Chapter 3 Notes

3.2 Construction of Stochastic Integrals

Let us consider a continuous, square-integrable martingale $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ equipped with the filtration $\mathcal{F}_t$ satisfying the usual conditions.
We assume $M_0 = 0$ a.s. P.

2.5 Problem Let X be a bounded, measurable, $\{\mathcal{F}_t\}$- adapted process. Let $0 < T < \infty$ be fixed. We wish to construct a sequence $\{X^{(k)}\}_{k=1}^{\infty}$

A. Simple Processes and Approximations

2.3 Defintion A process X is called **simple** if there exists a strictly increasing sequence of real numbers $\{t_n\}_{n=0}^{\infty}$ with $t_0 = 0$ and $\lim_{n\to\infty} t_n = \infty$, as well as a sequence of real random variables ξ_n

3.5 The Girsanov Theorem

A. The Basic Result

Throughout this section, we have a probability space $(\Omega, \mathcal{F}_t, P)$ and a d-dimensional Brownian motion $W = \{W_t = (W_t^{(1)}, W_t^{(2)}, ..., W_t^{(d)}, \mathcal{F}_t); 0 \leq t < \infty\}$ defined on it, with $P[W_0 = \mathbf{0}] = 1$. We assume that the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. Let $X = \{(X_t^{(1)}, X_t^{(2)}, ..., X_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ be a vector of measurable, adapted processes satisfying

$$P\left[\int_0^T (X_t^{(i)})^2 dt < \infty\right] = 1; \quad 1 \leq i \leq d, \quad 0 \leq T < \infty.$$

Then, for each i, the stochastic integral $I^{W^{(i)}}(X^{(i)})$ is defined, and is a member of $\mathcal{M}^{c,loc}$. We set

$$Z_t(X) \triangleq exp\left[\sum_{i=1}^{d} \int_0^t X_s^{(i)} - \frac{1}{2}\int_0^t \|X_s\|^2 ds\right].$$

Then, we have

$$Z_t(X) = 1 + \sum_{i=1}^{d} \int_0^t Z_s(X) X_s^{(i)} dW_s^{(i)},$$

which shows that $Z(X)$ is continuous local martingale, with $Z_0(X) = 1$.
If $Z(X)$ is a martingale, then by the martingale property, $E[Z_t(X)] = 1$.
Thus, we can define for each $0 \leq T < \infty$, a probability measure $\tilde{P}_T$ on $\mathcal{F}_T$ by

$$\tilde{P}_T(A) \triangleq \int_A Z_T(\omega) dP(\omega) = E[1_A Z_T(\omega)]; \quad A \in \mathcal{F}_T.$$

The family of probability measures $\{\tilde{P}_T; \quad 0 \leq t < \infty\}$ satisfies the consistency condition

$$\tilde{P}_T = \tilde{P}_t; \quad A \in \mathcal{F}_t, \quad 0 \leq t < T$$

which intuitively means that these probability measures are extensions of each other to a larger σ-algebra, while assigning the same probability to commonly known information (measurable sets).

$\because$ Let $A \in \mathcal{F}_t$. Then, $\tilde{P}(A) = E[1_A Z_T(X)] = E[1_A E[Z_T(X)|\mathcal{F}_t]] = E[1_A Z_t(X] = \tilde{P}_t(A)$ $\square$.

5.1 Theorem (Girsanov (1960), Cameron and Martin (1944)).
Assume that $Z(X)$ defined above is a martingale. Define a process $\tilde{W} = \{\tilde{W}_t = (\tilde{W}_t^{(1)}), ..., \tilde{W}_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ by

$$\tilde{W}_t^{(i)} \triangleq W_t^{(i)} - \int_0^t X_s^{(i)} ds; \quad 1 \leq t \leq d, \quad 0 \leq t < \infty.$$

For each fixed $T \in [0, \infty)$, the process $\{\tilde{W}_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \tilde{P}_T)$.

B. Proof and Ramifications

We denote by $\tilde{E}_T(\tilde{E})$ the expectation operator w.r.t. $\tilde{P}_T(\tilde{P})$.

5.3 Lemma Fix $0 \leq T < \infty$ and assume that $Z(X)$ is a martingale. If $0 \leq s \leq t \leq T$ and Y is an $\mathcal{F}_T$-measurable r.v. satisfying $\tilde{E}_T|Y| < \infty$, then we have the Bayes' rule:

$$\tilde{E}_T[Y|\mathcal{F}_s] = \frac{1}{Z_s(X)} E[Y Z_t(X)|\mathcal{F}_s], \quad a.s.\ P\ and\ \tilde{P}_T.$$

Proof) Fix $A \in \mathcal{F}_s$. Then,

$$\tilde{E}_T\left[1_A \frac{1}{Z_s(X)} E[Y Z_t(X)|\mathcal{F}_s]\right] = E\left[Z_t(X) 1_A \frac{1}{Z_s(X)} E[Y Z_t(X)|\mathcal{F}_s]\right] =$$

$$= E\left[E\{Z_t(X)1_A \frac{1}{Z_s(X)} E[YZ_t(X)|\mathcal{F}_s]|\mathcal{F}_s\}\right] = E[1_A E[YZ_t(X)|\mathcal{F}_s]] =$$

$$= E[1_A Y Z_t(X)] = \tilde{E}[1_A Y]. \quad \square$$

We denote by $\mathcal{M}_T^{c,loc}$ the class of continuous local martingales $M = \{M_t, \mathcal{F}_t; 0 \leq t \leq T\}$ on $\Omega, \mathcal{F}_T, P$ satisfying $P[M_0 = 0] = 1$, and define $\tilde{\mathcal{M}}_T^{c,loc}$ similarly, with P replaced by $\tilde{P}_T$.

5.4 Proposition Fix $0 \leq T < \infty$ and assume that $Z(X)$ is a martingale. If $M \in \mathcal{M}_T^{c,loc}$, then the process

$$\tilde{M}_t \triangleq M_t - \sum_{i=1}^d \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle, \ \mathcal{F}_t; \ 0 \leq t \leq T$$

is in $\tilde{\mathcal{M}}_T^{c,loc}$. If $N \in \mathcal{M}_T^{c,loc}$ and

$$\tilde{N}_t \triangleq N_t - \sum_{i=1}^d \int_0^t X_s^{(i)} d\langle N, W^{(i)}\rangle, \ \mathcal{F}_t; \ 0 \leq t \leq T,$$

then

$$\langle \tilde{M}, \tilde{N}\rangle_t = \langle M, N\rangle_t; \ 0 \leq t \leq T, \ a.s. \ P \text{ and } \tilde{P}_T,$$

where the cross-variations are computed under the appropriate measures.

Proof) We only consider the case where M and N are bounded martingales with bounded quadratic variations, and assume also that $Z_t(X)$ and $\sum_{j=1}^d \int_0^t (X_s^{(j)})^2 ds$ are bounded in t and ω; the general case can be reduced to this one by localization. By Proposition 2.14,

$$|\int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s|^2 \leq \langle M\rangle_t \int_0^t (X_s^{(i)})^2 ds,$$

and thus $\tilde{M}$ is also bounded.
From Problem 3.12, we have the integration by parts formula: If $X_t = X_0 + M_t + B_t$ and $Y_t = Y_0 + N_t + C_t$ are two semimartingales with $B_0 = C_0 = 0$ a.s., then

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle M, N\rangle_t.$$

Using the integration by parts formula, we have

$$Z_t(X)\tilde{M} = \int_0^t Z_u(X) dM_u + \sum_{i=1}^d \int_0^t \tilde{M}_u X_u^{(i)} Z_u(X) dW_u^{(i)},$$

which is a martingale under P.

$\because$ Consider the 1 dimensional case, where $i = 1$. Then, we have

$$Z_t(X)\tilde{M} = \int_0^t Z_s(X) d\tilde{M}_s + \int_0^t \tilde{M}_s dZ_s(X) + \int_0^t d\langle M, Z(X)\rangle_s =$$

$$= \int_0^t Z_s(X) dM_s - \int_0^t Z_s(X) X_s d\langle M, W\rangle_s + \int_0^t \tilde{M}_s Z_s(X) X_s dW_s +$$

$$+ \int_0^t Z_s(X) X_s d\langle M, W\rangle_s = \int_0^t Z_s(X) dM_s + \int_0^t \tilde{M}_s Z_s(X) X_s dW_s.$$

Therefore, for $0 \le s \le t \le T$, we have from Lemma 5.3:

$$\tilde{E}_T[\tilde{M}_t|\mathcal{F}_s] = \frac{1}{Z_s(X)} E[Z_t(X)\tilde{M}_t|\mathcal{F}_s] = \tilde{M}_s, \text{ a.s. } P \text{ and } \tilde{P}_T.$$

Therefore, $\tilde{M} \in \tilde{\mathcal{M}}^{c,loc}$. The integration by parts formula also implies:

$$\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t = \int_0^t \tilde{M}_u dN_u + \int_0^t \tilde{N}_u dM_u -$$

$$-\sum_{i=1}^d \left[\int_0^t \tilde{M}_u X_u^{(i)} d\langle N, W^{(i)}\rangle_u + \int_0^t \tilde{N}_u X_u^{(i)} d\langle M, W^{(i)}\rangle_u\right]$$

as well as

$$Z_t(X)[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t] = \int_0^t Z_u(X)\tilde{M}_u dN_u + \int_0^t Z_u(X)\tilde{N}_u dM_u +$$

$$+\sum_{i=1}^d \int_0^t [\tilde{M}_u\tilde{N}_u - \langle M, N\rangle_u] X_u^{(i)} Z_u(X) dW_u^{(i)}.$$

This last process is consequently a martingale under P, and so Lemma 5.3 implies that for $0 \le s \le t \le T$

$$\tilde{E}_T[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t|\mathcal{F}_s] = \tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t; \text{ a.s. } P \text{ and } \tilde{P}_T.$$

This proves that $\langle \tilde{M}, \tilde{N}\rangle_t = \langle M, N\rangle_t;\ 0 \le t \le T$, a.s. $\tilde{P}_T$ and P. $\square$

Proof of Theorem 5.1) We show that the continuous process $\tilde{W}$ on $(\Omega, \mathcal{F}_t \tilde{P}_T)$ satisfies the hypothesis of P.Levy's Theorem 3.16. Setting $M = W^{(j)}$ in Prop 5.4, we have $\tilde{M} = \tilde{W}_t^{(j)}$, thus $\tilde{W}^{(j)} \in \mathcal{M}_T^{c,loc}$. Setting $N = W^{(k)}$, we obtain

$$\langle \tilde{W}^{(j)}, \tilde{W}_t^{(k)}\rangle = \langle W^{(j)}, W^{(k)}\rangle_t = \delta_{j,k} t; \quad 0 \le t \le T \text{ a.s. } \tilde{P}_T \text{ and } P. \square$$

Let $\{M_t, \mathcal{F}_t; 0 \le t \le T\}$ be a continuous local martingale under P. With the assumption of Theorem 5.1, Prop 5.4 shows that M is a continuous semimartingale under $\tilde{P}_T$.

The converse is also true, if $\{\tilde{M}_t, \mathcal{F}_t; 0 \le t \le T\}$ is a continuous martingale under $\tilde{P}_T$, then Lemma 5.3 implies that for $0 \le s \le t \le T$:

$$E[Z_t(X)\tilde{M}_t|\mathcal{F}_s] = Z_s(X)\tilde{E}_T[\tilde{M}_t|\mathcal{F}_s] = Z_s(X)\tilde{M}_s \text{ a.s. } P \text{ and } \tilde{P}_T.$$

so $Z(X)\tilde{M}$ is a martingale under P.
If $\tilde{M} \in \mathcal{M}_T^{c,loc}$, a localization argument shows that $Z(X)\tilde{M} \in \mathcal{M}_T^{c,loc}$.
But $Z(X) \in \mathcal{M}^c$, so Ito's rule implies that $\tilde{M} = \frac{Z(X)\tilde{M}}{Z(X)}$ is a continuous semimartingale under P (c.f. Remark 3.4).
Thus, given $\tilde{M} \in \mathcal{M}_T^{c,loc}$, we have a decomposition

$$\tilde{M}_t = M_t + B_t; \quad 0 \le t \le T,$$

where $M \in \mathcal{M}_T^{c,loc}$ and B is of bounded variation with $B_0 = 0$, $P-$ a.s.
According to Prop 5.4, the process

$$\tilde{M}_t - (M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s) = B_t + \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s; \quad 0 \le t \le T,$$

is in $\tilde{\mathcal{M}}_T^{c,loc}$, and of being bounded variation, this process must be indistinguishable from the identity zero process (Problem 3.2). □

5.5 Proposition Under the hypotheses of Theorem 5.1, every $\tilde{M} \in \mathcal{M}_T^{c,loc}$ has the representation $\tilde{M}_t = M_t - \sum_{i=1}^d \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle$ for some $M \in \mathcal{M}_T^{c,loc}$.

C. Brownian Motion with Drift

Below, we discuss an interesting application of the Girsanov theorem: the distribution of passage times for Brownian motion with drift.
Consider a Brownian motion $W = \{W_t, \mathcal{F}_t; 0 \le t < \infty\}$ and the passage time $T_b = \inf\{t \ge 0;\ W_t = b\}$ to the level $b \ne 0$ has density and moment generating function (Remark 2.8.3), respectively:

$$P[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} exp\left[-\frac{b^2}{2t}\right] dt; \quad t > 0$$

$$Ee^{-\alpha T_b} = e^{-|b|\sqrt{2\alpha}}; \quad \alpha > 0.$$

For any real number $\mu \ne 0$, the process $\tilde{W} = \{\tilde{W}_t = W_t - \mu t, \mathcal{F}_t^W, 0 \le t < \infty\}$, is a Brownian motion under the unique measure $P^{(\mu)}$ which satisfies

$$P^{(\mu)} = E[1_A Z_t]; \quad A \in \mathcal{F}_t^W,$$

where $Z_t \triangleq exp(\mu W_t - \frac{1}{2}\mu^2 t)$ by Corollary 5.2. We say that, under $P^{(\mu)}$, $W_t = \mu t + \tilde{W}_t$ is a Brownian motion with drift μ. On the set $\{T_b \le t\} \in$

$\mathcal{F}_t^W \cap \mathcal{F}_{T_b}^W = \mathcal{F}_{t \cap T_b}^W$, we have $Z_{t \wedge T_b} = Z_{T_b}$, so the optional sampling theorem 1.3.22 and Problem 1.3.23(i) imply

$$P^{(\mu)}[T_b \le t] = E[1_{\{T_b \le t\}} Z_t] = E[1_{\{T_b \le t\}} E[Z_t | \mathcal{F}_{t \wedge T_b}^W]] = E[1_{\{T_b \le t\}} Z_{t \wedge T_b}] =$$

$$= E[1_{\{T_b \le t\}} Z_{T_b}] = E[1_{\{T_b \le t\}} e^{\mu b - \frac{1}{2}\mu^2 T_b}] = \int_0^t exp(\mu b - \frac{1}{2}\mu^2 s) P[T_b \in ds].$$

This relation has a several consequences. Firstly,

$$P^{(\mu)}[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} exp\left[-\frac{(b - \mu t)^2}{2t}\right] dt; \quad t > 0.$$

Second, letting $t \to \infty$, we have

$$P^{(\mu)}[T_b < \infty] = e^{\mu b} E[exp(-\frac{1}{2}\mu^2 T_b)],$$

so we obtain from the moment generating function that

$$P^{(\mu)}[T_b < \infty] = exp[\mu b - |\mu b|]].$$

In particular, a Brownian motion with drift $\mu \neq 0$ reaches level $b \neq 0$ with probability one if and only if μ and b have the same sign. If μ and b have opposite signs, the density $P^{(\mu)}[T_b \in dt]$ is defective, in the sense that $P^{(\mu)}[T_b < \infty] < 1$.s

D. The Notikov Condition

In order to use the Girsanov theorem effectively,