# 1 The Assignment

## 1.1 Time Dependent PDEs

In this section, we begin exploring time-dependent partial differential equations, specifically building up to the advection-diffusion equation by considering each case individually first, then combining them into one equation. We will also test different explicit and implicit finite differencing schemes to step forward in time and test the Courant-Fiedrichs-Lewy (CFL) condition, which is usually used to determine the size of the time step.

### 1.1.1 The Advection Equation

The one-dimensional advection equation is a function of concentration $u(t, x)$ over time:

$$u_t = cu_x \tag{1}$$

where $c$ is the speed of flow. In a similar vein to d'Alembert's solution to the wave equation, the exact solution to the 1D advection equation is found by shifting the initial conditions, $u(0, x)$, along the $x$-axis:

$$u(t, x) = u(0, x + ct) \tag{2}$$

This means that if $c$ is positive, the initial function is shifted to the left. The opposite is true if $c$ is negative, the initial function is shifted to the right.

To find a numerical solution to the advection equation, we applied four different finite difference schemes: forward and backward Euler, Lax-Wendroff, and Crank-Nicholson. These four schemes will be used on four different initial conditions, shown in table 1 with seven different sets of parameters, shown in table 2. The value $r$ is the Courant number, which is a result of the CFL condition. This is a way to determing the stability of the system. Generally, the Courant number is set to be less than one for the system to be stable.

| Case | Function |
|---|---|
| 0 | $u(0, x) = \sin(2x)$ |
| 1 | $u(0,x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ |
| 2 | $u(0, x) = \delta_{x,0}/\Delta x$ |
| 3 | $\exp(-4x^2)$ |

Table 1: The initial conditions for the advection equation.

| Trial | $c$ | $\Delta x$ | $\Delta t$ | $r = c\Delta t/\Delta x$ |
|---|---|---|---|---|
| 0 | 0.5 | 0.04 | 0.02 | 0.25 |
| 1 | 0.5 | 0.02 | 0.02 | 0.5 |
| 2 | 0.5 | 0.0137 | 0.02 | 0.728 |
| 3 | 0.5 | 0.0101 | 0.02 | 0.99 |
| 4 | 0.5 | 0.0099 | 0.02 | 1.11 |
| 5 | 0.5 | 0.02 | 0.01 | 0.25 |
| 6 | 0.5 | 0.02 | 0.04 | 1 |

Table 2: The sets of parameters used when computing the advection equation.

The first case, shown in figure 1 is a simple sinusoid. Since the pulse propogates to the left, the right side of the exact solution remains constant at the value of the boundary. The first trial, with $r = 0.25$, all methods are able to approximate the solution, though there are oscillations on the left side due to the boundary condition. Once $r >= 0.5$, the explicit forward Euler scheme no longer propogates the initial pulse correctly and flatlines at zero while the areas around the boundary increase to infinity. The smaller values of $\Delta x$ also increase the oscillations seen in the implicit methods. For trial 4, $r = 1.11$, and all methods become unstable except for the Crank-Nicholson scheme, which combines the explicit and implicit Euler methods.

The second case, shown in figure 2 is a Heaviside function centred at $x = 0$. Similarly to the first case, the forward Euler method is the most unstable, unable to approximate the solution in any of the cases. Trial 4 also shows once again that the Crank-Nicholson scheme is the most stable while the other methods lose their stability once $r >= 1$. However, the explicit Lax-Wendroff scheme is the most accurate and has the fewest oscillations as $\Delta x$ decreases.

The third case, shown in figure 3 is a delta function. The magnitude of the pulse is dependent on the value $\Delta x$, and is thus dependent on the parameters defined in the trial. Regardless of the magnitude of the pulse, it propogates the same way. Many of the properties seen in the previous case are present here as well: forward Euler is very unstable, Crank-Nicholson is the most stable, and Lax-Wendroff gives the most accurate result when the CFL condition is maintained. The unique aspects of this case are from the oscillations. In both the backward Euler and Crank-Nicholson schemes, there is a large oscillation on the other side of where the actual pulse is. The oscilations in the latter, however, become so large and numerous that it is difficult to tell where the important information is and which peaks are noise. This would also imply that the Crank-Nicholson scheme is very sensitive to sharp spikes in data while the Lax-Wendroff scheme is able to propogate them better.

The final case we are considering a a Gaussian distribution centred about $x = 0$. Since this case does not have sharp abrupt changes in slope, all the methods are able to propagate it a little better and the oscillations present in the implicit methods are greatly reduced.

#### 1.1.2 The Diffusion Equation

The pure diffusion equation describes an initial pulse which diffuses into the surrounding fluid. The one dimensional diffusion equation is:

$$u_t = \beta u_{xx} \tag{3}$$

where $\beta$ is the diffusion coefficient. The exact solution for $x$ over the interval $[0, 1]$ for the initial condition $u(0, x) = \sin(\pi x)$ and boundary conditions $u(t, 0) = u(t, 1) = 0$ is:

$$u(t, x) = e^{-\beta\pi^2 t} \sin(\pi x) \tag{4}$$

In order to solve the equation numerically, we can change the diffusion equation into a FD equation. Replacing the time derivative with a first order forward difference scheme and the spatial derivative by a centred difference scheme results in the equation:

$$u_i^{n+1} = u_i^n + r\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \tag{5}$$

where the Courant number is $r = \frac{\beta\Delta t}{(\Delta x)^2}$, $n$ is along the time axis and $j$ is along the $x$-axis. The conditions for stability can be found through von Neumann stability analysis. Starting from the finite difference equation, stepping forward in time and centred stepping in space:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \beta\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \tag{6}$$

we first assume that the each step can be represented as a product of an amplification factor $G^n$ and an exponential term so that $u_j^n = G^n e^{ikj\Delta x}$. This form of $u_j^n$ is then substituted into the FD equation and solved for $G^1$:

$$\begin{aligned} e^{ikj\Delta x}\frac{G^{n+1} - G^n}{\Delta t} &= \beta G^n \frac{e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}}{\Delta x^2} \\ G^1 - 1 &= \frac{\beta\Delta t}{\Delta x^2}\left(e^{-ik\Delta x} - 2 + e^{+ik\Delta x}\right) \\ \therefore G^1 &= (1 - 2r) + 2r\cos(k\Delta x) \end{aligned} \tag{7}$$

where $r = \frac{\beta\Delta t}{\Delta x^2}$ in the last substitution.

For stability, we impose the condition that $|G^n| \leq 1$. If the cosine term is equal to one ($\cos(k\Delta x) = 1$), then $|G^n| = 1$, which satisfies the stability condition. However, if the cosine term is equal to negative one, we use the fact that $r > 0$, so the stability condition becomes:

$$\begin{aligned} |G^n| &\leq 1 - 4r \\ -1 &\leq 1 - 4r \\ \therefore \Delta t \leq \Delta t_{max} &= \frac{\Delta x^2}{2\beta} \end{aligned} \tag{8}$$

When the diffusion equation was solved numerically for the conditions described in equation 4, it was run for 300 time steps, 51 grid points, $\beta = 1$, and $\Delta t = 0.9\Delta t_{max}$, the results shown in figure 5. Since the CFL conditions are met, the solution is stable and very closely approximates the exact solution, though the height of the peak is slightly overestimated.

#### 1.1.3 The Advection-Diffusion Equation

By combining the two previous sections on advection and diffusion, the advection-diffusion equation is:

$$u_t = \beta u_x x - cu_x \tag{9}$$

where the initial condition is advected with speed $c$ and diffusing with rate $\beta$. The finite difference equa-

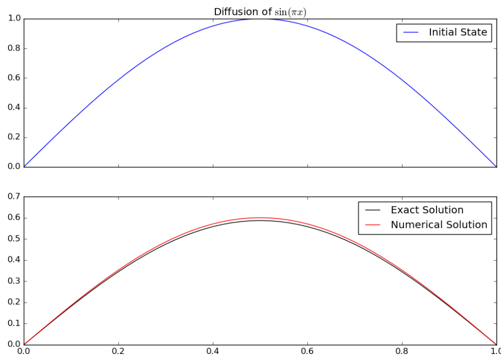


Figure 5: The diffusion of a simple sinusoid. The pulse is simply spread out and the peak falls.

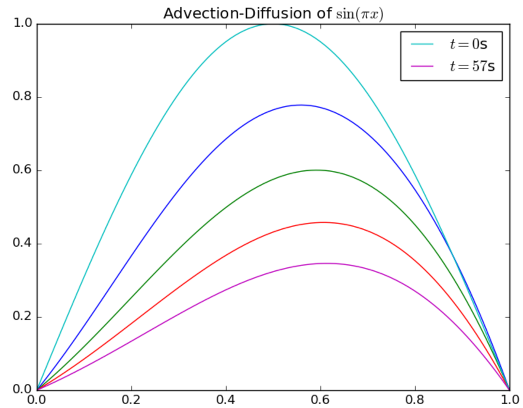


Figure 6: The advection-diffusion equation applied on a simple sinusoid. This is a combination of the advection and diffusion behaviour observed in the previous two parts.

tion, stepping forward in time, using the centred difference scheme for the second derivative and backward for the first derivative, is:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \beta \frac{u_{j+1}^n + 2u_j^n + u_{j-1}^n}{\Delta x^2} - c\frac{u_j^n - u_{j-1}^n}{\Delta x} \tag{10}$$

The von Neumann stability analysis is done similarly to before, subsituting $u_j^n = G^n e^{ikj\Delta x}$ and solving for $G^1$:

$$\begin{aligned} e^{ikj\Delta x}\frac{G^{n+1} - G^n}{\Delta t} &= \beta G^n \frac{e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}}{\Delta x^2} \\ &- cG^n \frac{e^{ikj\Delta x} e^{ik(j-1)\Delta x}}{\Delta x} \\ \frac{G^1 - 1}{\Delta t} &= \beta \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2} \\ &- c\frac{1 - e^{-ik\Delta x}}{\Delta x} \\ \therefore G^1 &= 1 + 2s\left(\cos(k\Delta x) - 1\right) \\ &+ r\left(e^{-ik\Delta x} - 1\right) \end{aligned} \tag{11}$$

Looking at the extreme values of $G^1$ and imposing the stability condition $|G^1| \leq 1$, the upper extreme when $\cos(k\Delta x) = 1$ results in $|G^1| = 1$. On the lower extreme, when $\cos(k\Delta x = -1$, the condition is:

$$\begin{aligned} -1 \leq 1 - 4s - 2r &= 1 - 2\Delta t\left(\frac{2\beta}{\Delta x^2} + \frac{2c}{\Delta x}\right) \\ \therefore \Delta t \leq \Delta t_{max} &= \frac{\Delta x^2}{c\Delta x + 2\beta} \end{aligned} \tag{12}$$

This time, the grid has 101 points, $c = 0.5$, $\beta = 0.1$, $\Delta t = 0.9\Delta t_{max}$, and the solution is run until $t = 57$s, shown in figure 6. The entire simulation took 2277 time steps. The behaviour is a combination of what was observed in the advection equation, since the peak has moved from $x = 0.5$ to $x = 0.61$, while the height of the peak has also decreased, resulting from the diffusion.

# 2 Discussion

The sections on the advection-diffusion serve to show the importance of the CFL conditions and the effect they have on the various schemes used to step forward in time. In general, the implicit methods (backward Euler and Crank-Nicholson) were more stable, even as the Courant number was increased to be over 1.

However, the implicit methods were also observed to exhibit high frequency oscillations at the boundary and from sharp abrupt changes in slope. Von Neumann stability analysis proved to be a reliable way to define an appropriate size for the time step.

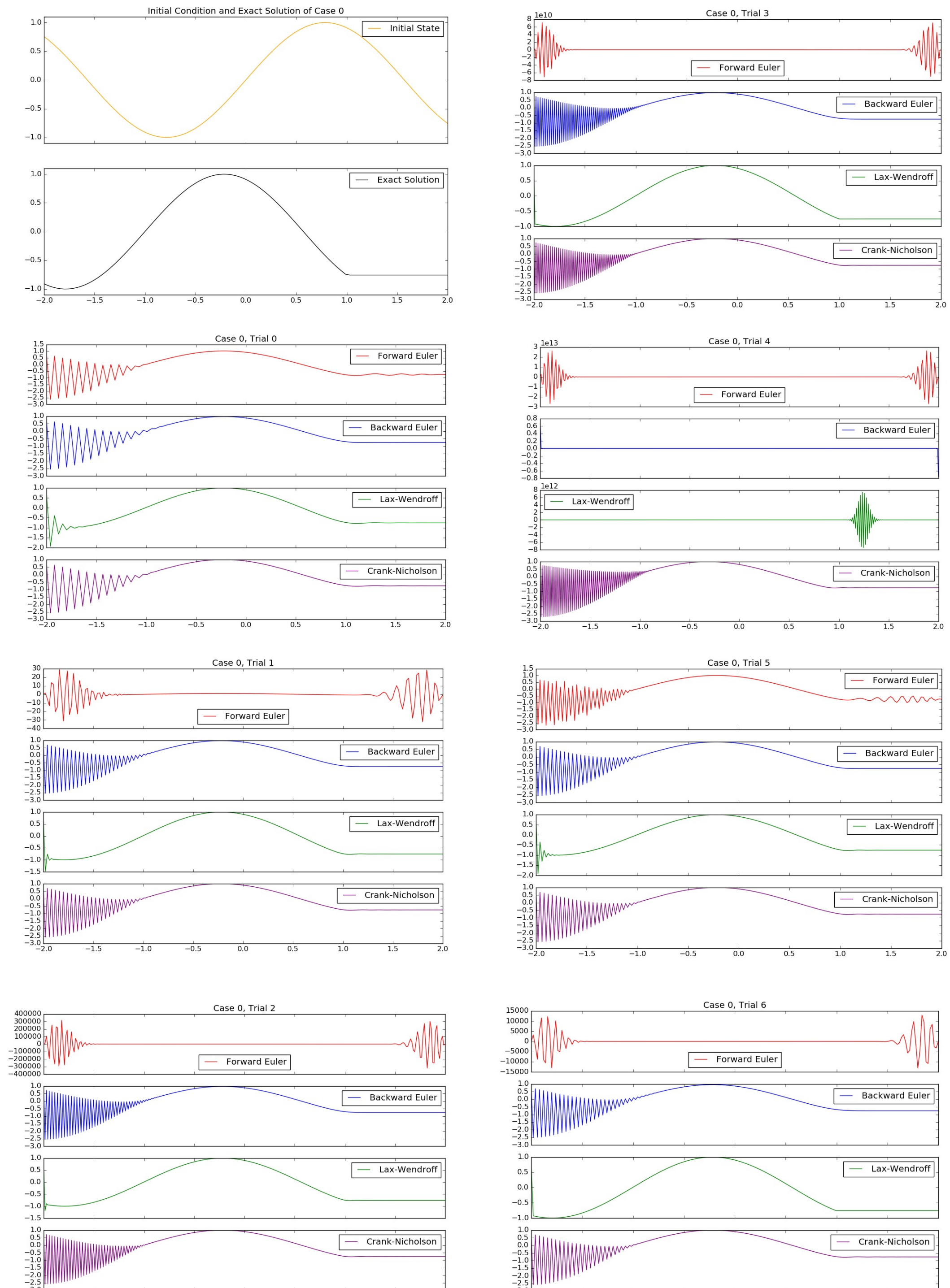




Figure 1: The numerical solution to case 0 from table 1 for seven trials, a simple sinusoid. The initial condition and exact solutions are shown in the top left and the solutions are are for $t = 2$.

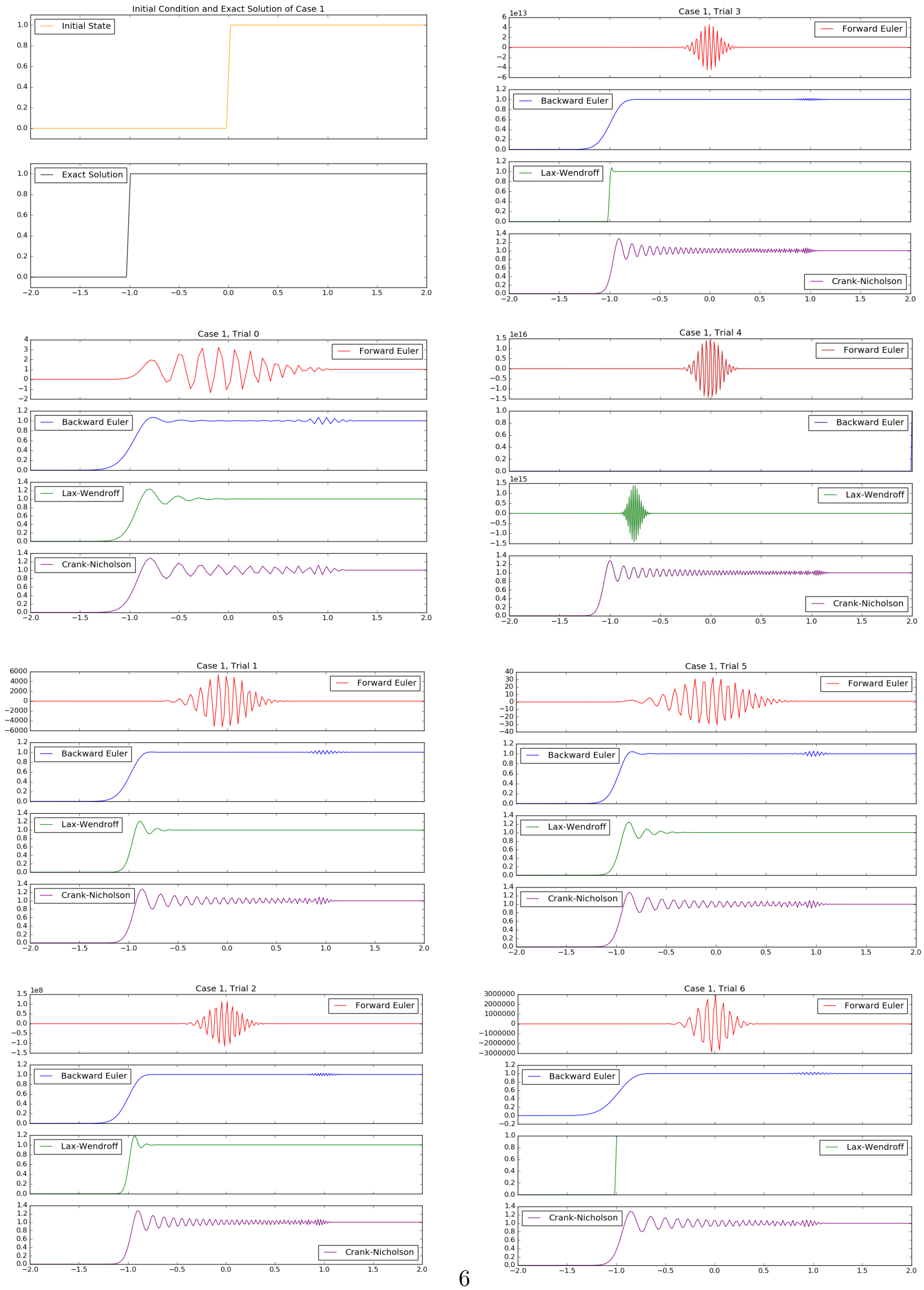




Figure 2: The numerical solution to case 1, a Heaviside function. The initial condition and exact solution are shown in the top left corner and the solutions are for $t = 2$.

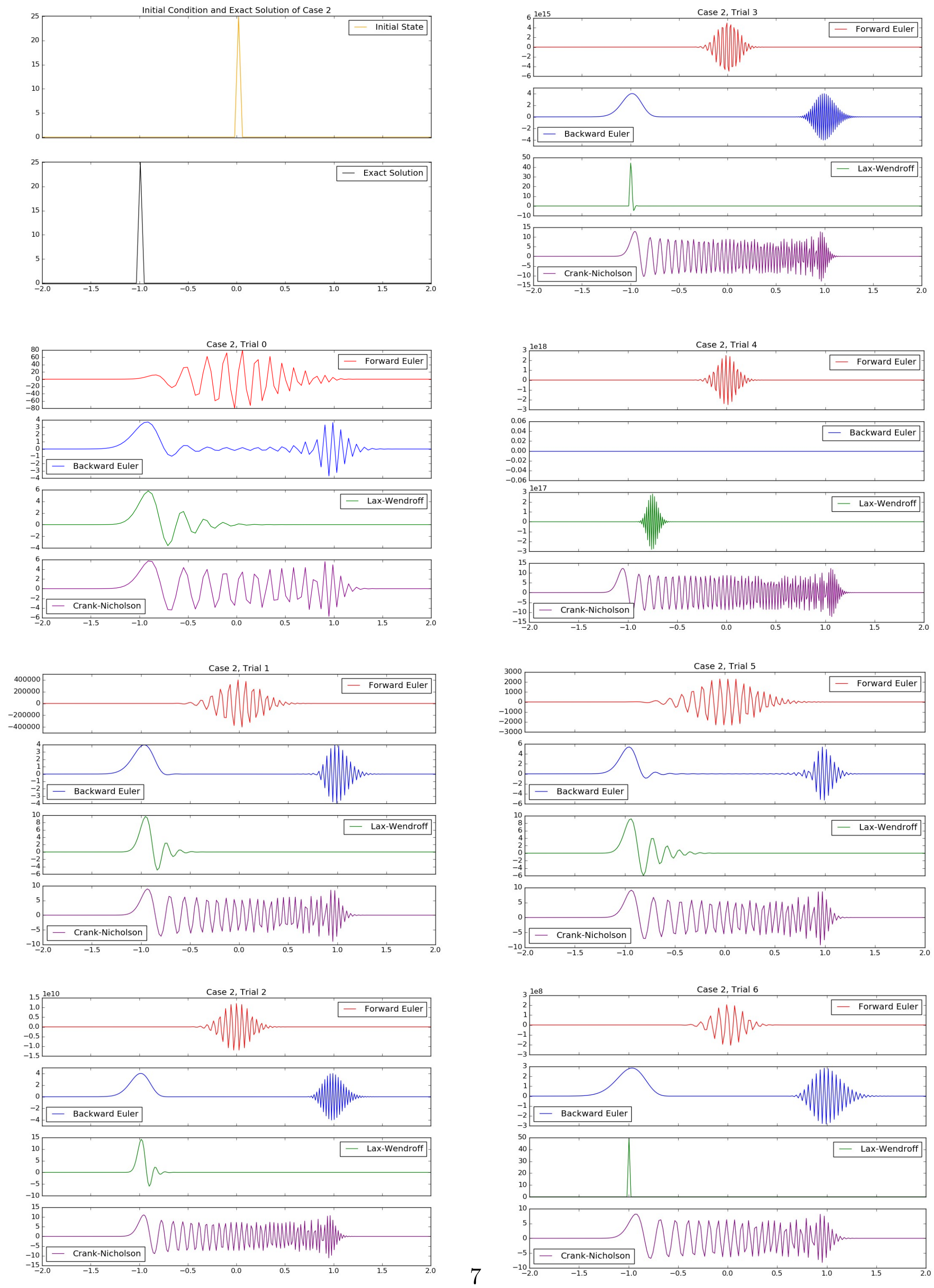




Figure 3: The numerical solution to case 2, a delta function. The initial condition and exact solution are shown in the top left corner and the solutions are for $t = 2$. It should be noted that the size of the pulse is dependent on the value of $\Delta x$ for each trial.

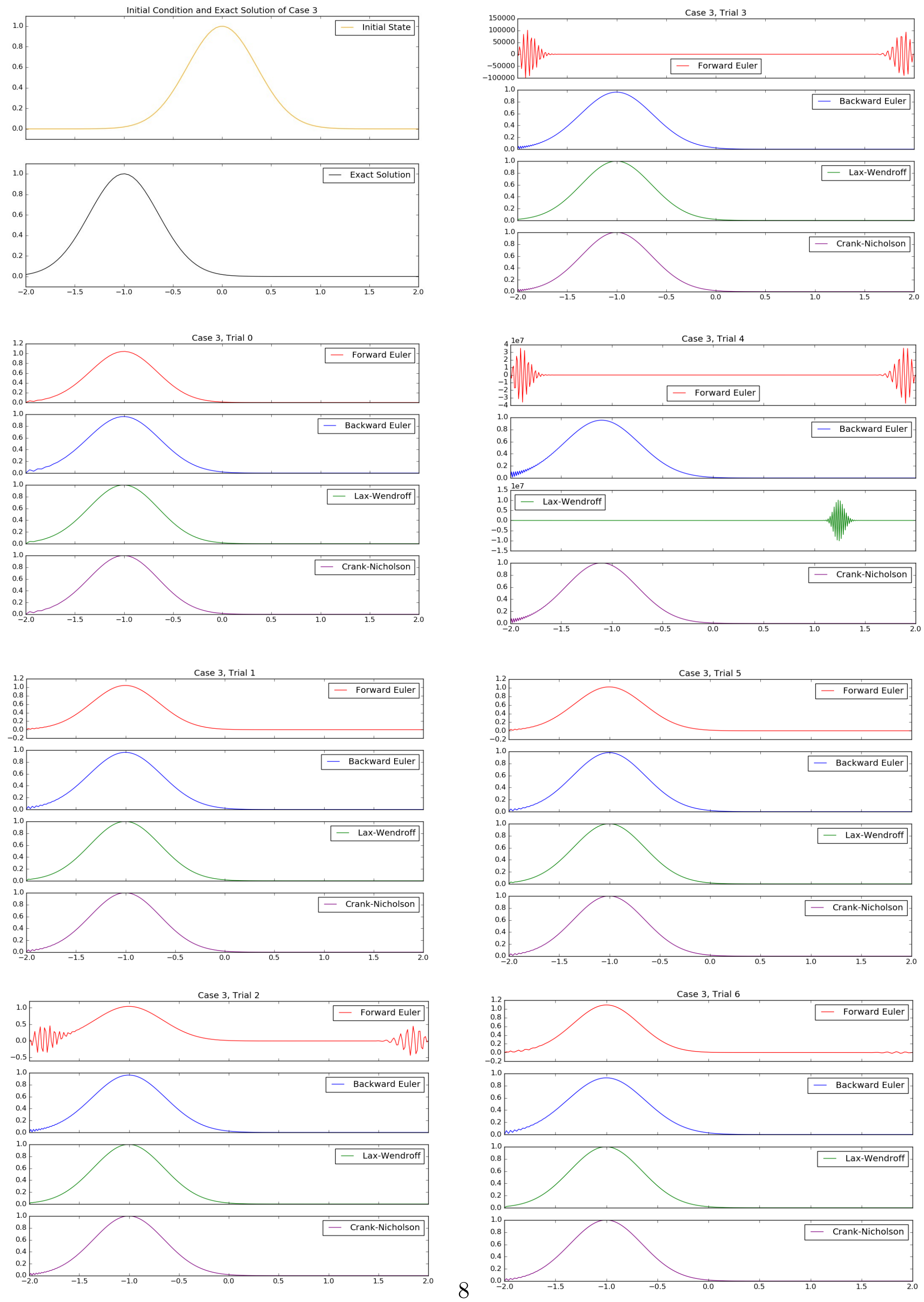




Figure 4: The numerical solution to case 3, a Gaussian distribution with its peak centred about $x = 0$. The initial condition and exact solution are shown in the top left corner and the solutions are for $t = 2$.