Chapter 5 Notes

5.1 Introduction

Consider a d-dimensional Markov family $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$, and assume that X has continuous paths. We suppose that the the relation

$$\lim_{t \downarrow 0} \frac{1}{t}[E^x f(X_t) - f(x)] = (\mathcal{A}f)(x); \quad \forall x \in \mathbb{R}^d \tag{1.1}$$

holds for every f in a suitable subclass of the space $C^2(\mathbb{R}^d)$ of real-valued twice-continuously differentiable functions on $\mathbb{R}^d$; the operator $\mathcal{A}f$ is given by

$$(\mathcal{A}f)(x) \triangleq \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} a_{ik}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i(x) \frac{\partial f(x)}{\partial x_i} \tag{1.2}$$

for suitable Borel-measurable functions $b_i, a_{ik} : \mathbb{R}^d \to \mathbb{R}, 1 \leq i, k \leq d$. The left-hand side of (1.1) is the infintesimal generator of the Markov family, applied to the test function f. On the other hand, the operator in (1.2) is called the second-order diffusion operator associated with the drift vector $b = (b_1, ..., b_d)$ and the diffusion matrix $a = \{a_{ik}\}_{1 \leq i,k \leq d}$ which is assumed to be symmetric and nonnegative-definite for every $x \in \mathbb{R}^d$.
The drift and diffusion coefficients can be interpreted in the following manner: fix $x \in \mathbb{R}^d$ and let $f_i(y) \triangleq y_i$, $f_{ik}(y) \triangleq (y_i - x_i)(y_k - x_k)$; $y \in \mathbb{R}^d$. Assuming that (1.1) holds for these test functions, we obtain

$$E^x[X_t^{(i)} - x_i] = tb_i(x) + o(t) \tag{1.3}$$

$$E^x[X_t^{(i)} - x_i] = tb_i(x) + o(t) \tag{1.4}$$

1.1 Definition. Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$ be a d-dimensional Markov family, such that
(i) X has a continuous sample paths;
(ii) relation (1.1) holds for every $f \in C^2(\mathbb{R}^d)$ which is bounded and has bounded first- and second- order derivatives;
(iii) relations (1.3), (1.4) holds for every $x \in \mathbb{R}^d$; and
(iv) the tenets (a)-(d) of Definition 2.6.3 are satisfied, but only for stopping times S.
Then X is called a (Kolmogorov-Feller) diffusion process.

5.2 Strong Solutions

In this section, we introduce the concept of a stochastic differential equation w.r.t. Brownian motion and its solution in the strong sense. We discuss the questions of existence and uniqueness of such solutions, as well as some of their elementary properties.

Let us start with Borel-measurable functions $b_i(t,x),\ \sigma_{ij}(t,x);\ 1 \le i \le d,\ 1 \le j \le r$, from $[0,\infty) \times \mathbb{R}^d$ into $\mathbb{R}$, and define the $(d \times 1)$ drift vector $b(t,x) = \{b_i(t,x)\}_{1\le i\le d}$ and the $(d \times r)$ disertion matrix $\sigma(t,x) = \{\sigma_{ij}(t,x)\}_{1\le i\le d; 1\le j\le r}$. The intent is to assign a meaning to the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t \tag{2.1}$$

written componentwise as

$$dX_t^{(i)} = b_i(t,X_t)dt + \sum_{j=1}^{r} \sigma_{ij}(t,X_t)dW_t^{(j)}; \quad 1 \le i \le d, \tag{2.1$'$}$$

where $W = \{W_t; 0 \le t < \infty\}$ is an r-dimensional Brownian motion and $X = \{X_t; 0 \le t < \infty\}$ is a suitable stochastic process with continuous sample paths and values in $\mathbb{R}^d$, the "solution" of the equation. The drift vector $b(t,x)$ and the dispersion matrix $\sigma(t,x)$ are the coefficients of this equation; the $(d \times d)$ matrix $a(t,x) \triangleq \sigma(t,x)\sigma^T(t,x)$ with elements

$$a_{ik}(t,x) \triangleq \sum_{j=1}^{r} \sigma_{ij}(t,x)\sigma_{kj}(t,x); \quad 1 \le i,k \le d \tag{2.2}$$

will be called the diffusion matrix.

A. Definitions