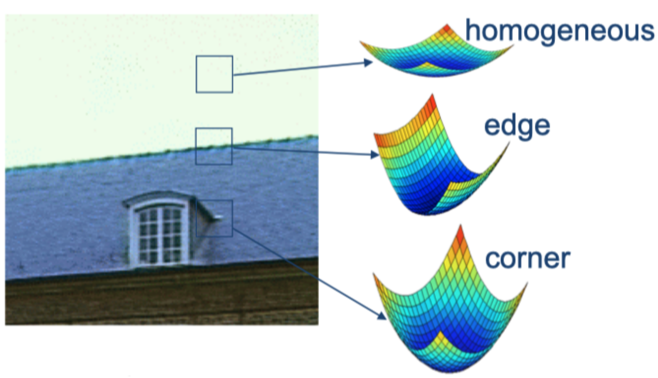
$f_x(x,y)$ – horizontal image gradient
$f_y(x,y)$ – vertical image gradient

$$\begin{aligned} S(\Delta x, \Delta y) &\approx \sum_{(x,y)\in window} \left[\begin{pmatrix} f_x(x,y) & f_y(x,y) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \right]^2 \\ &= \begin{pmatrix} \Delta x & \Delta y \end{pmatrix} \left(\sum_{(x,y)\in window} \begin{bmatrix} f_x^2(x,y) & f_x(x,y) f_y(x,y) \\ f_x(x,y) f_y(x,y) & f_y^2(x,y) \end{bmatrix} \right) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\ &= \begin{pmatrix} \Delta x & \Delta y \end{pmatrix} \mathbf{M} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \end{aligned}$$

- Iso-sensitivity curves are ellipses

4.3.1 Feature Point Extraction

We have that $SSD \simeq \delta^T M \delta$. Now if we shift our patterns over the picture, we assume it to change the following way:



Now we want to find points for which the following is large:

$$\min \delta^T M \delta \text{ for } ||\delta|| = 1$$

i.e. we want to maximize the eigenvalues of M.

Keypoint Detection *something something*

Often based on eigenvalues λ_1, λ_2 of "structure matrix" (aka "normal matrix" aka "second-moment matrix")

$$\mathbf{M} = \begin{bmatrix} \sum\limits_{(x,y)\in window} f_x^2(x,y) & \sum\limits_{(x,y)\in window} f_x(x,y)\,f_y(x,y) \\ \sum\limits_{(x,y)\in window} f_x(x,y)\,f_y(x,y) & \sum\limits_{(x,y)\in window} f_y^2(x,y) \end{bmatrix}$$

$f_x(x,y)$ – horizontal image gradient
$f_y(x,y)$ – vertical image gradient

Measure of "cornerness"

$$\begin{aligned} C(x,y) &= \det(\mathbf{M}) - k\cdot\left(trace(\mathbf{M})\right)^2 \\ &= \lambda_1\lambda_2 - k\cdot\left(\lambda_1+\lambda_2\right) \end{aligned}$$

[Harris, Stephens, 1988]