Chapter 4 Notes

4.2 Harmonic Functions and the Dirichlet Problem

A function $u : D \mapsto \mathbb{R}$ where D is an open subset of $\mathbb{R}^d$ is called **harmonic** in D if u is of class C^2 and $\Delta u \triangleq \sum_{i=1}^{d}(\frac{\partial^2 u}{\partial x_i^2}) = 0$ in D.
Throughout this section, $\{W_t, \mathcal{F}_t; 0 \le t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x\in\mathbb{R}^d}$ is a d-dimensional Brownian family and $\{\mathcal{F}_t\}$ satisfies the usual conditions. We denote by D an open set in $\mathbb{R}^d$ and introduce the stopping time (Problem 1.2.7)

$$\tau_D = \inf\{t \ge 0; W_t \in D^c\},$$

the time of first exit from D. The boundary of D will be denoted by ∂D, and $\bar{D} = D \cup \partial D$ is the closure of D. By Theorem 2.9.23, each component of W is a.s. unbounded, so

$$P^x[\tau_D < \infty] = 1; \quad \forall x \in D \subset \mathbb{R}^d,\ D \text{ bounded}.$$

Let $B_r \triangleq \{x \in \mathbb{R}^d; \|x\| < r\}$ be the open ball of radius r centered at the origin. The volume of this ball is

$$V \triangleq \frac{2r^d\pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})},$$

and its surface area is

$$S_r \triangleq \frac{2r^{d-1}\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \frac{d}{r}V_r.$$

We define a probability measure μ_r on ∂B_r by

$$\mu_r(dx) = P^0[W_{\tau_{B_r}} \in dx]; \quad r > 0.$$

In the integral notation, the above definition for $A \subset \partial B_r$ becomes:

$$\mu_r(A) = \int_{x\in A} P^0[W_{\tau_{B_r}} \in dx],$$

which we interpret as the probability of the Brownian motion W_t crossing the boundary ∂B_r by passing through points in A.

A. The mean-value property

Because of the rotational invariance of Brownian motion (Problem 3.3.18), the measure μ_r is also rotationally invariant and thus proportional to surface measure on ∂B_r. In particular, the Lebesgue integral of a function f over B_r can be written in iterated form as

$$\int_{B_r} f(x)dx = \int_0^r S_\rho \int_{\partial B_\rho} f(x)\mu_\rho(dx)d\rho.$$

2.1 Definition We say that the function $u : D \mapsto \mathbb{R}$ has the **mean-value property** if, for every $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have

$$u(a) = \int_{\partial B_r} u(a+x)\mu_r(dx).$$

Then we derive the consequence

$$u(a) = \frac{1}{V_r}\int_{B_r} u(a+x)dx.$$

$\because \int_{B_r} u(a+x)dx = \int_0^r S_\rho \int_{\partial B_\rho} u(a+x)\mu_\rho(dx)d\rho = \int_0^r S_\rho u(a+x)d\rho =$
$= u(a+x)\int_0^r S_\rho d\rho = u(a+x)V_r$
(the second inequality follows from the mean-value property)
of the mean-value property, which asserts that the mean integral value of u over a ball is equal to the value at the center.

2.2 Proposition If u is harmonic in D, then it has the mean-value property there.

Proof) With $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B} \subset D$, we have from Ito's formula:

$$u(W_{t\wedge\tau_{a+B_r}}) = u(W_0) + \sum_{i=1}^d \int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} + \frac{1}{2}\int_0^{t\wedge\tau_{a+B_r}} \Delta u(W_s)ds =$$

$$= u(W_0) + \sum_{i=1}^d \int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} \text{ where } 0 \le t < \infty,$$

since u is harmonic and $(\partial u/\partial x_i)$; $1 \le i \le d$, are bounded functions on $a + B_r$, the expectations under P^a of the stochastic integrals are all equal to 0. After taking these expectations on both sides and letting $t \to \infty$, we use

$$u(a) = E^a u(W_{\tau_{a+B_r}}) = \int_{\partial B_r} u(a+x)\mu_r(dx). \quad \square$$

2.3 Corollary (Maximum Principle) Suppose that u is harmonic in the open, connected domain D. If u achieves its supremum over D at some point in D, then u is identically constant.

Proof) Let $M = \sup_{x\in D} u(x)$, and let $D_M = \{x \in D; u(x) = M\}$. We assume that D_M is nonempty and show that $D_M = D$. Since u is continuous, $D_M = u^{-1}(\{M\}) \cap D$ is a closed set relative to D. But for $a \in D_M$, and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have the mean value property:

$$M = u(a) = \frac{1}{V_r}\int_{B_r} u(a+x)dx \leq \frac{1}{V_r}\int_{B_r} M dx = M,$$

which shows that $u = M$ on $a + B_r$.
Since $a \in D_M$ was arbitrary, and $a \in a + B_r \subset D_M$, we conclude D_M is open. Moreover, D is connected, either D_M or $D - D_M$ must be empty. □

For the sake of completeness, below is the converse of Proposition 2.2.

2.5 Proposition If u maps D into $\mathbb{R}$ and has the mean-value property, then u is of class C^∞ and harmonic.

Proof) We first prove that u is of class C^∞. For $\epsilon > 0$, let $g_\varepsilon : \mathbb{R}^d \to [0,\infty)$ be the C^∞ function

$$g_\varepsilon(x) = \begin{cases} c(\varepsilon)\exp\left[\frac{1}{\|x\|^2-\varepsilon^2}\right], & \|x\| < \varepsilon \\ 0, & \|x\| \geq \varepsilon \end{cases} \tag{1}$$

where $c(\varepsilon)$ is chosen so that

$$\int_{B_\varepsilon} g_\varepsilon(x)dx = \int_0^\varepsilon S_\rho \int_{\partial B_\rho} g_\varepsilon(x)\mu_\rho(dx)d\rho =$$

$$= c(\varepsilon)\int_0^\varepsilon S_\rho \int_{\partial B_\rho} \exp(\frac{1}{\|x\|^2-\varepsilon^2})\mu_\rho(dx)d\rho = c(\varepsilon)\int_0^\varepsilon S_\rho \exp(\frac{1}{\rho^2-\varepsilon^2})d\rho = 1.$$

For $\varepsilon > 0$ and $a \in D$ s.t. $a + \bar{B}_\varepsilon \subset D$, define

$$u_\varepsilon(a) \triangleq \int_{B_\varepsilon} u(a+x)g_\varepsilon(x)dx = \int_{\mathbb{R}^d} u(y)g_\varepsilon(y-a)dy.$$

From the second representation, u_ε is of class C^∞ on the open subset of D where it is defined. Furthermore, for every $a \in D$ there exists $\varepsilon > 0$ so that $a + \bar{B}_\varepsilon \subset D$; from mean-value property of u, we have

$$u_\varepsilon(a) = \int_{B_\varepsilon} u(a+x)g_\varepsilon(x)dx = c(\varepsilon)\int_0^\varepsilon S_\rho \int_{\partial B_\rho} u(a+x)\exp(\frac{1}{\rho^2-\varepsilon^2})\mu_\rho(dx)d\rho =$$

$$= c(\varepsilon)\int_0^\varepsilon S_\rho u(a)\exp(\frac{1}{\rho^2-\varepsilon^2})d\rho = u(a)$$

where the last equality is from the definition of $c(\varepsilon)$. Thus, u is also of class C^∞.

In order to show that $\Delta u = 0$ in D,we choose $a \in D$ and use a Taylor-series expansion in the neighborhood $a + \bar{B}_\varepsilon$,

$$u(a+y) = u(a) + \sum_{i=1}^{d} y_i \frac{\partial u}{\partial x_i}(a) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} y_i y_j \frac{\partial^2 u}{\partial x_i \partial x_j}(a) + o(\|y\|^2);\ y \in \bar{B}_\varepsilon,$$

where again $\varepsilon > 0$ is chosen so that $a + \bar{B}_\varepsilon \subset D$. Odd symmetry gives us

$$\int_{\partial B_\varepsilon} y_i \mu_\varepsilon(dy) = 0, \quad \int_{\partial B_\varepsilon} y_i y_j \mu_\varepsilon(dy) = 0; \quad i \neq j,$$

so integrating the above Taylor-expansion over ∂B_ε and using the mean-value property, we have

$$u(a) = \int_{\partial B_\varepsilon} u(a+y)\mu_\varepsilon(dy) = u(a) + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(a) \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) + o(\varepsilon^2).$$

But

$$\int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{1}{d}\sum_{i=1}^{d} \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{\varepsilon^2}{d},$$

thus we have

$$\frac{\varepsilon^2}{2d}\Delta u(a) + o(\varepsilon^2) = 0.$$

Dividing by ε^2 and letting $\varepsilon \downarrow 0$, we have $\Delta u(a) = 0$. $\square$

B. The Dirichlet problem

We take up now the Dirichlet problem (D, f) : with open $D \subset \mathbb{R}^d$ and $f : \partial D \to \mathbb{R}$ is a given continuous function, find a continuous function $u : \bar{D} \to \mathbb{R}$ s.t.

$$\Delta u = 0;\ \text{in } D$$

$$u = f;\ \text{on } \partial D.$$

Such a function, when it exists, will be called a solution to the Dirichlet problem (D, f). One may interpret $u(x)$ as the steady-state temperature at $x \in D$ when the boundary temperatures of D are specified by f.

The power of the probabilistic method is demonstrated by the fact that we can immediately write down a very likely solution to (D, f), namely

$$u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D},$$

provided that

$$E^x|f(W_{\tau_D})| < \infty; \quad \forall x \in D.$$

If $x \in \partial D$, then since $P^x[W_0 = x] = 1$, we have

$$u(x) = E^x f(W_{\tau_D}) = E^x f(W_0) = f(x).$$

Thus, u satisfies $u = f$ on ∂D. Furthermore, for $a \in D$ and B_r chosen so that $a + \bar{B}_r \subset D$, we have:

$$u(a) = E^a f(W_{\tau_D}) \overset{\text{tower}}{=} E^a\{E^a[f(W_{\tau_D})|\mathcal{F}_{\tau_{a+B_r}}]\} =$$

$$= E^a\{E^a[f(W_{\tau_D} - W_{\tau_{a+B_r}} + W_{\tau_{a+B_r}})|\mathcal{F}_{\tau_{a+B_r}}]\} =$$

$$= E^a\{u(W_{\tau_{a+B_r}})\} \overset{\text{def}}{=} \int_{\partial B_r} u(a+x)\mu_r(dx),$$

where the second last equality is from the strong Markov property of B.M.

Therefore, u has the mean-value property, and so it must satisfy $\Delta u = 0;$ in D. The only unresolved issue is whether u is continuous up to and including ∂D.
2.6 Proposition If $E^x|f(W_{\tau_D})| < \infty$ holds, then $u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D}$ is harmonic in D.

2.7 Proposition If f is bounded and

$$P^a[\tau_D < \infty] = 1; \quad \forall a \in D,$$

then any bounded solution to (D, f) has the representation $u(x) = E^x f(W_{\tau_D})$.

Proof) Let u be any bounded solution to (D, f), and let $D_n \triangleq \{x \in D; \inf_{y \in \partial D} \|x - y\| > \frac{1}{n}\}$. Then, D_n is an increasing sequence of subsets of D. From Ito's rule,

$$u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = u(W_0) + \sum_{i=1}^{d} \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)}; \quad 0 \le t < \infty, \quad n \ge 1.$$

Since $\frac{\partial u}{\partial x_i}$ is bounded in $\overline{B_n \cap D_n}$, we take expectations w.r.t P^a from both sides:

$$E^a u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = E^a(u(W_0)) = u(a);$$

where $0 \le t < \infty, \ n \ge 1, \ a \in D_n$.
As $t \to \infty, n \to \infty, P^a[\tau_D < \infty] = 1; \ \forall a \in D$ implies that $u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}})$ converges to $f(W_{\tau_D})$, a.s. P^a. The representation $u(x) = E^x f(W_{\tau_D}); \quad x \in \bar{D}$ follows from the bounded convergence theorem. □

In the light of Proposition 2.6 and 2.7, the existence of a solution to the Dirichlet problem boils down to the question of the continuity of u defined by

$E^x f(W_{\tau_D})$ at the boundary of D. We therefore undertake to characterize those points $a \in \partial D$ for which

$$\lim_{x\to a, x\in D} E^x f(W_{\tau_D}) = f(a)$$

holds for every bounded, measurable function $f : \partial D \to \mathbb{R}$ which is continuous at the point a.

2.9 Definition Consider the stopping time of the right-continuous filtration $\{\mathcal{F}_t\}$ given by $\sigma_D \triangleq \inf\{t > 0; W_t \in D^c\}$. We say that a point $a \in \partial D$ is regular for D if $P^a[\sigma_D = 0] = 1$, i.e., a Brownian motion path started at a does not immediately return to D and remain there for a nonempty time interval.

2.10 Remark A point $a \in \partial D$ is called irregular if $P^a[\sigma_D = 0] < 1$; however, the event $\{\sigma_D = 0\}$ belongs to $\mathcal{F}^W_{0+}$, and so the Blumenthal zero-one law (Theorem 2.7.17) gives for an irregular point $a : P^a[\sigma_D = 0] = 0$.

2.11 Remark The regularity is a local condition; i.e. $a \in \partial D$ is regular for D if and only if a is regular for $(a + B_r) \cap D$, for some $r > 0$.

2.12 Theorem Assume that $d \geq 2$ and fix $a \in \partial D$. The following are equivalent:
(i) $\lim_{x\to a, x\in D} E^x f(W_{\tau_D}) = f(a)$ holds for every bounded, measurable function $f : \partial D \to \mathbb{R}$ which is continuous at a;
(ii) a is regular for D;
(iii) for all $\varepsilon > 0$, we have

$$\lim_{x\to a, x\in D} P^x[\tau_D > \varepsilon] = 0.$$

Proof) We assume WLOG that $a = 0$, and begin by proving the implication $(i) \Rightarrow (ii)$ by contradiction. If the origin is irregular, then $P^0[\sigma_D = 0] = 0$ (Remark 2.10). Since a Brownian motion of dimension $d \geq 2$ never returns to its starting point (Prop 3.3.22), we have

$$\lim_{r\downarrow 0} P^0[W_{\tau_D} \in B_r] = P^0[W_{\tau_D} = 0] = 0.$$

Fix $r > 0$ for which $P^0[W_{\tau_D} \in B_r] < \frac{1}{4}$, and choose a sequence $\{\delta_n\}_{n=1}^{\infty}$ for which $0 < \delta_n < r$ for all n and $\delta_n \downarrow 0$. With $\tau_n \triangleq \inf\{t \geq 0; \|W_t\| \geq \delta_n\}$, we have $P^0[\tau_n \downarrow 0] = 1$, and thus $\lim_{n\to\infty} P^0[\tau_n < \sigma_D] = 1$. Furthermore, on the event $\{\tau_n < \sigma_D\}$ we have $W_{\tau_n} \in D$. For n large enough so that $P^0[\tau_n < \sigma_D] \geq \frac{1}{2}$ we may write

$$\frac{1}{4} > P^0[W_{\sigma_D} \in B_r] \geq P^0[W_{\sigma_D} \in B_r, \tau_n < \sigma_D] = E^0(1_{\{W_{\sigma_D}\in B_r\}} 1_{\{\tau_n<\sigma_D\}}) =$$

$$= E^0(1_{\{\tau_n<\sigma_D\}} E^0[1_{\{W_{\sigma_D}\in B_r\}}|\mathcal{F}_{\tau_n}]) = E^0(1_{\{\tau_n<\sigma_D\}} P^0[W_{\sigma_D} \in B_r|\mathcal{F}_{\tau_n}]) =$$

$$= \int_{D \cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r] P^0[\tau_n < \sigma_D, W_{\tau_n} \in dx] \geq \frac{1}{2} \inf_{x \in D \cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r],$$

for which we conclude that $P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2}$ for some $x_n \in D \cap B_{\delta_n}$. Now choose a bounded, continuous function $f : \partial D \to \mathbb{R}$ s.t. $f = 0$ outside B_r, $f \leq 1$ inside B_r, and $f(0) = 1$. For such a function we have

$$\overline{\lim}_{n \to \infty} E^{x_n} f(W_{\tau_D}) \leq \overline{\lim}_{n \to \infty} P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2} < f(0),$$

and (i) fails.
We next show that $(ii) \Rightarrow (iii)$. Observe first of all that for $0 < \delta < \varepsilon$, the function

$$g_\delta(x) \triangleq P^x[W_s \in D; \delta \leq s \leq \varepsilon] = E^x(P^{W_\delta}[\tau_D > \varepsilon - \delta]) =$$

$$= \int_{\mathbb{R}^d} P^y[\tau_D > \varepsilon - \delta] P^x[W_\delta \in dy]$$

is continuous in x. But

$$g_\delta(x) \downarrow g(x) \triangleq P^x[W_s \in D; 0 < s \leq \varepsilon] = P^x[\sigma_D > \varepsilon]$$

as $\delta \downarrow 0$, so g is upper semicontinuous. From this fact and the inequality $\tau_D \leq \sigma_D$, we conclude that $\overline{\lim}_{x \to 0} P^x[\tau_D > \varepsilon] \leq \overline{\lim}_{x \to 0} g(x) \leq g(0) = 0$, by (ii).

Finally, we prove $(iii) \Rightarrow (i)$. We know that for each $r > 0$, $P^x[\max_{0 \leq t \leq \varepsilon} \|W_t - W_0\| < r]$ does not depend on x and approaches one as $\varepsilon \downarrow 0$. But then

$$P^x[\|W_{\tau_D} - W_0\| < r] \geq P^x[\{\max_{0 \leq t \leq \varepsilon} \|W_t - W_0\| < r\} \cap \{\tau_D \leq \varepsilon\}] \geq$$

$$\geq P^0[\max_{0 \leq t \leq \varepsilon} \|W_t\| < r] - P^x[\tau_D > \varepsilon].$$

Letting $x \to 0$ $(x \in D)$ and $\varepsilon \downarrow 0$, successively, we obtain from (iii),

$$\lim_{x \to 0, x \in D} P^x[\|W_{\tau_D} - x\| < r] = 1; \quad 0 < r < \infty.$$

The continuity of f at the origin and its boundedness on ∂D gives $\lim_{x \to 0, x \in D} E^x f(W_{\tau_D}) = f(a)$. $\square$

C. Conditions for regularity

For many open sets D and boundary points $a \in \partial D$, we can convince ourselves intuitively that a Brownian motion originating at a will exit from $\bar{D}$ immediately, i.e., a is regular.

When $d = 2$, the center of a punctured disc is an irregular boundary point. The following development, culminating with Problem 2.16 shows that in $\mathbb{R}^2$, any irregular boundary point of D must be "isolated" in the sense that it cannot be connected to any other point outside D by a simple arc lying outside D.

2.13 Definition Let $D \subset \mathbb{R}^d$ be open and $a \in \partial D$. A **barrier** at a is a continuous function $v : \bar{D} \to \mathbb{R}$ which is harmonic in D, positive on $\bar{D} - \{a\}$, and equal to zero at a.

2.14 Example Let $D \subset B_r \subset \mathbb{R}^2$ be open, where $0 < r < 1$, and assume $(0,0) \in \partial D$. If a single valued, analytic branch of $\log(x_1 + ix_2)$ can be defined in $\bar{D} - (0,0)$, then

$$v(x_1, x_2) \triangleq \begin{cases} -\mathrm{Re}\frac{1}{\log(x_1+ix_2)} = -\frac{\log\sqrt{x_1^2+x_2^2}}{|\log(x_1+ix_2)|^2}; & (x_1, x_2) \in D - (0,0), \\ 0; & (x_1, x_2) = (0,0), \end{cases}$$

is a barrier at $(0,0)$. Indeed being the real part of an analytic solution, v is harmonic in D, and because $0 < \sqrt{x_1^2 + x_2^2} \leq r < 1$ in $\bar{D} - (0,0)$, v is positive on this set.

2.15 Proposition Let D be bounded and $a \in \partial D$. If there exists a barrier at a, then a is regular.

Proof) Let v be a barrier at a. We establish condition (i) of Theorem 2.12. With $f : \partial D \to \mathbb{R}$ bounded and continuous at a, define $M = \sup_{x \in \partial D} |f(x)|$. Choose $\varepsilon > 0$ and let $\delta > 0$ be s.t. $|f(x) - f(a)| < \varepsilon$ if $x \in \partial D$ and $\|x - a\| < \delta$. Choose k so that $kv(x) \geq 2M$ for $x \in \bar{D}$ and $\|x - a\| \geq \delta$.
We then have $|f(x) - f(a)| \leq \varepsilon + 2M \leq \varepsilon + kv(x)$; $x \in \partial D$, so

$$|E^x f(W_{\tau_D}) - f(a)| \leq E^x |f(W_{\tau_D}) - f(a)| \leq \varepsilon + kE^x v(W_{\tau_D}) = \varepsilon + kv(x); \quad x \in D$$

by Proposition 2.7. But v is continuous and $v(a) = 0$, so

$$\overline{\lim}_{x \to a, x \in D} |E^x f(W_{\tau_D}) - f(a)| \leq \varepsilon.$$

Finally, we let $\varepsilon \downarrow 0$ to obtain $\lim_{x \to a, x \in D} E^x f(W_{\tau_D}) = f(a)$. $\square$

2.17 Example (Lebesgue's Thorn) With $d = 3$ and $\{\varepsilon_n\}_{n=1^\infty}$ a sequence of positive numbers decreasing to zero, define

$$E = \{(x_1, x_2, x_3);\ -1 < x_1 < 1,\ x_2^2 + x_3^2 < 1\},$$

$$F_n = \{(x_1, x_2, x_3);\ 2^{-n} \leq x_1 \leq 2^{-n+1},\ x_2^2 + x_3^2 \leq \varepsilon_n\},$$

$$D = E - (\bigcup_{n=1}^{\infty} F_n).$$

Now $P^0[(W_t^{(2)}, W_t^{(3)}) = (0,0)$, for some $t > 0] = 0$ (Proposition 3.3.22), so the P^0-probability that $W = (W^{(1)}, W^{(2)}, W^{(3)})$ ever hits the compact set $K_n \triangleq$

$\{(x_1, x_2, x_3); 2^{-n} \le x_1 \le 2^{-n+1}, x_2 = x_3 = 0\}$ is zero. According to Problem 3.3.24, $\lim_{t\to\infty} \|W_t\| = \infty$ a.s. P^0, so for P^0-a.e. $\omega \in \Omega$, the path $t \mapsto W_t(\omega)$ remains bounded away from K_n. Thus, if ε_n is chosen sufficiently small, we can ensure that $P^0[W_t \in F_n$, for some $t \ge 0] \le 3^{-n}$. If W, beginning at the origin, does not return to D immediately, it must avoid D by entering $\bigcup_{n=1}^{\infty} F_n$. In other words,

$$P^0[\sigma_D = 0] \le P^0[W_t \in F_n, \text{ for some } t \ge 0 \text{ and } n \ge 1] \le \sum_{n=1}^{\infty} < 1. \quad \square$$

If the cusplike behavior is avoided, then the boundary points of D are regular, regardless of the dimension. To make this statement precise, let us define for $y \in \mathbb{R}^d - \{0\}$ and $0 \le \theta \le \pi$, the **cone** $C(y, \theta)$ with direction y and aperture θ by

$$C(y, \theta) = \{x \in \mathbb{R}^d; (x, y) \ge \|x\|\|y\| \cos\theta\}.$$

2.18 Definition We say that the point $a \in \partial D$ satisfies the **Zaremba's cone condition** if there exists $y \ne 0$ and $0 < \theta < \pi$ s.t. the translated cone $a + C(y, \theta)$ is contained in $\mathbb{R}^d - D$.

2.19 Theorem If a point $a \in \partial D$ satisfies the Zaremba's cone condition, then it is regular.

Proof) We assume WLOG that a is the origin and $C(y, \theta) \subset \mathbb{R}^d - D$, where $y \ne 0$ and $0 < \theta < \pi$. Because the change of variables $z = \frac{x}{\sqrt{t}}$ maps $C(y, \theta)$ onto itself, we have for any $t > 0$,

$$P^0[W_t \in C(y, \theta)] = \int_{C(y,\theta)} \frac{1}{(2\pi t)^{d/2}} \exp[-\frac{\|x\|^2}{2t}] dx =$$

$$= \int_{C(y,\theta)} \frac{1}{(2\pi)^{d/2}} \exp[-\frac{\|z\|^2}{2}] dz \triangleq q > 0,$$

where q is independent of t. Now, $P^0[\sigma_D \le t] \ge P^0[W_t \in C(y, \theta)] = q$, and letting $t \downarrow 0$, we conclude that $P^0[\sigma_D = 0] > 0$. Regularity follows from the Blumenthal zero-one law (Remark 2.10).

2.20 Remark If, for $a \in \partial D$ and some $r > 0$, the point a satisfies Zaremba's cone condition for the set $(a + B_r) \cap D$, then a is regular for D (Remark 2.11).

E. Supplementary Exercises

Problem 2.25

4.3 The One-Dimensional Heat Equation

Consider an infinite rod, insulated and extended along the x-axis of the (t,x) plane, and let $f(x)$ denote the temperature of the rod at time $t=0$ and location x. If $u(t,x)$ is the temperature of the rod at time $t \geq 0$ and position $x \in \mathbb{R}$, then, with appropriate choice of units, u will satisfy the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \tag{3.1}$$

with initial condition $u(0,x) = f(x); \quad x \in \mathbb{R}$.
Observe that the transition density

$$p(t;x,y) \triangleq \frac{1}{dy} P^x[W_t \in dy] = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}; \quad t > 0, \quad x, y \in \mathbb{R},$$

of the one-dimensional Brownian family satisfies the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}. \tag{3.2}$$

$$\because \frac{\partial p}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}\right] = -\frac{1}{2}\frac{1}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t} + \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}\left(\frac{(x-y)^2}{2t^2}\right);$$

$$\frac{\partial p}{\partial x} = \frac{1}{\sqrt{2\pi t}} \frac{-(x-y)}{t} e^{-(x-y)^2/2t} = \frac{-(x-y)}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t};$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{-1}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t} + \frac{-(x-y)}{\sqrt{2\pi t^3}} e^{-(x-y)^2/2t} \frac{-(x-y)}{t}.$$

Suppose then that $f : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function satisfying the condition

$$\int_{-\infty}^{\infty} e^{-ax^2} |f(x)| dx < \infty \tag{3.3}$$

for some $a > 0$. By Problem 3.1,

$$u(x) \triangleq E^x f(W_t) = \int_{-\infty}^{\infty} f(y) p(t;x,y) dy \tag{3.4}$$

is defined for $0 < t < \frac{1}{2a}$ and $x \in \mathbb{R}$, has derivatives of all orders, and satisfies the heat equation (3.1).

3.1. Problem Show that for any nonnegative integers n and m, under the assumption (3.3), we have

$$\frac{\partial^{n+m}}{\partial t^n \partial x^m} u(t,x) = \int_{-\infty}^{\infty} f(y) \frac{\partial^{n+m}}{\partial t^n \partial x^m} p(t;x,y) dy; \quad 0 < t < \frac{1}{2a}, \quad x \in \mathbb{R} \tag{3.5}$$

A. The Tychonoff uniqueness theorem

We call $p(t; x, y)$ a fundamental solution to the problem of finding a function u which satisfies the heat equation and agrees with the specified function f at time $t = 0$.

We shall say that a function $u : \mathbb{R}^m \to \mathbb{R}$ has continuous derivatives up to a certain order on a set G, if these derivatives exist and are continuous in the interior of G, and have continuous extensions on that part of the boundary ∂G which is included in G.

3.3 Theorem (Tychonoff (1935)). Suppose that the function u is $C^{1,2}$ on the strip $[0, T] \times \mathbb{R}$ and satisfies the heat equation (3.1) there, as well as the conditions

$$\lim_{t\downarrow 0, y\to x} u(t, y) = 0; \quad x \in \mathbb{R}, \tag{3.7}$$

$$\sup_{0<t\leq T} |u(t, x)| \leq K e^{ax^2}; \quad x \in \mathbb{R}, \tag{3.8}$$

for some positive constant K and a. Then, $u = 0$ on $[0, T] \times \mathbb{R}$.

3.4 Remark. If u_1 and u_2 satisfy the heat equation and (3.8), and

$$\lim_{t\downarrow 0, y\to x} u_1(t, y) = \lim_{t\downarrow 0, y\to x} u_2(t, y),$$

then Theorem 3.3 applied to $u_1 - u_2$ asserts that $u_1 = u_2$ on $(0, T) \times \mathbb{R}$.

3.5 Remark. Any probabilistic treatment of the heat equation involves a time-reversal. This is already suggested by the representation (3.4), in which the initial temperature function f evaluated at W_t rather than W_0.

Proof of Theorem 3.3) Let $T_y = \inf\{t \geq 0; W_t(\omega) = y\}$ be the passage time of W to y. Fix $x \in \mathbb{R}$, choose $n > |x|$, and let $R_n = T_n \wedge T_{-n}$. With $t \in [0, T)$ fixed and

$$v(\theta, x) \triangleq u(T - t - \theta, x); \quad 0 \leq \theta < T - t,$$

we have from Ito's rule, for $0 \leq s < T - t$,

$$\begin{aligned} u(T-t, x) = v(0, x) &= E^x v(s \wedge R_n, W_{s\wedge R_n}) = \\ = E^x[v(s, W_s) 1_{\{s<R_n\}}] &+ E^x[v(R_n, W_{R_n}) 1_{\{s\geq R_n\}}]. \end{aligned} \tag{3.9}$$

Now $|v(s, W_s)| 1_{\{s<R_n\}}$ is dominated by

$$\max_{0\leq s<T-t, |y|\leq n} |u(T - t - s, y)| \leq K e^{an^2}$$

and $v(s, W_s)$ converges P^x-a.s. to zero as $s \uparrow T - t$ by (3.7). Likewise, $|v(R_n, W_{R_n})| 1_{\{s\geq R_n\}}$ is dominated by $K e^{an^2}$. Letting $s \uparrow T - t$ in (3.9), we obtain from the bounded convergence theorem:

$$u(T - t, x) = E^x[v(R_n, W_{R_n}) 1_{\{R_n<T-t\}}].$$

Therefore, with $0 \le t < T, \quad |x| < n$,

$$|u(T-t,x)| \le Ke^{an^2}P^x[R_n < T-t] \le Ke^{an^2}P^x[R_n < T] \le$$
$$\le Ke^{an^2}(P^0[T_{n-x} < T] + P^0[T_{-n-x} < T]) =$$
$$= Ke^{an^2}(P^0[T_{n-x} < T] + P^0[T_{n+x} < T]) \le$$
$$\le Ke^{an^2}\sqrt{\frac{2}{n}}\left(\int_{(n-x)\sqrt{T}}^{\infty} e^{-z^2/2}dz + \int_{(n+x)/\sqrt{T}}^{\infty} e^{-z^2/2}dz\right),$$

where we have used the distribution function of passage time of Brownian motion. But from (2.9.20), we have $\lim_{n\to\infty} e^{an^2}\int_{(n\pm x)/\sqrt{T}}^{\infty} e^{-z^2/2}dz = 0$, provided $a < \frac{1}{2T}$.

Having proved the theorem for $a < \frac{1}{2T}$, we can extend it to the case where this inequality does not hold. Given a time interval $[0,T]$, choose $T_0 = 0 < T_1 < ... < T_n = T$ s.t. $a < \frac{1}{2(T_i - T_{i-1})}; \quad i = 1, ..., n$, and then show successively that $u = 0$ in each of the strips $(T_{i-1}, T_i]; \quad i = 1, ..., n$ by the above argument. $\square$

As a counter-example for the Tychonoff uniqueness theorem when the conditions are not satisfied, note that the function

$$h(t,x) \triangleq \frac{x}{t}p(t;x,0) = \frac{\partial}{\partial x}p(t;x,0); \quad t > 0, \quad x \in \mathbb{R}, \tag{3.10}$$

solves the heat equation (3.1) on every strip of the form $(0,T] \times \mathbb{R}$; furthermore, it satisfies condition (3.8) for every $0 < a < \frac{1}{2T}$, as well as (3.7) for every $x \neq 0$. However, the limit in (3.7) fails to exist for $x = 0$, although we do have $\lim_{t\downarrow 0} h(t,0) = 0$.

B. Nonnegative solutions of the heat equation

If the initial temperature f is nonnegative, as it always is if measured on the absolute scale, then the temperature should remain nonnegative for all $t > 0$; this is evident from the representation (3.4). Is it possible to characterize the nonnegative solutions of the heat equation? This was done by Widder (1944) who showed that such functions u have a representation

$$u(t,x) = \int_{-\infty}^{\infty} p(t;x,y)dF(y); \quad x \in \mathbb{R},$$

where $F : \mathbb{R} \to \mathbb{R}$ is nondecreasing (Corollary 3.7 $(i)'$, $(ii)'$). We extend Widder's work by providing probabilistic characterizations of nonnegative solutions to the heat equation in Corollary 3.7 $(iii)'$, $(iv)'$.

3.6 Theorem Let $v(t,x)$ be a nonnegative function defined on a strip $(0,T)\times\mathbb{R}$, where $0<T<\infty$. The following four conditions are equivalent:

(i) for some nondecreasing function $F:\mathbb{R}\to\mathbb{R}$,

$$v(t,x)=\int_{-\infty}^{\infty}p(T-t;x,y)dF(y);\quad 0<t<T,\quad x\in\mathbb{R}; \tag{3.11}$$

(ii) v is of class $C^{1,2}$ on $(0,T)\times\mathbb{R}$ and satisfies the "backward" heat equation

$$\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}=0 \tag{3.12}$$

on the strip;

(iii) for a Brownian family $\{W_s,\mathcal{F}_s;0\le s<\infty\}$, $(\Omega,\mathcal{F})$, $\{P^x\}_{x\in\mathbb{R}}$ and each fixed $t\in(0,T)$, $x\in\mathbb{R}$, the process $\{v(t+s,W_s),\mathcal{F}_s;0\le s<T-t\}$ is a martingale on $(\Omega,\mathcal{F},P^x)$;

(iv) for a Brownian family $\{W_s,\mathcal{F}_s;0\le s<\infty\}$, $(\Omega,\mathcal{F})$, $\{P^x\}_{x\in\mathbb{R}}$ we have

$$v(t,x)=E^x v(t+s,W_s);\quad 0<t\le t+s<T,\quad x\in\mathbb{R}. \tag{3.13}$$

Proof) $(i)\Rightarrow(ii)$. Since

$$\frac{\partial}{\partial t}p(T-t;x,y)+\frac{1}{2}\frac{\partial^2}{\partial x^2}p(T-t;x,y)=0,$$

we can prove the implication $(i)\Rightarrow(ii)$ by showing that the partial derivatives of v can be computed by differentiating under the integral in (3.11).
For $a>\frac{1}{2T}$, we have

$$\int_{-\infty}^{\infty}e^{-ay^2}dF(y)=\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty}p(\frac{1}{2a};0,y)dF(y)=\sqrt{\frac{\pi}{a}}v(T-\frac{1}{2a},0)<\infty.$$

This condition is analogous to (3.3) and allows us to proceed as in Problem 3.1:

$$\frac{\partial^{n+m}}{\partial t^n\partial x^m}v(t,x)=\int_{-\infty}^{\infty}f(y)\frac{\partial^{n+m}}{\partial t^n\partial x^m}p(t;x,y)dF(y);\quad 0<t<\frac{1}{2a},\quad x\in\mathbb{R}.$$

$(ii)\Rightarrow(iii)$, $(ii)\Rightarrow(iv)$.

We begin by applying Ito's rule to $v(t+s,W_s);0\le s<T-t$.

$$v(t+s,W_s)=v(t,W_0)+\int_0^s\frac{\partial}{\partial x}v(t+\sigma,W_\sigma)dW_\sigma+\int_0^s\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)v(t+\sigma,W_\sigma)d\sigma.$$

With $a<x<b$, we consider the passage times T_a and T_b and obtain:

$$v(t+(s\wedge T_a\wedge T_b),W_{s\wedge T_a\wedge T_b})=v(t,W_0)+\int_0^{s\wedge T_a\wedge T_b}\frac{\partial}{\partial x}v(t+\sigma,W_\sigma)dW_\sigma+$$

$$+\int_0^{s\wedge T_a\wedge T_b}\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)v(t+\sigma,W_\sigma)d\sigma.$$

Under the assumption (ii), the Lebesgue integral vanishes, as does the expectation of the stochastic integral because $\frac{\partial}{\partial x}v(t+\sigma,y)$ is bounded when $a\leq y\leq b$ and $0\leq\sigma\leq s<T-t$.

$$\therefore v(t,x)=E^x v(t+(s\wedge T_a\wedge T_b),W_{s\wedge T_a\wedge T_b}). \tag{3.14}$$

Letting $a\downarrow-\infty, b\uparrow\infty$ and relying on the nonnegativity of v and Fatou's lemma, we have

$$v(t,x)\geq E^x[\liminf_{a\downarrow-\infty,b\uparrow\infty} v(t+(s\wedge T_a\wedge T_b))]=E^x v(t+s,W_s);\quad 0<t\leq t+s<T, \tag{3.15}$$

Claim: Inequality (3.15) implies that for fixed $t\in(0,T)$ and $x\in\mathbb{R}$, the process $\{v(t+s,W_s),\mathcal{F}_s;0\leq s<T-t\}$ is a supermartingale on $(\Omega,\mathcal{F},P^x)$.

$\because$ For $0\leq s_1\leq s_2<T-t$, the Markov property (Proposition 2.5.13) yields

$$E^x[v(t+s_2,W_{s_2})|\mathcal{F}_{s_1}](\omega)=f(W_{s_1}(\omega))\quad\text{for } P^x\text{-a.e. }\omega\in\Omega, \tag{3.16}$$

where

$$f(y)\triangleq E^y v(t+s_2,W_{s_2-s_1}). \tag{3.17}$$

Prop 2.5.13: $P^x[X_{s+t}\in\Gamma|\mathcal{F}_s]=E^x f(X_s)\Rightarrow$

From (3.15), we have

$$E^y v(t+s_2,W_{s_2-s_1})\leq v(t+s_1,y),$$

and so for $0<t\leq t+s_1\leq t+s_2<T,\quad x\in\mathbb{R}:$

$$v(t+s_1,W_{s_1})\geq E^x[v(t+s_2,W_{s_2})|\mathcal{F}_{s_1}],\quad\text{a.s. } P^x. \tag{3.18}$$

Therefore, if the equality holds in (3.15), then $\{v(t+s,W_s),\mathcal{F}_s;0\leq s<T-t\}$ is a martingale. We now establish the reverse inequality.
We may write (3.14) as

$$v(t,x)=E^x[v(t+s,W_s)1_{\{s\leq T_a\wedge T_b\}}]+E^x[v(t+T_a,a)1_{\{T_a<s\wedge T_b\}}]$$
$$+E^x[v(t+T_b,b)1_{\{T_b<s\wedge T_a\}}]\leq E^x v(t+s,W_s)+$$
$$E^x[v(t+T_a,a)1_{\{T_a<s\}}]+E^x[v(t+T_b,b)1_{\{T_b<s\}}].$$

We will establish (3.13) as soon as we prove

$$\liminf_{b\to\infty}E^x[v(t+T_b,b)1_{\{T_b<s\}}]=0 \tag{3.19}$$

(a dual argument then shows that $\liminf_{a\to-\infty} E^x[v(t+T_a,a)1_{\{T_a<s\}}]=0$). For (3.19), it suffices to show that with $B>0$ large enough, we have

$$\int_B^\infty E^x[v(t+T_b,b)1_{\{T_b<s\}}]db<\infty.$$

We choose $x\in\mathbb{R}, 0<t<T$ and $0\le s<t$ so that $s+t<T$. From (2.6.3) and (3.10) we have

$$P^x[T_b\in d\sigma]=h(\sigma;b-x)d\sigma \quad b>x,\sigma>0.$$

$\because P^0[T_b\in dt]=\frac{|b|}{\sqrt{2\pi t^3}}e^{-b^2/2t}dt; \quad t>0.$

For $B\ge x$ sufficiently large, $h(\sigma,b-x)$ is an increasing function of $\sigma\in(0,s)$, provided $b\ge B$. Furthermore, for $r\in(s,t)$ and B perhaps larger, we have

$$h(s,b-x)\le\sqrt{\frac{r}{s^3}}p(r;x,b); \quad b\ge B.$$

It follows that

$$\int_B^\infty E^x[v(t+T_b,b)1_{\{T_b<s\}}]db=\int_B^\infty\int_0^s v(t+\sigma,b)h(\sigma,b-x)d\sigma db\le$$

$$\le\sqrt{\frac{r}{s^3}}\int_0^s\int_B^\infty v(t+\sigma,b)p(r;x,b)dbd\sigma\le\sqrt{\frac{r}{s^3}}\int_0^s E^x v(t+\sigma,W_r)d\sigma\le$$

$$\le\sqrt{\frac{r}{s^3}}\int_0^s v(t+\sigma-r,x)d\sigma<\infty,$$

where the next to last inequality is a consequence of (3.15). This proves (3.13) for $x\in\mathbb{R}, 0<t\le t+s<T$, as long as $s<t$.

We now remove the unwanted restriction $s<t$. We show by induction on the positive integers k that if

$$0<t\le t+s<T, \quad s<kt, \tag{3.20}$$

then

$$v(t,x)=E^x v(t+s,W_s); \quad x\in\mathbb{R}. \tag{3.21}$$

This will yield (3.13) for the range of values indicated there. We have just established that (3.20) implies (3..21) when $k=1$. Assume this implication holds for some $k\ge1$, so $\{v(t+s,W_s),\mathcal{F}_s;0\le s<kt\}$ is a martingale. Choose $s_2\in[kt,(k+1)t)$ and $s_1\in[0,kt)$ so that $0<s_2-s_1<t$. Then,

$$E^x v(t+s_2,W_{s_2})=E^x\{E^x[v(t+s_2,W_{s_2})|\mathcal{F}_{s_1}]\}=E^x v(t+s_1,W_{s_1})=v(t,x),$$

where we have used (3.16), (3.17) and the induction hypothesis in the form $E^y v(t+s_2,W_{s_2-s_1})=E^y v(t+s_1+(s_2-s_1),W_{s_2-s_1})=v(t+s_1,y)$ for the second equality.

$(iv) \Rightarrow (i)$

For $0 < \varepsilon < \frac{T}{4}$, $\frac{T}{2} < t < T$, $v(t,x) = E^x v(t+s, W_s)$ gives

$$v(t-\varepsilon, x) = E^x(t-\varepsilon+s, W_s) = E^x v(T-\varepsilon, W_{T-t}) = \int_{-\infty}^{\infty} \frac{p(T-t;x,y)}{p(\frac{T}{2};0,y)} dF_\varepsilon(y),$$

where F_ε is the nondecreasing function

$$F_\varepsilon(x) \triangleq \int_{-\infty}^{x} p\left(\frac{T}{2};0,y\right) v(T-\varepsilon, y)dy; \quad x \in \mathbb{R}.$$

Again, from $v(t,x) = E^x v(t+s, W_s)$, we have $F_\varepsilon(\infty) = E^0 v(T-\varepsilon, W_{T/2}) = E^0 v(T/2-\varepsilon+T/2, W_{T/2}) = v(T/2-\varepsilon, 0)$, and thus

$$\sup_{0<\varepsilon<T/4} F_\varepsilon(\infty) \le \max_{T/4 \le t \le T/2} v(t,0) < \infty.$$

By Helly's (selection) theorem , there exists a seq. $\varepsilon_1 >, .., > \varepsilon_k \downarrow 0$ and a nondecreasing function $F^* : \mathbb{R} \to [0,\infty)$ s.t. $\lim_{k\to\infty} F_{\varepsilon_k}(x) = F^*(x)$ for every x at which F^* is continuous.

$\because$ Helly's selection theorem: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of increasing functions mapping a real interval I into the real line $\mathbb{R}$, and suppose that it is uniformly bounded. Then, the sequence $(f_n)_{n\in\mathbb{N}}$ admits a pointwise convergent subsequence.

Because for fixed $x \in \mathbb{R}$ and $t \in ((T/2), T)$ the ratio $\frac{p(T-t;x,y)}{p((T/2);0,y)}$ is a bounded, continuous function of y, converging to 0 as $|y| \to \infty$, we have

$$v(t,x) = \lim_{k\to\infty} v(t-\varepsilon_k, x) = \lim_{k\to\infty} \int_{-\infty}^{\infty} \frac{p(T-t;x,y)}{p(\frac{T}{2};0,y)} dF_{\varepsilon_k}(y) =$$

$$= \int_{-\infty}^{\infty} \frac{p(T-t;x,y)}{p(\frac{T}{2};0,y)} dF^*(y)$$

by the extended Helly-Bray lemma.

$\because$ Helly-Bray lemma: If $F_n \to F$ and g is bounded and continuous a.s. F, then

$$Eg(X_n) = \int g dF_n \to \int g dF = Eg(X).$$

Defining $F(x) = \int_0^x \frac{dF^*(y)}{p((T/2);0,y)}$, we have (3.11) for $T/2 < t < T, x \in \mathbb{R}$.
If $0 < t \le T/2$, we choose $t_1 \in (T/2, T)$ and write

$$v(t,x) = E^x v(t+(t_1-t), W_{t_1-t}) = \int_{-\infty}^{\infty} p(t_1-t;x,y) v(t_1,y)dy =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(t_1-t;x,y) p(T-t_1;y,z) dy dF(z) =$$

$$= \int_{-\infty}^{\infty} p(T-t;x,z)dF(z). \quad \square$$

3.7 Corollary Let $u(t,x)$ be a nonnegative function defined on a strip $(0,T)\times\mathbb{R}$, where $0<T\le\infty$. The following four conditions are equivalent:
$(i)^{'}$ for some nondecreasing function $F:\mathbb{R}\to\mathbb{R}$,

$$u(t,x) = \int_{-\infty}^{\infty} p(t;x,y)dF(y); \quad 0<t<T, x\in\mathbb{R}; \tag{3.22}$$

$(ii)^{'}$ u is of class $C^{1,2}$ on $(0,T)\times\mathbb{R}$ and satisfies the heat equation (3.1) there;
$(iii)^{'}$ for a Brownian family $\{W_s,\mathcal{F}_s;0\le s<\infty\},(\Omega,\mathcal{F}),\{P^x\}_{x\in\mathbb{R}}$ and each fixed $t\in(0,T),x\in\mathbb{R}$, the process $\{u(t-s,W_s),\mathcal{F}_s;0\le s<t\}$ is a martingale on $(\Omega,\mathcal{F},P^x)$;
$(iv)^{'}$ for a Brownian family $\{W_s,\mathcal{F}_s;0\le s<\infty\},(\Omega,\mathcal{F}),\{P^x\}_{x\in\mathbb{R}}$ we have

$$u(t,x) = E^x u(t-s,W_s); \quad 0\le s<t<T, x\in\mathbb{R}. \tag{3.23}$$

Proof) If $T<\infty$, we obtain this corollary by defining $v(t,x)=u(T-t,x)$ and appealing to Theorem 3.6. If $T=\infty$, then for each integer $n\ge1$ we set $v_n(t,x)=u(n-t,x);0<t<n,x\in\mathbb{R}$. Applying Theorem 3.6 to each v_n we see that conditions $(ii)^{'}$, $(iii)^{'}$, and $(iv)^{'}$ are equivalent, they are implied by $(i)^{'}$ and they imply the existence, for any fixed $n\ge1$, of a nondecreasing function $F:\mathbb{R}\to\mathbb{R}$ s.t. (3.22) holds on $(0,n)\times\mathbb{R}$. For $t\ge n$, we have from (3.23):

$$u(t,x) = E^x u\left(\frac{n}{2},W_{t-n/2}\right) = \int_{-\infty}^{\infty} u\left(\frac{n}{2},z\right) p\left(t-\frac{n}{2};x,z\right) dz =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\left(\frac{n}{2};z,y\right) p\left(t-\frac{n}{2};x,z\right) dzdF(y) = \int_{-\infty}^{\infty} p(t;x,y)dF(y). \quad \square$$

Can we represent nonnegative solutions $v(t,x)$ of the backward heat equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} = 0$$

on the entire half-plane $(0,\infty)\times\mathbb{R}$, just as we did in Corollary 3.7 for nonnegative solutions $u(t,x)$ of the heat equation (3.1)? Certainly this cannot be achieved by a simple time-reversal on the results of Corollary 3.7. Instead, we can relate the functions u and v by the formula

$$v(t,x) = \sqrt{\frac{2\pi}{t}}\exp(\frac{x^2}{2t})u(\frac{1}{t},\frac{x}{t}); \quad 0<t<\infty, \quad x\in\mathbb{R}. \tag{3.24}$$

Claim: v satisfies (3.12) on $(0,\infty)\times\mathbb{R}$ if and only if u satisfies the heat equation (3.1) there.

3.9 Proposition (Robbins & Siegmund (1973)) Let $v(t,x)$ be a nonnegative function defined on the half-plane $(0,\infty)\times\mathbb{R}$. With $T=\infty$, conditions (ii), (iii), (iv) of Theorem 3.6 are equivalent to one another, and to $(i)^{'}$:

$$v(t,x)=\int_{-\infty}^{\infty}\exp(yx-\frac{1}{2}y^2t)dF(y);\quad 0<t<\infty,\ x\in\mathbb{R}. \tag{3.25}$$

Proof) The equivalence of (ii), (iii) and (iv) for $T=\infty$ follows from their equivalence for all finite T. If v is given by (3.25), then differentiation under the integral can be justified as in Theorem 3.6, and it results in

$$\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}=0.$$

If v satisfies (ii), then u given by (3.24) satisfies $(ii)^{'}$, and hence $(i)^{'}$ of Corollary 3.7. However, (3.24) and (3.22) reduce to (3.25). $\square$

C. Boundary Crossing Probabilities for Brownian motion

The representation (3.25) has rather unexpected consequences in the computation of boundary-crossing probabilities for Brownian motion. Let us consider consider a positive function $v(t,x)$ which is defined and of class $C^{1,2}$ on $(0,\infty)\times\mathbb{R}$, and satisfies the backward heat equation. Then v admits the representation (3.25) for some F, and differentiating under the integral we see that

$$\frac{\partial}{\partial t}v(t,x)=\int_{-\infty}^{\infty}-\frac{1}{2}y^2\exp(yx-\frac{1}{2}y^2t)dF(y)<0;\quad 0<t<\infty,\quad x\in\mathbb{R} \tag{3.26}$$

and that $v(t,\cdot)$ is convex for each $t>0$. In particular, $\lim_{t\downarrow 0}v(t,0)$ exists. We assume that this limit is finite, and, WLOG (by scaling if necessary) that

$$\lim_{t\downarrow 0}v(t,0)=1. \tag{3.27}$$

We also assume that

$$\lim_{t\to\infty}v(t,0)=0, \tag{3.28}$$

$$\lim_{x\to\infty}v(t,x)=\infty;\quad 0<t<\infty, \tag{3.29}$$

$$\lim_{x\to-\infty}v(t,x)=0,\quad 0<t<\infty. \tag{3.30}$$

(3.27)-(3.30) are satisfied if and only if F is a probability distribution function with $F(0+)=0$. We impose this condition, so that (3.25) becomes

$$v(t,x)=\int_{0+}^{\infty}\exp(yx-\frac{1}{2}y^2t)dF(y);\quad 0<t<\infty,\ x\in\mathbb{R}, \tag{3.31}$$

where $F(\infty) = 1,\ F(0+) = 0$. This representation shows that $v(t,\cdot)$ is strictly increasing, so for each $t > 0$ and $b > 0$ there is a unique number $A(t,b)$ s.t.

$$v(t, A(t,b)) = b. \tag{3.32}$$

Moreover, the function $A(\cdot, b)$ is continuous and strictly increasing (3.26). We may define $A(0,b) = \lim_{t\downarrow 0} A(t,b)$.

We shall show how one can compute the **probability that a Brownian path W, starting at the origin, will eventually cross the curve** $A(\cdot, b)$. The problem of computing the probability that a Brownian motion crosses a given, time-dependent continuous boundary $\{\psi(t);\ 0 \le t < \infty\}$ is thereby reduced to finding a solution v to the backward heat equation which also satisfies $(3.27) - (3.30)$ and $v(t,\psi(t)) = b;\ 0 \le t < \infty$, for some $b > 0$. In this generality, both problems are quite difficult; our point is that the probabilistic problem can be traded for a partial differential equation problem. We shall provide an explicit solution to both of them when the boundary is linear.

Let $\{W_t, \mathcal{F}_t; 0 \le t < \infty\}, (\Omega, \mathcal{F}), \{P^x\}_{x\in\mathbb{R}}$ be a Brownian family, and define

$$Z_t = v(t, W_t); \quad 0 < t < \infty.$$

For $0 < s < t$, we have from the Markov property and condition (iv) of Proposition 3.9:

$$E^0[Z_t|\mathcal{F}_s] = E^0[v(t,W_t)|\mathcal{F}_s] = f(W_s) = v(s, W_s) = Z_s, \quad a.s.\ P^0,$$

where $f(y) \triangleq E^y v(t, W_{t-s})$. In other words, $\{Z_t, \mathcal{F}_t; 0 < t < \infty\}$ is a continuous, nonnegative martingale on $(\Omega, \mathcal{F}, P^0)$. Let $\{t_n\}$ be a sequence of positive numbers with $t_n \downarrow 0$, and set $Z_0 = \lim_{n\to\infty} Z_{t_n}$. This limit exists, P^0-a.s. and is independent of the particular sequence $\{t_n\}$ chosen, (Proposition 1.3.14(i)). Being $\mathcal{F}^W_{0+}$-measurable, Z_0 must be a.s. constant (Theorem 2.7.17)

Theorem 2.7.17 (Blumenthal Zero-One Law). Let $\{B_t, \tilde{\mathcal{F}}_t; t \ge 0\}, (\Omega, \mathcal{F}), \{P^x\}_{x\in\mathbb{R}^d}$ be a d-dimensional Brownian family, where $\tilde{\mathcal{F}}_t \triangleq \bigcap_\mu \mathcal{F}^\mu_t$. If $F \in \tilde{\mathcal{F}}_0$, then for each $x \in \mathbb{R}^d$ we have either $P^x(F) = 0$ or $P^x(F) = 1$.

3.10 Lemma The extended process $Z \triangleq \{Z_t, \mathcal{F}_t; 0 \le t < \infty\}$ is continuous, nonnegative martingale under P^0 and satisfies $Z_0 = 1, Z_\infty = 0, P^0$-a.s.

Proof) Let $\{t_n\}$ be a sequence of positive numbers with $t_n \downarrow 0$. The sequence $\{Z_n\}_{n=1}^\infty$ is uniformly integrable (Problem 1.3.11, Remark 1.3.12), so by the Markov property for W, we have for all $t > 0$:

$$E^0[Z_t|\mathcal{F}_0] = E^0 Z_t = \lim_{n\to\infty} E^0 Z_{t_n} = E^0 Z_0 = Z_0.$$

This establishes that $\{Z_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a martingale.

Since $Z_\infty \triangleq \lim_{t\to\infty} Z_t$ exists P^0-a.s. (Problem 1.3.16), as does $Z_0 \triangleq \lim_{t\downarrow 0} Z_t$, it suffices to show that $\lim_{t\downarrow 0} Z_t = 1$ and $\lim_{t\to\infty} Z_t = 0$ in P^0-probability. For every finite $c > 0$, we shall show that

$$\lim_{t\downarrow 0} \sup_{|x|\le c\sqrt{t}} |v(t,x) - 1| = 0. \tag{3.33}$$

Indeed, for $t > 0$, $|x| \le c\sqrt{t}$:

$$\int_{0+}^{\infty} \exp\left(-yc\sqrt{t} - \frac{1}{2}y^2 t\right) dF(y) \le v(t,x) \le \int_{0+}^{\infty} \exp\left(yc\sqrt{t} - \frac{1}{2}y^2 t\right) dF(y). \tag{3.34}$$

Because $\pm yc\sqrt{t} - \frac{1}{2}y^2 t \le \frac{c^2}{2}$; $\forall y > 0$, the bounded convergence theorem implies that both integrals in (3.34) converge to 1, as $t \downarrow 0$, and (3.33) follows. Thus, for any $\varepsilon > 0$, we can find $t_{c,\varepsilon}$ depending on c and ε, s.t.

$$1 - \varepsilon < v(t,x) < 1 + \varepsilon; \quad |x| \le c\sqrt{t}, \quad 0 < t < t_{c,\varepsilon}.$$

Consequently, for $0 < t < t_{c,\varepsilon}$,

$$P^0[|Z_t - 1| > \varepsilon] = P^0[|v(t,W_t) - 1| > \varepsilon] \le P^0[|W_t| > c\sqrt{t}] = 2[1 - \Phi(c)],$$

where

$$\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz.$$

Letting first $t \downarrow 0$ and then $c \to \infty$, we conclude that $Z_t \to 1$ in probability as $t \downarrow 0$. A similar argument shows that

$$\lim_{t\to\infty} \sup_{|x|\le c\sqrt{t}} v(t,x) = 0, \tag{3.35}$$

and, using (3.35) instead of (3.33), one can also show that $Z_t \to 0$ in probability as $t \to \infty$. $\square$

It is now a fairly straightforward matter to apply Problem 1.3.28 to the martingale Z and obtain the probability that the Brownian path $\{W_t(\omega); 0 \le t < \infty\}$ ever crosses the boundary $\{A(t,b);\ 0 \le t < \infty\}$.

$\because$ Problem 1.3.28. Let $Z = \{Z_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous, nonnegative martingale with $Z_\infty \triangleq \lim_{t\to\infty} Z_t = 0$, a.s. P. Then, for every $s \ge 0, b > 0$:

$$(i)\ P\left[\sup_{t>s} Z_t \ge b | \mathcal{F}_s\right] = \frac{1}{b} Z_s, \quad \text{a.s. on } \{Z_s < b\}.$$

$$(ii)\ P\left[\sup_{t\ge s} Z_t \ge b\right] = P[Z_s \ge b] + \frac{1}{b} E[Z_s 1_{\{Z_s<b\}}].$$

3.12 Example. With $\mu > 0$, let $v(t,x) = \exp(\mu x - \mu^2 t/2)$, so $A(t,b) = \beta t + \gamma$, where $\beta = \frac{\mu}{2},\ \gamma = \frac{1}{\mu}\log b$. Then, $F(y) = 1_{[\mu,\infty)]}(y)$, and so for any $s > 0, \beta > 0, \gamma \in \mathbb{R}$, and Lebesgue-almost every $a < \gamma + \beta s$:

$$P^0[W_t \geq \beta t + \gamma, \text{for some } t \geq s | W_s = a] = e^{-2\beta(\gamma - a + \beta s)} \tag{3.38}$$

and for any $s > 0, \beta > 0$, and $\gamma \in \mathbb{R}$:

$$P^0[W_t \geq \beta t + \gamma, \text{for some } t \geq s | W_s = a] = 1 - \Phi(\frac{\gamma}{\sqrt{s}} + \beta\sqrt{s}) + e^{-2\beta\gamma}\Phi(\frac{\gamma}{\sqrt{s}} - \beta\sqrt{s}). \tag{3.39}$$

The observation that the time-inverted process Y of Lemma 2.9.4 $(Y_t = tW_{1/t};\ 0 < t < \infty,\ Y_t = 0 \text{ if } t = 0)$ is a Brownian motion allows one to cast (3.38) with $\gamma = 0$ into the following formula for the maximum of the so-called "tied-down" Brownian motion or "Brownian bridge":

$$P^0[\max_{0 \leq t \leq T} W_t \geq \beta | W_T = a] = e^{-2\beta(\beta - a)/T} \tag{3.40}$$

for $T > 0, \beta > 0$, a.e. $\alpha \leq \beta$, and (3.39) into a boundary-crossing probability on the bounded interval $[0,T]$:

$$P^0[W_t \geq \beta + \gamma t, \text{for some } t \in [0,T]] = 1 - \Phi(\gamma\sqrt{T} + \frac{\beta}{\sqrt{T}}) + e^{-2\beta\gamma}\Phi(\gamma\sqrt{T} - \frac{\beta}{\sqrt{T}}). \tag{3.41}$$

D. Mixed initial/boundary value problems

We now discuss the concept of temperatures in a semi-infinite rod and the relation of this concept to Brownian motion absorbed at the origin. Suppose that $f : (0,\infty) \to \mathbb{R}$ is a Borel-measurable function satisfying

$$\int_0^\infty e^{-ax^2}|f(x)|dx < \infty \tag{3.42}$$

for some $a > 0$. We define

$$u_1(t,x) \triangleq E^x[f(W_t)1_{\{T_0 > t\}}]; \quad 0 < t < \frac{1}{2a},\ x > 0. \tag{3.43}$$

The reflection principle gives us the formula (2.8.9)

$$P^x[W_t \in dy, T_0 > t] = p_-(t;x,y)dy \triangleq [p(t;x,y) - p(t;x,-y)]dy$$

for $t > 0, x > 0, y > 0$, and so

$$u_1(t,x) = \int_0^\infty f(y)p(t;x,y)dy - \int_{-\infty}^0 f(-y)p(t;x,-y)dy \tag{3.44}$$

which gives us a definition for u_1 valid on the whole strip $(0, \frac{1}{2a}) \times \mathbb{R}$. This representation is of the form (3.4) $(u(t,x) = E^x f(W_t) = \int_{-\infty}^{\infty} f(y)p(t;x,y)dy)$, where the initial datum f satisfies $f(y) = -f(-y); \quad y > 0$. Then, u_1 has derivatives of all orders, satisfies the heat equation, satisfies $f(x) = \lim_{t\downarrow 0, y\to x} u_1(t,y)$ at all continuity points of f, and

$$\lim_{t\downarrow 0, s\to t} u_1(s,x) = 0; \quad 0 < t < \frac{1}{2a}.$$

We may regard $u_1(t,x); 0 < t < \frac{1}{2a}, x \geq 0$, as the temperature in a semi-infinite rod along the nonnegative axis, when the end $x = 0$ is held at a constant temperature (equal to 0) and the initial temperature at $y > 0$ is $f(y)$.

4.4 The Formulas of Feynman and Kac

Consider the parabolic equation

$$\frac{\partial u}{\partial t} + ku = \frac{1}{2}\Delta u + g; \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \tag{4.1}$$

subject to the initial condition

$$u(0,x) = f(x); \quad x \in \mathbb{R}^d \tag{4.2}$$

for suitable functions $k : \mathbb{R}^d \to [0,\infty), g : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R}$.

In the special case that $g = 0$, we may define the Laplace transform

$$z_\alpha(x) \triangleq \int_0^\infty e^{-\alpha t} u(t,x)dt; \quad x \in \mathbb{R}^d,$$

and using the assumption that $\lim_{t\to\infty} e^{-\alpha t} u(t,x) = 0; \alpha > 0, x \in \mathbb{R}$, we have

$$\frac{1}{2}\Delta z_\alpha = \frac{1}{2}\int_0^\infty e^{-\alpha t}\Delta u dt = \lim_{T\to\infty}\int_0^T e^{-\alpha t}\frac{1}{2}\Delta u dt = \tag{4.3}$$

$$= \lim_{T\to\infty}\int_0^T e^{-\alpha t}(\frac{\partial u}{\partial t} + ku)dt = \lim_{T\to\infty}\int_0^T e^{-\alpha t}\frac{\partial u}{\partial t}dt + kz_\alpha =$$

$$= \lim_{T\to\infty}\left[\alpha\int_0^T e^{-\alpha t} u dt + e^{-\alpha T}u - f\right] + kz_\alpha = (\alpha + k)z_\alpha - f.$$

The stochastic representation for the solution z_α of the elliptic equation (4.3) is known as the Kac formula.
Throughout this section, $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}, (\{\Omega, \mathcal{F}\}), \{P^x\}_{x\in\mathbb{R}^d}$ is a d-dimensional Brownian family.

A. The multi-dimensional formula

4.1 Definition. Consider the continuouos function $f : \mathbb{R}^d \to \mathbb{R}, k : \mathbb{R}^d \to [0, \infty)$, and $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}$. Suppose that v is a continuous, real-valued function on $[0,T] \times \mathbb{R}^d$, of class $C^{1,2}$ on $[0,T) \times \mathbb{R}^d$, and satisfies

$$-\frac{\partial v}{\partial t} + kv = \frac{1}{2}\Delta v + g; \quad \text{on } [0,T) \times \mathbb{R}^d, \tag{4.4}$$

$$v(T,x) = f(x); \quad x \in \mathbb{R}^d. \tag{4.5}$$

Then the function v is said to be a solution of the Cauchy problem for the backward heat equation (4.4) with potential k and Lagrangian g, subject to the terminal condition (4.5).

4.2 Theorem (Feynman (1948), Kac (1949)). Let v be as in Definition 4.1 and assume that

$$\max_{0 \le t \le T} |v(t,x)| + \max_{0 \le t \le T} |g(t,x)| \le K e^{a\|x\|^2}; \quad x \in \mathbb{R}^d, \tag{4.6}$$

for some constant $K > 0$ and $0 < a < \frac{1}{2Td}$. Then v admits the stochastic representation

$$\begin{aligned} v(t,x) = E^x[f(W_{T-t}) \exp\Big\{-\int_0^{T-t} k(W_s)ds\Big\} + \\ + \int_0^{T-t} g(t+\theta, W_\theta) \exp\Big\{-\int_0^{\theta} k(W_s)ds\Big\} d\theta]; \quad 0 \le t \le T, x \in \mathbb{R}^d. \end{aligned} \tag{4.7}$$

In particular, such a solution is unique.

4.3 Remark. If $g \ge 0$ on $[0,T] \times \mathbb{R}^d$, then condition (4.6) may be replaced by

$$\max_{0 \le t \le T} |v(t,x)| \le K e^{a\|x\|^2}; \quad \forall x \in \mathbb{R}^d. \tag{4.8}$$

This leads to the following maximum principle for the Cauchy problem: if the continuous function $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is of class $C^{1,2}$ on $[0,T) \times \mathbb{R}^d$ and satisfies the growth condition (4.8), as well as the differential inequaity

$$-\frac{\partial v}{\partial t} + kv \ge \frac{1}{2}\Delta v \text{ on } [0,T) \times \mathbb{R}^d$$

with a continuous potential $k : \mathbb{R}^d \to [0, \infty)$, then $v \ge 0$ on $\{T\} \times \mathbb{R}^d$ implies $v \ge 0$ on $[0,T] \times \mathbb{R}^d$.

In other words, if the function v is nonnegative on the boundary, then it is nonnegative on the whole domain. This is because the solution (4.7) in this case is nonnegative, since $g \triangleq -\frac{\partial v}{\partial t} + kv - \frac{1}{2}\Delta v \ge 0$, $f(x) = v(T,x) \ge 0$, and the exponential function takes nonnegative values.

Proof of Theorem 4.2) Consider $Y(\theta) = v(t+\theta, W_\theta)\exp\Big\{\int_0^\theta k(W_s)ds\Big\}$.
Let $C(\theta) = \exp\Big\{\int_0^\theta k(W_s)ds\Big\}$, thus $Y(\theta) = v(t+\theta, W_\theta)C(\theta)$.
Using Ito's rule for Y, we have:

$$Y(\theta) = Y(0) + C(\theta)\int_0^\theta \frac{\partial}{\partial\theta}v(t+s, W_s)ds - C(\theta)\int_0^\theta k(W_s)v(t+s, W_s)ds +$$

$$+C(\theta)\int_0^\theta \sum_{i=1}^d \frac{\partial}{\partial x_i}v(t+s, W_s)dW_s^{(i)} + \frac{1}{2}C(\theta)\int_0^\theta \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}v(t+s, W_s)ds =$$

$$= v(t, W_0) + C(\theta)\left[-g(t+s, W_s)ds + \int_0^\theta \sum_{i=1}^d \frac{\partial}{\partial x_i}v(t+s, W_s)dW_s^{(i)}\right].$$

Writing this in differential form, $d\left[v(t+\theta, W_\theta)\exp\Big\{-\int_0^\theta k(W_s)ds\Big\}\right] =$

$$= \exp\Big\{-\int_0^\theta k(W_s)ds\Big\}\left[-g(t+\theta, W_\theta)d\theta + \sum_{i=1}^d \frac{\partial}{\partial x_i}v(t+\theta, W_\theta)dW_\theta^{(i)}\right].$$

Let $S_n = \inf\{t \geq 0; \|W_t\| \geq n\sqrt{d}\}$; $n \geq 1$. We choose $0 < r < T - t$ and integrate on $[0, r \wedge S_n]$; thus

$$v(t,x) = E^x \int_0^{r\wedge S_n} g(t+\theta, W_\theta)\exp\Big\{-\int_0^\theta k(W_s)ds\Big\}d\theta +$$

$$+E^x\left[v(t+S_n, W_{S_n})\exp\Big\{-\int_0^{S_n} k(W_s)ds\Big\}1_{\{S_n\leq r\}}\right]$$

$$+E^x\left[v(t+r, W_r)\exp\Big\{-\int_0^r k(W_s)ds\Big\}1_{\{S_n>r\}}\right].$$

The first term on the right-hand side converges to

$$E^x\int_0^{T-t} g(t+\theta, W_\theta)\exp\Big\{-\int_0^\theta k(W_s)ds\Big\}d\theta$$

as $n \to \infty$ and $r \uparrow T - t$, either by monotone convergence (if $g \geq 0$) or by dominated convergence (it is bounded in absolute value by $\int_0^{T-t}|g(t,\theta,W_\theta)|d\theta)$, which has finite expectation by virtue of (4.6). The second term is dominated by

$$E^x[|v(t+S_n, W_{S_n})|1_{\{S_n\leq T-t\}}] \leq Ke^{adn^2}P^x[S_n \leq T] \leq$$

$$\leq 2Ke^{adn^2}\sum_{j=1}^d P^x\left[\max_{0\leq t\leq T}|W_t^{(j)}| \geq n\right] \leq$$

$$\leq 2Ke^{adn^2}\sum_{j=1}^{d}\{P^x[W_T^{(j)} \geq n] + P^x[-W_T^{(j)} \geq n]\}.$$

where we have used (2.6.2): $(P^0[T_b < t] = 2P^0[B_t > b])$. But by (2.9.20),

$$e^{adn^2}P^x[\pm W_T^{(j)} \geq n] \leq e^{adn^2}\sqrt{\frac{T}{2\pi}}\frac{1}{n \mp x^{(j)}}e^{-(n\mp x^{(j)})^2/2T}$$

which converges to zero as $n \to \infty$, because $0 < a < \frac{1}{2Td}$. Again, by the dominated convergence theorem, the third term is shown to converge to $E^x[v(T, W_{T-t})\exp\{-\int_0^{T-t} k(W_s)ds\}]$ as $n \to \infty$ and $r \uparrow T - t$. $\square$

4.5 Corollary. Assume that $f : \mathbb{R}^d \to \mathbb{R}, k : \mathbb{R}^d \to [0,\infty)$, and $g : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ are continuous, and that the continuous function $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is of class $C^{1,2}$ on $(0,\infty) \times \mathbb{R}^d$ and satisfies (4.1), (4.2) (the solution of the parabolic equation). If for each finite $T > 0$ there exists constants $K > 0$ and $0 < a < \frac{1}{2Td}$ s.t.

$$\max_{0\leq t\leq T}|u(t,x)| + \max_{0\leq t\leq T}|g(t,x)| \leq Ke^{a\|x\|^2}; \quad \forall x \in \mathbb{R}^d,$$

then u admits the stochastic representation

$$u(t,x) = E^x[f(W_t)\exp\Big\{-\int_0^t k(W_s)ds\Big\}+ \\ + \int_0^t g(t-\theta, W_\theta)\exp\Big\{-\int_0^\theta k(W_s)ds\Big\}d\theta]; \quad 0 \leq t < \infty, x \in \mathbb{R}^d. \quad (4.9)$$

In the case $g = 0$ we can think of $u(t,x)$ in (4.1) as the temperature at time $t \geq 0$ at the point $x \in \mathbb{R}^d$ of a medium which is not a perfect heat conductor, but instead dissipates heat locally at rate k (heat flow with cooling). The Feynman-Kac formula (4.9) suggests that this situation is equivalent to Brownian motion with annihilation (killing) of particles at the same rate k; the probability that the particle survives up to time t, conditional on the path $\{W_s; 0 \leq s \leq t\}$, is then $\exp\{-\int_0^t k(W_s)ds\}$.

B. The one-dimensional formula

4.8 Definition A Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ is called piecewise-continuous if it admits left- and right- hands limit everywhere on $\mathbb{R}$ and it has only finitely many points of discontinuity in every bounded interval. We denote by D_f the set of discontinuity points of f. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called piecewise C^j if its derivatives $f^{(i)}$, $1 \leq i \leq j-1$ are continuous, and the derivative $f^{(j)}$ is piecewise-continuous.

4.9 Theorem (Kac (1951)) Let $f : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R} \to [0,\infty)$ be piecewise-continuous functions with

$$\int_{-\infty}^{\infty}|f(x+y)|e^{-|y|\sqrt{2\alpha}}dy < \infty; \quad \forall x \in \mathbb{R} \quad (4.16)$$

for some fixed constant $\alpha > 0$. Then the function z defined by

$$z(x) = E^x \int_0^\infty f(W_t) \exp\left\{ -\alpha t - \int_0^t k(W_s) ds \right\} dt \tag{4.14}$$

is piecewise C^2 and satisfies

$$(\alpha + k)z = \frac{1}{2} z^{''} + f; \quad \text{on } \mathbb{R} - (D_f \cup D_k). \tag{4.17}$$

4.10 Remark The Laplace transform computation

$$\int_0^\infty e^{-\alpha t} \frac{1}{\sqrt{2\pi t}} e^{-\xi^2/2t} dt = \frac{1}{\sqrt{2\alpha}} e^{-|\xi|\sqrt{2\alpha}}; \quad \alpha > 0,\ \xi \in \mathbb{R}$$

enables us to replace (4.16) by the equivalent condition

$$E^x \int_0^\infty e^{-\alpha t} |f(W_t)| dt < \infty, \quad x \in \mathbb{R}. \tag{4.16'}$$

Proof of Theorem 4.9) For a piecewise-continuous function g which satisfy condition (4.16), we introduce the resolvent operator G_α given by

$$(G_\alpha g)(x) \triangleq E^x \int_0^\infty e^{-\alpha t} g(W_t) dt = \int_{-\infty}^\infty \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\alpha t} e^{-\frac{(x-y)^2}{2t}} g(y) dt dy =$$

$$= \int_{-\infty}^\infty g(y) \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\alpha t} e^{-\frac{(x-y)^2}{2t}} dt dy = \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^\infty e^{-|x-y|\sqrt{2\alpha}} g(y) dy =$$

$$= \frac{1}{\sqrt{2\alpha}} \left[\int_{-\infty}^x e^{(y-x)|\sqrt{2\alpha}} g(y) dy + \int_x^\infty e^{(x-y)\sqrt{2\alpha}} g(y) dy \right]; \quad x \in \mathbb{R}.$$

Differentiating, we obtain

$$(G_\alpha g)^{'}(x) = \int_x^\infty e^{(x-y)\sqrt{2\alpha}} g(y) dy - \int_{-\infty}^x e^{(y-x)\sqrt{2\alpha}} g(y) dy; \quad x \in \mathbb{R},$$

$$(G_\alpha g)^{''}(x) = -2g(x) + 2\alpha (G_\alpha g)(x); \quad x \in \mathbb{R} - D_g. \tag{4.18}$$

Claim:

$$G_\alpha(kz) = G_\alpha f - z \tag{4.19}$$

and

$$G_\alpha(|kz|)(x) < \infty; \quad \forall x \in \mathbb{R}. \tag{4.20}$$

If we then write (4.18) successively with $g = f$ and $g = kz$ and subtract, we obtain:

$$(G_\alpha f)^{''}(x) = -2f(x) + 2\alpha (G_\alpha f)(x); \quad x \in \mathbb{R} - D_f$$

$$(G_\alpha (kz))^{''}(x) = -2kz + 2\alpha (G_\alpha (kz))(x); \quad x \in \mathbb{R} - D_{kz}$$

$$\therefore z^{''} = -2f + 2kz + 2\alpha z \Longleftrightarrow (\alpha + k)z = \frac{1}{2}z^{''} + f; \quad x \in \mathbb{R} - (D_f \cup D_{kz}).$$

By dominated convergence theorem, z is continuous, so $D_{kz} \subseteq D_k$. Integration of (4.17) yields the continuity of $z^{'}$.

In order to verify the claim (4.19), we start with the observation

$$0 \leq \int_0^t k(W_s) \exp\left\{-\int_s^t k(W_u)du\right\} ds = 1 - \exp\left\{-\int_0^t k(W_u)du\right\} \leq 1; \quad t \geq 0,$$

and so by the Fubini's theorem:

$$(G_\alpha f - z)(x) = E^x \int_0^\infty e^{-\alpha t}(1 - e^{-\int_0^t k(W_s)}) f(W_t)dt =$$

$$= E^x \int_0^\infty e^{-\alpha t} f(W_t) \int_0^t k(W_s) \exp\left\{-\int_s^t k(W_u)du\right\} dsdt =$$

$$= E^x \int_0^\infty \int_s^\infty e^{-\alpha t} f(W_t) k(W_s) \exp\left\{-\int_s^t k(W_u)du\right\} dtds =$$

$$= \int_0^\infty E^x \left[k(W_s) \int_s^\infty \exp\left\{-\alpha t - \int_s^t k(W_u)du\right\} f(W_t)dt\right] ds =$$

$$= \int_0^\infty E^x \left[k(W_s) \int_s^\infty \exp\left\{-\alpha t - \int_0^{t-s} k(W_{s+u})du\right\} f(W_t)dt\right] ds =$$

$$= \int_0^\infty e^{-\alpha s} E^x \left[k(W_s) \int_0^\infty \exp\left\{-\alpha t - \int_0^t k(W_{s+u})du\right\} f(W_{s+t})dt\right] ds =$$

$$= E^x \int_0^\infty e^{-\alpha s} k(W_s) E^x \left[\int_0^\infty \exp\left\{-\alpha t - \int_0^t k(W_{s+u})du\right\} f(W_{s+t})dt | \mathcal{F}_s\right] ds =$$

$$= E^x \int_0^\infty e^{-\alpha s} k(W_s) z(W_s) ds = (G_\alpha(kz))(x); \quad x \in \mathbb{R}.$$

where the second last equality is from the Markov property.
We may replace f by $|f|$ in (4.14) to obtain a nonnegative function $\hat{z} \geq |z|$, and just as earlier, we have

$$G_\alpha(|kz|)(x) \leq (G_\alpha(k\hat{z}))(x) = (G_\alpha(|f|) - \hat{z})(x) < \infty; \quad x \in \mathbb{R}. \quad \square$$

Here are some applications of Theorem 4.9.

4.11 Proposition (P.Levy's Arc-Sine Law for the Occupation Time of $(0, \infty)$). Let $\Gamma_+(t) \triangleq \int_0^t 1_{(0,\infty)}(W_s)ds$. Then,

$$P^0[\Gamma_+(t) \leq \theta] = \int_0^{\theta/t} \frac{ds}{\pi\sqrt{s(1-s)}} = \frac{2}{\pi} \arcsin\sqrt{\frac{\theta}{t}}; \quad 0 \leq \theta \leq t. \tag{4.21}$$

Proof) For $\alpha > 0,\ \beta > 0$, the function

$$z(x) = E^x \int_0^\infty \exp\left(-\alpha t - \beta \int_0^t 1_{(0,\infty)}(W_s)ds\right) dt$$

(with potential $k = \beta \cdot 1_{(0,\infty)}$) and Lagrangian $f = 1$) satisfies, according to Theorem 4.9, the equation

$$\alpha z(x) = \frac{1}{2} z^{''}(x) - \beta z(x) + 1; \quad x > 0,$$

$$\alpha z(x) = \frac{1}{2} z^{''}(x) + 1; \quad x < 0,$$

and the conditions

$$z(0+) = z(0-); \quad z^{'}(0+) = z^{'}(0-).$$

The unique bounded solution to the preceding equation has the form

$$z(x) = \begin{cases} Ae^{-x\sqrt{2(\alpha+\beta)}} + \frac{1}{\alpha+\beta}; & x > 0 \\ Be^{x\sqrt{2\alpha}} + \frac{1}{\alpha}; & x < 0. \end{cases}$$

The continuity of $z(\cdot)$ and $z^{'}(\cdot)$ at $x = 0$ allows us to solve for $A = (\sqrt{\alpha+\beta} - \sqrt{\alpha})/(\alpha+\beta)\sqrt{\alpha}$, so

$$z(0) = \int_0^\infty e^{-\alpha t} E^0 e^{-\beta\Gamma_+(t)} = \frac{1}{\sqrt{\alpha(\alpha+\beta)}}; \quad \alpha > 0, \beta > 0.$$

We have the related computation

$$\int_0^\infty e^{-\alpha t} \int_0^t \frac{e^{-\beta\theta}}{\pi\sqrt{\theta(t-\theta)}} d\theta dt = \int_0^\infty \int_0^t \frac{e^{-\beta\theta}}{\pi\sqrt{\theta}} \frac{e^{-\alpha t}}{\sqrt{t-\theta}} d\theta dt =$$

$$= \int_0^\infty \frac{e^{-\beta\theta}}{\pi\sqrt{\theta}} \int_\theta^\infty \frac{e^{-\alpha t}}{\sqrt{t-\theta}} dt d\theta = \frac{1}{\pi} \int_0^\infty \frac{e^{-(\alpha+\beta)\theta}}{\sqrt{\theta}} \int_0^\infty \frac{e^{-\alpha s}}{\sqrt{s}} ds d\theta = \frac{1}{\sqrt{\alpha(\alpha+\beta)}},$$

where $s = t - \theta$ and the last equality follows from

$$\int_0^\infty \frac{e^{-\gamma t}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{\gamma}}; \quad \gamma > 0. \tag{4.23}$$

The uniqueness of Laplace transforms implies

$$E^0 e^{-\beta\Gamma_+(t)} = \int_0^t \frac{e^{-\beta s}}{\pi\sqrt{s(t-s)}} ds.$$

thus, we have

$$P^0[\Gamma_+(t) \le \theta] = P^0[e^{-\beta\Gamma_+(t)} \ge e^{-\beta\theta}] =$$

$$= \int_0^{\theta/t} \frac{ds}{\pi\sqrt{s(1-s)}} = \frac{2}{\pi} \arcsin\sqrt{\frac{\theta}{t}}; \quad 0 \le \theta \le t. \quad \square$$

4.12 Proposition (Occupation Time of $(0,\infty)$ until First Hitting $b > 0$). For $\beta > 0, b > 0$, we have

$$E^0 \exp[-\beta\Gamma_+(T_b)] \triangleq E^0 \exp\left[-\beta\int_0^{T_b} 1_{(0,\infty)}(W_s)ds\right] = \frac{1}{\cosh b\sqrt{2\beta}}. \tag{4.23}$$

Proof) With $\Gamma_b(t) \triangleq \int_0^t 1_{(b,\infty)}(W_s)ds,\ \Gamma_+(t) \triangleq \int_0^t 1_{(0,\infty)}(W_s)ds$, positive numbers α, β, γ, and

$$z(x) \triangleq E^x \int_0^\infty 1_{(0,\infty)}(W_t)\exp(-\alpha t - \beta\Gamma_+(t) - \gamma\Gamma_b(t))dt,$$

we have

$$z(0) = E^0 \int_0^{T_b} \exp(-\alpha t - \beta\Gamma_+(t))d\Gamma_+(t) +$$

$$+E^0 \int_{T_b}^\infty \exp(-\alpha t - \beta\Gamma_+(t) - \gamma\Gamma_b(t))d\Gamma_+(t).$$

Since $\Gamma_b(t) > 0$ a.s.on $\{T_b < t\}$ (Problem 2.7.19), we have

$$\lim_{\gamma\uparrow\infty} z(0) = E^0 \int_0^{T_b} \exp(-\alpha t - \beta\Gamma_+(t))d\Gamma_+(t)$$

$$\lim_{\alpha\downarrow 0}\lim_{\gamma\uparrow\infty} z(0) = E^0 \int_0^{T_b} \exp(-\alpha t - \beta\Gamma_+(t))d\Gamma_+(t) \tag{4.24}$$

$$= \frac{1}{\beta}[1 - E^0 \exp(-\beta\Gamma_+(T_b))].$$

According to Theorem 4.9, the function $z(\cdot)$ is piecewise C^2 on $\mathbb{R}$ and satisfies the equation (with $\sigma = \alpha + \beta$):

$$\alpha z(x) = \frac{1}{2}z''(x); \quad x < 0,$$
$$\sigma z(x) = \frac{1}{2}z''(x) + 1; \quad 0 < x < b,$$
$$(\sigma+\gamma)z(x) = \frac{1}{2}z''(x) + 1; \quad x > b.$$

The unique bounded solution is of the form

$$z(x) = \begin{cases} Ae^{x\sqrt{2\alpha}}; & x < 0 \\ Be^{x\sqrt{2\sigma}} + Ce^{-x\sqrt{2\sigma}} + \frac{1}{\sigma}; & 0 < x < b. \\ De^{-x\sqrt{2(\sigma+\gamma)}} + \frac{1}{\sigma+\gamma}; & x > b. \end{cases}$$

Matching the values of $z(\cdot)$ and $z^{'}(\cdot)$ across the points $x=0$ and $x=b$, we obtain the values of the four constants A, B, C, and D. In particular, $z(0)=A$ is given by

$$2\frac{\frac{\sinh b\sqrt{2\sigma}}{\sqrt{2\sigma}}+\sqrt{\frac{\sigma+\gamma}{\sigma}}\left[\frac{\cosh b\sqrt{2\sigma}-1}{\sqrt{2\sigma}}\right]+\frac{1}{\sqrt{2(\sigma+\gamma)}}}{(\sqrt{2\alpha}+\sqrt{2(\sigma+\gamma)})\cosh b\sqrt{2\sigma}+\left(\sqrt{2\alpha(\frac{\sigma+\gamma}{\sigma})}+\sqrt{2\sigma}\right)\sinh b\sqrt{2\sigma}},$$

whence

$$\lim_{\gamma\uparrow\infty} z(0)=\frac{\sqrt{\frac{2}{\alpha+\beta}}(\cosh b\sqrt{2(\alpha+\beta)}-1)}{\sqrt{2(\alpha+\beta)\cosh b\sqrt{2(\alpha+\beta)}}+\sqrt{2\alpha}\sinh b\sqrt{2(\alpha+\beta)}}$$

$$\lim_{\alpha\downarrow 0}\lim_{\gamma\uparrow\infty} z(0)=\frac{1}{\beta}\left[1-\frac{1}{\cosh b\sqrt{2\beta}}\right].$$

The results (4.23) now follows from (4.24). $\square$