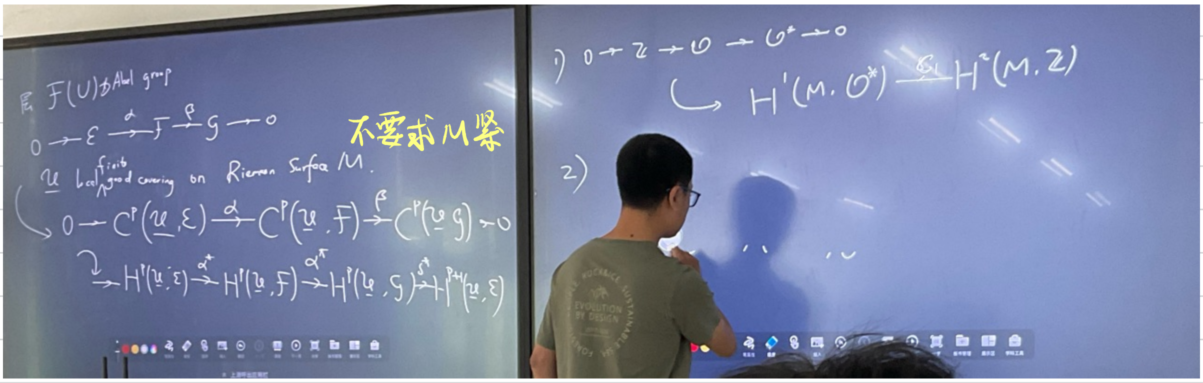
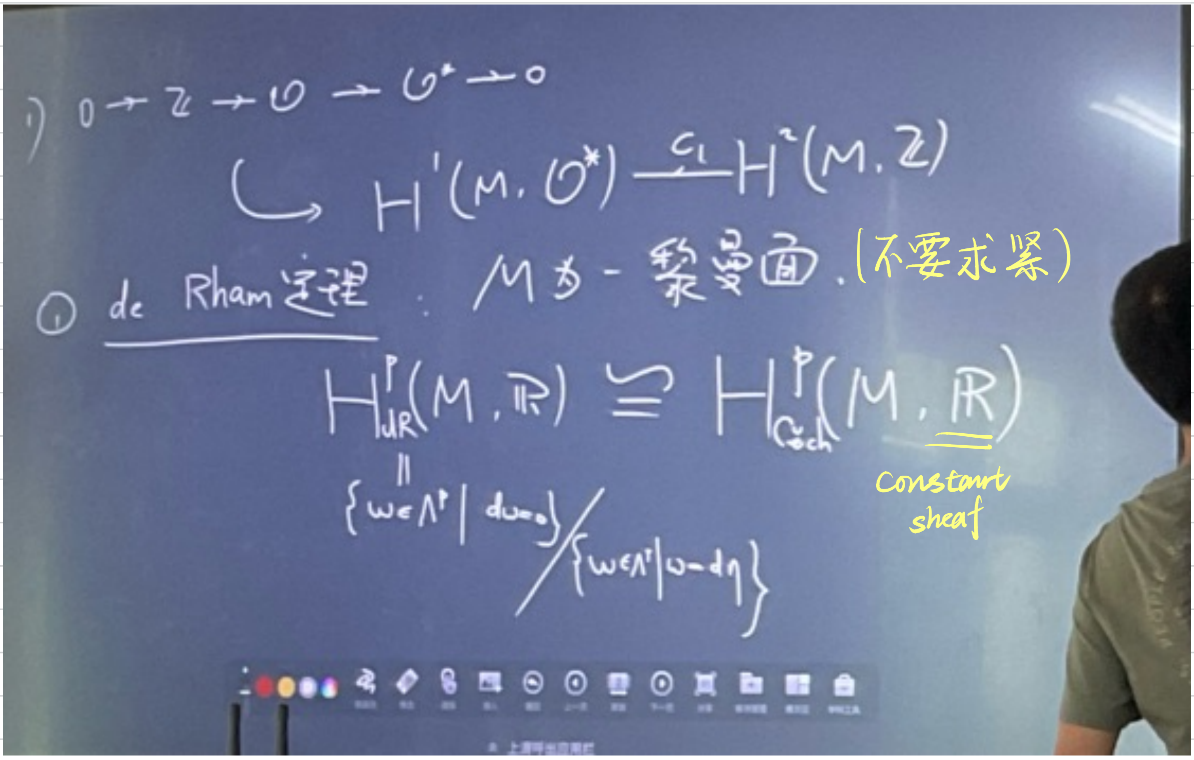
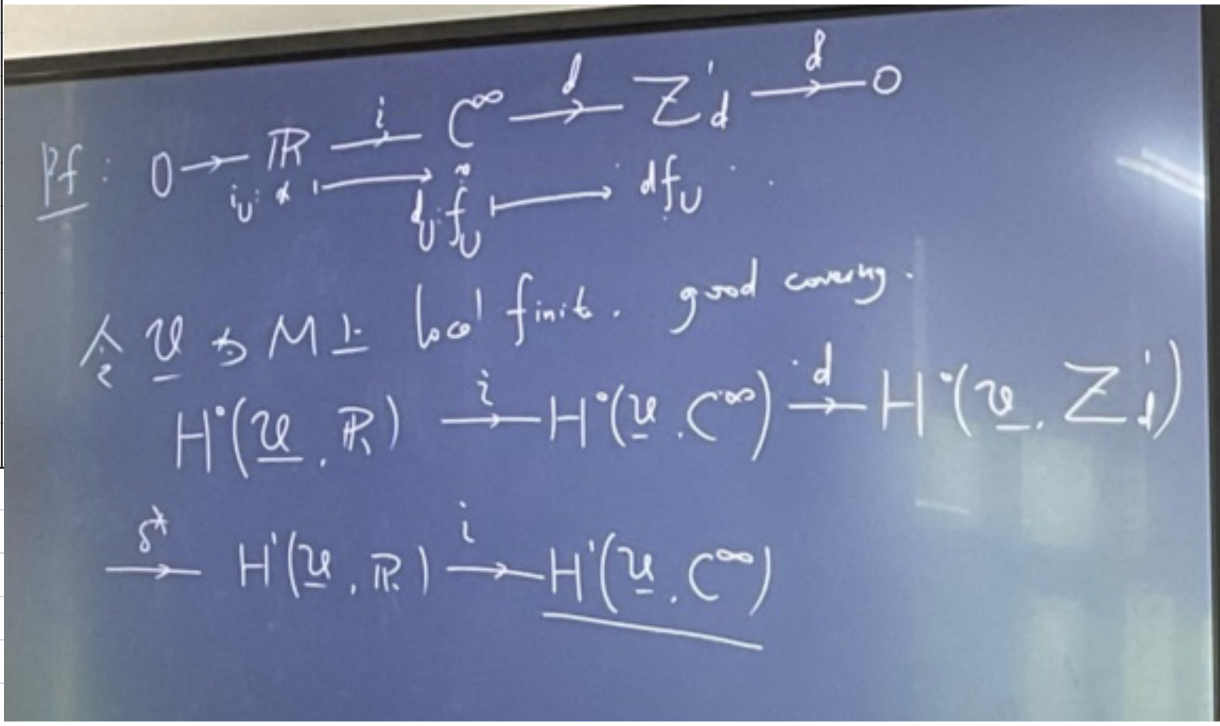
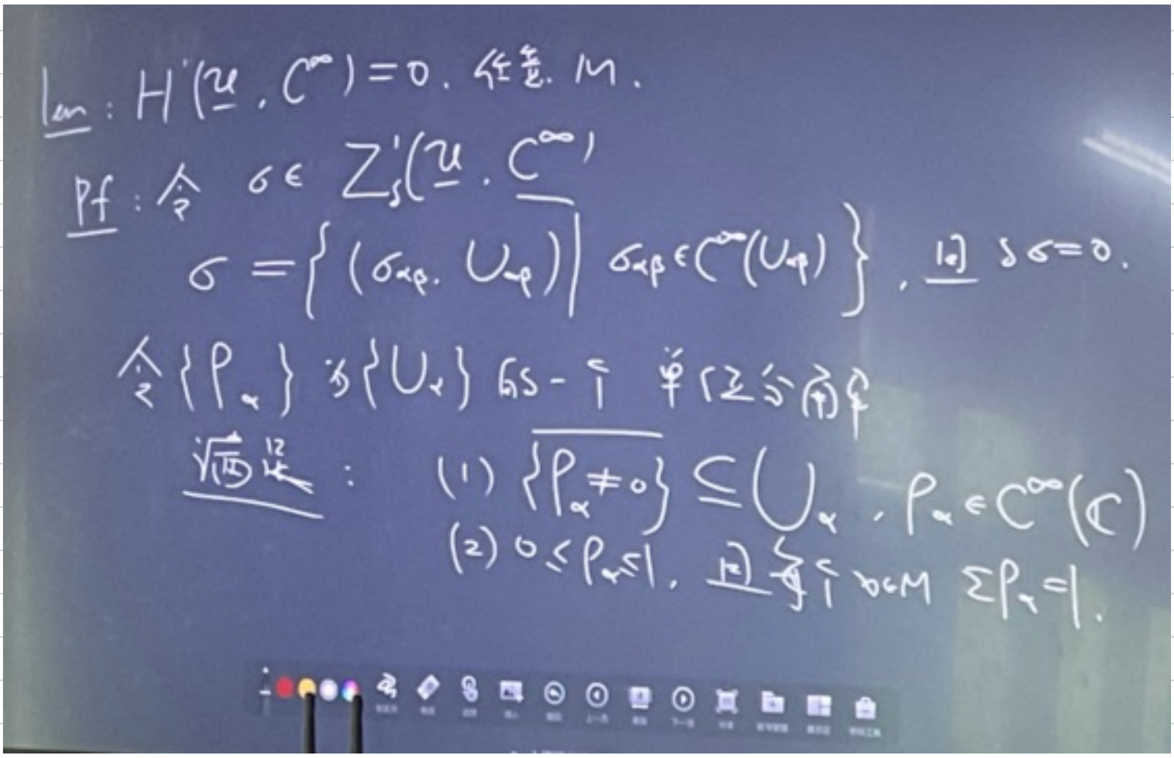
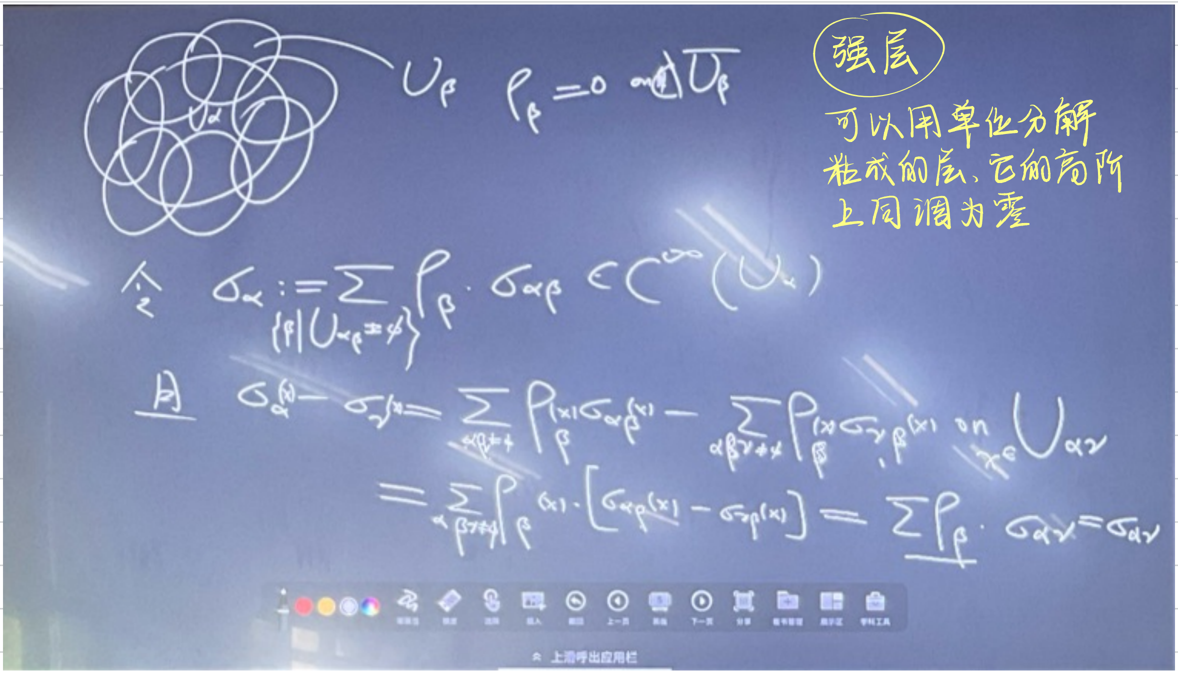


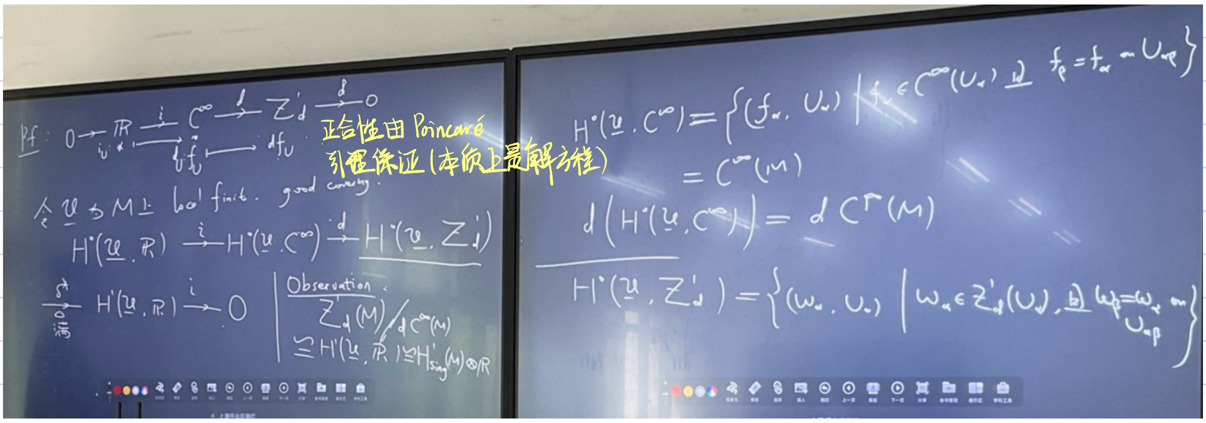


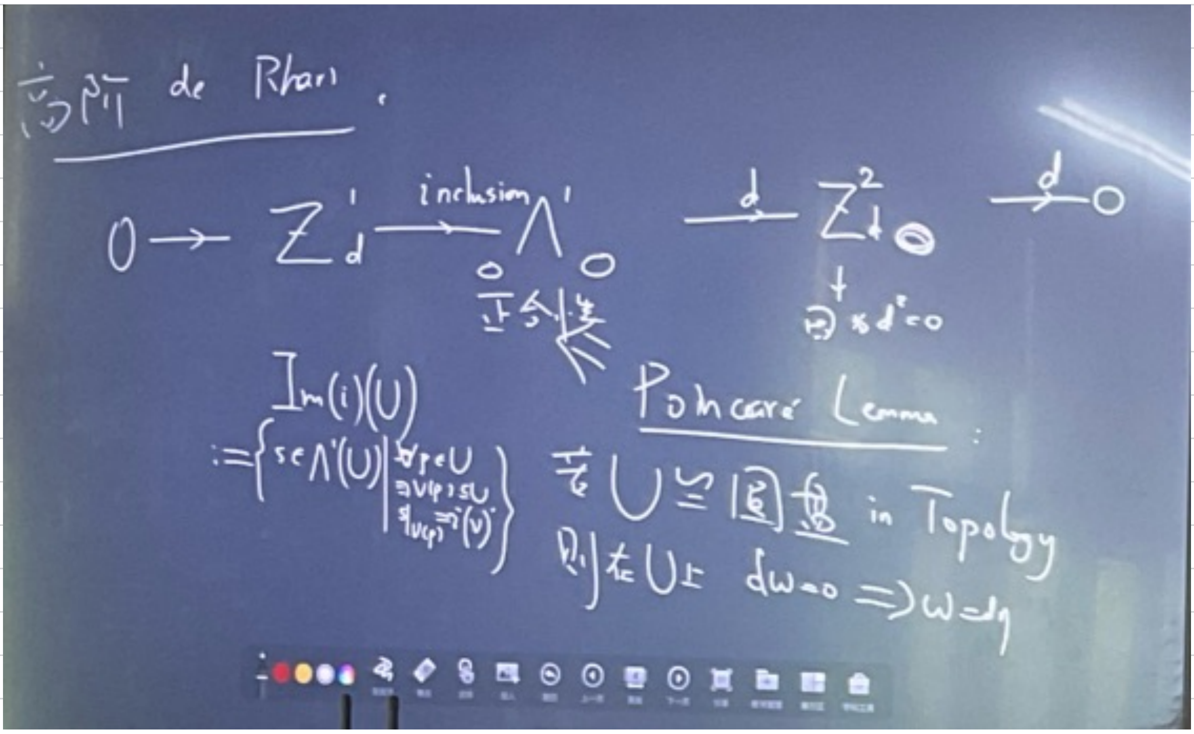
用很soft的办法证明

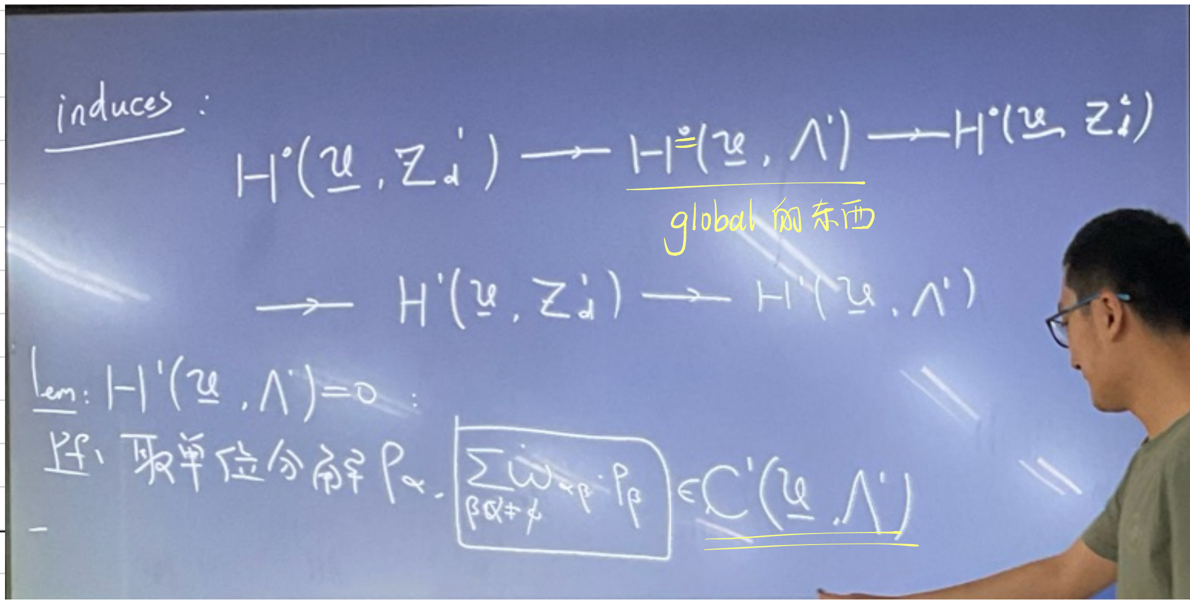



Lem: $H^1(\mathcal{U}, C^\infty) = 0$. 任意 M.
Pf: 令 $\sigma \in Z^1_s(\mathcal{U}, C^\infty)$
$\sigma = \{(\sigma_{\alpha\beta}, U_{\alpha\beta}) \mid \sigma_{\alpha\beta} \in C^\infty(U_{\alpha\beta})\}$, 且 $\delta\sigma = 0$.
令 $\{\rho_\alpha\}$ 为 $\{U_\alpha\}$ 的一个单位分解
满足: (1) $\overline{\{\rho_\alpha \neq 0\}} \subseteq U_\alpha$, $\rho_\alpha \in C^\infty(\mathbb{C})$
(2) $0 \le \rho_\alpha \le 1$, 且 对 $\forall x \in M$ $\sum \rho_\alpha = 1$.


U_α
U_β $\rho_\beta = 0$ on ∂U_β
强层
可以用单位分解粘成的层、它的高阶上同调为零
令 $\sigma_\alpha := \sum_{\{\beta \mid U_{\alpha\beta} \neq \emptyset\}} \rho_\beta \cdot \sigma_{\alpha\beta} \in C^\infty(U_\alpha)$
则 $\sigma_\alpha(x) - \sigma_\gamma(x) = \sum_{\alpha\beta\gamma \neq \emptyset} \rho_\beta(x)\sigma_{\alpha\beta}(x) - \sum_{\alpha\beta\gamma \neq \emptyset} \rho_\beta(x)\sigma_{\gamma\beta}(x)$ on $x \in U_{\alpha\gamma}$
$= \sum_{\alpha\beta\gamma \neq \emptyset} \rho_\beta(x) \cdot [\sigma_{\alpha\beta}(x) - \sigma_{\gamma\beta}(x)] = \sum \rho_\beta \cdot \sigma_{\alpha\gamma} = \sigma_{\alpha\gamma}$
上滑呼出应用栏


Pf: $0 \to \mathbb{R} \xrightarrow{i} C^\infty \xrightarrow{d} Z'_d \xrightarrow{d} 0$
正合性由 Poincaré 引理保证(本质上是解方程)
令 $\mathfrak{U}$ 为 M 上 locally finite. good covering.
$H^0(\mathfrak{U}, \mathbb{R}) \xrightarrow{i} H^0(\mathfrak{U}, C^\infty) \xrightarrow{d} H^0(\mathfrak{U}, Z'_d)$
$\to H'(\mathfrak{U}, \mathbb{R}) \xrightarrow{i} 0$
Observation:
$Z'_d(M)/dC^\infty(M) \cong H'(\mathfrak{U}, \mathbb{R}) \cong H'_{sing}(M)\otimes\mathbb{R}$
$H^0(\mathfrak{U}, C^\infty) = \{(f_\alpha, U_\alpha) \mid f_\alpha \in C^\infty(U_\alpha)$ 且 $f_\beta = f_\alpha$ on $U_{\alpha\beta}\}$
$= C^\infty(M)$
$d(H^0(\mathfrak{U}, C^\infty)) = dC^\infty(M)$
$H^0(\mathfrak{U}, Z'_d) = \{(\omega_\alpha, U_\alpha) \mid \omega_\alpha \in Z'_d(U_\alpha)$, 且 $\omega_\beta = \omega_\alpha$ on $U_{\alpha\beta}\}$


高阶 de Rham.
$0 \to Z'_d \xrightarrow{inclusion} \Lambda' \xrightarrow{d} Z^2_d \xrightarrow{d} 0$
正合性
因为 $d^2=0$
$Im(i)(U) := \{s \in \Lambda'(U) \mid \forall p \in U\ \exists V(p) \subseteq U\ s|_{V(p)} = i(v)\}$
Poincaré Lemma:
若 $U \cong$ 圆盘 in Topology
则在 U 上 $d\omega = 0 \Rightarrow \omega = d\eta$


induces:
$H^0(\mathfrak{U}, Z'_d) \to H^0(\mathfrak{U}, \Lambda') \to H^0(\mathfrak{U}, Z^2_d)$
global 的东西
$\to H'(\mathfrak{U}, Z'_d) \to H'(\mathfrak{U}, \Lambda')$
Lem: $H'(\mathfrak{U}, \Lambda') = 0$:
Pf: 取单位分解 ρ_α. $\boxed{\sum_{\beta\alpha\neq\phi} \omega_{\alpha\beta}\cdot\rho_\beta} \in C'(\mathfrak{U}, \Lambda')$

$$\Longrightarrow Z_d^2 \big/ d\Lambda^1 \cong H^1(\underline{\mathfrak{U}}, Z_d^1) \cong H^2_{\check{C}ech}(\underline{\mathfrak{U}}, \mathbb{R}) \cong H^2_{sing}(M)\otimes\mathbb{R}$$

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} C^\infty \xrightarrow{d} Z_d^1 \xrightarrow{d} 0$$

$$H^1(\underline{\mathfrak{U}},\mathbb{R}) \longrightarrow \underline{H^1(\underline{\mathfrak{U}}, C^\infty)} \longrightarrow H^1(\underline{\mathfrak{U}}, Z_d^1) \xrightarrow{\delta^*} H^2(\underline{\mathfrak{U}},\mathbb{R}) \longrightarrow \underline{H^2(\underline{\mathfrak{U}}, C^\infty)}$$

$$H^1(\underline{\mathfrak{U}}, C^\infty) = 0, \quad H^2(\underline{\mathfrak{U}}, C^\infty) = 0$$

$$\sum \sigma_{\alpha\beta\gamma}\rho_\beta \overset{\sigma_{\alpha\gamma}}{=} -\sum \sigma_{\alpha\beta\delta}\rho_\beta + \sum \sigma_{\delta\beta\gamma}\rho_\beta$$

2) Dolbeault 定理：

令M为黎曼面，$\underline{\mathfrak{U}}$ 为 l.f. good covering

$$H^1(\underline{\mathfrak{U}}, \mathcal{O}) \cong H^{0,1}_{\bar\partial}(M)$$

Pf: $0 \longrightarrow \mathcal{O} \xrightarrow{i} C^\infty \xrightarrow{\bar\partial_1} Z^{0,1}_{\bar\partial} \xrightarrow{\bar\partial_2} 0$

$$f \longmapsto \bar\partial f = \frac{\partial f}{\partial \bar z}\, d\bar z$$

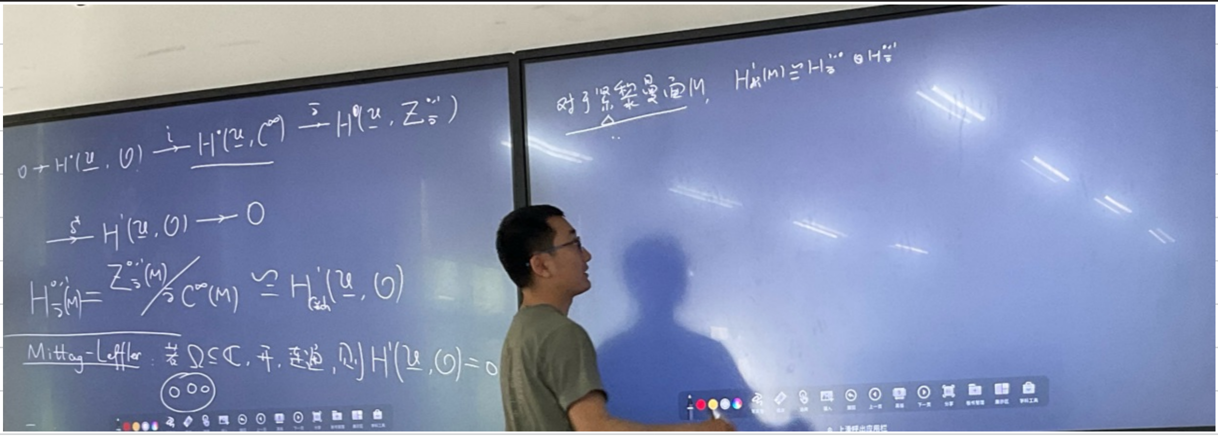
$\bar\partial$-Poincaré Lemma $\Longrightarrow$ $\mathrm{Im}\,\bar\partial_1 = \ker \bar\partial_2$

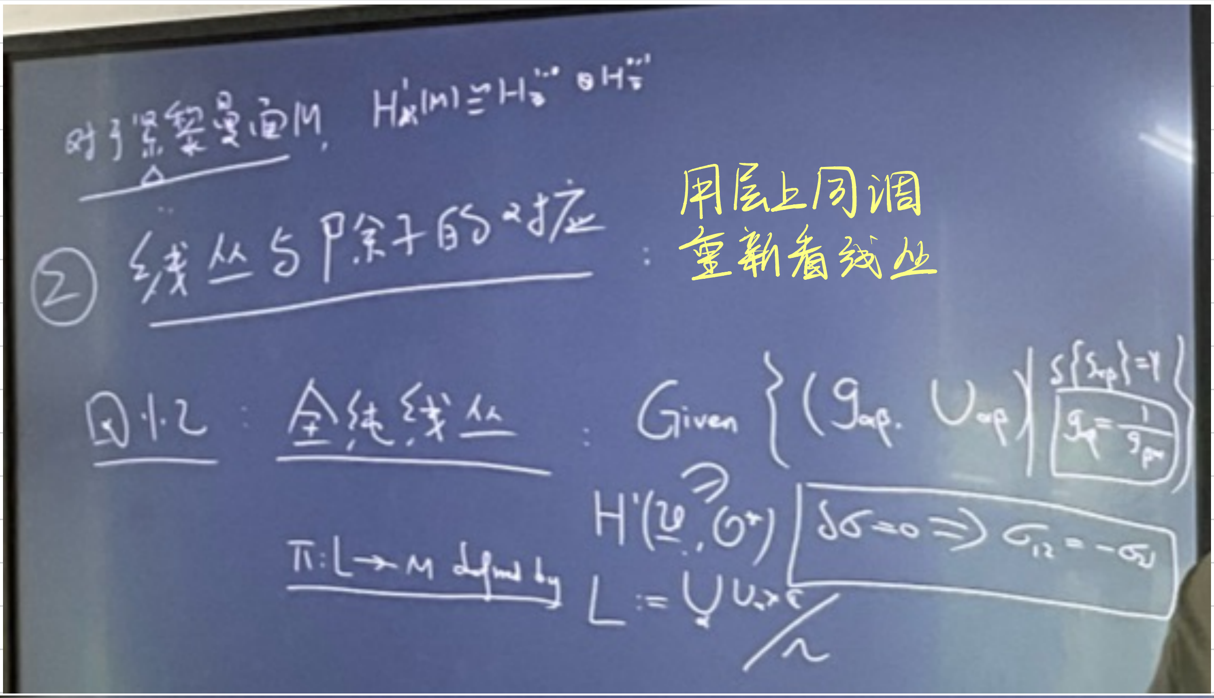
$$0 \to H^0(\underline{\mathfrak{U}}, \mathcal{O}) \xrightarrow{i} \underline{H^0(\underline{\mathfrak{U}}, C^\infty)} \xrightarrow{\bar\partial} H^0(\underline{\mathfrak{U}}, Z^{0,1}_{\bar\partial})$$

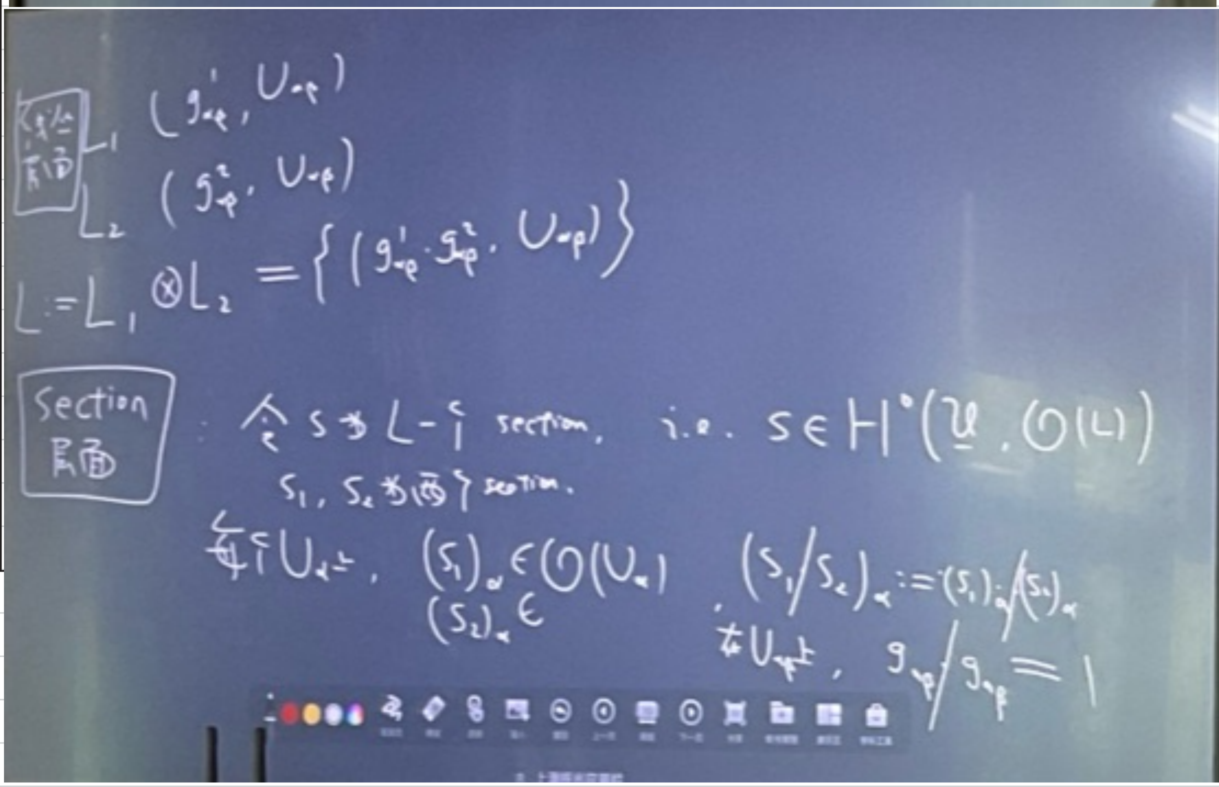
$$\xrightarrow{\delta^*} H^1(\underline{\mathfrak{U}}, \mathcal{O}) \longrightarrow 0$$

$$H^{0,1}_{\bar\partial}(M) = Z^{0,1}_{\bar\partial}(M) \big/ \bar\partial C^\infty(M) \cong H^1_{\check{C}ech}(\underline{\mathfrak{U}}, \mathcal{O})$$

Mittag-Leffler：若 $\Omega \subseteq \mathbb{C}$，开，连通，则 $H^1(\underline{\mathfrak{U}}, \mathcal{O}) = 0$.


0 → H¹(𝔘, 𝒪) →i H¹(𝔘, C^∞) →∂̄ H¹(𝔘, Z^{0,1}_∂̄)
→δ H¹(𝔘, 𝒪) → 0
H^{0,1}_∂̄(M) = Z^{0,1}_∂̄(M) / ∂̄C^∞(M) ≅ H¹_Čech(𝔘, 𝒪)
Mittag-Leffler: 若Ω⊆ℂ, 开, 连通, 则H¹(𝔘, 𝒪) = 0
对于紧黎曼面M, H¹_dR(M) ≅ H^{1,0}_∂̄ ⊕ H^{0,1}_∂̄


对于紧黎曼面M, H¹_dR(M) ≅ H^{1,0}_∂̄ ⊕ H^{0,1}_∂̄
用层上同调
重新看线丛
② 线丛与除子的对应
回1.2 全纯线丛: Given {(g_αβ, U_αβ) | δ{g_αβ} = 1, g_αβ = 1/g_βα}
H¹(𝔘, 𝒪*)
δσ = 0 ⇒ σ12 = −σ21
π: L → M defined by L := ∪ U_α × ℂ / ~


线丛层面
L1 (g¹_αβ, U_αβ)
L2 (g²_αβ, U_αβ)
L := L1 ⊗ L2 = {(g¹_αβ · g²_αβ, U_αβ)}
Section 层面
令s为L一个section, i.e. s ∈ H⁰(𝔘, 𝒪(L))
s1, s2为两个section.
每个U_α上, (s1)_α ∈ 𝒪(U_α), (s2)_α ∈
(s1/s2)_α := (s1)_α / (s2)_α
在U_αβ上, g_αβ / g_αβ = 1

若 $s_1, s_2 \in H^0(\underline{\mathcal{U}}, \mathcal{O}(L))$

定义 $\left(s_1/s_2\right)_\alpha := (s_1)_\alpha / (s_2)_\alpha \in \mathcal{M}^*(U_\alpha)$

且 在 $U_{\alpha\beta}$ 上，$\left(s_1/s_2\right)_\beta = \dfrac{(s_1)_\alpha \cdot g_{\alpha\beta}}{(s_2)_\alpha \cdot g_{\alpha\beta}} = \left(s_1/s_2\right)_\alpha$

i.e. $s_1/s_2 =: f \in \mathcal{M}^*(M)$、

反过来：固定一个 $s_0 \in H^0(\underline{\mathcal{U}}, \mathcal{O}(L))$. 任取 $f \in \mathcal{M}^*(M)$

进一步设 $f \cdot s_0$ 只有零点，

这时，$(f \cdot s_0)_{U_\alpha} = f \cdot (s_0)_{U_\alpha} \in \mathcal{O}(U_\alpha)$

且 $U_{\alpha\beta}$ 上，$(f \cdot s_0)_{U_\alpha} = f \cdot (s_0)_{U_\beta} \cdot g_{\beta\alpha}$

$= (f s_0)_{U_\beta} \cdot g_{\beta\alpha}$

i.e. $f \cdot s_0 \in H^0(\underline{\mathcal{U}}, \mathcal{O}(L))$

$$\Rightarrow H^0(\underline{\mathcal{U}}, \mathcal{O}(L)) = \{ f \in \mathcal{M}^*(M) \mid \underset{\substack{\parallel \\ \text{除极零点 一般都正}}}{(f \cdot s_0)} \geq 0 \}$$

$(f \cdot s_0) = (f) + (s_0)$ 证明：在每个 U_α 上，$(f \cdot s_0)|_{U_\alpha} = (f)|_{U_\alpha} + (s_0)|_{U_\alpha}$

如：$f(z) = \dfrac{(z-1)^2}{z}$ $\qquad s_0 = (z-2)^3$

$(f \cdot s_0) = \left(\frac{1}{z} \cdot (z-1)^2 \cdot (z-2)^3\right) = 3\{2\} + 2 \cdot \{1\} - 1 \cdot \{0\}$

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$$

$$\hookrightarrow H^0(\underline{\mathfrak{U}}, \mathcal{M}^*) \longrightarrow H^0(\underline{\mathfrak{U}}, \mathcal{M}^*/\mathcal{O}^*) \to H^1(\underline{\mathfrak{U}}, \mathcal{O}^*)$$

Lem1: $\mathrm{Div}(M) := \left\{ \sum_{i=1}^{N} n_i p_i \;\middle|\; p_i \in M,\ n_i \in \mathbb{Z} \right\}$

群，加法

$\varphi: \mathrm{Div}(M) \cong H^0(\underline{\mathfrak{U}}, \mathcal{M}^*/\mathcal{O}^*)$

Pf: 映射: $\varphi: D = \sum_{n_i \in \mathbb{Z}} n_i p_i \longmapsto \left\{ ([f_\alpha], U_\alpha) \;\middle|\; \begin{array}{l} f_\alpha \in \mathcal{M}^*(U_\alpha) \\ (f_\alpha) = \sum_{p_i \in U_\alpha} n_i p_i \end{array} \right\}$

well-defined: 取 $\underline{\mathfrak{U}}$ 足够细，每个 p_j 只含于一个 U_j.

$$f_j(z) := \begin{cases} (z - z_{p_j})^{n_j} \cdot g_j(z), \quad g_j(z) \in \mathcal{O}^*(U_j) & \text{if } p_j \in U_j \\ g_j(z) \in \mathcal{O}^*(U_j) & \text{if } \forall p_i \notin U_j \text{ 任何} \end{cases}$$

$\{([f_j], U_j)\}$ 即为所求.

$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$

$\hookrightarrow H^0(\underline{\mathfrak{U}}, \mathcal{M}^*) \longrightarrow H^0(\underline{\mathfrak{U}}, \mathcal{M}^*/\mathcal{O}^*) \to H^1(\underline{\mathfrak{U}}, \mathcal{O}^*)$

Lem1: $\mathrm{Div}(M) := \left\{ \sum_{i=1}^{N} n_i p_i \;\middle|\; p_i \in M,\ n_i \in \mathbb{Z} \right\}$

Weil divisor 群，加法 Cartier divisor

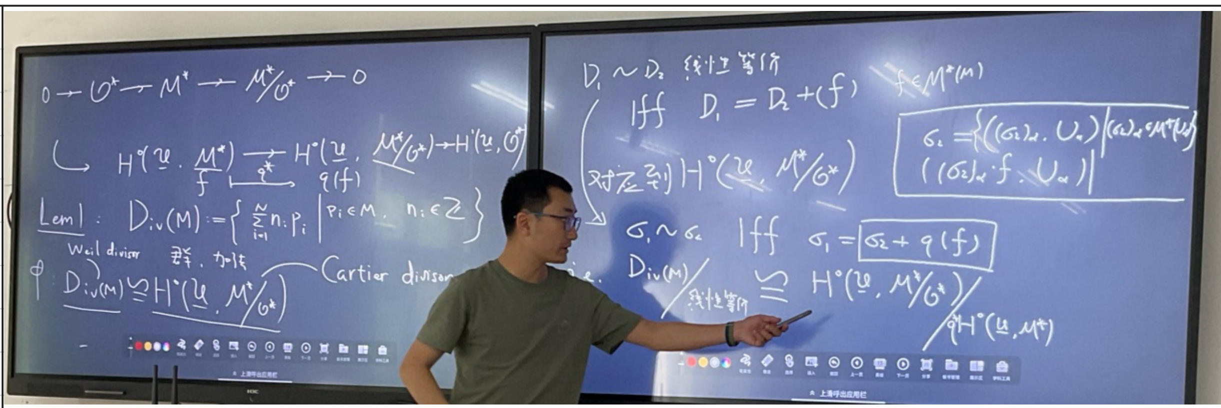
$\varphi: \mathrm{Div}(M) \cong H^0(\underline{\mathfrak{U}}, \mathcal{M}^*/\mathcal{O}^*)$

Pf: 映射: $\varphi: D = \sum_{n_i \in \mathbb{Z}} n_i p_i \longmapsto \left\{ ([f_\alpha], U_\alpha) \;\middle|\; \begin{array}{l} f_\alpha \in \mathcal{M}^*(U_\alpha) \\ (f_\alpha) = \sum_{p_i \in U_\alpha} n_i p_i \end{array} \right\}$

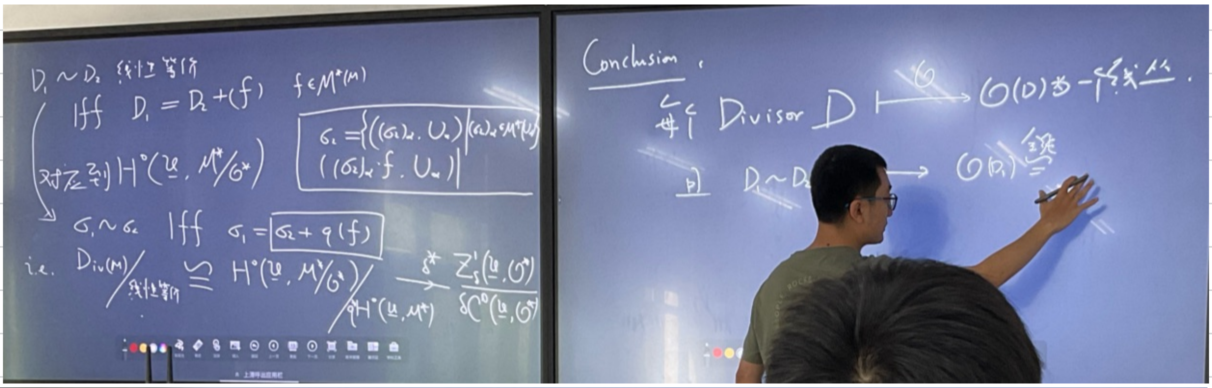
well-defined: 取 $\underline{\mathfrak{U}}$ 足够细，每个 p_j 只含于一个 U_j.

$f_j(z) := \begin{cases} (z - z_{p_j})^{n_j} \cdot g_j(z), \quad g_j(z) \in \mathcal{O}^*(U_j) & \text{if } p_j \in U_j \\ g_j(z) \in \mathcal{O}^*(U_j) & \text{if } \forall p_i \notin U_j \text{ 任何} \end{cases}$

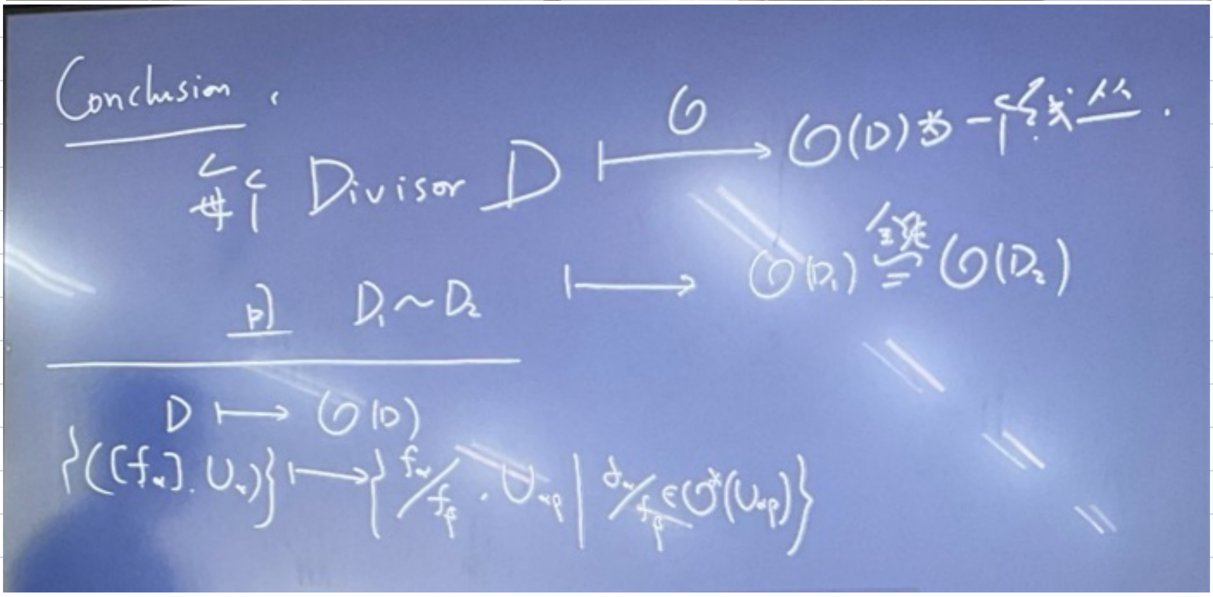
$\{([f_j], U_j)\}$ 即为所求.



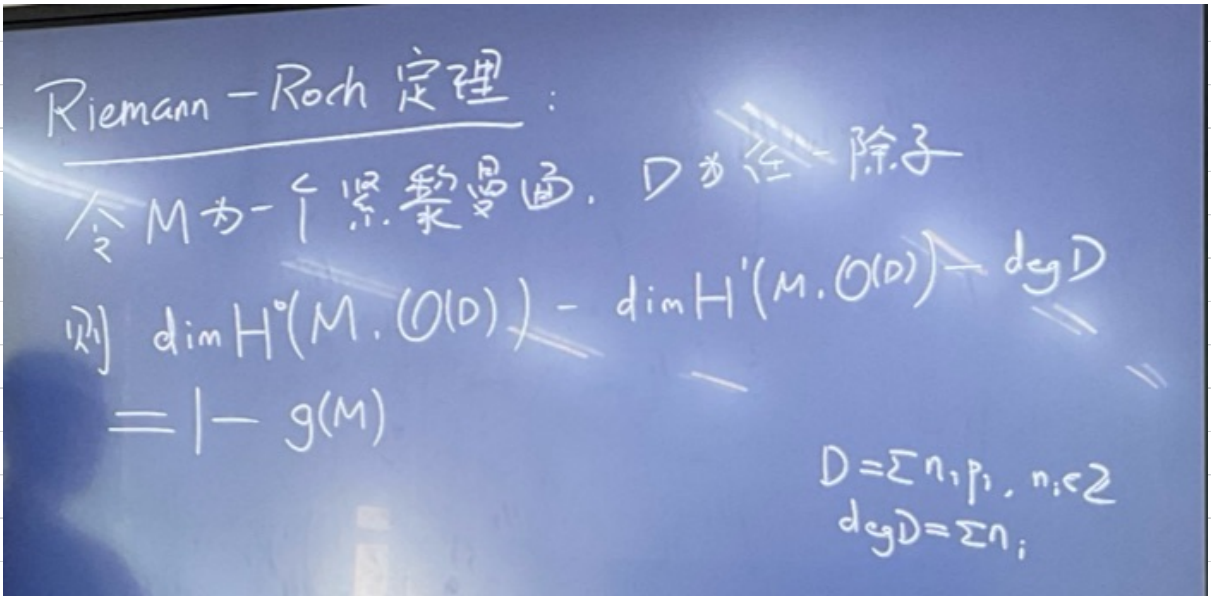
$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$
$H^0(\mathcal{U}, \mathcal{M}^*) \to H^0(\mathcal{U}, \mathcal{M}^*/\mathcal{O}^*) \to H^1(\mathcal{U}, \mathcal{O}^*)$
Lem1: $\mathrm{Div}(M) := \{ \sum_{i=1}^{N} n_i p_i \mid p_i \in M, n_i \in \mathbb{Z} \}$
Weil divisor 群，加法
Cartier divisor
$\mathrm{Div}(M) \cong H^0(\mathcal{U}, \mathcal{M}^*/\mathcal{O}^*)$
$D_1 \sim D_2$ 线性等价
iff $D_1 = D_2 + (f)$ $f \in \mathcal{M}^*(M)$
对应到 $H^0(\mathcal{U}, \mathcal{M}^*/\mathcal{O}^*)$
$\sigma_1 \sim \sigma_2$ iff $\sigma_1 = \sigma_2 + q(f)$
i.e. $\mathrm{Div}(M)/$线性等价 $\cong H^0(\mathcal{U}, \mathcal{M}^*/\mathcal{O}^*)/qH^0(\mathcal{U}, \mathcal{M}^*)$

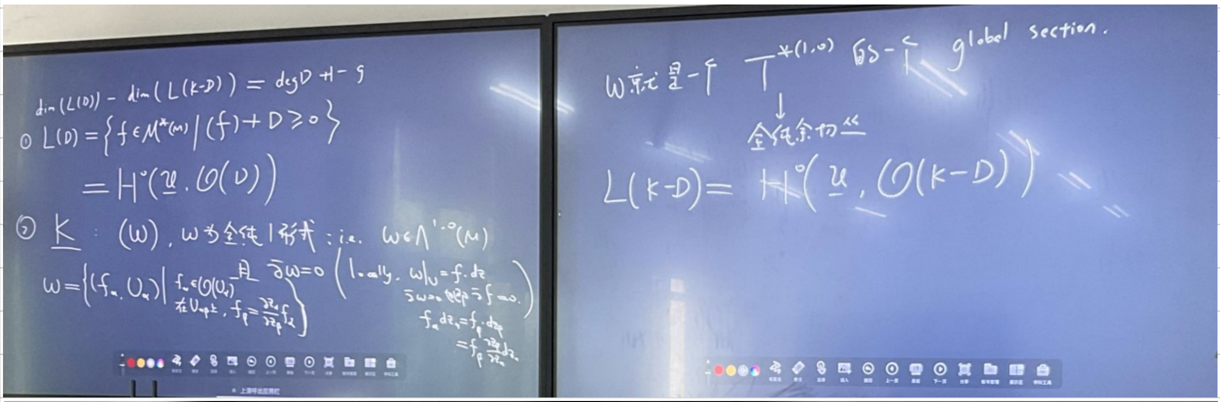


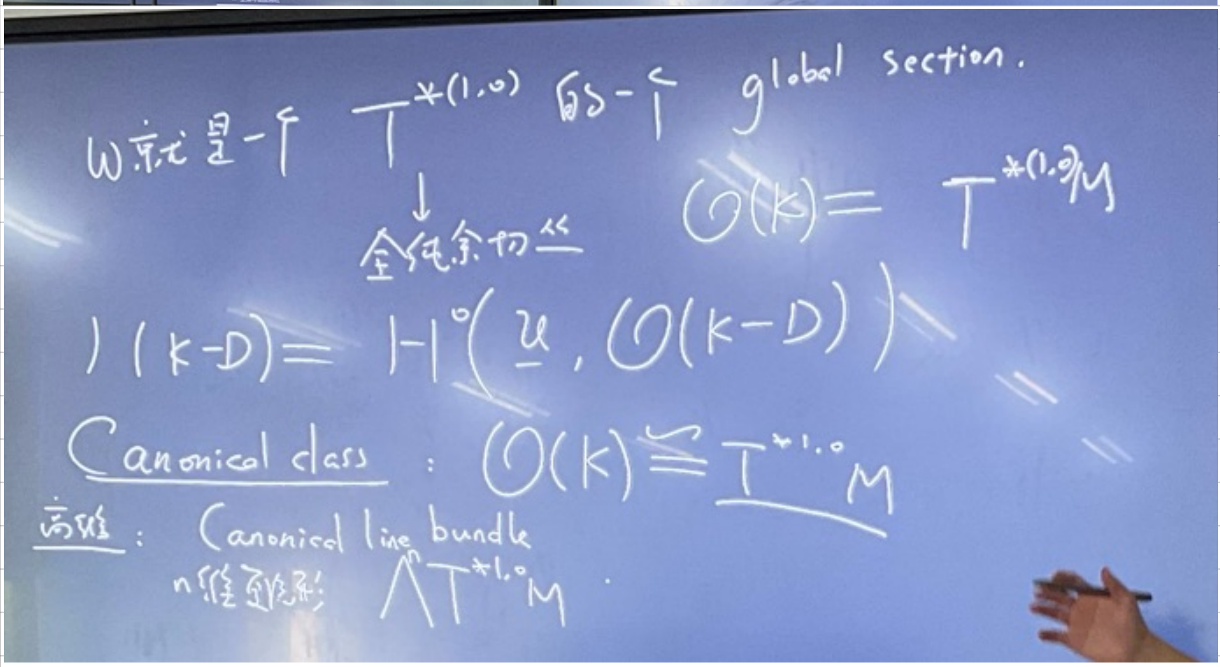
Conclusion:
每个 Divisor D
δ^* $Z^1(\mathcal{U}, \mathcal{O}^*)$



Conclusion:
每个 Divisor $D \xmapsto{\mathcal{O}} \mathcal{O}(D)$ 为一个线丛.
则 $D_1 \sim D_2 \longmapsto \mathcal{O}(D_1) \cong \mathcal{O}(D_2)$
$D \longmapsto \mathcal{O}(D)$
$\{([f_\alpha], U_\alpha)\} \longmapsto \{ f_\alpha / f_\beta , U_{\alpha\beta} \mid f_\alpha / f_\beta \in \mathcal{O}^*(U_{\alpha\beta}) \}$







Remark：
Canonical line bundle
上世纪记号混乱

$$h^0(M,\mathcal{O}(D)) := \dim_{\mathbb{C}} H^0(M,\mathcal{O}(D))$$

$$h^0(M,\mathcal{O}(D)) - h^0(M,\mathcal{O}(K-D)) = \deg D + 1 - g(M)$$

$$h^0(M,\mathcal{O}(D)) - h^1(M,\mathcal{O}(D)) - \deg D = 1 - \dim H_{\bar\partial}^{0,1}(M)$$

证明 $h^0(M.\mathcal{O}(D)) - h^1(M,\mathcal{O}(D))$

层正合列 $\Rightarrow$

$u:\Omega\to\mathbb{R}$ $\partial\bar\partial u = \frac{1}{4}\left[\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2\right]u$.

Hodge定理：紧曲面上一致椭圆方程的解空间有限维.

Hodge Decomposition：设M紧黎曼面，(1) $H_{\bar\partial}^{1,0} \cong \overline{H_{\bar\partial}^{0,1}}$

$\sigma \longmapsto \bar\sigma$

$\bar\partial\sigma = 0$

(2) $H_{dR}^1(M,\mathbb{C}) \cong H_{\bar\partial}^{1,0} \oplus H_{\bar\partial}^{0,1}$

$\overline{f\,dz} = \bar f\,d\bar z$

$\bar\partial(\bar f\,d\bar z) = \frac{\partial \bar f}{\partial \bar z}\,d\bar z\wedge d\bar z = 0$

推论 $\Rightarrow$ $\dim_{\mathbb{C}} H_{dR}^1 = 2g$

对于紧黎曼面 $\dim H_{\bar\partial}^{0,1} = g$

$H_1(M) = \{[\gamma] \mid \gamma:[0,1]\to M,\ \gamma(0)=\gamma(1),\ \gamma_1\sim\gamma_2 \text{ 若 } \gamma_1 \text{与} \gamma_2 \text{围住一个2维曲面片}\}$

$H_1(\mathbb{P}^1) = 0$

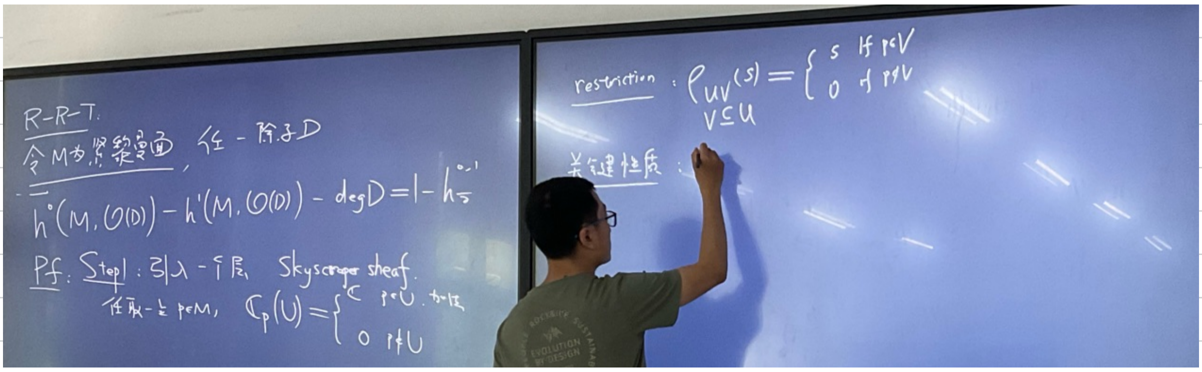
$\dim H_1(\mathbb{C}/\Lambda) = 2$

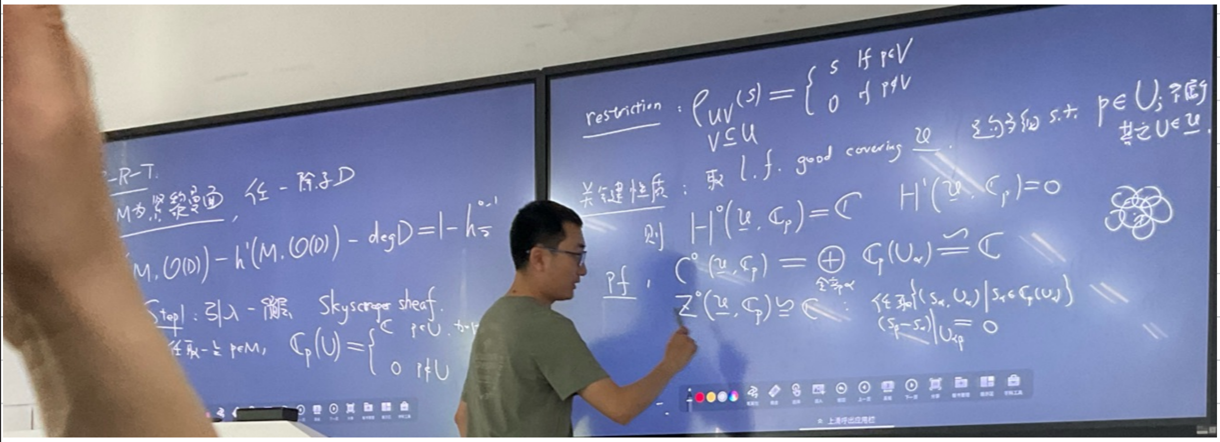
推论 $\Rightarrow$ $\dim_{\mathbb{C}} H_{dR}^1 = 2g$

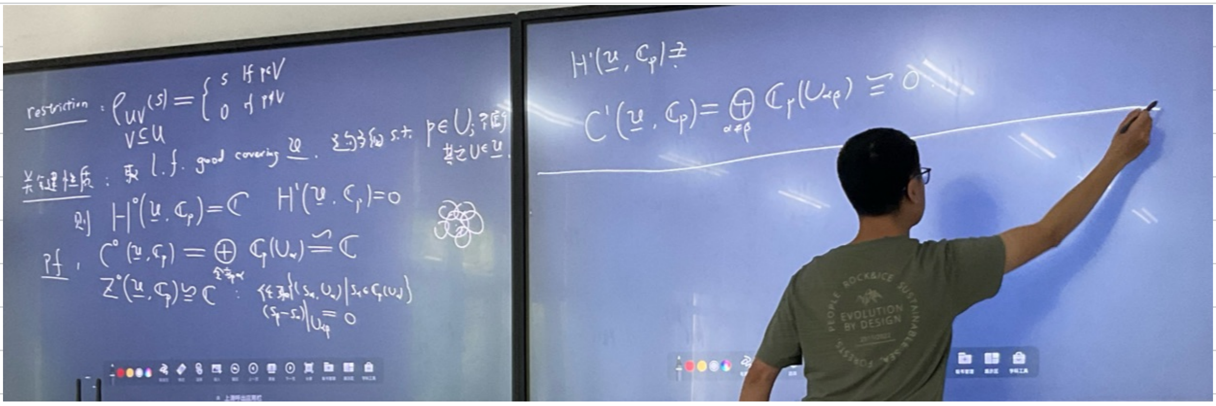
对于紧黎曼面 $\dim H_{\bar\partial}^{0,1} = g$

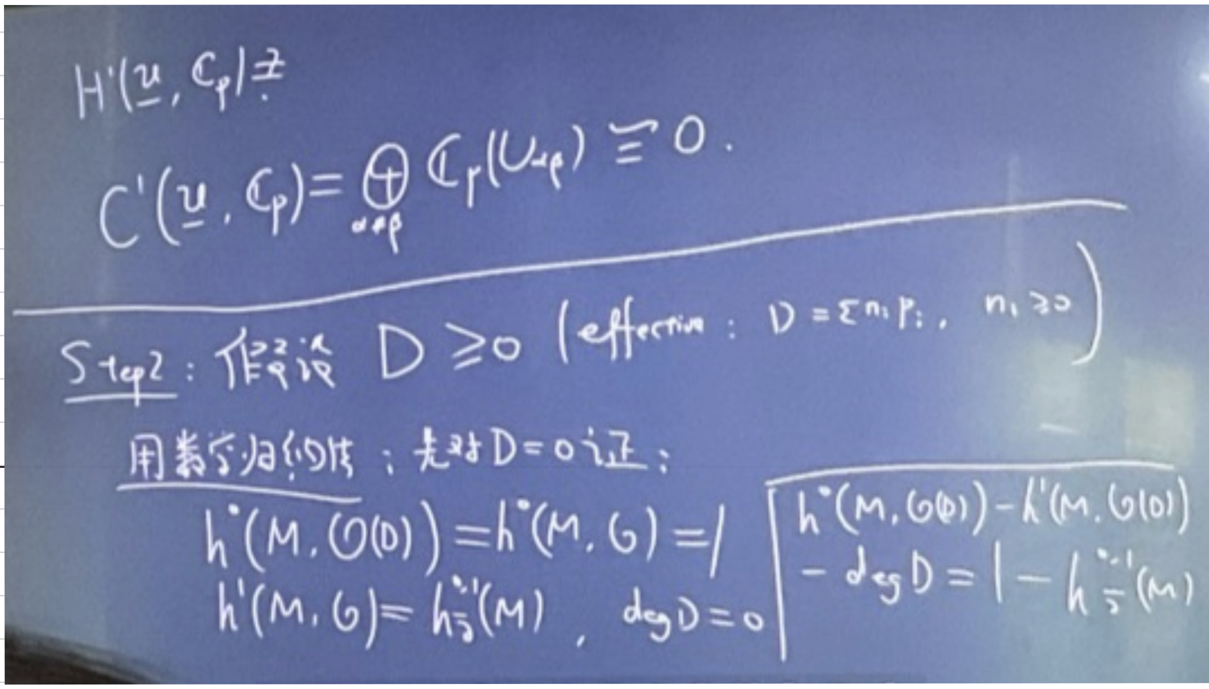
lem2：Serre dual Theorem

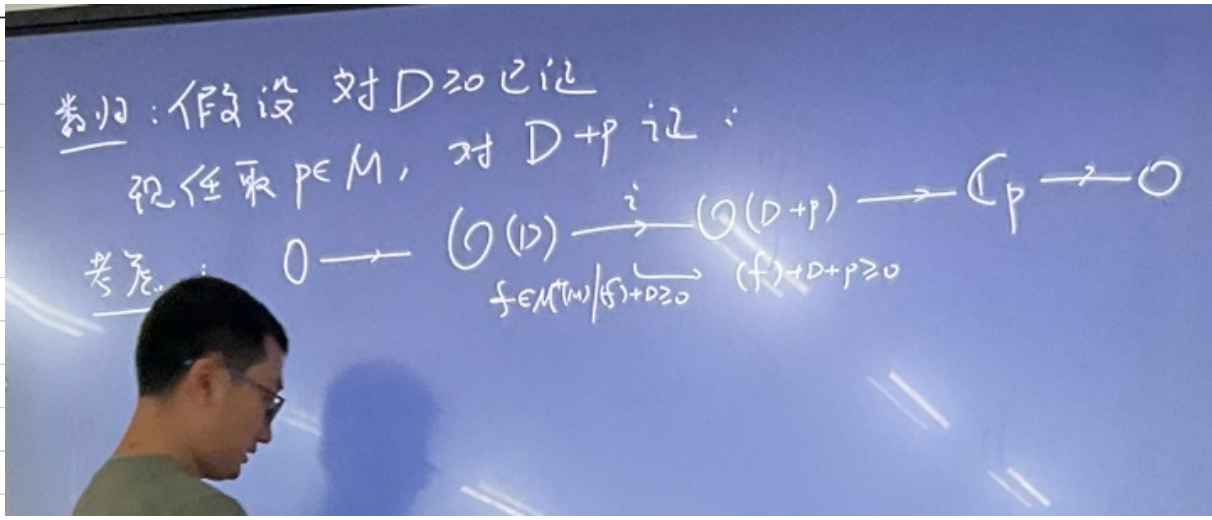
紧黎曼面，$H^0(M,\mathcal{O}(K-D)) \cong H^1(M,\mathcal{O}(D))$


R-R-T
令M为紧黎曼面，任一除子D
h°(M,O(D)) - h¹(M,O(D)) - degD = 1 - h^{0,1}
Pf: Step1: 引入一个层, Skyscraper sheaf.
任取一点 p∈M, C_p(U) = { C p∈U 加法; 0 p∉U
restriction: ρ_UV(s) = { s if p∈V; 0 if p∉V
V⊆U
关键性质:

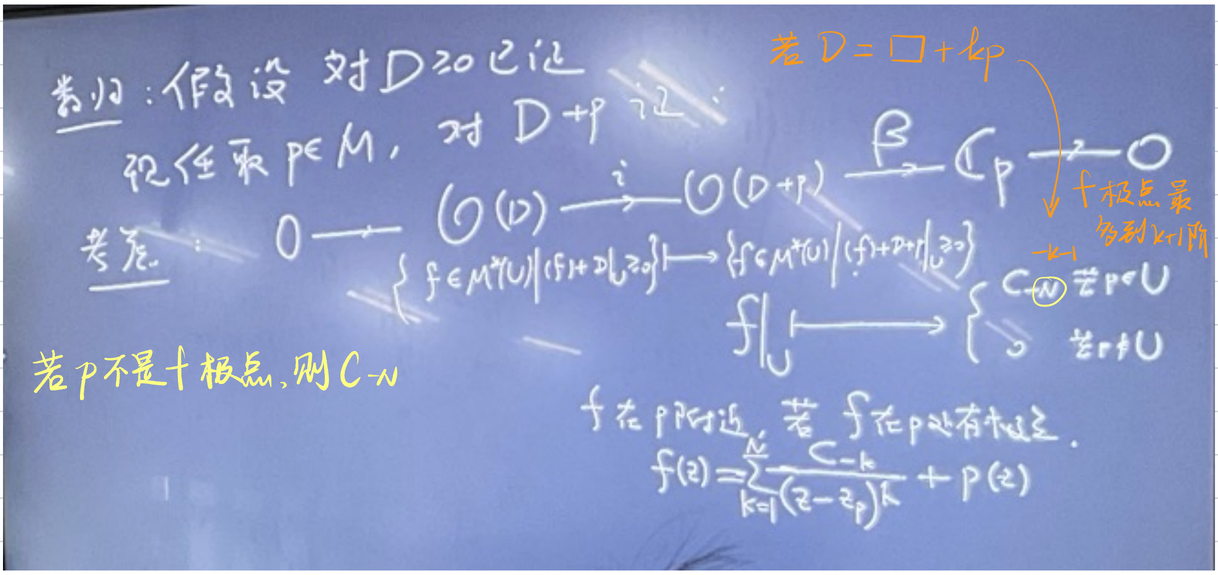

restriction: ρ_UV(s) = { s if p∈V; 0 if p∉V
V⊆U
关键性质: 取 l.f. good covering 𝔘
则 H°(𝔘, C_p) = C   H¹(𝔘, C_p) = 0
Pf: C°(𝔘, C_p) = ⊕ C_p(U_α) ≅ C
Z°(𝔘, C_p) ≅ C: 任取 {(s_α, U_α) | s_α ∈ C_p(U_α)}
(s_β - s_α)|U_αβ = 0


H¹(𝔘, C_p):
C¹(𝔘, C_p) = ⊕_{αβ} C_p(U_αβ) ≅ 0

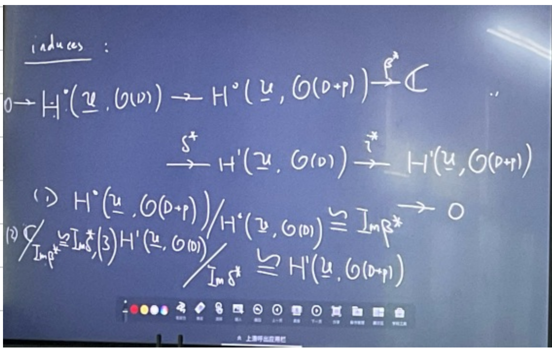

H¹(𝔘, C_p):
C¹(𝔘, C_p) = ⊕_{αβ} C_p(U_αβ) ≅ 0.
Step2: 假设 D ≥ 0 (effective: D = Σ n_i P_i, n_i ≥ 0)
用数学归纳法: 先对 D = 0 证:
h°(M, O(D)) = h°(M, O) = 1
h¹(M, O) = h^{0,1}(M), degD = 0
h°(M, O(D)) - h¹(M, O(D)) - degD = 1 - h^{0,1}(M)



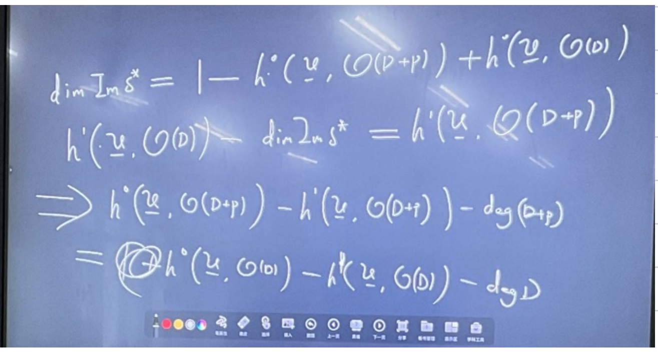
归纳：假设 对$D\geq 0$已证
现任取$p\in M$，对$D+p$证：
考虑：$0\longrightarrow \mathcal{O}(D)\xrightarrow{i}\mathcal{O}(D+p)\longrightarrow \mathbb{C}_p\longrightarrow 0$
$f\in \mathcal{M}^*(M)\,|\,(f)+D\geq 0$
$(f)+D+p\geq 0$



归纳：假设 对$D\geq 0$已证
现任取$p\in M$，对$D+p$证：
考虑：$0\longrightarrow \mathcal{O}(D)\xrightarrow{i}\mathcal{O}(D+p)\xrightarrow{\beta}\mathbb{C}_p\longrightarrow 0$
若$D=\square+kp$
f极点最多到k+1阶
$\{f\in \mathcal{M}^*(U)\,|\,(f)+D|_U\geq 0\}\longmapsto\{f\in \mathcal{M}^*(U)\,|\,(f)+D+p|_U\geq 0\}$
$f|_U\longmapsto\begin{cases} c_{-N} & 若p\in U\\ 0 & 若p\notin U\end{cases}$
若p不是f极点，则c_{-N}
f在p附近，若f在p处有极点，
$f(z)=\sum_{k=1}^{N}\frac{c_{-k}}{(z-z_p)^k}+p(z)$



induces：
$0\to H^0(\Sigma,\mathcal{O}(D))\to H^0(\Sigma,\mathcal{O}(D+p))\xrightarrow{\beta^*}\mathbb{C}$
$\xrightarrow{\delta^*}H^1(\Sigma,\mathcal{O}(D))\xrightarrow{i^*}H^1(\Sigma,\mathcal{O}(D+p))$
(1) $H^0(\Sigma,\mathcal{O}(D+p))/H^0(\Sigma,\mathcal{O}(D))\cong \mathrm{Im}\,\beta^*\to 0$
(2) $\mathbb{C}/\mathrm{Im}\,\beta^*\cong\mathrm{Im}\,\delta^*$
(3) $H^1(\Sigma,\mathcal{O}(D))/\mathrm{Im}\,\delta^*\cong H^1(\Sigma,\mathcal{O}(D+p))$



$\dim\mathrm{Im}\,\delta^*=1-h^0(\Sigma,\mathcal{O}(D+p))+h^0(\Sigma,\mathcal{O}(D))$
$h^1(\Sigma,\mathcal{O}(D))-\dim\mathrm{Im}\,\delta^*=h^1(\Sigma,\mathcal{O}(D+p))$
$\Longrightarrow h^0(\Sigma,\mathcal{O}(D+p))-h^1(\Sigma,\mathcal{O}(D+p))-\deg(D+p)$
$=h^0(\Sigma,\mathcal{O}(D))-h^1(\Sigma,\mathcal{O}(D))-\deg D$