

## Chapter 4 Notes

### 4.2 Harmonic Functions and the Dirichlet Problem

A function  $u : D \mapsto \mathbb{R}$  where  $D$  is an open subset of  $\mathbb{R}^d$  is called **harmonic** in  $D$  if  $u$  is of class  $C^2$  and  $\Delta u \triangleq \sum_{i=1}^d (\frac{\partial^2 u}{\partial x_i^2}) = 0$  in  $D$ .

Throughout this section,  $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ ,  $(\Omega, \mathcal{F})$ ,  $\{P^x\}_{x \in \mathbb{R}^d}$  is a  $d$ -dimensional Brownian family and  $\{\mathcal{F}_t\}$  satisfies the usual conditions. We denote by  $D$  an open set in  $\mathbb{R}^d$  and introduce the stopping time (Problem 1.2.7)

$$\tau_D = \inf\{t \geq 0; W_t \in D^c\},$$

the time of first exit from  $D$ . The boundary of  $D$  will be denoted by  $\partial D$ , and  $\bar{D} = D \cup \partial D$  is the closure of  $D$ . By Theorem 2.9.23, each component of  $W$  is a.s. unbounded, so

$$P^x[\tau_D < \infty] = 1; \quad \forall x \in D \subset \mathbb{R}^d, \quad D \text{ bounded.}$$

Let  $B_r \triangleq \{x \in \mathbb{R}^d; \|x\| < r\}$  be the open ball of radius  $r$  centered at the origin. The volume of this ball is

$$V \triangleq \frac{2r^d \pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})},$$

and its surface area is

$$S_r \triangleq \frac{2r^{d-1} \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \frac{d}{r} V_r.$$

We define a probability measure  $\mu_r$  on  $\partial B_r$  by

$$\mu_r(dx) = P^0[W_{\tau_{B_r}} \in dx]; \quad r > 0.$$

In the integral notation, the above definition for  $A \subset \partial B_r$  becomes:

$$\mu_r(A) = \int_{x \in A} P^0[W_{\tau_{B_r}} \in dx],$$

which we interpret as the probability of the Brownian motion  $W_t$  crossing the boundary  $\partial B_r$  by passing through points in  $A$ .

## A. The mean-value property

Because of the rotational invariance of Brownian motion (Problem 3.3.18), the measure  $\mu_r$  is also rotationally invariant and thus proportional to surface measure on  $\partial B_r$ . In particular, the Lebesgue integral of a function  $f$  over  $B_r$  can be written in iterated form as

$$\int_{B_r} f(x) dx = \int_0^r S_\rho \int_{\partial B_\rho} f(x) \mu_\rho(dx) d\rho.$$

**2.1 Definition** We say that the function  $u : D \mapsto \mathbb{R}$  has the **mean-value property** if, for every  $a \in D$  and  $0 < r < \infty$  s.t.  $a + \bar{B}_r \subset D$ , we have

$$u(a) = \int_{\partial B_r} u(a+x) \mu_r(dx).$$

Then we derive the consequence

$$u(a) = \frac{1}{V_r} \int_{B_r} u(a+x) dx.$$

$$\begin{aligned} \because \int_{B_r} u(a+x) dx &= \int_0^r S_\rho \int_{\partial B_\rho} u(a+x) \mu_\rho(dx) d\rho = \int_0^r S_\rho u(a+x) d\rho = \\ &= u(a+x) \int_0^r S_\rho d\rho = u(a+x) V_r \end{aligned}$$

(the second inequality follows from the mean-value property)

of the mean-value property, which asserts that the mean integral value of  $u$  over a ball is equal to the value at the center.

**2.2 Proposition** If  $u$  is harmonic in  $D$ , then it has the mean-value property there.

(Proof) With  $a \in D$  and  $0 < r < \infty$  s.t.  $a + \bar{B} \subset D$ , we have from Ito's formula:

$$\begin{aligned} u(W_{t \wedge \tau_{a+B_r}}) &= u(W_0) + \sum_{i=1}^d \int_0^{t \wedge \tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s) dW_s^{(i)} + \frac{1}{2} \int_0^{t \wedge \tau_{a+B_r}} \Delta u(W_s) ds = \\ &= u(W_0) + \sum_{i=1}^d \int_0^{t \wedge \tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s) dW_s^{(i)} \text{ where } 0 \leq t < \infty, \end{aligned}$$

since  $u$  is harmonic and  $(\partial u / \partial x_i); 1 \leq i \leq d$ , are bounded functions on  $a + B_r$ , the expectations under  $P^a$  of the stochastic integrals are all equal to 0. After taking these expectations on both sides and letting  $t \rightarrow \infty$ , we use

$$u(a) = E^a u(W_{\tau_{a+B_r}}) = \int_{\partial B_r} u(a+x) \mu_r(dx). \quad \square$$

**2.3 Corollary** (Maximum Principle) Suppose that  $u$  is harmonic in the open, connected domain  $D$ . If  $u$  achieves its supremum over  $D$  at some point in  $D$ ,

then  $u$  is identically constant.

Proof) Let  $M = \sup_{x \in D} u(x)$ , and let  $D_M = \{x \in D; u(x) = M\}$ . We assume that  $D_M$  is nonempty and show that  $D_M = D$ . Since  $u$  is continuous,  $D_M = u^{-1}(\{M\}) \cap D$  is a closed set relative to  $D$ . But for  $a \in D_M$ , and  $0 < r < \infty$  s.t.  $a + \bar{B}_r \subset D$ , we have the mean value property:

$$M = u(a) = \frac{1}{V_r} \int_{B_r} u(a+x) dx \leq \frac{1}{V_r} \int_{B_r} M dx = M,$$

which shows that  $u = M$  on  $a + B_r$ .

Since  $a \in D_M$  was arbitrary, and  $a \in a + B_r \subset D_M$ , we conclude  $D_M$  is open. Moreover,  $D$  is connected, either  $D_M$  or  $D - D_M$  must be empty.  $\square$

For the sake of completeness, below is the converse of Proposition 2.2.

**2.5 Proposition** If  $u$  maps  $D$  into  $\mathbb{R}$  and has the mean-value property, then  $u$  is of class  $C^\infty$  and harmonic.

Proof) We first prove that  $u$  is of class  $C^\infty$ . For  $\epsilon > 0$ , let  $g_\epsilon : \mathbb{R}^d \rightarrow [0, \infty)$  be the  $C^\infty$  function

$$g_\epsilon(x) = \begin{cases} c(\epsilon) \exp \left[ \frac{1}{\|x\|^2 - \epsilon^2} \right], & \|x\| < \epsilon \\ 0, & \|x\| \geq \epsilon \end{cases} \quad (1)$$

where  $c(\epsilon)$  is chosen so that

$$\begin{aligned} \int_{B_\epsilon} g_\epsilon(x) dx &= \int_0^\epsilon S_\rho \int_{\partial B_\rho} g_\epsilon(x) \mu_\rho(dx) d\rho = \\ &= c(\epsilon) \int_0^\epsilon S_\rho \int_{\partial B_\rho} \exp\left(\frac{1}{\|x\|^2 - \epsilon^2}\right) \mu_\rho(dx) d\rho = c(\epsilon) \int_0^\epsilon S_\rho \exp\left(\frac{1}{\rho^2 - \epsilon^2}\right) d\rho = 1. \end{aligned}$$

For  $\epsilon > 0$  and  $a \in D$  s.t.  $a + \bar{B}_\epsilon \subset D$ , define

$$u_\epsilon(a) \triangleq \int_{B_\epsilon} u(a+x) g_\epsilon(x) dx = \int_{\mathbb{R}^d} u(y) g_\epsilon(y-a) dy.$$

From the second representation,  $u_\epsilon$  is of class  $C^\infty$  on the open subset of  $D$  where it is defined. Furthermore, for every  $a \in D$  there exists  $\epsilon > 0$  so that  $a + \bar{B}_\epsilon \subset D$ ; from mean-value property of  $u$ , we have

$$\begin{aligned} u_\epsilon(a) &= \int_{B_\epsilon} u(a+x) g_\epsilon(x) dx = c(\epsilon) \int_0^\epsilon S_\rho \int_{\partial B_\rho} u(a+x) \exp\left(\frac{1}{\rho^2 - \epsilon^2}\right) \mu_\rho(dx) d\rho = \\ &= c(\epsilon) \int_0^\epsilon S_\rho u(a) \exp\left(\frac{1}{\rho^2 - \epsilon^2}\right) d\rho = u(a) \end{aligned}$$

where the last equality is from the definition of  $c(\varepsilon)$ . Thus,  $u$  is also of class  $C^\infty$ .

In order to show that  $\Delta u = 0$  in  $D$ , we choose  $a \in D$  and use a Taylor-series expansion in the neighborhood  $a + \bar{B}_\varepsilon$ ,

$$u(a + y) = u(a) + \sum_{i=1}^d y_i \frac{\partial u}{\partial x_i}(a) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d y_i y_j \frac{\partial^2 u}{\partial x_i \partial x_j}(a) + o(\|y\|^2); \quad y \in \bar{B}_\varepsilon,$$

where again  $\varepsilon > 0$  is chosen so that  $a + \bar{B}_\varepsilon \subset D$ . Odd symmetry gives us

$$\int_{\partial B_\varepsilon} y_i \mu_\varepsilon(dy) = 0, \quad \int_{\partial B_\varepsilon} y_i y_j \mu_\varepsilon(dy) = 0; \quad i \neq j,$$

so integrating the above Taylor-expansion over  $\partial B_\varepsilon$  and using the mean-value property, we have

$$u(a) = \int_{\partial B_\varepsilon} u(a + y) \mu_\varepsilon(dy) = u(a) + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}(a) \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) + o(\varepsilon^2).$$

But

$$\int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{1}{d} \sum_{i=1}^d \int_{\partial B_\varepsilon} y_i^2 \mu_\varepsilon(dy) = \frac{\varepsilon^2}{d},$$

thus we have

$$\frac{\varepsilon^2}{2d} \Delta u(a) + o(\varepsilon^2) = 0.$$

Dividing by  $\varepsilon^2$  and letting  $\varepsilon \downarrow 0$ , we have  $\Delta u(a) = 0$ .  $\square$

## B. The Dirichlet problem

We take up now the Dirichlet problem  $(D, f)$ : with open  $D \subset \mathbb{R}^d$  and  $f : \partial D \rightarrow \mathbb{R}$  is a given continuous function, find a continuous function  $u : \bar{D} \rightarrow \mathbb{R}$  s.t.

$$\Delta u = 0; \quad \text{in } D$$

$$u = f; \quad \text{on } \partial D.$$

Such a function, when it exists, will be called a solution to the Dirichlet problem  $(D, f)$ . One may interpret  $u(x)$  as the steady-state temperature at  $x \in D$  when the boundary temperatures of  $D$  are specified by  $f$ .

The power of the probabilistic method is demonstrated by the fact that we can immediately write down a very likely solution to  $(D, f)$ , namely

$$u(x) \triangleq E^x f(W_{\tau_D}); \quad x \in \bar{D},$$

provided that

$$E^x|f(W_{\tau_D})| < \infty; \quad \forall x \in D.$$

If  $x \in \partial D$ , then since  $P^x[W_0 = x] = 1$ , we have

$$u(x) = E^x f(W_{\tau_D}) = E^x f(W_0) = f(x).$$

Thus,  $u$  satisfies  $u = f$  on  $\partial D$ . Furthermore, for  $a \in D$  and  $B_r$  chosen so that  $a + \bar{B}_r \subset D$ , we have:

$$\begin{aligned} u(a) &= E^a f(W_{\tau_D}) \stackrel{\text{tower}}{=} E^a \{E^a[f(W_{\tau_D})|\mathcal{F}_{\tau_{a+B_r}}]\} = \\ &= E^a \{E^a[f(W_{\tau_D} - W_{\tau_{a+B_r}} + W_{\tau_{a+B_r}})|\mathcal{F}_{\tau_{a+B_r}}]\} = \\ &= E^a \{u(W_{\tau_{a+B_r}})\} \stackrel{\text{def}}{=} \int_{\partial B_r} u(a+x) \mu_r(dx), \end{aligned}$$

where the second last equality is from the strong Markov property of B.M.

Therefore,  $u$  has the mean-value property, and so it must satisfy  $\Delta u = 0$ ; in  $D$ . The only unresolved issue is whether  $u$  is continuous up to and including  $\partial D$ .

**2.6 Proposition** If  $E^x|f(W_{\tau_D})| < \infty$  holds, then  $u(x) \triangleq E^x f(W_{\tau_D})$ ;  $x \in \bar{D}$  is harmonic in  $D$ .

**2.7 Proposition** If  $f$  is bounded and

$$P^a[\tau_D < \infty] = 1; \quad \forall a \in D,$$

then any bounded solution to  $(D, f)$  has the representation  $u(x) = E^x f(W_{\tau_D})$ .

(Proof) Let  $u$  be any bounded solution to  $(D, f)$ , and let  $D_n \triangleq \{x \in D; \inf_{y \in \partial D} \|x - y\| > \frac{1}{n}\}$ . Then,  $D_n$  is an increasing sequence of subsets of  $D$ . From Ito's rule,

$$u(W_{t \wedge \tau_{B_n} \wedge \tau_{D_n}}) = u(W_0) + \sum_{i=1}^d \int_0^{t \wedge \tau_{B_n} \wedge \tau_{D_n}} \frac{\partial u}{\partial x_i}(W_s) dW_s^{(i)}; \quad 0 \leq t < \infty, \quad n \geq 1.$$

Since  $\frac{\partial u}{\partial x_i}$  is bounded in  $\overline{B_n \cap D_n}$ , we take expectations w.r.t  $P^a$  from both sides:

$$E^a u(W_{t \wedge \tau_{B_n} \wedge \tau_{D_n}}) = E^a(u(W_0)) = u(a);$$

where  $0 \leq t < \infty$ ,  $n \geq 1$ ,  $a \in D_n$ .

As  $t \rightarrow \infty, n \rightarrow \infty, P^a[\tau_D < \infty] = 1$ ;  $\forall a \in D$  implies that  $u(W_{t \wedge \tau_{B_n} \wedge \tau_{D_n}})$  converges to  $f(W_{\tau_D})$ , a.s.  $P^a$ . The representation  $u(x) = E^x f(W_{\tau_D})$ ;  $x \in \bar{D}$  follows from the bounded convergence theorem.  $\square$

In the light of Proposition 2.6 and 2.7, the existence of a solution to the Dirichlet problem boils down to the question of the continuity of  $u$  defined by

$E^x f(W_{\tau_D})$  at the boundary of  $D$ . We therefore undertake to characterize those points  $a \in \partial D$  for which

$$\lim_{x \rightarrow a, x \in D} E^x f(W_{\tau_D}) = f(a)$$

holds for every bounded, measurable function  $f : \partial D \rightarrow \mathbb{R}$  which is continuous at the point  $a$ .

**2.9 Definition** Consider the stopping time of the right-continuous filtration  $\{\mathcal{F}_t\}$  given by  $\sigma_D \triangleq \inf\{t > 0; W_t \in D^c\}$ . We say that a point  $a \in \partial D$  is regular for  $D$  if  $P^a[\sigma_D = 0] = 1$ , i.e., a Brownian motion path started at  $a$  does not immediately return to  $D$  and remain there for a nonempty time interval.

**2.10 Remark** A point  $a \in \partial D$  is called irregular if  $P^a[\sigma_D = 0] < 1$ ; however, the event  $\{\sigma_D = 0\}$  belongs to  $\mathcal{F}_{0+}^W$ , and so the Blumenthal zero-one law (Theorem 2.7.17) gives for an irregular point  $a : P^a[\sigma_D = 0] = 0$ .

**2.11 Remark** The regularity is a local condition; i.e.  $a \in \partial D$  is regular for  $D$  if and only if  $a$  is regular for  $(a + B_r) \cap D$ , for some  $r > 0$ .

**2.12 Theorem** Assume that  $d \geq 2$  and fix  $a \in \partial D$ . The following are equivalent:

- (i)  $\lim_{x \rightarrow a, x \in D} E^x f(W_{\tau_D}) = f(a)$  holds for every bounded, measurable function  $f : \partial D \rightarrow \mathbb{R}$  which is continuous at  $a$ ;
- (ii)  $a$  is regular for  $D$ ;
- (iii) for all  $\varepsilon > 0$ , we have

$$\lim_{x \rightarrow a, x \in D} P^x[\tau_D > \varepsilon] = 0.$$

(Proof) We assume WLOG that  $a = 0$ , and begin by proving the implication (i)  $\Rightarrow$  (ii) by contradiction. If the origin is irregular, then  $P^0[\sigma_D = 0] = 0$  (Remark 2.10). Since a Brownian motion of dimension  $d \geq 2$  never returns to its starting point (Prop 3.3.22), we have

$$\lim_{r \downarrow 0} P^0[W_{\tau_D} \in B_r] = P^0[W_{\tau_D} = 0] = 0.$$

Fix  $r > 0$  for which  $P^0[W_{\tau_D}] < \frac{1}{4}$ , and choose a sequence  $\{\delta_n\}_{n=1}^\infty$  for which  $0 < \delta_n < r$  for all  $n$  and  $\delta_n \downarrow 0$ . With  $\tau_n \triangleq \inf\{t \geq 0; \|W_t\| \geq \delta_n\}$ , we have  $P^0[\tau_n \downarrow 0] = 1$ , and thus  $\lim_{n \rightarrow \infty} P^0[\tau_n < \sigma_D] = 1$ . Furthermore, on the event  $\{\tau_n < \sigma_D\}$  we have  $W_{\tau_n} \in D$ . For  $n$  large enough so that  $P^0[\tau_n < \sigma_D] \geq \frac{1}{2}$  we may write

$$\begin{aligned} \frac{1}{4} &> P^0[W_{\sigma_D} \in B_r] \geq P^0[W_{\sigma_D} \in B_r, \tau_n < \sigma_D] = E^0(1_{\{W_{\sigma_D} \in B_r\}} 1_{\{\tau_n < \sigma_D\}}) = \\ &= E^0(1_{\{\tau_n < \sigma_D\}} E^0[1_{\{W_{\sigma_D} \in B_r\}} | \mathcal{F}_{\tau_n}]) = E^0(1_{\{\tau_n < \sigma_D\}} P^0[W_{\sigma_D} \in B_r | \mathcal{F}_{\tau_n}]) = \end{aligned}$$

$$= \int_{D \cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r] P^0[\tau_n < \sigma_D, W_{\tau_n} \in dx] \geq \frac{1}{2} \inf_{x \in D \cap B_{\delta_n}} P^x[W_{\tau_D} \in B_r],$$

for which we conclude that  $P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2}$  for some  $x_n \in D \cap B_{\delta_n}$ . Now choose a bounded, continuous function  $f : \partial D \rightarrow \mathbb{R}$  s.t.  $f = 0$  outside  $B_r$ ,  $f \leq 1$  inside  $B_r$ , and  $f(0) = 1$ . For such a function we have

$$\overline{\lim}_{n \rightarrow \infty} E^x f(W_{\tau_D}) \leq \overline{\lim}_{n \rightarrow \infty} P^{x_n}[W_{\tau_D} \in B_r] \leq \frac{1}{2} < f(0),$$

and (i) fails.

We next show that (ii)  $\Rightarrow$  (iii). Observe first of all that for  $0 < \delta < \varepsilon$ , the function

$$\begin{aligned} g_\delta(x) &\triangleq P^x[W_s \in D; \delta \leq s \leq \varepsilon] = E^x(P^{W_\delta}[\tau_D > \varepsilon - \delta]) = \\ &= \int_{\mathbb{R}^d} P^y[\tau_D > \varepsilon - \delta] P^x[W_\delta \in dy] \end{aligned}$$

is continuous in  $x$ . But

$$g_\delta(x) \downarrow g(x) \triangleq P^x[W_s \in D; 0 < s \leq \varepsilon] = P[\sigma_D > \varepsilon]$$

as  $\delta \downarrow 0$ , so  $g$  is upper semicontinuous. From this fact and the inequality  $\tau_D \leq \sigma_D$ , we conclude that  $\lim_{x \rightarrow 0} P^x[\tau_D > \varepsilon] \leq \lim_{x \rightarrow 0} g(x) \leq g(0) = 0$ , by (ii).

Finally, we prove (ii)  $\Rightarrow$  (i). We know that for each  $r > 0$ ,  $P^x[\max_{0 \leq t \leq \varepsilon} \|W_t - W_0\| < r]$  does not depend on  $x$  and approaches one as  $\varepsilon \downarrow 0$ . But then

$$\begin{aligned} P^x[\|W_{\tau_D} - W_0\| < r] &\geq P^x[\{\max_{0 \leq t \leq \varepsilon} \|W_t - W_0\| < r\} \cap \{\tau_D \leq \varepsilon\}] \geq \\ &\geq P^0[\max_{0 \leq t \leq \varepsilon} \|W_t\| < r] - P^x[\tau_D > \varepsilon]. \end{aligned}$$

Letting  $x \rightarrow 0$  ( $x \in D$ ) and  $\varepsilon \downarrow 0$ , successively, we obtain from (iii),

$$\lim_{x \rightarrow 0, x \in D} P^x[\|W_{\tau_D} - x\| < r] = 1; \quad 0 < r < \infty.$$

The continuity of  $f$  at the origin and its boundedness on  $\partial D$  gives  $\lim_{x \rightarrow 0, x \in D} E^x f(W_{\tau_D}) = f(a)$ .  $\square$

C. Conditions for regularity

E. Supplementary Exercises

### 4.3 The One-Dimensional Heat Equation

A. The Tychonoff uniqueness theorem

B. Nonnegative solutions of the heat equation

C. Boundary Crossing probabilities for Brownian motion

D. Mixed initial/boundary value problems

### 4.4 The Formulas of Feynman and Kac

A. The multi-dimensional formula

B. The one-dimensional formula