**Visual Computing - Lecture notes week 3**


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# 3. Convolution and Filtering

## 3.1 Linear Shift-Invariant Filtering

**Linear shift-invariant filtering** is about modifying pixels base don its *neighborhood*. Linear means that it should be a *linear combination* of neighbors. Shift-invariant means that we do the *same thing for each pixel*. This approach is useful for:

- Low-level image processing operations
- Smoothing and noise reduction
- Sharpening
- Detecting or enhancing features

### 3.1.1 Linear Filtering

$L$ is a **linear** operation if:

$$L[\alpha I_1 + \beta I_2] = \alpha L[I_1] + \beta L[I_2]$$

Linear operations can be written as:

$$I'(x,\, y) = \sum_{(i,\, j) \in \mathcal{N}(x,\, y)} K(x,\, y;\, i,\, j) I(i,\, j)$$

Where $I$ is the input image, $I'$ is the output of the operation, and $K$ is the **kernel** of the operation. $\mathcal{N}(m,\, n)$ denotes the *neighborhood* of $(m,\, n)$.

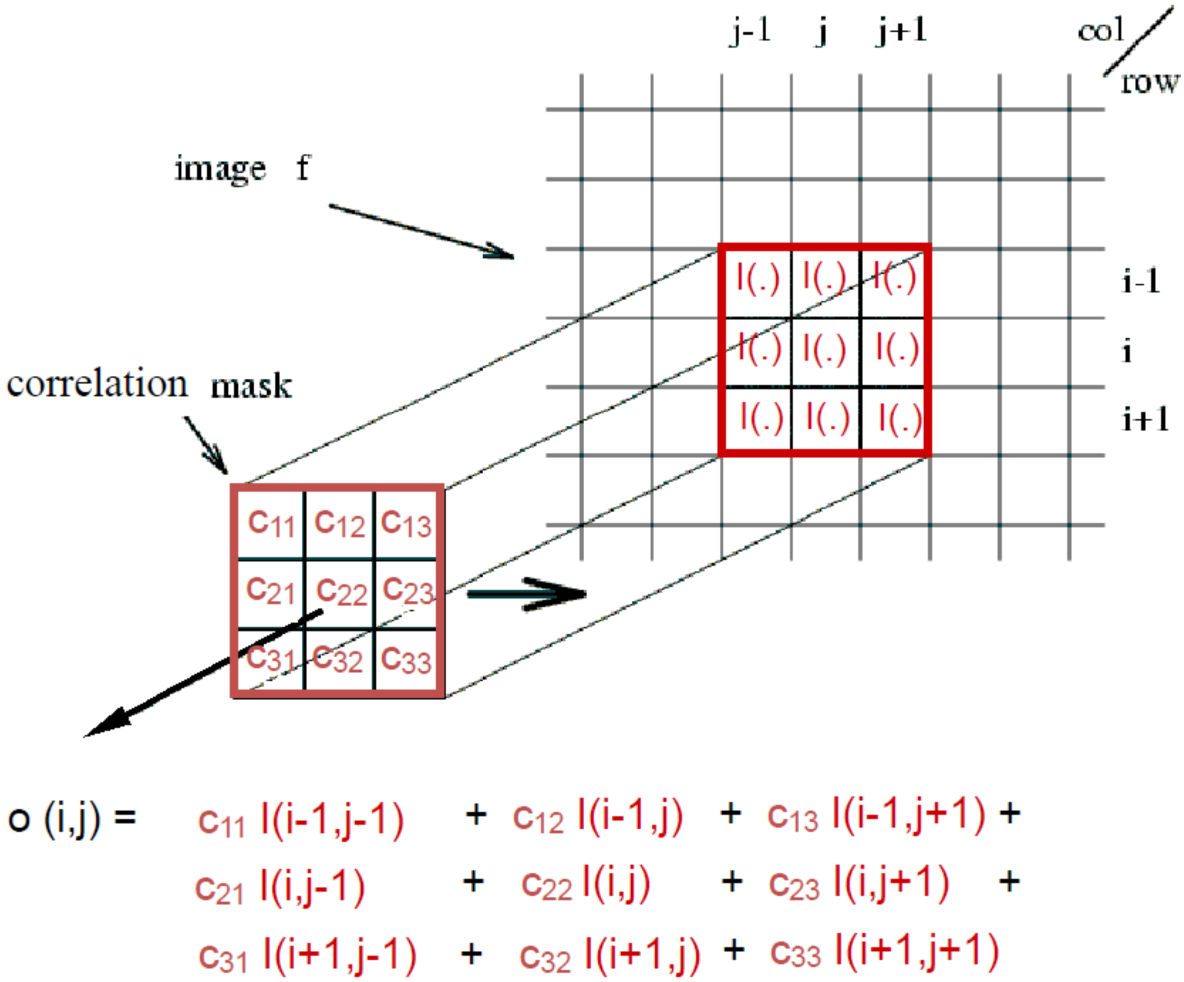
> Operations are **shift-invariant** if $K$ does *not* depend on $(x,\, y)$, i.e. we when using the same weights everywhere!

## 3.2 Correlation

In this approach, we take a **correlation mask** and apply it to an image.

> Correlation is as if we had a template (the mask), and search for it in our image.

This would look as follows:



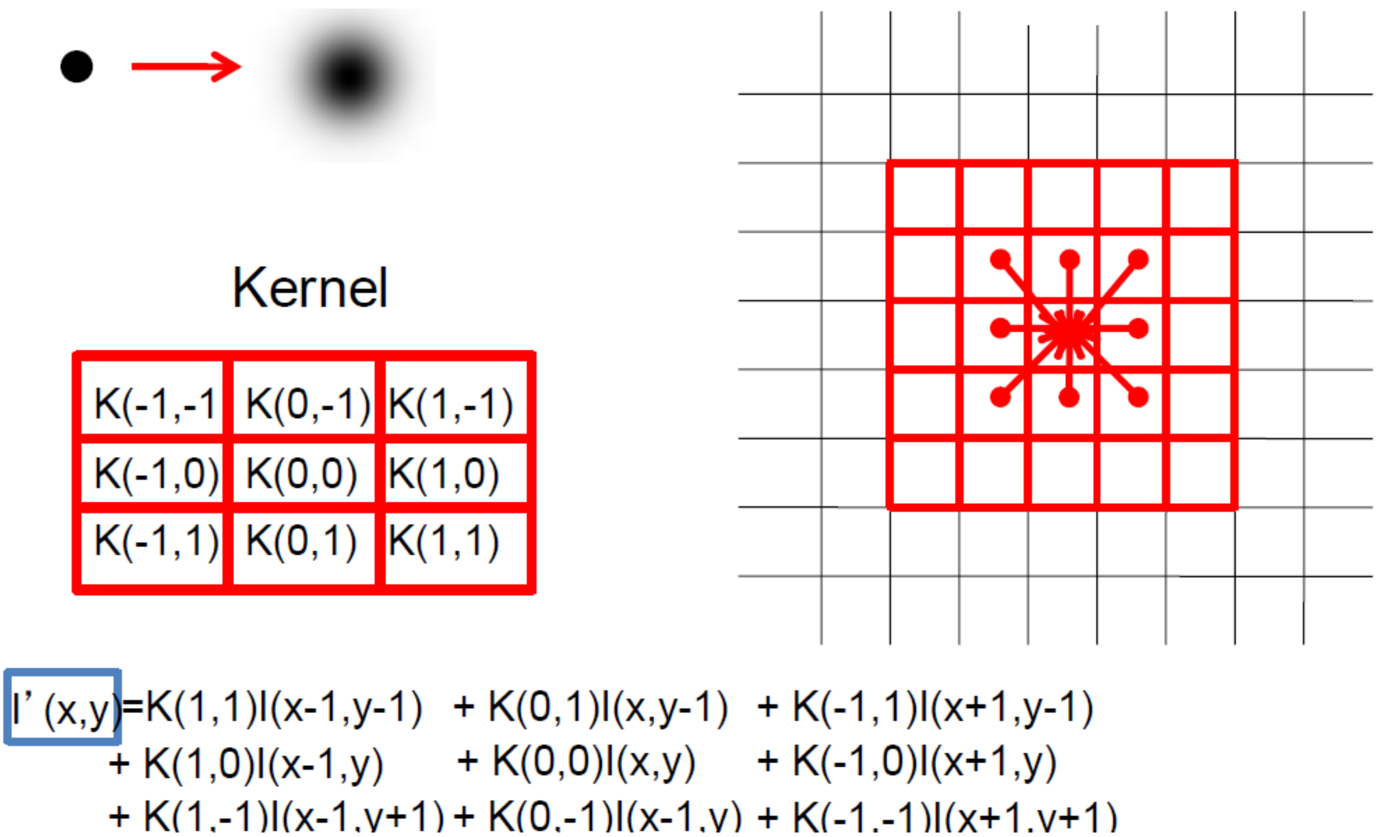

The linear operation of correlation looks as follows:

$$I' = K \circ I$$
$$I'(x, y) = \sum_{(i,\, j) \in \mathcal{N}(x,\, y)} K(i, j) I(x+i, y+j)$$

This represents the linear weights as an image.

## 3.3 Convolution

> Compared to correlation, where we looked at the neighborhood of a pixel and applied what we learned from the neighborhood to the single pixel, in convolution we look at a single pixel and apply what we can learn from it to its neighborhood.

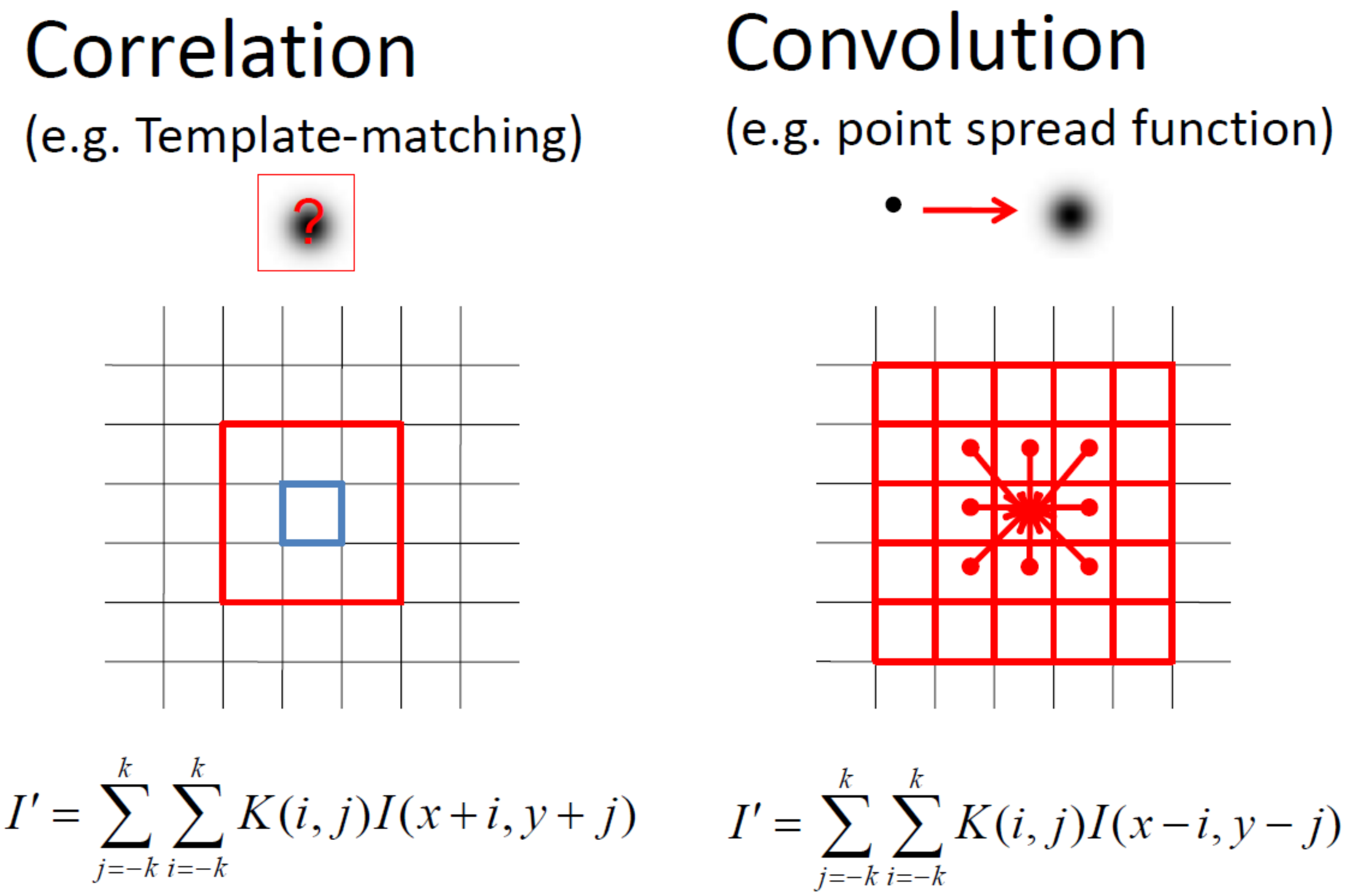



The linear operation of convolution is given by:

$$I' = K * I$$
$$I'(x, y) = \sum_{(i, j) \in \mathcal{N}(x, y)} K(i, j) I(x - i, y - j)$$

This too represents the linear weights as an image, it is actually the same as correlation, but with a reversed kernel.

## 3.3.1 Correlation vs Convolution




## 3.4 Separable Kernels

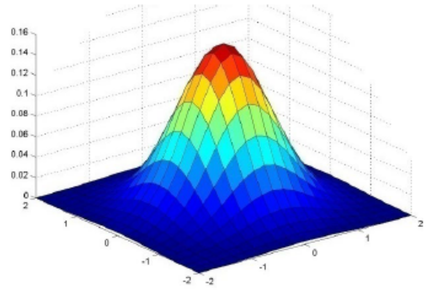
**Separable filters** can be written as $K(m, n) = f(m)g(n)$. For a rectangular neighborhood with size $(2M+1) \times (2N+1)$, $I'(m, n) = f * (g * I(\mathcal{N}(m, n)))$. We can rewrite this to:

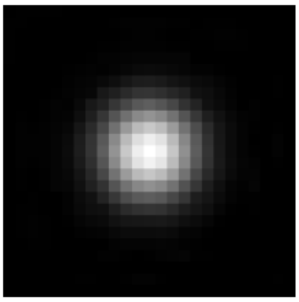
$$I''(m, n) = \sum_{j=-N}^{N} g(j) I(m, n-j)$$

$$I'(m, n) = \sum_{i=-M}^{M} f(i) I''(m-i, n)$$

## 3.5 Gaussian Kernel

The idea of the **Gaussian kernel** is that we weight the contributions of neighboring pixels by their nearness:





| | | | | |
|---|---|---|---|---|
| 0.003 | 0.013 | 0.022 | 0.013 | 0.003 |
| 0.013 | 0.059 | 0.097 | 0.059 | 0.013 |
| 0.022 | 0.097 | 0.159 | 0.097 | 0.022 |
| 0.013 | 0.059 | 0.097 | 0.059 | 0.013 |
| 0.003 | 0.013 | 0.022 | 0.013 | 0.003 |

5 x 5, σ = 1

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

### 3.5.1 Gaussian Smoothing Kernels

The amount of smoothing when using Gaussian kernels depends on $\sigma$ and on the window size.

The top 5 reasons to use Gaussian smoothing are:

1. Rotationally symmetric
2. Has a single lobe (neighbor's influence decreases monotonically)
3. Still one lobe in frequency domain (no corruption from high frequencies)
4. Simple relationship to $\sigma$
5. Easy to implement efficiently

## 3.6 Filter Examples

**Differential filters**

- *Prewitt operator:*

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

- *Sobel operator:*

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

**High-pass filters**

- *Laplacian operator:*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- *High-pass filter:*

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

# 4. Image Features

## 4.1 Template Matching

**Template matching** describes the problem of locating an object, described by a template $t(x,\, y)$, in the image $s(x,\, y)$. This is done by searching for the best match by minimizing mean-squared error:

$$\begin{aligned} E(p,\, q) &= \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} [s(x,\, y) - t(x-p,\, y-q)]^2 \\ &= \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} |s(x,\, y)|^2 + \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} |t(x,\, y)|^2 - 2 \cdot \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} s(x,\, y) \cdot t(x-p,\, y-q) \end{aligned}$$

Equivalently, we can *maximize* the **area correlation**:

$$r(p,\, q) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} s(x,\, y) \cdot t(x-p,\, y-q) = s(p,\, q) * t(-p,\, -q)$$

The area correlation is equivalent to the convolution of image $s(x,\, y)$ with impulse response $t(-x,\, -y)$.

## 4.2 Edge Detection

One idea, in a continuous-space, is to detect the local gradient:

$$|\mathrm{grad}(f(x,\, y))| = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$$

We mostly use the following **edge detection filters**:

Prewitt $\begin{pmatrix} -1 & 0 & 1 \\ -1 & [0] & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & [0] & 0 \\ 1 & 1 & 1 \end{pmatrix}$

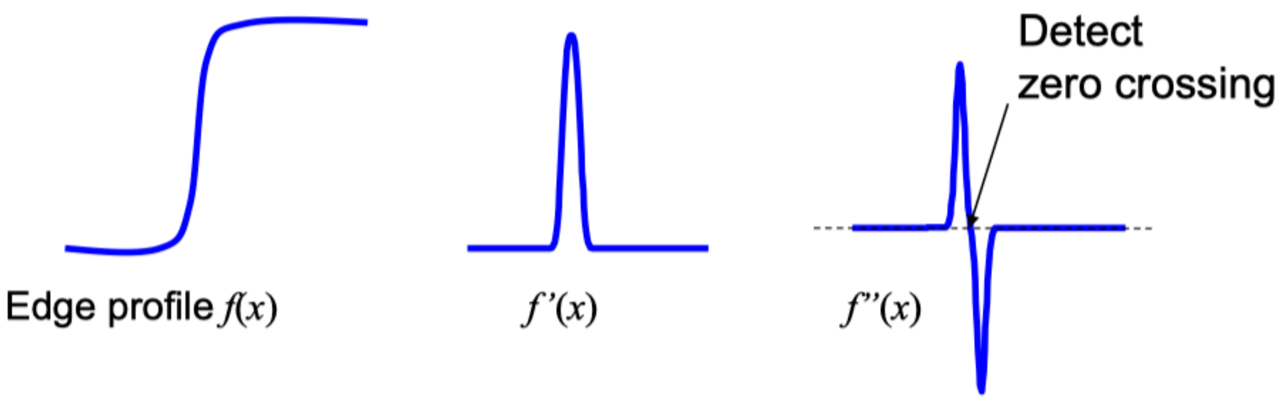
Sobel $\begin{pmatrix} -1 & 0 & 1 \\ -2 & [0] & 2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & -1 \\ 0 & [0] & 0 \\ 1 & 2 & 1 \end{pmatrix}$

Roberts $\begin{pmatrix} [0] & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} [1] & 0 \\ 0 & -1 \end{pmatrix}$

## 4.2.1 Laplacian Operator

The idea of a **Laplacian operator** is to detect discontinuities by considering the second derivative and searching for *zero-crossings* (those mark edge locations):

$$\nabla^2 f(x, y) = \frac{\partial^2 fx, y()}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2}$$




We can do a *discrete-space approximation* by convolution with a $3 \times 3$ impulse response:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

However, the Laplacian operator is sensitive to very fine detail and noise, so we might want to blur the image first.

### Laplacian of Gaussian

Blurring the image with Gaussian and Laplacian operator can be combined into convolution with **Laplacian of Gaussian operator** (LoG):

$$\mathrm{LoG}(x, y) = -\frac{1}{\pi\sigma^4}[1 - \frac{x^2 + y^2}{2\sigma^2}]exp(-\frac{x^2 + y^2}{2\sigma^2})$$

## 4.2.2 Canny Edge Detector

The **Canny edge detector** works with the following steps:

1. Smooth the image with a Gaussian filter
2. Compute the gradient magnitude and angle (Sobel, Prewitt, etc.):

$$M(x, y) = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} \quad \text{and} \quad \alpha(x, y) = \tan^{-1}(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x})$$

3. Apply nonmaxima suppression to gradient magnitude image
4. Double thresholding to detect strong and weak edge pixels
5. Reject weak edge pixels not connected with strong edge pixels

### Canny nonmaxima suppression

The **Canny nonmaxima suppression** works as follows:

1. Quantize the edge normal to one of the four directions: horizontal, -45deg, vertical, +45deg
2. If $M(x, y)$ is smaller than either of its neighbors in edge normal direction, then suppress is, else keep it

### Canny Thresholding

When using a Canny edge detector, we do double thresholding of the gradient magnitude:

- Strong edge: $M(x, y) \geq \Theta_{high}$
- Weak edge: $\Theta_{high} > M(x, y) \geq \Theta_{low}$

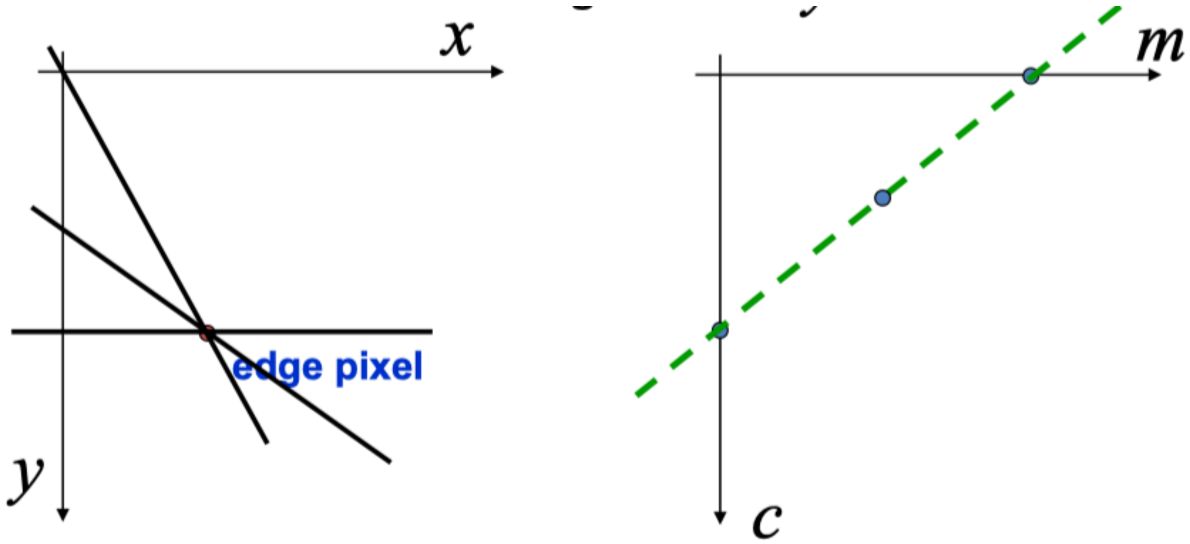
A typical setting for the thresholds would be $\frac{\Theta_{high}}{\Theta_{low}} \in [2, 3]$.

## 4.2.3 Hough Transform

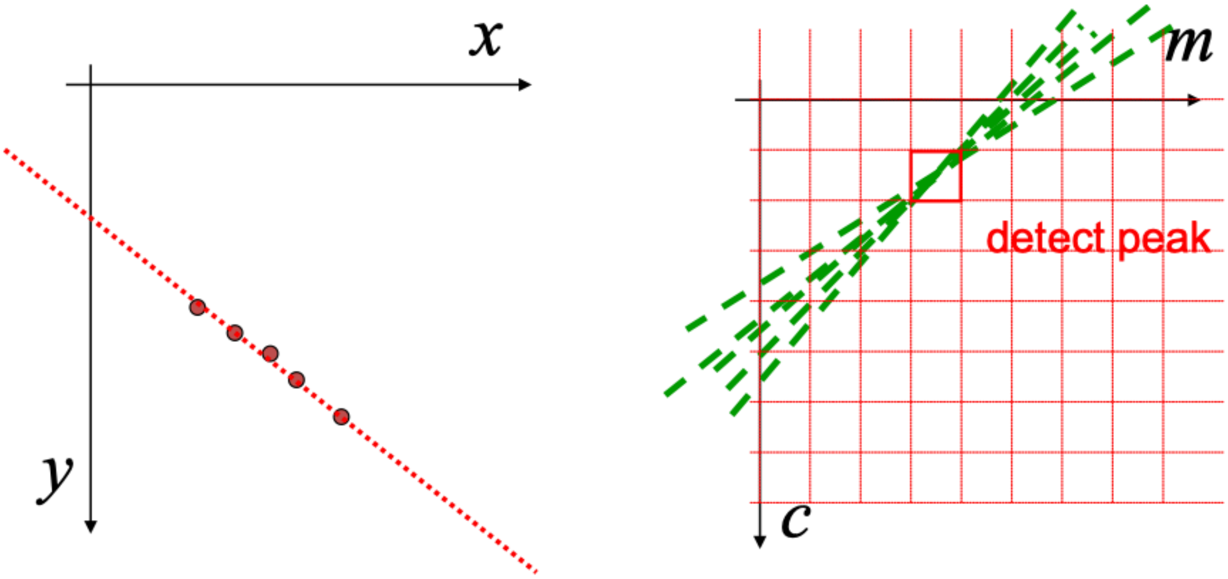
The **Hough transform** solves the problem of fitting a straight line (or curve) to a set of edge pixels. The Hough transform is a *generalized template matching technique*.

It works the following way:

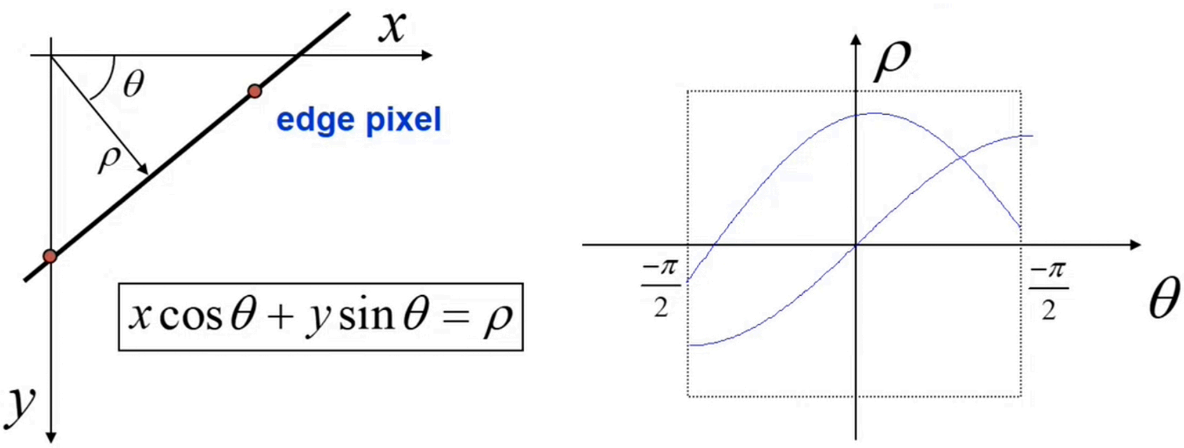
1. For an edge pixel in the $(x, y)$ plane we can draw the different lines that cross the edge pixel (all lines have the form $y = mx + c$). We can draw the $m$ and $c$ values in a $(m, c)$ plane and see that all those lines are linearly dependent:




2. If we have multiple edge pixels, we can do the same procedure for each of those, giving us a line in the $(m, c)$ plane for each edge pixel.
3. We then subdivide the $(m, c)$ plane into discrete "bins" and initialize the bin count of each bin to $0$. Each time a bin is crossed by one of the lines of the different edge pixels, we increase its count by one.
4. We then simply have to detect the peaks in the $(m, c)$ plane to get our fitted straight line:




We might encounter an infinite-slope problem, which can be avoided with an alternative parameterization:




## 4.3 Detecting Corner Points

Many applications benefit from features localized in $(x, y)$. If edges are well localized but only in one direction, we might want to **detect corners**.

The desirable properties of a corner detector are:

- accurate localization
- invariance against shift, rotation, scale, and brightness change
- robust against noise, high repeatability

*something something what patterns can be localized most accurately?*

- Local displacement sensitivity

$$S(\Delta x, \Delta y) = \sum_{(x,y)\in window} \left[ f(x,y) - f(x+\Delta x, y+\Delta y) \right]^2$$

- Linear approximation for small $\Delta x, \Delta y$

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

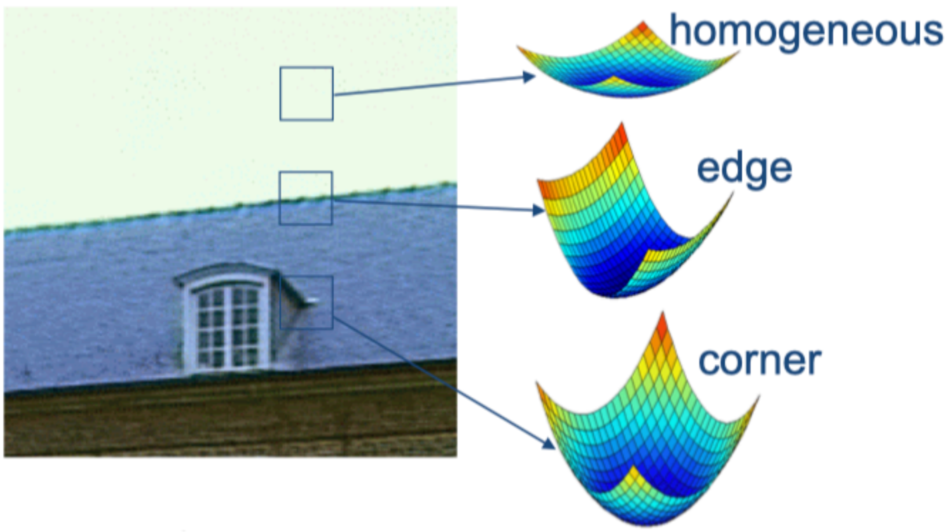
$f_x(x,y)$ – horizontal image gradient
$f_y(x,y)$ – vertical image gradient

$$\begin{aligned} S(\Delta x, \Delta y) &\approx \sum_{(x,y)\in window} \left[ \begin{pmatrix} f_x(x,y) & f_y(x,y) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \right]^2 \\ &= \begin{pmatrix} \Delta x & \Delta y \end{pmatrix} \left( \sum_{(x,y)\in window} \begin{bmatrix} f_x^2(x,y) & f_x(x,y) f_y(x,y) \\ f_x(x,y) f_y(x,y) & f_y^2(x,y) \end{bmatrix} \right) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\ &= \begin{pmatrix} \Delta x & \Delta y \end{pmatrix} \mathbf{M} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \end{aligned}$$

- Iso-sensitivity curves are ellipses

### 4.3.1 Feature Point Extraction

We have that $SSD \simeq \delta^T M \delta$. Now if we shift our patterns over the picture, we assume it to change the following way:




Now we want to find points for which the following is large:

$$\min \delta^T M \delta \text{ for } ||\delta|| = 1$$

i.e. we want to maximize the eigenvalues of $M$.

### Keypoint Detection

*something something*

Often based on eigenvalues $\lambda_1$, $\lambda_2$ of "structure matrix" (aka "normal matrix" aka "second-moment matrix")

$$\mathbf{M} = \begin{bmatrix} \sum_{(x,y)\in window} f_x^2(x,y) & \sum_{(x,y)\in window} f_x(x,y) f_y(x,y) \\ \sum_{(x,y)\in window} f_x(x,y) f_y(x,y) & \sum_{(x,y)\in window} f_y^2(x,y) \end{bmatrix}$$

$f_x(x,y)$ – horizontal image gradient
$f_y(x,y)$ – vertical image gradient

Measure of "cornerness"

$$C(x,y) = \det(\mathbf{M}) - k \cdot \left(trace(\mathbf{M})\right)^2 = \lambda_1 \lambda_2 - k \cdot (\lambda_1 + \lambda_2)$$

[Harris, Stephens, 1988]