Chapter 4 Notes

4.2 Harmonic Functions and the Dirichlet Problem

A function $u : D \mapsto \mathbb{R}$ where D is an open subset of $\mathbb{R}^d$ is called **harmonic** in D if u is of class C^2 and $\Delta u \triangleq \sum_{i=1}^{d}(\frac{\partial^2 u}{\partial x_i^2}) = 0$ in D.

Throughout this section, $\{W_t, \mathcal{F}_t; 0 \le t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x\in\mathbb{R}^d}$ is a d-dimensional Brownian family and $\{\mathcal{F}_t\}$ satisfies the usual conditions. We denote by D an open set in $\mathbb{R}^d$ and introduce the stopping time (Problem 1.2.7)

$$\tau_D = \inf\{t \ge 0; W_t \in D^c\},$$

the time of first exit from D. The boundary of D will be denoted by ∂D, and $\bar{D} = D \cup \partial D$ is the closure of D. By Theorem 2.9.23, each component of W is a.s. unbounded, so

$$P^x[\tau_D < \infty] = 1; \quad \forall x \in D \subset \mathbb{R}^d,\ D \text{ bounded}.$$

Let $B_r \triangleq \{x \in \mathbb{R}^d; \|x\| < r\}$ be the open ball of radius r centered at the origin. The volume of this ball is

$$V \triangleq \frac{2r^d \pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})},$$

and its surface area is

$$S_r \triangleq \frac{2r^{d-1}\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \frac{d}{r}V_r.$$

We define a probability measure μ_r on ∂B_r by

$$\mu_r(dx) = P^0[W_{\tau_{B_r}} \in dx]; \quad r > 0.$$

A. The mean-value property

Because of the rotational invariance of Brownian motion (Problem 3.3.18), the measure μ_r is also rotationally invariant and thus proportional to surface measure on ∂B_r. In particular, the Lebesgue integral of a function f over B_r can be written in iterated form as

$$\int_{B_r} f(x)dx = \int_0^r S_\rho \int_{\partial B_\rho} f(x)\mu_\rho(dx)d\rho.$$

2.1 Definition We say that the function $u : D \mapsto \mathbb{R}$ has the **mean-value property** if, for every $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B}_r \subset D$, we have

$$u(a) = \int_{\partial B_r} u(a+x)\mu_r(dx).$$

Then we derive the consequence

$$u(a) = \frac{1}{V_r}\int_{B_r} u(a+x)dx.$$

$\because \int_{B_r} u(a+x)dx = \int_0^r S_\rho \int_{\partial B_\rho} u(a+x)\mu_\rho(dx)d\rho = \int_0^r S_\rho u(a+x)d\rho =$
$= u(a+x)\int_0^r S_\rho d\rho = u(a+x)V_r$
(the second inequality follows from the mean-value property)
of the mean-value property, which asserts that the mean integral value of u over a ball is equal to the value at the center.

2.2 Proposition If u is harmonic in D, then it has the mean-value property there.

Proof) With $a \in D$ and $0 < r < \infty$ s.t. $a + \bar{B} \subset D$, we have from Ito's formula:

$$u(W_{t\wedge\tau_{a+B_r}}) = u(W_0) + \sum_{i=1}^{d}\int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} + \frac{1}{2}\int_0^{t\wedge\tau_{a+B_r}} \Delta u(W_s)ds =$$

$$= u(W_0) + \sum_{i=1}^{d}\int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} \text{ where } 0 \le t < \infty,$$

since u is harmonic and $(\partial u/\partial x_i)$; $1 \le i \le d$, are bounded functions on $a + B_r$, the expectations under P^a of the stochastic integrals are all equal to 0. After taking these expectations on both sides and letting $t \to \infty$, we use

$$u(a) = E^a u(W_{\tau_{a+B_r}}) = \int_{\partial B_r} u(a+x)\mu_r(dx). \quad \square$$

2.3 Corollary (Maximum Principle) Suppose that u is harmonic in the open, connected domain D. If u achieves its supremum over D at some point in D, then u is identically constant.

Proof)