

黎曼面解析映为射影代数曲线

代数曲线  $M \subset \mathbb{C}^2 \hookrightarrow \mathbb{C} \cup \mathbb{C}P^1$   
 $M = \{p \mid v_1 v_2 = 0\}$

定理:  $M$  连通, R.S. 亏格  $g$   
 $(2) \Rightarrow$  重:  $M \xrightarrow{\text{嵌入}} \mathbb{C}P^1$   
 $R-R \Rightarrow h^1(M, \mathcal{O}(1)) = g+2$   
 $f_1, \dots, f_{g+2}$

定理1. 设  $M$  是连通黎曼面,  
 有亚纯函数  $z: M \rightarrow \mathbb{C}P^1, dg(z)=n$   
 $f: M \rightarrow \mathbb{C}P^1, dg(f)=m$   
 则: 有函数  $\sigma_1(z), \sigma_2(z), \dots, \sigma_n(z)$   
 s.t.  $f^n + (-1)^1 \sigma_1(z) f^{n-1} + \dots + (-1)^n \sigma_n(z) f^0 \equiv 0$

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推论: 任何紧连通黎曼面  $M$ ,  
 $\exists$  解析映射  $\phi: M \rightarrow \mathbb{C}P^1$   
 且  $\phi(M)$  为齐次多项式零点  
 且  $\exists$  有限子集  $S \subset \phi(M), S_i = \phi^{-1}(S_i)$   
 s.t.  $\phi|_{M \setminus S}: M \setminus S \rightarrow \phi(M) \setminus S$  的解析同构

证:  $\phi: M \rightarrow \mathbb{C}P^1$   
 $p \mapsto [z(p): f(p)]$

解析:  $\bigcup_i \phi_i: U_i \rightarrow \mathbb{C}^2$   
 $\{(z, f(z)) \mid z \in U_i\}$   
 $\phi_i, z$  解析,  $f_i$  解析

$z, f_i$  存在!  
 $h^1(M, \mathcal{O}(g+2x)) \geq 1/g+g+2 = 3$

由刚才证明,  $\exists \sigma_1(z), \dots, \sigma_n(z)$   
 $f^n + (-1)^1 \sigma_1(z) f^{n-1} + \dots + (-1)^n \sigma_n(z) \equiv 0$   
 把  $f = \frac{z}{z'}$ ,  $z = \frac{z'}{z'}$   
 $\Rightarrow (\frac{z'}{z'})^n + (-1)^1 \sigma_1(\frac{z'}{z'}) (\frac{z'}{z'})^{n-1} + \dots + (-1)^n \sigma_n(\frac{z'}{z'}) = 0$   
 $\Rightarrow \hat{p}(z', z') = 0$

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$$\frac{1}{2} S := \left\{ \hat{p}(x^1, x^2) = \frac{\partial f}{\partial x^1} = \frac{\partial f}{\partial x^2} = \frac{\partial f}{\partial x^3} = c \right\}$$

$$S_c = \{ df(p) = df(p) = 0 \}$$

在  $M \setminus S_c$  上,  $2 \leq f \leq -1$  非 0

$$\Rightarrow \text{rank Jac}(q)|_{M \setminus S_c} = 1$$

Next

Next:  $\exists \pi: M \rightarrow \mathbb{C}P^2$  浸入映射.

准备工作:

Riemann Existence Theorem:

设  $p_1, \dots, p_n$  是连通黎曼面  $M$  上  $N$  个互异的点.

则  $\forall g_1, \dots, g_n \in \mathbb{C}^*$

必存在亚纯函数  $h: M \rightarrow \mathbb{C}P^1$

满足:

- $h(p_i) = c_i, i=1, \dots, n$
- $dh(p_i) \neq 0, i=1, \dots, n$
- $h \in \mathcal{O}(M)$

任取  $q \in M \setminus \{p_1, \dots, p_n\}$

令  $D_1 := (2g+2n)q - 2(p_1 + \dots + p_n) + 3p_i$   $deg D_1 = 2g+3$

$D_2 := (2g+2n)q - 2(p_1 + \dots + p_n) + 2p_i$   $deg D_2 = 2g+2$

$\mathcal{O}(D_2) \subset \mathcal{O}(D_1)$

$H^0(M, \mathcal{O}(D_1)) \subseteq H^0(M, \mathcal{O}(D_2))$

且  $h^*(M, \mathcal{O}(D_1)) = 1-g+2g+3$   $i.e. \exists f_i \in H^0(M, \mathcal{O}(D_1)) \setminus H^0(M, \mathcal{O}(D_2))$

$h^*(M, \mathcal{O}(D_2)) = 1-g+2g+2$

$f_i + (2g+2n)q - (\dots 2p_i \dots) + 3p_i \geq 0$

$f_i + (2g+2n)q - (\dots 2p_i \dots) + 2p_i \neq 0$

$\therefore \begin{cases} h_i(p_i) = 1, \text{ 且 } \exists 1 \text{ 阶极点.} \\ h_i(p_j) = 0, \quad j \neq i \\ h_i(p_j) \neq 0 \\ dh_i(p_i) = 0, \quad j \neq i \end{cases}$

令  $h_i := \frac{f_i}{1+f_i}$

$dh_i = \frac{1}{1+f_i} df_i - \frac{f_i}{(1+f_i)^2} df_i$

$f_i + (2g+2n)q - (\dots 2p_i \dots) + 3p_i \geq 0$

$f_i + (2g+2n)q - (\dots 2p_i \dots) + 2p_i \neq 0$

i.e.  $\begin{cases} h_i(p_i) = 1, \text{ 且 } \exists 1 \text{ 阶极点.} \\ h_i(p_j) = 0, \quad j \neq i \\ dh_i(p_i) \neq 0 \\ dh_i(p_j) = 0, \quad j \neq i \end{cases}$

令  $h_i := \frac{f_i}{1+f_i}$

$dh_i = \frac{1}{1+f_i} df_i - \frac{f_i}{(1+f_i)^2} df_i$

$h := \sum_{j=1}^N c_j h_j$

$\begin{cases} h(p_i) = c_i \\ dh(p_i) \neq 0 \end{cases}$

嵌入定理 2: 令  $M$  连通, 黎曼面.

则  $\exists \pi: M \rightarrow \mathbb{C}P^2$  浸入映射.

Pf:  $\exists \psi: M \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \rightarrow \mathbb{C}P^2$

$\psi \mapsto (\varphi, [\cdot : h(\varphi)])$   $(\mathbb{C}P^1, \mathbb{C}P^1)$

$h$  为:  $\varphi$  有有限集  $S$ ,  $S$  为有限点集.

设  $S = \{p_1, \dots, p_n\}$ ,  $h$  为  $M$  上亚纯函数,  $h(p_i) = i$ .

命题 1.2: 令  $M$  是  $\mathbb{R}^n$  的流形

则  $\exists \psi: M \rightarrow \mathbb{R}^n$  使得  $\psi$  是嵌入.

Pf:  $\exists \psi: M \rightarrow \mathbb{R}^n \times \mathbb{R}^n$   $i: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$   
 $p \mapsto (\phi(p), [h(p)])$   $(\phi(p), [h(p)]) \in \mathbb{R}^n \times \mathbb{R}^n$

$h$  为:  $q$  有邻域  $S$ ,  $S$  为有限子集.  
 设  $S = \{p_1, \dots, p_n\}$ ,  $h$  为  $M$  上连续函数,  $h(p_i) = i$ .

$\psi$  是单射: 任取  $p, q \in M$ ,  $\psi(p) \neq \psi(q)$

若  $p \in M \setminus \phi^{-1}(S)$  再若  $q \in M \setminus \phi^{-1}(S)$ , 则  $\phi(p) \neq \phi(q)$   
 再若  $q \in \phi^{-1}(S)$ ,  $\phi(p) \in \phi(M) \setminus S$   
 $\phi(p) \neq S$

若  $p \in \phi^{-1}(S)$ , 且  $q \in \phi^{-1}(S)$

假设  $\phi(p) = \phi(q)$  则设  $p = p_i$   $q = p_j$

则  $h(p) = i$   $h(q) = j$ .

$\psi$  是单射: 任取  $p, q \in M$ ,  $\psi(p) \neq \psi(q)$

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 $\phi(p) \neq S$

若  $p \in \phi^{-1}(S)$ , 且  $q \in \phi^{-1}(S)$

假设  $\phi(p) = \phi(q)$  则设  $p =$

则  $h(p) = i$   $h(q) = j$ .

$\psi$  是同构: It suffices to show

$\psi$  是双射且  $\psi^{-1}$  连续 (i.e.  $\text{rank } J\psi(p) = n$ ).

再 i.e.  $\nabla \psi \neq 0$  且  $\psi(p), \psi(q), h(p), h(q) \neq 0$ .

若  $p \in M \setminus \phi^{-1}(S)$ ,  $\psi$  非退化.

若  $q \in \phi^{-1}(S)$ ,  $d\psi(p) \neq 0$ .