

Riemann - Roch Theorem

可以 Hodge 分解的
复流形叫 Kähler 流形

Riemann-Roch Theorem
令 M 为紧, 连通的黎曼面. 任给除子 $D \in \text{Div}(M)$

Ex Appl: (1) $H_{JR}^{1,1}(M, \mathbb{C}) = H_{\mathbb{R}}^{1,1} \oplus H_{\mathbb{R}}^{1,1}$
(2) $H^1(M, \mathbb{C}) \cong H^1(M, \mathbb{C}(k))$
+ $H^{1,0} \cong \overline{H^{0,1}}$

完整版: $h'(m, \mathcal{O}(D)) - h'(m, \mathcal{O}(k-D))$
 $= \deg D + 1 - g(m)$

Pf: Step 1: 假设 $D \geq 0$ (i.e. $D = \sum_{i=1}^N n_i p_i$, $n_i \in \mathbb{Z}$ 且 $n_i > 0$)

用数学归纳法: $D=0$ 时, \checkmark

假设 $\deg D = k$ 时成立. 对 $D + P$ 进行讨论 (P 是任意点)

Step 1: 当 $D \geq 0$ 时

用数学归纳法

$$0 \rightarrow \mathcal{O}(D) \xrightarrow{i} \mathcal{O}(D+P) \xrightarrow{\beta} \mathbb{C} \rightarrow 0$$

$$f|_{(f)+D \geq 0} \mapsto f \mapsto \left\{ C_N \mid \begin{array}{l} \text{在 } r \text{ 附近, } f(z) = \sum_{k=1}^N \frac{1}{(z-z_r)^k} \\ \text{假设 } D \text{ 中有 } n-1 \text{ 个 } r. \end{array} \right\}$$

$$h(\underline{v}, \mathcal{O}(\underline{v})) - \dim(\Gamma_{\underline{v}} S^*) = h'(\underline{v}, \mathcal{O}(\underline{v}+1))$$

$$1 - \dim(\Gamma_m \beta_0) = \dim(\Gamma_m s^x)$$

$$-h''(u, \mathcal{O}(D+r)) + h'(u, \mathcal{O}(D))$$

$$= h'(u, 0(0)) - h'(u, 0(0)) - dy = 1 - h''(u)$$

Step 2.
讨论一般的除子 D

Step 2: 假设 $D = D_1 - D_2$, $D_1 \geq 0$, $D_2 \geq 0$
 归纳法: 对于 D_1 , \checkmark

$$0 \rightarrow \mathcal{O}(D_1 - p) \xrightarrow{i} \mathcal{O}(D_1) \xrightarrow{\beta} \mathbb{C}_p \rightarrow 0$$

$$f|_{(f)+D_1-p} \mapsto f \quad f \mapsto C_N$$

在 $p \in \mathbb{P}^1$ 上,
 $f(z) = \sum_{k=0}^N \frac{c_k}{(z-p)^k} + r(z)$
 在 D_1 上有 $(N+1)$ 个极点

induces: $0 \rightarrow H^0(\underline{u}, \mathcal{O}(D_1 - p)) \xrightarrow{i} H^0(\underline{u}, \mathcal{O}(D_1)) \xrightarrow{\beta} \mathbb{C}$
 $\xrightarrow{s^*} H^0(\underline{u}, \mathcal{O}(D_1 - p)) \rightarrow H^0(\underline{u}, \mathcal{O}(D_1)) \rightarrow 0$

$$h^1(\underline{u}, \mathcal{O}(D_1)) = h^0(\underline{u}, \mathcal{O}(D_1)) - \deg D_1$$

$$= h^1(\underline{u}, \mathcal{O}(D_1 - p)) = h^0(\underline{u}, \mathcal{O}(D_1 - p)) - (\deg D_1 - 1)$$

$$= 1 - h_{\underline{u}}^{0,1}(M)$$

$f: \Omega \xrightarrow{C^\infty} \mathbb{C}$ 光滑复值函数
全纯可推调和

$f \in C^0(\Omega) \Rightarrow \bar{\partial} f = 0 \Rightarrow \Delta f = 0$

共轭调和函数

Laplace 算子

令 $\Omega \subseteq \mathbb{C}$ - 开区域: 连通开集.

全纯: $Z = x + iy$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$

Given $f: \Omega \xrightarrow{C^\infty} \mathbb{C}$, $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$

并且: $\frac{\partial f}{\partial \bar{z}}(z) = 0 \Leftrightarrow f \in C^0(\Omega) \Leftrightarrow \bar{\partial} f = 0$

设 $f = u + iv$.

$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(u + iv)$

$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + \frac{i}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})$

Def: $(\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2 = \Delta$

$\partial f = \frac{\partial f}{\partial z} dz$, $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$, $d = \partial + \bar{\partial}$

$dz \wedge d\bar{z} = (dx + i dy) \wedge (dx - i dy) = 2i dx \wedge dy$

i.e. $\sqrt{-1} dz \wedge d\bar{z} = 2 dx \wedge dy$

$\sqrt{-1} \bar{\partial} f = \sqrt{-1} \frac{\partial f}{\partial \bar{z}} d\bar{z} = \sqrt{-1} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = 2 \frac{\partial f}{\partial \bar{z}} dx \wedge dy$

$\frac{\partial}{\partial z \partial \bar{z}} = \frac{1}{4}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$

$= \frac{1}{4}((\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2) = \frac{1}{4}\Delta$

Conclusion $\Rightarrow \sqrt{-1} \bar{\partial} f = \frac{1}{2} \Delta f dx \wedge dy$

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Conclusion $\Rightarrow \sqrt{-1} \bar{\partial} f = \frac{1}{2} \Delta f dx \wedge dy$

并且: $f \in C^0(\Omega) \Rightarrow \bar{\partial} f = 0 \Rightarrow \Delta f = 0$

设 $f = u + iv$, $u, v: \Omega \xrightarrow{C^\infty} \mathbb{R}$, $\Delta f = 0 \Rightarrow \Delta u = \Delta v = 0$

那么, 反过来, 设 $u: \Omega \xrightarrow{C^\infty} \mathbb{R}$, $\Delta u = 0$, 则 $\exists v: \Omega \rightarrow \mathbb{R}$, $u + iv \in C^0(\Omega)$

Ω 单连通

则存在, 称 v 为共轭调和函数.

证: 令 $\omega = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$, $d\omega = \Delta u \cdot dx \wedge dy$

因而 $d\omega = 0$, 由 $H^1 d\Omega(\Omega) = 0$ 知, $\omega = dv$.

Laplace 算子: Prop 1. $\sqrt{-1} \bar{\partial} u$ 全纯不交.

Ω 单连通
 2-形式, 称作闭 2-形式.

Pf: 令 $\omega = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$, $d\omega = \Delta u \cdot dx \wedge dy$
 因 $d\omega = 0$, 由 $H^2(\Omega) = 0$ 知, $\omega = du$.

Laplace 算子: Prop 1. $\sqrt{-1} \partial \bar{\partial} u$ 是实数.

Pf: $\sqrt{-1} \frac{\partial^2 u}{\partial z \partial \bar{z}} dz \wedge d\bar{z} = \sqrt{-1} \frac{\partial^2 u}{\partial z \partial \bar{z}} \cdot \frac{\partial^2 u}{\partial \bar{z} \partial z} dz \wedge d\bar{z} = \sqrt{-1} \frac{\partial^2 u}{\partial \bar{z} \partial z} dz \wedge d\bar{z}$

1. $\frac{\partial^2 u}{\partial z \partial \bar{z}}$
 $= \frac{\partial}{\partial \bar{z}} \left(\frac{\partial u}{\partial z} \frac{\partial \bar{z}}{\partial \bar{z}} \right)$
 $= \frac{\partial^2 u}{\partial \bar{z} \partial z} \cdot \frac{\partial \bar{z}}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial \bar{z}}$

i.e.: 任何光滑函数 u on M
 $\sqrt{-1} \partial \bar{\partial} u$ well-defined on M globally.
 $\sqrt{-1} \partial \bar{\partial} u \in Z^2(M)$

定理: 设 M 是紧, 连通黎曼曲面.

$P \in \Lambda^2(M, \mathbb{R})$, i.e. $P = a(x, y) dx \wedge dy$

则: $\exists u \in C^\infty(M)$ s.t. $\sqrt{-1} \partial \bar{\partial} u = P$

当且仅当 $\int_M P = 0$.
 (1.2.5) $\int_M \sqrt{-1} \partial \bar{\partial} u = \int_M \sqrt{-1} d \bar{\partial} u = \int_M \sqrt{-1} \bar{\partial}^2 u = 0$

解方程

$H_{\bar{\partial}}^{1,0}(M):$
 $T_M^{1,0} \xrightarrow{\bar{\partial}} T_M^{1,1} \xrightarrow{\bar{\partial}} 0$

$H_{\bar{\partial}}^{0,1}(M):$
 $T_M^{0,1} \xrightarrow{\bar{\partial}} T_M^{0,2} \xrightarrow{\bar{\partial}} 0$
 $C^\infty(M)$

推论 1: M 紧连通, 则

(1) $\sigma: H_{\bar{\partial}}^{1,0}(M) \rightarrow \overline{H_{\bar{\partial}}^{1,0}(M)}$ 是同构.
 $\omega \mapsto [\bar{\omega}]$

(2) $\Phi: H_{\bar{\partial}}^{1,0}(M) \oplus H_{\bar{\partial}}^{0,1}(M) \rightarrow H_{dR}^1(M, \mathbb{C})$
 $(\omega, \eta) \mapsto [\omega + \bar{\eta}]$

(3) $i: H_{\bar{\partial}}^{1,0}(M) \rightarrow H_{dR}^1(M, \mathbb{C})$ 是同构.

pf: (1): σ is well-defined:

$$H_{\frac{1}{2}}^{\cdot,\cdot}(M) = \overline{Z_{\frac{1}{2}}^{\cdot,\cdot}(M)}, \quad w \in Z_{\frac{1}{2}}^{\cdot,\cdot}(M), \quad w|_U = f dz$$

$$\sigma(w) = \bar{w} \in \overline{Z_{\frac{1}{2}}^{\cdot,\cdot}(M)} \quad \frac{\partial}{\partial \bar{z}} f = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{w}|_U = \bar{f} d\bar{z}$$

$$\text{i.e. } [\sigma(w)] \in \overline{H_{\frac{1}{2}}^{\cdot,\cdot}(M)} \quad \text{假设 } \bar{w}_1 = \bar{\partial} u + \bar{w}_2 \text{ 则}$$

$$\sigma \text{ 单: } [\bar{w}_1] = [\bar{w}_2] \Leftrightarrow w_1 = w_2$$

$$\sigma \text{ 满: 证明: } \forall [\theta] \in H_{\frac{1}{2}}^{\cdot,\cdot}(M), \exists \theta' \in [\theta], \text{ 满足 } \theta' \in Z_{\frac{1}{2}}^{\cdot,\cdot}(M)$$

it suffices to prove:

$$\text{固定 } \theta_0 \in [\theta], \text{ (0. 此外 for)}$$

$$\text{寻找 } \theta' = \theta_0 + \bar{\partial} u \text{ 满足 } \bar{\partial} \theta' = 0$$

$$\bar{\partial}(\theta_0 + \bar{\partial} u) = \bar{\partial} \theta_0 + \bar{\partial}^2 u = -\bar{\partial} \theta_0$$

$$\text{i.e. 寻找 } u \in C^\infty(M), \text{ 使得 } \bar{\partial}^2 u = -\bar{\partial} \theta_0$$

$$\int_M \bar{\partial} \theta_0 = \int_M d\theta_0 - \int_M \bar{\partial} \theta_0 = 0$$

$$(4): \Phi: H_{\frac{1}{2}}^{\cdot,\cdot}(M) \oplus H_{\frac{1}{2}}^{\cdot,\cdot}(M) \rightarrow H_{dR}^1(M)$$

$$(w, \eta) \mapsto w + \bar{\eta}$$

$$\Phi \text{ is well-defined: i.e. 证 } d(w + \bar{\eta}) = \bar{\partial} w + \bar{\partial} \bar{\eta} = 0$$

$$\frac{\partial}{\partial z} w + \frac{\partial}{\partial z} \bar{\eta} = 0$$

$$\Phi \text{ 单: i.e. 证 } \ker \Phi = 0$$

i.e. 令 (α, β) 有 $\alpha + \bar{\beta} = df$ 的 forms.

$$\text{locally, } (U, \varphi) \quad \alpha|_U = a dz \quad \bar{\beta}|_U = \bar{b} d\bar{z}$$

$$\text{on } U, \quad a dz + \bar{b} d\bar{z} = df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$\text{i.e. } \frac{\partial f}{\partial z} = a, \quad \frac{\partial f}{\partial \bar{z}} = \bar{b}$$

$$\text{又由 } \bar{\partial} a = 0 \text{ 和 } \frac{\partial f}{\partial \bar{z}} \text{ 全纯 i.e. } \bar{\partial} \bar{\partial} f = 0.$$

$$\text{又由 } \bar{\partial} \bar{\partial} f \text{ 值与 } (U, \varphi) \text{ 选取无关, 余, } \bar{\partial} \bar{\partial} f = 0 \text{ on } M. \Rightarrow f \text{ const.}$$

证满射
要用到解方程

证明满射
要用到解方程

$\bar{\partial}$ is surj.: 任取 $[\omega] \in H_{dR}^{1,0}(M, \mathbb{C})$
 $\exists \omega \in (\omega)$. $\omega = \tilde{\omega}_1 + \tilde{\omega}_2$, $\tilde{\omega}_1 \in \Lambda^{1,0} \tilde{\omega}_2 \in \Lambda^{0,1}$
 It suffices to find $\omega_0 \in (\omega)$, $\omega_0 = \omega_1 + \omega_2$,
 $\omega_1 \in \Lambda^{1,0}$, $\omega_2 \in H_0^{0,1}$
 i.e. 寻找 u s.t. $\omega_0 = \omega + du$
 $\bar{\partial}(\tilde{\omega}_1 + du) = 0$, $\bar{\partial}(\tilde{\omega}_2 + \bar{\partial}u) = 0$

$\bar{\partial}$ is surj.: 任取 $[\omega] \in H_{dR}^{1,0}(M, \mathbb{C})$
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 It suffices to find $\omega_0 \in (\omega)$, $\omega_0 = \omega_1 + \omega_2$,
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 i.e. 寻找 u s.t. $\omega_0 = \omega + du$
 $\bar{\partial}(\tilde{\omega}_1 + du) = 0$, $\bar{\partial}(\tilde{\omega}_2 + \bar{\partial}u) = 0$
 i.e. 解方程 $\bar{\partial}(\tilde{\omega}_1 + du) = 0$
 \Downarrow
 $\bar{\partial} du = -\bar{\partial} \tilde{\omega}_1$
 (2)

Correction
 Hodge Decomposition
 $H_{dR}^{1,0}(M) \rightarrow H_{dR}^{1,0}(M)$
 $Z_{dR}^{1,0} \rightarrow \bar{\partial}^{-1}$
 (2) $H_{dR}^{1,0}(M, \mathbb{C}) \cong H_{dR}^{1,0}(M) \oplus H_{dR}^{0,1}(M)$
 $([\omega + \bar{\partial}u]_{dR}^{1,0}) \mapsto ([\omega]_{dR}^{1,0})$

② Serre Duality

Step 1: 问自己 $\mathcal{O}(k-D)$ 是什么东西?
 $[K] = [(\omega)] \mid \omega \in Z_{dR}^{1,0}(M)$
 实际上 $[K] = T^{*(1,0)}M$. $\omega|_U = f \cdot dz$
 $[K-D] = T^{*(1,0)}M \otimes [-D] \xrightarrow{H^0(U, M^*/\mathcal{O}^*)} H^0(U, M^*/\mathcal{O}^*) \xrightarrow{f^*} H^0(U, \mathcal{O}^*)$
 转移函数 $(\frac{\partial z_i}{\partial z_j}, \frac{1}{z_j}, U_{ij})$
 $D_1 + D_2 \mapsto L_{D_1} \otimes L_{D_2}$
 $(\frac{\partial z_i}{\partial z_j}, \frac{1}{z_j}, U_{ij})$

(D) $H^0(U) = \{s: U \rightarrow L \mid \pi \circ s = id_U\}$
 $= \{f \in M^*(U) \mid (f) + D \geq 0\}$
 3) $\lambda \cdot f$ 是否
 $Z_{dR}^{1,0}(-D)(U) = \{f \cdot \omega \mid \omega \in Z_{dR}^{1,0}(U), (f \cdot \omega) + D \geq 0\}$
 Proposition: 令 $\Omega \subseteq \mathbb{C}$ 连通, 紧致.
 2) $\mathcal{O}(k-D)(\Omega) = Z_{dR}^{1,0}(-D)(\Omega)$
 Pf: $\mathcal{O}(k-D)(\Omega) = \{f \in M^*(\Omega) \mid (f) + k-D \geq 0\}$

Prop: 令 $\Omega \subseteq \mathbb{C}$ 连通, 紧致.
 2) $\mathcal{O}(k-D)(\Omega) = Z_{dR}^{1,0}(-D)(\Omega)$
 Pf: $\mathcal{O}(k-D)(\Omega) = \{f \in M^*(\Omega) \mid (f) + k-D \geq 0\}$
 $Z_{dR}^{1,0}(-D)(\Omega) = \{f \cdot \omega \mid \omega \in Z_{dR}^{1,0}(\Omega), (f \cdot \omega) - D \geq 0\}$

陈子利线丛

$H^1(\mathcal{U}, \mathcal{O})$ 中的元素是什么?

i.e. It suffices to prove:

$$H^1(\mathcal{U}, \mathcal{Z}_{\frac{1}{2}}^{1,0}(-D)) \cong H^1(\mathcal{U}, \mathcal{O}(D))$$

Step 2: $H^1(\mathcal{U}, \mathcal{Z}_{\frac{1}{2}}^{1,0}(-D)) \times H^1(\mathcal{U}, \mathcal{O}(D)) \rightarrow H^1(\mathcal{U}, \mathcal{O})$

$$(f \cdot \omega, \{(g_{-p}, U_{-p})\}) \mapsto (f \cdot g_{-p} \cdot \omega|_{U_{-p}})$$

Well-defined: $\nexists \pm U_{-p} \perp$

$$(f \cdot g_{-p} \cdot \omega|_{U_{-p}}) = (f) + (g_{-p}) + (\omega)|_{U_{-p}} \geq 0$$

因而 $f \cdot g_{-p} \cdot \omega|_{U_{-p}} = h_{-p} dz$

$$\{(h_{-p}, U_{-p})\} \in H^1(\mathcal{U}, \mathcal{O})$$

$\cong H_{\frac{1}{2}}^{0,1}(M) \cong$

Recall: $H^1(\mathcal{U}, \mathcal{O}) \cong H_{\frac{1}{2}}^{1,0}(M)$

i.e.: $\{(h_{-p}, U_{-p})\} \mapsto \eta$