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GEOMETRIC LEAST SQUARES FITTING OF CIRCLE AND ELLIPSE

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The least squares fitting of geometric features to given points minimizes the squares sum of error-of-fit in predefined measures. By the geometric fitting, the error distances are defined with the orthogonal, or shortest, distances from the given points to the geometric feature to be fitted. For the geometric fitting of circle and ellipse, robust algorithms are proposed which are based on the coordinate descriptions of the corresponding point on the circle/ellipse for the given point, where the connecting line of the two points is the shortest path from the given point to the circle/ellipse.

Keywords: Orthogonal distances fitting, circle fitting, ellipse fitting, orthogonal contacting point, singular value decomposition, nonlinear least squares, Gauss–Newton iteration.

1. INTRODUCTION

The fitting of geometric features to given points is desired in various fields of science and engineering. There is especially true of circle and ellipse, as these are practical geometric features for the application of image processing. In the past, for fitting problems the least squares method (LSM) has often been applied and has proven its usability in implementation and computing costs. In this paper we suggest robust algorithms for *least squares orthogonal distances fitting* of circle and ellipse to given points.

The LS fitting minimizes the squares sum of error-of-fit in predefined measures. There are two main categories of LS fitting problems for geometric features, algebraic and geometric, differentiated by the definition of the error distances.

By the *algebraic fitting* of a geometric feature, which is described by the implicit equation $F(\mathbf{x}, \mathbf{a}) = 0$ and the parameters vector $\mathbf{a} = (a_1, \dots, a_q)$, the error distances are defined with the deviations of the implicit equation from the expected value, i.e. zero, at each given point. The nonequality of the equation indicates there is some error-of-fit. Most publications for the LS fitting of circle[4,5] and ellipse[2,3,8,11,13,14] are concerned with the squares sum of algebraic distances or their modifications:

$$\sum_{i=1}^{m} e_i^2 = \sum_{i=1}^{m} [(x_i - x_c)^2 + (y_i - y_c)^2 - r^2]^2 \tag{1}$$

for the circle fitting, and

$$\sum_{i=1}^{m} e_i^2 = \sum_{i=1}^{m} (ax_i^2 + 2bx_iy_i + cy_i^2 + 2dx_i + 2ey_i + f)^2 \tag{2}$$

for the ellipse fitting.

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In spite of the advantages in implementation and computing costs, the algebraic fitting has drawbacks in accuracy, and in the physical interpretation of the fitting parameters and of the error-of-fit.[10,13] The known disadvantages of the algebraic fitting are[1,7,11,14]:

- The definition of error distances does not coincide with the measurement guideline.
- The parameters are dependent on the coordinate transformation, and biased.
- The fitting errors are unwillingly weighted.
- The fitting procedure sometimes ends in an unintended geometric feature.

By the *geometric fitting*, named also as the best fitting, the error distances are defined with the orthogonal, or shortest, distances from the given points to the geometric feature to be fitted. The geometric fitting is possibly the only solution to all the above problems of the algebraic fitting. The geometric fitting of circle and ellipse is a nonlinear problem and must be solved with iteration. For the geometric fitting of circle, there are some well established methods.[9,16] On the other hand, the geometric fitting of ellipse is a newly exploited field and is being continuously developed. Gander *et al.*[9] have proposed a geometric ellipse fitting algorithm in parametric form, which has a large number of fitting parameters. Consequently, it has an unnecessarily deteriorative performance of convergence. The paper of Cui *et al.*[6] describes a geometric ellipse fitting with a minimum variance estimator and parameter space decomposition technique.

In this paper, we propose alternative algorithms for the geometric fitting of circle and ellipse. Our algorithms are based on the coordinates description of the corresponding point on the circle/ellipse for the given point, where the connecting line of the two points is the shortest path from the given point to the circle/ellipse.

2. NONLINEAR LEAST SQUARES FITTING

Suppose that q parameters $\mathbf{a}$ are assumed to be related to $p(> q)$ measurements $\mathbf{X}$ according to:

$$\mathbf{X} = \mathbf{F}(\mathbf{a}) + \mathbf{e} \tag{3}$$

where $\mathbf{F}$ represents some nonlinear continuously differentiable observation functions of $\mathbf{a}$, and $\mathbf{e}$ denotes errors with zero mean whose influence is to be eliminated. The nonlinear least squares estimate of $\mathbf{a}$ given $\mathbf{X}$ minimizes the performance index

$$\sigma_0^2 = [\mathbf{X} - \mathbf{F}(\hat{\mathbf{a}})]^T[\mathbf{X} - \mathbf{F}(\hat{\mathbf{a}})]\,. \tag{4}$$

For convenience, the weighting matrix or noise covariance matrix is chosen as an identity matrix. The existence of the unbiased optimal solution, which is generally nonlinear in the estimate $\hat{\mathbf{a}}$, can be referenced from elsewhere.[15] There are various methods for the numerical solution of the above nonlinear estimation problem. We have chosen in this paper the Gauss–Newton iteration with initial parameters $\mathbf{a}_k$

and step-size parameter λ

$$\left.\frac{\partial \mathbf{F}}{\partial \mathbf{a}}\right|_{\mathbf{a}_k} \Delta \mathbf{a} = \mathbf{X} - \mathbf{F}(\mathbf{a}_k) \tag{5}$$

$$\mathbf{a}_{k+1} = \mathbf{a}_k + \lambda \Delta \mathbf{a} \tag{6}$$

with the terminating conditions[15]

$$\left\| \left.\left(\frac{\partial \mathbf{F}}{\partial \mathbf{a}}\right)^T \right|_{\mathbf{a}_k} [\mathbf{X} - \mathbf{F}(\mathbf{a}_k)] \right\| \approx 0 \quad \text{or} \quad \|\Delta \mathbf{a}\| \approx 0 \quad \text{or} \quad \sigma_0^2|_k - \sigma_0^2|_{k+1} \approx 0 \,. \tag{7}$$

The matrix of partial derivatives appearing in Eq. (5) is the Jacobian matrix $\mathbf{J}$

$$J_{ij} \equiv \frac{\partial F_i}{\partial a_j} \,. \tag{8}$$

For this type of numerical solution the user must supply the function values vector $\mathbf{F}$ and the Jacobian matrix $\mathbf{J}$ at the nearest corresponding points on the geometric feature from each given point. This is a necessary requirement for the least squares orthogonal distances fitting with the performance index of Eq. (4). Unfortunately, in the case of an ellipse fitting, it is not easy to locate the nearest point on an ellipse from a given point. We can compare the cases of circle and ellipse fitting in Secs. 3 and 4, respectively.

3. GEOMETRIC FITTING OF CIRCLE AND SPHERE

The circle/sphere in n-dimensional space ($n \geq 2$) with center at $\mathbf{X}_c$ and radius R can be described as below [Fig. 1(a)]

$$\|\mathbf{X} - \mathbf{X}_c\|^2 = R^2 \,. \tag{9}$$

For a given point $\mathbf{X}_i$, the nearest corresponding point $\mathbf{X}'_i$ on the circle/sphere is

$$\mathbf{X}'_i = \mathbf{X}_c + R \frac{\mathbf{X}_i - \mathbf{X}_c}{\|\mathbf{X}_i - \mathbf{X}_c\|} \tag{10}$$

with the orthogonal error distances vector

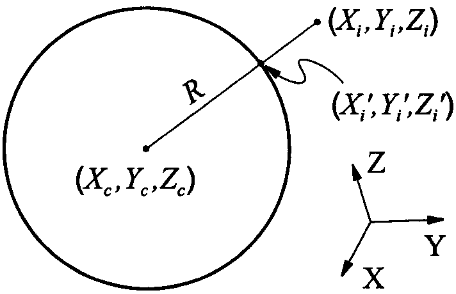
$$\mathbf{X}''_i = \mathbf{X}_i - \mathbf{X}'_i = [\|\mathbf{X}_i - \mathbf{X}_c\| - R] \frac{\mathbf{X}_i - \mathbf{X}_c}{\|\mathbf{X}_i - \mathbf{X}_c\|} \,. \tag{11}$$

From Eq. (10), the Jacobian matrix $\mathbf{J}$ for the Gauss–Newton iteration can be directly derived

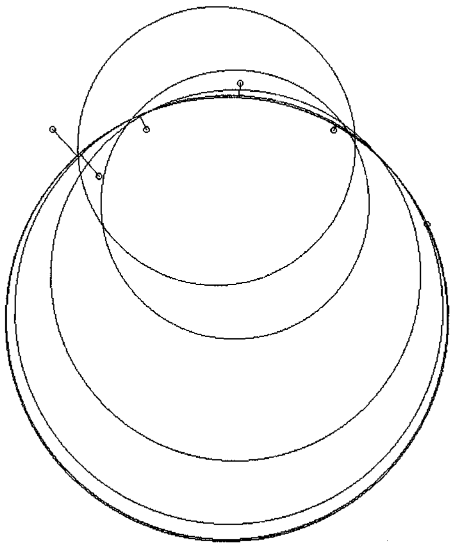
$$\mathbf{J}_{\mathbf{X}'_i,\mathbf{a}} = \frac{\partial \mathbf{X}_c}{\partial \mathbf{a}} + \frac{\mathbf{X}_i - \mathbf{X}_c}{\|\mathbf{X}_i - \mathbf{X}_c\|} \frac{\partial R}{\partial \mathbf{a}} - \frac{R}{\|\mathbf{X}_i - \mathbf{X}_c\|} \left[\mathbf{I} - \frac{(\mathbf{X}_i - \mathbf{X}_c)(\mathbf{X}_i - \mathbf{X}_c)^T}{\|\mathbf{X}_i - \mathbf{X}_c\|^2} \right] \frac{\partial \mathbf{X}_c}{\partial \mathbf{a}} \,. \tag{12}$$

We define the parameters vector $\mathbf{a}$ as

$$\mathbf{a}^T = (R, \mathbf{X}_c^T) \,. \tag{13}$$



(a)



(b)

Fig. 1. (a) Sphere in three-dimensional space with center $\mathbf{X}_c$ and radius R; (b) Geometric fitting of circle to the first six points in Table 1.

and, with the error distances vector $\mathbf{X}_i''$ of Eq. (11) and the Jacobian matrix $\mathbf{J}_{\mathbf{X}_i',\mathbf{a}}$ of Eq. (12) at each point $\mathbf{X}_i'$, we can construct $p = m \cdot n$ linear equations for the m given n-dimensional points. For the sphere fitting, the linear Eq. (5) looks like the following

$$\begin{pmatrix} J_{X_1',R} & J_{X_1',X_c} & J_{X_1',Y_c} & J_{X_1',Z_c} \\ J_{Y_1',R} & J_{Y_1',X_c} & J_{Y_1',Y_c} & J_{Y_1',Z_c} \\ J_{Z_1',R} & J_{Z_1',X_c} & J_{Z_1',Y_c} & J_{Z_1',Z_c} \\ \vdots & \vdots & \vdots & \vdots \\ J_{X_m',R} & J_{X_m',X_c} & J_{X_m',Y_c} & J_{X_m',Z_c} \\ J_{Y_m',R} & J_{Y_m',X_c} & J_{Y_m',Y_c} & J_{Y_m',Z_c} \\ J_{Z_m',R} & J_{Z_m',X_c} & J_{Z_m',Y_c} & J_{Z_m',Z_c} \end{pmatrix} \begin{pmatrix} \Delta R \\ \Delta X_c \\ \Delta Y_c \\ \Delta Z_c \end{pmatrix} = \begin{pmatrix} X_1'' \\ Y_1'' \\ Z_1'' \\ \vdots \\ X_m'' \\ Y_m'' \\ Z_m'' \end{pmatrix} \tag{14}$$

The initial parameters vector starting the Gauss–Newton iteration may be supplied from an algebraic circle fitting. Since the above algorithm is very robust, we can also take the center of gravitation and the RMS central distance.

$$\mathbf{X}_c = \overline{\mathbf{X}} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{X}_i, \qquad R = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\|\mathbf{X}_i - \overline{\mathbf{X}}\|^2}. \tag{15}$$

For an experimental example of the circle fitting, we have taken $m = 6$ coordinate pairs ($n = 2, q = 3$) in Table 1,[9] initial parameters $\mathbf{a}_0 = (R, X_c, Y_c)_0^T = (2.97, 4.50, 6.67)^T$ from Eq. (15), and step-size $\lambda = 1.3$. After 11 Gauss–Newton steps with norm of the terminal correction vector $\|\Delta\mathbf{a}\| = 5.32 \times 10^{-7}$, we have obtained [see Fig. 1(b)]

$$\hat{\mathbf{a}} = (4.71423, 4.73978, 2.98353)^T \quad \text{and} \quad \sigma_0^2 = 1.1080.$$

Table 1. Eight coordinate pairs.[9]

X	1	2	5	7	9	3	6	8
Y	7	6	8	7	5	7	2	4

A comparable algorithm, with initial parameters from an algebraic circle fitting, after 11 Gauss–Newton steps with $\|\Delta\mathbf{a}\| = 2.05 \times 10^{-6}$ delivers the same estimations.[9]

4. GEOMETRIC FITTING OF ELLIPSE

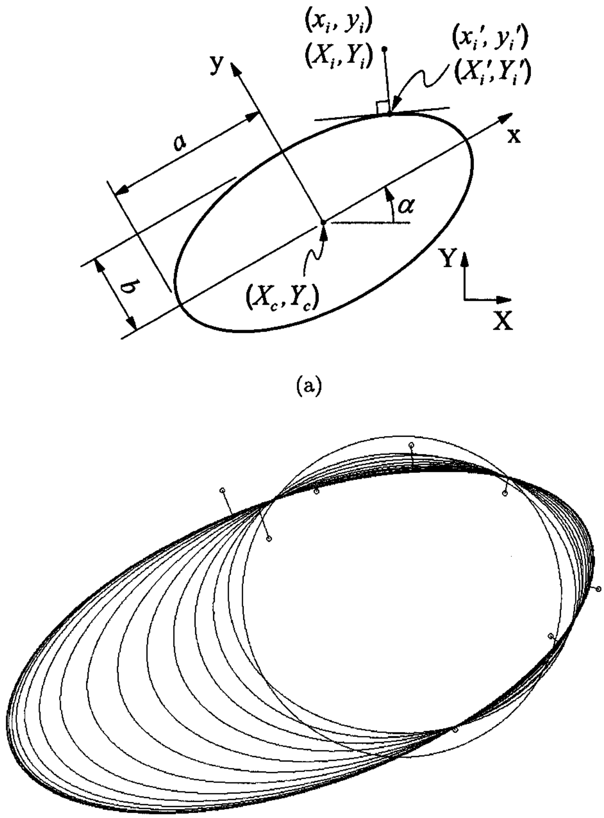
An ellipse in a plane can be uniquely described with $q = 5$ parameters, center coordinates X_c, Y_c, axis lengths $a, b (a \geq b)$ and pose angle $\alpha(-90° < \alpha \leq 90°)$ [see Fig. 2(a)]. For the least squares orthogonal distances fitting of ellipse with the algorithm described in Sec. 2, we must locate the nearest corresponding point $\mathbf{X}_i'$ (orthogonal contacting point) on the ellipse for the given point $\mathbf{X}_i$, and evaluate the Jacobian matrix at $\mathbf{X}_i'$. To locate the point $\mathbf{X}_i'$ on the ellipse, Gander *et al.*[9] have introduced additional m unknown parameters, where each point $\mathbf{X}_i'$ has an individual angular parameter to be simultaneously estimated with the five ellipse parameters. As a consequence, their linear system has a bulky and sparse Jacobian matrix and shows only a deteriorative convergence.

We have solved these problems of the orthogonal contacting point and Jacobian matrix through introducing a temporary coordinate system xy positioned at (X_c, Y_c) and rotated with the angle α. The coordinate transformation between the two coordinate systems xy and XY is [see Fig. 2(a)]

$$\mathbf{x} = \mathbf{R}(\mathbf{X} - \mathbf{X}_c) \quad \text{or} \tag{16}$$

$$\mathbf{X} = \mathbf{R}^{-1}\mathbf{x} + \mathbf{X}_c, \quad \text{where} \tag{17}$$

$$\mathbf{R} = \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \quad \text{and} \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \text{with} \quad C = \cos\alpha,\ S = \sin\alpha. \tag{18}$$



(a)

(b)

Fig. 2. (a) Ellipse in two-dimensional plane; (b) Geometric fitting of ellipse to the eight points in Table 1.

4.1. Orthogonal Contacting Point on Ellipse

In the xy system, three of five ellipse parameters disappear (X_c, Y_c and α) and the ellipse will be described only with the axis lengths a, b as below (standard position)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{19}$$

For a given point (x_i, y_i), the tangent line at the orthogonal contacting point (x, y) on the ellipse, and the connecting line of the two points are perpendicular to each other

$$\frac{dy}{dx} \cdot \frac{y_i - y}{x_i - x} = \frac{-b^2 x}{a^2 y} \cdot \frac{y_i - y}{x_i - x} = -1\,. \tag{20}$$

Rewrite Eqs. (19) and (20)

$$f_1(x, y) = \frac{1}{2}\,(a^2 y^2 + b^2 x^2 - a^2 b^2) = 0, \quad \text{and} \tag{21}$$

$$f_2(x, y) = b^2 x(y_i - y) - a^2 y(x_i - x) = 0\,. \tag{22}$$

The orthogonal contacting point (x, y) on the ellipse must simultaneously satisfy Eqs. (21) and (22) (orthogonal contacting conditions). Safaee-Rad *et al.*[14] have combined the two equations [(21) and (22)] into one quartic equation and chosen one solution which has the shortest connecting line length among the maximum four real solutions. We have solved this nonlinear problem using the generalized Newton method as below:

$$\mathbf{Q} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} b^2 x & a^2 y \\ (a^2 - b^2)y + b^2 y_i & (a^2 - b^2)x - a^2 x_i \end{pmatrix} \tag{23}$$

$$\mathbf{Q}_k \Delta\mathbf{x} = \mathbf{f}(\mathbf{x}_k) \quad \text{and} \tag{24}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \Delta\mathbf{x}\,. \tag{25}$$

We supply the initial point on the ellipse starting the iteration of Eqs. (24) and (25) as below, from the fact that the given point $\mathbf{x}_i$ and its nearest corresponding point $\mathbf{x}'_i$ on the ellipse lie in the same quadrant of the standard position.

$$\mathbf{x} = \begin{cases} \begin{pmatrix} x_i \\ \operatorname{sign}(y_i) \cdot \dfrac{b}{a}\sqrt{a^2 - x_i^2} \end{pmatrix}, & \text{if } \ |x_i| \le a \\ \begin{pmatrix} \operatorname{sign}(x_i) \cdot a \\ 0 \end{pmatrix}, & \text{if } \ |x_i| > a\,. \end{cases} \tag{26}$$

The iteration of Eqs. (24) and (25) with initial values from Eq. (26) terminates in 3–4 cycles with adequately accurate coordinates for the orthogonal contacting point.

After the given point $\mathbf{X}_i$ in XY system is transformed into $\mathbf{x}_i$ in xy system using Eq. (16), the orthogonal contacting point will be found by using the above generalized Newton method. Finally, we will have the point $\mathbf{X}'_i$ through the backward transformation of $\mathbf{x}'_i$ into XY system using Eq. (17). And the orthogonal error distances vector will be

$$\mathbf{X}''_i = \mathbf{X}_i - \mathbf{X}'_i\,. \tag{27}$$

4.2. Jacobian Matrix at the Orthogonal Contacting Point on Ellipse

If we define the parameters vector $\mathbf{a}$

$$\mathbf{a} = (X_c, Y_c, a, b, \alpha)^T\,, \tag{28}$$

from the derivatives of Eqs. (16) and (17) relatively to the parameters vector $\mathbf{a}$, we get

$$\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{a}}\right) = \begin{pmatrix} -C & -S & 0 & 0 & y_i \\ S & -C & 0 & 0 & -x_i \end{pmatrix}, \quad \text{and} \tag{29}$$

$$\mathbf{J}_{\mathbf{X}'_i,\mathbf{a}} = \left(\frac{\partial \mathbf{X}}{\partial \mathbf{a}}\right)\bigg|_{\mathbf{X}=\mathbf{X}'_i} = \mathbf{R}^{-1}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}}\right)\bigg|_{\mathbf{x}=\mathbf{x}'_i} + \begin{pmatrix} 1 & 0 & 0 & 0 & -xS - yC \\ 0 & 1 & 0 & 0 & xC - yS \end{pmatrix}\bigg|_{\mathbf{x}=\mathbf{x}'_i}. \tag{30}$$

The derivatives matrix appearing in the right-hand side of Eq. (30) is to be derived from Eqs. (21), (22) and (29), because the point $\mathbf{x}_i'$ is only implicitly known through the orthogonal contacting conditions. We differentiate f_1 and f_2 of Eqs. (21) and (22) relatively to the parameters vector $\mathbf{a}$

$$\frac{\partial f_{1,2}}{\partial \mathbf{a}} = \left(\frac{\partial f_{1,2}}{\partial X_c} \; \frac{\partial f_{1,2}}{\partial Y_c} \; \frac{\partial f_{1,2}}{\partial a} \; \frac{\partial f_{1,2}}{\partial b} \; \frac{\partial f_{1,2}}{\partial \alpha} \right) = \mathbf{0}\,. \tag{31}$$

After a series of substitution and reduction in Eqs. (29)–(31), we will get

$$\mathbf{J}_{\mathbf{X}_i',\mathbf{a}} = \mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{B}|_{\mathbf{x}=\mathbf{x}_i'} \tag{32}$$

where, $\mathbf{Q}$ is the Jacobian matrix of Eq. (23), and

$$\mathbf{B} = (\mathbf{B}_1 \quad \mathbf{B}_2 \quad \mathbf{B}_3 \quad \mathbf{B}_4 \quad \mathbf{B}_5), \text{ with}$$

$$\mathbf{B}_1 = \begin{pmatrix} b^2xC - a^2yS \\ b^2(y_i - y)C + a^2(x_i - x)S \end{pmatrix}, \mathbf{B}_3 = \begin{pmatrix} a(b^2 - y^2) \\ 2ay(x_i - x) \end{pmatrix}, \mathbf{B}_4 = \begin{pmatrix} b(a^2 - x^2) \\ -2bx(y_i - y) \end{pmatrix}$$

$$\mathbf{B}_2 = \begin{pmatrix} b^2xS + a^2yC \\ b^2(y_i - y)S - a^2(x_i - x)C \end{pmatrix}, \; \mathbf{B}_5 = \begin{pmatrix} (a^2 - b^2)xy \\ (a^2 - b^2)(x^2 - y^2 - xx_i + yy_i) \end{pmatrix}. \tag{33}$$

4.3. Orthogonal Distances Fitting of Ellipse

With the error distances vector $\mathbf{X}_i''$ of Eq. (27) and the Jacobian matrix $\mathbf{J}_{\mathbf{X}_i',\mathbf{a}}$ of Eq. (32) at each point $\mathbf{X}_i'$, we construct $p(= 2m)$ linear equations for m given two-dimensional points. For the ellipse fitting, the linear Eq. (5) look like:

$$\begin{pmatrix} J_{X_1',X_c} & J_{X_1',Y_c} & J_{X_1',a} & J_{X_1',b} & J_{X_1',\alpha} \\ J_{Y_1',X_c} & J_{Y_1',Y_c} & J_{Y_1',a} & J_{Y_1',b} & J_{Y_1',\alpha} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{X_m',X_c} & J_{X_m',Y_c} & J_{X_m',a} & J_{X_m',b} & J_{X_m',\alpha} \\ J_{Y_m',X_c} & J_{Y_m',Y_c} & J_{Y_m',a} & J_{Y_m',b} & J_{Y_m',\alpha} \end{pmatrix} \begin{pmatrix} \Delta X_c \\ \Delta Y_c \\ \Delta a \\ \Delta b \\ \Delta \alpha \end{pmatrix} = \begin{pmatrix} X_1'' \\ Y_1'' \\ \vdots \\ X_m'' \\ Y_m'' \end{pmatrix}. \tag{34}$$

The initial parameters vector starting the Gauss–Newton iteration may be supplied from an algebraic ellipse fitting, or from a circle fitting. When the parameters from a circle fitting are to be used as the initial values for the ellipse fitting, we set

$$(X_c, Y_c)_{\text{ellipse}} = (X_c, Y_c)_{\text{circle}}, \quad a = b = R \quad \text{and} \quad \alpha = 0\,. \tag{35}$$

If $a = b$ during the iteration, the elements of $\mathbf{B}_5$ in Eq. (33) are all zero, and consequently, the last column of the Jacobian matrix in Eq. (34) will be filled with zero. In this case, $\Delta\alpha$ will have the solution $\Delta\alpha = 0$ through the singular value decomposition (SVD) and succeeding backsubstitution.[12] During the iteration, if $a < b$ after an updating of the parameters, we simply exchange the two values and set $\alpha \leftarrow \alpha - \text{sign}(\alpha) \cdot \pi/2$.

For an experimental example of the ellipse fitting, we have taken eight coordinate pairs in Table 1,[9] initial parameters $\mathbf{a}_0 = (4.84, 4.80, 3.39, 3.39, 0)^T$ from the geometric circle fitting, and step size $\lambda = 1.2$. After 19 Gauss–Newton steps with norm of the terminal correction vector $\|\Delta\mathbf{a}\| = 4.17 \times 10^{-6}$, we have obtained [see Fig. 2(b)].

$$\hat{\mathbf{a}} = (2.69961, 3.81596, 6.51872, 3.03189, 0.35962)^T \quad \text{and} \quad \sigma_0^2 = 1.1719\,.$$

The comparable algorithm of Gander *et al.*[9] gives the same estimations only after 71 Gauss–Newton steps. Besides, they could not directly use the parameters from a circle fitting, because of the singularity in their Jacobian matrix by $a = b$. To bypass the singularity problem, they have used $a = R$, $b = R/2$. But an arbitrary modification of one parameter without proper adjustments to the others can cause a divergence!

5. DISCUSSION AND CONCLUSIONS

We have proposed new algorithms for the least squares orthogonal distances fitting of circle and ellipse. For the iterative circle fitting, the Jacobian matrix at the nearest point on the circle from a given point is directly available from the circle parameters and given point. The proposed circle fitting is very robust and we can simply use the center of gravitation and the RMS central distance as the initial parameter values. However, for the ellipse fitting, it is not easy to locate the nearest point on the ellipse from a given point, and to evaluate the Jacobian matrix. We have overcome these difficulties through a transforming the ellipse into the standard position and utilizing the orthogonal contacting conditions. The proposed ellipse fitting is robust and can take the parameters from a circle fitting without modification, and the comparison with other geometric ellipse fitting shows it has superior performances in convergence.

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Wolfgang Rauh received his Doctor degree in 1993 from the University of Stuttgart. After he had received his diploma in mechanical engineering from the University of Karlsruhe in 1984 he worked at the University of Stuttgart until 1989. He then became head of the group "Industrial Image Processing" at the Fraunhofer Institute for Manufacturing Engineering and Automation. Since 1991, he is head of the department "Information Processing" at the same institute, where a wide variety of projects in the area of image processing is carried out.