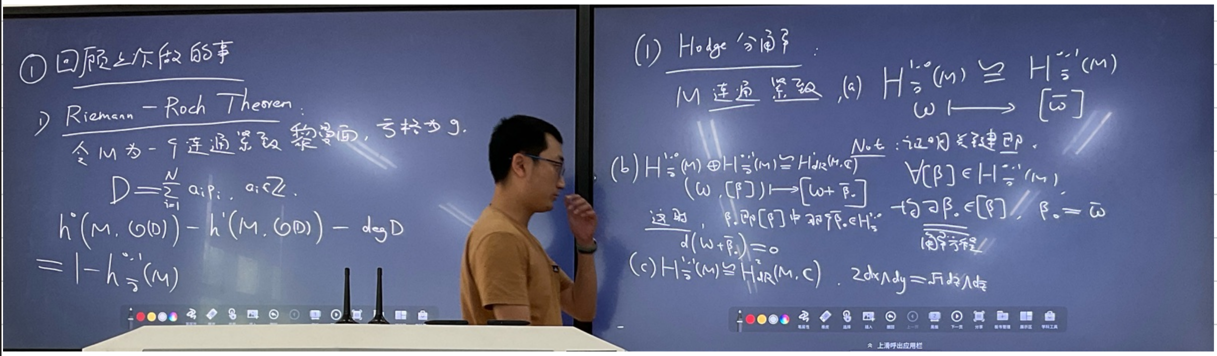
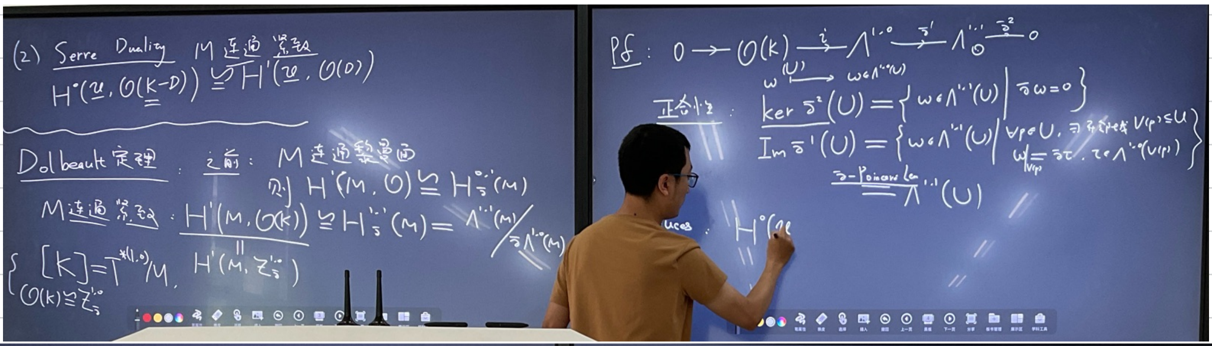
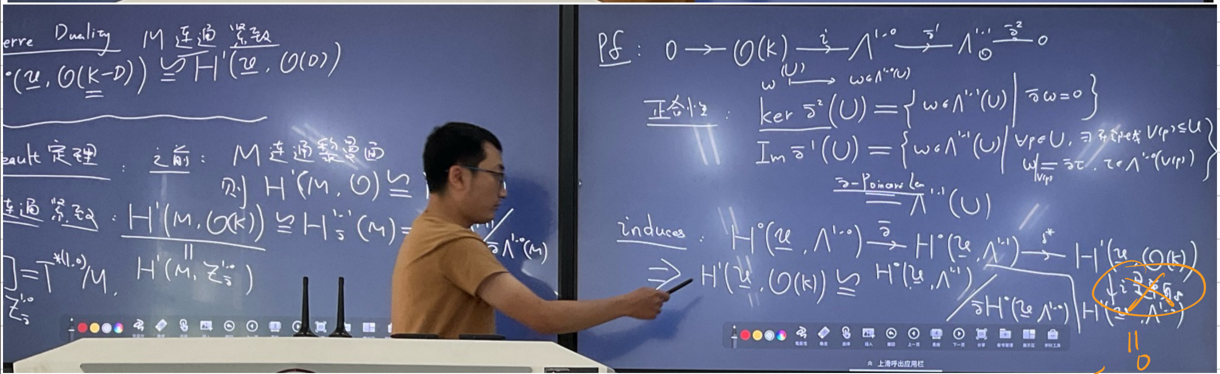
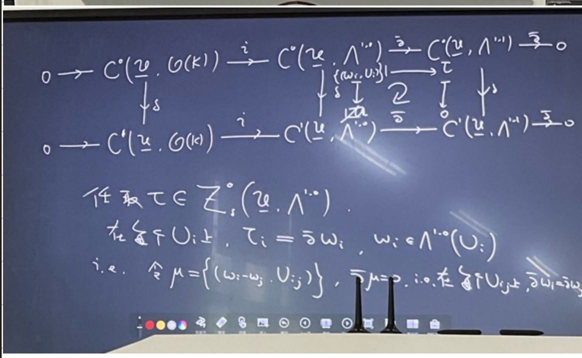
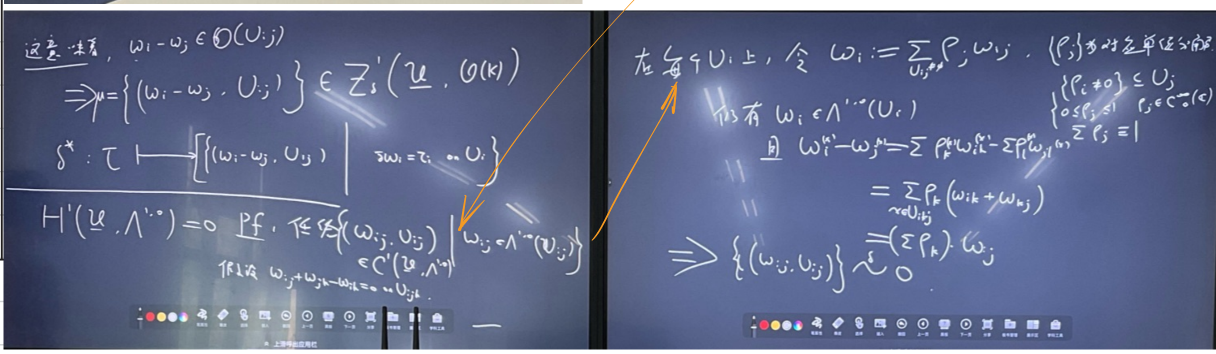
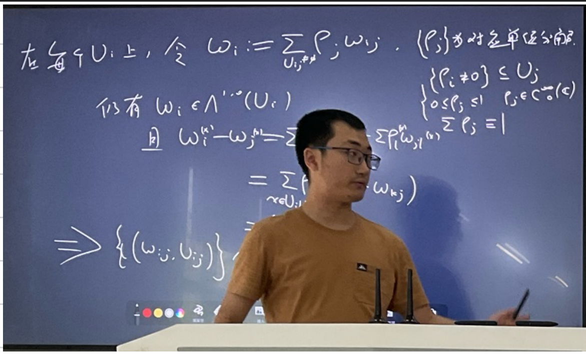

① 回顾上次所的事
1) Riemann-Roch Theorem:
令 M 为一个连通紧致黎曼面，亏格为 g.
$D=\sum_{i=1}^{N} a_i p_i,\ a_i\in\mathbb{Z}$.
$h^0(M,\mathcal{O}(D))-h^1(M,\mathcal{O}(D))-\deg D$
$=1-h^{0,1}_{\bar\partial}(M)$
(1) Hodge 分解:
M 连通 紧致, (a) $H^{1,0}_{\bar\partial}(M)\cong \overline{H^{0,1}_{\bar\partial}(M)}$
$\omega\longmapsto[\bar\omega]$
Not: 证明关键即,
(b) $H^{1,0}_{\bar\partial}(M)\oplus H^{0,1}_{\bar\partial}(M)\cong H^1_{dR}(M,\mathbb{C})$
$(\omega,[\beta])\longmapsto[\omega+\bar\beta_0]$
$\forall[\beta]\in H^{0,1}_{\bar\partial}(M)$, 一定 $\exists\beta_0\in[\beta]$, $\beta_0=\bar\omega$
调和方程
$d(\omega+\bar\beta_0)=0$
(c) $H^{1,1}_{\bar\partial}(M)\cong H^2_{dR}(M,\mathbb{C})$. $2dx\wedge dy=\sqrt{-1}dz\wedge d\bar z$

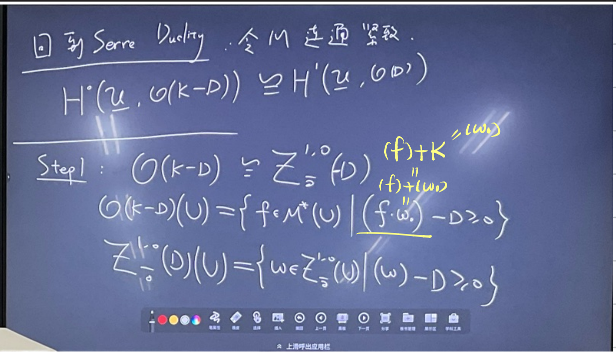

(2) Serre Duality M 连通 紧致
$H^0(\mathfrak{U},\mathcal{O}(K-D))\cong H^1(\mathfrak{U},\mathcal{O}(D))$
Dolbeault 定理: 之前: M 连通黎曼面
则 $H^1(M,\mathcal{O})\cong H^{0,1}_{\bar\partial}(M)$
M 连通 紧致: $H^1(M,\mathcal{O}(K))\cong H^{1,1}_{\bar\partial}(M)=\Lambda^{1,1}(M)/\bar\partial\Lambda^{1,0}(M)$
$[K]=T^{*(1,0)}M$, $H^1(M,Z^{1,0}_{\bar\partial})$
$\mathcal{O}(K)\cong Z^{1,0}_{\bar\partial}$
Pf: $0\to\mathcal{O}(K)\xrightarrow{i}\Lambda^{1,0}\xrightarrow{\bar\partial^1}\Lambda^{1,1}\xrightarrow{\bar\partial^2}0$
正合性: $\ker\bar\partial^2(U)=\{\omega\in\Lambda^{1,1}(U)\mid\bar\partial\omega=0\}$
$\mathrm{Im}\,\bar\partial^1(U)=\{\omega\in\Lambda^{1,1}(U)\mid\forall p\in U$, 存在邻域 $V(p)\subseteq U$, $\omega|_{V(p)}=\bar\partial\tau,\ \tau\in\Lambda^{1,0}(V(p))\}$
$\bar\partial$-Poincaré Lem
$=\Lambda^{1,1}(U)$


induces: $H^0(\mathfrak{U},\Lambda^{1,0})\xrightarrow{\bar\partial}H^0(\mathfrak{U},\Lambda^{1,1})\xrightarrow{\delta^*}H^1(\mathfrak{U},\mathcal{O}(K))$
$\Rightarrow H^1(\mathfrak{U},\mathcal{O}(K))\cong H^0(\mathfrak{U},\Lambda^{1,1})/\bar\partial H^0(\mathfrak{U},\Lambda^{1,0})$
$H^1(\mathfrak{U},\Lambda^{1,0})$
$=0$

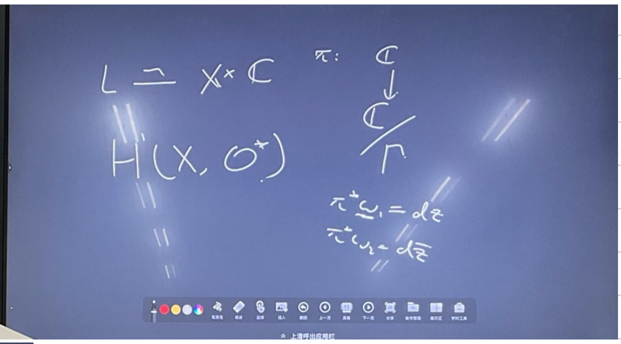

$0\to C^0(\mathfrak{U},\mathcal{O}(K))\xrightarrow{i}C^0(\mathfrak{U},\Lambda^{1,0})\xrightarrow{\bar\partial}C^0(\mathfrak{U},\Lambda^{1,1})\xrightarrow{\bar\partial}0$
$0\to C^1(\mathfrak{U},\mathcal{O}(K))\xrightarrow{i}C^1(\mathfrak{U},\Lambda^{1,0})\xrightarrow{\bar\partial}C^1(\mathfrak{U},\Lambda^{1,1})\xrightarrow{\bar\partial}0$
任取 $\tau\in Z^0(\mathfrak{U},\Lambda^{1,1})$.
在每个 U_i 上, $\tau_i=\bar\partial\omega_i$, $\omega_i\in\Lambda^{1,0}(U_i)$
i.e. 令 $\mu=\{(\omega_i-\omega_j,U_{ij})\}$, $\bar\partial\mu=0$. i.e. 在每个 U_{ij} 上, $\bar\partial\omega_i=\bar\partial\omega_j$

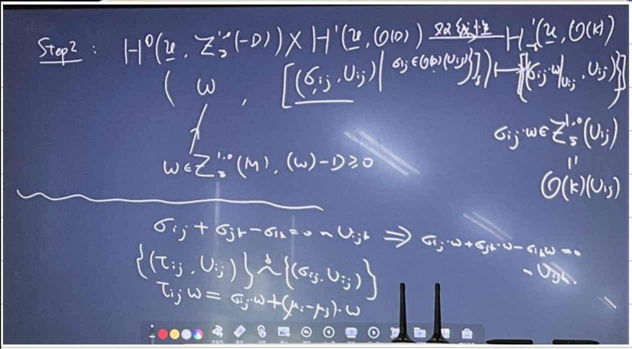

这意味着, $\omega_i-\omega_j\in\mathcal{O}(U_{ij})$
$\Rightarrow\mu=\{(\omega_i-\omega_j,U_{ij})\}\in Z^1(\mathfrak{U},\mathcal{O}(K))$
$\delta^*:\tau\longmapsto[\{(\omega_i-\omega_j,U_{ij})\mid\bar\partial\omega_i=\tau_i\ \text{on}\ U_i\}]$
$H^1(\mathfrak{U},\Lambda^{1,0})=0$ Pf. 任给 $\{(\omega_{ij},U_{ij})\mid\omega_{ij}\in\Lambda^{1,0}(U_{ij})\}\in C^1(\mathfrak{U},\Lambda^{1,0})$
假设 $\omega_{ij}+\omega_{jk}-\omega_{ik}=0$ on U_{ijk}.
在每个 U_i 上, 令 $\omega_i:=\sum_{U_{ij}\neq\emptyset}\rho_j\omega_{ij}$. $\{\rho_j\}$ 为对应单位分解.
$\{\rho_i\neq0\}\subseteq U_j$, $0\le\rho_j\le1$, $\rho_j\in C^\infty_0(\mathbb{C})$, $\sum\rho_j=1$
仍有 $\omega_i\in\Lambda^{1,0}(U_i)$
且 $\omega_i-\omega_j=\sum\rho_k\omega_{ik}-\sum\rho_l\omega_{jl}$
$=\sum_{\gamma\in U_{ij}}\rho_k(\omega_{ik}+\omega_{kj})$
$=(\sum\rho_k)\cdot\omega_{ij}$
$\Rightarrow\{(\omega_{ij},U_{ij})\}\overset{\delta}{\sim}0$

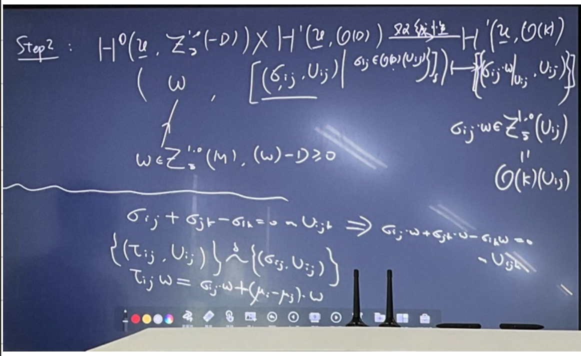


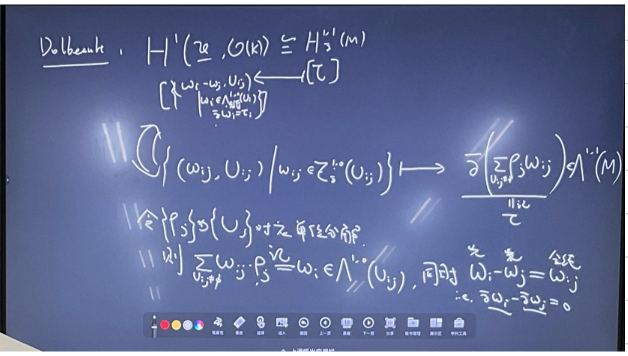

回到 Serre Duality
Step1

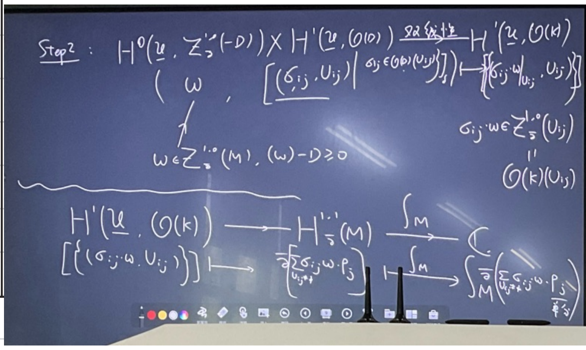

Cech sheaf
Singular
locally constant
Sheaves on manifolds

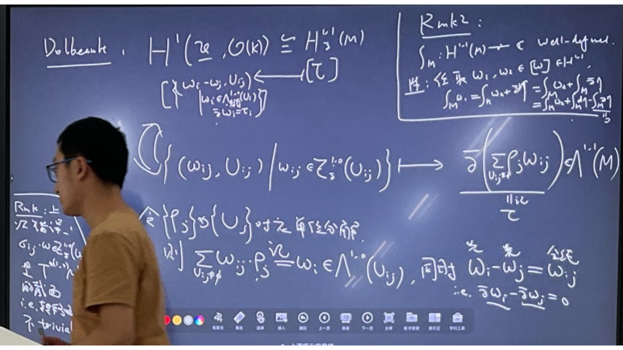


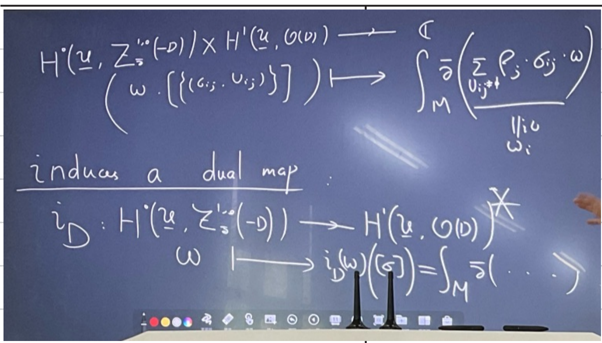

Step2

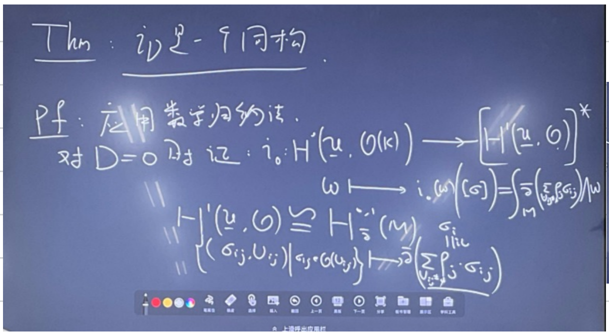

Step2

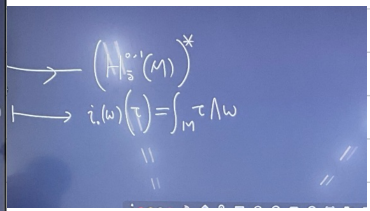

Dolbeault

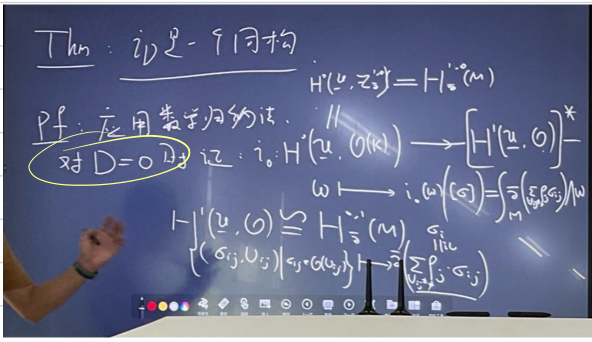

Step2

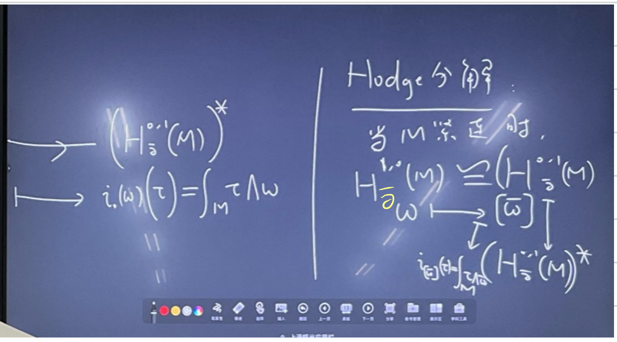

Dolbeault
Rmk2

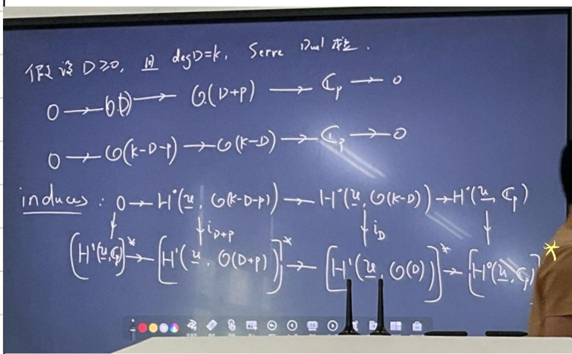


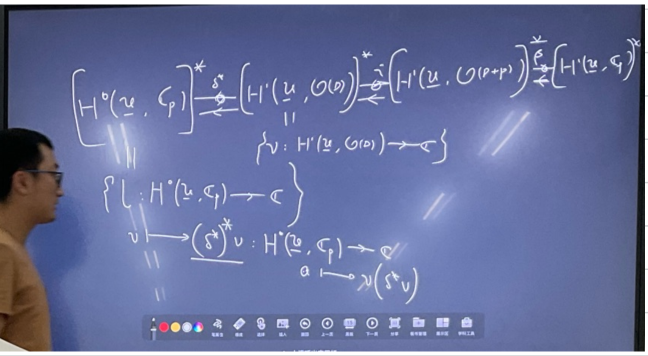












$A \xrightarrow{f} B \xrightarrow{g} C$ $\qquad \ker g = \mathrm{Im} f$

$C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$ $\qquad \ker f^* = \mathrm{Im} g^*$ 吗?

$\forall a \qquad f^*(b^*)(a) = b^*(f(a)) = 0$

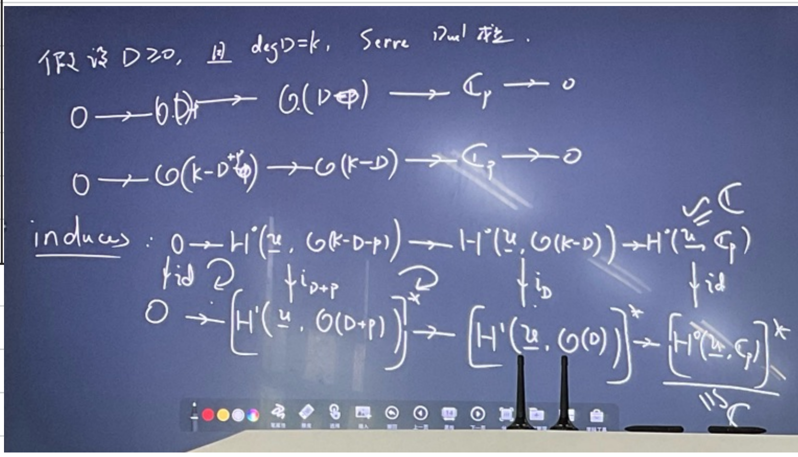
$b^* \in \ker f^* \Leftrightarrow b^*$ 在 $\underset{\ker g}{\underset{\parallel}{\mathrm{Im} f}}$ 上取零

$g^*(c^*)(b) = c^*(g(b))$

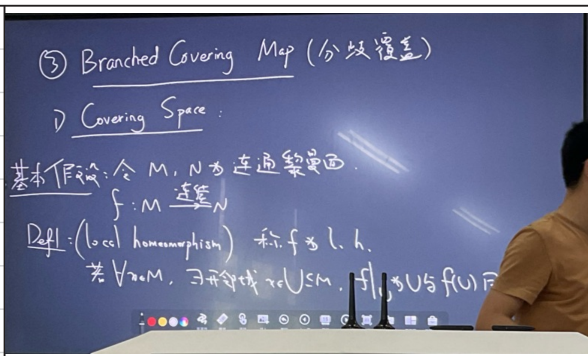
$b^* \in \ker f^* \Leftrightarrow b^*$ 在 $\ker g$ 上的取值为零

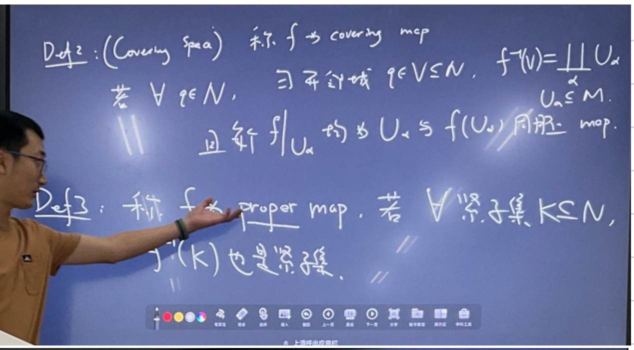
$c^*(g(b)) = b^*(b)$

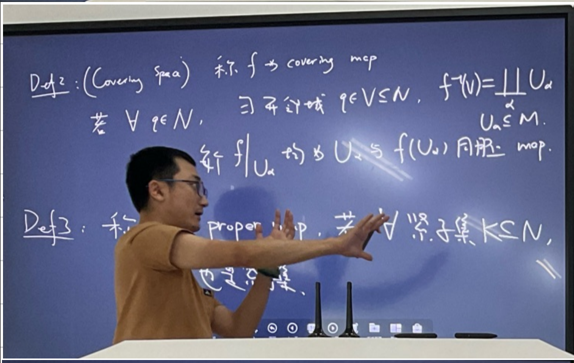
令 $c^*(g(b_i)) = b^*(b_i)$

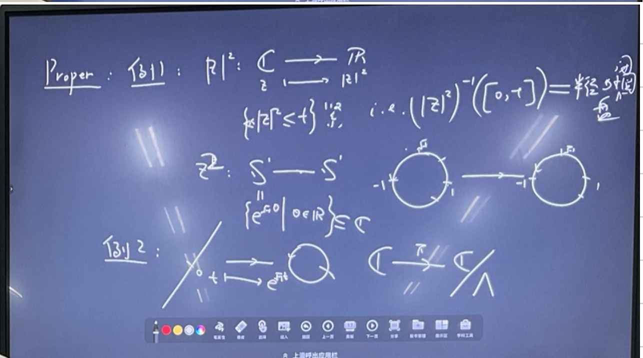


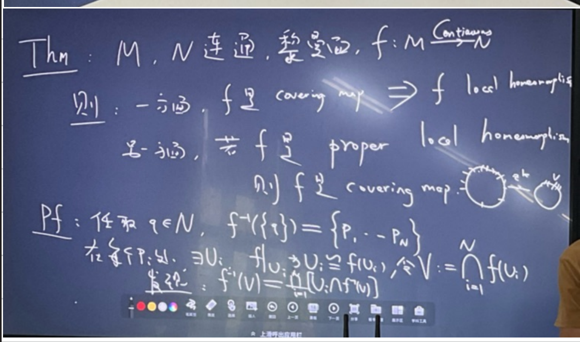
黎曼曲面上的解析几何

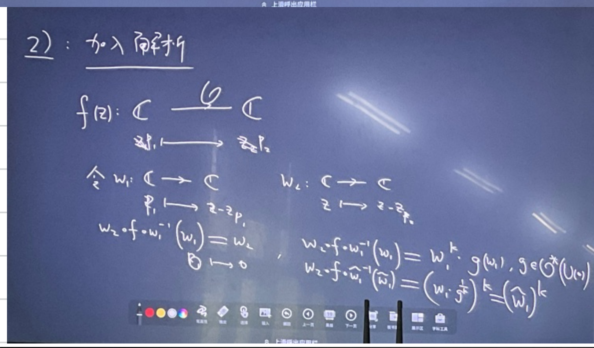


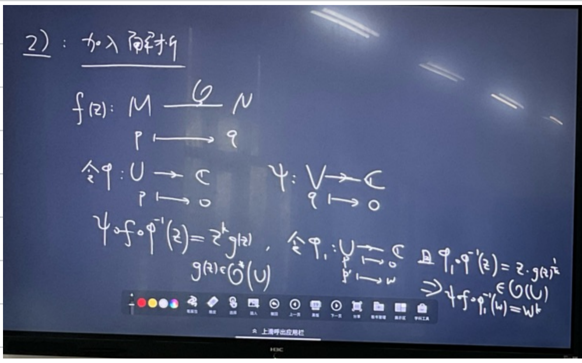


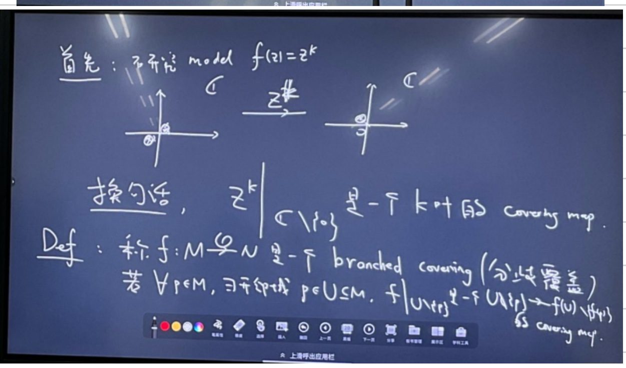












$$\psi\circ f\circ\varphi_1^{-1}(w)=(\psi\circ f\circ\varphi^{-1})(\varphi\circ\varphi_1^{-1}(w))$$
$$=\psi\circ f\circ\varphi^{-1}$$

$$[h(z)]^k=\psi\circ f\circ\varphi^{-1}(z)=z^k g(z)=\varphi_1\circ\varphi^{-1}(z)$$