

Chapter 4 Notes

4.2 Harmonic Functions and the Dirichlet Problem

A function $u : D \mapsto \mathbb{R}$ where D is an open subset of \mathbb{R}^d is called **harmonic** in D if u is of class C^2 and $\Delta u \triangleq \sum_{i=1}^d (\frac{\partial^2 u}{\partial x_i^2}) = 0$ in D .

Throughout this section, $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, (Ω, \mathcal{F}) , $\{P^x\}_{x \in \mathbb{R}^d}$ is a d -dimensional Brownian family and $\{\mathcal{F}_t\}$ satisfies the usual conditions. We denote by D an open set in \mathbb{R}^d and introduce the stopping time (Problem 1.2.7)

$$\tau_D = \inf\{t \geq 0; W_t \in D^c\},$$

the time of first exit from D . The boundary of D will be denoted by ∂D , and $\bar{D} = D \cup \partial D$ is the closure of D . By Theorem 2.9.23, each component of W is a.s. unbounded, so

$$P^x[\tau_D < \infty] = 1; \quad \forall x \in D \subset \mathbb{R}^d, \quad D \text{ bounded.}$$

Let $B_r \triangleq \{x \in \mathbb{R}^d; \|x\| < r\}$ be the open ball of radius r centered at the origin. The volume of this ball is

$$V \triangleq \frac{2r^d \pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})},$$

and its surface area is

$$S_r \triangleq \frac{2r^{d-1} \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \frac{d}{r} V_r.$$

We define a probability measure μ_r on ∂B_r by

$$\mu_r(dx) = P^0[W_{\tau_{B_r}} \in dx]; \quad r > 0.$$

In the integral notation, the above definition for $A \subset \partial B_r$ becomes:

$$\mu_r(A) = \int_{x \in A} P^0[W_{\tau_{B_r}} \in dx],$$

which we interpret as the probability of the Brownian motion W_t crossing the boundary ∂B_r by passing through points in A .

A. The mean-value property

Because of the rotational invariance of Brownian motion (Problem 3.3.18), the measure μ_r is also rotationally invariant and thus proportional to surface measure on ∂B_r . In particular, the Lebesgue integral of a function f over B_r can be written in iterated form as

$$\int_{B_r} f(x) dx = \int_0^r S_\rho \int_{\partial B_\rho} f(x) \mu_\rho(dx) d\rho.$$

2.1 Definition We say that the function $u : D \mapsto \mathbb{R}$ has the **mean-value property** if, for every $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have

$$u(a) = \int_{\partial B_r} u(a+x) \mu_r(dx).$$

Then we derive the consequence

$$u(a) = \frac{1}{V_r} \int_{B_r} u(a+x) dx.$$

$$\begin{aligned} \because \int_{B_r} u(a+x) dx &= \int_0^r S_\rho \int_{\partial B_\rho} u(a+x) \mu_\rho(dx) d\rho = \int_0^r S_\rho u(a+x) d\rho = \\ &= u(a+x) \int_0^r S_\rho d\rho = u(a+x) V_r \end{aligned}$$

(the second inequality follows from the mean-value property)

of the mean-value property, which asserts that the mean integral value of u over a ball is equal to the value at the center.

2.2 Proposition If u is harmonic in D , then it has the mean-value property there.

(Proof) With $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B} \subset D$, we have from Ito's formula:

$$\begin{aligned} u(W_{t \wedge \tau_{a+B_r}}) &= u(W_0) + \sum_{i=1}^d \int_0^{t \wedge \tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s) dW_s^{(i)} + \frac{1}{2} \int_0^{t \wedge \tau_{a+B_r}} \Delta u(W_s) ds = \\ &= u(W_0) + \sum_{i=1}^d \int_0^{t \wedge \tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s) dW_s^{(i)} \text{ where } 0 \leq t < \infty, \end{aligned}$$

since u is harmonic and $(\partial u / \partial x_i); 1 \leq i \leq d$, are bounded functions on $a + B_r$, the expectations under P^a of the stochastic integrals are all equal to 0. After taking these expectations on both sides and letting $t \rightarrow \infty$, we use

$$u(a) = E^a u(W_{\tau_{a+B_r}}) = \int_{\partial B_r} u(a+x) \mu_r(dx). \quad \square$$

2.3 Corollary (Maximum Principle) Suppose that u is harmonic in the open, connected domain D . If u achieves its supremum over D at some point in D ,

then u is identically constant.

Proof) Let $M = \sup_{x \in D} u(x)$, and let $D_M = \{x \in D; u(x) = M\}$. We assume that D_M is nonempty and show that $D_M = D$. Since u is continuous, $D_M = u^{-1}(\{M\}) \cap D$ is a closed set relative to D . But for $a \in D_M$, and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have the mean value property:

$$M = u(a) = \frac{1}{V_r} \int_{B_r} u(a+x) dx \leq \frac{1}{V_r} \int_{B_r} M dx = M,$$

which shows that $u = M$ on $a + B_r$.

Since $a \in D_M$ was arbitrary, and $a \in a + B_r \subset D_M$, we conclude D_M is open. Moreover, D is connected, either D_M or $D - D_M$ must be empty. \square

For the sake of completeness, below is the converse of Proposition 2.2.

2.5 Proposition If u maps D into \mathbb{R} and has the mean-value property, then u is of class C^∞ and harmonic.

Proof) We first prove that u is of class C^∞ . For $\epsilon > 0$, let $g_\epsilon : \mathbb{R}^d \rightarrow [0, \infty)$ be the C^∞ function

$$g_\epsilon(x) = \begin{cases} c(\epsilon) \exp \left[\frac{1}{\|x\|^2 - \epsilon^2} \right], & \|x\| < \epsilon \\ 0, & \|x\| \geq \epsilon \end{cases} \quad (1)$$

where $c(\epsilon)$ is chosen so that

$$\begin{aligned} \int_{B_\epsilon} g_\epsilon(x) dx &= \int_0^\epsilon S_\rho \int_{\partial B_\rho} g_\epsilon(x) \mu_\rho(dx) d\rho = \\ &= c(\epsilon) \int_0^\epsilon S_\rho \int_{\partial B_\rho} \exp\left(\frac{1}{\|x\|^2 - \epsilon^2}\right) \mu_\rho(dx) d\rho = c(\epsilon) \int_0^\epsilon S_\rho \exp\left(\frac{1}{\rho^2 - \epsilon^2}\right) d\rho = 1. \end{aligned}$$

For $\epsilon > 0$ and $a \in D$ s.t. $a + \bar{B}_\epsilon \subset D$, define

$$u_\epsilon(a) \triangleq \int_{B_\epsilon} u(a+x) g_\epsilon(x) dx = \int_{\mathbb{R}^d} u(y) g_\epsilon(y-a) dy.$$

From the second representation, u_ϵ is of class C^∞ on the open subset of D where it is defined. Furthermore, for every $a \in D$ there exists $\epsilon > 0$ so that $a + \bar{B}_\epsilon \subset D$; from mean-value property of u , we have

$$\begin{aligned} u_\epsilon(a) &= \int_{B_\epsilon} u(a+x) g_\epsilon(x) dx = c(\epsilon) \int_0^\epsilon S_\rho \int_{\partial B_\rho} u(a+x) \exp\left(\frac{1}{\rho^2 - \epsilon^2}\right) \mu_\rho(dx) d\rho = \\ &= c(\epsilon) \int_0^\epsilon S_\rho u(a) \exp\left(\frac{1}{\rho^2 - \epsilon^2}\right) d\rho = u(a) \end{aligned}$$

where the last equality is from the definition of $c(\varepsilon)$. Thus, u is also of class C^∞ .

In order to show that $\Delta u = 0$ in D , we choose $a \in D$ and use a Taylor-series expansion in the neighborhood $a + \bar{B}_\varepsilon$,

$$u(a + y) = u(a) + \sum_{i=1}^d y_i \frac{\partial u}{\partial x_i}(a) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d y_i y_j \frac{\partial^2 u}{\partial x_i \partial x_j}(a) + o(\|y\|^2); \quad y \in \bar{B}_\varepsilon,$$

where again $\varepsilon > 0$ is chosen so that $a + \bar{B}_\varepsilon \subset D$. Odd symmetry gives us

$$\int_{\partial B_\varepsilon} y_i \mu_\varepsilon(dy) = 0, \quad \int_{\partial B_\varepsilon} y_i y_j \mu_\varepsilon(dy) = 0; \quad i \neq j,$$

so integrating the above Taylor-expansion over ∂B_ε and using the mean-value property, we have

$$u(a) = \int_{\partial B_\varepsilon} u(a + y) \mu_\varepsilon(dy) = u(a) + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}(a) \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) + o(\varepsilon^2).$$

But

$$\int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{1}{d} \sum_{i=1}^d \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{\varepsilon^2}{d},$$

thus we have

$$\frac{\varepsilon^2}{2d} \Delta u(a) + o(\varepsilon^2) = 0.$$

Dividing by ε^2 and letting $\varepsilon \downarrow 0$, we have $\Delta u(a) = 0$. \square

B. The Dirichlet problem

We take up now the Dirichlet problem (D, f) : with open $D \subset \mathbb{R}^d$ and $f : \partial D \rightarrow \mathbb{R}$ is a given continuous function, find a continuous function $u : \bar{D} \rightarrow \mathbb{R}$ s.t.

$$\Delta u = 0; \quad \text{in } D$$

$$u = f; \quad \text{on } \partial D.$$

Such a function, when it exists, will be called a solution to the Dirichlet problem (D, f) . One may interpret $u(x)$ as the steady-state temperature at $x \in D$ when the boundary temperatures of D are specified by f .

The power of the probabilistic method is demonstrated by the fact that we can immediately write down a very likely solution to (D, f) , namely

$$u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D},$$

provided that

$$E^x|f(W_{\tau_D})| < \infty; \quad \forall x \in D.$$

If $x \in \partial D$, then since $P^x[W_0 = x] = 1$, we have

$$u(x) = E^x f(W_{\tau_D}) = E^x f(W_0) = f(x).$$

Thus, u satisfies $u = f$ on ∂D . Furthermore, for $a \in D$ and B_r chosen so that $a + \bar{B}_r \subset D$, we have:

$$\begin{aligned} u(a) &= E^a f(W_{\tau_D}) \stackrel{\text{tower}}{=} E^a \{E^a[f(W_{\tau_D})|\mathcal{F}_{\tau_{a+B_r}}]\} = \\ &= E^a \{E^a[f(W_{\tau_D} - W_{\tau_{a+B_r}} + W_{\tau_{a+B_r}})|\mathcal{F}_{\tau_{a+B_r}}]\} = \\ &= E^a \{u(W_{\tau_{a+B_r}})\} \stackrel{\text{def}}{=} \int_{\partial B_r} u(a+x) \mu_r(dx), \end{aligned}$$

where the second last equality is from the strong Markov property of B.M.

Therefore, u has the mean-value property, and so it must satisfy $\Delta u = 0$; in D . The only unresolved issue is whether u is continuous up to and including ∂D .

2.6 Proposition If $E^x|f(W_{\tau_D})| < \infty$ holds, then $u(x) \triangleq E^x f(W_{\tau_D})$; $x \in \bar{D}$ is harmonic in D .

2.7 Proposition If f is bounded and

$$P^a[\tau_D < \infty] = 1; \quad \forall a \in D,$$

then any bounded solution to (D, f) has the representation $u(x) = E^x f(W_{\tau_D})$.

(Proof) Let u be any bounded solution to (D, f) , and let $D_n \triangleq \{x \in D; \inf_{y \in \partial D} \|x - y\| > \frac{1}{n}\}$. Then, D_n is an increasing sequence of subsets of D . From Ito's rule,

$$u(W_{t \wedge \tau_{B_n} \wedge \tau_{D_n}}) = u(W_0) + \sum_{i=1}^d \int_0^{t \wedge \tau_{B_n} \wedge \tau_{D_n}} \frac{\partial u}{\partial x_i}(W_s) dW_s^{(i)}; \quad 0 \leq t < \infty, \quad n \geq 1.$$

Since $\frac{\partial u}{\partial x_i}$ is bounded in $\overline{B_n \cap D_n}$, we take expectations w.r.t P^a from both sides:

$$E^a u(W_{t \wedge \tau_{B_n} \wedge \tau_{D_n}}) = E^a(u(W_0)) = u(a);$$

where $0 \leq t < \infty$, $n \geq 1$, $a \in D_n$.

As $t \rightarrow \infty, n \rightarrow \infty, P^a[\tau_D < \infty] = 1$; $\forall a \in D$ implies that $u(W_{t \wedge \tau_{B_n} \wedge \tau_{D_n}})$ converges to $f(W_{\tau_D})$, a.s. P^a . The representation $u(x) = E^x f(W_{\tau_D})$; $x \in \bar{D}$ follows from the bounded convergence theorem. \square

In the light of Proposition 2.6 and 2.7, the existence of a solution to the Dirichlet problem boils down to the question of the continuity of u defined by

$E^x f(W_{\tau_D})$ at the boundary of D . We therefore undertake to characterize those points $a \in \partial D$ for which

$$\lim_{x \rightarrow a, x \in D} E^x f(W_{\tau_D}) = f(a)$$

holds for every bounded, measurable function $f : \partial D \rightarrow \mathbb{R}$ which is continuous at the point a .

2.9 Definition Consider the stopping time of the right-continuous filtration $\{\mathcal{F}_t\}$ given by $\sigma_D \triangleq \inf\{t > 0; W_t \in D^c\}$. We say that a point $a \in \partial D$ is regular for D if $P^a[\sigma_D = 0] = 1$, i.e., a Brownian motion path started at a does not immediately return to D and remain there for a nonempty time interval.

2.10 Remark A point $a \in \partial D$ is called irregular if $P^a[\sigma_D = 0] < 1$; however, the event $\{\sigma_D = 0\}$ belongs to \mathcal{F}_{0+}^W , and so the Blumenthal zero-one law (Theorem 2.7.17) gives for an irregular point $a : P^a[\sigma_D = 0] = 0$.

2.11 Remark The regularity is a local condition; i.e. $a \in \partial D$ is regular for D if and only if a is regular for $(a + B_r) \cap D$, for some $r > 0$.

2.12 Theorem Assume that $d \geq 2$ and fix $a \in \partial D$. The following are equivalent:

- (i) $\lim_{x \rightarrow a, x \in D} E^x f(W_{\tau_D}) = f(a)$ holds for every bounded, measurable function $f : \partial D \rightarrow \mathbb{R}$ which is continuous at a ;
- (ii) a is regular for D ;
- (iii) for all $\varepsilon > 0$, we have

$$\lim_{x \rightarrow a, x \in D} P^x[\tau_D > \varepsilon] = 0.$$

(Proof) We assume WLOG that $a = 0$, and begin by proving the implication (i) \Rightarrow (ii) by contradiction. If the origin is irregular, then $P^0[\sigma_D = 0] = 0$ (Remark 2.10). Since a Brownian motion of dimension $d \geq 2$ never returns to its starting point (Prop 3.3.22), we have

$$\lim_{r \downarrow 0} P^0[W_{\tau_D} \in B_r] = P^0[W_{\tau_D} = 0] = 0.$$

Fix $r > 0$ for which $P^0[W_{\tau_D} \in B_r] < \frac{1}{4}$, and choose a sequence $\{\delta_n\}_{n=1}^\infty$ for which $0 < \delta_n < r$ for all n and $\delta_n \downarrow 0$. With $\tau_n \triangleq \inf\{t \geq 0; \|W_t\| \geq \delta_n\}$, we have $P^0[\tau_n \downarrow 0] = 1$, and thus $\lim_{n \rightarrow \infty} P^0[\tau_n < \sigma_D] = 1$. Furthermore, on the event $\{\tau_n < \sigma_D\}$ we have $W_{\tau_n} \in D$. For n large enough so that $P^0[\tau_n < \sigma_D] \geq \frac{1}{2}$ we may write

$$\begin{aligned} \frac{1}{4} &> P^0[W_{\sigma_D} \in B_r] \geq P^0[W_{\sigma_D} \in B_r, \tau_n < \sigma_D] = E^0(1_{\{W_{\sigma_D} \in B_r\}} 1_{\{\tau_n < \sigma_D\}}) = \\ &= E^0(1_{\{\tau_n < \sigma_D\}} E^0[1_{\{W_{\sigma_D} \in B_r\}} | \mathcal{F}_{\tau_n}]) = E^0(1_{\{\tau_n < \sigma_D\}} P^0[W_{\sigma_D} \in B_r | \mathcal{F}_{\tau_n}]) = \end{aligned}$$

$$= \int_{D \cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r] P^0[\tau_n < \sigma_D, W_{\tau_n} \in dx] \geq \frac{1}{2} \inf_{x \in D \cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r],$$

for which we conclude that $P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2}$ for some $x_n \in D \cap B_{\delta_n}$. Now choose a bounded, continuous function $f : \partial D \rightarrow \mathbb{R}$ s.t. $f = 0$ outside B_r , $f \leq 1$ inside B_r , and $f(0) = 1$. For such a function we have

$$\overline{\lim}_{n \rightarrow \infty} E^{x_n} f(W_{\tau_D}) \leq \overline{\lim}_{n \rightarrow \infty} P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2} < f(0),$$

and (i) fails.

We next show that (ii) \Rightarrow (iii). Observe first of all that for $0 < \delta < \varepsilon$, the function

$$\begin{aligned} g_\delta(x) &\triangleq P^x[W_s \in D; \delta \leq s \leq \varepsilon] = E^x(P^{W_\delta}[\tau_D > \varepsilon - \delta]) = \\ &= \int_{\mathbb{R}^d} P^y[\tau_D > \varepsilon - \delta] P^x[W_\delta \in dy] \end{aligned}$$

is continuous in x . But

$$g_\delta(x) \downarrow g(x) \triangleq P^x[W_s \in D; 0 < s \leq \varepsilon] = P^x[\sigma_D > \varepsilon]$$

as $\delta \downarrow 0$, so g is upper semicontinuous. From this fact and the inequality $\tau_D \leq \sigma_D$, we conclude that $\overline{\lim}_{x \rightarrow 0} P^x[\tau_D > \varepsilon] \leq \overline{\lim}_{x \rightarrow 0} g(x) \leq g(0) = 0$, by (ii).

Finally, we prove (iii) \Rightarrow (i). We know that for each $r > 0$, $P^x[\max_{0 \leq t \leq \varepsilon} \|W_t - W_0\| < r]$ does not depend on x and approaches one as $\varepsilon \downarrow 0$. But then

$$\begin{aligned} P^x[\|W_{\tau_D} - W_0\| < r] &\geq P^x[\{\max_{0 \leq t \leq \varepsilon} \|W_t - W_0\| < r\} \cap \{\tau_D \leq \varepsilon\}] \geq \\ &\geq P^0[\max_{0 \leq t \leq \varepsilon} \|W_t\| < r] - P^x[\tau_D > \varepsilon]. \end{aligned}$$

Letting $x \rightarrow 0$ ($x \in D$) and $\varepsilon \downarrow 0$, successively, we obtain from (iii),

$$\lim_{x \rightarrow 0, x \in D} P^x[\|W_{\tau_D} - x\| < r] = 1; \quad 0 < r < \infty.$$

The continuity of f at the origin and its boundedness on ∂D gives $\lim_{x \rightarrow 0, x \in D} E^x f(W_{\tau_D}) = f(a)$. \square

C. Conditions for regularity

For many open sets D and boundary points $a \in \partial D$, we can convince ourselves intuitively that a Brownian motion originating at a will exit from \bar{D} immediately, i.e., a is regular.

When $d = 2$, the center of a punctured disc is an irregular boundary point. The following development, culminating with Problem 2.16 shows that in \mathbb{R}^2 ,

any irregular boundary point of D must be "isolated" in the sense that it cannot be connected to any other point outside D by a simple arc lying outside D .

2.13 Definition Let $D \subset \mathbb{R}^d$ be open and $a \in \partial D$. A **barrier** at a is a continuous function $v : \bar{D} \rightarrow \mathbb{R}$ which is harmonic in D , positive on $\bar{D} - \{a\}$, and equal to zero at a .

2.14 Example Let $D \subset B_r \subset \mathbb{R}^2$ be open, where $0 < r < 1$, and assume $(0, 0) \in \partial D$. If a single valued, analytic branch of $\log(x_1 + ix_2)$ can be defined in $\bar{D} - (0, 0)$, then

$$v(x_1, x_2) \triangleq \begin{cases} -\operatorname{Re} \frac{1}{\log(x_1 + ix_2)} = -\frac{\log \sqrt{x_1^2 + x_2^2}}{|\log(x_1 + ix_2)|^2}; & (x_1, x_2) \in D - (0, 0), \\ 0; & (x_1, x_2) = (0, 0), \end{cases}$$

is a barrier at $(0, 0)$. Indeed being the real part of an analytic solution, v is harmonic in D , and because $0 < \sqrt{x_1^2 + x_2^2} \leq r < 1$ in $\bar{D} - (0, 0)$, v is positive on this set.

2.15 Proposition Let D be bounded and $a \in \partial D$. If there exists a barrier at a , then a is regular.

Proof) Let v be a barrier at a . We establish condition (i) of Theorem 2.12. With $f : \partial D \rightarrow \mathbb{R}$ bounded and continuous at a , define $M = \sup_{x \in \partial D} |f(x)|$. Choose $\varepsilon > 0$ and let $\delta > 0$ be s.t. $|f(x) - f(a)| < \varepsilon$ if $x \in \partial D$ and $\|x - a\| < \delta$. Choose k so that $kv(x) \geq 2M$ for $x \in \bar{D}$ and $\|x - a\| \geq \delta$.

We then have $|f(x) - f(a)| \leq \varepsilon + 2M \leq \varepsilon + kv(x)$; $x \in \partial D$, so

$$|E^x f(W_{\tau_D}) - f(a)| \leq E^x |f(W_{\tau_D}) - f(a)| \leq \varepsilon + kE^x v(W_{\tau_D}) = \varepsilon + kv(x); \quad x \in D$$

by Proposition 2.7. But v is continuous and $v(a) = 0$, so

$$\overline{\lim}_{x \rightarrow a, x \in D} |E^x f(W_{\tau_D}) - f(a)| \leq \varepsilon.$$

Finally, we let $\varepsilon \downarrow 0$ to obtain $\lim_{x \rightarrow a, x \in D} E^x f(W_{\tau_D}) = f(a)$. \square

2.17 Example (Lebesgue's Thorn) With $d = 3$ and $\{\varepsilon_n\}_{n=1}^\infty$ a sequence of positive numbers decreasing to zero, define

$$E = \{(x_1, x_2, x_3); -1 < x_1 < 1, x_2^2 + x_3^2 < 1\},$$

$$F_n = \{(x_1, x_2, x_3); 2^{-n} \leq x_1 \leq 2^{-n+1}, x_2^2 + x_3^2 \leq \varepsilon_n\},$$

$$D = E - \left(\bigcup_{n=1}^\infty F_n \right).$$

Now $P^0[(W_t^{(2)}, W_t^{(3)}) = (0, 0), \text{ for some } t > 0] = 0$ (Proposition 3.3.22), so the P^0 -probability that $W = (W^{(1)}, W^{(2)}, W^{(3)})$ ever hits the compact set $K_n \triangleq$

$\{(x_1, x_2, x_3); 2^{-n} \leq x_1 \leq 2^{-n+1}, x_2 = x_3 = 0\}$ is zero. According to Problem 3.3.24, $\lim_{t \rightarrow \infty} \|W_t\| = \infty$ a.s. P^0 , so for P^0 -a.e. $\omega \in \Omega$, the path $t \mapsto W_t(\omega)$ remains bounded away from K_n . Thus, if ε_n is chosen sufficiently small, we can ensure that $P^0[W_t \in F_n, \text{ for some } t \geq 0] \leq 3^{-n}$. If W , beginning at the origin, does not return to D immediately, it must avoid D by entering $\bigcup_{n=1}^{\infty} F_n$. In other words,

$$P^0[\sigma_D = 0] \leq P^0[W_t \in F_n, \text{ for some } t \geq 0 \text{ and } n \geq 1] \leq \sum_{n=1}^{\infty} < 1. \quad \square$$

If the cusplike behavior is avoided, then the boundary points of D are regular, regardless of the dimension. To make this statement precise, let us define for $y \in \mathbb{R}^d - \{0\}$ and $0 \leq \theta \leq \pi$, the **cone** $C(y, \theta)$ with direction y and aperture θ by

$$C(y, \theta) = \{x \in \mathbb{R}^d; (x, y) \geq \|x\| \|y\| \cos \theta\}.$$

2.18 Definition We say that the point $a \in \partial D$ satisfies the **Zaremba's cone condition** if there exists $y \neq 0$ and $0 < \theta < \pi$ s.t. the translated cone $a + C(y, \theta)$ is contained in $\mathbb{R}^d - D$.

2.19 Theorem If a point $a \in \partial D$ satisfies the Zaremba's cone condition, then it is regular.

Proof) We assume WLOG that a is the origin and $C(y, \theta) \subset \mathbb{R}^d - D$, where $y \neq 0$ and $0 < \theta < \pi$. Because the change of variables $z = \frac{x}{\sqrt{t}}$ maps $C(y, \theta)$ onto itself, we have for any $t > 0$,

$$\begin{aligned} P^0[W_t \in C(y, \theta)] &= \int_{C(y, \theta)} \frac{1}{(2\pi t)^{d/2}} \exp\left[-\frac{\|x\|^2}{2t}\right] dx = \\ &= \int_{C(y, \theta)} \frac{1}{(2\pi)^{d/2}} \exp\left[-\frac{\|z\|^2}{2}\right] dz \triangleq q > 0, \end{aligned}$$

where q is independent of t . Now, $P^0[\sigma_D \leq t] \geq P^0[W_t \in C(y, \theta)] = q$, and letting $t \downarrow 0$, we conclude that $P^0[\sigma_D = 0] > 0$. Regularity follows from the Blumenthal zero-one law (Remark 2.10).

2.20 Remark If, for $a \in \partial D$ and some $r > 0$, the point a satisfies Zaremba's cone condition for the set $(a + B_r) \cap D$, then a is regular for D (Remark 2.11).

E. Supplementary Exercises

Problem 2.25

4.3 The One-Dimensional Heat Equation

Consider an infinite rod, insulated and extended along the x -axis of the (t, x) plane, and let $f(x)$ denote the temperature of the rod at time $t = 0$ and location x . If $u(t, x)$ is the temperature of the rod at time $t \geq 0$ and position $x \in \mathbb{R}$, then, with appropriate choice of units, u will satisfy the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad (3.1)$$

with initial condition $u(0, x) = f(x)$; $x \in \mathbb{R}$.

Observe that the transition density

$$p(t; x, y) \triangleq \frac{1}{dy} P^x[W_t \in dy] = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}; \quad t > 0, \quad x, y \in \mathbb{R},$$

of the one-dimensional Brownian family satisfies the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}. \quad (3.2)$$

$$\therefore \frac{\partial p}{\partial t} = \frac{\partial}{\partial t} \left[\frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \right] = -\frac{1}{2} \frac{1}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t} + \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \left(\frac{(x-y)^2}{2t^2} \right);$$

$$\frac{\partial p}{\partial x} = \frac{1}{\sqrt{2\pi t}} \frac{-(x-y)}{t} e^{-(x-y)^2/2t} = \frac{-(x-y)}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t};$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{-1}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t} + \frac{-(x-y)}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t} \frac{-(x-y)}{t}.$$

Suppose then that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel-measurable function satisfying the condition

$$\int_{-\infty}^{\infty} e^{-ax^2} |f(x)| dx < \infty \quad (3.3)$$

for some $a > 0$. By Problem 3.1,

$$u(x) \triangleq E^x f(W_t) = \int_{-\infty}^{\infty} f(y) p(t; x, y) dy \quad (3.4)$$

is defined for $0 < t < \frac{1}{2a}$ and $x \in \mathbb{R}$, has derivatives of all orders, and satisfies the heat equation (3.1).

3.1. Problem Show that for any nonnegative integers n and m , under the assumption (3.3), we have

$$\frac{\partial^{n+m}}{\partial t^n \partial x^m} u(t, x) = \int_{-\infty}^{\infty} f(y) \frac{\partial^{n+m}}{\partial t^n \partial x^m} p(t; x, y) dy; \quad 0 < t < \frac{1}{2a}, \quad x \in \mathbb{R} \quad (3.5)$$

A. The Tychonoff uniqueness theorem

We call $p(t; x, y)$ a fundamental solution to the problem of finding a function u which satisfies the heat equation and agrees with the specified function f at time $t = 0$.

We shall say that a function $u : \mathbb{R}^m \rightarrow \mathbb{R}$ has continuous derivatives up to a certain order on a set G , if these derivatives exist and are continuous in the interior of G , and have continuous extensions on that part of the boundary ∂G which is included in G .

3.3 Theorem (Tychonoff (1935)). Suppose that the function u is $C^{1,2}$ on the strip $[0, T] \times \mathbb{R}$ and satisfies the heat equation (3.1) there, as well as the conditions

$$\lim_{t \downarrow 0, y \rightarrow x} u(t, y) = 0; \quad x \in \mathbb{R}, \quad (3.7)$$

$$\sup_{0 < t \leq T} |u(t, x)| \leq K e^{ax^2}; \quad x \in \mathbb{R}, \quad (3.8)$$

for some positive constant K and a . Then, $u = 0$ on $[0, T] \times \mathbb{R}$.

3.4 Remark. If u_1 and u_2 satisfy the heat equation and (3.8), and

$$\lim_{t \downarrow 0, y \rightarrow x} u_1(t, y) = \lim_{t \downarrow 0, y \rightarrow x} u_2(t, y),$$

then Theorem 3.3 applied to $u_1 - u_2$ asserts that $u_1 = u_2$ on $(0, T) \times \mathbb{R}$.

3.5 Remark. Any probabilistic treatment of the heat equation involves a time-reversal. This is already suggested by the representation (3.4), in which the initial temperature function f evaluated at W_t rather than W_0 .

Proof of Theorem 3.3) Let $T_y = \inf\{t \geq 0; W_t(\omega) = y\}$ be the passage time of W to y . Fix $x \in \mathbb{R}$, choose $n > |x|$, and let $R_n = T_n \wedge T_{-n}$. With $t \in [0, T)$ fixed and

$$v(\theta, x) \triangleq u(T - t - \theta, x); \quad 0 \leq \theta < T - t,$$

we have from Ito's rule, for $0 \leq s < T - t$,

$$\begin{aligned} u(T - t, x) &= v(0, x) = E^x v(s \wedge R_n, W_{s \wedge R_n}) = \\ &= E^x [v(s, W_s) 1_{\{s < R_n\}}] + E^x [v(R_n, W_{R_n}) 1_{\{s \geq R_n\}}]. \end{aligned} \quad (3.9)$$

Now $|v(s, W_s)| 1_{\{s < R_n\}}$ is dominated by

$$\max_{0 \leq s < T-t, |y| \leq n} |u(T - t - s, y)| \leq K e^{an^2}$$

and $v(s, W_s)$ converges P^x -a.s. to zero as $s \uparrow T - t$ by (3.7). Likewise, $|v(R_n, W_{R_n})| 1_{\{s \geq R_n\}}$ is dominated by $K e^{an^2}$. Letting $s \uparrow T - t$ in (3.9), we obtain from the bounded convergence theorem:

$$u(T - t, x) = E^x [v(R_n, W_{R_n}) 1_{\{R_n < T-t\}}].$$

Therefore, with $0 \leq t < T$, $|x| < n$,

$$\begin{aligned}
|u(T-t, x)| &\leq K e^{an^2} P^x[R_n < T-t] \leq K e^{an^2} P^x[R_n < T] \leq \\
&\leq K e^{an^2} (P^0[T_{n-x} < T] + P^0[T_{-n-x} < T]) = \\
&= K e^{an^2} (P^0[T_{n-x} < T] + P^0[T_{n+x} < T]) \leq \\
&\leq K e^{an^2} \sqrt{\frac{2}{n}} \left(\int_{(n-x)/\sqrt{T}}^{\infty} e^{-z^2/2} dz + \int_{(n+x)/\sqrt{T}}^{\infty} e^{-z^2/2} dz \right),
\end{aligned}$$

where we have used the distribution function of passage time of Brownian motion. But from (2.9.20), we have $\lim_{n \rightarrow \infty} e^{an^2} \int_{(n \pm x)/\sqrt{T}}^{\infty} e^{-z^2/2} dz = 0$, provided $a < \frac{1}{2T}$.

Having proved the theorem for $a < \frac{1}{2T}$, we can extend it to the case where this inequality does not hold. Given a time interval $[0, T]$, choose $T_0 = 0 < T_1 < \dots < T_n = T$ s.t. $a < \frac{1}{2(T_i - T_{i-1})}$; $i = 1, \dots, n$, and then show successively that $u = 0$ in each of the strips $(T_{i-1}, T_i]$; $i = 1, \dots, n$ by the above argument. \square

As a counter-example for the Tychonoff uniqueness theorem when the conditions are not satisfied, note that the function

$$h(t, x) \triangleq \frac{x}{t} p(t; x, 0) = \frac{\partial}{\partial x} p(t; x, 0); \quad t > 0, \quad x \in \mathbb{R}, \quad (3.10)$$

solves the heat equation (3.1) on every strip of the form $(0, T] \times \mathbb{R}$; furthermore, it satisfies condition (3.8) for every $0 < a < \frac{1}{2T}$, as well as (3.7) for every $x \neq 0$. However, the limit in (3.7) fails to exist for $x = 0$, although we do have $\lim_{t \downarrow 0} h(t, 0) = 0$.

B. Nonnegative solutions of the heat equation

If the initial temperature f is nonnegative, as it always is if measured on the absolute scale, then the temperature should remain nonnegative for all $t > 0$; this is evident from the representation (3.4). Is it possible to characterize the nonnegative solutions of the heat equation? This was done by Widder (1944) who showed that such functions u have a representation

$$u(t, x) = \int_{-\infty}^{\infty} p(t; x, y) dF(y); \quad x \in \mathbb{R},$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing (Corollary 3.7 (i)', (ii)'). We extend Widder's work by providing probabilistic characterizations of nonnegative solutions to the heat equation in Corollary 3.7 (iii)', (iv)').

3.6 Theorem Let $v(t, x)$ be a nonnegative function defined on a strip $(0, T) \times \mathbb{R}$, where $0 < T < \infty$. The following four conditions are equivalent:

(i) for some nondecreasing function $F : \mathbb{R} \rightarrow \mathbb{R}$,

$$v(t, x) = \int_{-\infty}^{\infty} p(T - t; x, y) dF(y); \quad 0 < t < T, \quad x \in \mathbb{R}; \quad (3.11)$$

(ii) v is of class $C^{1,2}$ on $(0, T) \times \mathbb{R}$ and satisfies the "backward" heat equation

$$\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} = 0 \quad (3.12)$$

on the strip;

(iii) for a Brownian family $\{W_s, \mathcal{F}_s; 0 \leq s < \infty\}$, (Ω, \mathcal{F}) , $\{P^x\}_{x \in \mathbb{R}}$ and each fixed $t \in (0, T)$, $x \in \mathbb{R}$, the process $\{v(t + s, W_s), \mathcal{F}_s; 0 \leq s < T - t\}$ is a martingale on $(\Omega, \mathcal{F}, P^x)$;

(iv) for a Brownian family $\{W_s, \mathcal{F}_s; 0 \leq s < \infty\}$, (Ω, \mathcal{F}) , $\{P^x\}_{x \in \mathbb{R}}$ we have

$$v(t, x) = E^x v(t + s, W_s); \quad 0 < t \leq t + s < T, \quad x \in \mathbb{R}. \quad (3.13)$$

Proof) (i) \Rightarrow (ii). Since

$$\frac{\partial}{\partial t} p(T - t; x, y) + \frac{1}{2} \frac{\partial^2}{\partial x^2} p(T - t; x, y) = 0,$$

we can prove the implication (i) \Rightarrow (ii) by showing that the partial derivatives of v can be computed by differentiating under the integral in (3.11).

For $a > \frac{1}{2T}$, we have

$$\int_{-\infty}^{\infty} e^{-ay^2} dF(y) = \sqrt{\frac{\pi}{a}} \int_{-\infty}^{\infty} p\left(\frac{1}{2a}; 0, y\right) dF(y) = \sqrt{\frac{\pi}{a}} v\left(T - \frac{1}{2a}, 0\right) < \infty.$$

This condition is analogous to (3.3) and allows us to proceed as in Problem 3.1:

$$\frac{\partial^{n+m}}{\partial t^n \partial x^m} v(t, x) = \int_{-\infty}^{\infty} f(y) \frac{\partial^{n+m}}{\partial t^n \partial x^m} p(t; x, y) dF(y); \quad 0 < t < \frac{1}{2a}, \quad x \in \mathbb{R}.$$

(ii) \Rightarrow (iii), (ii) \Rightarrow (iv).

We begin by applying Ito's rule to $v(t + s, W_s); 0 \leq s < T - t$.

$$v(t + s, W_s) = v(t, W_0) + \int_0^s \frac{\partial}{\partial x} v(t + \sigma, W_\sigma) dW_\sigma + \int_0^s \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) v(t + \sigma, W_\sigma) d\sigma.$$

With $a < x < b$, we consider the passage times T_a and T_b and obtain:

$$v(t + (s \wedge T_a \wedge T_b), W_{s \wedge T_a \wedge T_b}) = v(t, W_0) + \int_0^{s \wedge T_a \wedge T_b} \frac{\partial}{\partial x} v(t + \sigma, W_\sigma) dW_\sigma +$$

$$+ \int_0^{s \wedge T_a \wedge T_b} \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) v(t + \sigma, W_\sigma) d\sigma.$$

Under the assumption (ii), the Lebesgue integral vanishes, as does the expectation of the stochastic integral because $\frac{\partial}{\partial x} v(t + \sigma, y)$ is bounded when $a \leq y \leq b$ and $0 \leq \sigma \leq s < T - t$.

$$\therefore v(t, x) = E^x v(t + (s \wedge T_a \wedge T_b), W_{s \wedge T_a \wedge T_b}). \quad (3.14)$$

Letting $a \downarrow -\infty, b \uparrow \infty$ and relying on the nonnegativity of v and Fatou's lemma, we have

$$v(t, x) \geq E^x \left[\liminf_{a \downarrow -\infty, b \uparrow \infty} v(t + (s \wedge T_a \wedge T_b)) \right] = E^x v(t + s, W_s); \quad 0 < t \leq t + s < T, \quad (3.15)$$

Claim: Inequality (3.15) implies that for fixed $t \in (0, T)$ and $x \in \mathbb{R}$, the process $\{v(t + s, W_s), \mathcal{F}_s; 0 \leq s < T - t\}$ is a supermartingale on $(\Omega, \mathcal{F}, P^x)$.

\therefore For $0 \leq s_1 \leq s_2 < T - t$, the Markov property (Proposition 2.5.13) yields

$$E^x[v(t + s_2, W_{s_2}) | \mathcal{F}_{s_1}](\omega) = f(W_{s_1}(\omega)) \quad \text{for } P^x\text{-a.e. } \omega \in \Omega, \quad (3.16)$$

where

$$f(y) \triangleq E^y v(t + s_2, W_{s_2 - s_1}). \quad (3.17)$$

Prop 2.5.13: $P^x[X_{s+t} \in \Gamma | \mathcal{F}_s] = E^x f(X_s) \Rightarrow$

From (3.15), we have

$$E^y v(t + s_2, W_{s_2 - s_1}) \leq v(t + s_1, y),$$

and so for $0 < t \leq t + s_1 \leq t + s_2 < T, \quad x \in \mathbb{R} :$

$$v(t + s_1, W_{s_1}) \geq E^x[v(t + s_2, W_{s_2}) | \mathcal{F}_{s_1}], \quad \text{a.s. } P^x. \quad (3.18)$$

Therefore, if the equality holds in (3.15), then $\{v(t + s, W_s), \mathcal{F}_s; 0 \leq s < T - t\}$ is a martingale. We now establish the reverse inequality.

We may write (3.14) as

$$\begin{aligned} v(t, x) &= E^x[v(t + s, W_s) 1_{\{s \leq T_a \wedge T_b\}}] + E^x[v(t + T_a, a) 1_{\{T_a < s \wedge T_b\}}] \\ &\quad + E^x[v(t + T_b, b) 1_{\{T_b < s \wedge T_a\}}] \leq E^x v(t + s, W_s) + \\ &\quad E^x[v(t + T_a, a) 1_{\{T_a < s\}}] + E^x[v(t + T_b, b) 1_{\{T_b < s\}}]. \end{aligned}$$

We will establish (3.13) as soon as we prove

$$\liminf_{b \rightarrow \infty} E^x[v(t + T_b, b) 1_{\{T_b < s\}}] = 0 \quad (3.19)$$

(a dual argument then shows that $\liminf_{a \rightarrow -\infty} E^x[v(t + T_a, a)1_{\{T_a < s\}}] = 0$). For (3.19), it suffices to show that with $B > 0$ large enough, we have

$$\int_B^\infty E^x[v(t + T_b, b)1_{\{T_b < s\}}]db < \infty.$$

We choose $x \in \mathbb{R}, 0 < t < T$ and $0 \leq s < t$ so that $s + t < T$. From (2.6.3) and (3.10) we have

$$P^x[T_b \in d\sigma] = h(\sigma; b - x)d\sigma \quad b > x, \sigma > 0.$$

$$\therefore P^0[T_b \in dt] = \frac{|b|}{\sqrt{2\pi}t^3}e^{-b^2/2t}dt; \quad t > 0.$$

For $B \geq x$ sufficiently large, $h(\sigma, b - x)$ is an increasing function of $\sigma \in (0, s)$, provided $b \geq B$. Furthermore, for $r \in (s, t)$ and B perhaps larger, we have

$$h(s, b - x) \leq \sqrt{\frac{r}{s^3}}p(r; x, b); \quad b \geq B.$$

It follows that

$$\begin{aligned} \int_B^\infty E^x[v(t + T_b, b)1_{\{T_b < s\}}]db &= \int_B^\infty \int_0^s v(t + \sigma, b)h(\sigma, b - x)d\sigma db \leq \\ &\leq \sqrt{\frac{r}{s^3}} \int_0^s \int_B^\infty v(t + \sigma, b)p(r; x, b)db d\sigma \leq \sqrt{\frac{r}{s^3}} \int_0^s E^x v(t + \sigma, W_r) d\sigma \leq \\ &\leq \sqrt{\frac{r}{s^3}} \int_0^s v(t + \sigma - r, x) d\sigma < \infty, \end{aligned}$$

where the next to last inequality is a consequence of (3.15). This proves (3.13) for $x \in \mathbb{R}, 0 < t \leq t + s < T$, as long as $s < t$.

We now remove the unwanted restriction $s < t$. We show by induction on the positive integers k that if

$$0 < t \leq t + s < T, \quad s < kt, \tag{3.20}$$

then

$$v(t, x) = E^x v(t + s, W_s); \quad x \in \mathbb{R}. \tag{3.21}$$

This will yield (3.13) for the range of values indicated there. We have just established that (3.20) implies (3.21) when $k = 1$. Assume this implication holds for some $k \geq 1$, so $\{v(t + s, W_s), \mathcal{F}_s; 0 \leq s < kt\}$ is a martingale. Choose $s_2 \in [kt, (k + 1)t)$ and $s_1 \in [0, kt)$ so that $0 < s_2 - s_1 < t$. Then,

$$E^x v(t + s_2, W_{s_2}) = E^x \{E^x[v(t + s_2, W_{s_2})|\mathcal{F}_{s_1}]\} = E^x v(t + s_1, W_{s_1}) = v(t, x),$$

where we have used (3.16), (3.17) and the induction hypothesis in the form $E^y v(t + s_2, W_{s_2-s_1}) = E^y v(t + s_1 + (s_2 - s_1), W_{s_2-s_1}) = v(t + s_1, y)$ for the second equality.

(iv) \Rightarrow (i)

For $0 < \varepsilon < \frac{T}{4}$, $\frac{T}{2} < t < T$, $v(t, x) = E^x v(t + s, W_s)$ gives

$$v(t - \varepsilon, x) = E^x(t - \varepsilon + s, W_s) = E^x v(T - \varepsilon, W_{T-t}) = \int_{-\infty}^{\infty} \frac{p(T - t; x, y)}{p(\frac{T}{2}; 0, y)} dF_{\varepsilon}(y),$$

where F_{ε} is the nondecreasing function

$$F_{\varepsilon}(x) \triangleq \int_{-\infty}^x p\left(\frac{T}{2}; 0, y\right) v(T - \varepsilon, y) dy; \quad x \in \mathbb{R}.$$

Again, from $v(t, x) = E^x v(t + s, W_s)$, we have $F_{\varepsilon}(\infty) = E^0 v(T - \varepsilon, W_{T/2}) = E^0 v(T/2 - \varepsilon + T/2, W_{T/2}) = v(T/2 - \varepsilon, 0)$, and thus

$$\sup_{0 < \varepsilon < T/4} F_{\varepsilon}(\infty) \leq \max_{T/4 \leq t \leq T/2} v(t, 0) < \infty.$$

By Helly's (selection) theorem, there exists a seq. $\varepsilon_1 > \dots > \varepsilon_k \downarrow 0$ and a nondecreasing function $F^* : \mathbb{R} \rightarrow [0, \infty)$ s.t. $\lim_{k \rightarrow \infty} F_{\varepsilon_k}(x) = F^*(x)$ for every x at which F^* is continuous.

\therefore Helly's selection theorem: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of increasing functions mapping a real interval I into the real line \mathbb{R} , and suppose that it is uniformly bounded. Then, the sequence $(f_n)_{n \in \mathbb{N}}$ admits a pointwise convergent subsequence.

Because for fixed $x \in \mathbb{R}$ and $t \in ((T/2), T)$ the ratio $\frac{p(T-t; x, y)}{p((T/2); 0, y)}$ is a bounded, continuous function of y , converging to 0 as $|y| \rightarrow \infty$, we have

$$\begin{aligned} v(t, x) &= \lim_{k \rightarrow \infty} v(t - \varepsilon_k, x) = \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} \frac{p(T - t; x, y)}{p(\frac{T}{2}; 0, y)} dF_{\varepsilon_k}(y) = \\ &= \int_{-\infty}^{\infty} \frac{p(T - t; x, y)}{p(\frac{T}{2}; 0, y)} dF^*(y) \end{aligned}$$

by the extended Helly-Bray lemma.

\therefore Helly-Bray lemma: If $F_n \rightarrow F$ and g is bounded and continuous a.s. F , then

$$Eg(X_n) = \int g dF_n \rightarrow \int g dF = Eg(X).$$

Defining $F(x) = \int_0^x \frac{dF^*(y)}{p((T/2); 0, y)}$, we have (3.11) for $T/2 < t < T$, $x \in \mathbb{R}$. If $0 < t \leq T/2$, we choose $t_1 \in (T/2, T)$ and write

$$\begin{aligned} v(t, x) &= E^x v(t + (t_1 - t), W_{t_1-t}) = \int_{-\infty}^{\infty} p(t_1 - t; x, y) v(t_1, y) dy = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(t_1 - t; x, y) p(T - t_1; y, z) dy dF(z) = \end{aligned}$$

$$= \int_{-\infty}^{\infty} p(T-t; x, z) dF(z). \quad \square$$

3.7 Corollary Let $u(t, x)$ be a nonnegative function defined on a strip $(0, T) \times \mathbb{R}$, where $0 < T \leq \infty$. The following four conditions are equivalent:

(i)' for some nondecreasing function $F : \mathbb{R} \rightarrow \mathbb{R}$,

$$u(t, x) = \int_{-\infty}^{\infty} p(t; x, y) dF(y); \quad 0 < t < T, x \in \mathbb{R}; \quad (3.22)$$

(ii)' u is of class $C^{1,2}$ on $(0, T) \times \mathbb{R}$ and satisfies the heat equation (3.1) there;

(iii)' for a Brownian family $\{W_s, \mathcal{F}_s; 0 \leq s < \infty\}$, (Ω, \mathcal{F}) , $\{P^x\}_{x \in \mathbb{R}}$ and each fixed $t \in (0, T)$, $x \in \mathbb{R}$, the process $\{u(t-s, W_s), \mathcal{F}_s; 0 \leq s < t\}$ is a martingale on $(\Omega, \mathcal{F}, P^x)$;

(iv)' for a Brownian family $\{W_s, \mathcal{F}_s; 0 \leq s < \infty\}$, (Ω, \mathcal{F}) , $\{P^x\}_{x \in \mathbb{R}}$ we have

$$u(t, x) = E^x u(t-s, W_s); \quad 0 \leq s < t < T, x \in \mathbb{R}. \quad (3.23)$$

Proof) If $T < \infty$, we obtain this corollary by defining $v(t, x) = u(T-t, x)$ and appealing to Theorem 3.6. If $T = \infty$, then for each integer $n \geq 1$ we set $v_n(t, x) = u(n-t, x); 0 < t < n, x \in \mathbb{R}$. Applying Theorem 3.6 to each v_n we see that conditions (ii)', (iii)', and (iv)' are equivalent, they are implied by (i)' and they imply the existence, for any fixed $n \geq 1$, of a nondecreasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ s.t. (3.22) holds on $(0, n) \times \mathbb{R}$. For $t \geq n$, we have from (3.23):

$$\begin{aligned} u(t, x) &= E^x u\left(\frac{n}{2}, W_{t-n/2}\right) = \int_{-\infty}^{\infty} u\left(\frac{n}{2}, z\right) p\left(t - \frac{n}{2}; x, z\right) dz = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p\left(\frac{n}{2}; z, y\right) p\left(t - \frac{n}{2}; x, z\right) dz dF(y) = \int_{-\infty}^{\infty} p(t; x, y) dF(y). \quad \square \end{aligned}$$

Can we represent nonnegative solutions $v(t, x)$ of the backward heat equation

$$\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} = 0$$

on the entire half-plane $(0, \infty) \times \mathbb{R}$, just as we did in Corollary 3.7 for nonnegative solutions $u(t, x)$ of the heat equation (3.1)? Certainly this cannot be achieved by a simple time-reversal on the results of Corollary 3.7. Instead, we can relate the functions u and v by the formula

$$v(t, x) = \sqrt{\frac{2\pi}{t}} \exp\left(\frac{x^2}{2t}\right) u\left(\frac{1}{t}, \frac{x}{t}\right); \quad 0 < t < \infty, \quad x \in \mathbb{R}. \quad (3.24)$$

Claim: v satisfies (3.12) on $(0, \infty) \times \mathbb{R}$ if and only if u satisfies the heat equation (3.1) there.

3.9 Proposition (Robbins & Siegmund (1973)) Let $v(t, x)$ be a nonnegative function defined on the half-plane $(0, \infty) \times \mathbb{R}$. With $T = \infty$, conditions (ii), (iii), (iv) of Theorem 3.6 are equivalent to one another, and to (i)' :

$$v(t, x) = \int_{-\infty}^{\infty} \exp(yx - \frac{1}{2}y^2t) dF(y); \quad 0 < t < \infty, \quad x \in \mathbb{R}. \quad (3.25)$$

Proof) The equivalence of (ii), (iii) and (iv) for $T = \infty$ follows from their equivalence for all finite T . If v is given by (3.25), then differentiation under the integral can be justified as in Theorem 3.6, and it results in

$$\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} = 0.$$

If v satisfies (ii), then u given by (3.24) satisfies (ii)', and hence (i)' of Corollary 3.7. However, (3.24) and (3.22) reduce to (3.25). \square

C. Boundary Crossing Probabilities for Brownian motion

The representation (3.25) has rather unexpected consequences in the computation of boundary-crossing probabilities for Brownian motion. Let us consider a positive function $v(t, x)$ which is defined and of class $C^{1,2}$ on $(0, \infty) \times \mathbb{R}$, and satisfies the backward heat equation. Then v admits the representation (3.25) for some F , and differentiating under the integral we see that

$$\frac{\partial}{\partial t} v(t, x) = \int_{-\infty}^{\infty} -\frac{1}{2} y^2 \exp(yx - \frac{1}{2} y^2 t) dF(y) < 0; \quad 0 < t < \infty, \quad x \in \mathbb{R} \quad (3.26)$$

and that $v(t, \cdot)$ is convex for each $t > 0$. In particular, $\lim_{t \downarrow 0} v(t, 0)$ exists. We assume that this limit is finite, and, WLOG (by scaling if necessary) that

$$\lim_{t \downarrow 0} v(t, 0) = 1. \quad (3.27)$$

We also assume that

$$\lim_{t \rightarrow \infty} v(t, 0) = 0, \quad (3.28)$$

$$\lim_{x \rightarrow \infty} v(t, x) = \infty; \quad 0 < t < \infty, \quad (3.29)$$

$$\lim_{x \rightarrow -\infty} v(t, x) = 0, \quad 0 < t < \infty. \quad (3.30)$$

(3.27)-(3.30) are satisfied if and only if F is a probability distribution function with $F(0+) = 0$. We impose this condition, so that (3.25) becomes

$$v(t, x) = \int_{0+}^{\infty} \exp(yx - \frac{1}{2} y^2 t) dF(y); \quad 0 < t < \infty, \quad x \in \mathbb{R}, \quad (3.31)$$

where $F(\infty) = 1$, $F(0+) = 0$. This representation shows that $v(t, \cdot)$ is strictly increasing, so for each $t > 0$ and $b > 0$ there is a unique number $A(t, b)$ s.t.

$$v(t, A(t, b)) = b. \quad (3.32)$$

Moreover, the function $A(\cdot, b)$ is continuous and strictly increasing (3.26). We may define $A(0, b) = \lim_{t \downarrow 0} A(t, b)$.

We shall show how one can compute the **probability that a Brownian path W , starting at the origin, will eventually cross the curve $A(\cdot, b)$** . The problem of computing the probability that a Brownian motion crosses a given, time-dependent continuous boundary $\{\psi(t); 0 \leq t < \infty\}$ is thereby reduced to finding a solution v to the backward heat equation which also satisfies (3.27) – (3.30) and $v(t, \psi(t)) = b$; $0 \leq t < \infty$, for some $b > 0$. In this generality, both problems are quite difficult; our point is that the probabilistic problem can be traded for a partial differential equation problem. We shall provide an explicit solution to both of them when the boundary is linear.

Let $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, (Ω, \mathcal{F}) , $\{P^x\}_{x \in \mathbb{R}}$ be a Brownian family, and define

$$Z_t = v(t, W_t); \quad 0 < t < \infty.$$

For $0 < s < t$, we have from the Markov property and condition (iv) of Proposition 3.9:

$$E^0[Z_t | \mathcal{F}_s] = E^0[v(t, W_t) | \mathcal{F}_s] = f(W_s) = v(s, W_s) = Z_s, \quad a.s. \ P^0,$$

where $f(y) \triangleq E^y v(t, W_{t-s})$. In other words, $\{Z_t, \mathcal{F}_t; 0 < t < \infty\}$ is a continuous, nonnegative martingale on $(\Omega, \mathcal{F}, P^0)$. Let $\{t_n\}$ be a sequence of positive numbers with $t_n \downarrow 0$, and set $Z_0 = \lim_{n \rightarrow \infty} Z_{t_n}$. This limit exists, P^0 -a.s. and is independent of the particular sequence $\{t_n\}$ chosen, (Proposition 1.3.14(i)). Being \mathcal{F}_{0+}^W -measurable, Z_0 must be a.s. constant (Theorem 2.7.17)

Theorem 2.7.17 (Blumenthal Zero-One Law). Let $\{B_t, \tilde{\mathcal{F}}_t; t \geq 0\}$, (Ω, \mathcal{F}) , $\{P^x\}_{x \in \mathbb{R}^d}$ be a d -dimensional Brownian family, where $\tilde{\mathcal{F}}_t \triangleq \bigcap_{\mu} \mathcal{F}_t^\mu$. If $F \in \tilde{\mathcal{F}}_0$, then for each $x \in \mathbb{R}^d$ we have either $P^x(F) = 0$ or $P^x(F) = 1$.

3.10 Lemma The extended process $Z \triangleq \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is continuous, nonnegative martingale under P^0 and satisfies $Z_0 = 1, Z_\infty = 0, P^0$ -a.s.

Proof) Let $\{t_n\}$ be a sequence of positive numbers with $t_n \downarrow 0$. The sequence $\{Z_n\}_{n=1}^\infty$ is uniformly integrable (Problem 1.3.11, Remark 1.3.12), so by the Markov property for W , we have for all $t > 0$:

$$E^0[Z_t | \mathcal{F}_0] = E^0 Z_t = \lim_{n \rightarrow \infty} E^0 Z_{t_n} = E^0 Z_0 = Z_0.$$

This establishes that $\{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale.

Since $Z_\infty \triangleq \lim_{t \rightarrow \infty} Z_t$ exists P^0 -a.s. (Problem 1.3.16), as does $Z_0 \triangleq \lim_{t \downarrow 0} Z_t$, it suffices to show that $\lim_{t \downarrow 0} Z_t = 1$ and $\lim_{t \rightarrow \infty} Z_t = 0$ in P^0 -probability. For every finite $c > 0$, we shall show that

$$\lim_{t \downarrow 0} \sup_{|x| \leq c\sqrt{t}} |v(t, x) - 1| = 0. \quad (3.33)$$

Indeed, for $t > 0$, $|x| \leq c\sqrt{t}$:

$$\int_{0+}^{\infty} \exp\left(-yc\sqrt{t} - \frac{1}{2}y^2t\right) dF(y) \leq v(t, x) \leq \int_{0+}^{\infty} \exp\left(yc\sqrt{t} - \frac{1}{2}y^2t\right) dF(y). \quad (3.34)$$

Because $\pm yc\sqrt{t} - \frac{1}{2}y^2t \leq \frac{c^2}{2}$; $\forall y > 0$, the bounded convergence theorem implies that both integrals in (3.34) converge to 1, as $t \downarrow 0$, and (3.33) follows. Thus, for any $\varepsilon > 0$, we can find $t_{c,\varepsilon}$ depending on c and ε , s.t.

$$1 - \varepsilon < v(t, x) < 1 + \varepsilon; \quad |x| \leq c\sqrt{t}, \quad 0 < t < t_{c,\varepsilon}.$$

Consequently, for $0 < t < t_{c,\varepsilon}$,

$$P^0[|Z_t - 1| > \varepsilon] = P^0[|v(t, W_t) - 1| > \varepsilon] \leq P^0[|W_t| > c\sqrt{t}] = 2[1 - \Phi(c)],$$

where

$$\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz.$$

Letting first $t \downarrow 0$ and then $c \rightarrow \infty$, we conclude that $Z_t \rightarrow 1$ in probability as $t \downarrow 0$. A similar argument shows that

$$\lim_{t \rightarrow \infty} \sup_{|x| \leq c\sqrt{t}} v(t, x) = 0, \quad (3.35)$$

and, using (3.35) instead of (3.33), one can also show that $Z_t \rightarrow 0$ in probability as $t \rightarrow \infty$. \square

It is now a fairly straightforward matter to apply Problem 1.3.28 to the martingale Z and obtain the probability that the Brownian path $\{W_t(\omega); 0 \leq t < \infty\}$ ever crosses the boundary $\{A(t, b); 0 \leq t < \infty\}$.

\therefore Problem 1.3.28. Let $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a continuous, nonnegative martingale with $Z_\infty \triangleq \lim_{t \rightarrow \infty} Z_t = 0$, a.s. P . Then, for every $s \geq 0, b > 0$:

$$(i) \quad P\left[\sup_{t \geq s} Z_t \geq b \mid \mathcal{F}_s\right] = \frac{1}{b} Z_s, \quad \text{a.s. on } \{Z_s < b\}.$$

$$(ii) \quad P\left[\sup_{t \geq s} Z_t \geq b\right] = P[Z_s \geq b] + \frac{1}{b} E[Z_s 1_{\{Z_s < b\}}].$$

3.12 Example. With $\mu > 0$, let $v(t, x) = \exp(\mu x - \mu^2 t/2)$, so $A(t, b) = \beta t + \gamma$, where $\beta = \frac{\mu}{2}$, $\gamma = \frac{1}{\mu} \log b$. Then, $F(y) = 1_{[\mu, \infty)}(y)$, and so for any $s > 0, \beta > 0, \gamma \in \mathbb{R}$, and Lebesgue-almost every $a < \gamma + \beta s$:

$$P^0[W_t \geq \beta t + \gamma, \text{ for some } t \geq s | W_s = a] = e^{-2\beta(\gamma - a + \beta s)} \quad (3.38)$$

and for any $s > 0, \beta > 0$, and $\gamma \in \mathbb{R}$:

$$P^0[W_t \geq \beta t + \gamma, \text{ for some } t \geq s | W_s = a] = 1 - \Phi\left(\frac{\gamma}{\sqrt{s}} + \beta\sqrt{s}\right) + e^{-2\beta\gamma} \Phi\left(\frac{\gamma}{\sqrt{s}} - \beta\sqrt{s}\right). \quad (3.39)$$

The observation that the time-inverted process Y of Lemma 2.9.4 ($Y_t = tW_{1/t}$; $0 < t < \infty$, $Y_t = 0$ if $t = 0$) is a Brownian motion allows one to cast (3.38) with $\gamma = 0$ into the following formula for the maximum of the so-called "tied-down" Brownian motion or "Brownian bridge":

$$P^0\left[\max_{0 \leq t \leq T} W_t \geq \beta | W_T = a\right] = e^{-2\beta(\beta - a)/T} \quad (3.40)$$

for $T > 0, \beta > 0$, a.e. $\alpha \leq \beta$, and (3.39) into a boundary-crossing probability on the bounded interval $[0, T]$:

$$P^0[W_t \geq \beta + \gamma t, \text{ for some } t \in [0, T]] = 1 - \Phi\left(\gamma\sqrt{T} + \frac{\beta}{\sqrt{T}}\right) + e^{-2\beta\gamma} \Phi\left(\gamma\sqrt{T} - \frac{\beta}{\sqrt{T}}\right). \quad (3.41)$$

D. Mixed initial/boundary value problems

We now discuss the concept of temperatures in a semi-infinite rod and the relation of this concept to Brownian motion absorbed at the origin. Suppose that $f : (0, \infty) \rightarrow \mathbb{R}$ is a Borel-measurable function satisfying

$$\int_0^\infty e^{-ax^2} |f(x)| dx < \infty \quad (3.42)$$

for some $a > 0$. We define

$$u_1(t, x) \triangleq E^x[f(W_t)1_{\{T_0 > t\}}]; \quad 0 < t < \frac{1}{2a}, \quad x > 0. \quad (3.43)$$

The reflection principle gives us the formula (2.8.9)

$$P^x[W_t \in dy, T_0 > t] = p_-(t; x, y) dy \triangleq [p(t; x, y) - p(t; x, -y)] dy$$

for $t > 0, x > 0, y > 0$, and so

$$u_1(t, x) = \int_0^\infty f(y)p(t; x, y) dy - \int_{-\infty}^0 f(-y)p(t; x, -y) dy \quad (3.44)$$

which gives us a definition for u_1 valid on the whole strip $(0, \frac{1}{2a}) \times \mathbb{R}$. This representation is of the form (3.4) $(u(t, x) = E^x f(W_t) = \int_{-\infty}^{\infty} f(y)p(t; x, y)dy)$, where the initial datum f satisfies $f(y) = -f(-y)$; $y > 0$. Then, u_1 has derivatives of all orders, satisfies the heat equation, satisfies $f(x) = \lim_{t \downarrow 0, y \rightarrow x} u_1(t, y)$ at all continuity points of f , and

$$\lim_{t \downarrow 0, s \rightarrow t} u_1(s, x) = 0; \quad 0 < t < \frac{1}{2a}.$$

We may regard $u_1(t, x); 0 < t < \frac{1}{2a}, x \geq 0$, as the temperature in a semi-infinite rod along the nonnegative axis, when the end $x = 0$ is held at a constant temperature (equal to 0) and the initial temperature at $y > 0$ is $f(y)$.

4.4 The Formulas of Feynman and Kac

A. The multi-dimensional formula

B. The one-dimensional formula