1. Motivation

Let

$$f_N(z)=\sum_{n=1}^{\infty} a_N(n)\mathrm{e}^{2\pi i n z}$$

be the weight 2 eta-quotient newform of level N. Let $p>3$ be a prime. Martin and Ono determined $a_N(p)$ for $N=27,32,36,64,144$ in terms of binary quadratic forms.

Conjecture 1.1. *Let $p>3$ be a prime. Then*

$$a_{14}(p)=-\left(\frac{p}{3}\right)\sum_{x=0}^{p-1}\left(\frac{x^3-75x-506}{p}\right)$$

$$a_{15}(p)=-\left(\frac{p}{3}\right)\sum_{x=0}^{p-1}\left(\frac{x^3-3x-322}{p}\right)$$

Conjecture 1.2. *For $n\in\mathbb{N}$ we have $a_{14}(2n)=-a_{14}(n)$ and so $a_{14}\left(2^k\right)=(-1)^k$ for $k\in\mathbb{N}$. In addition, $a_{24}\left(3^k\right)=(-1)^k$ for $k\in\mathbb{N}$*

2. Preparatory work

2.1. Elliptic curve.

One important conclusion (From Diamand) is:

$$a_1(E)=1, a_{p^e}(E)=p^e+1-\left|\widetilde{E}\left(\mathbf{F}_{p^e}\right)\right|,\quad e\geq 1 \tag{1}$$

More details are as followings:

Elliptic curve $E: y^2=x^3+a_2x^2+a_1x+a_0$, which is not singular, i.e. $\Delta(E)\neq 0$.

Another thing that is worth noticing is Affine model, which differs from project model in such way: $\mathbb{C}\to\bar{\mathbb{C}}$, for instance, solve the equations $x^2+1\equiv 0 \pmod 3$ in finite field $\mathbb{F}_3=\mathbb{Z}/3\mathbb{Z}$: in Affine model, it's unsolvable, but in Project model, it has one solution. Thus, the equation (1) becomes $a_{p^e}(E)=p^e-\left|\widetilde{E}\left(\mathbf{F}_{p^e}\right)\right|_{\text{Affine}},\quad e\geq 1$, and $\left|\widetilde{E}\left(\mathbf{F}_{p^e}\right)\right|_{\text{Affine}}$ is the number of solutions of $\{(x,y)/\mathbb{F}_{p^e}\in E\}$.

2.2. The modularity of $f_N(z)$.

Modularity Theorem is needed:

Theorem 2.1.

$$\sum_n a_E(n)q^n\in\mathcal{S}_2\left(\Gamma_0(N)\right)$$

, where $N=$ conductor(E).

The computation of conductor refers to Page.324 of Diamond.

3. Expected to do

1) Compute the conductor for $N=14$ and write down the concrete form of

$$\left|E\left(\mathbb{F}_p\right)\right|=\sum_{\substack{x\\ \Delta\neq 0}}\left(\left(\frac{E}{p}\right)+1\right)$$

2) Is $f_\chi=\sum a(n)\chi(n)q^n$ where $\chi:(\mathbb{Z}/M)^\times\to\mathbb{C}$ a modular form of weight k for congruence subgroup?

4. My work

4.1. Conductor. We need to evaluate the conductor of

$$E : y^2 = x^3 - 75x - 506$$

Doing a little algebra, we can get :

$$c_4 = 2^4 3^2 5^2, c_6 = 2^6 3^3 11 \cdot 23, \Delta = 2^{14} 3^6 7$$

An admissible change of variable $(x, y) = (2^2 x', 2^3 y')$ leads to :

$$c_4' = 3^2 5^2, c_6' = 3^3 11 \cdot 23, \Delta' = 2^2 3^6 7$$

i.e. the global minimal discriminant of E is $\Delta_{min}(E) = 2^2 3^6 7$, and it's difficult to evaluate δ_3 when we aim to get the algebraic conductor N_E of E. By Tate's algorithm [1], we can compute $N_E = 2 \cdot 3^2 \cdot 7$.

4.2. Modularity. Note that

$$f_\chi = \sum a(n)\chi(n)q^n = \sum_{r=0}^{M-1} \sum_{k=0}^{\infty} a(kM + r)\chi(r)q^{kM+r}$$

Consider

$$f_{\chi,M,r} = \sum_{k=0}^{\infty} a(kM + r)\chi(r)q^{kM+r} = \chi(r) \sum_{k=0}^{\infty} a(kM + r)q^{kM+r}$$

We have known that $\sum_{k=0}^{\infty} a(kM + r)q^{kM+r}$ can be represented as a linear combination of $f\left(\frac{\tau+r}{M}\right)$, combining Theorem 2.1 , and we can conclude that

$$f_\chi = \sum_{r \in (\mathbb{Z}/M\mathbb{Z})^\times} \chi(r) C_r f\left(\frac{\tau + r}{M}\right) \in \mathcal{S}_2\left(\Gamma_0(M^2 N_E)\right)$$

4.3. Verification. Notice that the condition "$p > 3$ be a prime" in the Conjecture, we get $a_{14}(2), a_{14}(3)$ from the expansion of $\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$. Based on conjecture1.1 and $a_{14}(2) = -1, a_{14}(3) = -2$, we use the recurrence relation

$$a_{p^e}(E) = a_p(E)a_{p^{e-1}}(E) - \mathbf{1}_E(p)pa_{p^{e-2}}(E), \quad e \geq 2 \tag{2}$$

to get all $a_{14}(p^e)$. Then for every $n = \prod_{p|n} p_1^{e_1} \cdots p_k^{e_k}, a(n) = \prod a\left(p_1^{e_1}\right) \cdots a\left(p_k^{e_k}\right)$.

With the modularity we have got, we can compute the upper bound

$$\frac{2}{12}\left[\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma_0(3^2\mathrm{N_E})\right]$$

However, I find that the conductor I evaluated is wrong in the process of running the program on MATLAB, because the results of $a_{14}(3^e)$ computed by the recurrence relation doesn't equal to the corresponding coefficient of 'Etas function'. There is no doubt that I have a misunderstanding on global minimal Weierstrass equation, I use the conductor $N_E = 2 \times 7$ in the program and complete the verification for all $n \leq 6408$, the new and smaller upper bound when $N_E = 14$.

Having proved

$$\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) = \sum_{n=1} a_E(n)q^n$$

with $a_{14}(1) = 1, a_{14}(2) = -1$ and recurrence relation, it's not hard to show $a_{14}(2^k) = (-1)^k$ by induction. Note that

$$a_{14}(2n) = a_{14}\left(2\prod_{p|n} 2^{e_1}\cdots p_k^{e_k}\right) = a_{14}(2^{e_1+1})a_{14}\left(\prod p_2^{e_2}\cdots\right) = -1\cdot a_{14}(2^{e_1})a_{14}\left(\prod p_2^{e_2}\cdots\right) = -a_{14}(n)$$

, which completes the the first two identities in Conjecture 1.2.

5. Updated on Jan. 29

Our goal: $\left(\frac{p}{3}\right) a_{14}(p) = a_E(p)$ for all p.

To use Sturm's Theorem, we need to show that

$$\left(\frac{n}{3}\right) a_{14}(n) = a_E(n)$$

holds for $n \leq \frac{2}{12}\left[\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma_0(\mathrm{N_E})\right]$.

But note that

$$\begin{aligned} &\left(\frac{p^e}{3}\right) a_{14}\left(p^e\right) \\ &= \left(\frac{p^e}{3}\right)\left(a_{14}(p)a_{14}\left(p^{e-1}\right) - 1_{14}(p)p\cdot a_{14}\left(p^{e-2}\right)\right) \\ &= \left(\frac{p}{3}\right) a_{14}(p)\left(\frac{p^{e-1}}{3}\right) a_{14}\left(p^{e-1}\right) - 1_{14}(p)\cdot p\cdot\left(\frac{p^{e-2}}{3}\right) a_{14}\left(p^{e-2}\right) \end{aligned} \tag{3}$$

i.e. a recurrence relation for $, e \geq 2$.

The difference between (2) and (3) is when $p = 3$ (The trivial conductor modula). But $a_E(3) = 0$ implies that $a_E\left(3^e\right) = 0, e \geq 1$ and $a_E(n) = 0, 3 \mid n$, which means $\left(\frac{p^e}{3}\right) a_{14}\left(p^e\right)$ and $a_E(p^e)$ have the same recurrence relation. With their multiplicativity, it suffices to show that

$$\left(\frac{p}{3}\right) a_{14}(p) = a_E(p)$$

holds for $p \leq \frac{2}{12}\left[\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma_0(\mathrm{N_E})\right]$.

References

[1] Joseph H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves.* Graduate Texts in Mathematics 151. Springer-Verlag,1994.