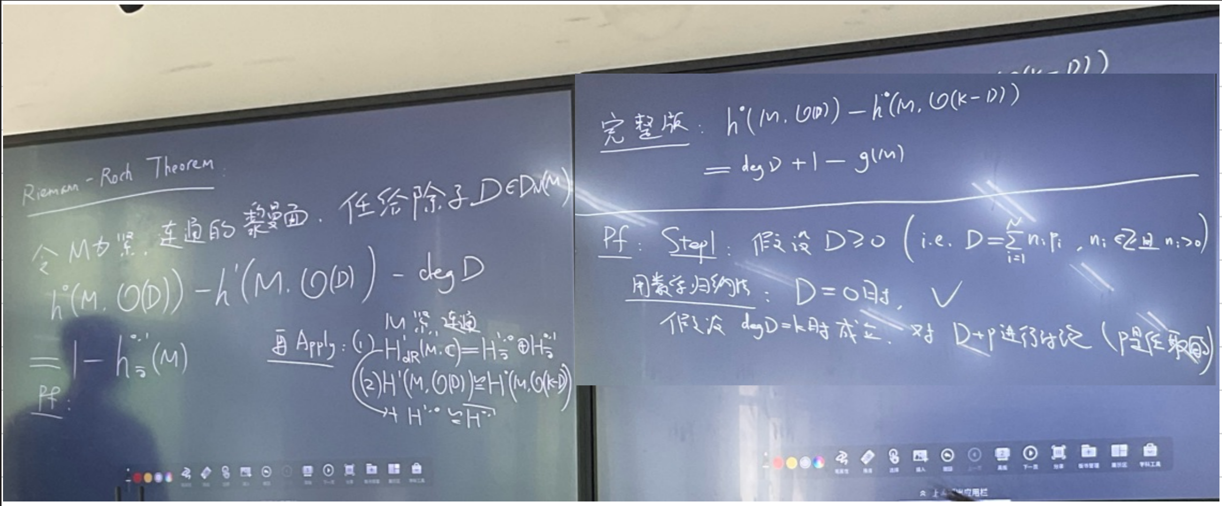
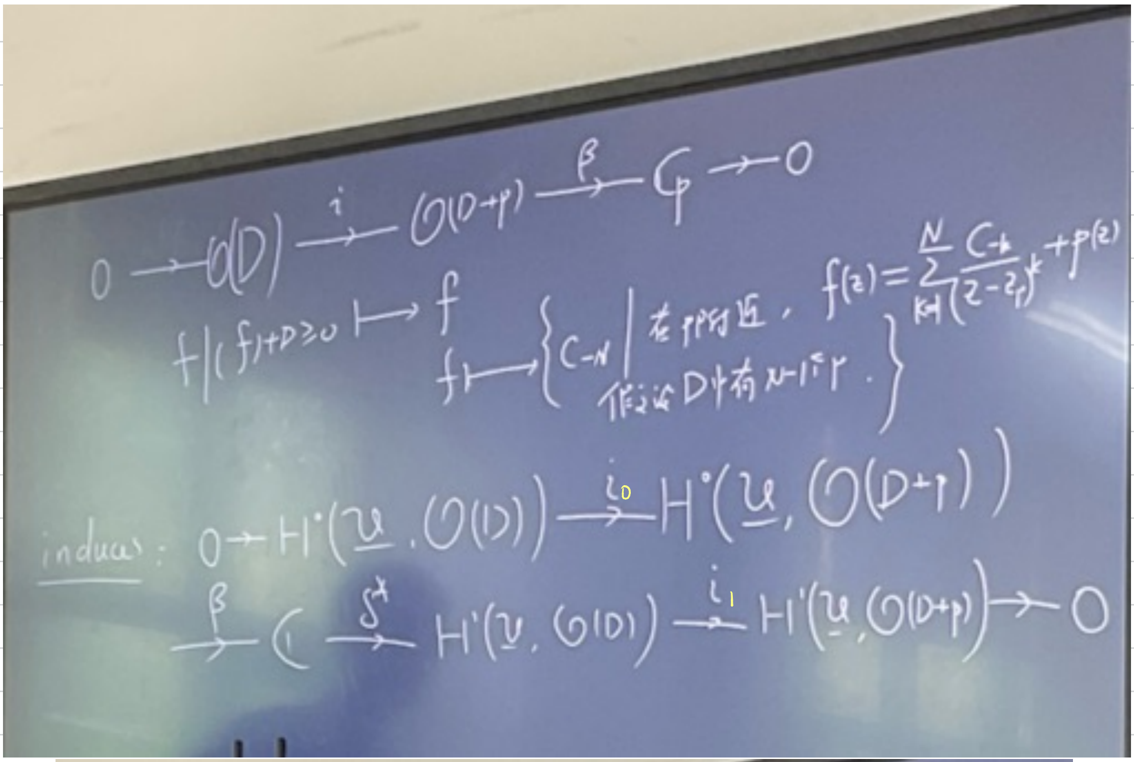
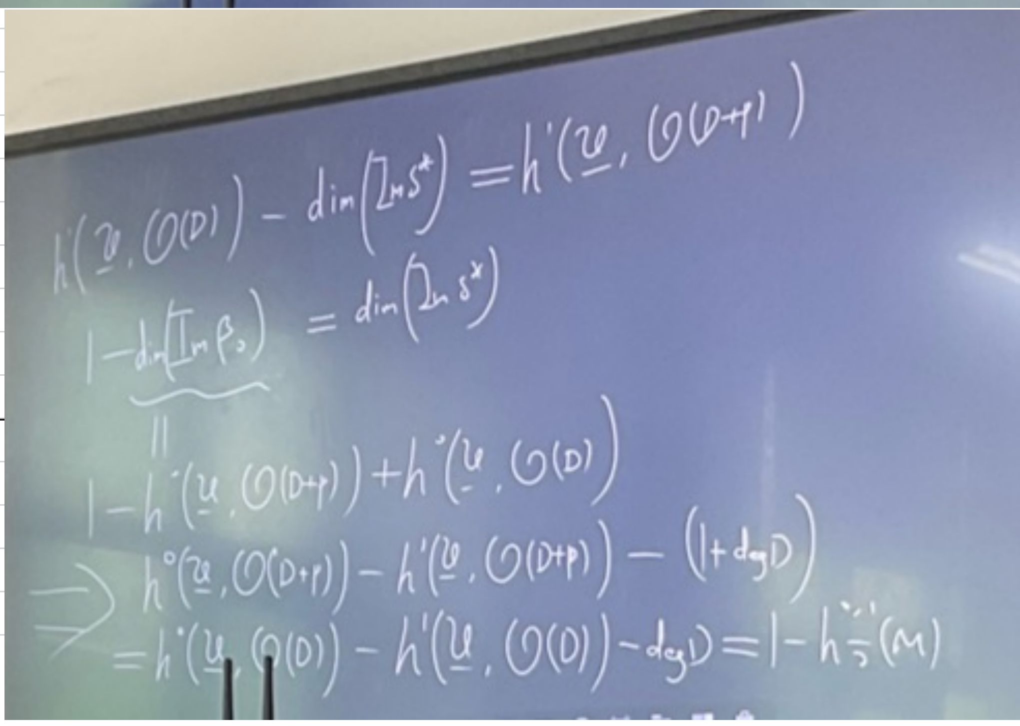
Riemann - Roch Theorem

可以 Hodge 分解的复流形叫 Kähler 流形

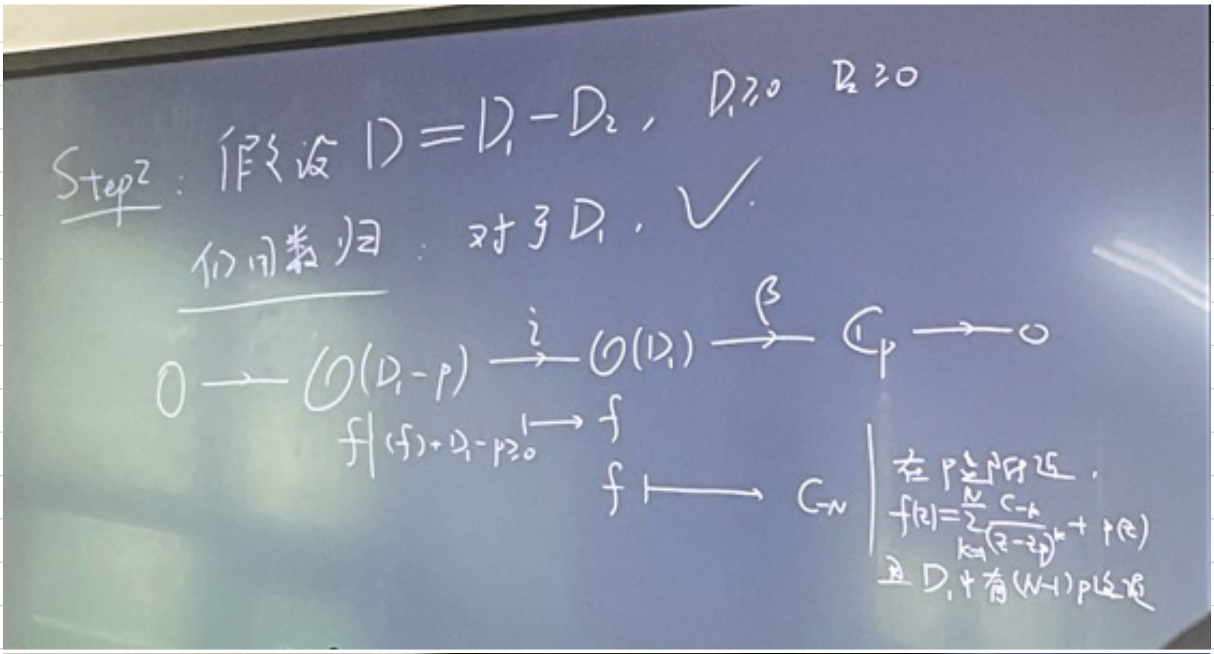


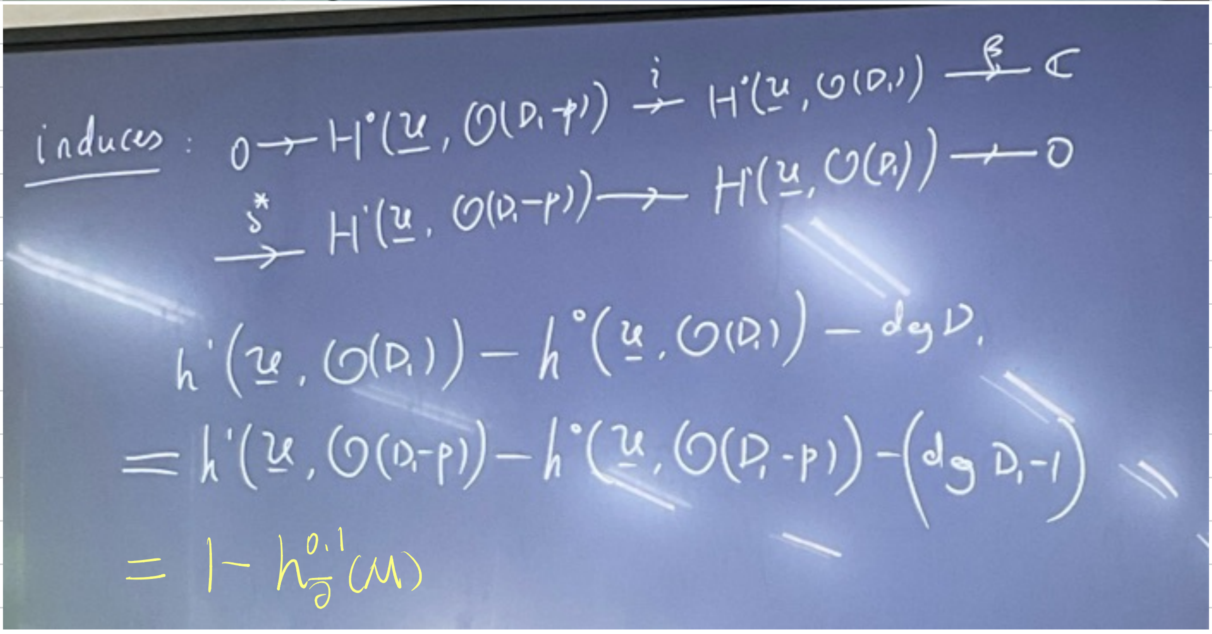
Step 1：当 $D \geq 0$ 时
用数学归纳法

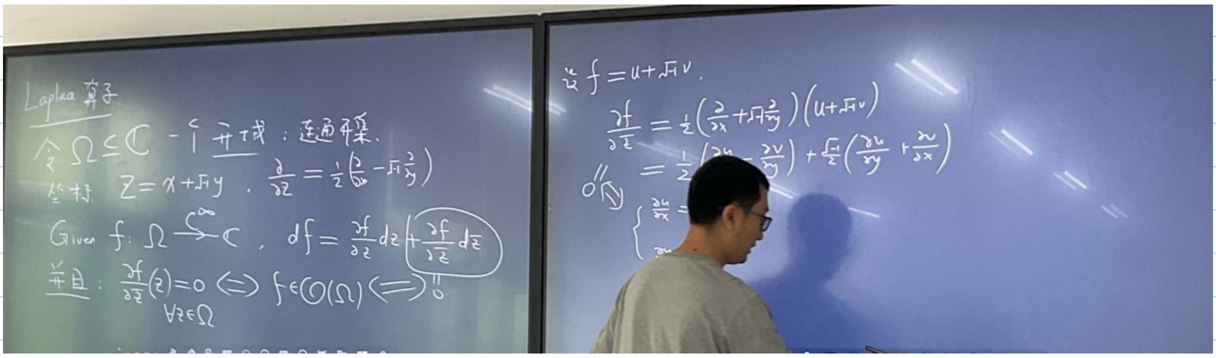




Step 2.
讨论一般的除子 D



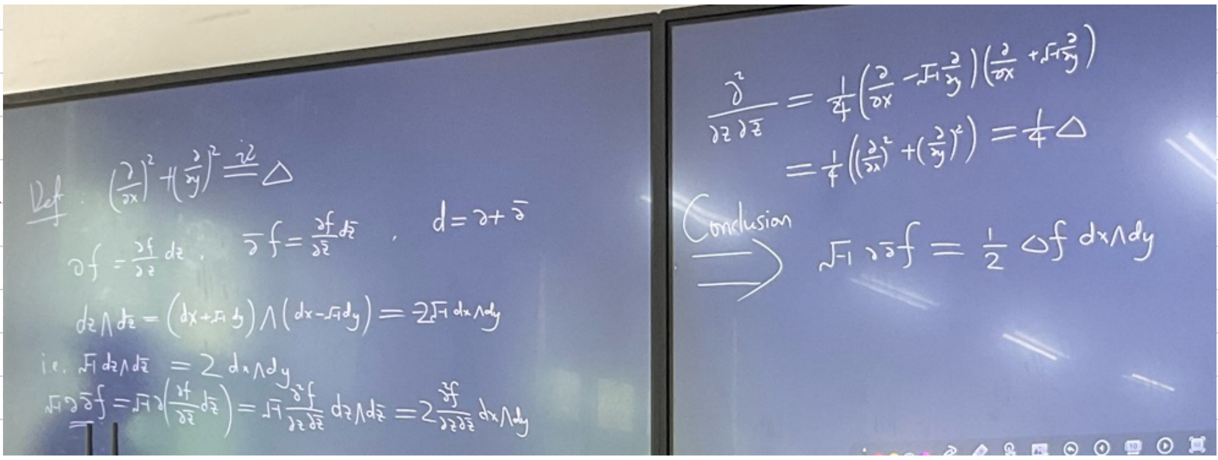




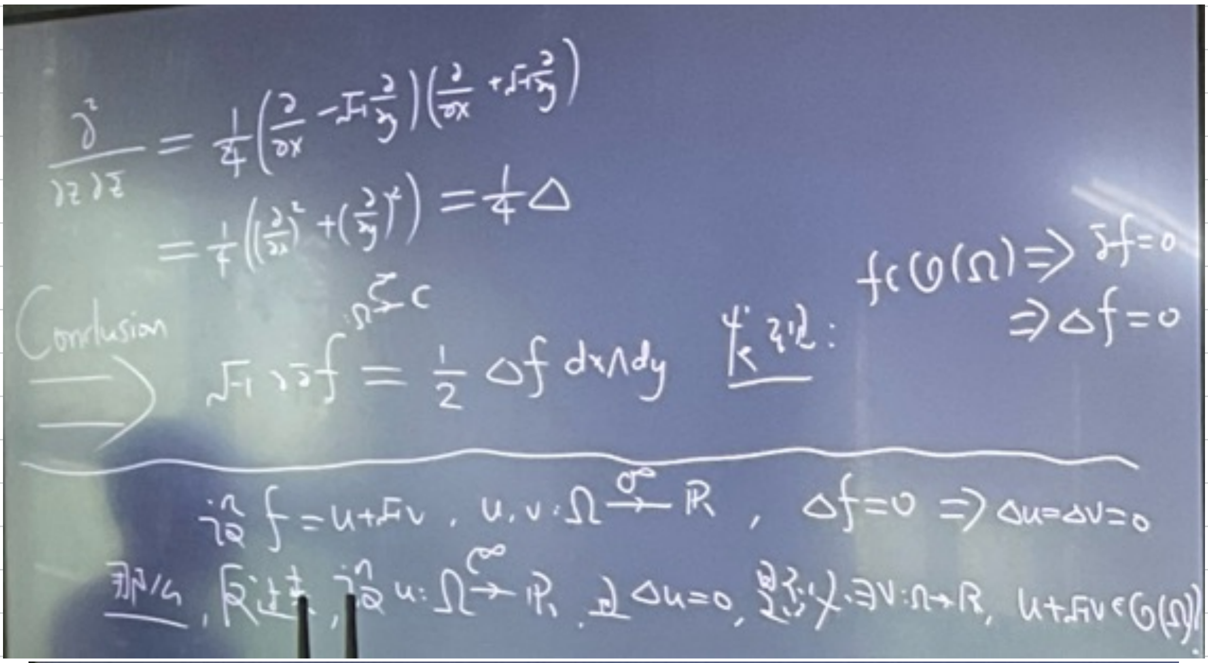
$f:\Omega\xrightarrow{C^\infty}\mathbb{C}$ 光滑复值函数

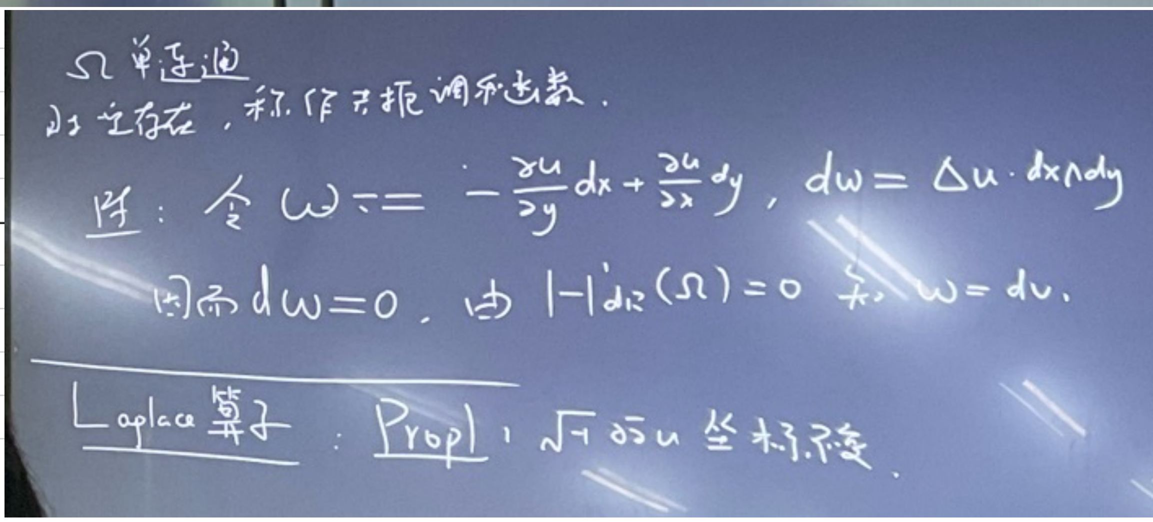
全纯 可推 调和

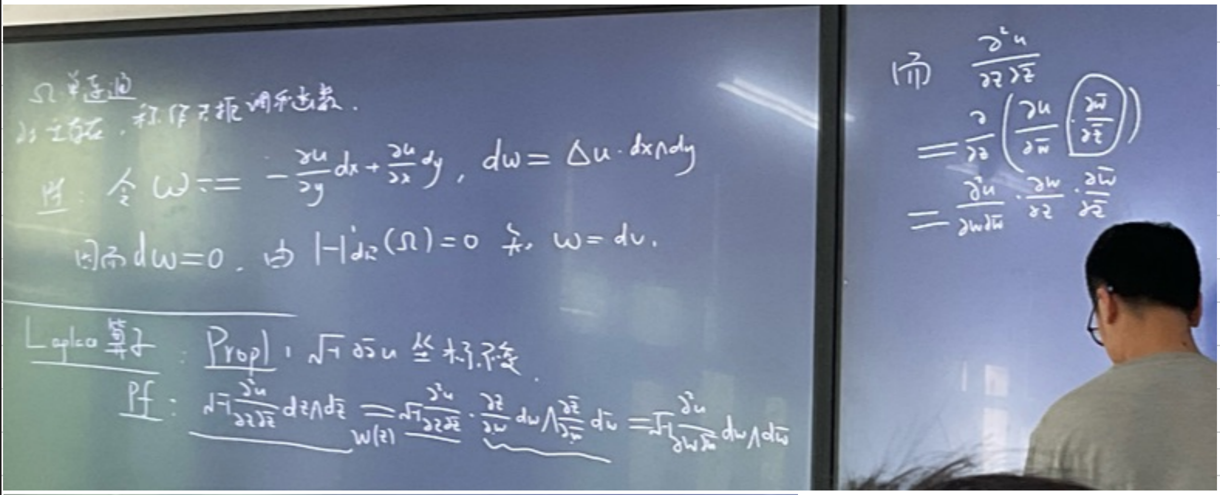
$f\in\mathcal{O}(\Omega)\Rightarrow\bar\partial f=0\Rightarrow\Delta f=0$

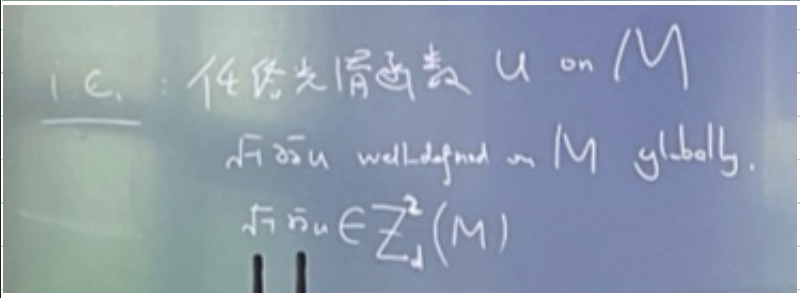


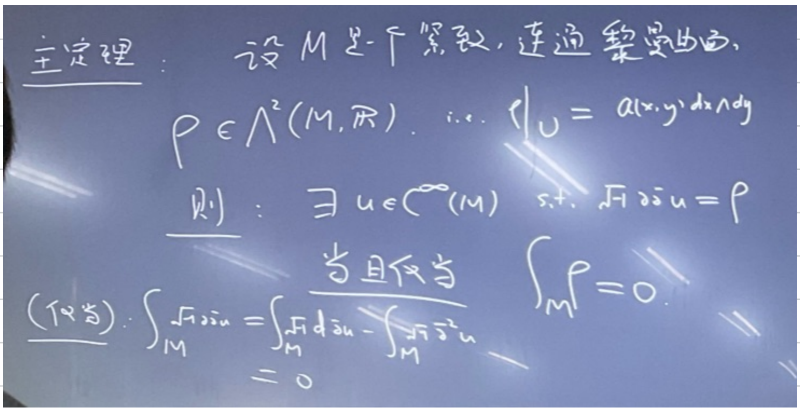
共轭调和函数







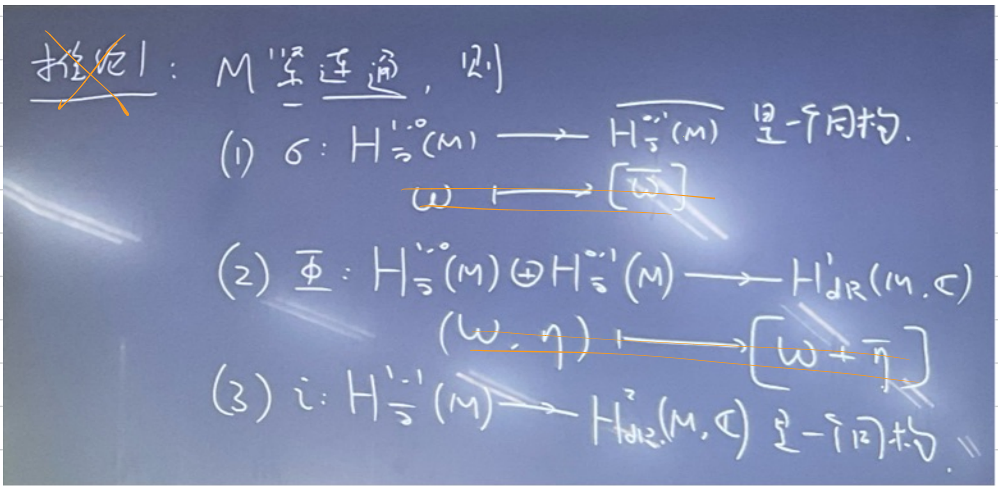




解方程

$H^{1,0}_{\bar\partial}(M)$:
$T^{1,0}M \xrightarrow{\bar\partial} T^{1,1}M \xrightarrow{\bar\partial} 0$

$H^{0,1}_{\bar\partial}(M)$:
$T^{0,0}M \xrightarrow{\bar\partial} T^{0,1}M \xrightarrow{\bar\partial} 0$
$T^{0,0}M = C^\infty(M)$



证: (1): σ is well-defined:

$H^{1,0}_{\bar\partial}(M) = Z^{1,0}_{\bar\partial}(M)$, $\omega\in Z^{1,0}_{\bar\partial}(M)$, $\omega|_U = f\,dz$, $\bar\partial f = 0$.

则 $\bar\omega|_U = \bar f\,d\bar z$

$\sigma(\omega) = \bar\omega \in Z^{0,1}_{\bar\partial}(M)$

i.e. $[\sigma(\omega)] \in H^{0,1}(M)$

假设 $\bar\omega_1 = \bar\partial u + \bar\omega_2$ 则

σ单: $[\bar\omega_1]=[\bar\omega_2] \Leftrightarrow \omega_1=\omega_2$

σ满: 即证明: $\forall [\theta]\in H^{0,1}_{\bar\partial}(M)$, $\exists!\,\theta'\in[\theta]$, 满足 $\bar\theta'\in Z^{1,0}_{\bar\partial}(M)$

It suffices to prove:

固定一个 $\theta_0\in[\theta]$, (θ_0 为 $d\bar z$-form)

寻找 $\theta' = \theta_0 + \bar\partial u$ 满足 $\bar\partial\,\bar\theta' = 0$

$$\bar\partial\,\bar\theta' = 0 \;=\; \bar\partial\left(\overline{\theta_0+\bar\partial u}\right) \;=\; \bar\partial(\bar\theta_0) + \bar\partial\partial\bar u$$

i.e. 寻找 $v\in C^\infty(M)$, $\sqrt{-1}\,\bar\partial\partial v = -\sqrt{-1}\,\bar\partial\bar\theta_0$

证满射
要用到解方程

$$\int_M \partial\theta_0 = \int_M d\theta_0 - \int_M \bar\partial\theta_0 = 0$$

(2): $\Phi: H^{1,0}_{\bar\partial}(M)\oplus \overline{H^{0,1}_{\bar\partial}(M)} \longrightarrow H^1_{dR}(M)$

$(\omega,\eta) \longmapsto \omega+\bar\eta$

Φ is well-defined: i.e. 证 $d(\omega+\bar\eta) = \partial\underbrace{\omega}_{f\,dz} + \bar\partial\underbrace{\bar\eta}_{\bar f\,d\bar z} = 0$

Φ单: i.e. 证 $\ker\Phi = 0$

i.e. 令 (α,β) 为 $\alpha+\bar\beta = df$ 的 forms.

locally, (U,φ), $\alpha|_U = a\,dz$, $\bar\beta|_U = \bar b\,d\bar z$

on U, $a\,dz + \bar b\,d\bar z = df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z$

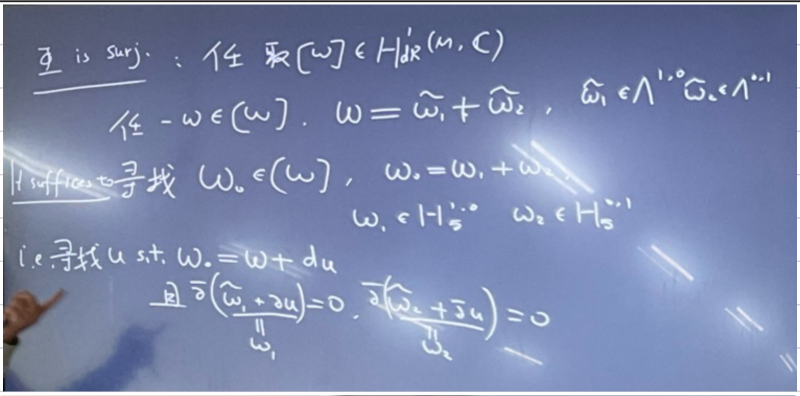
i.e. $\frac{\partial f}{\partial z} = a$, $\frac{\partial f}{\partial \bar z} = \bar b$

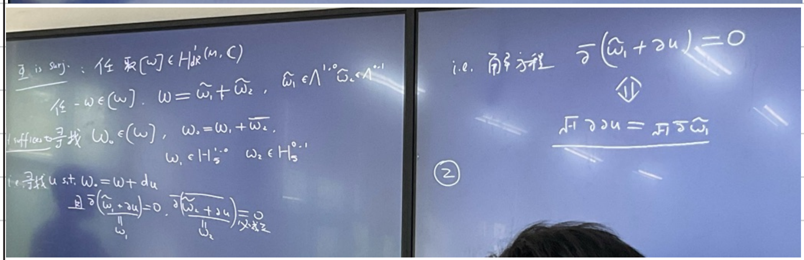
又由 $\bar\partial a = 0$ 知 $\frac{\partial f}{\partial z}$ 全纯 i.e. $\sqrt{-1}\partial\bar\partial f = 0$.

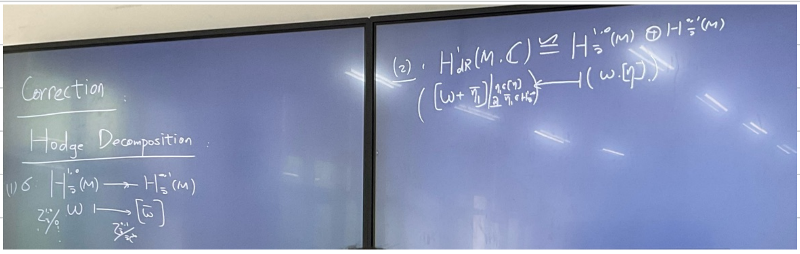
又由 $\sqrt{-1}\partial\bar\partial f$ 值与 (U,φ) 选取无关, 知 $\sqrt{-1}\partial\bar\partial f = 0$ on M.

$\Rightarrow f$ const.

证明满射
要用到解方程

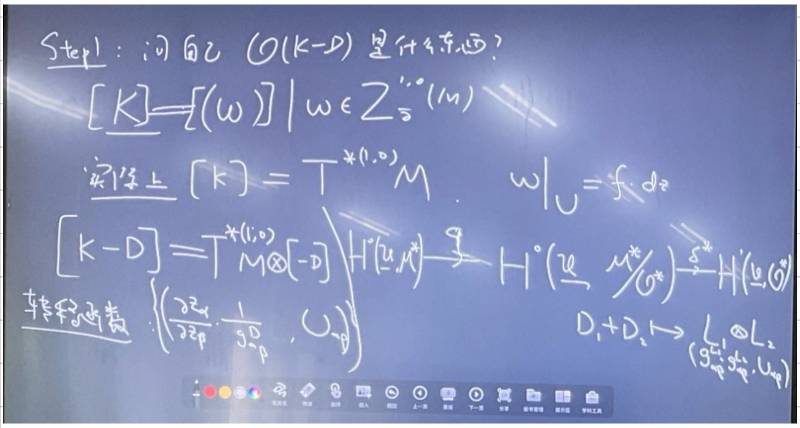


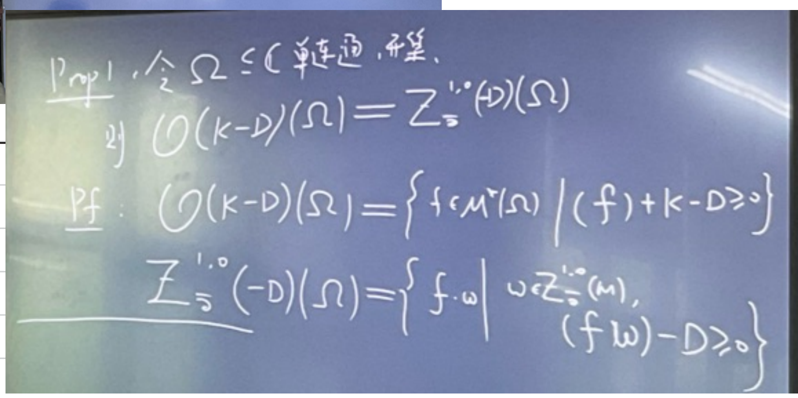




② Serre Duality

除子到线丛





$H^1(\mathcal{U},\mathcal{O})$ 中的元素是什么？

i.e. It suffices to prove:

$$H^1(\mathcal{U}, Z_{\bar\partial}^{1,0}(-D)) \cong H^1(\mathcal{U}, \mathcal{O}(D))$$

Step 2: $H^1(\mathcal{U}, Z_{\bar\partial}^{1,0}(-D)) \times H^1(\mathcal{U},\mathcal{O}(D)) \longrightarrow H^1(\mathcal{U},\mathcal{O})$

$$(f\cdot\omega, \{(g_{\alpha\beta}, U_{\alpha\beta})\}) \mapsto (f\cdot g_{\alpha\beta}\cdot\omega|_{U_{\alpha\beta}}, U_{\alpha\beta})$$

well-defined: 在 $U_{\alpha\beta}$ 上,

$$(f\cdot g_{\alpha\beta}\cdot\omega|_{U_{\alpha\beta}}) = (f)+(g_{\alpha\beta})+(\omega)|_{U_{\alpha\beta}} \geq 0$$

因而 $f\cdot g_{\alpha\beta}\cdot\omega|_{U_{\alpha\beta}} = h_{\alpha\beta}\, dz$,

$$\{(h_{\alpha\beta}, U_{\alpha\beta})\} \in H^1(\mathcal{U},\mathcal{O})$$

$\cong H^{0,1}_{\bar\partial}(M)$ ≶

Recall: $H^1(\mathcal{U},\mathcal{O}) \cong H^{1,0}_{\bar\partial}(M)$

i.e.: $\{(h_{\alpha\beta}, U_{\alpha\beta})\} \mapsto \eta$