BH-torus initial data

I. INTORDUCTION

There are two versions of the code: a standard one, which uses LAPACK and an improved version which calls PARDISO routines. The first one is portable and easy to compile, but offers limited grid sizes. The second one allows for much larger grids, but needs PARDISO, which only works on INTEL architecture (as far as I remember; it is a part of the MKL library).

The treatment of Einstein equations is based on [1]. Examples with magentized disks are given in [2, 3].

II. METRIC

We work in spherical coordinates (t, r, θ, φ). The metric (stationary, axially symmetric and circular) is assumed in the general form

$$g = -\alpha^2 dt^2 + \psi^4 e^{2q}(dr^2 + r^2 d\theta^2) + \psi^4 r^2 \sin^2\theta(\beta dt + d\varphi)^2, \tag{1}$$

where α, ψ, q, and β depend on r and θ.

In the puncture formalism, we use the following re-definitions:

$$\psi = \left(1 + \frac{r_\mathrm{s}}{r}\right) e^{\phi}, \tag{2}$$

$$\alpha\psi = \left(1 - \frac{r_\mathrm{s}}{r}\right) e^{-\phi} B. \tag{3}$$

$$\beta = \beta_\mathrm{K} + \beta_\mathrm{T}, \tag{4}$$

where

$$r_\mathrm{s} \equiv \frac{1}{2}\sqrt{m^2 - a^2}. \tag{5}$$

Here m and a are *parameters* that effectively specify the mass and the angular momentum of the black hole. Note that the lapse α reads

$$\alpha = \frac{1 - \frac{r_\mathrm{s}}{r}}{1 + \frac{r_\mathrm{s}}{r}} e^{-2\phi} B.$$

Writing the metrc (1) in the standard 3+1 form

$$g = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j, \tag{6}$$

we get

$$(\beta^r, \beta^\theta, \beta^\varphi) = (0, 0, \beta), \tag{7}$$

$$(\beta_r, \beta_\theta, \beta_\varphi) = (0, 0, \psi^4 r^2 \sin^2\theta\beta), \tag{8}$$

$$\gamma_{rr} = \psi^4 e^{2q}, \tag{9}$$

$$\gamma_{\theta\theta} = \psi^4 e^{2q} r^2, \tag{10}$$

$$\gamma_{\varphi\varphi} = \psi^4 r^2 \sin^2\theta. \tag{11}$$

The only non-vanishing components of the extrinsic curvature are

$$K_{r\varphi} = K_{\varphi r} = \frac{H_\mathrm{E} \sin^2\theta}{\psi^2 r^2} + \frac{1}{2\alpha}\psi^4 r^2 \sin^2\theta \partial_r \beta_\mathrm{T}, \tag{12}$$

$$K_{\theta\varphi} = K_{\varphi\theta} = \frac{H_\mathrm{F} \sin\theta}{\psi^2 r} + \frac{1}{2\alpha}\psi^4 r^2 \sin^2\theta \partial_\theta \beta_\mathrm{T}, \tag{13}$$

where $H_{\rm E}$ and $H_{\rm F}$ are defined as

$$H_{\rm E} = \frac{ma\left[(r_{\rm K}^2 - a^2)\Sigma_{\rm K} + 2r_{\rm K}^2(r_{\rm K}^2 + a^2)\right]}{\Sigma_{\rm K}^2}, \tag{14}$$

$$H_{\rm F} = -\frac{2ma^3 r_{\rm K}\sqrt{\Delta_{\rm K}}\cos\theta\sin^2\theta}{\Sigma_{\rm K}^2}. \tag{15}$$

Here $r_{\rm K}$, $\Sigma_{\rm K}$, and $\Delta_{\rm K}$ refer to standard expressions appearing in the Kerr metric:

$$r_{\rm K} = r\left(1 + \frac{m}{r} + \frac{m^2 - a^2}{4r^2}\right), \tag{16}$$

$$\Delta_{\rm K} = r_{\rm K}^2 - 2r_{\rm K} + a^2, \tag{17}$$

$$\Sigma_{\rm K} = r_{\rm K}^2 + a^2\cos^2\theta. \tag{18}$$

Note also that $\gamma^{ik}K_{ik} = 0$ (we are working in a the maximal slicing).

In the code we keep the following metric components: ϕ, q, B, $\beta_{\rm T}$, and $\beta_{\rm K}$.

If we need to go beneath the apparent horizon at $r = r_{\rm s}$, I would use the following symmetry described in [1]: Consider the transformation

$$\bar{r} = \frac{r_{\rm s}^2}{r} \tag{19}$$

(Eq. (49) in [1]). Then,

$$\phi(r) = \phi(\bar{r}), \tag{20}$$

$$B(r) = B(\bar{r}), \tag{21}$$

$$\beta_{\rm T}(r) = \beta_{\rm T}(\bar{r}), \tag{22}$$

$$\beta_{\rm K}(r) = \beta_{\rm K}(\bar{r}), \tag{23}$$

$$q(r) = q(\bar{r}). \tag{24}$$

Shibata also lists (see below for the fluid quantities):

$$\rho(r) = \rho(\bar{r}), \tag{25}$$

$$u_\varphi(r) = -u_\varphi(\bar{r}), \tag{26}$$

$$u^t(r) = -u^t(\bar{r}), \tag{27}$$

$$\alpha(r) = -\alpha(\bar{r}). \tag{28}$$

III. FLUID

The total energy-momentum tensor reads

$$T_{\mu\nu} = (\rho h + b^2)u_\mu u_\nu + \left(p + \frac{1}{2}b^2\right)g_{\mu\nu} - b_\mu b_\nu. \tag{29}$$

Here ρ is the rest-mass density, h denotes the specific enthalpy, u^μ are the components of the four-velocity of the fluid, and b^μ is the four-vector of the magnetic field. The fluid obeys the polytropic relation $p = K\rho^\Gamma$. Realistic equations of state were studied in [4] (but for non-magnetized fluids).

We keep $\Omega = u^\varphi/u^t$. The components u^t and u^φ, if needed, can be obtained as

$$u^t = \frac{1}{\sqrt{\alpha^2 - \psi^4 r^2\sin^2\theta(\Omega + \beta)^2}}, \tag{30}$$

$$u^\varphi = \Omega u^t. \tag{31}$$

We have $u^r = u^\theta = 0$.

Information about the magnetic field is contained in $b^2 = b_\mu b^\mu$. As with the four-velocity, $b^r = b^\theta = 0$. The remaining components can be obtained from

$$b_\varphi^2 = (u^t)^2\psi^4 r^2\sin^2\theta\alpha^2 b^2, \tag{32}$$

$$b_t = -\Omega b_\varphi. \tag{33}$$

IV. NUMERICAL GRID

The numerical grid spans over a finite spatial region $r_\mathrm{s} \le r \le r_\infty$, $0 \le \theta \le \pi/2$, where r_∞ is large, but finite. The grid is not equidistant. It is denser near the black hole horizon and also near the equatorial plane. The part of the code responsible for the construction of the grid is almost self-explanatory. In the polar direction, the grid is equidistant in $\mu = \cos\theta$.

```
! Grid resolution:

nr = 800
nt = 200

nrr = nr + 2
ntt = nt + 2
nra = 2
nrb = nr + 1
nta = 2
ntb = nt + 1

rs = sqrt(kerrm**2 - kerra**2)/2.0d0
pi = acos(-1.0d0)

rin = rs
dr  = rin/50.0d0
fac = 1.01d0

r(1) = rin

do i = 2, nrr
   r(i) = r(1) + (fac**(i-1) - 1.0d0)*dr/(fac - 1.0d0)
end do

do i = 1, nrr
write(*,*) i, r(i)
end do

write(*,*) 'inner boundary:        ', rs
write(*,*) 'outer boundary:        ', r(nrr)
write(*,*) 'outer boundary:        ', r(nrr)/rs, ' rs.'
write(*,*)
write(*,*) '------------------------------------'

dt1 = 0.5d0*pi/real(nt+1,kind(1.0d0))
dmu = 1.0d0/real(nt,kind(1.0d0))

do j = nta, ntb
   mu(j) = 1.0d0 + 0.5d0*dmu - real(j-1,kind(1.0d0))*dmu
end do

! Two types of grid in the angular direction

do j = nta, ntb
   t(j) = acos(mu(j))
!    t(j) = real(j-1,kind(1.0d0))*dt1

  if (t(j) .lt. tcut) then
    jcut = j
```

```
  end if

end do

t(nta-1) = 0.0d0
t(ntb+1) = 0.5d0*pi
```

V. OUTPUT

The output form can be easily adjusted to the preferred ASCII format. This is a sample output of ϕ, q and B. In the next file we store ρ and h.

```
open (110,file="phi_q_B.dat",form="formatted")
do i = nra-1, nrb+1
do j = nta-1, ntb+1
  write(110,*) r(i)*sin(t(j)), r(i)*cos(t(j)), phi(i,j), q(i,j), capitalb(i,j)
end do
  write(110,*)
end do
close(110)

open (112,file="rho_h.dat",form="formatted")
do i = nra-1, nrb+1
do j = nta-1, ntb+1
  write(112,*) r(i)*sin(t(j)), r(i)*cos(t(j)), rho(i,j), h(i,j)
end do
  write(112,*)
end do
close(112)
```

This form can easily be plotted with GNUPLOT. Sample GNUPLOT commands for plotting a disk picture are as follows (assuming the outer disk radius is set at $r = 35$):

```
set pm3d map
set size ratio 1.0
splot [0:35][0:35] 'rho_h.dat' u 1:2:4
```

VI. COMPILING AND RUNNING THE CODE

For simplicity the code is kept in one FORTRAN file `sho100.f95`. With LAPACK installed, I compile and run the code using GCC with the following command:

```
gfortran -fopenmp -O2 -llapack sho100.f95 -o sho100
sho100
```

The code runs for a fixed number of iterations (irrespectively of the achieved level of convergence), produces some output files and a restart file (RESTART.dat).

If no seed spacetime is given, one starts from the Kerr metric as a seed geometry. This is slightly tricky, due to the assumed (Keplerian) rotation law, i.e.,

$$j(\Omega) = -\frac{1}{2}\frac{d}{d\Omega}\ln\left\{1 - \left[a^2\Omega^2 + 3w^{\frac{4}{3}}\Omega^{\frac{2}{3}}(1-a\Omega)^{\frac{4}{3}}\right]\right\}, \tag{34}$$

where $j = u^t u_\varphi$. In principle, the parameter a in Eq. (34) should correspond to the assumed spin parameter of the black hole.

Our rotation law has the property that low-mass disks are geometrically thin. This property does not allow to start from Kerr metric exactly. A simple trick is to start the calculations assuming a different black hole spin value than the one assumed in Eq. (34). After some number of iterations, one can restart the calculations setting all spin parameters to the desired value.

In the attached example we build a disk model with the following parameters: $m = 1$, $a = 0$, $\Gamma = 4/3$, $r_1 = 8$, $r_2 = 35$ (inner an outer coordinate equatorial radii of the disk), $\rho_{\max} = 5 \times 10^{-5}$ (maximum of the rest-mass density within the disk; this is called `rho0` in the code), $n = 1$, $C_1 = 0.01$ (parameters controlling the distribution of the magnetic field; they are referred to as `nn` and `c2` in the code).

We start with 100 iterations, setting

```
kerrm   = 1.0d0
kerra   = 0.1d0*kerrm
restart = 0
niter   = 100
```

In the next step, the code is run for 2000 toiterations (say) starting from the restart file, with the parameters

```
kerrm   = 1.0d0
kerra   = 0.0d0*kerrm
restart = 1
niter   = 2000
```

VII. ELECTROMAGNETIC POTENTIAL

Working within the framework of ideal-magnetohydrodynamics, we assumed the dual of the electromagnetic tensor in the form

$$*F^{\mu\nu} = b^\mu u^\nu - b^\nu u^\mu. \tag{35}$$

The Hodge dual $*F_{\mu\nu}$ of a two-form $F_{\mu\nu}$ is defined as

$$*F_{\alpha\beta} = \frac{1}{2} F_{\mu\nu} \epsilon^{\mu\nu}{}_{\alpha\beta}, \tag{36}$$

where $\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-\det(g_{\mu\nu})}[\alpha,\beta,\gamma,\delta]$ and $[\alpha,\beta,\gamma,\delta]$ stands for 1 (respectively -1) if $(\alpha,\beta,\gamma,\delta)$ is an even (respectively odd) permutation of (0, 1, 2, 3), and 0 otherwise. Note that for a 2-form one has

$$**F_{\mu\nu} = -F_{\mu\nu}. \tag{37}$$

Hence,

$$\begin{aligned} F_{\mu\nu} &= -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu} * F^{\alpha\beta} = -\frac{1}{2}\sqrt{-\det(g_{\gamma\delta})}[\alpha,\beta,\mu,\nu] * F^{\alpha\beta} = -\frac{1}{2}\sqrt{-\det(g_{\gamma\delta})}[\alpha,\beta,\mu,\nu](b^\alpha u^\beta - b^\beta u^\alpha) \\ &= -\sqrt{-\det(g_{\gamma\delta})}[\alpha,\beta,\mu,\nu] b^\alpha u^\beta. \end{aligned}$$

The only non-zero components of b^μ and u^μ are b^t, b^φ, u^t, u^φ. Thus, the only non-vanishing components of $F_{\mu\nu}$ are $F_{r\theta} = -F_{\theta r}$. We have

$$F_{r\theta} = -F_{\theta r} = -\sqrt{-\det(g_{\gamma\delta})}(b^t u^\varphi - b^\varphi u^t). \tag{38}$$

The expression $b^t u^\varphi - b^\varphi u^t$ can be related to $b^2 = b_\mu b^\mu$ as follows. One has

$$b_t = g_{tt} b^t + g_{t\varphi} b^\varphi, \quad b_\varphi = g_{t\varphi} b^t + g_{\varphi\varphi} b^\varphi. \tag{39}$$

Solving these equations with respect to b^t and b^φ we get

$$b^t = \frac{-g_{\varphi\varphi} b_t + g_{t\varphi} b_\varphi}{\mathcal{L}}, \quad b^\varphi = \frac{g_{t\varphi} b_t - g_{tt} b_\varphi}{\mathcal{L}}, \tag{40}$$

where $\mathcal{L} = g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2$. From the condition $b_\mu u^\mu = 0$ we have

$$b_t = -\Omega b_\varphi, \tag{41}$$

where $\Omega = u^\varphi/u^t$. This gives

$$b^t u^\varphi - b^\varphi u^t = -\frac{b_\varphi u^t \left(g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2\right)}{\mathcal{L}} = \frac{b_\varphi}{\mathcal{L} u^t}, \tag{42}$$

where in the last step we have used the fact that

$$g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2 = -\frac{1}{(u^t)^2} \tag{43}$$

(this follows simply from the normalization of the four-velocity $u_\mu u^\mu = -1$). In the last step we note that the definition $b^2 = b_\mu b^\mu$ implies that

$$b_\varphi^2 = -(u^t)^2 \mathcal{L} b^2. \tag{44}$$

Hence,

$$b^t u^\varphi - b^\varphi u^t = -\mathrm{sgn}(b_\varphi)\frac{|b|}{\sqrt{-\mathcal{L}}} \tag{45}$$

(I should think about the sign of b_φ here). The determinant of the metric reads $\det(g_{\mu\nu}) = g_{\theta\theta} g_{rr} \mathcal{L}$. Hence, finally

$$F_{r\theta} = -F_{\theta r} = \mathrm{sgn}(b_\varphi)\sqrt{g_{rr} g_{\theta\theta}}|b| = \mathrm{sgn}(b_\varphi)\psi^4 e^{2q} r|b|. \tag{46}$$

The electromagnetic vector potential A_μ is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This means that

$$\partial_r A_\theta - \partial_\theta A_r = \mathrm{sgn}(b_\varphi)\psi^4 e^{2q} r|b|. \tag{47}$$

Is there any particular gauge that we should use? If not, one can, for instance, assume that $A_r \equiv 0$, and integrate (numerically) the resulting equation for A_θ. Note that the gauge freedom allows to add to the result a gradient of any smooth function Ψ, i.e., $\tilde{A}_r = \partial_r \Psi$, $\tilde{A}_\theta = \partial_\theta \Psi$, since $\partial_r \tilde{A}_\theta - \partial_\theta \tilde{A}_r = 0$.

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[1] M. Shibata, Rotating black hole surrounded by self-gravitating torus in the puncture framework, Phys. Rev. D 76, 064035 (2007).

[2] P. Mach, S. Gimeno-Soler, J. A. Font, A. Odrzywołek, Michał Piróg, Self-gravitating magnetized tori around black holes in general relativity, Phys. Rev. D 99, 104063 (2019).

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[4] W. Kulczycki, P. Mach, and E. Malec, Stationary massive disks around black holes: Realistic equation of state and bifurcation, Phys. Rev. D 104, 024005 (2021).