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GLM (STA4813)
Assignment 02

Question 1

(a) $f(x_i, \theta) = \binom{x_i + r - 1}{r-1} \theta^r (1-\theta)^x \quad x = 0, 1, \ldots$

$$f(x_i; \theta) = \exp\left(\log(1-\theta)^x + \log\theta^r + \log\binom{x_i+r-1}{r-1}\right)$$
$$= \exp\left(x\log(1-\theta) + r\log\theta + \log\binom{x_i+r-1}{r-1}\right)$$

Where :

$a(x) = x$
$b(\theta) = \log(1-\theta)$
$c(\theta) = \log(\theta)$
$d(x) = \binom{x_i + r - 1}{r-1}$

Therefore $f(x_i, \theta)$ belongs to the exponential Family of Probability Mass Function.

(b) $$E(X) = -\frac{c'(\theta)}{b'(\theta)} = -\left[\frac{1}{\theta} \div \frac{-1}{(1-\theta)}\right]$$
$$= \frac{(1-\theta)}{\theta}$$

Where

$c'(\theta) = \frac{1}{\theta}$; $b'(\theta) = \frac{-1}{1-\theta}$

$$Var(x) = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3}$$

$$= \left[\frac{1}{\theta(1-\theta)^2} + \frac{1}{\theta^2(1-\theta)}\right] \times \frac{(1-\theta)^3}{-1}$$

$$= \frac{-(1-\theta)^3}{\theta^2(1-\theta)^2}$$

$$= \frac{-(1-\theta)}{\theta^2} \quad ??$$

c) $U = a(x)\, b'(\theta) + c'(\theta)$

$$= x\left(\frac{-1}{1-\theta}\right) + \frac{1}{\theta}$$

$$= \frac{\theta(x-1) + (\theta+1)}{\theta(1-\theta)}$$

$$\approx \frac{\theta x - 2\theta + 1}{\theta(1-\theta)}$$

$$\approx \frac{\theta(x-2)+1}{\theta(1-\theta)}$$

Verify: $E(U) = b'(\theta)E(a(y)) + c'(\theta)$

$$= \left[\frac{-1}{1-\theta}\right]\left[\frac{(1-\theta)}{\theta}\right] + \frac{1}{\theta}$$

$$= -\frac{1}{\theta} + \frac{1}{\theta} = 0$$

(d) $J = E(-U) = E\left[\sum_{i=1}^{n} U_i'\right.$

$$= \sum_{i=0}^{n}\left[\frac{b''(\theta)c'(\theta)}{b'(\theta)} - c''(\theta)\right]$$

$$= \sum_{i=0}^{n}\left[\frac{\left(\frac{1}{(1-\theta)^2}\right)\left(\frac{1}{\theta}\right) - \left(\frac{-1}{\theta^2}\right)}{-\frac{1}{(1-\theta)}}\right]$$

$$= \sum_{i=0}^{n}\left(\frac{1-\theta}{\theta(1-\theta)^2} + \frac{1}{\theta^2}\right)$$

$$= \frac{n}{\theta^2(1-\theta)}$$

(e) $L(\theta|x) = \prod\binom{x_i+r-1}{r-1}\theta^r(1-\theta)^x$

$$\log(L(\theta|x)) = \sum\log\binom{x_i+r-1}{r-1} + r\log\theta + x\log(1-\theta)$$

$$= nr\log\theta + \sum_{i=1}^{n} x\log(1-\theta)$$

$$\frac{\partial}{\partial\theta}\log(L) = \frac{nr}{\theta} + \frac{\sum_{i=1}^{n} x(-1)}{1-\theta}$$

$$0 = \frac{nr}{\hat\theta} - \frac{\sum_{i=1}^{n} x}{1-\hat\theta}$$

$$(1-\hat\theta)nr = \hat\theta\sum_{i=0}^{n} x_i$$

$$nr = \hat\theta\left(\sum_{i=1}^{n} x_i + nr\right)$$

$$\hat\theta = \frac{nr}{\sum_{i=1}^{n} x + nr}$$

f) $W_n = \dfrac{\hat{\theta}_n - \theta_0}{\sqrt{Var(\hat{\theta}_n)}}$

$Var(\hat{\theta}_n) = \dfrac{\theta(1-\theta)}{n-r}$

$\therefore W = \dfrac{\hat{\theta}_n - \theta_0}{\sqrt{\dfrac{\theta_n(1-\theta_n}{nr}}}$

g) $\hat{\theta}_n \pm Z_{1-\alpha/2} \times Se(\hat{\theta}_n)$

$= \hat{\theta}_n \pm Z_{1-\alpha/2} \times \hat{\theta}_n$

Question 2:

Given $F(y,P) = P(1-P)^{y-1}$

(a) $\log(f(y;P)) = \log(P(\cancel{P}1-P)^{y-1})$
$= \log P + y\log(1-P) - \log(1-P)$

$\therefore F(y;P) = \exp(y\log(1-P) - \log(1-P) - \log(P))$

Where

$a(y) = y$

$b(P) = \log(1-P)$

$C(P) = \log(P)$

$d(y) = 0$

(b) $\mathcal{I} = E(-U') = E\left[\sum_{i=1}^{n} U_i'\right]$

$$= \sum_{i=1}^{n}\left[\frac{b''(p)\,c'(p)}{b'(p)} - c''(p)\right]$$

$$= \sum_{i=1}^{n}\left[\frac{\left(\frac{1}{(1-p)^2}\right)\left(\frac{1}{p}\right)}{\frac{-1}{1-p}} - \left(-\frac{1}{p^2}\right)\right]$$

$$= \sum_{i=1}^{n}\left(\frac{1}{p(1-p} + \frac{1}{p^2}\right)$$

$$= \frac{n}{p^2(1-p)}$$

c) $L(p|y) = \Pi(1-p)^{y-1}p$

$$\log(L(p|y)) = \sum_{i=1}^{n} \log(p(1-p)^{y-1})$$
$$= \sum_{i=1}^{n}(\log p + \log(1-p)^{y-1}$$
$$= n\log p + \log(1-p)\sum(y_i - n)$$

$$\frac{d}{dp}(\log(L(p|y))) = n/p + (-1/(1-p))\sum y_i - n$$

$$\Rightarrow n/\hat{p} + (-1/(1-\hat{p}))\sum(y_i - n) = 0$$
$$\Rightarrow \hat{p}\sum(y_i - n) = n(1-\hat{p})$$

$$\Rightarrow \hat{p} = \frac{n}{\sum y_i}$$

$$\therefore \hat{p} = 1/\bar{y} \approx \bar{y}^{-1}$$