

① 回顾上次做的事

② Riemann-Roch Theorem:

令 M 为 -1 连通紧黎曼面, 亏格为 g .

$$D = \sum_{i=1}^N a_i p_i, \quad a_i \in \mathbb{Z}$$

$$h^1(M, \mathcal{O}(D)) - h^0(M, \mathcal{O}(D)) = \deg D$$
$$= 1 - h^0(M, \mathcal{O}(D))$$

(1) Hodge 分解?

M 连通紧黎曼面, (a) $H^{1,0}(M) \cong H^{0,1}(M)$
 $\omega \mapsto \bar{\omega}$

(b) $H^{1,0}(M) \oplus H^{0,1}(M) \cong H^1(M, \mathbb{C})$ Note: 证明见复变 P9.
 $(\omega, [\beta]) \mapsto [\omega + \bar{\beta}] \quad \forall [\beta] \in H^{0,1}(M)$
这时, $\beta, \bar{\beta} \in [\beta]$ 中取 $\bar{\beta} \in H^{1,0}$ 则 $\omega + \bar{\beta} = \bar{\omega}$
 $d(\omega + \bar{\beta}) = 0$ (闭形式)
(c) $H^{1,0}(M) \cong H^1(M, \mathbb{C})$ 2-dim $\Lambda^1 = H^1 \Lambda^1$

(2) Serre Duality M 连通紧黎曼面
 $H^1(\mathcal{U}, \mathcal{O}(k-D)) \cong H^0(\mathcal{U}, \mathcal{O}(D))$

Dolbeault 定理

之前: M 连通黎曼面

$$H^1(M, \mathcal{O}) \cong H^{0,1}(M)$$

M 连通紧黎曼面 $H^1(M, \mathcal{O}(k)) \cong H^{0,1}(M) = \Lambda^{0,1}(M) / \bar{\partial} \Lambda^{0,1}(M)$

$$\begin{cases} [k] = T^{H^1(M)} \\ \mathcal{O}(k) \cong \mathbb{Z}_2 \end{cases}$$

Pf: $0 \rightarrow \mathcal{O}(k) \xrightarrow{i} \Lambda^{1,0} \xrightarrow{\bar{\partial}} \Lambda^{1,1} \xrightarrow{\bar{\partial}} 0$
 $\omega \mapsto \omega \wedge \Lambda^{1,1}$

正合性: $\ker \bar{\partial}^1(U) = \{ \omega \in \Lambda^{1,0}(U) \mid \bar{\partial} \omega = 0 \}$
 $\text{Im } \bar{\partial}^1(U) = \{ \omega \in \Lambda^{1,1}(U) \mid \forall p \in U, \exists \omega_i \in \Lambda^{1,0}(U_i) \text{ s.t. } \omega|_{U_i} = \bar{\partial} \omega_i, \tau \in \Lambda^{0,1}(U_i) \}$
 $\xrightarrow{\text{Poincaré Lemma}} \Lambda^{1,1}(U)$

induces: $H^1(\mathcal{U}, \Lambda^{1,0}) \xrightarrow{\bar{\partial}} H^1(\mathcal{U}, \Lambda^{1,1}) \xrightarrow{\bar{\partial}} H^1(\mathcal{U}, \mathcal{O}(k))$
 $\Rightarrow H^1(\mathcal{U}, \mathcal{O}(k)) \cong H^1(\mathcal{U}, \Lambda^{1,1}) / \bar{\partial} H^1(\mathcal{U}, \Lambda^{1,0})$

erre Duality M 连通紧黎曼面
 $H^1(\mathcal{U}, \mathcal{O}(k-D)) \cong H^0(\mathcal{U}, \mathcal{O}(D))$

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M 连通紧黎曼面 $H^1(M, \mathcal{O}(k)) \cong H^{0,1}(M) = \Lambda^{0,1}(M) / \bar{\partial} \Lambda^{0,1}(M)$

$$[k] = T^{H^1(M)} \\ \mathcal{O}(k) \cong \mathbb{Z}_2$$

Pf: $0 \rightarrow \mathcal{O}(k) \xrightarrow{i} \Lambda^{1,0} \xrightarrow{\bar{\partial}} \Lambda^{1,1} \xrightarrow{\bar{\partial}} 0$
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 $\Rightarrow H^1(\mathcal{U}, \mathcal{O}(k)) \cong H^1(\mathcal{U}, \Lambda^{1,1}) / \bar{\partial} H^1(\mathcal{U}, \Lambda^{1,0})$

$$\begin{array}{ccccccc} 0 & \rightarrow & C^0(\mathcal{U}, \mathcal{O}(k)) & \xrightarrow{i} & C^0(\mathcal{U}, \Lambda^{1,0}) & \xrightarrow{\bar{\partial}} & C^0(\mathcal{U}, \Lambda^{1,1}) \xrightarrow{\bar{\partial}} 0 \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 & \rightarrow & C^1(\mathcal{U}, \mathcal{O}(k)) & \xrightarrow{i} & C^1(\mathcal{U}, \Lambda^{1,0}) & \xrightarrow{\bar{\partial}} & C^1(\mathcal{U}, \Lambda^{1,1}) \xrightarrow{\bar{\partial}} 0 \end{array}$$

任取 $\tau \in Z^1(\mathcal{U}, \Lambda^{1,0})$

在 $\mathcal{U} \cap U_i$ 上, $\tau_i = \bar{\partial} \omega_i, \omega_i \in \Lambda^{1,0}(U_i)$

i.e. 令 $\mu = \{(\omega_i, \omega_j, U_{ij})\}, \tau = \bar{\partial} \mu$ i.e. 在 $\mathcal{U} \cap U_i$ 上, $\bar{\partial} \omega_i = \tau_i$

逆映射: $\omega_i - \omega_j \in \mathcal{O}(U_{ij})$
 $\Rightarrow \mu = \{(\omega_i - \omega_j, U_{ij})\} \in Z^1(\mathcal{U}, \mathcal{O}(k))$

$$\delta^*: \tau \mapsto \left\{ (\omega_i - \omega_j, U_{ij}) \mid \bar{\partial} \omega_i = \tau_i \text{ on } U_i \right\}$$

$H^1(\mathcal{U}, \Lambda^{1,0}) = 0$ Pf: 任取 $\{(\omega_i, \omega_j, U_{ij})\} \in C^1(\mathcal{U}, \Lambda^{1,0})$
任取 $\omega_j + \omega_k - \omega_l = 0$ on U_{jkl}

在 $\mathcal{U} \cap U_i$ 上, 令 $\omega_i := \sum_{j,k} p_{jk} \omega_j$ (p_{jk} 为适当常数)
则有 $\omega_i \in \Lambda^{1,0}(U_i)$
且 $\omega_i - \omega_j = \sum_k p_{jk} \omega_k - \sum_l p_{il} \omega_l, \sum_k p_{jk} = \sum_l p_{il} = 1$
 $= \sum_{k,l} (p_{jk} - p_{il}) \omega_k$
 $\Rightarrow \{(\omega_i, \omega_j, U_{ij})\} \in Z^1(\mathcal{U}, \mathcal{O}(k))$

在每一个 U_i 上, 令 $\omega_i := \sum_{j \neq i} p_j \omega_j$. $\{p_j\}$ 为互质分解.

仍有 $\omega_i \in \wedge^{1,0}(U_i)$

用 $\omega_i - \omega_j = \sum_{k \neq i,j} p_k \omega_k$

$\Rightarrow \{(\omega_{ij}, U_{ij})\}$

回到 Serre Duality. 令 M 连通紧致.

$H^0(U, \mathcal{O}(k-D)) \cong H^1(U, \mathcal{O}(D))$

Step 1: $\mathcal{O}(k-D) \cong \mathcal{Z}_S^{1,0}(D) \oplus \mathcal{K}^{1,0}(D)$

$\mathcal{O}(k-D)(U) = \{f \in \mathcal{M}^*(U) \mid (f, \omega_i) - D \geq 0\}$

$\mathcal{Z}_S^{1,0}(D)(U) = \{\omega \in \mathcal{Z}_S^{1,0}(U) \mid (\omega) - D \geq 0\}$

Coh. sheaf. \mathbb{Z} \nearrow \mathcal{F} locally constant.

Singular. $f: X \rightarrow Y$

$H^*(Y, \mathbb{Z}) \xrightarrow{f^*} H^*(X, \mathbb{Z})$

Sheaves on manifolds.

$L \cong X^* \mathbb{C}$

$\pi: \mathbb{C} \rightarrow \mathbb{P}^1$

$\pi^* \omega_1 = d\bar{z}$

$\pi^* \omega_2 = d\bar{z}$

Step 2: $H^0(U, \mathcal{Z}_S^{1,0}(D)) \times H^1(U, \mathcal{O}(D)) \xrightarrow{\text{Serre Duality}} H^1(U, \mathcal{O}(D))$

$(\omega, [(\sigma_{ij}, U_{ij}) \mid \sigma_{ij} \in \mathcal{O}(D)(U_{ij})]) \mapsto [(\sigma_{ij}, \omega_{ij}, U_{ij})]$

$\omega \in \mathcal{Z}_S^{1,0}(M), (\omega) - D \geq 0$

$\sigma_{ij} \omega \in \mathcal{Z}_S^{1,0}(U_{ij})$

$\mathcal{O}(k)(U_{ij})$

$\sigma_{ij} + \sigma_{jk} - \sigma_{ik} = 0 \sim U_{ijk} \Rightarrow \sigma_{ij} \omega + \sigma_{jk} \omega - \sigma_{ik} \omega = 0$

$\{(\tau_{ij}, U_{ij})\} \sim \{(\sigma_{ij}, U_{ij})\}$

$\tau_{ij} \omega = \sigma_{ij} \omega + (\mu_{ij} - \mu_j) \cdot \omega$

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Dolbeault: $H^1(U, \mathcal{O}(k)) \cong H^1_2(M)$

$[\begin{smallmatrix} \omega_{ij} - \omega_j, U_{ij} \\ \sigma_{ij} \in \mathcal{O}(D)(U_{ij}) \end{smallmatrix}] \mapsto [\tau]$

$\bigcup \{(\omega_{ij}, U_{ij}) \mid \omega_{ij} \in \mathcal{Z}_S^{1,0}(U_{ij})\} \mapsto \frac{\overline{\partial}(\sum_{i,j} p_j \omega_j)}{\mathbb{C}}$

令 $\{p_j\} \in \mathcal{U}$ 为互质分解.

则 $\sum_{i,j} \omega_{ij} p_j = \omega \in \wedge^{1,0}(U)$, 同时 $\omega - \omega_j = \omega_{ij}$

即 $\overline{\partial} \omega - \overline{\partial} \omega_j = 0$

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Dolbeault: $H^1(U, \mathcal{O}(k)) \cong H^1_2(M)$

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Remark: 上述过程为 Serre Duality 的推广.

$$H^i(\mathcal{U}, \mathbb{Z}_s^{(-D)} \times H^i(\mathcal{U}, \mathcal{O}(D))) \xrightarrow{\omega \mapsto \left(\sum_{j=1}^s p_j \cdot \sigma_j \cdot \omega \right)} \mathbb{C}$$

induces a dual map

$$i_D^*: H^i(\mathcal{U}, \mathbb{Z}_s^{(-D)}) \rightarrow H^i(\mathcal{U}, \mathcal{O}(D))^*$$

$$\omega \mapsto i_D^*(\omega) = \sum_M \tau \wedge \omega$$

Thm: i_D^2 - 同构

Pf: 应用数学归纳法.

对 $D=0$ 证: $i_0: H^i(\mathcal{U}, \mathcal{O}(0)) \rightarrow [H^i(\mathcal{U}, \mathcal{O})]^* \rightarrow (H_0^{i+1}(M))^*$

$$\omega \mapsto i_0(\omega) = \int_M \left(\sum_{j=1}^s p_j \sigma_j \right) \wedge \omega \mapsto i_0(\omega)(\tau) = \int_M \tau \wedge \omega$$

$$H^i(\mathcal{U}, \mathcal{O}) \cong H_0^{i+1}(M)$$

$$\left\{ \left(\sigma_{ij}, U_{ij} \right) \mid \sigma_{ij} \in \mathcal{O}(U_{ij}) \right\} \mapsto \left(\sum_{j=1}^s p_j \sigma_j \right)$$

$$[H^i(\mathcal{U}, \mathcal{O})]^* \rightarrow (H_0^{i+1}(M))^*$$

$$\mapsto i_0(\omega)(\tau) = \int_M \tau \wedge \omega$$

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$$H^i(\mathcal{U}, \mathcal{O}) \cong H_0^{i+1}(M)$$

$$\left\{ \left(\sigma_{ij}, U_{ij} \right) \mid \sigma_{ij} \in \mathcal{O}(U_{ij}) \right\} \mapsto \left(\sum_{j=1}^s p_j \sigma_j \right)$$

Hodge 分解

当 M 紧且

$$H_0^{i+1}(M) \cong (H_0^{i+1}(M))^*$$

$$H_0^{i+1}(M) \xrightarrow{\omega} [H_0^{i+1}(M)]^*$$

$$i_0(\omega) \xrightarrow{\tau} \int_M \tau \wedge \omega$$

假设 $D \geq 0$, 且 $\deg D = k$, Serre 对偶定理.

$$0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D+p) \rightarrow \mathcal{C}_p \rightarrow 0$$

$$0 \rightarrow \mathcal{O}(k-D) \rightarrow \mathcal{O}(k-D) \rightarrow \mathcal{C}_p \rightarrow 0$$

induces: $0 \rightarrow H^i(\mathcal{U}, \mathcal{O}(k-D+p)) \rightarrow H^i(\mathcal{U}, \mathcal{O}(k-D)) \rightarrow H^i(\mathcal{U}, \mathcal{C}_p)$

$$(H^i(\mathcal{U}, \mathcal{C}_p))^* \rightarrow (H^i(\mathcal{U}, \mathcal{O}(D+p)))^* \rightarrow (H^i(\mathcal{U}, \mathcal{O}(D)))^* \rightarrow (H^i(\mathcal{U}, \mathcal{C}_p))^*$$

$$[H^i(\mathcal{U}, \mathcal{C}_p)]^* \xrightarrow{\delta^*} [H^i(\mathcal{U}, \mathcal{O}(D))]^* \xrightarrow{\delta^*} [H^i(\mathcal{U}, \mathcal{O}(D+p))]^* \xrightarrow{\delta^*} [H^i(\mathcal{U}, \mathcal{C}_p)]^*$$

$$\{v: H^i(\mathcal{U}, \mathcal{O}(D)) \rightarrow \mathbb{C}\}$$

$$\{l: H^i(\mathcal{U}, \mathcal{C}_p) \rightarrow \mathbb{C}\}$$

$$v \mapsto (\delta^*)^* v: H^i(\mathcal{U}, \mathcal{C}_p) \rightarrow \mathbb{C}$$

$$a \mapsto \gamma(\delta^* v)$$

$$A \xrightarrow{f} B \xrightarrow{g} C \quad \ker g = \text{Im } f$$

$$C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \quad \ker f^* = \text{Im } g^* \text{ 吗?}$$

$$b^* \in \ker f^* \Leftrightarrow b^* \text{ 在 } \ker g \text{ 上取值全为零}$$

$$\forall a \quad f^*(b^*)(a) = b^*(f(a)) = 0$$

$$b^* \in \ker f^* \Leftrightarrow b^* \text{ 在 } \underset{\text{Im } g}{\text{Im } f} \text{ 上取零}$$

$$C^*(gcb) = b^*(cb)$$

$$g^*(C^*(cb)) = C^*(gcb)$$

$$\text{令 } C^*(gcb_i) = b^*(cb_i)$$

假设 $D \geq 0$, 且 $\deg D = k$, Serre 对偶定理.

$$0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D+p) \rightarrow \mathcal{C}_p \rightarrow 0$$

$$0 \rightarrow \mathcal{O}(k-D) \rightarrow \mathcal{O}(k-D) \rightarrow \mathcal{C}_p \rightarrow 0$$

induces: $0 \rightarrow H^i(\mathcal{U}, \mathcal{O}(k-D+p)) \rightarrow H^i(\mathcal{U}, \mathcal{O}(k-D)) \rightarrow H^i(\mathcal{U}, \mathcal{C}_p)$

$$0 \rightarrow [H^i(\mathcal{U}, \mathcal{O}(D+p))]^* \rightarrow [H^i(\mathcal{U}, \mathcal{O}(D))]^* \rightarrow [H^i(\mathcal{U}, \mathcal{C}_p)]^*$$

黎曼曲面上的 解析几何

③ Branched Covering Map (分枝覆盖)

Def: (Covering Space) 称 $f \rightarrow \text{covering map}$
 若 $\forall q \in N, \exists$ 开邻域 $q \in V \subseteq N, f(V) = \coprod_{U_n \subseteq M} U_n$
 且 $f|_{U_n}$ 均为 U_n 与 $f(U_n)$ 同胚 map.

Def: 称 $f \rightarrow \text{proper map}$, 若 \forall 紧子集 $K \subseteq N$, $f^{-1}(K)$ 也是紧子集.

基本作图: 令 M, N 为连通黎曼面.
 $f: M \rightarrow N$

Def: (local homeomorphism) 称 f 为 l.h.
 若 $\forall u \in M, \exists$ 开邻域 $u \in U \subseteq M, f|_U$ 为 U 与 $f(U)$ 同胚.

Def: (Covering Space) 称 $f \rightarrow \text{covering map}$
 若 $\forall q \in N, \exists$ 开邻域 $q \in V \subseteq N, f(V) = \coprod_{U_n \subseteq M} U_n$
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Def: 称 $f \rightarrow \text{proper map}$, 若 \forall 紧子集 $K \subseteq N$, $f^{-1}(K)$ 也是紧子集.

Thm: M, N 连通, 黎曼面, $f: M \xrightarrow{\text{Continuous}} N$
 则: 一方面, f 是 covering map $\Rightarrow f$ local homeomorphism
 另一方面, 若 f 是 proper local homeomorphism
 则 f 是 covering map.

Pf: 任取 $q \in N, f^{-1}(\{q\}) = \{p_1, \dots, p_n\}$
 在 $\forall p_i, \exists U_i, f|_{U_i} \cong U_i \cong f(U_i)$, 令 $V = \bigcap_{i=1}^n f(U_i)$
 且 $V \ni q, f^{-1}(V) = \bigcup_{i=1}^n U_i \cap f^{-1}(V)$

2): 加入解析性

$f(z): \mathbb{C} \xrightarrow{\phi} \mathbb{C}$
 $z \mapsto z^k$
 令 $w_i: \mathbb{C} \rightarrow \mathbb{C}$
 $p_i \mapsto z = p_i$
 $w_i \circ f \circ w_i^{-1}(w_i) = w_i$
 $w_i \circ f \circ w_i^{-1}(w_i) = (w_i, g^k) = (w_i)^k$

2): 加入解析性

$f(z): M \xrightarrow{\phi} N$
 $p \mapsto q$
 令 $\varphi: U \rightarrow \mathbb{C}$
 $p \mapsto 0$
 $\psi: V \rightarrow \mathbb{C}$
 $q \mapsto 0$
 $\psi \circ f \circ \varphi^{-1}(z) = z^k g(z)$, 令 $\varphi: U \rightarrow \mathbb{C}$
 $p \mapsto w$
 $\varphi \circ \varphi^{-1}(z) = z \cdot g(z)^{1/k}$
 $\Rightarrow \psi \circ f \circ \varphi^{-1}(w) = w^k$

首先: 解析 model $f(z) = z^k$

换句话说, $z^k: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$ 是一个 k 叶的 covering map.

Def: 称 $f: M \rightarrow N$ 是一个 branched covering (分枝覆盖)
 若 $\forall p \in M, \exists$ 开邻域 $p \in U \subseteq M, f|_{U \setminus \{p\}} \cong U \setminus \{p\} \rightarrow f(U) \setminus \{f(p)\}$ 是 covering map.

$$\psi \circ f \circ \varphi^{-1}(w) = (\psi \circ f \circ \varphi^{-1})(\varphi \circ \varphi^{-1}(w)) = \psi \circ f \circ \varphi^{-1}$$

$$[h(z)]^k = \psi \circ f \circ \varphi^{-1}(z) = z^k g(z) = \varphi \circ \varphi^{-1}(z)$$

