Finite Differences:
Staggered Grids

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Presentation 3

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Review: Finite Differences

- Finite differences is a way to evaluate differential equations discretely
- There are a few different ways that finite differences may be performed:
  - Forward

$$\partial_x^+ u = \frac{u(x+\Delta x)-u(x)}{\Delta x} = \frac{u_{k+1}-u_k}{h}$$

  - Reverse

$$\partial_x^- u = \frac{u(x)-u(x-\Delta x)}{\Delta x} = \frac{u_k-u_{k-1}}{h}$$

  - Centered

$$\partial_x u = \frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x} = \frac{u_{k+1}-u_{k-1}}{2h}$$

1^{st} Order DEs

- So far we have mainly dealt with DEs that are 2^{nd} order in space.
- 1^{st} order DEs have some peculiarities in terms of time effects

- Ex: The Continuity Equation

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \mathbf{j} = \sigma$$

- ρ – quantity density   j – flux   σ - generation

Continuity FDs

- We want to minimize error, so we will use the central difference method in space, and the forward difference in time. We will consider the case where $\sigma = 0$ in 2 space dimensions.

$$\frac{\partial \rho}{\partial t} = \frac{\partial j}{\partial x} + \frac{\partial j}{\partial y}$$

$$\frac{\rho_{m,n}^{l+1} - \rho_{m,n}^{l}}{\Delta t} = \frac{j_{m+1,n}^{l} - j_{m-1,n}^{l}}{\Delta x} + \frac{j_{m,n+1}^{l} - j_{m,n-1}^{l}}{\Delta y}$$

The Checkerboard Problem

- Also known as odd-even decoupling
- Consider some theoretical system where the flux has a known form:

$$j = j(x, y, t)$$

- Next, assume that when discretized, it oscillates between different values:

$$j^l_{m,n} = \frac{95}{2}\exp\left(\frac{\pi x_m}{\Delta x} + \frac{\pi y_n}{\Delta y} + \frac{\pi t^l}{2\Delta t}\right) + \frac{105}{2}$$

100	5	100	5	100
5	100	5	100	5
100	5	100	5	100
5	100	5	100	5
100	5	100	5	100
5	100	5	100	5
100		100		100

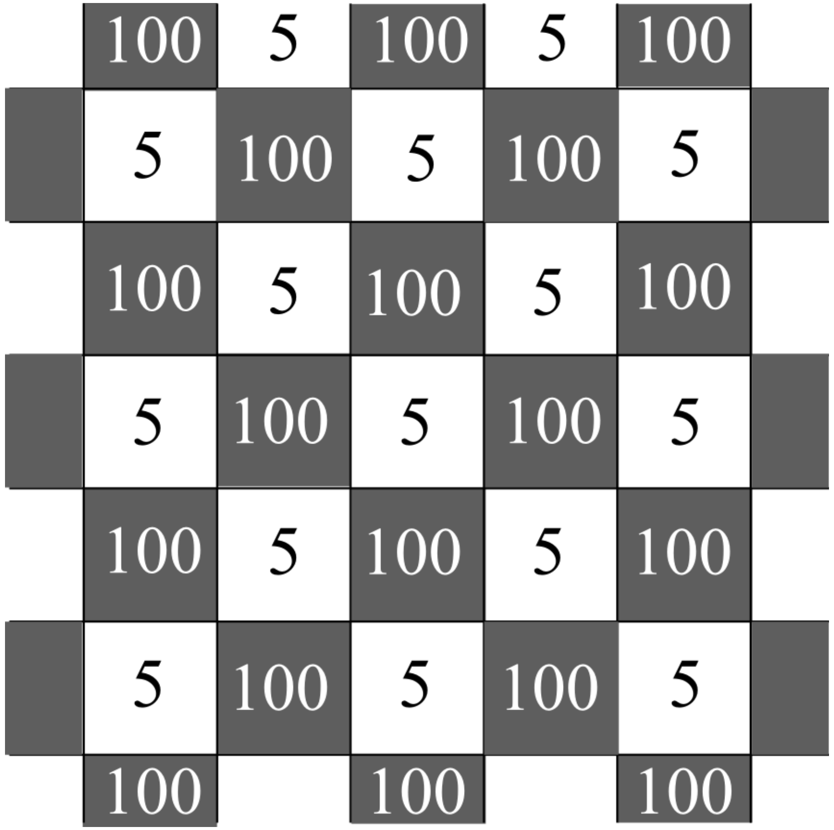
The Checkerboard Problem

- This produces the most simplistic checkerboard pattern.
- Analytically, we expect the density to behave as such:

$$\frac{\partial \rho}{\partial t} = \frac{\partial j}{\partial x} + \frac{\partial j}{\partial y}$$

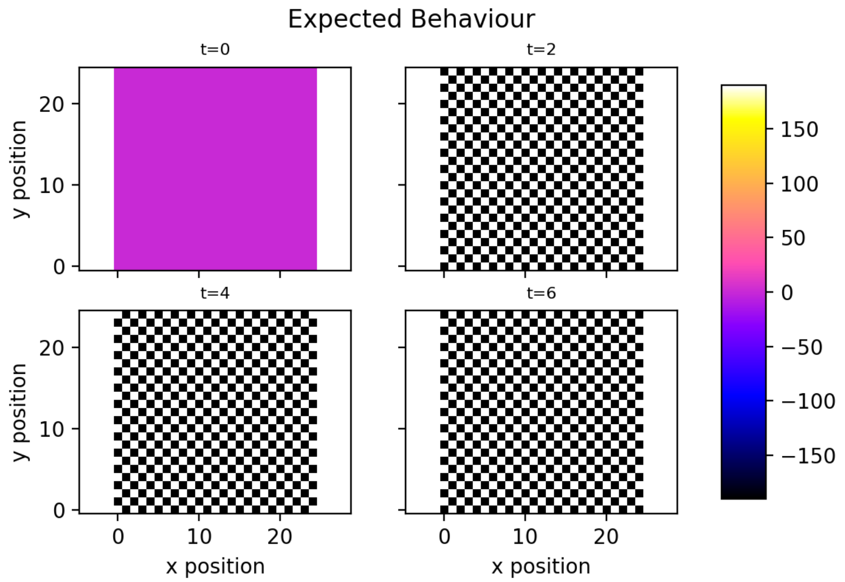
$$\frac{\partial \rho}{\partial t} = \frac{95 i \pi}{2} \exp\left(\frac{\pi x_m}{\Delta x} + \frac{\pi y_n}{\Delta y} + \frac{\pi t}{2\Delta t}\right)\left(\frac{1}{\Delta x} + \frac{1}{\Delta y}\right)$$

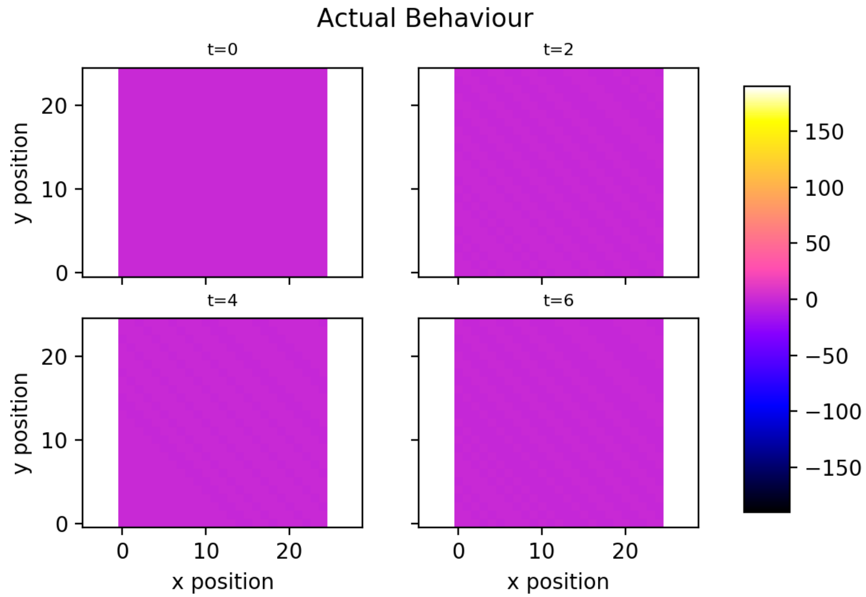
$$\rho^l_{m,n} = 190\Delta t \exp\left(\frac{\pi x_m}{\Delta x} + \frac{\pi y_n}{\Delta y} + \frac{\pi t}{2\Delta t}\right) \times \left(\frac{1}{\Delta x} + \frac{1}{\Delta y}\right) + \rho^0_{m,n}$$



Testing Checkerboard

- When we run our finite difference algorithm, the result isn't what we expect.
- It turns out our co-local coordinates respond poorly to high-frequency information and produce inaccurate results when exposed to high-frequencies



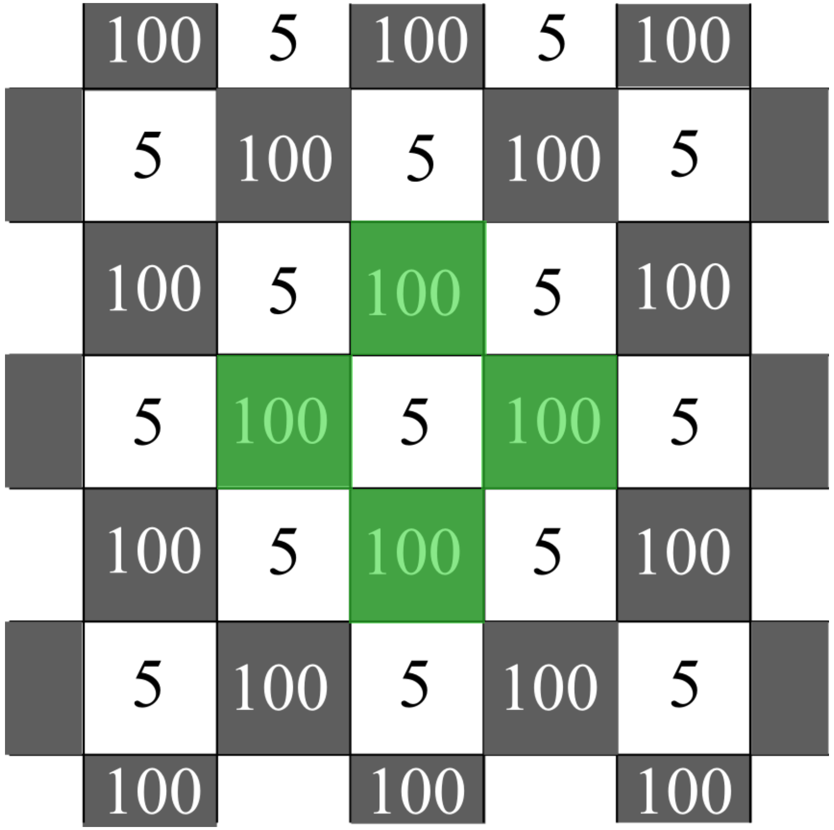


What's Going On

- Our 2 dimensional FD equation:

$$\frac{\rho^{l+1}_{m,n} - \rho^{l}_{m,n}}{\Delta t} = \frac{j^{l}_{m+1,n} - j^{l}_{m-1,n}}{2\Delta x} + \frac{j^{l}_{m,n+1} - j^{l}_{m,n-1}}{2\Delta y}$$

- Our equation only looks at adjacent cells, not the current cell
- If we are solving ρ for an 'even' cell, we are only looking at odd cells, and vice versa: we are odd-even decoupled



Checkerboard Consequences

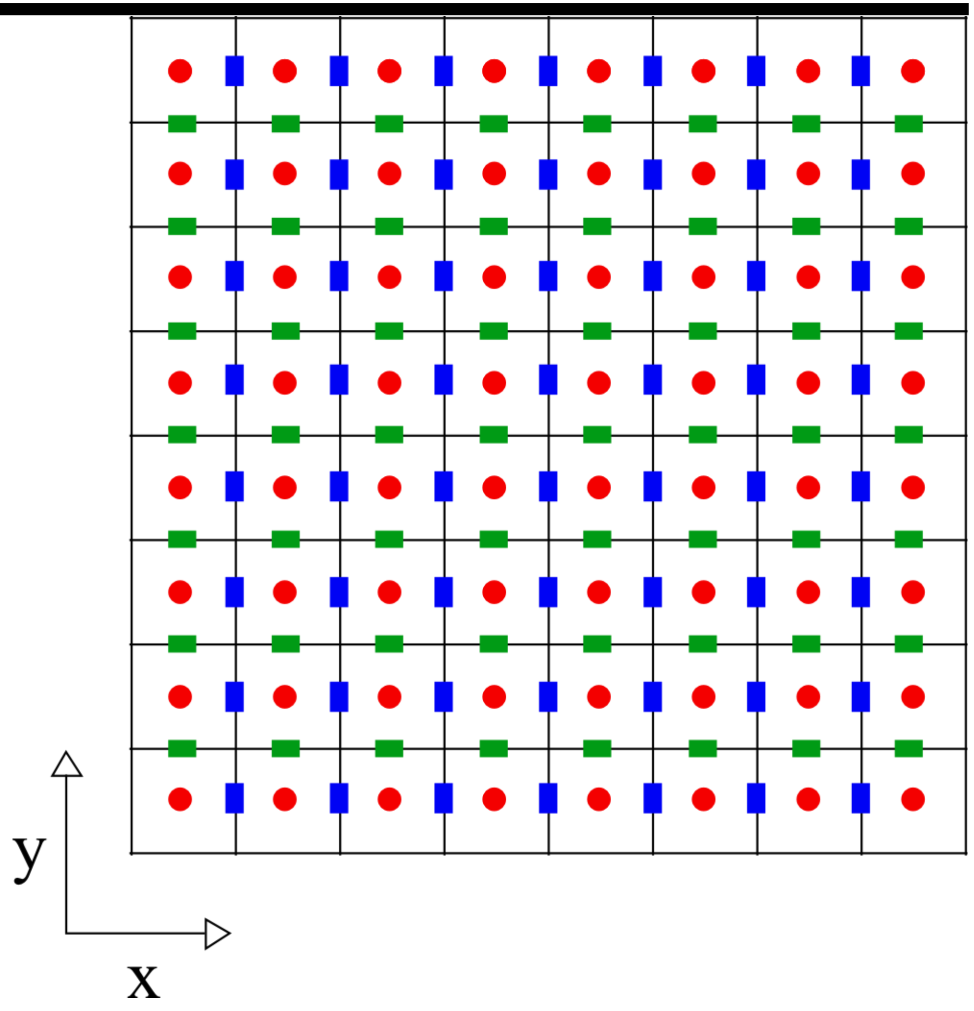
- Our checkerboard problem can occur anywhere there is alternating data. The same effect would occur if we forced ρ to oscillate and looked at j instead. Still more problems can occur if the pressure oscillates.
- In the hydrodynamics case where some wave should dissipate, it may get stuck in a high-frequency state and remain in the simulation

Possible solutions

- Increase the grid resolution
  - This is computationally inefficient
- Use only forward or reverse finite differences, not the centered difference
  - This will increase error in our results and defeats the purpose
- Use staggered grids
  - The accepted solution

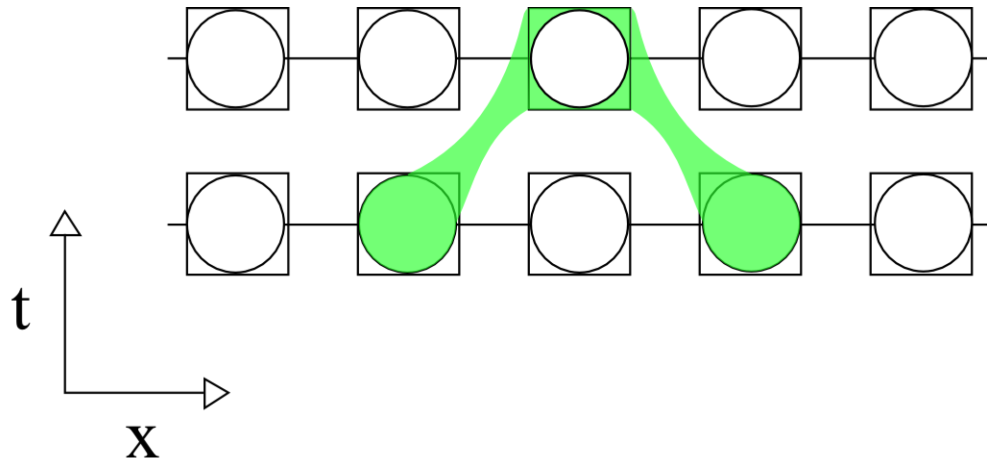
Staggered Grids

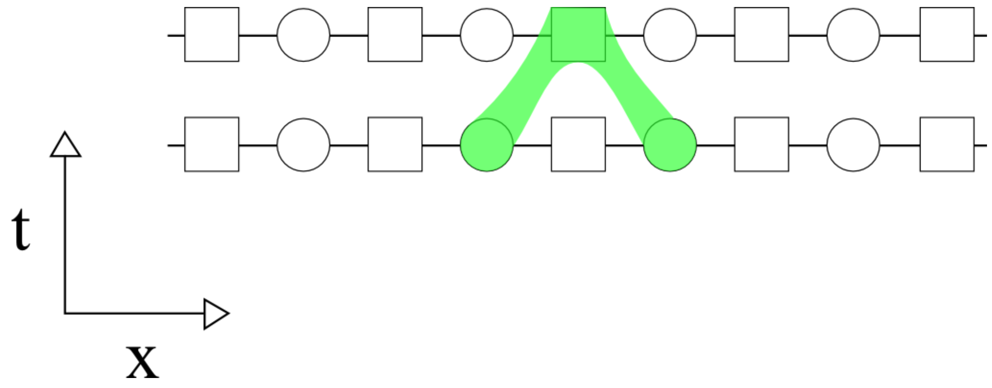
- When we have differential equations that rely on 1st order derivatives, use multiple grids
  - One for the scalar quantity
  - Additional one for each vector, shifted a half-step from the scalar grid.
- We can add further shifted grids for each additional first derivative



Stencils

- Using the staggered grid method, we can effectively sample two adjacent points of our flux quantity
- This uses the finite difference from a higher resolution while keeping the lower memory, lower resolution result

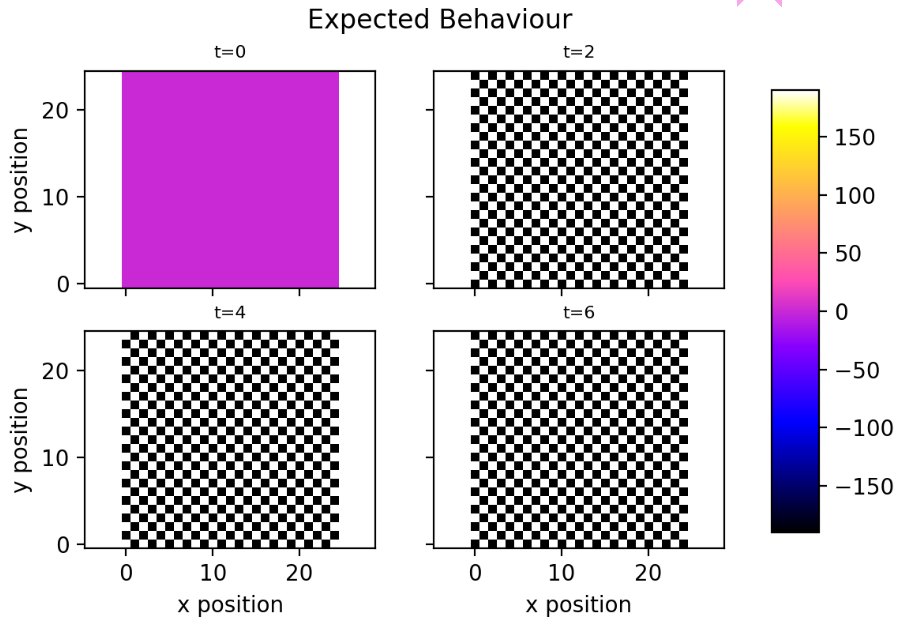


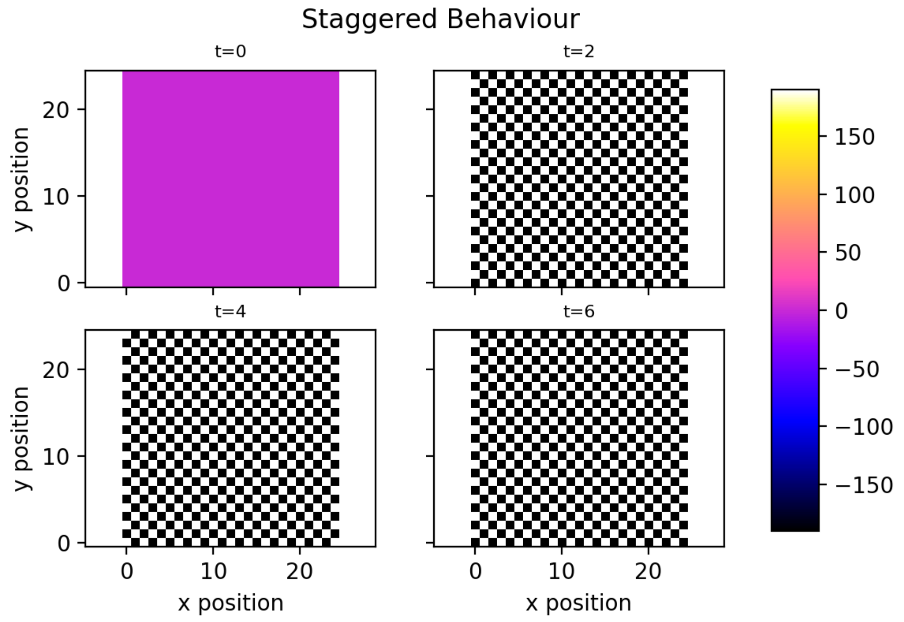


Checkerboard Revisited

- We can re-attempt the checkerboard problem using staggered grids.
- We can define a separate grid for the j values, with $j_{m+1/2,n}^{l}$ in between $\rho_{m,n}^{l}$ and $\rho_{m+1,n}^{l}$. Likewise, $j_{m,n+1/2}^{l}$ is between $\rho_{m,n}^{l}$ and $\rho_{m,n+1}^{l}$.

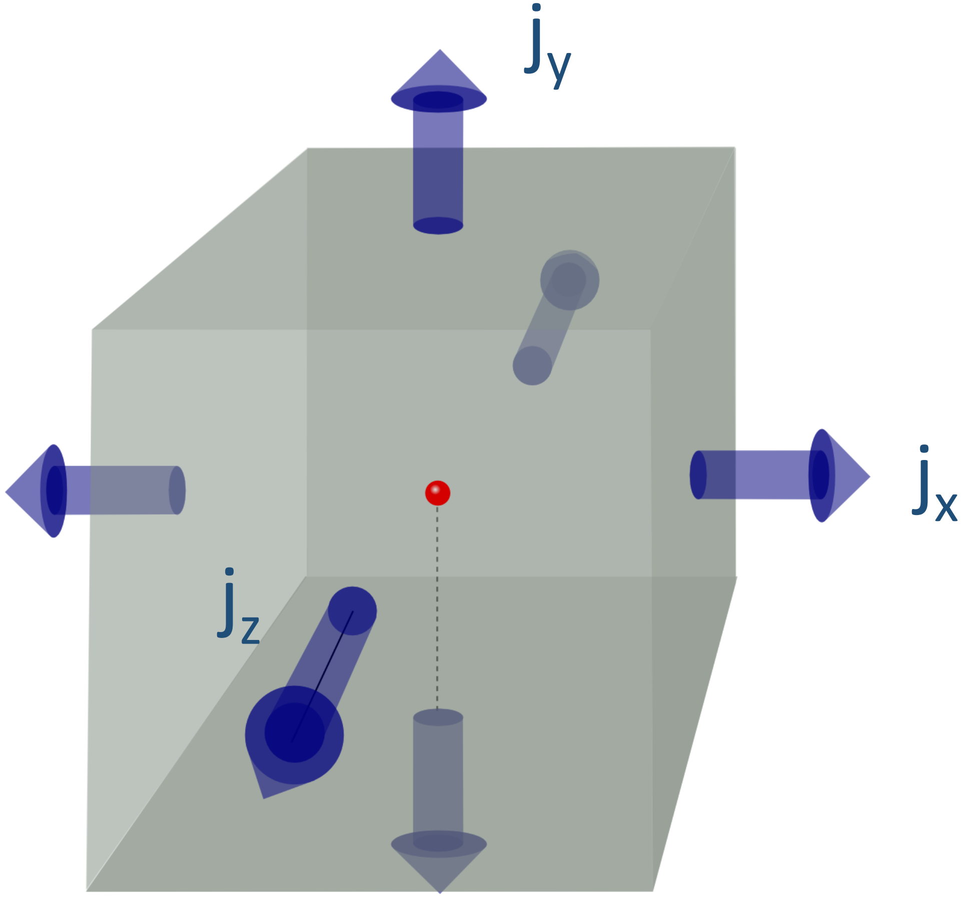
- The result matches our analytical derivation





3D Grids

- One benefit of staggered grids is intuitiveness.
- Each cell acts as a cubic volume, with the quantity at the centre, and a flux at each of the relevant cell walls.
- Some schemes with complicated sets of variables also place staggered values on cell edges, corners, etc.



Example: Shallow Water Equations

- These equations are derived from Navier-Stokes equations (with simplifications)

$$\frac{dh}{dt} + H\left(\frac{du}{dx} + \frac{dv}{dy}\right) = 0$$

$$\frac{du}{dt} - fv = -g\frac{dh}{dx}$$

$$\frac{dv}{dt} + fu = -g\frac{dh}{dy}$$

- Where h is the height of the fluid with velocities u and v in the x and y directions

Finite Difference Equations

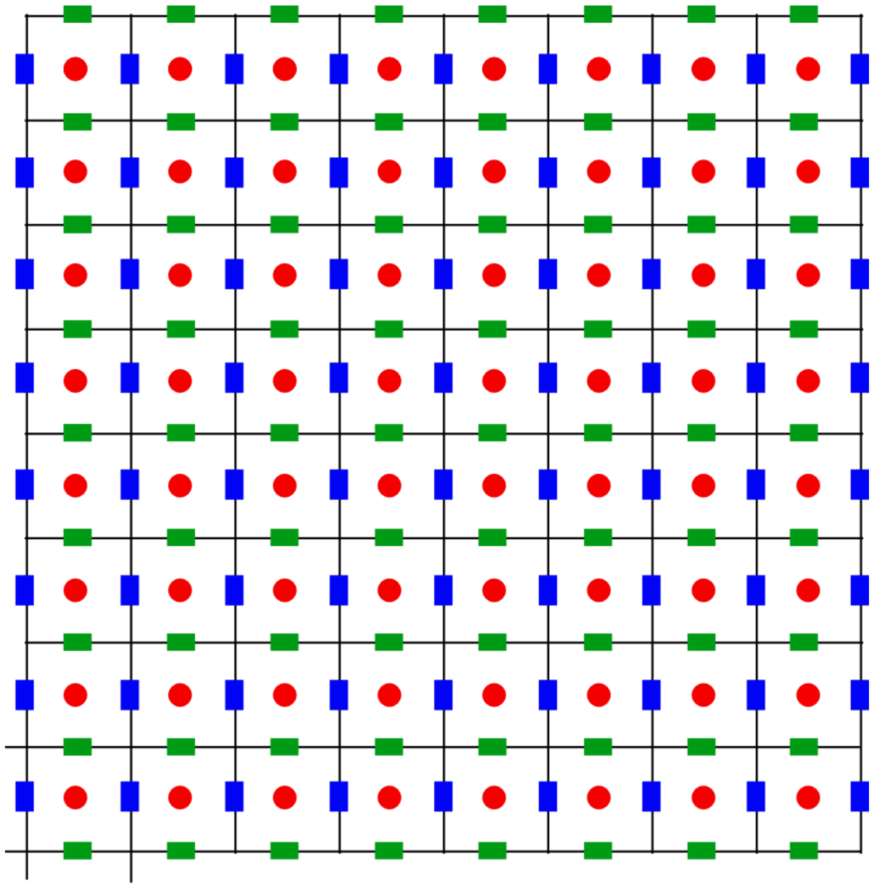
$$\frac{h_{x,y}^{t+1} - h_{x,y}^{t}}{\Delta t} = -H \left(\frac{u_{x+\frac{1}{2},y}^{t} - u_{x-\frac{1}{2},y}^{t}}{\Delta x} + \frac{v_{x,y+\frac{1}{2}}^{t} - v_{x,y-\frac{1}{2}}^{t}}{\Delta t} \right)$$

$$\frac{u_{x+\frac{1}{2},y}^{t+1} - u_{x+\frac{1}{2},y}^{t}}{\Delta t} = f v_{x+\frac{1}{2},y}^{t} - g \frac{h_{x+1,y}^{t} - h_{x,y}^{t}}{\Delta x}$$

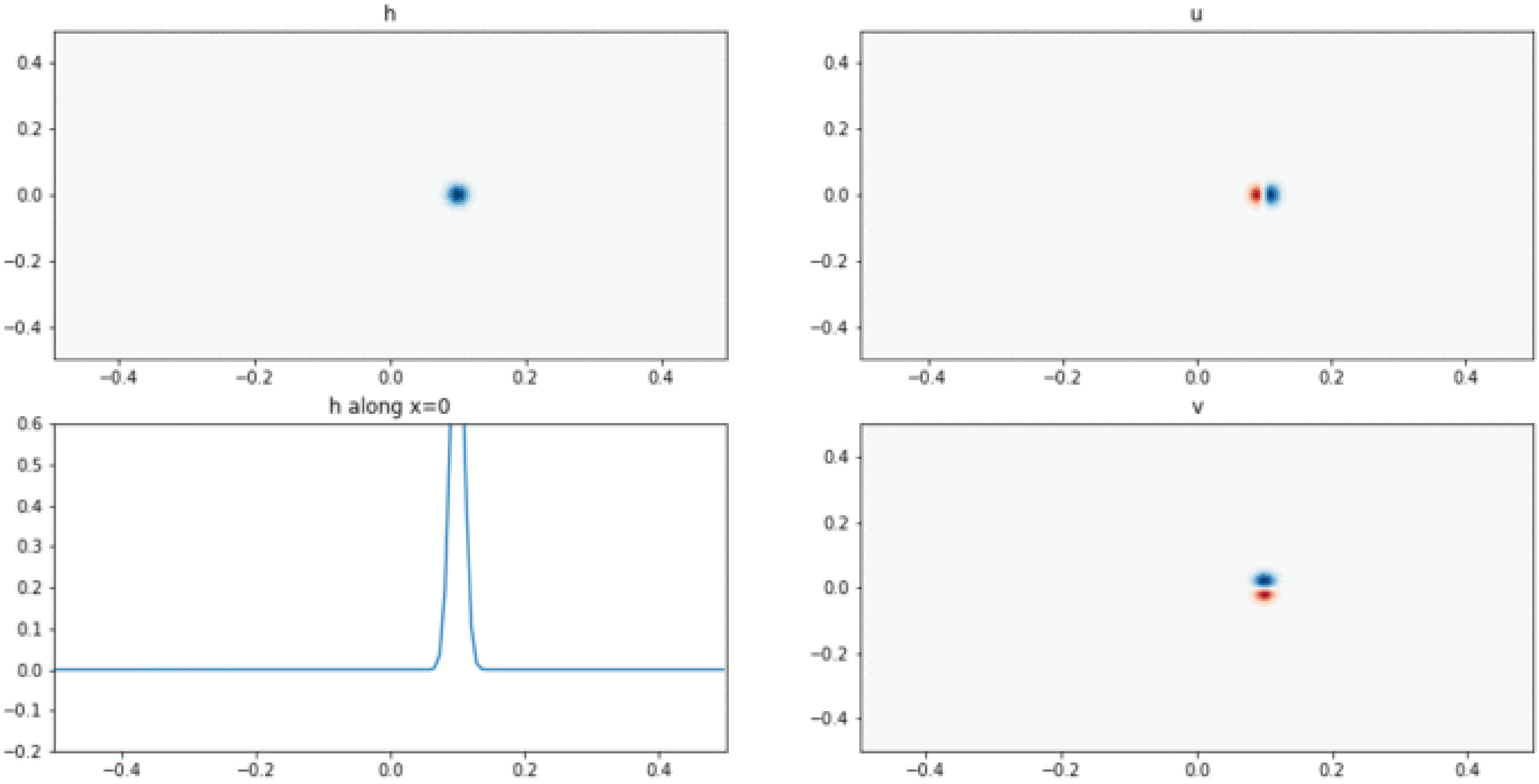
$$\frac{v_{x,y+\frac{1}{2}}^{t+1} - v_{x,y+\frac{1}{2}}^{t}}{\Delta t} = -f u_{x,y+\frac{1}{2}}^{t} - g \frac{h_{x,y+1}^{t} - h_{x,y}^{t}}{\Delta y}$$

Setting Up Grids

- Grids are created with ghost zones to set boundary conditions (in this case, boundary is zero)
- Velocity grids are used on the boundary, so they are actually rectangles
- The time derivatives depend of velocities depend on other velocity, so average is used



Results



Hydrodynamics

- Hydrodynamics are also subject to conservation equations – of mass and momentum specifically
- Hydrodynamics problems may be subject to high frequency components in shocks and sudden discontinuities – staggered grids are therefore preferred
- The code provided in class uses a set of three grid systems: one for scalars, and two for vectors (one for each space dimension).

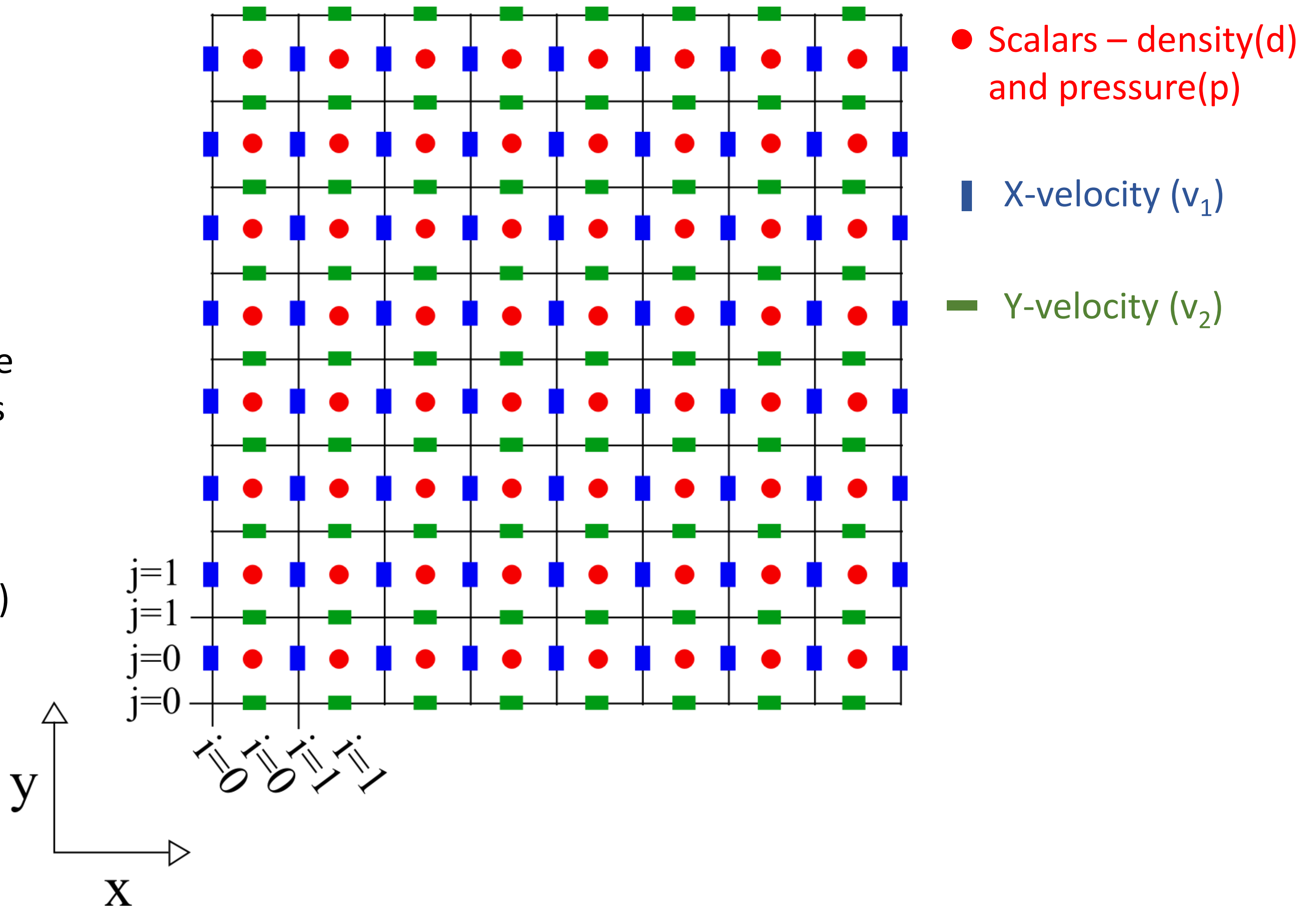
Hydrodynamics grid

- Indices for vectors run from 0 to N
- Indices for scalars run from 0 to N-1

- Example:

v1(i,j) is the x-velocity on the face between density values d(i,j) and d(i-1,j)

d(i,j) is the density between velocities v2(i,j+1) and v2(i,j)



Navier-Stokes equations

- Conservation of mass (Continuity):

$$\frac{d\rho}{dt} = -\rho\nabla \cdot \mathbf{v}$$

- Conservation of momentum (Euler's Equation):

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P$$

Navier-Stokes equations - FD

- Conservation of mass (Continuity):

$$\frac{\rho^{l+1}_{m,n} - \rho^{l}_{m,n}}{\Delta t} = -\rho^{l}_{m,n}\left(\frac{v^{l}_{1,m+\frac{1}{2},n} - v_{1,m-\frac{1}{2},n}}{\Delta x} + \frac{v^{l}_{2,m,n+\frac{1}{2}} - v^{l}_{2,m,n-\frac{1}{2}}}{\Delta y}\right)$$

- Conservation of momentum (Euler's Equation):

$$\frac{v^{l+1}_{1,m+\frac{1}{2},n} - v^{l}_{1,m+\frac{1}{2},n}}{\Delta t} = -\frac{1}{\rho_{m+\frac{1}{2},n}} \frac{P^{l}_{m+1,n} - P^{l}_{m,n}}{\Delta x}$$

Uh-oh

$$\frac{v^{l+1}_{2,m,n+\frac{1}{2}} - v^{l}_{2,m,n+\frac{1}{2}}}{\Delta t} = -\frac{1}{\rho_{m,n+\frac{1}{2}}} \frac{P^{l}_{m,n+1} - P^{l}_{m,n}}{\Delta y}$$

Mixed-Grid Problem

- So far in our general conservation equations and E&M equations, our different variables have always been related through a derivative
- Where our FD equation for d**v** depends on ρ and not dρ, we need a value of ρ at a location off its grid.
- We could subdivide the ρ grid, but at that point we re-introduce our original frequency problem. We can't stagger more, as that would require ad-infinitum staggering.
- We can use interpolation in some situations: this won't work when there are shocks or discontinuities in the data

Staggered vs. Co-Local

- While staggered grids are a common hydrodynamic method, some programs choose to use co-local. Below is a comparison of each.

Co-Local Grids	Staggered Grids
Intuitive Indices	Careful tracking of indices required: 3D spaces are particularly easy to mis-label
Evaluating Flux at the middle of a cell is unideal: we would prefer to evaluate it on a surface	Intuitive volumetric structure: flux occurs on cell walls
Odd-even decoupling may occur	Solves odd-even decoupling
Co-local grids make solving conservation of momentum easy	Staggered grids require some estimate of density at cell walls – may be very difficult

Summary

- Staggered Grids are commonly used when modelling 1st order differential equations when using finite differences
- Staggered Grids can overcome the checkerboard problem, and prevent the characteristic odd-even decoupling
- It is crucial that the relationships between the indices of different grids are understood by the coder
- Staggered grids can be used to solve the Navier-Stokes equations