

2016 年数学(一) 真题解析

一、选择题

(1) 【答案】 (C).

【解】 $\int_0^{+\infty} \frac{dx}{x^a(1+x)^b} = \int_0^1 \frac{dx}{x^a(1+x)^b} + \int_1^{+\infty} \frac{dx}{x^a(1+x)^b}$,

由 $\lim_{x \rightarrow 0^+} x^a \cdot \frac{1}{x^a(1+x)^b} = 1$ 且 $\int_0^1 \frac{dx}{x^a(1+x)^b}$ 收敛得 $a < 1$,

再由 $\lim_{x \rightarrow +\infty} x^{a+b} \cdot \frac{1}{x^a(1+x)^b} = 1$ 且 $\int_1^{+\infty} \frac{dx}{x^a(1+x)^b}$ 收敛得 $a+b > 1$,

即 $a < 1$ 且 $a+b > 1$, 应选(C).

(2) 【答案】 (D).

【解】 $F(x) = \int f(x) dx = \begin{cases} (x-1)^2 + C, & x < 1, \\ x(\ln x - 1) + C + 1, & x \geq 1. \end{cases}$

取 $C=0$ 得 $f(x)$ 的一个原函数为 $F(x) = \begin{cases} (x-1)^2, & x < 1, \\ x(\ln x - 1) + 1, & x \geq 1. \end{cases}$ 应选(D).

(3) 【答案】 (A).

【解】 设 $y_1 = (1+x^2)^2 - \sqrt{1+x^2}$, $y_2 = (1+x^2)^2 + \sqrt{1+x^2}$,

由线性微分方程解的结构得 $y_2 - y_1 = 2\sqrt{1+x^2}$ 为 $y' + p(x)y = 0$ 的解,

代入得 $\frac{2x}{\sqrt{1+x^2}} + p(x) \cdot 2\sqrt{1+x^2} = 0$, 解得 $p(x) = -\frac{x}{1+x^2}$;

再由线性微分方程解的结构, 得 $\frac{y_1 + y_2}{2} = (1+x^2)^2$ 为 $y' + p(x)y = q(x)$ 的解, 代入得

$4x(1+x^2) - \frac{x}{1+x^2} \cdot (1+x^2)^2 = q(x)$, 解得 $q(x) = 3x(1+x^2)$, 应选(A).

(4) 【答案】 (D).

【解】 $f(0) = 0$, $\lim_{x \rightarrow 0^-} f(x) = 0$, $\lim_{x \rightarrow 0^+} f(x) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

由 $f(0) = f(0-0) = f(0+0) = 0$ 得 $f(x)$ 在 $x=0$ 处连续.

由 $\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} \frac{x}{x} = 1$ 得 $f'_-(0) = 1$;

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{n}}{x} = \frac{1}{n},$$

由 $\frac{1}{n+1} < x < \frac{1}{n}$ 得 $\frac{n}{n+1} < \frac{x}{1} < 1$, 从而 $\lim_{x \rightarrow 0^+} \frac{x}{n} = 1$, 于是 $f'_+(0) = 1$,

因为 $f'_-(0) = f'_+(0) = 1$, 所以 $f(x)$ 在 $x=0$ 处可导, 应选(D).

(5) 【答案】 (C).

【解】 由 A 与 B 相似可知, 存在可逆矩阵 P , 使得 $P^{-1}AP = B$.

对 $P^{-1}AP = B$ 两边取转置得 $P^T A^T (P^{-1})^T = B^T$, 或 $[(P^T)^{-1}]^{-1} A^T [(P^T)^{-1}] = B^T$,

即 A^T 与 B^T 相似,(A) 正确;

由 $P^{-1}AP = B$ 得 $P^{-1}A^{-1}P = B^{-1}$, 即 A^{-1} 与 B^{-1} 相似,(B) 正确;

由 $P^{-1}AP = B$ 及 $P^{-1}A^{-1}P = B^{-1}$ 得 $P^{-1}(A + A^{-1})P = B + B^{-1}$,

即 $A + A^{-1}$ 与 $B + B^{-1}$ 相似,(D) 正确, 应选(C).

(6) 【答案】 (B).

【解】 二次型的矩阵为 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,

$$\text{由 } |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)^2(\lambda - 5) = 0 \text{ 得}$$

矩阵 A 的特征值为 $\lambda_1 = 5, \lambda_2 = \lambda_3 = -1$,

二次型的规范形为 $f(x_1, x_2, x_3) = 5y_1^2 - y_2^2 - y_3^2$,

从而 $f(x_1, x_2, x_3) = 2$ 表示的曲面为 $5y_1^2 - y_2^2 - y_3^2 = 2$, 该曲面表示双叶双曲面, 应选(B).

(7) 【答案】 (B).

【解】 由 $X \sim N(\mu, \sigma^2)$ 得 $\frac{X - \mu}{\sigma} \sim N(0, 1)$,

$$p = P\{X \leqslant \mu + \sigma^2\} = P\left\{\frac{X - \mu}{\sigma} \leqslant \sigma\right\} = \Phi(\sigma),$$

则 p 随着 σ 的增加而增加, 应选(B).

(8) 【答案】 (A).

【解】 方法一 $X \sim B\left(2, \frac{1}{3}\right)$, $Y \sim B\left(2, \frac{1}{3}\right)$,

$$E(X) = E(Y) = \frac{2}{3}, D(X) = D(Y) = \frac{4}{9}, E(XY) = 1 \times 1 \times P\{X=1, Y=1\} = \frac{2}{9},$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{2}{9} - \frac{4}{9} = -\frac{2}{9},$$

$$\text{则 } \rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{D(X)} \cdot \sqrt{D(Y)}} = -\frac{2}{9} \times \frac{9}{4} = -\frac{1}{2}, \text{ 应选(A).}$$

方法二 $P\{X=0\} = C_2^0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^2 = \frac{4}{9}$,

$$P\{X=1\} = C_2^1 \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}, \quad P\{X=2\} = C_2^2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^0 = \frac{1}{9},$$

$$X \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \end{pmatrix}, \text{ 同理 } Y \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \end{pmatrix},$$

$$E(X) = \frac{2}{3}, E(X^2) = \frac{8}{9}, D(X) = \frac{8}{9} - \frac{4}{9} = \frac{4}{9}, E(Y) = \frac{2}{3}, D(Y) = \frac{4}{9}.$$

$$P\{XY=1\}=P\{X=1,Y=1\}=\frac{2}{9},$$

$$P\{XY=0\}=\frac{7}{9}, \text{即 } XY \sim \begin{pmatrix} 0 & 1 \\ \frac{7}{9} & \frac{2}{9} \end{pmatrix}.$$

$$E(XY)=\frac{2}{9}, \text{Cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{2}{9}-\frac{4}{9}=-\frac{2}{9},$$

$$\rho_{XY}=\frac{\text{Cov}(X,Y)}{\sqrt{D(X)} \cdot \sqrt{D(Y)}}=-\frac{2}{9} \times \frac{9}{4}=-\frac{1}{2}.$$

二、填空题

(9) 【答案】 $\frac{1}{2}$.

【解】

$$\lim_{x \rightarrow 0} \frac{\int_0^x t \ln(1+t \sin t) dt}{1 - \cos x^2} = \lim_{x \rightarrow 0} \frac{\int_0^x t \ln(1+t \sin t) dt}{\frac{1}{2}x^4} = \lim_{x \rightarrow 0} \frac{x \ln(1+x \sin x)}{2x^3} = \frac{1}{2}.$$

(10) 【答案】 $j + (y-1)k$.

【解】 $\text{rot } A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & xy & z \end{vmatrix} = j + (y-1)k.$

(11) 【答案】 $-dx + 2dy$.

【解】 将 $x=0, y=1$ 代入得 $z=1$.

$(x+1)z - y^2 = x^2 f(x-z, y)$ 两边关于 x 求偏导得

$$z + (x+1)z'_x = 2xf(x-z, y) + x^2 f'_1(x-z, y) \cdot (1 - z'_x),$$

将 $x=0, y=1, z=1$ 代入得 $z'_x(0, 1) = -1$;

$(x+1)z - y^2 = x^2 f(x-z, y)$ 两边关于 y 求偏导得

$$(x+1)z'_y - 2y = x^2 [f'_1(x-z, y)(-z'_y) + f'_y(x-z, y)],$$

将 $x=0, y=1, z=1$ 代入得 $z'_y(0, 1) = 2$, 故 $dz|_{(0,1)} = -dx + 2dy$.

(12) 【答案】 $\frac{1}{2}$.

【解】 方法一 $\arctan x = x - \frac{x^3}{3} + o(x^3)$, $\frac{1}{1+ax^2} = 1 - ax^2 + o(x^2)$,

$$\text{则 } \arctan x - \frac{x}{1+ax^2} = \left(a - \frac{1}{3}\right)x^3 + o(x^3),$$

再由 $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + o(x^3)$ 得

$$\frac{f'''(0)}{3!} = a - \frac{1}{3}, \text{解得 } a = \frac{1}{2}.$$

方法二 $f'(x) = \frac{1}{1+x^2} - \frac{1-ax^2}{(1+ax^2)^2}$,

$$f''(x) = -\frac{2x}{(1+x^2)^2} + \frac{6ax - 2a^2x^3}{(1+ax^2)^3},$$

$$f'''(x) = -\frac{2-6x^2}{(1+x^2)^3} - \frac{(6a-6a^2x^2)(1+ax^2)-6ax(6ax-2a^2x^3)}{(1+ax^2)^4},$$

所以 $f'''(0) = -2 + 6a$, 由 $-2 + 6a = 1$ 得 $a = \frac{1}{2}$.

(13) 【答案】 $\lambda^4 + \lambda^3 + 2\lambda^2 + 3\lambda + 4$.

$$\begin{aligned} \text{【解】 } & \left| \begin{array}{cccc} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 4 & 3 & 2 & \lambda + 1 \end{array} \right| = \lambda \cdot \left| \begin{array}{ccc} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 3 & 2 & \lambda + 1 \end{array} \right| + \left| \begin{array}{ccc} 0 & -1 & 0 \\ 0 & \lambda & -1 \\ 4 & 2 & \lambda + 1 \end{array} \right| \\ &= \lambda \left[\lambda \left| \begin{array}{cc} -1 & 0 \\ 2 & \lambda + 1 \end{array} \right| + \left| \begin{array}{cc} 0 & -1 \\ 3 & \lambda + 1 \end{array} \right| \right] + \left| \begin{array}{cc} 0 & -1 \\ 4 & \lambda + 1 \end{array} \right| \\ &= \lambda [\lambda(\lambda^2 + \lambda + 2) + 3] + 4 \\ &= \lambda^4 + \lambda^3 + 2\lambda^2 + 3\lambda + 4. \end{aligned}$$

(14) 【答案】 (8.2, 10.8).

$$\text{【解】 } P\left\{-u_{0.025} < \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} < u_{0.025}\right\} = 0.95,$$

$$\text{得 } P\left\{\bar{x} - \frac{\sigma}{\sqrt{n}}u_{0.025} < \mu < \bar{x} + \frac{\sigma}{\sqrt{n}}u_{0.025}\right\} = 0.95,$$

$$\text{由 } \bar{x} + \frac{\sigma}{\sqrt{n}}u_{0.025} = 10.8 \text{ 得 } \frac{\sigma}{\sqrt{n}}u_{0.025} = 10.8 - \bar{x} = 1.3, \text{ 从而 } \bar{x} - \frac{\sigma}{\sqrt{n}}u_{0.025} = 8.2,$$

故 μ 的置信度为 0.95 的双侧置信区间为 (8.2, 10.8).

三、解答题

$$\begin{aligned} \text{【解】 } & \iint_D x \, dx \, dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_2^{2(1+\cos\theta)} r^2 \cos\theta \, dr \\ &= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^4\theta + 3\cos^3\theta + 3\cos^2\theta) \, d\theta \\ &= \frac{16}{3} \int_0^{\frac{\pi}{2}} (\cos^4\theta + 3\cos^3\theta + 3\cos^2\theta) \, d\theta \\ &= \frac{16}{3} (I_4 + 3I_3 + 3I_2) = 5\pi + \frac{32}{3}. \end{aligned}$$

(16) 【证明】 (I) 微分方程 $y'' + 2y' + ky = 0$ 的特征方程为 $\lambda^2 + 2\lambda + k = 0$,

$$\text{解得 } \lambda_1 = -1 + \sqrt{1-k}, \quad \lambda_2 = -1 - \sqrt{1-k},$$

因为 $0 < k < 1$, 所以 $\lambda_1 < 0, \lambda_2 < 0$, 从而 $\int_0^{+\infty} e^{\lambda_1 x} \, dx$ 与 $\int_0^{+\infty} e^{\lambda_2 x} \, dx$ 都收敛.

该方程的通解为 $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$,

由 $\int_0^{+\infty} y(x) \, dx = C_1 \int_0^{+\infty} e^{\lambda_1 x} \, dx + C_2 \int_0^{+\infty} e^{\lambda_2 x} \, dx$, 得 $\int_0^{+\infty} y(x) \, dx$ 收敛.

(II) 方法一 由 $\lambda_1 < 0, \lambda_2 < 0$ 得 $\lim_{x \rightarrow +\infty} y(x) = \lim_{x \rightarrow +\infty} (C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}) = 0$,

$$\lim_{x \rightarrow +\infty} y'(x) = \lim_{x \rightarrow +\infty} (C_1 \lambda_1 e^{\lambda_1 x} + C_2 \lambda_2 e^{\lambda_2 x}) = 0,$$

$$\text{于是 } \int_0^{+\infty} y(x) dx = -\frac{1}{k} \left[\int_0^{+\infty} y''(x) dx + 2 \int_0^{+\infty} y'(x) dx \right],$$

$$\text{再由 } \int_0^{+\infty} y''(x) dx = y'(x) \Big|_0^{+\infty} = -y'(0) = -1,$$

$$\int_0^{+\infty} y'(x) dx = y(x) \Big|_0^{+\infty} = -y(0) = -1,$$

$$\text{得 } \int_0^{+\infty} y(x) dx = \frac{3}{k}.$$

方法二 将 $y(0) = 1, y'(0) = 1$ 代入 $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$ 得 $\begin{cases} C_1 + C_2 = 1, \\ \lambda_1 C_1 + \lambda_2 C_2 = 1. \end{cases}$

$$\text{解得 } C_1 = \frac{1 - \lambda_2}{\lambda_1 - \lambda_2} = \frac{2 + \sqrt{1 - k}}{2\sqrt{1 - k}}, \quad C_2 = \frac{\lambda_1 - 1}{\lambda_1 - \lambda_2} = \frac{-2 + \sqrt{1 - k}}{2\sqrt{1 - k}},$$

$$\text{故 } \int_0^{+\infty} y(x) dx = -\left(\frac{C_1}{\lambda_1} + \frac{C_2}{\lambda_2}\right) = \frac{3}{k}.$$

(17) 【解】 方法一 由 $\frac{\partial f(x, y)}{\partial x} = (2x + 1)e^{2x-y}$ 得

$$\begin{aligned} f(x, y) &= \frac{1}{2} e^{-y} \int (2x + 1) d(e^{2x}) + \varphi(y) = \frac{1}{2} e^{-y} [(2x + 1)e^{2x} - 2 \int e^{2x} dx] + \varphi(y) \\ &= x e^{2x-y} + \varphi(y). \end{aligned}$$

由 $f(0, y) = y + 1$ 得 $\varphi(y) = y + 1$, 于是 $f(x, y) = x e^{2x-y} + y + 1$.

$$\frac{\partial f(x, y)}{\partial y} = -x e^{2x-y} + 1,$$

$$\text{令 } P(x, y) = \frac{\partial f(x, y)}{\partial x}, \quad Q(x, y) = \frac{\partial f(x, y)}{\partial y},$$

$$\frac{\partial P}{\partial y} = -(2x + 1)e^{2x-y}, \quad \frac{\partial Q}{\partial x} = -(2x + 1)e^{2x-y},$$

因为 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, 所以曲线积分与路径无关,

$$\text{于是 } I(t) = \int_0^1 (2x + 1)e^{2x} dx + \int_0^t (1 - e^{2-t}) dy = t + e^{2-t}.$$

由 $I'(t) = 1 - e^{2-t} = 0$ 得 $t = 2$,

因为 $I''(t) = e^{2-t} > 0$, 所以当 $t = 2$ 时, $I(t)$ 取最小值, 最小值为 $I(2) = 3$.

方法二 由 $\frac{\partial f(x, y)}{\partial x} = (2x + 1)e^{2x-y}$ 得 $f(x, y) = x e^{2x-y} + \varphi(y)$,

由 $f(0, y) = y + 1$ 得 $\varphi(y) = y + 1$, 从而 $f(x, y) = x e^{2x-y} + y + 1$.

$$\text{于是 } I(t) = \int_{L_t} df(x, y) = f(1, t) - f(0, 0) = e^{2-t} + t.$$

由 $I'(t) = 1 - e^{2-t} = 0$ 得 $t = 2$,

当 $t < 2$ 时, $I'(t) < 0$; 当 $t > 2$ 时, $I'(t) > 0$, 则 $t = 2$ 时 $I(t)$ 取最小值, 且最小值为

$$I(2) = 3.$$

(18) 【解】 由高斯公式得 $I = \iint_{\Sigma} (x^2 + 1) dy dz - 2y dz dx + 3z dx dy = \iiint_{\Omega} (2x + 1) dv$,

$$\text{而 } \iiint_{\Omega} dv = \frac{1}{3} \times \frac{1}{2} \times 2 \times 1 \times 1 = \frac{1}{3},$$

$$\begin{aligned} \iiint_{\Omega} x dv &= \int_0^1 x dx \int_0^{2(1-x)} dy \int_0^{1-x-\frac{y}{2}} dz = \int_0^1 x dx \int_0^{2(1-x)} \left(1 - x - \frac{y}{2}\right) dy \\ &= \int_0^1 x (1-x)^2 dx = \frac{1}{12}, \end{aligned}$$

$$\text{故 } I = \frac{1}{3} + 2 \times \frac{1}{12} = \frac{1}{2}.$$

(19) 【证明】

(I) $|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| = |f'(\xi)(x_n - x_{n-1})|$, 其中 ξ 介于 x_{n-1} 与 x_n 之间.

因为 $0 < f'(x) < \frac{1}{2}$, 所以 $|x_{n+1} - x_n| \leq \frac{1}{2} |x_n - x_{n-1}|$,

由递推关系得 $|x_{n+1} - x_n| \leq \frac{1}{2^{n-1}} |x_2 - x_1|$.

因为级数 $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} |x_2 - x_1|$ 收敛, 所以 $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ 收敛,

故级数 $\sum_{n=1}^{\infty} (x_{n+1} - x_n)$ 绝对收敛.

(II) 级数 $\sum_{n=1}^{\infty} (x_{n+1} - x_n)$ 的部分和为

$$S_n = (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{n+1} - x_n) = x_{n+1} - x_1,$$

因为级数 $\sum_{n=1}^{\infty} (x_{n+1} - x_n)$ 收敛, 即 $\lim_{n \rightarrow \infty} S_n$ 存在, 所以 $\lim_{n \rightarrow \infty} x_n$ 存在.

令 $\lim_{n \rightarrow \infty} x_n = a$, $x_{n+1} = f(x_n)$ 及函数 $f(x)$ 的连续性得 $a = f(a)$.

令 $\varphi(x) = x - f(x)$, 即 $x = a$ 为 $\varphi(x)$ 的零点.

因为 $\varphi(0) = -f(0) = -1$,

$\varphi(2) = 2 - f(2) = 1 - [f(2) - f(0)] = 1 - 2f'(\eta) > 0$, 其中 $\eta \in (0, 2)$,

所以 $\varphi(x)$ 在 $(0, 2)$ 内有零点.

又因为 $\varphi'(x) = 1 - f'(x) > 0$, 所以 $\varphi(x)$ 只有唯一的一个零点, 且位于 $(0, 2)$ 内,

于是 $0 < a < 2$, 即 $0 < \lim_{n \rightarrow \infty} x_n < 2$.

(20) 【解】 方法一

$$(\mathbf{A} \mid \mathbf{B}) = \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 2 & a & 1 & 1 & a \\ -1 & 1 & a & -a-1 & -2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & a+2 & 3 & -3 & a-4 \\ 0 & 0 & a-1 & 1-a & 0 \end{array} \right)$$

当 $a \neq -2$ 且 $a \neq 1$ 时,

$$(\mathbf{A} \mid \mathbf{B}) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & a+2 & 3 & -3 & a-4 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & \frac{3a}{a+2} \\ 0 & 1 & 0 & 0 & \frac{a-4}{a+2} \\ 0 & 0 & 1 & -1 & 0 \end{array} \right),$$

$$\mathbf{AX} = \mathbf{B} \text{ 有唯一解, } \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} 1 & \frac{3a}{a+2} \\ 0 & \frac{a-4}{a+2} \\ -1 & 0 \end{pmatrix};$$

$$\text{当 } a=1 \text{ 时, } (\mathbf{A} : \mathbf{B}) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & 3 & 3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

由 $r(\mathbf{A}) = r(\mathbf{A} : \mathbf{B}) = 2 < 3$ 得 $\mathbf{AX} = \mathbf{B}$ 有无数个解,

令 $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$,

$$\text{由 } \mathbf{X}_1 = k_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -k_1 - 1 \\ k_1 \end{pmatrix}, \quad \mathbf{X}_2 = k_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -k_2 - 1 \\ k_2 \end{pmatrix} \text{ 得}$$

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ -k_1 - 1 & -k_2 - 1 \\ k_1 & k_2 \end{pmatrix} (k_1, k_2 \text{ 为任意常数}).$$

$$\text{当 } a=-2 \text{ 时, } (\mathbf{A} : \mathbf{B}) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & -3 & 3 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right),$$

因为 $r(\mathbf{A}) \neq r(\mathbf{A} : \mathbf{B})$, 所以 $\mathbf{AX} = \mathbf{B}$ 无解.

$$\text{方法二} \quad |\mathbf{A}| = \begin{vmatrix} 1 & -1 & -1 \\ 2 & a & 1 \\ -1 & 1 & a \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ 0 & a+2 & 3 \\ 0 & 0 & a-1 \end{vmatrix} = (a+2)(a-1),$$

当 $a \neq -2$ 且 $a \neq 1$ 时, 因为 $r(\mathbf{A}) = r(\mathbf{A} : \mathbf{B}) = 3$, 所以 $\mathbf{AX} = \mathbf{B}$ 有唯一解,

$$\text{由 } (\mathbf{A} : \mathbf{B}) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & a+2 & 3 & -3 & -a-4 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & \frac{3a}{a+2} \\ 0 & 1 & 0 & 0 & \frac{a-4}{a+2} \\ 0 & 0 & 1 & -1 & 0 \end{array} \right),$$

得

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} 1 & \frac{3a}{a+2} \\ 0 & \frac{a-4}{a+2} \\ -1 & 0 \end{pmatrix};$$

$$\text{当 } a=1 \text{ 时, } (\mathbf{A} : \mathbf{B}) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & 3 & 3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

由 $r(\mathbf{A}) = r(\mathbf{A} : \mathbf{B}) = 2 < 3$ 得 $\mathbf{AX} = \mathbf{B}$ 有无数个解,

令 $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$,

由 $\mathbf{X}_1 = k_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -k_1 - 1 \\ k_1 \end{pmatrix}$, $\mathbf{X}_2 = k_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -k_2 - 1 \\ k_2 \end{pmatrix}$ 得

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ -k_1 - 1 & -k_2 - 1 \\ k_1 & k_2 \end{pmatrix} (k_1, k_2 \text{ 为任意常数}).$$

$$a = -2 \text{ 时}, (\mathbf{A} \vdash \mathbf{B}) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & -3 & 3 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right),$$

因为 $r(\mathbf{A}) \neq r(\mathbf{A} \vdash \mathbf{B})$, 所以 $\mathbf{AX} = \mathbf{B}$ 无解.

$$(21) \text{【解】 (I)} \text{ 由 } |\lambda\mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda & 1 & -1 \\ -2 & \lambda + 3 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda(\lambda + 1)(\lambda + 2) = 0 \text{ 得}$$

矩阵 \mathbf{A} 的特征值为 $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 0$.

$$\text{将 } \lambda_1 = -1 \text{ 代入 } (\lambda\mathbf{E} - \mathbf{A})\mathbf{X} = \mathbf{0}, \text{ 由 } -\mathbf{E} - \mathbf{A} = \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得 } \lambda_1 = -1 \text{ 对}$$

应的特征向量为 $\xi_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$;

$$\text{将 } \lambda_2 = -2 \text{ 代入 } (\lambda\mathbf{E} - \mathbf{A})\mathbf{X} = \mathbf{0}, \text{ 由 } -2\mathbf{E} - \mathbf{A} = \begin{pmatrix} -2 & 1 & -1 \\ -2 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得}$$

$\lambda_2 = -2$ 对应的特征向量为 $\xi_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$;

$$\text{将 } \lambda_3 = 0 \text{ 代入 } (\lambda\mathbf{E} - \mathbf{A})\mathbf{X} = \mathbf{0}, \text{ 由 } -\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得 } \lambda_3 = 0 \text{ 对应}$$

的特征向量为 $\xi_3 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$.

$$\text{令 } \mathbf{P} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \text{ 由 } \mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得}$$

$$\begin{aligned} \mathbf{A}^{99} &= \mathbf{P} \begin{pmatrix} (-1)^{99} & 0 & 0 \\ 0 & (-2)^{99} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} (-1)^{99} & 0 & 0 \\ 0 & (-2)^{99} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 2^{99}-2 & 1-2^{99} & 2-2^{98} \\ 2^{100}-2 & 1-2^{100} & 2-2^{99} \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

(II) 由 $\mathbf{B}^2 = \mathbf{BA}$ 得 $\mathbf{B}^{100} = \mathbf{B}^{98}\mathbf{B}^2 = \mathbf{B}^{99}\mathbf{A} = \cdots = \mathbf{BA}^{99}$,

$$\text{即 } (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3) = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \begin{pmatrix} 2^{99}-2 & 1-2^{99} & 2-2^{98} \\ 2^{100}-2 & 1-2^{100} & 2-2^{99} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{cases} \boldsymbol{\beta}_1 = (2^{99}-2)\boldsymbol{\alpha}_1 + (2^{100}-2)\boldsymbol{\alpha}_2 + 0\boldsymbol{\alpha}_3, \\ \boldsymbol{\beta}_2 = (1-2^{99})\boldsymbol{\alpha}_1 + (1-2^{100})\boldsymbol{\alpha}_2 + 0\boldsymbol{\alpha}_3, \\ \boldsymbol{\beta}_3 = (2-2^{98})\boldsymbol{\alpha}_1 + (2-2^{99})\boldsymbol{\alpha}_2 + 0\boldsymbol{\alpha}_3. \end{cases}$$

(22) 【解】 (I) 区域 D 的面积为 $A = \int_0^1 (\sqrt{x} - x^2) dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$,

$$\text{随机变量 } (X, Y) \text{ 的联合密度为 } f(x, y) = \begin{cases} 3, & (x, y) \in D, \\ 0, & (x, y) \notin D. \end{cases}$$

(II) 设 (U, X) 的联合分布函数为 $G(u, x)$,

$$G(0, \frac{1}{2}) = P\left\{U \leqslant 0, X \leqslant \frac{1}{2}\right\} = P\left\{X > Y, X \leqslant \frac{1}{2}\right\} = \int_0^{\frac{1}{2}} dx \int_{x^2}^x 3 dy = \frac{1}{4},$$

$$P\{U \leqslant 0\} = P\{X > Y\} = \int_0^1 dx \int_{x^2}^x 3 dy = \frac{1}{2},$$

$$P\left\{X \leqslant \frac{1}{2}\right\} = \int_0^{\frac{1}{2}} dx \int_{x^2}^{\sqrt{x}} 3 dy = \frac{\sqrt{2}}{2} - \frac{1}{8},$$

因为 $\frac{1}{4} \neq \frac{1}{2} \times \left(\frac{\sqrt{2}}{2} - \frac{1}{8}\right)$, 所以 U 与 X 不独立.

(III) 当 $z < 0$ 时, $F(z) = 0$;

当 $0 \leqslant z < 1$ 时,

$$F(z) = P\{Z \leqslant z\} = P\{U = 0, X \leqslant z\} = P\{X > Y, X \leqslant z\} = \int_0^z dx \int_{x^2}^x 3 dy = \frac{3}{2}z^2 - z^3;$$

当 $1 \leqslant z < 2$ 时,

$$\begin{aligned} F(z) &= P\{U + X \leqslant z\} = P\{U = 0, X \leqslant z\} + P\{U = 1, X \leqslant z-1\} \\ &= \frac{1}{2} + 2(z-1)^{\frac{3}{2}} - \frac{3}{2}(z-1)^2; \end{aligned}$$

当 $z \geqslant 2$ 时, $F(z) = 1$,

$$\text{故 } F(z) = \begin{cases} 0, & z < 0, \\ \frac{3}{2}z^2 - z^3, & 0 \leq z < 1, \\ \frac{1}{2} + 2(z-1)^{\frac{3}{2}} - \frac{3}{2}(z-1)^2, & 1 \leq z < 2, \\ 1, & z \geq 2. \end{cases}$$

(23) 【解】 (I) 总体 X 的分布函数为 $F(x) = \int_{-\infty}^x f(t) dt.$

当 $x < 0$ 时, $F(x) = 0$;

当 $x \geq \theta$ 时, $F(x) = 1$;

当 $0 \leq x < \theta$ 时, $F(x) = \int_0^x \frac{3x^2}{\theta^3} dx = \frac{x^3}{\theta^3},$

$$\text{即 } F(x) = \begin{cases} 0, & x < 0, \\ \frac{x^3}{\theta^3}, & 0 \leq x < \theta, \\ 1, & x \geq \theta. \end{cases}$$

设 T 的分布函数为 $F_T(t)$, 则

$$\begin{aligned} F_T(t) &= P\{T \leq t\} = P\{\max\{X_1, X_2, X_3\} \leq t\} \\ &= P\{X_1 \leq t, X_2 \leq t, X_3 \leq t\} \\ &= P\{X_1 \leq t\}P\{X_2 \leq t\}P\{X_3 \leq t\} \\ &= P^3\{X \leq t\} = F^3(t) = \begin{cases} 0, & t < 0, \\ \frac{t^9}{\theta^9}, & 0 \leq t < \theta, \\ 1, & t \geq \theta. \end{cases} \end{aligned}$$

随机变量 T 的概率密度为 $f_T(t) = \begin{cases} \frac{9t^8}{\theta^9}, & 0 < t < \theta, \\ 0, & \text{其他.} \end{cases}$

$$(II) E(aT) = aE(T) = a \int_0^\theta t \cdot \frac{9t^8}{\theta^9} dt = \frac{9a}{10}\theta,$$

由 $E(aT) = \theta$ 得 $a = \frac{10}{9}$, 即当 $a = \frac{10}{9}$ 时, aT 为参数 θ 的无偏估计.