Chapter 4 Notes

4.2 Harmonic Functions and the Dirichlet Problem

A function $u : D \mapsto \mathbb{R}$ where D is an open subset of $\mathbb{R}^d$ is called **harmonic** in D if u is of class C^2 and $\Delta u \triangleq \sum_{i=1}^{d}(\frac{\partial^2 u}{\partial x_i^2}) = 0$ in D.
Throughout this section, $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$ is a d-dimensional Brownian family and $\{\mathcal{F}_t\}$ satisfies the usual conditions. We denote by D an open set in $\mathbb{R}^d$ and introduce the stopping time (Problem 1.2.7)

$$\tau_D = \inf\{t \geq 0; W_t \in D^c\},$$

the time of first exit from D. The boundary of D will be denoted by ∂D, and $\bar{D} = D \cup \partial D$ is the closure of D. By Theorem 2.9.23, each component of W is a.s. unbounded, so

$$P^x[\tau_D < \infty] = 1; \quad \forall x \in D \subset \mathbb{R}^d,\ D \text{ bounded}.$$

Let $B_r \triangleq \{x \in \mathbb{R}^d; \|x\| < r\}$ be the open ball of radius r centered at the origin. The volume of this ball is

$$V \triangleq \frac{2r^d \pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})},$$

and its surface area is

$$S_r \triangleq \frac{2r^{d-1}\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \frac{d}{r}V_r.$$

We define a probability measure μ_r on ∂B_r by

$$\mu_r(dx) = P^0[W_{\tau_{B_r}} \in dx]; \quad r > 0.$$

In the integral notation, the above definition for $A \subset \partial B_r$ becomes:

$$\mu_r(A) = \int_{x \in A} P^0[W_{\tau_{B_r}} \in dx],$$

which we interpret as the probability of the Brownian motion W_t crossing the boundary ∂B_r by passing through points in A.

A. The mean-value property

Because of the rotational invariance of Brownian motion (Problem 3.3.18), the measure μ_r is also rotationally invariant and thus proportional to surface measure on ∂B_r. In particular, the Lebesgue integral of a function f over B_r can be written in iterated form as

$$\int_{B_r} f(x)dx = \int_0^r S_\rho \int_{\partial B_\rho} f(x)\mu_\rho(dx)d\rho.$$

2.1 Definition We say that the function $u : D \mapsto \mathbb{R}$ has the **mean-value property** if, for every $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have

$$u(a) = \int_{\partial B_r} u(a+x)\mu_r(dx).$$

Then we derive the consequence

$$u(a) = \frac{1}{V_r}\int_{B_r} u(a+x)dx.$$

$\because \int_{B_r} u(a+x)dx = \int_0^r S_\rho \int_{\partial B_\rho} u(a+x)\mu_\rho(dx)d\rho = \int_0^r S_\rho u(a+x)d\rho =$
$= u(a+x)\int_0^r S_\rho d\rho = u(a+x)V_r$
(the second inequality follows from the mean-value property)
of the mean-value property, which asserts that the mean integral value of u over a ball is equal to the value at the center.

2.2 Proposition If u is harmonic in D, then it has the mean-value property there.

Proof) With $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B} \subset D$, we have from Ito's formula:

$$u(W_{t\wedge\tau_{a+B_r}}) = u(W_0) + \sum_{i=1}^d \int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} + \frac{1}{2}\int_0^{t\wedge\tau_{a+B_r}} \Delta u(W_s)ds =$$

$$= u(W_0) + \sum_{i=1}^d \int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} \text{ where } 0 \le t < \infty,$$

since u is harmonic and $(\partial u/\partial x_i)$; $1 \le i \le d$, are bounded functions on $a + B_r$, the expectations under P^a of the stochastic integrals are all equal to 0. After taking these expectations on both sides and letting $t \to \infty$, we use

$$u(a) = E^a u(W_{\tau_{a+B_r}}) = \int_{\partial B_r} u(a+x)\mu_r(dx). \quad \square$$

2.3 Corollary (Maximum Principle) Suppose that u is harmonic in the open, connected domain D. If u achieves its supremum over D at some point in D, then u is identically constant.

Proof) Let $M = \sup_{x\in D} u(x)$, and let $D_M = \{x \in D; u(x) = M\}$. We assume that D_M is nonempty and show that $D_M = D$. Since u is continuous, $D_M = u^{-1}(\{M\}) \cap D$ is a closed set relative to D. But for $a \in D_M$, and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have the mean value property:

$$M = u(a) = \frac{1}{V_r}\int_{B_r} u(a+x)dx \le \frac{1}{V_r}\int_{B_r} M dx = M,$$

which shows that $u = M$ on $a + B_r$.
Since $a \in D_M$ was arbitrary, and $a \in a + B_r \subset D_M$, we conclude D_M is open. Moreover, D is connected, either D_M or $D - D_M$ must be empty. $\square$

For the sake of completeness, below is the converse of Proposition 2.2.

2.5 Proposition If u maps D into $\mathbb{R}$ and has the mean-value property, then u is of class C^∞ and harmonic.

Proof) We first prove that u is of class C^∞. For $\epsilon > 0$, let $g_\varepsilon : \mathbb{R}^d \to [0,\infty)$ be the C^∞ function

$$g_\varepsilon(x) = \begin{cases} c(\varepsilon)\exp\left[\frac{1}{\|x\|^2-\varepsilon^2}\right], & \|x\| < \varepsilon \\ 0, & \|x\| \ge \varepsilon \end{cases} \tag{1}$$

where $c(\varepsilon)$ is chosen so that

$$\int_{B_\varepsilon} g_\varepsilon(x)dx = \int_0^\varepsilon S_\rho \int_{\partial B_\rho} g_\varepsilon(x)\mu_\rho(dx)d\rho =$$

$$= c(\varepsilon)\int_0^\varepsilon S_\rho \int_{\partial B_\rho} \exp(\frac{1}{\|x\|^2-\varepsilon^2})\mu_\rho(dx)d\rho = c(\varepsilon)\int_0^\varepsilon S_\rho \exp(\frac{1}{\rho^2-\varepsilon^2})d\rho = 1.$$

For $\varepsilon > 0$ and $a \in D$ s.t. $a + \bar{B}_\varepsilon \subset D$, define

$$u_\varepsilon(a) \triangleq \int_{B_\varepsilon} u(a+x)g_\varepsilon(x)dx = \int_{\mathbb{R}^d} u(y)g_\varepsilon(y-a)dy.$$

From the second representation, u_ε is of class C^∞ on the open subset of D where it is defined. Furthermore, for every $a \in D$ there exists $\varepsilon > 0$ so that $a + \bar{B}_\varepsilon \subset D$; from mean-value property of u, we have

$$u_\varepsilon(a) = \int_{B_\varepsilon} u(a+x)g_\varepsilon(x)dx = c(\varepsilon)\int_0^\varepsilon S_\rho \int_{\partial B_\rho} u(a+x)\exp(\frac{1}{\rho^2-\varepsilon^2})\mu_\rho(dx)d\rho =$$

$$= c(\varepsilon)\int_0^\varepsilon S_\rho u(a)\exp(\frac{1}{\rho^2-\varepsilon^2})d\rho = u(a)$$

where the last equality is from the definition of $c(\varepsilon)$. Thus, u is also of class C^∞.

In order to show that $\Delta u = 0$ in D,we choose $a \in D$ and use a Taylor-series expansion in the neighborhood $a + \bar{B}_\varepsilon$,

$$u(a+y) = u(a) + \sum_{i=1}^{d} y_i \frac{\partial u}{\partial x_i}(a) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} y_i y_j \frac{\partial^2 u}{\partial x_i \partial x_j}(a) + o(\|y\|^2);\ y \in \bar{B}_\varepsilon,$$

where again $\varepsilon > 0$ is chosen so that $a + \bar{B}_\varepsilon \subset D$. Odd symmetry gives us

$$\int_{\partial B_\varepsilon} y_i \mu_\varepsilon(dy) = 0, \quad \int_{\partial B_\varepsilon} y_i y_j \mu_\varepsilon(dy) = 0; \quad i \neq j,$$

so integrating the above Taylor-expansion over ∂B_ε and using the mean-value property, we have

$$u(a) = \int_{\partial B_\varepsilon} u(a+y)\mu_\varepsilon(dy) = u(a) + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(a) \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) + o(\varepsilon^2).$$

But

$$\int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{1}{d}\sum_{i=1}^{d} \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{\varepsilon^2}{d},$$

thus we have

$$\frac{\varepsilon^2}{2d}\Delta u(a) + o(\varepsilon^2) = 0.$$

Dividing by ε^2 and letting $\varepsilon \downarrow 0$, we have $\Delta u(a) = 0$. $\square$

B. The Dirichlet problem

We take up now the Dirichlet problem (D, f) : with open $D \subset \mathbb{R}^d$ and $f : \partial D \to \mathbb{R}$ is a given continuous function, find a continuous function $u : \bar{D} \to \mathbb{R}$ s.t.

$$\Delta u = 0;\ \text{in } D$$

$$u = f;\ \text{on } \partial D.$$

Such a function, when it exists, will be called a solution to the Dirichlet problem (D, f). One may interpret $u(x)$ as the steady-state temperature at $x \in D$ when the boundary temperatures of D are specified by f.

The power of the probabilistic method is demonstrated by the fact that we can immediately write down a very likely solution to (D, f), namely

$$u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D},$$

provided that

$$E^x|f(W_{\tau_D})| < \infty; \quad \forall x \in D.$$

If $x \in \partial D$, then since $P^x[W_0 = x] = 1$, we have

$$u(x) = E^x f(W_{\tau_D}) = E^x f(W_0) = f(x).$$

Thus, u satisfies $u = f$ on ∂D. Furthermore, for $a \in D$ and B_r chosen so that $a + \bar{B}_r \subset D$, we have:

$$u(a) = E^a f(W_{\tau_D}) \overset{\text{tower}}{=} E^a\{E^a[f(W_{\tau_D})|\mathcal{F}_{\tau_{a+B_r}}]\} =$$

$$= E^a\{E^a[f(W_{\tau_D} - W_{\tau_{a+B_r}} + W_{\tau_{a+B_r}})|\mathcal{F}_{\tau_{a+B_r}}]\} =$$

$$= E^a\{u(W_{\tau_{a+B_r}})\} \overset{\text{def}}{=} \int_{\partial B_r} u(a+x)\mu_r(dx),$$

where the second last equality is from the strong Markov property of B.M.

Therefore, u has the mean-value property, and so it must satisfy $\Delta u = 0$; in D. The only unresolved issue is whether u is continuous up to and including ∂D.
2.6 Proposition If $E^x|f(W_{\tau_D})| < \infty$ holds, then $u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D}$ is harmonic in D.

2.7 Proposition If f is bounded and

$$P^a[\tau_D < \infty] = 1; \quad \forall a \in D,$$

then any bounded solution to (D, f) has the representation $u(x) = E^x f(W_{\tau_D})$.

Proof) Let u be any bounded solution to (D, f), and let $D_n \triangleq \{x \in D; \inf_{y \in \partial D} \|x - y\| > \frac{1}{n}\}$. Then, D_n is an increasing sequence of subsets of D. From Ito's rule,

$$u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = u(W_0) + \sum_{i=1}^{d} \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)}; \quad 0 \le t < \infty, \quad n \ge 1.$$

Since $\frac{\partial u}{\partial x_i}$ is bounded in $\overline{B_n \cap D_n}$, we take expectations w.r.t P^a from both sides:

$$E^a u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = E^a(u(W_0)) = u(a);$$

where $0 \le t < \infty,\ n \ge 1,\ a \in D_n$.
As $t \to \infty, n \to \infty, P^a[\tau_D < \infty] = 1;\ \forall a \in D$ implies that $u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}})$ converges to $f(W_{\tau_D})$, a.s. P^a. The representation $u(x) = E^x f(W_{\tau_D}); \quad x \in \bar{D}$ follows from the bounded convergence theorem. □

In the light of Proposition 2.6 and 2.7, the existence of a solution to the Dirichlet problem boils down to the question of the continuity of u defined by

$E^x f(W_{\tau_D})$ at the boundary of D. We therefore undertake to characterize those points $a \in \partial D$ for which

$$\lim_{x\to a, x\in D} E^x f(W_{\tau_D}) = f(a)$$

holds for every bounded, measurable function $f : \partial D \to \mathbb{R}$ which is continuous at the point a.

2.9 Definition Consider the stopping time of the right-continuous filtration $\{\mathcal{F}_t\}$ given by $\sigma_D \triangleq \inf\{t > 0; W_t \in D^c\}$. We say that a point $a \in \partial D$ is regular for D if $P^a[\sigma_D = 0] = 1$, i.e., a Brownian motion path started at a does not immediately return to D and remain there for a nonempty time interval.

2.10 Remark A point $a \in \partial D$ is called irregular if $P^a[\sigma_D = 0] < 1$; however, the event $\{\sigma_D = 0\}$ belongs to $\mathcal{F}^W_{0+}$, and so the Blumenthal zero-one law (Theorem 2.7.17) gives for an irregular point $a : P^a[\sigma_D = 0] = 0$.

2.11 Remark The regularity is a local condition; i.e. $a \in \partial D$ is regular for D if and only if a is regular for $(a + B_r) \cap D$, for some $r > 0$.

2.12 Theorem Assume that $d \geq 2$ and fix $a \in \partial D$. The following are equivalent:
(i) $\lim_{x\to a, x\in D} E^x f(W_{\tau_D}) = f(a)$ holds for every bounded, measurable function $f : \partial D \to \mathbb{R}$ which is continuous at a;
(ii) a is regular for D;
(iii) for all $\varepsilon > 0$, we have

$$\lim_{x\to a, x\in D} P^x[\tau_D > \varepsilon] = 0.$$

Proof) We assume WLOG that $a = 0$, and begin by proving the implication $(i) \Rightarrow (ii)$ by contradiction. If the origin is irregular, then $P^0[\sigma_D = 0] = 0$ (Remark 2.10). Since a Brownian motion of dimension $d \geq 2$ never returns to its starting point (Prop 3.3.22), we have

$$\lim_{r\downarrow 0} P^0[W_{\tau_D} \in B_r] = P^0[W_{\tau_D} = 0] = 0.$$

Fix $r > 0$ for which $P^0[W_{\tau_D} \in B_r] < \frac{1}{4}$, and choose a sequence $\{\delta_n\}_{n=1}^{\infty}$ for which $0 < \delta_n < r$ for all n and $\delta_n \downarrow 0$. With $\tau_n \triangleq \inf\{t \geq 0; \|W_t\| \geq \delta_n\}$, we have $P^0[\tau_n \downarrow 0] = 1$, and thus $\lim_{n\to\infty} P^0[\tau_n < \sigma_D] = 1$. Furthermore, on the event $\{\tau_n < \sigma_D\}$ we have $W_{\tau_n} \in D$. For n large enough so that $P^0[\tau_n < \sigma_D] \geq \frac{1}{2}$ we may write

$$\frac{1}{4} > P^0[W_{\sigma_D} \in B_r] \geq P^0[W_{\sigma_D} \in B_r, \tau_n < \sigma_D] = E^0(1_{\{W_{\sigma_D} \in B_r\}} 1_{\{\tau_n < \sigma_D\}}) =$$

$$= E^0(1_{\{\tau_n < \sigma_D\}} E^0[1_{\{W_{\sigma_D} \in B_r\}} | \mathcal{F}_{\tau_n}]) = E^0(1_{\{\tau_n < \sigma_D\}} P^0[W_{\sigma_D} \in B_r | \mathcal{F}_{\tau_n}]) =$$

$$= \int_{D\cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r] P^0[\tau_n < \sigma_D, W_{\tau_n} \in dx] \geq \frac{1}{2} \inf_{x\in D\cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r],$$

for which we conclude that $P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2}$ for some $x_n \in D \cap B_{\delta_n}$. Now choose a bounded, continuous function $f : \partial D \to \mathbb{R}$ s.t. $f = 0$ outside B_r, $f \leq 1$ inside B_r, and $f(0) = 1$. For such a function we have

$$\overline{\lim}_{n\to\infty} E^{x_n} f(W_{\tau_D}) \leq \overline{\lim}_{n\to\infty} P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2} < f(0),$$

and (i) fails.
We next show that $(ii) \Rightarrow (iii)$. Observe first of all that for $0 < \delta < \varepsilon$, the function

$$g_\delta(x) \triangleq P^x[W_s \in D; \delta \leq s \leq \varepsilon] = E^x(P^{W_\delta}[\tau_D > \varepsilon - \delta]) =$$

$$= \int_{\mathbb{R}^d} P^y[\tau_D > \varepsilon - \delta] P^x[W_\delta \in dy]$$

is continuous in x. But

$$g_\delta(x) \downarrow g(x) \triangleq P^x[W_s \in D; 0 < s \leq \varepsilon] = P^x[\sigma_D > \varepsilon]$$

as $\delta \downarrow 0$, so g is upper semicontinuous. From this fact and the inequality $\tau_D \leq \sigma_D$, we conclude that $\overline{\lim}_{x\to 0} P^x[\tau_D > \varepsilon] \leq \overline{\lim}_{x\to 0} g(x) \leq g(0) = 0$, by (ii).

Finally, we prove $(iii) \Rightarrow (i)$. We know that for each $r > 0$, $P^x[\max_{0\leq t\leq\varepsilon} \|W_t - W_0\| < r]$ does not depend on x and approaches one as $\varepsilon \downarrow 0$. But then

$$P^x[\|W_{\tau_D} - W_0\| < r] \geq P^x[\{\max_{0\leq t\leq\varepsilon} \|W_t - W_0\| < r\} \cap \{\tau_D \leq \varepsilon\}] \geq$$

$$\geq P^0[\max_{0\leq t\leq\varepsilon} \|W_t\| < r] - P^x[\tau_D > \varepsilon].$$

Letting $x \to 0$ ($x \in D$) and $\varepsilon \downarrow 0$, successively, we obtain from (iii),

$$\lim_{x\to 0, x\in D} P^x[\|W_{\tau_D} - x\| < r] = 1; \quad 0 < r < \infty.$$

The continuity of f at the origin and its boundedness on ∂D gives $\lim_{x\to 0, x\in D} E^x f(W_{\tau_D}) = f(a)$. $\square$

C. Conditions for regularity

For many open sets D and boundary points $a \in \partial D$, we can convince ourselves intuitively that a Brownian motion originating at a will exit from $\bar{D}$ immediately, i.e., a is regular.

When $d = 2$, the center of a punctured disc is an irregular boundary point. The following development, culminating with Problem 2.16 shows that in $\mathbb{R}^2$, any irregular boundary point of D must be "isolated" in the sense that it cannot be connected to any other point outside D by a simple arc lying outside D.

2.13 Definition Let $D \subset \mathbb{R}^d$ be open and $a \in \partial D$. A **barrier** at a is a continuous function $v : \bar{D} \to \mathbb{R}$ which is harmonic in D, positive on $\bar{D} - \{a\}$, and equal to zero at a.

2.14 Example Let $D \subset B_r \subset \mathbb{R}^2$ be open, where $0 < r < 1$, and assume $(0,0) \in \partial D$. If a single valued, analytic branch of $\log(x_1 + ix_2)$ can be defined in $\bar{D} - (0,0)$, then

$$v(x_1, x_2) \triangleq \begin{cases} -\mathrm{Re}\frac{1}{\log(x_1+ix_2)} = -\frac{\log\sqrt{x_1^2+x_2^2}}{|\log(x_1+ix_2)|^2}; & (x_1, x_2) \in D - (0,0), \\ 0; & (x_1, x_2) = (0,0), \end{cases}$$

is a barrier at $(0,0)$. Indeed being the real part of an analytic solution, v is harmonic in D, and because $0 < \sqrt{x_1^2 + x_2^2} \le r < 1$ in $\bar{D} - (0,0)$, v is positive on this set.

2.15 Proposition Let D be bounded and $a \in \partial D$. If there exists a barrier at a, then a is regular.

Proof) Let v be a barrier at a. We establish condition (i) of Theorem 2.12. With $f : \partial D \to \mathbb{R}$ bounded and continuous at a, define $M = \sup_{x \in \partial D} |f(x)|$. Choose $\varepsilon > 0$ and let $\delta > 0$ be s.t. $|f(x) - f(a)| < \varepsilon$ if $x \in \partial D$ and $\|x - a\| < \delta$. Choose k so that $kv(x) \ge 2M$ for $x \in \bar{D}$ and $\|x - a\| \ge \delta$.
We then have $|f(x) - f(a)| \le \varepsilon + 2M \le \varepsilon + kv(x)$; $x \in \partial D$, so

$$|E^x f(W_{\tau_D}) - f(a)| \le E^x |f(W_{\tau_D}) - f(a)| \le \varepsilon + kE^x v(W_{\tau_D}) = \varepsilon + kv(x); \quad x \in D$$

by Proposition 2.7. But v is continuous and $v(a) = 0$, so

$$\overline{\lim}_{x \to a, x \in D} |E^x f(W_{\tau_D}) - f(a)| \le \varepsilon.$$

Finally, we let $\varepsilon \downarrow 0$ to obtain $\lim_{x \to a, x \in D} E^x f(W_{\tau_D}) = f(a)$. $\square$

2.17 Example (Lebesgue's Thorn) With $d = 3$ and $\{\varepsilon_n\}_{n=1^\infty}$ a sequence of positive numbers decreasing to zero, define

$$E = \{(x_1, x_2, x_3);\ -1 < x_1 < 1,\ x_2^2 + x_3^2 < 1\},$$

$$F_n = \{(x_1, x_2, x_3);\ 2^{-n} \le x_1 \le 2^{-n+1},\ x_2^2 + x_3^2 \le \varepsilon_n\},$$

$$D = E - (\bigcup_{n=1}^{\infty} F_n).$$

Now $P^0[(W_t^{(2)}, W_t^{(3)}) = (0,0)$, for some $t > 0] = 0$ (Proposition 3.3.22), so the P^0-probability that $W = (W^{(1)}, W^{(2)}, W^{(3)})$ ever hits the compact set $K_n \triangleq$

$\{(x_1, x_2, x_3); 2^{-n} \leq x_1 \leq 2^{-n+1}, x_2 = x_3 = 0\}$ is zero. According to Problem 3.3.24, $\lim_{t\to\infty} \|W_t\| = \infty$ a.s. P^0, so for P^0-a.e. $\omega \in \Omega$, the path $t \mapsto W_t(\omega)$ remains bounded away from K_n. Thus, if ε_n is chosen sufficiently small, we can ensure that $P^0[W_t \in F_n$, for some $t \geq 0] \leq 3^{-n}$. If W, beginning at the origin, does not return to D immediately, it must avoid D by entering $\bigcup_{n=1}^{\infty} F_n$. In other words,

$$P^0[\sigma_D = 0] \leq P^0[W_t \in F_n, \text{ for some } t \geq 0 \text{ and } n \geq 1] \leq \sum_{n=1}^{\infty} < 1. \quad \square$$

If the cusplike behavior is avoided, then the boundary points of D are regular, regardless of the dimension. To make this statement precise, let us define for $y \in \mathbb{R}^d - \{0\}$ and $0 \leq \theta \leq \pi$, the **cone** $C(y, \theta)$ with direction y and aperture θ by

$$C(y, \theta) = \{x \in \mathbb{R}^d; (x, y) \geq \|x\|\|y\| \cos\theta\}.$$

2.18 Definition We say that the point $a \in \partial D$ satisfies the **Zaremba's cone condition** if there exists $y \neq 0$ and $0 < \theta < \pi$ s.t. the translated cone $a + C(y, \theta)$ is contained in $\mathbb{R}^d - D$.

2.19 Theorem If a point $a \in \partial D$ satisfies the Zaremba's cone condition, then it is regular.

Proof) We assume WLOG that a is the origin and $C(y, \theta) \subset \mathbb{R}^d - D$, where $y \neq 0$ and $0 < \theta < \pi$. Because the change of variables $z = \frac{x}{\sqrt{t}}$ maps $C(y, \theta)$ onto itself, we have for any $t > 0$,

$$P^0[W_t \in C(y, \theta)] = \int_{C(y,\theta)} \frac{1}{(2\pi t)^{d/2}} \exp[-\frac{\|x\|^2}{2t}]dx =$$

$$= \int_{C(y,\theta)} \frac{1}{(2\pi)^{d/2}} \exp[-\frac{\|z\|^2}{2}]dz \triangleq q > 0,$$

where q is independent of t. Now, $P^0[\sigma_D \leq t] \geq P^0[W_t \in C(y, \theta)] = q$, and letting $t \downarrow 0$, we conclude that $P^0[\sigma_D = 0] > 0$. Regularity follows from the Blumenthal zero-one law (Remark 2.10).

2.20 Remark If, for $a \in \partial D$ and some $r > 0$, the point a satisfies Zaremba's cone condition for the set $(a + B_r) \cap D$, then a is regular for D (Remark 2.11).