Chapter 3 Notes

3.2 Construction of Stochastic Integrals

Let us consider a continuous, square-integrable martingale $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ equipped with the filtration $\mathcal{F}_t$ satisfying the usual conditions.
We assume $M_0 = 0$ a.s. P.

A. Simple Processes and Approximations

Similar to the construction of the Lebesgue integrals, we start with the construction of stochastic integrals w.r.t. simple functions and extend the result to more general class of functions.

2.3 Definition A process X is called **simple** if there exists a strictly increasing sequence of real numbers $\{t_n\}_{n=0}^{\infty}$ with $t_0 = 0$ and $\lim_{n\to\infty} t_n = \infty$, as well as a sequence of real random variables $\{\xi_n\}_{n=0}^{\infty}$ and a nonrandom constant $C < \infty$ with $\sup_{n\geq 0} |\xi_n(\omega)| \leq C$, for every $\omega \in \Omega$, s.t. ξ_n is $\mathcal{F}_{t_n}$ measurable for every $n \geq 0$ and

$$X_t(\omega) = \xi_0(\omega)1_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i(\omega)1_{(t_i,t_{i+1}]}; \quad 0 \leq t < \infty \quad \omega \in \Omega.$$

The class of all simple processes will be denoted by $\mathcal{L}_0$. Because the members of $\mathcal{L}_0$ are progressively measurable and bounded, we have $\mathcal{L}_0 \subseteq \mathcal{L}^*(M) \subseteq \mathcal{L}(M)$.

For a simple function $X \in \mathcal{L}_0$, we define the stochastic integral as a martingale transform:

$$I_t(X) \triangleq \sum_{i=0}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n}) =$$

$$= \sum_{i=0}^{\infty} \xi_i(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}), \quad 0 \leq t < \infty,$$

where $n \geq 0$ is the unique integer for which $t_n \leq t < t_{n+1}$.

2.4 Lemma Let X be a bounded, measurable, $\mathcal{F}_t$-adapted process. Then there exists a sequence $\{X^{(m)}\}_{m=1}^{\infty}$ of simple processes s.t.

$$\sup_{T>0} \lim_{m\to\infty} E\int_0^T |X_t^{(m)} - X_t|^2 dt = 0.$$

Proof) We shall show how to construct, for each $T > 0$, a sequence $\{X^{(n,T)}\}_{n=1}^{\infty}$ of simple processes s.t.

$$\lim_{n\to\infty} E\int_0^T |X_t^{(n,T)} - X_t|^2 = 0.$$

Thus, for each positive integer m, there is another integer n_m s.t.

$$E\int_0^m |X_t^{(n_m,m)} - X_t|^2 \leq \frac{1}{m},$$

and the sequence $\{X^{(n_m,n))}\}_{m=1}^{\infty}$ satisfies for any $T > 0$,

$$E\int_0^T |X_t^{(n_m,m)} - X_t|^2 dt \leq E\int_0^m |X_t^{(n_m,m)} - X_t|^2 dt$$

for $m \geq T$, thus

$$\sup_{T>0} \lim_{m\to\infty} E\int_0^T |X_t^{(n_m,m)} - X_t|^2 dt \leq \sup_{T>0} \lim_{m\to\infty} E\int_0^m |X_t^{(n_m,m)} - X_t|^2 dt =$$

$$= \lim_{m\to\infty} E\int_0^m |X_t^{(n_m,m)} - X_t|^2 dt = 0.$$

Henceforth, T is a fixed, positive number.

(a) Suppose that X is continuous, then sequence of simple processes

$$X_t^{(n)}(\omega) \triangleq X_0(\omega) 1_{\{0\}}(t) + \sum_{k=0}^{2^n-1} X_{\frac{kT}{2^n}}(\omega) 1_{(\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]}(t); \quad n \geq 1$$

converges pointwise to $X_t(\omega)$ as $n \to \infty$ by the continuity of X, thus by the dominated convergence theorem,

$$\lim_{n\to\infty} E\int_0^T |X_t^{(n)} - X_t|^2 = 0.$$

(b) Now suppose that X is a progressively measurable; we consider the continuous, progressively measurable processes

$$F_t(\omega) \triangleq \int_0^{t\wedge T} X_s(\omega) ds; \quad \tilde{X}_t^{(m)} \triangleq m[F_t(\omega) - F_{(t-\frac{1}{m})\vee 0}(\omega)]; \; m \geq 1,$$

for $t \geq 0 \quad \omega \in \Omega$ (cf, Problem 1.2.19). Then by part (a), for each $m \geq 1$, there exists a sequence of simple processes $\{\tilde{X}^{(m,n)}\}_{n=1}^{\infty}$ s.t. $\lim_{n\to\infty} E\int_0^t |\tilde{X}_t^{(m,n)} - \tilde{X}_t^{(m)}|^2 dt = 0$.
Let us consider the $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable product set

$$A \triangleq \{(t,\omega) \in [0,T] \times \Omega; \quad \lim_{m\to\infty} \tilde{X}_t^{(m)}(\omega) = X_t(\omega)\}^c.$$

For each $\omega \in \Omega$, the cross section $A_\omega \triangleq \{t \in [0,T];\ (t,\omega) \in A\}$ is in $\mathcal{B}([0,T])$ and, according to the fundamental theorem of calculus, has Lebesgue measure 0.

$\because$ For fixed $\omega \in \Omega$, we want to show that $\mu(A_\omega) = 0,\ a.s.$, which is equivalent to showing $\lim_{m\to\infty} \tilde{X}_t^{(m)}(\omega) = X_t(\omega)$ $a.s.\ P$. Let $h = \frac{1}{m}$. Then,

$$\lim_{m\to\infty} \tilde{X}_t^{(m)}(\omega) = \lim_{m\to\infty} \frac{F_t(\omega) - F_{(t-\frac{1}{m})\vee 0}(\omega)}{\frac{1}{m}} = \lim_{h\to 0} \frac{F_t(\omega) - F_{(t-h)\vee 0}(\omega)}{h} = X_t(\omega)$$

by the fundamental theorem of calculus for Lebesgue integrals.

The bounded convergence theorem now gives $\lim_{m\to\infty} E\int_0^T |\tilde{X}_t^{(m)} - X_t|^2 dt = 0$ and so a sequence $\{\tilde{X}^{(m,n_m)}\}_{m=1}^{\infty}$ of bounded, simple processes can be chosen, for which

$$\lim_{m\to\infty} E\int_0^T |\tilde{X}_t^{(m,n_m)} - X_t|^2 dt = 0.$$

(c) Finally, let X be a measurable and adapted. We cannot guarantee immediately that the continuous process $F = \{F_t;\ 0 \leq t < \infty\}$ defined in part (b) is progressively measurable, because we do not know whether it is adapted. We do know, however, that the process X has a progressively measurable modification Y (Proposition 1.1.12) and now we show that the progressively measurable process $\{G_t \triangleq \int_0^{t\wedge T} Y_s ds, \mathcal{F}_t; 0 \leq t \leq T\}$ is a modification of F.
For the measurable process $\eta_t(\omega) = 1_{\{X_t(\omega)\neq Y_t(\omega)\}};\ 0 \leq t \leq T,\ \omega \in \Omega$, we have from Fubini: $E\int_0^T \eta_t(\omega)dt = \int_0^T P[X_t(\omega) \neq Y_t(\omega)] = 0$. Therefore, $\int_0^T \eta_t(\omega)dt = 0$ for $P-a.e.,\ \omega \in \Omega$. Now $\{F_t \neq G_t\}$ is contained in the event $\{\omega;\ \int_0^T \eta_t(\omega)dt) > 0\}$, G_t is $\mathcal{F}_t$-measurable, and, by the assumption, $\mathcal{F}_t$ contains all subsets of P-null events. Therefore, F_t is also $\mathcal{F}_t$ measurable, hence adapted to $\mathcal{F}_t$. Adaptability and continuity imply progressive measurability, and we can repeat the same argument as in part (b).

2.5 Problem Let X be a bounded, measurable, $\{\mathcal{F}_t\}$- adapted process. Let $0 < T < \infty$ be fixed. We wish to construct a sequence $\{X^{(k)}\}_{k=1}^{\infty}$

2.6 Proposition If the function $t \mapsto \langle M\rangle_t(\omega)$ is absolutely continuous w.r.t. Lebesgue measure for P-a.e. $\omega \in \Omega$, then $\mathcal{L}_0$ is dense in $\mathcal{L}$ w.r.t. the metric [X].

Proof) If $X \in \mathcal{L}$ is bounded, then Lemma 2.4 guarantees the existence of a bounded sequence $\{X^{(m)}\}$ of simple processes satisfying $\sup_{T>0} \lim_{m\to\infty} E \int_0^T |\tilde{X}_t^{(m)} - X_t|^2 dt = 0$. From these, we extract a subsequence $\{X^{(m_k)}\}$, s.t. the set

$$\{(t,\omega) \in [0,\infty) \times \Omega;\ \lim_{k\to\infty} X_t^{(m_k)}(\omega) = X_t(\omega)\}^c$$

has a product measure zero. The absolute continuity of $t \mapsto \langle M\rangle_t(\omega)$ and the bounded convergence theorem imply

$$[X^{(m_k)} - X] = \sum_{n=1}^{\infty} \frac{1}{2^n}(1 \wedge [X^{(m_k)} - X]_n) \to 0 \text{ as } k \to \infty,$$

where $[X]_n^2 = E\int_0^n X_t^2 d\langle M\rangle_t$.

3.3 The Change-of-Variable Formula

One of the most important tools in the study of stochastic processes of the martingale type is the change-of-variables formula, or Ito's rule. It provides an integral-differential calculus for the sample paths of such processes. Let us consider a basic probability space $(\Omega, \mathcal{F}, P)$ with an associated filtration $\{\mathcal{F}_t\}$ which we always assume to satisfy the usual conditions.

3.1. Definition A **continuous semimartingale** $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is an adapted process which has the decomposition, P *a.s.*,

$$X_t = X_0 + M_t + B_t; \quad 0 \le t < \infty,$$

where $M = \{M_t, \mathcal{F}_t; o \le t < \infty\} \in \mathcal{M}^{c,loc}$ and $B = \{B_t, \mathcal{F}_t; 0 \le t < \infty\}$ is the difference of continuous, nondecreasing, adapted processes $A_t^{\pm}, \mathcal{F}_t; 0 \le t < \infty$

$$B_t = A_t^+ - A_t^-; \quad 0 \le t < \infty,$$

with $A_0^{\pm} = 0$, P *a.s.* We shall always assume that this is the minimal decomposition of B; in other words, A_t^+ is the positive variation of B on $[0,t]$ and A_t^- is the negative variation. The total variation of B on $[0,t]$ is then $B_t \triangleq A_t^+ + A_t^-$.

3.2. Problem Let $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous semimartingale with decomposition $X_t = X_0 + M_t + B_t; \quad 0 \le t < \infty$. Suppose that X has another decomposition

$$X_t = X_0 + \tilde{M}_t + \tilde{B}_t; \quad 0 \le t < \infty,$$

where $\tilde{M} \in \mathcal{M}^{c,loc}$ and $\tilde{B}$ is a continuous, adapted process which has finite total variation on each of the bounded interval $[0,t]$. Prove that $P - a.s.$,

$$M_t = \tilde{M}_t, \quad B_t = \tilde{B}_t, \quad 0 \le t < \infty.$$

Proof) Suppose that such a decomposition for X_t exists for all $0 \le t < \infty$. Then, we have

$$X_0 + M_t + B_t = X_0 + \tilde{M}_t + \tilde{B}_t \Leftrightarrow M_t - \tilde{M}_t = \tilde{B}_t - B_t.$$

The LHS of the above equality is a difference of continuous local martingales in P, thus it is a continuous local martingale. The RHS of the equation is a difference of functions of bounded variation, thus it is of bounded variation. Thus, $M_t - \tilde{M}_t \in \mathcal{M}^{c,loc} \cap BV \Rightarrow M_t - \tilde{M}_t = 0,\ P - a.s.$
$\therefore\ M_t = \tilde{M}_t, \quad B_t = \tilde{B}_t, \quad 0 \le t < \infty,\ P - a.s.$

A. The Ito Rule

Ito's formula states that a "smooth function" of a continuous semimartingale is a continuous semimartingale, and it provides its decomposition.

3.3 Theorem (Ito (1944), Kunita & Watanabe (1967)) Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 and let $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous semimartingale with decomposition $X_t = X_0 + M_t + B_t; \quad 0 \le t < \infty$. Then, $P - a.s.$,

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)dM_s + \int_0^t f^{'}(X_s)dB_s + \frac{1}{2}\int_0^t f^{''}(X_s)d\langle M\rangle_s,\ 0 \le t < \infty.$$

3.4 Remark For fixed ω and $t > 0$, the function $X_s(\omega)$ is bounded for $0 \le s \le t$, so $f^{'}(X_s(\omega))$ is bounded on this interval. It follows that $\int_0^t f^{'}(X_s)dM_s$ is defined, and its stochastic integral is a continuous, local martingale. The other two terms are Lebesgue-Stieltjes integrals, so as functions of the upper limit of integration, are of bounded variation.
$\therefore \{f(X_t), \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous semimartingale.

3.5 Remark The Ito's formula is often written in differential notation:

$$df(X_t) = f^{'}(X_t)dM_t + f^{'}(X_t)dB_t + \frac{1}{2}f^{''}(X_t)d\langle M\rangle_t =$$

$$= f^{'}dX_t + \frac{1}{2}f^{''}(X_t)d\langle M\rangle_t, \quad 0 \le t < \infty.$$

This is the "chain rule" for stochastic calculus.

Proof of Theorem 3.3) The proof will be accomplished in several steps.

Step 1) Localization.

3.4 Representations of Continuous Martingales in Terms of Brownian Motion

In this section, we motivate the idea that Brownian motion is the fundamental continuous martingale by showing how to represent other continuous martingales in terms of it.

4.1 Remark Our first representation theorem involves the notion of the extension of a probability space. Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be an adapted process on some $(\Omega, \mathcal{F}_t, P)$. We may need a d-dimensional Brownian motion independent of X, but because $(\Omega, \mathcal{F}_t, P)$ may not be rich enough to support this Brownian motion, we must extend the probability space to construct this. Let $(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{P})$ be another probability space, on which we consider a d-dimensional Brownian motion $\hat{B} = (\hat{B}_t, \hat{\mathcal{F}}_t; 0 \leq t < \infty)$, set $\tilde{\Omega} \triangleq \Omega \times \hat{\Omega}$, $\mathcal{G} \triangleq \mathcal{F} \bigotimes \hat{\mathcal{F}}$, $\tilde{P} \triangleq P \times \hat{P}$, and define a new filtration by $\mathcal{G}_t \triangleq \mathcal{F}_t \bigotimes \hat{\mathcal{F}}_t$. The latter may not satisfy the usual conditions, so we augment it and make it right-continuous by defining

$$\tilde{\mathcal{F}}_t \triangleq \bigcap_{s>t} \sigma(\mathcal{G}_s \cup \mathcal{N}),$$

where $\mathcal{N}$ is the collection of $\tilde{P}$-null sets in $\tilde{\mathcal{G}}$. We also complete $\tilde{\mathcal{G}}$, by defining $\tilde{\mathcal{F}} = \sigma(\tilde{\mathcal{G}} \cup \mathcal{N})$. We may extend X and B to $\{\tilde{\mathcal{F}}_t\}$-adapted processes on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by defining for $(\omega, \hat{\omega}) \in \tilde{\Omega}$,

$$\tilde{X}_t(\omega, \hat{\omega}) = X_t(\omega), \quad \tilde{B}_t(\omega, \hat{\omega}) = B_t(\hat{\omega}).$$

Then, $\tilde{B} = \{\tilde{B}_t, \tilde{\mathcal{F}}_t; 0 \leq t < \infty\}$ is a d-dimensional Brownian motion, independent of $\tilde{X} = \{\tilde{X}_t, \tilde{\mathcal{F}}_t; 0 \leq t < \infty\}$. Indeed, $\tilde{B}$ is independent of the extension to $\tilde{\Omega}$ of any $\mathcal{F}$-measurable r.v. on Ω. To simplify notation, we write X and B instead of $\tilde{X}$ and $\tilde{B}$ in the context of extensions.

A. Continuous Local Martingales as Stochastic Integrals with Respect to Brownian Motion

If $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion and X is a measurable, adapted process with $P[\int_0^t X_s^2 ds < \infty] = 1$ for every $0 \leq t < \infty$, then the stochastic integral $I_t(X) = \int_0^t X_s dW_s$ is a continuous local martingale with quadratic variation process $\langle I(X) \rangle_t = \int_0^t X_s^2 ds$, which is an absolutely continuous function of t, P a.s. Our first representation result provides the converse of this statement.

4.2 Theorem Suppose $M = \{M_t = (M_t^{(1)}, ..., M_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ is defined on $(\Omega, \mathcal{F}, P)$ with $M^{(i)} \in \mathcal{M}^{c,loc}$, $1 \leq i \leq d$. Suppose also that for $1 \leq i, j \leq d$, the cross-variation $\langle M^{(i)}, M^{(j)} \rangle_t(\omega)$ is an absolutely continuous function of t for P-almost every ω. Then there is an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of $(\Omega, \mathcal{F}, P)$ on which is defined a d-dimensional Brownian motion $W = \{W_t = (W_t^{(1)}, ..., W_t^{(d)}), \tilde{\mathcal{F}}_t; 0 \leq$

$t < \infty\}$, and a matrix $X = \{(X_t^{(i,k)})_{i,k=1}^d, \tilde{\mathcal{F}}_t; 0 \le t < \infty\}$ of measurable, adapted processes with

$$(4.1) \quad \tilde{P}\left[\int_0^t (X_s^{(i,k)})^2 ds < \infty\right] = 1; \quad 1 \le i,k \le d; \quad 0 \le t < \infty,$$

s.t. we have, $\tilde{P}$-a.s., the representations

$$(4.2) \quad M_t^{(i)} = \sum_{k=1}^d \int_0^t X_s^{(i,k)} dW_s^{(k)}; \quad 1 \le i \le d, \quad 0 \le t < \infty,$$

$$(4.3) \quad \langle M^{(i)}, M^{(j)}\rangle_t = \sum_{k=1}^d \int_0^t X_s^{(i,k)} ds; \quad 1 \le i,j \le d, \quad 0 \le t < \infty.$$

Proof) We prove this theorem by a random, time-dependent rotation of coordinates which reduces it to d separate, one-dimensional cases. Let

$$z_t^{i,j} = z_t^{j,i} = \frac{d}{dt}\langle M^{(i)}, M^{(j)}\rangle_t =$$
$$= \lim_{n\to\infty} n[\langle M^{(i)}, M^{(j)}\rangle_t - \langle M^{(i)}, M^{(j)}\rangle_{(t-\frac{1}{n})^+}],$$

so that the matrix-valued process $Z = \{Z_t = (z_t^{i,j})_{i,j=1}^d, \mathcal{F}_t; 0 \le t < \infty\}$ is symmetric and progressively measurable. For $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$, we have

$$\sum_{i=1}^d \sum_{j=1}^d \alpha_i z_t^{i,j} \alpha_j = \frac{d}{dt}(\langle \sum_{i=1}^d \alpha_i M^{(i)}, \sum_{j=1}^d \alpha_j M^{(j)}\rangle_t) = \frac{d}{dt}\langle \sum_{i=1}^d \alpha_i M^{(i)}\rangle_t \ge 0,$$

since $\langle ., .\rangle_t$ is a bilinear form, thus Z_t is positive-semidefinite for Lebesgue-almost every t, P-a.s. Any symmetric, positive-semidefinite matrix Z can be diagonalized by an orthogonal matrix Q, i.e., $Q^{-1} = Q^T$, so that $Q^{-1}ZQ = \Lambda$ and Λ is diagonal with the (nonnegative) eigenvalues of Z as its diagonal elements. To obtain Q and Λ as Borel-measurable functions of Z, we start with a progressively measurable, symmetric, positive-semidefinite matrix process Z, and so there exist progressively measurable, matrix-valued processes $\{Q_t(\omega) = (q_t^{i,j})_{i,j=1}^d; \mathcal{F}_t; 0 \le t < \infty\}$ and $\{\Lambda_t(\omega) = (\delta_{ij}\lambda_t^i(\omega))_{i,j=1}^d, \mathcal{F}_t; 0 \le t < \infty\}$ such that for Lebesgue-almost every t, we have

$$\sum_{k=1}^d q_t^{k,i} q_t^{k,j} = \sum_{k=1}^d q_t^{i,k} q_t^{j,k} = \delta_{i,j}; \quad 1 \le i,j \le d,$$

$$\sum_{k=1}^d \sum_{l=1}^d q_t^{k,i} z_t^{k,l} q_t^{l,j} = \delta_{ij}\lambda_t^i; \quad 1 \le i,j \le d,$$

a.s. P. From the first equation, when $i = j$ we have $(q_t^{k,i})^2 \le 1$, so

$$\int_0^t (q_t^{k,i})^2 d\langle M^{(k)}\rangle_s \le \langle M^{(k)}\rangle_t < \infty,$$

and we can define continuous local martingales by the prescription

$$N_t^{(i)} \triangleq \sum_{k=1}^{d} \int_0^t q_s^{k,i} dM_s^{(k)}; \quad 1 \le i \le d, \quad 0 \le t < \infty.$$

Then, we have by Proposition 2.17, a.s. P,

$$\langle N^{(i)}, N^{(j)} \rangle_t = \sum_{k=1}^{d} \sum_{l=1}^{d} \int_0^t q_s^{k,i} q_s^{l,j} d\langle M^{(k)}, M^{(l)} \rangle_s =$$

$$= \sum_{k=1}^{d} \sum_{l=1}^{d} \int_0^t q_s^{k,i} z_s^{k,l} q_s^{l,j} ds = \delta_{ij} \int_0^t \lambda_s^i ds.$$

In particular,

$$\int_0^t \lambda_s^i ds = \langle N^{(i)} \rangle_t < \infty.$$

We now represent the vector of local martingales $N = \{N_t^{(1)}, ..., N_t^{(d)}, \mathcal{F}_t; 0 \le t < \infty\}$ as a vector of stochastic integrals on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, which supports a d-dimensional Brownian motion $B = \{B_t = (B_t^{(1)}, ..., B_t^{(d)}), \tilde{\mathcal{F}}_t; 0 \le t < \infty\}$ independent of N. Since

$$\int_0^t 1_{\{\lambda_s^i > 0\}} \frac{1}{\lambda_s^i} d\langle N^{(i)} \rangle_s = \int_0^t 1_{\{\lambda_s^i > 0\}} ds \le t,$$

we can define continuous, local martingales

$$W_t^{(i)} \triangleq \int_0^t 1_{\{\lambda_s^i > 0\}} \frac{1}{\sqrt{\lambda_s^i}} dN_s^{(i)} + \int_0^t 1_{\{\lambda_s^i = 0\}} dB_s^{(i)}; \quad 1 \le i \le d.$$

From Problem 1.5.26, we have

$$\langle W^{(i)}, W^{(j)} \rangle_t = \delta_{ij} t, \quad 1 \le i, j \le d \quad 1 \le t < \infty,$$

so according to P.Levy's theorem (Theorem 3.16), $W = \{W_t = (W_t^{(1)}, ..., W_t^{(d)}), \tilde{\mathcal{F}}_t; 0 \le t < \infty\}$ is a d-dimensional Brownian motion. Moreover,

$$\int_0^t \sqrt{\lambda_s^i} dW_s^{(i)} = \int_0^t 1_{\{\lambda_s^i > 0\}} dN_s^{(i)} = N_t^{(i)}, \quad 1 \le i, j \le d \quad 0 \le t < \infty,$$

because the martingale $\int_0^t 1_{\{\lambda_s^i = 0\}} dN^{(i)}$, having quadratic variation

$$\int_0^t 1_{\{\lambda_s^i = 0\}} d\langle N^{(i)} \rangle_s = \int_0^t 1_{\{\lambda_s^i = 0\}} \lambda_s^i ds = 0,$$

is itself identically zero.

Having thus obtained the stochastic integral representation for N in terms of d-dimensional Brownian motion W, we invert the rotation of coordinates $N_t^{(i)}$ to obtain a representation for M. We first observe that for $1 \le i, k \le d$,

$$\int_0^t (q_s^{i,k})^2 \lambda_s^k ds \le \int_0^t \lambda_s^k ds = \langle N^{(k)} \rangle_s < \infty; \quad 0 \le t < \infty$$

so with $X_t^{(i,k)} \triangleq q_t^{i,k} \sqrt{\lambda_t^k}$, condition (4.1) holds. Furthermore,

$$\sum_{k=1}^d \int_0^t X_s^{(i,k)} dW_s^{(k)} = \sum_{k=1}^d \int_0^t q_s^{i,k} dN_s^{(k)} = \sum_{j=1}^d \sum_{k=1}^d \int_0^t q_s^{i,k} q_s^{j,k} dM_s^{(j)} = \tag{4.12}$$

$$= \sum_{j=1}^d \delta_{ij} \int_0^t dM_s^{(j)} = M_t^{(i)},$$

which establishes (4.2). Equation (4.3) is an immediate consequence of (4.2). □

4.3 Remark If for P-a.e. $\omega \in \Omega$, the matrix-valued process $Z_t(\omega) = (z_t^{i,j}(\omega))_{i,j=1}^d$ has a constant rank r, $1 \le r \le d$, for Lebesgue-almost every t, then the Brownian motion W used in the representation (4.2) can be chosen to be r-dimensional, and there is no need to introduce the extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{P})$. Indeed, we may take $\lambda_t^1, ..., \lambda_t^r$ to be the r strictly positive eigenvalues of Z_t, and replace the definition of $W_t^{(i)}$ in the proof of Theorem 4.2 by

$$W_t^{(i)} = \int_0^t \frac{1}{\sqrt{\lambda_s^i}} dN_s^{(i)}; \quad 1 \le i \le r.$$

Since $N_t^{(i)} = 0;\ r+1 \le i \le d,\ 0 \le t < \infty$, (4.12) becomes

$$\sum_{k=1}^r \int_0^t X_s^{(i,k)} dW_s^{(k)} = \sum_{k=1}^r \int_0^t q_s^{i,k} dN_s^{(k)} = M_t^{(i)}, \quad 1 \le i \le d.$$

Because this definition for $W_t^{(i)}$ defines $W^{(1)}, ..., W^{(r)}$ without reference to Brownian motion B, there is no need to extend the original probability space.

B. Continuous Local Martingales as Time-Changed Brownian Motions

Our next representation result requires us to consider the inverse of the quadratic variation of a continuous local martingale; because such a quadratic variation may not be strictly increasing, we begin with a problem describing this situation in some detail.

4.6 Theorem (Time-Change for Martingales [Dambis (1965), Dubins & Schwartz (1965)]) Let $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\} \in \mathcal{M}^{c,loc}$ satisfy $\lim_{t\to\infty}\langle M\rangle_t = \infty$ a.s. P. Define, for each $0 \le s < \infty$, the stopping time

$$T(s) = \inf\{t \ge 0; \langle M\rangle_t > s\}.$$

Then the time-changed process

$$B_s \triangleq M_{T(s)}, \quad \mathcal{G}_s \triangleq \mathcal{F}_{T(s)}; \quad 0 \le s < \infty$$

is a standard one-dimensional Brownian motion. In particular, the filtration $\{\mathcal{G}_s\}$ satisfies the usual conditions and we have, a.s. P:

$$M_t = B_{\langle M\rangle_t} \quad 0 \le t < \infty.$$

Proof)

3.5 The Girsanov Theorem

A. The Basic Result

Throughout this section, we have a probability space $(\Omega, \mathcal{F}_t, P)$ and a d-dimensional Brownian motion $W = \{W_t = (W_t^{(1)}, W_t^{(2)}, ..., W_t^{(d)}, \mathcal{F}_t); 0 \le t < \infty\}$ defined on it, with $P[W_0 = \mathbf{0}] = 1$. We assume that the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. Let $X = \{(X_t^{(1)}, X_t^{(2)}, ..., X_t^{(d)}), \mathcal{F}_t; 0 \le t < \infty\}$ be a vector of measurable, adapted processes satisfying

$$P\left[\int_0^T (X_t^{(i)})^2 dt < \infty\right] = 1; \quad 1 \le i \le d, \quad 0 \le T < \infty.$$

Then, for each i, the stochastic integral $I^{W^{(i)}}(X^{(i)})$ is defined, and is a member of $\mathcal{M}^{c,loc}$. We set

$$Z_t(X) \triangleq exp\left[\sum_{i=1}^d \int_0^t X_s^{(i)} - \frac{1}{2}\int_0^t \|X_s\|^2 ds\right].$$

Then, we have

$$Z_t(X) = 1 + \sum_{i=1}^d \int_0^t Z_s(X) X_s^{(i)} dW_s^{(i)},$$

which shows that $Z(X)$ is continuous local martingale, with $Z_0(X) = 1$.
If $Z(X)$ is a martingale, then by the martingale property, $E[Z_t(X)] = 1$.
Thus, we can define for each $0 \le T < \infty$, a probability measure $\tilde{P}_T$ on $\mathcal{F}_T$ by

$$\tilde{P}_T(A) \triangleq \int_A Z_T(\omega) dP(\omega) = E[1_A Z_T(\omega)]; \quad A \in \mathcal{F}_T.$$

The family of probability measures $\{\tilde{P}_T; \quad 0 \le t < \infty\}$ satisfies the consistency condition

$$\tilde{P}_T = \tilde{P}_t; \quad A \in \mathcal{F}_t, \quad 0 \le t < T$$

which intuitively means that these probability measures are extensions of each other to a larger σ-algebra, while assigning the same probability to commonly known information (measurable sets).

$\because$ Let $A \in \mathcal{F}_t$. Then, $\tilde{P}(A) = E[1_A Z_T(X)] = E[1_A E[Z_T(X)|\mathcal{F}_t]] = E[1_A Z_t(X] = \tilde{P}_t(A)$ $\square$.

5.1 Theorem (Girsanov (1960), Cameron and Martin (1944)).
Assume that $Z(X)$ defined above is a martingale. Define a process $\tilde{W} = \{\tilde{W}_t = (\tilde{W}_t^{(1)}, ..., \tilde{W}_t^{(d)}), \mathcal{F}_t; 0 \le t < \infty\}$ by

$$\tilde{W}_t^{(i)} \triangleq W_t^{(i)} - \int_0^t X_s^{(i)} ds; \quad 1 \le i \le d, \quad 0 \le t < \infty.$$

For each fixed $T \in [0, \infty)$, the process $\{\tilde{W}_t, \mathcal{F}_t; 0 \le t \le T\}$ is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \tilde{P}_T)$.

B. Proof and Ramifications

We denote by $\tilde{E}_T(\tilde{E})$ the expectation operator w.r.t. $\tilde{P}_T(\tilde{P})$.

5.3 Lemma Fix $0 \le T < \infty$ and assume that $Z(X)$ is a martingale. If $0 \le s \le t \le T$ and Y is an $\mathcal{F}_T$-measurable r.v. satisfying $\tilde{E}_T|Y| < \infty$, then we have the Bayes' rule:

$$\tilde{E}_T[Y|\mathcal{F}_s] = \frac{1}{Z_s(X)} E[Y Z_t(X)|\mathcal{F}_s], \quad a.s.\ P\ and\ \tilde{P}_T.$$

Proof) Fix $A \in \mathcal{F}_s$. Then,

$$\tilde{E}_T\left[1_A \frac{1}{Z_s(X)} E[Y Z_t(X)|\mathcal{F}_s]\right] = E\left[Z_t(X) 1_A \frac{1}{Z_s(X)} E[Y Z_t(X)|\mathcal{F}_s]\right] =$$

$$= E\left[E\{Z_t(X) 1_A \frac{1}{Z_s(X)} E[Y Z_t(X)|\mathcal{F}_s]|\mathcal{F}_s\}\right] = E[1_A E[Y Z_t(X)|\mathcal{F}_s]] =$$

$$= E[1_A Y Z_t(X)] = \tilde{E}[1_A Y]. \quad \square$$

We denote by $\mathcal{M}_T^{c,loc}$ the class of continuous local martingales $M = \{M_t, \mathcal{F}_t; 0 \le t \le T\}$ on $\Omega, \mathcal{F}_T, P$ satisfying $P[M_0 = 0] = 1$, and define $\tilde{\mathcal{M}}_T^{c,loc}$ similarly, with P replaced by $\tilde{P}_T$.

5.4 Proposition Fix $0 \le T < \infty$ and assume that $Z(X)$ is a martingale. If $M \in \mathcal{M}_T^{c,loc}$, then the process

$$\tilde{M}_t \triangleq M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle, \ \mathcal{F}_t; \ 0 \le t \le T$$

is in $\tilde{\mathcal{M}}_T^{c,loc}$. If $N \in \mathcal{M}_T^{c,loc}$ and

$$\tilde{N}_t \triangleq N_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle N, W^{(i)}\rangle, \ \mathcal{F}_t; \ 0 \le t \le T,$$

then

$$\langle \tilde{M}, \tilde{N}\rangle_t = \langle M, N\rangle_t; \ 0 \le t \le T, \ a.s. \ P \text{ and } \tilde{P}_T,$$

where the cross-variations are computed under the appropriate measures.

Proof) We only consider the case where M and N are bounded martingales with bounded quadratic variations, and assume also that $Z_t(X)$ and $\sum_{j=1}^d \int_0^t (X_s^{(j)})^2 ds$ are bounded in t and ω; the general case can be reduced to this one by localization. By Proposition 2.14,

$$|\int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s|^2 \le \langle M\rangle_t \int_0^t (X_s^{(i)})^2 ds,$$

and thus $\tilde{M}$ is also bounded.
From Problem 3.12, we have the integration by parts formula: If $X_t = X_0 + M_t + B_t$ and $Y_t = Y_0 + N_t + C_t$ are two semimartingales with $B_0 = C_0 = 0$ a.s., then

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle M, N\rangle_t.$$

Using the integration by parts formula, we have

$$Z_t(X)\tilde{M} = \int_0^t Z_u(X) dM_u + \sum_{i=1}^{d} \int_0^t \tilde{M}_u X_u^{(i)} Z_u(X) dW_u^{(i)},$$

which is a martingale under P.

$\because$ Consider the 1 dimensional case, where $i = 1$. Then, we have

$$Z_t(X)\tilde{M} = \int_0^t Z_s(X) d\tilde{M}_s + \int_0^t \tilde{M}_s dZ_s(X) + \int_0^t d\langle M, Z(X)\rangle_s =$$

$$= \int_0^t Z_s(X) dM_s - \int_0^t Z_s(X) X_s d\langle M, W\rangle_s + \int_0^t \tilde{M}_s Z_s(X) X_s dW_s +$$

$$+ \int_0^t Z_s(X) X_s d\langle M, W\rangle_s = \int_0^t Z_s(X) dM_s + \int_0^t \tilde{M}_s Z_s(X) X_s dW_s.$$

Therefore, for $0 \le s \le t \le T$, we have from Lemma 5.3:

$$\tilde{E}_T[\tilde{M}_t|\mathcal{F}_s] = \frac{1}{Z_s(X)} E[Z_t(X)\tilde{M}_t|\mathcal{F}_s] = \tilde{M}_s, \text{ a.s. } P \text{ and } \tilde{P}_T.$$

Therefore, $\tilde{M} \in \tilde{\mathcal{M}}^{c,loc}$. The integration by parts formula also implies:

$$\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t = \int_0^t \tilde{M}_u dN_u + \int_0^t \tilde{N}_u dM_u -$$

$$- \sum_{i=1}^d \left[\int_0^t \tilde{M}_u X_u^{(i)} d\langle N, W^{(i)}\rangle_u + \int_0^t \tilde{N}_u X_u^{(i)} d\langle M, W^{(i)}\rangle_u \right]$$

as well as

$$Z_t(X)[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t] = \int_0^t Z_u(X)\tilde{M}_u dN_u + \int_0^t Z_u(X)\tilde{N}_u dM_u +$$

$$+ \sum_{i=1}^d \int_0^t [\tilde{M}_u\tilde{N}_u - \langle M, N\rangle_u] X_u^{(i)} Z_u(X) dW_u^{(i)}.$$

This last process is consequently a martingale under P, and so Lemma 5.3 implies that for $0 \le s \le t \le T$

$$\tilde{E}_T[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t|\mathcal{F}_s] = \tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t; \text{ a.s. } P \text{ and } \tilde{P}_T.$$

This proves that $\langle \tilde{M}, \tilde{N}\rangle_t = \langle M, N\rangle_t;\ 0 \le t \le T$, a.s. $\tilde{P}_T$ and P. $\Box$

Proof of Theorem 5.1) We show that the continuous process $\tilde{W}$ on $(\Omega, \mathcal{F}_t \tilde{P}_T)$ satisfies the hypothesis of P.Levy's Theorem 3.16. Setting $M = W^{(j)}$ in Prop 5.4, we have $\tilde{M} = \tilde{W}_t^{(j)}$, thus $\tilde{W}^{(j)} \in \tilde{\mathcal{M}}_T^{c,loc}$. Setting $N = W^{(k)}$, we obtain

$$\langle \tilde{W}^{(j)}, \tilde{W}_t^{(k)}\rangle = \langle W^{(j)}, W^{(k)}\rangle_t = \delta_{j,k} t; \quad 0 \le t \le T \text{ a.s. } \tilde{P}_T \text{ and } P. \ \Box$$

Let $\{M_t, \mathcal{F}_t; 0 \le t \le T\}$ be a continuous local martingale under P. With the assumption of Theorem 5.1, Prop 5.4 shows that M is a continuous semimartingale under $\tilde{P}_T$.
The converse is also true, if $\{\tilde{M}_t, \mathcal{F}_t; 0 \le t \le T\}$ is a continuous martingale under $\tilde{P}_T$, then Lemma 5.3 implies that for $0 \le s \le t \le T$:

$$E[Z_t(X)\tilde{M}_t|\mathcal{F}_s] = Z_s(X)\tilde{E}_T[\tilde{M}_t|\mathcal{F}_s] = Z_s(X)\tilde{M}_s \text{ a.s. } P \text{ and } \tilde{P}_T.$$

so $Z(X)\tilde{M}$ is a martingale under P.
If $\tilde{M} \in \mathcal{M}_T^{c,loc}$, a localization argument shows that $Z(X)\tilde{M} \in \mathcal{M}_T^{c,loc}$.
But $Z(X) \in \mathcal{M}^c$, so Ito's rule implies that $\tilde{M} = \frac{Z(X)\tilde{M}}{Z(X)}$ is a continuous semimartingale under P (c.f. Remark 3.4).
Thus, given $\tilde{M} \in \mathcal{M}_T^{c,loc}$, we have a decomposition

$$\tilde{M}_t = M_t + B_t; \quad 0 \le t \le T,$$

where $M \in \mathcal{M}_T^{c,loc}$ and B is of bounded variation with $B_0 = 0$, $P-$ a.s.
According to Prop 5.4, the process

$$\tilde{M}_t - (M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s) = B_t + \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s; \quad 0 \le t \le T,$$

is in $\tilde{\mathcal{M}}_T^{c,loc}$, and of being bounded variation, this process must be indistinguishable from the identity zero process (Problem 3.2). $\square$

5.5 Proposition Under the hypotheses of Theorem 5.1, every $\tilde{M} \in \mathcal{M}_T^{c,loc}$ has the representation $\tilde{M}_t = M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle$ for some $M \in \mathcal{M}_T^{c,loc}$.

C. Brownian Motion with Drift

Below, we discuss an interesting application of the Girsanov theorem: the distribution of passage times for Brownian motion with drift.
Consider a Brownian motion $W = \{W_t, \mathcal{F}_t; 0 \le t < \infty\}$ and the passage time $T_b = \inf\{t \ge 0;\ W_t = b\}$ to the level $b \ne 0$ has density and moment generating function (Remark 2.8.3), respectively:

$$P[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} exp\left[-\frac{b^2}{2t}\right] dt; \quad t > 0$$

$$Ee^{-\alpha T_b} = e^{-|b|\sqrt{2\alpha}}; \quad \alpha > 0.$$

For any real number $\mu \ne 0$, the process $\tilde{W} = \{\tilde{W}_t = W_t - \mu t, \mathcal{F}_t^W, 0 \le t < \infty\}$, is a Brownian motion under the unique measure $P^{(\mu)}$ which satisfies

$$P^{(\mu)} = E[1_A Z_t]; \quad A \in \mathcal{F}_t^W,$$

where $Z_t \triangleq exp(\mu W_t - \frac{1}{2}\mu^2 t)$ by Corollary 5.2. We say that, under $P^{(\mu)}$, $W_t = \mu t + \tilde{W}_t$ is a Brownian motion with drift μ. On the set $\{T_b \le t\} \in \mathcal{F}_t^W \cap \mathcal{F}_{T_b}^W = \mathcal{F}_{t \cap T_b}^W$, we have $Z_{t\wedge T_b} = Z_{T_b}$, so the optional sampling theorem 1.3.22 and Problem 1.3.23(i) imply

$$P^{(\mu)}[T_b \le t] = E[1_{\{T_b \le t\}} Z_t] = E[1_{\{T_b \le t\}} E[Z_t | \mathcal{F}_{t\wedge T_b}^W]] = E[1_{\{T_b \le t\}} Z_{t\wedge T_b}] =$$

$$= E[1_{\{T_b \le t\}} Z_{T_b}] = E[1_{\{T_b \le t\}} e^{\mu b - \frac{1}{2}\mu^2 T_b}] = \int_0^t exp(\mu b - \frac{1}{2}\mu^2 s) P[T_b \in ds].$$

This relation has a several consequences. Firstly,

$$P^{(\mu)}[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} exp\left[-\frac{(b-\mu t)^2}{2t}\right] dt; \quad t > 0.$$

Second, letting $t \to \infty$, we have

$$P^{(\mu)}[T_b < \infty] = e^{\mu b} E[exp(-\frac{1}{2}\mu^2 T_b)],$$

so we obtain from the moment generating function that

$$P^{(\mu)}[T_b < \infty] = exp[\mu b - |\mu b|]].$$

In particular, a Brownian motion with drift $\mu \neq 0$ reaches level $b \neq 0$ with probability one if and only if μ and b have the same sign. If μ and b have opposite signs, the density $P^{(\mu)}[T_b \in dt]$ is defective, in the sense that $P^{(\mu)}[T_b < \infty] < 1$.s

D. The Novikov Condition

In order to use the Girsanov theorem effectively, we need some fairly general conditions under which the process $Z(X)$ is a martingale. This process is a local martingale because it satisfies the integral equation. Indeed, with

$$T_n \triangleq \inf\{t \geq 0;\ max_{1\leq i\leq d} \int_0^t (Z_s(X)X_s^{(i)})^2 ds = n\},$$

the stopped process $Z^{(n)} \triangleq \{Z_t^{(n)} \triangleq Z_{t\wedge T_n},\ \mathcal{F}_t; 0 \leq t < \infty\}$ are martingales. Consequently, we have

$$E[Z_{t\wedge T_n}|\mathcal{F}_s] = Z_{s\wedge T_n};\ 0 \leq s \leq t,\ n \geq 1,$$

and using Fatou's lemma as $n \to \infty$ we obtain $E[Z_t|\mathcal{F}_s] =$
$= E[\liminf_{n\to\infty} Z_{t\wedge T_n}|\mathcal{F}_s] \leq \liminf_{n\to\infty} E[Z_{t\wedge T_n}|\mathcal{F}_s] = Z_s;\ 0 \leq s \leq t$. In other words, $Z(X)$ is always a supermartingale and is a martingale if and only if

$$EZ_t(X) = 1; \quad 0 \leq t < \infty$$

(Problem 1.3.25).

5.12 Proposition Let $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be in $\mathcal{M}^{c,loc}$ and define

$$Z_t = \exp[M_t - \frac{1}{2}\langle M\rangle_t];\ 0 \leq t < \infty.$$

If

$$E[\exp\{\frac{1}{2}\langle M\rangle_t\}] < \infty;\ 0 \leq t < \infty,$$

then $EZ_t = 1;\ 0 \leq t < \infty$.

Proof) Let $T(s) = \inf\{t \geq 0;\ \langle M\rangle_t > s\}$, so the time-changed process $B_s \triangleq M_{T(s)}$ is a Brownian motion (Theorem 4.6, Problem 4.7). For $b < 0$, we define the stopping time for $\{\mathcal{G}_s\}$:

$$S_b = \inf\{s \geq 0;\ B_s - s = b\}.$$

Problem 5.7 yields the Wald identity $E[\exp(B_{S_b} - \frac{1}{2}S_b)] = 1$, whence
$E[\exp(\frac{1}{2}S_b)] = e^{-b}$. Consider the exponential martingale $\{Y_s \triangleq \exp(B_s -$

$\frac{1}{2}\langle B\rangle_s) = \exp(B_s - \frac{s}{2}), \mathcal{G}_s; 0 \le s < \infty\}$ and define $\{N_s \triangleq Y_{s\wedge S_b}, \mathcal{G}_s; 0 \le s < \infty\}$. According to Problem 1.3.24 (i), N is a martingale, and because $P[S_b < \infty] = 1$ we have

$$N_\infty = \lim_{s\to\infty} N_s = \exp(B_{S_b} - \frac{1}{2}S_b).$$

Therefore, from Fatou's lemma, $E[N_\infty|\mathcal{F}_s] = E[\liminf_{t\to\infty} N_t|\mathcal{F}_s] \le$ $\le \liminf_{t\to\infty} E[N_t|\mathcal{F}_s] = N_s$ thus $N = \{N_s, \mathcal{G}_s; 0 \le s \le \infty\}$ is a supermartingale with a last element. However, $EN_\infty = 1 = EN_0$ by the Wald's identity, so $N = \{N_s, \mathcal{G}_s; 0 \le s \le \infty\}$ has a constant expectation; thus N is actually a martingale with a last element (Problem 1.3.25). This allows us to use the optional sampling theorem 1.3.22 to conclude that for any stopping time R of the filtation $\{\mathcal{G}_s\}$:

$$E[\exp\{B_{R\wedge S_b} - \frac{1}{2}(R \wedge S_b)\}] = 1.$$

Now let us fix $t \in [0,\infty)$ and recall from the proof of Theorem 4.6 that $\langle M\rangle_t$ is a stopping time of $\{\mathcal{G}_s\}$. It follows that for $b < 0$:

$$E[\exp\{B_{\langle M\rangle_t\wedge S_b} - \frac{1}{2}(\langle M\rangle_t \wedge S_b)\}] =$$

$$= E[1_{\{S_b\le\langle M\rangle_t\}}]\exp(b + \frac{1}{2}S_b)] + E[1_{\{\langle M\rangle_t<S_b\}}\exp(M_t - \frac{1}{2}\langle M\rangle)_t] = 1.$$

The first expectation in this identity is bounded above by $e^b E[\exp(\frac{1}{2}\langle M\rangle_t)]$, which converges to zero as $b \downarrow -\infty$, thanks to the assumption $E[\exp\{\frac{1}{2}\langle M\rangle_t\}] < \infty$. As $b \downarrow -\infty$, the second expectation converges to EZ_t because of the monotone convergence theorem. Therefore, $EZ_t = 1;\ 0 \le t < \infty$. $\square$

5.13 Corollary (Novikov (1972). Let $W = \{(W_t^{(1)}, ..., W_t^{(n)}),\ \mathcal{F}_t;\ 0 \le t < \infty\}$ be a d-dimensional Brownian Motion, and let $X = (X_t^{(1)}, ..., X_t^{(d)}), \mathcal{F}_t;\ 0 \le t < \infty\}$ be a vector of measurable, adapted processes satisfying $P[\int_0^T (X_t^{(i)})^2 dt < \infty] = 1;\ 1 \le i \le d,\ 0 \le T < \infty$. If

$$E\left[\exp(\frac{1}{2}\int_0^T \|X_s\|^2 ds)\right] < \infty;\ 0 \le T < \infty,$$

then $Z(X)$ is a martingale.

Proof) Note that $\langle I^M(X^{(i)})\rangle_t = \int_0^t (X_u^{(i)})^2 d\langle M\rangle_u$ for $1 \le i \le d$, thus