The category of representations of a concrete category as a functor category

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$$\begin{tikzcd}
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& & & & \\
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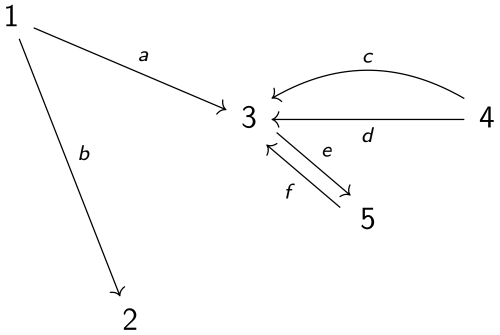

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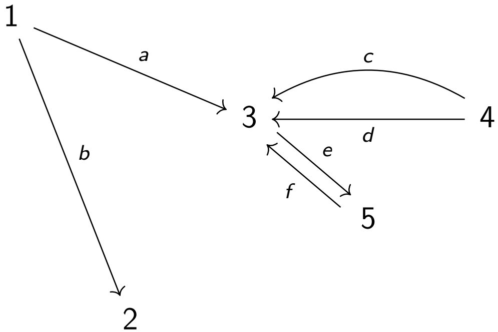

For example

$$
\begin{aligned}
s(a) = s(b) &= 1 \\
t(b) &= 2 \\
t(a) = t(c) = t(d) = t(f) &= 3
\end{aligned}
$$

etc.

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So in the quiver q we have

$$q_0 = \{1, 2, 3, 4, 5\}$$

and

$$q_1 = \{a, b, c, d, e, f\}$$

Another map relates the arrows with the objects. That is the Hom-set, i.e. set of morphisms, between two objects (order matters):

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$$\mathrm{Hom}(1, 3) = \{a\}$$
$$\mathrm{Hom}(4, 3) = \{c, d\}$$
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$$\begin{tikzcd} 1 \arrow[r, "b"] \arrow[loop below, "a"] & 2 \arrow[loop below, "c"] \end{tikzcd}$$

$$\mathrm{End}(1) = \{a\}$$
$$\mathrm{End}(2) = \{c\}$$
$$\mathrm{Hom}(1, 2) = \{b\}$$

You now know what a quiver is.

A <u>category</u> $\mathcal{C}$ is a quiver with two further maps:

($\mathbb{1}$) For every object $X \in \mathcal{C}_0$ there is the <u>identity map</u>

$$\begin{aligned} \mathbb{1} : \mathcal{C}_0 &\longrightarrow \mathcal{C}_1 \\ X &\longmapsto \mathbb{1}_X : X \longrightarrow X \end{aligned}$$

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(μ) For two composable morphisms φ and $\psi \in \mathcal{C}_1$, i.e. with $t(\varphi) = s(\psi)$ there is the composition map

$$\begin{aligned} \mu &: \mathcal{C}_1 \times \mathcal{C}_1 \longrightarrow \mathcal{C}_1 \\ \varphi &: A \longrightarrow B \\ \psi &: B \longrightarrow C \\ (\varphi, \psi) &\longmapsto \mu(\varphi, \psi) := \varphi\psi : A \longrightarrow C \end{aligned}$$

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4. $\mathbb{1}_{s(\varphi)}\varphi = \varphi = \varphi\mathbb{1}_{t(\varphi)}$, i.e. the identity is a left and right unit of the composition.

These properties make each endomorphism set $\mathrm{End}(M)$ for $M \in \mathcal{C}$ together with the composition into a monoid, called the endomorphism monoid $(\mathrm{End}(M), \mu)$.

So when you define a category, you always answer the four questions

- What are the objects?
- What are the morphisms? Especially what are the identity morphisms?
- How do you compose morphisms?
- Why is the composition associative? Why is the identity a unit for the composition?

You now know what a category is

A small example for a category: The symmetric group on two objects S_2.

$$\begin{tikzcd} \{1,2\} \arrow[loop left, "\mathbb{1}_{\{1,2\}}"] \arrow[loop right, "(1,2)"] \end{tikzcd}$$

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The rule that the composition of $(1,2)$ with itself results in the identity $\mathbb{1}_{\{1,2\}}$ makes sure there are only 2 morphisms in total. This is an example of a category which is a sub-category of the category $\textsc{Sets}$ with sets as objects and functions between sets as morphisms. Between those categories lies the category $\textsc{FinSets}$ in which the objects are finite sets and morphisms are functions between finite sets.

As another example of a finite subcategory of $\textsc{FinSets}$, take the category with two objects that are sets with 3 elements.

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As another example of a finite subcategory of $\textsc{FinSets}$, take the category with two objects that are sets with 3 elements. The endomorphisms at each objects are the cyclic permutations on the 3 elements $\{1,2,3\}$ and on the 3 elements $\{4,5,6\}$.

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\{1,2,3\} \arrow[loop above, "(2{,}3{,}1)"] \arrow[loop left, "(3{,}2{,}1)"] \arrow[loop below, "\mathbb{1}_{\{1,2,3\}}"] & & \{4,5,6\} \arrow[loop above, "(5{,}6{,}4)"] \arrow[loop right, "(6{,}5{,}4)"] \arrow[loop below, "\mathbb{1}_{\{4,5,6\}}"]
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\end{tikzcd}$$

The endomorphism monoid on each object is the goup C_3, i.e. the cyclic group on three elements. This also means that each endomorphism is invertible. We thus call this category $C_3 C_3$.

You may have noticed that this picture is not complete when we look back to the axioms for a category: We have three endomorphisms at the first object and three endomorphisms at the second object. Since we also have a morphism from the first to the second object, we also need all the possible compositions of those morphisms.

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Of the missing two morphisms from $\{1,2,3\}$ to $\{4,5,6\}$, the first one is mapping $1 \mapsto 5, 2 \mapsto 6$ and $3 \mapsto 4$, where the last one is mapping $1 \mapsto 6, 2 \mapsto 4$ and $3 \mapsto 5$.

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The categories S_2, $C_3 C_3$, $\textsc{Set}$, $\textsc{FinSets}$ can themselves all be considered objects in a greater category, $\textsc{Cat}$, i.e. <u>the category of categories</u>.

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$$\begin{tikzcd}
S_2 \arrow[rr] & & S_3 \arrow[r] & C_3 \arrow[r] \arrow[d] & C_3C_3 \arrow[dl] \\
& \textsc{Sets} \arrow[rr, shift left=2] & & \textsc{FinSets} \arrow[ul] \arrow[ll] \arrow[ll, shift left=-2] &
\end{tikzcd}$$

The categories S_2, C_3C_3, $\textsc{Set}$, $\textsc{FinSets}$ can themselves all be considered objects in a greater category, $\textsc{Cat}$, i.e. <u>the category of categories</u>.

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$$

We know what the objects are in $\textsc{Cat}$. But what are the morphisms? What is meant with an arrow from $\textsc{Sets}$ to $\textsc{FinSets}$?

You now know the objects in the category $\textsc{Cat}$ of all categories.

A functor $F : \mathcal{C} \to \mathcal{D}$, between categories $\mathcal{C}$ and $\mathcal{D}$, consists of the following data:

- An object $Fc \in \mathcal{D}_0$, for each object $c \in \mathcal{C}_0$.
- A function $Ff : Fc \to Fc' \in \mathcal{D}_1$, for each morphism $f : c \to c' \in \mathcal{C}_1$, so that the source and target of Ff are, respectively, equal to F applied to the source or target of f, in other words, $s(Ff) = Fs(f)$ and $t(Ff) = Ft(f)$.

The assignments are required to satisfy the following two functoriality axioms:

- For any composable pair $f : M \to N, g : N \to L \in \mathcal{C}_1, F\,f \cdot F\,g = F(f \cdot g)$.
- For each object $c \in \mathcal{C}_0, F(1_c) = 1_{Fc}$.

So with functors you always answer the four questions

- How does it work on objects?
- How does it work on morphisms?
- Why does it respect composition?
- Why does it respect identity morphisms?

You now know what a functor is.

You now know the objects and morphisms in the category CAT of all categories.

Let us now take a look at just two categories, $\mathcal{C}$ and $\mathcal{D}$ as objects in Cat.
The Hom-set $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ of all functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ is itself a category, called the <u>functor category</u>.
This makes the functors $F, G, H : \mathcal{C} \longrightarrow \mathcal{D}$ objects in $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ when before they were considered morphisms.

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As you can imagine, we are again looking for the morphisms in this category, i.e. what are morphisms between functors?

Let us now take a look at just two categories, $\mathcal{C}$ and $\mathcal{D}$ as objects in CAT.
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This makes the functors $F, G, H : \mathcal{C} \longrightarrow \mathcal{D}$ objects in $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ when before they were considered morphisms.

$$F \xRightarrow{\alpha} G \xRightarrow{\beta} H$$

As you can imagine, we are again looking for the morphisms in this category, i.e. what are morphisms between functors?

You now know the objects in the category $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ of all functors between categories $\mathcal{C}$ and $\mathcal{D}$.

Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha : F \Rightarrow G$ consists of:

- a morphism $\alpha_c : Fc \to Gc \in \mathcal{D}_1$ for each object $c \in \mathcal{C}_0$, the collection of which define the components of the natural transformation, so that, for any morphism $f : c \to c' \in \mathcal{C}_1$, the following square of morphisms in $\mathcal{D}$

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"] & Gc' \end{tikzcd}$$

  commutes, i.e., has a common composite $Fc \to Gc' \in \mathcal{D}_1$. This means explicitly that

$$Ff\alpha_{c'} = \alpha_c Gf, \forall f : c \to c' \in \mathcal{C}_1.$$

You now know what a natural transformation is.

You now know the objects and morphisms in the category $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ of all functors between categories $\mathcal{C}$ and $\mathcal{D}$.

Now that we know what a functor category is, we want to work towards a special kind of functor category: Given a finite concrete category $\mathcal{C}$ whose endomorphism monoids are explicitly cyclic, we can calculate its $\mathbb{k}$-linear closure, i.e. the $\mathbb{k}$-Algebroid $\mathcal{A}$.
Then we can calculate the category of $\mathbb{k}$-linear functors from the $\mathbb{k}$-Algebroid $\mathcal{A}$ into the matrix category $\mathbb{k}\text{-}\mathrm{Mat}$ over the same field $\mathbb{k}$.

$$\mathrm{Hom}_{\mathbb{k}}(\mathcal{A}, \mathbb{k}\text{-}\mathrm{Mat})$$

category of $\mathbb{k}$-linear functors $\mathrm{Hom}_{\mathbb{k}}(\mathcal{A}, \mathbb{k}\text{-}\mathrm{Mat})$

$$\begin{array}{c}\mathrm{FinSets}\\ \big\downarrow\\ \text{finite concrete category } \mathcal{C}\end{array}$$

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$$
\begin{tikzcd}
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\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \text{field}\, \Bbbk \arrow[dll] \\
\Bbbk\text{-Algebroid}\, \mathcal{A} \arrow[dr] & & \\
 & \text{category of } \Bbbk\text{-linear functors}\, \mathrm{Hom}_{\Bbbk}(\mathcal{A}, \Bbbk\text{-Mat}) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{FinSets} \arrow[d] & & \\
\text{finite concrete category}\,\mathcal{C} \arrow[d] & & \\
\text{finitely presented category}\,\mathrm{fp}\mathcal{C} \arrow[d] & & \text{field}\,\Bbbk \arrow[dll] \\
\Bbbk\text{-Algebroid}\,\mathcal{A} \arrow[dr] & & \\
& \text{category of } \Bbbk\text{-linear functors}\,\mathrm{Hom}_{\Bbbk}(\mathcal{A}, \Bbbk\text{-Mat}) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{FinSets} \arrow[d] & & \\
\text{finite concrete category } \mathcal{C} \arrow[dd] \arrow[dr] \arrow[drr] & & \\
& \text{quiver } q \arrow[dl] & \text{relations } \mathtt{rel} \arrow[dll] \\
\text{finitely presented category } \mathrm{fp}\mathcal{C} \arrow[d] & & \text{field } \mathbb{k} \arrow[dll] \\
\mathbb{k}\text{-Algebroid } \mathcal{A} \arrow[dr] & & \\
& \text{category of } \mathbb{k}\text{-linear functors } \mathrm{Hom}_{\mathbb{k}}(\mathcal{A}, \mathbb{k}\text{-Mat}) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{FinSets} \arrow[d] & & \\
\text{finite concrete category}\,\mathcal{C} \arrow[dd] \arrow[dr] \arrow[drr] & & \\
& \text{quiver}\,q \arrow[dl] & \text{relations}\,\texttt{rel} \arrow[dll] \\
\text{finitely presented category}\,\mathrm{fp}\mathcal{C} \arrow[d] & & \text{field}\,\mathbb{k} \arrow[dll] \arrow[d] \\
\mathbb{k}\text{-}\mathrm{Algebroid}\,\mathcal{A} \arrow[dr] & & \mathbb{k}\text{-}\mathrm{Mat} \arrow[dl] \\
& \text{category of $\mathbb{k}$-linear functors}\,\mathrm{Hom}_{\mathbb{k}}(\mathcal{A}, \mathbb{k}\text{-}\mathrm{Mat}) &
\end{tikzcd}
$$

Category $\mathcal{C}$	$\mathrm{Hom}_{\Bbbk}(\mathcal{A}, \Bbbk\text{-Mat})$	$\Bbbk$-**mat**
Objects $\mathcal{C}_0$	$\Bbbk$-linear functors $F : \mathcal{A} \to \Bbbk$-**mat** $c \in \mathcal{A}_0 \mapsto Fc \in \Bbbk$-$\mathbf{mat}_0$ $\varphi \in \mathcal{A}_1 \mapsto F\varphi \in \Bbbk$-$\mathbf{mat}_1$	natural numbers $\mathbb{N}_0$
Morphisms $\mathcal{C}_1$	natural transformations, components are matrices	$m \times n$-matrices
Composition: $\varphi : A \to B$, $\psi : B \to C$ $\varphi\psi : A \to C$	 $\eta : F \Rightarrow G$, $\eta_c : Fc \to Gc$ $\varepsilon : G \Rightarrow H$, $\varepsilon_c : Gc \to Hc$ $\eta\varepsilon : F \Rightarrow H$, $\eta_c\varepsilon_c : Fc \to Hc$ (component-wise)	matrix multiplication

Category $\mathcal{C}$	$\mathrm{Hom}_{\Bbbk}(\mathcal{A}, \Bbbk\text{-}\mathrm{Mat})$	$\Bbbk$-**mat**
Direct Sum:		
Let $I = \{1, \ldots, N\}$ be a finite set	$\{F_i\}_{i \in I}$ a family of objects in $\mathrm{Hom}_{\Bbbk}(\mathcal{A}, \Bbbk\text{-}\mathrm{Mat})_0$	$\{n_i\}_{i \in I}$ a family of objects in $\Bbbk$-**mat**$_0$
	$F : c \mapsto \sum_{i=1}^{N} F_i(c)$	$n = \sum_{i=1}^{N} n_i$
	at each object the sum of natural numbers at that object	sum of natural numbers

Projections: $\pi_i : F \to F_i$ with components

$$(\pi_i)_c := \begin{pmatrix} 0_{F_{<i}(c), F_i(c)} \\ 1_{F_i(c)} \\ 0_{F_{>i}(c), F_i(c)} \end{pmatrix}$$

Coprojections: $\iota_i : F_i \to F$ with components

$$(\iota_i)_c := \begin{pmatrix} 0_{F_i(c),\, F_{<i}(c)} & 1_{F_i(c)} & 0_{F_i(c),\, F_{>i}(c)} \end{pmatrix}$$

For a morphism $a : c \to c' \in \mathcal{A}_1$ we have a family of morphisms $\{F_i a : F_i c \to F_i c'\}_{i \in I}$. Then the direct sum F is defined on morphisms as

$$Fa := \sum_{i \in I} (\pi_i)_c F_i a (\iota_i)_{c'} : Fc \to Fc'$$

which satisfies

$$\begin{aligned}
(\iota_i)_c Fa (\pi_i)_{c'} &= (\iota_i)_c \sum_{j \in I} (\pi_j)_c F_j a (\iota_j)_{c'} (\pi_i)_{c'} \\
&= \sum_{j \in I} (\iota_i)_c (\pi_j)_c F_j a (\iota_j)_{c'} (\pi_i)_{c'} \\
&= \sum_{j \in I} (\delta_{i,j})_c F_j a (\delta_{j,i})_{c'} \\
&= 1_{F_i c} F_i a 1_{F_i c'} \\
&= F_i a
\end{aligned}$$

One step in the decomposition algorithm looks like this:

$$
\begin{tikzcd}
 & F & \\
F \arrow[ur, bend left, "\eta"] & I \oplus K = F \arrow[u] & F \arrow[ul, bend right, "\eta"'] \\
I \arrow[u, "\iota"] \arrow[r, dash] & \oplus \arrow[u, dash] \arrow[r, dash] & K \arrow[u, "\kappa"']
\end{tikzcd}
$$

Then we have two morphisms ι, κ with target F and sources I, K such that $I \oplus K = F$.

Algorithm 1: `DecomposeOnceByRandomEndomorphism`

Input : a functor F in a functor category

Output : a pair $[\iota : I \to F, \kappa : K \to F]$ of morphisms such that $I \oplus K = F$ with $I \neq 0$ and $K \neq 0$ or `fail` if it was unable to further decompose F;

$d := \max\{\dim_{\Bbbk} Fc\}_{c \in \mathcal{A}_0}$;

if $d = 0$ **then**

  return `fail`; `// the zero representation is indecomposable`

end

$\mathcal{B} = [\beta_1, \ldots, \beta_h]$ is a $\Bbbk$-basis of $\mathrm{Hom}_{\mathrm{Hom}_{\Bbbk}(\mathcal{A}, \Bbbk\text{-Mat})}(F, F)$;

add $0_{F,F}$ to $\mathcal{B}$;

$n := \lfloor \log_2(d) \rfloor + 1$;

for $b \in [h+1, h, \ldots, 2]$ **do**

  $\alpha := \beta_b + \mathrm{random}(\mathbb{k}) \cdot \beta_{b-1}$; `// a heuristic ansatz for a random endomorphism`

  for $i \in [1, \ldots, n]$ **do**

    $\alpha_2 := \alpha^2$;

    `/* We do not expect the exponentiation to produce an idempotent, still this is a very cheap test: */`

    if $\alpha = \alpha_2$ **then**

      break;

    end

    $\alpha := \alpha_2$;

  end

  if $\alpha = 0$ **then**

    continue ; `// try another endomorphism`

  end

for $b \in [h+1, h, \ldots, 2]$ **do**

  $\ldots$

  $\kappa := \mathrm{KernelEmbedding}(\alpha)$;

  if $\kappa = 0$ **then**

    continue ; `// try another endomorphism`

  end

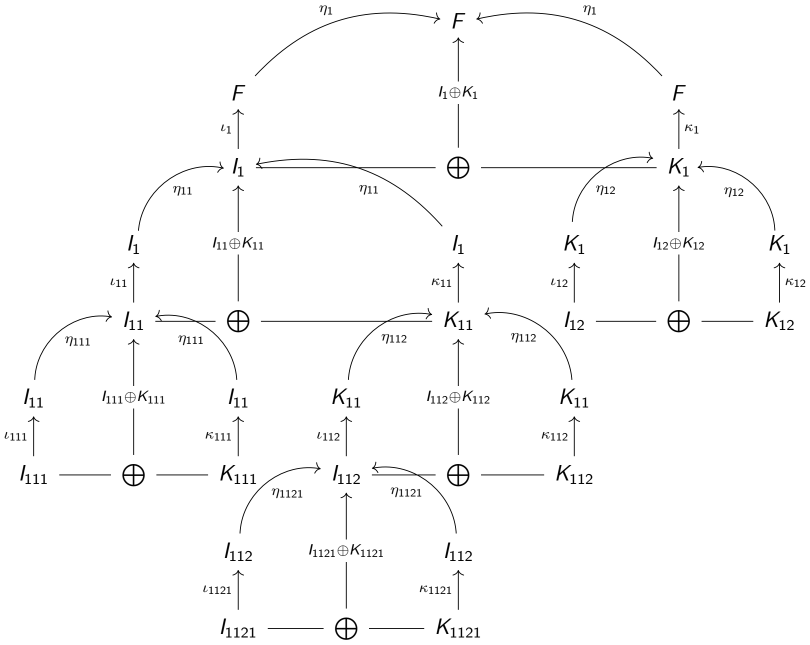
  $\iota := \mathrm{ImageEmbedding}(\alpha)$;

  return $[\iota, \kappa]$;

end

return `fail`; `// The input functor` F `is indecomposable with a high probability.`

Direct sum decomposition



Algorithm 2: `WeakDirectSumDecomposition`

Input : a functor F in a functor category

Output : a list $[\eta_i : F_i \to F]$ of embeddings.

```
queue := [1_F];
summands := ∅;
while queue ≠ ∅ do
    let η = remove(queue);
    result :=
     DecomposeOnceByRandomEndomorphism(s(η));
    if result = fail then        // s(η) indecomposable
        add η to summands;
    else
        [ι, κ] = result;
        append [ιη, κη] to queue;
    end
end
return summands;
```

Hom-based invariants:

`EmbeddingOfSumOfImagesOfAllMorphisms`:

$$\begin{tikzcd} \bigoplus_{i \in \mathtt{hom}} M \arrow[rr, "u_{\mathrm{out}}(\mathtt{hom})"] \arrow[dr, two heads, "\pi"'] & & N \\ & I \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "b"] \arrow[loop below, "a"] & 2 \arrow[loop below, "c"]
\end{tikzcd}$$