If the metric is $\gamma_{\hat{a}\hat{b}}$ with hatted indices being spherical coordinates and we want γ_{ab} with unhatted indices being Cartesian coordinates, then we need to perform a transformation using the Jacobian matrix J:

$$\begin{aligned}\gamma_{ab} &= \frac{\partial x^{\hat{a}}}{\partial x^a}\frac{\partial x^{\hat{b}}}{\partial x^b}\gamma_{\hat{a}\hat{b}} \\ &= \frac{\partial x^{\hat{a}}}{\partial x^a}\gamma_{\hat{a}\hat{b}}\frac{\partial x^{\hat{b}}}{\partial x^b} \\ &= J^{x^{\hat{a}}}_{x^a}\gamma_{\hat{a}\hat{b}}J^{x^{\hat{b}}}_{x^b} \\ &= (J^T\gamma)_{a\hat{b}}J^{x^{\hat{b}}}_{x^b} \\ &= (J^T\gamma J)_{ab}\end{aligned} \tag{1}$$

This can be expressed as a matrix multiplication:

$$\gamma_{ab} = J^T\gamma J = \sum_{k=1}^{3}\sum_{l=1}^{3} J^T_{ik}\gamma_{kl}J_{lj} = \sum_{k=1}^{3}\sum_{l=1}^{3} J_{ki}\gamma_{kl}J_{lj} \tag{2}$$

where $\frac{\partial x^{\hat{a}}}{\partial x^a}$ are the components of the Jacobian matrix J. Specifically, the Jacobian matrix elements are the partial derivatives of the spherical coordinates with respect to the Cartesian coordinates:

$$J^{\hat{a}}_{a} = \frac{\partial x^{\hat{a}}}{\partial x^a} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix}$$

The elements of the Jacobian matrix J are:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r}, & \frac{\partial r}{\partial y} &= \frac{y}{r}, & \frac{\partial r}{\partial z} &= \frac{z}{r}, \\ \frac{\partial \theta}{\partial x} &= \frac{xz}{r^2\sqrt{x^2+y^2}}, & \frac{\partial \theta}{\partial y} &= \frac{yz}{r^2\sqrt{x^2+y^2}}, & \frac{\partial \theta}{\partial z} &= \frac{\sqrt{x^2+y^2}}{r^2}, \\ \frac{\partial \phi}{\partial x} &= \frac{-y}{x^2+y^2}, & \frac{\partial \phi}{\partial y} &= \frac{x}{x^2+y^2}, & \frac{\partial \phi}{\partial z} &= 0. \end{aligned} \tag{3}$$

The Jacobian matrix J is:

$$J = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{\cos(\phi)z}{r^2} & \frac{\sin(\phi)z}{r^2} & -\frac{\sqrt{x^2+y^2}}{r^2} \\ -\frac{\sin(\phi)}{r\sin(\theta)} & \frac{\cos(\phi)}{r\sin(\theta)} & 0 \end{pmatrix} \tag{4}$$

The metric tensor $\gamma_{\hat{a}\hat{b}}$ in spherical coordinates is given by:

$$\gamma = \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix} \tag{5}$$

where:

$$\gamma_{11} = \psi^4 \exp(2q), \quad \gamma_{22} = \psi^4 \exp(2q) r^2, \quad \gamma_{33} = \psi^4 r^2 \sin^2(\theta) \tag{6}$$

The expression $J^T \gamma J$ in indices form is:

$$\begin{aligned}
(J^T\gamma J)_{11} &= J_{11}\gamma_{11}J_{11} + J_{21}\gamma_{22}J_{21} + J_{31}\gamma_{33}J_{31} \\
(J^T\gamma J)_{12} &= J_{11}\gamma_{11}J_{12} + J_{21}\gamma_{22}J_{22} + J_{31}\gamma_{33}J_{32} \\
(J^T\gamma J)_{13} &= J_{11}\gamma_{11}J_{13} + J_{21}\gamma_{22}J_{23} + J_{31}\gamma_{33}J_{33} \\
(J^T\gamma J)_{22} &= J_{12}\gamma_{11}J_{12} + J_{22}\gamma_{22}J_{22} + J_{32}\gamma_{33}J_{32} \\
(J^T\gamma J)_{23} &= J_{12}\gamma_{11}J_{13} + J_{22}\gamma_{22}J_{23} + J_{32}\gamma_{33}J_{33} \\
(J^T\gamma J)_{33} &= J_{13}\gamma_{11}J_{13} + J_{23}\gamma_{22}J_{23} + J_{33}\gamma_{33}
\end{aligned} \tag{7}$$

Finally, the resulting elements of the matrix multiplication, written element-wise, are:

$$\gamma_{xx} = \frac{e^{2q}\psi^4 x^2}{r^2} + \frac{e^{2q}\psi^4 z^2 \cos^2(\phi)}{r^2} + \psi^4 \sin^2(\phi) \tag{8}$$

$$\gamma_{xy} = \frac{e^{2q}\psi^4 xy}{r^2} - \psi^4 \cos(\phi)\sin(\phi) + \frac{e^{2q}\psi^4 z^2 \cos(\phi)\sin(\phi)}{r^2} \tag{9}$$

$$\gamma_{xz} = \frac{e^{2q}\psi^4 xz}{r^2} - \frac{e^{2q}\psi^4 z\sqrt{x^2+y^2}\cos(\phi)}{r^2} \tag{10}$$

$$\gamma_{yy} = \frac{e^{2q}\psi^4 y^2}{r^2} + \psi^4 \cos^2(\phi) + \frac{e^{2q}\psi^4 z^2 \sin^2(\phi)}{r^2} \tag{11}$$

$$\gamma_{yz} = \frac{e^{2q}\psi^4 yz}{r^2} - \frac{e^{2q}\psi^4 z\sqrt{x^2+y^2}\sin(\phi)}{r^2} \tag{12}$$

$$\gamma_{zz} = \frac{e^{2q}\psi^4 (x^2+y^2)}{r^2} + \frac{e^{2q}\psi^4 z^2}{r^2} \tag{13}$$

The K matrix is:

$$K = \begin{pmatrix} 0 & 0 & K_{r\varphi} \\ 0 & 0 & K_{\theta\varphi} \\ K_{r\varphi} & K_{\theta\varphi} & 0 \end{pmatrix} \tag{14}$$

The expressions for $K_{r\varphi}$ and $K_{\theta\varphi}$ on the positive z-axis are:

$$K_{r\varphi} = K_{\varphi r} = \frac{H_E \sin^2\theta}{\psi^2 r^2} + \frac{1}{2\alpha}\psi^4 r^2 \sin^2\theta \partial_r \beta_T \tag{15}$$

$$K_{\theta\varphi} = K_{\varphi\theta} = \frac{H_F \sin\theta}{\psi^2 r} + \frac{1}{2\alpha}\psi^4 r^2 \sin^2\theta \partial_\theta \beta_T \tag{16}$$

The expression $J^T K J$ in indices form is:

$$(J^T K J)_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} J_{ik} K_{kl} J_{lj} \tag{17}$$

The final outputs, element-wise, are:

$$(J^T K J)_{11} = J_{11} J_{31} K_{13} + J_{21} J_{31} K_{23} + J_{31}(J_{11} K_{13} + J_{21} K_{23}) \tag{18}$$

$$(J^T K J)_{12} = J_{12} J_{31} K_{13} + J_{22} J_{31} K_{23} + J_{32}(J_{11} K_{13} + J_{21} K_{23}) \tag{19}$$

$$(J^T K J)_{13} = J_{13} J_{31} K_{13} + J_{23} J_{31} K_{23} + J_{33}(J_{11} K_{13} + J_{21} K_{23}) \tag{20}$$

$$(J^T K J)_{21} = J_{11} J_{32} K_{13} + J_{21} J_{32} K_{23} + J_{31}(J_{12} K_{13} + J_{22} K_{23}) \tag{21}$$

$$(J^T K J)_{22} = J_{12} J_{32} K_{13} + J_{22} J_{32} K_{23} + J_{32}(J_{12} K_{13} + J_{22} K_{23}) \tag{22}$$

$$(J^T K J)_{23} = J_{13} J_{32} K_{13} + J_{23} J_{32} K_{23} + J_{33}(J_{12} K_{13} + J_{22} K_{23}) \tag{23}$$

$$(J^T K J)_{31} = J_{11} J_{33} K_{13} + J_{21} J_{33} K_{23} + J_{31}(J_{13} K_{13} + J_{23} K_{23}) \tag{24}$$

$$(J^T K J)_{32} = J_{12} J_{33} K_{13} + J_{22} J_{33} K_{23} + J_{32}(J_{13} K_{13} + J_{23} K_{23}) \tag{25}$$

$$(J^T K J)_{33} = J_{13} J_{33} K_{13} + J_{23} J_{33} K_{23} + J_{33}(J_{13} K_{13} + J_{23} K_{23}) \tag{26}$$

The final outputs on the positive z-axis are:

$$\begin{aligned}
K_{xx} &= -\frac{((2\alpha H_E + \partial_r \beta_T \psi^6 r^4)x + z\cos(\phi)(\partial_\theta \beta_T \psi^6 r^3 + 2\alpha H_F \csc(\theta)))\sin(\phi)\sin(\theta)}{\alpha \psi^2 r^4} \\
K_{xy} &= \frac{(2\alpha H_E + \partial_r \beta_T \psi^6 r^4)(x\cos(\phi) - y\sin(\phi))\sin(\theta) + z\cos(2\phi)(2\alpha H_F + \partial_\theta \beta_T \psi^6 r^3 \sin(\theta))}{2\alpha \psi^2 r^4} \\
K_{xz} &= \frac{(-2\alpha H_E z + \psi^6 r^3(\partial_\theta \beta_T \sqrt{x^2+y^2} - \partial_r \beta_T r z) + 2\alpha H_F \sqrt{x^2+y^2}\csc(\theta))\sin(\phi)\sin(\theta)}{2\alpha \psi^2 r^4} \\
K_{yy} &= \frac{\cos(\phi)((2\alpha H_E + \partial_r \beta_T \psi^6 r^4)y + z(\partial_\theta \beta_T \psi^6 r^3 + 2\alpha H_F \csc(\theta))\sin(\phi))\sin(\theta)}{\alpha \psi^2 r^4} \\
K_{yz} &= -\frac{\cos(\phi)(-2\alpha H_E z + \psi^6 r^3(\partial_\theta \beta_T \sqrt{x^2+y^2} - \partial_r \beta_T r z) + 2\alpha H_F \sqrt{x^2+y^2}\csc(\theta))\sin(\theta)}{2\alpha \psi^2 r^4} \\
K_{zz} &= 0
\end{aligned} \tag{27}$$

The expressions for $K_{r\varphi}$ and $K_{\theta\varphi}$ on the negative z-axis are:

$$K_{r\varphi} = K_{\varphi r} = \frac{H_E \sin^2\theta}{\psi^2 r^2} + \frac{1}{2\alpha}\psi^4 r^2 \sin^2\theta \partial_r \beta_T \tag{28}$$

$$K_{\theta\varphi} = K_{\varphi\theta} = \frac{H_F \sin\theta}{\psi^2 r} - \frac{1}{2\alpha}\psi^4 r^2 \sin^2\theta \partial_\theta \beta_T \tag{29}$$

The elements of the extrinsic curvature matrix K on the negative z-axis are:

$$\begin{aligned}
K_{xx} &= -\frac{((2\alpha H_E + \partial_r\beta_T\psi^6 r^4)x + z\cos(\phi)(-\partial_\theta\beta_T\psi^6 r^3 + 2\alpha H_F\csc(\theta)))\sin(\phi)\sin(\theta)}{\alpha\psi^2 r^4} \\
K_{xy} &= \frac{(2\alpha H_E + \partial_r\beta_T\psi^6 r^4)(x\cos(\phi) - y\sin(\phi))\sin(\theta) + z\cos(2\phi)(2\alpha H_F - \partial_\theta\beta_T\psi^6 r^3\sin(\theta))}{2\alpha\psi^2 r^4} \\
K_{xz} &= -\frac{\sin(\phi)(-2\alpha H_F\sqrt{x^2+y^2} + (2\alpha H_E z + \psi^6 r^3(\partial_\theta\beta_T\sqrt{x^2+y^2} + \partial_r\beta_T r z))\sin(\theta))}{2\alpha\psi^2 r^4} \\
K_{yy} &= \frac{\cos(\phi)((2\alpha H_E + \partial_r\beta_T\psi^6 r^4)y + z(-\partial_\theta\beta_T\psi^6 r^3 + 2\alpha H_F\csc(\theta))\sin(\phi))\sin(\theta)}{\alpha\psi^2 r^4} \\
K_{yz} &= \frac{\cos(\phi)(-2\alpha H_F\sqrt{x^2+y^2} + (2\alpha H_E z + \psi^6 r^3(\partial_\theta\beta_T\sqrt{x^2+y^2} + \partial_r\beta_T r z))\sin(\theta))}{2\alpha\psi^2 r^4} \\
K_{zz} &= 0
\end{aligned} \tag{30}$$

The expressions for $K_{r\varphi}$ and $K_{\theta\varphi}$ on the horizon are:

$$K_{r\varphi} = K_{\varphi r} = \frac{H_E \sin^2\theta}{\psi^2 r^2} \tag{31}$$

$$K_{\theta\varphi} = K_{\varphi\theta} = \frac{H_F \sin\theta}{\psi^2 r} \tag{32}$$

The elements of the extrinsic curvature matrix K on the horizon are:

$$\begin{aligned}
K_{xx} &= -\frac{2\sin(\phi)(H_F z\cos(\phi) + H_E x\sin(\theta))}{\psi^2 r^4} \\
K_{xy} &= \frac{H_F z\cos(2\phi) + H_E(x\cos(\phi) - y\sin(\phi))\sin(\theta)}{\psi^2 r^4} \\
K_{xz} &= \frac{\sin(\phi)(H_F\sqrt{x^2+y^2} - H_E z\sin(\theta))}{\psi^2 r^4} \\
K_{yy} &= \frac{2\cos(\phi)(H_F z\sin(\phi) + H_E y\sin(\theta))}{\psi^2 r^4} \\
K_{yz} &= \frac{\cos(\phi)(-H_F\sqrt{x^2+y^2} + H_E z\sin(\theta))}{\psi^2 r^4} \\
K_{zz} &= 0
\end{aligned} \tag{33}$$