Chapter 4 Notes

4.2 Harmonic Functions and the Dirichlet Problem

A function $u : D \mapsto \mathbb{R}$ where D is an open subset of $\mathbb{R}^d$ is called **harmonic** in D if u is of class C^2 and $\Delta u \triangleq \sum_{i=1}^{d}(\frac{\partial^2 u}{\partial x_i^2}) = 0$ in D.
Throughout this section, $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x\in\mathbb{R}^d}$ is a d-dimensional Brownian family and $\{\mathcal{F}_t\}$ satisfies the usual conditions. We denote by D an open set in $\mathbb{R}^d$ and introduce the stopping time (Problem 1.2.7)

$$\tau_D = \inf\{t \geq 0; W_t \in D^c\},$$

the time of first exit from D. The boundary of D will be denoted by ∂D, and $\bar{D} = D \cup \partial D$ is the closure of D. By Theorem 2.9.23, each component of W is a.s. unbounded, so

$$P^x[\tau_D < \infty] = 1; \quad \forall x \in D \subset \mathbb{R}^d,\ D \text{ bounded}.$$

Let $B_r \triangleq \{x \in \mathbb{R}^d; \|x\| < r\}$ be the open ball of radius r centered at the origin. The volume of this ball is

$$V \triangleq \frac{2r^d\pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})},$$

and its surface area is

$$S_r \triangleq \frac{2r^{d-1}\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \frac{d}{r}V_r.$$

We define a probability measure μ_r on ∂B_r by

$$\mu_r(dx) = P^0[W_{\tau_{B_r}} \in dx]; \quad r > 0.$$

In the integral notation, the above definition for $A \subset \partial B_r$ becomes:

$$\mu_r(A) = \int_{x\in A} P^0[W_{\tau_{B_r}} \in dx],$$

which we interpret as the probability of the Brownian motion W_t crossing the boundary ∂B_r by passing through points in A.

A. The mean-value property

Because of the rotational invariance of Brownian motion (Problem 3.3.18), the measure μ_r is also rotationally invariant and thus proportional to surface measure on ∂B_r. In particular, the Lebesgue integral of a function f over B_r can be written in iterated form as

$$\int_{B_r} f(x)dx = \int_0^r S_\rho \int_{\partial B_\rho} f(x)\mu_\rho(dx)d\rho.$$

2.1 Definition We say that the function $u : D \mapsto \mathbb{R}$ has the **mean-value property** if, for every $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have

$$u(a) = \int_{\partial B_r} u(a+x)\mu_r(dx).$$

Then we derive the consequence

$$u(a) = \frac{1}{V_r}\int_{B_r} u(a+x)dx.$$

$\because \int_{B_r} u(a+x)dx = \int_0^r S_\rho \int_{\partial B_\rho} u(a+x)\mu_\rho(dx)d\rho = \int_0^r S_\rho u(a+x)d\rho =$
$= u(a+x)\int_0^r S_\rho d\rho = u(a+x)V_r$
(the second inequality follows from the mean-value property)
of the mean-value property, which asserts that the mean integral value of u over a ball is equal to the value at the center.

2.2 Proposition If u is harmonic in D, then it has the mean-value property there.

Proof) With $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B} \subset D$, we have from Ito's formula:

$$u(W_{t\wedge\tau_{a+B_r}}) = u(W_0) + \sum_{i=1}^d \int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} + \frac{1}{2}\int_0^{t\wedge\tau_{a+B_r}} \Delta u(W_s)ds =$$

$$= u(W_0) + \sum_{i=1}^d \int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} \text{ where } 0 \le t < \infty,$$

since u is harmonic and $(\partial u/\partial x_i)$; $1 \le i \le d$, are bounded functions on $a + B_r$, the expectations under P^a of the stochastic integrals are all equal to 0. After taking these expectations on both sides and letting $t \to \infty$, we use

$$u(a) = E^a u(W_{\tau_{a+B_r}}) = \int_{\partial B_r} u(a+x)\mu_r(dx). \quad \square$$

2.3 Corollary (Maximum Principle) Suppose that u is harmonic in the open, connected domain D. If u achieves its supremum over D at some point in D, then u is identically constant.

Proof) Let $M = \sup_{x\in D} u(x)$, and let $D_M = \{x \in D; u(x) = M\}$. We assume that D_M is nonempty and show that $D_M = D$. Since u is continuous, $D_M = u^{-1}(\{M\} \cap D)$ is a closed set relative to D. But for $a \in D_M$, and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have the mean value property:

$$M = u(a) = \frac{1}{V_r}\int_{B_r} u(a+x)dx \leq \frac{1}{V_r}\int_{B_r} M dx = M,$$

which shows that $u = M$ on $a + B_r$.
Since $a \in D_M$ was arbitrary, and $a \in a + B_r \subset D_M$, we conclude D_M is open. Moreover, D is connected, either D_M or $D - D_M$ must be empty. □

For the sake of completeness, below is the converse of Proposition 2.2.

2.5 Proposition If u maps D into $\mathbb{R}$ and has the mean-value property, then u is of class C^∞ and harmonic.

Proof) We first prove that u is of class C^∞. For $\epsilon > 0$, let $g_\varepsilon : \mathbb{R}^d \to [0, \infty)$ be the C^∞ function

$$g_\varepsilon(x) = \begin{cases} c(\varepsilon)\exp\left[\frac{1}{\|x\|^2-\varepsilon^2}\right], & \|x\| < \varepsilon \\ 0, & \|x\| \geq \varepsilon \end{cases} \tag{1}$$

where $c(\varepsilon)$ is chosen so that

$$\int_{B_\varepsilon} g_\varepsilon(x)dx = \int_0^\varepsilon S_\rho \int_{\partial B_\rho} g_\varepsilon(x)\mu_\rho(dx)d\rho =$$

$$= c(\varepsilon)\int_0^\varepsilon S_\rho \int_{\partial B_\rho} \exp(\frac{1}{\|x\|^2-\varepsilon^2})\mu_\rho(dx)d\rho = c(\varepsilon)\int_0^\varepsilon S_\rho \exp(\frac{1}{\rho^2-\varepsilon^2})d\rho = 1.$$

For $\varepsilon > 0$ and $a \in D$ s.t. $a + \bar{B}_\varepsilon \subset D$, define

$$u_\varepsilon(a) \triangleq \int_{B_\varepsilon} u(a+x)g_\varepsilon(x)dx = \int_{\mathbb{R}^d} u(y)g_\varepsilon(y-a)dy.$$

From the second representation, u_ε is of class C^∞ on the open subset of D where it is defined. Furthermore, for every $a \in D$ there exists $\varepsilon > 0$ so that $a + \bar{B}_\varepsilon \subset D$; from mean-value property of u, we have

$$u_\varepsilon(a) = \int_{B_\varepsilon} u(a+x)g_\varepsilon(x)dx = c(\varepsilon)\int_0^\varepsilon S_\rho \int_{\partial B_\rho} u(a+x)\exp(\frac{1}{\rho^2-\varepsilon^2})\mu_\rho(dx)d\rho =$$

$$= c(\varepsilon)\int_0^\varepsilon S_\rho u(a)\exp(\frac{1}{\rho^2-\varepsilon^2})d\rho = u(a)$$

where the last equality is from the definition of $c(\varepsilon)$. Thus, u is also of class C^∞.

In order to show that $\Delta u = 0$ in D,we choose $a \in D$ and use a Taylor-series expansion in the neighborhood $a + \bar{B}_\varepsilon$,

$$u(a+y) = u(a) + \sum_{i=1}^{d} y_i \frac{\partial u}{\partial x_i}(a) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} y_i y_j \frac{\partial^2 u}{\partial x_i \partial x_j}(a) + o(\|y\|^2);\ y \in \bar{B}_\varepsilon,$$

where again $\varepsilon > 0$ is chosen so that $a + \bar{B}_\varepsilon \subset D$. Odd symmetry gives us

$$\int_{\partial B_\varepsilon} y_i \mu_\varepsilon(dy) = 0, \quad \int_{\partial B_\varepsilon} y_i y_j \mu_\varepsilon(dy) = 0; \quad i \neq j,$$

so integrating the above Taylor-expansion over ∂B_ε and using the mean-value property, we have

$$u(a) = \int_{\partial B_\varepsilon} u(a+y)\mu_\varepsilon(dy) = u(a) + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(a) \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) + o(\varepsilon^2).$$

But

$$\int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{1}{d}\sum_{i=1}^{d} \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{\varepsilon^2}{d},$$

thus we have

$$\frac{\varepsilon^2}{2d}\Delta u(a) + o(\varepsilon^2) = 0.$$

Dividing by ε^2 and letting $\varepsilon \downarrow 0$, we have $\Delta u(a) = 0$. $\square$

B. The Dirichlet problem

We take up now the Dirichlet problem (D, f) : with open $D \subset \mathbb{R}^d$ and $f : \partial D \to \mathbb{R}$ is a given continuous function, find a continuous function $u : \bar{D} \to \mathbb{R}$ s.t.

$$\Delta u = 0;\ \text{in } D$$

$$u = f;\ \text{on } \partial D.$$

Such a function, when it exists, will be called a solution to the Dirichlet problem (D, f). One may interpret $u(x)$ as the steady-state temperature at $x \in D$ when the boundary temperatures of D are specified by f.

The power of the probabilistic method is demonstrated by the fact that we can immediately write down a very likely solution to (D, f), namely

$$u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D},$$

provided that

$$E^x|f(W_{\tau_D})| < \infty; \quad \forall x \in D.$$

By the definition of τ_D, u satisfies $u = f$ on ∂D. Furthermore, for $a \in D$ and B_r chosen so that $a + \bar{B}_r \subset D$, we have from strong Markov property:

$$u(a) = E^a f(W_{\tau_D}) = E^a\{E^a[f(W_{\tau_D})|\mathcal{F}_{\tau_{a+B_r}}]\} =$$

$$= E^a\{u(W_{\tau_{a+B_r}})\} = \int_{\partial B_r} u(a+X)\mu_r(dx).$$

Therefore, u has the mean-value property, and so it must satisfy $\Delta u = 0$; in D. The only unresolved issue is whether u is continuous up to and including ∂D.

2.6 Proposition If $E^x|f(W_{\tau_D})| < \infty$ holds, then $u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D}$ is harmonic in D.

2.7 Proposition If f is bounded and

$$P^a[\tau_D < \infty] = 1; \quad \forall a \in D,$$

then any bounded solution to (D, f) has the representation $E^x f(W_{\tau_D})$.

Proof) Let u be any bounded solution to (D, f), and let $D_n \triangleq \{x \in D; \inf_{y \in \partial D} \|x - y\| > \frac{1}{n}\}$. From Ito's rule, we have

$$u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = u(W_0) + \sum_{i=1}^{d} \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)}; \quad 0 \le t < \infty, \quad n \ge 1.$$

Since $\frac{\partial u}{\partial x_i}$ is bounded in $B_n \bar{\cap} D_n$, we take expectations and conclude that

$$u(a) = E^a u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}); \quad 0 \le t < \infty, \quad n \ge 1, \quad a \in D_n.$$

As $t \to \infty, n \to \infty$, $P^a[\tau_D < \infty] = 1; \quad \forall a \in D$ implies that $u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}})$ converges to $f(W_{\tau_D})$, a.s. P^a. The representation $u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D}$ follows from the bounded convergence theorem. $\square$