Visual Computing - Notes Week 9

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3. Transforms

3.1 Introduction

3.1.1 Linear Transforms

In computer graphics, we are mainly concerned about **linear transforms.** This is due to:

- Computationally speaking, linear transforms and linear maps are easy to solve
- Linear transforms are still very powerful
- All maps can be approximated as linear maps (though sometimes only over a short distance, or small amount of time)
- Composition of linear transformations is linear, leading to uniform representation of transformations

3.1.2 Algebraic Definition

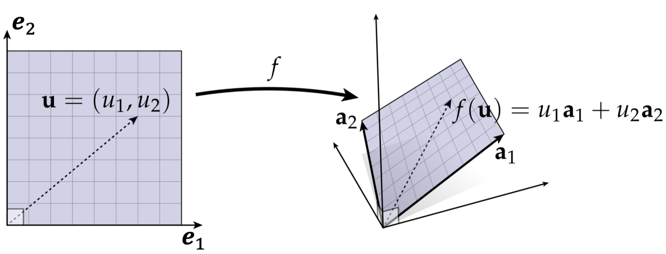
A map f is *linear* if it maps vectors to vectors, and if for all vectors u, v and scalars α we have:

$$f(u+v) = f(u) + f(v) f(\alpha u) = \alpha f(u)$$

For maps between $\mathbb{R}^m$ and $\mathbb{R}^n$, we can give an even more explicit definition:

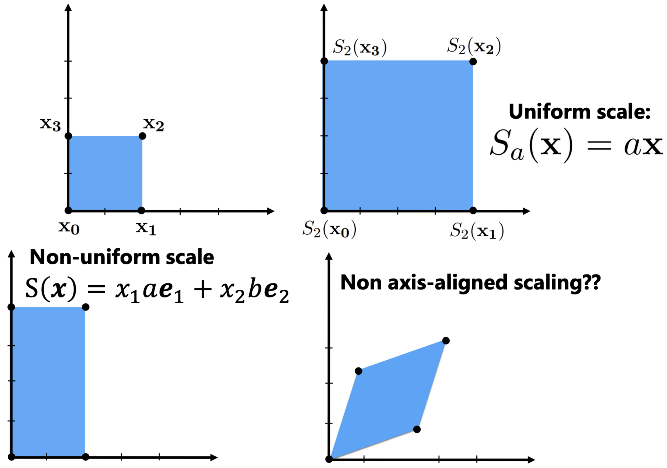
If a map can be expressed as $f(u) = \sum_{i=1}^{m} u_i a_i$, with fixed vectors a_i, then it is linear.

Example:



u is a linear combination of e_1 and e_2. $f(u)$ is the *same* linear combination, but of a_1 and a_2, and we have that $a_1 = f(e_1)$ and $a_2 = f(e_2)$.

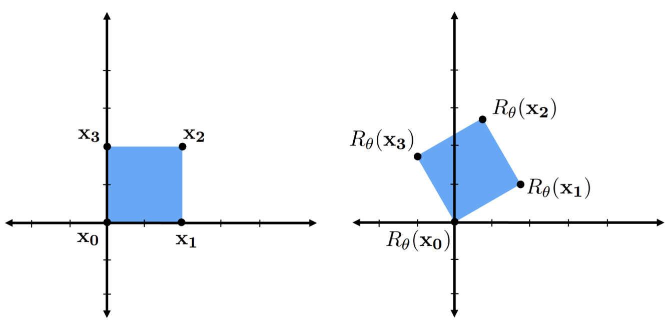
3.2 Scale



Scaling is simply defined by either scalar multiplication of the whole vector (*uniform scale*) or scalar multiplication of specific basis vectors (*non-uniform scale*).

$$S(x) = x_1 a e_1 + x_2 b e_2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot x$$

3.3 Rotation



Mathematically, **rotations** can be defined by the following two formulae:

$$R_\theta(e_1) = (\cos\theta, \sin\theta) = a_1 R_\theta(e_2) = (-\sin\theta, \cos\theta) = a_2$$

Which leads us, due to the linearity of rotations, to the following simple formula:

$$R_\theta(x) = x_1 a_1 + x_2 a_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot x$$

Remark: Rotation around any other point than the origin is *not linear,* since for linearity, the origin must map to the origin!

3.4 Reflection

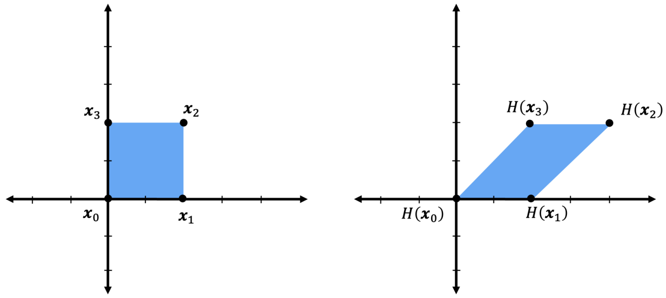
For now, we only consider **reflection** about the y-axis and x-axis. They are expressed as:

- $Re_y(x) = x_1 e_x \cdot (-1) + x_2 e_y$
- $Re_x(x) = x_1 e_x + x_2 e_2 \cdot (-1)$

Those special cases are actually simple *non-uniform scales.*

3.5 Shear

A **shear operation** (in the x direction) is done by moving the upper edge along the x-axis by some defined amount.

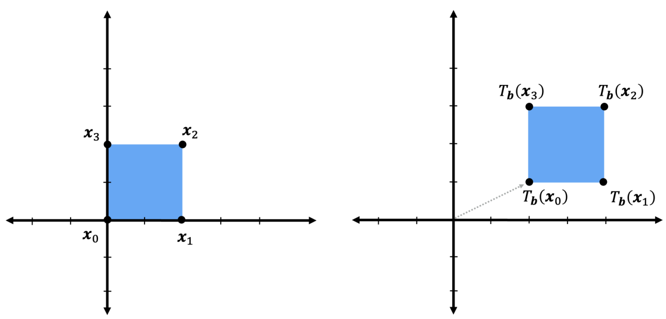


Mathematically, this operation is defined through:

$$H_a(x) = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} a \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot x$$

3.6 Translation

Translation describes mappings of the following form:



We can denote this transformation by:

$$T_b(x) = x_1 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_2 \begin{bmatrix} ? \\ ? \end{bmatrix}$$

such that $T_b(x) = x + b$ for some translation vector b.

Remark: Translation is *not linear*, but affine.

3.7 2D Homogeneous Coordinates (2D-H)

3.7.1 Introduction

The key idea with **2D-H coordinates** is to *lift* our 2D points into the 3D space. For example, our 2D point (x_1, x_2) is represented as:

$$(x_1, x_2) \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

This also leads to the change, that our 2D transforms are now represented by 3×3 matrices instead of 2×2.

Example: The previously seen 2D rotation in homogeneous coordinates is defined by:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

3.7.2 Translation In 2D-H Coordinates

One of our main interests in 2D-H coordinates is that we are able to define non-linear maps in 2D as linear maps in 3D/2D-H coordinates!

The translation operation expressed in 2D-H, i.e. as a 3×3 matrix multiplication, is given by:

$$T_b(x) = x + b = \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + b_1 x_3 \\ x_2 + b_2 x_3 \\ x_3 \end{bmatrix}$$

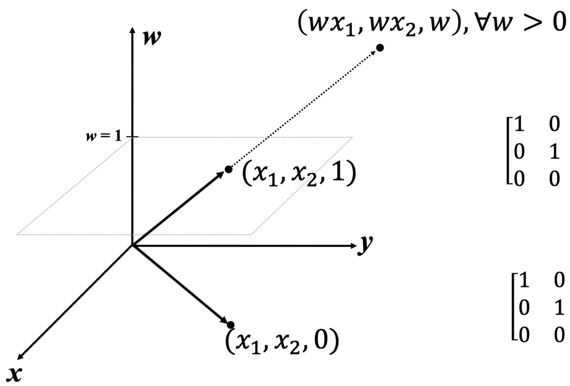
In our homogeneous coordinates, translation is a linear transformation!

Remark: x_3 is usually set to 1.

3.7.3 Points vs. Vectors

In computer graphics we often have to distinguish between *points* and *vectors.*

We define a vector to have $x_3 = 0$ in 2D-H and a point to have $x_3 \neq 0$ in 2D-H. To get from a point in 2D-H back to 2D, we simply divide all components by x_3.



$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

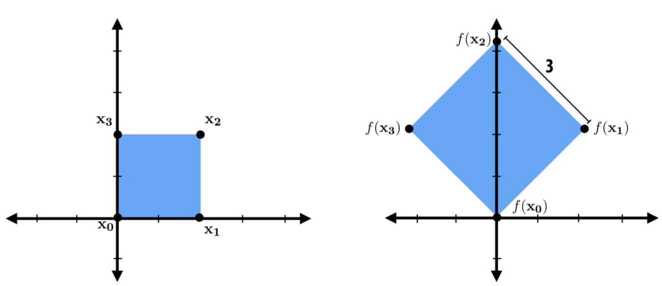
vs

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

3.8 Composition Of Linear Transformations

We can **compose linear transforms** via matrix multiplication. This enables for simple and efficient implementation, since we can reduce a complex chain of transforms to a single matrix.

Example: We take a look at the following transform: $R_{\pi/4} S_{[1.5,\, 1.5]} x$



3.9 Moving To 3D (And 3D-H)

Similar to 2D, we represent 3D transforms as 3×3 matrices and 3D-H transforms as 4×4 matrices.

Example:

- A *scale* in 3D is given by:

$$S_S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_z \end{bmatrix}, \quad S_S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A *shear*, in x and based on the y, z position, is given by:

$$H_{x,\,d} = \begin{bmatrix} 1 & d_y & d_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_{x,\,d} = \begin{bmatrix} 1 & d_y & d_z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A *translation* by vector b:

$$T_b = \begin{bmatrix} 1 & 0 & 0 & b_x \\ 0 & 1 & 0 & b_y \\ 0 & 0 & 1 & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotation about x-axis:

$$R_{x,\,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Rotation about y-axis:

$$R_{y,\,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

- Rotation about z-axis:

$$R_{z,\,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Perspective Projection Transformations, Geometry and Texture Mapping

4.1 Perspective Projection Transformations

4.1.1 Basic Perspective Projection

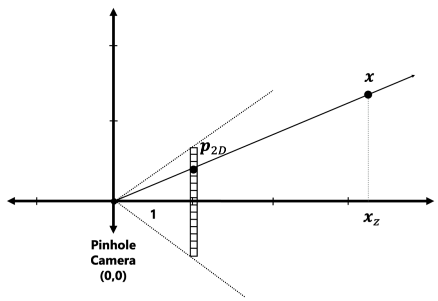
When doing basic perspective projection, the desired *perspective projection result* (some 2D point), is given by:

$$p_{2D} = (\frac{x_x}{x_z},\, \frac{x_y}{x_z})$$

The procedure for a basic perspective projection, follows 4 steps:

1. Input point in 3D-h: $x = (x_x,\, x_y,\, x_z,\, 1)$

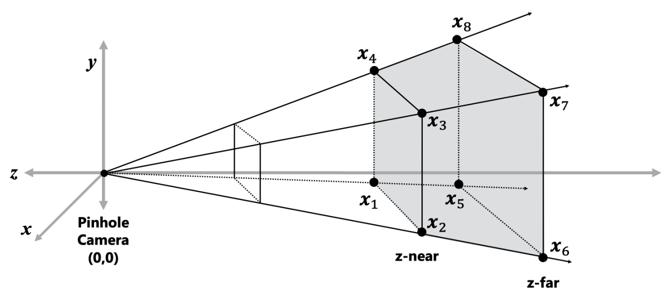
2. Applying map to get the projected point in 3D-H: $Px = (x_x,\, x_y,\, x_z,\, x_z)$
3. Point projected to 2D-H by dropping the z coordinate: $p_{2D-H} = (x_x,\, x_y,\, x_z)$
4. Point in 2D by homogeneous divide: $p_{2D} = (\frac{x_x}{x_z},\, \frac{x_y}{x_z})$



$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4.1.2 The View Frustum

The **view frustum** denotes the region in space that will appear on the screen.



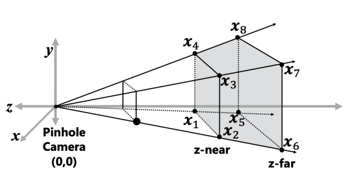
We want a transformation that maps view frustum to a unit cube, such that computing screen coordinates in that space becomes trivial.

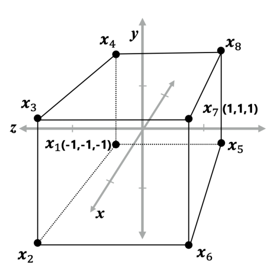
Define the following properties:

- $\theta :$ The field of view in the y direction ($h = 2 \cdot \tan\left(\frac{\theta}{2}\right)$)
- $f = \cdot\left(\frac{\theta}{2}\right)$
- $r :$ The aspect ratio, i.e. $\frac{\text{width}}{\text{height}}$

Then, we can define the *transformation matrix from frustum to unit cube* as:

$$P = \begin{bmatrix} \frac{f}{r} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{zfar+znear}{znear-zfar} & \frac{2 \cdot zfar \cdot znear}{znear-zfar} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$





4.2 Geometry

4.2.1 Implicit Representations of Geometry

In an **implicit representation**, points aren't known directly, but satisfy some relationship. For example, we might define a unit sphere as all points x such that $x^2 + y^2 + z^2 = 1$. Implicit surfaces make some tasks easy, such as deciding whether some point is inside or outside our implicit surface.

4.2.2 Explicit Representations of Geometry

In an **explicit representation,** all points are given directly. For example, the points on a sphere are $(\cos u \sin v, \sin u \sin v, \cos v)$ for $0 \leq u < 2\pi$ and $0 \leq v < \pi$. There are many explicit representations in graphics, such as:

- Triangle meshes
- Polygon meshes
- Point clouds
- etc.

Explicit surfaces make some tasks easy, such as sampling. However, they also make some tasks hard, such as deciding whether a given point is inside or outside our surface.