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Abstract.

1. Introduction

In this work, we aim to establish all the Ramanujan–Mordell type formulas that can be parametrized by moonshine modular curves.

Here are some initial steps we need to take.

(1) For the levels N listed in [1, Table 3], read off the ones that are levels of theta functions $\theta_Q(\tau)$. For a complete account of properties of $\theta_Q(\tau)$, see Theorem 1.1 of Wang and Pei's book, which can be found on the file-tree of this folder.
(2) For the theta functions $\theta_Q(\tau)$ given in Step 1, work out a linear combination $F_k(\tau)$ of Eisenstein series, so that $F_k(\tau)/\theta_Q(\tau)^{2k}$ or $F_k(\tau)/\theta_Q(\tau)^k$ depending on Q being unary or binary has poles supported at the cusp $[i\infty]$ only with order as small as possible. Bases for Eisenstein series can be found in Section 5.3 of W. Stein's book, which can be found on the file-tree.

Given a quadratic form $Q = Q(\vec{x})$ of rank n. It is known that

$$\theta_Q(\tau) = \sum_{\vec{x}\in\mathbb{Z}^n} q^{Q(\vec{x})}$$

is a holomorphic modular form of weight $n/2$ for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with some character. For details, look up Chapter 1 of Wang and Pei's book. According to the decomposition theorem of modular forms, we know that every holomorphic modular form can be written as the sum of Eisenstein series and cusp forms. For integral weight bigger than 1, there are canonical constructions for Eisenstein series, that is to tell, the subspace of Eisenstein series is essentially fully known. Look up Chapter 5 of Stein's book Modular forms, a Computational approach. In this project, for some quadratic form Q, we aim to find the decomposition of $\theta_Q(\tau)$.

For example, take $Q = \vec{x}\vec{x}^T = x_1^2 + \cdots + x_{2k}^2$ the usual Euclidean inner product of rank $2k$ with $k \geq 1$. One can check by the book of Wang and Pei that $\theta_Q(\tau)$ is holomorphic modular form of weight k for $\Gamma_0(4)$ (having three cusps $[i\infty]$, $[\frac{1}{2}]$ and $[0]$) with quadratic character $\left(\frac{-4}{d}\right)$, is non-vanishing in $\mathbb{H}$ and has a multiple-zero at the cusp $\frac{1}{2}$ of order N. Though the curve $(\Gamma_0(4)\backslash\mathbb{H})^*$ is of genus zero, for some convenience, Mordell considered a "small" curve, say, $\Gamma_0(4)+\backslash\mathbb{H}$, where $\Gamma_0(4)+ = \langle\Gamma_0(4), \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}\rangle$ called the Fricke group of $\Gamma_0(4)$, so that only two of those three cusps need to be considered, say, $[i\infty]$ and $[\frac{1}{2}]$. Using Eisenstein series for $\Gamma_0(4)$, Mordell constructed an Eisenstein series $E(k,\tau)$ of weight k depending on k and τ such that

$$E(k,\tau)/\theta_Q(\tau)$$

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is modular function for $\Gamma_0(4)+$. For example, if the subspace of Eisenstein series for $\Gamma_0(4)$ is given by $\bigoplus \mathbb{C}E_i(\tau)$, it suffices to find coefficients c_i such that

$$\frac{\sum c_i E_i(\tau)}{\theta_Q(\tau)}$$

is invariant under the Fricke involution $\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$. So this shall yield some interrelations between c_i's, from which one can deduce the desired linear combination of $E_i(\tau)$ such that its quotient by $\theta_Q(\tau)$ is invariant under $\Gamma_0(4)+$.

Following this, one can tell that

$$\frac{E(k,\tau)}{\theta_Q(\tau)} = C_{-N} q_{1/2}^{-N} + \cdots \text{(at the cusp } [\tfrac{1}{2}])$$

is a meromorphic function on the curve $\Gamma_0(4) + \backslash\mathbb{H}$ with only a pole at $[\frac{1}{2}]$ of order at most N. Since the curve is of genus zero, its function field is singly generated, say, $x(\tau)$. See [1, Table]. By an FLT, one can choose $x(\tau)$ to be with a simple pole at the cusp $[\frac{1}{2}]$, i.e., $x(\tau) = q_{1/2}^{-1} + O(1)$. Therefore, $\frac{E(k,\tau)}{\theta_Q(\tau)}$ can be written as a polynomial in $x(\tau)$.

(1) Find all the primitive positive definite quadratic forms $p(X,Y) = aX^2 + bXY + cY^2$ such that

$$\theta_Q(\tau) = \sum_{\vec{x}\in\mathbb{Z}^{2n}} q^{Q(\vec{x})} = \left(\sum q^{p(X,Y)}\right)^n$$

is of level N that is on the Table 3 of [1], where $Q(\vec{x})$ is the n-copy of $p(X,Y)$, i.e., $Q = p(X_1, X_2) + \cdots + p(X_{2n-1}, X_{2n})$. For example, $p(X,Y) = X^2+Y^2$ is the case considered in Ramanujan-Mordell and is of level 4. For another example, $p(X,Y) = X^2 + 3Y^2$ is of level 12 and was considered in the work of Cooper, Kane and Ye.

(2) For each of such N's, identify the space of Eisenstein series for $\Gamma_0(N)$, i.e., give a basis for the space.

References

[1] T. Huber, D. Schultz and D. Ye, *Ramanujan–Sato series for $\frac{1}{\pi}$*, Acta Arithmetica

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