# Representations of a concrete category as objects in the functor category

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# 1 Introduction

$$\mathbf{Quiv} \to^{CatClosure} \leftarrow_{U} \mathbf{Cats} \to^{k-Algebroid} \leftarrow_{U} \mathbf{k} - \mathbf{Cats} \to^{AdditiveClosure} \leftarrow_{U} \mathbf{k} - \mathbf{Cats}^{\oplus}$$

# 2 A short overview of the tools used

GAP, QPA / QPA2, Catreps, CAP, homalg_project

# 3 The categories Cat and Quiv

This section serves two purposes: On the one hand, it is an introduction to quivers and category theory. On the other hand it introduces concrete categories which we want to represent, and all the additional constructions that are needed to that goal.

## 3.1 Introduction to quivers and category theory

In this section, we first want to define the category **Quiv** and how it is the prototype for the category **Cats**. In order to describe the category **Quiv** of quivers, we first have to define what a category is and for this we need the definition of a quiver. Lateron we will revisit this definition as we can define quivers as the objects in the quiver category **Quiv**.

**Definition 3.1.1.** (Quiver)
A <u>directed graph</u> or <u>quiver</u> $q$ consists of a class of <u>objects</u> (or <u>vertices</u>) $q_0 = \mathrm{Obj}\, q$ and a class of <u>morphisms</u> (or <u>arrows</u>) $q_1 = \mathrm{Mor}\, q$ together with two defining maps

$$s, t \colon q_1 \rightrightarrows q_0$$

$s$ called <u>source</u> and $t$ called <u>target</u>.

In the next definition we are giving a new characterization for $q_1$ by looking at all arrows between two fixed objects.

**Definition 3.1.2.** (Hom-set of a (locally) small quiver)

(1) Given two objects $M, N \in q_0$ we write $\mathrm{Hom}_q(M,N)$ or $q(M,N)$ for the fiber $(s,t)^{-1}(\{(M,N)\})$ of the product map $(s,t) : q_1 \longrightarrow q_0 \times q_0$ over the pair $(M,N) \in q_0 \times q_0$. This is the class of all morphisms with source $= M$ and target $= N$. We indicate this by writing $\varphi : M \longrightarrow N$ or $M \xrightarrow{\varphi} N$. Hence $q_1$ is the disjoint union $\dot{\bigcup}_{M,N \in q_0} \mathrm{Hom}_q(M,N) = q_1$. As usual we define $\mathrm{End}_q(M) := \mathrm{Hom}_q(M,M)$.

(2) If the class $\mathrm{Hom}_q(M,N)$ is a <u>set</u> for all pairs $(M,N)$ then we call the quiver <u>locally small</u>. We therefore talk about <u>Hom-sets</u>. If additionally, $q_0$ is a set, then the quiver is called <u>small</u>.

**Example 3.1.3.** (Quiver with 2 objects and 3 morphisms)

$$\begin{tikzcd} 1 \arrow[r, "b"] \arrow[loop below, "a"] & 2 \arrow[loop below, "c"] \end{tikzcd}$$

The objects of this quiver $q$ are $q_0 = \{1,2\}$, and the morphisms are $q_1 = \{a,b,c\}$ with
$s(a) = 1 = t(a)$, $s(c) = 2 = t(c)$ and $s(b) = 1, t(b) = 2$.
Thus $\mathrm{End}_q(1) = \{a\}, \mathrm{End}_q(2) = \{c\}$ and $\mathrm{Hom}_q(1,2) = \{b\}$ whereas $\mathrm{Hom}_q(2,1) = \varnothing$.

In `QPA` this quiver is encoded as `q(2)[a:1->1,b:1->2,c:2->2]` where the first `(2)` in parentheses stands for the total number of objects.

**Definition 3.1.4.** (Composable arrows; path in a quiver)
Since we already have the source and target maps, we say two arrows $a, b \in q_1$ are composable if $t(a) = s(b)$ or $t(b) = s(a)$. In this case we can write a sequence of composable arrows $p = a_1 a_2 \cdots a_n$ where $t(a_i) = s(a_{i+1})$ for $i = 1, \ldots, n-1$. We call this sequence a path from $s(a_1)$ to $t(a_n)$ and the integer $n \in \mathbb{Z}_{\geq 0}$ the length $l(p)$ of the path $p$. Although it's not an arrow, we can define the source and target of a path $p = a_1 \cdots a_n$ as $s(p) := s(a_1)$ and $t(p) := t(a_n)$. A path $p = a_1 \cdots a_n$ with $s(a_1) = t(a_n)$, i.e. $s(p) = t(p)$, is called cyclic.
For an endomorphism $a \in \operatorname{End}_q(M)$ we write $a^n$ for $aa \cdots a$ ($n$ times). In the case of $n = 0$ an empty path whose source and target are the vertex $i \in q_0$ is called the trivial path at $i$ and is denoted $e_i$. Note that the composition of paths $e_i e_i$ has length zero starting at $i$ therefore $e_i^2 = e_i$.

**Lemma 3.1.5.** *Let $Q$ be a quiver. If there is a path of length at least $|Q_0|$, then there are cyclic paths, and thus infinitely many paths.[2]*

*Proof.* Assume that there exists a path of length greater or equal to $|Q_0|$. Then there exists a path of length $n = |Q_0|$, say $\alpha_1 \cdots \alpha_n$. Consider the vertices $x_i = s(\alpha_i)$ for $1 \leq i \leq n$ and $x_{n+1} = t(\alpha_n)$. Then these are $n+1$ vertices, thus there has to exist $i < j$ with $x_i = x_j$. Let $\omega = \alpha_i \cdots \alpha_{j-1}$, this is a path with source and target $x_i = x_j$, thus a cyclic path. But then $\omega^m$ is a path for any natural number $m$. The path $\omega$ has length $j - i \geq 1$, thus $\omega^m$ has length $m(j-i)$. This shows that these paths are pairwise different. $\square$

**Example 3.1.6.** (A quiver with no cycles)

$$\begin{tikzcd}
2 \arrow[r, "\psi"] \arrow[dr, "\psi\rho"] & 3 \arrow[d, "\rho"] \\
1 \arrow[u, "\varphi"] \arrow[ur, "\varphi\psi"] \arrow[r, "\varphi\psi\rho"'] & 4
\end{tikzcd}$$

The longest path $1 \to 2 \to 3 \to 4$ has length 3. If after the object 4 another arrow would go to either $1, 2, 3$ or 4 itself, we would have a cyclic path and thus infinitely many paths.

**Definition 3.1.7.** (Category)
A category $\mathcal{C}$ is a quiver with two further maps:

(id) The identity map $1_{()}$ mapping every object $X \in \mathcal{C}_0$ to its identity morphism $1_X$:

$$\mathcal{C}_0 \xrightarrow{1} \mathcal{C}_1$$

($\mu$) And for any two composable morphisms $\varphi$ and $\psi \in \mathcal{C}_1$, i.e. with $t(\varphi) = s(\psi)$, the composition map $\mu$, which maps $\varphi, \psi \in \mathcal{C}_1 \times \mathcal{C}_1$ to $\mu(\varphi, \psi) \in \mathcal{C}_1$ which we also write as $\varphi\psi$.

$$\mathcal{C}_1 \times \mathcal{C}_1 \xrightarrow{\mu} \mathcal{C}_1$$

The defining properties for 1 and $\mu$ are:

(1) $s(1_M) = M = t(1_M)$, i.e.
$1_M \in \operatorname{End}_{\mathcal{C}} \forall M \in \mathcal{C}$.

(2) $s(\varphi\psi) = s(\varphi)$ and
$t(\varphi\psi) = t(\psi)$
for all composable morphisms $\varphi, \psi \in \mathcal{C}$.

$$\mu : \operatorname{Hom}_{\mathcal{C}}(M, L) \times \operatorname{Hom}_{\mathcal{C}}(L, N) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(M, N)$$

(3) $(\varphi\psi)\rho = \varphi(\psi\rho)$ [associativity of composition]

(4) $1_{s(\varphi)}\varphi = \varphi = \varphi 1_{t(\varphi)}$ [unit property]
The identity is a left and right unit of the composition.

**Definition 3.1.8.** (Finite and concrete categories)

(1) A finite category is a category with a finite set of objects and a finite set of morphisms.

(2) A concrete category is a category whose objects have underlying sets and whose morphisms are functions between these underlying sets. Otherwise it's called an abstract category.

Clearly every finite category is a small category.

# 4 Finite concrete categories

As we have seen in the previous section, a quiver $q$ with a path of length greater than $|q_0|$ must have loops and is thus infinite. We will construct finite concrete categories by paying attention that the arrows between different objects are only one-directional, thus we have a partial order on the set of objects.

The following algorithm takes two integers n and m as arguments and gives a finite concrete category as output with n objects which are each FinSets with m elements. The generating endomorphisms are each a permutation of order m, while the non-endomorphisms are bijective mappings from each object $c \in \mathcal{C}_0$ to all later objects $c' \in \mathcal{C}, c' > c$ with the obvious order.

```
##
InstallMethod( ConcreteCategoryForCAP,
        "for two integers",
        [ IsInt, IsInt ],

  function( n, m )
local objects, gmorphisms, permute, j, k, list, C;
  objects := [];
  for j in [1..n] do
    objects[j] := FinSet([1+(j-1)*m..j*m]);
  od;
  gmorphisms := [];
  permute := function(o, j, m)
    local r;
    r := RemInt( o+1, m );
    if r > 0 then
      return (r+(j-1)*m);
    else
      return j*m;
    fi;
  end;
  for j in [1..n] do
    for k in [j..n] do
if j = k then
    Add( gmorphisms, MapOfFinSets( objects[j],
List( objects[j], o-> [o, permute(o,j,m) ] ),
objects[k]) );
else # k > j
Add( gmorphisms, MapOfFinSets( objects[j],
List( objects[j], o -> [o, o+(k-j)*m] ),
objects[k]) );
fi;
od;
  od;

    DeactivateCachingOfCategory( FinSets );
    CapCategorySwitchLogicOff( FinSets );
    DisableSanityChecks( FinSets );
```

```
    C := Subcategory( FinSets, "A finite concrete category" : overhead := false, FinalizeCategory :=

DeactivateCachingOfCategory( C );
    CapCategorySwitchLogicOff( C );
    DisableSanityChecks( C );

SetFilterObj( C, IsFiniteConcreteCategory );

AddIsAutomorphism( C,
      function( alpha )
        return IsAutomorphism( UnderlyingCell( alpha ) );
    end );

AddInverse( C,
      function( alpha )
        return Inverse( UnderlyingCell( alpha ) ) / CapCategory( alpha );
    end );

SetSetOfObjects( C, List( objects, o-> o / C ) );
SetSetOfGeneratingMorphisms( C, List( gmorphisms, g-> g / C ) );

    Finalize( C );

    return C;
end );
```

# 5 The categories Quiv, Cat, FinSets, k-Mat, CatReps and the Functor Category

In this section, we first want to define the category **Quiv** and how it is the prototype for the category **Cats**. We want to restrict ourselves to finite concrete categories, which brings us to the category **FinSets**. Our goal is to represent concrete categories, for this we need the source and target categories of our representations. The source category is **k-Algebroids** which we compute algorithmically from a concrete category. The target category of our category representations is **k-Mat**. The category where our category representations lie in is **CatReps** for which we show that it's a subcategory of the **Functor Category**.

## 5.1 Additional structure on the Hom-set of a category

**Example 5.1.1.** A group G defines a category BG with a single object. The group elements are its morphisms, which are all automorphisms of the single object. The identity element $e \in G$ acts as the identity morphism for the unique object in this category. The hom-set of that category is itself a group.

This example can be generalized to categories where the hom-set is a ring or an R-algebra. But for this we need a commutative ring R and thus the category R-Mod.

**Definition 5.1.2.** Ab-Category An Ab-category is a category in which all homomorphism sets are abelian groups. That means in addition to the composition of morphisms $\mu : \mathcal{C}_1 \times \mathcal{C}_1 \longrightarrow \mathcal{C}_1$ we have another binary operation $+ : \mathcal{C}_1 \times \mathcal{C}_1 \longrightarrow \mathcal{C}_1$ , that distributes over the composition, i.e.

$$\mu(f+g,h) = \mu(f,h) + \mu(g,h)$$

.

The concept of a functor is central in category theory. It is how the objects and morphisms of two categories relate to one another.

**Definition 5.1.3.** (Functor)
In the category **Cat** which has categories as objects, functors are the morphisms between these objects. Let $\mathcal{C}, \mathcal{D} \in \operatorname{Obj} \mathbf{Cat}$ be categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between $\mathcal{C}$ and $\mathcal{D}$ consists of the following data:

(1) For every object $c \in \mathcal{C}_0$ there is an object $Fc \in \mathcal{D}$.

(2) For every morphism $f \in \mathcal{C}_1$, i.e. $c \xrightarrow{f} c'$ there is a morphism $Ff \in \mathcal{D}_1$ with $Fc \xrightarrow{Ff} Fc'$, i.e. $s(Ff) = F s(f)$ and $t(Ff) = F t(f)$

Functors are compatible with the identity map and the composition map:

(3) For every object $c \in \mathcal{C}_0$ we have $1_c \in \mathcal{C}_1$ and for the functor $F$ we demand that $F 1_c = 1_{Fc} \in \mathcal{D}_1$.

(4) For every pair of morphisms $f, g \in \mathcal{C}_1$ with $t(f) = s(g)$ we have $fg \in \mathcal{C}_1$ and we demand that $Fg\, Fg = F gf$.

With the definition of a category and the category of functors finished, we can come back and use them to define the category of quivers **Quiv**.

**Definition 5.1.4.** (The category **Quiv**)[3] Let the Kronecker category $\mathcal{K}$ be the category with two objects, 0 and 1, and two non-identity morphisms, s and t $1 \underset{t}{\overset{s}{\rightrightarrows}} 0$. Let **FinSets** be the category of finite sets with morphisms being maps between those sets. The category of quivers **Quiv** is the category of functors from $\mathcal{K}$ to **FinSets**. For a quiver $q \in \operatorname{Obj} \mathbf{Quiv}$ we write $q_x$ for the image of $x \in \{0, 1\}$ under $q$. The images under $q$ of the morphisms $s$ and $t$ are again denoted by $s$ and $t$.

As we have seen, every category is a quiver, but in general, to become a category, a quiver is lacking identity morphisms and the composition of morphisms. To be more precise, there is a functor $U$ from the category of categories CAT to the category of quivers Quiv, called the underlying quiver or forgetful functor.

$$\text{Cat} \xrightarrow{\quad U \quad} \text{Quiv}$$

mapping every object $M \in \mathcal{C}_0$ to the same objects in $q_0$, mapping every arrow $\varphi \in \mathcal{C}_1$ to an arrow $a \in q_1$, respecting source and target, but forgetting the special role of the identity morphisms and of the composition morphisms.

**Example 5.1.5.** (Underlying quiver)

$$
\begin{tikzcd}
B \arrow[loop above, "1_B"] \arrow[r, "\psi"] \arrow[dr, "\psi\rho" near start] & C \arrow[loop above, "1_C"] \arrow[d, "\rho"] \\
A \arrow[loop below, "1_A"] \arrow[u, "\varphi"] \arrow[ur, "\varphi\psi"' near start] \arrow[r, bend left, "(\varphi\psi)\rho"] \arrow[r, bend right, "\varphi(\psi\rho)"'] & D \arrow[loop below, "1_D"]
\end{tikzcd}
\quad \xRightarrow{U} \quad
\begin{tikzcd}
2 \arrow[loop above, "h"] \arrow[r, "b"] \arrow[dr, "e" near start] & 3 \arrow[loop above, "i"] \arrow[d, "c"] \\
1 \arrow[loop below, "g"] \arrow[u, "a"] \arrow[ur, "d"' near start] \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] & 4 \arrow[loop below, "j"]
\end{tikzcd}
$$

In the category on the left, associativity of composition guaranteed that $(\varphi\psi)\rho = \varphi(\psi\rho)$, so those two arrows were already the same, so they are mapped to the same arrow $f = U((\varphi\psi)\rho) = U(\varphi(\psi\rho))$ in the quiver on the right. We didn't have to draw both arrows for $f$, but since they are equal, there is still only one arrow in the hom-set $\mathrm{Hom}_q(1,4) = \{f, f\} = \{f\}$.
All the other identities are not preserved under the forgetful functor, e.g. $d$ doesn't know what it has to do with $a$ and $b$ apart from $s(d) = s(a)$ and $t(d) = t(b)$. Especially the former identity arrows are now just endomorphisms with no defining property.
The paths $g^2 f, gf$ and $fj^3$ are all different, while in the category, they all simplify to $1_A 1_A(\varphi\psi)\rho = 1_A(\varphi\psi)\rho = (\varphi\psi)\rho 1_D 1_D 1_D = (\varphi\psi)\rho$ due to the unit property and associativity.

**Definition 5.1.6.** (Ab-category) An Ab-category is a category in which all homomorphism sets are abelian groups, and composition distributes over addition.
In other words, A category $\mathcal{C}$ is an Ab-category if for every pair of objects $M, N \in \mathcal{C}_0$, $(\mathrm{Hom}_{\mathcal{C}}(M,N), +)$ is an abelian group (with the neutral element called zero morphism), and for all morphisms $\gamma, \delta \in \mathrm{Hom}_{\mathcal{C}}(M,N), \alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(N,L)$

$$(\gamma + \delta)\alpha = \gamma\alpha + \delta\alpha \text{ and}$$

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta.$$

Note that every hom-set has its own unique zero morphism. E.g. in $\mathrm{Mat}_{\mathbb{Q}}$ the 2-by-3 zero-matrix $0 \in \mathrm{Hom}(2,3)$ is different from the 4-by-4 zero-matrix $0 \in \mathrm{Hom}(4,4)$.

**Definition 5.1.7.** (Initial object, terminal object, zero object)

**Example 5.1.8.**

**Definition 5.1.9.** (Kernel of a morphism

**Definition 5.1.10.** (Abelian category)

**Definition 5.1.11.** (k-linear category)

Quiver -¿ CAT: U: forget 1, forget composition
search $U^{-1}$
Beispiel für Adjunktion
Path Algebra:

# 6 Functors and natural transformations

## 6.1 Functors map one category to another

**Example 6.1.1.** (Identity Functor) bla

**Example 6.1.2.** (Forgetful functor) bla

**Definition 6.1.3.** (full functor; faithful functor)

### 6.2 Natural transformations are morphisms between functors

# 7 Yoneda's Lemma: Completion and cocompletion of a category

## 7.1 Embedding categories

**Lemma 7.1.1.** *(Yoneda's Lemma)*

*Proof.* □

Projective objects?

$$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}\begin{pmatrix}1\to5, & 2\to6\\ 3\to7, & 4\to8\end{pmatrix}\begin{pmatrix}6 & 8\end{pmatrix}\begin{pmatrix}5\to9\\ 6\to10\\ 7\to11\\ 8\to12\end{pmatrix}\begin{pmatrix}9 & 10\end{pmatrix}\begin{pmatrix}11 & 12\end{pmatrix}\begin{pmatrix}9\to13, & 10\to14\\ 11\to15 & 12\to16\end{pmatrix}\mathrm{id}$$

## 7.2 Yoneda Projective

Consider the concrete category

$$\begin{tikzcd}
\{1,2,3,4\} \arrow[loop above, "(1{,}2{,}3{,}4)"] \arrow[rr, "\begin{pmatrix}1\to5{,}2\to6{,}\\3\to7{,}4\to8\end{pmatrix}"] \arrow[d] \arrow[drr] & & \{5,6,7,8\} \arrow[loop above, "(6{,}8)"] \arrow[d, bend left, "\begin{pmatrix}5\to9\\6\to10\\7\to11\\8\to12\end{pmatrix}"'] \arrow[dll] \\
\{13,14,15,16\} \arrow[loop below, "\mathrm{id}"] & & \{9,10,11,12\} \arrow[ll, "\begin{pmatrix}9\to13{,}10\to14{,}\\11\to15{,}12\to16\end{pmatrix}"] \arrow[loop below, "(9{,}10)(11{,}12)"]
\end{tikzcd}$$

and its $\mathbb{K}$-Algebroid kq

$$\begin{tikzcd}
1 \arrow[loop above, "a"] \arrow[r, "b"] \arrow[dr, "c"] \arrow[d, bend right, "d"'] & 2 \arrow[loop above, "e"] \arrow[dl, "g"] \arrow[d, bend left, "f"] \\
4 \arrow[loop left, "j"] & 3 \arrow[l, "i"] \arrow[loop right, "h"]
\end{tikzcd}$$

together with the relations

$$[a^4-(1), e^2-(2), h^2-(3), j^1-(4), bf-c, bef-ach, bg-d, ci-d, achi-beg, a^3beg-chi, fi-g]$$

The resulting category algebra has dimension 43.

We can look at the submodule of the category algebra consisting of all arrows starting at `kq.1`. This is what the function `YonedaProjective( CatReps, kq.1 )` gives us:

```
proj1 := YonedaProjective( CatReps, kq.1 ); <(1)->4, (2)->8, (3)->8, (4)->8; (a)->4x4,
(b)->4x8, (c)->4x8, (d)->4x8, (e)->8x8, (f)->8x8, (g)->8x8, (h)->8x8, (i)->8x8, (4)->8x8>
```

The number 4 associated with object (1) tells us that the submodule of all arrows starting and ending at (1) has dimension 4. Its basis is the set of paths $\{a, a^2, a^3, a^4 = (1)\}$.

Likewise in

```
proj4 := YonedaProjective( CatReps, kq.4 ); <(1)->0, (2)->0, (3)->0, (4)->1; (a)->0x0,
(b)->0x0, (c)->0x0, (d)->0x1, (e)->0x0, (f)->0x0, (g)->0x1, (h)->0x0, (i)->0x1, (4)->1x1>
```

The submodule of all arrows starting at (4) is only of dimension 1, since it's already the identity arrow $\{j = (4)\}$.

Dimension of the (quotient of the) path algebra is 43. Sum of all dimensions of the yoneda projectives on each objects is 43.

**Definition 7.2.1.** (Yoneda projective) Yoneda's projective representation given by the object $o$ is the submodule of the category algebra consisting of all arrows starting at $o$.

Conjecture:

Dimension of the path algebra = Sum of dimensions of the yoneda projectives on each object.

What does the yoneda projective mean???

Function that creates examples for concrete categories so that I can check my conjecture.

# 8 Functors and natural transformations

## 8.1 Functors act on objects and morphisms of a category

## 8.2 Natural transformations are morphisms between functors

## 8.3 Representations are Functors into a matrix category

## 8.4 Finite concrete categories

Yonedas Einbettungs-Lemma: Fehlende Limiten bzw. Kolimiten exitieren nach der Einbettung.

Einbettung in Kategorien, die mehr Limiten haben als die Zielkategorie.

"(Ko-)Vervollständigung" der Kategorie (Completion / Cocompletion)

Quiver = unvollständige Struktur einer Kategorie Erzeugendensystem einer Kategorie.

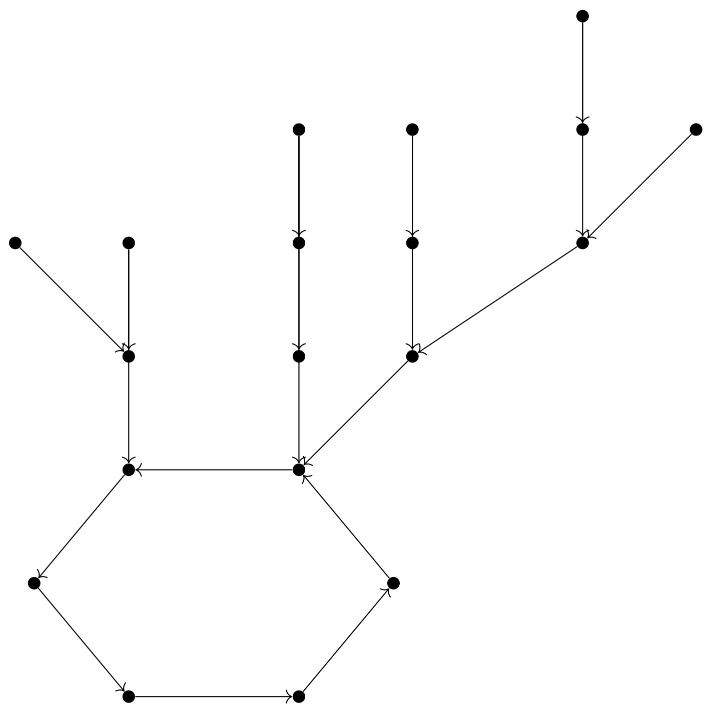
K-linearer Abschluss einer Kategorie

Pfadalgebra = Kategorien-Algebra path algebra = 1 Object, welches eine Algebra ist. Dabei verliert man wieder die Informationen über die mehreren Objekte.

So wie Menge ein Erz-system eines Monoid.

# 9 Relations of the Algebroid

## 9.1 Relations of endomorphisms



**Lemma 9.1.1** ($\sigma$-Lemma)**.** *For each endomorphism $f$ in a finite concrete category $C$ there exist $m, n \in \mathbb{N}$ such that $f^{(m+n)} = f^m$.*

Beschreibung der Algorithmen
WeakDirectSumDecomposition ¡– Tiefensuche. Objekte (Funktoren) in indecomposable Functors.

**Example 9.1.2.** (Representation of a concrete category)

$$\begin{tikzcd}
5 \arrow[r, "\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}"] \arrow[loop below, "\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}"] & 4 \arrow[loop below, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}"]
\end{tikzcd}$$

$$nine \Uparrow$$

$$\begin{tikzcd}
1 \arrow[r, "b"] \arrow[loop below, "a"] & 2 \arrow[loop below, "c"]
\end{tikzcd}$$

$$\begin{tikzcd}
\{1, 2, 3\} \arrow[r, "(4{,}5{,}6)"] \arrow[loop below, "(2{,}1{,}3)"] & \{4, 5, 6\} \arrow[loop below, "(5{,}6{,}4)"]
\end{tikzcd}$$

$$F(a)\eta_1 = \eta_1 G(a) F(b)\eta_2 = \eta_1 G(b)$$

# 10 $\mathbb{K}$-linear Category (Algebroid)

Group: Category with one object.
Groupoid: A small category in which every morphism is an isomorphism.
Algebroid
EmbeddingOfSumOfImages
What is an Algebroid? Bialgebroid?

# 11 Additive Category

# 12 Abelian Category

# 13 The Category of Categories

# 14 The Categories of Functors

# 15 The Representation of a Category

# 16 Representation

Grundidee von FunctorCategory
Standard-Monoidale Struktur von der Zielkategorie z.B. TensorUnit(C)

# 17 Algorithms

```
    AddInverse( C,
      function( alpha )
        return Inverse( UnderlyingCell( alpha ) ) / CapCategory( alpha );
    end );

    c := ConcreteCategory( L );

    C!.ConcreteCategoryRecord := c;

    objects := List( c.objects, FinSet );

    SetSetOfObjects( C, List( objects, o -> o / C ) );

    SetSetOfGeneratingMorphisms( C, List( c.generators, g -> ConvertToMapOfFinSets( obje

    Finalize( C );

    return C;

end );

##
InstallMethod( ConcreteCategoryForCAP,
        "for two integers",
        [ IsInt, IsInt ],

  function( n, m )
        local objects, gmorphisms, permute, j, k, list, C;
  objects := [];
  for j in [1..n] do
    objects[j] := FinSet([1+(j-1)*m..j*m]);
  od;
  gmorphisms := [];
  permute := function(o, j, m)
    local r;
    r := RemInt( o+1, m );
    if r > 0 then
      return (r+(j-1)*m);
```

```
    else
      return j*m;
    fi;
  end;
  for j in [1..n] do
    for k in [j..n] do
                  if j = k then
                      Add( gmorphisms, MapOfFinSets( objects[j],
                                  List( objects[j], o-> [o, permute(o,j,m) ] ),
                                  objects[k]) );
                  else # k > j
                          Add( gmorphisms, MapOfFinSets( objects[j],
                                  List( objects[j], o -> [o, o+(k-j)*m] ),
                                  objects[k]) );
                  fi;
        od;
  od;

    DeactivateCachingOfCategory( FinSets );
    CapCategorySwitchLogicOff( FinSets );
    DisableSanityChecks( FinSets );

    C := Subcategory( FinSets, "A finite concrete category" : overhead := false, Finaliz

        DeactivateCachingOfCategory( C );
    CapCategorySwitchLogicOff( C );
    DisableSanityChecks( C );

        SetFilterObj( C, IsFiniteConcreteCategory );

        AddIsAutomorphism( C,
      function( alpha )
        return IsAutomorphism( UnderlyingCell( alpha ) );
    end );

        AddInverse( C,
      function( alpha )
        return Inverse( UnderlyingCell( alpha ) ) / CapCategory( alpha );
    end );

        SetSetOfObjects( C, List( objects, o-> o / C ) );
        SetSetOfGeneratingMorphisms( C, List( gmorphisms, g-> g / C ) );

    Finalize( C );
```

We want the endomorphism relations so that the path algebra is finite-dimensional and we get a finite Gröbner basis.

Proof that algorithm is correct Proof that it terminates.

Wir haben BasisOfExternalHom benutzt um Decompose in CAP umzusetzen um EmbeddingOfSubRepresentation umzusetzen um WeakDirectSumDecomposition umzusetzen.

## Notes

**Algorithm 1:** `RightQuiverFromConcreteCategory`

**Input :** a finite concrete category $C$ with $n$ objects
**Output :** the right quiver $q(n)$

let $Obj$ be the set of objects of $C$;
let $n := Length(Obj)$;
let $gMor$ be the set of generating morphisms of $C$;
let $A$ be the empty set and let $i := 1$;
**foreach** *morphism mor in gMor* **do**
    let $A_{i,1}$ be the position of $Source(mor)$ in $Obj$;
    let $A_{i,2}$ be the position of $Range(mor)$ in $Obj$;
    let $i := i + 1$;
**end**
let $q$ be the right quiver with vertices $\{1, \ldots, n\}$ and arrows $A$.
**return** q;

**Algorithm 2:** `RelationsOfEndomorphisms`

**Input:** a commutative ring $k$ and a finite concrete category $C$
**Output:** the endomorphism relations of the category $C$

let $q :=$ `RightQuiverFromConcreteCategory`$(C)$;
let $kq$ be the path algebra generated by $k$ and $q$;
let $gMor$ be the set of generating morphisms of $C$;
let $A := Arrows(q)$;
let $relsEndo$ be the empty set;
**foreach** $i = 1, \ldots, Length(gMor)$ **do**
    let $mor := gMor_i$ **if** *mor is not an endomorphism* **then**
        continue;
    **end**
    let $m := 0$ and let $powers$ be the empty set;
    let $foundEqual$ be false;
    **while** $mor^m \notin powers$ **do**
        let $n := 1$;
        **while** $\neg foundEqual$ **do**
            **if** $mor^{(m+n)} = mor^m$ **then**
                Add the relation $kq.(A_i)^{(m+n)} - kq.(A_i)^m$ to relsEndo;
                foundEqual := true;
            **end**
            n := n+1;
        **end**
        Add $mor^m$ to powers;
        m := m+1;
    **end**
**end**
**return** *relsEndo*;

## References

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[3] Jan Geuenich. https://hss.ulb.uni-bonn.de/2017/4681/4681.pdf